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Invariant Theory
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R. V. GAMKRELIDZE
V. L. POPOV
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Algebraic Theory of Locally Nilpotent Derivations

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To my wife Sheryl
and our wonderful children,
Jenna, Kathryn, and Ella Marie,
whom I love very much.

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Evansville, Indiana *Gene Freudenburg*
May 2006

Contents

Introduction

But, in the further development of a branch of mathematics, the human mind, encouraged by the success of its solutions, becomes conscious of its independence. It evolves from itself alone, often without appreciable influence from without, by means of logical combination, generalization, specialization, by separating and collecting ideas in fortunate new ways, new and fruitful problems, and appears then itself as the real questioner.

David Hilbert*, Mathematical Problems*

The study of locally nipotent derivations and $\mathbb{G}_a$-actions has recently emerged from the long shadows of other branches of mathematics, branches whose provenance is older and more distinguished. The subject grew out of the rich environment of Lie theory, invariant theory, and differential equations, and continues to draw inspiration from these and other fields.

At the heart of the present exposition lie sixteen principles for locally nilpotent derivations, laid out in *Chapter 1*. These provide the foundation upon which the subsequent theory is built. As a rule, we would like to distinguish which properties of a locally nilpotent derivation are due to its being a "derivation", and which are special to the condition "locally nilpotent". Thus, we first consider general properties of derivations. The sixteen First Principles which follow can then be seen as belonging especially to the locally nilpotent derivations.

Of course, one must choose one's category. While $\mathbb{G}_a$-actions can be investigated in a characteristic-free environment, locally nilpotent derivations are, by nature, objects belonging to rings of characteristic zero. Most of the basic results about derivations found in *Chap. 1* are stated for a commutative k-domain B, where k is a field of characteristic zero. *Chapter 2* establishes further properties of locally nilpotent derivations when certain additional divisorial properties are assumed. The main such properties are the ascending chain condition on principal ideals, the highest common factor property, and unique factorization into irreducibles.

In discussing geometric aspects of the subject, it is also generally assumed that B is affine, and that the underlying field k is algebraically closed. The associated geomtery falls under the rubric of affine algebraic geometry. Miyanishi writes: "There is no clear definition of affine algebraic geometry. It is one branch of algebraic geometry which deals with the affine spaces and the polynomial rings, hence affine algebraic varieties as subvarieties of the affine spaces and finitely generated algebras as the residue rings of the polynomial rings" [208]. Due to their obvious importance, special attention is given throughout the book to polynomial rings and affine spaces $\mathbb{A}^n_k$.

Chapter 3 explores the case of polynomial rings over k. Here, the jacobian derivations are of central importance. Makar-Limanov's Theorem asserts that every locally nilpotent derivation of a polynomial ring is a rational kernel multiple of a jacobian derivation. The set of polynomials which define the jacobian derivation then give a transcendence basis for the kernel. The reader will also find in this chapter a wide range of examples, from that of Bass and Nagata originating in the early 1970s, up to the important new examples of de Bondt discovered in late 2004.

Chapter 4 looks at the case of polynomial rings in two variables, first over a field k, and then over other base rings. An elementary proof of Rentschler's Theorem is given, which is then applied to give proofs for Jung's Theorem and the Structure Theorem for the planar automorphism group. This effectively classifies all locally nilpotent derivations of $k[x, y]$, and likewise all algebraic $\mathbb{G}_a$-actions on the plane $\mathbb{A}^2$. *Chapter 5* documents the tremendous progress in our understanding of the three-dimensional case which has been made over the past two decades, beginning with the Bass-Nagata example and Miyanishi's Theorem. We now have a large catalogue of interesting and instructive examples, in addition to the impressive Daigle-Russell classification in the homogeneous case, and Kaliman's classification of the free $\mathbb{G}_a$-actions. These feats notwithstanding, a meaningful classification of the locally nilpotent derivations of $k[x, y, z]$ remains elusive. A promising tool toward such classification is the local slice construction.

Chapter 6 examines the case of linear actions of $\mathbb{G}_a$ on affine spaces, and it is here that the oldest literature on the subject of $\mathbb{G}_a$-actions can be found. One of the main results of the chapter is the Maurer-Weitzenböck Theorem, a classical result showing that a linear action of $\mathbb{G}_a$ on $\mathbb{A}^n$ has a finitely generated ring of invariants.[1]

Nagata's famous counterexamples to the Fourteenth Problem of Hilbert showed that the Maurer-Weitzenböck Theorem does not generalize to higher-dimensional groups, i.e., it can happen that a linear $\mathbb{G}_a^m$-action on affine space has a non-finitely generated ring of invariants when $m > 1$. It can also happen that a non-linear $\mathbb{G}_a$-action has non-finitely generated invariant ring, and these form the main topic of *Chapter 7*. The key examples are due to P. Roberts in dimension seven (1990), and to the author and Daigle in dimension five (1998). The chapter features a proof of non-finiteness for both of these.

Chapter 8 discusses various algorithms associated with locally nilpotent derivations, most importantly, the van den Essen algorithm for calculating kernels of finite type. Then, *Chapter 9* introduces the Makar-Limanov and

[1] This result is commonly attributed only to R. Weitzenböck, but after reading Armand Borel's *Essays in the History of Lie Groups and Algebraic Groups*, it becomes clear that L. Maurer should receive at least equal credit.

Derksen invariants of a ring, and illustrates how they can be applied. The concluding chapter, *Chapter 10*, shows how locally nilpotent derivations can be found and used in a variety of important problems, such as the Cancellation Problem and Embedding Problem. In particular, the reader will find in this chapter a relatively short proof that, for an affine surface X, the condition $X \times \mathbb{C} = \mathbb{C}^3$ implies $X = \mathbb{C}^2$. This proof is due to Crachiola and Makar-Limanov, and is further evidence of the power and importance of locally nilpotent derivations in the study of affine algebraic geometry.

In addition to the numerous articles found in the *Bibliography*, there are four larger works which I used in preparing this manuscript. These are the books of Nowicki (1994) and van den Essen (2000), and the extensive lecture notes written by Makar-Limanov (1998) and Daigle (2003). I have also received preliminary versions of two monographs whose aim is to survey recent progress in affine algebraic geometry, with particular attention to the role of locally nilpotent derivations and $\mathbb{G}_a$-actions. These are due to Kaliman [155] and Miyanishi [208]. In addition, I found in the books of Kraft (1985), Popov (1992), Grosshans (1997), Borel (2001), Derksen and Kemper (2002), and Dolgachev (2003) a wealth of pertinent references and historical background regarding invariant theory.

The reader will find that this book focuses on the algebraic aspects of locally nilpotent derivations, as the book's title indicates. The subject is simply too large and diverse to include a complete geometric treatment in a volume of this size. The manuscripts of Kaliman and Miyanishi mentioned in the preceding paragraph will serve to fill this void.

It is my intention that the material of this book appeal to as wide an audience as possible, and I believe that the style of writing and choice of topics reflect this intention. In particular, I have endeavored to make the exposition reasonably "self-contained". It is my hope that the reader will find as much fascination and reward in the subject as have I.

Historical Overview

The study of locally nilpotent derivations in its present form appears to have emerged in the 1960s, and was first made explicit in the work of several mathematicians working in France, including Dixmier, Gabriel and Nouazé, and Rentschler. Their motivation came from the areas of Lie algebras and Lie groups, where the connections between derivations, vector fields, and group actions were well-explored.

The study of linear $\mathbb{G}_a$-actions goes back at least as far as Hilbert in the late Nineteenth Century, who already calculated the invariants of the basic actions up to integral closure (see [131], §10, Note 1). In 1899, Maurer outlined his proof showing the finite generation of invariant rings for one-dimensional group actions. In 1932, Weitzenböck gave a more complete version of Maurer's proof, which used ideas of P. Gordan and M. Roberts dating to 1868 and

1871, respectively, in addition to the theory developed by Hilbert (see *Chap. 6*). Remarkably, in their paper dating to 1876, Gordan and M. Nöther studied certain systems of differential operators, and were led to investigate special kinds of non-linear $\mathbb{G}_a$-actions on $\mathbb{C}^n$, though they did not use this language. See *Chap. 3* and *Chap. 6* below.

It seems that the appearance of Nagata's counterexamples to Hilbert's Fourteenth Problem in 1958 spurred a renewed interest in $\mathbb{G}_a$-actions and more general unipotent actions, since the theorem of Maurer and Weitzenböck could then be seen in sharp contrast to the case of higher-dimensional vector group actions. It was shortly thereafter, in 1962, that Seshadri published his well-known proof of the Maurer-Weitzenböck result. Nagata's 1962 paper [237] contains significant results about connected unipotent groups acting on affine varieties, and his classic Tata lecture notes [238] appeared in 1965. The case of algebraic $\mathbb{G}_a$-actions on affine varieties was considered by Bialynicki-Birula in the mid-1960s [20, 21, 22]. In 1966, Hadziev published his famous theorem [138], which is a finiteness result for the maximal unipotent subgroups of reductive groups. The 1969 article of Horrocks [146] considered connectedness and fundamental groups for certain kinds of unipotent actions, and the 1973 paper of Hochschild and Mostow [144] remains a standard reference for unipotent actions. Grosshans began his work on unipotent actions in the early 1970s; his 1997 book [131] provides an excellent overview of the subject. Another notable body of research from the 1970s is due to Fauntleroy, whose focus was on invariant theory associated to $\mathbb{G}_a$-modules in arbitrary characteristic [105, 106, 107, 108]. The papers of Pommerening also began to appear in the late 1970s (see [131, 250]), and Tan's algorithm for computing invariants of basic $\mathbb{G}_a$-actions apppeared in 1989. These developments are traced in *Chap. 6* below.

In a famous paper published in 1968, Rentschler classified the locally nilpotent derivations of the polynomial ring in two variables over a field of characteristic zero, and pointed out how this gives the equivalent classification of all the algebraic $\mathbb{G}_a$-actions on the plane $\mathbb{A}^2$ (see *Chap. 4*). This article is highly significant, in that it was the first publication devoted to the study of certain locally nilpotent derivations (even though its title mentions only $\mathbb{G}_a$-actions). Indeed, Rentschler's landmark paper crystallized the definitions and concepts for locally nilpotent derivations in their modern form, and further provided a compelling illustration of their importance, namely, a simple proof of Jung's Theorem using locally nilpotent derivations.

It must be noted that the classification of planar $\mathbb{G}_a$-actions in characteristic zero was first given by Ebey in 1962 [93]. Ebey's paper clearly deserves more recognition than it receives. Of the more than 300 works listed in the *Bibliography* of this book, only the 1966 paper of Shafarevich [276] cites it (and this is where I recently discovered it). The paper was an outgrowth of Ebey's thesis, written under the direction of Max Rosenlicht. Rather than using the standard theorems of Jung (1942) or van der Kulk (1953) on planar automorphisms, the author used an equivalent result of Engel, dating to 1958.

The crucial Slice Theorem appeared in the 1967 paper of Gabriel and Nouazé [125], which is cited in Rentschler's paper. This result is foreshadowed in the 1965 paper of Lipman [187] (Thm. 2). Other proofs of the Slice Theorem were given by Dixmier in 1974 ([86], 4.7.5), Miyanishi in 1978 ([213], 1.4) and Wright in 1981 ([311], 2.1). In Dixmier's proof we find the implicit definition and use of what is herein referred to as the Dixmier map. Wright's proof also uses such a construction. The first explicit definition and use of this map is found in van den Essen [98], 1993, and in Deveney and Finston [74], 1994. Arguably, the Dixmier map is to unipotent actions what the Reynolds operator is to reductive group actions (see [100], 9.2).

Certainly, one main source of interest for the study of locally nilpotent derivations was, and continues to be, the Jacobian Conjecture. This famous problem and its connection to derivations is briefly described in *Chap. 3* below, and is thoroughly investigated in the book of van den Essen [100]. It seems likely that the conjecture provided, at least partly, the motivation behind Vasconcelos's Theorem on locally nilpotent derivations, which appeared in 1969; see *Chap. 1.* In Wright's paper (mentioned above), locally nilpotent derivations also play a central role in his discussion of the conjecture.

There are not too many papers about locally nilpotent derivations or $\mathbb{G}_a$-actions from the decade of the 1970s. A notable exception is found in the work of Miyanishi, who was perhaps the first researcher to systematically investigate $\mathbb{G}_a$-actions throughout his career. Already in 1968, his paper [209] dealt with locally finite higher iterative derivations. These objects were first defined by Hasse and Schmidt [141] in 1937, and serve to generalize the definition of locally nilpotent derivations in order to give a correspondence with $\mathbb{G}_a$-actions in arbitrary characteristic. Miyanishi's 1971 paper [210] is about planar $\mathbb{G}_a$-actions in positive characteristic, giving the analogue of Rentschler's Theorem in this case. His 1973 paper [211] uses $\mathbb{G}_a$-actions to give a proof of the cancellation theorem of Abhyankar, Eakin and Heinzer. In his 1978 book [213], Miyanishi entitled the first section "Locally nilpotent derivations" (Sect. 1.1). In these few pages, Miyanishi organized and proved many of the fundamental properties of locally nilpotent derivations: The correspondence of locally nilpotent derivations and exponential automorphisms (Lemma 1.2); the fact that the kernel is factorially closed (Lemma 1.3.1); the Slice Theorem (Lemma 1.4), and its local version (Lemma 1.5). While these results already existed elsewhere in the literature, this publication constituted an important new resource for the study of locally nilpotent derivations. A later section of the book, called "Locally nilpotent derivations in connection with the cancellation problem" (Sect. 1.6), proved some new cases in which the cancellation problem has a positive solution, based on locally nilpotent derivations. In addition, Miyanishi's 1980 paper [214] and 1981 book [215] include some of the earliest results about $\mathbb{G}_a$-actions on $\mathbb{A}^3$. Ultimately, his 1985 paper [217] outlined the proof of his well-known theorem about invariant rings of $\mathbb{G}_a$-actions on $\mathbb{A}^3$ (see *Chap. 5* below). In many other papers, Miyanishi used $\mathbb{G}_a$-actions extensively in the classification of surfaces, characterization of affine spaces, and the like.

In 1984, Bass produced a non-triangularizable $\mathbb{G}_a$-action on $\mathbb{A}^3$, based on the automorphism published by Nagata in 1972 (see *Chap. 5* below). This example, together with the 1985 theorem of Miyanishi, marked the beginning of the current generation of research on $\mathbb{G}_a$-actions and locally nilpotent derivations. The entire subject seems to have gathered momentum in the late 1980s, with important new results of Popov, Snow, M. Smith, Winkelmann, and Zurkowski [253, 281, 282, 283, 306, 315, 316]. [2] This trend continued in the early 1990s, especially in several papers due to van den Essen, and Deveney and Finston, who began a more systematic approach to the study of locally nilpotent derivations. Paul Roberts' counterexample to the Fourteenth Problem of Hilbert appeared in 1990, and it was soon realized that his example was the invariant ring of a $\mathbb{G}_a$-action on $\mathbb{A}^7$. The 1994 book of Nowicki [247] included a chapter about locally nilpotent derivations. The book of van den Essen, published in 2000, is about polynomial automorphisms and the Jacobian Conjecture, and takes locally nilpotent derivations as one of its central themes.

By the mid-1990s, Daigle, Kaliman, Makar-Limanov, and Russell began making significant contributions to our understanding of the subject. The introduction by Makar-Limanov in 1996 of the ring of absolute constants (now called the Makar-Limanov invariant) brought widespread recognition to the study of locally nilpotent derivations as a powerful tool in understanding affine geometry and commutative ring theory. Extensive (unpublished) lecture notes on the subject of locally nilpotent derivations from Makar-Limanov and from Daigle were written in 1998 and 2003, respectively. Papers of Kaliman which appeared in 2004 contain important new results about $\mathbb{C}^+$-actions on threefolds, bringing to bear a wide range of tools from topology and algebraic geometry.

The Makar-Limanov invariant is currently one of the central themes in the classification of algebraic surfaces. In particular, families of surfaces having a trivial Makar-Limanov invariant have been classified by Bandman and Makar-Limanov, Daigle and Russell, Dubouloz, and Gurjar and Miyanishi [8, 61, 92, 134]. Already in 1983, Bertin [17] had studied surfaces which admit a $\mathbb{C}^+$-action.

By the late-1990s, locally nilpotent derivations also began to appear in some thesis work, especially from the Nijmegen School, i.e., students of van den Essen at the University of Nijmegen: Berson, Bikker, de Bondt (in progress), Derksen, Eggermont, Holtackers, Hubbers, Ivanenko, Janssen, Maubach, van Rossum, and Willems (in progress). Two students of Daigle at the University of Ottawa, Khoury and Z. Wang, wrote their dissertations on the subject of locally nilpotent derivations. It appears that Wang's 1999 PhD thesis holds the distinction of being the first devoted to the subject of locally nilpotent

[2] My own work in this area began in 1993, and I "went to school" on these papers.

derivations. Likewise, Crachiola wrote his thesis under the direction of Makar-Limanov at Wayne State University; and the thesis of Jorgenson was supervised by Finston at New Mexico State University. These, at least, are the ones of which I am aware.

As mentioned, the study of locally nilpotent derivations is also motivated by certain problems in differential equations. El Kahoui writes:

> A classical application of derivations theory is the study of various questions such as first integrals and invariant algebraic sets for ordinary polynomial differential systems over the reals or the complexes....Very often, the study of practical questions, arising for example from differential equations, leads to dealing with derivations over abstract rings, sometimes even nonreduced, of characteristic zero. One of the fundamental questions in this topic is to describe their rings of constants. ([94], Introduction)

It was proved by Coomes and Zurkowski [38] that, over $k = \mathbb{C}$, a polynomial vector field $f = (f_1, ..., f_n)$ has a polynomial flow if and only if the corresponding derivation $f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}$ is locally finite. (See also [99, 247].)

The foregoing brief overview is by no means a complete account of the subject's development. Significant work in this area from many other researchers can be found in the *Bibliography*, much of which is discussed in the following chapters. In a conversation with the author in 2003 concerning locally nilpotent derivations and $\mathbb{G}_a$-actions, A. Bialynicki-Birula remarked: "I believe that we are just at the beginning of our understanding of this wonderful subject."

1

First Principles

Let B denote a commutative k-domain, where k is any field of characteristic zero. Then B^* denotes the group of units of B and $\text{frac}(B)$ denotes the field of fractions of B. Further, $\text{Aut}(B)$ denotes the group of ring automorphisms of B, and $\text{Aut}_k(B)$ denotes the group of automorphisms of B as a k-algebra. If $S \subset B$ is any subset, then S^C denotes the complement $B - S$. If $A \subset B$ is a subring, then $\text{tr.deg.}_A B$ denotes the transcendence degree of $\text{frac}(B)$ over $\text{frac}(A)$. Given $x \in B$, the principal ideal of B generated by x will be denoted by either xB or (x); the ideal generated by $x_1, ..., x_n \in B$ is $(x_1, ..., x_n)$. The ring of $n \times n$ matrices with entries in B is indicated by $\mathcal{M}_n(B)$. The transpose of a matrix M is M^T.

The term **affine k-domain** will mean a commutative k-domain which is finitely generated as a k-algebra. The standard notations $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are used throughout to denote the fields of rational, real, and complex numbers, respectively. Likewise, $\mathbb{Z}$ denotes the integers, $\mathbb{N}$ is the set of non-negative integers, and $\mathbb{Z}_+$ is the set of positive integers. S_n will denote the symmetric group on n letters.

1.1 Basic Definitions for Derivations

This section endeavors to catalog all the basic definitions and notations commonly found in the literature relating to locally nilpotent derivations.

By a **derivation** of B, we mean any function $D : B \to B$ which satisfies the following conditions: For all $a, b \in B$,

(C.1) $D(a + b) = Da + Db$
(C.2) $D(ab) = aDb + bDa$

Condition (C.2) is usually called the **Leibniz rule** or **product rule**. The set of all derivations of B is denoted by $\text{Der}(B)$. If A is any subring of B, then $\text{Der}_A(B)$ denotes the subset of all $D \in \text{Der}(B)$ with $D(A) = 0$. The set

$\ker D = \{b \in B | Db = 0\}$ is the **kernel** of D. Given $D \in \mathrm{Der}(B)$, four facts of fundamental importance can be seen immediately.

(C.3) $\ker D$ is a subring of B for any $D \in \mathrm{Der}(B)$.
(C.4) The subfield $\mathbb{Q} \subset k$ has $\mathbb{Q} \subset \ker D$ for any $D \in \mathrm{Der}(B)$.
(C.5) $\mathrm{Aut}(B)$ acts on $\mathrm{Der}(B)$ by conjugation: $\alpha \cdot D = \alpha D \alpha^{-1}$.
(C.6) Given $b \in B$ and $D, E \in \mathrm{Der}(B)$, if $[D, E] = DE - ED$, then bD, $D + E$, and $[D, E]$ are again in $\mathrm{Der}(B)$.

Verification of properties (C.3)-(C.6) is an easy exercise.

We are especially interested in $\mathrm{Der}_k(B)$, called the **k-derivations** of B. For k-derivations, the conditions above imply that D is uniquely defined by its image on any set of generators of B as a k-algebra, and that $\mathrm{Der}_k(B)$ forms a Lie algebra over k. If A is a subring of B containing k, then $\mathrm{Der}_A(B)$ is a Lie subalgebra of $\mathrm{Der}_k(B)$.

Given $D \in \mathrm{Der}(B)$, let $A = \ker D$. We define several terms and notations for D.

- Given $n \geq 0$, D^n denotes the n-fold composition of D with itself, where it is understood that D^0 is the identity map.
- A commonly used alternate term for the kernel of D is the **ring of constants** of D, with alternate notation B^D.
- The **image** of D is denoted DB.
- The B-ideal generated by the image DB is denoted (DB).
- The A-ideal $A \cap DB$ is the **plinth ideal**[1] of D, denoted $\mathrm{pl}(D)$. (See *Prop. 1.8.*)
- An ideal $I \subset B$ is an **integral ideal** for D if and only if $DI \subset I$ [220]. (Some authors call such I a **differential ideal**, e.g. [247].)
- An element $f \in B$ is an **integral element** for D if and only if fB is an integral ideal for D.
- D is **reducible** if and only if there exists a non-unit $b \in B$ such that $DB \subset b \cdot B$. Otherwise, D is **irreducible**.
- Any element $s \in B$ with $Ds = 1$ is called a **slice** for D. Any $s \in B$ such that $Ds \in \ker D$ and $Ds \neq 0$ is called a **local slice** for D. (Some authors use the term **pre-slice** instead of local slice, e.g. [100].)
- Given $b \in B$, we say D is **nilpotent at** b if and only if there exists $n \in \mathbb{Z}_+$ with $D^n b = 0$.
- The set of all elements of B at which D is nilpotent is denoted $\mathrm{Nil}(D)$.

1.1.1 Polynomial Rings and Algebraic Elements

For a ring A, the polynomial ring in one variable t over A is defined in the usual way, and is denoted by $A[t]$. It is also common to write $A^{[1]}$ for this ring. More generally, polynomial rings over a coefficient ring are defined as follows: If A is any commutative ring, then $A^{[0]} := A$, and for $n \geq 0$, $A^{[n+1]} := A^{[n]}[t]$,

[1] The term plinth commonly refers to the base of a column or statue.

where t is a variable over $A^{[n]}$. We say that R is a **polynomial ring in** n **variables over** A if and only if $A \subset R$ and R is A-isomorphic to $A^{[n]}$. In this case, we simply write $R = A^{[n]}$.

Given a subring $A \subset B$, an element $t \in B$ is **algebraic** over A if there exists nonzero $P \in A^{[1]}$ such that $P(t) = 0$. If P can be chosen to be monic over A, then t is an **integral** element over A. The **algebraic closure** of A in B is the subring $\bar{A}$ of B consisting of all $t \in B$ which are algebraic over A. A is said to be **algebraically closed** in B if $\bar{A} = A$. B is an **algebraic extension** of A if $\bar{A} = B$. The terms **integrally closed** subring, **integral closure**, and **integral extension** are defined analogously.

Recall that a subring $A \subset B$ is **factorially closed** in B if and only if, given nonzero $f, g \in B$, the condition $fg \in A$ implies $f \in A$ and $g \in A$. Other terms used for this property are **saturated** and **inert**. Note that the condition "in B" is important in this definition. For example, if B is an integral domain and $f \in B - A$, then $ff^{-1} = 1 \in A$, but $f \notin A$. Nonetheless, when the ambient ring B is understood, we will often say simply that A is factorially closed. When A is factorially closed in B, then $A^* = B^*$, A is algebraically closed in B, and every irreducible element of A is irreducible in B. As we will see, factorially closed subrings play an important role in the subject at hand.

1.1.2 Localizations

Let $S \subset B - \{0\}$ be any multiplicatively closed subset. Then $S^{-1}B \subset \mathrm{frac}(B)$ denotes the localization of B at S, i.e.,

$$S^{-1}B = \{ab^{-1} \in \mathrm{frac}(B) \mid a \in B \ , \ b \in S\} \ .$$

In case $S = \{f^i\}_{i \geq 0}$ for some nonzero $f \in B$, then B_f denotes $S^{-1}B$. Likewise, if $S = B - \mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of B, then $B_{\mathfrak{p}}$ denotes $S^{-1}B$.

1.1.3 Degree Functions

A **degree function** on B is any map $\deg : B \to \mathbb{N} \cup \{-\infty\}$ such that, for all $f, g \in B$, the following conditions are satisfied.

(1) $\deg(f) = -\infty \Leftrightarrow f = 0$
(2) $\deg(fg) = \deg(f) + \deg(g)$
(3) $\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$

Here, it is understood that $(-\infty) + (-\infty) = -\infty$, and $(-\infty) + n = -\infty$ for all $n \in \mathbb{N}$. Likewise, $-\infty < n$ for all $n \in \mathbb{N}$. It is an easy exercise to show:

- equality holds in condition (3) if $\deg(f) \neq \deg(g)$.
- $B_0 := \{b \in B \mid \deg(b) \leq 0\}$ is a factorially closed subring of B.
- $B^* \subset B_0$

Remark 1.1. In some cases, it is advantageous to use a degree function which takes values in $\mathbb{Z} \cup \{-\infty\}$. In such cases, however, the degree zero elements may no longer form a factorially closed subring or contain all units.

1.1.4 Homogeneous Derivations

Suppose B is a graded ring $B = \oplus_{i \in I} B_i$, where I is an ordered abelian semigroup, each B_i is a $\mathbb{Q}$-module, and $B_i B_j \subset B_{i+j}$ for every $i, j \in I$. If we label the given grading of B by ω, then elements of the submodules B_i are called **ω-homogeneous elements** of B, and if $f \in B_i$, then the **ω-degree of** f is i.

A derivation $D \in \mathrm{Der}(B)$ which respects this grading is called an **ω-homogeneous derivation**. Specifically, we mean that there exists $d \in I$ such that $DB_i \subset B_{i+d}$ for each $i \in I$. The element $d \in I$ is called the **ω-degree of** D. Observe that if D is ω-homogeous and $f \in B$ decomposes as $f = \sum_{i \in I} f_i$ for $f_i \in B_i$, then $Df = 0$ if and only if $Df_i = 0$ for every i. This is because the decomposition of Df into homogeneous summands is $\sum_{i \in I} Df_i$ when D is ω-homogeneous.

Our main interest lies in the case $I = \mathbb{Z}^n$ or $I = \mathbb{N}^n$ for some $n \geq 1$.

1.1.5 The Graded Ring Associated to a Filtration

If B (a commutative k-domain) admits a $\mathbb{Z}$-filtration by k-vector subspaces, then it is possible to construct from B a $\mathbb{Z}$-graded ring $\mathrm{Gr}(B)$, together with a natural function $B \to \mathrm{Gr}(B)$. In addition, each $D \in \mathrm{Der}_k(B)$ respecting this filtration is associated to a homogeneous derivation $\mathrm{gr}(D) \in \mathrm{Der}_k(\mathrm{Gr}(B))$, and it is often easier to work with $\mathrm{gr}(D)$ than with D. The present treatment of these ideas follows closely their presentation by Makar-Limanov in [190].

By a **$\mathbb{Z}$-filtration** of B we mean a collection $\{B_i\}_{i \in \mathbb{Z}}$ of subsets of B with the following properties.

1. Each B_i is a vector space over k.
2. $B_j \subset B_i$ whenever $j \leq i$.
3. $B = \cup_{i \in \mathbb{Z}} B_i$
4. $B_i B_j \subset B_{i+j}$ for all $i, j \in \mathbb{Z}$.

The filtration will be called a **proper $\mathbb{Z}$-filtration** if the following two properties also hold.

5. $\cap_{i \in \mathbb{Z}} B_i = \{0\}$
6. If $a \in B_i \cap B_{i-1}^C$ and $b \in B_j \cap B_{j-1}^C$, then $ab \in B_{i+j} \cap B_{i+j-1}^C$.

Note that any degree function on B will give a proper $\mathbb{Z}$-filtration. Note also that we could define filtrations with $\mathbb{Z}$ replaced by any ordered abelian semigroup.

For k-vector spaces $W \subset V$, the notation V/W will denote the k-vector space V modulo W in the usual sense. Suppose $B = \cup B_i$ is a proper $\mathbb{Z}$-filtration, and define the **associated graded algebra** $\mathrm{Gr}(B)$ as follows. The k-additive structure on $\mathrm{Gr}(B)$ is given by

$$\mathrm{Gr}(B) = \oplus_{n \in \mathbb{Z}} B_n / B_{n-1} \ .$$

Consider elements $a + B_{i-1}$ belonging to B_i/B_{i-1}, and $b + B_{j-1}$ belonging to B_j/B_{j-1}, where $a \in B_i$ and $b \in B_j$. Their product is the element of B_{i+j}/B_{i+j-1} defined by

$$(a + B_i/B_{i-1})(b + B_j/B_{j-1}) = ab + B_{i+j-1} \ .$$

Now extend this multiplication to all of $\mathrm{Gr}(B)$ by the distributive law.

Note that, because of axiom 6, $\mathrm{Gr}(B)$ is a commutative k-domain.

Because of axiom 5, for each nonzero $a \in B$, the set $\{i \in \mathbb{Z} | a \in B_i\}$ has a minimum, which will be denoted $\iota(a)$. The natural map $\rho : B \to \mathrm{Gr}(B)$ is the one which sends each nonzero $a \in B$ to its class in B_i/B_{i-1}, where $i = \iota(a)$. We also define $\rho(0) = 0$.

Given $a \in B$, observe that $\rho(a) = 0$ if and only if $a = 0$. Note further that ρ is a multiplicative map, but is not an algebra homomorphism, since it does not generally respect addition.

In case B is already a $\mathbb{Z}$-graded ring, then B admits a filtration relative to which B and $\mathrm{Gr}(B)$ are canonically isomorphic via ρ. In particular, if $B = \oplus_{i\in\mathbb{Z}} A_i$, then a proper $\mathbb{Z}$-filtration is defined by $B_i = \oplus_{j\le i} A_j$.

Example 1.2. Let $B = k[x]$, a univariate polynomial ring over k, and let B_i consist of polynomials of degree at most i ($i \ge 0$). Then $k[x] = \cup B_i$ is a $\mathbb{Z}$-filtration (with $B_i = \{0\}$ for $i < 0$), and $\mathrm{Gr}(k[x]) = \oplus_{i\ge 0} kx^i \cong k[x]$.

Example 1.3. Let $B = k(x)$, a univariate rational function field over k. Given nonzero $p(x), q(x) \in k[x]$, define the degree of $p(x)/q(x)$ to be $\deg p(x) - \deg q(x)$. Let B_i consist of functions of degree at most i. Then $\mathrm{Gr}(k(x)) = k[x, x^{-1}]$, the ring of Laurent polynomials.

Now suppose $B = \cup B_i$ is a proper $\mathbb{Z}$-filtration. Given $D \in \mathrm{Der}_k(B)$, we say that D **respects the filtration** if there exists an integer t such that, for all $i \in \mathbb{Z}$, $D(B_i) \subset B_{i+t}$. Define a function $\mathrm{gr}(D) : \mathrm{Gr}(B) \to \mathrm{Gr}(B)$ as follows.

If $D = 0$, then $\mathrm{gr}(D)$ is the zero map.

If $D \neq 0$, choose t to be the least integer such that $D(B_i) \subset B_{i+t}$ for all $i \in \mathbb{Z}$. Then given $i \in \mathbb{Z}$, define

$$\mathrm{gr}(D) : B_i/B_{i-1} \to B_{i+t}/B_{i+t-1}$$

by the rule $\mathrm{gr}(D)(a+B_{i-1}) = Da+B_{i+t-1}$. Now extend $\mathrm{gr}(D)$ to all of $\mathrm{Gr}(B)$ by linearity. It is an easy exercise to check that $\mathrm{gr}(D)$ satisfies the product rule, and is therefore a homogeneous k-derivation of $\mathrm{Gr}(B)$. The reader should note that $\mathrm{gr}(D) = 0$ if and only if $D = 0$. In addition, observe that, by definition,

$$\rho(\ker D) \subset \ker(\mathrm{gr}(D)) \ .$$

Remark 1.4. Given $a \in B$, the notation $\mathrm{gr}(a)$ is commonly used to denote the image $\rho(a)$. In doing so, one must be careful to distinguish $\mathrm{gr}(D)(a)$ from $\mathrm{gr}(Da)$.

1.1.6 Locally Finite and Locally Nilpotent Derivations

A derivation $D \in \mathrm{Der}(B)$ is said to be **locally finite** if and only if for each $f \in B$, the $\mathbb{Q}$-vector space spanned by the images $\{D^n f | n \geq 0\}$ is finite dimensional. Equivalently, there exists a monic polynomial $p(t) \in \mathbb{Q}[t]$ (depending on f) such that $p(D)(f) = 0$.

A derivation $D \in \mathrm{Der}(B)$ is said to be **locally nilpotent** if and only if to each $f \in B$, there exists $n \in \mathbb{Z}_+$ (depending on f) such that $D^n f = 0$, i.e., if and only if $\mathrm{Nil}(D) = B$. Thus, the locally nilpotent derivations are special kinds of locally finite dervivations. Let $\mathrm{LND}(B)$ denote the set of all $D \in \mathrm{Der}(B)$ which are locally nilpotent. Important examples of locally nilpotent derivations are the familiar partial derivative operators on a polynomial ring. If A is a subring of B, define $\mathrm{LND}_A(B) := \mathrm{Der}_A(B) \cap \mathrm{LND}(B)$.

As mentioned in the *Introduction*, the derivations investigated in this book are the locally nilpotent derivations. Apart from being an interesting and important topic in its own right, the study of locally nilpotent derivations is motivated by their connection to algebraic group actions. Specifically, the condition "locally nilpotent" imposed on a derivation corresponds precisely to the condition "algebraic" imposed on the corresponding group action. This is explained in *Sect. 1.5* below.

For a discussion of derivations in a more general setting, the reader is referred to the books of Northcott [245] and Nowicki [247]. The topic of locally finite derivations is explored in Chap. 9 of Nowicki's book; in Chap. 1.3 of van den Essen's book [100]; and in papers of Zurkowski [315, 316].

1.1.7 The Degree Function Induced by a Derivation

The **degree function** ν_D induced by a derivation D is a simple yet indispensable tool in working with D, especially in the locally nilpotent case. Given $D \in \mathrm{Der}(B)$ and $f \in \mathrm{Nil}(D)$, we know that $D^n f = 0$ for $n \gg 0$. If $f \neq 0$, define

$$\nu_D(f) = \min\{n \in \mathbb{N} \mid D^{n+1} f = 0\} \ .$$

In addition, define $\nu_D(0) = -\infty$. It is shown in *Prop. 1.9* that $\mathrm{Nil}(D)$ is a subalgebra of B and ν_D is a degree function on $\mathrm{Nil}(D)$. Thus, if D is locally nilpotent, ν_D induces a proper $\mathbb{Z}$-filtration $B = \cup_{i \in \mathbb{N}} B_i$ which D respects, where $B_i = \{f \in B | \nu_D(f) \leq i\}$. In this case, note that $B_0 = \ker D$, and that each element of $B_1 \cap B_0^C$ is a local slice.

Another common notation for ν_D is $\deg_D$. One reason for choosing not to use the latter notation here is that one often uses several degree functions simultaneously while working with derivations, and it can be awkward to keep track of the meaning of the deg symbol. The notation ν_D is similar to that introduced earlier by Zurkowski.

1.1.8 The Exponential and Dixmier Maps

Given $D \in \mathrm{LND}(B)$, the **exponential function** determined by D is $\exp D : B \to B$, where

$$\exp D(f) = \sum_{i \geq 0} \frac{1}{i!} D^i f \ .$$

Likewise, for any local slice $r \in B$ of D, the **Dixmier map** induced by r is $\pi_r : B \to B_{Dr}$, where

$$\pi_r(f) = \sum_{i \geq 0} \frac{(-1)^i}{i!} D^i f \frac{r^i}{(Dr)^i} \ .$$

Here, B_{Dr} denotes localization at Dr. Note that, since D is locally nilpotent, both $\exp D$ and π_r are well-defined. These definitions rely on the fact that B contains $\mathbb{Q}$.

1.1.9 The Derivative of a Polynomial

If A is a subring of B, and $B = A[t] \cong A^{[1]}$ for some $t \in B$, the **derivative** of B relative to the pair (A, t) is the derivation $(\frac{d}{dt})_A \in \mathrm{Der}_A(B)$ uniquely defined by $(\frac{d}{dt})_A(t) = 1$. (As mentioned, a derivation is uniquely determined by its image on a generating set.) Usually, if the subring A is understood, we denote this derivation more simply by $\frac{d}{dt}$; in this case, given $P(t) \in A[t]$, we also define

$$P'(t) := \frac{d}{dt}(P(t)) \ .$$

Likewise, given $n \geq 0$, the notations

$$P^{(n)}(t) \quad \text{and} \quad \frac{d^n P}{dt^n}$$

each denotes the n-fold composition

$$\left(\frac{d}{dt}\right)^n (P(t)) \ .$$

Note that it is possible that $B = \tilde{A}[t] = A[t]$ for subrings $A \neq \tilde{A}$ (or even $A \not\cong \tilde{A}$), in which case $(\frac{d}{dt})_A \neq (\frac{d}{dt})_{\tilde{A}}$. It can also happen that $B = A[t] = A[s]$ for elements $s \neq t$, in which case $(\frac{d}{dt})_A \neq (\frac{d}{ds})_A$. So one must be careful. See [2].

1.2 Basic Facts about Derivations

At the beginning of this chapter, two defining conditions (C.1) and (C.2) for a k-derivation D of B are given, which imply further conditions (C.3)-(C.6). We now examine the next layer of consequences implied by these conditions.

Proposition 1.5. *Let $D \in \mathrm{Der}(B)$ be given, and let $A = \ker D$.*

(a) *$D(ab) = aDb$ for all $a \in A, b \in B$. Therefore, D is an A-module endomorphism of B.*
(b) power rule: *For any $t \in B$ and $n \geq 1$, $D(t^n) = nt^{n-1}Dt$.*
(c) quotient rule: *If $g \in B^*$ and $f \in B$, then $D(fg^{-1}) = g^{-2}(gDf - fDg)$.*
(d) higher product rule: *For any $a, b \in B$ and any integer $m \geq 0$,*

$$D^m(ab) = \sum_{i+j=m} \binom{m}{i} D^i a D^j b \ .$$

Proof. Property (a) is immediately implied by (C.1) and (C.2).

To prove (b), proceed by induction on n, the case $n = 1$ being clear. Given $n \geq 2$, assume by induction that $D(t^{n-1}) = (n-1)t^{n-2}Dt$. By the product rule (C.2),

$$D(t^n) = tD(t^{n-1}) + t^{n-1}Dt = t \cdot (n-1)t^{n-2}Dt + t^{n-1}Dt = nt^{n-1}Dt \ .$$

So (b) is proved. Part (c) follows from the equation

$$Df = D(g \cdot fg^{-1}) = gD(fg^{-1}) + fg^{-1}Dg \ .$$

Finally, (d) is easily proved by inductive application of the product rule (C.2), together with (C.1) and (C.4). □

Proposition 1.6. *Suppose A is a subring of B and $t \in B$ is transcendental over A. If $P(t) \in A[t]$ is given by $P(t) = \sum_{0 \leq i \leq m} a_i t^i$ for $a_i \in A$, then*

$$P'(t) = \sum_{1 \leq i \leq m} i a_i t^{i-1} \ .$$

where $P'(t) = (\frac{d}{dt})_A(P(t))$.

Proof. By parts (a) and (b) above, we have, for $1 \leq i \leq m$,

$$\frac{d}{dt}(a_i t^i) = a_i \frac{d}{dt}(t^i) = a_i(it^{i-1}) \ .$$

By now applying the additive property (C.1), the desired result follows. □

The proof of the following corollary is an easy exercise.

Corollary 1.7. (Taylor's Formula) *Let A be a subring of B. Given $s, t \in B$, and $P \in A^{[1]}$ of degree $n \geq 0$,*

$$P(s+t) = \sum_{i=0}^{n} \frac{P^{(i)}(s)}{i!} t^i \ .$$

Proposition 1.8. *Let $D \in \mathrm{Der}(B)$ and let $A = \ker D$.*

(a) *$DB \cap \ker D$ is an ideal of $\ker D$ (the plinth ideal).*
(b) *Any ideal of B generated by elements of A is an integral ideal for D.*
(c) **chain rule:** *If $P \in A^{[1]}$ and $t \in B$, then $D(P(t)) = P'(t)Dt$.*
(d) *A is an algebraically closed subring of B.*

Proof. For (a), since $D : B \to B$ is an A-module homomorphism, both A and DB are A-submodules of B. Thus, $A \cap DB$ is an A-submodule of A, i.e., an ideal of A. Part (b) is immediately implied by *Prop. 1.5 (a)*. Likewise, part (c) is easily implied by *Prop. 1.5 (a,b,c)*.

For (d), suppose $t \in B$ is an algebraic element over A, and let $P \in A^{[1]}$ be a nonzero polynomial of minimal degree such that $P(t) = 0$. Then part (b) implies $0 = D(P(t)) = P'(t)Dt$. If $Dt \neq 0$, then $P'(t) \neq 0$ as well, by minimality of P. Since B is a domain, this is impossible. Therefore, $Dt = 0$. □

Note that $P'(t)$ in part (b) above means evaluation of P' as defined on $A^{[1]}$.

Proposition 1.9. (See also [246]) *Let $D \in \mathrm{Der}(B)$ be given.*

(a) *$\nu_D(Df) = \nu_D(f) - 1$ whenever $f \in \mathrm{Nil}(D) - \ker(D)$.*
(b) *$\mathrm{Nil}(D)$ is a $\mathbb{Q}$-subalgebra of B.*
(c) *ν_D is a degree function on $\mathrm{Nil}(D)$.*

Proof. For the given elements f and g, set $m = \nu_D(f)$ and $n = \nu_D(g)$. Assume $fg \neq 0$, so that $m \geq 0$ and $n \geq 0$. Since $0 = D^{m+1}f = D^m(Df)$, it follows that $Df \in \mathrm{Nil}(D)$. Assertion (a) now follows by definition of ν_D.

In addition, if $\mu = \max\{m, n\}$, then $D^{\mu+1}(f+g) = D^{\mu+1}f + D^{\mu+1}g = 0$. So $\mathrm{Nil}(D)$ is closed under addition. This equation also implies that, for all $f, g \in \mathrm{Nil}(D)$, $\nu_D(f+g) \leq \max\{\nu_D(f), \nu_D(g)\}$.

By the higher product rule, we also see that

$$D^{m+n+1}(fg) = \sum_{i+j=m+n+1} \binom{m+n+1}{i} D^i f D^j g \ .$$

If $i + j = m + n + 1$ for non-negative i and j, then either $i > m$ or $j > n$. Thus, $D^i f D^j g = 0$, implying $D^{m+n+1}(fg) = 0$. Therefore, $\mathrm{Nil}(D)$ is closed under multiplication, and forms a subalgebra of B, and (b) is proved.

The reasoning above shows that $\nu_D(fg) \leq m + n$, and further shows that $D^{m+n}(fg) = \frac{(m+n)!}{m!n!} D^m f D^n g \neq 0$. Therefore, $\nu_D(fg) = m + n$, and (c) is proved. □

Note that the converse of part (c) in *Prop. 1.8* above is also true for fields:

Proposition 1.10. (See Nowicki [247], 3.3.2) *Let $K \subset L$ be fields of characteristic zero. The following are equivalent.*

(a) *There exists $d \in \mathrm{Der}(L)$ such that $K = \ker d$.*
(b) *K is algebraically closed in L.*

From this Nowicki further characterizes all k-subalgebras of B which are kernels of k-derivations.

Proposition 1.11. ([247], 4.1.4) *Let B be an affine k-domain, and $A \subset B$ a k-subalgebra. The following are equivalent.*

(a) *There exists $D \in \mathrm{Der}_k(B)$ such that $A = \ker D$.*
(b) *A is integrally closed in B and $\mathrm{frac}(A) \cap B = A$.*

We also have:

Proposition 1.12. *Suppose B is an algebraic extension of the subring B'. If $D, E \in \mathrm{Der}(B)$ and $Df = Ef$ for every $f \in B'$, then $D = E$.*

Proof. We have that $D - E \in \mathrm{Der}(B)$, and that $\ker(D-E)$ contains B'. Since $\ker(D-E)$ is algebraically closed in B, it contains the algebraic closure of B' in B, i.e., $B \subset \ker(D-E)$. This means $D - E = 0$, and thus $D = E$. $\square$

The following classical result is due to Seidenberg. The reader is referred to [272] or [100], 1.2.15, for its proof.

Proposition 1.13. (Seidenberg's Theorem) *If B is a noetherian domain, let $K = \mathrm{frac}(B)$ and let $\mathcal{B} \subset K$ be the integral closure of B in K. Given $D \in \mathrm{Der}(K)$, if $DB \subset B$, then $D\mathcal{B} \subset \mathcal{B}$.*

Seidenberg's theorem indicates the relation between derivations and integral extensions. One also encounters derivations in relation to localizations, quotients, completions, extensions of the base field, tensor products, etc.

Let S be any multiplicatively closed subset of B not containing 0, and let $D \in \mathrm{Der}_k(B)$. Then D extends uniquely to a **localized derivation** $S^{-1}D$ on the localization $S^{-1}B$ via the quotient rule above. Note that $(S \cap \ker D)^{-1}(\ker D) \subset \ker(S^{-1}D)$, with equality when $S \subset \ker D$.

As a matter of notation, if $S = \{f^i\}_{i \geq 0}$ for some nonzero $f \in B$, then D_f denotes the induced derivation $S^{-1}D$ on B_f. Likewise, if $S = B - \mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of B, then $D_{\mathfrak{p}}$ denotes the induced derivation $S^{-1}D$ on $B_{\mathfrak{p}}$.

Similarly, suppose $D \in \mathrm{Der}_k(B)$ and $I \subset B$ is an integral ideal of D. Then the pair (D, I) induces a well-defined **quotient derivation** D/I on the quotient B/I in an obvious way: $D/I([b]) = [Db]$, where $[b]$ denotes the congruence class of $b \in B$, modulo I. Conversely, any ideal $I \subset B$ for which D/I is well-defined is an integral ideal of D. Miyanishi gives the following basic properties of integral ideals for derivations.

Proposition 1.14. (Lemma 1.1 of [220]) *Suppose that B is a commutative noetherian k-domain, and let $\delta \in \mathrm{Der}_k(B)$. If $I, J \subset B$ are integral ideals for δ, then:*

(a) *$I + J$, IJ, and $I \cap J$ are integral ideals for δ.*
(b) *Every prime divisor $\mathfrak{p}$ of I is an integral ideal for δ.*
(c) *The radical $\sqrt{I}$ is an integral ideal for δ.*

As to **extension of scalars**, Nowicki gives:

Proposition 1.15. ([247], 5.1.1 and 5.1.3) *Let $D \in \mathrm{Der}_k(B)$ be given, and suppose $k \subset k'$ is a field extension. Let $B' = B \otimes_k k'$ and $D' = D \otimes 1 \in \mathrm{Der}_{k'}(B')$. Then*

(a) *$B^D \otimes_k k'$ and $(B')^{D'}$ are isomorphic as k'-algebras.*
(b) *B^D is a finitely generated k-algebra if and only if $(B')^{D'}$ is a finitely generated k'-algebra.*

This same result in the context of group actions is given by Nagata in [240], 8.9 and 8.10. See also [100], 1.2.7.

Remark 1.16. Regarding more general tensor products, suppose that R_1 and R_2 are commutative k-algebras, and $D_i \in \mathrm{Der}_k(R_i)$ $(i = 1, 2)$. Then $D_1 \otimes D_2$ is defined in the standard way as a k-module endomorphism of $R_1 \otimes_k R_2$, but it is not generally a derivation.

Remark 1.17. Observe that, while $\ker D$ is algebraically closed in B for $D \in \mathrm{Der}(B)$, $\mathrm{Nil}(D)$ may fail to be. For example, if δ is the extension of the derivative $\frac{d}{dx}$ on $k[x]$ to the field $k(x)$, then $\mathrm{Nil}(\delta) = k[x]$ in $k(x)$. As another example (due to Daigle), consider the power series ring $B = \mathbb{Q}[[x]]$ and its natural derivation $D = \frac{d}{dx}$: we have $\ker D = \mathbb{Q}$ and $\mathrm{Nil}(D) = \mathbb{Q}[x]$. Thus, $\mathrm{Nil}(D)$ is not even integrally closed in B, since $\sqrt{1+x} \notin \mathrm{Nil}(D)$, whereas $1 + x \in \mathrm{Nil}(D)$.

1.3 Group Actions

In the purely algebraic situation, it is advantageous to consider the general category of commutative k-domains. In the geometric setting, however, our primary interest relates to affine k-varieties X over an algebraically closed field k. The classical case is when $k = \mathbb{C}$.

So assume in this section that k is an algebraically closed field (of characteristic zero). We will consider affine varieties X over k, endowed with the Zariski topology. The coordinate ring of X, or ring of regular functions, is indicated by either $k[X]$ or $\mathcal{O}(X)$. If B is an affine k-domain, then $X = \mathrm{Spec}(B)$ is the corresponding affine variety. In particular, affine n-space over k will be denoted by $\mathbb{A}^n_k$, or simply $\mathbb{A}^n$ when the ground field k is understood. We also wish to consider algebraic groups G over k. The reader can find these standard definitions in many sources, some of which are given at the end of this section.

Suppose that G is an algebraic k-group, and that G acts algebraically on the affine k-variety X.[2] In this case, X is called a **G-variety**. G acts by automorphisms on the coordinate ring $B = k[X]$, and the **ring of invariants** for the action is

[2] Note that the terms **regular action** and **rational action** are also used in the literature to indicate algebraic actions.

$$B^G = \{f \in B | g \cdot f = f \text{ for all } g \in G\} \ .$$

Some authors also refer to B^G as the **fixed ring** of the action. An element $f \in B^G$ is called an **invariant function** for the action. Likewise, $f \in B$ is called a **semi-invariant** for the action if there exists a character $\chi : G \to k^*$ such that $g \cdot f = \chi(g)f$ for all $g \in G$. In this case, χ is the **weight** of the semi-invariant f. Certain important groups, like the special linear group $SL_2(k)$ and the additive group k^+ of the field k, have no nontrivial characters.

The set of **fixed points** for the action is

$$X^G = \{x \in X | g \cdot x = x \text{ for all } g \in G\} \ .$$

The action is **fixed point free**, or simply **free**, if X^G is empty.

The **orbit** of $x \in X$ is $\{g \cdot x | g \in G\}$, denoted by $G \cdot x$ or $\mathcal{O}_x$. The consideration of orbits leads naturally to the important (and subtle) question of forming quotients. Questions about orbits are at the heart of Geometric Invariant Theory.

Since we wish to navigate within the category of affine varieties, we define the **categorical quotient** for the action (if it exists) to be an affine variety Z, together with a morphism $\pi : X \to Z$, satisying: (a) π is constant on the orbits, and (b) for any other affine variety Z' with morphism $\phi : X \to Z'$ which is constant on the orbits, ϕ factors uniquely through π. The categorical quotient is commonly denoted by $X /\!\!/ G$.

A categorical quotient is a **geometric quotient** if the points of the underlying space correspond to the orbits of G on X. The geometric quotient is commonly denoted by X/G. Dolgachev writes:

> The main problem here is that the quotient space X/G may not exist in the category of algebraic varieties. The reason is rather simple. Since one expects that the canonical projection $f : X \to X/G$ is a regular map of algebraic varieties and so has closed fibres, all orbits must be closed subsets in the Zariski topology of X. This rarely happens when G is not a finite group. (Introduction to [87])

A third kind of quotient is the **algebraic quotient** $Y = \mathrm{Spec}(B^G)$. While B^G is not necessarily an affine ring, Winkelmann has shown that it is always at least quasi-affine, i.e., the ring of regular functions on an open subset of an affine variety [307]. The function $X \to Y$ induced by the inclusion $B^G \hookrightarrow B$ is the **algebraic quotient map**.

In many situations, B^G is indeed an affine ring, and the algebraic quotient map is a morphism of affine k-varieties. In this case, the categorical quotient $X /\!\!/ G$ exists and equals the algebraic quotient $Y = \mathrm{Spec}(B^G)$. (This is easily verified using the universal mapping property of $X /\!\!/ G$, and the fact that the inclusion map $B^G \to B$ is injective.) So hereafter in this book, the terms **quotient** and **quotient map** will mean the algebraic/categorical quotient and its associated morphism, with the underlying assumption that B^G is finitely generated.

Of course, it may happen that the invariant functions B^G do not separate orbits, so that even when $\mathcal{O}(X/G)$ exists, it may not equal B^G, since the geometric and algebraic quotients may not admit a bijective correspondence. Dolgachev points out the simple example of $G = GL_n(k)$ acting on $\mathbb{A}^n$ in the natural way: This action has two orbits, whereas $B^G = k$.

If G acts on two varieties X and X', then a morphism $\phi : X \to X'$ is called **equivariant** relative to these two actions if and only if $\phi(g \cdot x) = g \cdot \phi(x)$ for all $g \in G$ and $x \in X$.

The G-action $\rho : G \times X \to X$ is called **proper** if and only if the morphism $G \times X \to X \times X$, $(g, x) \mapsto (x, \rho(g, x))$, is proper as a map of algebraic k-varieties (see [87], 9.2). If $k = \mathbb{C}$, then properness has its usual topological meaning, i.e., the inverse image of a compact set is compact.

The action ρ is called **locally finite** if and only if the linear span of the orbit of every $f \in k[X]$ is a finite-dimensional vector space over k.

In case the underlying space X is a k-vector space $X = V$ and G acts by vector space automorphisms $G \to GL(V)$, then ρ is said to be a **linear** action, and V is called a G**-module**.

Given $n \geq 1$, let U_n denote the subgroup of $GL_n(k)$ consisting of upper triangular matrices with each diagonal entry equal to 1. A linear algebraic group $G \subset GL_n(k)$ is called **unipotent** if and only if it is conjugate to a subgroup of U_n; equivalently, the only eigenvalue of G is 1. G is **reductive** if and only if G is connected and contains no nontrivial connected normal unipotent subgroup. It is well-known that, when G is a reductive group acting on X, then $k[X]^G$ is affine (so the categorical and algebraic quotients are the same), and the quotient map is always surjective and separates closed orbits (see Kraft [174]).

Regarding group actions, our primary interest is in algebraic $\mathbb{G}_a$-actions, where $\mathbb{G}_a$ denotes the **additive group** of the field k. Also important are the algebraic $\mathbb{G}_m$-actions, where $\mathbb{G}_m$ denotes the **multiplicative group of units** of k. Other common notations are $\mathbb{G}_a(k)$ or k^+ for $\mathbb{G}_a$, and $\mathbb{G}_m(k)$,$k^\times$, or k^* for $\mathbb{G}_m$. Under the assumption k is algebraically closed, $\mathbb{G}_a$ and $\mathbb{G}_m$ are the only irreducible algebraic k-groups of dimension 1 (20.5 of [147]). For non-algebraically closed fields, there may be other such groups, for example, the circle group S^1 over the field $\mathbb{R}$ of real numbers (which is reductive).

Any group $\mathbb{G}_m^n$ $(n \geq 1)$ is called an **algebraic torus** of dimension n. Likewise, the group $\mathbb{G}_a^n$ $(n \geq 1)$ is called the **vector group** of dimension n. The algebraic tori $\mathbb{G}_m^n$ are reductive, and the vector groups $\mathbb{G}_a^n$ are unipotent.

Suppose $\rho : \mathbb{G}_a \times X \to X$ is an algebraic $\mathbb{G}_a$-action on the algebraic variety X, noting that $\mathbb{G}_a \cong \mathbb{A}^1$. Then $\mathbb{G}_a$ also acts on the coordinate ring $k[X]$.

- ρ is **equivariantly trivial** or **globally trivial** if and only if there exists an affine variety Y and an equivariant isomorphism $\phi : X \to Y \times \mathbb{A}^1$, where $\mathbb{G}_a$ acts trivially on Y, and $\mathbb{G}_a$ acts on $\mathbb{A}^1$ by $t \cdot x = x + t$.

- ρ is **locally trivial** if and only if there exists a covering $X = \cup_{i=1}^{n} X_i$ by Zariski open sets such that the action restricts to an equivariantly trivial action on each X_i.

Observe that these definitions can be extended to other categories. For example, we are assuming $\mathbb{G}_a$ acts by algebraic automorhisms, with ϕ being an algebraic isomorphism; but when $k = \mathbb{C}$, $\mathbb{G}_a$ may act by holomorphic automorphisms, with ϕ being a holomorphic isomorphism.

On the subject of affine k-varieties, there are a number of good references, including Harris [139], Hartshorne [140], Kunz [178], or Miyanishi [219]. For references to actions of algebraic groups and classical invariant theory, the reader should see Bass [13], Derksen and Kemper [71], Dieudonné and Carrell [83], Dolgachev [87], Humphreys [147], Kraft [174], Kraft and Procesi [177], Popov [254], together with the nice review of Popov's book written by Schwarz [271], or van den Essen [100], Chap. 9. The article of Greuel and Pfister [129] and the book of Grosshans [131] focus on the invariant theory of unipotent groups. The latter includes a wealth of historical references.

If X is an affine variety, elements of the Lie algebra $\mathrm{Der}_k(k[X])$ can be viewed as **vector fields** on X with polynomial coefficients, and can be used to study X; see the article of Siebert [280]. For a discussion of locally finite group actions relative to derivations and vector fields, see Drensky and Yu [90], Draisma [88], or Cohen and Draisma [36].

1.4 First Principles for Locally Nilpotent Derivations

We next turn our attention to the locally nilpotent case, with the ongoing assumption that k is any field of characteristic zero, and B is a commutative k-domain.

Principle 1. *Suppose* $D \in \mathrm{LND}(B)$*.*

(a) $\ker D$ *is factorially closed.*
(b) $B^* \subset \ker D$*. In particular,* $\mathrm{LND}(B) = \mathrm{LND}_k(B)$*.*
(c) *If* $D \neq 0$*, then* D *admits a local slice* $r \in B$*.*
(d) $\mathrm{Aut}_k(B)$ *acts on* $\mathrm{LND}(B)$ *by conjugation.*

Proof. By *Prop. 1.9*, ν_D is a degree function, and it was observed earlier that for any degree function, the set of degree-zero elements forms a factorially closed subring containing B^*. This is the content of (a) and (b).

For (c), choose $b \in B$ such that $Db \neq 0$, and set $n = \nu_D(b) \geq 1$. Then $D^n b \neq 0$ and $D^{n+1} b = 0$, so we may take $r := D^{n-1} b$.

Part (d) is due to the observation that $(\alpha D \alpha^{-1})^n = \alpha D^n \alpha^{-1}$ for any $\alpha \in \mathrm{Aut}_k(B)$ and $n \geq 0$. $\square$

Note that, while derivations of fields are of interest and importance, the foregoing result shows that the only locally nilpotent derivation of a field is the zero derivation.

Corollary 1.18. *If K is a field of characteristic 0, then* $\mathrm{LND}(K) = \{0\}$.

Remark 1.19. The kernel of a locally finite derivation may fail to be factorially closed. For example, on the polynomial ring $k[x,y]$ define a k-derivation D by $Dx = y$ and $Dy = x$. Then $D(x^2 - y^2) = 0$, whereas neither $D(x-y)$ nor $D(x+y)$ is 0. Likewise, there are factorially closed subrings $A \subset B$ with $\mathrm{tr.deg.}_A B = 1$ which are not the kernel of any locally nilpotent derivation. For example, we may take $A = k[x^2 - y^3]$ in $B = k[x,y]$ (see [49] p.226, first remark).

Principle 2. *Let S be any subset of B which generates B as k-algebra, and let $D \in \mathrm{Der}_k(B)$. Then*

$$D \in \mathrm{LND}(B) \quad \Leftrightarrow \quad S \subset \mathrm{Nil}(D) \ .$$

Proof. This follows immediately from the fact that $\mathrm{Nil}(D)$ is a subalgebra (*Prop. 1.9(b)*). □

Suppose B is finitely generated over $A = \ker D$, namely, $B = A[x_1, ..., x_n]$. Then this result implies that $D \in \mathrm{LND}(B)$ if and only if there exists $N \in \mathbb{Z}$ such that $D^N x_i = 0$ for each i.

Principle 3. *Suppose $D \in \mathrm{LND}(B)$ is nonzero, and set $A = \ker D$. If $P \in A^{[1]}$ and $t \in B$ are such that neither t nor $P(t)$ is zero, then*

$$\nu_D(P(t)) = (\deg P) \cdot \nu_D(t) \ .$$

Proof. The case $\nu_D(t) = 0$ is trivial, so assume $\nu_D(t) > 0$. For each $a \in A$ ($a \neq 0$) and $i \in \mathbb{N}$ we have $\nu_D(at^i) = i\nu_D(t)$. Thus, the nonzero terms in $P(t)$ have distinct ν_D-degrees, and the desired conclusion follows. □

Principle 4. *Given an ideal I of B, and given $D \in \mathrm{LND}(B)$, $D \neq 0$, D induces a well-defined element $D/I \in \mathrm{LND}(B/I)$ if and only if I is an integral ideal of D. In addition, if I is an integral ideal of D which is a maximal ideal of B, then $DB \subset I$.*

Proof. Only the second assertion requires demonstration. If I is a maximal ideal, then B/I is a field, and thus $\mathrm{LND}(B/I) = \{0\}$. In particular, $D/I = 0$, so $DB \subset I$. □

Principle 5. *Let $D \in \mathrm{LND}(B)$ and $f_1, ..., f_n \in B$ ($n \geq 1$) be given. Suppose there exists a permutation $\sigma \in S_n$ such that $Df_i \in f_{\sigma(i)}B$ for each i. Then in each orbit of σ there is an i with $Df_i = 0$.*

Proof. Suppose $Df_i \neq 0$ for each i, and choose $a_1, ..., a_n \in B$ such that $Df_i = a_i f_{\sigma(i)}$. Then $\nu_D(f_i) \geq 1$ and $\nu_D(a_i) \geq 0$ for each i. It follows that, for each i,

$$\nu_D(f_i) - 1 = \nu_D(Df_i) = \nu_D(a_i f_{\sigma(i)}) = \nu_D(a_i) + \nu_D(f_{\sigma(i)}) \geq \nu_D(f_{\sigma(i)}) \ .$$

Therefore,

$$\sum_{1\le i\le n} \nu_D(f_i) - n \ge \sum_{1\le i\le n} \nu_D(f_{\sigma(i)}) \ ,$$

which is absurd, since the two summations have the same value. Therefore, $Df_i = 0$ for at least one i. Now apply this result to the decomposition of σ into disjoint cycles, and the desired result follows. □

The case $n = 1$ above is especially important.

Corollary 1.20. *If $Df \in fB$ for $D \in \mathrm{LND}(B)$ and $f \in B$, then $Df = 0$.*

Principle 6. (Prop. 1 of [116]) *Suppose B is a commutative k-domain, and $D \in \mathrm{LND}(B)$. Assume that D is extended to a derivation D^* of the ring $B[t] = B^{[1]}$. Then D^* is locally nilpotent if and only if $D^*t \in B$.*

Proof. If $D^*t \in B$, then since $B \subset \mathrm{Nil}(D^*)$, it follows that $t \in \mathrm{Nil}(D^*)$. So in this case, D^* is locally nilpotent by *Princ. 2* above.

Conversely, assume D^* is locally nilpotent, but that $D^*t \notin B$. Choose N such that $(D^*)^N t \notin B$, but $(D^*)^{N+1} t \in B$, which is possible since D^* is locally nilpotent. Then $P(t) := (D^*)^N t$ is of positive t-degree. Suppose $\deg_t P(t) = m \ge 1$ and $\deg_t D^*t = n \ge 1$, and write $P(t) = \sum_i b_i t^i$ for $b_i \in B$. Then

$$D^*(P(t)) = P'(t)D^*t + \sum_i (Db_i)t^i \ ,$$

which belongs to B, and thus has t-degree 0. It follows that $(m-1)+n \le m$, so $n = 1$, i.e., D^*t is linear in t. This implies that $\deg_t (D^*)^i t \le 1$ for all $i \ge 0$, and in particular we must have $m = 1$. Write $P(t) = at+b$ and $D^*t = ct+d$ for $a,b,c,d \in B$ and $a,c \ne 0$. Then $D^*(at+b) = (ac+Da)t+ad+Db$ belongs to B, meaning that $ac + Da = 0$. But then $Da \in aB$, so by the preceding corollary, $Da = 0$. But then $ac = 0$ as well, a contradiction. Therefore, $D^*t \in B$. □

Principle 7. *Given $D \in \mathrm{Der}_k(B)$, and given nonzero $f \in B$,*

$$fD \in \mathrm{LND}(B) \quad \Leftrightarrow \quad D \in \mathrm{LND}(B) \text{ and } f \in \ker D \ .$$

Proof. Suppose $fD \in \mathrm{LND}(B)$ but $\mathrm{Nil}(D) \ne B$. Then $D \ne 0$. Set $N = \nu_{fD}(f) \ge 0$, and choose $g \in B - \mathrm{Nil}(D)$. It follows that $g \ne 0$, $\nu_{fD}(g) \ge 0$, and $\nu_{fD}(D^n g) \ge 0$ for all $n \ge 1$. On the one hand, we have

$$\nu_{fD}(f \cdot D^n g) = \nu_{fD}((fD)(D^{n-1}g)) = \nu_{fD}(D^{n-1}g) - 1 \ .$$

On the other hand, we see that

$$\nu_{fD}(f \cdot D^n g) = \nu_{fD}(f) + \nu_{fD}(D^n g) = N + \nu_{fD}(D^n g) \ .$$

Therefore,

$$\nu_{fD}(D^n g) = \nu_{fD}(D^{n-1}g) - (N+1) \quad \text{for all } n \ge 1 \ .$$

This implies

$$\nu_{fD}(D^n g) = \nu_{fD}(g) - n(N+1) \ ,$$

which is absurd since it means ν_{fD} has values in the negative integers. Therefore, $D \in \mathrm{LND}(B)$. To see that $f \in \ker D$, note that $(fD)(f) \in fB$. By *Cor. 1.20*, it follows that $f \in \ker(fD)$; and since B is a domain, $\ker(fD) = \ker D$.

The converse is immediate. □

Remark 1.21. Nowicki gives an example to show that the result above may fail for a non-reduced ring. Let $R = \mathbb{Q}[x]/(x^3) = \mathbb{Q}[\bar{x}]$, and let $d \in \mathrm{Der}_{\mathbb{Q}}(R)$ be defined by $d\bar{x} = \bar{x}^2$. Then $d, \bar{x}d \in \mathrm{LND}(R)$, but $\bar{x} \notin \ker d$ ([247], 8.1.3).

Principle 8. *Suppose $B = A[t]$, where A is a subring of B and t is transcendental over A. Then*

(a) $\frac{d}{dt} \in \mathrm{LND}_A(A[t])$
(b) $\ker(\frac{d}{dt}) = A$
(c) $\mathrm{LND}_A(A[t]) = A \cdot \frac{d}{dt}$

Proof. Part (a) is an immediate consequence of *Prop. 1.6*, since this shows that each application of $\frac{d}{dt}$ reduces degree in $A[t]$ by one.

By definition, $A \subset \ker(\frac{d}{dt})$. Conversely, suppose $P(t) \in \ker(\frac{d}{dt})$. If $\deg P \geq 1$, then since this kernel is algebraically closed, it would follow that $t \in \ker(\frac{d}{dt})$, a contradiction. Therefore, $\ker(\frac{d}{dt}) = A$.

For (c), let $D \in \mathrm{LND}_A(B)$ be given, $D \neq 0$. By *Prop. 1.8(c)*, for any $p(t) \in A[t]$, $D(p(t)) = p'(t)Dt$. Consequently, $D = Dt\frac{d}{dt}$. Since both D and $\frac{d}{dt}$ are locally nilpotent, *Princ. 7* implies that $Dt \in A$. Therefore, $\mathrm{LND}_A(A[t]) \subseteq A \cdot \frac{d}{dt}$. The reverse inclusion is implied by *Princ. 7*. □

Principle 9. *Let $S \subset B - \{0\}$ be a multiplicatively closed set, and let $D \in \mathrm{Der}_k(B)$ be given. Then*

$$S^{-1}D \in \mathrm{LND}(S^{-1}B) \quad \Leftrightarrow \quad D \in \mathrm{LND}(B) \text{ and } S \subset \ker D \ .$$

In this case, $\ker(S^{-1}D) = S^{-1}(\ker D)$.

Proof. Suppose $S^{-1}D$ is locally nilpotent, noting that D is clearly locally nilpotent in this case. Since $S \subset (S^{-1}B)^* \subset \ker(S^{-1}D)$, it follows that $S \subset \ker D$.

Conversely, suppose D is locally nilpotent and $S \subset \ker D$. Let $f/g \in S^{-1}B$ be given. Since $g \in \ker D$, it follows immediately from the quotient rule that $g^{-1} \in \ker(S^{-1}D)$. Thus,

$$(S^{-1}D)^n(f/g) = g^{-1}(S^{-1}D)^n(f) = g^{-1}D^n f = 0 \quad \text{for} \quad n \gg 0 \ .$$

Therefore, $S^{-1}D$ is locally nilpotent.

Assuming $S \subset \ker D$, it follows that

$$S^{-1}D(f/g) = 0 \Leftrightarrow (1/g)Df = 0 \Leftrightarrow Df = 0 \Leftrightarrow f/g \in S^{-1}(\ker D) \ .$$

□

Principle 10. *Suppose* $D \in \mathrm{LND}(B)$.

(a) $\exp D \in \mathrm{Aut}_k(B)$
(b) *If* $[D, E] = 0$ *for* $E \in \mathrm{LND}(B)$, *then* $D + E \in \mathrm{LND}(B)$ *and*

$$\exp(D+E) = \exp D \circ \exp E \ .$$

(c) *The subgroup of* $\mathrm{Aut}_k(B)$ *generated by* $\{\exp D | D \in \mathrm{LND}(B)\}$ *is normal.*

Proof. Since every function D^i is additive, $\exp D(f) = \sum_{i \geq 0} \frac{1}{i!} D^i f$ is an additive function. To see that $\exp D$ respects multiplication, suppose $f, g \in B$ are nonzero, with $\nu_D(f) = m$ and $\nu_D(g) = n$. Then $D^i f = D^j g = 0$ for $i > m$ and $j > n$, and

$$\begin{aligned}
(\exp D)(f) \cdot (\exp D)(g) &= \left(\sum_{0 \leq i \leq m} \frac{1}{i!} D^i f \right) \cdot \left(\sum_{0 \leq j \leq n} \frac{1}{j!} D^j g \right) \\
&= \sum_{0 \leq i+j \leq m+n} \frac{1}{i!j!} D^i f D^j g \\
&= \sum_{0 \leq i+j \leq m+n} \frac{1}{(i+j)!} \binom{i+j}{j} D^i f D^j g \\
&= \sum_{0 \leq t \leq m+n} \frac{1}{t!} \left(\sum_{i+j=t} \binom{i+j}{j} D^i f D^j g \right) \\
&= \sum_{0 \leq t \leq m+n} \frac{1}{t!} D^t (fg) \\
&= (\exp D)(fg) \ .
\end{aligned}$$

The penultimate line follows from the preceding line by the higher product rule. Thus, $\exp D$ is an algebra homomorphism.

Next, let $f \in B$ be given, and choose $m \geq 0$ so that $D^m f = E^m f = 0$. Set $n = 2m$. Since D and E commute,

$$(D+E)^n (f) = \sum_{i+j=n} \binom{n}{i} D^i E^j (f) \ .$$

For each term of this sum, either $i \geq m$ or $j \geq m$, and it follows that $D^i E^j (f) = E^j D^i (f) = 0$ for each pair i, j. Therefore, $D + E \in \mathrm{LND}(B)$. Further, by using this same expansion for $(D+E)^n$, the proof that $\exp(D+E) = \exp D \circ \exp E$ now follows formally exactly as above. Thus, (b) is proved.

In addition, note that by *Princ. 7*, $-D \in \mathrm{LND}(B)$. Thus, by part (b), it follows that

$$\exp D \circ \exp(-D) = \exp(-D) \circ \exp D = \exp 0 = I \ .$$

Therefore, $\exp D$ is an automorphism, and (a) is proved.

Finally, part (c) follows from the observation that

$$\alpha(\exp D)\alpha^{-1} = \exp(\alpha D\alpha^{-1})$$

for any $\alpha \in \mathrm{Aut}_k(B)$, and that $\alpha D\alpha^{-1}$ is again locally nilpotent. □

Principle 11. *Let $D \in \mathrm{LND}(B)$ be given, $D \neq 0$, and set $A = \ker D$. Choose a local slice $r \in B$ of D, and let $\pi_r : B \to B_{Dr}$ denote the Dixmier map defined by r.*

(a) $\pi_r(B) \subset A_{Dr}$
(b) π_r *is a k-algebra homomorphism.*
(c) $\ker \pi_r = rB_{Dr} \cap B$
(d) $B_{Dr} = A_{Dr}[r]$
(e) *The transcendence degree of B over A is 1.*

Proof. Consider first the case $Ds = 1$ for some $s \in B$.

For (a), recall the definition

$$\pi_s(h) = \sum_{i\geq 0} \frac{(-1)^i}{i!} D^i h s^i \ .$$

From this, one verifies immediately that $D(\pi_s(h)) = 0$ for all $h \in B$. Therefore, $\pi_s(B) \subset A = A_{Ds}$.

For (b), let t be transcendental over B, and extend D to $B[t]$ via $Dt = 0$. Let $\iota : B \hookrightarrow B[t]$ be inclusion, and let $\epsilon : B[t] \to B$ be the evaluation map $\epsilon(t) = s$. By principles *7* and *10*, $\exp(-tD)$ is an automorphism of $B[t]$. In addition, $\pi_s = \epsilon \circ \exp(-tD) \circ \iota$. Therefore, π_s is a homomorphism.

For (c), note that $\pi_s(s) = s-(Ds)s = 0$. Therefore, $\pi(sB) = 0$. Conversely, if $\pi_s(f) = 0$, then since $\pi_s(f) = f + sb$ for some $b \in B$, we conclude that $f \in sB$. Therefore, $\ker(\pi_s) = sB$ when $Ds = 1$.

Next, since the kernel of D on $B[t]$ equals $A[t]$, π_s extends to a homomorphism $\pi_s : B[t] \to A[t]$. Define the homomorphism $\phi : B \to A[s]$ by $\phi = \epsilon \circ \pi_s \circ \exp(tD) \circ \iota$. Specifically, ϕ is defined by

$$\phi(g) = \sum_{n\geq 0} \frac{1}{n!} \pi_s(D^n g)s^n \ .$$

Then ϕ is a surjection, since $\phi(a) = a$ for $a \in A$, and $\phi(s) = s$. Also, if $\phi(g) = 0$, then since s is transcendental over A by *Prop. 1.8*, it follows that each coefficient of $\phi(g) \in A[s]$ is zero. If $g \neq 0$, then the highest-degree coefficient of $\phi(g)$ equals $(1/n!)\pi_s(D^n g)$, where $n = \nu_D(g) \geq 0$. Thus, $D^n g \in \ker \pi_s = sB$, and since also $D^n g \in A - 0$, we conclude that $s \in A$ (since A is factorially closed). But $s \notin A$, so it must be that $g = 0$. Therefore, ϕ is an isomorphism, and (d) is proved.

We have now proved (a)-(d) in the special case $Ds = 1$.

For the general case, suppose that, for the local slice r, $Dr = f \in A$. Let D_f denote the extension of D to B_f. Then $s := r/f$ is a slice of D_f. Since π_r is the restriction to B of the homomorphism $\pi_s : B_f \to B_f$, it follows that π_r is a homomorphism. The kernel is $sB_f \cap B = rB_f \cap B$, and $B_f = A_f[s] = A_f[r]$. Therefore, results (a)-(d) hold in the general case.

Finally, (e) follows immediately from (d). □

Several important corollaries are implied by this result.

Corollary 1.22. (Slice Theorem) *Suppose $D \in \mathrm{LND}(B)$ admits a slice $s \in B$, and let $A = \ker D$. Then:*

(a) $B = A[s]$ *and* $D = \frac{d}{ds}$
(b) $A = \pi_s(B)$ *and* $\ker \pi_s = sB$
(c) *If B is affine, then A is affine.*

Proof. Immediate. □

Corollary 1.23. *Let B be any commutative k-domain, and let $D \in \mathrm{LND}(B)$. Let δ denote the extension of D to a derivation of the field* $\mathrm{frac}(B)$ *(so δ is not locally nilpotent if $D \neq 0$). Then* $\ker \delta = \mathrm{frac}(\ker D)$.

Proof. Let $A = \ker D$. If r is a local slice of D and $f = Dr$, then $B \subset B_f = A_f[r]$. We therefore have $\mathrm{frac}(B) \subset \mathrm{frac}(A)(r) \subset \mathrm{frac}(B)$, which implies $\mathrm{frac}(B) = \mathrm{frac}(A)(r)$. Now suppose $\delta g = 0$ for $g \in \mathrm{frac}(B)$, and write $g = P(r)$ for the rational function P having coefficients in $\mathrm{frac}(A)$. Then $0 = P'(r)\delta r$, and since $\delta r \neq 0$, $P'(r) = 0$. It follows that $g = P(r) \in \mathrm{frac}(A)$, which shows $\ker \delta \subset \mathrm{frac}(\ker D)$. The reverse containment is obvious. □

Corollary 1.24. *If S is a commutative k-domain such that* $\mathrm{tr.deg.}_k S = 1$ *and* $\mathrm{LND}(S) \neq \{0\}$, *then $S = K^{[1]}$, where K is a field algebraic over k. If, in addition, k is algebraically closed, then $S = k^{[1]}$.*

Proof. Suppose $\delta \in \mathrm{LND}(S)$ is nonzero, and set $K = \ker \delta$. We have $k \subset K \subset S$, where $\mathrm{tr.deg.}_K S = 1$. Therefore, $\mathrm{tr.deg.}_k K = 0$, i.e., K is an algebraic field extension of k. If $r \in S$ is a local slice of δ, then $\delta r \in K^*$, so $s = (\delta r)^{-1} r$ is a slice of δ. It follows that $S = K[s]$. □

Another immediate implication of *Princ. 11*, combined with *Prop. 1.9*, is the following degree formula. (See also Cor. 2.2 of [74] and 1.3.32 of [100].)

Corollary 1.25. *Given $D \in \mathrm{LND}(B)$, set $L = \mathrm{frac}(B)$ and $K = \mathrm{frac}(B^D)$. If $b \in B$ and $b \notin B^D$, then $\nu_D(b) = [L : K(b)]$.*

The next result examines the case in which two kernels coincide, and is due to Daigle.

Principle 12. (Lemma 1.1 of [49]) *Suppose $D, E \in \mathrm{LND}(B)$ are such that $A := \ker D = \ker E$. Then there exist nonzero $a, b \in A$ such that $aD = bE$.*

Proof. We may assume $D, E \neq 0$. By *Princ. 11 (d)*, there exists $c \in A$ and $t \in B$ such that $B_c = A_c[t]$. By *Princ. 9*, $\ker(D_c) = \ker(E_c) = A_c$. Therefore, by *Princ. 8*, $D_c = \beta \cdot (d/dt)$ and $E_c = \alpha \cdot (d/dt)$ for some $\alpha, \beta \in A_c$. Consequently, $\alpha D_c = \beta E_c$. Choose $n \in \mathbb{Z}_+$ so that $a := c^n \alpha$ and $b := c^n \beta$ belong to A. Then $aD_c = bE_c$. Restriction to B gives the desired result. □

A second result of Daigle is the following. In case B is an affine ring, it characterizes those subrings of B which occur as the kernel of some locally nilpotent derivation.

Principle 13. (Prop. 1.4 of [49]) *Let A be a subalgebra of B other than B itself, $S = A - \{0\}$, and $K = S^{-1}A$, the field of fractions of A. Consider the following statements.*

1. $A = \ker D$ *for some* $D \in \mathrm{LND}(B)$.
2. $S^{-1}B = K^{[1]}$ *and* $A = K \cap B$.

In all cases, (1) implies (2). If, in addition, B is finitely generated over A, then (2) implies (1).

Proof. That (1) implies (2) follows immediately from part (d) of *Princ. 11*, together with *Princ. 9*. Conversely, assume (2) holds and that $B = A[b_1, ..., b_n]$ for some $b_i \in B$. Since $\frac{d}{dt}(b_i) \in K[t]$ for each i, there exists $s \in S$ so that $s\frac{d}{dt}(B) \subset B$. Since $s \in K$, $s\frac{d}{dt}$ is locally nilpotent. If D denotes the restriction of $s\frac{d}{dt}$ to B, it follows that D is also locally nilpotent, and $\ker D = B \cap \ker(s\frac{d}{dt}) = B \cap K = A$. □

Principle 14. (Compare to Princ. II of [97]) *Suppose B is the graded ring $B = \oplus_{i \in \mathbb{Z}} B_i$, and let $D \in \mathrm{LND}(B)$ be given. Suppose that, for integers $m \leq n$, D admits a decomposition $D = \sum_{m \leq i \leq n} D_i$, where each $D_i \in \mathrm{Der}_k(B)$ is homogeneous of degree i relative to this grading, and where $D_m \neq 0$ and $D_n \neq 0$.*

(a) $D_m, D_n \in \mathrm{LND}(B)$
(b) *If $f \in \ker D$ and $f = \sum_{u \leq i \leq v} f_i$ for $f_i \in B_i$, then $f_u \in \ker D_m$ and $f_v \in \ker D_n$.*

Proof. Given $t \in \mathbb{Z}_+$, the function D^t is a sum of functions of the form $D_{i_1} D_{i_2} \cdots D_{i_t}$, where $m \leq i_j \leq n$ for each j. In particular, every such "monomial" is a homogeneous function on B, and the highest degree summand appearing in D^t is $(D_n)^t$.

Suppose $F \in B$ is homogeneous but not 0, and set $t = \nu_D(F) + 1 \geq 1$. Then the highest-degree homogeneous summand of $D^t F$ equals $(D_n)^t F$, and since $0 = D^t F$, it follows that $(D_n)^t F = 0$. Therefore, the subalgebra $\mathrm{Nil}(D_n)$ contains every homogeneous element of B, and it follows that $\mathrm{Nil}(D_n) = B$. Likewise, the lowest-degree homogeneous summand of $D^t F$ is $(D_m)^t F = 0$, which implies D_m is locally nilpotent. So (a) is proved.

Similarly, we have

$$0 = Df = \sum_{\substack{m \le i \le n \\ u \le j \le v}} D_i f_j \ .$$

Each term $D_i f_j$ is homogeneous, and the degree of $D_n f_v$ exceeds that of any other term. Therefore, $D_n f_v = 0$. Likewise, the term of least degree is $D_m f_u$, which must also be zero. $\square$

Principle 15. (Lemma 4 of [193], and Lemma 3 of [190]) *Let $B = \cup_{i \in \mathbb{Z}} B_i$ be a proper $\mathbb{Z}$-filtration, and suppose $D \in \mathrm{LND}(B)$ respects this filtration. Then $\mathrm{gr}(D) \in \mathrm{LND}(\mathrm{Gr}(B))$.*

Proof. We may assume $D \neq 0$. Let δ denote the derivation $\mathrm{gr}(D)$ on $\mathrm{Gr}(B)$, and let ρ denote the natural mapping of B into $\mathrm{Gr}(B)$. It suffices to show that δ is locally nilpotent on the generating sets B_i/B_{i-1} of $\mathrm{Gr}(B)$.

Because D respects the filtration, we may consider the least integer t so that $DB_i \subset B_{i+t}$ for every i. Given nonzero $a \in B$, let $\iota(a) = i$, which means that $a \in B_i \cap B_{i-1}^C$. By hypothesis, $Da \in B_{i+t}$, so $\iota(Da) \le i + t$. If $\iota(Da) < i + t$, then $Da \in B_{i+t-1}$ and $\delta(a + B_{i-1}) = Da + B_{i+t-1} = 0$. Otherwise, $\iota(Da) = i + t$, meaning $\delta(\rho(a)) = \rho(Da))$. By iteration, we have that either $\delta^n(\rho(a)) = 0$, or $\delta^n(\rho(a)) = \rho(D^n a))$. Since D is locally nilpotent, we conclude that $\delta^{n+1}(\rho(a)) = 0$ for $n = \nu_D(a)$. $\square$

The final basic principle in our list is due to Vasconcelos; the reader is referred to [300] for its proof. (Note: Vasconcelos's definition of "locally finite" is the same as the present definition of locally nilpotent.) Other proofs appear in van den Essen [100], 1.3.37; and in Wright [311], Prop. 2.5.

Principle 16. (Vasconcelos's Theorem) *Suppose $R \subset B$ is a subring over which B is integral. If $D \in \mathrm{Der}_k(B)$ restricts to a locally nilpotent derivation of R, then $D \in \mathrm{LND}(B)$.*

The proof given by Vasconcelos shows that if $L = \mathrm{frac}(\ker D)$ and $K = \mathrm{frac}(\ker D|_R)$, then L is a finite extension of K. Thus, if T is the set of nonzero elements of $\ker D$, and S is the set of nonzero elements of $\ker(D|_R)$, then $T^{-1}B = L[t]$ and $S^{-1}R = K[t]$, where $t \in R$ is a local slice, and $D = \frac{d}{dt}$ in each case.

Observe that the condition "integral" in Vasconcelos's theorem cannot be weakened to "algebraic", since B may be a nontrivial algebraic extension of $\mathrm{Nil}(D)$.

1.4.1 Remarks

Remark 1.26. For any derivation D, an exponential map $\exp(tD) : B[[t]] \to B[[t]]$ can be defined by $\exp(tD)(f) = \sum_{i \ge 0}(1/i!)(D^i f)t^i$, where t is transcendental over B. Again, this is a ring automorphism, and the proof is identical to the one above. This map can be useful in proving that a given derivation is locally finite or locally nilpotent, as for example in [300].

Remark 1.27. One difficulty in working with locally nilpotent derivations is that $\mathrm{LND}(B)$ admits no obvious algebraic structure. For example, for the standard derivative $\frac{d}{dt}$ on the polynomial ring $k[t]$, we have seen that $\frac{d}{dt}$ is locally nilpotent, whereas $t\frac{d}{dt}$ is not. Thus, $\mathrm{LND}(B)$ is not closed under multiplication by elements of B, and does not form a B-module.

Likewise, if $k[x,y]$ is a polynomial ring in two variables over k, the derivations $D_1 = y(\partial/\partial x)$ and $D_2 = x(\partial/\partial y)$ are locally nilpotent, where $\partial/\partial x$ and $\partial/\partial y$ denote the usual partial derivatives. However, neither $D_1 + D_2$ nor $[D_1, D_2]$ is locally nilpotent. So $\mathrm{LND}(B)$ is also not closed under addition or bracket multiplication.

1.5 $\mathbb{G}_a$-Actions

In this section, assume that the field k is algebraically closed (still of characteristic zero). Let B be an affine k-domain, and let $X = \mathrm{Spec}(B)$ be the corresponding affine variety.

Given $D \in \mathrm{LND}(B)$, by combining *Princ. 7* and *Princ. 10*, we obtain a group homomorphism

$$\eta : (\ker D, +) \to \mathrm{Aut}_k(B)\ , \eta(f) = \exp(fD)\ .$$

In addition, if $D \neq 0$, then η is injective. Restricting η to the subgroup $\mathbb{G}_a = (k, +)$, we obtain the algebraic representation $\eta : \mathbb{G}_a \hookrightarrow \mathrm{Aut}_k(B)$. Geometrically, this means that D induces the faithful algebraic $\mathbb{G}_a$-action $\exp(tD)$ on X $(t \in k)$.

Conversely, let $\rho : \mathbb{G}_a \times X \to X$ be an algebraic $\mathbb{G}_a$-action over k. Then ρ induces a derivation $\rho'(0)$, where differentiation takes places relative to $t \in \mathbb{G}_a$.

To be more precise, at the level of coordinate rings, $\rho^* : B \to B[t]$ is a k-algebra homomorphism (since ρ is a morphism of algebraic k-varieties). Given $t \in k$ and $f \in B$, denote the action of t on f by $t \cdot f$. Define $\delta : B \to B$ by the composition

$$B \xrightarrow{\rho^*} B[t] \xrightarrow{d/dt} B[t] \xrightarrow{t=0} B$$

i.e., $\delta = \epsilon \frac{d}{dt}\rho^*$, where ϵ denotes evaluation at $t = 0$.

Proposition 1.28. $\delta \in \mathrm{LND}(B)$.

Proof. To see this, we first verify conditions (C.1) and (C.2). Condition (C.1) holds, since δ is composed of k-module homomorphisms. For (C.2), observe that, given $a \in B$, if $\rho^*(a) = P(t) \in B[t]$, then for each $t_0 \in k$, $t_0 \cdot a = P(t_0)$. In particular, $a = 0 \cdot a = P(0) = \epsilon\rho^*(a)$. Therefore, given $a, b \in B$:

$$\delta(ab) = \epsilon\frac{d}{dt}(\rho^*(a)\rho^*(b)) = \epsilon\left(\rho^*(a)\frac{d}{dt}\rho^*(b) + \rho^*(b)\frac{d}{dt}\rho^*(a)\right) = a\delta b + b\delta a$$

So condition (C.2) holds, and δ is a derivation.

To see that δ is locally nilpotent, let $f \in B$ be given, and suppose $\rho^*(f) = P(t) = \sum_{0 \le i \le n} f_i t^i$ for $f_i \in B$. For general $s, t \in k$, we have

$$(s+t) \cdot f = s \cdot (t \cdot f) = \sum_{0 \le i \le n} (s \cdot f_i) t^i \ .$$

On the other hand, it follows from Taylor's formula that

$$(s+t) \cdot f = P(s+t) = \sum_{0 \le i \le n} \frac{P^{(i)}(s)}{i!} t^i \ .$$

Equating coefficients yields: $s \cdot f_i = (1/i!) P^{(i)}(s)$ for all $s \in k$.

We now proceed by induction on the t-degree of $\rho^*(f)$. If the degree is zero, then $\delta(f) = P'(0) = 0$, and thus $f \in \mathrm{Nil}(\delta)$. Assume $g \in \mathrm{Nil}(\delta)$ whenever the degree of $\rho^*(g)$ is less than n. Then $\delta(f) = P'(0) = f_1$ and $\deg \rho^*(f_1) = \deg P'(s) = n - 1$. Thus, $\delta(f) \in \mathrm{Nil}(\delta)$, which implies $f \in \mathrm{Nil}(\delta)$ as well. $\square$

The reader can check that $D = (\exp(tD))'(0)$, and conversely $\rho = \exp(t\rho'(0))$. For other proofs, see [131], §8; [100], 9.5.2; and [47], §4.

In summary, there is a bijective correspondence between $\mathrm{LND}(B)$ and the set of all algebraic $\mathbb{G}_a$-actions on $X = \mathrm{Spec}(B)$, where $D \in \mathrm{LND}(B)$ induces the action $\exp(tD)$, and where the action ρ induces the derivation $\delta = \rho'(0)$, as described above. In addition, the kernel of the derivation coincides with the invariant ring of the corresponding action:

$$\ker D = B^{\mathbb{G}_a} \ ,$$

since $Df = 0$ if and only if $\exp(tD)(f) = f$ for all $t \in k$.

With this, many of the algebraic results we have established can be translated into geometric language. For example, *Cor. 1.24* becomes:

Corollary 1.29. *Let C be an affine curve over k (an algebraically closed field of characteristic zero). If C admits a nontrivial algebraic $\mathbb{G}_a$-action, then $C = \mathbb{A}^1$.*

Note that this result is also implied by a classical theorem of Rosenlicht, which asserts that every orbit of a unipotent group acting algebraically on a quasiaffine algebraic variety is closed [265]. For a short proof of Rosenlicht's theorem, see [28], Prop. 4.10.

Consider the algebraic $\mathbb{G}_a$-action $\exp(tD)$ on the variety $X = \mathrm{Spec}(B)$. Note that, since $\mathbb{G}_a \cong \mathbb{A}^1$, the orbit of the $\mathbb{G}_a$-action $\mathcal{O}_x = \{t \cdot x | t \in k\}$ for $x \in X$ is either a line $\mathbb{A}^1$ or a single point. By Rosenlicht's theorem, these orbits are closed in X, and when the action is nontrivial, the union of the one-dimensional orbits forms a Zariski-dense open subset of X. As to the fixed points of the $\mathbb{G}_a$-action, it is easy to see that these are defined by the ideal (DB) generated by the image of D; we denote this set by either $\mathrm{Fix}D$ or $X^{\mathbb{G}_a}$. The $\mathbb{G}_a$-action is fixed-point free if and only if $1 \in (DB)$. At the opposite extreme, suppose D is reducible, meaning $DB \subset fB$ for some non-unit $f \in B$. Then $V(f) \subset X^{\mathbb{G}_a} \subset X$, where $V(f)$ denotes the hypersurface of X defined by f. This means that D has the form fD' for some $D' \in \mathrm{LND}(B)$ and $f \in \ker D$ (*Princ. 7*).

An early and important result on the fixed points of $\mathbb{G}_a$-actions is due to Białynicki-Birula.

Theorem 1.30. [22] *If X is irreducible and affine, and* $\dim X \geq 1$, *then the algebraic action of any connected unipotent group G on X has no isolated fixed points.*

See also [20, 21, 123, 146] for related results.

Consider the geometric implications of *Princ. 11.* If $D \in \mathrm{LND}(B)$ has local slice $r \in B$, set $A = \ker D$ and $f = Dr$. Then $B_f = A_f[r]$ and the extension of D to B_f equals $\frac{d}{dr}$. Thus, the induced $\mathbb{G}_a$-action $\exp(tD)$ on $X = \mathrm{Spec}B$ restricts to an equivariantly trivial action on the principal open set U_f defined by f. Likewise, if $f_1, ..., f_n \in \mathrm{pl}(D) = A \cap DB$ and satisfy $f_1B + \cdots + f_nB = B$, then the principal open sets U_{f_i} cover X, and the $\mathbb{G}_a$-action on X is locally trivial (hence fixed-point free) relative to these open sets. And finally, if D admits a slice, then $X = Y \times \mathbb{A}^1$ for $Y = \mathrm{Spec}A$, and the action of $\mathbb{G}_a$ on X is equivariantly trivial relative to this decomposition: $t \cdot (y, z) = (y, z + t)$.

To summarize these algebro-geometric connections:

- free $\mathbb{G}_a$-action $\Leftrightarrow 1 \in (DB)$
- locally trivial $\mathbb{G}_a$-action $\Leftrightarrow 1 \in (\mathrm{pl}(D))$
- equivariantly trivial $\mathbb{G}_a$-action $\Leftrightarrow 1 \in DB$

Here, $(\mathrm{pl}(D))$ denotes the B-ideal generated by the A-ideal $\mathrm{pl}(D) = A \cap DB$. See also Thm. 2.5 of [81].

Remark 1.31. The bijection between locally nilpotent derivations and $\mathbb{G}_a$-actions described above remains valid over any field k of characteristic zero. The proofs would require a more general geometric setting.

Remark 1.32. A general fact connecting derivations to group actions should be mentioned, namely, the result of Nowicki [248], which asserts that for a polynomial ring B, if G is a connected algebraic group which acts algebraically on B, then there exists $D \in \mathrm{Der}_k(B)$ with $\ker D = B^G$. In particular, this means B^G is an algebraically closed subring of B. Derksen [66] constructed a derivation whose kernel coincides with the fixed ring of the group action in Nagata's famous counterexample to Hilbert's Fourteenth Problem (see *Chap. 6*).

Remark 1.33. An early result of Nagata (Thm. 4.1 of [237], 1962) is that the invariant ring of a $\mathbb{G}_a$-action on a factorial affine variety V has the following property: If every unit of $k[V]$ belongs to $k[V]^{\mathbb{G}_a}$, and if $f \in k[V]^{\mathbb{G}_a}$, then each prime factor of f belongs to $k[V]^{\mathbb{G}_a}$. In particular, $k[V]^{\mathbb{G}_a}$ is a UFD. Here, the characteristic of the field k is arbitrary. See also Lemma 1 of [210].

2

Further Properties of Locally Nilpotent Derivations

In this chapter, the purpose of the first three sections is to investigate derivations in the case B has one or more nice divisorial properties, in addition to the on-going assumption that B is a commutative k-domain, where k is a field of characteristic zero. Subsequent sections discuss the defect of a derivation, exponential automorphisms, and construction of kernel elements. The term unique factorization domain is abbreviated by UFD, and principal ideal domain by PID.

2.1 Irreducible Derivations

First, B is said to satisfy the **ascending chain condition (ACC) on principal ideals** if and only if every infinite chain $(b_1) \subset (b_2) \subset (b_3) \subset \cdots$ of principal ideals of B eventually stabilizes. Note that every UFD and every commutative noetherian ring satisfies this condition.

Lemma 2.1. *If B satisfies the ACC on principal ideals, so does $B^{[n]}$ for every $n \geq 0$.*

Proof. By induction, it suffices to show that $B^{[1]}$ has the ACC on principal ideals. Suppose

$$(p_1(t)) \subset (p_2(t)) \subset (p_3(t)) \subset \cdots$$

is an infinite chain of of principal ideals, where $p_i(t) \in B[t]$ (t an indeterminate over B). Since B is a domain, the degrees of the $p_i(t)$ must stabilize, so we may assume (truncating the chain if necessary) that for some positive integer d, $\deg_t p_i(t) = d$ for all i. (If $d = 0$ this is already a chain in B.) Thus, given i, there exist $e_i \in B$ with $p_i(t) = e_i p_{i+1}(t)$. For each integer m with $0 \leq m \leq d$, let $p_i^{(m)}$ denote the coefficient of t^m in $p_i(t)$. Equating coefficients, we have $p_i^{(m)} = e_i p_{(i+1)}^{(m)}$, which yields

$$(p_1^{(m)}) \subset (p_2^{(m)}) \subset (p_3^{(m)}) \subset \cdots$$

and this is an ascending chain of principal ideals in B. By the ACC, each such chain stabilizes, and since there are only finitely many such chains, we conclude that the given chain in $B[t]$ also stabilizes. $\square$

Next, we say that B is a **highest common factor ring**, or **HCF-ring**, if and only if the intersection of any two principal ideals of B is again principal. Examples of HCF-rings are: a UFD, a valuation ring, or a polynomial ring over a valuation ring.

Note that a UFD is an HCF-ring which also satisfies the ACC on principal ideals.

Recall that $D \in \mathrm{Der}_k(B)$ is irreducible if and only if DB is contained in no proper principal ideal. We will show that for commutative k-domains satisfying the ACC on principal ideals, a derivation is always a multiple of an irreducible derivation.

Proposition 2.2. (See also [47], Lemma 2.18) *Let $\delta \in \mathrm{Der}_k(B)$ and $\delta \neq 0$.*

(a) *If B satisfies the ACC for principal ideals, then there exists an irreducible $D \in \mathrm{Der}_k(B)$ and $a \in B$ such that $\delta = aD$.*
(b) *If B is an HCF-ring, and if $aD = bE$ for $a, b \in B$ and irreducible k-derivations D and E, then $(a) = (b)$.*
(c) *If B is a UFD and $\delta = aD$ for irreducible D and $a \in B$, then D is unique up to multiplication by a unit.*

Proof. Note first that, for any commutative k-domain B, if $D \in \mathrm{Der}_k(B)$ has $DB \subset aB$ for $a \in B$ and $a \neq 0$, then there exists $D' \in \mathrm{Der}_k(B)$ such that $D = aD'$. To see this, let $\Delta \in \mathrm{Der}_k(\mathrm{frac}B)$ be given by $\Delta = \frac{1}{a}D$. Then Δ is well-defined, and restricts to B, so we may take D' to be the restriction of Δ to B.

To prove part (a), suppose δ is not irreducible. Then $\delta B \subset a_1 B$ for some non-unit $a_1 \in B$. So there exists $D_1 \in \mathrm{Der}_k(B)$ with $\delta = a_1 D_1$, and since B is a domain, $\ker \delta = \ker D_1$. If D_1 is irreducible, we are done. Otherwise, continue in this way to obtain a sequence of derivations D_i, and non-units $a_i \in B$, such that $\ker D_i = \ker \delta$ for each i, and

$$\delta = a_1 D_1 = a_1 a_2 D_2 = a_1 a_2 a_3 D_3 = \cdots .$$

The process terminates after n steps if any D_n is irreducible, and part (a) will follow.

Otherwise this chain is infinite, with the property that every a_i is a non-unit of B. In this case, choose $f \in B$ not in $\ker \delta$. By the ACC on principal ideals, the chain

$$(\delta f) \subset (D_1 f) \subset (D_2 f) \subset (D_3 f) \subset \cdots$$

eventually stabilizes: $(D_n f) = (D_{n+1} f)$ for all $n \gg 0$. Therefore, there exists a sequence of units $u_i \in B$ with $u_n D_{n+1} f = D_n f = a_{n+1} D_{n+1} f$. Since

$D_{n+1}f \neq 0$, this implies $u_n = a_{n+1}$, i.e., every a_n is a unit for $n \gg 0$. We arrive at a contradiction, meaning this case cannot occur. So part (a) is proved.

For part (b), set $T = aD = bE$. Since B is an HCF-ring, there exists $c \in B$ with $aB \cap bB = cB$. Therefore, $TB \subset cB$, and there exists a k-derivation F of B such that $T = cF$. Write $c = as = bt$ for $s, t \in B$. Then $cF = asF = aD$ implies $D = sF$ (since B is a domain), and likewise $cF = btF = bE$ implies $E = tF$. By irreducibility, s and t are units of B, and thus $(a) = (b)$. So part (b) is proved.

Finally, part (c) follows immediately from parts (a) and (b), and the unique factorization hypothesis, since a UFD is both a ring satisfying the ACC on principal ideals and an HCF-ring. □

Corollary 2.3. *If B is a UFD, $D \in \mathrm{LND}(B)$ is irreducible, and $A = \ker D$, then $\mathrm{LND}_A(B) = \{aD | a \in A\}$.*

See Ex. 2.16 and 2.17 in [47] for examples in which the conclusions of these results fail for other rings.

2.2 Minimal Local Slices

Minimal local slices for a locally nilpotent derivation D are defined, and their basic properties discussed. The number of equivalence classes of minimal local slices can be a useful invariant of D. This number is closely related to the plinth ideal $DB \cap \ker D$ of D.

We assume throughout this section that B is a commutative k-domain which satisfies the ACC on principal ideals.

Fix $D \in \mathrm{LND}(B)$, $D \neq 0$, and set $A = \ker D$. Let $B = \cup_{i \geq 0} B_i$ be the filtration of B induced by D, i.e., nonzero elements $f \in B_i$ have $\nu_D(f) \leq i$. Note that the set of local slices for D is $B_1 - B_0$. An equivalence relation is defined on B_1 via:

$$r \sim s \quad \Leftrightarrow \quad A[r] = A[s] \ .$$

In particular, all kernel elements are equivalent.

Proposition 2.4. **(a)** *B satisfies the ACC on subalgebras of the form $A[r]$, $r \in B_1 - B_0$.*

(b) *Given $r_0 \in B_1 - B_0$, the set*

$$\{A[r] \mid r \in B_1 - B_0,\ A[r_0] \subseteq A[r]\} \ ,$$

partially ordered by set inclusion, contains at least one maximal element. Moreover, if $A[r]$ is maximal for this set, then $A[r]$ is also a maximal element of the superset $\{A[s] \mid s \in B_1 - B_0\}$, partially ordered by set inclusion.

Proof. Suppose $A[r_1] \subseteq A[r_2] \subseteq A[r_3] \subseteq \cdots$ for $r_i \in B_1 - B_0$. Given $i \geq 1$, since $r_i \in A[r_{i+1}] \cong A^{[1]}$, the degree of r_i as a polynomial in r_{i+1} (over A)

equals 1: otherwise $Dr_i \notin A$ (*Princ. 3*). For each $i \geq 1$, write $r_i = a_i r_{i+1} + b_i$ for some $a_i, b_i \in A$. Then, for each $i \geq 1$, $Dr_i = a_i \cdot Dr_{i+1}$. We thus obtain an ascending chain of principal ideals,

$$(Dr_1) \subseteq (Dr_2) \subseteq (Dr_3) \subseteq \cdots \ .$$

Since this chain must eventually stabilize, we conclude that all but finitely many of the a_i are units of B. It follows that $A[r_n] = A[r_{n+1}]$ for $n \gg 0$. This proves (a).

To prove (b), just use (a), and apply Zorn's Lemma. $\square$

We say that $r \in B_1$ is a **minimal local slice** for D if and only if $A[r]$ is a maximal element of the set $\{A[s] \mid s \in B_1\}$. The set of minimal local slices of D is denoted $\min(D)$, which by the proposition is non-empty if $D \neq 0$.

Proposition 2.5. *Let $\sigma \in B_1 - B_0$ be given. Then $\sigma \in \min(D)$ if and only if every $s \sim \sigma$ is irreducible.*

Proof. If $\sigma \in \min(D)$ factors as $\sigma = ab$, then $1 = \nu_D(ab) = \nu_D(a) + \nu_D(b)$, which implies either $\nu_D(a) = 0$ and $\nu_D(b) = 1$, or $\nu_D(a) = 1$ and $\nu_D(b) = 0$. Thus, either $a \in A$ and $b \in B_1$, or $a \in B_1$ and $b \in A$. Assuming $a \in A$ and $b \in B_1$, if a is not a unit of B, then $A[\sigma]$ is properly contained in $A[b]$, which is impossible. Therefore $a \in B^*$, and σ is irreducible. Since every $s \sim \sigma$ is also in $\min(D)$, every such s is also irreducible.

Suppose $\sigma \notin \min(D)$. Then there exists $r \in B_1$ such that $A[\sigma]$ is properly contained in $A[r]$. Since both σ and r are local slices, σ has degree 1 as a polynomial in r, i.e., $\sigma = ar + b$ for $a, b \in A$, $a \neq 0$, and $a \notin B^*$. Thus, $\sigma \sim ar$, which is reducible. $\square$

Proposition 2.6. *Let $r \in B_1 - B_0$ be given. If Dr is irreducible in B, then either $r \in \min(D)$ or D admits a slice (or both).*

Proof. If $r \notin \min(D)$, then $r = as + b$ for some $s \in \min(D)$ and some non-unit $a \in A$. Thus, $Dr = a \cdot Ds$, and since Dr is irreducibile, $Ds \in B^*$. $\square$

Proposition 2.7. *Given $D \in \mathrm{LND}(B)$ with $A = \ker D$, the following are equivalent.*

1. *D has a unique minimal local slice (up to equivalence).*
2. *The plinth ideal $DB \cap A$ is a principal ideal of A.*

Proof. Suppose that D has only one minimal local slice $r \in B$, up to equivalence, and let $a \in DB \cap A$ be given. Then there exists a local slice p of D such that $Dp = a$. By hypothesis, there exist $b, c \in A$ with $p = br + c$. Thus, $a = Dp = bDr \in (Dr)A$. Therefore, $DB \cap A = Dr \cdot A$.

Conversely, suppose $DB \cap A = fA$ for some $f \in A$, and let a minimal local slice r be given. Since, in particular $f \in DB$, there exists $s \in B$ with $Ds = f$. Clearly, s is also a local slice. If $Dr = fg$ for some $g \in A$, then $r - sg \in A$, which implies $A[r] \subset A[s]$. Since r is minimal, $A[r] = A[s]$, i.e., r and s are equivalent. $\square$

2.3 Three Lemmas about UFDs

The following three lemmas about UFDs are recorded here for future use. The first lemma and its proof seem to be well-known, although I could not find a reference.

Notice that the first lemma is valid in any characteristic. In addition, notice that we do not need to assume that B is an affine ring for the second and third lemmas.

Lemma 2.8. *Suppose k is an algebraically closed field. If B is an affine UFD over k with* $\mathrm{tr.deg.}_k B = 1$ *and $B^* = k^*$, then $B = k^{[1]}$.*

Proof. Let X denote the curve $\mathrm{Spec}(B)$. Since B is a UFD, X is normal (hence smooth), and its class group $\mathrm{Cl}(X)$ is trivial (see, for example, [140], II.6.2). Embed X in a complete nonsingular algebraic curve Y as an open subset. Then there are points $P_i \in Y$, $1 \leq i \leq n$, such that $Y - X = \{P_1, ..., P_n\}$. Since X is affine, $n \geq 1$.

Let F dentote the subgroup of $\mathrm{Cl}(Y)$ generated by the divisor classes $[P_1], ..., [P_n]$. Then $\{0\} = \mathrm{Cl}(X) = \mathrm{Cl}(Y)/F$, meaning $\mathrm{Cl}(Y)$ is finitely generated. It is known that if C is any complete nonsingular curve over k which is not rational, then $\mathrm{Cl}(C)$ is not finitely generated. (This follows from the fact that the jacobian variety of C is a divisible group; see Mumford [232], p. 62.) Therefore, Y is rational, which implies $Y = \mathbb{P}^1$ (the projective line over k).

It follows that X is the complement of n points in $\mathbb{P}^1$, which is isomorphic to the complement of $n-1$ points of $\mathbb{A}^1$. Therefore, $B = \mathcal{O}(X)$ has the form $k[t]_{f(t)}$ for some $f \in k[t] \cong k^{[1]}$. Since $B^* = k^*$, it follows that $B = k[t]$. $\square$

Lemma 2.9. *Suppose k is an algebraically closed field of characteristic zero. If B is a UFD over k with* $\mathrm{tr.deg.}_k B = 2$*, then every irreducible element of* $\mathrm{LND}(B)$ *has a slice.*

Proof. Suppose $D \in \mathrm{LND}(B)$ is irreducible, and set $A = \ker D$. By *Prop. 2.4*, D has a minimal local slice y.

Suppose $Dy \notin B^*$. Then there exists irreducible $x \in B$ dividing Dy. Since A is factorially closed, $x \in A$.

Let $\bar{D} = D \pmod{x}$ on $\bar{B} = B \pmod{x}$. Since D is irreducible, $\bar{D} \neq 0$. In addition, $\mathrm{tr.deg.}_k \bar{B} = 1$. By *Cor. 1.24*, it follows that $\bar{B} = k^{[1]}$ and $\ker \bar{D} = k$. Since $\bar{D}\bar{y} = 0$, we have that $y \in xB + k$. Write $y = xz + \lambda$ for some $z \in B$ and $\lambda \in k$. Then $y - \lambda = xz$ is irreducible, by *Prop. 2.5*. But this implies $z \in B^* \subset A$, and thus $y = xz + \lambda \in A$, a contradiction.

Therefore $Dy \in B^*$, and D has a slice. $\square$

Lemma 2.10. *Suppose k is an algebraically closed field of characteristic zero. If B is a UFD over k with degree function* deg *such that $B_0 = k$, where B_0 is the subring of elements of degree at most 0, and if $t \in B$ is of minimal positive degree, then $k[t]$ is factorially closed in B.*

Proof. Given $\lambda \in k$, suppose $t-\lambda = ab$ for $a, b \in B$. Then $\deg a + \deg b = \deg t$, which implies that either $\deg a = 0$ or $\deg b = 0$ by minimality of $\deg t$. Therefore, either $a \in k^*$ or $b \in k^*$, meaning $t - \lambda$ is irreducible.

Now suppose $cd \in k[t]$ for $c, d \in B$. Then there exist $\mu, \lambda_i \in k$ $(1 \leq i \leq n)$ such that $cd = \mu \prod_i (t-\lambda_i)$. Since this is a factorization of cd into irreducibles, it follows that every irreducible factor of c and d is of the form $t-\lambda_i$. Therefore, $c, d \in k[t]$. $\square$

2.4 The Defect of a Derivation

The main purpose of this section is to prove the following property for locally nilpotent derivations, which is due to Daigle (unpublished).

Theorem 2.11. *Suppose that B is a commutative k-domain, of finite transcendence degree over k. Then for any pair $D \in \mathrm{Der}_k(B)$ and $E \in \mathrm{LND}(B)$, D respects the filtration of B induced by E. Consequently, $\mathrm{gr}(D)$ is a well-defined derivation of $\mathrm{Gr}(B)$ relative to this filtration, which is locally nilpotent if D is locally nilpotent.*

It should be noted that a similar and likewise important result is given by Wang in his thesis.

Theorem 2.12. ([302], Cor. 2.2.7) *Suppose B is a finitely generated commutative k-domain which is $\mathbb{Z}$-graded. Then for any $D \in \mathrm{LND}(B)$, D respects the induced $\mathbb{Z}$-filtration of B.*

Suppose B is a commutative k-domain equipped with a degree function $\deg : B \to \mathbb{N} \cup \{-\infty\}$, and let B_0 denote the set of degree-zero elements, together with 0. Recall that B_0 is a factorially closed k-subalgebra of B, with $B^* \subset B_0$.

In addition to the degree function, let $D \in \mathrm{Der}(B)$ be given. Together, these define an associated **defect function** $\mathrm{def} : B \to \mathbb{Z} \cup \{-\infty\}$, namely,

$$\mathrm{def}(b) = \deg(Db) - \deg(b) \quad \text{for} \quad b \neq 0 \ , \quad \text{and} \quad \mathrm{def}(0) = -\infty \ .$$

Likewise, for any non-empty subset $S \subset B$, $\mathrm{def}(S)$ is defined by $\sup_{b \in S} \mathrm{def}(b)$. Note that $\mathrm{def}(S)$ takes its values in $\mathbb{Z} \cup \{\pm\infty\}$. The **defect** of D relative to deg is then defined to be $\mathrm{def}(B)$, and is denoted by $\mathrm{def}(D)$.

The reason for defining the defect of a derivation is that, if $B = \cup_{i \in \mathbb{Z}} B_i$ is the filtration of B induced by the degree function deg, then D (nonzero) respects this filtration if and only if $\mathrm{def}(D)$ is finite. The defect has the following basic properties.

Lemma 2.13. *Let $a, b \in B$, and let S be a non-empty subset of B.*

(a) $\mathrm{def}(S) = -\infty$ *if and only if* $S \subset \ker D$.
(b) $\mathrm{def}(D) = -\infty$ *if and only if* $D = 0$.

(c) *D is homogeneous relative to* deg *if and only if* def *is constant on* $B - 0$.
(d) $\text{def}(ab) \le \max\{\text{def}(a), \text{def}(b)\}$, *with equality when* $\text{def}(a) \ne \text{def}(b)$.
(e) $\text{def}(a^n) = \text{def}(a)$ *for all positive integers* n.
(f) *If* $a \in \ker D$, *then* $\text{def}(ab) = \text{def}(b)$.
(g) *If* $\deg(a) < \deg(b)$, *then* $\text{def}(a+b) \le \max\{\text{def}(a), \text{def}(b)\}$.
(h) *If* $a, b \in B_0$, *then* $\text{def}(a+b) \le \max\{\text{def}(a), \text{def}(b)\}$.

Proof. Following is a proof of item (g); verification of the others is left to the reader.

$$\begin{aligned}
\text{def}(a+b) &= \deg(D(a+b)) - \deg(a+b) \\
&= \deg(Da + Db) - \deg(b) \\
&\le \max\{\deg(Da), \deg(Db)\} - \deg(b) \\
&= \max\{\deg(Da) - \deg(b), \deg(Db) - \deg(b)\} \\
&\le \max\{\deg(Da) - \deg(a), \deg(Db) - \deg(b)\} \\
&= \max\{\text{def}(a), \text{def}(b)\} \ .
\end{aligned}$$

□

The defect was used by Makar-Limanov in [189] to study locally nilpotent derivations, and independently by Wang in his thesis, which also contains the following result. (As above, B_0 denotes the subalgebra of degree-0 elements.)

Proposition 2.14. ([302], Lemma 2.2.5, (4)) *For any transcendence basis S of B_0 over k,*

$$\text{def}(B_0) = \text{def}(S) \ .$$

In particular, if B_0 is finitely generated over k, then $\text{def}(B_0) < \infty$.

Proof. Note first that, since $S \subset k[S]$, we have $\text{def}(S) \le \text{def}(k[S])$. Conversely, let $f \in k[S]$ be given. Then there exist $t_1, ..., t_n \in S$ such that f is a finite sum of monomials of the form $at_1^{e_1} \cdots t_n^{e_n}$, where $a \in k^*$ and $e_i \in \mathbb{N}$. From the properties in the lemma above, it follows that

$$\text{def}(f) \le \max_{1 \le i \le n} \text{def}(t_i) \le \text{def}(S) \ .$$

Therefore, $\text{def}(k[S]) \le \text{def}(S)$, meaning $\text{def}(k[S]) = \text{def}(S)$. So if $B_0 = k[S]$, we are done.

Otherwise, choose $x \in B_0$ not in $k[S]$. By hypothesis, there exist $a_0, ..., a_n \in k[S]$ such that, if T is indeterminate over $k[S]$ and $P(T) = \sum_i a_i T^i$, then $P(x) = 0$. Choose P of minimal positive T-degree with this property, so that $P'(x) \ne 0$, and set $Q(T) = \sum_i (Da_i) T^i$. Then by the product rule, $0 = D(P(x)) = Q(x) + P'(x)Dx$, which implies

$$\deg(Dx) = \deg(P'(x)Dx) = \deg Q(x) \le \max\{\deg(Da_0), ..., \deg(Da_n)\} \ .$$

Since $\text{def}(b) = \deg(Db)$ for elements b of B_0, it follows that

$$\mathrm{def}(x) \leq \max\{\mathrm{def}(a_0), ..., \mathrm{def}(a_n)\} \leq \mathrm{def}(k[S]) = \mathrm{def}(S) \ .$$

Therefore, $\mathrm{def}(B_0) \leq \mathrm{def}(S)$, meaning $\mathrm{def}(B_0) = \mathrm{def}(S)$. $\square$

Corollary 2.15. *Suppose $B = A[T] = A^{[1]}$ for some subring A of B which is of finite transcendence degree over k. Then, relative to T-degrees,* $\mathrm{def}(D)$ *is finite for every nonzero $D \in \mathrm{Der}_k(B)$.*

Proof. Let $M = \max\{\mathrm{def}(A), \mathrm{def}(T)\}$, which is finite by the result above. Suppose $f(T) \in A[T]$ has degree $n \geq 1$, and write $f = \sum_i a_i T^i$ for $a_i \in A$. By property (g) from the lemma above, we have $\mathrm{def}(f) \leq \max_{0 \leq i \leq n} \mathrm{def}(a_i T^i)$. Using the other properties in the lemma, it follows that

$$\mathrm{def}(f) \leq \max\{\mathrm{def}(a_0), ..., \mathrm{def}(a_n), \mathrm{def}(T)\} \leq M \ .$$

Therefore, $\mathrm{def}(D) \leq M < \infty$. $\square$

Finally, we turn our attention to the case in which the degree function on B is determined by a locally nilpotent derivation E:

$$\deg(b) = \nu_E(b) \quad \text{for} \quad b \in B \ .$$

We can now prove the result of Daigle given at the beginning of this section.

Proof of Thm. 2.11. We need to prove that, relative to the degree function ν_E on B defined by E, the defect $\mathrm{def}(D) < \infty$ for every $D \in \mathrm{Der}_k(B)$. In case $D = 0$, this is clear, so assume $D \neq 0$. If $A = \ker E$, and if $r \in B$ is a local slice of D, then $B_f = A_f[r]$, where $f = Dr$. Let D_f denote the extension of D to B_f, and let Def denote the defect on B_f defined by degrees in r and the derivation D_f. Then by *Cor. 2.15* above, $\mathrm{Def}(D_f)$ is finite.

Note that for any $b \in B$, $\nu_E(b)$ equals the r-degree of b as an element of $A_f[r]$. It follows that, for every nonzero $b \in B$,

$$\mathrm{def}(b) = \nu_E(Db) - \nu_E(b) = \deg_r(D_f b) - \deg_r(b) = \mathrm{Def}(b) \leq \mathrm{Def}(D_f) \ .$$

Therefore, $\mathrm{def}(D) \leq \mathrm{Def}(D_f) < \infty$. $\square$

In fact, even more can be said. The following result is implicit in the work of Makar-Limanov; see [189], Lemma 2, and its proof. To paraphrase, it says that the defect, which measures the jump in degree after applying D, achieves its maximum already on the subalgebra of degree-0 elements.

Corollary 2.16. *Suppose B is of finite transcendence degree over k. Let $D \in \mathrm{Der}_k(B)$ and $E \in \mathrm{LND}(B)$ be nonzero, such that* $\ker D \neq \ker E$*. If a defect function on B is defined by*

$$\mathrm{def}(b) = \nu_E(Db) - \nu_E(b) \ ,$$

then $\mathrm{def}(D) = \mathrm{def}(\ker E) < \infty$.

Proof. Let $A = \ker E$, noting that $A \not\subset \ker D$ (otherwise $A = \ker D$ by considering transcendence degrees and algebraic closure). By *Thm. 2.11*, we conclude that

$$-\infty < \mathrm{def}(A) \leq \mathrm{def}(B) < \infty \ .$$

We will show that if $\mathrm{def}(A) < \mathrm{def}(B)$, then $\mathrm{def}(A) < \mathrm{def}(b)$ for every $b \notin A$, which is patently absurd, since $\mathrm{def}(b) = -\infty$ when $b \in \ker D$. We will repeatedly use the properties of defect stated in the lemma above.

Assume $\mathrm{def}(A) < \mathrm{def}(B)$, and choose a local slice r of E. We first need to establish that $\mathrm{def}(r) > \mathrm{def}(A)$.

By hypothesis, there exists $\beta \in B$ such that $\mathrm{def}(A) < \mathrm{def}(\beta)$. For $n \geq 1$, there exist $c, c_0, c_1, ..., c_n \in A$ such that $c\beta = c_0 + c_1 r + \cdots c_n r^n$. Since $\mathrm{def}(c) < \mathrm{def}(\beta)$, we have $\mathrm{def}(c\beta) = \mathrm{def}(\beta)$. Therefore,

$$\mathrm{def}(\beta) = \mathrm{def}(c\beta) = \mathrm{def}(c_0 + c_1 r + \cdots + c_n r^n) \leq \max_i \{\mathrm{def}(c_i r^i)\} \ ,$$

since these terms are strictly increasing in degree. Therefore, there exists at least one $i \geq 1$ such that $\mathrm{def}(\beta) \leq \mathrm{def}(c_i r^i)$. It follows that

$$\mathrm{def}(A) < \mathrm{def}(\beta) \leq \mathrm{def}(c_i r^i) \leq \max\{\mathrm{def}(c_i), \mathrm{def}(r^i)\} \leq \max\{\mathrm{def}(A), \mathrm{def}(r)\} \ ,$$

meaning $\mathrm{def}(A) < \mathrm{def}(r)$.

Next, let $P(T) \in A[T] = A^{[1]}$ be given, where $\deg_T P(T) = m \geq 1$. Suppose

$$P(T) = a_0 + a_1 T + \cdots + a_m T^m \text{ and } Q(T) = (Da_0) + (Da_1)T + \cdots + (Da_m)T^m \ ,$$

where $a_0, ..., a_m \in A$. Set $b = P(r)$, which is not in A. Since $\nu_E(b) = \nu_E(P(r)) = m$, we have

$$\mathrm{def}(b) = \mathrm{def}(P(r)) = \nu_E(D(P(r))) - m = \nu_E(Q(r) + P'(r)Dr) - m \ .$$

Since $\nu_E(r) = 1$,

$$\nu_E(P'(r)Dr) = \nu_E(P'(r)) + \nu_E(Dr) = (m-1) + \mathrm{def}(r) + \nu_E(r) = \mathrm{def}(r) + m \ .$$

On the other hand, for each i $(0 \leq i \leq m)$, we have

$$\nu_E((Da_i)r^i) = \nu_E(Da_i) + \nu_E(r^i) = \mathrm{def}(a_i) + \nu_E(a_i) + i \leq \mathrm{def}(A) + m \ ,$$

since each $\nu_E(a_i) = 0$. Since $\mathrm{def}(A) < \mathrm{def}(r)$, it follows that $\nu_E(Q(r)) < \nu_E(P'(r)Dr)$. Consequently,

$$\begin{aligned}\mathrm{def}(b) &= \nu_E(P'(r)Dr) - m \\ &= \nu_E(P'(r)) + \nu_E(Dr) - m \\ &= (m-1) + \mathrm{def}(r) + \nu_E(r) - m \\ &= \mathrm{def}(r) > \mathrm{def}(A) \ .\end{aligned}$$

Finally, let $\gamma \in B$ be given, $\gamma \notin A$. Then for some $a \in A$ and some $P(T) \in A[T]$ of T-degree at least one, $a\gamma = P(r)$. From the preceding discussion, we have

$$\mathrm{def}(A) < \mathrm{def}(r) = \mathrm{def}(a\gamma) \leq \max\{\mathrm{def}(a), \mathrm{def}(\gamma)\} \leq \max\{\mathrm{def}(A), \mathrm{def}(\gamma)\} ,$$

which implies $\mathrm{def}(\gamma) > \mathrm{def}(A)$. This completes the proof. □

In order to illustrate the necessity of assuming that the ring is of finite transcendence degree over k, consider $\mathcal{B} = k[x_1, x_2, ...]$, the ring of polynomials in a countably infinite number of variables x_i. Define $D \in \mathrm{Der}_k(\mathcal{B})$ and $E \in \mathrm{LND}(\mathcal{B})$ by $Dx_n = x_{2n}$ for all $n \geq 1$; and by $Ex_1 = 0$, and $Ex_n = x_{n-1}$ for $n \geq 2$. Then using degrees determined by E, we see that for all $n \geq 1$,

$$\mathrm{def}(x_n) = \nu_E(Dx_n) - \nu_E(x_n) = 2n - n = n .$$

See also Remark 5 (p.21) of [302].

2.5 Exponential Automorphisms

Given an automorphism $\varphi \in \mathrm{Aut}_k(B)$, φ is an **exponential automorphism** if and only if $\varphi = \exp D$ for some $D \in \mathrm{LND}(B)$. It is natural to ask whether a given automorphism is exponential. A complete answer to this question, with detailed proofs, is given by van den Essen in Sect. 2.1 of [100]; see also the article of Gabriel and Nouazé [125], Sect. 3.5, and the book of Nowicki [247], 6.1.4. Here is a brief summary of van den Essen's treatment.

Given a ring homomorphism $f : B \to B$, define the map $E : B \to B$ by $E = f - I$, where I denotes the identity map. Then for any $a, b \in B$,

$$E(ab) = aEb + bEa + (Ea)(Eb) = aEb + f(b)Ea .$$

We say that E is an **f-derivation** of B. E is said to be **locally nilpotent** if and only if to each $b \in B$, there exists a positive integer n with $E^n b = 0$.

In case E is locally nilpotent, define the map $\log(I + E) : B \to B$ by

$$\log(I + E) = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} E^n ,$$

which is well-defined since E is a locally nilpotent f-derivation.

Proposition 2.17. (Prop. 2.1.3 of [100]) *Let $f : B \to B$ be a ring homomorphism, and set $E = f - I$.*

(a) *f is an exponential automorphism if and only if E is a locally nilpotent f-derivation.*
(b) *If E is locally nilpotent and $D = \log(I + E)$, then $D \in \mathrm{LND}(B)$ and $f = \exp D$.*

Of course, there may be simpler criteria showing that an automorphism φ is not an exponential automorphism. For example, $\exp D$ cannot have finite order when $D \neq 0$, since $(\exp D)^n = \exp(nD)$.

2.6 Wronskians and Kernel Elements

In this section, Wronskian determinants associated to a derivation are defined, and some of their basic properties are given. They are especially useful for constructing constants for derivations. The proofs for this section are elementary, and most are left to the reader. The assumption that B is a commutative k-domain continues.

Given $D \in \mathrm{Der}_k(B)$ with $A = \ker D$, and given $\mathbf{f} = (f_1, ..., f_n) \in B^n$, let $D\mathbf{f} = (Df_1, ..., Df_n) \in B^n$. The **Wronskian** of $\mathbf{f}$ relative to D is

$$W_D(\mathbf{f}) = \det \begin{pmatrix} \mathbf{f} \\ D\mathbf{f} \\ \vdots \\ D^{n-1}\mathbf{f} \end{pmatrix} .$$

Observe that W_D is A-linear in each argument f_i.

Proposition 2.18. *Let M be a square matrix of order n with entries in B, and let $M_1, ..., M_n$ denote the rows of M. Then*

$$D|M| = \sum_{i=1}^{n} \det \begin{pmatrix} M_1 \\ \vdots \\ DM_i \\ \vdots \\ M_n \end{pmatrix} .$$

Corollary 2.19. (See [37], 7.3, ex.8)

$$DW_D(\mathbf{f}) = \det \begin{pmatrix} \mathbf{f} \\ D\mathbf{f} \\ \vdots \\ D^{n-2}\mathbf{f} \\ D^n\mathbf{f} \end{pmatrix}$$

Corollary 2.20. *If $D^{n+1}f_i = 0$ for each i, then for $1 \le i \le n$,*

$$D^{n-i}W_D(\mathbf{f}) = \det \begin{pmatrix} \mathbf{f} \\ D\mathbf{f} \\ \vdots \\ \widehat{D^i\mathbf{f}} \\ \vdots \\ D^n\mathbf{f} \end{pmatrix}$$

Corollary 2.21. *If $D^n f_i = 0$ for each i, then $W_D(\mathbf{f}) \in A$.*

Corollary 2.22. *If* $D^n f_i = 0$ *for* $i = 1, ..., n-1$*, then*

$$DW_D(\mathbf{f}) = D^n f_n \cdot W_D(f_1, ..., f_{n-1}) \ .$$

Proposition 2.23. *For any* $g \in B$*,* $W_D(g\,\mathbf{f}) = g^n W_D(\mathbf{f})$*.*

Proof. From the generalized product rule,

$$D^n(fg) = \sum_{i=0}^{n} \binom{n}{i} D^i f D^{n-i} g$$

Thus, the matrix

$$H = \begin{pmatrix} gf_1 & \cdots & gf_n \\ D(gf_1) & \cdots & D(gf_n) \\ \vdots & & \vdots \\ D^{n-1}(gf_1) & \cdots & D^{n-1}(gf_n) \end{pmatrix}$$

may be factored as $H = GF$, where

$$G = \begin{pmatrix} g & 0 & 0 & 0 & \cdots & 0 \\ Dg & g & 0 & 0 & \cdots & 0 \\ D^2 g & 2Dg & g & 0 & \cdots & 0 \\ D^3 g & 3D^2 g & 3Dg & g & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ D^{n-1} g & c_1 D^{n-2} g & c_2 D^{n-3} g & c_3 D^{n-4} g & \cdots & g \end{pmatrix}, \ F = \begin{pmatrix} \mathbf{f} \\ D\mathbf{f} \\ \vdots \\ D^{n-1}\mathbf{f} \end{pmatrix},$$

and c_i denotes the binomial coefficient $\binom{n-1}{i}$. It follows that

$$W_D(g\,\mathbf{f}) = |H| = |G| \cdot |F| = g^n W_D(\mathbf{f}) \ .$$

□

Note that Wronskians were used in [120] to construct certain kernel elements, as in *Cor. 2.21.*

The following variant of the Wronskian can also be used to construct kernel elements. It is especially useful when a derivation admits a large number of inequivalent local slices.

Suppose $D \in \mathrm{Der}_k(B)$ has local slices $z_{ij} \in B$ $(1 \le i \le n-1, 1 \le j \le n)$ which satisfy the following additional condition: If $\mathbf{z}_i = (z_{i1}, ..., z_{in})$ for $1 \le i \le n-1$, there exist $a_1, ..., a_n, y_1, ..., y_n \in B^D$ such that $D\mathbf{z}_i = a_i \mathbf{y}$, where $\mathbf{y} = (y_1, ..., y_n)$. Then for the matrix

$$M = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n-1,1} & z_{n-1,2} & \cdots & z_{n-1,n-1} \\ y_1 & y_2 & \cdots & y_n \end{pmatrix},$$

we have $\det M \in B^D$. This fact is immediate from *Prop. 2.18*.

This method is used in *Chap. 6* below to construct certain homogeneous invariants for $\mathbb{G}_a^2$-actions of Nagata type.

2.7 The Star Operator

As mentioned, there is generally no clear algebraic structure for $\mathrm{LND}(B)$. It will be remembered that $\mathrm{Aut}_k(B)$ acts on $\mathrm{LND}(B)$ by conjugation. Thus, one obvious binary operation to consider on this set is

$$(D_1, D_2) \to (\exp D_2)D_1(\exp(-D_2)) \ .$$

But since this does not take one outside the conjugacy class of D_1, it is not particularly useful.

However, a *rational* automorphism of B might conjugate $D \in \mathrm{LND}(B)$ to another element of $\mathrm{LND}(B)$, and herein lies a chance to get something new. The **star operator** $*$ defined below makes use of this idea. It is not defined on all $\mathrm{LND}(B)$, but rather on subsets $\mathrm{LND}_f(B)$, where $f \in B$. (Here, $\mathrm{LND}_f(B)$ denotes elements $D \in \mathrm{LND}(B)$ with $Df = 0$.) We also require that B is affine.

To define this operation, let $f \in B$ be such that $\mathrm{LND}_f(B)$ contains a nonzero element, and let $d, \delta \in \mathrm{LND}_f(B)$ be given. Extend d and δ to derivations $\hat{d}, \hat{\delta} \in \mathrm{LND}(B_f)$.

- Set $\alpha = \exp(f^{-1}\hat{\delta}) \in \mathrm{Aut}_k(B_f)$
- Set $\Delta = \alpha\hat{d}\alpha^{-1} \in \mathrm{LND}_k(B_f)$
- Choose $n \geq 0$ minimal so that $f^n\Delta(B) \subset B$. This is possible, since B is finitely generated.

We now define $d * \delta$ to be the restriction of $f^n\Delta$ to B. It follows that:

- $d * \delta \in \mathrm{LND}_f(B)$
- $\ker(d * \delta) = \alpha(\ker \hat{d}) \cap B$
- $d * \delta$ is irreducible if d is irreducible.

Observe that, in general, $d * 0 = d$, whereas $0 * d = 0$.

3

Polynomial Rings

This chapter investigates locally nilpotent derivations in the case B is a polynomial ring in a finite number of variables over a field k of characteristic zero. Equivalently, we are interested in the algebraic actions of $\mathbb{G}_a$ on $\mathbb{A}_k^n$.

3.1 Variables, Automorphisms, and Gradings

If $B = k^{[n]}$, then there exist polynomials $x_1, ..., x_n \in B$ such that $B = k[x_1, ..., x_n]$. Any such set $\{x_1, ..., x_n\}$ is called a **system of variables** or a **coordinate system** for B. Any subset $\{x_1, ..., x_i\}$ of a system of variables is called a **partial system of variables** for B ($1 \le i \le n$). A polynomial $f \in B$ is called a **variable** or **coordinate function** for B if and only if f belongs to some system of variables for B. Quite often, we will write $k[\mathbf{x}]$ in place of $k[x_1, ..., x_n]$, where $\mathbf{x} = (x_1, ..., x_n)$.

$k^{(n)}$ denotes the **field of rational functions in** n **variables over** k, which is the quotient field of $k^{[n]}$. If $k^{[n]} = k[x_1, ..., x_n]$, then $k^{(n)} = k(x_1, ..., x_n)$.

The group of algebraic k-automorphisms of $B = k^{[n]}$ is called the **general affine group** or **affine Cremona group** in dimension n, and is denoted $GA_n(k)$. This group may be viewed as an infinite-dimensional algebraic group over k (see [168]). If $B = k[x_1, ..., x_n]$ then the familiar general linear group $GL_n(k)$ can be realized as a subgroup of $GA_n(k)$, namely, as those elements which restrict to a linear transformation of the k-vector subspace

$$V = kx_1 \oplus \cdots \oplus kx_n \subset B \ .$$

Elements of $GL_n(k)$ are called the **linear automorphisms** of B. Note that linearity depends on the choice of coordinates. However, due to work of Kraft and Schwarz, it is known that there is only one conjugacy class of $GL_n(k)$ in $GA_n(k)$. Thus, we say that a group representation $\rho : G \to GA_n(k)$ is **linearizable** if and only if ρ factors through a rational representation:

$$G \to GL_n(k) \hookrightarrow GA_n(k) \ .$$

Given a coordinate system $B = k[x_1, ..., x_n]$, an automorphism $F \in GA_n(k)$ is given by $F = (F_1, ..., F_n)$, where $F_i = F(x_i) \in B$. The **triangular automorphisms** or **Jonquières automorphisms** are those of the form $F = (F_1, ..., F_n)$, where $F_i \in k[x_1, ..., x_i]$.[1] The triangular automorphisms form a subgroup, denoted $BA_n(k)$, which is the generalization of the Borel subgroup in the theory of finite-dimensional representations.

The **tame subgroup** of $GA_n(k)$ is the subgroup generated by $GL_n(k)$ and $BA_n(k)$. Its elements are called **tame automorphisms**. It is known that for $n \leq 2$, every element of $GA_n(k)$ is tame (see *Chap. 4*), whereas non-tame automorphisms exist in $GA_3(k)$ (see [278, 279]).

As to gradings of polynomial rings, we are mainly interested in $\mathbb{Z}^m$-gradings for some $m \geq 1$. In particular, suppose $B = k[x_1, ..., x_n]$. Let $I = \mathbb{Z}^m$ for some $m \geq 1$, and set $J = \mathbb{Z}^n$. Given a homomorphism $\alpha : J \to I$, define the function $\deg_\alpha$ on the set of monomials by $\deg_\alpha(x_1^{e_1} \cdots x_n^{e_n}) = \alpha(e_1, ..., e_n)$. Given $i \in I$, let B_i be the k-module generated by monomials μ with $\deg_\alpha(\mu) = i$ (a module over an empty basis is understood to be 0). For instance, if $I = \mathbb{Z}$ and $\alpha(e_1, ..., e_n) = \sum e_i$, then the induced grading is called the **standard grading** of B, relative to the coordinate system $(x_1, ..., x_n)$. Likewise, if $\alpha(e_1, ..., e_n) = e_1 \in \mathbb{Z}$, then B is graded according to its usual degree relative to x_1.

3.2 Derivations of Polynomial Rings

3.2.1 Some Definitions

Given $D \in \mathrm{Der}_k(B)$, define the **corank** of D to be the maximum integer i such that there exists a partial system of variables $\{x_1, ..., x_i\}$ of B contained in $\ker D$. In other words, the corank of D is the maximal number of variables within the same system annihilated by D. Denote the corank of D by $\mathrm{corank}(D)$. Define the **rank** of D by $\mathrm{rank}(D) = n - \mathrm{corank}(D)$. By definition, the rank and corank are invariants of D, in the sense that these values do not change after conjugation by an element of $GA_n(k)$. The rank and corank were first defined in [116].

A k-derivation D of B is said to be **rigid** when the following condition holds: If $\mathrm{corank}(D) = i$, and if $\{x_1, ..., x_i\}$ and $\{y_1, ..., y_i\}$ are partial systems of variables of B contained in $\ker D$, then $k[x_1, ..., x_i] = k[y_1, ..., y_i]$. This definition is due to Daigle [48].

[1] Ernest Jean Philippe Fauque de Jonquières (1820-1901) was a career officer in the French navy, achieving the rank of vice-admiral in 1879. He learned advanced mathematics by reading works of Poncelet, Chasles, and other geometers. In 1859, he introduced the planar transformations $(x, y) \to \left(x, \frac{a(x)y+b(x)}{c(x)y+d(x)}\right)$, where $ad - bc \neq 0$. These were later studied by Cremona.

By a **linear derivation** of $B = k[x_1, ..., x_n]$ we mean any $D \in \text{Der}_k(B)$ such that D restricts to a linear transformation of the k-vector subspace $V = kx_1 \oplus \cdots \oplus kx_n \subset B$. Equivalently, a linear derivation is homogeneous of degree zero in the standard sense. Note that linearity depends on the choice of coordinates on B. By a **linearizable derivation** of B we mean any $D \in \text{Der}_k(B)$ which is linear relative to some system of coordinates on B, i.e., conjugate to a linear derivation.

Likewise, for $B = k[x_1, ..., x_n]$, set $\mathbf{x} = (x_1, ..., x_n)$. We say that $D \in \text{Der}_k(B)$ is **quasi-linear** if and only if there exists a matrix $M \in \mathcal{M}_n(\ker D)$ such that $D(\mathbf{x}) = M\mathbf{x}$. (Here, it is understood that $D(\mathbf{x}) = (Dx_1, ..., Dx_n)$.) Then D is locally nilpotent if and only if M is a nilpotent matrix.

If $B = k[x_1, ..., x_n]$, we say D is a **triangular derivation** of B if and only if $Dx_i \in k[x_1, ..., x_{i-1}]$ for $i = 2, ..., n$ and $Dx_1 \in k$. Note that triangularity depends on the choice of coordinates on B. By a **triangularizable derivation** of B we mean any $D \in \text{Der}_k(B)$ which is triangular relative to some system of coordinates on B, i.e., conjugate to a triangular derivation. As we will see, the triangular derivations form a very large and important class of locally nilpotent derivations of polynomial rings. Several of the main examples and open questions discussed below involve triangular derivations. In many respects, the triangular derivations provide an archetype for the study of $\text{LND}(k^{[n]})$.

For polynomial rings, other natural categories of derivations to study are the following: Let D be a k-derivation of the polynomial ring $B = k[x_1, ..., x_n]$.

1. D is a **monomial derivation** if each image Dx_i is a monomial in $x_1, ..., x_n$.
2. D is an **elementary derivation** if, for some j with $1 \le j \le n$, $Dx_i = 0$ for $1 \le i \le j$, and $Dx_i \in k[x_1, ..., x_j]$ if $j + 1 \le i \le n$.
3. D is a **nice derivation**[2] if $D^2 x_i = 0$ for each i.
4. D is a **simple monomial derivation** if $Dx_1 = 0$ and $Dx_i = x_{i-1}^{e_i}$ for positive integers e_i $(2 \le i \le n)$.

These definitions depend on the coordinate system chosen. Note that any nice derivation is locally nilpotent, and that any elementary derivation is both triangular and nice.

3.2.2 Partial Derivatives

Given a system of variables on the polynomial ring $B = k[x_1, ..., x_n]$, a natural set of derivations on B is the set of **partial derivatives relative to** $(x_1, ..., x_n)$. In particular, $\partial_{x_i} \in \text{Der}_k(B)$ is defined by the rule $\partial_{x_i}(x_j) = \delta_{ij}$ (Kronecker delta). Another common notation for ∂_{x_i} is $\frac{\partial}{\partial x_i}$, and if $f \in B$, f_{x_i} denotes $\partial_{x_i}(f)$.

[2] Van den Essen gives a more exclusive definition of a nice derivation. See [100], 7.3.12.

Note that ∂_{x_i} is locally nilpotent for each i, since $B = A[x_i]$ for $A = k[x_1, ..., \widehat{x_i}, ..., x_n]$, and $\partial_{x_i}(A) = 0$. Note also that the meaning of ∂_{x_i} depends on the entire system of variables to which x_i belongs. For example, in the two-dimensional case, $k[x,y] = k[x, y+x]$, and $\partial_x(y+x) = 1$ relative to (x,y), whereas $\partial_x(y+x) = 0$ relative to $(x, y+x)$. In general, we will say $D \in \mathrm{LND}(B)$ is a **partial derivative** if and only if there exists a system of coordinates $(y_1, ..., y_n)$ on B relative to which $D = \partial_{y_1}$.

It is easy to see that, as a B-module, $\mathrm{Der}_k(B)$ is freely generated by $\{\partial_{x_1}, ..., \partial_{x_n}\}$, and that this is a basis of commuting derivations. In particular, given $D \in \mathrm{Der}_k(B)$,

$$D = \sum_{1 \leq i \leq n} D(x_i)\partial_{x_i} \ .$$

To verify this expression for D, it suffices to check equality for each x_i, and this is obvious. Note that the rank of D is the minimal number of partial derivatives needed to express D in this form. Thus, elements of $\mathrm{Der}_k(B)$ having rank one are precisely those of the form $f\partial_{x_1}$ for $f \in B$, relative to some system of coordinates $(x_1, ..., x_n)$ for B.

Example 3.1. On the polynomial ring $B = k^{[n]} = k[x_1, ..., x_n]$, define the derivation

$$D = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \ .$$

If $N = \prod_{i=1}^{n-1} i^i$, then

$$W_D(x_1^{n-1}, ..., x_n^{n-1}) = N \cdot \det \begin{pmatrix} x_1^{n-1} & \cdots & x_n^{n-1} \\ x_1^{n-2} & \cdots & x_n^{n-2} \\ \vdots & & \vdots \\ x_1 & \cdots & x_n \\ 1 & \cdots & 1 \end{pmatrix} = N \cdot \prod_{i>j}(x_i - x_j) \ ,$$

i.e., the Vandermonde determinant of $x_1, ..., x_n$ may be realized as a Wronskian. □

The partial derivatives ∂_{x_i} also extend (uniquely) to the field $K = k(x_1, ..., x_n)$ by the quotient rule, although they are no longer locally nilpotent on all K:

$$\mathrm{Nil}(\partial_{x_i}) = k(x_1, ..., \widehat{x_i}, ..., x_n)[x_i] \ .$$

In this case, we see that $\mathrm{Der}_k(K)$ is a vector space over K of dimension n, with basis $\partial_{x_1}, ..., \partial_{x_n}$. More generally:

Proposition 3.2. *If L is a field of finite transcendence degree n over k, then $\mathrm{Der}_k(L)$ is a vector space over L of dimension n.*

Proof. Suppose $k \subset k(x_1, ..., x_n) \subset L$ for algebraically independent x_i, and set $K = k(x_1, ..., x_n)$. Suppose $D \in \mathrm{Der}_k(L)$ and $t \in L$ are given, and let $P \in K[T] = K^{[1]}$ be the minimal polynomial of t. Suppose $P(T) = \sum_i a_i T^i$. Then $0 = D(P(t)) = P'(t)Dt + \sum_i D(a_i)t^i$. Since $P'(t) \neq 0$, this implies

$$Dt = -(P'(t))^{-1} \sum_i D(a_i)t^i \ ,$$

meaning that D is completely determined by its values on K. Conversely, this same formula shows that every $D \in \mathrm{Der}_k(K)$ can be uniquely extended to L.

In particular, the partial derivatives ∂_{x_i} extend uniquely to L. If $f \in K$ and $D \in \mathrm{Der}_k(L)$, then $Df = f_{x_1}Dx_1 + \cdots + f_{x_n}Dx_n$. We conclude that

$$\mathrm{Der}_k(L) = \mathrm{span}_L\{\partial_{x_1}, ..., \partial_{x_n}\} \ .$$

If $a_1\partial_{x_1} + \cdots a_n\partial_{x_n} = 0$ for $a_i \in L$, then evaluation at x_i shows that $a_i = 0$. Therefore, the partial derivatives are linearly independent over L, and the dimension of $\mathrm{Der}_k(L)$ equals n. $\square$

Proposition 3.3. (Multivariate Chain Rule) *Suppose $D \in \mathrm{Der}_k(K)$ for $K = k(x_1, ..., x_n)$, and $f_1, ..., f_m \in K$. Then for any $g \in k(y_1, ..., y_m) = k^{(m)}$,*

$$D\big(g(f_1, , , .f_m)\big) = \frac{\partial g}{\partial y_1}(f_1, ..., f_m) \cdot Df_1 + \cdots + \frac{\partial g}{\partial y_m}(f_1, ..., f_m) \cdot Df_n \ .$$

Proof. By the product rule, it suffices to assume $g \in k[y_1, ..., y_m]$. In addition, by linearity, it will suffice to show the formula in the case g is a monomial: $g = y_1^{e_1} \cdots y_m^{e_m}$ for $e_1, ..., e_m \in \mathbb{N}$.

From the product rule and the univariate chain rule, we have that

$$\begin{aligned}
\frac{\partial}{\partial x_j}(f_1^{e_1} \cdots f_m^{e_m}) &= \sum_i (f_1^{e_1} \cdots \widehat{f_i^{e_i}} \cdots f_m^{e_m}) \frac{\partial}{\partial x_j}(f_i^{e_i}) \\
&= \sum_i e_i (f_1^{e_1} \cdots f_i^{e_i - 1} \cdots f_m^{e_m})(f_i)_{x_j} \ .
\end{aligned}$$

Since $D = Dx_1 \frac{\partial}{\partial x_1} + \cdots + Dx_n \frac{\partial}{\partial x_n}$, we have

$$\begin{aligned}
D(f_1^{e_1} \cdots f_m^{e_m}) &= \sum_j \frac{\partial}{\partial x_j}(f_1^{e_1} \cdots f_m^{e_m}) \cdot Dx_j \\
&= \sum_j \sum_i (f_i)_{x_j} (e_i f_1^{e_1} \cdots f_i^{e_i - 1} \cdots f_m^{e_m}) \cdot Dx_j \\
&= \sum_i \sum_j (f_i)_{x_j} (e_i f_1^{e_1} \cdots f_i^{e_i - 1} \cdots f_m^{e_m}) \cdot Dx_j \\
&= \sum_i (e_i f_1^{e_1} \cdots f_i^{e_i - 1} \cdots f_m^{e_m}) \sum_j (f_i)_{x_j} \cdot Dx_j \\
&= \sum_i (e_i f_1^{e_1} \cdots f_i^{e_i - 1} \cdots f_m^{e_m}) Df_i
\end{aligned}$$

$$= \sum_i \frac{\partial g}{\partial x_i}(f_1^{e_1} \cdots f_m^{e_m}) \cdot Df_i \ .$$

□

In addition, using partial derivatives, we define for any $D \in \mathrm{Der}_k(K)$, or or $D \in \mathrm{Der}_k(B)$, the **divergence** of D:

$$\mathrm{div}(D) = \sum_{i=1}^{n} \partial_{x_i}(Dx_i) \ .$$

Nowicki [247] defines D to be **special** if $\mathrm{div}(D) = 0$.

The use of partial derivatives also allows us to describe homogeneous decompositions of derivations relative to $\mathbb{Z}$-gradings of $B = k[x_1, ..., x_n] = k^{[n]}$. Given $w = (w_1, ..., w_n) \in \mathbb{Z}^n$, B is naturally $\mathbb{Z}$-graded by $B = \oplus_{i \in \mathbb{Z}} B_i$, where B_i is the k-vector space generated by monomials $x_1^{a_1} \cdots x_n^{a_n}$ with $a_1 w_1 + \cdots a_n w_n = i$.

Proposition 3.4. (Prop. 5.1.14 of [100]) *Let nonzero $w \in \mathbb{Z}^n$ be given. Then every nonzero $D \in \mathrm{Der}_k(B)$ admits a unique decomposition $D = \sum_{i \in \mathbb{Z}} D_i$, where D_i is homogeneous of degree i relative to the grading of B induced by w, and $D_i = 0$ for all but finitely many $i \in \mathbb{Z}$.*

Proof. There exist $f_1, ..., f_n \in B$ such that $D = \sum f_i \partial_{x_i}$. Since each monomial x_i is homogeneous relative to the w-grading, each partial derivative ∂_{x_i} is a homogeneous derivation (relative to w). Note that generally, the degree of ∂_{x_i} will vary with i. Each coefficient function f_i admits a decomposition into w-homogeneous summands; suppose $f_i = \sum_j f_{ij}$. Then each summand $f_i \partial_{x_i}$ can be decomposed as a finite sum of w-homogeneous derivations, namely, $f_i \partial x_i = \sum_j f_{ij} \partial_{x_i}$. Therefore, $D = \sum_{i,j} f_{ij} \partial_{x_i}$, and by gathering terms of the same degree, the desired result follows. □

3.2.3 Jacobian Derivations

Let $B = k[x_1, ..., x_n] = k^{[n]}$. The **jacobian matrix** of $f_1, ..., f_m \in B$ is the $m \times n$ matrix of partial derivatives

$$\mathcal{J}(f_1, ..., f_m) := \frac{\partial(f_1, ..., f_m)}{\partial(x_1, ..., x_n)} = \big((f_i)_{x_j}\big) \ .$$

Note that the jacobian matrix depends on the system of coordinates x_i. When $m = n$, the **jacobian determinant** of $f_1, ..., f_n \in B$ is $\det \mathcal{J}(f_1, ..., f_n) \in B$.

Suppose $k[y_1, ..., y_m] = k^{[m]}$, and let $F : k[y_1, ..., y_m] \to k[x_1, ..., x_n]$ be a k-algebra homomorphism. Then the jacobian matrix of F is $\mathcal{J}(F) = \mathcal{J}(f_1, ..., f_m)$, where $f_i = F(y_i)$, and the jacobian determinant of F is $\det \mathcal{J}(F)$.[3] In addition, suppose $A = (a_{ij})$ is a matrix with entries a_{ij}

[3] Many authors use DF to denote the jacobian matrix of F, but we prefer to reserve D for derivations.

in $k[y_1,...,y_m]$. Then $F(A)$ denotes the matrix $(F(a_{ij}))$ with entries in $k[x_1,...,x_m]$.

Given k-algebra homomorphisms

$$k[z_1,...,z_l] \xrightarrow{G} k[y_1,...,y_m] \xrightarrow{F} k[x_1,...,x_n] \ ,$$

the **chain rule for jacobian matrices** is:

$$\mathcal{J}(G \circ F) = F(\mathcal{J}(G)) \cdot \mathcal{J}(F) \ ,$$

where $\cdot$ denotes matrix multiplication. This follows from the multivariate chain rule above. Note that if $\mathcal{J}(G)$ is a square matrix, then we have

$$\det F\left(\mathcal{J}(G)\right) = F\left(\det \mathcal{J}(G)\right) \ .$$

Observe that the standard properties of determinants imply:

> $\det \mathcal{J}$ is a k-derivation of B in each one of its arguments.

In particular, suppose $f_1,...,f_{n-1} \in B$ are given, and set $\mathbf{f} = (f_1,...,f_{n-1})$. Then $\mathbf{f}$ defines $\Delta_{\mathbf{f}} \in \mathrm{Der}_k(B)$ via

$$\Delta_{\mathbf{f}}(g) := \det \mathcal{J}(f_1,...,f_{n-1},g) \quad (g \in B) \ .$$

$\Delta_{\mathbf{f}}$ is called the **jacobian derivation** of B determined by $\mathbf{f}$.

Observe that the definitions of jacobian matrices and jacobian derivations also extend to the rational function field $K = k(x_1,...,x_n)$.

It is well-known that, if $F = (f_1,...,f_n)$ is a system of variables for B, then

$$\det \mathcal{J}(F) = \det \frac{\partial(f_1,...,f_n)}{\partial(x_1,...,x_n)} = \Delta_{\mathbf{f}}(f_n) \in k^* \ .$$

This is easily seen from the chain rule: By definition, F admits a polynomial inverse F^{-1}, and $I = FF^{-1}$ implies that

$$1 = \det\big(F(\mathcal{J}(F^{-1})) \cdot \mathcal{J}(F)\big) = \det F(\mathcal{J}(F^{-1})) \det \mathcal{J}(F) \ ,$$

meaning $\det \mathcal{J}(F)$ is a unit of B.

In the other direction lurks the famous **Jacobian Conjecture**, which can be formulated in the language of derivations: Suppose $\mathbf{f} = (f_1,...,f_{n-1})$ for $f_i \in B$.

> If $\Delta_{\mathbf{f}}$ has a slice s, then $k[f_1,...,f_{n-1},s] = B$. Equivalently, if $\Delta_{\mathbf{f}}$ has a slice, then $\Delta_{\mathbf{f}}$ is locally nilpotent and $\ker \Delta_{\mathbf{f}} = k[f_1,...,f_{n-1}]$.

See van den Essen [100], Chap. 2, for further details about the Jacobian Conjecture.

Following are several lemmas about jacobian derivations, which will be used to prove important facts about locally nilpotent derivations of polynomial rings.

Lemma 3.5. *Given $f_1, ..., f_{n-1} \in K$, set $\mathbf{f} = (f_1, ..., f_{n-1})$.*

(a) $\Delta_{\mathbf{f}} = 0$ *if and only if $f_1, ..., f_{n-1}$ are algebraically dependent.*
(b) *If $\Delta_{\mathbf{f}} \neq 0$, then* $\ker \Delta_{\mathbf{f}}$ *is the algebraic closure of $k(f_1, ..., f_{n-1})$ in K.*
(c) *For any $g \in K$, $\Delta_{\mathbf{f}}(g) = 0$ if and only if $f_1, ..., f_{n-1}, g$ are algebraically dependent.*

Proof. (following [190]) To prove part (a), suppose $f_1, ..., f_{n-1}$ are algebraically dependent. Let $P(t)$ be a polynomial with coefficients in the field $k(f_2, ..., f_{n-1})$ of minimal degree such that $P(f_1) = 0$. Then

$$0 = \Delta_{(P(f_1), f_2, ..., f_{n-1})} = P'(f_1)\Delta_{(f_1, f_2, ..., f_{n-1})} = P'(f_1)\Delta_{\mathbf{f}} \ .$$

By minimality of degree, $P'(f_1) \neq 0$, so $\Delta_{\mathbf{f}} = 0$.

Conversely, suppose $f_1, ..., f_{n-1}$ are algebraically independent, and choose $f_n \in K$ transcendental over $k(f_1, ..., f_{n-1})$. Then for each i, x_i is algebraic over $k(f_1, ..., f_n)$, and there exists $P_i \in k[y_1, ..., y_{n+1}] = k^{[n+1]}$ such that $P_i(f_1, ..., f_n, x_i) = 0$. Now $\partial P_i/\partial y_{n+1} \neq 0$, since otherwise P_i gives a relation of algebraic dependence for $f_1, ..., f_n$. We may assume the degree of P_i is minimal in y_{n+1}, so that $\partial P_i/\partial y_{n+1}$ is nonzero when evaluated at $(f_1, ..., f_n, x_i)$.

By the chain rule, for each i and each j,

$$0 = \partial_{x_j} P_i(f_1, ..., f_n, x_i) = \sum_{1 \le s \le n} (P_i)_s (f_s)_{x_j} + (P_i)_{n+1}(x_i)_{x_j} \ ,$$

where $(P_i)_s$ denotes $\frac{\partial P_i}{\partial y_s}(f_1, ..., f_n, x_i)$. In matrix form, this becomes

$$0 = \begin{pmatrix} (P_i(f_1, ..., f_n, x_i))_{x_1} \\ \vdots \\ (P_i(f_1, ..., f_n, x_i))_{x_n} \end{pmatrix} = M \begin{pmatrix} (P_i)_1 \\ \vdots \\ (P_i)_n \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ (P_i)_{n+1} \\ \vdots \\ 0 \end{pmatrix} \ ,$$

where $M = \mathcal{J}(f_1, ..., f_n)$. Let $e_i = (0, ..., 1, ...0) \in K^n$ be the standard basis vectors ($1 \le i \le n$). The image of M as a linear operator on K^n is spanned by $(P_1)_{n+1}e_1, ..., (P_n)_{n+1}e_n$, and since $(P_i)_{n+1} \neq 0$ for each i, we conclude that M is surjective. Therefore, $\det M = \Delta_{\mathbf{f}}(f_n) \neq 0$. So part (a) is proved.

To prove (b), note first that, under the hypothesis $\Delta_{\mathbf{f}} \neq 0$, part (a) implies $f_1, ..., f_{n-1}$ are algebraically independent. This means that the transcendence degree of $k(f_1, ..., f_{n-1})$ equals $n-1$. Since $k(f_1, ..., f_{n-1}) \subset \ker \Delta_{\mathbf{f}}$, we have that $\ker \Delta_{\mathbf{f}}$ is the algebraic closure of $k(f_1, ..., f_{n-1})$ in K.

To prove (c), suppose first that $f_1, ..., f_{n-1}, g$ are algebraically independent. Then $f_1, ..., f_{n-1}$ are algebraically independent, and $\ker \Delta_{\mathbf{f}}$ is an algebraic extension of $k(f_1, ..., f_{n-1})$. Since g is transcendental over $k(f_1, ..., f_{n-1})$, it is also transcendental over $\ker \Delta_{\mathbf{f}}$. Therefore, $\Delta_{\mathbf{f}}(g) \neq 0$.

Conversely, suppose $f_1, ..., f_{n-1}, g$ are algebraically dependent. If $f_1, ..., f_{n-1}$ are algebraically independent, the same argument used above shows that

$g \in \ker \Delta_{\mathbf{f}}$. And if $f_1, ..., f_{n-1}$ are algebraically dependent, then $\Delta_{\mathbf{f}}$ is the zero derivation, by part (a). $\square$

Lemma 3.6. (Lemma 6 of [190]) *Suppose* $K = k(x_1, ..., x_n) = k^{(n)}$ *and* $D \in \mathrm{Der}_k(K)$ *has* $\mathrm{tr}\deg_k(\ker D) = n - 1$. *Then for any set* $\mathbf{f} = (f_1, ..., f_{n-1})$ *of algebraically independent elements of* $\ker D$*, there exists* $a \in K$ *such that* $D = a\Delta_{\mathbf{f}}$.

Proof. First, $\ker D = \ker \Delta_{\mathbf{f}}$, since each is equal to the algebraic closure of $k(f_1, ..., f_{n-1})$ in K. Choose $g \in K$ so that $Dg \neq 0$. Define $a = Dg(\Delta_{\mathbf{f}} g)^{-1}$. Then $D = a\Delta_{\mathbf{f}}$ when restricted to the subfield $k(f_1, ..., f_{n-1}, g)$. Since $Dg \neq 0$, g is transcendental over $\ker D$, hence also over $k(f_1, ..., f_{n-1})$. Thus, K is an algebraic extension of $k(f_1, ..., f_{n-1}, g)$. By *Prop. 1.12* we conclude that $D = a\Delta_{\mathbf{f}}$ on all of K. $\square$

Lemma 3.7. (Lemma 7 of [190]) *For* $n \geq 2$*, let* $K = k(x_1, ..., x_n) = k^{(n)}$. *Given* $f_1, ..., f_{n-1} \in K$ *algebraically independent, set* $\mathbf{f} = (f_1, ..., f_{n-1})$. *If* $\mathbf{g} = (g_1, ..., g_{n-1})$ *for* $g_i \in \ker \Delta_{\mathbf{f}}$*, then there exists* $a \in \ker \Delta_{\mathbf{f}}$ *such that* $\Delta_{\mathbf{g}} = a\Delta_{\mathbf{f}}$.

Proof. If $\Delta_{\mathbf{g}} = 0$, we can take $a = 0$. So assume $\Delta_{\mathbf{g}} \neq 0$, meaning that $g_1, ..., g_{n-1}$ are algebraically independent. In particular, $g_i \notin k$ for each i.

Since $\mathrm{tr.deg.}_k \ker \Delta_{\mathbf{f}} = n - 1$, the elements $f_1, ..., f_{n-1}, g_1$ are algebraically dependent. Let $P \in k[T_1, ..., T_n] = k^{[n]}$ be such that $P(\mathbf{f}, g_1) = 0$. The notation P_i will denote the partial derivative $\partial P/\partial T_i$. Then we may assume that $P_n(\mathbf{f}, g_1) \neq 0$; otherwise replace P by P_n. Likewise, by re-ordering the f_i if necessary, we may assume that $P_1(\mathbf{f}, g_1) \neq 0$. Since a jacobian determinant is a derivation in each argument, it follows that

$$0 = \Delta_{(P(\mathbf{f},g_1),f_2,...,f_{n-1})} = P_1(\mathbf{f}, g_1)\Delta_{(g_1,f_2,...,f_{n-1})} + P_n(\mathbf{f}, g_1)\Delta_{(f_1,f_2,...,f_{n-1})} \ .$$

Thus, $\Delta_{(g_1,f_2,...,f_{n-1})} = a\Delta_{\mathbf{f}}$ for some nonzero $a \in \ker \Delta_{\mathbf{f}}$.

If $n = 2$ we are done. Otherwise $n \geq 3$, and we may assume inductively that for some i with $1 \leq i \leq n - 2$ we have

$$\Delta_{(g_1,...,g_i,f_{i+1}...,f_{n-1})} = b\Delta_{\mathbf{f}}$$

for some nonzero $b \in \ker \Delta_{\mathbf{f}}$. Then $g_1, ..., g_i, f_{i+1} ..., f_{n-1}$ are algebraically independent, since the derivation they define is nonzero. Choose $Q \in k[T_1, ..., T_n]$ with $Q(g_1, ..., g_i, f_{i+1}, ..., f_{n-1}, g_{i+1}) = 0$, noting that $Q_n \neq 0$ (otherwise Q is a dependence relation for $g_1, ..., g_i, f_{i+1}, ..., f_{n-1}$). By re-ordering the f_i if necessary, we may assume that $Q_{i+1}(g_1, ..., g_i, f_{i+1}, ..., f_{n-1}, g_{i+1}) \neq 0$. As above, we have

$$\begin{aligned} 0 &= \Delta_{(g_1,...,g_i,Q(*),f_{i+2},...,f_{n-1})} \\ &= Q_{i+1}(*)\Delta_{(g_1,...,g_{i+1},f_{i+2},...,f_{n-1})} + Q_n(*)\Delta_{(f_1,...,f_{n-1})} \ , \end{aligned}$$

where $(*)$ denotes the input $(g_1, ..., g_i, f_{i+1}, ..., f_{n-1}, g_{i+1})$. This together with the inductive hypothesis allows us to conclude that

$$\Delta_{(g_1,...,g_{i+1},f_{i+2}...,f_{n-1})} = c\Delta_{\mathbf{f}}$$

for some nonzero $c \in \ker \Delta_{\mathbf{f}}$. This completes the proof. □

Lemma 3.8. *If $\Delta_{\mathbf{f}}$ is a jacobian derivation of $k^{(n)}$, then* $\mathrm{div}(\Delta_{\mathbf{f}}) = 0$.

Proof. Suppose $k^{(n)} = k(x_1, ..., x_n)$. For given x_i, *Prop. 2.18* implies that

$$\partial_{x_i}\left(\Delta_{\mathbf{f}}(x_i)\right) = \sum_{j=1}^{n} \Delta_{(f_1,...,(f_j)_{x_i},...,f_{n-1})}(x_i) \ .$$

Therefore,

$$\mathrm{div}(\Delta_{\mathbf{f}}) = \sum_{1\le i,j\le n} \Delta_{(f_1,...,(f_j)_{x_i},...,f_{n-1})}(x_i) \ .$$

Expanding these determinants, we see that

$$\mathrm{div}(\Delta_{\mathbf{f}}) = \sum_{\sigma\in S_n} \mathrm{sign}(\sigma)(f_1)_{y_1}(f_2)_{y_2}\cdots(f_j)_{y_jy_n}\cdots(f_{n-1})_{y_{n-1}} \ ,$$

where $\sigma = (y_1, ..., y_n)$ is a permutation of $(x_1, ..., x_n)$. Since $(f_j)_{y_jy_n} = (f_j)_{y_ny_j}$, terms corresponding to $(y_1, ..., y_j, ..., y_n)$ and $(y_1, ..., y_n, ..., y_j)$ cancel each other out, their signs being opposite. Therefore, the entire sum is 0. □

An additional fact about jacobian derivations is due to Daigle. It is based on the following result; the reader is referred to the cited paper for its proof.

Proposition 3.9. (Cor. 3.10 of [49]) *Let $f_1, ..., f_m \in B = k[x_1, ..., x_n] = k^{[n]}$ be given. Set $A = k[f_1, ..., f_m]$ and $M = \mathcal{J}(f_1, ..., f_m)$. Suppose $I \subset B$ is the ideal generated by the $d \times d$ minors of M, where d is the transcendence degree of A over k. If A is factorially closed in B, then* $\mathrm{height}(I) > 1$.

Corollary 3.10. (Cor. 2.4 of [49]) *Suppose $f_1, ..., f_{n-1} \in B = k[x_1, ..., x_n] = k^{[n]}$ are algebraically independent, and set $\mathbf{f} = (f_1, ..., f_{n-1})$. If $k[f_1, ..., f_{n-1}]$ is a factorially closed subring of B, then $\Delta_{\mathbf{f}}$ is irreducible, and* $\ker \Delta_{\mathbf{f}} = k[f_1, ..., f_{n-1}]$.

Proof. Since $\Delta_{\mathbf{f}} \neq 0$, we have that $\ker \Delta_{\mathbf{f}}$ is equal to the algebraic closure of $k[f_1, ..., f_{n-1}]$ in B. By hypothesis, $k[f_1, ..., f_{n-1}]$ is factorially closed, hence also algebraically closed in B. Therefore $\ker \Delta_{\mathbf{f}} = k[f_1, ..., f_{n-1}]$.

Let I be the ideal generated by the image of $\Delta_{\mathbf{f}}$, namely,

$$I = (\Delta_{\mathbf{f}}(x_1), ..., \Delta_{\mathbf{f}}(x_n)) \ .$$

Since the images $\Delta_{\mathbf{f}}(x_i)$ are precisely the $(n-1)\times(n-1)$ minors of the jacobian matrix $\mathcal{J}(f_1, ..., f_{n-1})$, the foregoing proposition implies that $\mathrm{height}(I) > 1$. Therefore, I is contained in no principal ideal other than B itself, and $\Delta_{\mathbf{f}}$ is irreducible. □

This, of course, has application to the locally nilpotent case, as we will see. However, not all derivations meeting the conditions of this corollary are locally nilpotent. For example, it was pointed out in *Chap. 1* that $k[x^2 - y^3]$ is factorially closed in $k[x, y]$, but is not the kernel of any locally nilpotent derivation of $k[x, y]$.

Another key fact about jacobians is given by van den Essen.

Proposition 3.11. (1.2.9 of [100]) *Let k be a field of characteristic zero and let $F = (F_1, ..., F_n)$ for $F_i \in k[x_1, ..., x_n] = k^{[n]}$. Then the rank of $\mathcal{J}(F)$ equals* tr $\deg_k k(F)$.

Here, the **rank** of the jacobian matrix is defined to be the maximal order of a nonzero minor of $\mathcal{J}(F)$.

Remark 3.12. It was observed that the jacobian determinant of a system of variables in a polynomial ring is always a unit of the base field. This fact gives a very useful way to construct locally nilpotent derivations of polynomial rings, as follows. Let $B = k[x_1, ..., x_n]$ for $n \geq 2$. Given i with $1 \leq i \leq n-1$, let $K = k(x_1, ..., x_i)$, and suppose $f_{i+1}, ..., f_{n-1} \in B$ satisfy $K[x_{i+1}, ..., x_n] = K[f_{i+1}, ..., f_{n-1}, g]$ for some $g \in B$. Define $D \in \mathrm{Der}_k(B)$ by

$$D = \Delta_{(x_1, ..., x_i, f_{i+1}, ..., f_{n-1})} \ ,$$

and let E denote the extension of D to $K[x_{i+1}, ..., x_n]$. Since $E(f_j) = 0$ for each j and $E(g) \in K^*$, it follows that E is locally nilpotent. Therefore, D is also locally nilpotent.

Example 3.13. Let $B = \mathbb{C}[x, y, z, u] = \mathbb{C}^{[4]}$, and define

$$p = yu + z^2 \ , \ v = xz + yp \ , \text{ and } w = x^2u - 2xzp - yp^2 \ .$$

The **Vénéreau polynomials** are $f_n := y + x^n v$, $n \geq 1$. The preceding remark can be used to prove that f_n is an x-variable of B when $n \geq 3$.

First, define a $\mathbb{C}(x)$-derivation θ of $\mathbb{C}(x)[y, z, u]$ by

$$\theta y = 0 \ , \ \theta z = x^{-1}y \ , \ \theta u = -2x^{-1}z \ ,$$

noting that $\theta p = 0$. Then

$$y = \exp(p\theta)(y) \ , \ v = \exp(p\theta)(xz) \ , \ \text{and } w = \exp(p\theta)(x^2u) \ .$$

It follows that, for all $n \geq 1$,

$$\begin{aligned} \mathbb{C}(x)[y, z, u] &= \mathbb{C}(x)[y, xz, x^2u] \\ &= \mathbb{C}(x)[y, v, w] \\ &= \mathbb{C}(x)[y + x^n v, v, w] \\ &= \mathbb{C}(x)[f_n, v, w] \ . \end{aligned}$$

Next, assume $n \geq 3$, and define a derivation d of B by $d = \Delta_{(x,v,w)}$. Since $\mathbb{C}(x)[y, v, w] = \mathbb{C}(x)[y, z, u]$, it follows from the preceding remark that d is locally nilpotent. And since $dx = dv = 0$, we have that $x^{n-3}vd$ is also locally nilpotent. In addition, it is easily checked that $dy = x^3$. Therefore,

$$\exp(x^{n-3}vd)(x) = x \quad \text{and} \quad \exp(x^{n-3}vd)(y) = y + x^{n-3}vd(y) = y + x^n v = f_n \ .$$

Set $P_n = \exp(x^{n-3}vd)(z)$ and $Q_n = \exp(x^{n-3}vd)(u)$. Then $\mathbb{C}[x, f_n, P_n, Q_n] = \mathbb{C}[x, y, z, u]$.

It remains an open question whether f_1 or f_2 are x-variables of B, or even variables. The Vénéreau polynomials are further explored in *Chap. 10* below.

3.2.4 Homogenizing a Derivation

Suppose $B = k[x_1, ..., x_n] = k^{[n]}$, and $D \in \mathrm{Der}_k(B)$ is given, $D \neq 0$. Set $A = \ker D$. Write $Dx_i = f_i(x_1, ..., x_n)$ for $f_i \in B$, and set $d = \max_i \deg(Dx_i)$, where degrees are taken relative to the standard grading of B. The **homogenization** of D is the derivation $D^H \in \mathrm{Der}_k(B[w])$ defined by

$$D^H(w) = 0 \quad \text{and} \quad D^H(x_i) = w^d f_i(\tfrac{x_1}{w}, ..., \tfrac{x_n}{w}) \ ,$$

where w is an indeterminate over B. Note that D^H is homogeneous of degree $d-1$, relative to the standard grading of $B[w]$, and $D^H \bmod (w-1) = D$ as derivations of B. In addition, if D is (standard) homogeneous to begin with, then $D^H(x_i) = Dx_i$ for every i.

In order to give further properties of D^H relative to D, we first extend D to the derivation $\mathcal{D} \in \mathrm{Der}_k(B[w, w^{-1}])$ defined by $\mathcal{D}b = Db$ for $b \in B$, and $\mathcal{D}w = 0$. Note that $\ker \mathcal{D} = A[w, w^{-1}]$, and that if $D \in \mathrm{LND}(B)$, then $\mathcal{D} \in \mathrm{LND}(B[w, w^{-1}])$.

Next, define $\alpha \in \mathrm{Aut}_k(B[w, w^{-1}])$ by $\alpha(x_i) = \frac{x_i}{w}$ and $\alpha(w) = w$, noting that $\alpha\mathcal{D}\alpha^{-1} \in \mathrm{Der}_k(B[w, w^{-1}])$. In particular,

$$\alpha\mathcal{D}\alpha^{-1}(x_i) = \alpha\mathcal{D}(wx_i) = w\alpha(Dx_i) = wf_i(\tfrac{x_1}{w}, ..., \tfrac{x_n}{w}) \ .$$

Therefore, $w^{d-1} \cdot \alpha\mathcal{D}\alpha^{-1}(x_i) = D^H(x_i)$, that is, D^H equals the restriction of $w^{d-1}\alpha\mathcal{D}\alpha^{-1}$ to $B[w]$. We thus conclude that D^H has the following properties.

1. D^H is homogeneous of degree $d-1$ in the standard grading of $B[w]$.
2. $\ker(D^H) = \ker(\alpha\mathcal{D}\alpha^{-1}) \cap B[w] = \alpha(A[w, w^{-1}]) \cap B[w]$
3. If $p : B[w] \to B$ is evaluation at $w = 1$, then $p(\ker D^H) = \ker D$.
4. If D is irreducible, then D^H is irreducible.
5. If $D \in \mathrm{LND}(B)$, then $D^H \in \mathrm{LND}_w(B[w])$.

Since $D^H \equiv D$ modulo $(w-1)$, the assignment $D \mapsto D^H$ is an injective function from $\mathrm{LND}(B)$ into the subset of standard homogeneous elements of $\mathrm{LND}_w(B[w])$. This is not, however, a bijective correspondence, since D^H will never be of the form wE for $E \in \mathrm{LND}_w(B[w])$.

Homogenizations are used in *Chap. 8* to calculate kernel elements of D, especially property (3) above.

3.2.5 Other Base Rings

Observe that many of the definitions given for $k^{[n]}$ naturally generalize to the rings $A^{[n]}$ for non-fields A. In this case, we simply include the modifier **over** A. For example, if $B = A[x_1, ..., x_n]$, we refer to **variables of** B **over** A as those $f \in B$ such that $B = A[f]^{[n-1]}$. Likewise, **partial derivatives over** A, **jacobian derivations over** A, **linear derivations over** A, and **triangular derivations over** A are defined as elements of $\mathrm{Der}_A(B)$ in the obvious way.

3.3 Group Actions on $\mathbb{A}^n$

Given $f \in B = k^{[n]}$, the variety in $\mathbb{A}^n$ defined by f will be denoted by $V(f)$. Likewise, if $I \subset B$ is an ideal, the variety defined by I is $V(I)$.

The group of algebraic automorphisms of $\mathbb{A}^n$ is anti-isomorphic to $GA_n(k)$, in the sense that $(F_1 \circ F_2)^* = F_2^* \circ F_1^*$ in $GA_n(k)$ when F_1 and F_2 are automorphisms of $\mathbb{A}^n$. Thus, we identify these two groups with one another.

If an algebraic k-group G acts algebraically on affine space $X = \mathbb{A}^n$, we also define the **rank** of the G-action exactly as rank was defined for a derivation, i.e., the least integer $r \geq 0$ for which there exists a coordinate system $(x_1, ..., x_n)$ on $k[X]$ such that $k[x_{r+1}, ..., x_n] \subset k[X]^G$.

The G-action on $X = \mathbb{A}^n$ is a **linear action** if and only if G acts by linear automorphisms. The action is a **triangular action** if and only if G acts by triangular automorphisms. And the action is a **tame action** if and only if G acts by tame automorphisms. Similarly, the action is **linearizable** if it is conjugate to a linear action, and **triangularizable** if it is conjugate to a triangular action.

The case in which the ring of invariants is a polynomial ring over k is quite important. For example, if H is a normal subgroup of G, and if $k[X]^H = k^{[m]}$ for some m, then G/H acts on the affine space $\mathbb{A}^m$ defined by $k[X]^H$, and this action can be quite interesting. This is the idea behind the main examples of *Chap. 7* and *Chap. 10* below.

Following are some particulars when the group $\mathbb{G}_a$ acts on affine space. Let a $\mathbb{G}_a$-action on $\mathbb{A}^n$ be given by

$$\rho : \mathbb{G}_a \times \mathbb{A}^n \to \mathbb{A}^n \quad \text{where} \quad \rho(t, \mathbf{x}) = (F_1(t, \mathbf{x}), ..., F_n(t, \mathbf{x}))$$

for functions F_i, and $\mathbf{x} = (x_1, ..., x_n)$ for coordinate functions x_i on $\mathbb{A}^n$.

- ρ is algebraic if and only if $F_i \in k[t, x_1, ..., x_n] \cong k^{[n+1]}$ for each i.
- ρ is linear if and only if each F_i is a linear polynomial in $x_1, ..., x_n$ over $k[t]$.
- ρ is triangular if and only if $F_i \in k[t, x_1, ..., x_i]$ for each i.
- ρ is **quasi-algebraic** if and only if $F_i(t_0, \mathbf{x}) \in k[x_1, ..., x_n]$ for each $t_0 \in k$ and each i. (See [282].)

- ρ is **holomorphic** if and only if $k = \mathbb{C}$ and each F_i is a holomorphic function on $\mathbb{C}^{n+1}$.

Of course, $\exp(tD)$ is a linear algebraic $\mathbb{G}_a$-action if and only if D is a linear locally nilpotent derivation (i.e., given by a nilpotent matrix), and $\exp(tD)$ is a triangular $\mathbb{G}_a$-action if and only if D is a triangular derivation.

3.3.1 Translations

The simplest algebraic $\mathbb{G}_a$-action on $X = \mathbb{A}^n$ is a **translation**, meaning that for some system of coordinates $(x_1, ..., x_n)$, the action is given by

$$t \cdot (x_1, ..., x_n) = (x_1 + t, x_2, ..., x_n) = \exp(t\partial_{x_1}) \ .$$

Clearly, a translation is fixed-point free, and admits a geometric quotient: $X/\mathbb{G}_a = X/\!/\mathbb{G}_a \cong \mathbb{A}^{n-1}$.

In case $n = 1$, the locally nilpotent derivations of $k[x]$ are those of the form $c\frac{d}{dx}$ for some $c \in k$ (*Princ. 8*). So translations are the only algebraic $\mathbb{G}_a$-actions on the affine line: $t \cdot x = x + tc$.

3.3.2 Planar Actions

The simplest linear $\mathbb{G}_a$-action on the plane comes from the standard representation of $\mathbb{G}_a$ on $V = \mathbb{A}^2$ via matrices

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \quad (t \in k) \ .$$

The algebraic quotient $V/\!/\mathbb{G}_a$ is the line $\mathrm{Spec}(k[x])$. If $\pi : V \to V/\!/\mathbb{G}_a$ is the quotient map, then the fiber $\pi^{-1}(\lambda)$ over any $\lambda \in V/\!/\mathbb{G}_a$ is the line $x = \lambda$, which is a single orbit if $\lambda \neq 0$, and a line of fixed points if $\lambda = 0$. In this case, the geometric quotient $V/\mathbb{G}_a$ does not exist.

More generally, a triangular action on $\mathbb{A}^2$ is defined by

$$t \cdot (x, y) = (x, y + tf(x)) = \exp(tD)$$

for any $f(x) \in k[x]$, where $D = f(x)\partial_y$. In case $k = \mathbb{C}$, define a planar $\mathbb{G}_a$-action by the orthogonal matrices

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \quad (t \in \mathbb{C}) \ .$$

This is not an algebraic action, although it is quasi-algebraic, locally finite, and holomorphic. It is the exponential of the locally finite derivation $x\partial_y - y\partial_x$ on $\mathbb{C}[x, y]$.

3.4 Locally Nilpotent Derivations of Polynomial Rings

One of the foundational facts about locally nilpotent derivations of polynomial rings is the following, which is due to Makar-Limanov (Lemma 8 of [190]).

Theorem 3.14. (Makar-Limanov's Theorem) *Let $D \in \mathrm{LND}(B)$ be irreducible, where $B = k^{[n]}$. Let $f_1, ..., f_{n-1}$ be $n-1$ algebraically independent elements of* $\ker D$, *and set $\mathbf{f} = (f_1, ..., f_{n-1})$. Then there exists $a \in \ker D$ such that $\Delta_{\mathbf{f}} = aD$. In particular, $\Delta_{\mathbf{f}} \in \mathrm{LND}(B)$.*

In case $n \leq 3$, even stronger properties hold; see *Thm. 5.6* below.

The proof below follows that of Makar-Limanov, using the lemmas proved earlier concerning jacobian derivations.

Proof. Let S be the set of nonzero elements of $A = \ker D$, and let K be the field $S^{-1}A$. Then D extends to a locally nilpotent derivation $S^{-1}D$ of $S^{-1}B$. By *Princ. 13*, we have that $K = \ker(S^{-1}D)$, and $S^{-1}B = K[r] = K^{[1]}$ for some local slice r of D. Therefore $(S^{-1}B)^* = K^*$.

Extend D to a derivation D' on all $\mathrm{frac}(B)$ via the quotient rule. (Note: D' is not locally nilpotent.) From *Cor. 1.23*, we have that $\ker D' = K$.

By *Lemma 3.6*, there exists $\eta \in \mathrm{frac}(B)$ such that $D' = \eta\Delta_{\mathbf{f}}$. Note that $\Delta_{\mathbf{f}}$ restricts to a derivation of B.

Suppose $\eta = b/a$ for $a, b \in B$ with $\gcd(a, b) = 1$. Write $\Delta_{\mathbf{f}} = c\delta$ for $c \in B$ and irreducible $\delta \in \mathrm{Der}_k(B)$. Then $aD = bc\delta$, and by *Prop. 2.3* we have that $(a) = (bc)$. Since $\gcd(a, b) = 1$, this means $b \in B^*$, so we may just as well assume $b = 1$. Therefore, $\Delta_{\mathbf{f}} = aD$. The key fact to prove is that $a \in \ker D$.

Let $g_1, ..., g_n \in S^{-1}B$ be given, and consider the jacobian determinant $\det \mathcal{J}(g_1, ..., g_n) \in \mathrm{frac}(B)$. We claim that $\det \mathcal{J}(g_1, ..., g_n)$ is contained in the principal ideal $aS^{-1}B$ of $S^{-1}B$.

Since $S^{-1}B = K[r]$, each g_i can be written as a finite sum $g_i = \sum a_{ij}r^j$ for $a_{ij} \in K$ and $j \geq 0$. Therefore, $\det \mathcal{J}(g_1, ..., g_n)$ is a sum of functions of the form $\det \mathcal{J}(a_1r^{e_1}, ..., a_nr^{e_n})$ for $a_i \in K$ and $e_i \geq 0$. By the product rule, for each i we also have

$$\det \mathcal{J}(a_1r^{e_1}, ..., a_nr^{e_n}) = \\ a_i \det \mathcal{J}(a_1r^{e_1}, ..., r^{e_i}, ..., a_nr^{e_n}) + r^{e_i} \det \mathcal{J}(a_1r^{e_1}, ..., a_i, ..., a_nr^{e_n}).$$

So $\det \mathcal{J}(g_1, ..., g_n)$ may be expressed as a sum of functions of the form $q \det \mathcal{J}(b_1, ..., b_n)$, where $q \in S^{-1}B$, and either $b_i \in K$ or $b_i = r^{e_i}$ for $e_i \geq 1$. If every $b_i \in K$, then $b_1, ..., b_n$ are linearly dependent, and this term will be zero. Likewise, if $b_i = r^{e_i}$ and $b_j = r^{e_j}$ for $i \neq j$, then $b_1, ..., b_n$ are linearly dependent, and this term is zero. Therefore, by reordering the b_i if necessary, any nonzero summand $q \det \mathcal{J}(b_1, ..., b_n)$ is of the form $q \det \mathcal{J}(a_1, ..., a_{n-1}, r^e) = q\Delta_{\mathbf{a}}(r^e)$, where $q \in S^{-1}B$, $a_i \in K$, $\mathbf{a} = (a_1, ..., a_{n-1})$, and $e \geq 1$. By *Lemma 3.7*, there exists $h \in \ker \Delta_{\mathbf{f}} = K$ such that $\Delta_{\mathbf{a}} = h\Delta_{\mathbf{f}}$ for some $h \in K$. In particular, $\Delta_{\mathbf{a}}$ resticts to $S^{-1}B$. Since $\Delta_{\mathbf{f}}(y) \in aB$ for all $y \in B$, it follows that $q\Delta_{\mathbf{a}}(r^e) \in ahS^{-1}B = aS^{-1}B$ (since

h is a unit). Since $\det \mathcal{J}(g_1, ..., g_n)$ is a sum of such functions, we conclude that $\det \mathcal{J}(g_1, ..., g_n) \in aS^{-1}B$ for *any* $g_1, ..., g_n \in S^{-1}B$, as claimed.

In particular, if $B = k[x_1, ..., x_n]$, then $1 = \det \mathcal{J}(x_1, ..., x_n) \in aS^{-1}B$, implying that $a \in (S^{-1}B)^* = K^*$. But this means $a \in B \cap K = \ker D$. □

Quite recently, Makar-limanov generalized this result in [191] to give a description of the locally nilpotent derivations of any commutative affine $\mathbb{C}$-domain. He writes that his goal is "to give a standard form for an lnd on the affine domains. This form is somewhat analogous to a matrix representation of a linear operator" (p.2). The theorem he proves is the following.

Theorem 3.15. (Generalized Makar-Limanov Theorem) *Let I be a prime ideal of $B = \mathbb{C}^{[n]}$, and let R be the factor ring B/I, with standard projection $\pi : B \to R$. Given $D \in \mathrm{LND}(R)$, there exist elements $f_1, ..., f_{n-1} \in B$ and nonzero elements $a, b \in R^D$ such that, for every $g \in B$,*

$$aD(\pi(g)) = b\pi(\mathcal{J}(f_1, ..., f_{n-1}, g)) \ .$$

Another way to express the conclusion of this theorem is that $aD = b(\Delta_{\mathbf{f}}/I)$, where $\mathbf{f} = (f_1, ..., f_{n-1})$. The reader is referred to Makar-Limanov's paper for the general proof.

The Makar-Limanov Theorem implies the following.

Corollary 3.16. (Prop. 1.3.51 of [100]) *If $B = k^{[n]}$ and $D \in \mathrm{LND}(B)$, then* $\mathrm{div}(D) = 0$.

Proof. Choose algebraically independent $f_1, ..., f_{n-1} \in \ker D$. There exists an irreducible $\delta \in \mathrm{LND}(B)$ and $c \in \ker D$ such that $D = c\delta$. According to the theorem above, there also exists $a \in \ker D$ such that $a\delta = \Delta_{\mathbf{f}}$. Therefore, $D = (c/a)\Delta_{\mathbf{f}}$, so by the product rule, together with *Lemma 3.8*, we have

$$\mathrm{div}(D) = (c/a)\mathrm{div}(\Delta_{\mathbf{f}}) + \sum_i \partial_{x_i}(c/a)\Delta_{\mathbf{f}}(x_i) = 0 + \Delta_{\mathbf{f}}(c/a) = 0 \ .$$

□

The next two results are due to Daigle.

Lemma 3.17. (Prop. 1.2 of [49]) *Let A be a subalgebra of B such that B has transcendence degree 1 over A. If $D, E \in \mathrm{Der}_A(B)$, then there exist $a, b \in B$ for which $aD = bE$.*

Proof. Let $K = \mathrm{frac}(A)$ and $L = \mathrm{frac}(B)$. By *Prop. 3.2*, the dimension of $\mathrm{Der}_K(L)$ as a vector space over L is equal to one. Therefore, if S is the set of nonzero elements of B, then $S^{-1}D$ and $S^{-1}E$ are linearly dependent over K, and consequently $aD = bE$ for some $a, b \in B$. □

Proposition 3.18. (Cor. 2.5 of [49]) *Suppose $B = k^{[n]}$, and $D \in \mathrm{LND}(B)$ has $\ker D \cong k^{[n-1]}$. If $\ker D = k[f_1, ..., f_{n-1}]$ and $\mathbf{f} = (f_1, .., f_{n-1})$, then $\Delta_{\mathbf{f}}$ is irreducible and locally nilpotent, and $D = a\Delta_{\mathbf{f}}$ for some $a \in \ker D$.*

Proof. Let $A = \ker D$. Since A is factorially closed, the fact that $\Delta_{\mathbf{f}}$ is irreducible follows from *Cor. 3.10* above. By *Lemma 3.17*, there exist $a, b \in B$ such that $bD = a\Delta_{\mathbf{f}}$, since D and $\Delta_{\mathbf{f}}$ have the same kernel. We may assume $\gcd(a,b) = 1$. Then $\Delta_{\mathbf{f}} B \subset bB$, implying that b is a unit. So we may assume $b = 1$. The fact that $\Delta_{\mathbf{f}}$ is locally nilpotent and $a \in A$ now follows from *Princ. 7*. $\square$

In the other direction, we would like to know whether, if $\mathbf{f} = (f_1, ..., f_{n-1})$ for $f_i \in B$, the condition that $\Delta_{\mathbf{f}}$ is irreducible and locally nilpotent always implies $\ker \Delta_{\mathbf{f}} = k[f_1, ..., f_{n-1}]$. But this is a hard question. For example, the truth of this property for $n = 3$ would imply the truth of the two-dimensional Jacobian Conjecture!

To see this, we refer to Miyanishi's Theorem in *Chap. 5*, which asserts that the kernel of any nonzero locally nilpotent derivation of $k^{[3]}$ is isomorphic to $k^{[2]}$. Suppose $A = k[f,g]$ is the kernel of a locally nilpotent derivation of $k^{[3]}$. Let $u, v \in k[f,g]$ have the property that $\det \frac{\partial(u,v)}{\partial(f,g)}$ is a nonzero constant. We have

$$\Delta_{(u,v)} = \det \frac{\partial(u,v)}{\partial(f,g)} \Delta_{(f,g)}$$

which we know to be irreducible and locally nilpotent. If the above property were true, it would follow that $A = \ker \Delta_{(u,v)} = k[u,v]$.

The section concludes with facts about rank.

Proposition 3.19. (Lemma 3 of [116]) *Suppose $B = k^{[n]}$ and $D \in \mathrm{Der}_k(B)$ is linear relative to the coordinate system $(x_1, ..., x_n)$ on B. Set $V = kx_1 \oplus \cdots \oplus kx_n$.*

(a) $\mathrm{rank}(D)$ *equals the rank of D as a linear operator on V.*
(b) $D \in \mathrm{LND}(B)$ *if and only if D is nilpotent as a linear operator on V.*

Proof. Suppose that $\mathrm{corank}(D) = m$, and let η denote the nullity of D as a linear operator on V. Let $(f_1, ..., f_n)$ be a system of variables on B for which $f_1, ..., f_m \in \ker D$. Suppose the standard $\mathbb{Z}$-grading of B is $B = \oplus B_i$, and let L_i denote the homogeneous summand of f_i coming from B_1. In other words, $L_i \in V$ is the linear part f_i in the usual sense.

Since each $Dx_i \in V$, it follows that D is homogeneous of degree 0 in the standard grading. Thus, since $Df_1 = \cdots Df_m = 0$, we also have $DL_1 = \cdots = DL_m = 0$, and it follows that $\eta \geq m$.

Conversely, let $v_1, ..., v_\eta \in V$ be linearly independent vectors annihilated by D. Since $(v_1, ..., v_\eta)$ is a partial system of variables on B, it follows that $\eta \leq m$. This proves (a), and (b) is clear. $\square$

In his thesis, Wang [302] (Lemma 2.3.5) gives the equivalent statement: With the notation and hypotheses of the proposition above,

$$\dim_k(V \cap \ker D) = \mathrm{corank}(D) \; .$$

3.5 Slices in Polynomial Rings

Questions about slices are always important. Such questions are discussed in *Chap. 10*. But for polynomial rings, we have the following fundamental result.

Proposition 3.20. *Suppose* $B = k^{[n]}$ *and* $D \in \mathrm{LND}(B)$ *has* $Ds = 1$ *for* $s \in B$.

(a) *s is a variable of $B[w] = k^{[n+1]}$.*
(b) *If $B/sB = k^{[n-1]}$, then D is a partial derivative.*

Proof. Let $A = \ker D \subset B$. By the Slice Theorem, $B = A[s]$ and $\pi_s(B) = A$, where π_s is the Dixmier map defined by s. Extend D to $D^* \in \mathrm{LND}(B[w])$ by setting $D^*w = 0$. Then $\ker D^* = A[w]$. Since w is transcendental over A, we have $A[w] \cong A[s] = B = k^{[n]}$. So there exist $g_1, ..., g_n \in B[w]$ such that $A[w] = k[g_1, ..., g_n]$. Therefore,

$$B[w] = A[s][w] = A[w][s] = k[g_1, ..., g_n, s] \ ,$$

and s is a variable of $B[w]$.

In addition, we have that $A \cong B/sB$ by the Slice Theorem. Thus, if $B/sB = k^{[n-1]}$, then $B = A[s]$ implies that s is a variable of B. $\square$

Note that the condition of part (b) holds if s is a variable. Part (a) appears only as part of a proof in [198], namely, in the proof of Thm. 1.2. But it clearly deserves to be highlighted. A crucial question is:

If $s \in B$ is a variable of $B[w]$, does it follow that s is a variable of B?

A negative answer to this question would imply a negative solution to either the Embedding Problem or Cancellation Problem. Potential examples of such phenomena are provided by the Vénéreau polynomials $f_1, f_2 \in \mathbb{C}^{[4]}$: These are known to be variables of $\mathbb{C}^{[5]}$, but it is an open question whether they are variables of $\mathbb{C}^{[4]}$. See *Chap. 10* for details.

In summary, we obtain the following useful principle: Suppose $D \in \mathrm{LND}(k^{[n]})$ has a slice s.

(1) $\ker D$ is n-generated.
(2) If D^* extends D trivially to $k^{[n+1]}$, then $\ker D^*$ is n-generated.
(3) If s is a variable of $k^{[n]}$, then $\ker D$ is $(n-1)$-generated.

3.6 Triangular Derivations and Automoprhisms

Fix a coordinate system $B = k[x_1, ..., x_n]$. Define subgroups $H_i, K_i \subset BA_n(k)$, $i = 1, ..., n$, by

$$H_i = \{h \in BA_n(k) | h(x_j) = x_j, 1 \le j \le n - i\}$$
$$K_i = \{g \in BA_n(k) | g(x_j) = x_j, i + 1 \le j \le n\} = BA_i(k).$$

Then for each i, K_i acts on H_i by conjugation, and $BA_n(k) = H_i \rtimes K_{n-i}$.

Proposition 3.21. *Suppose $B = k^{[n]}$ and $D \in \mathrm{Der}_k(B)$ is triangular in some coordinate system. Then $D \in \mathrm{LND}(B)$. In addition, if $n \geq 2$, then $\mathrm{rank}(D) \leq n - 1$.*

Proof. We argue by induction on n for $n \geq 1$, the case $n = 1$ being obvious. For $n \geq 2$, note that since D is triangular, D restricts to a triangular derivation of $k[x_1, ..., x_{n-1}]$. By induction, D is locally nilpotent on this subring. In particular, $Dx_n \in k[x_1, ..., x_{n-1}] \subset \mathrm{Nil}(D)$, which implies $x_n \in \mathrm{Nil}(D)$. Therefore, D is locally nilpotent on all B.

Now suppose $n \geq 2$. If $Dx_1 = 0$ we are done, so assume $Dx_1 = c \in k^*$. Choose $f \in k[x_1]$ so that $Dx_2 = f'(x_1)$. Then $D(cx_2 - f(x_1)) = 0$, and this is a variable of B. $\square$

We next describe the factorization of triangular automorphisms into unipotent and semi-simple factors. (See [90] for a related result.)

Proposition 3.22. *Every triangular automorphism of $k^{[n]}$ is of the form $\exp T \circ L$, where L is a diagonal matrix and T is a triangular derivation.*

Proof. If $F \in BA_n(k)$, then $F \circ L$ is unipotent triangular for some diagonal matrix L. So it suffices to assume F is unipotent, i.e., of the form

$$F = (x_1, x_2 + f_2(x_1), x_3 + f_3(x_1, x_2), ..., x_n + f_n(x_1, ..., x_{n-1}))$$

for polynomials f_i. We show by induction on n that the map $F - I = (0, f_2, ..., f_n)$ is locally nilpotent, the case $n = 1$ being obvious. (Observe that $(F - I)(c) = 0$ for $c \in k$.)

Let $A = k[x_1, ..., x_{n-1}]$, and suppose by induction that $F - I$ restricts to a locally nilpotent map on A. Then it suffices to show that $F - I$ is nilpotent at every polynomial of the form ax_n^t ($a \in A$). One easily obtains the formula

$$(F - I)^m (ax_n^t) = (F - I)^m (a) x_n^t + (\text{lower } x_n \text{ terms}) \ .$$

By induction, $(F - I)^m(a) = 0$ for $m \gg 0$. Since the x_n-degree is thus lowered, we eventually obtain $(F - I)^M (ax_n^t) = 0$ for $M \gg 0$. It follows that $F - I$ is locally nilpotent on all B. Thus, *Prop. 2.17* implies $F = \exp D$ for $D = \log(I + (F - I)) \in \mathrm{LND}(B)$. $\square$

Observe that, for triangular derivations D_1, D_2 of $B = k^{[n]}$, $D_1 + D_2$ is again triangular, hence locally nilpotent. In general, however, D_1 and D_2 do not commute, and $\exp D_1 \exp D_2 \neq \exp(D_1 + D_2)$. Nonetheless, the product on the left is an exponential automorphism.

Corollary 3.23. *If D_1 and D_2 are triangular k-derivations of $B = k[x_1, ..., x_n]$, then there exists a triangular k-derivation E such that $\exp D_1 \exp D_2 = \exp E$.*

Proof. Since $\exp D_1 \exp D_2$ is triangular, it equals $\exp E \circ L$ for triangular E and diagonal L; and it is clear that in this case $L = I$ (identity). $\square$

See also the proof of Cor. 3 in [90].

The main theorem of this section is the following.

Theorem 3.24. *If $F \in BA_n(k)$ has finite order, then there exists $L \in GL_n(k)$ and a triangular $D \in \mathrm{LND}(B)$ such that $F = \exp(-D)L \exp D$.*

The linearizability of finite-order triangular automorphisms was first proved by Ivanenko in [151]. The proof presented below makes use of exponential automorphisms to give a much shorter demonstration. Whether a general element of finite order in $GA_n(k)$ can be linearized remains an open problem.

The proof is based on the following more general fact.

Proposition 3.25. *Let R be a UFD containing k, let $D \in \mathrm{LND}(R)$, and let $\lambda \in \mathrm{Aut}_k(R)$ have finite order $m \geq 2$. Set $A = \ker D$ and $\gamma = \exp D \circ \lambda$. Suppose*

(1) $\lambda(a) \in A$ *for all* $a \in A$.
(2) $\lambda(a) = a$ *for all* $a \in A^*$.
(3) γ *has finite order* m

Then there exists $E \in \mathrm{LND}(R)$ such that $\ker E = A$ and $\gamma = \exp(-E)\lambda \exp E$.

Proof. Write $D = f\Delta$ for irreducible $\Delta \in \mathrm{LND}(R)$ and $f \in A$. Since $\ker \Delta = \ker(\lambda^{-1}\Delta\lambda) = A$ by hypothesis (1), we conclude from *Princ. 12*, together with the fact that R is a UFD and Δ is irreducible, that $\lambda^{-1}\Delta\lambda = c\Delta$ for some $c \in A^*$. By hypothesis, $\lambda(c) = c$, and thus $\lambda^{-i}\Delta\lambda^i = c^i\Delta$ for each $i \in \mathbb{Z}$. It follows that for each $i \in \mathbb{Z}$, $\lambda^{-i}D\lambda^i = \lambda^{m-i}(f)c^i\Delta$. In particular, $D = \lambda^{-m}D\lambda^m = c^m D$, so $c^m = 1$.

Set $E = g\Delta$ for undetermined $g \in A$. Then

$$\begin{aligned} &\exp(-E)\lambda\exp(E) = (\exp D)\lambda \quad \text{if and only if} \\ &\exp(-E)\exp(\lambda E\lambda^{-1}) = \exp D \quad \text{if and only if} \\ &\exp((\lambda(g)c^{-1} - g)\Delta) = \exp(f\Delta). \end{aligned}$$

So we need to solve for $g \in A$ which satisfies the equation $f = c^{-1}\lambda(g) - g$. We find a solution $g \in \mathrm{span}_{k[c]}\{f, \lambda(f), \lambda^2(f), ..., \lambda^{m-1}(f)\} \subset A$. (Note that $k[c]$ is a field.)

First, if $\gamma_i := \lambda^{-i}(\exp D)\lambda^i$, then

$$1 = \gamma^m = (\exp D \circ \lambda)^m = \gamma_m\gamma_{m-1}\cdots\gamma_2\gamma_1 \ .$$

Since $\gamma_i = \exp(\lambda^{m-i}(f)c^i\Delta)$, it follows that

$$\exp(h\Delta) = 1 \quad \text{for} \quad h = \sum_{i=1}^{m} \lambda^{m-i}(f)c^i \ .$$

Therefore, $h = 0$, and we may eliminate $\lambda^{m-1}(f)$ from the spanning set above.

Next, for undetermined coefficients $a_i \in k[c]$, consider $g = a_1 f + a_2\lambda(f) + \cdots + a_{m-1}\lambda^{m-2}(f)$. Then $c^{-1}\lambda(g) - g$ equals

$$-a_1 f+(c^{-1}a_1-a_2)\lambda(f)+\cdots+(c^{-1}a_{m-2}-a_{m-1})\lambda^{m-2}(f)+c^{-1}a_{m-1}\lambda^{m-1}(f)\ .$$

Since $h = 0$, we have that $c^{-1}a_{m-1}\lambda^{m-1}(f)$ equals

$$-c^{-2}a_{m-1}f - c^{-3}a_{m-1}\lambda(f) - \cdots - c^{-(m-1)}a_{m-1}\lambda^{m-3}(f) - a_{m-1}\lambda^{m-2}(f)\ .$$

Combining these gives that $c^{-1}\lambda(g) - g$ equals

$$\begin{aligned}&(-a_1 - c^{-2}a_{m-1})f + (c^{-1}a_1 - a_2 - c^{-3}a_{m-1})\lambda(f) + \cdots\\ &\cdots+(c^{-1}a_{m-3}-a_{m-2}-c^{-(m-1)}a_{m-1}\lambda^{m-3}(f)+(c^{-1}a_{m-2}-2a_{m-1})\lambda^{m-2}(f).\end{aligned}$$

So we need to solve for a_i such that $M(a_1, a_2, ..., a_{m-1})^T = (1, 0, ..., 0)^T$ for

$$M = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & -c^{-2} \\ c^{-1} & -1 & 0 & \cdots & 0 & -c^{-3} \\ 0 & c^{-1} & -1 & \cdots & 0 & -c^{-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -c^{m-1} \\ 0 & 0 & 0 & \cdots & c^{-1} & -2 \end{pmatrix}_{(m-1)\times(m-1)} .$$

It is easily checked that $|M| \neq 0$. For example, replace row 2 by c^{-1}(row 1)+(row 2); then replace row 3 by c^{-1}(row 2)+(row 3); and so on. Eventually, we obtain the non-singular upper-triangular matrix

$$N = \begin{pmatrix} -1 & 0 & 0 & \cdots & -c^{-2} \\ 0 & -1 & 0 & \cdots & -2c^{-3} \\ 0 & 0 & -1 & \cdots & -3c^{-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -m \end{pmatrix} .$$

Therefore, we can solve for g, and thereby conjugate γ to λ. $\square$

Proof of Theorem 3.24. Let m be the order of F. We have that $BA_n(k) = H_1 \rtimes K_{n-1}$, so we can write $F = hg$ for $g \in K_{n-1}$ and $h \in H_1$. Then $1 = F^m = (gh)^m = g^m h'$ for some $h' \in H_1$, which implies $g^m = h' = 1$. By induction, there exists a triangular derivation D with $Dx_n = 0$ and $\tilde{g} := \exp(-D)g\exp D \in GL_n(k) \cap K_{n-1}$. Thus, $\exp(-D)F\exp D = \tilde{h}\tilde{g}$ for $\tilde{h} := \exp(-D)h\exp D \in H_1$. So it suffices to assume from the outset that $F = hg$ for linear $g \in K_{n-1}$ and $h \in H_1$.

If $h = (x_1, ..., x_{n-1}, ax_n + f(x_1, ..., x_{n-1}))$, then

$$h = \exp\left(f\frac{\partial}{\partial x_n}\right) \circ (x_1, ..., x_{n-1}, ax_n)\ .$$

Thus, $F = \exp(f\frac{\partial}{\partial x_n})L$, where $L = (x_1, ..., x_{n-1}, ax_n)g \in GL_n(k)$. Note that L restricts to $\ker(\frac{\partial}{\partial x_n}) = k[x_1, ..., x_{n-1}]$. By the preceding proposition, the theorem now follows. $\square$

We also have:

Proposition 3.26. *For the polynomial ring $B = k[x_1, ..., x_n]$, every monomial derivation $D \in \mathrm{LND}(B)$ is triangular relative to some ordering of $x_1, ..., x_n$.*

Proof. We may assume, with no loss of generality, that

$$\nu_D(x_1) \leq \nu_D(x_2) \leq \cdots \leq \nu_D(x_n) \ .$$

Given i, write $Dx_i = ax_1^{e_1} \cdots x_n^{e_n} \neq 0$ for $a \in k$ and $e_i \geq 0$. If $Dx_i \neq 0$, then $\nu_D(x_i) - 1 = \sum_{j=1}^{n} e_j \nu_D(x_j)$. Due to the ordering above, this is only possible if $e_j = 0$ for $j \geq i$. Therefore, $Dx_i \in k[x_1, ..., x_{i-1}]$ for every i. □

We will see that triangular monomial derivations provide us with important examples.

3.7 Homogeneous Locally Nilpotent Derivations

The first result of this section will be stated for any commutative k-domain B, though we are most interested in the case B is a polynomial ring.

Suppose $D \in \mathrm{LND}(B)$ is homogeneous of degree d relative to some $\mathbb{Z}$-grading of B. This is equivalent to giving an algebraic action of the group $\mathbb{G}_a \rtimes \mathbb{G}_m$ on $X = \mathrm{Spec}(B)$, where the action of $\mathbb{G}_m$ on $\mathbb{G}_a = \mathrm{Spec}(k[x])$ is given by $t \cdot x = t^d x$. This is further equivalent to giving $D \in \mathrm{LND}(B)$ and an action $\mathbb{G}_m \to \mathrm{Aut}_k(B)$, $t \to \lambda_t$, such that $\lambda_t^{-1} D \lambda_t = t^d D$ for all t. The homogeneous polynomials $f \in B_i$ are the semi-invariants $f \in B$ for which $t \cdot f = t^i f$ $(t \in \mathbb{G}_m)$.

Assume these conditions hold.

Proposition 3.27. *Under the hypotheses above, if $s \in \mathbb{G}_m$ has finite order m not dividing d, then $\exp D \circ \lambda_s$ is conjugate to λ_s. In particular,*

$$(\exp D \circ \lambda_s)^m = 1 \ .$$

Proof.

$$\exp\left(\tfrac{s^d}{1-s^d} D\right) (\exp D) \lambda_s \exp\left(-\tfrac{s^d}{1-s^d} D\right) = \lambda_s$$

□

The second result of this section is about kernels of homogeneous derivations.

Proposition 3.28. *Suppose $D \in \mathrm{LND}(B)$, $D \neq 0$, is homogeneous relative to some $\mathbb{N}$-grading $\oplus_{i \in \mathbb{N}} B_i$ of $B = k^{[n]}$. If $B_0 \cap \ker D = k$ and $\ker D$ is a polynomial ring, then $\ker D = k[g_1, ..., g_{n-1}]$ for homogeneous g_i.*

This is immediately implied by the following more general fact about $\mathbb{N}$-gradings.

Proposition 3.29. (Lemma 7.6 of [47]) *Let $A = k^{[r]}$ for $r \geq 1$ and let $A = \oplus_{i\in\mathbb{N}} A_i$ be a grading such that $A_0 = k$. If $A = k[f_1, ..., f_m]$ for homogeneous f_i, then there is a subset $\{g_1, ..., g_r\}$ of $\{f_1, ..., f_m\}$ with $A = k[g_1, ..., g_r]$.*

Proof. Consider a subset $\{g_1, ..., g_s\}$ of $\{f_1, ..., f_m\}$ satisfying $A = k[g_1, ..., g_s]$ and minimal with respect to this property; in particular, $\deg g_i > 0$ for all i. Let $R = k[T_1, ..., T_s] = k^{[s]}$ with grading $R = \oplus_{i\in\mathbb{N}} R_i$ determined by $R_0 = k$ and $\deg T_i = \deg g_i$. Then the surjective k-homomorphism $e : R \to A$, $e(\varphi) = \varphi(g_1, ..., g_s)$, is homogeneous of degree zero.

Suppose that the prime ideal $\mathfrak{p} = \ker e$ is not zero. Note that $(T_1, ..., T_s)$ contains $\mathfrak{p}$, meaning that the variety $V(\mathfrak{p}) \subset \mathbb{A}^s$ contains the origin. Since the origin is a smooth point (A is smooth over k), and since $\mathfrak{p}$ is generated by its homogeneous elements, the jacobian (smoothness) condition implies that some homogeneous $\varphi \in \mathfrak{p}$ contains a term λT_j ($\lambda \in k^*$). Since φ is homogeneous and $\deg T_i > 0$ for all i, $\varphi - \lambda T_j \in k[T_1, ..., T_{j-1}, T_{j+1}, ..., T_s]$. Thus, $g_j \in k[g_1, ..., g_{j-1}, g_{j+1}, ..., g_s]$, contradicting minimality of $\{g_1, ..., g_s\}$.

Therefore, $\ker e = 0$, and e is an isomorphism, implying $s = r$. □

Corollary 3.30. *If $B = k^{[n]}$ and if $\mathbb{G}_m$ acts algebraically on $\mathbb{A}^n$ in such a way that $B^{\mathbb{G}_m} = k$, then the action is linearizable.*

Proof. The action induces a $\mathbb{Z}$-grading of B for which elements of B_i are semi-invariants of weight i. In particular, $B_0 = B^{\mathbb{G}_m}$. If $f \in B_i$ and $g \in B_j$ for $i < 0$ and $j > 0$, then $f^j g^{-i} \in B_0$, a contradiction. Therefore, we can assume any non-constant semi-invariant has strictly positive weight. So the grading on B induced by the $\mathbb{G}_m$-action is an $\mathbb{N}$-grading: $B = \oplus_{i\in\mathbb{N}} B_i$.

Suppose $B = k[x_1, ..., x_n]$. Given i ($1 \leq i \leq n$), we can write $x_i = \sum_{j\in\mathbb{N}} f_{ij}$, where $f_{ij} \in B_j$. So B is generated as a k-algebra by finitely many homogeneous polynomials f_{ij}. By the preceding result, there exist homogeneous $g_1, ..., g_n \in B$ such that $B = k[g_1, ..., g_n]$, i.e., $(g_1, ..., g_n)$ is a system of semi-invariant variables for B. □

3.8 Symmetric Locally Nilpotent Derivations

Quite often, a special property belonging to a $\mathbb{G}_a$-action is equivalent to the condition that that the action can be embedded in a larger algebraic group action. For example, we saw above how homogeneity equates to an action of $\mathbb{G}_a \rtimes \mathbb{G}_m$. Another important condition to consider is symmetry. The symmetric group S_n acts naturally on the polynomial ring $B = k[x_1, ..., x_n]$ by permutation of the variables x_i. Define $D \in \mathrm{Der}_k(B)$ to be **fully symmetric** if and only if $D\sigma = \sigma D$ for each $\sigma \in S_n$. To give $D \in \mathrm{LND}(B)$ fully symmetric is equivalent to giving an algebraic action of $\mathbb{G}_a \times S_n$ on $\mathbb{A}^n$, where S_n acts in the standard way on $\mathbb{A}^n$.

Example 3.31. $E = \sum_{i=1}^{n} \partial_{x_i}$ is fully symmetric and locally nilpotent, and $\ker E = k[x_1 - x_2, x_2 - x_3, ..., x_{n-1} - x_n]$. Note that E is a partial derivative. If $f \in \ker E \cap B^{S_n}$, then fE is also fully symmetric and locally nilpotent.

Proposition 3.32. *Let $\mathbb{Z}_2$ act on $B = k[x_1, ..., x_n]$ by transposing x_1 and x_2, and fixing $x_3, ..., x_n$. If $D \in \mathrm{LND(B)}$ commutes with this $\mathbb{Z}_2$-action, then $D(x_1 - x_2) = 0$.*

Proof. Let $\tau \in \mathbb{Z}_2$ transpose x_1 and x_2, fixing $x_3, ..., x_n$, and let $Dx_1 = F(x_1, x_2)$ for $F \in k[x_3, ..., x_n]^{[2]}$. Then $Dx_2 = D(\tau x_1) = \tau D x_1 = F(x_2, x_1)$. This implies

$$D(x_1 - x_2) = F(x_1, x_2) - F(x_2, x_1) \in (x_1 - x_2) \cdot B \ .$$

By *Cor. 1.20*, we conclude that $D(x_1 - x_2) = 0$. $\square$

Now suppose D is a fully symmetric locally nilpotent derivation. Then $D(x_i - x_j) = 0$ for all i, j, so $k[x_1 - x_2, x_2 - x_3, ..., x_{n-1} - x_n] \subset \ker D$. Consequently, the derivations fE above are the only fully symmetric locally nilpotent derivations.

Corollary 3.33. *If $D \in \mathrm{LND}(B)$ is fully symmetric and $D \neq 0$, then* $\mathrm{rank}(D) = 1$.

3.9 Some Important Early Examples

This section illustrates the fact that the triangular derivations of polynomial rings already provide a rich variety of important examples.

In 1972, Nagata [239] published an example of a polynomial automorphism of $\mathbb{A}^3$ which he conjectured is not tame. Later, Bass embedded Nagata's automorphism as an element of a one-parameter subgroup of polynomial automophisms of $\mathbb{A}^3$, gotten by exponentiating a certain non-linear locally nilpotent derivation of $k[x, y, z]$. It was known at the time that every unipotent group of polynomial automorphisms of the plane is triangular in some coordinate system (see *Chap. 4*). In sharp contrast to the situation for the plane, Bass showed that the subgroup he constructed could not be conjugated to the triangular subgroup. Then Popov generalized Bass's construction to produce non-triangularizable $\mathbb{G}_a$-actions on $\mathbb{A}^n$ for every $n \geq 3$. These discoveries initiated the exploration of a new world of algebraic representations $\mathbb{G}_a \hookrightarrow GA_n(k)$.

Remark 3.34. Some of the examples below exhibit, without explanation, the kernel of the derivation under consideration. Methods for calculating kernels of locally nilpotent derivations are discussed in *Chap. 8* below.

3.9.1 Bass's Example

([12], 1984) The example of Bass begins with the basic linear derivation of $k[x,y,z]$, namely, $\Delta = x\partial_y + 2y\partial_z$. Then $\ker \Delta = k[x,F]$, where $F = xz - y^2$. Note that $D := F\Delta$ is also a locally nilpotent derivation of $k[x,y,z]$, and the corresponding $\mathbb{G}_a$-action on $\mathbb{A}^3$ is

$$\alpha_t := \exp(tD) = (x, y + txF, z + 2tyF + t^2xF^2) \ .$$

Nagata's automorphism is α_1. The fixed point set of this action is the cone $F = 0$, which has an isolated singularity at the origin. On the other hand, Bass observed that any triangular automorphism $(x, y + f(x), z + g(x,y))$ has a cylindrical fixed point set, i.e., defined by $f(x) = g(x,y) = 0$, which (if non-empty) has the form $C \times \mathbb{A}^1$ for some variety C. In general, an affine variety X is called a **cylindrical variety** if $X = Y \times \mathbb{A}^1$ for some affine variety Y. Since a cylindrical variety can have no isolated singularities, it follows that α_t cannot be conjugated to any triangular set of automoprhisms.

3.9.2 Popov's Examples

([253], 1987) Generalizing Bass's approach, Popov pointed out that the fixed-point set of any triangular $\mathbb{G}_a$-action on $\mathbb{A}^n$ is a cylindrical variety, whereas the hypersurface defined by a non-degenerate quadratic form is not a cylindrical variety. So to produce non-triangularizable examples in higher dimensions, it suffices to find $D \in \mathrm{LND}(k^{[n]})$ such that $\ker D$ contains a non-degenerate quadratic form h; then $\exp(thD)$ is a non-triangularizable $\mathbb{G}_a$-action. In even dimensions, let $B = k[x_1, ..., x_n, y_1, ..., y_n]$, and define D by

$$\begin{aligned}
&Dx_1 = 0,\ Dx_2 = x_1,\ Dx_3 = x_2,\ \dots\ , Dx_n = x_{n-1}, \\
&Dy_1 = y_2,\ Dy_2 = y_3,\ \dots\ , Dy_{n-1} = y_n,\ Dy_n = 0.
\end{aligned}$$

Then D is a triangular (linear) derivation, and $Dh = 0$ for the non-degenerate quadratic form $h = \sum_{i=1}^{n} (-1)^{i+1} x_i y_i$. For odd dimensions at least 5, start with D above, and extend D to trivially to $k[x_1, ..., x_n, y_1, ..., y_n, z]$; that is, $Dz = 0$. Then $h + z^2$ is a non-degenrate quadratic form annihilated by D.

3.9.3 Smith's Example

([281], 1989) At the conclusion of his paper, Bass asked whether the $\mathbb{G}_a$-action he gave on $\mathbb{A}^3$ is **stably tame**, i.e., whether the action becomes tame when extended trivially to $\mathbb{A}^4$. M. Smith gave a positive answer to this question by first showing the following.

Lemma 3.35. (Smith's Formula) *Let $D \in \mathrm{LND}(B)$ for $B = k^{[n]}$ and let $f \in \ker D$ be given. Extend D to $B[w]$ by $Dw = 0$, and define $\tau \in GA_{n+1}(k)$ by $\tau = \exp(f\partial_w)$. Then*

$$\exp(fD) = \tau^{-1} \exp(-wD) \tau \exp(wD) \ .$$

Proof. Since τ fixes B, $\tau D = D\tau$, so $\tau^{-1}(-wD)\tau = \tau^{-1}(-w)D = (f-w)D$. Applying the exponential now gives

$$\exp(fD)\exp(-wD) = \exp((f-w)D)$$
$$= \exp\big(\tau^{-1}(-wD)\tau\big) = \tau^{-1}\exp(-wD)\tau.$$

□

Applying this lemma with $f = tF$ and $D = \Delta$ from Bass's example yields the following tame factorization for the example of Bass-Nagata: for $t \in \mathbb{G}_a$,

$$\exp(tD) = (x, y + txF, z + 2tyF + t^2xF^2, w)$$
$$= (x, y, z, w-tF)(x, y-wx, z-2wy+w^2x, w)(x, y, z, w+tF)(x, y+wx, z+2wy + w^2x, w).$$

Lemma 3.36. *This $\mathbb{G}_a$-action on $\mathbb{A}^4$ is not triangularizable.*

Proof. Note first that the rank of D on $k^{[4]}$ is clearly 2. Let $X \subset \mathbb{A}^4$ be the set of fixed points. Then $X = C \times \mathbb{A}^1$ for a singular cone C, and the singularities of X form a line. Suppose $k[x, y, z, w] = k[a, b, c, d]$ and that D is triangular in the latter system of coordinates, with $Da = 0$ and $Db \in k[a]$. The ideal defining X is (Db, Dc, Dd), and thus $X \subset V(Db)$. If $Db \neq 0$, this is a union of parallel coordinate hyperplanes, implying $X \subset H$ for a coordinate hyperplane H. Since this is clearly impossible, $Db = 0$. We also have $X \subset V(Dc)$, where $Dc \in k[a, b]$. If $Dc \neq 0$, this implies $X = Y \times \mathbb{A}^2$, where Y is a component of the curve in $\mathrm{Spec}(k[a, b])$ defined by Dc. But this also cannot occur, since then the singularities of X would be of dimension 2. Thus, $Dc = 0$. But this would imply that the rank of D is 1, a contradiction. Therefore, D extended to $k[x, y, z, w]$ cannot be conjugated to a triangular derivation by any element of $GA_4(k)$. □

So in dimension 4 (and likewise in higher dimensions), there exist $\mathbb{G}_a$-actions which are tame but not triangularizable. It is an important open question whether every tame $\mathbb{G}_a$-action on $\mathbb{A}^3$ can be triangularized. It goes to the structure of the tame subgroup. It should by noted that the recent work of Shestakov and Umirbaev [278, 279] implies that the Nagata automorphism α_1 above is *not* tame as an element of $GA_3(k)$.

3.9.4 Winkelmann's Example 1

([306], 1990) In this groundbreaking paper, Winkelmann investigates $\mathbb{C}^+$-actions on $\mathbb{C}^n$ which are fixed-point free, motivated by questions about the quotients of such actions. In dimension 4, he defines an action $\exp(tD)$, where D is the triangular derivation on $\mathbb{C}[x, y, z, w]$ defined by

$$Dx = 0\,,\ Dy = x\,,\ Dz = y\,,\ Dw = y^2 - 2xz - 1\ .$$

Then $\exp(tD)$ defines a free algebraic $\mathbb{C}^+$-action on $\mathbb{C}^4$, but the orbit space is not Hausdorff in the natural topology (Lemma 8).[4] In particular, D is not a partial derivative, i.e., the action is not a translation. Winkelmann calculates this kernel explicitly: $\ker D = \mathbb{C}[x, f, g, h]$, where

$$f = y^2 - 2xz \ , \ g = xw + (1-f)y \ , \text{ and } xh = g^2 - f(1-f)^2 \, .$$

In particular, $\ker D$ is the coordinate ring of a singular hypersurface in $\mathbb{C}^4$. This implies $\text{rank}(D) = 3$, since if the rank were 1 or 2, the kernel would be a polynomial ring (see *Chap. 4*). In [283], Snow gives the similar example

$$Ex = 0 \, , \ Ey = x \, , \ Ez = y \, , \ Ew = 1 + y^2 \ .$$

and also provides a simple demonstration that the topological quotient is non-Hausdorff (Example 3.5). (It is easy to show that D and E are conjugate.) In [100], van den Essen considers E, and indicates that E does not admit a slice, a condition which is *a priori* independent of the fact that the corresponding quotient is not an affine space (Example 9.5.25). And in [81], Sect. 3, Deveney, Finston, and Gehrke consider E as well, showing that the associated $\mathbb{C}^+$-action $\exp(tE)$ on $\mathbb{C}^4$ is not proper.

The properties of a group action being fixed-point free, proper, or locally trivial are discussed in [76, 81]. In the case $\mathbb{G}_a$ acts algebraically on affine space $\mathbb{C}^n$, these papers provide a simple algebraic characterization of properness, and show that these three properties are related in the following way:

$$\text{locally trivial} \quad \Rightarrow \quad \text{proper} \quad \Rightarrow \quad \text{fixed-point free}$$

and these implications are strict. They also point out that for a proper $\mathbb{G}_a$-action on $\mathbb{C}^n$, the topological space of orbits is necessarily Hausdorff; and that for a locally trivial action, the orbits can be separated by invariant functions.

3.9.5 Winkelmann's Example 2

([306], 1990) On $B = \mathbb{C}[u, v, x, y, z] = \mathbb{C}^{[5]}$, define the triangular derivation F by

$$Fu = Fv = 0 \, , \ Fx = u \, , \ Fy = v \, , \ Fz = 1 + (vx - uy) \ .$$

Then $Fx, Fy, Fz \in \ker F$ and $(Fx, Fy, Fz) = (1)$, which implies $\exp(tF)$ is a locally trivial $\mathbb{C}^+$-action on $\mathbb{C}^5$. The kernel of F is presented in [81], namely

$$\ker F = \mathbb{C}[u, v, vx - uy, x + x(vx - uy) - uz, y + y(vx - uy) - vz]$$

To see that the associated $\mathbb{C}^+$-action on $\mathbb{C}^5$ is not globally trivial (i.e., that F does not admit a slice), note that F is homogeneous of degree 0 relative

[4] Winkelmann calls the set of orbits for this action with the quotient topology a "quotient space", and uses the notation $\mathbb{C}^n/G$. But this differs from the algebraic quotient $\text{Spec}(\ker D)$, and is also not a geometric quotient.

to the $\mathbb{C}^*$-action $(tu, t^{-1}v, tx, t^{-1}y, z)$. We thus have an action of $\mathbb{C}^+ \times \mathbb{C}^*$ on $\mathbb{C}^5$. The invariant ring of the $\mathbb{C}^*$-action is $B_0 = \mathbb{C}[uv, xy, vx, uy, z]$, the ring of degree-0 elements. Therefore F restricts to B_0. If F has a slice in B, then by homogeneity there exists a slice $s \in B_0$. But the ideal generated by the image of F restricted to B_0 equals $(vx + uy, uv, 1 + vx - uy)$, which does not contain 1, meaning F has no slice in B_0. (The fixed-point set of the induced $\mathbb{C}^+$-action on Spec(B_0) is of dimension one.) Therefore F has no slice in B.

In [153], Jorgenson asks: Is there a triangular $\mathbb{G}_a$-action on $\mathbb{C}^4$ that is locally trivial but not equivariantly trivial? (Question 2)

3.9.6 Example of Deveney and Finston

([77], 1995) Again in dimension 5, define δ on $\mathbb{C}[u, v, x, y, z]$ by

$$\delta u = \delta v = 0 \ , \ \delta x = u \ , \ \delta y = v \ , \ \delta z = 1 + uy^2 \ .$$

The authors show that $\exp(t\delta)$ is a proper $\mathbb{C}^+$-action on $\mathbb{C}^5$ and that $\ker \delta$ is isomorphic to the ring

$$\mathbb{C}[u_1, u_2, u_3, u_4, u_5]/(u_2u_5 - u_1^2u_4 - u_3^3 - 3u_1u_3) \ ,$$

which is the coordinate ring of a singular hypersurface $Y \subset \mathbb{C}^5$. If $p : \mathbb{C}^5 \to Y$ is the quotient morphism, then fibers of p over singular points of Y are two-dimensional, which clearly implies that the action is not locally trivial.

For another example of a free $\mathbb{G}_a$-action on an affine variety which does not admit a geometric quotient, see Derksen [67], 4.1.

3.9.7 Bass's Question

The section concludes with a question of Bass [12], which is still open.

> If k is an algebraically closed field, is the automorphism group $GA_n(k)$ generated by one-parameter subgroups, i.e., by images of algebraic homomorphisms from $\mathbb{G}_a$ and $\mathbb{G}_m$?

3.10 The Homogeneous Dependence Problem

In a fascinating paper [128] dating from 1876, Paul Gordan and Max Nöther investigated the vanishing of the Hessian determinant of an algebraic form, using the language of systems of differential operators. In particular, the question they consider is the following. Suppose $h \in \mathbb{C}[x_1, ..., x_n]$ is a homogeneous polynomial whose **Hessian determinant** is identically zero:

$$\det \left(\frac{\partial^2 h}{\partial x_i \partial x_j} \right)_{ij} = 0 \ .$$

Does it follow that h is degenerate, i.e., that $h \in \mathbb{C}[Tx_1, ..., Tx_{n-1}]$ for some $T \in GL_n(\mathbb{C})$? They prove that the answer is yes when $n = 3$ and $n = 4$, and garner some partial results for the case $n = 5$.

In the course of their proof, the authors consider changes of coordinates involving a parameter $\lambda \in \mathbb{C}$:

> *Die Functionen $\Phi(x)$, gebildet für die Argumente $x + \lambda\xi$, sind unabhängig von λ:*
>
> $$\Phi(x + \lambda\xi) = \Phi(x) \ . \quad \text{(p.550)}^5$$

Here, x denotes a vector of coordinates $(x_1, ..., x_n)$, and ξ a vector of homogeneous polynomials. In modern terms, the association $\lambda \cdot x = x + \lambda\xi$ gives a $\mathbb{C}^+$-action on $\mathbb{C}^n$ (where $\lambda \in \mathbb{C}$), and the functions Φ are its invariants. The authors continue:

> *Ist eine solche ganze Function Φ das Product zweier ganzen Functionen*
>
> $$\Phi = \phi(x) \cdot \psi(x)$$
>
> *so sind auch die Factoren selbst Functionen Φ.* (p. 551)[6]

We recognize this as the property that the ring of invariants of a $\mathbb{C}^+$-action is factorially closed. In effect, Gordan and Nöther studied an important type of $\mathbb{C}^+$-action on $\mathbb{C}^n$, which we will now describe in terms of derivations.

Let $B = k[x_1, ..., x_n] = k^{[n]}$, and let $D \in \mathrm{LND}(B)$ be given, $D \neq 0$. The **Homogeneous Dependence Problem** for locally nilpotent derivations asks:

> If D is standard homogeneous and has the property that $D^2 x_i = 0$ for each i, is the rank of D always strictly less than n? Equivalently, does there exist a linear form $L \in B$ with $DL = 0$, i.e., are the images Dx_i linearly dependent?

For such a derivation D, note that the $\mathbb{G}_a$-action is simply

$$\exp(tD) = (x_1 + tDx_1, ..., x_n + tDx_n) \ ,$$

and these are precisely the kinds of coordinate changes considered by Gordan and Nöther. Note that, given i,

$$D \circ \exp D(x_i) = D(x_i + Dx_i) = Dx_i + D^2 x_i = Dx_i \ .$$

On the other hand, $Dx_i \in \ker D$ means that $\exp D(Dx_i) = Dx_i$. Therefore, D and $\exp D$ commute. This in turn implies that, if we write $F = \exp D = x + H$,

[5] "The functions $\Phi(x)$, constructed for the arguments $x + \lambda\xi$, are independent of λ."

[6] "If such a function Φ is a product of two entire functions $\Phi = \phi(x)\psi(x)$, then so also are the factors themselves functions Φ."

where $x = (x_1, ..., x_n)$ and $H = (Dx_1, ..., Dx_n)$, then $H \circ H = 0$. Herein lies the connection to the work of Gordan and Nöther.

In their paper, Gordan and Nöther effectively proved that the answer to the Homogeneous Dependence Problem is yes when $n = 3$ or $n = 4$. In fact, they showed that in these cases there exist two independent linear forms, L and M, with $DL = DM = 0$, which implies that the rank of D is 1 when $n = 3$, and at most 2 when $n = 4$.

In the modern era, Wang proved in his 1999 thesis (Prop. 2.4.4) that if $D \in \text{LND}(k[x_1, x_2, x_3])$ has the property that $D^2 x_i = 0$ for each i, then $\text{rank}(D) \leq 1$ [302, 303]. So in the case of dimension 3, the homogeneity condition can be removed. A simple proof of Wang's result is given in *Chap. 5* below. Wang further proved that, in dimension 4, the rank of a homogeneous derivation having $D^2 x_i = 0$ for each i could not equal 3 (Lemma 2.5.2). Then in 2000, Derksen constructed an example of such a derivation D in dimension 8 whose rank is 7, thereby showing that the stronger result of Gordan and Nöther (i.e., that the kernel contains two independent linear forms) does not generalize. Finally in 2004, de Bondt found a way to construct counterexamples to the Homogeneous Dependence Problem in all dimensions $n \geq 6$ by using derivations of degree 4. So the Homogeneous Dependence Problem remains open only for the case $n = 5$. The examples of Derksen and de Bondt are discussed below.

At the time of their work, neither Wang nor Derksen seems to have been aware of the paper of Gordan and Nöther. Rather, it is an example of an important question resurfacing. The Gordan-Nöther paper was brought to the author's attention only recently by van den Essen, and its existence was made known to him by S. Washburn. Van den Essen was interested in its connections to his study of the Jacobian Conjecture; see [23, 24, 25, 104] for a discussion of these connections, and some new positive results for this conjecture. The recent article of DeBondt [64] gives a modern proof of the results of Gordan and Nöther, in addition to some partial results in dimension 5.

3.10.1 Construction of Examples

We construct, for each $N \geq 8$, a family of derivations D of the polynomial ring $k[x_1, ..., x_N]$ with the property that $D^2 x_i = 0$ for each i. The example of Derksen belongs to this family.

Given $m \geq 1$, let $B = k[s_1, ..., s_m] = k^{[m]}$ and let $\delta \in \text{LND}(B)$ be such that $\delta^2 s_i = 0$ for each i (possibly $\delta = 0$). Let $u \in B^\delta = \ker \delta$ be given ($u \neq 0$). Extend δ to $B[t] = B^{[1]}$ by setting $\delta t = 0$.

Next, given $n \geq 3$, choose an $n \times n$ skew-symmetric matrix M with entries in $B[t]^\delta$, i.e., $M \in \mathcal{M}_n(B[t]^\delta)$ and $M^T = -M$. Also, let $\mathbf{v} \in (B[t]^\delta)^n$ be a nonzero vector in the kernel of M.

Next, let $\mathbf{x} = (x_1, ..., x_n)$, $\mathbf{y} = (y_1, ..., y_n)$, and z be indeterminates over $B[t]$, so that $B[t, \mathbf{x}, \mathbf{y}, z] = k^{[m+2n+2]}$. Note that $m + 2n + 2 \geq 9$. Extend δ to a locally nilpotent derivation of this larger polynomial ring by setting

$$\delta\mathbf{x} = u\mathbf{v} \quad , \quad \delta\mathbf{y} = M\mathbf{x} \quad \text{and} \quad \delta z = u^{-1}\delta(\langle\mathbf{x},\mathbf{y}\rangle) \ .$$

Here, it is understood that for vectors $\mathbf{a} = (a_1, ..., a_n)$ and $\mathbf{b} = (b_1, ..., b_n)$, the statement $\delta\mathbf{a} = \mathbf{b}$ means $\delta a_i = b_i$ for each i. In addition, $\langle\mathbf{a},\mathbf{b}\rangle$ denotes the inner product of $\mathbf{a}$ and $\mathbf{b}$. Observe the **product rule for inner products**:

$$\delta(\langle\mathbf{a},\mathbf{b}\rangle) = \langle\delta\mathbf{a},\mathbf{b}\rangle + \langle\mathbf{a},\delta\mathbf{b}\rangle \ .$$

It is clear from the definition that $\delta^2\mathbf{x} = 0$. In addition,

$$\delta^2\mathbf{y} = \delta(M\mathbf{x}) = M(\delta\mathbf{x}) = M(u\mathbf{v}) = uM\mathbf{v} = 0 \ .$$

Further, since M is skew-symmetric, we have $0 = \langle\mathbf{x}, M\mathbf{x}\rangle = \langle\mathbf{x},\delta\mathbf{y}\rangle$. Therefore,

$$\delta(\langle\mathbf{x},\mathbf{y}\rangle) = \langle\delta\mathbf{x},\mathbf{y}\rangle + \langle\mathbf{x},\delta\mathbf{y}\rangle = \langle\delta\mathbf{x},\mathbf{y}\rangle - \langle\mathbf{x},\delta\mathbf{y}\rangle \in \ker\delta \ .$$

It follows that δz is a well-defined polynomial (since u divides $\delta\mathbf{x}$), and $\delta^2 z = 0$. In addition, if $F = uz - \langle\mathbf{x},\mathbf{y}\rangle$, then $\delta F = 0$.

Since F does not involve t, the kernel element $t - F$ is a variable. It follows that

$$B[t,\mathbf{x},\mathbf{y},z]/(t-F) = B[\mathbf{x},\mathbf{y},z] = k^{[m+2n+1]} \ ,$$

and that the derivation $D := \delta \bmod (t-F)$ has the property that $D^2\mathbf{x} = D^2\mathbf{y} = D^2 z = 0$.

3.10.2 Derksen's Example

This example appears in [100], 7.3, Exercise 6. It uses the minimal values $m = 1$ and $n = 3$ from the construction above, so that $m + 2n + 1 = 8$. Derksen found this example by considering the exterior algebra associated to three linear derivations.

First, let δ be the zero derivation of $B = k[s] = k^{[1]}$, and choose $u = s$. The extension of δ to $k[s,t]$ is also zero. Choose

$$\mathbf{v} = \begin{pmatrix} t^2 \\ s^2t \\ s^4 \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} 0 & s^4 & -s^2t \\ -s^4 & 0 & t^2 \\ s^2t & -t^2 & 0 \end{pmatrix} \ .$$

With these choices, we get the derivation D on the polynomial ring

$$k[s,x_1,x_2,x_3,y_1,y_2,y_3,z] = k^{[8]}$$

defined by $Ds = 0$,

$$D\mathbf{x} = \begin{pmatrix} sF^2 \\ s^3F \\ s^5 \end{pmatrix} \ , \quad D\mathbf{y} = \begin{pmatrix} 0 & s^4 & -s^2F \\ -s^4 & 0 & F^2 \\ s^2F & -F^2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \ ,$$

and $Dz = F^2y_1 + s^2Fy_2 + s^4y_3$, where F is the quadratic form

$$F = sz - (x_1y_1 + x_2y_2 + x_3y_3) \ .$$

Observe that D is homogeneous, of degree 4. To check that s is the only linear form in the kernel of D (up to scalar multiples), let V_i denote the vector space of forms of degree i in these 8 variables, and let $W \subset V_5$ denote the subspace generated by the monomials appearing in the image of $D : V_1 \to V_5$. Then it suffices to verify that the linear map $D : V_1 \to W$ has a one-dimensional kernel, and this is easily done with standard methods of linear algebra. We conclude that the rank of D is 7. □

3.10.3 De Bondt's Examples

Theorem 3.37. (De Bondt [65]) *For $n \geq 3$, let $B = k^{[2n]} = k[x_1, y_1, ..., x_n, y_n]$, and define $D \in \mathrm{Der}_k(B)$ by*

$$Dx_i = fgx_i - g^2y_i \quad \text{and} \quad Dy_i = f^2x_i - fgy_i \ ,$$

where $f = x_1y_2 - x_2y_1$ and $g = x_1y_3 - x_3y_1$. Then:

(a) *D is standard homogeneous of degree 4.*
(b) *$f, g \in \ker D$*
(c) *$D^2x_i = D^2y_i = 0$ for each i, and therefore $D \in \mathrm{LND}(B)$.*
(d) $\mathrm{rank}(D) = 2n$.

Proof. Let $R = k[a, b] = k^{[2]}$ and let $N \in \mathcal{M}_2(R)$ be given by

$$N = \begin{pmatrix} ab & -b^2 \\ a^2 & -ab \end{pmatrix} \ .$$

Then $N^2 = 0$.

Let $\mathcal{B} = R[x_1, y_1, ..., x_n, y_n] = k^{[2n+2]}$. Define R-linear $\mathcal{D} \in \mathrm{LND}_R(\mathcal{B})$ by

$$\mathcal{D} = \begin{pmatrix} N & 0 & \cdots & 0 \\ 0 & N & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N \end{pmatrix}_{(2n \times 2n)} \ .$$

Then for each i,

$$\mathcal{D}x_i = abx_i - b^2y_i \ , \ \mathcal{D}y_i = a^2x_i - aby_i \ , \quad \text{and} \quad \mathcal{D}^2x_i = \mathcal{D}^2y_i = 0 \ .$$

In addition, for every pair i, j, we have

$$\mathcal{D}(x_iy_j) = x_i(a^2x_j - aby_j) + y_j(abx_i - b^2y_i) = a^2x_ix_j - b^2y_iy_j = \mathcal{D}(x_jy_i) \ ,$$

which implies $x_iy_j - x_jy_i \in \ker \mathcal{D}$ for each pair i, j.

Set $f = x_1y_2 - x_2y_1$ and $g = x_1y_3 - x_3y_1$. The crucial observation is that f and g are kernel elements not involving a or b. Thus, $(a - f, b -$

$g, x_1, ..., y_n)$ is a triangular system of coordinates on $\mathcal{B}$. If $I \subset \mathcal{B}$ is the ideal $I = (a - f, b - g)$, then $B := \mathcal{B} \bmod I$ is isomorphic to $k^{[2n]}$, and we may take $B = k[x_1, y_1, ..., x_n, y_n]$. Since $a - f$ and $b - g$ belong to $\ker \mathcal{D}$, the ideal I is an integral ideal of $\mathcal{D}$, and we have that $D := \mathcal{D} \bmod I$ is well-defined, locally nilpotent, and homogeneous on B.

It remains to show that $\operatorname{rank}(D) = 2n$. If $Dv = 0$ for a variable $v \in B$, then by homogeneity, there exists a linear form $L = \sum(a_i x_i + b_i y_i)$ for scalars a_i, b_i such that $DL = 0$. But then $\sum(a_i Dx_i + b_i Dy_i) = 0$. So it suffices to show that the images $Dx_1, Dy_1, ..., Dx_n, Dy_n$ are linearly independent.

To this end, define a vector of univariate polynomials

$$\mathbf{t} = (t, t^2, t^3, t^4 - 1, t^5 - 1, t^6, ..., t^{2n}) ,$$

noting that $f(\mathbf{t}) = -t$ and $g(\mathbf{t}) = -t^2$. Then for each i,

$$\deg_t Dy_i(\mathbf{t}) = 2i + 3 \quad \text{and} \quad \deg_t Dx_i(\mathbf{t}) = 2i + 4 .$$

Since these degrees are all distinct for $1 \le i \le n$, it follows that these polynomials are linearly independent. $\square$

Note that de Bondt's derivations are quasi-linear, in addition to being nice derivations.

In order to exhibit an example in odd dimension $2n + 1$ for $n \ge 3$, let $k^{[2n+1]} = B[z]$, and extend D to this ring. In particular, Dz should satisfy: (1) $Dz \in \ker D$, (2) $\deg Dz = 5$, and (3) Dz is not in the span of $Dx_1, ..., Dy_n$. For example, $h = x_2 y_3 - x_3 y_2 \in \ker D$, so we may take $Dz = h(f x_n - g y_n)$. Then $Dz \in \ker D$ and $\deg Dz = 5$. Moreover, $\deg_t Dz(\mathbf{t}) = 2n + 7$, so Dz is independent of the other images.

3.10.4 A Rank-4 Example in Dimension 5

In the notation of de Bondt's examples, consider the case $n = 2$: Let $\mathcal{B} = k[a, b, x_1, y_1, x_2, y_2] = k^{[6]}$ and $R = k[a, b]$. In this case, replace the matrix N with

$$N' = \begin{pmatrix} ab^2 & -b^4 \\ a^2 & -ab^2 \end{pmatrix} .$$

This defines an R-linear $\mathcal{D} \in \mathrm{LND}_R(\mathcal{B})$, namely,

$$\mathcal{D} = \begin{pmatrix} N' & 0 \\ 0 & N' \end{pmatrix}_{4 \times 4} .$$

Note that we still have $f = x_1 y_2 - x_2 y_1 \in \ker \mathcal{D}$. Set $E = \mathcal{D} \bmod (a - f)$ on $B = \mathcal{B} \bmod (a - f) = k^{[5]}$. Then E is standard homogeneous of degree 4, and satisfies

$$E^2 b = E^2 x_1 = E^2 y_1 = E^2 x_2 = E^2 y_2 = 0 .$$

In addition, the rank of E is 4. To see this, it suffices to show that the images Ex_1, Ey_1, Ex_2, Ey_2 are linearly independent. As above, evaluate these polynomials at $\mathbf{t} = (1, t, t^2 - 1, t^3, t^4)$. Then

$$Ex_1(\mathbf{t}) = t^4 - t^2 + 1 \ , \ Ey_1(\mathbf{t}) = t^7 - t^5 + t^3 \ , \ Ex_2(\mathbf{t}) = t^6 - t^4 \ , \ Ey_2(\mathbf{t}) = t^9 - t^7 \ .$$

Therefore, Ex_1, Ey_1, Ex_2, Ey_2 are linearly independent.

4

Dimension Two

In this chapter, we examine locally nilpotent R-derivations of $R[x, y] = R^{[2]}$ for certain rings R containing $\mathbb{Q}$. This set is denoted $\mathrm{LND}_R(R[x, y])$.

We begin with the case R is a field, and here the main fact is due to Rentschler from 1968 [260].

Theorem 4.1. (Rentschler's Theorem) *Let k be a field of characteristic zero. If $D \in \mathrm{LND}(k[x, y])$ is nonzero, then there exists $f \in k[x]$ and a tame automorphism $\alpha \in GA_2(k)$ such that $\alpha D \alpha^{-1} = f(x)\partial_y$.*

Geometrically, this says that every planar $\mathbb{G}_a$-action is conjugate, by a tame automorphism, to a triangular action $t \cdot (x, y) = (x, y + tf(x))$.[1] Rentschler also showed that his theorem implies Jung's Theorem, which appeared in 1942 [154], and which asserts that every plane automorphism is tame in the characteristic zero case. Rentschler's proof of Jung's Theorem is a compelling illustration of the importance of locally nilpotent derivations and $\mathbb{G}_a$-actions in the study of affine algebraic geometry.

Jung's Theorem was the predecessor of the well-known Structure Theorem for the group $GA_2(k)$. Recall that $GA_2(k)$ denotes the full group of algebraic automorphisms of the plane $\mathbb{A}^2_k$; $Af_2(k)$ is the affine linear subgroup, whose elements are products of linear maps $L \in GL_2(k)$ and translations $T = (x + a, y + b)$, $a, b \in k$; and $BA_2(k)$ is the subgroup of triangular automorphisms $\alpha = (ax + b, cy + f(x))$, $a, c \in k^*, b \in k, f \in k[x]$. A tame automorphism is composed of linear and triangular factors.

Theorem 4.2. (Structure Theorem) *For any field k, $GA_2(k)$ has the amalgamated free product structure*

$$GA_2(k) = Af_2(k) *_B BA_2(k) ,$$

where $B = Af_2(k) \cap BA_2(k)$.

[1] As noted in the *Introduction*, this description of the planar $\mathbb{G}_a$-actions was first given by Ebey in 1962 [93]. The statement about tame automorphisms is not explicit in his paper, but can be inferred from the proof.

In his 1992 paper [312], Wright gives the following description of the evolution of this result.

> That GA_2 is generated by Af and BA was first proved by Jung [154] for k of characteristic zero. Van der Kulk [298] generalized this to arbitary characteristic and proved a factorization theorem which essentially gives the amalgamated free product structure, although he did not state it in this language. Nagata [239] seems to be the first to have stated and proved the assertion as it appears above. The techniques in these proofs require that k be algebraically closed. However, it is not hard to deduce the general case from this (see [308]).
> Some fairly recent proofs have been given which use purely algebraic techniques, and for which it is not necessary to assume k is algebraically closed [82, 207]. (p. 283)

In this same paper, Wright gives another proof of the Structure Theorem, based on Serre's tree theory [274].

While the paper of Nagata mentioned by Wright dates to 1972, it should be noted that the Structure Theorem was stated without proof in the well-known 1966 paper of Shafarevich [276], Thm. 7. It should also be noted that another early proof of Jung's Theorem was given by Engel [96]. Moreover, Jung's Theorem is implied by the famous Embedding Theorem of Abhyankar and Moh [3], and Suzuki [289]. Makar-Limanov has recently given a new proof of the Embedding Theorem, using jacobian derivations [188].

Combining the Structure Theorem with Serre's theory gives a complete description of all planar group actions: If G is an algebraic group acting algebraically on $\mathbb{A}^2$, given by $\phi : G \to GA_2(k)$, then $\phi(G)$ is conjugate to a subgroup of either $GL_2(k)$ or $BA_2(k)$ (see [283]). The key fact used here comes from combinatorial group theory: Any subgroup of an amalgamated free product $A *_C B$ having bounded length can be conjugated into either A or B (Thm. 8 of [274]). If G is an algebraic subgroup of $GA_2(k)$, then it is of bounded degree; and Wright pointed out that, in $GA_2(k)$, bounded degree implies bounded length (see Kambayashi [168], Lemma 4.1 and Thm. 4.3). Earlier results on planar group actions appear in [20, 93, 135, 149, 210, 310].

In particular, planar actions of reductive groups can be conjugated to linear actions, and actions of unipotent groups can be conjugated to triangular actions. This is true regardless of the characteristic of k. We thus recover Rentschler's Theorem from the Structure Theorem. We also get the following description of planar $\mathbb{G}_a$-actions in any characteristic.

Theorem 4.3. *For any field k, an algebraic action of $\mathbb{G}_a$ on $\mathbb{A}^2_k$ is conjugate to a triangular action.*

Section 1 gives a self-contained proof of Rentschler's Theorem, Jung's Theorem, and the Structure Theorem (in the case k is of characteristic 0). *Section 2* then discusses $\mathrm{LND}_R(R[x,y])$ for integral domains R containing $\mathbb{Q}$. In contrast to Rentschler's Theorem, it turns out that when R is not a field, most elements of $\mathrm{LND}_R(R[x,y])$ are not triangularizable, i.e., not conjugate to $f(x)\partial_y$ for $f \in R[x]$ via some R-automorphism (see *Sect. 5.3* of *Chap. 5*). However, we show that if R is a highest common factor (HCF) ring, then it is still true that the kernel of a locally nilpotent R-derivation of $R[x,y]$ is isomorphic to $R^{[1]}$.

We are especially interested in how this theory applies to polynomial rings over k. We consider locally nilpotent derivations of $k[x_1, ..., x_n] \cong k^{[n]}$ of rank at most 2: If we suppose $Dx_1 = \cdots Dx_{n-2} = 0$, and if $R = k[x_1, ..., x_{n-2}]$, then $D \in \mathrm{LND}_R(R[x_{n-1}, x_n])$. Of particular importance is *Thm. 4.16*, which implies that a fixed-point free $\mathbb{G}_a$-action on $\mathbb{A}^n$ of rank at most 2 must be conjugate to a coordinate translation: $t \cdot (x_1, ..., x_n) = (x_1, ..., x_{n-1}, x_n + t)$.

Section 2 is largely a generalization of the work of Daigle and Freudenburg found in [54]. Bhatwadekar and Dutta [18] have studied $\mathrm{LND}_R(R[x,y])$ for a different class of rings R, namely, R is a noetherian integral domain containing $\mathbb{Q}$. In contrast to the situation for HCF-rings, their work shows that the kernel of such a derivation need not be isomorphic to $R^{[1]}$, or even finitely generated over R.

Example 4.4. (Example 3.6 of [18]) Let $R = \mathbb{C}[x,y,z]_{(x,y,z)}/(F)$, where $F = y^2z - x^3 + xz^2$, and $\mathbb{C}[x,y,z]_{(x,y,z)}$ denotes homogeneous localization at the ideal (x,y,z). Note that F defines a nonsingular elliptic curve in the projective plane. The authors show that there exists $D \in \mathrm{LND}_R(R[x,y])$ whose kernel is isomorphic to the symbolic Rees algebra $\oplus_{n\geq 0} P^{(n)}T^n$, where P is a height-one prime of R for which no symbolic power $P^{(n)}$ is principal ($n \geq 1$). In particular, $\ker D$ is not finitely generated over R.

A simpler pathological example is the following (Example 2.5 of [14], Example 3.2.3 of [205], and Example 6.3 of [47]).

Example 4.5. Take $R = \mathbb{C}[t^2, t^3] \subset \mathbb{C}[t]$ and define the R-derivation D on $R[x,y]$ by $Dx = t^2$ and $Dy = t^3$. Then $D \in \mathrm{LND}_R(R[x,y])$, but the kernel of D is not finitely generated over $\mathbb{C}$. To see this, note that D is the restriction of a derivation of $\mathbb{C}[t,x,y]$ whose kernel is $\mathbb{C}[t,f]$, where $f = y - tx$. Thus, $\ker D = \mathbb{C}[t,f] \cap R[x,y] = R[ft^2, f^2t^2, ...]$, which is not finitely generated.

The convention used for composing elements of $GA_2(k)$ is:

$$\begin{aligned}&\big(f_1(x,y), f_2(x,y)\big) \circ \big(g_1(x,y), g_2(x,y)\big)\\ &\quad = \big(f_1(g_1(x,y), g_2(x,y)), f_2(g_1(x,y), g_2(x,y))\big)\ .\end{aligned}$$

Throughout this chapter, a generalized notion of partial derivative will be useful. Relative to the ring $R[x,y]$, the notation ∂_x and ∂_y will mean the R-derivations uniquely defined by

$$\partial_x(x) = 1\ ,\ \partial_x(y) = 0 \quad \text{and} \quad \partial_y(x) = 0\ ,\ \partial_y(y) = 1\ .$$

It is easy to see that ∂_x and ∂_y are locally nilpotent. Likewise, if $R[x,y] = R[u,v]$, we have ∂_u, ∂_v, and so on. Notice that ∂_x depends not only on x, but on the choice of y as well. In other words, if $R[x,y] = R[x,y']$, then ∂_x relative to (x,y') may differ from ∂_x relative to (x,y).

4.1 The Polynomial Ring in Two Variables over a Field

The main goal of this section is to give a self-contained elementary proof of Rentschler's Theorem. The proof presented here is a modified version of the proof of Makar-Limanov in [190].

Rentschler's original proof uses the Newton polygon of a derivation relative to the standard grading of $k[x,y]$, and deals first with the fixed-point free case. Other proofs along the same lines as Rentschler's proof are given in [91, 100]. The proof given below differs from these in several significant ways: It does not use Newton polygons, it exploits non-standard gradings of $k[x,y]$, and the fixed-point free case is not treated separately. A discussion of Newton polygons is given at the end of this chapter.

For the remainder of this section, let $A = k[x,y]$, the polynomial ring in two variables over a field k of characteristic zero. We argue only from first principles. In particular, recall the following four properties from *Chap. 1*: Suppose $D \in \mathrm{LND}(A)$ and $D \neq 0$.

(P1) $\ker D$ is a factorially closed subring of A. (*Princ. 1*)

(P2) Any pair $a,b \in \mathbb{Z}$ determines a $\mathbb{Z}$-grading ω on A via $\deg_\omega(x^i y^j) = ai+bj$. Let $f \in \ker D$ be given. If $\bar{D}$ is the highest homogeneous summand of D relative to ω, and if $\bar{f}$ is the highest homogeneous summand of f, then $\bar{D} \in \mathrm{LND}(A)$ and $\bar{f} \in \ker \bar{D}$. (*Princ. 14*)

(P3) $\mathrm{LND}_{k[x]}(A) = k[x] \cdot \partial_y$ (*Princ. 8*)

(P4) If $Df \in gA$ and $Dg \in fA$ for $f,g \in A$, then either $Df = 0$ or $Dg = 0$. (*Princ. 5*)

In addition to these properties, we require the following fact about bivariate polynomials.

Lemma 4.6. *Let a and b be positive integers such that $a\mathbb{Z}+b\mathbb{Z} = \mathbb{Z}$, and define a grading ω on $A = k[x,y]$ by $\deg_\omega(x) = a$ and $\deg_\omega(y) = b$. Then $f \in A$ is ω-homogeneous if and only if there exists a standard homogeneous polynomial $g \in A$ such that $f = x^i y^j g(x^b, y^a)$ for integers i and j with $0 \le i < b$ and $0 \le j < a$.*

Proof. If $a = b = 1$, there is nothing to prove, so assume $ab > 1$. Let G denote the cyclic group $\mathbb{Z}_a \times \mathbb{Z}_b = \mathbb{Z}_{ab}$, and suppose G is generated by t. Then G acts on A by $t \cdot (x,y) = (t^a x, t^b y)$, and $A^G = k[x^b, y^a]$. Viewing A as an A^G-module, we can decompose A into semi-invariant spaces

$$A = \oplus_{\substack{0 \le i < b \\ 0 \le j < a}} x^i y^j A^G \ ,$$

where the weight of an element of $x^i y^j A^G$ is $ai+bj$. If $f \in A$ is ω-homogeneous, it is a semi-invariant of this G-action, and thus $f = x^i y^j g(x^b, y^a)$ for some $i,j \ge 0$ and some $g \in A$. Since f is ω-homogeneous, g must be standard homogeneous. $\square$

4.1.1 Proof of Rentschler's Theorem

Suppose $D \in \text{LND}(A)$ is given, where $D \neq 0$.

Consider first the case that $Dx \neq 0$ and $Dy \neq 0$. In this case, choose non-constant $f \in \ker D$. We may assume f belongs to the maximal ideal (x, y); otherwise replace f by $f(x, y) - f(0, 0)$. We will show:

(∗) There exists an $\mathbb{N}$-grading of A relative to which the highest-degree homogeneous summand $\bar{f}$ of f has the form $\bar{f} = d(x + cy^r)^s$ or $\bar{f} = d(y + cx^r)^s$ $(c, d \in k^*, r, s \in \mathbb{N})$.

To prove this, write $f = p(x) + q(y) + xyF$ for some $p(x) \in xk[x]$, $q(y) \in yk[y]$, and $F \in A$. If $p = 0$ or $q = 0$, then either $f \in yA$ or $f \in xA$. By (P1), we would have either $Dy = 0$ or $Dx = 0$, a contradiction. Therefore, if $m = \deg p(x)$ and $n = \deg q(y)$, then $m \geq 1$ and $n \geq 1$. Set $e = \gcd(m, n)$, $a = n/e$, and $b = m/e$. Define a grading ω on A by declaring that $\deg_\omega(x) = a$ and $\deg_\omega(y) = b$. Then $\deg_\omega p(x) = am = bn = \deg_\omega q(y)$, and it follows that $\deg_\omega f \geq am$. Let $\bar{f}, \bar{F}$, and $\bar{D}$ denote the highest-degree homogeneous summands of f, F, and D, respectively, relative to ω. Then $\bar{f} \neq 0$ and $\bar{D} \neq 0$. By (P2), $\bar{D}$ is locally nilpotent, and $\bar{D}\bar{f} = 0$.

If $\deg_\omega f > am$, then $\bar{f} = xy\bar{F}$. Since $\bar{F} \neq 0$ in this case, it follows from (P1) that $\bar{D}x = \bar{D}y = 0$, and thus $\bar{D} = 0$, a contradiction. Therefore, $\deg_\omega f = am$, which implies that $\bar{f} = ux^m + vy^n + xy\bar{F}$ for some $u, v \in k$. In addition, $u \neq 0$ and $v \neq 0$, since these are the highest-degree coefficients of $p(x)$ and $q(y)$. It follows that $\bar{D}x \neq 0$ and $\bar{D}y \neq 0$. For example, if $\bar{D}x = 0$, then $\bar{D}(\bar{f} - ux^m) = 0$. But $\bar{f} - ux^m$ is not zero and belongs to yA, meaning that $\bar{D}y = 0$ as well, a contradiction.

Combining this form with that of *Lemma 4.6* above, we conclude that $\bar{f} = g(x^b, y^a)$ for some $g \in A$ which is non-constant and standard homogeneous. Let K denote the algebraic closure of k. In $K[x, y]$, $g(x, y)$ factors as a product of linear polynomials, and thus $\bar{f}$ factors as $\bar{f} = \prod_{i=1}^{e}(c_i x^b + d_i y^a)$ for some $c_i, d_i \in K$. (If any c_i or d_i were 0, then either u or v above would also be 0.)

Let δ be the extension of $\bar{D}$ to $K[x, y]$. Then δ is locally nilpotent, since $\delta^t x = \bar{D}^t x$ and $\delta^t y = \bar{D}^t y$ for all $t \geq 0$. By (P1), we have that $\delta(c_i x^b + d_i y^a) = 0$ for each i. If any two of these are linearly independent, then $\delta(x^b) = \delta(y^a) = 0$, which would imply $\delta x = \delta y = 0$, a contradiction. Therefore, there exist $c, d \in K^*$ such that $\bar{f} = d(cx^b + y^a)^e$. Since $\delta(cx^b + y^a) = 0$, it follows that

$$cbx^{b-1}\delta x = -ay^{a-1}\delta y \quad \Rightarrow \quad c = -\frac{ay^{a-1}\delta y}{bx^{b-1}\delta x} \in k(x, y) \cap K = k \; .$$

Therefore, $cx^b + y^a \in A$ and $\bar{D}(cx^b + y^a) = 0$. If $a > 1$ and $b > 1$, this implies $\bar{D}x \in yA$ and $\bar{D}y \in xA$. By (P4), either $\bar{D}x = 0$ or $\bar{D}y = 0$, a contradiction. Therefore, either $a = 1$ or $b = 1$, and (∗) is proved.

Next, suppose $\bar{f} = d(y + cx^b)^e$, and define a triangular automorphism $\alpha = (x, y - cx^b)$. Set $D' = \alpha D \alpha^{-1}$. Then D' is a locally nilpotent derivation containing $\alpha(f)$ in its kernel, and $\alpha(f)$ belongs to the ideal (x,y). The crucial observation is that $\deg_x \alpha(f) < \deg_x f$, whereas $\deg_y \alpha(f) = \deg_y f$.

In the same way, if $\bar{f} = d(x+cy^a)^e$, let $D' = \beta D \beta^{-1}$ for $\beta = (x - cy^a, y)$. In this case, $\deg_y \beta(f) < \deg_y f$ and $\deg_x \alpha(f) = \deg_x f$.

Now if $D'x \neq 0$ and $D'y \neq 0$, then the same argument given above can be applied to the derivation D' and the polynomial $\alpha(f)$, in order to lower the degree of the kernel element in either x or y. Since this process of lowering degree cannot continue indefinitely, we eventually obtain a tame automorphism γ such that either $\gamma D \gamma^{-1}(x) = 0$ or $\gamma D \gamma^{-1}(y) = 0$. By applying the transposition (y,x) in the latter case, we may assume γ is a tame automorphism such that $\gamma D \gamma^{-1}(x) = 0$.

It follows by (P3) that $\gamma D \gamma^{-1} \in \mathrm{LND}_{k[x]}(A) = k[x] \cdot \partial_y$. This completes the proof of Rentschler's Theorem. $\square$

Recall that, for $f \in k[x,y]$, Δ_f denotes the derivation $\Delta_f h = f_x h_y - f_y h_x$ for $h \in k[x,y]$.

Corollary 4.7. *Let $D \in \mathrm{Der}_k(k[x,y])$ be given. Then $D \in \mathrm{LND}(k[x,y])$ if and only if D is of the form $D = \Delta_f$, where $f \in k[v]$ for some variable v of $k[x,y]$.*

Proof. Assume D is locally nilpotent. If $D = 0$, then $D = \Delta_0$. If $D \neq 0$, then by Rentschler's Theorem, there exists a system of variables (u,v) for $k[x,y]$ relative to which $D = g(v)\partial_u$ for some $g(v) \in k[v]$. Since (u,v) is an automorphism of $k[x,y]$, $\frac{\partial(u,v)}{\partial(x,y)} = c$ for some $c \in k^*$. We may assume $c = -1$: otherwise replace v by $-c^{-1}v$. It follows that $\Delta_v v = 0$ and $\Delta_v u = 1$. Choose $f(v) \in k[v]$ so that $f'(v) = g(v)$. Then $\Delta_f = f'(v)\Delta_v$, which implies $\Delta_f u = Du = g(v)$ and $\Delta_f v = Dv = 0$. Therefore $D = \Delta_f$.

Conversely, if $f \in k[v]$ for some variable v, then (as above) there exists $u \in k[x,y]$ with $k[u,v] = k[x,y]$, $\Delta_v v = 0$ and $\Delta_v u = 1$. Since $\Delta_f = f'(v)\Delta_v$ and $f'(v) \in \ker \Delta_v$, it follows that Δ_f is locally nilpotent. $\square$

4.1.2 Proof of Jung's Theorem

Every automorphism (i.e., change of coordinates) of A defines a pair of locally nilpotent derivations of A, namely, the partial derivatives relative to the new coordinate functions. Thus, in the characteristic zero case, Rentschler's Theorem is a powerful tool in the study of plane automorphisms.

Theorem 4.8. (Jung's Theorem) *The group $GA_2(k)$ of algebraic automorphisms of $k[x,y]$ is generated by its linear and triangular subgroups, $GL_2(k)$ and $BA_2(k)$.*

Proof. (following Rentschler [260]) If $(F,G) \in GA_2(k)$, then $\partial_F \in \mathrm{LND}(A)$ and $\ker(\partial_F) = k[G]$. By Rentschler's Theorem, there exists tame $\varphi \in GA_2(k)$ such that $\varphi^{-1}\partial_F\varphi = f(x)\partial_y$ for some $f \in k[x]$. In fact, since $\partial_F(F) = 1$, the ideal generated by the image of ∂_F is $(1) = A$. Therefore, $A = fA$, so $f \in A^* = k^*$.

Note that $\ker(\partial_y) = k[x] = k[\varphi^{-1}(G)]$, which implies $G = a\varphi(x) + b$ for $a, b \in k$, $a \neq 0$. In addition, $\partial_y(\varphi^{-1}(F)) = f^{-1}$, which implies $F = f^{-1}\varphi(y) + g(\varphi(x))$ for some $g \in k[x]$. We thus have

$$(F, G) = (f^{-1}\varphi(y) + g(\varphi(x)), a\varphi(x) + b) ,$$

which is the composition of φ with a triangular automorphism. Therefore (F, G) is tame. □

4.1.3 Proof of the Structure Theorem

From Jung's Theorem we deduce the full Structure Theorem. The theorem asserts that $GA_2(k)$ admits a kind of unique factorization property. Our proof follows Wright in [309], Prop. 7. See also [312] for another proof.

Generally, a group G is an **amalgamated free product** of two of its subgroups if and only if G is the homomorphic image of the free product of two groups. Specifically, if $f(G_1 * G_2) = G$ for groups G_1 and G_2 and epimorphism f, we write $G = A *_C B$, where $A = f(G_1)$, $B = f(G_2)$, and $C = A \cap B$. Equivalently, $G = A *_C B$ means that A, B and $C = A \cap B$ are subgroups of G satisfying the following condition.

> Let $\mathcal{A}$ and $\mathcal{B}$ be systems of nontrivial right coset representatives of A and B, respectively, modulo C. Then every $g \in G$ is uniquely expressible as $g = ch_1 \cdots h_n$, where $c \in C$, and the h_i lie alternately in $\mathcal{A}$ and $\mathcal{B}$.

Note that, in this case, if $C = \{1\}$, then G is simply the free product of A and B.

For the group $G = GA_2(k)$, consider subgroups

$$A = Af_2(k) = \{(a_1x+b_1y+c_1, a_2x+b_2y+c_2) | a_i, b_i, c_i \in k; a_1b_2-a_2b_1 \neq 0\}$$

$$B = BA_2(k) = \{(ax + b, cy + f(x)) | a, c \in k^*; b \in k; f(x) \in k[x]\}$$
$$C = A \cap B = \{(ax + b, cx + dy + e) | a, d \in k^*; b, c, e \in k\} \ .$$

Suppose $\alpha = (a_1x + b_1y + c_1, a_2x + b_2y + c_2) \in A - C$. Then $b_1 \neq 0$, and we have

$$\alpha = (b_1x + c_1, b_2x + (a_2b_1 - b_2a_1)b_1^{-1}y + c_2) \cdot (a_1b_1^{-1}x + y, x) \ .$$

Likewise, suppose $\beta = (ax+b, cy+f(x)) \in B-C$. Write $f(x) = r+sx+x^2g(x)$ for some $r, s \in k$ and $g(x) \in k[x]$. Then

$$\beta = (ax + b, cy + r + sx) \cdot (x, y + x^2 \cdot c^{-1}g(x)) \ .$$

Therefore, we may choose the following sets of nontrivial coset representatives for A and B, respectively, modulo C:

$$\mathcal{A} = \{(tx + y, x) | t \in k\} \quad \text{and} \quad \mathcal{B} = \{(x, y + x^2 \cdot f(x)) | f(x) \in k[x], f \neq 0\} .$$

In addition, observe the semi-commuting relation among elements of C and $\mathcal{A}$:

$$(tx + y, x)(ax + b, cx + dy + e) = (dx + (tb + e), ay + b)(d^{-1}(ta + c)x + y, x)$$

Likewise, among elements of C and $\mathcal{B}$ we have

$$\begin{aligned}(x, y + f(x))(ax + b, cx + dy + e) &= (ax + b, cx + dy + e)(x, y + d^{-1}f(ax + b)) \\ &= (ax + b, (c + ds)x + dy + (dr + e))(x, y + x^2h(x)),\end{aligned}$$

where $d^{-1}f(ax + b) = r + sx + x^2h(x)$ for some $r, s \in k$. It follows that any element κ belonging to the subgroup generated by A and B can be expressed as a product

$$\kappa = ch_1 \cdots h_n \ , \tag{4.1}$$

where $c \in C$, and the h_i lie alternately in $\mathcal{A}$ or $\mathcal{B}$. It remains to check the uniqueness of such factorization.

Consider a product of the form

$$\varphi = \gamma_s \alpha_{s-1} \gamma_{s-1} \cdots \alpha_1 \gamma_1 \quad (s \geq 1) \ ,$$

where, for each i, $\alpha_i = (t_i x + y, x) \in \mathcal{A}$, and $\gamma_i = (x, y + f_i(x)) \in \mathcal{B}$. For any $(F, G) \in GA_2(k)$, where $F, G \in k[x, y]$, define the **degree** of (F, G) to be $\max\{\deg F, \deg G\}$. Set $d_i = \deg \gamma_i$ $(1 \leq i \leq s)$.

Lemma 4.9. *In the notation above, if $\varphi = (F, G)$ and $s \geq 1$, then $\deg G > \deg F$ and $\deg \varphi = d_1 d_2 \cdots d_s$.*

Proof. For $s = 1$, this is clear. For $s > 1$, assume by induction that

$$\psi := \gamma_{s-1} \alpha_{s-2} \gamma_{s-2} \cdots \alpha_1 \gamma_1 = (P, Q)$$

satisfies $\deg(\psi) = \deg Q = d_1 \cdots d_{s-1} > \deg P$. Then

$$\alpha_{s-1} \psi = (t_{s-1} P + Q, P) = (R, P)$$

where $\deg(\alpha_{s-1}\psi) = \deg R = d_1 \cdots d_{s-1} > \deg P$. Therefore

$$\varphi = \gamma_s \alpha_{s-1} \psi = (R, P + f_s(R)) = (F, G) \ .$$

Since $\deg R > \deg P$ and $\deg f_s \geq 2$, we see that $\deg(\varphi) = \deg G = (\deg f_s)(\deg R) = d_1 \cdots d_{s-1} d_s > \deg F$. $\square$

In order to finish the proof of the Structure Theorem, we must show that the factorization (4.1) of κ above is unique, and for this it suffices to assume $\kappa = (x, y)$ (identity). Suppose $(x, y) = ch_1 \cdots h_n$, where $c \in C$, and the h_i lie alternately in $\mathcal{A}$ or $\mathcal{B}$. By the preceding lemma, $1 = (\deg h_1) \cdots (\deg h_n)$. Since $\deg h_i > 1$ for each $h_i \in \mathcal{B}$, we conclude that $n \leq 1$. If $n = 1$, then $(x, y) = ch_1$ for $h_1 \in \mathcal{A}$, which is impossible since then $c = h_1^{-1} \in C$. Thus, $n = 0$, and $c = (x, y)$.

This completes the proof of the Structure Theorem:

$$GA_2(k) = Af_2(k) *_C BA_2(k) \ .$$

4.1.4 Remark about Fields of Positive Characteristic

Just after Rentschler's Theorem appeared, Miyanishi took up the question of $\mathbb{G}_a$-actions on the plane when the underlying field k is of positive characteristic. He proved the following.

Theorem 4.10. ([210] 1971) *Let k be an algebraically closed field of positive characteristic p. Then any $\mathbb{G}_a$-action on $\mathbb{A}^2_k$ is equivalent to an action of the form*

$$t \cdot (x,y) = \left(x, y + tf_0(x) + t^p f_1(x) + \cdots + t^{p^n} f_n(x)\right)$$

where $t \in \mathbb{G}_a(k)$, $(x,y) \in k^2$, and $f_0(x), ..., f_n(x) \in k[x]$.

It seems that this result does not receive as much attention as it deserves, as it completely characterizes the planar $\mathbb{G}_a$-actions in positive characteristic.

4.2 Locally Nilpotent R-Derivations of $R[x,y]$

In this section, we consider $\text{LND}_R(R[x,y])$ for various rings R, where $R[x,y]$ denotes the polynomial ring in two variables over R.

4.2.1 Kernels for $\text{LND}_R(R[x,y])$

Consider first rings R having the property:

> (†) R is an integral domain containing $\mathbb{Q}$, and for every nonzero $D \in \text{LND}_R(R[x,y])$, $\ker D = R^{[1]}$.

We have seen that even if R is an affine (rational) integral domain containing $\mathbb{Q}$, it may fail to satisfy (†).

Recall that an integral domain R is a highest common factor ring, or HCF-ring, if and only if it has the property that the intersection of two principal ideals is again principal. Examples of HCF-rings are: a UFD, a valuation ring, or a polynomial ring over a valuation ring. These form a large and useful class of rings R, and we show that those containing $\mathbb{Q}$ also satisfy property (†).

Theorem 4.11. *Let R be an HCF-ring containing $\mathbb{Q}$. If $D \in \text{LND}_R(R[x,y])$ and $D \neq 0$, then there exists $P \in R[x,y]$ such that* $\ker D = R[P]$.

To prove this, we refer to the following classical result from commutative algebra, due to Abhyankar, Eakin, and Heinzer (1972).

Proposition 4.12. (Prop. 4.8 of [2]) *Let R be an HCF-ring, and suppose A is an integral domain of transcendence degree one over R and that $R \subset A \subset R^{[n]}$ for some $n \geq 1$. If A is an inert (factorially closed) subring of $R^{[n]}$, then $A = R^{[1]}$.*

We now give the proof of the theorem.

Proof. Let K denote the field of fractions of R. Then D extends to a K-derivation D_K of $K[x,y]$, since $D_K(x) = Dx$ and $D_K(y) = Dy$ define D_K uniquely. In addition, D_K is locally nilpotent, since $D_K^r(x) = D_K^r(y) = 0$ for r sufficiently large. By Rentschler's Theorem, there exists $P' \in K[x,y]$ such that $\ker D_K = K[P']$. Since $\ker D = \ker D_K \cap R[x,y]$, the transcendence degree of $\ker D$ over R is 1. We thus have $R \subset \ker D \subset R[x,y]$, with $\ker D$ factorially closed. By *Prop. 4.12*, we can find $P \in \ker D$ with $\ker D = R[P]$. $\square$

Theorem 4.13. *Let R be an HCF-ring containing $\mathbb{Q}$, let $B = R[x,y] = R^{[2]}$, and let $K = \mathrm{frac}(R)$. If $D \in \mathrm{LND}_R(B)$ is irreducible, then there exist $P, Q \in B$ such that $K[P,Q] = K[x,y]$, $\ker D = R[P]$, and $DQ \in R$.*

Proof. Continuing the notation above, we have $\ker D_K = K[P']$. Suppose $\ker D = R[P]$ for $P \in B$. Then $\ker D_K = K[P]$. To see this, note that $aP' \in B$ for some nonzero $a \in R$, which implies $aP' \in \ker D \subset K[P]$. But then $K[P'] \subset K[P] \subset \ker D_K$, so $K[P] = K[P']$.

Also by Rentshler's Theorem, there exists $Q' \in K[x,y]$ and $f(P) \in K[P]$ such that $K[P,Q'] = K[x,y]$ and

$$D_K = f(P)\frac{\partial}{\partial Q'}$$

relative to this coordinate system. Since D is irreducible, D_K is also irreducible, meaning that $\deg_P f(P) = 0$. Therefore, $D_K Q' \in K^*$. Choose nonzero $b \in R$ so that $Q := bQ' \in B$. Then $K[P,Q] = K[x,y]$, and $DQ = D_K Q = bD_K Q' \in R$. $\square$

In light of the examples cited at the beginning of this chapter, the theorem above is a very strong and useful result, which to a large extent governs the behavior of elements of $\mathrm{LND}_R(R[x,y])$ when R is an HCF-ring containing $\mathbb{Q}$.

This leaves open the question:

> If R is an integral domain which is not an HCF-ring, does there always exist a nonzero $D \in \mathrm{LND}_R(R[x,y])$ whose kernel is not isomorphic to $R^{[1]}$?

Given $F \in B$, define $F_x = \partial_x(F)$ and $F_y = \partial_y(F)$. If δ is any R-derivation of B, then $\delta F = F_x\delta x + F_y\delta y$. So if $\delta x = F_y$ and $\delta y = -F_x$, then $\delta F = 0$. This motivates the following definition.

$$\text{Given } F \in B, \quad \Delta_F := F_y\partial_x - F_x\partial_y$$

One obvious question:

> For which $F \in R[x,y]$ is the induced R-derivation Δ_F locally nilpotent?

For HCF-rings we have the following answer.

Theorem 4.14. *Assume R is an HCF-ring containing $\mathbb{Q}$, $B = R[x,y] = R^{[2]}$, and $K = \text{frac}(R)$. Given $F \in B$, Δ_F is locally nilpotent if and only if there exist $P, Q \in B$ such that $F \in R[P]$ and $K[P,Q] = K[x,y]$. Moreover, every $D \in \text{LND}_R(B)$ equals Δ_F for some $F \in B$.*

Proof. If $\Delta_F \in \text{LND}_R(B)$, then $P, Q \in B$ with the stated properties exist, by the preceding theorem. Conversely, if $F = \varphi(P)$ for some univariate polynomial φ, then $\Delta_F = \varphi'(P) \cdot \Delta_P$. It thus suffices to show Δ_P is locally nilpotent, since $\varphi'(P) \in \ker(\Delta_P)$.

On $K[x,y]$, we have

$$(\Delta_P)_K(P) = 0 \quad \text{and} \quad (\Delta_P)_K(Q) = P_y Q_x - P_x Q_y \ .$$

Thus, $(\Delta_P)_K(Q)$ equals the jacobian determinant $\frac{\partial(P,Q)}{\partial(x,y)}$, and since (P,Q) defines a K-automorphism of $K[x,y]$, it follows that $(\Delta_P)_K(Q) \in K^*$. Therefore $(\Delta_P)_K$ is locally nilpotent, which implies Δ_P is also.

Finally, let $D \in \text{LND}_R(B)$ be given. If $D = 0$, then $D = \Delta_0$. If $D \neq 0$, then there exist $P, Q \in B$ such that $\ker D = R[P]$, $K[P,Q] = K[x,y]$, and $DQ = g(P)$ for some $g \in R[P]$, according to the theorem above. It follows that $\ker D_K = K[P]$ and $D_K = g(P)\partial_Q$. Choose $F \in R[P]$ so that $F'(P) = g(P)$. Then $(\Delta_F)_K = g(P)(\Delta_P)_K = g(P)\partial_Q = D_K$, which implies that $D = \Delta_F$ on B. $\square$

This result relates to the **divergence** $\text{div}(D)$ of D: For any base ring R and any $D \in \text{Der}_R(R[x,y])$, $\text{div}(D) = \partial_x(Dx) + \partial_y(Dy)$. Therefore $\text{div}(\Delta_F) = 0$ for any $F \in R[x,y]$. Moreover, if R is an HCF-ring, then $\text{div}(D) = 0$ for all $D \in \text{LND}_R(R[x,y])$.

In view of *Cor. 2.3*, we get a fairly complete description of the locally nilpotent R-derivations of $R[x,y]$ in the case R is a UFD.

Corollary 4.15. *If R is a UFD, $K = \text{frac}(R)$, and $B = R[x,y] = R^{[2]}$, define the subset $\mathcal{L} \subset B$ by*

$$\mathcal{L} = \{P \in B \mid K[x,y] = K[P]^{[1]}, \text{gcd}_B(P_x, P_y) = 1\} \ .$$

Then $\text{LND}_R(R[x,y]) = \{f\Delta_P | P \in \mathcal{L}, f \in R[P]\}$. Moreover, the irreducible elements of $\text{LND}_R(R[x,y])$ are precisely $\{\Delta_P | P \in \mathcal{L}\}$.

4.2.2 The Case $(DB) = B$

Another key result about locally nilpotent R-derivations of $R[x,y]$ is the following. It concerns a larger class of rings than HCF-rings, but a smaller class of derivations than $\text{LND}_R(R[x,y])$.

Theorem 4.16. *Let R be any commutative $\mathbb{Q}$-algebra, and let $B = R[x,y] = R^{[2]}$. Given $D \in \text{LND}_R(R[x,y])$, the following conditions are equivalent.*

(1) *$(DB) = B$, where (DB) is the B-ideal generated by DB.*

(2) *There exists* $s \in B$ *with* $Ds = 1$

In addition, when these conditions hold, $\ker D = R^{[1]}$.

This result was first proved by Daigle and the author in [54] (1998) for the case R is a UFD containing $\mathbb{Q}$. Since $k^{[n]}$ is a UFD, this could then be applied to questions about free $\mathbb{G}_a$-actions on affine space (see *Cor. 4.23* below). Shortly thereafter, Bhatwadekar and Dutta showed that the result is also true when R is a normal noetherian domain containing $\mathbb{Q}$ [18] (1997). Ultimately, Berson, van den Essen, and Maubach proved the theorem in the general form above. Their work was motivated, in part, by certain questions relating to the Jacobian conjecture, questions in which one cannot always assume that the base ring R is even a domain. The complete proof uses results found in [16]; see *Remark 4.18* below.

The proof given here is for the case R is a UFD, and is a modified version of the one in [54]; see also Thm. 6.7 of [47]. We first need the following lemma.

Lemma 4.17. (Lemma 2.6 of [54]) *Suppose* R *is an integral domain containing* $\mathbb{Q}$*, and* $F \in R[x,y]$*. If* Δ_F *is locally nilpotent and if the ideal* (F_x, F_y) *contains 1, then* $\ker \Delta_F = R[F]$.

Proof. Let K be the quotient field of R, and extend Δ_F to the locally nilpotent derivation δ on $K[x,y]$. By Rentschler's Theorem, $\ker \delta = K[G]$ for some $G \in K[x,y]$. Write $F = \varphi(G)$ for a univariate polynomial φ with coefficients in K. Choose $u, v \in R[x,y]$ so that $uF_x + vF_y = 1$. Then $1 = u\varphi'(G)G_x + v\varphi'(G)G_y$, which implies that $\varphi'(G)$ is a unit, and the degree of φ is 1. Therefore, $K[F] = K[\varphi(G)] = K[G]$.

Now $\ker \Delta_F = \ker \delta \cap R[x,y] = K[F] \cap R[x,y]$. Suppose that $K[F] \cap R[x,y]$ is not contained in $R[F]$. Choose $\lambda(T) \in K[T]$ (T an indeterminate) of minimal degree so that $\lambda(F) \in R[x,y]$ but $\lambda(F) \notin R[F]$. Note that since $\partial_x, \partial_y \in \mathrm{LND}_R(R[x,y])$, it follows that $\partial_x(\lambda(F)) = \lambda'(F)F_x$ and $\partial_y(\lambda(F)) = \lambda'(F)F_y$ belong to $R[x,y]$.

Choose $u, v \in R[x,y]$ such that $uF_x + vF_y = 1$. Then $\lambda'(F) = u\lambda'(F)F_x + v\lambda'(F)F_y \in R[x,y]$. By the assumption of minimality on the degree of λ, it follows that $\lambda'(F) \in R[F]$. This implies that the only coefficient of $\lambda(T)$ not in R is the degree-zero coefficient $\lambda(0)$, i.e., if $\mu(T) := \lambda(T) - \lambda(0)$, then $\mu(T) \in R[T]$. But then $\lambda(0) = \lambda(F) - \mu(F) \in R[x,y] \cap K = R$, a contradiction. $\square$

We now give the proof of the theorem.

Proof. (UFD case) The implication (2) $\Rightarrow$ (1) is clear.

Conversely, suppose (1) holds. Since R is a UFD, there exists $P \in R[x,y]$ with $\ker D = R[P]$, and since D is irreducible, we may assume $D = \Delta_P$. By the preceding results of this section, we know that there exists $Q \in R[x,y]$ such that $DQ \in R$ and $DQ \neq 0$. Choose a minimal local slice ρ so that $(\ker D)[Q] \subset (\ker D)[\rho]$; this is possible, since $R[x,y]$ is also a UFD, and thus satisfies the ACC on principal ideals (see *Sect. 6* of *Chap. 1*). Then clearly $D\rho \in R$, so we may assume Q itself is a minimal local slice.

If $DQ \notin R^*$, there exists a prime element $q \in R$ dividing DQ. Set $\bar{R} = R/qR$, and let $\pi : R[x, y] \to \bar{R}[x, y]$ be the extension of the projection $R \to \bar{R}$ which sends $x \to x$ and $y \to y$. If $h = \pi(P)$, then $\Delta_h \pi = \pi \Delta_P = \pi D$, and by induction $\Delta_h^n \pi = \pi D^n$ for $n \geq 1$. Thus, given $\sigma \in \bar{R}[x, y]$, if $\sigma = \pi(\tau)$ for $\tau \in R[x, y]$, then $\Delta_h^n(\sigma) = \Delta_h^n(\pi(\tau)) = \pi D^n(\tau) = 0$ for n sufficiently large. Therefore, Δ_h is locally nilpotent.

Suppose $1 = uP_x + vP_y$ for $u, v, \in R[x, y]$. Then $1 = \pi(uP_x + vP_y) = \pi(u)h_x + \pi(v)h_y$. By *Lemma 4.17*, it follows that $\ker \Delta_h = \bar{R}[h] = \pi(R[P])$. Now $DQ \in qR$ means $0 = \pi(DQ) = \Delta_h \pi(Q)$. Therefore $\pi(Q) \in \pi(R[P])$, i.e., there exists $f(P) \in R[P]$ and $Q' \in R[x, y]$ such that $Q - f(P) = qQ'$, violating the condition that Q is minimal.

Therefore $DQ \in R^*$. Setting $s = (DQ)^{-1}Q$, we have $Ds = 1$. $\square$

Remark 4.18. The purpose of this remark is to give the reader a sketch for the proof of *Thm. 4.16* in its most general form. In [16], Berson, van den Essen, and Maubach show that *Thm. 4.16* holds with the added assumption that $\mathrm{div}(D) = 0$ (Thm. 3.5). In addition, they show (Prop. 2.8) that if Ω is an integral domain containing $\mathbb{Q}$, then each locally nilpotent Ω-derivation of $\Omega^{[n]}$ has divergence zero. So this already proves the theorem when the base ring is an integral domain containing $\mathbb{Q}$. Now use the following two facts: Let R be any $\mathbb{Q}$-algebra, N the nilradical of R, and $\delta \in \mathrm{LND}(R^{[n]})$ for some n. Then (1) $\mathrm{div}(\delta) \in N$; and (2) If the quotient derivation D/N has a slice in $(R/N)^{[n]}$, then D has a slice in $R^{[n]}$. The proof of (1) follows immediately from the domain case by considering D mod $\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of R. The proof of (2) follows from Lemma 3.3.3 of van Rossum's thesis [299]. Finally, to complete the proof of the theorem, consider D/N on $(R/N)[x, y]$. Then $\mathrm{div}(D/N) = 0$ by (1), and therefore D/N has a slice. By (2), it follows that D itself has a slice.

4.2.3 Other Results for R-Derivations of $R[x, y]$

Also in the paper of Berson, van den Essen, and Maubach is the following result, which is related to their investigation of the Jacobian Conjecture.

Proposition 4.19. (Thm. 4.1 of [16]) *Let R be any commutative $\mathbb{Q}$-algebra. If $D \in \mathrm{Der}_R(R[x, y])$ is surjective and has divergence zero, then $D \in \mathrm{LND}_R(R[x, y])$.*

This result was shown earlier by Stein [285] in the case R is a field,[2] and by Berson [14] (Thm. 3.6) in the case R is a commutative noetherian $\mathbb{Q}$-algebra.

Clearly, the condition $\mathrm{div}\, D = 0$ cannot be removed. Cerveau proved:

[2] The authors of the paper [16] mistakenly omitted the divergence condition when they quoted the result of Stein in their introduction.

Proposition 4.20. (Thm. 5.1 of [34]) *If $D \in \mathrm{Der}_{\mathbb{C}}(\mathbb{C}[x,y])$ is surjective, then D is conjugate to $\delta_c = cx\partial_x + \partial_y$ for some $c \in k$.*

Take for example $c = 1$. We have $\delta_1^n(x) = x$ for all $n \geq 0$, so δ_1 is not locally nilpotent. On the other hand, δ_1 is surjective. To see this, it suffices to show that every monomial $x^m y^n$ is in its image, and this is accomplished by induction on n. If $m = 0$, then $\delta_1(\frac{1}{n+1}y^{n+1}) = y^n$, so we may assume $m > 0$. If $n = 0$, then $\delta_1(\frac{1}{m}x^m) = x^m$. So assume $m > 0$ and $n > 0$, and that every monomial $x^m y^a$ is in the image for $0 \leq a \leq n-1$. In particular, select $f \in \mathbb{C}[x,y]$ with $\delta_1 f = x^m y^{n-1}$. Then

$$\delta_1(x^m y^n) = mx^m y^n + nx^m y^{n-1} = mx^m y^n + \delta_1(nf) \ ,$$

which implies $\delta_1(\frac{1}{m}(x^m y^n - nf)) = x^m y^n$.

Finally, it was stated in the *Introduction* that we would like to understand which properties of a locally nilpotent derivation come from its being a "derivation", and which are special to the condition "locally nilpotent". For this comparison, we quote the following result, which is due to Berson.

Proposition 4.21. (Prop. 2.3 of [14]) *Let R be a UFD, and let*

$$D \in \mathrm{Der}_R(R[x,y]) \ ,$$

where $R[x,y] = R^{[2]}$. If $D \neq 0$, then $\ker D = R[f]$ *for some $f \in R[x,y]$.*

This generalizes the earlier result of Nagata and Nowicki in case R is a field ([241], Thm. 2.8). The reader should note that, for a derivation which is not locally nilpotent, it is possible that $\ker D = R$, i.e., $f \in R$. Nowicki gives an example of $d \in \mathrm{Der}_k(k[x,y])$ for a field k with $k[x,y]^d = k$ ([247], 7.3.1).

Another proof of Berson's result is given in [94], although the quotation of the Nagata-Nowicki theorem in the abstract of that paper is incorrect.

4.2.4 A Note on Vector Group Actions

The theorem of Abhyankar, Eakin, and Heinzer used in this section can also be used to prove the following.

Proposition 4.22. (Lemma 2 of [211]) *Suppose that the vector group $G = \mathbb{G}_a^{n-1}$ acts faithfully algebraically on $\mathbb{A}^n$. Then $k[\mathbb{A}^n]^G = k^{[1]}$.*

Proof. Let $A \subset k^{[n]}$ be the ring of invariants $k[\mathbb{A}^n]^G$. Then A, being the ring of common invariants for successive commuting $\mathbb{G}_a$-actions, is factorially closed. In addition, since $\dim G = n-1$ and the action is faithful, the transcendence degree of A over k is 1. By *Thm. 4.12*, $A = k^{[1]}$. $\square$

4.3 Rank-Two Derivations of Polynomial Rings

In working with polynomial rings, a natural invariant of a given derivation is its rank. Specifically, let B denote the polynomial ring in n variables over k for $n \geq 2$. From *Chap. 3*, recall that a k-derivation D of B has rank at most 2 if and only if there exist $x_1, ..., x_{n-2} \in B$ such that $B = k[x_1, ..., x_n]$ and $k[x_1, ..., x_{n-2}] \subset \ker D$. In this case, D is an R-derivation of $B = R^{[2]}$, where $R = k[x_1, ..., x_{n-2}]$, a UFD. Owing to the results of the preceding sections, the set of $D \in \mathrm{LND}(B)$ having rank at most 2 constitutes one class of locally nilpotent derivations of B which are fairly well understood. For example, *Thm. 4.13* implies that, for such D, $\ker D$ is always a polynomial ring over k, a very strong property.

Suppose $D \in \mathrm{LND}(B)$ is of rank at most 2 on B, and also satisfies the condition $(DB) = B$. Geometrically, this implies that the corresponding $\mathbb{G}_a$-action on $\mathbb{A}^n$ is fixed-point free. By *Thm. 4.16*, there exists a slice $s \in B$ of D, i.e., $Ds = 1$. Suppose $\ker D = R[P]$ for $P \in B$. By the Slice Theorem, $B = R[P, s] = k[x_1, ..., x_{n-2}, P, s]$, meaning that P and s are variables of B, and $D = \partial_s$ relative to this coordinate system on B. Combined with the results above, we get the following corollaries of *Thm. 4.16*.

Corollary 4.23. *For $n \geq 3$, every rank-two algebraic action of $\mathbb{G}_a$ on $\mathbb{A}^n$ has fixed points.*

Corollary 4.24. (Variable Criterion) *For $n \geq 2$, let $R = k^{[n-2]}$, $K = \mathrm{frac}(R)$, and $B = R[x, y] = k^{[n]}$. Given $P \in B$, the following are equivalent.*

(1) $K[x, y] = K[P]^{[1]}$ *and* $(P_x, P_y) = (1)$
(2) $B = R[P]^{[1]}$

Proof. If (1) holds, then Δ_P has $(\Delta_P B) = B$. By *Thm. 4.16*, there exists $Q \in R[x, y]$ with $\Delta_P Q = 1$. By *Lemma 4.17*, $\ker \Delta_P = R[P]$. It follows from the Slice Theorem that $B = R[P, Q]$, so condition (2) holds.

Conversely, if (2) holds, then $B = R[P, Q]$ for some Q, and thus $K[P, Q] = K[x, y]$. Since (P, Q) defines a R-automorphism of $R[x, y]$, we have

$$\frac{\partial(P, Q)}{\partial(x, y)} = P_x Q_y - P_y Q_x \in R^* \ ,$$

implying that $(P_x, P_y) = (1)$. □

Example 4.25. In [35], Choudary and Dimca show that the hypersurfaces $X \subset \mathbb{C}^4$ defined by equations of the form $f = 0$ for

$$f = x + x^{d-1} y + y^{d-1} z + t^d \quad (d \geq 1)$$

are coordinate hyperplanes (i.e., f is a variable of $k[x, y, z, t]$ when $k = \mathbb{C}$). We show the same holds for any field k of characteristic zero. First, if $K = k(y, t)$, then f is a triangular K-variable of $K[x, z]$. Second,

$$(f_x\,,\,f_z) = (1 + (d-1)x^{d-2}y\,,\,y^{d-1}) = (1)\ ,$$

so the conditions of the Variable Criterion are satisfied. □

Even for $n = 3$, the rank-two case is already of interest. Here, we consider locally nilpotent R-derivations of $B = R[y,z]$, where $R = k[x]$. Note that R is not only a UFD in this case, but even a PID. This class of derivations includes all triangular derivations of B (in an appropriate coordinate system). Until recently, it was not known whether any $D \in \mathrm{LND}(k[x,y,z])$ could have rank 3. Such examples were found in 1996; see *Chap. 5*. In other words, there exist $D \in \mathrm{LND}(k[x,y,z])$ which cannot be conjugated to $\mathrm{LND}_R(R[y,z])$.

Example 4.26. In his 1988 paper [282], Snow proved a special case of *Cor. 4.23*, namely, that a free triangular action of $\mathbb{G}_a$ on $\mathbb{A}^3$ is conjugate to a coordinate translation. In Example 3.4 of [283], Snow defines $D \in \mathrm{LND}_R(R[y,z])$ by $Dy = x^2$ and $Dz = 1 - 2xy$, where $R = k[x]$. This satisfies $(DB) = B$, so there exist $u, v \in B$ such that $B = R[u,v]$, $\ker D = R[u]$, and $Dv = 1$. To find such u, just consider the image of z under the Dixmier map π_y relative to the local slice y:

$$\pi_y(z) \ = \ z - Dz\frac{y}{Dy} + \frac{1}{2}D^2z\frac{y^2}{(Dy)^2} \ = \ -x^{-2}(y - xF)\ ,$$

where $F = xz + y^2$. We see that, if $K = k(x)$, then $K[y,z] = K[y,F] = K[y - xF, F]$, so $u := y - xF$ is a variable of $K[y,z]$. It follows that $\ker D = R[y - xF]$. To find v, note that $0 = Du = Dy - xDF \Rightarrow DF = x$; and since $u \equiv y (\mathrm{mod}\, x)$ and $F \equiv y^2 (\mathrm{mod}\, x)$, x divides $F - u^2$, where $D(F - u^2) = x$. Thus, if $v := x^{-1}(F - u^2)$, then $Dv = 1$. Specifically, $v = z + 2yF - xF^2$. In other words, $D = \alpha(\partial_z)\alpha^{-1}$, where α is the automorphism of B defined by

$$\alpha = (x, u, v) = (x, y - xF, z + 2yF - xF^2)\ .$$

We recognize this as the famous Nagata automorphism of $k[x,y,z]$; see *Chap. 3*.

4.3.1 Applications to Line Embeddings

As above, set $B = k[x,y,z]$. In [1], Abhyankar introduced the algebraic embeddings $\theta_n : \mathbb{A}^1 \hookrightarrow \mathbb{A}^3$ defined by $\theta_n(t) = (t + t^{n+2}, t^{n+1}, t^n)$. For $n \leq 4$, it was shown by Craighero [44, 45] that θ_n is **rectifiable**, i.e., there exist $u, v, w \in k[x,y,z]$ such that $B = k[u,v,w]$ and $\theta_n^* : B \to k[t]$ is given by

$$\theta_n^*(u) = \theta_n^*(v) = 0 \quad \text{and} \quad \theta_n^*(w) = t\ .$$

Another proof of rectifiability for $n = 3, 4$ is given by Bhatwadekar and Roy in [19].

Note that, over the real number field, the image of θ_n is topologically a line (unknot), since this image is contained in the surface $y^n = z^{n+1}$, which can be continuously deformed to a coordinate plane. The question is whether this curve can be straightened algebraically.

The proof of Bhatwadekar and Roy is based on the following result, which is due to Russell.

Proposition 4.27. (Prop. 2.2 of [266]) *Let $P \in k[x, y, z]$ have the form*

$$P = x + f(x, z)z + \lambda z^s y \ ,$$

where $f \in k[x, z]$, $s \in \mathbb{Z}$ is non-negative, and $\lambda \in k^*$. Then P is a variable of $k[x, y, z]$.

Proof. Set $R = k[z]$ and $K = k(z)$. Then P is a triangular K-variable of $K[x, y]$, and $(P_x, P_y) = (1 + f_x z, z^s) = (1)$. It follows from the Variable Criterion that P is a variable of $k[x, y, z]$. □

Consider $\theta_3 = (t + t^5, t^4, t^3)$. We seek P of the form above such that $P(t+t^5, t^4, t^3) = t$, and one easily finds that $P := x-(x^2-2z^2)z+z^3y$ works. By the proposition above, P is a variable, and therefore, θ_3 is rectifiable. Let us find $u, v \in B$ so that $B = k[u, v, P]$ and the kernel of $\theta_3^* : B \to k[t]$ equals (u, v).

We first need to find Q with $\Delta_P Q = 1$. Note that

$$\ker \Delta_P = k[z, P] = k[z, x + zF + 2z^3] = k[z, x + zF] \ ,$$

where $F = z^2 y - x^2$. Thus,

$$0 = \Delta_P(x + zF) = \Delta_P x + z\Delta_P F = z^3 + z\Delta_P F \quad \Rightarrow \quad \Delta_P F = -z^2 \ .$$

One then checks that $G := zy + 2xF + zF^2$ has $\Delta_P G = -z$. Likewise, if $Q := -(y + 2xG + z^2G^2)$, then $\Delta_P Q = 1$. Thus, $B = k[z, P, Q]$.

Finally, suppose $\theta_3^*(Q) = f(t)$ for $f(t) \in k[t]$. If

$$u := z - P^3 \quad \text{and} \quad v := Q - f(P) \ ,$$

then $B = k[u, v, P]$ and $\ker \theta_3^* = (u, v)$.

The foregoing method of finding Q is *ad hoc*. In [15], Berson and van den Essen give an algorithm for finding Q such that $R[x, y] = R[P, Q]$, given that $R[x, y] = R[P]^{[1]}$.

Next, consider θ_4. It is not hard to discover P of the form described in the proposition above which satisfies $P(t + t^6, t^5, t^4) = t$, namely,

$$P := x + (2x^3z - x^2 - 2xz^4 - 5z^3)z - 4z^4y \ .$$

Since P is a variable of $k[x, y, z]$, θ_4 is rectifiable. However, to also find kernel generators for θ_4^* which are variables, as we did for θ_3^*, is far more complicated in this case. To this end, the interested reader is invited to implement the aforementioned algorithm of Berson and van den Essen.

It remains an open question whether, for any $n \geq 5$, θ_n can be rectified. However, it appears unlikely that a variable of the form described in the proposition can be used to straighten θ_5.

As a final remark about line embeddings, note that the rectifiable embeddings are precisely those defined by $\alpha(t,0,0)$, where $\alpha \in GA_3(k)$. If we could describe all locally nilpotent derivations of $k[x,y,z]$, then one would at least get a description of all the exponential embeddings $\alpha(t,0,0)$, where α is the exponential of a locally nilpotent derivation. The embeddings θ_3 and θ_4 are of this type. Consider also the automorphism $\exp(D)$ of $\mathbb{A}^3$, where D is the $(2,5)$ derivation of $k[x,y,z]$ discusssed in *Chap. 5*: By setting $x = 0$, $y = t$, and $z = 0$, we obtain the very simple rectifiable embedding $\phi : \mathbb{A}^1 \to \mathbb{A}^3$ defined by $\phi(t) = (t^5, t + t^{13}, t^9 + t^{21})$. It seems unlikely that a straightening automorphism for ϕ could be found directly from its definition.

4.4 Automorphisms Preserving Lattice Points

In this chapter, we have considered rings $R[x,y]$, where the base ring R contains $\mathbb{Q}$. However, there is at least one other important ring of characteristic zero which should be mentioned in this chapter, namely, $\mathbb{Z}[x,y]$, the ring of bivariate polynomials with integral coefficients. Its group of automorphisms is $GA_2(\mathbb{Z}) = \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}[x,y])$, elements of which map the lattice points $\mathbb{Z}^2 \subset \mathbb{R}^2$ into themselves.

Regarding locally nilpotent derivations of $\mathbb{Z}[x,y]$, many of the properties which hold for k-domains carry over to these as well: $\mathrm{LND}(\mathbb{Z}[x,y]) = \mathrm{LND}_{\mathbb{Z}}(\mathbb{Z}[x,y])$, kernels are factorially closed and of the form $\mathbb{Z}[P]$, etc. However, one key difference is that $D \in \mathrm{LND}(\mathbb{Z}[x,y])$ does not generally give a well-defined automorphism by the exponential map. For example, if $D = \partial_x + x\partial_y$, then $D \in \mathrm{LND}(\mathbb{Z}[x,y])$, but $\exp D = (x+1, y+2x+\frac{1}{2})$, which does not preserve lattice points. Similarly, the Dixmier map may not be defined. Clearly, what we require is not only that D be locally nilpotent, but also that $D^n x, D^n y \in (n!)$ for all $n \geq 0$.

The group $SL_2(\mathbb{Z}) \subset GA_2(\mathbb{Z})$ plays a central role in the study of lattices and modular forms; see Serre [273]. Serre points out that $SL_2(\mathbb{Z}) = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$, where $\mathbb{Z}_n$ denotes the cyclic group $\mathbb{Z}/n\mathbb{Z}$. Specifically, $\mathbb{Z}_4$ and $\mathbb{Z}_6$ are generated by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix},$$

respectively, and $\mathbb{Z}_2 = \{\pm I\}$.

As in the Structure Theorem over fields, the full automorphism group $GA_2(\mathbb{Z})$ also admits a description as an amalgamated free product, though this description is more complicated. In his paper [309], in the Corollary to

Thm. 5, Wright describes the decomposition of the group $GA_2(K)$ for any principal ideal domain K. His main interest is in the case $K = k^{[1]}$, but the result holds for $\mathbb{Z}$ as well.

4.5 Newton Polygons

Rentschler's original proof of *Thm. 4.1* used the Newton polygon of a locally nilpotent derivation of $k[x, y]$. One could also give a proof using the Newton polygon of a kernel element in place of the Newton polygon of the derivation. This section gives some basic facts about these two types of Newton polygons. As before, let $A = k[x, y]$.

4.5.1 Newton Polygon of a Polynomial

Given $f \in A$, suppose $f = \sum_{i,j \geq 0} a_{ij} x^i y^j$ $(a_{ij} \in k)$. Define the **support** of f by $\mathrm{Supp}(f) = \{(i, j) | a_{ij} \neq 0\}$. Define the **Newton polygon** of f to be the convex hull of $\mathrm{Supp}(f) \cup \{(0, 0)\}$, denoted by $\mathrm{Newt}(f)$. Observe that both the support and Newton polygon of f depend on our choice of coordinates x, y in A.

Lemma 4.28. *Given nonzero $D \in \mathrm{LND}(A)$, and given non-constant $f \in \ker D$, $\mathrm{Newt}(f)$ is a triangle T with vertices of the form $(0, 0)$, $(m, 0)$, and $(0, n)$, and either m divides n or n divides m.*

This lemma represents a special case of a more general fact, which will be proved in the *Appendix* to this chapter. It could also be deduced directly from Rentschler's Theorem.

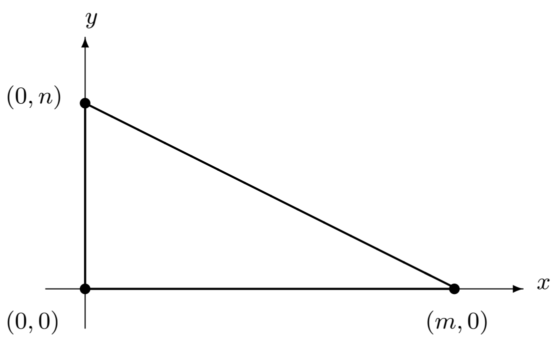

Fig. 4.1. Newton Polygon of $f \in \ker D$

4.5.2 Newton Polygon of a Derivation

The Newton polygon $\mathrm{Newt}(D)$ of a locally nilpotent derivation of A is defined in the following way.

The ring $A = k[x, y]$ admits a natural $\mathbb{Z}^2$-grading, namely,

$$A = \sum_{(i,j)\in\mathbb{Z}^2} A_{(i,j)} \ ,$$

where $A_{(i,j)} = k \cdot x^i y^j$ if $i, j \geq 0$, and otherwise $A_{(i,j)} = 0$. Any $D \in \mathrm{LND}(A)$ which respects this grading will be called standard homogeneous (for a $\mathbb{Z}^2$-grading). In particular, this means that there exists $(c, d) \in \mathbb{Z}^2$ such that $DA_{(a,b)} \subset A_{(a,b)+(c,d)}$ for every $(a, b) \in \mathbb{Z}^2$. The pair (c, d) is called the degree of D. For example, the partial derivatives ∂_x and ∂_y are standard homogeneous, of degrees $(-1, 0)$ and $(0, -1)$ respectively.

Accordingly, any $D \in \mathrm{LND}(A)$ can be decomposed as

$$D = \sum_{(i,j)\in\mathbb{Z}^2} D_{(i,j)} \ ,$$

where $D_{(i,j)}$ is homogeneous of degree (i, j), and $D_{(i,j)} = 0$ if $i < -1$ or $j < -1$. To see this, write $D = a\partial_x + b\partial_y$ for some $a, b \in A$. Write a and b as

$$a = \sum_{\substack{i\geq -1\\ j\geq 0}} a_{ij}x^{i+1}y^j \quad \text{and} \quad b = \sum_{\substack{i\geq 0\\ j\geq -1}} b_{ij}x^iy^{j+1} \ ,$$

where $a_{ij}, b_{ij} \in k$, with $a_{ij} = 0$ if $j = -1$ and $b_{ij} = 0$ if $i = -1$. Then

$$D_{(i,j)} = a_{ij}x^{i+1}y^j\partial_x + b_{ij}x^iy^{j+1}\partial_y \ .$$

Define the **support** of D by

$$\mathrm{Supp}(D) = \{(i, j) \in \mathbb{Z}^2 | D_{(i,j)} \neq 0\} \ ,$$

and define the **Newton polygon** of D to be the convex hull of $\mathrm{Supp}(D) \cup \{(-1, -1)\}$. Note that every vertex of $\mathrm{Newt}(D)$ other than $(-1, -1)$ belongs to $\mathrm{Supp}(D)$.

Lemma 4.29. *If $D \neq 0$, then* $\mathrm{Newt}(D)$ *is equal to the triangle with vertices $(-1, -1), (\lambda, -1)$, and $(-1, \mu)$, where*

$$\lambda = \max\{i|D_{(i,-1)} \neq 0\} \cup \{-1\} \ , \quad \mu = \max\{j|D_{(-1,j)} \neq 0\} \cup \{-1\} \ .$$

To prove Rentschler's Theorem from here, one shows that the Newton polygon can be reduced by a triangular automorphism. On the other hand, this lemma can be deduced from Rentschler's Theorem.

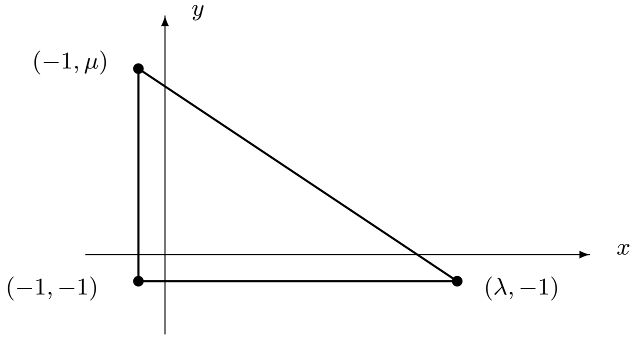

Fig. 4.2. Newton polygon of D

4.6 Appendix: Newton Polytopes

We have seen that Newton polygons were used by Rentschler to study locally nilpotent derivations of $k^{[2]}$. It is natural to define higher dimensional analogues of these, and investigate their properties relative to locally nilpotent derivations of polynomial rings. Such investigation is the subject of papers by Hadas, Makar-Limanov, and Derksen [70, 136, 137].

Let $f \in B = k^{[n]}$ be given. Relative to a coordinate system $B = k[x_1, ..., x_n]$, we define the **Newton polytope** of f, denoted $\mathrm{Newt}(f)$, as follows. Write

$$f = \sum \alpha_e x_1^{e_1} \cdots x_n^{e_n} \ , \quad e = (e_1, ..., e_n) \in \mathbb{Z}^n \ , \ e_i \geq 0 \ , \ \alpha_e \in k \ .$$

The **support** of f, denoted $\mathrm{Supp}(f)$, equals $\{e \in \mathbb{Z}^n | \alpha_e \neq 0\}$; and $\mathrm{Newt}(f)$ equals the convex hull of $\mathrm{Supp}(f) \cup \{0\}$ in $\mathbb{Q}^n$. Note that this definition depends on the choice of coordinates for B.

For such polynomials, the following very special property emerges.

Proposition 4.30. (Thm. 3.2 of [137]) *Suppose $B = k[x_1, ..., x_n] = k^{[n]}$, $D \in \mathrm{LND}(B)$, $D \neq 0$, and $f \in \ker D$. Then every vertex of $\mathrm{Newt}(f)$ is contained in a coordinate hyperplane. More precisely, if $q = (q_1, ..., q_n) \in \mathbb{Q}^n$ is a vertex of $\mathrm{Newt}(f)$, then $q_i = 0$ for at least one i.*

Proof. Suppose $q \neq 0$. By convexity, there exists a hyperplane $H \subset \mathbb{Q}^n$ such that $H \cap \mathrm{Newt}(f) = \{q\}$. We may suppose that H is defined by the equation

$$\sum_{1 \leq i \leq n} a_i y_i = d \ , \quad y = (y_1, ..., y_n) \in \mathbb{Q}^n \ , \ a_i, d \in \mathbb{Z} \ , \ d > 0 \ .$$

Since $0 \in \mathrm{Newt}(f)$, we conclude that $\sum a_i y_i \leq d$ for $y \in \mathrm{Newt}(f)$, with equality only at q.

Note that H determines a grading on B, namely, the degree of the monomial $x_1^{t_1} \cdots x_n^{t_n}$ is $\sum_{i=1}^{n} a_i t_i$. For this grading, write $B = \oplus_{i \in \mathbb{Z}} B_i$; then $f = \sum_{i=0}^{d} f_i$ for $f_i \in B_i$, and f_d is the monomial supported at q. In addition, if D' is the highest-degree homogeneous summand of D relative to this grading, then according to *Princ. 14*, $D' \in \mathrm{LND}(B)$ and $f_d \in \ker D'$. Since $D \neq 0$, $D' \neq 0$ as well. If each q_i were strictly positive, then since $\ker D'$ is factorially closed and f_d is a monomial, we would have that $x_i \in \ker D'$ for each i, implying $D' = 0$. Therefore, at least one of the q_i equals 0. □

Remark 4.31. Any variable f of $B = k^{[n]}$ is in the kernel of some nonzero $D \in \mathrm{LND}(B)$, for example, $D = \partial_g$ where (f, g) is a partial system of variables for B. Thus, any variable possesses the property described in the theorem above. In fact, it is shown in [70] that, regardless of the characteristic of the field k, the Newton polytope of an invariant of an algebraic $\mathbb{G}_a$-action on $\mathbb{A}^n$ has all its vertices on coordinate hyperplanes (Thm. 3.1). Thus, this property applies to any variable, regardless of characteristic.

A second property of Newton polytopes involves its edges. Continuing the assumptions and notations above, an edge E of $\mathrm{Newt}(f)$ is called an **intrusive edge** if it is contained in no coordinate hyperplane of $\mathbb{Q}^n$. (Such edges are called **trespassers** in [70].) If E is the intrusive edge joining vertices p and q, then $e(E)$, or just e, will denote the vector $e = p - q$ (or $q - p$, it doesn't matter).

Proposition 4.32. (Thm. 4.2 of [137]) *Suppose $f \in \ker D$ for $D \in \mathrm{LND}(B)$ and $D \neq 0$. Let E be an intrusive edge of* $\mathrm{Newt}(f)$*, where $e = (e_1, ..., e_n) = p - q$ for vertices $p = (p_1, ..., p_n)$ and $q = (q_1, ..., q_n)$ of* $\mathrm{Newt}(f)$*. Then:*

(1) *There exist distinct integers r and s such that $p_r = 0$, $q_s = 0$, and either $e \in q_r \cdot \mathbb{Z}^n$ or $e \in p_s \cdot \mathbb{Z}^n$*
(2) $\min_{1 \leq i \leq n}\{e_i\} < 0$ *and* $\max_{1 \leq i \leq n}\{e_i\} > 0$

Proof. (following [137]) We first make several simplifying assumptions and observations.

- Given i, set $B_i = k[x_1, ..., \hat{x_i}, ..., x_n]$. Note that $f \notin \cup_{i=1}^{n} B_i$, since the Newton polytope of an element of B_i has no intrusive edges.
- Suppose this result holds for algebraically closed fields of characteristic zero, and let L denote the algebraic closure of k. If D_L denotes the extension of D to $B_L = L[x_1, ..., x_n]$, then $D_L \in \mathrm{LND}(B_L)$ and $f \in \ker(D_L)$. Note that the Newton polytope of f relative to D or D_L is the same. So from now on we assume k is algebraically closed.
- To simplify notation, for $u \in \mathbb{Z}^n$, x^u will denote the monomial $x_1^{u_1} \cdots x_n^{u_n}$.
- After a permutation of the variables, we may assume that $x_1, ..., x_\gamma \in \ker D$ and $x_{\gamma+1}, ..., x_n \notin \ker D$, where γ is an integer with $0 \leq \gamma \leq n$.

We now proceed with the proof.

To show (1), first choose a $\mathbb{Q}$-linear function $\lambda : \mathbb{Q}^n \to \mathbb{R}$ such that $\ker(\lambda) = \langle e \rangle$. (For example, choose a projection $\mathbb{Q}^n \to \mathbb{Q}^{n-1}$ mapping E to a single point, and then pick your favorite copy of $\mathbb{Q}^{n-1}$ in $\mathbb{R}$, such as $\mathbb{Q}[\zeta]$ for a real algebraic number ζ of degree $n-1$.) Thus, λ is constant on E, and we may further assume that $\lambda(E) = \max \lambda(\mathrm{Newt}(f)) \geq 0$. Such λ induces a grading $B = \oplus_{\rho \in \mathbb{R}} B_\rho$ in which the degree of the monomial x^u equals $\lambda(u)$. Elements of B_ρ will be called λ-homogeneous of λ-degree ρ.

As before, if D' is the highest λ-homogeneous summand of D, and if f' is the highest λ-homogeneous summand of f, then D' is locally nilpotent, $D'(f') = 0$, and $\mathrm{Newt}(f')$ is the convex hull of $0, p$ and q. So we might just as well assume $D = D'$ and $f = f'$.

Choose $\epsilon \in \mathbb{Q}^n$ which spans the kernel of λ. We may assume that $\epsilon \in \mathbb{Z}^n$, and that the entries of ϵ have no common factor. It follows that every λ-homogeneous element of B can be written as $x^\alpha P(x^\epsilon)$, where $\alpha \in \mathbb{Z}^n$ has non-negative entries, and P is a univariate polynomial over k. (Note that some entries of ϵ can be negative.) In particular, $f = x^\alpha P(x^\epsilon)$ for some α and some P. This implies that we may write

$$f = x^\alpha \prod_i (x^\epsilon - t_i) = x^w \prod_i (x^u - t_i x^v)$$

where $t_i \in k$, $u, v, w \in \mathbb{Z}^n$ have non-negative entries, and $u_i v_i = 0$ for each i. Moreover, since every x_i appears in f, it follows that $u_i + v_i + w_i > 0$ for each i.

Since e is an intrusive edge, f is divisible by at least one factor of the form $x^u - tx^v$, $t \in k$, meaning $x^u - tx^v \in \ker D$. If also $x^u - sx^v$ divides f for $s \neq t$, then we have $x^u, x^v \in \ker D$. Since $\ker D$ is factorially closed, $x^w \in \ker D$ as well. But then since $u_i + v_i + w_i > 0$, it would follow that $x_i \in \ker D$ for every i (again since $\ker D$ is factorially closed), which is absurd. Therefore $\ker D$ contains $x^u - tx^v$ for exactly one value of t, and $t \neq 0$. Altogether, this implies $f = cx^w(x^u - tx^v)^m$ for some $c \in k^*$ and positive integer m.

In the same way, if $g \in \ker D$ is any other λ-homogeneous element, then $g = dx^\omega(x^u - tx^v)^\mu$, where $d \in k$, $\omega \in \mathbb{Z}^n$ has non-negative entries, and μ is a non-negative integer. Since $\ker D$ is factorially closed, $x^\omega \in k[x_1, ..., x_\gamma]$. It follows that $\ker D = k[x_1, ..., x_\gamma, x^u - tx^v]$, and since the transcendence degree of $\ker D$ is $n-1$, we have $\gamma + 1 = n - 1$, or $\gamma = n - 2$.

Let $K = k(x_1, ..., x_{n-2})$. Then D extends to a locally nilpotent K-derivation of $R = K[x_{n-1}, x_n]$, so by Rentschler's Theorem $x^u - tx^v$ is a K-variable of R. As such, it must have a degree-one term in either x_{n-1} or x_n over K.

Let us assume $u_n = 1$ (so $v_n = 0$). Recalling that $\epsilon = u - v$, we have that $\epsilon_n = 1$. Now $f = x^w(x^u - tx^v)^m$, where $w_n = 0$. It follows that $p = mu + w$ and $q = mv + w$. Since $v_n = 0$, we conlude that $q_n = 0$ as well. Since $e = p - q$ is an integral multiple of ϵ, and $e_n = p_n$, we conclude that $e = p_n\epsilon$. The other cases are similar. This completes the proof of (1).

By the preceding result, both p and q have at least one 0 entry, and since e is an intrusive edge, we can find r and s with $p_r = 0$, $q_s = 0$, and $r \neq s$. Thus, $e_r = p_r - q_r = -q_r \leq 0$ and $e_s = p_s - q_s = p_s \geq 0$, and (2) follows. □

As noted in [137], these two results provide a quick way to determine that certain polynomials cannot be annihilated by a nonzero locally nilpotent derivation. For example, if $a, b, c \in \mathbb{Z}$ have $a, b, c > 1$ and $\gcd(a, b, c) = 1$, then $f = x^a + y^b + z^c$, $g = x^a y^b + z^c$, and $h = x^{a+1} y^b + x^a z^c + 1$ are elements of $k[x, y, z]$ not in the kernel of any nonzero $D \in \mathrm{LND}(k[x, y, z])$. In the first case, $\mathrm{Newt}(f)$ has the intrusive edge $(a, -b, 0)$. In the second case, $\mathrm{Newt}(g)$ has the intrusive edge $(a, b, -c)$. An in the third case, $\mathrm{Newt}(h)$ has vertices $p = (a + 1, b, 0)$ and $q = (a, 0, c)$, so that $e = (1, b, -c)$ is an intrusive edge which belongs to neither $b\mathbb{Z}^3$ nor $c\mathbb{Z}^3$. Consequently, the surfaces in $\mathbb{A}^3$ defined by f, g and h are not stabilized by any $\mathbb{G}_a$-action on $\mathbb{A}^3$.

Example 4.33. Let $B = k[x, y, z] = k^{[3]}$, and define $G \in B$ by

$$G = x^2 z^3 - 2xy^2 z^2 + y^4 z + 2x^3 yz - 2x^2 y^3 + x^5 \ .$$

In *Chap. 5* it is shown that G is irreducible, and that there exist nonzero $D \in \mathrm{LND}(B)$ with $DG = 0$. Since G is homogeneous in the standard sense, of degree 5, its support is contained in the plane H defined by $x_1 + x_2 + x_3 = 5$. Let Q be the convex hull of $\mathrm{Supp}(G)$, which is contained in H. It is easy to see that the vertex set of Q is

$$\{(2, 0, 3), (0, 4, 1), (2, 3, 0), (5, 0, 0)\} \ ,$$

so Q is a quadrilateral, and $\mathrm{Newt}(G)$ consists of a cone over Q. Thus, $\mathrm{Newt}(G)$ has 5 faces. Two of these faces lie in coordinate planes, and 3 do not (call these **intrusive faces**). There are two intrusive edges, namely, E_1 joining $(0, 4, 1)$ and $(2, 0, 3)$, and E_2 joining $(0, 4, 1)$ and $(2, 3, 0)$. They define vectors $e_1 = (-2, 4, -2)$ and $e_2 = (-2, 1, 1)$.

One can also consider higher-dimensional faces of Newton polytopes for polynomials f annihilated by locally nilpotent derivations, and thereby get further conditions on $\mathrm{Newt}(f)$. The last section of [70] gives some results in this direction.

Remark 4.34. Define $f = x^2 + y^2 + z^2 \in \mathbb{C}[x, y, z]$, and $D \in \mathrm{LND}(\mathbb{C}[x, y, z])$ by

$$Dx = -z \ , \ Dy = -iz \ , \ Dz = x + iy \ .$$

Then $Df = 0$. Likewise, $f \in \mathbb{R}[x, y, z]$, but it is shown in *Chap. 9* below that there is no nonzero $\delta \in \mathrm{LND}(\mathbb{R}[x, y, z])$ with $\delta f = 0$. This example points out the limitations of the the information provided by $\mathrm{Newt}(f)$.

5

Dimension Three

As mentioned in the *Introduction*, a big impetus was given to the study of $\mathbb{G}_a$-actions by the appearance of Bass's 1984 paper, which showed that, in contrast to the situation for $\mathbb{A}^2$, there exist algebraic actions of $\mathbb{G}_a$ on $\mathbb{A}^3$ which cannot be conjugated to a triangular action. Since then, our understanding of the dimension three case has expanded dramatically, though it remains far from complete.

Parallel to recent developments for unipotent actions is the theorem of Koras and Russell, which asserts that every algebraic $\mathbb{C}^*$-action on $\mathbb{C}^3$ can be linearized [173]. Their result was then generalized to the following.

> *Every algebraic action of a connected reductive group G on $\mathbb{C}^3$ can be linearized.*

See Popov [255]. On the other hand, the question whether every algebraic action of a finite group on $\mathbb{C}^3$ can be linearized remains open.

In the study of $\mathbb{G}_a$-actions on $\mathbb{A}^n$, the dimension-three case stands between the fully developed theory in dimension two, and the wide open possibilities in dimension four. The fundamental theorems of this chapter show that many important features of planar $\mathbb{G}_a$-actions carry over to dimension three: Every invariant ring is a polynomial ring; the quotient map is always surjective; and free $\mathbb{G}_a$-actions are translations. It turns out that none of these properties remains generally true in dimension four. In addition, there exist locally nilpotent derivations in dimension three of maximal rank 3, and these have no counterpart in dimension two. Such examples will also be explored in this chapter.

Throughout this chapter, k is a field of characteristic zero, and B will denote the three dimensional polynomial ring $k[x, y, z]$ over k.

5.1 Miyanishi's Theorem

In the late 1970s, Fujita, Miyanishi, and Sugie succeeded in proving the Cancellation Theorem for surfaces, which asserts that if Y is an affine surface over $\mathbb{C}$, and if $Y \times \mathbb{C}^n \cong \mathbb{C}^{n+2}$ for some $n \geq 0$, then $Y \cong \mathbb{C}^2$ [124, 222]. Key elements of their proof provided the foundation for Miyanishi's subsequent proof that the quotient Y of a nontrivial $\mathbb{C}^+$-action on $\mathbb{C}^3$ is isomorphic to $\mathbb{C}^2$. In each case, the crucial step is to show that the surface Y contains a cylinderlike open set, i.e., an open set of the form $K \times \mathbb{C}$ for some curve K. This is very close (in fact, equivalent) to saying that Y admits a $\mathbb{C}^+$-action.

The theorem of Miyanishi, which appeared in 1985, established the single most important fact about locally nilpotent derivations and $\mathbb{G}_a$-actions in dimension three.

Theorem 5.1. (Miyanishi's Theorem) *Let k be a field of characteristic zero. Then the kernel of any nonzero locally nilpotent derivation of $k^{[3]}$ is a polynomial ring $k^{[2]}$.*

This theorem means that there exist $f, g \in k[x, y, z]$, algebraically independent over k, such that the kernel is $k[f, g]$. But in contrast to the situation for dimension 2, it is not necessarily the case that f and g form a partial system of variables, i.e., $k[x, y, z] \neq k[f, g]^{[1]}$ generally.

Miyanishi's geometric result is equivalent to the case $k = \mathbb{C}$ of the theorem. Applying Kambayashi's theorem on plane forms gives the proof for any field k of characteristic zero (see *Subsect. 5.1.1* below). Miyanishi's first attempts to prove the theorem appeared in [214] (1980) and [215] (1981), but the arguments were flawed. Then in 1985, Miyanishi sketched a correct proof for the field $k = \mathbb{C}$ in his paper [217]. Certain details for the complete proof were later supplied by Sugie's 1989 paper [287].

An independent proof of Miyanishi's Theorem, in a somewhat more general form, was given recently by Kaliman and Saveliev [164]. Specifically, they show that if X is a smooth contractible affine algebraic threefold over $\mathbb{C}$ which admits a nontrivial $\mathbb{C}^+$-action, then the affine surface $S = X /\!\!/ \mathbb{C}^+$ is smooth and contractible. If, in addition, X admits a dominant morphism from $\mathbb{C}^2 \times \Gamma$ for some curve Γ (for example, $X = \mathbb{C}^3$), then $S \cong \mathbb{C}^2$. Their main theorem is given below.

In the early 1990s, Zurkowski proved an important special case of Miyanishi's Theorem in the manuscript [315]. Zurkowski was apparently unaware of Miyanishi's result, as Miyanishi's work is not cited in the paper, and Zurkowski describes his result as "a step towards extending results of [Rentschler] to three dimensional space" (p. 3). Specifically, what Zurkowski showed was that if k is an algebraically closed field of characteristic zero, and if $D \neq 0$ is a locally nilpotent derivation of $B = k[x, y, z]$ which is homogeneous relative to some positive grading on B, then $\ker D = k[f, g]$ for homogeneous $f, g \in B$. The term **positive grading** means that $B = \oplus_{i \geq 0} B_i$, $B_0 = k$, and x, y, and z are homogeneous. In his master's thesis, Holtackers [145] gives a more streamlined version of Zurkowski's proof. And in [26], Bonnet gives a third proof for the case $k = \mathbb{C}$, also quite similar to Zurkowski's. None of these three proofs was published.

A complete proof of Miyanishi's Theorem will not be attempted here, since both of the existing proofs use geometric methods which go beyond the scope of this book. Miyanishi's proof makes extensive use of the theory of surfaces with negative logarithmic Kodaira dimension, much of which was developed for the proof of cancellation for surfaces. The proof of Kaliman and Saveliev requires some algebraic topology, Phrill-Brieskorn theorems on quotient singularities, and a theorem of Taubes from gauge theory.

The section is organized in the following way. First, the reduction to the field $k = \mathbb{C}$ is given, using Kambayashi's Theorem. Then some general properties of the quotient map are discussed, followed by a description of Miyanishi's proof. Finally, a complete proof of Zurkowski's result is given for the case $k = \mathbb{C}$. This proof follows Zurkowski's algebraic arguments in the main, with simplifications based on some of the theory established earlier in this text, and concludes with Miyanishi's topological argument to shorten the overall length.

5.1.1 Kambayashi's Theorem

To deduce the general case of Miyanishi's Theorem from the case $k = \mathbb{C}$ we need the following result, which was proved by Kambayashi in [168]; see also Shafarevich [276], Thm. 9.

Theorem 5.2. (Kambayashi's Theorem) *Let k and K be fields such that K is a separably algebraic extension of k. Suppose R is a commutative k-algebra for which $K \otimes_k R \cong K^{[2]}$. Then $R \cong k^{[2]}$.*

Kambayashi's proof relies on the Structure Theorem for $GA_2(k)$.

Now suppose k is any field of characteristic zero, and that $D \in \mathrm{LND}(B)$ for $B = k^{[3]}$, $D \neq 0$. Following is the argument given by Daigle and Kaliman in [58].

Let $k_0 \subseteq k$ be the subfield of k generated over $\mathbb{Q}$ by the coefficients of the polynomials Dx, Dy, and Dz, and set $B_0 = k_0[x, y, z]$. Then D restricts to $D_0 \in \mathrm{LND}(B_0)$. Since k_0 has the form $\mathbb{Q}(F)$ for a finite set F, it is isomorphic to a subfield of $\mathbb{C}$. If we assume $k_0 \subset \mathbb{C}$, then we may extend D_0 to $D' \in \mathrm{LND}(\mathbb{C}[x, y, z])$. By Miyanishi's Theorem for the complex field, together with *Prop. 1.15*, we have

$$\mathbb{C} \otimes_{k_0} \ker D_0 = \ker D' = \mathbb{C}^{[2]} .$$

By Kambayashi's Theorem, it follows that $\ker D_0 = k_0^{[2]}$. Therefore,

$$\ker D = k \otimes_{k_0} \ker D_0 = k^{[2]} .$$

5.1.2 Basic Properties of the Quotient Morphism

Let X be an affine threefold over $\mathbb{C}$ which is factorial, i.e., the coordinate ring $\mathcal{O}(X)$ is a UFD. Suppose X admits a nontrivial algebraic action of $\mathbb{C}^+$, and let $Y = X/\!/\mathbb{C}^+$ denote the quotient of this action. By results of *Chap. 1*, Y is normal and $\text{tr.deg.}_{\mathbb{C}}\mathcal{O}(Y) = 2$. By the Zariski Finiteness Theorem, which can be found in *Chap. 6*, it follows that Y is affine. From this, standard theory from commutative algebra implies that Y is regular in codimension 1, i.e., the singularities of Y form a finite set (see for example [178]).

Let $\pi : X \to Y$ be the corresponding quotient morphism. Again from *Chap. 1*, recall that there exists a curve $\Gamma \subset Y$, determined by the image of a local slice for the corresponding locally nilpotent derivation, together with an open set $U = \pi^{-1}(Y - \Gamma) \subset X$ such that $U \cong (Y - \Gamma) \times \mathbb{C}$, and $\pi : U \to Y - \Gamma$ is the standard projection.

Lemma 5.3. (Lemma 2.1 of [160]; Lemma 1 of [176]) *Under the above hypotheses:*

(a) *Every non-empty fiber of π is of dimension 1.*
(b) *If $C \subset Y$ is a closed irreducible curve, then $\pi^{-1}(C) \subset X$ is an irreducible surface.*
(c) *$Y - \pi(X)$ is finite.*

Proof. Let $B = \mathcal{O}(X)$ and $A = \mathcal{O}(X)^{\mathbb{C}^+} \subset B$. Assume that, for some $y \in Y$, the fiber $\pi^{-1}(y)$ has an irreducible component Z of dimension 2. Then Z is defined by a single irreducible function $f \in B$. Since Z is stable under the $\mathbb{C}^+$-action, f is a non-constant invariant function, i.e., f is an irreducible element of A. Any other invariant $g \in A$ is constant on Z, meaning that $g = fh + c$ for some $h \in B$ and $c \in \mathbb{C}$. Since A is factorially closed, $h \in A$ as well. In other words, $g = fh + c$ is an equation in A, which implies that $A/fA = \mathbb{C}$. But this is impossible, since A is an affine UFD of dimension 2 over $\mathbb{C}$, and $f \in A$ is irreducible. This proves (a).

By part (a), $\pi^{-1}(C)$ must have an irreducible component of dimension 2. The irreducible closed curve $C \subset Y$ is defined by an irreducible function $q \in \mathcal{O}(Y)$, which lifts to the irreducible function $q\pi \in A$. Since A is factorially closed in B, this function is irreducible in B as well, meaning that $\pi^{-1}(C)$ is reduced and irreducible. This proves (b).

Finally, let $W \subset Y$ denote the Zariski closure of $Y - \pi(X)$. If $Y - \pi(X)$ is not finite, then W is a curve, and $F := \pi^{-1}(W)$ is of dimension 2. This implies that $\pi(F)$ is of dimension 1, which contradicts the fact that $\pi(F)$ is contained in the finite set $W \cap \pi(X)$. Therefore, $Y - \pi(X)$ is finite. $\square$

5.1.3 Description of Miyanishi's Proof

Let Y be a smooth algebraic surface over $\mathbb{C}$. Then Y can be completed to a smooth projective surface V in such a way that the divisor at infinity,

$D = V - Y$, consists of smooth curves with simple normal crossings. Y is said to have **logarithmic Kodaira dimension** $\bar{\kappa}(Y) = -\infty$ if $|n(D + K_V)| = \emptyset$ for every $n > 0$. Here, K_V denotes the canonical divisor of V. The property $\bar{\kappa}(Y) = -\infty$ is independent of the completion V, and is thus an invariant of Y.

One of the more important facts relating to the logarithmic Kodaira dimension is the following. First, an open subset $U \subset Y$ is called a **cylinderlike open set** if $U \cong K \times \mathbb{C}$ for some curve K. Second, a **Platonic $\mathbb{C}^*$-fibration** is a surface of the form $\mathbb{C}^2/G - \{0\}$, where G is a finite non-abelian subgroup of $GL_2(\mathbb{C})$ acting linearly on $\mathbb{C}^2$, and $\mathbb{C}^2/G$ denotes the quotient. If Y (a smooth surface which is not necessarily affine) has $\bar{\kappa}(Y) = -\infty$, then either (1) Y contains a cylinderlike open set, or (2) there exists a curve $\Gamma \subset Y$ such that $Y - \Gamma$ is isomorphic to the complement of a finite subset of a Platonic $\mathbb{C}^*$-fibration. If Y is affine, then case (1) holds. This classification is due largely to Miyanishi and Tsunoda [225, 226]. See also Iitaka [150] for further details about the logarithmic Kodaira dimension.

Now suppose that a nontrivial algebraic $\mathbb{C}^+$-action on $X = \mathbb{C}^3$ is given, and let Y denote the quotient $X/\!\!/\mathbb{C}^+$. As above, we conclude that Y is a normal affine surface, and that the set $Y' \subset Y$ of singular points of Y is finite.

Let $\pi : X \to Y$ be the quotient map, and let $X' = \pi^{-1}(Y')$. By the preceding lemma, X' is a union of finitely many curves. It is easy to show that there exists a coordinate plane $H \subset X$ which intersects X' in a finite number of points. If $H_0 = H - X'$ and $Y_0 = Y - Y'$, then π restricts to a dominant morphism of smooth surfaces $H_0 \to Y_0$. Since H_0 is the complement of a finite number of points in a plane, $\bar{\kappa}(H_0) = -\infty$. Because H_0 dominates Y_0, it follows that $\bar{\kappa}(Y_0) = -\infty$ as well.

Miyanishi next shows that, if Y' is non-empty, then Y is isomorphic to a quotient $\mathbb{C}^2/G$ for some nontrivial planar action of a finite group $G \subset GL_2(\mathbb{C})$. In particular, $Y' = \{0\}$. Let $X_0 = X - \pi^{-1}(0)$. The topological universal cover for $Y_0 = Y - \{0\}$ is $Z = \mathbb{C}^2 - \{0\}$, where the general fiber of the covering map $p : Z \to Y_0$ consists of $|G|$ points. Therefore, the restriction $\pi : X_0 \to Y_0$ factors through Z, i.e., $X_0 \to Z \to Y_0$. But this is impossible, since there exist open sets $U \subset X$ and $V \subset Y$ such that $U \cong V \times \mathbb{C}$, and $\pi : U \to V$ is the standard projection. In particular, the fiber of π over an element of $V \cap Y_0$ is a single line. Therefore, Y must be smooth.

Miyanishi's argument next uses the fact that a smooth affine surface Y over $\mathbb{C}$ with $\bar{\kappa}(Y) = -\infty$ contains a cylinderlike open set. This part of the argument constitutes a key ingredient in the Fujita-Miyanishi-Sugie proof of the Cancellation Theorem for surfaces, which is discussed in *Chap. 10* below.

In summary, Y is an affine surface over $\mathbb{C}$ such that $\mathcal{O}(Y)$ is a UFD, $\mathcal{O}(Y)^* = \mathbb{C}^*$, and Y contains a cylinderlike open set. (We only needed smoothness to get at this latter condition.) By the Miyanishi characterization of the plane, it follows that $Y \cong \mathbb{C}^2$. This characterization is stated and proved in *Thm. 9.9* below.

The reader is referred to the original articles [217, 287] for further details.

5.1.4 Proof of Miyanishi's Theorem: Positive Homogeneous Case

Suppose that D is a nonzero locally nilpotent derivation of $B = \mathbb{C}[x, y, z]$ which is homogeneous with respect to some positive system of weights on (x, y, z). The goal is to show that there exist homogeneous $f, g \in B$ such that $\ker D = \mathbb{C}[f, g]$.

Set $A = \ker D$, and suppose that $A \not\cong \mathbb{C}^{[2]}$. It follows that $\text{rank}(D) = 3$ (see *Thm. 4.11*). We will assume throughout that

$$0 < \deg x \leq \deg y \leq \deg z \quad \text{and} \quad \gcd(\deg x, \deg y, \deg z) = 1 \ .$$

Set $L = \mathbb{C}(x, y, z)$ and $K = \text{frac}(A)$. Extend D to $D_L \in \text{Der}_{\mathbb{C}}(L)$. Then $\ker D_L = K$ (*Cor. 1.23*). Let K^{hom} denote the set of nonzero homogeneous elements of K, i.e., elements u/v, where u and v are nonzero homogeneous elements of A. Then $\deg(u/v) = \deg u - \deg v$.

By hypothesis, we are given a faithful algebraic action of the group $\mathbb{G}_a \rtimes \mathbb{G}_m$ on B. Thus, the $\mathbb{G}_m$-action restricts to a nontrivial action on A, which in turn extends to a $\mathbb{G}_m$-action on K. The invariant ring $K_0 = K^{\mathbb{G}_m}$ is a subfield of K. Since $\text{tr.deg.}_{\mathbb{C}} K = 2$, it follows that $\text{tr.deg.}_{\mathbb{C}} K_0 \leq 1$. Choose $u, v \in A$ which are algebraically independent and homogeneous. Then $u^{\deg v} v^{-\deg u}$ belongs to K_0, but is not constant. Therefore, $\text{tr.deg.}_{\mathbb{C}} K_0 = 1$. Since $K_0 \subset \mathbb{C}(x, y, z)$, it follows by Lüroth's Theorem that $K_0 = \mathbb{C}(\zeta)$ for some $\zeta \in K_0$.

Write $\zeta = F/G$ for $F, G \in A$ homogeneous of the same degree $n > 0$. We may assume $\gcd(F, G) = 1$, since if $F = TF_1$ and $G = TG_1$ for $T, F_1, G_1 \in B$, then T, F_1 and G_1 are again homogeneous elements of A (recall that A is factorially closed). Let $h \in A$ be homogeneous and irreducible. Then $h^n/G^\eta \in K_0$, where $\eta = \deg h$. This implies that there exist standard-homomgeneous polynomials $\phi, \psi \in \mathbb{C}[X, Y] = \mathbb{C}^{[2]}$ such that $h^n/G^\eta = \phi(F, G)/\psi(F, G)$. There exist $\alpha_i, \beta_i \in \mathbb{C}$ such that

$$h^n \psi(F, G) = G^\eta \phi(F, G) = \prod_i (\alpha_i F + \beta_i G) \ .$$

Since h is irreducible, h divides $\alpha_i F + \beta_i G$ for at least one i. If h divides linearly independent $\alpha_i F + \beta_i G$ and $\alpha_j F + \beta_j G$, then h divides $\gcd(F, G) = 1$, a contradiction. Therefore, $h^n = (\alpha F + \beta G)^\ell$ for some $\alpha, \beta \in \mathbb{C}$ and $\ell \geq 1$. In particular, the only irreducible factor of $\alpha F + \beta G$ is h. We conclude that for each irreducible homogeneous $h \in A$,

$$h^c = \alpha F + \beta G \quad \text{for some} \quad c \geq 1 \ . \tag{5.1}$$

Choose an irreducible factor f of F. Then $f^a = \alpha F + \beta G$ for some $\alpha, \beta \in \mathbb{C}$ and $a \geq 1$. Since $\gcd(F, G) = 1$, it follows that $\beta = 0$, and we may assume $\alpha = 1$. Arguing similarly for G, we conclude that there exist irreducible $f, g \in A$ such that

$$F = f^a \text{ and } G = g^b \quad \text{for some} \quad a, b \geq 1 \ . \tag{5.2}$$

In addition, $\gcd(a,b)=1$. To see this, set $t=\gcd(a,b)$, and write $a=ta'$ and $b=tb'$ for positive integers a',b'. If $\zeta'=f^{a'}/g^{b'}\in K_0$, then $(\zeta')^t=\zeta$, which implies $t=1$.

Since $A\neq\mathbb{C}[f,g]$ and A is factorially closed, there exists an irreducible $h\in A$ not belonging to $\mathbb{C}[f,g]$. Combining (5.1) and (5.2), we conclude that $h^c=\alpha f^a+\beta g^b$ for some $\alpha,\beta\in\mathbb{C}^*$ and some $c>1$. We may rescale this equation to obtain

$$f^a+g^b+h^c=0\ . \tag{5.3}$$

Since $\mathbb{C}(h^c/g^b)=\mathbb{C}(\zeta+1)=\mathbb{C}(\zeta)$ and $\mathbb{C}(h^c/f^a)=\mathbb{C}(\zeta^{-1}+1)=\mathbb{C}(\zeta)$, the roles of f,g, and h are symmetric. Therefore, the integers a,b, and c are pairwise relatively prime.

Let $E\in\mathrm{LND}(B)$ be any partial derivative of B. By applying E to equation (5.3), we obtain a second equation, and the two equations can be written in matrix form.

$$\begin{pmatrix} f & g & h \\ aEf & bEg & cEh \end{pmatrix}\begin{pmatrix} f^{a-1} \\ g^{b-1} \\ h^{c-1}\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \tag{5.4}$$

If the rows of the first matrix were proportional, then $Ef\in fB$ would imply $f\in\ker E=\mathbb{C}^{[2]}$, and likewise $g,h\in\ker E$. But A is an algebraic extension of $\mathbb{C}[f,g,h]$, which would imply $A=\ker E$ and $\mathrm{rank}(D)=1$, a contradiction. Therefore, the two rows of this matrix are not proportional, and the system can be solved (over L):

$$\begin{pmatrix} f^{a-1} \\ g^{b-1} \\ h^{c-1}\end{pmatrix}=\frac{u_E}{v_E}\begin{pmatrix} cgEh-bhEg \\ ahEf-cfEh \\ bfEg-agEf\end{pmatrix}\ , \tag{5.5}$$

where $u_E,v_E\in B$ are relatively prime (and depend on E). This means u_E divides f^{a-1}, g^{b-1}, and h^{c-1}, which have no common factor, so we may assume $u_E=1$.

Set $\sigma=\deg f+\deg g+\deg h$. When $E=\partial_x$, the first equation of (5.5) yields the inequality

$$(a-1)\deg f\leq\deg g+\deg h-\deg x-\deg v_E\leq\deg g+\deg h-1\ ,$$

which implies $\deg f\leq\frac{\sigma-1}{a}$. Likewise, $\deg g\leq\frac{\sigma-1}{b}$ and $\deg h\leq\frac{\sigma-1}{c}$. By addition, we thus obtain

$$\sigma\leq\frac{\sigma-1}{a}+\frac{\sigma-1}{b}+\frac{\sigma-1}{c}=(\sigma-1)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ .$$

This implies

$$1<1+\frac{1}{\sigma-1}\leq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ . \tag{5.6}$$

Since a,b,c and are pairwise relatively prime, it is easy to conclude that either

$$1 \in \{a,b,c\} \quad \text{or} \quad \{a,b,c\} = \{2,3,5\} . \tag{5.7}$$

Let $p \in A$ be any other irreducible homogeneous element not belonging to $\mathbb{C}[f,g]$. The same reasoning used above shows that

$$p^5 = \lambda f^2 + \mu g^3 \quad \text{for some} \quad \lambda, \mu \in \mathbb{C} . \tag{5.8}$$

Combining (5.3) and (5.8) yields

$$p^5 = (\lambda - \mu) f^2 - \mu h^5 .$$

which by (5.7) is only possible if $\lambda = \mu$. Therefore, $p \in \mathbb{C}[f,g,h]$, and it follows that $A = \mathbb{C}[f,g,h]$. This means that if $a = 1$, then $A = \mathbb{C}[g,h]$, a contradiction, and likewise if $b = 1$ or $c = 1$.

Therefore, $\{a,b,c\} = \{2,3,5\}$, and we conclude:

$$A = \mathbb{C}[f,g,h] \cong \frac{\mathbb{C}[X,Y,Z]}{(X^2 + Y^3 + Z^5)} . \tag{5.9}$$

This ring is isomorphic to the coordinate ring of the quotient variety $\mathbb{C}^2/G$, where $G \subset SL_2(\mathbb{C})$ is a binary icosahedral group acting on the plane $\mathbb{C}^2$; see [140], Exercise 5.8. [1] The argument of Miyanishi now gives a contradiction: If $S = \mathrm{Spec}(A)$, then the quotient map $p : \mathbb{C}^2 \to S$ for the G-action is a finite morphism of order $|G|$; $0 \in S$ is the unique singular point; and the restriction $p : \mathbb{C}^2 - \{0\} \to S - \{0\}$ is the topological universal cover of $S - \{0\}$. It follows that, if $\pi : \mathbb{C}^3 \to S$ is the quotient morphism for the $\mathbb{G}_a$-action, then there exists a morphism $\rho : \mathbb{C}^3 \to \mathbb{C}^2$ such that $\pi = p\rho$. But this is impossible, since there exist open sets $U \subset X$ and $V \subset S$ such that $U \cong V \times \mathbb{C}$, and $\pi : U \to V$ is the standard projection. In particular, the fiber of π over an element of $V \cap S_0$ is a single line.

Specifically, it was the assumption that $A \neq \mathbb{C}^{[2]}$ which led to this contradiction. Therefore, $A = \mathbb{C}[P,Q]$ for some $P,Q \in B$. By *Prop. 3.28*, we may choose P,Q to be homogeneous. $\square$

5.1.5 The Type of a Standard Homogeneous Derivation

Observe that the theorem just proved does not claim $\deg f$ and $\deg g$ are relatively prime. Indeed, this may be false. For example, consider the grading of B given by $\deg x = 2$, $\deg y = 3$, and $\deg z = 4$, and let D denote the standard linear derivation $D = x\partial_y + y\partial_z$. Then D is homogeneous for this grading, and $\ker D = k[x,P]$ for $P = 2xz - y^2$, where $\deg x = 2$ and $\deg P = 6$.

There is, however, an important case in which the degrees of the generators are known to be coprime.

[1] At this point of his proof, Zurkowski continues with a purely algebraic proof involving several cases. By using Miyanishi's topological argument, the proof is much shorter.

Proposition 5.4. (Thm. 2.2 of [51]) *Suppose $D \in \mathrm{LND}(B)$ is homogeneous relative to some positive grading of $B = k[x,y,z]$, and $\ker D = k[f,g]$ for homogeneous f and g. If the integers $\deg x$, $\deg y$, and $\deg z$ are pairwise relatively prime, then $\deg f$ and $\deg g$ are also relatively prime.*

Note that this result applies to the standard homogeneous case. In this case, the pair $(\deg f, \deg g)$ is uniquely associated to D (up to order), giving rise to the following definition.

Definition 5.5. If $D \in \mathrm{LND}(B)$ is homogeneous in the standard grading of B, and $\ker D = k[f,g]$ for homogeneous polynomials f and g, then D is of **type** $(\deg f, \deg g)$, where $\deg f \leq \deg g$.

In particular, if type (e_1, e_2) occurs, then Daigle's result implies $\gcd(e_1, e_2) = 1$, though only certain relatively prime pairs of integers can occur. The set of pairs which actually occur is known, and is given in the remark which concludes the paper [60]. In this paper, Daigle and Russell give a complete classification of affine rulings of weighted projective planes, and such rulings are closely related to homogeneous locally nilpotent derivations of $k^{[3]}$. In particular, they associate to standard homogeneous D the two projective plane curves defined by its homogeneous kernel generators f and g.

5.2 Other Fundamental Theorems in Dimension Three

Let $f, g \in B$ be given. In keeping with the notation of *Chap. 3*, $\Delta_{(f,g)}$ will denote the jacobian derivation

$$\Delta_{(f,g)}h = \frac{\partial(f,g,h)}{\partial(x,y,z)} \ .$$

Note that we could also use gradient notation to write

$$\Delta_{(f,g)}(x,y,z) = \nabla_f \wedge \nabla_g \ .$$

If $D \in \mathrm{LND}(B)$ has $\ker D = k[f,g]$ for some $f, g \in B$, then we know from *Lemma 3.7* that $\Delta_{(f,g)}$ is locally nilpotent, $\ker \Delta_{(f,g)} = \ker D$, and there exist $a, b \in k[f,g]$ such that $a\Delta_{(f,g)} = bD$. In [49], Daigle proved the following stronger result.

Theorem 5.6. (Jacobian Formula) *Given $D \in \mathrm{LND}(B)$, $D \neq 0$, choose $f, g \in B$ such that $\ker D = k[f,g]$. Then $D = \lambda\Delta_{(f,g)}$ for some $\lambda \in \ker D$.*

Next, if $D \in \mathrm{LND}(B)$, let $\pi : \mathbb{A}^3 \to \mathbb{A}^2$ denote the quotient map, induced by the inclusion $\ker D \hookrightarrow B$. An important fact about π is due to Bonnet [27].

Theorem 5.7. (Bonnet's Theorem) *If k is a field of characteristic zero, and if $\pi : \mathbb{A}^3 \to \mathbb{A}^2$ is the quotient map associated to a nontrivial algebraic $\mathbb{G}_a$-action on $\mathbb{A}^3$, then π is surjective.*

Bonnet's proof is for $k = \mathbb{C}$, using a very nice topological argument. The general form of the theorem was then deduced by Daigle and Kaliman [58], where surjectivity for the general case refers to π as a map from Spec(B) to Spec$(\ker D)$. See Bonnet [27] for an example of a $\mathbb{G}_a$-action on $\mathbb{A}^4$ with quotient isomorphic to $\mathbb{A}^3$, but non-surjective quotient morphism.

In [160], Kaliman proved the following theorem for $k = \mathbb{C}$; the general case is deduced in [58].

Theorem 5.8. (Kaliman's Theorem) *Let k be a field of characteristic zero. Every free algebraic $\mathbb{G}_a$-action on $\mathbb{A}^3$ is a translation in a suitable polynomial coordinate system. Equivalently, if $B = k^{[3]}$ and $D \in \mathrm{LND}(B)$, and if $(DB) = B$ (the ideal generated by the image), then $Ds = 1$ for some $s \in B$.*

Special cases of this result were proved earlier in [54, 79, 175, 282]. In his paper, Kaliman also gives a proof of Bonnet's Theorem in a more general setting. It should be noted that in dimension higher than 3, there exist free algebraic $\mathbb{G}_a$-actions which are not conjugate to a translation (see *Example 3.9.4*).

Also contained in the work of Bonnet, Daigle, and Kaliman is the following (see [58], Thm. 1).

Theorem 5.9. (Principal Ideal Theorem) *Let k be a field of characteristic zero, $B = k^{[3]}$, $D \in \mathrm{LND}(B)$, and $A = \ker D$.*

(a) *B is faithfully flat as an A-module.*
(b) *The plinth ideal $A \cap DB$ of D is a principal ideal of A.*

In the language of *Chap. 2*, this implies that a locally nilpotent derivation of $k^{[3]}$ has a unique minimal local slice (up to equivalence), namely, we may choose $r \in B$ with $Dr = h$, where $A \cap DB = hA$. Then $B_h = A_h[r]$. Geometrically, this means that if $V \subset \mathbb{A}^2$ is the complement of the curve C defined by $h \in A$, and if $U \subset \mathbb{A}^3$ is the complement of the surface S defined by $h \in B$, then U is equivariantly isomorphic to $V \times \mathbb{A}^1$. In particular, the fiber of the quotient map $\pi : \mathbb{A}^3 \to \mathbb{A}^2$ lying over any point of V is a line (a single orbit, isomorphic to $\mathbb{G}_a$). Many of the remaining mysteries of the dimension three case thus lie hidden in the morphism $\pi : S \to C$.

An additional fact concerning locally nilpotent derivations in dimension three is the following. The proof of this result uses a more general fact, which is stated and proved in the *Appendix* section at the end of this chapter.

Theorem 5.10. (Intersection of Kernels) *Given nonzero $D, E \in \mathrm{LND}(B)$, exactly one of the following 3 statements is true.*

1. $\ker D \cap \ker E = k$
2. *There exist $f, g, h \in B$ such that $\ker D = k[f, g]$, $\ker E = k[f, h]$, and $\ker D \cap \ker E = k[f]$.*
3. $\ker D = \ker E$

Proof. Set $A_1 = \ker D$ and $A_2 = \ker E$, and assume $A_1 \cap A_2 \neq k$ and $A_1 \neq A_2$. We know that A_2 is factorially closed and isomorphic to $k^{[2]}$. Taking $S = A_2$ and $R = k$ in *Cor. 5.39* below, it follows that we can choose $f \in B$ such that $A_1 \cap A_2 = k[f]$ and $A_2 = k[f]^{[1]}$. By symmetry, $A_1 = k[f]^{[1]}$ as well. $\square$

Corollary 5.11. *Suppose $D, E \in \mathrm{LND}(B)$ are nonzero and have distinct kernels. If $\ker D = k[f]^{[1]}$ and $Ef = 0$, then $\ker E = k[f]^{[1]}$.*

Proof. By the preceding theorem, there exist $\tilde{f}, g, h \in B$ with $\ker D = k[\tilde{f}, g]$ and $\ker E = k[\tilde{f}, h]$. Since the kernels of D and E are distinct, $Eg \neq 0$. If $\tilde{f} = P(g)$ for $P \in k[f]^{[1]}$, then then $0 = E\tilde{f} = P'(g)Eg$, which implies $P'(g) = 0$, i.e., $\tilde{f} \in k[f]$. Therefore $\ker E = k[\tilde{f}, h] \subset k[f, h] \subset \ker E$, so $\ker E = k[f, h]$. $\square$

In calculating kernels, the following condition can be useful.

Proposition 5.12. (Kernel Criterion) *Suppose $a, b \in B = k[x, y, z]$ are such that $\Delta_{(a,b)}$ is locally nilpotent and nonzero. Then the following are equivalent.*

1. $k[a, b] = \ker \Delta_{(a,b)}$
2. $\Delta_{(a,b)}$ *is irreducible, and* $\ker \Delta_{(a,b)} \subset k(a, b)$.

Proof. The implication (1) implies (2) follows from *Prop. 3.18*. Conversely, assume (2) holds. By Miyanishi's Theorem, there exist $u, v \in B$ such that $\ker \Delta_{(a,b)} = k[u, v]$. It follows that

$$\Delta_{(a,b)} = \frac{\partial(a, b)}{\partial(u, v)} \cdot \Delta_{(u,v)} \ .$$

Since $\Delta_{(a,b)}$ is irreducible, $\frac{\partial(a,b)}{\partial(u,v)} \in k^*$, i.e., (a, b) is a Jacobian pair for $k[u, v]$. Since $k(a, b) = k(u, v)$, the inclusion $k[a, b] \hookrightarrow k[u, v]$ is birational. It is well known that the Jacobian Conjecture is true in the birational case, and we thus conclude $k[a, b] = k[u, v]$.[2] $\square$

Another result in dimension three is the following, which was proved by Wang in his thesis [302]; see also [303]. Because of the Principal Ideal Theorem, it is now possible to give a much shorter proof than that originally presented by Wang.

Theorem 5.13. (Wang's Theorem) *Let $B = k[x, y, z]$, and suppose $D \in \mathrm{LND}(B)$ is such that $D^2x = D^2y = D^2z = 0$. Then* $\mathrm{rank}(D) \leq 1$.

[2] The same reasoning yields yet another equivalent formulation of the two-dimensional Jacobian Conjecture: Given $a, b \in B$, if $\Delta_{(a,b)}$ is irreducible, then $\ker \Delta_{(a,b)} = k[a, b]$.

Proof. It suffices to assume D is irreducible. Let $A = \ker D$, and let $I \subset A$ be the plinth ideal $I = DB \cap A$ of D. Then by the Principal Ideal Theorem, I is a principal ideal of A, namely, $I = aA$ for some $a \in A$. Since $Dx, Dy, Dz \in I$ by hypothesis, a is a common divisor of these three elements, and therefore $DB \subset aB$. By irreducibility, $a \in B^*$. Therefore, $I = B$, and D has a slice s. It follows that $B = A[s]$, and since $A = k^{[2]}$, we conclude that the rank of D is 1. □

Recall that derivations with $D^2x = D^2y = D^2z = 0$ are called nice.

To conclude this section, we give the result of Kaliman and Saveliev in its full generality.

Theorem 5.14. [164] *Let X be a smooth contractible complex affine algebraic threefold with a nontrivial algebraic $\mathbb{C}^+$-action on it, and let $S = X/\!/\mathbb{C}^+$ be its algebraic quotient.*

(a) *X is rational, and S is a smooth contractible surface.*
(b) *If X admits a dominant morphism from a threefold of the form $C \times \mathbb{C}^2$, then $S = \mathbb{C}^2$.*
(c) *If the action is free, then it is equivariantly trivial.*

Note that if both conditions (b) and (c) are satisfied, then $X = \mathbb{C}^3$.

In [176], Kraft also considers $\mathbb{C}^+$-actions on a smooth contractible threefold X. He gives a proof for the smoothness of the quotient map $\pi : X \to S$ under certain additional conditions. Using this smoothness property, together with some topological considerations due to Kaliman, he gives a short proof that a free $\mathbb{C}^+$-action on X is a translation, under the additional assumption that the quotient S is smooth. This implies Kaliman's Theorem for $X = \mathbb{C}^3$.

Remark 5.15. There might be other classes of threefolds to which Miyanishi's Theorem could generalize. For example, $X = SL_2(k)$ is smooth and factorial, but when $k = \mathbb{C}$, it is not contractible. On the other hand, it seems likely that every nonzero $D \in \text{LND}(k[X])$ has $\ker D = k^{[2]}$. In working with affine 3-space, the existence of 3 independent locally nilpotent derivations (namely, the partial derivatives) is of central importance. Geometrically, these are translations in 3 independent directions. Likewise, $SL_2(k)$ has 4 fundamental actions: Realize $\mathbb{G}_a$ as the subgroup of $SL_2(k)$ consisting of upper (respectively, lower) triangular matrices with ones on the diagonal. Then both left and right muliplication by elements of $\mathbb{G}_a$ define $\mathbb{G}_a$-actions, specifically:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b+ta \\ c & d+tc \end{pmatrix}, \quad \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+tc & b+td \\ c & d \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} a+tb & b \\ c+td & d \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c+ta & d+tb \end{pmatrix}$$

These actions are conjugate to one another, fixed point free, and have $X/\!/\mathbb{G}_a = \mathbb{A}^2$; this quotient is calculated in *Chap. 6* below. However, in contrast to the situation for three-dimensional affine space, the quotient morphism $\pi : X \to \mathbb{A}^2$ is not surjective, since $\pi^{-1}(0)$ is an empty fiber. For example, the quotient map for the first of these actions is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to (a, c) \ .$$

On the other hand, notice that the geometric quotient does exist: $X/\mathbb{G}_a = \mathbb{A}^2 - \{(0,0)\}$. A conjectural generalization of Miyanishi's Theorem is given in *Chap. 9* below, where the class of threefolds used includes both $\mathbb{A}^3$ and $SL_2(k)$.

5.3 Questions of Triangularizability and Tameness

5.3.1 Triangularizability

The non-triangularizable $\mathbb{G}_a$-action of Bass was discussed already in *Chap. 3*. Specifically, it takes the form $\exp(tFD)$, where D is the basic linear derivation $D = x\partial_y + 2y\partial_z$ on $B = k[x, y, z]$, and $F = xz - y^2$. Subsequently, more general classes of non-triangularizable locally nilpotent derivations of $k^{[n]}$ for $n \geq 3$ appeared in [116, 253]. In [283], Snow speculated:

> Perhaps the fixed point set being 'cylindrical' is the only obstruction for a $\mathbb{C}$ action to be triangular. (p.169)

However, Example 4.3 of [54], showed that this is not the case: The authors construct $D \in \mathrm{LND}(B)$ which is non-triangularizable of rank two, and whose corresponding set of fixed points is a line. In particular, D is irreducible, making this example quite different than that of Bass. Ultimately, Daigle produced:

Theorem 5.16. (Cor. 3.4 of [48]) *Let $D \in \mathrm{LND}(B)$ be irreducible of rank 2, where $B = k[x, y, z]$ and $\ker D = k[x, P]$ for some $P \in B$. The following are equivalent.*

1. *D is triangularizable*
2. *D is triangularizable over $k[x]$*
3. *there exists a variable Q of B such that $k(x)[P, Q] = k(x)[y, z]$*

The reader will note that the difference between condition (3) of this theorem and the general rank-2 case is the requirement that Q is a k-variable of $k[x, y, z]$, rather than the weaker condition that Q is a $k(x)$-variable of $k(x)[y, z]$.

Example 5.17. (Example 3.5 of [48]) If $P = y + \frac{1}{4}(xz + y^2)$, define $D = \Delta_{(x,P)}$. Then D is an irreducible rank-two locally nilpotent derivation of B which is not triangularizable. Its fixed points are defined by the ideal $(DB) = (x, 1 +$

y^3), a union of three lines. Note that if T is the triangular derivation defined by $Tx = 0$, $Ty = x$, and $Tz = 1 + y^3$, then the fixed points of D and T agree in the strongest possible sense, in that they are defined by precisely the same ideals. See also van den Essen [100], 9.5.17.

It thus became evident that most elements of $\mathrm{LND}(B)$ are not triangularizable, and that one should focus on other aspects of the subject.

5.3.2 Tameness

As mentioned earlier, it is now known that there exist non-tame algebraic automorphisms of $B = k[x, y, z]$. In particular, it is known that the Nagata automorphism is not tame. Let $TA_3(k) \subset GA_3(k)$ denote the tame subgroup. As in *Chap. 3*, given $t \in k$, let $\alpha_t = \exp(tFD)$ denote Bass's $\mathbb{G}_a$-action on $\mathbb{A}^3$, where α_1 is the Nagata automorphism. Then $G \cap TA_3(k) = \{1\}$, where $G = \{\alpha_t | t \in k\}$.

On the other hand, any triangular derivation T has

$$\{\exp(tT) | t \in k\} \subset BA_3(k) \subset TA_3(k) \ .$$

The following question is open.

Is every *tame* $\mathbb{G}_a$-action on $\mathbb{A}^3$ conjugate to a triangular action?

In other words, given nonzero $E \in \mathrm{LND}(B)$, does $\exp E \in TA_3(k)$ imply that E is conjugate to a triangular derivation? By way of comparison, recall from *Chap. 3* that Bass's $\mathbb{G}_a$-action becomes tame, but not triangularizable, when extended to $\mathbb{A}^4$. We want to know whether this situation can also occur in dimension 3.

Example 5.18. (Example 1.13 of [115]) Set $\lambda = (y, z, x) \in GA_3(k)$, and define $\alpha, \beta \in BA_3(k)$ by

$$\alpha = (x, y, z + x^3 - y^2) \quad , \quad \beta = (x, y + x^2, z + x^3 + \tfrac{3}{2}xy) \ .$$

If $\gamma = \alpha\lambda\beta\lambda^{-1}$, then γ is tame, and

$$\gamma = (x + z^2, y + z^3 + \tfrac{3}{2}xz, z - y^2 + x(x^2 - 3yz) + \tfrac{1}{4}z^2(3x^2 - 8yz)) \ .$$

As an automorphism of $\mathbb{A}^3$, the fixed point set of γ is the cuspidal cubic defined by $z = 0$ and $x^3 - y^2 = 0$, which has an isolated singularity at the origin. By Popov's criterion, γ cannot be conjugated into the triangular subgroup $BA_3(k)$. However, using van den Essen's result (*Prop. 2.17*), it can also be shown that γ is not an exponential automorphism.

5.4 The Homogeneous $(2,5)$ Derivation

In [116], Question 2, we asked: Do there exist locally nilpotent derivations of the polynomial ring $k[x_1, ..., x_n]$ having maximal rank n? It was known at the time that, for $n = 1$, the answer is positive, and for $n = 2$ the answer is negative (by Rentschler's Theorem). But for $n \geq 3$ the answer was not known. In 1996, the author produced the following example of a locally nilpotent derivation on $B = k[x, y, z]$ having rank three; it appeared in [119].

Define polynomials

$$F = xz - y^2 \ , \ G = zF^2 + 2x^2yF + x^5 \ , \ R = x^3 + yF \ .$$

Define the k-derivation $\Delta : B \to B$ by $\Delta = \Delta_{(F,G)}$. Observe that Δ is irreducible, and homogeneous of degree 4 in the standard grading. It is easily checked that $\Delta^3 x = \Delta^7 y = \Delta^{11} z = 0$, and therefore Δ is locally nilpotent. In Bass's example we saw that the subring $k[x, F]$ is the kernel of the standard linear derivation of B. Therefore, by *Cor. 5.11*, we conclude that $\ker \Delta = k[F, g]$ for some homogeneous $g \in B$. In particular, $G \in k[F, g]$, and by considering degrees, we conclude that $\deg g$ is either 1 or 5. If $\deg g = 1$, then (by homogeneity) G is in the linear span $\langle g^5, g^3F, gF^2 \rangle$, which implies g divides G. Since G is irreducible, we conclude $\deg g = 5$, and that G is in the linear span of g itself. Therefore, $\ker \Delta = k[F, G]$. Δ is called the **homogeneous** $(2, 5)$ **derivation** of B.

Now suppose $h \in B$ is a variable of B, and $\Delta h = 0$. The linear part h_1 of h is nonzero, and by homogeneity, $\Delta h_1 = 0$ as well. But it is clear that $k[F, G]$ can contain no polynomial of degree 1, so $\ker \Delta$ does not contain a variable. In other words, the rank of Δ is 3. This implies that Δ is not triangularizable, since any triangularizable derivation annihilates a variable. We have thus proved:

Theorem 5.19. $\Delta \in \mathrm{LND}(B)$, $\ker \Delta = k[F, G]$, *and* $\mathrm{rank}\,(\Delta) = 3$.

The polynomial R is a minimal local slice of Δ, with $\Delta R = -FG$. Geometrically, this means that if $\pi : \mathbb{A}^3 \to \mathbb{A}^2$ is the quotient map for the corresponding $\mathbb{G}_a$-action, then π is an equivariant projection over the complement of the two lines of $\mathbb{A}^2$ defined by $FG = 0$. The fiber over the origin is the line of fixed points, defined by $x = y = 0$. Over points $(a, 0)$ for $a \neq 0$, the fiber consists of two lines (orbits) in the surface $G = 0$; and over $(0, b)$, the fiber consists of five lines (orbits) in the surface defined by $F = 0$.

As mentioned, Δ is not a triangularizable derivation. However, it may be viewed as the quotient of a triangular derivation in dimension six. First, observe the relation $F^3 + R^2 = xG$. Applying Δ yields $2R(-FG) = G\Delta x$, and thus $\Delta x = -2FR$. Likewise, applying Δ to the equation $R = x^3 + yF$ gives $-FG = 3x^2(-2FR) + F\Delta y$, so $\Delta y = 6x^2R - G$. And Δz is gotten from $0 = \Delta F = z\Delta x + x\Delta z - 2y\Delta y$. In summary:

$$\Delta x = -2FR \ , \ \Delta y = 6x^2R - G \ , \ \Delta z = 2x(5yR + F^2) \ .$$

Theorem 5.20. *Define a triangular derivation T on $R = k[u, v, w, x, y, z] = k^{[6]}$ by*

$$Tu = Tv = 0\,,\ Tw = -uv\,,\ Tx = -2uw\,,\ Ty = 6wx^2 - v\,,\ Tz = 2x(u^2 + 5yw)\ .$$

Define the ideal $I = (u - F, v - G, w - R)$. Then $TI \subset I$; $R \bmod I \cong k^{[3]}$; and $\Delta = T \bmod I$.

Proof. The latter two conclusions are obvious once it is shown that $TI \subset I$. To show this, define another derivation D of R by

$$D = \frac{\partial(u, v, f, g, h, \cdot)}{\partial(u, v, w, x, y, z)}\ ,$$

where

$$f = u - (xz - y^2);\ \ g = v - (u^2z + 2ux^2y + x^5);\ \ h = w - (x^3 + uy)\ .$$

Since $f, g, h \in \ker D$, $DI \subset I$, and it is clear that $D \bmod I = \Delta$ on $R \bmod I \cong B$. (But D is not *a priori* locally nilpotent.) Direct calculation shows that, modulo I,

$$Dw \equiv Tw\,,\ Dx \equiv Tx\,,\ Dy \equiv Ty\,,\ Dz \equiv Tz\ .$$

Thus, $0 = Df = Tf + \kappa$ for some $\kappa \in I$, implying $Tf \in I$. Likewise, $Tg, Th \in I$, so $TI \subset I$. $\square$

Note that, since $T(w - (x^3 + uy)) = 0$, the rank of T is 3. Geometrically, this result means that the triangular $\mathbb{G}_a$-action on $\mathbb{A}^6$ defined by T restricts to a $\mathbb{G}_a$-action on the coordinate threefold $X \subset \mathbb{A}^6$ defined by I, and this action is equivalent to Δ on $\mathbb{A}^3$.

We also have:

Theorem 5.21. [119] *Let Δ be the homogenous $(2,5)$ derivation of $B = k[x_1, x_2, x_3]$. Given $n \geq 4$, extend Δ to Δ° on $B[x_4, ..., x_n] = k^{[n]}$ by setting $\Delta^\circ x_i = x_{i-1}^5$, $4 \leq i \leq n$. Then Δ° is homogeneous and locally nilpotent of rank n.*

Remark 5.22. Consider the rank-4 derivation Δ° defined on $k[x_1, x_2, x_3, x_4]$, as above. At this writing, it is not known what $\ker \Delta^\circ$ is, or even whether this kernel is finitely generated. Existing algorithms are inconclusive, due to the size of calculations involved. Clearly, a method other than *brute force* is needed.

5.5 Local Slice Constructions

In an effort to understand and generalize the $(2,5)$ example above, the author defined local slice constructions in [118], which brought into view large new families of rank-3 elements of LND(B). Also working just after the appearance of the $(2,5)$ example, Daigle used a geometric approach, quite different from the method of local slice constructions, to find additional rank-3 examples. At the time, it appeared that the two methods produced the same examples.

The present section describes local slice constructions. The following section discusses some of the geometric theory developed by Daigle and Russell, and its connection to local slice constructions.

5.5.1 Definition and Main Facts

Given $D \in \mathrm{LND}(B)$, consider the following condition.

There exist $f, g, r \in B$ and $P \in k^{[1]}$ such that

$$(*) \qquad \ker D = k[f,g] \text{ and } Dr = g \cdot P(f) \neq 0.$$

Note that this condition does not depend on any particular system of coordinates for B. To date, we know of no nonzero $D \in \mathrm{LND}(B)$ which fails to satisfy $(*)$.

Lemma 5.23. *Assume D satisfies $(*)$, and set $S = k[f] - 0$. Then for any local slice $r' \in B$ of D,*

$$D(r'/g) \in S \quad \Leftrightarrow \quad S^{-1}A[r'] = S^{-1}A[r] \ .$$

Proof. If $D(r'/g)$ belongs to S, then $rDr' - r'Dr \in A$. Thus, for some nonzero $a, b \in k[f]$, $gar' - gbr \in A$. Since A is factorially closed, $ar' - br \in A$. Therefore, $S^{-1}A[r'] = S^{-1}A[r]$.

Conversely, suppose $S^{-1}A[r'] = S^{-1}A[r]$ for some $r' \in B$. Then $r' = cr + d$ for $c \in (S^{-1}A)^* = k(f)^*$ and $d \in S^{-1}A$. Thus,

$$D(r'/g) = cD(r/g) \in k(f)^* \cap (1/g)B = S \ .$$

□

Now assume that D is irreducible, and D satisfies $(*)$ for some local slice r not belonging to gB. Since D is irreducible, we may assume $D = \Delta_{(f,g)}$ (Jacobian Formula). Let $\bar{B}$ denote the domain B/gB, and let $\bar{D} = D \bmod gB \in \mathrm{LND}(\bar{B})$, noting that $\bar{D}r = 0$. Since $\ker \bar{D}$ is the algebraic closure of $k[f]$ in $\bar{B}$, we conclude that there exists $\phi \in k[f]^{[1]}$ such that $\phi(r) \in gB$. If we choose ϕ to be of minimal r-degree, such that $\phi(r)$ is irreducible in $k[f, r]$, then ϕ is unique up to nonzero constant multiples. Suppose $h = g^{-1}\phi(r) \in B$.

Theorem 5.24. (Thm. 2 of [118]) *In the above notation,*

(a) $\Delta_{(f,h)} \in \mathrm{LND}(B)$
(b) $\Delta_{(f,h)}r = -h \cdot P(f)$
(c) *If $\Delta_{(f,h)}$ is irreducible, then* $\ker \Delta_{(f,h)} = k[f,h]$

Proof. Let $\delta = \Delta_{(f,h)}$. Since $\Delta_{(f,gh)} = g \cdot \Delta_{(f,h)} + h \cdot \Delta_{(f,g)}$, it follows that

$$g \cdot \delta = \Delta_{(f,\phi(r))} - h \cdot D = \phi'(r) \cdot \Delta_{(f,r)} - h \cdot D \ .$$

Therefore, $g \cdot \delta r = -h \cdot Dr$, which implies $\delta r = -h \cdot (Dr/g) = -hP(f)$. So (b) is proved.

Since $\delta r \neq 0$, r is transcendental over $K = k(f,h)$, i.e., $K[r] \cong K^{[1]}$. Since $g = \phi(r)/h$, we have $g \in K[r]$; and since $k[f] \cap gB = \{0\}$, $\deg_r g \geq 1$ and $g \notin K[r]^*$.

We claim that g is irreducible in $K[r]$. Since $gh = \phi(f,r)$, it suffices to show that ϕ is irreducible in $K[r]$. However, ϕ was chosen to be irreducible in $k[f,r] \cong k^{[2]}$, hence it is also irreducible in $k[f,h,r] \cong k^{[3]}$. Since ϕ is not in K, it follows easily that ϕ is irreducible in $K[r]$. Consequently, g is also irreducible in $K[r]$.

It follows that $g \cdot K[r]$ is a maximal ideal of $K[r]$, and thus

$$g \cdot K[x,y,z] \cap K[r] = g \cdot K[r] \ .$$

Set $T = \{g^n \cdot a(f) | n \geq 0,\ a \in k[f] - 0\}$, and let $A = \ker D = k[f,g]$. Then $T^{-1}A[r] = T^{-1}B$. Given $b \in B$, choose n so that $g^n b \in k(f)[g,r] \subset K[r]$. Then, using the above ideal equality inductively, we obtain $b \in K[r]$. Therefore, $B \subset K[r]$. Since δ is locally nilpotent on $K[r]$, part (a) is proved.

To prove (c), suppose δ is irreducible. Since $\ker D = k[f,g]$ and $\delta f = 0$, *Cor. 5.11* implies that $\ker \delta = k[f,\eta]$ for some $\eta \in B$. If $h = p(\eta)$ for $p \in k[f]^{[1]}$, then $\delta = p'(\eta)\Delta_{(f,\eta)}$. Since δ is irreducible, $p'(\eta) \in k^*$, implying that $h = a\eta + b(f)$ for some $a \in k^*$ and $b \in k[f]$. But then $k[f,h] = k[f,\eta]$. $\square$

The procedure by which $\Delta_{(f,h)}$ is obtained from D is called a **local slice construction**. Specifically, we say $\Delta_{(f,h)}$ is obtained by local slice construction from the data $(f,g,r) \in B^3$. An important observation is that local slice constructions do not require any homogeneity conditions.

Note that when $\Delta_{(f,h)}$ is obtained from $\Delta_{(f,g)}$ using data (f,g,r), then $\Delta_{(f,g)}$ is obtained from $\Delta_{(f,h)}$ using data (f,h,r). To continue the process inductively, we may, by the lemma above, replace r with any r' for which $S^{-1}A[r'] = S^{-1}A[r]$.

It may also happen that the original derivation D admits a local slice r such that $Dr = fg$. Then $\Delta_{(f,h)}r = -fh$. Thus, to continue the process inductively, we may use data (h,f,r) instead of (f,h,r).

Example 5.25. Let D denote the standard linear derivation of $B = k[x,y,z]$:

$$D = x\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial z}$$

If $F, G, R \in B$ are defined as before, then the $(2,5)$-derivation Δ is obtained from D by a local slice construction with data (F,x,R). In particular, $\ker D = k[x,F]$ and $\ker \Delta = k[F,G]$, so F is a common kernel generator; $DR = xF$ and $\Delta R = -FG$, so R is a common local slice; and the algebraic relation between these four polynomials is $F^3 + R^2 = xG$.

5.5.2 Examples of Fibonacci Type

Using local slice constructions, we describe a sequence of homogeneous locally nilpotent derivations of B which plays a central role in the classification of the standard homogeneous elements of $\mathrm{LND}(B)$. In keeping with the notation of [118], define polynomials $F = xz - y^2$ and $r = -R = -(x^3 + yF)$; and inductively define H_n by[3]

$$H_0 = -y \ , \ H_1 = x \ , \ H_2 = F \ , \ H_{n-1}H_{n+1} = H_n^3 + r^{a_n} \ ,$$

where $a_n = \deg H_n$. The fact that $H_n \in B$ for all n was shown in [118]. The sequence of degrees a_n is given by every other element of the Fibonacci sequence, namely, $a_{n+1} = 3a_n - a_{n-1}$.

Define a sequence δ_n of derivations of B by $\delta_n := \Delta_{(H_n, H_{n+1})}$.

Theorem 5.26. (Sect. 4.2 of [118]) *For each $n \geq 1$,*

1. *δ_n is irreducible, locally nilpotent, and homogeneous*
2. $\ker \delta_n = k[H_n, H_{n+1}]$*, and δ_n is of type* (a_n, a_{n+1})
3. *$\delta_n r = -H_n H_{n+1}$*
4. *δ_{n+1} is obtained from δ_n by a local slice construction, using the data (H_n, H_{n-1}, r)*

For example, the partial derivative ∂_z equals δ_0, with kernel $k[x,y] = k[H_0, H_1]$; the standard linear derivation is δ_1, with kernel $k[x,F] = k[H_1, H_2]$; and the homogeneous $(2,5)$ derivation is δ_2, with kernel $k[F,G] = k[H_2, H_3]$.

5.5.3 Type $(2, 4m+1)$

Starting with any δ_n, there is a large derived family of standard homogeneous derivations D having $\ker D = k[H_n]^{[1]}$. To illustrate, start with the $(1,2)$ example δ_1. For $m \geq 1$, set $r_m = x^{2m+1} + F^m y$, which is a homogeneous local slice of δ_1. Since $\delta_1 r_m = xF^m$, we can carry out a local slice construction with the data (F, x, r_m) to obtain a homogeneous locally nilpotent derivation with kernel $k[F, G_m]$, where G_m is homogeneous of degree $4m+1$, namely, $G_m = zF^{2m} + 2x^{2m}F^m y + x^{4m+1}$.

5.5.4 Triangular Derivations

Let T denote any triangular derivation of $B = k[x,y,z]$. Then $\ker T = k[x,P]$ for $P \in B$ of the form $P = a(x)z + b(x,y)$. The polynomial $r = yP$ is a local slice of the partial derivative $\delta_0 = \partial_z$, with $\delta_0 r = a(x)y$. The derivation $T = \Delta_{(x,P)}$ is gotten by local slice construction using the data (x, y, r). In other words:

[3] The definition of H_1 was inadvertently omitted from the final printing of the original article [118].

> The set of derivations obtained from a partial derivative by a single local slice construction is precisely the set of all triangular derivations of B.

Of course, this statement only makes sense in the context of a fixed coordinate system.

5.5.5 Rank Two Derivations

Proposition 5.27. *Every irreducible locally nilpotent derivation of B of rank at most two can be transformed to a partial derivative by a sequence of local slice constructions.*

Proof. Let $D \in \mathrm{LND}(B)$ be irreducible, with $\mathrm{rank}(D) \leq 2$, and suppose $Dx = 0$. Set $K = k(x)$. By *Thm. 4.13*, there exist $P, Q \in B$ such that $K[P,Q] = K[y,z]$, $\ker D = k[x,P]$, and $DQ \in k[x]$. Moreover, the ideal generated by the image of D is (P_y, P_z), and if $(P_y, P_z) = (1)$, then Q may be chosen so that $k[x,P,Q] = B$ and $D = \partial/\partial Q$ (*Thm. 4.16*).

We proceed by induction on $\deg_K P$.

Consider first the case $\deg_K P = 1$. If $P = ay + bz$ for $a, b \in k[x]$, then $(P_y, P_z) = (a,b)$. Since (a,b) is principal, and since D is irreducible, we conclude that $(a,b) = (1)$. Therefore, D is already a partial derivative in this case, as in the preceding paragraph.

Assume $\deg_K P > 1$. If $\deg_K Q \geq \deg_K P$, then the structure theory for $GA_2(K)$ implies that there exists $Q' \in K[y,z]$ such that $K[P,Q'] = K[y,z]$ and $\deg_K P > \deg_K Q'$; see *Subsect. 4.1.3* above. Moreover, since $K[P,Q] = K[P,Q']$, we must have $\gamma Q' = \alpha Q + \beta(P)$ for some nonzero $\alpha, \gamma \in k[x]$ and some $\beta \in k[x,P]$. Thus, $\gamma DQ' = \alpha DQ \in k[x]$, which implies $DQ' \in k[x]$. So it is no loss of generality to assume $\deg_K P > \deg_K Q$. (Recall that $D = \Delta_{(x,P)}$ up to multiplication by elements of k^*, and we are therefore free to replace Q by Q' in the argument, since doing so does not affect the definition of D.) In addition, if QB is not a prime ideal of B, there exists $\ell \in k[x]$ dividing Q such that $(Q/\ell)B$ is prime. This is because Q is a K-variable. So it is no loss of generality to further assume Q is irreducible in B.

Observe that D satisfies condition $(*)$ in *Subsect. 5.5.1*, since $r := PQ$ has $Dr = P \cdot DQ$ and $DQ \in k[x]$. Consider $D' := \Delta_{(x,Q)}$. By *Thm. 5.24*, D' is again locally nilpotent, and since $D'x = 0$, it is of rank at most two. Since Q is both irreducible and a K-variable, it follows that D' is irreducible. Therefore, $\ker D' = k[x,Q]$. Since $\deg_K Q < \deg_K P$, we may (by induction) assume that D' can be transformed into a partial derivative by a finite sequence of local slice constructions. Since D is obtained from D' by a single local slice construction, we conclude that D can be transformed into a partial derivative by a finite sequence of local slice constructions. □

5.6 The Homogeneous Case

In this section, we consider derivations of $B = k[x, y, z]$ which are homogeneous relative to a system of positive integral weights $\omega = (\omega_1, \omega_2, \omega_3)$ on B. The standard grading is $\omega = (1, 1, 1)$. The subset of all homogeneous elements of $\mathrm{LND}(B)$ relative to ω will be denoted $\mathrm{LND}_\omega(B)$. Recall that if $D \in \mathrm{LND}_\omega(B)$ and $D \neq 0$, then $\ker D = k[f, g]$ for homogeneous f and g. (That two generators suffice is Miyanishi's Theorem; that f and g can be chosen to be homogeneous follows from *Prop. 3.28.*)

Given ω, $\mathbb{P}^2_\omega$ will denote the **weighted projective plane** $\mathrm{Proj}(B)$ over the algebraic closure of k, and C_f, C_g will denote the projective curves defined by f and g. In case $\omega = (1, 1, 1)$, $\mathbb{P}^2$ will denote standard projective plane.

As we have seen, $\mathrm{LND}_\omega(B)$ is a large and interesting class of derivations, even for standard weights, and one would like to classify them in some meaningful way. For example:

> Can every positive-homogeneous locally nilpotent derivation of B be obtained from a partial derivative via a finite sequence of local slice constructions?

Shortly after the appearance of the $(2, 5)$ example, Daigle translated the problem of understanding homogeneous derivations into geometric language (Thm. 3.5 of [50]).[4]

Theorem 5.28. (Two Lines Theorem) *Suppose $\omega = (\omega_1, \omega_2, \omega_3)$ is a system of positive weights on B such that $\gcd(\omega_1, \omega_2, \omega_3) = 1$, and suppose $f, g \in B$ are homogeneous relative to ω, with $\gcd(\deg f, \deg g) = 1$. The following are equivalent.*

1. *There exists $D \in \mathrm{LND}_\omega(B)$ with $\ker D = k[f, g]$.*
2. *f and g are irreducible, and $\mathbb{P}^2_\omega - (C_f \cup C_g)$ is a surface which is isomorphic to the complement of two lines in $\mathbb{P}^2$.*

In order to prove this, Daigle shows that the two conditions are each equivalent to a third, namely:

3. $B_{(fg)} = (A_{(fg)})^{[1]}$, where $A = k[f, g]$.

The subscript (fg) here denotes homogeneous localization. Specifically, since f and g are homogeneous, B_{fg} is a graded ring, and $B_{(fg)}$ is the degree-zero component of B_{fg}. A key fact in showing the equivalence is that, when (3) holds, $A_{(fg)}$ is equal to the Laurent polynomial ring $k[t, t^{-1}]$, where $t = f^{\deg g}/g^{\deg f}$.

In some sense, the Two Lines Theorem replaces the problem of describing all ω-homogeneous elements of $\mathrm{LND}(k^{[3]})$ by the following problem, which belongs to the theory of algebraic surfaces:

[4] The case $\gcd(\deg f, \deg g) \neq 1$ is also described in this paper.

Find all pairs of curves C_1, C_2 in $\mathbb{P}^2_\omega$ such that the complement of $C_1 \cup C_2$ is isomorphic to the complement of two lines in $\mathbb{P}^2$.

If $C_1, C_2 \subset \mathbb{P}^2_\omega$ is such a pair of curves, the isomorphism from the complement of two lines in $\mathbb{P}^2$ to $\mathbb{P}^2_\omega - (C_1 \cup C_2)$ extends to a birational isomorphism $\sigma : \mathbb{P}^2 \to \mathbb{P}^2_\omega$, and this σ can be factored into a finite succession of blow-ups and blow-downs.

In order to illustrate this idea, consider the locally nilpotent derivations δ_0, δ_1, and δ_2 defined in *5.5.2*. These derivations are of Fibonacci Type, with degree type $(1,1)$, $(1,2)$, and $(2,5)$, respectively. In this case, the grading is the standard one, so $\omega = (1,1,1)$ and $\mathbb{P}^2_\omega = \mathbb{P}^2$. We have $\ker \delta_0 = k[x,y]$, and the two projective curves C_x, C_y are already lines. Likewise, $\ker \delta_1 = k[x,F]$ as above, and the complements of $C_x \cup C_y$ and $C_x \cup C_F$ in $\mathbb{P}^2$ are isomorphic. An explicit birational isomorphism of π_1 of $\mathbb{P}^2$ is given in *Fig. 5.1*, where the arrows ($\downarrow$) denote a blowing-down along a curve of self-intersection (-1) (so the inverse is a blowing up at the indicated point).

The numerical labels indicate the self-intersection number of the labeled curve, and the surfaces $\mathbb{F}_n$ are the **Hirzebruch surfaces** ($n \geq 1$); see [312] for a discussion of these surfaces.

Likewise, if $\ker \delta_2 = k[F,G]$ as above, then the complements of $C_x \cup C_F$ and $C_F \cup C_G$ in $\mathbb{P}^2$ are isomorphic, and *Fig. 5.2* illustrates the explicit isomorphism π_2. Note that C_F is smooth; C_G has a cusp; and the two curves intersect tangentially at this point. Note also that π_2 collapses C_F to a point, and maps C_x to C_G.

A significant portion of this theory can be translated into a problem of combinatorics, using for example weighted dual trees. In this way, Daigle and Russell eventually proved that finding all curves C_f and C_g which satisfy condition (2) of the Two Lines Theorem is equivalent to finding all affine rulings of the weighted projective plane $\mathbb{P}^2_\omega$ [59]; and then in [60], they give a complete description of all affine rulings of $\mathbb{P}^2_\omega$. Following is their definition of affine ruling, as found in [59].

Definition 5.29. Let X be a complete normal rational surface, and let Λ be a one-dimensional linear system on X without fixed components. Then Λ is an **affine ruling** of X if there exist nonempty open subsets $U \subset X$ and $\Gamma \subset \mathbb{P}^1$ such that $U \cong \Gamma \times \mathbb{A}^1$ and such that the projection morphism $\Gamma \times \mathbb{A}^1 \to \Gamma$ determines Λ.

Daigle writes: “There is a rich interplay between the theory of algebraic surfaces and homogeneous locally nilpotent derivations of $k[x,y,z]$” ([47], p. 35). Indeed, the results of Daigle and Russell are of broad significance in the study of algebraic surfaces, and the impressive geometric machinery and theory they develop has implications far beyond the study of $\mathbb{G}_a$-actions on $\mathbb{A}^3$. Their work effectively provides a complete classification of the positive homogeneous locally nilpotent derivations of B, where the local slice construction in the algebraic theory corresponds to a certain kind of birational modification of surfaces

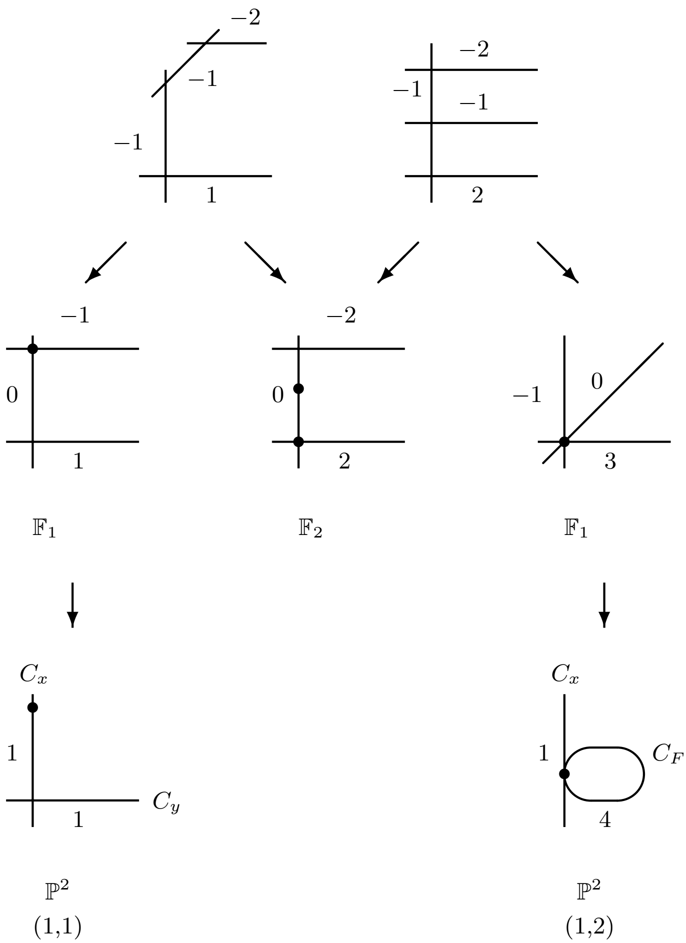

Fig. 5.1. Birational Map π_1 of $\mathbb{P}^2$

in the geometric theory. In particular, their work implies an affirmative answer to the question asked above, namely, whether every positive-homogeneous locally nilpotent derivation of $k^{[3]}$ can be obtained from a partial derivative by a finite sequence of local slice constructions.

Theorem 5.30. [46] *If ω is a positive system of weights on B, and if $D, E \in \mathrm{LND}_\omega(B)$ are irreducible, then D can be transformed to E via a finite sequence of local slice constructions.*

In the case of standard weights, the derivations of Fibonacci Type play a central role.

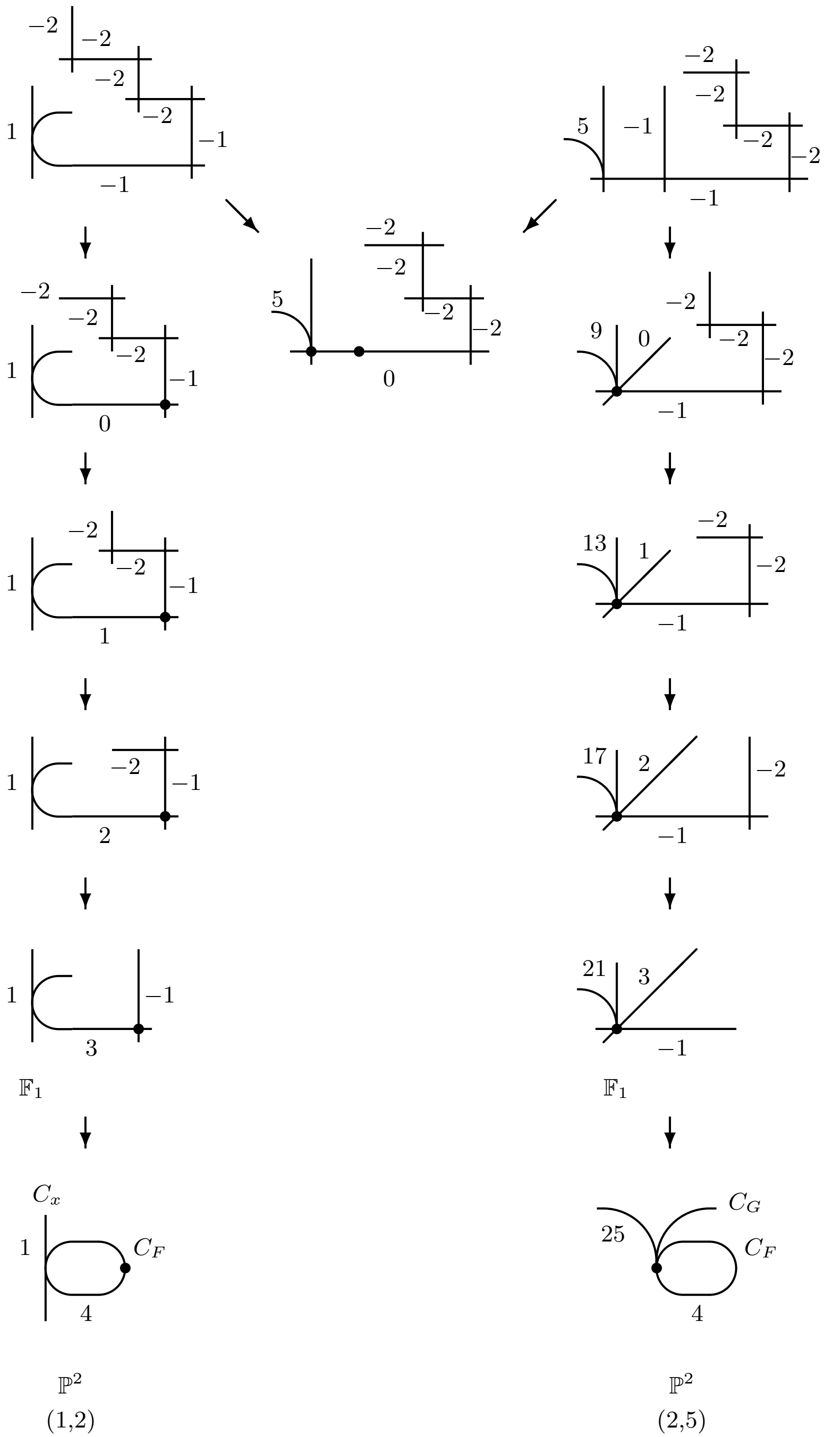

Fig. 5.2. Birational Map π_2 of $\mathbb{P}^2$

Theorem 5.31. *If $D \in \mathrm{LND}_\omega(B)$ for standard weights $\omega = (1,1,1)$, and $D \neq 0$, then, up to change of coordinates,* $\ker D = k[H_n]^{[1]}$ *for one of the polynomials H_n defined above.*

These two results are valid over an algebraically closed field k of characteristic zero. The second result, while unpublished, is due to Daigle, and can be proved using the results of [60, 61].

Remark 5.32. It is not surprising that some of the projective plane curves encountered here in the context of locally nilpotent derivations appeared in earlier work on classification of curves. For example, the quintic curve G used in the $(2,5)$ example was studied by Yoshihara [313]. Yoshihara's example motivated the work of Miyanishi and Sugie in [223], who studied reduced plane curves D whose complement $\mathbb{P}^2 - D$ has logarithmic Kodaira dimension $-\infty$. They remark: "So far, we have only one example for D of the second kind. That is, a quintic rational curve with only one cusp of multiplicity 2 which was obtained by H. Yoshihara." In his review of this paper (MR 82k:14013), Gizatulin asserts the existence of a family of curves C_i "of the second kind" whose degrees are the Fibonacci numbers 1,2,5,13,... . In particular, C_3 is Yoshihara's quintic. It appears, however, that Gizatulin never published the details of his examples.

5.7 Graph of Kernels and Generalized Local Slice Constructions

As we have seen, the study of $\mathrm{LND}_\omega(B)$ can be reduced to a problem in dimension two, where the tools of surface theory can be applied. What about the general case? As mentioned, local slice constructions do not require any kind of homogeneity, thus providing a point of departure for investigating the full set $\mathrm{LND}(B)$.

We are interested in subrings A of B which occur as the kernel of *some* $D \in \mathrm{LND}(B)$, rather than in the specific derivation D of which A is the kernel. With this in mind, we will say that $D \in \mathrm{LND}(B)$ is a **realization** of its kernel. In [118], Sect. 5, we define the graph Γ, where

$$\mathrm{vert}(\Gamma) = \{\ker D | D \in \mathrm{LND}(B), D \neq 0\}$$

and where two distinct vertices $\ker D$ and $\ker D'$ are joined by an edge if and only if D' can be obtained from D by a single local slice construction. Subsequently, Daigle in [53] generalized the graph Γ to a graph $\underline{\mathrm{KLND}}(\mathcal{B})$ defined for any integral domain $\mathcal{B}$ of characteristic zero, by first distinguishing in $\mathcal{B}$ certain subrings of codimension 2. The graph he defines is an invariant of the ring $\mathcal{B}$, and the group of automorphisms of $\mathcal{B}$ acts on it in a natural way. In case $\mathcal{B} = B = k^{[3]}$, Daigle's definition holds that neighboring vertices in

$\underline{\text{KLND}}(B)$ admit both a common kernel generator and a common local slice, and it turns out that $\Gamma = \underline{\text{KLND}}(B)$ in this case.

This graph is related to Daigle's generalization of the local slice construction. According to Daigle:

> This generalization produces new insight into the local slice construction. In particular, we find that that process is essentially a two-dimensional affair and that it is intimately related to Danielewski surfaces. ([53], p.1)

Here, a surface defined by a polynomial of the form $xz - \phi(z) \in k[x, y, z]$ is called a **special Danielewski surface** over k; these will be discussed in *Chap. 9* below. (Daigle refers to these simply as Danielewski surfaces.) If R is the coordinate ring of a special Danielewski surface, then any triple $(x, y, z) \in R^3$ such that $R = k[x, y, z]$ and $xy \in k[z] - k$ is called a **coordinate system** of R. If $R \subset \mathcal{B}$ for some commutative k-domain $\mathcal{B}$, then $\mathcal{B}_R$ denotes localization of $\mathcal{B}$ at the nonzero elements of R.

Now suppose $\mathcal{B}$ is a k-affine UFD. Suppose there exists an element $w \in \mathcal{B}$ and subrings $R \subset A \subset \mathcal{B}$ which satisfy the following two conditions.

(i) $A = \ker D$ for some irreducible $D \in \text{LND}(\mathcal{B})$
(ii) $A_R = K[Dw] = K^{[1]}$, where $K = R_R = \text{frac}(R)$

Proposition 5.33. (Prop. 9.12.1 of [47]) *In the above notation, $\mathcal{B}_R$ is a special Danielewski surface over K, and there exists $\tilde{v} \in \mathcal{B}$ such that $(Dw, \tilde{v}, w)$ is a coordinate system of $\mathcal{B}_R$. Moreover, for any pair $u, v \in \mathcal{B}$ such that $A_R = K[u]$ and (u, v, w) is a coordinate system of $\mathcal{B}_R$, the ring $A' = K[v] \cap \mathcal{B}$ is the kernel of a locally nilpotent derivation of $\mathcal{B}$.*

In this case, we say that A' is obtained from the triple (A, R, w) by a local slice construction. When $\mathcal{B} = B = k^{[3]}$, this procedure is equivalent to the local slice construction as originally defined.

Any subring R of $\mathcal{B}$ satisfying conditions (i) and (ii) above for some A and w will be called a **Daigle subring** of $\mathcal{B}$.

Example 5.34. (Ex. 9.14 of [47]) For the ring $\mathcal{B} = k[u, v, x, y, z] = k^{[5]}$, define elements

$$s = vx - uy \ , \ t = uz - x(s + 1) \ , \quad \text{and} \quad w = xt \ ,$$

and define subrings

$$A = \ker \partial_z = k[u, v, x, y] \quad \text{and} \quad R = k[u, v, s] \ .$$

Then $\partial_z w = ux$. Set $K = \text{frac}(R) = k(u, v, s)$. Then $A_R = K[x] = K[\partial_z w]$, so the triple (A, R, w) satisfies conditions (i) and (ii) above. Therefore, $\mathcal{B}_R$ is a Danielewski surface over K. In fact, $\mathcal{B}_R = K[x, z] = K[x, t] = K^{[2]}$. Therefore, the triple (x, t, w) is a coordinate system of $\mathcal{B}_R$. By the proposition, the ring $A' = K[t] \cap \mathcal{B}$ is the kernel of some $D \in \text{LND}(\mathcal{B})$. In particular,

$$Du = Dv = 0 \ , \ Dx = u \ , \ Dy = v \ , \ \text{and } Dz = 1 + s \ .$$

This is precisely the derivation of Winkelmann discussed earlier in *Example 3.9.5*, where the kernel is given explicitly.

The section concludes with the following natural question.

> Can every irreducible locally nilpotent derivation of $B = k[x,y,z]$ be obtained from a partial derivative via a finite sequence of local slice constructions? Equivalently, does every connected component of Γ contain a vertex which is the kernel of a partial derivative of B?

5.8 $\mathbb{G}_a^2$-Actions

Based on the following recent result of Kaliman, it is possible to describe all algebraic actions of $\mathbb{G}_a^2$ on $\mathbb{A}^3$, or equivalently, all commuting pairs $D, E \in \mathrm{LND}(k[x,y,z])$.

Theorem 5.35. [159] *Suppose $f : \mathbb{C}^3 \to \mathbb{C}^1$ is a polynomial function. If infinitely many fibers of f are isomorphic to $\mathbb{C}^2$, then f is a variable of $\mathbb{C}[x,y,z]$.*

This was generalized in [58] (Thm. 3) to all fields k of characteristic zero. This allows us to prove:

Proposition 5.36. *If $B = k[x,y,z]$, and if $D, E \in \mathrm{LND}(B)$ are nonzero, have distinct kernels, and are such that $DE = ED$, then there exists a variable $f \in B$ such that $\ker D \cap \ker E = k[f]$.*

Proof. Since D and E commute, E restricts to a nonzero locally nilpotent derivation on $A = \ker D = k^{[2]}$. By Rentschler's Theorem, there exists $f \in A$ such that $A = k[f]^{[1]}$ and $\ker D \cap \ker E = \ker(E|_A) = k[f]$. Therefore, the quotient map for the $\mathbb{G}_a^2$-action is of the form $F : \mathbb{A}^3 \to \mathbb{A}^1$, and is given by evaluation of the polynomial f.

The inclusions $k[f] \to A \to B$ give a factorization of F as the composition $H : \mathbb{A}^3 \to \mathbb{A}^2$ and $G : \mathbb{A}^2 \to \mathbb{A}^1$. By the Slice Theorem, there are open sets $U \subset \mathbb{A}^3$ and $V \subset \mathbb{A}^2$ such that $U = V \times \mathbb{A}^1$, $H(U) = V$, and $H : U \to V$ is a projection. Likewise, there exist open sets $V' \subset \mathbb{A}^2$ and $W \subset \mathbb{A}^1$ such that $V' = W \times \mathbb{A}^1$, $G(V') = W$, and $G : V' \to W$ is a projection. Thus, if $U' = H^{-1}(V \cap V')$ and $W' = G(V \cap V')$, it follows that $U' = W' \times \mathbb{A}^2$, $F(U') = W'$, and $F : U' \to W'$ is a projection. In particular, every fiber of F over a point of W' is isomorphic to $\mathbb{A}^2$. By the result of Kaliman, f is variable of B. $\square$

It should be noted that, in his thesis [205] and later in [204], Maubach also recognized this application of Kaliman's result.

This proposition indicates that a rank-three $\mathbb{G}_a$-action on $\mathbb{A}^3$ cannot be extended to a $\mathbb{G}_a^2$-action. Nonetheless, there are actions of $\mathbb{G}_a^2$ on $\mathbb{A}^3$ which are not conjugate to a triangular action. For example, let $P, Q \in B$ be any pair such that $k(x)[P, Q] = k(x)[y, z]$. Define $k(x)$-derivations Δ_P and Δ_Q as in *Chap. 4*. Then

$$\Delta_P = f(x)\partial_Q \quad \text{and} \quad \Delta_Q = g(x)\partial_P \qquad (f(x), g(x) \in k[x]) \ .$$

We see that Δ_P and Δ_Q commute on $k(x)[y, z]$, and restrict to B. Moreover, by *Thm. 5.16*, Δ_P is triangularizable if and only if Q is a variable of B, and likewise Δ_Q is triangularizable if and only P is a variable of B.

With a bit more work, one can show that such "neighboring pairs" of rank-two derivations provide a description of all $\mathbb{G}_a^2$-actions on $\mathbb{A}^3$.

5.9 Appendix: An Intersection Condition

The goal of this section is to prove the following fact. The theorem and its proof are due to the author and Daigle.

Theorem 5.37. *Let U be a UFD, and let R be a subring of U containing $\mathbb{Q}$. Let D be a nonzero locally nilpotent R-derivation of U, with $A = \ker(D)$. Suppose S is a subring of U such that:*

1. *$S = R[u, v] \cong R^{[2]}$ for some $u, v \in U$.*
2. *S is factorially closed in U.*
3. *$R \subset S \cap A \subset S$, but $R \neq S \cap A$ and $S \cap A \neq S$.*

Then there exists $w \in S$ such that $S \cap A = R[w]$ and $K[u, v] = K[w]^{[1]}$, where $K = \mathrm{frac}(R)$.

To prove this, some preliminaries are required.

Let $D : U \to U$ be a derivation of an integral domain U, and let $d : S \to S$ be a derivation of a subring S. Then D is a **quasi-extension** of d if there exists a nonzero $t \in U$ such that $Ds = t \cdot ds$ for all $s \in S$. Observe that if D is a quasi-extension of d, then $S \cap \ker D = \ker d$.

Lemma 5.38. *Let U be an integral domain containing $\mathbb{Q}$, and let $D : U \to U$ be a derivation which is a quasi-extension of a derivation $d : S \to S$ for some subring S. If D is locally nilpotent on U, then d is locally nilpotent on S.*

Proof. Suppose to the contrary that d is not locally nilpotent, and choose $s \in S$ for which $d^n s \neq 0$ for all $n \geq 1$. Then $\nu_D(d^n s) > 0$ for all $n \geq 0$. By hypothesis, there exists $t \in B$ such that $D = td$ on S. Set $\tau = \nu_D(t)$, noting that $\tau \geq 0$ (since $t \neq 0$). For every $n \geq 1$ we have

$$D(d^{n-1}s) = td(d^{n-1}s) = t \cdot d^n s \ .$$

Applying $\nu_D(\cdot)$ to each side of this equation yields:

$$\nu_D(d^{n-1}s) - 1 = \tau + \nu_D(d^n s) \ \Rightarrow \ \nu_D(d^n s) = \nu_D(d^{n-1}s) - (\tau + 1) \ .$$

By induction, we obtain: $\nu_D(d^n s) = \nu_D(s) - n(\tau + 1)$. But this implies $\nu_D(d^n s) < 0$ for $n \gg 0$, a contradiction. Therefore, d is locally nilpotent. $\square$

We are now ready to prove the theorem.

Proof. Let $\sigma \in S \cap A$, $\sigma \notin R$, be given, and write $\sigma = f(u,v)$ for $f \in R[u,v]$. Then

$$0 = D\sigma = f_u Du + f_v Dv \ .$$

Consider first the case when neither Du nor Dv is 0: Set $t = \gcd(f_u, f_v) \in U$, and choose $a, b \in U$ such that $f_u = tb$ and $f_v = ta$. Since S is factorially closed, it follows that $a, b \in S$. Therefore, $aDv = -bDu$, and we conclude that a divides Du. Set $r = Du/a$.

Define an R-derivation d on S by $ds := as_u - bs_v$ $(s \in S)$. Given $s \in S$, we have:

$$\begin{aligned} aDs &= a(s_u Du + s_v Dv) \\ &= as_u Du + s_v(aDv) \\ &= as_u Du - s_v(bDu) \\ &= (as_u - bs_v)Du \\ &= dsDu \ . \end{aligned}$$

Therefore, $Ds = rds$ for all $s \in S$. Note that $d \neq 0$ and $r \neq 0$, since otherwise $S \subset A$. We conclude that, if neither Du nor Dv is zero, then D is a quasi-extension of d

Consider next the case $Du = 0$ or $Dv = 0$. We may assume $Dv = 0$, in which case $Du \neq 0$ (otherwise $S \subset A$). In this case, let $d = \partial/\partial u$ and $r = Du$. Then for every $s \in S$, $Ds = s_u Du = rds$. So in either case, D is a quasi-extension of some nonzero d on S.

By the preceding lemma, d is locally nilpotent on $S = R[u,v]$. By *Thm. 4.13*, there exists $w \in S$ and $\alpha \in R[w]$ such that $d = \alpha\Delta_w$ and $\ker d = R[w]$, where Δ_w is the locally nilpotent R-derivation on $R[u,v]$ defined by $\Delta_w(h) = h_u w_v - h_v w_u$. Consequently, $R[w] \subset S \cap A$.

Conversely, let $\psi \in S \cap A$ be given. Then

$$0 = D\psi = rd\psi \quad \Rightarrow \quad d\psi = 0 \quad \Rightarrow \quad \psi \in \ker d = R[w] \ .$$

Therefore, $S \cap A = R[w]$. Moreover, *Thm. 4.13* shows that $K[u,v] = K[w]^{[1]}$. $\square$

An immediate consequence of *Thm. 5.37* is the following.

Corollary 5.39. *If, in addition to the hypotheses of Thm. 5.37, R is a field, then there exists $w \in S$ such that*

$$S \cap A = R[w] \quad \text{and} \quad S = R[w]^{[1]} \ .$$

Another consequence is:

Corollary 5.40. (Thm. 2 of [117]) *Let D be a locally nilpotent k-derivation of the polynomial ring $k[x_1, ..., x_n]$, $n \geq 2$, and suppose that $k[x_1, x_2] \cap \ker D \neq k$. Then either $Dx_1 = Dx_2 = 0$, or there exists $g \in k[x_1, x_2]$ such that $k[x_1, x_2] = k[g]^{[1]}$ and $k[x_1, x_2] \cap \ker D \subset k[g] \subset \ker D$.*

6

Linear Actions of Unipotent Groups

In this chapter and the next, we investigate one of the most basic questions one can ask about locally nilpotent derivations and $\mathbb{G}_a$-actions:

> **(Finiteness Problem)** Let k be any field of characteristic 0. If D is a locally nilpotent k-derivation of the polynomial ring $k^{[n]}$, is the kernel of D finitely generated as a k-algebra? Equivalently, if $\mathbb{G}_a$ acts algebraically on $\mathbb{A}^n$, is the ring of invariants finitely generated?

This represents a special case of Hilbert's Fourteenth Problem. It turns out that, if $n \leq 3$ the answer is *yes*, and if $n \geq 5$ the answer is generally *no*. Remarkably, only when $n = 4$ is the answer not known!

Invariant theory originally concerned itself with groups of vector space transformations, so the linear algebraic $\mathbb{G}_a$-actions were the first $\mathbb{G}_a$-actions to be studied. They have a long and interesting history. While fairly simple when compared to other algebraic $\mathbb{G}_a$-actions, the linear ones remain a fascinating object of study, and appear in a variety of problems.

The present chapter examines linear $\mathbb{G}_a$-actions, and more generally, linear actions of vector groups $\mathbb{G}_a^m$, or products $\mathbb{G}_a^m \rtimes \mathbb{G}_a$, on $\mathbb{A}^n$. Here, there are two main results: (1) The theorem of Maurer and Weitzenböck, which asserts that (in the characteristic zero case) the invariant ring of a linear $\mathbb{G}_a$-action on affine space is finitely generated. (2) The examples of Nagata and others, which show that the invariant ring of a higher-dimensional vector group, acting linearly on affine space, need not be finitely generated.

After some discussion of Hilbert's Fourteenth Problem in *Sect. 1*, a proof of the Maurer-Weitzenböck Theorem is presented in *Sect. 2*, based on the Finiteness Theorem. The remainder of the chapter is devoted to discussion of the vector group actions of Nagata and others, in addition to some recent examples involving non-commutative unipotent groups.

6.1 The Finiteness Theorem

In modern terminology, the famous Fourteenth Problem of Hilbert is as follows.

> For a field k, let $k^{[n]}$ denote the polynomial ring in n variables over k, and let $k^{(n)}$ denote its field of fractions. If K is a subfield of $k^{(n)}$ containing k, is $K \cap k^{[n]}$ finitely generated over k?

The main case of interest at the time of Hilbert was that of invariant rings for algebraic subgroups of $GL_n(\mathbb{C})$ acting on $\mathbb{C}^n$ as a vector space. But one can also consider the case of invariant rings of more general algebraic group actions on varieties:

> For a field k, suppose the linear algebraic k-group G acts algebraically on an affine k-variety V. Is the invariant ring $k[V]^G$ finitely generated?

It turns out that the answer depends on the type of group G which is acting. For reductive groups, the Fourteenth Problem has a positive answer.

> **(Finiteness Theorem)** *If k is any field, and G is a reductive k-group acting by algebraic automorphisms on an affine k-variety V, then the algebra of invariants $k[V]^G$ is finitely generated over k.*

In the late 1950s, Nagata published his celebrated counterexamples to Hilbert's Fourteenth Problem [235, 236]. One of these examples uses the unipotent group $\mathbb{G}_a^{13}$ acting linearly on $\mathbb{A}^{32}$, and Nagata proves that the invariant ring of this action is not finitely generated. In the language of derivations, this can be realized by 13 commuting linear triangular derivations D_i of the polynomial ring $k^{[32]}$ for which the subring $\cap_i \ker D_i$ is not finitely generated. Nagata's results are valid for any field k which is not an algebraic extension of a finite field.

The central idea in proving the Finiteness Theorem is due to Hilbert, whose original proof was for $SL_n(\mathbb{C})$. The full proof of the theorem represents the culmination of the efforts of many mathematicians over the past century, most recently for certain cases in positive characteristic. The case of finite groups was settled by E. Noether in her famous papers of 1916 ([243], characteristic 0) and 1926 ([244], positive characteristic).

In his lectures [238], Nagata formulated the following generalization of the Finiteness Theorem.

> The following properties of the linear algebraic group G are equivalent:
> (a) For all algebraic actions of G on an affine algebraic variety X, the algebra $k[X]^G$ is finitely generated over k. (b) G is reductive.

That (a) implies (b) was proved in 1979 by Popov [251]. See also the Appendix to Chap. 1 of [234].

The Finiteness Theorem is the main tool used in the proof of the Maurer-Weitzenböck Theorem presented below. In fact, for this proof, we only need the Finiteness Theorem for the group $G = SL_2(k)$, which had already been established by Gordan in 1868. While we do not include a proof of the Finiteness Theorem, several accounts of the theorem and its proof can be found in the literature. For example, the recent book of van den Essen [100] gives a very accessible proof in the case k is algebraically closed of characteristic zero (Chap. 9). The reader is also referred to the article of Humphreys [148], which provides an introductory survey of reductive group actions, and to the monograph of Popov [254], which gives a more extended treatment of the subject; each contains insightful historical background and a good list of pertinent references. The article of Mumford [233] is also required reading for anyone interested in the subject. Other standard references include [13, 83, 112, 234, 240, 242].

In view of the Finiteness Theorem, the question of finite generation for reductive groups has been replaced by other questions about invariant rings. For example, in what has come to be called **computational invariant theory**, the idea is to determine degree bounds for a system of generators, or to find algorithms which produce minimal generating sets, for invariant rings of reductive group actions; see [71].

The speech delivered by Hilbert in 1900 to the International Congress included 10 of his 23 famous problems; the speech and all the problems were later published in [142]. In contrast to its influence on mathematics in the following century, this speech bears the unassuming title *Mathematische Probleme.*[1] In 1903, the speech and problems appeared in English translation in [143]. In 1974, the American Mathematical Society sponsored a special Symposium on the mathematical consequences of Hilbert's problems. The volume [233] contains the proceedings of that symposium, as well as the English translation of Hilbert's speech. The purpose of the Symposium was "to focus upon those areas of importance in contemporary mathematical research which can be seen as descended in some way from the ideas and tendencies put forward by Hilbert in his speech" (from the Introduction). In particular, the volume contains one paper discussing each of the 23 problems, written by 23 of the most influential mathematicians of the day. The paper for Problem Fourteen was written by Mumford, *op. cit.*

6.2 Linear $\mathbb{G}_a$-Actions

Suppose k is a field of characteristic 0. Recall that a linear derivation D of a polynomial ring $k[x_1, ..., x_n]$ restricts to the vector space $kx_1 \oplus \cdots \oplus kx_n$, and may thus be specified by an $n \times n$ matrix M. Moreover, D is locally nilpotent if and only if M is a nilpotent matrix, and in this case, $\mathbb{G}_a$ acts by linear transformations on $\mathbb{A}^n$, namely, those given by the unipotent matrices $\exp(tM)$, $t \in k$. The most important fact about the invariant rings for these actions is the following.

[1] *Mathematical Problems*

Theorem 6.1. (Maurer-Weitzenböck Theorem) *If k is a field of characteristic zero, and if $\mathbb{G}_a$ acts algebraically on $X = \mathbb{A}^n_k$ by linear transformations, then $k[X]^{\mathbb{G}_a}$ is finitely generated.*

Actually, both Maurer [206] in 1899, and Weitzenböck [304] in 1932, thought they had proved finite generation for any algebraic group acting linearly on $\mathbb{C}^n$, but each made essentially the same mistake. However, their proofs for actions of the one-dimensional groups were sound. In [29], V.4, A. Borel gives a detailed exposition of the history of this result, highlighting the contributions of Maurer to invariant theory. Borel writes:

> Maurer's next publication [206] is an unfortunate one, since he sketches what he claims to be a proof of a theorem on the finiteness of invariants for any (connected) linear Lie group, a statement we know to be false. However, it also contains some interesting results, with correct proofs, including one which nowadays is routinely attributed to Weitzenböck (although the latter refers to [Maurer] for it). (p. 111)

Regarding Weitzenböck's knowledge of Maurer's earlier paper, Borel writes:

> He views its results as valid. His goal is to give a full proof, rather than just a sketch. (p. 112)

It was Hermann Weyl who, in reviewing Weitzenböck's paper in 1932, found a gap. Borel continues:

> The theorem in that [one-dimensional] case is nowadays attributed to Weitzenböck, probably beginning with Weyl, but this seems unjustified to me. The proof is quite similar to Maurer's, to which Weitzenböck refers explicitly. In particular, in the most important case of a nilpotent transformation, there is the same reduction to a theorem of P. Gordan. It is true that Maurer limits himself to regular transformations. However, his argument extends trivially to the case where the given Lie algebra is commutative, spanned by one nilpotent transformation and several diagonalizable ones, with integral eigenvalues, and Maurer proved that the smallest regular algebra containing a given linear transformation is of that form. But, surely, this is not the reason for that misnomer. Simply, [Maurer's paper] had been overlooked. (p. 113)

Eventually, Seshadri gave a proof of the theorem in his 1962 paper [275], where he "brings out clearly the underlying idea of Weitzenböck's proof" (p.404). Nagata included a proof of the Maurer-Weitzenböck Theorem in his classic *Lectures on the Fourteenth Problem of Hilbert* from 1965, based on Seshadri's ideas ([238], Chap. IV). Grosshans proved the Maurer-Weitzenböck Theorem in his book [131] in the context of more general group actions, and Tyc [297] gave a more algebraic version of Seshadri's proof in the case $k = \mathbb{C}$.

6.2.1 Basic $\mathbb{G}_a$-Actions

The **elementary nilpotent matrix** in dimension n over the field k is the $n \times n$ matrix E_n given by

$$E_n = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n \times n} .$$

A **basic** $\mathbb{G}_a$**-action** on $V = \mathbb{A}^n$ is any action which is conjugate by an element of $GL_n(k)$ to $\exp(tE_n)$, $t \in k$; see also Tan [291]. Likewise, viewing E_n as an element of $\mathrm{LND}(k^{[n]})$, we say that any $D \in \mathrm{LND}(k^{[n]})$ which is conjugate to E_n by an element of $GL_n(k)$ is a **basic linear derivation**.

The importance of the basic actions lies in the fact that, in the characteristic zero case, any linear algebraic $\mathbb{G}_a$-action on affine space is isomorphic to a direct sum of basic actions. This is easily seen by considering the Jordan normal form of a nilpotent matrix. Mauer, Weitzenböck, and Seshadri used this fact to show that, in the characteristic zero case, every linear $\mathbb{G}_a$-action is **fundamental**, meaning that it factors through a representation of $SL_2(k)$. (Onoda [249] refers to these as **standard** $\mathbb{G}_a$-actions.) This is the key fact underlying the proof of finite generation in the linear case. In fact, the proof shows that the invariant ring $k[V]^{\mathbb{G}_a}$ is isomorphic to the ring of invariants of an $SL_2(k)$-action on a larger polynomial algebra. By the Finiteness Theorem, it follows that $k[V]^{\mathbb{G}_a}$ is finitely generated.

It should be noted that, in positive characteristic, there exist linear $\mathbb{G}_a$-actions on $\mathbb{A}^n$ which are not fundamental. An example is given by Fauntleroy in [106], though he also shows that the invariant ring for this example is finitely generated. This explains why Seshadri's proof does not work for arbitrary linear $\mathbb{G}_a$-actions in positive characteristic. The question of finite generation for linear algebraic $\mathbb{G}_a$-actions in positive characteristic remains open.

Recall that if $M \in \mathfrak{gl}_n(k)$ is any nilpotent matrix, then its Jordan form is given by

$$M = \begin{pmatrix} E_{n_1} & 0 & \cdots & 0 \\ 0 & E_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_{n_r} \end{pmatrix}_{n \times n} ,$$

where the integers n_i satisfy $n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$ and $n_1 + n_2 + \cdots + n_r = n$. It follows that any linear algebraic $\mathbb{G}_a$-action is conjugate to a direct sum of basic $\mathbb{G}_a$-actions. Consequently:

> The number of conjugacy classes of linear algebraic $\mathbb{G}_a$-actions on $\mathbb{A}^n$ is equal to the number of partitions of the integer n.

6.2.2 A Proof of the Maurer-Weitzenböck Theorem

We assume throughout that k is a field of characteristic zero. The proof is based on the following general fact.

Proposition 6.2. (Thm. 1.2 of [130]) *Let G be a reductive k-group, and suppose $H \subset G$ is an algebraic subgroup for which the invariant ring $k[G]^H$ is finitely generated, where H acts on G by right multiplication. Then for any affine G-variety V, the corresponding ring of invariants $k[V]^H$ is finitely generated.*

Proof. (following Grosshans) Consider the action of the group $G \times H$ on the variety $G \times V$ defined by $(g,h) \cdot (a,v) = (gah^{-1}, g \cdot v)$, where $g \cdot v$ denotes the given action of G on V. We calculate the invariant ring $k[G \times V]^{G \times H}$ in two different ways.

First, since the action of $1 \times H$ on V (or, more properly, on $1 \times V$) is trivial, it follows that

$$k[G \times V]^{1 \times H} = (k[G] \otimes k[V])^{1 \times H} = k[G]^H \otimes k[V] \ .$$

See, for example, Lemma 1 (p. 7) of Nagata [238]. Since both $k[G]^H$ and $k[V]$ are affine rings (by hypothesis), $k[G]^H \otimes k[V]$ is also affine. And since G is reductive, it follows that

$$k[G \times V]^{G \times H} = (k[G] \otimes k[V])^{G \times H} = \left((k[G] \otimes k[V])^{1 \times H}\right)^{G \times 1}$$

is finitely generated.

Second, consider the equality

$$k[G \times V]^{G \times H} = \left(k[G \times V]^{G \times 1}\right)^{1 \times H} \ .$$

Let $f \in k[G \times V]^{G \times 1}$. Then for all $(g,1) \in G \times 1$ and $(a,v) \in G \times V$, we have

$$f(a,v) = f((g,1) \cdot (a,v)) = f(ga,v) \ .$$

Since the action of G on itself by left multiplication is transitive, we conclude that f is a function of v alone. Therefore, f is also invariant under the action of $1 \times H$ if and only if $f(a,v) = f(a, h \cdot v)$ for all $h \in H$. It follows that $k[G \times V]^{G \times H} \cong k[V]^H$. $\square$

Next, consider the group $G = SL_2(k)$, represented as 2×2 matrices of determinant 1. Its Lie algebra is $\mathfrak{g} = \mathfrak{s}l_2(k)$, the 2×2 matrices of trace 0. Any copy of $\mathbb{G}_a$ in $SL_2(k)$ is the exponential of a one-dimensional nilpotent Lie subalgebra of $\mathfrak{g}$. For example, we can take $H = \exp(tN)$ for $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $t \in k$.

Proposition 6.3. *Let $\mathbb{G}_a$ act on $SL_2(k)$ by right multiplication. Then*

$$k[SL_2(k)]^{\mathbb{G}_a} \cong k^{[2]} \ .$$

Proof. Let $B = k[x,y,z,w] = k^{[4]}$, and define the triangular derivation D on B by

$$Dx = 0\ ,\ Dy = x\ ,\ Dz = 0\ , Dw = z\ .$$

Then $\text{rank}(D) \leq 2$, and since D is clearly irreducible and not a partial derivative, it follows that $\text{rank}(D) = 2$. *Theorem 4.13* implies that $\ker D = k[x,z,xw-yz]$. Let I be the ideal $I = (xw - yz - 1)$, and set $\bar{B} = k[x,y,z,w] \bmod I$. Then $\bar{D} := D \bmod I$ is an element of $\text{LND}(\bar{B})$.

Note that $\bar{B} = k[SL_2(k)]$. We thus get the $\mathbb{G}_a$-action $\exp(t\bar{D})$ on $SL_2(k)$ defined by

$$t \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b+ta \\ c & d+tc \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\ ,$$

which is right-multiplication by elements of $\mathbb{G}_a$. Therefore, $k[SL_2(k)]^{\mathbb{G}_a} = \ker \bar{D}$.

To calculate this kernel, let $K = k(x,z)$, noting that

$$\bar{B}_K := K[y,w]/(xw - yz - 1) \cong K^{[1]}\ .$$

Since $\bar{D}x = \bar{D}z = 0$, it follows that $\bar{D}$ extends to a locally nilptent K-derivation $\bar{D}_K$ of $\bar{B}_K$. Therefore, $\ker \bar{D}_K = K$, which implies

$$\ker \bar{D} = K \cap \bar{B} = k[x,z] \cong k^{[2]}\ .$$

□

Next, let $k[x,y] = k^{[2]}$ and write $k[x,y] = \oplus_{n\geq 0} V_n$, where $V_n \cong \mathbb{A}^{n+1}$ is the vector space of forms of degree n relative to the standard grading of $k[x,y]$. Any linear action of a linear algebraic group G on $k[x,y]$ is homogeneous, in the sense that the action restricts to a G-action on each V_n. Thus, from a linear G-action on $\mathbb{A}^2$ we obtain a corresponding linear G-action on $V_n = \mathbb{A}^{n+1}$ for all $n \geq 1$. In this case, V_n is called the **G-module of binary forms of degree n.**

Proposition 6.4. *Every linear algebraic action of $\mathbb{G}_a$ on $\mathbb{A}^n$ is fundamental.*

Proof. We need to show that every linear algebraic $\mathbb{G}_a$-action on $\mathbb{A}^n$ is the restriction of an $SL_2(k)$-action.

For each $n \geq 1$, let V_n denote the $SL_2(k)$-module of binary forms of degree n. Let $\mathbb{G}_a \subset SL_2(k)$ denote the subgroup

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad (t \in k)\ .$$

Then the restricted action of $\mathbb{G}_a$ on V_n is exactly the basic action of $\mathbb{G}_a$ on $\mathbb{A}^{n+1}$. To see this, consider the basic action of $\mathbb{G}_a$ on $\mathbb{A}^2$ defined by $t \cdot (x,y) = (x, y+tx)$. Given $n \geq 2$, take as a basis of V_n the monomials $X_i = x^{n-i}y^i$

$(0 \le i \le n)$. The induced $\mathbb{G}_a$-action on V_n is $t \cdot X_i = x^{n-i}(y+tx)^i$. What is the corresponding locally nilpotent derivation D of $k[X_0,...,X_n]$? Recall that, since $\exp(tD)(X_i) = X_i + tDX_i + \cdots$, we are looking for the degree-one coefficient relative to t, i.e., $d/dt(x^{n-i}(y+tx)^i)$ evaluated at $t=0$. This is easily calculated, and we get $DX_i = ix^{n-(i-1)}y^{i-1} = iX_{i-1}$ if $i \ge 1$, and $DX_0 = 0$. This is the basic linear derivation. Therefore, every basic $\mathbb{G}_a$-action is fundamental.

As remarked earlier, any linear algebraic $\mathbb{G}_a$-action on affine space is isomorphic to a direct sum of basic actions. In particular, suppose $W = \mathbb{A}^n$ is a $\mathbb{G}_a$-module, with Jordan block decomposition $W = W_1 \oplus \cdots \oplus W_m$, where the $\mathbb{G}_a$-action on W restricts to the basic action on each W_i for which $\dim W_i \ge 2$, and restricts to the identity action on any W_i for which $\dim W_i = 1$. Define an $SL_2(k)$-action on W by letting $SL_2(k)$ act on each affine space W_i: If $\dim W_i \ge 2$, let $SL_2(k)$ act on W_i in the way described above; and if $\dim W_i = 1$, use the trivial action of $SL_2(k)$. Then the given $\mathbb{G}_a$-action is a restriction of this $SL_2(k)$-action. $\square$

Combining this proposition with the two which precede it, we get a proof of the Maurer-Weitzenböck Theorem.

In fact, more can be said in this case regarding *Prop. 6.2*, using the fact that $k[SL_2(k)]^{\mathbb{G}_a} = k[V_1] \cong k^{[2]}$. The proof of the proposition shows that

$$k[V_n]^{\mathbb{G}_a} = \left(k[SL_2(k) \times V_n]^{SL_2(k)}\right)^{\mathbb{G}_a} = k[SL_2(k) \times V_n]^{SL_2(k)\times\mathbb{G}_a} \ .$$

The second calculation then shows

$$\begin{aligned} k[SL_2(k) \times V_n]^{SL_2(k)\times\mathbb{G}_a} &= \left(k[SL_2(k) \times V_n]^{\mathbb{G}_a}\right)^{SL_2(k)} \\ &= \left(k[V_n] \otimes k[SL_2(k)]^{\mathbb{G}_a}\right)^{SL_2(k)} \\ &= (k[V_n] \otimes k[V_1])^{SL_2(k)} \\ &= k[V_n \times V_1]^{SL_2(k)} \ . \end{aligned}$$

In other words, the invariant ring of a linear algebraic $\mathbb{G}_a$-action on $\mathbb{A}^n$ is isomorphic to the ring of invariants of a certain $SL_2(k)$-action on $\mathbb{A}^{n+2}$. According to Derksen and Kemper, this isomorphism was proved in 1861 by M. Roberts [261]. See Example 2.5.2 of [71], and Example 3.6 of [257].

For a nice overview of the classical theory of binary forms, see the lecture notes of Dixmier [85].

6.2.3 Further Remarks about Linear $\mathbb{G}_a$-Actions

Remark 6.5. The main idea used in the proof of *Prop. 6.2* is called the **transfer principle** or **adjunction argument**, which asserts that, if H is a closed subgroup of an algebraic group G, then $(k[G/H] \otimes k[X])^G \cong k[X]^H$ for any G-module X. In Chap. 2 of his book [131], Grosshans gives a nice historical outline of the transfer principle, followed by its proof and various applications; a statement and proof of the transfer principle in its most general form can be found in Popov [252]. Grosshans writes:

> Roughly speaking, it allows information on $k[G/H]$ to be transferred to W^H. For example, suppose that G is reductive and that $k[G/H]$ is finitely generated. Let $W = A$ be a finitely generated, commutative k-algebra on which G acts rationally. Then using the transfer principle and Theorem A, we see that A^H is finitely generated. The most important instance of this occurs when $H = U$ is a maximal unipotent subgroup of a unipotent group G.... In the study of binary forms, H is taken to be a maximal unipotent subgroup of $SL(2,\mathbb{C})$. The transfer theorem in this context was proved by M. Roberts in 1871 [*sic*] and describes the relationship between "covariants", i.e., the algebra $(\mathbb{C}[G/H] \otimes A)^G$, and "semi-invariants", the algebra A^H. (From the Introduction to Chap. 2)

Specifically, any maximal unipotent subgroup U of $G = SL_2(k)$ is one-dimensional, i.e., $U = \mathbb{G}_a$. Thus, the Maurer-Weitzenböck Theorem is a special case of the following more general fact. See also Thm. 9.4 of [131].

Theorem 6.6. (Hadziev [138], 1966) *Let G be a reductive group and let U be a maximal unipotent subgroup of G. Let A be a finitely generated, commutative k-algebra on which G acts rationally. Then A^U is finitely generated over k.*

The theorem of Hadziev was generalized by Grosshans in [130]. Another proof of the transfer principle is given in [257].

Remark 6.7. It is a fascinating exercise to compute the invariants of a particular linear $\mathbb{G}_a$-action on $\mathbb{A}^n$. Tan [291] presented an algorithm for calculating the invariants of the basic actions, and his paper includes several illustrative examples. Nowicki also calculated the ring of invariants for several linear $\mathbb{G}_a$-actions on affine space, including some which are not basic; see 6.8 and 6.9 of [247]. We will study algorithms for finding rings of invariants, and will provide examples, in *Chap. 8* below.

Remark 6.8. In [297], Tyc proves:

Theorem 6.9. *Let $B = \mathbb{C}[x_1, ..., x_n] = \mathbb{C}^{[n]}$, and let $D \in \mathrm{LND}(B)$ preserve the vector space $W = \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_n$. Then*

1. *$\ker D$ is a Gorenstein ring.*
2. *$\ker D$ is a polynomial ring if and only if $W = W_0 \oplus W'$ for subspaces W_0 and W' of W for which $D(W_0) = 0$, $D(W') \subset W'$, and the Jordan matrix of $D : W' \to W'$ is one of the following:*

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Gorenstein property was announced earlier (without proof) by Onoda in [249].

Remark 6.10. For *any* linear derivation $D \in \mathrm{Der}_k(k[x_1,...,x_n])$, both $\ker D$ and $\mathrm{Nil}D$ are finitely generated; see 6.2.2 and 9.4.7 of Nowicki [247].

Remark 6.11. In the papers [105, 106, 107], Fauntleroy studies linear $\mathbb{G}_a$-actions from the geometric viewpoint, with particular attention to the case in which the ground field is of positive characteristic. For example, in the first of these papers, he shows that if the fixed point set of a linear $\mathbb{G}_a$-action on affine space is a hyperplane, then the ring of invariants is finitely generated. The recent article of Tanimoto [294] also gives some cases in which the ring of invariants of a linear $\mathbb{G}_a$-action on affine space is finitely generated, under the assumption that the ground field is algebraically closed of positive characteristic.

6.3 Linear Counterexamples to the Fourteenth Problem

In the statement of the Maurer-Weitzenböck Theorem, one cannot generally replace the group $\mathbb{G}_a$ with higher-dimensional vector groups $\mathbb{G}_a^n$. This section will discuss certain rational representations of $\mathbb{G}_a^n$ for which the ring of invariants is not finitely generated, beginning with the famous examples of Nagata. Among the linear counterexamples discussed below is an action of $\mathbb{G}_a^{12}$ on $\mathbb{A}^{19}$ due to A'Campo-Neuen. More recently, unipotent groups other than vector groups have been used to give smaller linear counterexamples, and these are also discussed. For example, the author has constructed a linear action of the group $\mathbb{G}_a^4 \rtimes \mathbb{G}_a$ on $\mathbb{A}^{11}$ for which the invariant ring is not finitely generated. Complete proofs for these two examples are given in the next chapter.

6.3.1 Examples of Nagata

The first counterexamples to Hilbert's Fourteenth Problem were presented by Nagata in 1958. Prior to the appearance of Nagata's examples, Rees [259] constructed a counterexample to Zariski's generalization of the Fourteenth Problem, which asks:

> Let R be a normal affine ring over a field k. If L is a field with $k \subseteq L \subseteq \mathrm{frac}(R)$, is $R \cap L$ an affine ring?

In Rees's example, $\mathrm{frac}(R \cap L)$ contains the function field of a non-singular cubic projective plane curve, and cannot therefore be a counterexample to Hilbert's problem. But Rees' example was very important in its own right, and indicated that counterexamples to Hilbert's problem might be found in a similar fashion.

Shortly thereafter, Nagata discovered two counterexamples to Hilbert's problem. In [235], he describes the situation as follows. (By "original 14-th problem", he means the specific case using fixed rings for linear actions of algebraic groups.)

> In 1958, the writer found at first a counter-example to the 14-th problem and then another example which is a counter-example to the original 14-th problem. This second example was announced at the International Congress in Edinburgh (1958). Though the first example is in the case where dim $K = 4$, in the second example dim K is equal to 13. Then the writer noticed that the first example is also a counter-example to the original 14-th problem. (p. 767)

How did Nagata find these examples? As Steinberg [286] points out, the heart of Nagata's method is to relate the structure of the ring of invariants to an interpolation problem in the projective plane, namely, that for each $m \geq 1$, there does not exist a curve of degree $4m$ having multiplicity at least m at each of 16 general points of the projective plane. Steinberg writes: "Nagata's ingenious proof of this is a tour de force but the results from algebraic geometry that he uses are by no means elementary" (p.377).

The foundation of this geometric approach to the problem was laid by Zariski in the early 1950s. His idea was to look at rings of the form $R(D)$, where D is a positive divisor on some non-singular projective variety X, and $R(D)$ is the ring of rational functions on X with poles only on D. Mumford writes:

> In his penetrating article [314], Zariski showed that Hilbert's rings $K \cap k[x_1, ..., x_n]$ were isomorphic to rings of the form $R(D)$ for a suitable X and D; asked more generally whether all the rings $R(D)$ might not be finitely generated; and proved $R(D)$ finitely generated if dim $X = 1$ or 2....Unfortunately, it was precisely by focusing so clearly the divisor-theoretic content of Hilbert's 14th problem that Zariski cleared the path to counter-examples. [233]

In the example constructed by Rees, X is birational to $\mathbb{P}^2 \times C$ for an elliptic curve C; and in Nagata's examples, X is the surface obtained by blowing up $\mathbb{P}^2$ at 16 general points. For further detail, the reader is referred to Mumford's article, *op. cit.*, as well as Nagata's 1965 lectures on the subject [238].

It is well worth recording the positive result in Zariski's landmark 1954 paper, as mentioned by Mumford.

> **(Zariski's Finiteness Theorem)** *For a field k, let A be an affine normal k-domain, and let K be a subfield of* $\mathrm{frac}(A)$ *containing k. If* $\mathrm{tr.deg.}_k K \leq 2$, *then $K \cap A$ is finitely generated over k.*

For example, this allows us to conclude that, if X is an algebraic k-variety and $\dim X \leq 3$, then $k[X]^G$ is finitely generated for any algebraic group G acting algebraically on X: If tr $\deg_k k[X]^G \leq 2$, apply Zariski's theorem; and if tr $\deg_k k[X]^G = 3$, then $k[X]$ is algebraic over $k[X]^G$, and G is necessarily finite. In this case, Hilbert's Finiteness Theorem applies. Details of this reasoning are provided in the *Appendix* at the end of this chapter.

6.3.2 Examples of Steinberg and Mukai

In 1997, Steinberg [286] published a lucid exposition of Nagata's original constructions, and modified Nagata's approach to obtain linear counterexamples of reduced dimension. Subsequently, Mukai [227] generalized this geometric approach even further to give entire families of counterexamples, including some in yet smaller dimension.

Let the vector group $U = \mathbb{G}_a^n$ be represented on $\mathbb{A}^{2n}$ by

$$(t_1, ..., t_n) \mapsto \begin{pmatrix} I & 0 \\ M & I \end{pmatrix} \quad \text{for} \quad M = \begin{pmatrix} t_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t_n \end{pmatrix} .$$

This is called a **standard vector group representation**. Likewise, let the torus $\mathbb{G}_m^{2n}$ be represented on $\mathbb{A}^{2n}$ by

$$(c_1, ..., c_{2n}) \mapsto \begin{pmatrix} c_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_{2n} \end{pmatrix} .$$

This is called a **standard torus representation**. We let $T = \mathbb{G}_m^{2n-1} \subset \mathbb{G}_m^{2n}$ denote the subgroup $T = \{(c_1, ..., c_{2n}) \mid c_1 \cdots c_{2n} = 1\}$. Since the standard actions of $\mathbb{G}_a^n$ and $\mathbb{G}_m^{2n}$ semi-commute (the torus normalizes the vector group), we obtain a representation of UT on $\mathbb{A}^{2n}$.

The examples of Nagata, Steinberg, and Mukai each uses a subgroup $G \subset U$ or $G \subset UT$, acting on $\mathbb{A}^{2n}$ in the way described above. According to Mukai, Nagata showed that "the invariant ring S^G with respect to a general linear subspace $G \subset \mathbb{C}^n$ of codimension 3 was not finitely generated for $n = 16$" (p.1). Here, $\mathbb{C}^n$ indicates $\mathbb{G}_a^n$ over the field $k = \mathbb{C}$.

In Steinberg's paper, the main example is for $n = 9$, where he considers the subgroup $G = \mathbb{G}_a^6$ of U defined in the following way: In case char $k = 0$, choose $a_1, ...a_9 \in k$ such that $a_i \neq a_j$ for $i \neq j$, and $\sum a_i \neq 0$, and let $G \subset \mathbb{G}_a^9$ be the subgroup for which $\sum t_i = \sum a_i t_i = \sum a_i^3 t_i = 0$. In case char $k > 0$, choose distinct $a_1, ..., a_9 \in k$ so that $\prod a_i$ is neither 0 nor any root of 1, and let $G \subset \mathbb{G}_a^9$ be the subgroup for which $\sum t_i = \sum a_i t_i = \sum (a_i^2 - a_i^{-1}) t_i = 0$. Steinberg shows that the action of GT on $\mathbb{A}^{18}$ has non-finitely generated ring of invariants (Thm. 1.2), which implies (by the Finiteness Theorem) that the invariant ring of G is also non-finitely generated. The examples of Nagata and Steinberg are valid over any field k which is not a locally finite field. A **locally finite field** is defined to be any algebraic extension of a finite field.

Subsequently, Mukai proved the following result.

Theorem 6.12. (Mukai's Theorem) *Let $\mathbb{C}^n$ act on $\mathbb{C}^{2n}$ by the standard action. If $G \subset \mathbb{C}^n$ is a general linear subspace of codimension $r < n$, then the ring of G-invariant functions is finitely generated if and only if*

$$\frac{1}{r} + \frac{1}{n-r} > \frac{1}{2} .$$

(The proof of 'only if' for this theorem is given in [227]; for the proof of 'if', see [228].) It follows that, if $S = \mathcal{O}(\mathbb{C}^{2n}) = \mathbb{C}^{[2n]}$, then S^G is not finitely generated if $\dim G = m \geq 3$ and $n \geq m^2/(m-2)$. Thus, there exist linear algebraic actions of $\mathbb{G}_a^3$ on $\mathbb{C}^{18}$ and of $\mathbb{G}_a^4$ on $\mathbb{C}^{16}$ whose rings of invariants are not finitely generated. At the time of their appearance, these were the smallest linear counterexamples to the Fourteenth Problem, both in terms of the dimension of the group which acts ($m = 3$), and the dimension of the space which is acted upon ($2n = 16$). Subsequently, both Tanimoto and Freudenburg found linear counterexamples using smaller affine spaces, namely, $\mathbb{A}^{13}$ and $\mathbb{A}^{11}$, respectively; these are discussed below. As we will see in the next chapter, even smaller counterexamples can be found if we consider more general (non-linear) actions.

The papers of Nagata, Steinberg, and Mukai are largely self-contained. In particular, Steinberg's two main lemmas (2.1 and 2.2) provide the crucial link between an interpolation problem in the projective plane and the structure of certain fixed rings. The group associated with a cubic curve plays an important role in this approach to the problem. Steinberg goes on to discuss the status of the classical geometric problem lying at the heart of this approach, which is of interest in its own right, described by him as follows:

> Find the dimension of the space of all polynomials (or curves) of a given degree with prescribed multiplicities at the points of a given finite set in general position in the plane, thus also determine if there is a curve, i.e., a nonzero polynomial, in the space and if the multiplicity conditions are independent. (p. 383)

The recent paper of Kuttler and Wallach [186] also gives an account of these ideas, in addition to generalizations of some of Steinberg's results. See also Mukai [230] and Roé [264].

6.3.3 Examples of A'Campo-Neuen and Tanimoto

In her paper [4] (1994), A'Campo-Neuen used a non-linear counterexample to Hilbert's Fourteenth Problem which had been published earlier by Roberts to construct a counterexample arising as the fixed ring of a linear action of $G = \mathbb{G}_a^{12}$ on $\mathbb{A}^{19}$. Her example is valid for any field k of characteristic 0. Apparently, this was the first linear counterexample to be produced after those of Nagata, a span of 36 years!

As in the examples of Nagata, her example is gotten by restriction of a standard vector group action to a certain subgroup. In particular, given $(t_1, ..., t_{12}) \in G$, the G-action is defined explicitly by the lower triangular matrix

$$\begin{pmatrix} I & 0 \\ M^T & I \end{pmatrix}$$

of order 19, where the identities are of order 4 and 15 respectively, and M is the 4×15 matrix

$$M = \begin{pmatrix} t_1 & t_2 & 0 & t_3 & t_4 & 0 & t_5 & t_6 & 0 & t_7 & t_8 & t_9 & t_{10} & t_{11} & 0 \\ t_{12} & t_1 & t_2 & 0 & 0 & 0 & 0 & 0 & 0 & t_{12} & t_7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_{12} & t_3 & t_4 & 0 & 0 & 0 & 0 & 0 & t_8 & t_9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t_{12} & t_5 & t_6 & 0 & 0 & 0 & 0 & t_{10} & t_{11} \end{pmatrix} .$$

A'Campo-Neuen's proof is quite elegant, and is given in *Sect. 7.3* of the next chapter. This proof uses locally nilpotent derivations of polynomial rings, and is very different from the geometric proofs of Nagata, Steinberg, and Mukai.

In 2004, Tanimoto [295] imitated the methods of A'Campo-Neuen to give two linear counterexamples to the Fourteenth Problem, one of which uses affine space of dimension 13. His examples are based on the non-linear counterexamples of Freudenburg and Daigle [55], and Freudenburg [120]. However, in order to utilize these earlier examples, Tanimoto realized that it was necessary to consider non-abelian unipotent group actions. In particular, he gives a counterexample in which the group $\mathbb{G}_a^7 \rtimes \mathbb{G}_a$ acts linearly on $\mathbb{A}^{13}$, and another in which $\mathbb{G}_a^{18} \rtimes \mathbb{G}_a$ acts linearly on $\mathbb{A}^{27}$.

Here are the particulars for the smaller of these two actions. Let $\mu = (\mu_0, ..., \mu_6)$ denote an element of $V = \mathbb{G}_a^7$, and define a linear $\mathbb{G}_a$-action on V by ${}^t\mu = \exp(tD)(\mu)$, where D is the nilpotent matrix whose Jordan block form is

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & E_3 & 0 \\ 0 & 0 & E_3 \end{pmatrix}_{7\times 7} \quad \text{for} \quad E_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} .$$

This action defines semi-addition on $\mathbb{G}_a^7 \rtimes \mathbb{G}_a$, namely,

$$(t, \mu)(t', \mu') = (t + t', \mu + {}^t\mu') .$$

Tanimoto gives seven commuting linear triangular derivations $\Delta_0, ..., \Delta_6$ of the polynomial ring

$$B = k[w, x, s_1, t_1, u_1, s_2, t_2, u_2, s_3, t_3, u_3, v_1, v_2] = k^{[13]}$$

whose exponentials give the linear action of $\mathbb{G}_a^7$ on $\mathbb{A}^{13}$, namely,

$$\begin{aligned} &\Delta_0 = x\frac{\partial}{\partial v_2} + w\frac{\partial}{\partial v_1} , \Delta_1 = x\frac{\partial}{\partial s_3} - w\frac{\partial}{\partial s_2} , \Delta_2 = x\frac{\partial}{\partial t_3} - w\frac{\partial}{\partial t_2} , \\ &\Delta_3 = x\frac{\partial}{\partial u_3} - w\frac{\partial}{\partial u_2} , \Delta_4 = x\frac{\partial}{\partial s_2} + w\frac{\partial}{\partial s_1} , \Delta_5 = x\frac{\partial}{\partial t_2} + w\frac{\partial}{\partial t_1} , \\ &\Delta_6 = x\frac{\partial}{\partial u_2} + w\frac{\partial}{\partial u_1} . \end{aligned}$$

Combined with another derivation

$$\Delta = x\frac{\partial}{\partial s_1} + s_1\frac{\partial}{\partial t_1} + t_1\frac{\partial}{\partial u_1} + s_2\frac{\partial}{\partial t_2} + t_2\frac{\partial}{\partial u_2} + s_3\frac{\partial}{\partial t_3} + t_3\frac{\partial}{\partial u_3} + x\frac{\partial}{\partial v_1} ,$$

these eight induce the full action of $\mathbb{G}_a^7 \rtimes \mathbb{G}_a$ on $\mathbb{A}^{13}$. For the interested reader, Tanimoto also gives the action in matrix form.

6.3.4 A Linear Counterexample in Dimension Eleven

Quite recently, the author constructed a family of linear counterexamples to the Fourteenth Problem in which, for each integer $n \geq 4$, the unipotent group $\Gamma_n = \mathbb{G}_a^n \rtimes \mathbb{G}_a$ acts on $V_n = \mathbb{A}^{2n+3}$ by linear transformations, and $k[V_n]^{\Gamma_n}$ is not finitely generated [113]. The smallest of these is for the group $\mathbb{G}_a^4 \rtimes \mathbb{G}_a$ acting on $\mathbb{A}^{11}$. To date, this is the smallest affine space for which a linear counterexample is known to exist. The specific action in this case is described in the following theorem.

Theorem 6.13. (Thm. 4.1 of [113]) *Let*

$$B = k[w, x, s_1, s_2, t_1, t_2, u_1, u_2, v_1, v_2, z] = k^{[11]} ,$$

and define commuting linear triangular derivations T_1, T_2, T_3, T_4 on B by

$$T_1 = x\frac{\partial}{\partial s_2} - w\frac{\partial}{\partial s_1} , \quad T_2 = x\frac{\partial}{\partial t_2} - w\frac{\partial}{\partial t_1} ,$$
$$T_3 = x\frac{\partial}{\partial u_2} - w\frac{\partial}{\partial u_1} , \quad T_4 = x\frac{\partial}{\partial v_2} - w\frac{\partial}{\partial v_1} .$$

Define a fifth linear triangular derivation Θ, which semi-commutes with the T_i:

$$\Theta = x\frac{\partial}{\partial s_1} + s_1\frac{\partial}{\partial t_1} + t_1\frac{\partial}{\partial u_1} + u_1\frac{\partial}{\partial v_1} + s_2\frac{\partial}{\partial t_2} + t_2\frac{\partial}{\partial u_2} + u_2\frac{\partial}{\partial v_2} + x\frac{\partial}{\partial z} .$$

Let $\mathfrak{g}$ be the Lie algebra generated by T_1, T_2, T_3, T_4 and Θ. Then the group $\Gamma = \exp \mathfrak{g} \cong \mathbb{G}_a^4 \rtimes \mathbb{G}_a$ acts on $V = \mathbb{A}^{11}$ by linear transformations, and $k[V]^\Gamma$ is not finitely generated.

The proof for this example is based on recent work of Kuroda involving non-linear $\mathbb{G}_a$-actions, and will be outlined in the next chapter.

6.4 Linear $\mathbb{G}_a^2$-Actions

There remain two natural questions about linear actions of unipotent groups on affine space: Suppose that the unipotent group G of dimension m acts linearly algebraically on the affine space $V = \mathbb{A}^n$ in such a way that the ring of invariants $k[V]^G$ is not finitely generated.

1. What is the minimal dimension $m = \mu$ which can occur in this situation?
2. What is the minimal dimension $n = \nu$ which can occur in this situation?

The examples of Mukai show that $\mu \leq 3$, whereas the theorem of Maurer-Weitzenböck implies $\mu \geq 2$. Likewise, the example of Freudenburg shows that $\nu \leq 11$, while Zariski's Theorem implies $\nu \geq 5$.

In particular, the first question reduces to a single case when G is abelian.

> Is the ring of invariants of a linear $\mathbb{G}_a^2$-action on $\mathbb{A}^n$ always finitely generated? Equivalently, if M and N are commuting nilpotent matrices, is $\ker M \cap \ker N$ finitely generated?

In order to investigate this question, let $U \subset GL_n(k)$ denote the maximal unipotent subgroup consisting of upper triangular matrices with ones on the diagonal. Let $\mathfrak{u} \subset \mathfrak{gl}_n(k)$ denote the Lie algebra of U, i.e., upper triangular matrices with zeros on the diagonal. Note that $\mathfrak{u}$ is a nilpotent Lie algebra, consisting of nilpotent elements. Since $\mathbb{G}_a^2$ is a unipotent group, every rational representation $\mathbb{G}_a^2 \subset GL_n(k)$ can be conjugated to U. Thus, we need to consider the two-dimensional Lie subalgebras $\mathfrak{h}$ of $\mathfrak{u}$.

6.4.1 Actions of Nagata Type

As mentioned, the unipotent groups G studied by Nagata, Steinberg, and Mukai are subgroups of standard representations of vector groups. They are given by a set of commuting matrices $D_1, ..., D_m$, where $m = \dim G$, and where $D_i^2 = 0$ for each i. As derivations, each D_i is a nice derivation. It is natural to begin the study of linear $\mathbb{G}_a^2$-actions with this type.

Suppose D and E are given by

$$D = \begin{pmatrix} C & 0 & \cdots & 0 \\ 0 & C & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} \lambda_1 C & 0 & \cdots & 0 \\ 0 & \lambda_2 C & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n C \end{pmatrix}$$

where

$$C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ,$$

the dimension of D and E is $2n \times 2n$, and $\lambda_i \in \mathbb{C}$. Then $DE = ED$ and $D^2 = E^2 = 0$. In [231], the authors define the corresponding $\mathbb{G}_a^2$-action to be of **Nagata type**. In his earlier paper [227], Mukai had already shown that the invariant ring of such an action is isomorphic to the total coordinate ring of the blow-up of $\mathbb{P}^{n-3}$ at n points ($n \geq 4$). The later paper then sketches how to use this fact to show their main result:

Theorem 6.14. (Cor. 1 of [231]) *The ring of invariants for a $\mathbb{G}_a^2$-action of Nagata type is finitely generated.*

See also [229]. In [32], Castravet and Tevelev give another proof of this result, using geometric methods similar to those of Mukai. Specifically, they show that if $\mathbb{G}_a^2$ acts on $\mathbb{C}^{2n}$ by an action of Nagata type, then the algebra of invariants is generated by 2^{n-1} invariant functions which they define explicitly using determinants (Thm. 1.1). The following is a simple construction of such invariants quite similar to that of Castravet and Tevelev.

Let $B = k[x_1, y_1, ..., x_n, y_n]$ for $n = 2m + 1$ and $m \geq 2$, and view D and E as derivations of B:

$$Dy_i = Ey_i = 0 \quad , \quad Dx_i = y_i \quad , \quad \text{and} \quad Ex_i = \lambda_i y_i \ .$$

If $z_{ij} := x_i y_j - x_j y_i$ for $1 \le i, j \le n$, then $Dz_{ij} = 0$ and $Ez_{ij} = (\lambda_i - \lambda_j) y_i y_j$ for every pair i, j. Introduce a $\mathbb{Z}^2$-grading on B by declaring $\deg x_i = (-1, 1)$ and $\deg y_i = (1, 0)$ for each i. Then D and E are homogeneous of degreee $(2, -1)$, and $\deg z_{ij} = (0, 1)$ for each i, j.

Proposition 6.15. *For this $\mathbb{G}_a^2$-action on $\mathbb{A}^{2n}$, there exist invariants of degree $(1, d)$ for each $d = 0, 1, ..., m$.*

Proof. Define $\mathbf{y} = (y_{m+1}, y_{m+2}, ..., y_{2m+1})$. Then there exist scalars $c_{rs} \in \mathbb{Q}[\lambda_{ij}]$ such that, if

$$\mathbf{z}_i = (c_{i,1} z_{i,m+1}, c_{i,2} z_{i,m+2}, ..., c_{i,m+1} z_{i,2m+1}) \quad (1 \le i \le m) \ ,$$

then $E\mathbf{z}_i = t_i y_i \mathbf{y}$ for some $t_i \in k$.

Now construct an $(m+1) \times (m+1)$ matrix K such that the i-th row equals $\mathbf{z}_i$ for $1 \le i \le m$, and such that the last row equals $\mathbf{y}$. Then $\deg(\det K) = (1, m)$ and $E(\det K) = 0$ (see *Sect. 2.6*). To construct invariants of degree $(1, d)$ for $d < m$, just consider minor and sub-minor determinants of K. □

6.4.2 Actions of Basic Type

A rational represtentation $\mathbb{G}_a^2 \subset GL_n(k)$ is of **basic type** if it admits a restriction to the basic $\mathbb{G}_a$-action on $\mathbb{A}^n$. More specifically, this will mean that there exists $M \in GL_n(k)$ such that the representation of $\mathbb{G}_a^2$ is given by the exponential of the Lie algebra $kE_n + kM$, where $E_n \in \mathfrak{u}$ is the elementary nilpotent matrix in dimension n. In particular, M commutes with E_n.

Let $Z(E_n, \mathfrak{u})$ denote the **centralizer** of E_n in $\mathfrak{u}$, i.e., elements of $\mathfrak{u}$ which commute with E_n under multiplication.

Proposition 6.16. *A basis of $Z(E_n, \mathfrak{u})$ is $\{E_n, E_n^2, ..., E_n^{n-1}\}$.*

Proof. Recall first that E_n is the matrix with ones on the first super-diagonal and zeros elsewhere, and likewise E_n^i is the matrix with ones the ith super-diagonal with zeros elsewhere. Therefore, $E_n, E_n^2, ..., E_n^{n-1}$ are linearly independent. Let $\mathfrak{g} \subset \mathfrak{u}$ denote $\mathfrak{g} = kE_n + kE_n^2 + \cdots kE_n^{n-1}$.

Given $M \in \mathfrak{u}$, write

$$M = \begin{pmatrix} 0 & \alpha \\ 0 & A \end{pmatrix} = \begin{pmatrix} A' & \alpha' \\ 0 & 0 \end{pmatrix} \ .$$

where A is the $(1, 1)$-minor submatrix of M, A' is the (n, n)-minor submatrix of M, α is the corresponding row matrix of length $(n-1)$, and α' is the corresponding column matrix of length $(n-1)$. Then by comparing elements of M lying on its superdiagonals, we conclude that $M \in \mathfrak{g}$ if and only if $A = A'$.

Write

$$E_n = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \ ,$$

where I is the identity matrix of order $(n-1)$, which is the $(n,1)$-minor of E_n. Then

$$E_n M = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \alpha \\ 0 & A \end{pmatrix} = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$$

while

$$M E_n = \begin{pmatrix} A' & \alpha' \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & A' \\ 0 & 0 \end{pmatrix} .$$

Therefore, $M \in Z(E_n, \mathfrak{u})$ if and only if $A = A'$ if and only if $M \in \mathfrak{g}$. □

The nilpotent subalgebra $\mathfrak{g} \subset \mathfrak{u}$ appearing in the proof above will be called the **superdiagonal algebra**. The corresponding unipotent Lie group $G \subset U$ will be called the **superdiagonal subgroup** of $GL_n(k)$. Note that $G \cong \mathbb{G}_a^{n-1}$.

Next, let $Z(E_n) = Z(E_n, \mathfrak{gl}_n(k))$ denote the full centralizer of E_n.

Corollary 6.17. *A basis for $Z(E_n)$ is $\{I, E_n, E_n^2, ..., E_n^{n-1}\}$.*

Proof. Let $P \in \mathfrak{gl}_n(k)$ be given. The condition $PE_n = E_n P$ immediately implies P is upper triangular, and that its diagonal entries are equal. Thus, it is possible to write $P = cI + M$ for $M \in \mathfrak{u}$ and $c \in k$. Then $(I+M)E_n = E_n(I+M)$ implies $ME_n = E_n M$, so by the proposition, M is a linear combination of $E_n, E_n^2, ..., E_n^{n-1}$. Therefore, P is a linear combination of $I, E_n, E_n^2, ..., E_n^{n-1}$. □

An immediate consequence is the following.

Corollary 6.18. *If a rational representation $\mathbb{G}_a^2 \subset GL_n(k)$ admits a restriction to the basic $\mathbb{G}_a$-action on $\mathbb{A}^n$, then $\mathbb{G}_a^2 \subset G$, where G is the superdiagonal subgroup of $GL_n(k)$.*

The centralizer of a general element of $\mathfrak{u}$ can be similarly described, but the description is more complicated. For each postive integer j, let E_j denote the (upper triangular) elementary nilpotent matrix of order j. Given positive integers $i \geq j \geq r$, let $E_{(i,j,r)}$ denote the $i \times j$ matrix formed by E_j^r in the first j rows, and zeros elsewhere. Note that $E_{(j,j,1)} = E_j$.

Given $N \in \mathfrak{u}$, suppose N has Jordan block form

$$N = \begin{pmatrix} E_{s_1} & 0 & \cdots & 0 \\ 0 & E_{s_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_{s_\lambda} \end{pmatrix} ,$$

where the integers s_i satisfy $s_1 \geq s_2 \geq \cdots \geq s_\lambda$ and $s_1 + \cdots + s_\lambda = n$. Then $Q \in Z(N, \mathfrak{u})$ if and only if Q has the block form

$$Q = \begin{pmatrix} Q_{(1,1)} & Q_{(1,2)} & Q_{(1,3)} & \cdots & Q_{(1,\lambda)} \\ 0 & Q_{(2,2)} & Q_{(2,3)} & \cdots & Q_{(2,\lambda)} \\ 0 & & Q_{(3,3)} & \cdots & Q_{(3,\lambda)} \\ \vdots & \vdots & \vdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & Q_{(\lambda,\lambda)} \end{pmatrix} ,$$

where $Q_{(i,j)}$ belongs to $\oplus_{1\le r<s_j} kE_{(s_i,s_j,r)}$. Details are left to the reader.

Remark 6.19. The variety defined by pairs of commuting nilpotent matrices of a fixed dimension has been studied. In particular, it was shown by Baranovsky [10] that this variety is irreducible (2001). The paper of Basili [11] contains a very nice historical survey of this problem, and an elementary proof of irreducibility (2003). And the article of Schröer [270] studies certain subvarieties of this variety relative to their irreducible components (2004).

Remark 6.20. From one point of view, the reason that the invariant ring in Nagata's example is not finitely generated is that, while the vector group itself has a very simple structure, its embedding in $GL(V)$ is complicated with respect to the coordinate lines in $GL(V)$ constituted by by all 1-dimensional unipotent root subgroups relative to a fixed maximal torus T of $GL(V)$. This philosophy led to the Popov-Pommerening Conjecture . The conjecture claims that in the opposite case, where a unipotent subgroup H of $GL(V)$ is generated by some of these root subgroups (or equivalently, where H is normalized by T), the invariant algebra of H is finitely generated. This conjecture has been confirmed in many important special cases, but remains open in its full generality. It is one of the main problems in the invariant theory of linear actions of unipotent groups. For details, the reader is referred to [5, 131, 254].

6.5 Appendix: Finite Group Actions

The following fact is well-known, and is provided here for the readers' convenience. The statement of the proposition and the proof given here are due to Daigle (unpublished).

Proposition 6.21. *Suppose k is a field, and B is a finitely generated commutative k-domain. Let G be a group of algebraic k-automorphisms of B (i.e., G acts faithfully on B). Then the following are equivalent.*

(1) *G is finite*
(2) *B is integral over B^G*
(3) *B is algebraic over B^G*

Proof. We first show that, for given $b \in B$, the following are equivalent.

(4) The orbit $\mathcal{O}_b$ is finite
(5) b is integral over B^G
(6) b is algebraic over B^G

(4) $\Rightarrow$ (5): If $\mathcal{O}_b$ is finite, define the monic polynomial $f(x) \in k[x]$ by

$$f(x) = \prod_{a\in\mathcal{O}_b} (x-a) \ .$$

Then $f \in B^G[x]$ and $f(b) = 0$, and (5) follows.

(5) $\Rightarrow$ (6): Obvious.

(6) $\Rightarrow$ (4): If $h(x) \in B^G[x]$ and $h(b) = 0$ for nonzero h, choose $a \in \mathcal{O}_b$, and suppose $a = g \cdot b$ for $g \in G$. Then

$$h(a) = h(g \cdot b) = g \cdot h(b) = g \cdot 0 = 0 \ ,$$

i.e., every $a \in \mathcal{O}_b$ is a root of h. Since B is a domain, the number of roots of h is finite, and (4) follows.

Therefore (4),(5), and (6) are equivalent.

(1) $\Rightarrow$ (2): Choose $b \in B$. Since G is finite, $\mathcal{O}_b$ is finite, and therefore b is integral over B^G. So B is integral over B^G.

(2) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (1): Since $k \subset B^G$, we have that B is finitely generated over B^G. Write $B = B^G[x_1, ..., x_n]$, and define a function

$$\begin{aligned} G &\to \mathcal{O}_{x_1} \times \cdots \times \mathcal{O}_{x_n} \\ g &\mapsto (g \cdot x_1, ..., g \cdot x_n) \ , \end{aligned}$$

Now each element x_i is algebraic over B^G, meaning that each orbit $\mathcal{O}_{x_i}$ is finite. Therefore, the set $\mathcal{O}_{x_1} \times \cdots \times \mathcal{O}_{x_n}$ is also finite. In addition, the function above is injective, since the automorphism g of B is completely determined by its image on a set of generators. Therefore G is finite. $\square$

7

Non-Finitely Generated Kernels

This chapter shows that the solution to the Finiteness Problem for non-linear algebraic $\mathbb{G}_a$-actions is, in general, negative. In particular, we will explore the famous examples of Paul Roberts and some of the rich theory which has flowed from them. These were the first examples to (in effect) show that the kernel of a locally nilpotent derivation on a polynomial ring is not always finitely generated.

7.1 Roberts' Examples

In the mid-1980s, Roberts was studying the examples of Rees and Nagata from a point of view somewhat different than that presented above. The main idea of this approach is to consider a ring R which is the symbolic blow-up of a prime ideal P in a commutative Noetherian ring A. What this means is that R is isomorphic to a graded ring of the form $\oplus_{n\geq 0}P^{(n)}$, where $P^{(n)}$ denotes the nth symbolic power of P, defined as

$$P^{(n)} = \{x \in A \mid xy \in P^n \text{ for some } y \notin P\} \ .$$

Rees used symbolic blow-ups in constructing his counterexample to Zariski's Problem. In his 1985 paper [262], Roberts writes:

> In a few nice cases the symbolic blow-up of P is a Noetherian ring or, equivalently, a finitely generated A-algebra. In general, however, $\oplus P^{(n)}$ is not Noetherian. The first example of this is due to Rees.

It was in this paper that Roberts constructed a new counterexample to Zariski's problem similar to Rees's example, but having somewhat nicer properties. Subsequently, in a 1990 paper [263] Roberts constructed an important new counterexample to Hilbert's Fourteenth Problem along similar lines. In the latter paper, Roberts gave the following description of these developments.

> In his example, Rees takes R to be the coordinate ring of the cone over an elliptic curve and shows that if P is the prime ideal corresponding to a point of infinite order then the ring $\oplus_{n\geq 0}P^{(n)}$ is not finitely generated and is a counterexample to Zariski's problem. Shortly thereafter Nagata gave a counterexample to Hilbert's original problem, and, in fact gave a counterexample which was a ring of invariants of a linear group acting on a polynomial ring, which is the special case which motivated the original problem. In his example a similar construction to that of Rees was used in which P was not prime, but was the ideal defining sixteen generic lines through the origin in affine space of three dimensions. The proof was based on the existence of points of infinite order on elliptic curves.
> But this did not totally end the story. Rees's example uses a ring which is not regular, and Nagata's uses an ideal which is not prime; Cowsick then asked whether there were examples in which the ring was regular and the ideal prime. Such an example was given in Roberts [262]. However, this still did not totally finish the problem, since this example was based on that of Nagata and made crucial use of the fact that when the ring was completed the ideal broke up into pieces and did not remain prime.

Roberts proceeded to construct an example of a prime ideal in a complete regular local ring (a power series ring in seven variables) whose symbolic blow-up is not finitely generated.

Explicitly, Roberts takes k to be any field of characteristic 0 and $B = k^{[7]} = k[X,Y,Z,S,T,U,V]$, and defines a graded $k[X,Y,Z]$-module homomorphism $\phi : B \to B$. He proves that the kernel of ϕ is not finitely generated over k. This construction is then "completed" to give the example in terms of symbolic blow-ups.

In the paper, ϕ is defined explicitly by its effect on monomials in S,T,U,V. Though Roberts does not use the language of derivations, one recognizes from his description of these images that ϕ is equivalent to the triangular k-derivation $\mathcal{D}$ of B defined by

$$\mathcal{D} = X^{t+1}\partial_S + Y^{t+1}\partial_T + Z^{t+1}\partial_U + (XYZ)^t\partial_V \ ,$$

where $t \geq 2$. According to Roberts, this example originated in his study of Hochster's Monomial Conjecture, which had been proved for any field of characteristic 0. The conjecture asserted that for any local ring of dimension 3 with system of parameters X,Y,Z, and for any non-negative integer t, the monomial $X^tY^tZ^t$ is not in the ideal generated by the monomials X^{t+1}, Y^{t+1}, and Z^{t+1}.

It was shortly after its appearance that A'Campo-Neuen [4] recognized that Roberts' example could be realized as the invariant ring of an algebraic (but non-linear) $\mathbb{G}_a$-action on $\mathbb{A}^7$. This was recognized independently by Deveney and Finston in [75] at about the same time, and they give a different proof that the kernel of the derivation $\mathcal{D}$ above is not finitely generated in the case $t = 2$.

It was observed by van den Essen and Janssen [102] that, because of Roberts' example, there exist locally nilpotent derivations with non-finitely generated kernels in all dimensions higher than 7 as well. For example, if $\mathcal{D}$ is the derivation on $k[x_1, ..., x_7]$ associated with Roberts' example, and if $B^{\mathcal{D}}$ is its kernel, then the extension of $\mathcal{D}$ to $k[x_1, ..., x_n]$ ($n \geq 8$) obtained by setting $\mathcal{D}x_i = 0$ for $8 \leq i \leq n$ has kernel equal to $B^{\mathcal{D}}[x_8, ..., x_n]$, which is also not finitely generated. Another family of counterexamples in higher (odd) dimensions was given by Kojima and Miyanishi in [171]. They consider the triangular derivations λ on polynomial rings $k[x_1, ..., x_n, y_1, ..., y_n, z]$ defined by $\lambda(x_i) = 0$, $\lambda(y_i) = x_i^{t+1}$, and $\lambda(z) = (x_1 \cdots x_n)^t$. They prove that, for each $n \geq 3$ and $t \geq 2$, the kernel of λ is not finitely generated. Since Roberts' examples are included in this family as the case $n = 3$, their paper provides a new proof for Roberts' example as well.

Actually, it was Derksen [66] who first recognized a connection between counterexamples to Hilbert's problem and derivations, but the derivations he uses are in general not locally nilpotent. In particular, he constructs a derivation of the polynomial ring in 32 variables whose kernel coincides with the fixed ring of Nagata's example.

In [120], the author gave a triangular derivation in dimension 6 with non-finitely generated kernel, and then used this to give an independent proof for Roberts' example when $t = 2$. This proof is more constructive than those preceding it, using the Dixmier map in a crucial way. Then in [55], the author and Daigle presented a triangular derivation in dimension 5 with non-finitely generated kernel. In the present work, the reader will find two proofs for this dimension 5 example. The first proof, presented in the following section, is similar to that given by the author in [120]. This proof constitutes an algorithm for constructing the necessary sequence of kernel elements, a sequence which cannot be contained in any finitely generated subring of the kernel. The main step of this proof is to show the existence of a certain sequence of polynomials in four variables (*Thm. 7.1*). The second proof, given in the *Appendix* to this chapter, is a recent proof due to van den Essen. The first proof has the advantage of being readily adapted to the explicit construction of invariants, while the second proof has the advantage of being somewhat shorter. It is also shown in this chapter how non-finite generation for the Roberts example is deduced from the dimension 5 example via the Finiteness Theorem (*Thm. 7.7*). Other proofs for Roberts' example may be found in [100, 183, 201]; see also [179].

For the remainder of this chapter, we assume the underlying field k is of characteristic zero.

7.2 Counterexample in Dimension Five

7.2.1 The Basic Action in Dimension Four

Let $R = k[a, s, t, u]$ denote the polynomial ring in four variables, and let D be the basic linear derivation on R, i.e., $D = a\partial_s + s\partial_t + t\partial_u$. In *Example 8.9* below, it is calculated that $\ker D = k[a, F, G, H]$, where

$$F = 2at - s^2\,,\; G = 3a^2u - 3ast + s^3\,,\;\text{ and } a^2H = G^2 - F^3\;.$$

Observe that D is triangular and homogeneous, of degree $(0, -1)$, with respect to the following $\mathbb{Z}^2$-grading on R:

$$\deg a = (1, 0)\,,\quad \deg s = (1, 1)\,,\quad \deg t = (1, 2)\,,\quad \deg u = (1, 3)\;.$$

Define a sequence $t_n \in k[a]$ as follows:

$$t_1 = a\,,\quad t_2 = 1\,,\quad t_3 = a\,,\quad \text{and}\quad t_n = t_{n-3}\quad \text{for } n \geq 4\;.$$

This is a periodic sequence, with period 3. In addition, define a sequence of degrees by $\delta_0 = (0, 0)$, and $\delta_n = (0, n) + \sum_{j=1}^{n} \deg t_j$ for $n \geq 1$.

The purpose of this subsection is to prove the following theorem. The proof presented here is along the same lines as that found in [120], adapted to dimension 4.

Theorem 7.1. *There exists a sequence of homogeneous polynomials $w_n \in R$ $(n \geq 0)$ such that $w_0 = 1$; $w_1 = s$; $Dw_n = t_n w_{n-1}$ for all $n \geq 1$; and $\deg w_n = \delta_n$ for all n.*

Proof. Fix $w_0 = 1$ throughout. Suppose $w_1, ..., w_n$ $(n \geq 3)$ is any sequence of homogeneous polynomials in R with the property that $\deg w_m = \delta_m$ and $Dw_m = t_m w_{m-1}$ $(1 \leq m \leq n)$; and $w_m \in aR$ whenever $m \equiv 1 \pmod 3$ and $m > 1$. The proof follows in three steps.

Step 1: The Functions $\lambda_{(m,i)}$.

For every integer $m \geq 3$, define $\lambda_{(m,0)} = \lambda_{(m,m)} = 1$. We show the following: Given m with $3 \leq m \leq n+1$, and given i with $1 \leq i \leq m-1$, there is a unique $\lambda_{(m,i)} \in \{1, a, a^{-1}\}$ such that $\lambda_{(m,i)} w_i w_{m-i} \in R$ and $\deg(\lambda_{(m,i)} w_i w_{m-i}) = \delta_m$.

Note that the $\mathbb{Z}^2$-grading on R extends to a $\mathbb{Z}^2$-grading on $R[a^{-1}]$. Since $1 \leq i \leq m-1$, we see that:

$$\begin{aligned}\deg(\lambda_{(m,i)} w_i w_{m-i}) = \delta_m &\Leftrightarrow \deg \lambda_{(m,i)} + \delta_i + \delta_{m-i} = \delta_m \\ &\Leftrightarrow \deg \lambda_{(m,i)} + \sum_{j=1}^{i} \deg t_j + \sum_{j=1}^{m-i} \deg t_j = \sum_{j=1}^{m} \deg t_j \\ &\Leftrightarrow \deg \lambda_{(m,i)} + \sum_{j=1}^{i} \deg t_j = \sum_{j=m-i+1}^{m} \deg t_j\;.\end{aligned}$$

Since $\deg t_\ell + \deg t_{\ell+1} + \deg t_{\ell+2} = (2,0)$ for every $\ell \geq 1$, we may eliminate terms from each side of this last equality in groups of three. If $i \equiv 0 \pmod 3$, this leaves:

$$\deg \lambda_{(m,i)} = (0,0) \quad \Leftrightarrow \quad \lambda_{(m,i)} = 1 \ .$$

If $i \equiv 1 \pmod 3$, we have:

$$\begin{aligned} \deg \lambda_{(m,i)} + (1,0) &= \deg t_m \in \{(1,0),(0,0)\} \\ &\Leftrightarrow \deg \lambda_{(m,i)} \in \{(0,0),(-1,0)\} \\ &\Leftrightarrow \lambda_{(m,i)} \in \{1, a^{-1}\} \ . \end{aligned}$$

And if $i \equiv 2 \pmod 3$, then:

$$\begin{aligned} \deg \lambda_{(m,i)} + (1,0) &= \deg t_{m-1} + \deg t_m \in \{(1,0),(2,0)\} \\ &\Leftrightarrow \deg \lambda_{(m,i)} \in \{(0,0),(1,0)\} \\ &\Leftrightarrow \lambda_{(m,i)} \in \{1, a\} \ . \end{aligned}$$

The only case in which $\lambda_{(m,i)}$ fails to be a polynomial is the case $\lambda_{(m,i)} = a^{-1}$, which occurs if and only if $i \equiv 1 \pmod 3$ and $m \equiv 2 \pmod 3$. In this case, if $i \geq 4$, then by hypothesis, w_i is divisible by a; and if $i = 1$, then $m - i \geq 4$ and $m - i \equiv 1 \pmod 3$, so w_{m-i} is divisible by a. Therefore, $\lambda_{(m,i)} w_i w_{m-i}$ is always a polynomial.

Step 2: The Polynomials $\zeta_{(m,i)}$.

Define the index set $J \subset \mathbb{Z}^2$ by

$$J = \{(m,i) \ : \ 3 \leq m \leq n+1 \ , \ 1 \leq i \leq \tfrac{m}{2}\} \ \cup \ \{(m,0) \ : \ 3 \leq m \leq n\} \ ,$$

noting that w_i and w_{m-i} are defined for each $(m,i) \in J$. We define polynomials

$$\zeta_{(m,i)} = \zeta_{(m,m-i)} = \lambda_{(m,i)} w_i w_{m-i} \qquad (m,i) \in J \ . \tag{7.1}$$

Note that $\zeta_{(m,i)} \in aR$ whenever $m \equiv 1 \pmod 3$ and $i > 1$. To see this, observe that, in this case, $\lambda_{(m,i)} \in R$. If $i \equiv 0 \pmod 3$, then $m - i \equiv 1 \pmod 3$ and $m - i > 1$, implying $w_{m-i} \in aR$; if $i \equiv 1 \pmod 3$, then $w_i \in aR$ (since $i > 1$); and if $i \equiv 2 \pmod 3$, then $\lambda_{(m,i)} = a$.

Applying D to equation (7.1) above yields:

$$D\zeta_{(m,i)} = t_{m-i} \lambda_{(m,i)} w_i w_{(m-1)-i} + t_i \lambda_{(m,i)} w_{i-1} w_{(m-1)-(i-1)} \ ,$$

where $4 \leq m \leq n+1$ and $1 \leq i \leq m$. Observe that

$$\delta_m = \deg(\lambda_{(m,i)} w_i w_{m-i}) = \deg \lambda_{(m,i)} + \delta_i + \delta_{m-i} \ .$$

Since $\delta_\ell = (0,\ell) + \sum_{j=1}^{\ell} \deg t_j$ for $\ell \geq 0$, it follows that

$$\begin{aligned}&\deg(t_{m-i}\lambda_{(m,i)}) - \deg(t_m\lambda_{(m-1,i)})\\ &= [\deg t_{m-i} + (\delta_m - \delta_i - \delta_{m-i})] - [\deg t_m + (\delta_{m-1} - \delta_i - \delta_{(m-1)-i})]\\ &= (0,0)\ .\end{aligned}$$

Likewise,

$$\begin{aligned}&\deg(t_i\lambda_{(m,i)}) - \deg(t_m\lambda_{(m-1,i-1)})\\ &= [\deg t_i + (\delta_m - \delta_i - \delta_{m-i})] - [\deg t_m + (\delta_{m-1} - \delta_{i-1} - \delta_{m-i})]\\ &= (0,0)\ .\end{aligned}$$

Now the t_i and $\lambda_{(m,i)}$ belong to $k[a,a^{-1}]$, and the $\mathbb{Z}^2$-grading of $R[a^{-1}]$ restricts to a $\mathbb{Z}^2$-grading on $k[a,a^{-1}]$. In particular, homogeneous elements of $k[a,a^{-1}]$ of the same degree are equal (up to constants), and it follows that:

$$t_{m-i}\lambda_{(m,i)} = t_m\lambda_{(m-1,i)} \quad \text{and} \quad t_i\lambda_{(m,i)} = t_m\lambda_{(m-1,i-1)}\ .$$

Consequently:

$$D\zeta_{(m,i)} = t_m\left(\zeta_{(m-1,i)} + \zeta_{(m-1,i-1)}\right) \quad (m \geq 4,\ 1 \leq i \leq m)\ . \tag{7.2}$$

Step 3: The Polynomials w_n.

We construct the sequence w_n inductively in blocks of six. The reader is reminded that $w_0 = 1$, that $\zeta_{(n,0)}$ denotes w_n, and that $\zeta_{(n,i)} = \zeta_{(n,n-i)}$.

Define w_1 through w_7 as follows:

$$\begin{aligned} w_1 = s\,, \quad w_2 = t\,, \quad w_3 = au\\ w_4 = \tfrac{1}{2}(2\zeta_{(4,1)} - \zeta_{(4,2)})\,, \quad w_5 = \tfrac{1}{5}(3\zeta_{(5,1)} - \zeta_{(5,2)})\\ w_6 = \tfrac{1}{7}(2\zeta_{(6,1)} + \zeta_{(6,2)} - \zeta_{(6,3)})\,, \quad w_7 = \tfrac{1}{7}(2\zeta_{(7,2)} - \zeta_{(7,3)})\end{aligned}$$

Then $w_4, w_7 \in aR$, and $Dw_n = t_n w_{n-1}$ for $1 \leq n \leq 7$, as desired.

Inductively, suppose that $w_1, w_2, ..., w_{6m-5}$ ($m \geq 2$), are homogeneous polynomials in R such that $\deg w_i = \delta_i$ and $Dw_i = t_i w_{i-1}$ for $1 \leq i \leq (6m-5)$; and $w_i \in aR$ for every $i > 1$ such that $i \equiv 1 \,(\mathrm{mod}\, 3)$.

Set $n = 6m-4$, which is even, and define

$$w_n = \zeta_{(n,1)} - \zeta_{(n,2)} + \zeta_{(n,3)} - \cdots + (-1)^{\frac{n}{2}}\zeta_{(n,\frac{n-2}{2})} + \tfrac{1}{2}(-1)^{\frac{n+2}{2}}\zeta_{(n,\frac{n}{2})}\ . \tag{7.3}$$

Applying D to w_n, it follows immediately from equation (7.2) above that $Dw_n = t_n w_{n-1}$. This calculation uses the fact that $\zeta_{(n-1,\frac{n}{2})} = \zeta_{(n-1,\frac{n-2}{2})}$.

For notational convenience, "reset" n so that $n = 6m-2$, which is also even. Observe that $\zeta_{(n,i)}$ is defined, and $\zeta_{(n,i)} \in aR$, so long as $2 \leq i \leq \frac{n}{2}$. For rational unknowns c_i, set

$$w_n = c_1\zeta_{(n,2)} + c_2\zeta_{(n,3)} + \cdots + c_{\frac{n-2}{2}}\zeta_{(n,\frac{n}{2})}\ , \tag{7.4}$$

noting that $w_n \in aR$. Applying D twice, it follows from equation (7.2) that

$$\begin{aligned}(t_n t_{n-1})^{-1} D^2 w_n = c_1 w_{n-2} &+ (2c_1 + c_2)\zeta_{(n-2,1)} \\ &+(c_1 + 2c_2 + c_3)\zeta_{(n-2,2)} + (c_2 + 2c_3 + c_4)\zeta_{(n-2,3)} \\ &+\cdots \\ &+(c_{\frac{n-8}{2}} + 2c_{\frac{n-6}{2}} + c_{\frac{n-4}{2}})\zeta_{(n-2,\frac{n-6}{2})} \\ &+ (c_{\frac{n-6}{2}} + 2c_{\frac{n-4}{2}} + 2c_{\frac{n-2}{2}})\zeta_{(n-2,\frac{n-4}{2})} \\ &+ (c_{\frac{n-4}{2}} + 2c_{\frac{n-2}{2}})\zeta_{(n-2,\frac{n-2}{2})} \ .\end{aligned}$$

This calculation uses the fact that $\zeta_{(n-2,\frac{n}{2})} = \zeta_{(n-2,\frac{n-4}{2})}$. Replacing w_{n-2} as in equation (7.3) above, we obtain the system $Ac = y$, where $c = (c_1, ..., c_{\frac{n-2}{2}})^T$, $y = (1, -1, 1, ..., (-1)^{\frac{n-2}{2}}, \frac{1}{2}(-1)^{\frac{n}{2}})^T$, and A is the square matrix of order $\frac{n-2}{2}$ given by:

$$A = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ -1 & 0 & 1 & 2 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & & \vdots & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & & & \ddots & \ddots & \ddots & \vdots & \vdots \\ (-1)^{\frac{n-4}{2}} & 0 & \cdots & \cdots & \cdots & \cdots & 1 & 2 & 1 & 0 \\ (-1)^{\frac{n-2}{2}} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 2 & 2 \\ \frac{1}{2}(-1)^{\frac{n}{2}} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 1 & 2 \end{pmatrix} .$$

Since $|A| = n - 1$, the system $Ac = y$ can be solved for c. In fact, it is easy to solve: The alternating sum of the rows of A produces a 0 in every entry but the first, which yields $c_1 = \frac{n-3}{n-1}$; then $c_2 = 1 - 3c_1$, $c_3 = -1 - 2c_2$, etc. Note that c_i is an *alternating* sequence of rationals: $c_1 > 0$, $c_2 < 0$, etc.

As for $w_{n-1} = w_{6m-3}$, note that $t_n = a$ divides Dw_n (by equation (7.2)), so we simply set $w_{n-1} = a^{-1}Dw_n$. Since $D^2 w_n = a^2 w_{n-2}$, it follows that $Dw_{n-1} = aw_{n-2}$, as desired.

Next, set $n = 6m - 1$, which is odd. For rational unknowns d_i, define

$$w_n = d_1\zeta_{(n,1)} + d_2\zeta_{(n,2)} + \cdots + d_{\frac{n-1}{2}}\zeta_{(n,\frac{n-1}{2})} \ . \tag{7.5}$$

Applying D to w_n, and replacing w_{n-1} as in equation (7.4) above, we obtain the system $Bd = c'$, where $c' = (0, c_1, c_2, ..., c_{\frac{n-3}{2}})^T$, $d = (d_1, d_2, ..., d_{\frac{n-1}{2}})^T$, and B is the square matrix of order $\frac{n-1}{2}$ given by

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ c_1 & 1 & 1 & 0 & \cdots & \cdots & 0 \\ c_2 & 0 & 1 & 1 & \cdots & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ c_{\frac{n-5}{2}} & 0 & \cdots & \cdots & 0 & 1 & 1 \\ c_{\frac{n-3}{2}} & 0 & \cdots & \cdots & 0 & 0 & 1 \end{pmatrix} .$$

The determinant of B equals $1 - \sum(-1)^{i+1}c_i$, where i ranges from 1 to $\frac{n-3}{2}$. Because the c_i alternate,

$$\sum(-1)^{i+1}c_i \geq c_1 - c_2 = 4c_1 - 1 = \frac{3n-14}{n-2} > 2 \quad (\text{since} \quad n \geq 11) .$$

Therefore, $|B| < -1$, and the system $Bd = c'$ may be solved for d. Again, it is easy to solve: The alternating sum of the rows of B produces a 0 in every entry but the first, which yields $d_1 = -d_2 = -(c_1 - c_2 + c_3 - \cdots)/|B|$; then $d_3 = (c_2 - c_3 + c_4 - \cdots)/|B|$, $d_4 = (c_3 - c_4 + c_5 - \cdots)/|B|$, etc. Note that d_i is alternating: $d_1 > 0$, $d_2 < 0$, etc.

Finally, reset $n = 6m+1$, which is also odd. Observe that $\zeta_{(n,i)}$ is defined, and $\zeta_{(n,i)} \in aR$, so long as $2 \leq i \leq \frac{n-1}{2}$. For rational unknowns e_i, set

$$w_n = e_1\zeta_{(n,2)} + e_2\zeta_{(n,3)} + \cdots + e_{\frac{n-3}{2}}\zeta_{(n,\frac{n-1}{2})} ,$$

noting that $w_n \in aR$. As before, apply D to w_n twice, using equations (7.2) and (7.5) above, to obtain the system $Ce = d$, where $e = (e_1, e_2, ..., e_{\frac{n-3}{2}})^T$, and C is the square matrix of order $\frac{n-3}{2}$ given by:

$$C = \begin{pmatrix} (d_1+2) & 1 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ (d_2+1) & 2 & 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ d_3 & 1 & 2 & 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ d_4 & 0 & 1 & 2 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & & \vdots & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & & & \ddots & \ddots & \ddots & \vdots & \vdots \\ d_{\frac{n-7}{2}} & 0 & \cdots & \cdots & \cdots & \cdots & 1 & 2 & 1 & 0 \\ d_{\frac{n-5}{2}} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 2 & 1 \\ d_{\frac{n-3}{2}} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 1 & 3 \end{pmatrix} .$$

This calculation uses the fact that $\zeta_{(n-2,\frac{n-1}{2})} = \zeta_{(n-2,\frac{n-3}{2})}$. The determinant of C equals $(n - 2 + \sum(-1)^{i+1}(n-2i-2)d_i)$, as i ranges from 1 to $\frac{n-3}{2}$. Since d_i is an alternating sequence of rationals, with $d_1 > 0$, we conclude that $|C| > 0$, and that the system can be solved for e.

As for $w_{n-1} = w_{6m}$, proceed as for w_{6m-3}: a divides Dw_n, and we set $w_{n-1} = a^{-1}Dw_n$. Since $D^2w_n = a^2w_{n-2}$, it follows that $Dw_{n-1} = aw_{n-2}$, as desired. This completes the proof of the theorem. □

An immediate consequence of *Thm. 7.1* is the following formula, which is needed below.

$$D^{3i} w_{3m} = a^{2i} w_{3(m-i)} \qquad (m \geq 1, 0 \leq i \leq m) \tag{7.6}$$

Example 7.2. The first seven w_i are listed above. The next block of six may be constructed as in the proof above, and is as follows.

$$\begin{aligned}
w_8 &= \tfrac{1}{2}\left(2\zeta_{(8,1)} - 2\zeta_{(8,2)} + 2\zeta_{(8,3)} - \zeta_{(8,4)}\right)\\
w_9 &= \tfrac{1}{9}\left(7\zeta_{(9,1)} - 5\zeta_{(9,2)} + 3\zeta_{(9,3)} - \zeta_{(9,4)}\right)\\
w_{10} &= \tfrac{1}{9}\left(7\zeta_{(10,2)} - 12\zeta_{(10,3)} + 15\zeta_{(10,4)} - 8\zeta_{(10,5)}\right)\\
w_{11} &= \tfrac{1}{33}\left(42\zeta_{(11,1)} - 42\zeta_{(11,2)} + 35\zeta_{(11,3)} - 23\zeta_{(11,4)} + 8\zeta_{(11,5)}\right)\\
w_{12} &= \tfrac{1}{117}\left(84\zeta_{(12,1)} - 42\zeta_{(12,2)} + 35\zeta_{(12,4)} - 58\zeta_{(12,5)} + 33\zeta_{(12,6)}\right)\\
w_{13} &= \tfrac{1}{117}\left(84\zeta_{(13,2)} - 126\zeta_{(13,3)} + 126\zeta_{(13,4)} - 91\zeta_{(13,5)} + 33\zeta_{(13,6)}\right)
\end{aligned}$$

7.2.2 Dimension Five

Our next goal is to prove the following.

Theorem 7.3. (Daigle and Freudenburg [55]) *Let Δ be the triangular monomial derivation of $k[x,s,t,u,v] = k^{[5]}$ defined by*

$$\Delta = x^3\partial_s + s\partial_t + t\partial_u + x^2\partial_v \ .$$

Then the kernel of Δ is not finitely generated as a k-algebra.

In order to prove *Thm. 7.3*, we make use of the following general result (see Lemma 2.1 of [55]).

Lemma 7.4. (Non-Finiteness Criterion) *Let $K = \oplus_{i\in\mathbb{N}} K_i$ be a graded k-domain such that $K_0 = k$, and let δ be a homogeneous locally nilpotent k-derivation of K. Given homogeneous $\alpha \in \ker\delta$ which is not in the image of δ, let $\tilde{\delta}$ be the extension of δ to $K[T]$ defined by $\tilde{\delta}T = \alpha$, where T is a variable over K. Suppose β_n is a sequence of non-zero elements of $\ker\tilde{\delta}$ having leading T-coefficients $b_n \in K$. If $\deg b_n$ is bounded, but $\deg_T \beta_n$ is not bounded, then $\ker\tilde{\delta}$ is not finitely generated over k.*

Proof. Let $M[T]$ be the extension to $K[T]$ of the maximal ideal $M = \oplus_{i>0} K_i$ of K. Set $m = \deg\alpha - \deg\delta$, and for every integer n, define $K[T]_n = \sum_{i\in\mathbb{N}} K_{n-mi}T^i$. Then $\oplus_{n\in\mathbb{Z}} K[T]_n$ is a $\mathbb{Z}$-grading of $K[T]$, and $\tilde{\delta}$ is homogeneous.

If $\phi \in \ker\tilde{\delta}$ is homogeneous, then $\phi = \sum \phi_i T^i$ for homogeneous $\phi_i \in K$. Since $\tilde{\delta}(\phi) = 0$, it follows easily from the product rule that $\delta(\phi_{i-1}) = -i\alpha\phi_i$ for $i > 0$. Thus, $\phi_i \notin k^*$ for $i > 0$, since otherwise $\alpha = \delta(-i^{-1}\phi_i^{-1}\phi_{i-1})$, which is in the image of δ. So if $i > 0$, then $\phi_i \in M$, since each ϕ_i is homogeneous. Since also $\phi_0 \in k + M$, we conclude that $\phi \in k + M[T]$.

Now for a general element $\psi \in \ker \tilde{\delta}$, write $\psi = \sum \psi_n$, where $\psi_n \in K[T]_n$. Since $\tilde{\delta}$ is homogeneous, we conclude that each $\psi_n \in \ker \tilde{\delta}$ as well. Therefore, $\ker \tilde{\delta} \subset k + M[T]$.

Finally, suppose $\{\beta_n\} \subset k[f_1, ..., f_N]$ for $f_i \in \ker \tilde{\delta}$. Then f_i belongs to $k + M[T]$ for each i, and it is no loss of generality to assume that each $f_i \in M[T]$. This implies that any monomial expression $f_{i_1} f_{i_2} \cdots f_{i_s}$ belongs to $M^s[T]$. Note that every K-coefficient of an element in $M^s[T]$ has degree at least s. Since $\deg b_n$ is bounded, there is a finite set F of monomial expressions $f_{i_1} f_{i_2} \cdots f_{i_s}$ such that $\{\beta_n\} \subset \langle F \rangle$, the k-linear span of F. However, the T-degrees in $\langle F \rangle$ are bounded, whereas the T-degrees for the sequence β_n are unbounded, a contradiction. Therefore, the sequence β_n is not contained in any finitely generated subset of $\ker \tilde{\delta}$. $\square$

As above, let D be the basic derivation on $R = k[a, s, t, u]$, with elements t_n and w_n as in *Thm. 7.1*. Let x be an integral element over R with $x^3 = a$, and let v be transcendental over R. Then $S := R[x, v] = k[x, s, t, u, v] = k^{[5]}$. Moreover, Δ restricts to both R and $k[x, s, t, u]$, and $\Delta|_R = D$.

The proof of *Thm. 7.3* now consists of combining *Lemma 7.4* with the following.

Lemma 7.5. *Let $A = \ker \Delta$, and let $\pi_v : S \to A_{\Delta v}$ be the Dixmier map for Δ induced by the local slice v. For each $m \geq 1$, $(-1)^{3m}(3m)!\pi_v(xw_{3m})$ is an element of A of the form*

$$xv^{3m} + \sum_{i=0}^{3m-1} \beta_i v^i \qquad (\beta_i \in k[x, s, t, u]) \ .$$

Proof. From *Thm. 7.1* and formula (7.6) above, we see that

$$\begin{aligned}
\Delta^{3i}(xw_{3m}) &= x(\Delta v)^{3i} w_{3(m-i)} \qquad (0 \leq i \leq m); \\
\Delta^{3i+1}(xw_{3m}) &= x^2(\Delta v)^{3i+1} w_{3(m-i)-1} \quad (0 \leq i < m); \\
\Delta^{3i+2}(xw_{3m}) &= (\Delta v)^{3i+2} w_{3(m-i)-2} \quad (0 \leq i < m) \ .
\end{aligned}$$

In particular, $(\Delta v)^j$ divides $\Delta^j(xw_{3m})$ for every j $(0 \leq j \leq 3m)$, and it follows that

$$\pi_v(xw_{3m}) = \sum_{j=0}^{3m} \frac{(-1)^j}{j!} \Delta^j(xw_{3m}) \frac{v^j}{(\Delta v)^j}$$

is a polynomial, i.e., belongs to A. Since its leading v-term is $\frac{(-1)^{3m}}{(3m)!} xv^{3m}$, the desired result follows. $\square$

Remark 7.6. By also considering $\pi_v(w_n)$ for $n = 3m + 1$ or $n = 3m + 2$, one obtains, for *all* $n \geq 1$, an element of $\ker \Delta$ whose leading v-term is xv^n.

7.2.3 Dimension Seven

Given $n \geq 3$, let $\mathcal{D}_n$ be the Roberts derivation on $B = k[X,Y,Z,S,T,U,V]$ defined by
$$\mathcal{D}_n = X^n\partial_S + Y^n\partial_T + Z^n\partial_U + (XYZ)^{n-1}\partial_V \ .$$
This derivation has many symmetries. For example, consider the faithful action of the torus $\mathbb{G}_m^3$ on B defined by
$$(\lambda,\mu,\nu)\cdot(X,Y,Z,S,T,U,V) = (\lambda X, \mu Y, \nu Z, \lambda^n S, \mu^n T, \nu^n U, (\lambda\mu\nu)^{n-1}V) \ .$$
This torus action commutes with $\mathcal{D}_n$.

Similarly, there is an obvious action of the symmetric group S_3 on B, with orbits $\{X,Y,Z\}$, $\{S,T,U\}$, and $\{V\}$. Specifically, S_3 is generated by $\sigma = (Z,X,Y,U,S,T,V)$ and $\tau = (X,Z,Y,S,U,T,V)$. Again, this action commutes with $\mathcal{D}_n$.

It is easily checked that S_3 acts on $\mathbb{G}_m^3$ by conjugation, and we thus obtain an action of $\mathbb{G}_m^3 \rtimes S_3$ on B. Specifically, $\tau(\lambda,\mu,\nu)\tau = (\lambda,\nu,\mu)$ and $\sigma(\lambda,\mu,\nu)\sigma^{-1} = (\nu,\lambda,\mu)$.

Since the $\mathbb{G}_m^3$-action does not have many invariants, consider instead the subgroup H of $\mathbb{G}_m^3$ defined by $\lambda\mu\nu = 1$. H is a 2-dimensional torus, and the group $G := H \rtimes S_3$ acts on B. The invariant ring of H is generated by monomials, namely,
$$B^H = k[XYZ,\ STU,\ X^nZ^nT,\ X^nY^nU,\ Y^nZ^nS,\ Y^nSU,\ X^nTU,\ Z^nST,\ V] \ .$$
Since H is normal in G, the fixed ring of the G-action is
$$B^G = (B^H)^{S_3} = k[x,s,t,u,v] \ ,$$
where
$$x = XYZ\,,\ \ s = X^nY^nU + X^nZ^nT + Y^nZ^nS\,,$$
$$t = X^nTU + Y^nSU + Z^nST\,,\ \ u = STU\,,\ \ v = V\,.$$
The main point to observe is that B^G is a polynomial ring: $B^G \cong k^{[5]}$. Since the action of G commutes with $\mathcal{D}_n$, it follows that $\mathcal{D}_n$ restricts to a locally nilpotent derivation of B^G. In particular,
$$\mathcal{D}_n x = 0\,,\ \mathcal{D}_n s = 3x^n\,,\ \mathcal{D}_n t = 2s\,,\ \mathcal{D}_n u = t\,,\ \mathcal{D}_n v = x^{n-1}\,. \tag{7.7}$$
This leads to:

Theorem 7.7. *Let $\mathcal{D}_3$ be the Roberts derivation on $B = k[X,Y,Z,S,T,U,V]$ defined by*
$$\mathcal{D}_3 = X^3\partial_S + Y^3\partial_T + Z^3\partial_U + (XYZ)^2\partial_V \ .$$
The kernel of $\mathcal{D}_3$ is not finitely generated.

Proof. From (7.7) we see that $\mathcal{D}_3$ restricted to $B^G = k^{[5]}$ is manifestly the same derivation as in *Thm. 7.3*, and we conclude that the invariant ring $B^{G\times\mathbb{G}_a} = (B^G)^{\mathbb{G}_a}$ is not finitely generated in this case. Suppose $B^{\mathbb{G}_a}$ were finitely generated. Then the reductive group G acts on the variety $\mathrm{Spec}(B^{\mathbb{G}_a})$, and by Hilbert's Finiteness Theorem the invariant ring $(B^{\mathbb{G}_a})^G = B^{\mathbb{G}_a\times G}$ would be finitely generated, a contradiction. Therefore, $B^{\mathbb{G}_a}$ is not finitely generated. □

Of course, Roberts showed non-finite generation of $\ker \mathcal{D}_n$ for all $n \geq 3$, but the proof given here for $n = 3$ does not generalize. However, by assuming Roberts' result we can, in turn, show the following.

Theorem 7.8. *Let μ_n be the triangular monomial derivation of $k[x,s,t,u,v]$ defined by*

$$\mu_n = x^{n+1}\partial_s + s\partial_t + t\partial_u + x^n\partial_v \ .$$

If $n \geq 2$, then the kernel of μ_n is not finitely generated as a k-algebra.

Proof. According to Lemma 3 of [263], there exists a sequence of homogeneous elements $\alpha_m \in \ker \mathcal{D}_n$ such that the leading V-term of α_m is XV^m. Let β_m denote the product of all elements in the orbit of α_m under the action of S_3. Then $\{\beta_m\} \subset B^G$ and β_m has leading V-term $(XYZ)^2V^{6m}$. This implies that the kernel of μ_n contains a sequence whose leading v-term is x^2v^{6m}. Applying the Non-Finiteness Criterion now implies that $\ker \mu_n$ is not finitely generated. □

Observe that $\mu_n(s - xv) = 0$ and that $k[x,s,t,u,v] = k[x, s - xv, t, u, v]$. Let κ be the polynomial automorphism of $k[x,s,t,u,v]$ fixing x,t,u,v and mapping $s \mapsto s + xv$. Then

$$\kappa\mu_n\kappa^{-1} = x^n\partial_v + (s + xv)\partial_t + t\partial_u \ ,$$

which gives the following.

Corollary 7.9. *Let $\mathcal{O} = k[a,b] = k^{[2]}$, and let d_n be the triangular $\mathcal{O}$-derivation of $\mathcal{O}[x,y,z] = k^{[5]}$ defined by*

$$d_n = a^n\partial_x + (ax + b)\partial_y + y\partial_z \ .$$

If $n \geq 2$, then the kernel of d_n is not finitely generated as an $\mathcal{O}$-algebra.

One may thus view the derivation d_n on $k^{[5]}$ as Roberts' derivation $\mathcal{D}_n$ on $k^{[7]}$ with all of the (obvious) symmetries removed.

Remark 7.10. It is now easy to find triangular derivations of $k^{[6]}$ with non-finitely generated kernel. For example, if d_n on $k[x,s,t,u,v]$ is as in *Cor. 7.9*, extend d_n to $k[x,s,t,u,v,w]$ by $d_n w = 0$. Another example is found in [120], where a direct proof is given that the triangular monomial derivation

$$\delta = x^3\partial_s + y^3 s\partial_t + y^3 t\partial_u + x^2y^2\partial_v$$

on $k[x,y,s,t,u,v]$ has non-finitely generated kernel. Note that, in dimension 5, $\Delta_3 = \delta \bmod (y - 1)$ on $k[x,s,t,u,v]$.

Remark 7.11. In [183], Kuroda calculated generating sets and Hilbert series for the kernels of the Roberts derivations $\mathcal{D}_n$. Tanimoto [293] likewise calculated generating sets and Hilbert series for the example of Freudenburg in dimension 6 above, and also for the example Δ of Daigle and Freudenburg in dimension 5.

7.3 Proof for A'Campo-Neuen's Example

We can now give a proof that the fixed ring of A'Campo-Neuen's linear $\mathbb{G}_a^{12}$-action on $\mathbb{A}^{19}$ is not finitely generated. The matrix form of this action was given in the preceding chapter. A'Campo-Neuen's proof is quite elegant, and we follow it here, emphasizing the role of commuting locally nilpotent derivations.

Let $Q = k[w,x,y,z] = k^{[4]}$. The proof uses (repeatedly) the fact that, given $f \in k[x,y,z]$, the triangular Q-derivation of $Q[\lambda,\mu] = Q^{[2]}$ defined by

$$\theta = w\partial_\lambda + f(x,y,z)\partial_\mu$$

has

$$\ker\theta = Q[w\mu - f\lambda] = Q^{[1]}\ .$$

This equaltiy is an immediate consequence of the results of *Chap. 4*.

Let $\Omega = Q[\mathbf{s},\mathbf{t},\mathbf{u},\mathbf{v}] = k^{[19]}$, where

$$\mathbf{x} = (x,y,z)\ ,\ \mathbf{s} = (s_1,s_2,s_3)\ ,\ \mathbf{t} = (t_1,t_2,t_3)\ ,\ \mathbf{u} = (u_1,u_2,u_3)\ ,$$
$$\text{and } \mathbf{v} = (v_1,v_2,v_3,v_4,v_5,v_6)\ .$$

Define commuting linear derivations $D_1, ..., D_{12} \in \mathrm{LND}_Q(\Omega)$ as follows:

$$\begin{array}{lll}
D_1 = w\partial_{v_5} + z\partial_{v_6} & D_2 = w\partial_{v_4} + z\partial_{v_5} & D_3 = w\partial_{v_3} + y\partial_{v_4} \\
D_4 = w\partial_{v_2} + y\partial_{v_3} & D_5 = w\partial_{v_1} + x\partial_{v_2} & D_6 = w\partial_{u_2} + z\partial_{u_3} \\
D_7 = w\partial_{u_1} + z\partial_{u_2} & D_8 = w\partial_{t_2} + y\partial_{t_3} & D_9 = w\partial_{t_1} + y\partial_{t_2} \\
D_{10} = w\partial_{s_2} + x\partial_{s_3} & D_{11} = w\partial_{s_1} + x\partial_{s_2} &
\end{array}$$

and

$$D_{12} = x\partial_{s_1} + y\partial_{t_1} + z\partial_{u_1} + x\partial_{v_1}\ .$$

Given i, $1 \le i \le 12$, let $\Omega_i = (((\Omega^{D_1})^{D_2})\cdots)^{D_i}$. Then successive use of the the kernel calculation above shows that

$$\begin{aligned}
\Omega_1 &= k^{[18]} = Q[\mathbf{s},\mathbf{t},\mathbf{u},v_1,v_2,v_3,v_4,v_5']\ ,\ \text{where } v_5' = zv_5 - wv_6 \\
\Omega_2 &= k^{[17]} = Q[\mathbf{s},\mathbf{t},\mathbf{u},v_1,v_2,v_3,v_4']\ ,\ \text{where } v_4' = z^2v_4 - wv_5' \\
\Omega_3 &= k^{[16]} = Q[\mathbf{s},\mathbf{t},\mathbf{u},v_1,v_2,v_3']\ ,\ \text{where } v_3' = yz^2v_3 - wv_4' \\
\Omega_4 &= k^{[15]} = Q[\mathbf{s},\mathbf{t},\mathbf{u},v_1,v_2']\ ,\ \text{where } v_2' = y^2z^2v_2 - wv_3' \\
\Omega_5 &= k^{[14]} = Q[\mathbf{s},\mathbf{t},\mathbf{u},V]\ ,\ \text{where } V = xy^2z^2v_1 - wv_2' \\
\Omega_6 &= k^{[13]} = Q[\mathbf{s},\mathbf{t},u_1,u_2',V]\ ,\ \text{where } u_2' = zu_2 - wu_3
\end{aligned}$$

$$
\begin{aligned}
\Omega_7 &= k^{[12]} = Q[\mathbf{s}, \mathbf{t}, U, V] \ , \ \text{where } U = z^2 u_1 - w u_2' \\
\Omega_8 &= k^{[11]} = Q[\mathbf{s}, t_1, t_2', U, V] \ , \ \text{where } t_2' = y t_2 - w t_3 \\
\Omega_9 &= k^{[10]} = Q[\mathbf{s}, T, U, V] \ , \ \text{where } T = y^2 t_1 - w t_2' \\
\Omega_{10} &= k^{[9]} = Q[s_1, s_2', T, U, V] \ , \ \text{where } s_2' = x s_2 - w s_3 \\
\Omega_{11} &= k^{[8]} = Q[S, T, U, V] \ , \ \text{where } S = x^2 s_1 - w s_2' \ .
\end{aligned}
$$

Finally, the effect of D_{12} on Ω_{11} is

$$
D_{12} S = x^3 \ , \ D_{12} T = y^3 \ , \ D_{12} U = z^3 \ , \ D_{12} V = (xyz)^2 \ ,
$$

and this is just the Roberts derivation extended to $k^{[8]}$ by $D_{12}(w) = 0$. Therefore, $\Omega_{12} = B^{\mathcal{D}}[w] = (B^{\mathcal{D}})^{[1]}$, where $B^{\mathcal{D}}$ denotes the kernel of the Roberts derivation, and this implies Ω_{12} is not finitely generated.

Remark 7.12. The recent paper of Tanimoto [292] gives the following generalization of A'Campo-Neuen's result. For $n \geq 2$, let δ be an elementary monomial derivation of $B = k^{[n]}$ which does not have a slice. There exist integers m, N and a linear representation of $G = \mathbb{G}_a^m$ on $X = \mathbb{A}^N$ such that $k[X]^G \cong (\ker \delta)^{[1]}$ (Cor. 1.4 of [292]).

7.4 Quotient of a $\mathbb{G}_a$-Module

The main purpose of this section is to prove the following result involving sums of basic $\mathbb{G}_a$-actions.

Given $n \geq 2$, let $\mathbb{G}_a$ act on $\mathbb{A}^n$ by the basic action, and let V_n denote $\mathbb{A}^n$ as a $\mathbb{G}_a$-module with this action. Specifically, assume that the action is given by the exponential of the elementary nilpotent matrix E_n.

Theorem 7.13. (Thm 3.1 of [113]) *Given $N \geq 5$, let $k[\mathbf{u}, \mathbf{v}] = k^{[N+2]}$ denote the coordinate ring of the $\mathbb{G}_a$-module $V_2 \oplus V_N$, where $\mathbf{u} = (u_1, u_2)$ with $E_2(u_1) = 0$, and $\mathbf{v} = (v_1, ..., v_N)$ with $E_N(v_1) = 0$. Let $X \subset \mathbb{A}^{N+2}$ denote the coordinate hypersurface defined by the invariant function $v_1 - u_1^2$. Then $\mathbb{G}_a$ acts on X by restriction, and $k[X]^{\mathbb{G}_a}$ is not finitely generated.*

The proof of the theorem is based on recent work of Kuroda. This result is already quite interesting in its own right; in addition, the result is used in the next section to give a proof for the linear counterexample in dimension 11 which was discussed in *Chap. 6*.

7.4.1 Generalized Roberts Derivations

The recent paper of Kuroda [183] studies certain kinds of elementary monomial derivations. In it, the author gives an inequality determined by the defining monomials of such a derivation, and proves that when this inequality is satisfied, the ring of constants is not finitely generated (Thm. 1.3).

Of particular interest is the following special case. Let $B = k[\mathbf{x}, \mathbf{y}, z] = k^{[2n+1]}$, where $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_n)$ for $n \geq 1$. Given an integer $t \geq 1$, define the **generalized Roberts derivation** $D_{(t,n)}$ on B by:

$$D_{(t,n)}(\mathbf{x}) = 0 \; ; \; D_{(t,n)}(y_i) = x_i^{t+1} \; (1 \leq i \leq n); \text{ and } D_{(t,n)}(z) = (x_1 \cdots x_n)^t \; .$$

Theorem 7.14. *If $n \geq 3$ and $t \geq 2$, or if $n \geq 4$ and $t \geq 1$, then the ring of constants of $D_{(t,n)}$ is not finitely generated.*

The original examples of Roberts were $D_{(t,3)}$ for $t \geq 2$. Then Kojima and Miyanishi proved the result for the case $n \geq 3$ and $t \geq 2$ [171]. Finally, Kuroda proved all cases of the theorem, including the case $n \geq 4$ and $t \geq 1$ ([183], Cor. 1.5). It is this last case which is new, and which serves as the catalyst for the new counterexample in dimension 11: We will consider the Roberts derivations $D_{(1,n)}$ for $n \geq 4$.

It should be noted that Kurano showed that the ring of constants for $D_{(1,3)}$ is generated by 9 elements [179].

7.4.2 Proof of Theorem 7.13

Given $n \geq 4$, let δ_n denote the Roberts derivation $D_{(1,n)}$ on the polynomial ring $k[\mathbf{x}, \mathbf{y}, z] = k^{[2n+1]}$, where $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_n)$. Specifically,

$$\delta_n = x_1^2 \partial_{y_1} + \cdots + x_n^2 \partial_{y_n} + (x_1 \cdots x_n) \partial_z \; .$$

By Kuroda's result, the ring of constants of δ_n is not finitely generated.

This derivation has many symmetries. First, there is an obvious action of the torus $\mathbb{G}_m^n$ on $k[\mathbf{x}, \mathbf{y}, z]$ which commutes with δ_n, namely, given $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{G}_m^n$,

$$\lambda \cdot x_i = \lambda_i x_i \; (1 \leq i \leq n); \; \lambda \cdot y_i = \lambda_i^2 y_i \; (1 \leq i \leq n); \text{ and } \lambda \cdot z = (\lambda_1 \cdots \lambda_n) z \; .$$

There is also an obvious action of the symmetric group S_n on $k[\mathbf{x}, \mathbf{y}, z]$ commuting with δ_n: Given $\sigma \in S_n$, define

$$\sigma(x_i) = x_{\sigma(i)} \quad (1 \leq i \leq n), \quad \sigma(y_i) = y_{\sigma(i)} \quad (1 \leq i \leq n), \quad \text{and} \quad \sigma(z) = z \; .$$

Since the actions of the torus $\mathbb{G}_m^n$ and the symmetric group S_n semi-commute with each other, we obtain an action of the group $\mathbb{G}_m^n \rtimes S_n$ on $k[\mathbf{x}, \mathbf{y}, z]$. (In this group, S_n acts on $\mathbb{G}_m^n$ by conjugation.)

Again, the full torus action does not have many invariant functions, and so we consider instead the action of the subgroup $H \subset \mathbb{G}_m^n$ consisting of elements λ such that $\lambda_1 \cdots \lambda_n = 1$. Note that H is also a torus, of dimension $n-1$. We thus obtain an action of the group $G := H \rtimes S_n$ on $k[\mathbf{x}, \mathbf{y}, z]$.

It is easily calculated that the ring of invariants $k[\mathbf{x}, \mathbf{y}, z]^H$ is generated by the set of monomials

$$\{x_1 \cdots x_n\} \cup \{T_1 \cdots T_n \,|\, T_i \in \{x_i^2, y_i\}, 1 \leq i \leq n\} \cup \{z\} \; .$$

Let s_i denote the sum of monomials in this set of $\mathbf{y}$-degree i $(0 \leq i \leq n)$:

$$
\begin{aligned}
&s_0 = x_1 \cdots x_n \\
&s_1 = \textstyle\sum_{1 \le i \le n} (x_1 \cdots \hat{x}_i \cdots x_n)^2 y_i \\
&s_2 = \textstyle\sum_{1 \le i < j \le n} (x_1 \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_n)^2 y_i y_j \\
&\vdots \\
&s_{n-2} = \textstyle\sum_{1 \le i < j \le n} (x_i x_j)^2 y_1 \cdots \hat{y}_i \cdots \hat{y}_j \cdots y_n \\
&s_{n-1} = \textstyle\sum_{1 \le i \le n} x_i^2 y_1 \cdots \hat{y}_i \cdots y_n \\
&s_n = y_1 \cdots y_n
\end{aligned}
$$

Since H is normal in G, the fixed ring of the G-action is

$$k[\mathbf{x}, \mathbf{y}, z]^G = (k[\mathbf{x}, \mathbf{y}, z]^H)^{S_n} = k[s_0, s_1, ..., s_n, z] = k^{[n+2]} \ .$$

Since the action of G commutes with δ_n, it follows that δ_n restricts to a locally nilpotent derivation of this polynomial ring:

$$
\begin{aligned}
&\delta_n(s_0) = 0 \ , \\
&\delta_n(s_1) = n s_0^2 \ , \\
&\delta_n(s_2) = \binom{n}{2} s_1 \ , \\
&\vdots \\
&\delta_n(s_i) = \binom{n}{i} s_{i-1} \ , \\
&\vdots \\
&\delta_n(s_n) = s_{n-1} \\
&\delta_n(z) = s_0
\end{aligned}
$$

Let $\mathbf{s} = (s_0, s_1, ..., s_n)$, and let Δ_n denote the restriction of δ_n to $k[\mathbf{s}, z]$. Up to scalar multiples, the induced $\mathbb{G}_a$-action $\exp(t\Delta_n)$ on $\mathbb{A}^{n+2} = \mathrm{Spec}(k[\mathbf{s}, z])$ is precisely the $\mathbb{G}_a$-action on $X = \mathbb{A}^{N+1}$ described in *Thm. 7.13*, where $N = n + 1$. It remains to show that the ring of constants $k[\mathbf{s}, z]^{\Delta_n}$ is not finitely generated.

In order to do this, we use the Non-Finiteness Criterion (*Lemma 7.4*). According to Lemma 2.2 of Kuroda [183], there exists a sequence $\alpha_i \in k[\mathbf{x}, \mathbf{y}, z]^{\delta_n}$ such that, for some positive integer e, α_i has the form $\alpha_i = x_1^e z^i +$ (terms of lower degree in z) for each $i \ge 1$. We may assume each α_i is homogeneous (i.e., semi-invariant) relative to the given torus action; otherwise replace α_i by the homogeneous summand of α_i containing $x_1^e z^i$, which is also in the kernel of δ_n. Given $i \ge 1$, let β_i denote the product of all elements in the orbit of α_i under the action of S_n. Then $\beta_i \in k[\mathbf{s}, z]^{\Delta_n}$ for each i, and has leading z-term of the form

$$\left((x_1^e z^i)(x_2^e z^i) \cdots (x_n^e z^i)\right)^{(n-1)!} = s_0^{e(n-1)!} z^{in!} \ .$$

Therefore, the set of coefficient degrees $\{\deg s_0^{e(n-1)!}\}$ is bounded, while the set $\{\deg_z z^{in!}\}_{i \ge 1}$ is unbounded. This is the first condition for non-finite generation.

The second condition is that $\delta_n(z) = s_0$ does not belong to the image of the restriction of Δ_n to the subring $k[\mathbf{s}]$. But this is obvious, since s_0 does not belong to the ideal $(\delta_n(\mathbf{s})) = (s_0^2, s_1, ..., s_{n-1})$. By the Non-Finiteness Criterion, it follows that the ring of constants $k[\mathbf{s}, z]^{\Delta_n}$ is not finitely generated. $\square$

7.5 Proof for the Linear Example in Dimension Eleven

This section gives a proof of *Thm. 6.13*, stated in the preceding chapter, continuing the notation of the theorem.

Let $A = k[x, s, t, u, v, z] = k^{[6]}$, and define the triangular derivation d on A by

$$dx = 0 \ , \ ds = x^2 \ , \ dt = s \ , \ du = t \ , \ dv = u \ , \ dz = x \ .$$

By the results of the preceding section, A^d is not finitely generated.

Recall that $B = k[w, x, s_1, s_2, t_1, t_2, u_1, u_2, v_1, v_2, z] = k^{[11]}$. The common ring of constants for the commuting derivations T_1, T_2, T_3, T_4 of B is

$$(((B^{T_4})^{T_3})^{T_2})^{T_1} = k[w, x, xs_1 + ws_2, xt_1 + wt_2, xu_1 + wu_2, xv_1 + wv_2, z] \cong k^{[7]} \ .$$

This is due to the simple fact that, for any base ring R, if γ is the R-derivation of $R[X, Y] = R^{[2]}$ defined by $\gamma X = a$ and $\gamma Y = -b$ for some $a, b \in R$ (not both 0), then $R[X, Y]^\gamma = R[bX + aY] \cong R^{[1]}$. This fact is an easy consequence of the results in *Chap. 4*. Applying this four times in succession gives the equality above.

Recall that Θ is a fifth linear triangular derivation of B which semi-commutes with the T_i. In particular, in the Lie algebra of k-derivations of B, we have relations

$$[T_1, \Theta] = T_2 \ , \quad [T_2, \Theta] = T_3 \ , \quad [T_3, \Theta] = T_4 \ , \quad [T_4, \Theta] = 0 \ . \qquad (7.8)$$

So these 5 linear derivations form a Lie algebra $\mathfrak{g}$.

Let $\mathfrak{h} \subset \mathfrak{g}$ denote the subalgebra

$$\mathfrak{h} = kT_1 \oplus kT_2 \oplus kT_3 \oplus kT_4 \ ,$$

noting that $\mathfrak{g} = \mathfrak{h} \oplus k\Theta$. The group $\Gamma := \exp \mathfrak{g}$ acts linearly on $\mathbb{A}^{11}$. Let Ω denote the subgroup $\Omega = \exp \mathfrak{h}$, which is isomorphic to the vector group $\mathbb{G}_a^4$.

The equations (7.8) above show that the adjoint $[\cdot, \Theta]$ defines the basic linear derivation of $\mathfrak{h} \cong \mathbb{A}^4$. It follows that Ω is normal in Γ; the subgroup $\exp t[\cdot, \Theta] \subset \Gamma$ acts on Ω; and $\Gamma \cong \mathbb{G}_a^4 \rtimes \mathbb{G}_a$.

This means that Θ restricts to the subring $B^\Omega = k^{[7]}$, and is given by:

$\Theta w = 0$
$\Theta x = 0$
$\Theta(xs_1 + ws_2) = x^2$

$\Theta(xt_1 + wt_2) = xs_1 + ws_2$
$\Theta(xu_1 + wu_2) = xt_1 + wt_2$
$\Theta(xv_1 + wv_2) = xu_1 + wu_2$
$\Theta z = x$

This is manifestly the same as the derivation d, defined at the beginning of this section, extended to the ring $A[w] = A^{[1]}$ by $dw = 0$. Therefore,

$$B^{\Gamma} = (B^{\Omega})^{\Theta} \cong A[w]^d = A^d[w] \cong (A^d)^{[1]} ,$$

and this ring is not finitely generated over k. □

Remark 7.15. Using *Thm. 7.13*, the same reasoning yields, for each integer $n \geq 4$, a linear action of the group $\Gamma_n = \mathbb{G}_a^n \rtimes \mathbb{G}_a$ on $V_n = \mathbb{A}^{2n+3}$ such that $k[V_n]^{\Gamma_n}$ is not finitely generated.

7.6 Kuroda's Examples in Dimensions Three and Four

Nagata posed the following question in his 1959 paper [235] as "Problem 2".

> Let K be a subfield of the field $k(x_1, ..., x_n)$ such that $\text{tr.deg.}_k K = 3$.
> Is $K \cap k[x_1, ..., x_n]$ always finitely generated?

The counterexample in Nagata's paper has $\text{tr.deg.}_k K = 4$, as does the counterexample of Steinberg (Thm. 1.2 of [286]). Also, the kernels of the derivations d_n (above) have $\text{tr.deg.}_k(\ker d_n) = 4$. On the other hand, Rees's counterexample to the Zariski problem has fixed ring of dimension three.

Quite recently, Nagata's problem was settled in the negative by Kuroda.

Kuroda's first counterexamples are subfields of $k(\mathbf{x}) = k(x_1, x_2, x_3, x_4) = k^{(4)}$. Let γ and δ_{ij} be integrers $(1 \leq i \leq 3, 1 \leq j \leq 4)$ such that

$$\gamma \geq 1 \ , \ \delta_{ij} \geq 1 \text{ if } 1 \leq j \leq 3 \ , \text{ and } \delta_{i,4} \geq 0 \text{ if } 1 \leq i \leq 3 \ .$$

Let $k(\Pi)$ denote the subfield of $k(\mathbf{x})$ generated by

$$\Pi_1 = x_4^{\gamma} - x_1^{-\delta_{1,1}} x_2^{\delta_{1,2}} x_3^{\delta_{1,3}} x_4^{\delta_{1,4}}$$

$$\Pi_2 = x_4^{\gamma} - x_1^{\delta_{2,1}} x_2^{-\delta_{2,2}} x_3^{\delta_{2,3}} x_4^{\delta_{2,4}}$$

$$\Pi_2 = x_4^{\gamma} - x_1^{\delta_{3,1}} x_2^{\delta_{3,2}} x_3^{-\delta_{3,3}} x_4^{\delta_{3,4}} \ .$$

Let $k[\mathbf{x}] = k[x_1, x_2, x_3, x_4]$.

Theorem 7.16. (Kuroda [182], Thm 1.1) *If*

$$\frac{\delta_{1,1}}{\delta_{1,1} + \min\{\delta_{2,1}, \delta_{3,1}\}} + \frac{\delta_{2,2}}{\delta_{2,2} + \min\{\delta_{3,2}, \delta_{1,2}\}} + \frac{\delta_{3,3}}{\delta_{3,3} + \min\{\delta_{1,3}, \delta_{2,3}\}} < 1$$

then $k(\Pi) \cap k[\mathbf{x}]$ is not finitely generated over k.

Kuroda also shows that $k(\Pi) \cap k[\mathbf{x}]$ cannot be the kernel of any locally nilpotent derivation of $k[\mathbf{x}]$.

Nonetheless, there does exist $D \in \mathrm{Der}_k(k[\mathbf{x}])$ with $\ker D = k(\Pi) \cap k[\mathbf{x}]$. For the simplest symmetric example, take $k(\Pi) = k(f, g, h)$ for

$$f = w - x^{-1}y^3z^3 \ , \ g = w - x^3y^{-1}z^3 \ , \ h = w - x^3y^3z^{-1} \ .$$

Note that the rational inequality of Kuroda's theorem is satisfied: $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} < 1$. The jacobian derivation $\Delta_{(f,g,h)} \in \mathrm{Der}_k(k(\mathbf{x}))$ restricts to $k[\mathbf{x}]$, namely, $\Delta_{(f,g,h)} = 4x_1x_2x_3D$, where

$$\begin{aligned} &Dx_i = x_i(5x_i^4 - \varphi) \quad \text{for } \varphi = x_1^4 + x_2^4 + x_3^4 \quad (1 \leq i \leq 3), \text{ and} \\ &Dx_4 = -20(x_1x_2x_3)^3 \ . \end{aligned}$$

That $\ker D = k(\Pi) \cap k[\mathbf{x}]$ can be proved using [185].

Kuroda's second family of examples have members which are subfields L of $K = k(x_1, x_2, x_3) = k^{(3)}$, i.e., K is an algebraic extension of L, but $L \cap k[x_1, x_2, x_3]$ is not finitely generated. These appear in [184]. By the result proved in the *Appendix* to the preceding chapter, together with the Finiteness Theorem, it follows that $L \cap k[x_1, x_2, x_3]$ cannot be the ring of invariants of any algebraic group action on $\mathbb{A}^3$.

Given positive integers γ and δ_{ij} $(i, j = 1, 2)$, let $k(H)$ denote the subfield of K generated by

$$\begin{aligned} H_1 &= x_1^{\delta_{2,1}}x_2^{-\delta_{2,2}} - x_1^{-\delta_{1,1}}x_2^{\delta_{1,2}} \\ H_2 &= x_3^{\gamma} - x_1^{-\delta_{1,1}}x_2^{\delta_{1,2}} \\ H_3 &= 2x_1^{\delta_{2,1}-\delta_{1,1}}x_2^{\delta_{1,2}-\delta_{2,2}} - x_1^{-2\delta_{1,1}}x_2^{2\delta_{1,2}} \ . \end{aligned}$$

Theorem 7.17. (Kuroda [184], Thm. 1.1) *If*

$$\frac{\delta_{1,1}}{\delta_{1,1} + \delta_{2,1}} + \frac{\delta_{2,2}}{\delta_{2,2} + \delta_{1,2}} < \frac{1}{2} \ ,$$

then $K(H) \cap k[x_1, x_2, x_3]$ *is not finitely generated over* k.

Kuroda does in fact use some theory of locally nilpotent derivations in his proofs. These examples bear study, especially in dimension four, in the effort to decide whether kernels of locally nilpotent derivations of $k^{[4]}$ are finitely generated.

See also [180, 181].

7.7 Locally Trivial Examples

In [80], Deveney and Finston show how to use known counterexamples to the Fourteenth Problem in order to construct a class of locally trivial $\mathbb{G}_a$-actions on factorial affine $\mathbb{C}$-varieties Y such that $k[Y]^{\mathbb{G}_a}$ is not finitely generated. Similarly, Jorgenson [153] used Rees's counterexample to the Zariski Problem to construct an example of a normal affine $\mathbb{C}$-variety X of dimension 4 which admits a locally trivial $\mathbb{G}_a$-action such that $k[X]^{\mathbb{G}_a}$ is not finitely generated. His paper includes the following results.

Theorem 7.18. (Thm 3.1 of [153]) *Let A be a normal affine k-domain, where k is an algebraically closed field of characteristic 0. Let L be a field such that $k \subset L \subset \text{frac}(A)$, and set $R = L \cap A$. Then there exists a normal affine $\mathbb{G}_a$-variety $X \subset \mathbb{A}^n$ (for some $n \geq 1$) such that the $\mathbb{G}_a$-action on X is locally trivial, and $R = k[X]^{\mathbb{G}_a}$.*

7.8 Some Positive Results

Naturally, one would like to know the solution to the Finiteness Problem in dimension four:

> Is the kernel of every locally nilpotent derivation of $k^{[4]}$ finitely generated?

To date this question is open. The examples discussed above are all triangular, so in dimension four we first examine triangular derivations, and here the answer is positive.

Theorem 7.19. (Daigle and Freudenburg [57]) *Let k be an algebraically closed field of characteristic zero, and let R be a k-affine Dedekind domain or a localization of such a ring. The kernel of any triangular R-derivation of $R[x, y, z]$ is finitely generated as an R-algebra.*

The proof of this result uses a certain property of polynomials proved by Sathaye in [269]. This theorem easily implies a positive answer to our question.

Corollary 7.20. *If T is a triangular derivation of $k[w, x, y, z]$, then the kernel of T is finitely generated.*

A special case of the corollary was earlier proved in [202], namely, the case T is triangular monomial.

Finite generation notwithstanding, $\ker T$ may be very complicated.

Theorem 7.21. (Daigle and Freudenburg [56]) *For each integer $n \geq 3$, there exists a triangular derivation of $k[w, x, y, z]$ whose kernel, though finitely generated, cannot be generated by fewer than n elements.*

The actual construction of such derivations is a bit complicated, and the reader should see the article for details.

It should be noted that *Thm. 7.19* fails for more general rings R. For example, if $R = k[a, b]$, a polynomial ring in two variables over k, then the derivation d_n of *Cor. 7.9* is a triangular R-derivation of $R[x, y, z]$ with non-finitely generated kernel.

Following are two positive results concerning special classes of derivations.

Theorem 7.22. (van den Essen and Janssen [102]) *Let D be an elementary derivation of $B = k[x_1, ..., x_n]$ for which $Dx_1 = \cdots = Dx_i = 0$ and $Dx_j \in k[x_1, ..., x_i]$ for $j > i$.*

(a) *If either $i \le 2$ or $n - i \le 2$ then* $\ker D$ *is finitely generated.*
(b) *If $1 \in (DB)$, then* $\ker D$ *is a polynomial ring.*

Theorem 7.23. (Khoury [169]) *For $n \le 6$, the kernel of every elementary monomial derivation of $k^{[6]}$ is generated by at most 6 elements.*

In his thesis [205], Maubach asked the following question, which is still open at this writing. Define a monomial derivation D on $B = k[x, y, z, u, v]$ by

$$D = x\partial_y + y\partial_z + z\partial_u + u^2\partial_v \ .$$

Is the kernel of D finitely generated?

7.9 Winkelmann's Theorem

From a geometric point of view, counterexamples to Hilbert's problem show that, in general, the ring of invariants of an algebraic group acting on an affine variety need not be the coordinate ring of an affine variety. However, recent work of Winkelmann asserts that these rings are always at least quasi-affine, that is, isomorphic to the coordinate ring of a Zariski-open subset of an affine variety.

Theorem 7.24. (Thm. 1 of [307]) *Let k be a field and R an integrally closed k-algebra. Then the following properties are equivalent.*

1. *There exists a reduced irreducible quasi-affine k-variety U such that $R \cong k[U]$.*
2. *There exists a reduced irreducible affine k-variety V and a regular $\mathbb{G}_a$-action on V such that $R \cong k[V]^{\mathbb{G}_a}$.*
3. *There exists a reduced irreducible k-variety W and a subgroup $G \subset \mathrm{Aut}_k(W)$ such that $R \cong k[W]^G$.*

To illustrate this theorem, Winkelmann considers the dimension-five counterexample Δ defined and discussed in this chapter (see *Thm. 7.3* above). Let $V \subset \mathbb{A}^6$ be the affine subvariety defined by points $(w_1, ..., w_6)$ such that

$$w_3 = w_2w_4 - w_1w_5 \quad \text{(a coordinate hypersurface)} \ , \quad \text{and} \quad w_1^2w_6 = w_2^3 + w_3^2 \ .$$

Then the set of singularities $\mathrm{Sing}(V)$ of V is defined by $w_1 = w_2 = w_3 = 0$, and $\ker \Delta \cong k[V - \mathrm{Sing}(V)]$ (Lemma 12).

7.10 Appendix: Van den Essen's Proof

Following is van den Essen's recent proof that the kernel of $\Delta \in \mathrm{LND}(k^{[5]})$ is not finitely generated (*Thm. 7.3* above), as found in [101]. This proof has the advantage of being fairly short, while the proof given above can be more readily adapted to the explicit construction of invariants.

Let D_1 denote the de-homogenization $x = 1$ of the basic linear derivation D of $k[x, s, t, u]$. Specifically, D_1 is the derivation of $A := k[s, t, u]$ defined by

$$D_1 = \partial_s + s\partial_t + t\partial_u \ .$$

Notice that D_1 is a partial derivative, with slice s and kernel $k[t_1, u_1]$, where

$$t_1 = t - \tfrac{1}{2}s^2 \quad \text{and} \quad u_1 = u - st + \tfrac{1}{3}s^3 \ .$$

Proposition 7.25. *Let* $e : \mathbb{N} \to \mathbb{N}$ *be given by*

$$e(3l) = 2l \quad \text{and} \quad e(3l + 1) = e(3l + 2) = 2l + 1 \quad (l \geq 0) \ .$$

Then there exists a sequence $c_i \in k[s, t, u]$, $i \geq 0$, *such that* $c_0 = 1$, $Dc_i = c_{i-1}$, *and* $\deg c_i \leq e(i)$ *for* $i \geq 1$.

Here, deg denotes standard degrees on $k[s, t, u]$. Compare to *Thm. 7.1* above.

Proof. Consider on A the grading w defined by

$$w(s) = 1 \ , \ w(t) = 2 \ , \ \text{and } w(u) = 3 \ ,$$

and write $A = \oplus_{i \geq 0} A_i$ accordingly. Degrees for this grading are denoted by w-deg. Then $D_1(A_n) \subset A_{n-1}$ for all $n \geq 1$. By induction on n we construct the functions $c_n \in A$.

Assume that c_n is already constructed. Write $c_n = \sum_{i=0}^{n} H_{n-i}s^i$, with $H_{n-i} \in A_{n-i} \cap A^{D_1}$ (this is possible since $A = A^{D_1}[s]$ and $c_n \in A_n$). Then

$$\tilde{c}_{n+1} := \sum_{i=0}^{n} \tfrac{1}{i+1} H_{n-i}s^{i+1} \in A_{n+1}$$

and $D_1(\tilde{c}_{n+1}) = c_n$. By *Lemma 7.26* below, there exists $h \in A_{n+1} \cap A^{D_1}$ such that $c_{n+1} := \tilde{c}_{n+1} - h$ satisfies $\deg c_{n+1} \leq e(n + 1)$. □

Lemma 7.26. *If* $f \in A_{n+1}$ *is such that deg* $D_1 f \leq e(n)$, *then there exists* $h \in A_{n+1} \cap A^{D_1}$ *such that deg* $(f - h) \leq e(n + 1)$.

Proof. (i) Let $n = 3l$ (the cases $n = 3l+1$ and $n = 3l+2$ are treated similarly) and let M be the k-span of all $f \in A_{n+1}$ such that $\deg D_1 f \leq 2l = e(3l)$. Write $f = \sum \alpha_{ijk}s^i t^j u^k$, with $i + 2j + 3k = 3l + 1$ and $\alpha_{ijk} \in k$. Then

$$D_1 f = \sum_{i+2j+3k=3l+1} (i\alpha_{ijk} + (j + 1)\alpha_{i-2,j+1,k} + (k + 1)\alpha_{i-1,j-1,k+1})s^{i-1}t^j u^k \ .$$

So

$$(*) \quad \deg D_1 f \le 2l \quad \Leftrightarrow \quad i\alpha_{ijk} + (k+1)\alpha_{i-1,j-1,k+1} + (j+1)\alpha_{i-2,j+1,k} = 0$$

for all i, j, k satisfying $i + 2j + 3k = 3l + 1$ and $(i-1) + j + k \ge 2l + 1$ i.e. $i + j + k \ge 2l + 2$. For such a triple we have $i > 0$. Hence by $(*)$ each α_{ijk} is a linear combination of certain α_{pqr}'s with $p + q + r < i + j + k$. Consequently each α_{ijk} is a linear combination of the α_{pqr}'s satisfying $p + q + r = 2l + 2$. Since there are $[\frac{l-1}{2}] + 1$ of them (just solve the equations $p + 2q + 3r = 3l + 1$ and $p + q + r = 2l + 2$) it follows that $\dim \pi(M) \le [\frac{l-1}{2}] + 1$, where for any $g \in A$ $\pi(g)$ denotes the sum of all monomials of g of degree $\ge 2l + 2$.

(ii) Put $N := A^{D_1} \cap A_{n+1}$. Then N is the k-span of all "monomials"

$$n_p := {t_1}^{3p+2} {u_1}^{l-(2p+1)} \ , \ \text{where } 0 \le p \le [\tfrac{l-1}{2}] \ .$$

We claim that the $\pi(n_p)$ are linearly independent over k. It then follows from (i) and the fact that $\pi(N) \subset \pi(M)$ that $\pi(N) = \pi(M)$, which proves the lemma.

(iii) To prove this claim, put

$$w_p := (-2)^{3p+2} 3^{l-(2p+1)} \pi(n_p)|_{t=0, u=\frac{1}{3}s} = \pi((s^2)^{3p+2}(s + s^3)^{l-(2p+1)}) \ .$$

Observe that

$$(s^2)^{3p+2}(s + s^3)^{l-(2p+1)} = \sum_{j=0}^{l-(2p+1)} \binom{l-(2p+1)}{j} s^{3l+1-2j} \ .$$

Since $3l + 1 - 2j \ge 2l + 2$ iff $0 \le j \le [\frac{l-1}{2}]$ we get

$$w_p = \sum_{j=0}^{[\frac{l-1}{2}]} \binom{l-(2p+1)}{j} s^{3l+1-2j} \ .$$

Then the linear independence of the w_p (and hence of the $\pi(n_p)$) follows since $\det\left(\binom{l-(2p+1)}{j}\right)_{0 \le p, j \le [\frac{l-1}{2}]} \ne 0$. $\square$

Proof. (*of Thm. 7.3.*) Recall that Δ is defined on $k[x, s, t, u, v]$ by

$$\Delta = x^3 \partial_s + s\partial_t + t\partial_u + x^2 \partial_v \ .$$

(i) Define for all $i \ge 0$

$$a_i := x^{2i+1} c_i \left(\frac{s}{x^3}, \frac{t}{x^3}, \frac{u}{x^3} \right) \ .$$

Then one easily verifies that $\Delta a_i = x^2 a_{i-1}$ for all $i \ge 1$ and that

$$F_n := \sum_{i=0}^{n}(-1)^i \frac{n!}{(n-i)!} a_i v^{n-i} \in B^{\Delta} \quad (n \geq 1) \ .$$

Suppose now that B^{Δ} is finitely generated by $g_1, \dots, g_m$ over k. We may assume that $g_i(0) = 0$ for all i. Write $g_i = \sum g_{ij} v^j$, with $g_{ij} \in k[x, s, t, u]$. By (ii) below we get that $g_{ij} \in (x, s, t, u)$ for all i, j. Let d denote the maximum of the v-degrees of all g_i. Consider $F_{d+1} = xv^{d+1}$+lower degree v-terms, as above. So $F_{d+1} \in B^{\Delta} = k[g_1, \dots, g_m]$. Looking at the coefficient of v^{d+1}, we deduce that $x \in (x, s, t, u)^2$, a contradiction.

(ii) To prove that $g_{ij} \in (x, s, t, u)$ for all i, j it suffices to show that if $g = \sum g_j v^j \in B^{\Delta}$ satisfies $g(0) = 0$ then each $g_j \in (x, s, t, u)$. First, clearly $g_0 \in (x, s, t, u)$. So let $j \geq 1$.. From $\Delta g = 0$ we get $j g_j x^2 = \Delta(-g_{j-1}) \in \Delta(k[x, s, t, u]) \subset (x^3, s, t)$ for all $j \geq 1$. If $g_j(0) \in k^*$, then $x^2 \in (x^3, s, t, ux^2)$, a contradiction. So $g_j(0) = 0$, i.e., $g_j \in (x, s, t, u)$. $\square$

8

Algorithms

We have seen that the invariant ring of a $\mathbb{G}_a$-action on an affine variety need not be finitely generated as a k-algebra. But in many important cases, most notably in the linear case, the invariant ring is known to be finitely generated, and in these cases it is desirable to have effective means of calculating invariants. In this chapter, we consider constructive invariant theory for $\mathbb{G}_a$-actions, beginning with the classical linear case.

The explicit determination of the rings of invariants for basic $\mathbb{G}_a$-actions was first taken up by Fauntleroy in the 1977 paper [105]. Fauntleroy considered these actions over an algebraically closed field (in any characteristic). However, it was later shown by Tan (1989) that "the finite sets claimed to be generating sets in [Fauntleroy's paper] are not generating sets" [291]. Tan's paper gives an algorithm for calculating generators for the rings of invariants of the basic $\mathbb{G}_a$-actions, again in the case k is an algebraically closed field. Tan then illustrates his algorithm with several interesting examples.

Based on Tan's ideas, van den Essen (1993) developed an algorithm to calculate rings of invariants for a more general class of $\mathbb{G}_a$-actions. For any field k of characteristic zero, and for any finitely generated commutative k-domain B, the algorithm of van den Essen calculates $\ker D$ for any $D \in \mathrm{LND}(B)$, under the assumption that $\ker D$ is finitely generated [98]. Thus, the algorithm already provides an effective way to calculate a set of generators for the ring of invariants of a linear $\mathbb{G}_a$-action on $\mathbb{A}^n$ in the characteristic zero case, even when the underlying field is not algebraically closed. Van den Essen's algorithm relies heavily on the theory and computation of Gröbner bases. The algorithm can also be used to show finite generation, since termination of the algorithm in a finite number of steps means that a generating set has been calculated, and this will be a finite set.

For polynomial rings, van den Essen's algorithm seems particularly well-suited for calculating kernels of triangular derivations, which is already a quite complicated and important case. Nonetheless, despite its utility in the study of locally nilpotent derivations, the algorithm does not predict the number of steps required to calculate a given kernel, making the algorithm impractical in certain cases. It can happen that a finitely generated kernel can be computed easily by *ad hoc* methods, whereas the algorithm runs for several days on a computer algebra system without reaching a conclusion in attempting to calculate the same kernel. Such an example is the homogeneous $(2,5)$-example in dimension three, which was discussed in *Chap. 5.*

In an effort to address some of these difficulties, Maubach (2001) found an algorithm to compute generators of the kernel of any k-derivation of the polynomial ring $k[x_1,...,x_n]$ up to a certain predetermined degree bound. It is based on the idea that a homogeneous derivation D restricts to a vector space mapping on the subspace of forms of fixed degree d. Then one can calculate the kernel of each such restriction by linear algebra, rather than using Gröbner bases. Of course, one cannot be sure to get all kernel elements in this way. But in his thesis [205] of 2003, Maubach points out that one can use an abbreviated form of the van den Essen algorithm (which he calls the *kernel-check algorithm*) to see whether a given set of kernel elements is a generating set. He describes the situation as follows.

> The major drawback of the Essen algorithm is that in practice it is not very fast for most locally nilpotent derivations. The major drawback of the homogeneous algorithm is that it cannot answer the question if found generators are sufficient. However, if we use the homogeneous algorithm to compute generators, and then use the kernel-check algorithm ... to decide if these actually generate the whole kernel, then generally this is a fast way (p. 42).

Maubach discusses an example for which he made calculations using the MAGMA computer algebra system. His algorithm calculated generators up to certain degree within 22 seconds, and an additional 2 seconds were used in the kernel-check algorithm to verify that these generated the entire kernel. Applying only the van den Essen algorithm used 65 minutes.

In addition to the kernel algorithm, van den Essen gives in his book [100] two additional algorithms related to locally nilpotent derivations. The first is the *image membership algorithm.* Assuming again that B is a finitely generated commutative k-domain, $D \in \text{LND}(B)$, and $\ker D$ is finitely generated, this algorithm decides whether a given element $a \in B$ belongs to the image DB, and if so gives $b \in B$ with $Db = a$. The second is the *extendibility algorithm.* For the polynomial ring $B = k[x_1,...,x_n]$, this gives a way of deciding, by means of locally nilpotent derivations, whether a given set of polynomials $f_1,...,f_{n-1} \in B$ forms a partial system of variables, i.e., whether there exists $f_n \in B$ such that $B = k[f_1,...,f_{n-1},f_n]$. If a positive conclusion is reached, then the image algorithm can be used to find f_n. These two algorithms are discussed in *Sect. 8.2* and *Sect. 8.5* below.

A more geometric approach to computing invariants is taken by de Salas in [268]. In particular, he gives an algorithm for computing invariants for unipotent group actions over an algebraically closed field, but only relative to subvarieties where the quotient exists. His description is the following.

> The general theory of invariants is reduced, firstly, to cases where G is either geometrically reductive or unipotent. Furthermore, computation of invariants by the action of a smooth, connected group is reduced to computation of the invariants by the action of the additive group. Indeed, if $B \subset G$ is a Borel subgroup and E is a linear representation of G, then it is known that $E^G = E^B$ and hence computation of the invariants is reduced to the case of a solvable group. It is therefore enough to give a method for computing invariants for the additive group $\mathbb{G}_a$ and the multiplicative group $\mathbb{G}_m$. However the latter is very simple and one is therefore reduced to the case of the additive group $\mathbb{G}_a$. (From the Introduction)

The reader is referred to de Salas's article for details.

8.1 Van den Essen's Algorithm

This section is largely a restatement of van den Essen's own exposition of the algorithm, found in §1.4 of his book [100].

The algorithm is based on the Slice Theorem. Suppose $B = k[b_1, ..., b_n]$, a finitely generated commutative k-domain. Let $D \in \mathrm{LND}(B)$ be nonzero, and assume that $A := \ker D$ is known to be finitely generated. The input of the algorithm is $(D; b_1, ..., b_n; r)$, where $r \in B$ is any local slice of D.

Select a local slice $r \in B$, and set $f = Dr$. Then we know that $B_f = A_f[r] = A_f^{[1]}$. Moreover, if D_f denotes the extension of D to B_f, then $D_f(s) = 1$ for $s = r/f$, and $\ker D_f = A_f = k[\pi_s(b_1), ..., \pi_s(b_n), 1/f]$ by the Slice Theorem, where π_s is the Dixmier map on B_f. Choose $e_i \geq 0$ so that $f^{e_i}\pi_s(b_i) \in A$. Then A is an algebraic extension of the subring $A_0 := k[f^{e_1}\pi_s(b_1), ..., f^{e_n}\pi_s(b_n), f]$, and

$$A_0 \subset A \subset A_0[1/f] \ .$$

Given $m \geq 1$, define inductively A_m to be the subalgebra of B generated by the set

$$\{h \in B | fh \in A_{m-1}\} \ .$$

Since A is factorially closed, it follows that $A_0 \subset A_m \subset A$ for each m.

Theorem 8.1. *In the notation above:*

(a) *A_m is a finitely generated k-subalgebra of A for every $m \geq 0$.*
(b) *A is finitely generated if and only if $A = A_m$ for some m.*
(c) *If $A_m = A_{m+1}$ for some $m \geq 0$, then $A = A_m$.*

Proof. Part (a) is proved by induction on m, the case $m = 0$ being clear. Given $m \geq 1$ assume $A_{m-1} = k[g_1, ..., g_l]$. Let J be the ideal in $k[Y] = k[Y_1, ..., Y_l] = k^{[l]}$ of polynomials P such that $P(g_1, ..., g_l) \in fB$. By the Hilbert Basis Theorem, there exist $P_1, ..., P_s \in k[Y]$ such that $J = (P_1, ..., P_s)$. Let $h_1, ..., h_s$ be elements of A for which $P_i(g_1, ..., g_l) = fh_i$; then $A_{m-1}[h_1, ..., h_s] \subset A_m$, and we wish to see that the reverse inclusion also holds. But this is clear: If h is a generator of A_m, then there exists $F \in J$ with $F(g_1, ..., g_l) = fh$. Choosing $Q_1, ..., Q_s \in k[Y]$ with $F = \sum Q_i P_i$, we have $fh = \sum Q_i(g_1, ..., g_l) fh_i$, implying $h = \sum Q_i(g_1, ..., g_l) h_i$. So (a) is proved.

To prove (b), suppose $A = k[t_1, ..., t_N]$. Since $A \subset A_0[1/f]$, there exists a non-negative integer m such that $\{f^m t_1, f^m t_2, ..., f^m t_N\} \subset A_0$. For every j, $1 \leq j \leq N$, we see that $f^{m-1} t_j \in A_1$, $f^{m-2} t_j \in A_2$, and so on, until finally we arrive at $t_j \in A_m$. Therefore, $A \subset A_m$, meaning $A = A_m$.

To prove (c), assume $A_m = A_{m+1}$. If $h \in A_{m+2}$, then $fh \in A_{m+1} = A_m$, so in fact $A_{m+2} = A_{m+1}$. By induction, we have that $A_M = A_m$ for all $M \geq m$. Since every element of A belongs to A_M for some $M \geq 0$, the conclusion of (c) follows. $\square$

This result provides the theoretical basis for the **van den Essen kernel algorithm**, which is applied in three steps.

- Step 1. Use the Dixmier map to write down the initial subring A_0.
- Step 2. Given A_m for $m \geq 0$, calculate A_{m+1}.
- Step 3. Decide if $A_{m+1} = A_m$. If so, stop; if not, repeat Step 2 for A_{m+1}.

Step 2 is achieved by calculating a set of generators $P_1, ..., P_s$ of the ideal $J \subset k[Y_1, ..., Y_l]$. Suppose $A_m = k[g_1, ..., g_l]$, and let $\bar{B}$ denote B modulo fB. If $\bar{g}_i$ is the residue class of g_i in $\bar{B}$, then

$$J = \{P \in k[Y] | P(\bar{g_1}, ..., \bar{g_l}) = \bar{0}\} .$$

Then one can use standard Gröbner basis calculations to find generators of J; see, for example the *relation algorithm* in Appendix C of van den Essen's book. Once we find $J = (P_1, ..., P_s)$, we have that $P_i(g_1, ..., g_l) = fh_i$ for some $h_i \in B$. Since B is a domain, the h_i are uniquely determined. If B is a polynomial ring, we have that $h_i = f^{-1} P_i(g_1, ..., g_l)$. In more general rings, one can again use Gröbner bases in order to calculate the h_i explicitly; see p.39 of van den Essen's book for details.

As for Step 3, one uses another standard Gröbner basis calculation known as the *membership algorithm*. This algorithm will decide whether the generator h_i of A_{m+1} belongs to the subalgebra A_m, and if so, it computes a polynomial $Q_i \in k[Y]$ so that $h_i = Q_i(g_1, ..., g_l)$. The membership algorithm is described in Appendix C of van den Essen's book.

As remarked by Maubach, "One of the great strengths of the [van den Essen] algorithm is to be able to determine if one has sufficient generators" ([205], p.32). Maubach gives an abbreviated version of the algorithm, to be used for checking a given finite set of kernel elements, and calls this the **kernel-check algorithm**. Its output is simply yes or no, depending on whether or not the given set generates the kernel over k. The algorithm proceeds in the following two steps.

$B = k[b_1, ..., b_n]$, $D \in \text{LND}(B)$, and $f \in \ker D$ are as above. A set $\{f_1, ..., f_m\} \subset \ker D$ is given.

- Step 1. Find generators $P_1, ..., P_s$ for the ideal

$$J = \{P \in k^{[m]} | P(f_1, ..., f_m) \in fB\} \ .$$

- Step 2. If $f^{-1}P_i(f_1, ..., f_m) \in k[f_1, ..., f_m]$ for each i, then $\ker D = k[f_1, ..., f_m]$, and the output is yes; otherwise the output is no.

8.2 Image Membership Algorithm

Based on the kernel algorithm, the image membership algorithm decides whether $a \in DB$ for given $D \in \mathrm{LND}(B)$ and $a \in B$. In addition, if $a \in DB$, then the algorithm gives $b \in B$ for which $Db = a$. Following is a brief description of the algorithm.

Continuing the notation above, we assume $D \in \mathrm{LND}(B)$ is given, and $A = \ker D$ is finitely generated. Suppose $a \in B$ is given. We continue to assume r is a local slice, $Dr = f$, and $s = r/f$. Then $B_f = A_f[s]$ and $D = d/ds$ on B_f. The degree of a as a polynomial in s equals $m := \nu_D(a)$. By integration, there exists $\beta \in B_f$ of degree $m+1$ such that $D\beta = a$. Thus, $g := f^{m+1}\beta \in B$ and $Dg = f^{m+1}a$. For the sake of computations, van den Essen gives the explicit formula

$$\beta = \sum_{0 \le i \le m} \frac{(-1)^i}{(i+1)!} D^i(a) s^{i+1} \ .$$

Since A is finitely generated, it can be represented as $A = k[X_1, ..., X_N]/I$ for some positive integer N, where $k[X_1, ..., X_N] = k^{[N]}$, and $I \subset k[X_1, ..., X_N]$ is an ideal. If $A = k[f_1, ..., f_l]$ has been computed, then I can be found by the relation algorithm. Suppose $I = (H_1, ..., H_t)$, and suppose $F, F_i, G \in k[X_1, ..., X_N]$ are such that $\bar{F} = f$, $\bar{F}_i = f_i$, and $\bar{G} = g$, where $\bar{(\cdot)}$ denotes congruence class modulo I. Let $J \subset k[X_1, ..., X_N, Y_1, ..., Y_N] = k^{[2N]}$ be the ideal

$$J = (Y_1 - F_1, ..., Y_l - F_l, F^{m+1}, H_1, ..., H_t) \ .$$

Finally, we let $\tilde{G}$ be the normal form of G relative to an appropriately chosen Gröbner basis of J.

Proposition 8.2. (1.4.15 of [100]) *In the above notation, $a \in DB$ if and only if $\tilde{G} \in k[Y_1, ..., Y_N]$. In this case, the element $b := (g - \tilde{G}((f_1, ..., f_l))/f^{m+1}$ satisfies $Db = a$.*

For proofs and further details about the algorithm, the reader is referred to van den Essen [100], Sect. 1.4.

8.3 Criteria for a Derivation to be Locally Nilpotent

In both the kernel and image algorithms, one uses $\nu_D(q)$ for elements $q \in B$, but there is no way to predict how large this might be for arbitrary choice of q. Motivated by this consideration, van den Essen [98] posed the **Recognition Problem**:

> Let B be a finitely generated k-algebra and $D \in \mathrm{Der}_k(B)$. Give an algorithm to decide if D is locally nilpotent.

As a step towards achieving this, he gives the following; see [98] and [100], 1.4.17.

Proposition 8.3. (Partial Nilpotency Criterion) *Let $B = k[b_1, ..., b_n]$, and suppose $D \in \mathrm{Der}_k(B)$ has the property that the transcendence degree of B over $\ker D$ equals 1. Given a transcendence basis $x_1, ..., x_{t-1}$ of $\ker D$, set*

$$N = \max_i \{[\mathrm{frac}(B) : k(x_1, ..., x_{t-1}, b_i)] | Db_i \neq 0\} \ ,$$

which is finite. Then D is locally nilpotent if and only if $D^{N+1}(b_i) = 0$ for every i.

The proof is a direct application of *Cor. 1.25*. The key to using this criterion is to find the $t-1$ algebraically independent kernel elements.

In the case $B = k[x, y] = k^{[2]}$, van den Essen gives a complete solution to the Recognition Problem.

Proposition 8.4. (1.3.52 of [100]) *Let $B = k[x, y]$, and let $D \in \mathrm{Der}_k(B)$ be given, $D \neq 0$. Set*

$$d = \max\{\deg_x(Dx), \deg_x(Dy), \deg_y(Dx), \deg_y(Dy)\} \ .$$

Then D is locally nilpotent if and only if $D^{d+2}x = D^{d+2}y = 0$.

Proof. If the generators x and y belong to $\mathrm{Nil}(D)$, then D is locally nilpotent (*Princ. 2*).

Conversely, suppose D is locally nilpotent. We may assume that D is irreducible, since the value of d will only increase when D is multiplied by a kernel element, whereas $\nu_D(x)$ and $\nu_D(y)$ will not change.

By Rentschler's Theorem, there exist $P, Q \in B$ such that $B = k[P, Q]$, $\ker D = k[P]$, and $DQ \in k[P]$. Since D is irreducible, we may write $D = P_y\partial_x - P_x\partial_y$ (*Cor. 4.7*). Observe that

$$\deg_y P = [k(x, y) : k(x, P)] = [k(P, Q) : k(x, P)] = \deg_Q x = \nu_D(x)$$

and

$$\deg_x P = [k(x, y) : k(P, y)] = [k(P, Q) : k(P, y)] = \deg_Q y = \nu_D(y) \ .$$

In each case, the last equality is due to the fact that Q is a local slice. Thus:

$$\begin{aligned}
&\deg_x(Dx) = \deg_x(P_y) \leq \deg_x(P) = \nu_D(y)\\
&\deg_y(Dx) = \deg_y(P_y) = \deg_y(P) - 1 = \nu_D(x) - 1\\
&\deg_x(Dy) = \deg_x(P_x) = \deg_x(P) - 1 = \nu_D(y) - 1\\
&\deg_y(Dy) = \deg_y(P_x) \leq \deg_y(P) = \nu_D(x)
\end{aligned}$$

The desired result now follows from the definition of ν_D. □

The analogous criterion in higher dimensions is not valid. For example, let $D \in \mathrm{LND}(k[x,y,z])$ be the $(2,5)$-example discussed in *Chap. 5*. Then the least integer N so that $D^N(x) = D^N(y) = D^N(z) = 0$ is $N = 11$. On the other hand, D is homogeneous of degree 4, meaning that Dx, Dy, and Dz is each of total degree 5.

However, for homogeneous derivations in dimension 3 a bound was given by Holtackers.

Proposition 8.5. ([145], Thm. 3.1) *Let $B = k[x,y,z] = k^{[3]}$ and let $D \in \mathrm{Der}_k(B)$ be standard homogeneous, $D \neq 0$. Let e denote the greatest integer $e \leq \frac{1}{4}(\deg D + 3)^2 + 1$. Then D is locally nilpotent if and only if $D^{e+1}x = D^{e+1}y = D^{e+1}z = 0$.*

Proof. As above, if the generators x, y, and z belong to $\mathrm{Nil}(D)$, then D is locally nilpotent.

Conversely, assume D is locally nilpotent, and that $\ker D = k[f,g]$ for homogeneous f and g. It is no loss of generality to assume D is irreducible. Then by the Jacobian Formula we may also assume that $D = \Delta_{(f,g)}$.

A general formula for field extensions is

$$[k(x_1, ..., x_n) : k(F_1, ..., F_n)] \leq (\deg F_1) \cdots (\deg F_n) \ .$$

(See [100], Prop. B.2.7, and [145], Prop. 1.12.) Set

$$N = [k(x,y,z) : k(f,g,x)] \leq (\deg f)(\deg g) \ .$$

By the Partial Nilpotency Criterion, $D^{N+1}x = 0$.

Since f and g are homogeneous,

$$\deg D + 1 = \deg(Dx) = \deg(f_y g_z - f_z g_y) = \deg f + \deg g - 2 \leq d \ ,$$

where $d = \max\{\deg Dx, \deg Dy, \deg Dz\}$. Thus, setting $a = \deg f$ and $b = \deg g$, we wish to maximize the quantity $(ab+1)$ subject to the condition $a+b = \deg D + 3$. Viewing $ab + 1$ as quadratic in a, the result follows immediately. □

In fact, Holtackers gives a more general formula, namely, for the weighted-homogeneous case.

Note that, in the general dimension 3 case, the obstruction is that the degrees $\deg(f), \deg(g)$ are not always uniformly bounded by a polynomial function of

$$d = \max\{\deg(f_y g_z - f_z g_y), \deg(f_x g_z - f_z g_x), \deg(f_x g_y - f_y g_x)\} \ .$$

8.4 Maubach's Algorithm

This algorithm calculates generators of the kernel of a homogeneous derivation up to a certain predetermined degree. The derivation involved need not be locally nilpotent. In addition, Maubach describes a procedure for using the homogeneous algorithm to calculate kernel elements of a non-homogeneous derivation. This is accomplished by homogenizing the given derivation. In the present treatment, we will content ourselves with a brief description of these ideas. The reader is referred to [203, 205] for the proofs.

8.4.1 The Homogeneous Algorithm

Let B be a commutative k-domain graded by the additive semi-group $I = \mathbb{N}^q$ for some $q \geq 1$. In addition, if $B = \oplus_{\alpha \in I} B_\alpha$, suppose that each B_α is a vector space of finite dimension over k, and $B_0 = k$.

For elements of I, declare $\beta \leq \alpha$ in I if and only if $\beta_i \leq \alpha_i$ for each i, $1 \leq i \leq q$. Likewise, declare $\beta < \alpha$ if and only if $\beta \leq \alpha$ and $\beta \neq \alpha$. Define

$$B_{\leq \alpha} = \sum_{\beta \leq \alpha} B_\beta \quad \text{and} \quad B_{< \alpha} = \sum_{\beta < \alpha} B_\beta \ .$$

Let $D \in \mathrm{Der}_k(B)$ be homogeneous relative to this grading, $D \neq 0$. Given $\alpha \in I$, we say that a subset $\mathcal{F} = \{F_1, ..., F_s\} \subset B_{\leq \alpha}$ is a **good set for** α (relative to D) if it satisfies:

(1) each $F_i \in B_\beta$ for some $\beta \leq \alpha$
(2) $k[\mathcal{F}] \cap B_{\leq \alpha} = (\ker D) \cap B_{\leq \alpha}$
(3) $F_i \notin k[F_1, ..., \hat{F}_i, ..., F_s]$ for each i

Likewise, $\mathcal{F}$ is a **good set for** $< \alpha$ when:

(1)$'$ each $F_i \in B_\beta$ for some $\beta < \alpha$
(2)$'$ $k[\mathcal{F}] \cap B_{< \alpha} = (\ker D) \cap B_{< \alpha}$
(3)$'$ $F_i \notin k[F_1, ..., \hat{F}_i, ..., F_s]$ for each i

Note that, since $A_0 = k$, a good set for 0 is the empty set.

Given a degree bound α, the algorithm calculates finite sets $\mathcal{F}_\beta \subset B_\beta$ such that their union gives a good set for α. The main tool is the following induction step, which, given a good set $\mathcal{F}$ for $< \alpha$, calculates a set $\mathcal{F}_\alpha \subset B_\alpha$ such that $\mathcal{F}_\alpha \cup \mathcal{F}$ is a good set for α; this will be the set of "kernel generators up to degree α", in the sense of condition (2) for good sets above.

Proposition 8.6. (3.17 and 3.18 of [205]) *Let $\alpha \in I$ be given.*

(a) *Suppose we have a collection of sets $\{\mathcal{F}_\beta | \beta < \alpha\}$ with the property that, for each $\beta < \alpha$, $\cup_{\gamma \leq \beta} \mathcal{F}_\gamma$ is a good set for β. Then $\cup_{\beta < \alpha} \mathcal{F}_\beta$ is a good set for $< \alpha$.*

(b) *Suppose we have a collection of sets $\{\mathcal{F}_\beta | \beta < \alpha\}$ with the property that, for each $\beta < \alpha$, $\cup_{\beta<\alpha}\mathcal{F}_\beta$ is a good set for $< \alpha$. Then we can construct a finite set $\mathcal{F}_\alpha \subset B_\alpha$ such that $\cup_{\beta\leq\alpha}\mathcal{F}_\beta$ is a good set for α.*

By this result, a good set for α can be calculated inductively from the empty set, which is a good set for 0.

The key step is to construct the set $\mathcal{F}_\alpha$ in part (b) of the proposition. To this end, let $A = \ker D$ and write $A = \oplus_{\alpha\in I} A_\alpha$. Let $\mathcal{F} = \cup_{\beta<\alpha}\mathcal{F}_\beta$, as in part (b), which is a good set for $< \alpha$. If $\mathcal{F} = \{F_1, ..., F_s\}$, then

$$k[\mathcal{F}] \cap A_\alpha = \sum_{e\in E} k \cdot F_1^{e_1} \cdots F_s^{e_s} \ ,$$

where $E = \{e \in \mathbb{N}^s | F_1^{e_1} \cdots F_s^{e_s} \in A_\alpha\}$. Choose a subset $J \subset E$ so that the set $\mathcal{B} = \{F_1^{e_1} \cdots F_s^{e_s} | e \in J\}$ is a basis for the vector space $k[\mathcal{F}] \cap A_\alpha$. Then $\mathcal{B}$ is in the kernel K of the linear map $D : B_\alpha \to B_{\alpha+\delta}$ (where δ is the degree of D), and we take $\mathcal{F}_\alpha$ to be the completion of $\mathcal{B}$ to a basis of K.

In addition, Maubach shows:

Proposition 8.7. (3.1.10 of [205]) *Suppose that the preceding algorithm produces the set $\mathcal{F} = \{F_1, ..., F_s\}$ which is a good set α ($\alpha \in I$), and assume that it is verified that* $\ker D = k[\mathcal{F}]$. *If also* $\ker D = k[g_1, ..., g_t]$, *then $s \leq t$.*

8.4.2 Application to Non-homogeneous Derivations

Next, Maubach shows how to apply the homogeneous algorithm to non-homogeneous derivations in the case B is a polynomial ring.

Suppose $B = k[x_1, ..., x_n] = k^{[n]}$ and $B[w] = k^{[n+1]}$. Recall from *Chap. 3* that every $D \in \mathrm{Der}_k(B)$ admits a homogenization $D^H \in \mathrm{Der}_k(B[w])$, and that if $p : B[w] \to B$ is the evaluation map $p(f(w)) = f(1)$, then $p(\ker D^H) = \ker D$. So the idea is simply to apply the homogeneous algorithm to D^H to produce a subset $\mathcal{F} \subset \ker(D^H)$ of generators up to a certain degree. Then $p(\mathcal{F}) \subset \ker D$. If $k[\mathcal{F}] = \ker(D^H)$, then $k[p(\mathcal{F})] = \ker D$.

Maubach points out one pitfall of this approach: It may happen that $\ker(D^H)$ is not finitely generated, but $\ker D$ is finitely generated. He gives the following example.

Define D on $k[x, s, t, u, v] = k^{[5]}$ by

$$D = \partial_s + (sx^2)\partial_t + (tx^2)\partial_u + x\partial_v \ .$$

Since D has a slice s, $\ker D$ is finitely generated. On the other hand, D^H on $k[w, x, s, t, u, v]$ is given by

$$D^H = w^3\partial_s + (sx^2)\partial_t + (tx^2)\partial_u + (xw^2)\partial_v \ ,$$

and $\ker(D^H)$ is not finitely generated. This is because $D^H \bmod (x - 1) = E$, where E is the triangular derivation of $k[w, s, t, u, v]$ defined by

$$E = w^3\partial_s + s\partial_t + t\partial_u + w^2\partial_v \ ,$$

and it was shown in *Chap. 6* that the kernel of E is not finitely generated.

8.5 Extendibility Algorithm

The next algorithm is also due to van den Essen, and addresses the **Extendibility Problem**:

> Let $B = k[x_1, ..., x_n] = k^{[n]}$, and suppose $f_1, ..., f_{n-1} \in B$. Give an algorithm to decide if $(f_1, ..., f_{n-1})$ constitutes a partial system of coordinates, i.e., if there exists $f_n \in B$ such that $(f_1, ..., f_n) \in GA_n(k)$; and if so, find f_n. (3.2.10 of [100])

Set $\mathbf{f} = (f_1, ..., f_{n-1})$. If $f_1, ..., f_{n-1}$ can be extended to an automorphism $(f_1, ..., f_n)$, then $\Delta_{\mathbf{f}} \in \mathrm{LND}(B)$, where

$$\Delta_{\mathbf{f}}(f_n) \in k^* \quad \text{and} \quad \ker \Delta_{\mathbf{f}} = k[f_1, ..., f_{n-1}] \ .$$

In other words, $\Delta_{\mathbf{f}}$ is a partial derivative in the appropriate coordinate system. By the Slice Theorem, the converse is also true: If $\Delta_{\mathbf{f}}$ is locally nilpotent and admits a slice s, and if $\ker \Delta_{\mathbf{f}} = k[f_1, ..., f_{n-1}]$, then $B = k[f_1, ..., f_{n-1}, s]$. This equivalence is the basis of the **extendibility algorithm**, described here in four steps.

- Step 1. Check that the f_i are algebraically independent, which amounts to verifying that at least one image $\Delta_{\mathbf{f}}(x_i)$ is nonzero.
- Step 2. If $\Delta_{\mathbf{f}} \neq 0$, use the Partial Nilpotency Criterion to see if $\Delta_{\mathbf{f}}$ is locally nilpotent. This requires calculating each degree

$$[k(x_1, ..., x_n) : k(f_1, ..., f_{n-1}, x_i)]$$

  such that $\Delta_{\mathbf{f}} x_i \neq 0$. (Van den Essen indicates how to calculate these in the case k is algebraically closed.)
- Step 3. If $\Delta_{\mathbf{f}}$ is locally nilpotent, check whether $\Delta_{\mathbf{f}}$ has a slice: Use the image membership algorithm to see if 1 belongs to the image of $\Delta_{\mathbf{f}}$. If so, the membership algorithm will produce a slice s, and by the Slice Theorem, $B = \ker \Delta_{\mathbf{f}}[s]$.
- Step 4. Finally, decide whether $\ker \Delta_{\mathbf{f}} = k[f_1, ..., f_{n-1}]$ by the kernel-check algorithm. Alternatively, use 3.2.1 of van den Essen [100] to check whether $(f_1, ..., f_{n-1}, s)$ is an automorphism of B.

8.6 Examples

The purpose of this section is to illustrate the use of the van den Essen algorithm by calculating the kernel for each of several specific locally nilpotent derivations of polynomial rings. The first three examples are the basic linear derivations D_3, D_4, and D_5, and here the kernel algorithm reduces to the procedure originally outlined by Tan.

Example 8.8. Let $B = k[x, y, z] = k^{[3]}$, and consider the basic linear derivation D_3. The element y is a local slice, with $D_3 y = x$, and the initial subring of the kernel produced by the algorithm is

$$A_0 = k[\pi_y(x), \pi_y(y), x\pi_y(z)] = k[x, 0, xz - y^2] .$$

Next, J is the ideal in $k[Y_1, Y_2]$ of polynomials P such that $P(x, xz - y^2) \in xB$. Then $P(0, -y^2) \in xB$, which implies that $P(0, -y^2) = 0$, and thus $J \subset Y_1 \cdot k[Y_1, Y_2]$. So if h is a generator of A_1 and $xh = P(x, xz - y^2)$, then $xh = xQ(x, xz - y^2)$ for some Q, meaning $h = Q(x, xz - y^2) \in A_0$. Therefore, $A_1 = A_0$, and the algorithm terminates after two steps: $\ker D_3 = k[x, xz - y^2]$.

Note that the rank of D_3 is 2, so the results of *Chap. 4* already imply $\ker D_3 = k[x, P]$, where $D_3 y = P_z$ and $D_3 z = -P_y$. So this particular kernel could more easily have been found by integration, which yields $P = xz - y^2$.

Example 8.9. Let $B = k[x, y, z, u] = k^{[4]}$, and consider the basic linear derivation D_4. Then y is a local slice, with $D_4 y = x$, and

$$A_0 = k[\pi_y(x), \pi_y(y), x\pi_y(z), x^2\pi_y(u)] = k[x, 0, f, g] ,$$

where $f = xz - y^2$ and $g = x^2 u - 3xyz + 2y^3$. Note that $g^2 + 4f^3 \in xB$, and this is the only such algebraic relation between f and g. In fact, $g^2 + 4f^3 = x^2 h$, where

$$h = x^2u^2 - 6xyzu + 4y^3u + 4xz^3 - 3y^2z^2 .$$

Therefore, $A_1 = k[x, f, g, xh]$ and $A_2 = k[x, f, g, h]$. Modulo x, we see that $\bar{f} = -y^2$, $\bar{g} = 2y^3$, and $\bar{h} = 4y^3u - 3y^2z^2$. This means that $\bar{h}$ is transcendental over A_2 modulo x, so there are no new algebraic relations which can be formed with h to yield a multiple of x. We conclude that $\ker D_4 = k[x, f, g, h]$. In particular, $\ker D_4$ is not a polynomial ring. Geometrically, it is the coordinate ring of the singular hypersurface $X^2U - Z^2 - 4Y^3 = 0$ in $\mathbb{A}^4$.

Example 8.10. Let $B = k[x, y, z, u, v] = k^{[5]}$, and consider the basic linear derivation D_5. Then y is a local slice, with $D_5 y = x$, and

$$A_0 = k[\pi_y(x), \pi_y(y), x\pi_y(z), x^2\pi_y(u), x^3\pi_y(v)] = k[x, 0, f, g, p] ,$$

where

$$f = xz - y^2 \ , \ g = x^2u - 3xyz + 2y^3 \ , \text{ and } p = x^3v - 4x^2yu + 6xy^2z - 3y^4 .$$

Modulo x we have $\bar{f} = -y^2$, $\bar{g} = 2y^3$, and $\bar{p} = -3y^4$. The ideal of relations $J_1 \subset k[X, Y, Z]$ among these three (mod x) is

$$J_1 = (4X^3 + Y^2, 3X^2 + Z, 27Y^4 + 16Z^3) = (4X^3 + Y^2, 3X^2 + Z) .$$

We have $4f^3 + g^2 = x^2 h$ for h as above, and $3f^2 + p = x^2 q$ for $q = xv - 4yu + 3z^2$. Therefore, $A_1 = k[x, f, g, p, xh, xq]$ and $A_2 = k[x, f, g, p, h, q]$. The ideal of relations $J_2 \subset k[X, Y, Z, U, V]$ among these five (mod x) is

$$J_2 = (J_1, U - XV) = (4X^3 + Y^2, 3X^2 + Z, U - XV) \ .$$

We have that $h - fq = xr$ for $r = xu^2 - 2yzu + z^3 - xzv - y^2v$, and thus $A_3 = A_2[r] = k[x,f,g,p,h,q,r]$. Since r mod x is transcendental over A_2 mod x, the algorithm terminates. In addition, we have $h = fq + xr$ and $p = x^2q - 3f^2$. It follows that

$$\ker D_5 = k[x,f,g,p,h,q,r] = k[x,f,g,q,r] \ .$$

In [249], Onoda shows that this ring is isomorphic to

$$k[x,y,z,u,v]/(x^3v + y^3 + z^2 + x^2yu) \ .$$

Example 8.11. The kernel algorithm can also be applied successively to vector group actions. For example, consider the 2-dimensional commutative nilpotent Lie algebra $\mathfrak{g}$ consisting of matrices

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ s & 0 & 0 & 0 & 0 \\ t & 2s & 0 & 0 & 0 \\ 0 & 3t & 3s & 0 & 0 \\ 0 & 0 & 6t & 4s & 0 \end{pmatrix} \quad (s,t \in k) \ .$$

This corresponds to the vector space generated by the commuting linear derivations D_5 and E on $B = k[x,y,z,u,v] = k^{[5]}$, where

$$Ex = Ey = 0 \ , \ Ez = x \ , \ Eu = 3z \ , \ \text{and } Ev = 6u \ .$$

In addition, $\exp(\mathfrak{g}) = \mathbb{G}_a^2 \subset GL_5(k)$. Therefore, continuing the notation of the preceding example, if $C = \ker D_5 = k[x,f,g,q,r]$, then E restricts to C, and $B^{\mathbb{G}_a^2} = C^E$.

To implement the algorithm, note that

$$Ex = 0 \ , \ Ef = x^2 \ , \ Eg = 0 \ , \ Eq = 12f \ , \ \text{and } Er = -xq \ .$$

So f is the local slice to be used. At the first stage of the algorithm, we have

$$C_0 = k[\pi_f(x), \pi_f(f), \pi_f(g), x^2\pi_f(q), x^3\pi_f(r)] \ ,$$

which reduces to

$$C_0 = k[x, 0, g, p - 3f^2, g^2] = k[x, g, p - 3f^2] \ .$$

Modulo x, we have $g = 2y^3$ and $p - 3f^2 = -6y^4$. Therefore, x divides $2(p - 3f^2)^3 + 27g^4$, and in fact it is easily calculated (with a computer algebra system) that

$$2(p - 3f^2)^3 + 27g^4 = x^3H$$

for some $H \in C$, where H is congruent modulo x to $216y^6(vy^2 - 2yzu + z^3)$. It follows that

$$B^{\mathbb{G}_a^2} = C^E = k[x, g, p - 3f^2, H] \ .$$

Example 8.12. It can also happen that a kernel in dimension 4 is a polynomial ring, for example, the kernel of a partial derivative. Another example is $T := \Delta_{(x,f,g)}$, where $f = xz - y^2$ and $g = xu - yz$. T is triangular, and has $Ty = x^2$. Since f and g are already algebraically independent modulo x, the algorithm terminates after one step: $\ker T = k[x, f, g]$. Note that T is clearly not a partial derivative.

Example 8.13. It is shown in [56] that, in dimension 4, there exists for any integer $n \geq 3$ a triangular derivation of $B = k[x, y, z, u] = k^{[4]}$ whose kernel cannot be generated by fewer than n elements. The proof is based on the kernel algorithm. In the cited paper, an explicit example is constructed whose kernel cannot be generated by fewer than 6 elements, and in all likelihood at least 7 elements are required in any system of kernel generators. The following is a simpler example of the same type of construction.

Let $f = x^2z + xy + y^4$ and $g = x^2u + y^6$. Then $\Delta_{(x,f,g)}$ is triangular, and is divisible by x^2. Set

$$\delta = x^{-2}\Delta_{(x,f,g)} \ .$$

At the first stage of the kernel algorithm, $A_0 = k[x, f, g]$. Modulo x, $\bar{f} = y^4$ and $\bar{g} = y^6$. Thus, $f^3 - g^2 \in xB$, and we set $h = x^{-1}(f^3 - g^2)$ to obtain $A_1 = k[x, f, g, h]$. Direct calculation shows $\bar{h} = 3y^9$. The ideal of relations $J \subset k[X, Y, Z]$ between y^4, y^6, and $3y^9$ is

$$J = (X^3 - Y^2, 81X^9 - Z^4, 9Y^3 - Z^2) = (X^3 - Y^2, 9Y^3 - Z^2) \ .$$

Set $\ell = x^{-1}(9g^3 - h^2)$, so that $A_2 = k[x, f, g, h, \ell]$. Direct calculation shows that, modulo x, $\bar{\ell} = 12(y^{15}u - y^{17}z - y^{15})$, which is transcendental over $\bar{A}_2$. This implies that the algorithm terminates: $\ker \delta = k[x, f, g, h, \ell]$. One sees easily that this subring of B is not a polynomial ring, and therefore requires at least 4 generators. The conjecture is that this kernel cannot be generated by fewer than 5 elements, i.e., that the given set of generators is a minimal set. To prove this, one would show first that x can be included in any minimal set of generators; and second that the quotient ring $k[y^4, y^6, y^9][t]$ cannot be generated by 3 elements, where t is an indeterminate over $k[y]$. Note that the only system of integer weights relative to which δ is homogeneous is $(3, 1, -2, 0)$, meaning that the result of Maubach (*Prop.8.7*) does not apply: The homogeneous algorithm requires non-negative weights.

8.7 Remarks

Remark 8.14. It appears that the first accurate calcuation of $\ker D_5$ is found in the Cerezo's 1987 paper [33], a wonderful hand-written manuscript which tabulates invariants of nilpotent matrices in low dimension, as well as their orbits and nullcones, and the Poincarè series of their invariant rings. The book of Grosshans [131] includes a calculation of the kernels of the basic linear derivations D_2, D_3, D_4, and D_5 (pp. 56-58). For an alternate calculation of $\ker D_5$, see Nowicki [247] (Example 6.8.4).

According to Nowicki, "Cerezo also computed a system of generators of minimal length for the case $n = 6$ (the minimal length is 23, the degrees of these generators go up to 18)" ([247],p. 73). Subsequently, by considering its associated Poincarè series, Onoda showed that, for the basic linear derivation D_6 in dimension 6, $\ker D_6$ is not a complete intersection ([249], Cor. 3.5). It should be noted that Nowicki also calculated rings of invariants for certain non-basic linear $\mathbb{G}_a$-actions on $\mathbb{A}^n$; see [247], Chap. 6.

Remark 8.15. Tan's paper [291] concludes by calculating a few invariant rings in low dimension. For example, consider the basic action of $\mathbb{G}_a$ on $\mathbb{A}^4_k$. Tan shows that in characteristic 0 and 3, the invariant ring is minimally generated by four polynomials, whereas in characteristic 2, the invariant ring is 3-generated, i.e., is a polynomial ring.

Remark 8.16. In [89], Drensky and Genov give an algorithm to calculate the Hilbert series of the invariant ring of a linear action of $\mathbb{G}_a$ on a vector space.

9

The Makar-Limanov and Derksen Invariants

Locally nilpotent derivations are useful though rather elusive objects. Though on "majority" of rings we do not have them at all, when we have them it is rather hard to find them and even harder to find all of them or to give some qualitative statements. Even for polynomial rings we do not know too much.

Leonid Makar-Limanov, *Introduction* to [191]

In March of 1994, a meeting entitled "Workshop on Open Algebraic Varieties" was held at McGill University. This meeting was organized by Peter Russell, who at the time was working with Mariusz Koras to solve the Linearization Problem for $\mathbb{C}^*$-actions on $\mathbb{C}^3$. A key remaining piece of their work was to decide whether certain hypersurfaces in $\mathbb{C}^4$ were algebraically isomorphic to $\mathbb{C}^3$. The simplest such threefold $X \subset \mathbb{C}^4$ is given by zeros of the polynomial $f \in k[x,y,z,t]$ defined by

$$f = x + x^2y + z^2 + t^3 \ .$$

X resembles $\mathbb{C}^3$ in many ways, and had resisted previous attempts to discern its nature: It is a smooth contractible factorial affine threefold which admits a dominant morphism from $\mathbb{C}^3$. In this case, it is even known that X is diffeomorphic to $\mathbb{R}^6$ (see Dimca [84] and Kaliman [158]). By way of comparison, every normal affine surface which is homeomorphic to $\mathbb{C}^2$ is isomorphic to $\mathbb{C}^2$ (see [175]).

One of the participants in the McGill meeting was Leonid Makar-Limanov, who, upon his arrival, announced that he had discovered a proof that X is not isomorphic to $\mathbb{C}^3$. He explained his proof in his talk, and distributed a preprint of his paper to the participants. The proof was rather lengthy, but the main idea was ingeneously simple: Show that for every $D \in \mathrm{LND}(k[X])$, it must be that $Dx = 0$. Since no such non-constant regular function exists for $\mathbb{C}^3$, it follows that $X \not\cong \mathbb{C}^3$.

This breakthrough of Makar-Limanov did not entirely complete the proof of linearization for $\mathbb{C}^*$-actions, but provided the crucial new idea which allowed this to happen. A revised version of Makar-Limanov's original paper was published in 1996 [193]. In their 1997 paper [163], Makar-Limanov and Kaliman dealt with the full class of Russell-Koras threefolds, allowing completion of the linearization proof. Following is the abstract of that paper.

> P. Russell and M. Koras classified all smooth affine contractible threefolds with hyperbolic $\mathbb{C}^*$-action and quotient isomorphic to that of the corresponding linear action on the tangent space at the unique fixed point. It is not clear from their description whether there exist nontrivial Russell-Koras threefolds that are isomorphic to $\mathbb{C}^3$. They showed that this question arises naturally in connection with the problem of linearizing a $\mathbb{C}^*$-action on $\mathbb{C}^3$. We prove that none of the nontrivial Russell-Koras threefolds are isomorphic to $\mathbb{C}^3$.

Two papers of Koras and Russell [172, 173] give details for their proof of linearization for $\mathbb{C}^*$-actions on $\mathbb{C}^3$, and the article [161] provides an overview of their work.

These ideas led Makar-Limanov to formulate the definition and basic theory of a subtle new invariant for an algebraic variety X, which he called the **ring of absolute constants** of the variety, or **AK invariant**, denoted $AK(X)$. This is defined to be the ring of regular functions on X which remain invariant under all algebraic $\mathbb{G}_a$-actions on X. Other researchers have adopted the term **Makar-Limanov invariant**, or **ML invariant**, denoted $ML(X)$, and this is the terminology and notation to be used in the present treatment.

Similarly, let B be any integral domain containing $\mathbb{Q}$, and define its Makar-Limanov invariant by

$$AK(B) = ML(B) = \bigcap_{D \in \mathrm{LND}(B)} \ker D \ .$$

$AK(B)$ is sometimes also called the **absolute kernel** of B. If B is the coordinate ring of an affine variety X over a field k of characteristic zero, then the notations $AK(X) = AK(B)$ and $ML(X) = ML(B)$ are used interchangeably.

We can immediately make the following observations: If B is an integral domain containing $\mathbb{Q}$, then

(1) $B^* \subset ML(B)$ (see *Princ. 1*)
(2) $ML(K) = K$ if K is a field
(3) $ML(B)$ is a characteristic subring of B.

The meaning of this last property is that $ML(B)$ is mapped into itself by every element of $\mathrm{Aut}_k(B)$.

It is also easy to see that when $B = k[x_1, ..., x_n]$, a polynomial ring of dimension n, then $ML(B) = k$, since for the partial derivatives ∂_{x_i}, we have

$$\ker \partial_{x_1} \cap \cdots \cap \ker \partial_{x_n} = k \ .$$

What Makar-Limanov showed was that, for the polynomial $f \in k^{[4]}$ above, if $R = (k^{[4]} \bmod f)$, then $ML(R) = k[x] \cong k^{[1]}$.

In [42], the authors define a ring B to be a **rigid ring** if $ML(B) = B$, i.e., B admits no nontrivial locally nilpotent derivation. (This is not to be confused with the term rigid derivation, defined earlier.) Likewise, define a variety X to be a **rigid variety** if $ML(X) = k[X]$. For example, property (3) above asserts that every field is rigid. We have also seen a class of rigid rings which are not fields: The only one-dimensional affine k-domain which admits a nonzero locally nilpotent derivation is $k^{[1]}$ (*Cor. 1.24*). In addition, any variety whose automorphism group is finite is necessarily rigid, for example, a smooth cubic hypersurface in dimension at least 4.

It should be noted that, more than two decades prior to the introduction of the definition of rigid rings by Crachiola and Makar-Limanov, Miyanishi studied certain rigid rings relative to the Cancellation Problem, although he did not give these rings a special name. See *Sect. 9.7.2* below.

Shortly after Makar-Limanov's proof, Derksen defined a similar invariant, and used it to give another proof that the Russell-Koras threefold above is not an affine space. Specifically, for the ring B, define the **Derksen invariant** $\mathcal{D}(B)$ of B to be the subalgebra of B generated by the sets $\{\ker D | D \in \text{LND}(B), D \neq 0\}$. In other words, $\mathcal{D}(B)$ is the smallest subalgebra of B containing the kernel of every nonzero locally nilpotent derivation of B. For example, if $B = k[x_1, ..., x_n] = k^{[n]}$, then $x_i \in \mathcal{D}(B)$ for each i, and thus $\mathcal{D}(B) = B$.

One of the main goals of this chapter is to prove that $ML(X) = k[x]$ for the threefold X above. To this end we first calculate $ML(S)$ for Danielewski surfaces S. We also look at characterizations of the plane and of special Danielewski surfaces using the ML invariant, in addition to stability properties of the ML invariant. The chapter concludes with a section which briefly surveys some recent progress in the classification of algebraic surfaces, where the Makar-Limanov invariant is a key tool.

The Makar-Limanov and Derksen invariants are among the more important and promising new tools emerging from the study of locally nilpotent derivations over the last two decades. Their usefulness in applications to geometric questions has already been amply demonstrated. Clearly, there remain many intriguing questions about these invariants.

9.1 Danielewski Surfaces

The complex algebraic surfaces defined by equations of the form $x^n z = p(y)$ have been studied in many different contexts, increasingly so over the past 15 years. Of particular note is their connection to the cancellation problem. In 1989, it was shown by Danielewski [62] that if W_n is the surface defined by $x^n z = y^2 - y$, then W_1 and W_2 are not isomorphic as algebraic varieties, but that $W_m \times \mathbb{C} \cong W_n \times \mathbb{C}$ for all pairs of positive integers m and n. Then in [110], Fieseler showed that $W_m \not\cong W_n$ if $m \neq n$. See also the article of Wilkens [305].

More generally, a **Danielewski surface** over the field k is defined to be any surface S which is algebraically isomorphic to the surface in $\mathbb{A}^3$ defined by an equation of the form $x^n z = p(y)$, where $n \geq 0$, and $p(y) \in k[y]$ is non-constant. Note that the particular equations $x^n z - y$ and $z - p(y)$ define coordinate planes in $\mathbb{A}^3$. Note also that we do not require S to be smooth, i.e., $p(y)$ and $p'(y)$ might have a common divisor which is non-constant. In this case, the normalization of S may or may not be a Danielewski surface.

One very strong property of Danielewski surfaces is that they admit non-trivial algebraic $\mathbb{G}_a$-actions, owing to the fact that the triangular derivation $x^n\partial_y + p'(y)\partial_z$ on the polynomial ring $k[x,y,z]$ annihilates $x^n z - p(y)$. In particular, a Danielewski surface is not rigid. This is an important consideration, for example, in understanding the automorphism group $\mathrm{Aut}_k(S)$. In fact, Makar-Limanov has calculated both $ML(S)$ and $\mathrm{Aut}_k(S)$ for all Danielewski surfaces S, and gives conditions as to when two Danielewski surfaces are isomorphic; see [192, 195].

The primary fact about Danielewski surfaces relative to their $\mathbb{G}_a$-actions is the following.

Theorem 9.1. *Let B denote the ring $k[x,y,z]$ with relation $x^n z = p(y)$, where $n \in \mathbb{N}$ and $p(y) \in k[y]$.*

(a) *If $n \leq 1$ or if $\deg p(y) = 1$, then $ML(B) = k$.*
(b) *If $n \geq 2$ and $\deg p(y) \geq 2$, then $ML(B) = k[x]$. Moreover, $\ker D = k[x]$ for every nonzero $D \in \mathrm{LND}(B)$.*

Proof. Define $\delta \in \mathrm{LND}(B)$ by $\delta(x) = 0$ and $\delta(y) = x^n$. Then $\ker\delta = k[x]$, so we have $ML(B) \subset k[x]$ in all cases.

In case $n = 1$, define $\epsilon = \alpha\delta\alpha^{-1}$, where α is the automorphism of B interchanging x and z. Then $\ker\epsilon = k[z]$, and thus $ML(B) = k$ when $n = 1$.

In case either $n = 0$ or $\deg p(y) = 1$, the polynomial $X^n Z - p(Y)$ is a variable of $k[X,Y,Z] = k^{[3]}$, which implies that $B = k^{[2]}$. Thus, we also have $ML(B) = k$ in this case, and (a) is proved.

Suppose $n \geq 2$ and $\deg p(y) \geq 2$. Define a grading on B by assigning degrees $\deg(x) = -1$ and $\deg(y) = 0$. Then $\deg(z) = n$.

Let $D \in \mathrm{LND}(B)$, $D \neq 0$, and let $f \in \ker D$ be given. Assume that $Dx \neq 0$. By the relation $x^n z = p(y)$, it is possible to write f as a sum of monomials of the form $x^a q(y)$ for $a \geq 0$, and $x^a z^b q(y)$ for $0 \leq a < n$ and $b \geq 0$.

If $\deg f < 0$, then x appears in every monomial of f, which implies $f \in xB$. But then $Dx = 0$, a contradiction. So $\deg f \geq 0$.

Let $\bar{D}$ and $\bar{f}$ denote the highest homogeneous summands of D and f, respectively. Then $\deg\bar{f} \geq 0$ as well.

Suppose $\deg\bar{f} = 0$. Then $\bar{f}$ is an invariant of the k^*-action on B defined by the degree function (i.e., for $t \in k^*$, $t \cdot x = t^{-1}x$ and $t \cdot y = y$). Thus, $\bar{f} \in B^{k^*} = k[x^n z, y] = k[y]$. But then $\bar{D}y = 0$, which implies $\bar{D}(x^n z) = 0$, and this in turn implies that $\bar{D}x = \bar{D}z = 0$, a contradiction.

Therefore, $\deg\bar{f} > 0$. By homogeneity, this means that z appears in every monomial of $\bar{f}$, and thus $\bar{f} \in zB$. Consequently, $\bar{D}z = 0$. It follows that $\bar{D}$ extends to a locally nilpotent K-derivation of $K[x,y]$, where $K = k(z)$. But this is the coordinate ring of a curve C over K, and C is not a line, since

$n \geq 2$ and $\deg p(y) \geq 2$. Therefore, the only locally nilpotent derivation of $K[x, y]$ is 0, which is a contradiction since $\bar{D} \neq 0$.

So the only possiblity is that $Dx = 0$. Since $k[x]$ is algebraically closed in B, part (b) is proved. □

Note that this immediately implies that the surfaces $xz = p(y)$ and $x^n z = q(y)$ are not algebraically isomorphic when $n \geq 2$ and $\deg q(y) \geq 2$.

Accordingly, we define a Danielewski surface S to be **special** if and only if $ML(S) = k$. This is equivalent to the condition that S is isomorphic to a surface in $\mathbb{A}^3$ given by an equation of the form $xz = p(y)$. So for example a plane is a special Danielewski surface. The special Danielewski surfaces are important for a number of reasons, including the fact that they have a relatively large automorphism group.

9.2 A Preliminary Result

Before proceeding with a proof of Makar-Limanov's result, we need to establish a couple of preliminary facts.

Lemma 9.2. *Let B be any commutative k-domain, and suppose $B[x] = B^{[1]}$. Suppose that*

$$c_1 u^m + c_2 v^n \in B - 0 \ ,$$

where $c_1, c_2 \in B - 0$, $u, v \in B[x]$, and $m, n \in \mathbb{N}$. If $m \geq 2$ and $n \geq 2$, then $u, v \in B$.

Proof. It suffices to assume B is a field; otherwise, replace B with $\text{frac}(B)$.

Suppose $m, n \geq 2$, and write $c_1 u^m + c_2 v^n = t \in B^*$, which implies that u and v are relatively prime. Differentiation by x yields

$$mc_1 u^{m-1} u' + nc_2 v^{n-1} v' = 0 \ ,$$

which implies $u' \in vB[x]$ and $v' \in uB[x]$. By *Princ. 5*, it follows that either $u' = 0$ or $v' = 0$. Thus, either $u \in B$ or $v \in B$. But then since $c_1 u^m + c_2 v^n \in B$, we must have both $u \in B$ and $v \in B$. □

Lemma 9.3. (Lemma 2 of [196]) *Let $m, n \in \mathbb{N}$ be larger than 1. Let B be a commutative k-domain, let $D \in \text{LND}(B)$ nonzero, and set $A = \ker D$. Suppose $D(c_1 a^m + c_2 b^n) = 0$, where $a, b \in B$, $c_1, c_2 \in B^D$ are both nonzero, and $c_1 a^m + c_2 b^n \neq 0$. Then $Da = Db = 0$.*

Proof. Let $r \in B$ be a local slice of D. Then $B_{Dr} = A_{Dr}[r]$, and we can write $a = u(r)$ and $b = v(r)$ for univariate polynomials u and v having coefficients in A_{Dr}. By the preceding lemma, it follows that $u, v \in A_{Dr}$, which implies $Da = Db = 0$. □

Proposition 9.4. *Let B be a commutative k-domain, where k is any field of characteristic zero, and let $D \in \mathrm{LND}(B)$, $D \neq 0$. Suppose that there exist $f, g \in B$, together with positive integers m and n, and a non-constant standard homogeneous polynomial $P \in k[x,y] = k^{[2]}$, such that $P(f^m, g^n) \in \ker D - \{0\}$. Then at least one of the following conditions must be true.*

(1) $Df = Dg = 0$
(2) $P \in k[x]$ *(which implies $Df = 0$).*
(3) $P \in k[y]$ *(which implies $Dg = 0$).*
(4) $m = 1$ *and* $P(f, g^n) = a(f + bg^n)^e$ *for some* $a \in k^*$, $b \in k$, *and* $e \geq 1$.
(5) $n = 1$ *and* $P(f^m, g) = a(g + bf^m)^e$ *for some* $a \in k^*$, $b \in k$, *and* $e \geq 1$.

Proof. Assume that either $Df \neq 0$ or $Dg \neq 0$; otherwise (1) holds, and there is nothing to prove.

Let K denote the algebraic closure of k. In $K[x,y]$, $P(x,y)$ factors as a product of linear polynomials, and thus $P(f^m, g^n)$ factors as

$$P(f^m, g^n) = \prod_{i=1}^{e} (c_i f^m + d_i g^n) \quad (c_i, d_i \in K \text{ and } e \geq 1) \ .$$

Let δ be the extension of D to $B_K := (K \otimes_k B)$. Then δ is locally nilpotent, since $B \subset \mathrm{Nil}(\delta)$, and B_K is generated by B over K. We have that $\delta(c_i f^m + d_i g^n) = 0$ for each i. If any two of these factors are linearly independent, then $\delta(f^m) = \delta(g^n) = 0$, which would imply $\delta f = Df = 0$ and $Dg = \delta g = 0$, a contradiction. Therefore, there exist $c, d \in K$ such that $P(f^m, g^n) = (cf^m + dg^n)^e$, where either $c \neq 0$ or $d \neq 0$ (or both).

If $c = 0$, then $P \in k[y]$, and case (3) holds. If $d = 0$, then $P \in k[x]$, and case (2) holds.

Assume $cd \neq 0$, which implies $Df \neq 0$ and $Dg \neq 0$. Then $P(f^m, g^n) = a(f^m + bg^n)^e$ for some $a \in K^*$ and $b \in K$. Since $\delta(f^m + bg^n) = 0$, it follows that

$$mf^{m-1}\delta f = -bng^{n-1}\delta g \quad \Rightarrow \quad b = -\frac{mf^{m-1}Df}{ng^{n-1}Dg} \in \mathrm{frac}(B) \cap K = k \ .$$

Therefore $(f^m + bg^n) \in B$ and $D(f^m + bg^n) = 0$. If $m > 1$ and $n > 1$, the preceding lemma would imply that $Df = Dg = 0$, a contradiction. Therefore, either $m = 1$ or $n = 1$. $\square$

Note that this proposition generalizes certain facts demonstrated in the proof of Rentschler's Theorem, which was given in *Chap. 4*. The reader should also compare it to *Cor. 5.40*.

In their recent paper [162], Kaliman and Makar-Limanov describe methods for calculation of $ML(A)$ for affine domains A over $\mathbb{C}$. In particular, they give the following result which generalizes *Lemma 9.3* above.

Proposition 9.5. ([162], Cor. 2.1) *Suppose:*

(1) *A is an affine $\mathbb{C}$-domain, and $a, b \in A$ are algebraically independent.*
(2) *$p \in \mathbb{C}[x, y] = \mathbb{C}^{[2]}$ is non-constant, irreducible, and not a variable.*

Then for every nonzero $D \in \mathrm{LND}(A)$, $p(a, b) \in A^D$ implies $a \in A^D$ and $b \in A^D$.

9.3 The Threefold $x + x^2y + z^2 + t^3 = 0$

In this section, we prove the result of Makar-Limanov stated in the chapter's introduction. As mentioned, Makar-Limanov's original proof was rather long and technical, and relied on jacobian derivations. Eventually, he streamlined his arguments and wrote a shorter proof, which appeared in [196]. These proofs were given for the field $k = \mathbb{C}$.

In his thesis [68], Derksen introduced the definition of the invariant $\mathcal{D}(R)$, and showed that $\mathcal{D}(R) \neq R$ for the ring R which Makar-limanov had considered. His proof follows the ideas of Makar-Limanov, placing them in a more geometric framework.

In [40], Crachiola extends the definition of the Makar-Limanov invariant to any field, and writes: "The purpose of the present paper is to place the AK invariant in a characterstic free environment" (p.2). Rather than using locally nilpotent derivations, he defines a class of exponential maps and uses the intersection of their fixed rings to define the invariant. However, the main results of this paper also assume that the underlying field is algebraically closed.

The proof given for the theorem below is a variation of Makar-Limanov's second proof. The main difference is that the proof given here is valid for any field of characteristic zero.

Implicitly, what Makar-Limanov first shows is the following.

Theorem 9.6. *Let k be any field of characterstic zero, and let R denote the ring $R = k[x, y, z, t]$, where $x + x^2y + z^2 + t^3 = 0$. Then $\mathcal{D}(R) = k[x, z, t]$. In particular, R is not algebraically isomorphic to $\mathbb{A}^3_k$.*

Proof. Note first that

$$k[x, z, t] \subset R \subset k[x, x^{-1}, z, t] \ , \ \text{where} \quad y = -x^{-2}(x + z^2 + t^3) \ .$$

Introduce a degree function on $k[x, x^{-1}, z, t]$ by declaring that $\deg x = -1$, and $\deg z = \deg t = 0$. Then $\deg y = 2$. This degree function induces a proper $\mathbb{Z}$-filtration

$$R = \cup_{i \in \mathbb{Z}} R_i \ , \ \text{where} \quad R_i = \{f \in R | \deg f \leq i\} \ ,$$

Let $\mathrm{Gr}(R) = \oplus_{i \in \mathbb{Z}}(R_i/R_{i-1})$ be the graded ring associated to the filtration of R, and let $\mathrm{gr} : R \to \mathrm{Gr}(R)$ be the natural map of R into $\mathrm{Gr}(R)$. Set

$$X = \mathrm{gr}(x)\ ,\ Y = \mathrm{gr}(y)\ ,\ Z = \mathrm{gr}(z)\ ,\ \text{and}\quad T = \mathrm{gr}(t)\ .$$

Now x^2y, z^2, and t^3 each lies in $(R_0 \setminus R_{-1})$, whereas their sum does not:

$$x^2y + z^2 + t^3 = -x \in R_{-1}\ .$$

It follows that $X^2Y + Z^2 + T^3 = 0$ in $\mathrm{Gr}(R)$. (N.b.: This does not mean that $\mathrm{gr}(x) = 0$, since gr is not an algebra map.)

We will show $\mathrm{Gr}(R) = k[X, Y, Z, T]$. Consider any element of R of the form cx^ay^b, where $c \in k[z, t]$ and $a, b \in \mathbb{N}$, noting that every $r \in R$ can be expressed as a sum of such terms. If $a \geq 2b$, then

$$cx^ay^b = cx^{a-2b}(x^2y)^b \in k[x, z, t]\ .$$

If $a < 2b$, write $a = 2n + \delta$, where $\delta = 0$ or 1. Then $n < b$, and

$$cx^ay^b = cx^{\delta}y^{b-n}(x^2y)^n = cx^{\delta}(-x - z^2 - t^3)^ny^{b-n}\ .$$

So in this case, if $n > 0$, then the degree in y used to express this function can be reduced. It follows that every $r \in R$ can be expressed in the form

$$r = p(x, z, t) + v(y, z, t) + x \cdot w(y, z, t)$$

for polynomials p, v, and w. By moving the pure z, t-parts of v and w to p, we can bring r to the form

$$r = p(x, z, t) + y \cdot v(y, z, t) + xy \cdot w(y, z, t)\ .$$

If $v \neq 0$, then $\deg(yv)$ is a positive even integer; and if $w \neq 0$, then $\deg(xyw)$ is a positive odd integer. In particular, the degrees of p, yv, and xyw are distinct, which implies

$$\mathrm{gr}(r) \in \{\mathrm{gr}(p), \mathrm{gr}(yv), \mathrm{gr}(xyw)\} \subset k[X, Y, Z, T]\ .$$

So $\mathrm{Gr}(R) = k[X, Y, Z, T]$.

In addition, this same argument shows that if $r \in R$ and $r \notin k[x, z, t]$, then $\deg(r) > 0$, and $\mathrm{gr}(r) \notin k[X, Z, T]$, since elements of $k[X, Z, T]$ cannot have positive degree. Therefore,

$$\mathrm{gr}^{-1}(k[X, Z, T]) \subset k[x, z, t]\ .$$

Now suppose that $D \in \mathrm{LND}(R)$ is given, $D \neq 0$, and let $f \in \ker D$ be given. Consider the case that $f \notin k[x, z, t]$; then $F := \mathrm{gr}(f) \notin k[X, Z, T]$.

Let $\delta = \mathrm{gr}(D)$ be the homogeneous derivation of $\mathrm{Gr}(R)$ associated to D. By *Princ. 15*, we know that $\delta \in \mathrm{LND}(\mathrm{Gr}(R))$, and that $\delta F = 0$. By definition of the map gr, F is a homogeneous element of $\mathrm{Gr}(R)$.

View F as an element of $S[X, Y]$, where $S = k[Z, T]$. Write

$$F = X^aY^bQ(X, Y)$$

where $a, b \in \mathbb{N}$ and $Q \in S[X,Y]$, but $Q \notin X \cdot S[X,Y]$ and $Q \notin Y \cdot S[X,Y]$. Then $Q(X,Y)$ is also homogeneous. Write

$$Q(X,Y) = \lambda X^c + \mu Y^d + XY \cdot G$$

for nonzero elements λ, μ of S, $c, d \in \mathbb{N}$, and $G \in S[X,Y]$. Then $-c = 2d$, which implies $c = d = 0$, i.e., $Q \in S$.

Therefore, we can write

$$F = X^a Y^b g(Z,T) \quad (a, b \in \mathbb{N}, \ , \ g \in k[Z,T]) \ .$$

If $a \geq 2b$, then $F = X^{a-2b}(X^2Y)^b g(Z,T) \in k[X,Z,T]$, a contradiction. Therefore, $0 \leq a < 2b$. In particular, $b \geq 1$, which implies $\delta Y = 0$.

Consider another system of weights on $\mathrm{Gr}(R)$ given by

$$\omega(X) = 6 \ , \ \omega(Y) = -6 \ , \ \omega(Z) = 3 \ , \ \omega(T) = 2 \ .$$

Let $\bar{\delta}$ denote the highest homogeneous summand of δ relative to the induced grading, noting that $\bar{\delta}$ is locally nilpotent. Since Y is homogeneous, $\bar{\delta}(Y) = 0$. Choose $H \in \ker(\bar{\delta})$ which is algebraically independent of Y, which is possible, since $\ker(\bar{\delta})$ is of transcendence degree 2 over k. Also, assume H is homogeneous relative to both gradings of $\mathrm{gr}(R)$, which is possible since $\ker(\bar{\delta})$ is generated by homogeneous elements. Then H has the form $H = X^a Y^b h(Z,T)$ for some $a, b \in \mathbb{N}$ and homogeneous $h \in k[Z,T]$. By algebraic independence, we may assume $H = X^a h(Z,T)$, which is non-constant.

Suppose $a \geq 1$, so that $\bar{\delta}X = 0$. Then also $\bar{\delta}(Z^2 + T^3) = 0$. But then *Lemma 9.3* would imply $\bar{\delta}Z = \bar{\delta}T = 0$, i.e., $\bar{\delta} = 0$ identically, which is not the case. Therefore $\bar{\delta}X \neq 0$, and $a = 0$, meaning $H = h(Z,T)$. According to *Lemma 4.6*, there exists a standard homogeneous polynomial $P \in k^{[2]}$ such that $h(Z,T) = P(Z^2, T^3)$. By *Prop. 9.4*, it follows that either $\bar{\delta}Z = 0$ or $\bar{\delta}T = 0$ (or both).

Let $K = k(Z)$ if $\bar{\delta}Z = 0$, and $K = k(T)$ if $\bar{\delta}T = 0$. Then $\bar{\delta}$ extends to a locally nilpotent K-derivation of $K[X,Y,Z,T]$, which is the coordinate ring of a non-special Danielewski surface over K. By *Thm. 9.1* above, it follows that $\bar{\delta}X = 0$.[1] However, this contradicts the earlier conclusion that $\bar{\delta}X \neq 0$.

The only possibility, therefore, is that $f \in k[x,z,t]$. This proves $\mathcal{D}(R) \subset k[x,z,t]$.

To complete the proof, define $D_1, D_2 \in \mathrm{LND}(R)$ by

$$D_1 x = D_1 z = 0 \quad \text{and} \quad D_1 t = -x^2 \ ;$$

and

$$D_2 x = D_2 t = 0 \quad \text{and} \quad D_2 z = -x^2 \ .$$

Then manifestly $k[x,z,t] \subset \mathcal{D}(R)$. $\square$

[1] This illustrates why it is desirable to remove, when possible, the restriction that the underlying field be algebraically closed.

Corollary 9.7. *Let k be any field of characteristic zero, and let R denote the ring $R = k[x, y, z, t]$, where $x + x^2y + z^2 + t^3 = 0$. Then $ML(R) = k[x]$.*

Proof. Let $D \in \text{LND}(R)$ be nonzero, and suppose $Dx \neq 0$. Choose $f, g \in \ker D$ which are algebraically independent. By the foregoing result, $f, g \in k[x, z, t]$. Write

$$f = xf_1(x, z, t) + f_2(z, t) \quad \text{and} \quad g = xg_1(x, z, t) + g_2(z, t) \ .$$

Note that $f_2(z, t)$ and $g_2(z, t)$ are algebraically independent in R. Otherwise, there is a bivariate polynomial P over k with $P(f_2, g_2) = 0$. But then $P(f, g) \in xR$, which implies $x \in \ker D$, a contradiction.

Continuing the notation of the preceding proof, let δ be the associated derivation of $\text{Gr}(R)$. Since $\deg xf_1(x, z, t)$ and $\deg xg_1(x, z, t)$ are negative, we have $\deg f = \deg f_2$ and $\deg g = \deg g_2$. It follows that

$$\text{gr}(f) = \text{gr}(f_2(z, t)) = f_2(Z, T) \quad \text{and} \quad \text{gr}(g) = \text{gr}(g_2(z, t)) = g_2(Z, T) \ ,$$

and these images are algebraically independent elements of $\ker \delta$. (The restriction $\text{gr} : k[z, t] \to k[Z, T]$ *is* an algebra isomorphism.) Since $k[Z, T]$ is the algebraic closure of $k[f_2(Z, T), g_2(Z, T)] \subset \ker \delta$, it follows that $k[Z, T] \subset \ker \delta$. But then $0 = \delta(X^2Y + Z^2 + T^3) = \delta(X^2Y)$, which implies $\delta = 0$, a contradiction.

So the only possibility is that $Dx = 0$.

Conversely, we see that for D_1 and D_2 as above,

$$\ker D_1 \cap \ker D_2 = k[x] \ .$$

Therefore, $ML(R) = k[x]$. $\square$

Corollary 9.8. *For the ring R as above, let $D \in \text{LND}(R)$, where $D \neq 0$, and set $L = k[x, x^{-1}]$. Then there exists $P \in k[x, z, t] \subset R$ such that P is an L-variable of $L[z, t] = L^{[2]}$, and $\ker D = k[x, P] = k^{[2]}$.*

Proof. Recall that we may view R as a subset of $L[z, t]$. Since $Dx = 0$, we can extend D to a locally nilpotent L-derivation D_L of $L[z, t]$. By the results of *Chap. 4*, we know that $\ker D_L = L[P]$ for some $P \in L[z, t]$ which is is an L-variable of $L[z, t]$. We may assume P belongs to $k[x, z, t]$, and is irreducible in R. Thus, $\ker D = k[x, z, t] \cap k[x, x^{-1}, P] = k[x, P]$. $\square$

9.4 Characterizing $k[x, y]$ by LNDs

An important general problem of commutative algebra is to give conditions which imply that a given ring is a polynomial ring. In 1971, C.P. Ramanujam characterized the affine plane over $\mathbb{C}$ as the only nonsingular algebraic surface that is contractible and simply connected at infinity [258]. The first algebraic characterization was given by Miyanishi in 1975: If k is an algebraically closed field (of any characteristic), and if X is a smooth affine factorial surface over k with trivial units which admits a nontrivial $\mathbb{G}_a$-action, then $X = \mathbb{A}^2$ [212].

Several equivalent conditions for a ring to be $k[x,y]$ are given in the next theorem. Its proof is based on the three lemmas about UFDs proved in *Chap. 2*.

Theorem 9.9. *Let k be an algebraically closed field of characteristic zero, and suppose B is a UFD of transcendence degree 2 over k. Then the following are equivalent.*

(1) *B is affine, $B^* = k^*$, and there exist $f \in B$ and a subring $R \subset B_f$ such that $B_f = R^{[1]}$.*
(2) *B is affine, $B^* = k^*$, and B is not rigid.*
(3) *$ML(B) = k$.*
(4) *There exists a degree function* deg *on B together with nonzero $D \in$ LND(B) such that $\deg f > 0$ for every non-constant $f \in B^D$.*
(5) *$B = k^{[2]}$*

Geometrically, condition (1) says that the surface $S = \mathrm{Spec}(B)$ contains a cylinderlike open set. Notice that neither (3) nor (4) assumes, *a priori*, that B is finitely generated. The implication (1) ⇒ (5) is due to Swan [290], 1979. The implication (2) ⇒ (5) is Miyanishi's 1975 result [212]. The implication (3) ⇒ (5) was shown by Makar-Limanov in 1998 [190], Lemma 19. See also [220], Thm. 2.6; [221], Thm. 2.21; and [95], Thm. 3.1.

Proof. That (5) implies the other four conditions is clear. We will show:

$$(1) \Rightarrow (2) \Rightarrow (5) \quad \text{and} \quad (3) \Rightarrow (4) \Rightarrow (5)\ .$$

(1) ⇒ (2). Suppose $B = k[a_1, ..., a_n]$. By hypothesis, $B_f = R[t]$ for some $t \in B_f$, and we may assume $t \in B$. We thus have $\frac{d}{dt} \in \mathrm{LND}(B_f)$ and $\frac{df}{dt} = 0$ (since f is a unit of B_f). Choose $N \geq 0$ so that $f^N \frac{d}{dt}(a_i) \in B$ for each i, and set $\delta = f^N \frac{d}{dt}$. Then $\delta \in \mathrm{LND}(B_f)$, and δ restricts to B. Therefore, $\delta|_B$ is a nonzero element of LND(B), since $\delta t = f^N \neq 0$.

(2) ⇒ (5). Since B is not rigid, there exists nonzero $D \in \mathrm{LND}(B)$, and by *Prop. 2.2*, we may assume D is irreducible (since B is a UFD). By *Lemma 2.9*, D has a slice y. By the Slice Theorem, $B = A[y]$, where $A = \ker D$. So it will suffice to show $A = k^{[1]}$.

Define the field $K = \mathrm{frac}(A)$. Since B is a finitely generated UFD and tr.deg.$_k K = 1$, we conclude by Zariski's Finiteness Theorem that $A = K \cap B$ is finitely generated. Therefore, A is an affine UFD of transcendence degree 1 over k. In addition, the units of A are trivial, since $B^* = k^*$. By *Lemma 2.8*, $A = k^{[1]}$.

(3) ⇒ (4). By hypothesis, there exist nonzero $D, E \in \mathrm{LND}(B)$ with $B^D \cap B^E = k$. Suppose $f \in B^D$ is non-constant; since tr.deg.$_k B^D = 1$, B^D is the algebraic closure of $k[f]$ in B. If $\nu_E(f) = 0$, then

$$f \in B^E \quad \Rightarrow \quad k[f] \subset B^E \quad \Rightarrow \quad B^D \subset B^E \ ,$$

and this is impossible. Therefore, $\nu_E(f) > 0$ for every non-constant $f \in B^D$.

(4) $\Rightarrow$ (5). We may assume D is irreducible. By *Lemma 2.9*, D has a slice y. By the Slice Theorem, $B = A[y]$, where $A = B^D$. So it will suffice to show $A = k^{[1]}$.

Since $D \neq 0$, $\text{tr.deg.}_k B^D = 1$, meaning $B^D \neq k$. Choose $x \in B^D$ of minimal positive degree. By *Lemma 2.10*, $k[x]$ is factorially closed, and therefore algebraically closed, in B^D. Since B^D is algebraic over $k[x]$, we conclude that $B^D = k[x]$. $\square$

Note that these results may no longer be true when the field k is not algebraically closed. For example, consider $B = k[x,y,z] : xz = p(y)$, the coordinate ring of a special Danielewski surface. Then $ML(B) = k$. In addition, B is a UFD if and only if $p(y)$ is irreducible. So if $p(y)$ is irreducible and of degree at least 2, then B is a UFD which is not a polynomial ring.

For another example, consider the standard linear derivation $D = x\partial_y + 2y\partial_z$ on $k[x,y,z]$. Its kernel is $k[x,f]$ for $f = xz - y^2$. Set $K = k(f)$, a non-algebraically closed field. Then D extends to a nontrivial locally nilpotent derivation of $K[x,y,z]$, a UFD of transcendence degree two over K. It can be checked that $K[x,y,z] \ncong K^{[2]}$. On the other hand, if we consider standard degrees on $K[x,y,z]$, then $\ker D$ contains $1/f$ and x^2/f, which have degrees -2 and 0, respectively. So the degree hypothesis of condition (4) above is not satisfied in this example.

Application.

The ring $B = k[x,y,z] : x^2 + y^3 + z^5 = 0$ is rigid for any field k of characteristic zero. To see this, note first that it is well-known B is a UFD. In addition, it suffices to assume k is algebraically closed, since any nonzero element of $\text{LND}(B)$ induces a nonzero element of $\text{LND}(\bar{k} \otimes B)$, where $\bar{k}$ denotes the algebraic closure of k. Observe that B has a degree function defined by

$$\deg x = 15 \quad , \quad \deg y = 10 \quad , \quad \text{and} \quad \deg z = 6 \ ,$$

relative to which $\deg b \leq 0$ if and only if $b \in k$. Therefore, if B were not rigid, the result above would imply that $B \cong k^{[2]}$, which is absurd.

Makar-Limanov also proves:

Proposition 9.10. ([190], Lemma 16) *If B is a commutative $\mathbb{C}$-domain with* $\text{tr.deg.}_{\mathbb{C}} B = 2$ *and $ML(B) = \mathbb{C}$, then B is isomorphic to a subring of $\mathbb{C}(x)[y]$.*

Makar-Limanov points out that his characterization of the plane does not generalize to rings of transcendence degree three: For example, if $X = SL_2(\mathbb{C})$, then $k[X]$ is a UFD, and we saw in *Remark 5.15* that $ML(X) = \mathbb{C}$. However, the following seems reasonable.

Conjecture. Suppose the field k is algebraically closed. If X is a factorial algebraic threefold over k, and $ML(X) = k$, then $\ker D = k^{[2]}$ for every nonzero $D \in \text{LND}(k[X])$.

Kaliman posed the following related question.

Let X be a smooth contractible algebraic $\mathbb{C}$-variety of dimension 3, with $ML(X) = \mathbb{C}$. Is X isomorphic to $\mathbb{C}^3$? (Problem 1, p. 7 of [31])

Remark 9.11. Recall Kambayahsi's Theorem (*Chap. 5*), which says that if R is a commutative k-algebra and $K \otimes_k R = K^{[2]}$ (K a separable algebraic field extension of k), then $R = k^{[2]}$. Thus, for non-algebraically closed fields k, one could give a characterization of $k[x, y]$ similar to the one above by replacing the condition "k is algebraically closed" with the condition "$\bar{k} \otimes_k B$ is a UFD, where $\bar{k}$ is the algebraic closure of k".

Remark 9.12. In [216, 218], Miyanishi gives an algebraic characterization of affine 3-space. See also [166].

9.5 Characterizing Danielewski Surfaces by LNDs

In Section 2 of his recent article [53], Daigle gives two important new characterizations of the special Danielewski surfaces in terms of their locally nilpotent derivations. These are stated and applied below; the reader is referred to the article for proofs. To paraphrase, the first of these says that an affine surface S is a special Danielewski surface if and only if $k[S]$ admits two distinct locally nilpotent derivations having a common local slice, and whose kernels are polynomial rings. The second asserts that S is a special Danielewski surface if and only if $k[S]$ is a UFD which admits a k-*simple* derivation. Section 4 of the paper is a significant investigation of the graph of kernels $\underline{\text{KLND}}(\mathcal{B})$, where $\mathcal{B}$ is a commutative k-domain of transcendence degree 2 over k. (This graph is discussed in *Chap. 5*.)

It should be noted that Bandman and Makar-Limanov proved earlier that any smooth hypersurface S of $\mathbb{C}^3$ such that $ML(S) = \mathbb{C}$ is a special Danielewski surface [8]. It should also be noted that, unlike the characterizations of $k[x, y]$ in the preceding section, Daigle's characterizations do not require the underlying field to be algebraically closed.

9.5.1 Two Characterizations

Theorem 9.13. ([53], Thm. 2.5) *Let k be a field of characteristic zero, and let R be any commutative k-domain. Let $D_1, D_2 : R \to R$ be locally nilpotent k-derivations which satisfy:*

(a) $\ker D_1 = k^{[1]}$ *and* $\ker D_2 = k^{[1]}$ *but* $\ker D_1 \neq \ker D_2$
(b) *There exists $y \in R$ such that $D_i y \in \ker D_i \setminus k$ for each i*

Set $x_1 = D_1 y$ *and* $x_2 = D_2 y$. *Then* $\ker D_1 = k[x_1]$, $\ker D_2 = k[x_2]$, *and* R *is isomorphic to the ring* $k[X_1, X_2, Y]/(X_1 X_2 - \phi(Y))$ *for some* $\phi \in k[Y]$.

Observe that x_1, x_2 and y are algebraically dependent in R, and the equation $X_1 X_2 - \phi(Y)$ is their (essentially unique) dependence relation.

For the second characterization, Daigle gives the following definition. Let B be any commutative k-domain of transcendence degree 2 over k. Then $D \in \mathrm{Der}_k(B)$ is **k-simple** if and only if D is locally nilpotent, irreducible, and there exists $y \in B$ such that $\ker D = k[Dy]$.

Theorem 9.14. ([53], Thm. 2.6) *Let B be a UFD of transcendence degree 2 over k. If B admits a k-simple derivation, then B is the coordinate ring of a special Danielewski surface over k.*

Daigle writes, "This work started as an attempt to understand the process known as the local slice construction" (p.37). Regarding the locally nilpotent derivations of $k[x, y, z] = k^{[3]}$, he writes that "a crucial rôle is played by the polynomials $f \in k[x, y, z]$ whose generic fiber is a Danielewski surface", i.e., $k(f)[x, y, z]$ is a special Danielewski surface over the field $k(f)$ (p.77).

9.5.2 Application to Embedding Questions

In [121], we use Daigle's first characterization, together with the derivations of Fibonacci type defined in *Chap.5*, to construct non-equivalent embeddings of certain special Danielewski surfaces in $\mathbb{A}^3$. Specifically, we say that two embeddings $i, j : S \hookrightarrow \mathbb{A}^3$ of a surface S are **equivalent** if there exists $\alpha \in GA_3(k)$ with $j = \alpha \circ i$. Otherwise, the embeddings are **non-equivalent**.

Let $\{H_n\} \subset k[x, y, z]$ be the sequence of polynomials defined in *Chap.5*, meaning that $k[H_n, H_{n+1}]$ is the kernel of a locally nilpotent derivation of $k[x, y, z]$ of Fibonacci type. Recall that these have a common local slice r, and satisfy

$$H_{n-1} H_{n+1} = H_n^3 + r^{d_n} \quad (d_n \in \mathbb{N}) \quad \text{and} \quad D_n r = H_n H_{n+1} \ .$$

Given $a \in k$, let $(Y_n)_a \subset \mathbb{A}^3$ be the surface defined by the fiber $H_n - a$.

Theorem 9.15. (Thm. 6 of [121]) *Let the integer $n \geq 3$ be given.*

(a) *For each $a \in \mathbb{C}^*$, $(Y_n)_a$ is isomorphic to the special Danielewski surface in $\mathbb{A}^3$ defined by $xz = y^{d_n} + a^3$.*
(b) *The zero fiber $(Y_n)_0$ is not a Danielewski surface.*

Proof. Let δ_1, δ_2 be the locally nilpotent derivations of $B = k[x, y, z]$ whose kernels are $k[H_{n-1}, H_n]$ and $k[H_n, H_{n+1}]$, respectively. Let D_1, D_2 be the corresponding locally nilpotent quotient derivations on the ring

$$\bar{B} := B \bmod (H_n - a) \ .$$

Then it is easy to check that $\ker D_1 = k[\bar{H}_{n-1}]$ and $\ker D_2 = k[\bar{H}_{n+1}]$. In addition, $\delta_1 r = H_{n-1}H_n$ and $\delta_2 r = H_n H_{n+1}$, implying $D_1(\bar{r}) = a\bar{H}_{n-1}$ and $D_2(\bar{r}) = a\bar{H}_{n+1}$. Moreover, we have $H_{n-1}H_{n+1} = H_n^3 + r^{d_n}$, so that in $\bar{B}$,

$$\bar{H}_{n-1}\bar{H}_{n+1} = a^3 + \bar{r}^{d_n} \ .$$

Using the theorem of Daigle, we conclude that $\bar{B}$ is isomorphic to the ring $B/(xz - y^{d_n} - a^3)$, and (a) is proved.

Now consider the locally nilpotent derivation $\Delta := D_1 \pmod{H_n}$ on the ring B mod H_n. Since $H_{n-2}H_n = H_{n-1}^3 + r^{d_{n-1}}$, it follows that $\ker \Delta = k[\bar{H}_{n-1}, \bar{r}]$, where $\bar{H}_{n-1}^3 + \bar{r}^{d_{n-1}} = 0$. In particular, $\ker \Delta$ is not a polynomial ring. However, the work of Makar-Limanov shows that the kernel of a locally nilpotent derivation of the coordinate ring of a Danielewski surface is always a polynomial ring. Therefore, $\mathrm{Spec}(B \bmod H_n)$ is not a Danielewski surface, and (b) is proved. □

Corollary 9.16. (Cor. 2 of [121]) *Let $n \geq 3$ and $a \in k^*$ be given. Let $Z \subset \mathbb{A}^3$ be the Danielewski surface defined by $xz = y^{d_n} - 1$, and let $(Y_n)_a \subset \mathbb{A}^3$ be the surface defined by $H_n = a$. Then Z and $(Y_n)_a$ are isomorphic as algebraic varieties, but their embeddings in $\mathbb{A}^3$ are non-equivalent.*

Question 9.17. In case S is the special Danielewski surface defined by $xz = y^2$, do there exist non-equivalent embeddings of S in $\mathbb{A}^3$?

Remark 9.18. In the same paper [121], we give non-equivalent embeddings for all the non-special Danielewski surfaces. In addition, the paper gives the first example of two smooth algebraic hypersurfaces in $\mathbb{C}^3$ which are algebraically non-isomorphic, but holomorphically isomorphic. Locally nilpotent derivations are a central tool in the exposition. The reader is referred to the article for details.

9.6 LNDs of Special Danielewski Surfaces

In [52], Daigle describes completely what are all the locally nilpotent derivations of a special Danielewski surface. The following theorem, which is the main result of his paper, gives this description.

9.6.1 Transitivity Theorem

Let $B = k[x_1, x_2, y]$, where $x_1x_2 = \phi(y)$ for some univariate polynomial ϕ. Define $\delta \in \mathrm{LND}(B)$ by $\delta(x_1) = 0$ and $\delta(y) = x_1$. Given $f \in k[x_1]$, let Δ_f denote the exponential automorphism $\Delta_f = \exp(f\delta)$. Note that:

- $\Delta_{f+g} = \Delta_f \Delta_g$
- $\Delta_f \delta \Delta_f^{-1} = \delta$

In addition, let τ be the automorphism of B interchanging x_1 and x_2, and let G denote the subgroup of $\mathrm{Aut}_k(B)$ generated by τ and all Δ_f, $f \in k[x_1]$. Finally, set

$$\mathrm{KLND}(B) = \{\ker D | D \in \mathrm{LND}(B), D \neq 0\} ,$$

noting that G acts on this set by $\alpha \cdot A = \alpha(A)$.

Theorem 9.19. (Transivity Theorem) *The action of G on $\mathrm{KLND}(B)$ is transitive.*

Since a plane is a special Danielewski surface, this result is, in fact, a generalization of Rentschler's Theorem, where the derivation δ plays the role of a partial derivative.

Corollary 9.20. *Given nonzero $D \in \mathrm{LND}(B)$, there exists $\theta \in G$ such that*

$$\theta D \theta^{-1} = \delta .$$

As in the proof of Jung's Theorem, this implies a kind of tameness for the automorphism group of the surface.

9.6.2 An Example over the Reals

We saw in *Chap. 4* that the Newton polytope of a polynomial can, in some cases, indicate that the polynomial is not in the kernel of a locally nilpotent derivation. However, the Newton polytope pays no attention to the underlying field, and is therefore inadequate in many situations to make such determination. This is illustrated in the following example.

For this section, let $f \in \mathbb{Q}[X,Y,Z] = \mathbb{Q}^{[3]}$ be the polynomial

$$f = X^2 + Y^2 + Z^2 .$$

Then there is a nonzero locally nilpotent derivation T of $\mathbb{C}[X,Y,Z]$ with $Tf = 0$, namely,

$$TX = -Z \ , \ TY = -iZ \ , \ TZ = X + iY \ .$$

However, as mentioned in *Chap. 4*, there is no $D \in \mathrm{LND}(\mathbb{R}[X,Y,Z])$ with $Df = 0$, except $D = 0$. This fact is a consequence of the main result of this section, which follows.

Theorem 9.21. *Let $\mathbb{C}[X,Y,Z] = \mathbb{C}^{[3]}$ and let $f = X^2 + Y^2 + Z^2$. If*

$$B = \mathbb{C}[X,Y,Z]/(f) \quad \text{and} \quad B' = \mathbb{R}[X,Y,Z]/(f) ,$$

then $ML(B') = B'$, while $ML(B) = \mathbb{C}$. In particular,

$$ML(\mathbb{C} \otimes_{\mathbb{R}} B') \neq \mathbb{C} \otimes_{\mathbb{R}} ML(B') .$$

Note that $\operatorname{Spec} B$ is a special Danielewski surface over $\mathbb{C}$. However, this result shows that $\operatorname{Spec} B'$ is not a Danielewski surface over $\mathbb{R}$.

In order to prove the theorem, a preliminary result is needed.

Write $B = \mathbb{C}[x, y, z]$ and $B' = \mathbb{R}[x, y, z]$, where x, y, and z denote the congruence classes modulo f of X, Y, and Z, respectively.

Put a grading on B and B' by declaring that $\deg x = \deg y = \deg z = 1$. Call this the **standard grading** of each ring. Write $B = \oplus B_i$ and $B' = \oplus (B')_i$ accordingly. Then $B_1 = \mathbb{C}x \oplus \mathbb{C}y \oplus \mathbb{C}z$ and $(B')_1 = \mathbb{R}x \oplus \mathbb{R}y \oplus \mathbb{R}z$. We will say that $D \in \mathrm{LND}(B)$ is **homogeneous** if it respects the standard grading of B, and **linear** if it is homogeneous of degree 0 (i.e., $D : B_1 \to B_1$). Likewise, say $D' \in \mathrm{LND}(B')$ is **homogeneous** if it respects the standard grading of B', and **linear** if it is homogeneous of degree 0 (i.e., $D' : (B')_1 \to (B')_1$).

Define $x_1 = x + iz$ and $x_2 = x - iz$. Then

$$B = \mathbb{C}[x_1, x_2, y] \quad , \quad x_1x_2 + y^2 = 0 \quad , \quad \text{and} \quad B_1 = \mathbb{C}x_1 \oplus \mathbb{C}x_2 \oplus \mathbb{C}y \ .$$

Let δ be the **standard linear derivation** of B, namely, $\delta x_1 = 0$ and $\delta y = x_1$. Let G be the group described in the Transivity Theorem, and let Γ be the subgroup of G generated by τ and all Δ_t for $t \in \mathbb{C}$. Note that Γ may be viewed as a subgroup of the orthogonal group $O_3(\mathbb{C})$.

Here is a corollary to the Transivity Theorem.

Corollary 9.22. *Let $D \in \mathrm{LND}(B)$ be given. The following are equivalent.*

(1) *D is irreducible and homogeneous*
(2) *$D = \gamma\delta\gamma^{-1}$ for some $\gamma \in \Gamma$*
(3) *D is linear*

Proof. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (1) are clear. Assume D is irreducible and homogeneous. By the Transivity Theorem, we have $D = \alpha(h\delta)\alpha^{-1}$ for some $\alpha \in G$ and $h \in \mathbb{C}[x_1]$. Since D is irreducible, $h \in \mathbb{C}^*$, so we can assume $D = \alpha\delta\alpha^{-1}$.

Let $\gamma \in \Gamma$ be given, along with $t \in \mathbb{C}^*$ and integer $n \geq 1$. Set $T = \Delta_f(\gamma\delta\gamma^{-1})\Delta_f^{-1}$ for $f = tx_1^n$, and suppose that T is homogeneous. Set

$$L = \gamma\delta\gamma^{-1}(x_1) = ax_1 + bx_2 + cy \quad \text{and} \quad M = \gamma\delta\gamma^{-1}(y) = a'x_1 + b'x_2 + c'y \ ,$$

where $a, b, c, a', b', c' \in \mathbb{C}$. Then

$$T(x_1) = \Delta_f(L) = ax_1 + b(x_2 + 2tyx_1^n + t^2x_1^{2n+1}) + c(y + tx_1^{n+1}) \ .$$

If either $b \neq 0$ or $c \neq 0$, then by homogeneity, $1 = n + 1$, a contradiction. Therefore $b = c = 0$, so $\gamma\delta\gamma^{-1}(x_1) = ax_1$. But this implies $\gamma\delta\gamma^{-1}(x_1) = 0$.

In the same way $b' = c' = 0$, so $\gamma\delta\gamma^{-1}(y) = a'x_1$. Since any $\mathbb{C}$-derivation of B is determined by its images on x_1 and y, we conclude that $\gamma\delta\gamma^{-1} = a'\delta$. Therefore, $T = \Delta_f(a'\delta)\Delta_f^{-1} = a'\delta$.

The other possibility is that $n = 0$, and then $\Delta_f \in \Gamma$ already.

By induction, we conclude that D is in all events conjugate to δ by some element of Γ. $\square$

Proof. (of Thm. 9.21) Let $D' \in \mathrm{LND}(B')$ be given, $D' \neq 0$. To prove the result, it suffices to assume D' is homogeneous and irreducible. Extend D' to $D \in \mathrm{LND}(B)$. Then D is also homogeneous and irreducible. By the preceding result, D is linear, and thus $Dx = D'x$, $Dy = D'y$, and $Dz = D'z$ are also linear. So D' is linear.

We can view $\exp(D')$ as an element of the orthogonal group $O_3(\mathbb{R})$. Let $o(3, \mathbb{R})$ denote the real Lie algebra corresponding to $O_3(\mathbb{R})$. Then $M \in o(3, \mathbb{R})$ if and only if $M + M^T = 0$. Since $D' \in o(3, \mathbb{R})$, we conclude that $D' + (D')^T = 0$. Therefore D' has the form

$$D' = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \quad (a, b, c \in \mathbb{R}) \ .$$

The characteristic polynomial of D' is $|D' - \lambda I| = -(\lambda^3 + (a^2 + b^2 + c^2)\lambda)$. Since D' is also a nilpotent matrix, its only eigenvalue (in $\mathbb{C}$) is 0. Therefore, $a^2 + b^2 + c^2 = 0$, implying $a = b = c = 0$, a contradiction. $\square$

In principle, it should also be the case that the unit sphere $S^2 \subset \mathbb{R}^3$ defined by $f = 1$ should admit no nontrivial $\mathbb{G}_a$-action: The orbits are closed, hence compact, hence of dimension 0. To see this algebraically, suppose that D is a locally nilpotent derivation of $k[S^2]$, and put weights on $k[S^2]$ by declaring that $\deg x = \deg y = \deg z = 1$. The idea is that if $D \neq 0$, then $\mathrm{gr}(D)$ is a nonzero locally nilpotent derivation of the ring $B' = \mathrm{Gr}(k[S^2])$ above, which is impossible. The only fact to check is that any such D respects the filtration of $k[S^2]$, and this follows from the result of Wang mentioned earlier (*Thm. 2.12*).

9.7 Further Properties of the ML Invariant

There are many natural questions involving the Makar-Limanov invariant. For example, for a ring B, what is the relation between $ML(B)$ and $ML(B^{[n]})$? Or, can we describe those surfaces X with $ML(X) = \mathbb{C}$? Such surfaces have a fairly large group of automorphisms. These questions have been asked by Makar-Limanov in several papers, who succeeded in giving answers for many important cases. The recent thesis of Crachiola [39] deals with similar questions. This section will discuss progress on these questions due to Makar-Limanov, Crachiola, Miyanishi, and others. Discussion of similar questions, especially for surfaces, can also be found in *Chap.* **??** below.

Throughout this section, B is assumed to be a finitely generated commutative k-domain.

9.7.1 Stability Properties

The following result says that for a rigid ring B, adding a variable gives only the expected outcome. Other proofs are found in [189, 190, 302, 41].

Proposition 9.23. *If B is rigid, then $ML(B^{[1]}) = B$.*

Proof. Let $B[x] = B^{[1]}$, noting that $ML(B[x]) \subset B$ by considering d/dx.

$B[x]$ is $\mathbb{N}$-graded by $B[x] = \oplus_{i\geq 0} C_i$, where $C_i = Bx^i$. Let $D \in \mathrm{LND}(B[x])$ be nonzero. Since B is finitely generated over k, we may apply *Thm. 2.11* to conclude that D respects the filtration of $B[x]$. (Note that the grading of $B[x]$ is induced by the degree function ν_E associated to the locally nilpotent derivation $E = d/dx$.) Let $d \in \mathbb{Z}$ be such that $DC_i \subset C_{i+d}$ for all $i \in \mathbb{N}$, and let $\delta = \mathrm{gr}(D)$ on $\mathrm{Gr}(B[x]) \cong B[x]$. Then δ is locally nilpotent, and homogeneous of degree d.

Suppose $\delta x = 0$. Then for every $n \in \mathbb{Z}$, we have that $\delta_n := \delta \bmod (x-n)$ is a locally nilpotent derivation of $B[x]/(x-n) = B$. By hypothesis, this implies $\delta_n = 0$ for each $n \in \mathbb{Z}$, i.e., $\delta(B[x]) \subset (x-n)B[x]$ for all $n \in \mathbb{Z}$. Since $\delta \neq 0$, there exists $t \in B[x]$ not in the kernel of δ. Write $\delta t = P(x)$ for some nonzero $P \in B^{[1]}$. Then $P(x) \in (x-n)B[x]$ for all $n \in \mathbb{Z}$ implies $P(n) = 0$ for every $n \in \mathbb{Z}$, which is absurd, since B is a domain (P cannot have an infinite number of roots in the field $\mathrm{frac}(B)$). Therefore, $\delta x \neq 0$.

By homogeneity, $\delta x = bx^n$ for some $b \in B$ and $n \geq 0$. In fact, $n = 0$, since otherwise $\delta x \in xB[x]$ would imply $\delta x = 0$. Therefore, $\delta x \in B$, and $d = -1$. Consequently, $DB = DC_0 \subset C_{-1} = \{0\}$, and $B \subset ML(B[x])$. □

Another way to say this is that $ML(B^{[1]}) = ML(B)$ when B is rigid. It follows that $ML(B^{[1]}) = ML(B)$ whenever the transcendence degree of B over k is 1. This is generalized by Makar-Limanov to the following.

Proposition 9.24. *If* $\mathrm{tr.deg.}_k B = 1$, *then* $ML(B^{[n]}) = ML(B)$ *for all* $n \geq 0$.

For a complete proof, see the Theorem (p. 51) of [190]; see also the main theorem of [42].

As a corollary to this important result, Makar-Limanov obtains the cancellation theorem of Abhyankar, Eakin, and Heinzer.

Proposition 9.25. [2, 190] *Let Γ_1 and Γ_2 be two algebraic curves over $\mathbb{C}$. If $\Gamma_1 \times \mathbb{C}^n \cong \Gamma_2 \times \mathbb{C}^n$ for some $n \geq 1$, then $\Gamma_1 \cong \Gamma_2$.*

See the cited monograph for a proof.

In general, however, $ML(B^{[1]}) \neq ML(B)$. For example, if W_n is the Danielewski surface defined by the equation $x^n z = y^2 - 1$ for $n \geq 1$, then $W_m \times \mathbb{A}^1 \cong W_n \times \mathbb{A}^1$, but $W_m \not\cong W_n$ if $m \neq n$. We have seen that $ML(W_n) = k[x]$ when $n \geq 2$, and $ML(W_1) = k$. Therefore,

$$ML(W_n \times \mathbb{A}^1) \neq ML(W_n) \quad \text{if} \quad n \geq 2 ,$$

since for all $n \geq 1$, $ML(W_n \times \mathbb{A}^1) = ML(W_1 \times \mathbb{A}^1) = k$. In [194], Makar-Limanov has worked out three independent locally nilpotent derivations of the coordinate ring $k[W_2 \times \mathbb{A}^1]$, as follows. The coordinate ring is $k[x, y, z, t]$, where $x^2 z = y^2 - 1$. Two obvious locally nilpotent derivations are d/dt, and D defined by $Dx = Dt = 0$ and $Dy = x^2$. A third such derivation E, where $Ex \neq 0$, is defined by the jacobian determinant

$$Ef = \frac{\partial(x^2 z - y^2, t^2 x + 2ty + xz, t^3 x + 3t^2 y + 3txz + yz, f)}{\partial(x, y, z, t)} .$$

So one would like to have more general conditions under which $ML(B^{[1]}) = ML(B)$. Makar-Limanov conjectured in [194] that this is the case whenever B is a UFD.

The comparison of $ML(B^{[n]})$ with $ML(B)$ has already been used by Crachiola and Makar-Limanov in giving new (and shorter) proofs for certain cases of the Cancellation Problem. The first case, when $\text{tr.deg.}_k B = 1$, was mentioned above; a second case, when $\text{tr.deg.}_k B = 2$ and $n = 1$, is discussed in the next chapter. The crucial question in this regard is the following.

For $n \geq 1$, if B is a rigid UFD over k, does $ML(B^{[n]}) = ML(B)$?

In his thesis, Crachiola gives an affirmative answer in the case B is a regular rigid UFD of transcendence degree 2 over k ([39], Cor. 5.20). See also [9] for a discussion of stability questions relative to the ML-invariant.

9.7.2 Miyanishi's Results on Strong Invariance

In his 1978 book [213], Miyanishi included a section entitled "Locally nilpotent derivations in connection with the cancellation problem" (Sect. 1.6). In this section, he is primarily interested in cancellation questions relative to certain rigid rings, though he does not use this terminolgy. Miyanishi uses the more general notion of locally finite higher derivations in place of locally nilpotent derivations, so that his results apply in any characteristic. In particular, in characteristic zero, a ring is rigid if and only if it has no nontrivial locally finite *iterative* higher derivation. One can thereby extend the definition of rigidity to k-algebras in any characteristic.

A k-algebra A is defined to be **strongly invariant** if and only if the following property holds: Given a k-algebra B, and indeterminates $x_1, ..., x_n$ and $y_1, ..., y_n$ $(n \geq 1)$, if

$$\theta : A[x_1, ..., x_n] \to B[y_1, ..., y_n]$$

is a k-algebra isomorphism, then $A \cong B$. Miyanishi shows:

Proposition 9.26. ([213], Lemma 6.3 and Prop. 6.6.1) *Let A be a k-algebra. If A has no locally finite higher derivation, then A is strongly invariant. If A is an affine k-domain and $\dim_k A = 1$, then A is strongly invariant if and only if A is rigid.*

In particular, this result implies the cancellation theorem of Abhyankar, Eakin, and Heinzer mentioned above. Note that the condition that A has no locally finite higher derivation is, *a priori*, stronger than the condition that A is rigid. Miyanishi gives an example showing that a strongly invariant ring might still admit a nontrivial locally finite higher derivation (Example 6.4.3).

9.7.3 Examples of Crachiola and Maubach

As observed, for a polynomial ring B, we have $\mathcal{D}(B) = B$ and $ML(B) = k$. In proving that the coordinate ring R of the Russell threefold discussed above is not a polynomial ring, we showed both $\mathcal{D}(R) \neq R$ and $ML(R) \neq k$. In [43], Crachiola and Maubach show that the Derksen and Makar-Limanov invariants are independent of one another.

Specifically, they first construct an affine noetherian $\mathbb{C}$-domain S of dimension two for which $ML(S) = \mathbb{C}$ and $\mathcal{D}(S) \neq S$. This ring is defined by

$$\begin{aligned} S &= \mathbb{C}[x^2, x^3, y^3, y^4, y^5, x^{i+1}y^{j+1}] \quad (i, j \in \mathbb{N}) \\ &= \mathbb{C}[x^2, x^3, y^3, y^4, y^5, xy, x^2y, xy^2, x^2y^2, xy^3, x^2y^3, xy^4, x^2y^4] \ . \end{aligned}$$

Similarly, they construct a ring S' with the property that $ML(S') \neq \mathbb{C}$, but $\mathcal{D}(S') = S'$. In particular, let A be any commutative $\mathbb{C}$-domain of transcendence degree one over $\mathbb{C}$, other than $\mathbb{C}^{[1]}$. Recall that any such ring is rigid. Then by *Prop. 9.23* above, we have that $ML(A[x_1, ..., x_n]) = ML(A) = A \neq \mathbb{C}$. On the other hand, when $n \geq 2$, the partial derivatives relative to A show that $\mathcal{D}(A[x_1, ..., x_n]) = A[x_1, ..., x_n]$. So we may take $S' = A[x_1, ..., x_n]$ for $n \geq 2$.

9.8 Further Results in the Classification of Surfaces

A current successful trend in the study of algebraic surfaces is to use $\mathbb{G}_a$-actions and the Makar-Limanov invariant to classify surfaces. This section will survey some of these results.

In this chapter, we have seen characterizations for the plane over an algebraically closed field, in addition to two characterizations of the special Danielewski surfaces due to Daigle. These results are based on the existence of nontrivial $\mathbb{G}_a$-actions on the surface. The reader will recall from *Chap. 2* that, if X is a factorial affine surface over an algebraically closed field k, then every irreducible element of $\text{LND}(\mathcal{O}(X))$ has a slice. If, in addition, X is not rigid and $\mathcal{O}(X)$ has trivial units, then $X = \mathbb{A}^2$; this is Miyanishi's characterization of the plane.

Recently, a series of similar and related results have been published which aim to classify certain normal affine surfaces X which admit a nontrivial $\mathbb{G}_a$-action. In case X admits at least two independent $\mathbb{G}_a$-actions (i.e., $ML(X) =$

k), then even more can be said. Dubouloz defines X to be an **ML-surface** if $ML(X) = k$ [92]. Apart from a plane, we have seen such surfaces in the form of special Danielewski surfaces S, defined by $xz = f(y)$ for non-constant f. This surface admits two independent (conjugate) $\mathbb{G}_a$-actions.

Some of the earliest work in this direction was done by Gizatullin [127] (1971), who studied surfaces which are **geometrically quasihomogeneous**. By definition, such a surface has an automorphism group with a Zariski open orbit whose complement is finite. See also [63, 126]. Another early paper, about surfaces which admit a $\mathbb{G}_a$-action, is due to Bertin [17] (1983). According to Dubouloz [92], the 2001 paper [8] of Bandman and Makar-Limanov represents the rediscovery of a link between nonsingular ML-surfaces and Gizatullin's geometrically quasihomogeneous surfaces. Dubouloz writes:

> More precisely, they have established that, on a nonsingular ML-surface V, there exist at least two nontrivial algebraic $\mathbb{C}_+$-actions that generate a subgroup H of the automorphism group $\mathrm{Aut}(V)$ of V such that the orbit $H \cdot v$ of a general closed point $v \in V$ has finite complement. By Gizatullin, such a surface is rational and is either isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ or can be obtained from a nonsingular projective surface $\tilde{V}$ by deleting an ample divisor of a special form, called a *zigzag*. This is just a linear chain of nonsingular rational curves. Conversely, a nonsingular surface V completable by a zigzag is rational and geometrically quasihomogeneous. In addition, if V is not isomorphic to $\mathbb{C}^* \times \mathbb{A}^1$ then it admits two independent $\mathbb{C}_+$-actions. More precisely, Bertin showed that if V admits a $\mathbb{C}_+$-action then this action is unique uless V is completable by a zigzag. (From the Introduction to [92])

In this paper from 2004, Dubouloz generalizes these earlier results by giving the following geometric characterization of the ML-surfaces in terms of their boundary divisors.

Theorem 9.27. [92] *Let V be a normal affine surface over $\mathbb{C}$ that is not isomorphic to $\mathbb{C}^* \times \mathbb{A}^1$. Then V is completable by a zigzag if and only if $ML(V) = \mathbb{C}$.*

In particular, Dubouloz has removed the condition that V be nonsingular.

The characterization given by Bandman and Makar-Limanov was for smooth affine rational surfaces with trivial Makar-Limanov invariant, embedded in $\mathbb{C}^3$ as a hypersurface. Their conclusion is that these must be Danielewski surfaces, given by equations $xz = y^m - 1$ in a suitable coordinate system (x, y, z), with $m \geq 1$.

One type of surface which draws attention is a **homology plane**, defined to be a smooth algebraic surface X over $\mathbb{C}$ whose homology groups $H_i(X;\mathbb{Z})$ are trivial for $i > 0$. For example, the affine plane $\mathbb{A}^2$ over $\mathbb{C}$ is the unique homology plane X with $\bar{\kappa}(X) = -\infty$ (see [208]). Similarly, X is a **$\mathbb{Q}$-homology plane** if X is a smooth algebraic surface defined over $\mathbb{C}$ such that $H_i(X;\mathbb{Q}) = (0)$ for $i > 0$. Finally, the definition of a **log $\mathbb{Q}$-homology plane** X coincides with that of a $\mathbb{Q}$-homology plane, except that X is permitted to have certain kinds of singular points (at worst quotient singularities). Homology planes share many properties with the affine plane. One motivation to study them comes from their connection to the plane Jacobian Conjecture. See §3.3 of [208], as well as [224], for details about homology planes.

In [200] Masuda and Miyanishi considered the case of a surface X which is a $\mathbb{Q}$-homology plane. In this case X must be affine and rational. If X is also not rigid, then the orbits are the fibers of an $\mathbb{A}^1$-fibration $X \to \mathbb{A}^1$, which implies that $\bar{\kappa}(X) = -\infty$. In the strongest case, X is a $\mathbb{Q}$-homology plane which is an ML-surface, and then the authors conclude that X is isomorphic to the quotient of one of the surfaces $xz = y^m - 1$ under a suitable free action of the cyclic group $\mathbb{Z}_m$.

Subsequently, Gurjar and Miyanishi [133] extended these results to the case in which X is a log $\mathbb{Q}$-homology plane. Their main result in this regard is that if X is a log $\mathbb{Q}$-homology plane, then $ML(X) = \mathbb{C}$ if and only if the fundamental group at infinity, $\pi_1^\infty(X)$, is a finite cyclic group.

Another recent paper classifying ML-surfaces is due to Daigle and Russell [61]. They work over an algebraically closed field k, and if $k = \mathbb{C}$ then the class of surfaces they consider is the class of log $\mathbb{Q}$-homology planes with trivial Makar-Limanov invariant. Specifically, they consider the class $\mathcal{M}_0$ of normal affine surfaces U over k satisfying: (i) $ML(U) = k$ and (ii) $\text{Pic}(U_s)$ is a finite group, where U_s denotes the smooth part of U, and $\text{Pic}(U_s)$ denotes its Picard group. They show that every $U \in \mathcal{M}_0$ can be realized as an open subset of some weighted projective plane, and give precise conditions as to when any two such surfaces are isomorphic (Thm. A and Thm. B). In particular, surfaces U and U' belonging to this class are isomorphic if and only if the equivalence class of the weighted graphs at infinity and the resolution graphs of singularities are the same for U and U'. The authors also classify the $\mathbb{G}_a$-actions on these surfaces. For many of these surfaces, the analogue of Daigle's Transitivity Theorem for Danielewski surfaces does not hold. However, Theorem C indicates that the number of orbits in the set

$$\{\ker D | D \in \text{LND}(\mathcal{O}(U)), D \neq 0\}$$

under the action of $\text{Aut}_k\mathcal{O}(U)$ is at most 2. This theorem also gives necessary and sufficient conditions for the action to be transitive. Some of Daigle and Russell's results are based on their earlier papers [59, 60].

The reader is referred to the recent monograph of Miyanishi [208], which provides an excellent overview of recent progress in the classification of open algebraic surfaces. $\mathbb{G}_a$-actions and locally nilpotent derivations constitute one of the major themes of his exposition.

10

Slices, Embeddings and Cancellation

This chapter investigates the following question about locally nilpotent derivations of polynomial rings.

Slice Problem. If $D \in \mathrm{LND}(k^{[n]})$ has a slice, is $\ker D \cong k^{[n-1]}$?

Recall that even the Jacobian Conjecture can be stated as a problem about derivations of polynomial rings having a slice; see *Chap. 3*.

The main point of departure is the Slice Theorem (*Thm. 1.22*), which asserts that when $D \in \mathrm{LND}(B)$ has a slice (B a commutative k-domain), $A = \ker D$ satisfies $B = A[s]$, $D = d/ds$, and $\pi_s(B) = A$ for the Dixmier map π_s determined by s. Conversely, note that if $B = A[s]$, where s is transcendental over A, then the derivation $D = d/ds$ of B defined by $DA = 0$ and $Ds = 1$ is locally nilpotent, and $\ker D = A$. Thus, the Slice Problem is equivalent to a version of the famous **Cancellation Problem**:

> If X is an affine variety and $X \times \mathbb{A}^1 \cong \mathbb{A}^{n+1}$, does it follow that $X \cong \mathbb{A}^n$? Equivalently, if A is an affine ring and $A^{[1]} \cong k^{[n+1]}$, is $A \cong k^{[n]}$?

For $n = 1$, this problem was solved by Abhyankar, Eakin and Heinzer, as discussed in the preceding chapter. For $n = 2$, a positive solution is due to Fujita, Miyanishi, and Sugie in the characteristic zero case, and to Russell in the case of positive characteristic; see [124, 222, 267]. For $n \geq 3$, this remains an open problem. This version of the Cancellation Problem will be called the **Cancellation Problem for Affine Spaces**.

In this chapter, *Sect. 1* gives some positive results for locally nilpotent derivations of polynomial rings when the image generates the unit ideal. Geometrically, these correspond to fixed-point free $\mathbb{G}_a$-actions on affine space. *Section 2* discusses two important new positive results of K. Masuda on locally nilpotent derivations having a slice, which thus constitutute new cases in which the Cancellation Problem for Affine Spaces has a positive solution. This section also includes a new proof, due to Crachiola and Makar-Limanov, for one case of the cancellation theorem for surfaces, namely, that if V is an affine surface over an algebraically closed field k such that $V \times \mathbb{A}^1 \cong \mathbb{A}^3$, then $V \cong \mathbb{A}^2$. The authors write:

> Even for the special case we are considering, the only known proofs are the original one and a recent proof of Rajendra Gurjar [132] which relies on the topological methods of Mumford-Ramanujam. These are beautiful proofs which use many ideas, making them not quite self-contained for some readers. So our intention is to present a more self-contained purely algebraic proof of the Cancellation Theorem and to narrow the gap between the formulation and the proof. We also hope that the algebraic approach will be easier to use in the case of higher dimension. (From the Introduction of [41])

Their proof illustrates how powerful the theory of locally nilpotent derivations can be. Indeed, based on theory already established in this book, the proof presented below consists of a single short paragraph.

Section 3 gives an explicit formula for the torus action associated with any locally nilpotent derivation having a slice. *Section 4* explores the fascinating constructions of Asanuma, who used embeddings of affine spaces to construct torus actions on $\mathbb{A}^n$. The purpose of this section is to give a self-contained treatment of the main constructions and proofs in Asanuma's work by translating them into the language of locally nilpotent derivations. *Section 5* considers the Vénéreau polynomials and their relation to the Embedding Problem. And *Sect. 6* concludes the chapter with a few open problems related to the topics of this chapter.

An algebraic embedding $g : \mathbb{A}^m \to \mathbb{A}^n$ is **rectifiable** if and only if there exists a system of coordinate functions $f_1, ..., f_n$ on $\mathbb{A}^n$ such that $g(\mathbb{A}^m)$ is defined by the ideal $(f_{m+1}, ..., f_n)$.

Embedding Problem Is every algebraic embedding $g : \mathbb{A}^m \to \mathbb{A}^n$ rectifiable?

The answer is known to be affirmative if $n > 2m + 1$ [44, 152, 156, 284], or if $m = 1$ and $n = 2$ [3, 289]. There are presently no confirmed counterexamples for the field $k = \mathbb{C}$, but conjectural counterexamples $\mathbb{C}^1 \to \mathbb{C}^3$ and $\mathbb{C}^3 \to \mathbb{C}^4$ are discussed in this chapter.

A very nice discussion of the Cancellation Problem, Embedding Problem, and Jacobian Conjecture can be found in Kraft's paper [175].

10.1 Some Positive Results

10.1.1 Free Actions on Affine Spaces

As we have already seen, there exist fixed-point free $\mathbb{G}_a$-actions on affine space which are not translations. Such an action, defined by van den Essen, was discussed in *Example 3.9.4*. There are, however, several cases in which a free $\mathbb{G}_a$-action on affine space is known to be conjugate to a translation.

Let $B = k^{[n]}$ for $n \geq 1$, and suppose $D \in \mathrm{LND}(k^{[n]})$. We have:

1. If $(DB) = (1)$ and $\mathrm{rank}(D) \leq 2$, then D has a slice (*Thm. 4.16*).
2. If $(DB) = (1)$ and $n \leq 3$, then D has a slice (Kaliman's Theorem, *5.8*).
3. If $(DB) = (1)$, $\ker D = k^{[n-1]}$, and $\exp(tD)$ is a locally trivial $\mathbb{G}_a$-action, then D has a slice (see [78]).
4. If $(DB) = (1)$ and D is elementary, then D has a slice. (*Thm. 7.22*).

In fact, in each of these cases, the kernel of D is a polynomial ring, implying that the Slice Problem also has a positive solution in these cases, and that D is a partial derivative relative to some system of coordinates on B.

Recall that if $D \in \mathrm{LND}(k^{[n]})$ has a slice s, and s is a variable of $k^{[n]}$, then D is a partial derivative (*Prop. 3.20*). So this is one case in which the Slice Problem has a positive solution. Geometrically, this implies that *any* algebraic $\mathbb{G}_a$-action on $\mathbb{A}^n$ may be viewed as the projection of a translation in $\mathbb{A}^{n+1}$. More specifically, there is a coordinate translation on $\mathbb{A}^{n+1}$ and a projection $p : \mathbb{A}^{n+1} \to \mathbb{A}^n$ such that p is equivariant relative to the actions on $\mathbb{A}^{n+1}$ and $\mathbb{A}^n$. To see this, let $B = k^{[n]}$. Given $D \in \mathrm{LND}(B)$, extend D to D^* on $B[s] = k^{[n+1]}$ by setting $D^*s = 1$. Then D^* is a partial derivative relative to some coordinate system on $k^{[n+1]}$, and D^* restricts to B.

10.1.2 Theorem of K. Masuda

Two important special cases of the Slice Problem were settled recently by K. Masuda, as presented in the following theorem. The main idea is to reinterpret the kernel of the derivation as the ring of invariants of a torus action, and use standard theory of algebraic G-vector bundles. An algebraic vector bundle $p : X \to Y$ for affine G-varieties X and Y is called an **algebraic G-vector bundle** if p is G-equivariant, and G acts on the fibers of p by vector space isomorphisms (linear transformations). An algebraic G-vector bundle $p : X \to Y$ is called **trivial** if $X \cong Y \times V$ equivariantly for some G-module V, and p is the corresponding projection $Y \times V \to Y$. If Y is an affine space, then the well-known result of Quillen and Suslin asserts that every algebraic vector bundle over Y is trivial [256, 288].

Theorem 10.1. (K. Masuda's Theorem)([199], Thm. 3.2 and Thm. 3.4) *Let k be an algebraically closed field of characteristic zero and $B = k^{[n]}$. Suppose $D \in \mathrm{LND}(B)$ has a slice.*

(a) *If D is triangular, then D is a partial derivative.*
(b) *If $k = \mathbb{C}$ and $\mathrm{rank}(D) \leq 3$, then D is a partial derivative.*

It should be noted that, for the case $B = k^{[4]}$, part (b) of this theorem was proved earlier in [69], Cor. 4.5.

In both parts (a) and (b) of the theorem, the hypotheses cannot be weakened to "fixed-point free" instead of "having a slice". The example of Winkelmann, discussed in *Chap. 3*, shows this, since it is given by a rank-3 triangular derivation of $k^{[4]}$, and the induced $\mathbb{G}_a$-action is fixed-point free. On the other hand, it was shown that if $\mathrm{rank}(D) \leq 2$, then the hypothesis that the induced $\mathbb{G}_a$-action is fixed-point free does imply that D is a partial derivative (*Cor. 4.23*).

Proof. Part (a): We proceed by induction on n. By Rentschler's Theorem, the result holds for $n \leq 2$, so we may assume $n \geq 3$.

Let $A = \ker D$. Then $B = A[s]$, and this gives an obvious $\mathbb{G}_m$-action on B, namely, $B^{\mathbb{G}_m} = A$ and $t \cdot s = ts$. By hypothesis, $Dx_1 \in k$. If $Dx_1 \in k^*$, then D is a partial derivative by *Prop. 3.20.* So we may assume $Dx_1 = 0$.

Given $\lambda \in k$, let $\bar{B} = B \bmod (x_1 - \lambda)$ and $\bar{D} = D \bmod (x_1 - \lambda) \in \mathrm{LND}(\bar{B})$. Given $f \in B$, let $\bar{f}$ denote the class of f in $\bar{B}$. Then $\bar{D}$ is a triangular derivation of $\bar{B} = k[\bar{x_2}, ..., \bar{x_n}] \cong k^{[n-1]}$, and $\bar{D}\bar{s} = 1$. By the Slice Theorem, $\bar{B} = (\ker \bar{D})[\bar{s}]$, and by the inductive hypothesis, $\ker \bar{D} = k^{[n-2]}$. In particular, $\bar{s}$ is a variable of $\bar{B} = k^{[n-1]}$.

Let $X = \mathbb{A}^n$, and let the morphism $p : X \to \mathbb{A}^1$ be induced by the inclusion $k[x_1] \to B$. Then p is both $\mathbb{G}_a$- and $\mathbb{G}_m$-equivariant. Let F_λ denote the fiber $p^{-1}(\lambda)$. Since x_1 is a variable of B, $F_\lambda \cong \mathbb{A}^{[n-1]}$, and this fiber has induced actions of $\mathbb{G}_a$ and $\mathbb{G}_m$. By the preceding paragraph, the action of $\mathbb{G}_m$ on $\bar{B}$ is simply multiplication of the variable $\bar{s}$. Therefore, every fiber F_λ is a $\mathbb{G}_m$-module, and p is an algebraic $\mathbb{G}_m$-vector bundle. Since the action of $\mathbb{G}_m$ on the base $\mathbb{A}^1$ is trivial, the standard theory of algebraic G-vector bundles now implies that X is equivariantly isomorphic to $X^{\mathbb{G}_m} \times V$ for a one-dimensional $\mathbb{G}_m$-module V, where the fixed point set $X^{\mathbb{G}_m}$ has the structure of an algebraic vector bundle over the base $\mathbb{A}^1$ (see Masuda's article for details). As such, $X^{\mathbb{G}_m}$ is isomorphic to an affine space $\mathbb{A}^{n-1}$, by the Quillen-Suslin Theorem. Since $\mathcal{O}(X^{\mathbb{G}_m}) = A$, we conclude that $\ker D = k^{[n-1]}$. This completes the proof of part (a).

Part (b): In this case, assume that $R = \mathbb{C}^{[n-3]}$ for $n \geq 3$, $B = R^{[3]} = \mathbb{C}^{[n]}$, $D \in \mathrm{LND}_R(B)$, and $Ds = 1$ for $s \in B$. If $A = \ker D$, then $R \subset A$ and $B = A[s]$. Let $X = \mathrm{Spec}(B) = \mathbb{C}^n$ and $Y = \mathrm{Spec}(R) = \mathbb{C}^{n-3}$, and let $q : X \to Y$ be the morphism induced by the inclusion $R \to B$. As before, there are induced $\mathbb{C}^*$-actions on X and Y relative to which q is equivariant, where $B^{\mathbb{C}^*} = A$, and the action of $t \in \mathbb{C}^*$ on s is ts.

Given $\xi \in Y$, let Φ_ξ denote the fiber $q^{-1}(\xi)$, noting that $\Phi_\xi \cong \mathbb{C}^3$. Since the $\mathbb{C}^*$-action on Y is trivial, the $\mathbb{C}^*$-action on X restricts to Φ_ξ, i.e., Φ_ξ is a $\mathbb{C}^*$-variety. By the Koras-Russell Theorem, the $\mathbb{C}^*$-action on Φ_ξ is linearizable, i.e., Φ_ξ is a $\mathbb{C}^*$-module (see the Introduction to *Chap. 5*). Therefore, $q : X \to Y$ is a $\mathbb{C}^*$-vector bundle with trivial action on the base. Again using standard results from the theory of algebraic G-vector bundles, it follows that this is a trivial $\mathbb{C}^*$-vector bundle, and that $X \cong Y \times W$ equivariantly for a three-dimensional $\mathbb{C}^*$-module W, where $W = W^{\mathbb{C}^*} \times Z$ for a one-dimensional $\mathbb{C}^*$-module Z, and

$W^{\mathbb{C}^*} \cong \mathbb{C}^2$. Therefore, $X \cong \mathbb{C}^{n-1} \times Z$ equivariantly, and $X^{\mathbb{C}^*} \cong \mathbb{C}^{n-1}$. (Again, see Masuda's article for details of this reasoning.) Therefore, $\ker D = \mathbb{C}^{[n-1]}$. $\square$

10.1.3 The ML-Invariant and Cancellation

As noted earlier, Makar-Limanov proved the cancellation theorem for curves by using the fact that $ML(B^{[n]}) = ML(B)$ when $\text{tr.deg.}_k B = 1$. In his thesis, Crachiola used the fact that $ML(B^{[1]}) = ML(B)$ for rigid rings B to give a new proof of one instance of the affine cancellation theorem for surfaces (Example 4.8 of [39]). More recently, Crachiola and Makar-Limanov gave another version of this argument [41]. Their proof uses the Makar-Limanov invariant defined in arbitrary characteristic; the proof given here will be only for the case of characteristic zero. Note that their work provides a proof of Miyanishi's Theorem in the case that a locally nilpotent derivation D of $k^{[3]}$ admits a slice.

Proposition 10.2. *If X is an affine surface over an algebraically closed field k, and $X \times \mathbb{A}^1 \cong \mathbb{A}^3$, then $X \cong \mathbb{A}^2$.*

Proof. (characteristic $k = 0$) Let $R = \mathcal{O}(X)$ and $B = \mathcal{O}(X \times \mathbb{A}^1) = k^{[3]}$. By hypothesis, there exists $t \in B$ such that $B = R[t]$. Suppose R is rigid. Then by *Prop. 9.23*, $ML(R[t]) = R$. However, we have $ML(R[t]) = ML(B) = k$, a contradiction since $\text{tr.deg.}_k R = 2$. Therefore, R is not rigid. In addition, R is a UFD, since it is the kernel of $d/dt \in \text{LND}(R[t]) = \text{LND}(B)$. Since we also clearly have $R^* = k^*$, it follows from part (1) of *Thm. 9.9* that $R \cong k^{[2]}$. $\square$

Remark 10.3. In their recent paper [111], Finston and Maubach construct affine threefolds X_1 and X_2 such that $X_1 \not\cong X_2$, but $X_1 \times \mathbb{C} \cong X_2 \times \mathbb{C}$. The main difference between these and the original counterexamples to cancellation discovered by Danielewski is that X_1 and X_2 are singular but factorial (i.e., their coordinate rings are UFDs). The examples are constructed as the total spaces for principal $\mathbb{G}_a$-bundles over certain surfaces, and one of their main tools is the Makar-Limanov invariant. A principal $\mathbb{G}_a$-bundle with affine total space X_i arises from a locally trivial algebraic $\mathbb{G}_a$-action on X_i. In the cases they consider, the quotient $X_i/\mathbb{G}_a$ has the structure of a quasi-affine variety, and X_i has the structure of a principal $\mathbb{G}_a$-bundle over $X_i/\mathbb{G}_a$.

10.2 Torus Action Formula

Any affine ring $B = A[s]$ (s transcendental over A) has an obvious locally nilpotent derivation D with a slice, namely, $DA = 0$ and $Ds = 1$. Likewise, there are obvious actions of $\mathbb{G}_a$ and $\mathbb{G}_m$: If $t \in \mathbb{G}_a$, then ${}^t a = a$ for $a \in A$ and ${}^t s = s + t$; and if $t \in \mathbb{G}_m$, then $t \cdot a = a$ for $a \in A$ and $t \cdot s = t^n s$ for some

$n \in \mathbb{Z}$. An explicit formula for the $\mathbb{G}_a$-action is given by $\exp(tD)$. In this case, there is also an explicit formula for the action of the torus $\mathbb{G}_m$ in terms of D.

Given $b \in B$, if we write $b = P(s)$ for some $P \in A^{[1]}$, then of course $t \cdot b = P(t^n s)$. This relies on our ability to write b as a polynomial in s over A, which is achieved in the following way.

Lemma 10.4. *Let B be a commutative k-domain, and let $D \in \mathrm{LND}(B)$ have a slice $s \in B$. Then for $b \in B$,*

$$b = \sum_{n \geq 0} \frac{1}{n!} \pi_s(D^n b) s^n \ .$$

Proof. For each $i \geq 0$, define the function $F_i : B \to A$ as follows: Given $b \in B$, suppose $b = \sum_{n \geq 0} a_i s^i$ for $a_i \in A$. Then $F_i(b) = a_i$.

Recall that the kernel of π_s is the ideal sB, and that $\pi_s(a) = a$ for all $a \in A$. Therefore, $a_0 = \pi_s(a_0) = \pi_s(b)$. Moreover, it is clear that for $i \geq 1$,

$$F_{i-1}(Db) = ia_i = iF_i(b) \quad \Rightarrow \quad F_i(b) = \frac{1}{i} F_{i-1}(b) \quad (i \geq 1) \ .$$

By induction, it follows that, for all $n \geq 0$, $F_n(b) = \frac{1}{n!} \pi_s(D^n b)$. $\square$

However, it is possible to give a more direct formula for the torus action which does not rely on finding the coefficients of an element over the kernel A.

Specifically, let $\rho : B \times \mathbb{G}_m \to B$ denote the $\mathbb{G}_m$-action on B defined by $\rho(a,t) = a$ for $a \in A$, and $\rho(s,t) = t^n s$. Given $t \in \mathbb{G}_m$, let $\rho_t : B \to B$ denote the restriction of ρ to $B \times \{t\}$. Let λ denote an indeterminate over B, and extend D to $B[\lambda]$ via $D\lambda = 0$. Then D (extended) is locally nilpotent on $B[\lambda]$, and $\exp(\lambda D)$ is a well-defined automorphism of $B[\lambda]$.

Theorem 10.5. *In the notation above, $\rho_t = \exp(-\lambda D)|_{\lambda = (1-t^n)s}$.*

Proof. Given $t \in \mathbb{G}_m$, define $\beta_t = \exp(-\lambda D)|_{\lambda=(1-t^n)s}$. The main fact to show is that, given $u, v \in \mathbb{G}_m$, $\beta_u \beta_v = \beta_{uv}$, i.e., this defines an action β of $\mathbb{G}_m$ on B. Since the fixed ring of β is A, and since $\beta_t(s) = \rho_t(s)$ for all t, it will then follow that $\beta = \rho$.

Introduce a second indeterminate μ. Given $Q \in B[\mu]$, let $\epsilon_Q : B[\lambda, \mu] \to B[\lambda, \mu]$ denote evaluation at Q, i.e., $\epsilon_Q(f) = f$ for $f \in B[\mu]$ and $\epsilon_Q(\lambda) = Q$. Likewise, if $R \in B[\lambda]$, let δ_R denote evaluation at R, i.e., $\delta_R(g) = g$ for $g \in B[\lambda]$ and $\delta_R(\mu) = R$. Note that if α is any automorphism of $B[\lambda, \mu]$ such that $\alpha(B[\mu]) \subset B[\mu]$ and $\alpha(\lambda) = \lambda$, then $\alpha \epsilon_Q \alpha^{-1} = \epsilon_{\alpha(Q)}$. A similar formula holds for δ_R.

Let $u, v \in \mathbb{G}_m$ be given. Then

$$\beta_u \beta_v = \epsilon_s \circ \exp(-(1-u^n)\lambda D) \circ \delta_s \circ \exp(-(1-v^n)\mu D) \ .$$

If $\alpha = \exp(-(1-u^n)\lambda D)$, then

$$\alpha \circ \delta_s = \delta_{\alpha(s)} \circ \alpha = \delta_{(s-(1-u^n)\lambda)} \circ \alpha \ .$$

Therefore

$$\begin{aligned}\beta_u\beta_v &= \epsilon_s \circ \delta_{(s-(1-u^n)\lambda)} \circ \exp(-(1-u^n)\lambda D) \circ \exp(-(1-v^n)\mu D) \\ &= \epsilon_s \circ \delta_{(s-(1-u^n)\lambda)} \circ \exp\Big(-\big((1-u^n)\lambda + (1-v^n)\mu\big)D\Big) \ .\end{aligned}$$

Now

$$\begin{aligned}\epsilon_s \circ \delta_{(s-(1-u^n)\lambda)}\Big((1-u^n)\lambda + (1-v^n)\mu\Big) &= \epsilon_s\Big((1-u^n)\lambda \\ &\qquad +(1-v^n)(s-(1-u^n)\lambda)\Big) \\ &= \epsilon_s\Big((1-v^n)s + v^n((1-u^n)\lambda\Big) \\ &= (1-v^n)s + v^n(1-u^n)s \\ &= (1-u^nv^n)s \\ &= \epsilon_s\Big((1-u^nv^n)\lambda\Big) \ .\end{aligned}$$

Therefore

$$\beta_u\beta_v = \epsilon_s \circ \exp(-(1-u^nv^n)\lambda D) = \beta_{uv} \ .$$

□

Remark 10.6. The reader is warned that some authors would write

$$\exp(-(1-t^n)sD)$$

in place of the evaluation notation used in this formula. To do so is technically incorrect, but might be accepted as a convenient abuse of notation: Since s is not in the kernel of D, $(1-t^n)sD$ is not locally nilpotent, and its exponential is *not* an algebraic automorphism of B.

Example 10.7. The simple action of $\mathbb{G}_m$ on $\mathbb{A}^n$ given by

$$t(x_1, ..., x_n) = (x_1, ..., x_{n-1}, t^N x_n)$$

is of the form $\exp(-\lambda D)|_{\lambda=(1-t^N)x_n}$, where $D = \frac{\partial}{\partial x_n}$.

10.3 Asanuma's Torus Actions

In a remarkable paper published in 1999, Asanuma showed how non-rectifiable embeddings $\mathbb{R}^1 \hookrightarrow \mathbb{R}^3$ (e.g., knots) could be used to construct non-linearizable algebraic actions of real tori $\mathbb{R}^*$ and $(\mathbb{R}^*)^2$ on $\mathbb{R}^5$ [7]. These were the first examples in which a commutative reductive k-group admits a non-linearizable algebraic action on affine space, where k is a field of characteristic 0. Later, using completely different methods, the author and Moser-Jauslin found a non-linearizable action of the circle group $S^1 = SO_2(\mathbb{R})$ on $\mathbb{R}^4$ [122]. It remains an open question whether, over the field $k = \mathbb{C}$, every algebraic action of a commutative reductive group on $\mathbb{C}^n$ can be linearized; for a discussion of the linearization problem for complex reductive groups, see [175].

The main result Asanuma proves is this.

Theorem 10.8. (Cor. 6.3 of [7]) *Let k be an infinite field. If there exists a non-rectifiable closed embedding $\mathbb{A}^m \to \mathbb{A}^n$, then there exist non-linearizable faithful algebraic $(k^*)^r$-actions on $\mathbb{A}^{1+n+m}$ for each $r = 1, ..., 1+m$.*

To put his theorem to use, Asanuma uses an earlier result of Shastri.

Theorem 10.9. [277] *Every open knot type admits a polynomial representation in $\mathbb{R}^3$.*

Therefore, for topological reasons, there exist non-rectifiable algebraic embeddings of $\mathbb{R}$ into $\mathbb{R}^3$. For example, Shastri gives the following polynomial parametrization of the trefoil knot:

$$\phi(v) = (v^3 - 3v, v^4 - 4v^2, v^5 - 10v) \ .$$

Note that ϕ also defines an algebraic embedding of $\mathbb{C}^1$ into $\mathbb{C}^3$. It is presently unknown whether its image can be conjugated to a coordinate line by an algebraic automorphism of $\mathbb{C}^3$. However, Kaliman proved that the image of any algebraic embedding of $\mathbb{C}^1$ in $\mathbb{C}^3$ can be conjugated to a coordinate line by a *holomorphic* automorphism of $\mathbb{C}^3$; see [157].

Asanuma uses a purely algebraic approach in his constructions. The main idea in his paper is to associate a certain Rees algebra to an embedding $\phi : k^m \hookrightarrow k^n$, namely, the Rees algebra of the ideal $\ker(\phi^* : k^{[n]} \to k^{[m]})$.

In the present treatment, we consider only embeddings $\phi : k^1 \hookrightarrow k^3$. However, the arguments presented here can easily be extended to the more general case of Asanuma's theorem. We associate to ϕ a certain triangular derivation D on $k^{[5]}$, and show that D induces a torus action in the manner of *Thm. 10.5* above. Geometrically, the fixed-point set L is isomorphic to a line k^1, and the quotient Q is isomorphic to k^3. The crucial fact is that the canonical embedding of L into Q is precisely the embedding ϕ. It should be noted that van den Essen and van Rossum also recognized derivations implicit in Asanuma's work; see [103].

10.3.1 The Derivation Associated to an Embedding

Let $\phi : k^1 \to k^3$ be an algebraic embedding, given by $\phi(v) = (f(v), g(v), h(v))$. Specifically, this means that ϕ is injective and $\phi'(v) \neq 0$ for all v (see van den Essen [100], Cor. B.2.6). Let ϕ^* denote the corresponding ring homomorphism $\phi^* : k[x, y, z] \to k[v]$, given by $\phi^*(p(x, y, z)) = p(\phi(v))$. Since ϕ is an embedding, ϕ^* is surjective, i.e., there exists $F \in k[x, y, z]$ such that $\phi^*(F) = v$.

Set $\mathbf{x} = (x, y, z)$, and let $B = k^{[5]} = k[u, v, \mathbf{x}]$. Define a triangular derivation D on B by

$$Du = 0 \ , \ Dv = -u \ , \ D\mathbf{x} = \phi'(v) \tag{10.1}$$

and let $A = \ker D$. Observe that $D(u\mathbf{x}+\phi(v)) = 0$, and thus $F(u\mathbf{x}+\phi(v)) \in A$. Observe also that (by Taylor's Formula) there exists $s \in B$ such that

$$F(u\mathbf{x} + \phi(v)) = us + F(\phi(v)) = us + v \ .$$

Therefore, $0 = D(us+v) = uDs - u$, which implies $Ds = 1$, i.e., D has a slice and $B = A[s]$.

From the Slice Theorem, we conclude that the kernel of D is generated by the images under π_s of the system of variables $u, v, \mathbf{x}$, namely,

$$\ker D = k[u, v + us, \mathbf{x} - \textstyle\sum_{i \geq 1} \frac{1}{i!}\phi^{(i)}(v)u^{i-1}s^i\,] \ . \tag{10.2}$$

So the kernel, which is of transcendence four over k, is generated by 5 polynomials. In general, it is unknown whether this number can always reduced to four, or equivalently, whether s is a variable of B. However, if the field k is algebraically closed, then $\ker D = k^{[4]}$, since D is triangular (K. Masuda's Theorem). In any case, recall that s is known to be a variable of $B[t] = k^{[6]}$ (*Prop. 3.20*).

For example, using Shastri's parametrization of the trefoil knot, given above, it is easy to calculate the slice s explicitly. As Shastri points out (p.14), if $F = yz - x^3 - 5xy + 2z - 7x$, then $F(\phi(v)) = v$, and thus

$$s = u^{-1}(F(u\mathbf{x} + \phi(v)) - v) \ .$$

Of course, when ϕ is rectifiable, D is conjugate to a partial derivative. In this case, finding a system of variables for the kernel is equivalent to finding a system of coordinates in $\mathbb{A}^3$ relative to which $\phi(\mathbb{A}^1)$ is a coordinate line.

Theorem 10.10. *Suppose $\phi : \mathbb{A}^1 \to \mathbb{A}^3$ is a rectifiable embedding, and let D be the induced triangular derivation (10.1) of $k[u, v, \mathbf{x}]$. If $k[x, y, z] = k[F, G, H]$, where $\phi^*(F) = v$ and $\ker \phi^* = (G, H)$, then*

$$\ker D = k[u, F(u\mathbf{x} + \phi(v)), \tfrac{1}{u}G(u\mathbf{x} + \phi(v)), \tfrac{1}{u}H(u\mathbf{x} + \phi(v))] \ .$$

In particular, D is conjugate to a partial derivative.

Proof. Set $A = \ker D$, and let w denote the slice of D derived from F, i.e., $v + uw = F(u\mathbf{x} + \phi(v))$. Set

$$m = \tfrac{1}{u}G(u\mathbf{x} + \phi(v)) \ , \ n = \tfrac{1}{u}H(u\mathbf{x} + \phi(v)) \ .$$

Then $m, n \in A$, since A is factorially closed.

Let $R \subset A$ denote the subring

$$\begin{aligned} R &= k[u\mathbf{x} + \phi(v)] \\ &= k[F(u\mathbf{x} + \phi(v)), G(u\mathbf{x} + \phi(v)), H(u\mathbf{x} + \phi(v))] \\ &= k[v + uw, um, un] \ . \end{aligned}$$

Set $\mathbf{X} = \pi_w(\mathbf{x})$. Since π_w fixes elements of A, and since $\pi_w(\phi(v)) = \phi(v+uw)$, we see that

$$u\mathbf{X} + \phi(v + uw) = \pi_w(u\mathbf{x} + \phi(v)) = u\mathbf{x} + \phi(v) \ .$$

Therefore, $u\mathbf{X}+\phi(v+uw) \in R^3$ implies $u\mathbf{X} \in R^3$. Note that the R-ideal $uB \cap R$ is generated over R by um and un. Therefore we can write $u\mathbf{X} = (um)\mathbf{p} + (un)\mathbf{q}$ for some $\mathbf{p}, \mathbf{q} \in R^3$. But then $\mathbf{X} = m\mathbf{p}+n\mathbf{q} \in k[u, v+uw, m, n]^3$, which implies

$$A = \pi_w(B) = k[u, v + uw, \mathbf{X}] \subset k[u, v + uw, m, n] \subset A \ .$$

□

10.3.2 Two-Dimensional Torus Action

Let $G = G_1 \times G_2$ denote the two-dimensional torus $\mathbb{G}_m^2$, where each $G_i \cong \mathbb{G}_m$. Define an algebraic action of G on $\mathbb{A}^5$ in the following way.

- G_1 acts as in *Thm. 10.5*, namely, the action of $\theta \in G_1$ is given by $\exp(\lambda D)|_{\lambda=(1-\theta)s}$. Observe that $B^{G_1} = A$.
- G_2 acts linearly, namely, the action of $t \in G_2$ is given by $t(u, v, \mathbf{x}) = (t^{-1}u, v, t\mathbf{x})$.

If the G_2-action is extended to $B[\lambda]$ by declaring that $t \cdot \lambda = t\lambda$, then λD is homogeneous of degree 0 relative to the G_2-action. It follows that the actions of G_1 and G_2 commute (see *Sect. 3.7*), and we thus obtain an action of the torus G on $\mathbb{A}^5$. Explicitly, this action is given by:

$$(\theta, t)(u, v, \mathbf{x}) = \left(t^{-1}u, v + (1-\theta)us, t\Big(\mathbf{x} - \sum_{i\geq 1} \tfrac{1}{i!}\phi^{(i)}(v)u^{i-1}(1-\theta)^i s^i\Big) \right) \ . \tag{10.3}$$

Theorem 10.11. *For the G-action above:*

1. $B^G = k[u\mathbf{x} + \phi(v)] \cong k^{[3]}$
2. *The fixed point set* $\mathrm{Fix}(G)$ *is the line defined by the ideal* $I = (u, \mathbf{x})$.
3. *The canonical embedding* $\mathrm{Fix}(G) \hookrightarrow \mathrm{Spec}(B^G)$ *is equivalent to the embedding* $\phi : k^1 \hookrightarrow k^3$.

Proof. First, $B^G = (B^{G_2})^{G_1} = k[v, u\mathbf{x}]^{G_1}$. Now $k[v, u\mathbf{x}] \cong k^{[4]}$, and the derivation D restricts to a mapping $\delta : k[v, u\mathbf{x}] \to uk[v, \mathbf{x}]$, where $\ker \delta = k[v, u\mathbf{x}]^{G_1}$.

Define the triangular derivation Δ on $k[v, u\mathbf{x}]$ by $\Delta v = -1$ and $\Delta(u\mathbf{x}) = \phi'(v)$. Then $\delta = u\Delta$. Since $k[v, u\mathbf{x}] = k[v, u\mathbf{x} + \phi(v)]$, it follows that $\ker \delta = \ker \Delta = k[u\mathbf{x} + \phi(v)]$. This proves (1), and (2) is obvious.

As for (3), the canonical embedding of the fixed points into the quotient is induced by the composition $B^G \hookrightarrow B \to B/I$. The image of $p(u\mathbf{x}+\phi(v)) \in B^G$ equals $p(\phi(v)) \bmod I$, and (3) now follows. □

10.3.3 The One-Dimensional Torus Action

Let $H \subset G$ denote the one-dimensional torus consisting of pairs $(t, t^{-1}) \in G$.

Theorem 10.12. *For the H-action above:*

1. $B^H = k[u\mathbf{x} + \phi(v), s] \cong k^{[4]}$;
2. *The fixed point set* $\mathrm{Fix}(H)$ *is the surface defined by the ideal* $J = (u, \mathbf{x} - \phi'(v)s)$, *and* $\mathrm{Fix}(H) \cong k^2$;
3. *The canonical embedding* $\mathrm{Fix}(H) \hookrightarrow \mathrm{Spec}(B^H)$ *is equivalent to the embedding* $1 \times \phi : k^2 \hookrightarrow k^4$.

Proof. Note that, since D and H commute, D restricts to B^H, and H restricts to $\ker D$. Denote the restriction of D to B^H by d. Then $\ker d = (\ker D)^H = (B^{G_1})^H = B^G$ (since $G = G_1 \times H$). In addition, since $s \in B^H$ and $ds = 1$, it follows that $B^H = (\ker d)[s] = B^G[s]$. So (1) follows by *Theorem 10.11.*

For (2), note first the condition $t^{-1}u = u$ for all $t \in k^*$ implies $u = 0$. Second, $v = v + (1 - t^{-1})us$, which imposes no new condition, since $u = 0$. Similarly, since $u = 0$, we see from the formula (10.3) that

$$\mathbf{x} = t\Big(\mathbf{x} - \phi'(v)(1 - t^{-1})s\Big) \ \Rightarrow \ (1 - t)\mathbf{x} = -t\phi'(v)(1 - t^{-1})s = (1 - t)\phi'(v)s \ .$$

Item (2) now follows by cancelling $(1 - t)$ on each side.

As for (3), the canonical embedding of the fixed points into the quotient is induced by the composition $B^H \hookrightarrow B \to B/J$. Since $B = A[s]$, we see from line (10.2) that B/J is generated by the class of v and s, so $B/J \cong k^{[2]}$. Since the image of $p(u\mathbf{x} + \phi(v), s) \in B^H$ equals $p(\phi(v), s) \bmod J$, (3) now follows. □

Corollary 10.13. *If the embedding $\phi : \mathbb{A}^1 \hookrightarrow \mathbb{A}^3$ is not rectifiable, then the induced algebraic actions of $H = \mathbb{G}_m$ and $G = \mathbb{G}_m^2$ on $\mathbb{A}^5$ are not linearizable.*

Proof. If the $\mathbb{G}_m$-action were linearizable, it would be conjugate to the induced tangent space action, in which case the fixed points would be embedded in the quotient as a linear subspace. In particular, this embedding would be rectifiable, contradicting part (3) of *Thm. 10.12.* Therefore, the $\mathbb{G}_m$-action cannot be linearizable. Since it is a restriction of the $\mathbb{G}_m^2$-action, the $\mathbb{G}_m^2$-action is also not linearizable. □

Remark 10.14. There exist non-rectifiable holomorphic embeddings of $\mathbb{C}$ into $\mathbb{C}^n$ for all $n \geq 2$. By methods similar to Asanuma, Derksen and Kutzschebauch use these embeddings to show that, for every nontrivial complex reductive Lie group G, there exists an effective non-linearizable holomorphic action of G on some affine space. See their article [72] for details.

10.4 Vénéreau Polynomials

Let $B = \mathbb{C}[x,y,z,u] = \mathbb{C}^{[4]}$. The **Vénéreau polynomials** $f_n \in B$ were first defined explicitly by Vénéreau in his thesis [301]. They evolved out of his work, together with Kaliman and Zaidenberg, on questions related to cancellation and embedding problems in the affine setting. Their papers [165, 167] discuss the origin and importance of the Vénéreau polynomials.

Recall from *Example 3.13* that $f_n = y + x^n v$, where

$$p = yu + z^2 \ , \ v = xz + yp \ , \text{ and } w = x^2u - 2xzp - yp^2 \ .$$

It was shown in that example that f_n is an x-variable of B for each $n \geq 3$. This was first shown by Vénéreau in his thesis, using other methods. He also showed that, for all $n \geq 1$, every polynomial of the form $f_n - \lambda(x)$ defines a hypersurface in $\mathbb{C}^4$ which is isomorphic to $\mathbb{C}^3$, where $\lambda(x) \in \mathbb{C}[x]$. It remains an important open question whether f_1 and f_2 are variables of B. If not, it would give a counterexample to the Embedding Problem.

This section studies the Vénéreau polynomials $f := f_1$ and $g := f_2$, working more generally over any field k of characteristic zero.

10.4.1 A Proof that f Defines an Affine Space

We prove the following result by means of locally nilpotent derivations.

Theorem 10.15. *Let $\mathcal{B} = k[x,y,z,u]$, where $y + x(xz + y(yu + z^2)) = 0$. Then $\mathcal{B} = k[x]^{[2]}$.*

Proof. Define $w_1 = xu - 2zp + vp^2$. Note the relations $y + xv = 0$ and $x^2p = v^2 + xyw_1$. Thus, $v = xv_1$ and $y = -x^2v_1$ for $v_1 = z - vp$. Note also that $\mathcal{B} = k[x, v_1, z, u]$.

Define $D \in \text{LND}(\mathcal{B})$ by

$$Dx = Dv_1 = 0 \ , \ Dz = -x^3v_1^2 \ , \ Du = 1 - 2xv_1z \ ,$$

and set $A = \ker D$. Then $Dy = 0$, $Dp = y$, and $Dw_1 = x$.

Note that $z \equiv v_1$ and $p \equiv v_1^2$ modulo $x\mathcal{B}$, which implies $w_1 + 2v_1^3 \in x\mathcal{B}$. Therefore, if $s \in \mathcal{B}$ is such that $w_1 + 2v_1^3 = xs$, then $Ds = 1$. By the Slice Theorem, $\mathcal{B} = A[s]$, and $A = k[x, v_1, \pi_s(z), \pi_s(u)]$. We show that, if $A' = k[x, v_1]$, then $A = A'$.

Since $s\mathcal{B} \subset \ker \pi_s$, we have that $0 = \pi_s(xs) = \pi_s(w_1 + 2v_1^3) = \pi_s(w_1) + 2v_1^3$, so $\pi_s(w_1) \in A'$.

Since $x^2 p = v^2 + xyw_1 = x^2 v_1^2 - x^3 v_1 w_1$, we have $p = v_1^2 - xv_1 w_1$. This implies $\pi_s(p) \in A'$. Likewise, $z = v_1 + xv_1 p$, which implies $\pi_s(z) \in A'$.

Finally,

$$\begin{aligned} xu = w_1 + 2zp - vp^2 &= w_1 + 2(v_1 + vp)(v_1^2 - xv_1 w_1) - xv_1 p^2 \\ &= (w_1 + 2v_1^3) + 2(v_1^2 vp - xv_1^2 w_1 - xv_1 vpw_1) - xv_1 p^2 \\ &= xs + 2x(v_1^3 p - v_1^2 w_1 - v_1 vpw_1) - xv_1 p^2 \ , \end{aligned}$$

so $u = s + 2(v_1^3 p - v_1^2 w_1 - v_1 vpw_1) - v_1 p^2$. This implies $\pi_s(u) \in A'$. Therefore, $A' = A$, and $\mathcal{B} = k[x, v_1, s]$ $\square$

A very similar argument can be used to show that, for all $\lambda(x) \in k[x]$, the hypersurfaces $f - \lambda(x) = 0$ and $g - \lambda(x) = 0$ are isomorphic to $\mathbb{A}^3$.

10.4.2 Stable Variables

Let $B = k^{[n]}$ for some $n \geq 1$. An element $F \in B$ is called a **stable variable** or **stable coordinate** of B if F is a variable of $B[t] = k^{[n+1]}$. (This terminology comes from [198].) As remarked, it is unknown whether f or g is a variable of $B = k[x, y, z, u]$. However, we have the following result. Again, locally nilpotent derivations are the central tool in its proof.

Theorem 10.16. (Freudenburg [114]) *Let $B = k[x, y, z, u] = k^{[4]}$ and $B[t] = k^{[5]}$. Then f and g are x-variables of $B[t]$.*

Proof. Define $T \in B[t]$ by $T = xt + p$, observing that both $v - yT$ and $w + vT$ belong to $xB[t]$. Let $V, W' \in B[t]$ be such that $xV = v - yT$ and $xW' = w_1 + vT$. Then $W' + VT$ also belongs to $xB[t]$, and we choose $W \in B[t]$ so that $xW = W' + VT$. By direct calculation,

$$V = z - yt \ , \ W' = xu - zp + vt \ , \ x^2 W = w_1 + 2vT - yT^2 \ , \ W_1 = u + 2zt - yt^2 \ .$$

On the subring $k[y, z, u, t]$, define a locally nilpotent derivation δ by

$$\delta y = \delta t = 0 \ , \quad \delta z = -y \ , \quad \delta u = 2z \ .$$

Then

$$V = \exp(t\delta)(z) \quad \text{and} \quad W = \exp(t\delta)(u) \ ,$$

and therefore $k[y, z, u, t] = k[y, V, W, t]$. In addition, since $\ker \delta = k[y, t, p]$, it follows that $p = yu + z^2 = yW + V^2$.

We next wish to define a locally nilpotent derivation D on $B[t]$. To simplify notation, when $h_1, ..., h_5 \in B[t]$, let $\partial(h_1, h_2, h_3, h_4, h_5)$ denote the jacobian determinant of $h_1, ..., h_5$ relative to the coordinate system (x, y, z, u, t). Note that ∂ is a derivation in each of its arguments.

Define $D = \partial(x, \cdot, v, W, T)$. Then D is locally nilpotent, since $Dx = 0$ and

$$k(x)[y, v, W, T] = k(x)[y, z, u, t] \ .$$

In addition:

$$\begin{aligned}
Dy &= \partial(x, y, v, W, T) \\
&= x^{-1}\partial(x, y, v, W' + VT, T) && \text{since} \quad W = x^{-1}(W' + VT) \\
&= x^{-1}\partial(x, y, v, W', T) && \text{since} \quad VT = x^{-1}(vT - yT^2) \in k(x, y, v, T) \\
&= x^{-2}\partial(x, y, v, w + vT, T) && \text{since} \quad W' = x^{-1}(w + vT) \\
&= x^{-2}\partial(x, y, v, w, T) && \text{since} \quad vT \in k(x, y, v, T) \\
&= x^{-2}\partial(x, y, v, w, xt + p) && \text{since} \quad T = xt + p \\
&= x^{-2}\partial(x, y, v, w, xt) && \text{since} \quad p \text{ is algebraic over } k[x, y, v, w] \\
&= x^{-1}\partial(x, y, v, w, t) \\
&= x^{-1}d(y) = x^{-1}x^3 = x^2 \ .
\end{aligned}$$

The derivation d appearing in the last line is $d = \Delta_{(x,v,w)}$, which was used in *Example 3.13* above. It follows that a partial system of variables on $B[t]$ is given by:

$$\exp(vD)(x) = x \quad \text{and} \quad \exp(vD)(y) = y + vDy = y + x^2v = g \ .$$

Therefore, g is an x-variable of $B[t]$.

Next, set $T_1 = T$. Then $v - fT_1$ belongs to $xB[t]$, and we let $V_1 \in B[t]$ be such that $xV_1 = v - fT_1$. In particular, $V_1 = V - vp - xvt$. We have

$$\begin{aligned}
x^2 &= \partial(x, y, v, W, T_1) \\
&= \partial(x, y + xv, v, W, T_1) \\
&= \partial(x, f, v, W, T_1) \\
&= \partial(x, f, v - fT_1, W, T_1) \\
&= \partial(x, f, xV_1, W, T_1) \\
&= x\partial(x, f, V_1, W, T_1) \ .
\end{aligned}$$

so $\partial(x, f, V_1, W, T_1) = x$.

Given $F \in B[t]$, let $F(0)$ denote evaluation at $x = 0$. Since $f(0) = y$, $T_1(0) = p$, and $V_1(0) = V - yp^2$, it follows that for $V_2 := V_1 + fT_1^2$ we have $V_2(0) = V$. Thus, $(fW + V_2^2)(0) = yW + V^2 = p$, which implies that $(T_1 - (fW + V_2^2))(0) = 0$. Let $T_2 \in B[t]$ be such that $xT_2 = T_1 - (fW + V_2^2)$. Then

$$\begin{aligned}
x &= \partial(x, f, V_1, W, T_1) \\
&= \partial(x, f, V_1 + fT_1^2, W, T_1) \\
&= \partial(x, f, V_2, W, T_1) \\
&= \partial(x, f, V_2, W, T_1 - (fW + V_2^2)) \\
&= \partial(x, f, V_2, W, xT_2) \\
&= x\partial(x, f, V_2, W, T_2) \ .
\end{aligned}$$

Therefore, $\partial(x, f, V_2, W, T_2) = 1$. Now $k(x, f, V_2, W, T_2) = k(x, y, z, u, t)$, and it is well-known that the Jacobian Conjecture holds in the birational case. Therefore,

$$k[x, f, V_2, W, T_2] = k[x, y, z, u, t] \ ,$$

and f is an x-variable of $B[t]$ $\square$

See the article [114] for further results about the Vénéreau polynomials.

10.5 Open Questions

In this section, let $\phi : \mathbb{R}^1 \hookrightarrow \mathbb{R}^3$ denote the trefoil knot embedding given above.

Question 10.17. If D is the triangular derivation of $\mathbb{R}^{[5]}$ associated to ϕ, is $\ker D$ a polynomial ring?

If the answer here is negative, Asanuma's constructions provide a counterexample to the Cancellation Problem over the field of real numbers. One strategy would be to calculate the kernel over $\mathbb{C}$, which *is* a polynomial ring, and see whether the four generators can be given with real coefficients. A negative answer would also give a counterexample to the analogue of Kambayashi's Theorem in higher dimension, since

$$\mathbb{C}^{[4]} = \ker (\mathbb{C} \otimes_{\mathbb{R}} D) = \mathbb{C} \otimes_{\mathbb{R}} \ker D \ .$$

The **Shastri ideal** associated to a line embedding $\psi : k^1 \to k^3$ equals the kernel of $\psi^* : k[x, y, z] \to k[v]$. In particular, let I denote the Shastri ideal associated to ϕ, and suppose ϕ is given by $\phi(v) = (f(v), g(v), h(v))$. If $F = yz - x^3 - 5xy + 2z - 7x$, then $F(\phi(v)) = v$, and

$$I = (x - f(F), y - g(F), z - h(F)) \ .$$

Question 10.18. (Complete Intersection) If $I \subset \mathbb{R}[x, y, z]$ is the Shastri ideal associated to ϕ, what explicit polynomials $p, q \in \mathbb{R}[x, y, z]$ satisfy $I = (p, q)$?

We next consider the complexification of ϕ.

Question 10.19. Let $\Phi : \mathbb{C}^1 \to \mathbb{C}^3$ be the extension $\Phi = \mathbb{C} \otimes_{\mathbb{R}} \phi$. Is Φ algebraically rectifiable?

Note that there is no longer any topological obstruction to the rectifiability of Φ. As mentioned, Kaliman even showed Φ is holomorphically rectifiable. The reader is referred to the nice article of Bhatwadekar and Roy [19] for further discussion of embedded lines.

Next, observe that for the Asanuma actions of $\mathbb{G}_m$ and $\mathbb{G}_m^2$ (over $\mathbb{R}$) on $\mathbb{R}^5$, there exist restricted actions of $\mathbb{Z}_2$ and $\mathbb{Z}_2^2$ on $\mathbb{R}^5$.

Question 10.20. Are the algebraic actions of $\mathbb{Z}_2$ and $\mathbb{Z}_2^2$ on $\mathbb{R}^5$ induced by ϕ linearizable?

In characteristic zero, it is an important open question whether any finite abelian group can act in a non-linearizable way on $\mathbb{A}^n$. In positive characteristic, Asanuma produced an example of a non-linearizable algebraic K-action, where K is a product of two finite cyclic groups [6].

The final question of this chapter concerns a special type of the $\mathbb{G}_m$-actions considered in this chapter. Suppose $D \in \mathrm{LND}(k[x_1, ..., x_n])$ has a slice s, and suppose also that D is nice, i.e., $D^2 x_i = 0$ for each i. In this case, the induced $\mathbb{G}_m$-action on $\mathbb{A}^n$ defined by *Thm. 10.5* is also called **nice**, and has the simple form $t \cdot x_i = x_i - (1-t)sDx_i$.

Question 10.21. Can every nice $\mathbb{G}_m$-action on $\mathbb{A}^n$ be linearized?

11

Epilogue

It is by the solution of problems that the investigator tests the temper of his steel; he finds new methods and new outlooks, and gains a wider and freer horizon. ...for he who seeks for methods without having a definite problem in mind seeks for the most part in vain.

David Hilbert, *Mathematical Problems*

What are the big problems which will guide future research into locally nilpotent derivations? Certainly, any answer I give reflects my own point of view, and we have good reason to "expect the unexpected". There is tremendous room for further research into the locally nilpotent derivations of other commutative rings, like power series rings, or integral domains not containing a field, or of non-commutative rings, including Lie algebras, quantum polynomial rings, universal enveloping algebras, and the like. Some of these investigations have been started by Makar-Limanov (see [190, 197]). Indeed, the study of locally nilpotent derivations in the commutative setting has its roots in the study locally nilpotent derivations of Lie algebras, as found in the work of Dixmier, or Gabriel and Nouazé. We also had a glimpse of this in *Chap. 7*, where the proof for the linear counterexample to Hilbert's problem in dimension 11 exploited a locally nilpotent derivation of a certain four-dimensional Lie algebra. It will doubtless be profitable to study locally nilpotent derivations in families, rather than just as individual objects, and this could lead to forming a powerful Lie theory for $\mathrm{LND}(B)$. In addition, much more can be said about the geometric aspects of the subject, including the case of positive characteristic, or the study of $\mathbb{G}_a$-actions on complete varieties, or the study of quotients and quotient maps for $\mathbb{G}_a$-actions.

Many open questions, ranging from specific cases to broader themes, have already been posed and discussed in the foregoing chapters. A solution to the Embedding Problem or Cancellation Problem for affine spaces would reverberate across the whole of algebra, and we have seen how locally nilpotent derivations might play a role in their solution. Following are several additional directions for future inquiry.

11.1 Rigidity of Kernels for Polynomial Rings

Given $n \geq 2$, does there exist any $D \in \mathrm{LND}(k^{[n]})$ such that $\ker D$ is rigid? Note that if the answer were known to be negative for $n = 3$, independent of

Miyahishi's Theorem, then Miyanishi's Theorem would follow as a corollary, based on the characterization of $k[x, y]$ given above.

11.2 The Extension Property

For a ring R (choose your category), a subring S has the **extension property** if and only if:

> For every $D \in \text{Der}(R)$, if D restricts to S and $D|_S$ is locally nilpotent, then $D \in \text{LND}(R)$.

This definition is motivated by Vasconcelos's Theorem, which says that when R is an integral extension of S, then S has the extension property. Is the converse true? Again, the Jacobian problem is related to this property: For a polynomial ring $R = k[x_1, ..., x_n]$, if $S = k[f_1, ..., f_n]$ is a jacobian subring which also has the extension property, then $S = R$. So one would like to know what properties of S imply that S has the extension property.

11.3 Nilpotency Criterion

If B is a finitely generated k-domain and $D \in \text{Der}_k(B)$, does $D \in \text{LND}(B)$? To date, I know of no specific example where the answer to this question is not known. This question was discussed in *Chap. 8*, where van den Essen's Partial Nilpotency Criterion was given. That criterion is based on finding a transcendence basis for the kernel of D, which leaves the seemingly weaker question: Given $D \in \text{LND}(B)$, if $\text{tr.deg.}_k B \geq 2$, can we construct even one non-constant kernel element? Experience indicates that when B is equipped with a degree function and the degree of $f \in B$ is known, then it should be possible to determine an integer N such that $D^N f = 0$, thereby yielding $D^{N-1} f \in B^D$. At present such bounds are not generally known.

11.4 Calculating the Makar-Limanov Invariant

As mentioned, the recent paper [162] of Kaliman and Makar-Limanov gives certain methods for calculating the Makar-Limanov invariant of affine $\mathbb{C}$-varieties. But this remains a difficult calculation in general. One of their results, *Prop. 9.5* above, is the following. Let B be an affine $\mathbb{C}$-domain, $a, b \in B$ algebraically independent over $\mathbb{C}$, and $D \in \text{LND}(B)$. Suppose $D(p(a, b)) = 0$, where $p \in \mathbb{C}[x, y]$ is non-constant, irreducible, and not a variable. Then $Da = Db = 0$. This result is an important tool in their calculation of $ML(B)$ for certain rings B.

Notice that the conditions on the polynomial p are precisely *equivalent* to the condition that $\delta p \neq 0$ for every nonzero $\delta \in \text{LND}(\mathbb{C}[x, y])$: Suppose

$\delta f = 0$ for nonzero $\delta \in \mathrm{LND}(B)$ and non-constant $f \in B$. By Rentschler's Theorem, $f \in \mathbb{C}[z]$ for some variable z. Thus, if f is irreducible, then f is a linear function of z, hence a variable.

Motivated by this observation, the following conjectural generalization of their result seems reasonable, and could be a tool in calculating the Makar-Limanov invariant.

> **Conjecture.** Let k be an algebraically closed field of characteristic zero, and let B be an affine k-domain. Suppose $a_1, ..., a_n \in B$ are algebraically independent ($n \geq 1$), and let nonzero $D \in \mathrm{LND}(B)$ be given. Suppose $D(p(a_1, ..., a_n)) = 0$, where $p \in k^{[n]}$ and $\delta p \neq 0$ for all nonzero $\delta \in \mathrm{LND}(k^{[n]})$. Then $Da_i = 0$ for each $i = 1, ..., n$.

11.5 Relative Invariants

Following are two natural variations of the Makar-Limanov and Derksen invariants which might be useful in the study of commutative rings. In particular, the first of these may be used to determine that a ring is not a polynomial ring over a given subring.

For a commutative k-algebra B, and subalgebra $A \subset B$, define the **relative Makar-Limanov invariant** associated to A by

$$ML_A(B) = \bigcap_{D \in \mathrm{LND}_A(B)} B^D .$$

Likewise, define the **relative Derksen invariant** $\mathcal{D}_A(B)$ associated to A to be the subalgebra of B generated by the kernels of nonzero elements of $\mathrm{LND}_A(B)$.

Next, define the set $\mathrm{LND}^*(B) \subset \mathrm{LND}(B)$ by

$$\mathrm{LND}^*(B) = \{D \in \mathrm{LND}(B) \mid Ds = 1 \text{ for some } s \in B\} .$$

Then

$$ML^*(B) = \bigcap_{D \in \mathrm{LND}^*(B)} B^D ,$$

and $\mathcal{D}^*(B)$ is the subalgebra of B generated by the kernels of nonzero elements of $\mathrm{LND}^*(B)$. Note that if $B = k^{[n]}$, then $ML^*(B) = k$ and $\mathcal{D}^*(B) = B$.

11.6 Structure of LND(B)

As observed earlier, one difficulty in working with locally nilpotent derivations is that $\mathrm{LND}(B)$ admits no obvious algebraic structure. It is natural to ask: Given $D, E \in \mathrm{LND}(B)$ and $f \in B$, under what conditions are fD, $D + E$, and $[D, E]$ (respectively) locally nilpotent? A complete answer for the first of these three cases is provided by *Princ. 7*, namely, fD is locally nilpotent if and only if $f \in \ker D$. As to the second case, a partial answer is given in *Princ. 10*, namely, that if $[D, E] = 0$ (i.e., D and E commute), then the sum $D + E$ is locally nilpotent. However, it can also happen that $[D, E] \neq 0$ and $D + E \in \mathrm{LND}(B)$ (for example, triangular derivations of polynomial rings). See also the main result of [109]. As to the third case, which asks when $[D, E]$ is locally nilpotent, this appears to be completely open. A more complete understanding of the behavior of locally nilpotent derivations relative to these operations might be found by investigating the classical formula of Campbell-Baker-Hausdorf, as found in [30, 170]; see also [296].

A more geometric approach to understanding $\mathrm{LND}(B)$ is to consider the action of an algebraic group G on the set $\mathcal{X} = \mathrm{LND}(B)$, where the notion of an algebraic action is defined in some natural way. For example, if G is a finite subgroup of $\mathrm{Aut}_k(B)$, then G acts on $\mathcal{X}$ by $g \cdot D = gDg^{-1}$. The fixed set $\mathcal{X}^G$ consists of locally nilpotent derivations D of B such that $Dg = gD$ for all $g \in G$.

11.7 Maximal Subalgebras

For a commutative k-domain B, we have seen that $\mathrm{Der}_k(B)$ forms a Lie algebra over k, while $\mathrm{LND}(B)$ does not. Nonetheless, it is natural to study the Lie subalgebras $\mathfrak{g}$ of $\mathrm{LND}(B)$, by which we mean subsets $\mathfrak{g}$ of $\mathrm{LND}(B)$ which are subalgebras of $\mathrm{Der}_k(B)$. For example, we have mainly been interested in one-dimensional subalgebras, generated by a single element $D \in \mathrm{LND}(B)$. This induces the larger subalgebra $A \cdot D$, where $A = \ker D$. Likewise, if $D_1, ..., D_n \in \mathrm{LND}(B)$ commute, then their k-span is a subalgebra of finite dimension. We might also ask: What are the **maximal** subalgebras of $\mathrm{LND}(B)$, i.e., subalgebras $\mathfrak{m}$ of $\mathrm{LND}(B)$ with the property that if $\mathfrak{m} \subseteq \mathfrak{g}$ for another subalgebra $\mathfrak{g}$ of $\mathrm{LND}(B)$, then $\mathfrak{m} = \mathfrak{g}$. For the polynomial ring $B = k[x_1, ..., x_n]$, a natural candidate is the triangular subalgebra

$$\mathfrak{T} = k\partial_{x_1} \oplus k[x_1]\partial_{x_2} \oplus \cdots \oplus k[x_1, ..., x_{n-1}]\partial_{x_n} .$$

If $\mathfrak{T}$ is a maximal subalgebra, is it the unique maximal subalgebra up to conjugation? Note that we have seen earlier that

$$\mathrm{Der}_k(B) = B\partial_{x_1} \oplus B\partial_{x_2} \oplus \cdots \oplus B\partial_{x_n} .$$

11.8 Invariants of a Sum

This question will be stated for polynomial rings, though it could also be stated more generally. Let $k[\mathbf{x}] = k^{[n]}$ and $k[\mathbf{y}] = k^{[m]}$ be polynomial rings, where $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_m)$, and let $D \in \mathrm{LND}(k[\mathbf{x}])$ and

$E \in \text{LND}(k[\mathbf{y}])$ be given. Extend D and E to locally nilpotent derivations of the polynomial ring $k[\mathbf{x}, \mathbf{y}] = k^{[n+m]}$ by setting $D\mathbf{y} = 0$ and $E\mathbf{x} = 0$. Then $DE = ED = 0$, which implies that $D + E$ is locally nilpotent. Describe $\ker(D + E)$ in terms of $\ker D$ and $\ker E$. In particular, is $\ker(D + E)$ finitely generated if both $\ker D$ and $\ker E$ are? Notice that $\ker(D + E)$ does contain cross elements, for example, if $r \in k[\mathbf{x}]$ is a local slice of D and $s \in k[\mathbf{y}]$ is a local slice of E, then $rEs - sDr$ belongs to $\ker(D + E)$.

Note that $\ker D$ and $\ker E$ are subalgebras of $\ker(D + E)$. However, $\ker(D + E)$ is, in general, strictly larger than $\ker D \otimes_k \ker E$, since the latter is the invariant ring of the $\mathbb{G}_a^2$-action defined by the commutative Lie algebra $kD + kE$ (see Nagata [238], Lemma 1). Thus, the transcendence degree of $\ker D \otimes_k \ker E$ over k generally equals $n - 2$.

This question is motivated by the linear locally nilpotent derivations D. We saw that, due to the Jordan form of a matrix, such derivations can always be written as a sum $D = D_1 + \cdots + D_t$, where the polynomial ring $k[\mathbf{x}]$ decomposes as $k[\mathbf{x}_1, ..., \mathbf{x}_t]$ for $\mathbf{x}_i = (x_{i1}, ..., x_{ij_i})$, and where D_i restricts to the basic derivation of $k[\mathbf{x}_i]$, with $D_i\mathbf{x}_j = 0$ when $i \neq j$.

11.9 Finiteness Problem for Extensions

Suppose B is a finitely generated commutative k-domain, and suppose $D \in \text{LND}(B)$ is such that $\ker D$ is finitely generated. If D is extended to a derivation D^* on the ring $B[t] = B^{[1]}$, then D^* is locally nilpotent if and only $D^*t \in B$ (*Princ. 6*). Assuming this is the case, what conditions on D^*t guarantee that $\ker D^*$ is also finitely generated?

This question is also motivated by the linear locally nilpotent derivations of polynomial rings, in particular, the basic (triangular) ones. A good understanding of this problem, together with the preceding problem, might yield a proof of the Maurer-Weitzenböck Theorem which does not rely on the Finiteness Theorem for reductive groups.

11.10 Geometric Viewpoint

It might be profitable to think about a $\mathbb{G}_a$-action as a point belonging to a variety or scheme. For example, if $B = k[x_1, ..., x_n] = k^{[n]}$, then the set of all linear k-derivations of B is an affine space, since we may view a linear element $D \in \text{Der}_k(B)$ as a matrix: $D \in M_n(k) = \mathbb{A}^{n^2}$. The condition that $D = (x_{ij})$ be locally nilpotent (a nilpotent matrix) is that $(D^n)_{ij} = 0$ for $1 \leq i, j \leq n$, where the functions $(D^n)_{ij} \in k[x_{ij} | 1 \leq i, j \leq n]$ are the component functions of D^n. These conditions can be written down explicitly: If $f(\lambda) = \sum_{i=0}^n f_i \lambda^i$ is the characteristic polynomial of D, then the condition that D be nilpotent is precisely that D belong to the subvariety $X := V(f_0, ..., f_{n-1}) \subset \mathbb{A}^{n^2}$. X is the **variety of nilpotent matrices** in dimension n^2. Note that $GL_n(k)$

acts on X by conjugation. In addition, since the polynomials defining X are algebraically independent, $\dim X = n^2 - n$. For instance, when $n = 2$, write

$$D = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} .$$

Then the subvariety of nilpotents is $V(x_{11}+x_{22}, x_{11}x_{22}-x_{12}x_{21}) \subset \mathbb{A}^4$, which is a special Danielewski surface.

Now suppose more generally that the polynomial ring B is graded by $B = \oplus_{i\geq 0} B_i$, where each B_i is a finite dimensional vector space over k. Given homogeneous $D \in \mathrm{Der}_k(B)$, set $d_i = \deg Dx_i$, $1 \leq i \leq n$. Then $D \in B_{d_1} \times \cdots \times B_{d_n}$, which is an affine space, and we may ask what conditions imply that D is locally nilpotent.

Likewise, given $N > 0$, we may consider triangular $\delta \in \mathrm{Der}_k(B)$ such that $\deg \delta(x_i) \leq N$ for each i, $1 \leq i \leq n$. (So $\delta \in \mathrm{LND}(B)$.) In this case, is the condition "$\ker \delta$ is finitely generated" an open condition?

In the same vein, notice that a $\mathbb{G}_a^m$-action on $\mathbb{A}^n$ may be specified by an $m \times n$ matrix over $k[x_1, ..., x_n]$. If the action is linear, the entries of the matrix are linear forms.

11.11 Paragonic Varieties

For an affine ring B, we can ask whether B admits any locally nilpotent derivations (i.e., if B is rigid or not). But we can also ask whether B itself is isomorphic to the kernel of a locally nilpotent derivation of $k^{[n]}$ for some n. For example, in the proof of the homogeneous case of Miyanishi's Theorem, it was important to show that $B = k[x, y, z]/(x^2 + y^3 + z^5)$ could not be such a kernel. It was later also shown that this ring is rigid. Similar problems are well-studied for other groups. For example, an affine **toric variety** corresponds to the invariant ring of a linear action of a torus on $\mathbb{C}^n$.

Define the affine k-variety X to be a **paragonic variety** if and only if there exist integers $m, n \geq 1$ and an algebraic action of the vector group $G = \mathbb{G}_a^m$ on $\mathbb{A}^n$ such that $k[X] \cong k[\mathbb{A}^n]^G$. Note that a paragonic variety X has several fundamental properties. For example, X is irreducible, factorial, and admits a dominant morphism $\mathbb{A}^n \to X$. In addition, $\mathbb{A}^n$ is a paragonic variety for all $n \geq 0$.

The singular threefold $x^2y + z^2 + t^3 = 0$ in $\mathbb{A}^4$ is a paragonic variety, since its coordinate ring is the invariant ring of the basic action of $\mathbb{G}_a$ on $\mathbb{A}^4$. It is an open question whether the Russell threefold X_0, defined by the similar equation $x + x^2y + z^2 + t^3 = 0$, is a paragonic variety. A proof that X_0 is non-paragonic would give another proof that $X_0 \not\cong \mathbb{A}^3$. A related question, posed by Kaliman, is whether $X_0 \times \mathbb{A}^1$ is isomorphic to $\mathbb{A}^4$.

In general, we would like to know whether a nontrivial paragonic variety can ever be rigid. Note that the first section of this chapter asks this question for a special kind of paragonic variety.

11.12 Stably Triangular $\mathbb{G}_a$-Actions

Given a $\mathbb{G}_a$-action $\rho : \mathbb{G}_a \times \mathbb{A}^n \to \mathbb{A}^n$, we say that ρ is **stably triangular** if and only if ρ extends to a triangular action on some larger affine space. More precisely, this means that there exists an embedding $\epsilon : \mathbb{A}^n \to \mathbb{A}^N$ for some $N \geq n$, together with a triangular $\mathbb{G}_a$-action $\tau : \mathbb{G}_a \times \mathbb{A}^N \to \mathbb{A}^N$, such that ρ is the restriction of τ to $\epsilon(\mathbb{A}^n)$. (Note that we do not require ϵ to be a coordinate embedding.) Equivalently, this means that, for the corresponding element $D \in \mathrm{LND}(k^{[n]})$, there exists an extension of D to a triangular derivation T on some larger polynomial algebra $B = k^{[N]}$, together with an integral ideal $I \subset k^{[N]}$, such that $B/I = k^{[n]}$ and $T/I = D$. Note that under this definition, triangularizable implies stably triangular.

Question: Is every algebraic $\mathbb{G}_a$-action on $\mathbb{A}^n$ stably triangular?

For example, we saw how the homogeneous $(2,5)$-example of *Chap. 5* is stably triangular. In particular, the $\mathbb{G}_a$-action ρ on $\mathbb{A}^3$ can be extended to a triangular action τ on $\mathbb{A}^6$ in such a way that ρ is the restriction of τ to a coordinate subvariety $X \subset \mathbb{A}^6$, where $X \cong \mathbb{A}^3$ (*Thm. 5.20*).

A positive solution to this problem would reiterate the premise that the triangular derivations provide an archetype for the study of locally nilpotent derivations of polynomial rings.

11.13 Extending $\mathbb{G}_a$-Actions to Larger Group Actions

Does there exist a non-triangularizable $\mathbb{G}_a$-action on $\mathbb{C}^n$ which extends to an action of some reductive group G? A possible example would involve $SL_2(\mathbb{C})$ acting on $\mathbb{C}^n$. If such an action can be found, can the $\mathbb{G}_a$-action be chosen to be fixed-point free?

A similar question was posed by Bass in [13]: If a reductive group G acts on $\mathbb{A}^n$, and if $U \subset G$ is a maximal unipotent subgroup, can the action of U be triangularized "as a step toward linearizing the action of G"? (p. 5)

Note that a non-triangularizable $\mathbb{G}_a$-action on $\mathbb{C}^3$ cannot be extended to an action of a reductive group G, since $\mathbb{G}_a \subset G_0$ (the connected component of the identity). G_0 is again reductive, and every algebraic action of a connected reductive group on $\mathbb{C}^3$ can be linearized; see *Chap. 5.*

11.14 Variable Criterion

In *Chap. 4*, locally nilpotent derivations were used to prove the Variable Criterion for polynomial rings (*Cor. 4.24*). Specifically, it asserts that if $F \in R[x,y]$ is a variable over $\mathrm{frac}(R)$, and $(F_x, F_y) = (1)$, then F is an R-variable, where $R = k^{[n]}$. Give necessary and sufficient conditions that $F \in R[x,y,z]$ be an R-variable of $R[x,y,z]$. In particular, we must have $(F_x, F_y, F_z) = (1)$. Such considerations arose in connection to the Vénéreau polynomials in *Chap. 10.*

11.15 Bass's Question on Rational Triangularization

The final question asked by Bass in his 1984 paper [12] is the following.

> If a unipotent group G acts on $\mathbb{A}_k^n$, can the action be rationally triangularized, i.e., can we write $k(x_1, ..., x_n) = k(y_1, ..., y_n)$ so that each subfield $k(y_1, ..., y_i)$ is G-invariant?

This problem remains generally open. It was considered by Deveney and Finston in their papers [73, 74], where they gave several positive results. Working over the field $k = \mathbb{C}$, they observed that a $\mathbb{G}_a$-action is rationally triangularizable if and only if the quotient field of the invariant ring is a pure transcendental extension of $\mathbb{C}$; that any $\mathbb{G}_a$-action on $\mathbb{A}^n$ becomes rationally triangularizable if one more variable is added; and that any $\mathbb{G}_a$-action on $\mathbb{A}^n$ for $n \leq 4$ is rationally triangularizable.

11.16 Popov's Questions

The following questions of V. Popov appear in [31].

Working over an algebraically closed field k of characteristic zero, Popov defines the subgroup $GA_n^*(k)$ of $GA_n(k)$ as those automorphisms of jacobian determinant 1 (the "volume-preserving" elements). A subgroup G of $GA_n^*(k)$ is called **∂-generated** if G is generated by exponential automorphisms, and **finitely ∂-generated** if there exists a finite set of elements $d_1, ..., d_N \in \mathrm{LND}(k^{[n]})$, $N \geq 1$, such that G is generated by $\{\exp(fd_i) | f \in \ker d_i, 1 \leq i \leq N\}$. He goes on to give several examples of finitely ∂-generated subgroups, namely, any connected semisimple algebraic subgroup of $GA_n^*(k)$, the group of translations, and the triangular subgroup of $GA_n^*(k)$.

> **Question 1.** Is $GA_n^*(k)$ ∂-generated? If yes, is it finitely ∂-generated?

Clearly, $GA_n^*(k)$ is ∂-generated for $n = 1$, and it follows from the Structure Theorem that this is true for $n = 2$ as well. Note that for $n = 3$, Nagata's automorphism is not tame, but it is exponential.

A second question posed by Popov is:

> **Question 2.** Let $D, E \in \mathrm{LND}(k^{[n]})$, and let $G \subset GA_n(k)$ be the minimal closed subgroup containing the groups $\{\exp(tD) | t \in k\}$ and $\{\exp(tE) | t \in k\}$. When is G of finite dimension?

Here, "closed" means closed with respect to the structure of $GA_n(k)$ as an infinite dimensional algebraic group, as defined by Shafarevich in [276]. See also Kambayashi [168].

11.17 Miyanishi's Question

Let $D \in \mathrm{LND}(k^{[n]})$ be nonzero, and let $A = \ker D$. Are all projective modules over A free? (Section 1.2 of [208])

References

1. S. Abhyankar, *Lectures on expansion techniques in algebraic geometry*, Tata Inst. Fund. Res. Lectures on Math. and Phys., vol. 57, Tata Inst. Fund. Res., Bombay, 1977, (Notes by B. Singh).
2. S. Abhyankar, P. Eakin, and W. Heinzer, *On the uniqueness of the coefficient ring in a polynomial ring*, J. Algebra **23** (1972), 310–342.
3. S. Abhyankar and T.T. Moh, *Embeddings of the line in the plane*, J. Reine Angew. Math. **276** (1975), 148–166.
4. A. A'Campo-Neuen, *Note on a counterexample to Hilbert's fourteenth problem given by P. Roberts*, Indag. Math., N.S. **5** (1994), 253–257.
5. I. V. Arzhantsev, *Affine embeddings of homogeneous spaces*, preprint 2005, arXiv:math.AG/0503563 v1.
6. T. Asanuma, *Non-linearizable algebraic actions on $\mathbb{A}^n$*, J. Algebra **166** (1994), 72–79.
7. ———, *Non-linearizable algebraic k^*-actions on affine spaces*, Invent. Math. **138** (1999), 281–306.
8. T. Bandman and L. Makar-Limanov, *Affine surfaces with $AK(S) = \mathbb{C}$*, Michigan Math. J. **49** (2001), 567–582.
9. ———, *Non-stability of AK-invariant*, Michigan Math. J. **53** (2005), 263–281.
10. V. Baranovsky, *The variety of pairs of commuting nilpotent matrices is irreducible*, Transform. Groups **6** (2001), 3–8.
11. R. Basili, *On the irreducibility of commuting varieties of nilpotent matrices*, J. Algebra **268** (2003), 58–80.
12. H. Bass, *A non-triangular action of $\mathbb{G}_a$ on $\mathbb{A}^3$*, J. Pure Appl. Algebra **33** (1984), 1–5.
13. ———, *Algebraic group actions on affine spaces*, Contemp. Math. **43** (1985), 1–23.
14. J. Berson, *Derivations on polynomial rings over a domain*, Master's thesis, Univ. Nijmegen, The Netherlands, 1999.
15. J. Berson and A. van den Essen, *An algorithm to find a coordinate's mate*, J. Symbolic Comput. **36** (2003), 835–843.
16. J. Berson, A. van den Essen, and S. Maubach, *Derivations having divergence zero on $R[x, y]$*, Israel J. Math. **124** (2001), 115–124.
17. J. Bertin, *Pinceaux de droites et automorphismes des surfaces affines*, J. Reine Angew. Math. **341** (1983), 32–53.

18. S.M. Bhatwadekar and A.K. Dutta, *Kernel of Locally Nilpotent R-Derivations of* $R[X, Y]$, Trans. Amer. Math. Soc. **349** (1997), 3303–3319.
19. S.M. Bhatwadekar and A. Roy, *Some results on embedding of a line in 3-space*, J. Algebra **142** (1991), 101–109.
20. A. Bialynicki-Birula, *Remarks on the action of an algebraic torus on* k^n, Bull. Acad. Pol. Sci. **14** (1966), 177–181.
21. ______, *Remarks on the action of an algebraic torus on* k^n*, II*, Bull. Acad. Pol. Sci. **15** (1967), 123–125.
22. ______, *On fixed point schemes of actions of multiplicative and additive groups*, Topology **12** (1973), 99–103.
23. M. de Bondt and A. van den Essen, *Nilpotent symmetric Jacobian matrices and the Jacobian Conjecture*, J. Pure Appl. Algebra **193** (2004), 61–70.
24. ______, *The Jacobian Conjecture: linear triangularization for homogeneous polynomial maps in dimension three*, J. Algebra **294** (2005), 294–306.
25. ______, *A reduction of the Jacobian Conjecture to the symmetric case*, Proc. Amer. Math. Soc. **133** (2005), 2201–2205.
26. P. Bonnet, *A proof of Miyanishi's result for homogeneous locally nilpotent derivations on* $\mathbb{C}[x_1, x_2, x_3]$, unpublished note, 2004, 5 pages.
27. ______, *Surjectivity of quotient maps for algebraic* $(\mathbb{C}, +)$*-actions*, Transform. Groups **7** (2002), 3–14.
28. A. Borel, *Linear Algebraic Groups*, Graduate Texts in Mathematics, vol. 126, Springer-Verlag, 1991, Berlin, Heidelberg, New York.
29. ______, *Essays in the History of Lie Groups and Algebraic Groups*, History of Mathematics, vol. 21, Amer. Math. Soc. and London Math. Soc., 2001, Providence, London.
30. N. Bourbaki, *Lie groups and lie algebras*, Elements of Math. Ch.1-3, Springer-Verlag, 1989, Berlin, New York.
31. Collected by G. Freudenburg and P. Russell, *Open problems in affine algebraic geometry*, Contemp. Math. **369** (2005), 1–30.
32. A. Castravet and J. Tevelev, *Hilbert's 14-th Problem and Cox rings*, preprint, 2005, avail. at arXiv:math.AG/0505337v1.
33. A. Cerezo, *Table des invariants algebriques et rationnels d'une matrice nilpotente de petite dimension*, Tech. Report 146, Universite de Nice Prepub. Math., France, 1987.
34. D. Cerveau, *Dérivations surjectives de l'anneau* $\mathbb{C}[x, y]$, J. Algebra **195** (1997), 320–335.
35. A. Choudary and A. Dimca, *Complex hypersurfaces diffeomorphic to affine spaces*, Kodai Math. J. **12** (1994), 171–178.
36. A.M. Cohen and J. Draisma, *From Lie algebras of vector fields to algebraic actions*, Transform. Groups **8** (2003), 51–68.
37. P. M. Cohn, *Algebra, Vol. 1 (Second Ed.)*, John Wiley and Sons, 1982, Chichester, New York.
38. B. Coomes and V. Zurkowski, *Linearization of polynomial flows and spectra of derivations*, J. Dynamics Differential Equations **3** (1991), 29–66.
39. A. Crachiola, *On the AK-Invariant of Certain Domains*, Ph.D. thesis, Wayne State University, Detroit, Michigan, 2004.
40. ______, *The hypersurface* $x + x^2y + z^2 + t^3 = 0$ *over a field of arbitrary characteristic*, Proc. Amer. Math. Soc. **134** (2006), 1289–1298.
41. A. Crachiola and L. Makar-Limanov, *An algebraic proof of a cancellation theorem for surfaces*, preprint 2005, 9 pages.

42. ———, *On the rigidity of small domains*, J. Algebra **284** (2005), 1–12.
43. A. Crachiola and S. Maubach, *The Derksen invariant vs. the Makar-Limanov invariant*, Proc. Amer. Math. Soc. **131** (2003), 3365–3369.
44. P. C. Craighero, *A result on m-flats in* $\mathbb{A}_k^n$, Rend. Sem. Mat. Univ. Padova **75** (1986), 39–46.
45. ———, *A remark on Abhyankar's space lines*, Rend. Sem. Mat. Univ. Padova **80** (1988), 87–93.
46. D. Daigle, *Classification of homogeneous locally nilpotent derivations of* $k[x,y,z]$, preprint 2003.
47. ———, *Locally Nilpotent Derivations*, Lecture Notes for the 26th Autumn School of Algebraic Geometry, Łukęcin, Poland, September 2003. Avail. at `http://aix1.uottawa.ca/ ddaigle/`.
48. ———, *A necessary and sufficient condition for triangulability of derivations of* $k[x,y,z]$, J. Pure Appl. Algebra **113** (1996), 297–305.
49. ———, *On some properties of locally nilpotent derivations*, J. Pure Appl. Algebra **114** (1997), 221–230.
50. ———, *Homogeneous locally nilpotent derivations of* $k[x,y,z]$, J. Pure Appl. Algebra **128** (1998), 109–132.
51. ———, *On kernels of homogeneous locally nilpotent derivations of* $k[x,y,z]$, Osaka J. Math. **37** (2000), 689–699.
52. ———, *On locally nilpotent derivations of* $k[x_1,x_2,y]/(\phi(y)-x_1x_2)$, J. Pure Appl. Algebra **181** (2003), 181–208.
53. ———, *Locally nilpotent derivations and Danielewski surfaces*, Osaka J. Math. **41** (2004), 37–80.
54. D. Daigle and G. Freudenburg, *Locally nilpotent derivations over a UFD and an application to rank two locally nilpotent derivations of* $k[X_1,\ldots,X_n]$, J. Algebra **204** (1998), 353–371.
55. ———, *A counterexample to Hilbert's Fourteenth Problem in dimension five*, J. Algebra **221** (1999), 528–535.
56. ———, *A note on triangular derivations of* $k[X_1,X_2,X_3,X_4]$, Proc. Amer. Math. Soc. **129** (2001), 657–662.
57. ———, *Triangular derivations of* $k[X_1,X_2,X_3,X_4]$, J. Algebra **241** (2001), 328–339.
58. D. Daigle and S. Kaliman, *A note on locally nilpotent derivations and variables of* $k[x,y,z]$, preprint 2004, 9 pages.
59. D. Daigle and P. Russell, *Affine rulings of normal rational surfaces*, Osaka J. Math. **38** (2001), 37–100.
60. ———, *On weighted projective planes and their affine rulings*, Osaka J. Math. **38** (2001), 101–150.
61. ———, *On log* $\mathbb{Q}$*-homology planes and weighted projective planes*, Canad. J. Math. **56** (2004), 1145–1189.
62. W. Danielewski, *On the cancellation problem and automorphism groups of affine algebraic varieties*, Preprint, Warsaw, 1989.
63. V. I. Danilov and M. H. Gizatullin, *Automorphisms of affine surfaces*, Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), 523–565.
64. M. de Bondt, *Homogeneous quasi-translations and an article of P. Gordan and M. Nöther*, preprint, 2005, 16 pages.
65. ———, *Quasi-translations and counterexamples to the Homogeneous Dependence Problem*, preprint, 2004, 10 pages.

66. H. Derksen, *The kernel of a derivation*, J. Pure Appl. Algebra **84** (1993), 13–16.
67. ———, *Quotients of algebraic group actions*, Automorphisms of affine spaces (Dordrecht) (A. van den Essen, ed.), Kluwer, 1995, pp. 191–200.
68. ———, *Constructive Invariant Theory and the Linearization Problem*, Ph.D. thesis, Univ. Basel, 1997.
69. H. Derksen, A. van den Essen, and P. van Rossum, *The cancellation problem in dimension four*, Tech. Report 0022, Dept. of Mathematics, Univ. Nijmegen, The Netherlands, 2000.
70. H. Derksen, O. Hadas, and L. Makar-Limanov, *Newton polytopes of invariants of additive group actions*, J. Pure Appl. Algebra **156** (2001), 187–197.
71. H. Derksen and G. Kemper, *Computational Invariant Theory*, Springer Verlag, Berlin, Heidelberg, New York, 2002.
72. H. Derksen and F. Kutzschebauch, *Nonlinearizable holomorphic group actions*, Math. Ann. **311** (1998), 41–53.
73. J. K. Deveney and D. R. Finston, *Rationally triangulable automorphisms*, J. Pure and Appl. Algebra **72** (1991), 1–4.
74. ———, *Fields of $\mathbb{G}_a$ invariants are ruled*, Canad. Math. Bull. **37** (1994), 37–41.
75. ———, *$\mathbb{G}_a$-actions on $\mathbb{C}^3$ and $\mathbb{C}^7$*, Comm. Algebra **22** (1994), 6295–6302.
76. ———, *Algebraic aspects of additive group actions on complex affine space*, Automorphisms of affine spaces (Dordrecht) (A. van den Essen, ed.), Kluwer, 1995, pp. 179–190.
77. ———, *A proper $\mathbb{G}_a$-action on $\mathbb{C}^5$ which is not locally trivial*, Proc. Amer. Math. Soc. **123** (1995), 651–655.
78. ———, *On locally trivially $\mathbb{G}_a$-actions*, Transform. Groups **2** (1997), 137–145.
79. ———, *Free $\mathbb{G}_a$-actions on $\mathbb{C}^3$*, Proc. Amer. Math. Soc. **128** (1999), 131–138.
80. ———, *$\mathbb{G}_a$-invariants and slices*, Comm. Algebra **30** (2002), 1437–1447.
81. J. K. Deveney, D. R. Finston, and M. Gehrke, *$\mathbb{G}_a$-actions on $\mathbb{C}^n$*, Comm. Algebra **12** (1994), 4977–4988.
82. W. Dicks, *Automorphisms of the polynomial ring in two variables*, Publ. Sec. Mat. Univ. Autónoma Barcelona **27** (1983), 155–162.
83. J. Dieudonné and J. Carrell, *Invariant theory, old and new*, Adv. Math. **4** (1970), 1–80.
84. A. Dimca, *Hypersurfaces in $\mathbb{C}^{2n}$ diffeomorphic to $\mathbb{R}^{4n-2}$ ($n \geq 2$)*, Max-Plank Institute, preprint, 1990.
85. J. Dixmier, *Lectures on Binary Forms*, West Chester University of Pennsylvania (Notes by F. Grosshans), 1986.
86. ———, *Enveloping Algebras*, North-Holland Publishing, Amsterdam, New York, Oxford, 1977.
87. I. Dolgachev, *Lectures on Invariant Theory*, London Math. Soc. Lect. Notes Series, vol. 296, Cambridge University Press, Cambridge, UK, 2003.
88. J. Draisma, *Lie algebras of vector fields*, Ph.D. thesis, Technische Universiteit Eindhoven (The Netherlands), 2002.
89. V. Drensky and G. K. Genov, *Multiplicities of Schur functions with applications to invariant theory and PI-algebras*, C. R. Acad. Bulgare Sci. **57** (2004), 5–10.
90. V. Drensky and J.-T. Yu, *Exponential automorphisms of polynomial algebras*, Comm. Algebra **26** (1998), 2977–2985.
91. L.M. Druzkowski and J. Gurycz, *An elementary proof of the tameness of polynomial automorphisms of k^2*, Univ. Iagel. Acta Math. **35** (1997), 251–260.
92. A. Dubouloz, *Completions of normal affine surfaces with a trivial Makar-Limanov invariant*, Michigan Math. J **52** (2004), 289–308.

93. S. Ebey, *The operation of the universal domain on the plane*, Proc. Amer. Math. Soc. **13** (1962), 722–725.
94. M. El Kahoui, *Constants of derivations in polynomial rings over unique factorization domains*, Proc. Amer. Math. Soc. **132** (2004), 2537–2541.
95. ______, *UFDs with commuting linearly independent locally nilpotent derivations*, J. Algebra **289** (2005), 446–452.
96. W. Engel, *Ganze Cremona-Transformationen von Primzahlgrad in der Ebene*, Math. Ann. **136** (1958), 319–325.
97. A. van den Essen, *Locally finite and locally nilpotent derivations with applications to polynomial flows and morphisms*, Proc. Amer. Math. Soc. **116** (1992), 861–871.
98. ______, *An algorithm to compute the invariant ring of a $\mathbb{G}_a$-action on an affine variety*, J. Symbolic Comp. **16** (1993), 551–555.
99. ______, *Locally finite and locally nilpotent derivations with applications to polynomial flows, morphisms, and $\mathbb{G}_a$-actions, II*, Proc. Amer. Math. Soc. **121** (1994), 667–678.
100. ______, *Polynomial Automorphisms and the Jacobian Conjecture*, Birkhauser, Boston, 2000.
101. ______, *A simple solution of Hilbert's fourteenth problem*, Colloq. Math. **105** (2006), 167–170.
102. A. van den Essen and T. Janssen, *Kernels of elementary derivations*, Tech. Report 9548, Dept. of Mathematics, Univ. Nijmegen, The Netherlands, 1995.
103. A. van den Essen and P. van Rossum, *Triangular derivations related to problems on affine n-space*, Proc. Amer. Math. Soc. **130** (2001), 1311–1322.
104. A. van den Essen and S. Washburn, *The Jacobian Conjecture for symmetric Jacobian matrices*, J. Pure Appl. Algebra **189** (2004), 123–133.
105. A. Fauntleroy, *Linear $\mathbb{G}_a$-actions on affine spaces and associated rings of invariants*, J. Pure Appl. Algebra **9** (1977), 195–206.
106. ______, *On Weitzenböck's theorem in positive characteristic*, Proc. Amer. Math. Soc. **64** (1977), 209–213.
107. ______, *Algebraic and algebro-geometric interpretations of Weitzenbock's problem*, J. Algebra **62** (1980), 21–38.
108. A. Fauntleroy and A. Magid, *Proper $\mathbb{G}_a$-actions*, Duke J. Math. **43** (1976), 723–729.
109. M. Ferrero, Y. Lequain, and A. Nowicki, *A note on locally nilpotent derivations*, J. Pure Appl. Algebra **79** (1992), 45–50.
110. K.-H. Fieseler, *On complex affine surfaces with $\mathbb{C}^+$-actions*, Comment. Math. Helvetici **69** (1994), 5–27.
111. D. Finston and S. Maubach, *The automorphism group of certain factorial threefolds and a cancellation problem*, preprint, 2006.
112. J. Fogarty, *Invariant Theory*, Benjamin, New York, 1969.
113. G. Freudenburg, *A linear counterexample to the Fourteenth Problem of Hilbert in dimension eleven*, Proc. Amer. Math. Soc. (to appear).
114. ______, *The Vénéreau polynomials relative to $\mathbb{C}^*$-fibrations and stable coordinates*, to appear in "Affine Algebraic Geometry", Proceedings of the 2004 Conference to honor the retirement of M. Miyanishi.
115. ______, *One-parameter subgroups and the triangular subgroup of the affine Cremona group*, Automorphisms of Affine Spaces (Dordrecht) (A. van den Essen, ed.), Kluwer, 1995, pp. 201–213.

116. ______, *Triangulability criteria for additive group actions on affine space*, J. Pure and Appl. Algebra **105** (1995), 267–275.
117. ______, *A note on the kernel of a locally nilpotent derivation*, Proc. Amer. Math. Soc. **124** (1996), 27–29.
118. ______, *Local slice constructions in $K[X,Y,Z]$*, Osaka J. Math. **34** (1997), 757–767.
119. ______, *Actions of $\mathbb{G}_a$ on $\mathbb{A}^3$ defined by homogeneous derivations*, J. Pure Appl. Algebra **126** (1998), 169–181.
120. ______, *A counterexample to Hilbert's Fourteenth Problem in dimension six*, Transform. Groups **5** (2000), 61–71.
121. G. Freudenburg and L. Moser-Jauslin, *Embeddings of Danielewski surfaces*, Math. Z. **245** (2003), 823–834.
122. ______, *Real and rational forms of certain $O_2(\mathbb{C})$-actions, and a solution to the Weak Complexification Problem*, Transform. Groups **9** (2004), 257–272.
123. A. Fujiki, *The fixed point set of $\mathbb{C}$ actions on a compact complex space*, Osaka J. Math. **32** (1995), 1013–1022.
124. T. Fujita, *On Zariski problem*, Proc. Japan Acad. **55A** (1979), 106–110.
125. P. Gabriel and Y. Nouazé, *Idéaux premiers de l'algébre enveloppante d'une algébre de Lie nilpotente*, J. Algebra **6** (1967), 77–99.
126. M. H. Gizatullin, *Invariants of incomplete algebraic surfaces that can be obtained by means of completions*, Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 485–497.
127. ______, *Quasihomogeneous affine surfaces*, Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 1047–1071.
128. P. Gordan and M. Nöther, *Über die algebraische Formen, deren Hesse'sche Determinante identisch verschwindet*, Math. Ann. **10** (1876), 547–568.
129. G.-M. Greuel and G. Pfister, *Geometric quotients of unipotent group actions*, Proc. London Math. Soc. **67** (1993), 75–105.
130. F. Grosshans, *The invariants of unipotent radicals of parabolic subgroups*, Invent. Math. **73** (1983), 1–9.
131. F. D. Grosshans, *Algebraic Homogeneous Spaces and Invariant Theory*, Lect. Notes in Math., vol. 1673, Springer Verlag, 1997.
132. R. Gurjar, *A topological proof of a cancellation theorem for $\mathbb{C}^2$*, Math. Z. **240** (2002), 83–94.
133. R. Gurjar and M. Miyanishi, *Automorphisms of affine surfaces with $\mathbb{A}^1$-fibrations*, Michigan Math. J. **53** (2005), 33–55.
134. R. V. Gurjar and M. Miyanishi, *On the Makar-Limanov invariant and fundamental group at infinity*, preprint 2002.
135. A. Gutwirth, *The action of an algebraic torus on an affine plane*, Trans. Amer. Math. Soc. **105** (1962), 407–414.
136. O. Hadas, *On the vertices of Newton polytopes associated with an automorphism of the ring of polynomials*, J. Pure Appl. Algebra **76** (1991), 81–86.
137. O. Hadas and L. Makar-Limanov, *Newton polytopes of constants of locally nilpotent derivations*, Comm. Algebra **28** (2000), 3667–3678.
138. D. Hadziev, *Some problems in the theory of vector invariants*, Soviet Math. Dokl. **7** (1966), 1608–1610.
139. J. Harris, *Algebraic Geometry: A First Course*, GTM, vol. 133, Springer-Verlag, 1992.
140. R. Hartshorne, *Algebraic Geometry*, GTM, vol. 52, Springer-Verlag, 1977.

141. H. Hasse and F. K. Schmidt, *Noch eine Begründung der Theorie der höheren Differentialquotienten in einem algebraischen Functionenkörper einer Unbestimmten*, J. Reine Angew. Math. **177** (1937), 215–237.
142. D. Hilbert, *Mathematische Probleme*, Archiv der Math. und Physik **1** (1901), 44–63, 213–237.
143. ______, *Mathematical Problems*, Bull. Amer. Math. Soc. **8** (1902), 437–479.
144. G. Hochschild and G. D. Mostow, *Unipotent groups in invariant theory*, Proc. Nat. Acad. Sci. USA **70** (1973), 646–648.
145. D. Holtackers, *On kernels of homogeneous derivations*, Master's thesis, Univ. Nijmegen, The Netherlands, 2003.
146. G. Horrocks, *Fixed point schemes of additive group actions*, Topology **8** (1969), 233–242.
147. J. E. Humphreys, *Linear Algebraic Groups*, Springer-Verlag (Berlin, Heidelberg, New York), 1981.
148. J. E. Humpreys, *Hilbert's Fourteenth Problem*, Amer. Math. Monthly **70** (1978), 341–353.
149. T. Igarashi, *Finite subgroups of the automorphism group of the affine plane*, Master's thesis, Osaka University, 1977.
150. S. Iitaka, *Algebraic geoemetry: An introduction to birational geoemtry of algebraic varieties*, Graduate Texts Math., vol. 76, Springer-Verlag (Berlin, Heidelberg, New York), 1982.
151. N. Ivanenko, *Some classes of linearizable polynomial maps*, J. Pure Appl. Algebra **126** (1998), 223–232.
152. Z. Jelonek, *The extension of regular and rational embeddings*, Math. Ann. **277** (1987), 113–120.
153. K. Jorgenson, *A note on a class of rings found as* $\mathbb{G}_a$*-invariants for locally trivial actions on normal affine varieties*, Rocky Mountain J. Math. **34** (2004), 1343–1352.
154. H. W. E. Jung, *Über ganze birationale Transformationen der Ebene*, J. Reine Angew. Math. **184** (1942), 161–174.
155. S. Kaliman, *Actions of* $\mathbb{C}^*$ *and* $\mathbb{C}_+$ *on affine algebraic varieties*, preliminary monograph 2006.
156. ______, *Extensions of isomorphisms between affine algebraic subvarieties of* k^n *to automorphisms of* k^n, Proc. Amer. Math. Soc. **113** (1991), 325–334.
157. ______, *Isotopic embeddings of affine algebraic varieties into* $\mathbb{C}^n$, Contemp. Math. **137** (1992), 291–295.
158. ______, *Smooth contractible hypersurfaces in* $\mathbb{C}^n$ *and exotic algebraic structures on* $\mathbb{C}^3$, Math. Z. **214** (1993), 499–510.
159. ______, *Polynomials with general* $\mathbb{C}^2$*-fibers are variables*, Pacific J. Math. **203** (2002), 161–189.
160. ______, *Free* $\mathbb{C}^+$*-actions on* $\mathbb{C}^3$ *are translations*, Invent. Math. **156** (2004), 163–173.
161. S. Kaliman, M. Koras, L. Makar-Limanov, and P. Russell, $\mathbb{C}^*$*-actions on* $\mathbb{C}^3$ *are linearizable*, Electron. Res. Announc. Amer. Math. Soc. **3** (1997), 63–71.
162. S. Kaliman and L. Makar-Limanov, *AK-invariant of affine domains*, to appear in "Affine Algebraic Geometry", Proceedings of the 2004 Conference to honor the retirement of M. Miyanishi.
163. ______, *On the Russell-Koras contractible threefolds*, J. Algebraic Geom. **6** (1997), 247–268.

164. S. Kaliman and N. Saveliev, *$\mathbb{C}^+$-actions on contractible threefolds*, Michigan Math. J. **52** (2004), 619–625.
165. S. Kaliman, S. Vénéreau, and M. Zaidenberg, *Simple birational extensions of the polynomial ring $\mathbb{C}^{[3]}$*, Trans. Amer. Math. Soc. **356** (2004), 509–555.
166. S. Kaliman and M. Zaidenberg, *Miyanishi's characterization of the affine 3-space does not hold in higher dimensions*, Ann. Inst. Fourier, Grenoble **50** (2000), 1649–1669.
167. ———, *Vénéreau polynomials and related fiber bundles*, J. Pure Applied Algebra **192** (2004), 275–286.
168. T. Kambayashi, *Automorphism group of a polynomial ring and algebraic group action on an affine space*, J. Algebra **60** (1979), 439–451.
169. J. Khoury, *On some properties of elementary derivations in dimension six*, J. Pure Appl. Algebra **156** (2001), 69–79.
170. A. W. Knapp, *Lie Groups Beyond an Introduction*, Birkhäuser (Second Edition), 2002, Boston, Basel, Stuttgart.
171. H. Kojima and M. Miyanishi, *On P. Roberts' counterexample to the fourteenth problem of Hilbert*, J. Pure Appl. Algebra **122** (1997), 247–268.
172. M. Koras and P. Russell, *Contractible threefolds and $\mathbb{C}^*$-actions on $\mathbb{C}^3$*, J. Algebraic Geom. **6** (1997), 671–695.
173. ———, *$\mathbb{C}^*$-actions on $\mathbb{C}^3$: The smooth locus of the quotient is not of hyperbolic type*, J. Algebraic Geom. **8** (1999), 603–694.
174. H. Kraft, *Geometrische Methoden in der Invariantentheorie*, Vieweg-Verlag, 1985, Braunschweig.
175. ———, *Challenging problems on affine n-space*, Semináire Bourbaki **802** (1995), 295–317.
176. ———, *Free $\mathbb{C}^+$-actions on affine threefolds*, Contemp. Math., vol. 369, pp. 165–175, American Mathematical Society, Providence, RI, 2005.
177. H. Kraft and C. Procesi, *Classical Invariant Theory: A Primer*, 1996, avail. at `www.math.unibas-ch`.
178. E. Kunz, *Introduction to Commutative Algebra and Algebraic Geometry*, Birkhäuser, 1985, Boston, Basel, Stuttgart.
179. K. Kurano, *Positive characteristic finite generatiion of symbolic Rees algebra and Roberts' counterexamples to the fourteenth problem of Hilbert*, Tokyo J. Math. **16** (1993), 473–496.
180. S. Kuroda, *Hilbert's fourteenth problem and algebraic extensions*, preprint 2006.
181. ———, *Hilbert's fourteenth problem and invariant fields of finite groups*, preprint 2006.
182. ———, *A counterexample to the Fourteenth Problem of Hilbert in dimension four*, J. Algebra **279** (2004), 126–134.
183. ———, *A generalization of Roberts' counterexample to the Fourteenth Problem of Hilbert*, Tohoku Math. J. **56** (2004), 501–522.
184. ———, *A counterexample to the Fourteenth Problem of Hilbert in dimension three*, Michigan Math. J. **53** (2005), 123–132.
185. ———, *Fields defined by locally nilpotent derivations and monomials*, J. Algebra **293** (2005), 395–406.
186. J. Kuttler and N. Wallach, *Representations of SL_2 and the distribution of points in $\mathbb{P}^n$*, Prog. Math., vol. 220, pp. 355–373, Birkhäuser (Boston, Basel, Berlin), 2004, In: Noncommutative Harmonic Analysis.
187. J. Lipman, *Free derivation modules on algebraic varieties*, Amer. J. Math. **87** (1965), 874–898.

188. L. Makar-Limanov, *Abhyankar-Moh-Suzuki, new proof*, preprint 2004, 14 pages.
189. ______, *Facts about cancelation*, preprint 1997, 6 pages.
190. ______, *Locally nilpotent derivations, a new ring invariant and applications*, Lecture notes, Bar-Ilan University, 1998. Avail. at `http://www.math.wayne.edu/~lml/`.
191. ______, *Locally nilpotent derivations of affine domains*, MPIM Preprint Series 2004-92. Avail. at `www.mpim-bonn.mpg.de`.
192. ______, *On the group of automorphisms of a class of surfaces*, Israel J. Math. **69** (1990), 250–256.
193. ______, *On the hypersurface $x + x^2y + z^2 + t^3 = 0$ in $\mathbb{C}^4$ or a $\mathbb{C}^3$-like threefold which is not $\mathbb{C}^3$*, Israel J. Math. **96** (1996), 419–429.
194. ______, *AK invariant, some conjectures, examples and counterexamples*, Ann. Polon. Math. **76** (2001), 139–145.
195. ______, *On the group of automorphisms of a surface $x^ny = p(z)$*, Israel J. Math. **121** (2001), 113–123.
196. ______, *Again $x + x^2y + z^2 + t^3 = 0$*, Contemp. Math., vol. 369, pp. 177–182, American Mathematical Society, Providence, RI, 2005.
197. L. Makar-Limanov and A. Nowicki, *On the rings of constants for derivations of power series rings in two variables*, Colloq. Math. **87** (2001), 195–200.
198. L. Makar-Limanov, P. van Rossum, V. Shpilrain, and J.-T. Yu, *The stable equivalence and cancellation problems*, Comment. Math. Helv. **79** (2004), 341–349.
199. K. Masuda, *Torus actions and kernels of locally nilpotent derivations with slices*, preprint 2005, 10 pages.
200. K. Masuda and M. Miyanishi, *The additive group actions on $\mathbb{Q}$-homology planes*, Ann. Inst. Fourier (Grenoble) **53** (2003), 429–464.
201. S. Maubach, *Hilbert 14 and related subjects*, Master's thesis, Catholic Univ. Nijmegen, 1998.
202. ______, *Triangular monomial derivations on $k[x_1, x_2, x_3, x_4]$ have kernel generated by at most four elements*, J. Pure Appl. Algebra **153** (2000), 165–170.
203. ______, *An algorithm to compute the kernel of a derivation up to a certain degree*, Ann. Polon. Math. **76** (2001), 147–158.
204. ______, *The commuting derivations conjecture*, J. Pure Appl. Algebra **179** (2003), 159–168.
205. ______, *Polynomial endomorphisms and kernels of derivations*, Ph.D. thesis, Univ. Nijmegen, The Netherlands, 2003.
206. L. Maurer, *Über die Endlichkeit der Invariantensysteme*, Sitzungsber. Math.-Phys. Kl. Kgl. Bayer. Akad. Wiss. München **29** (1899), 147–175.
207. J. H. McKay and S. Wang, *An elementary proof of the automorphism theorem for the polynomial ring in two variables*, J. Pure Appl. Algebra **52** (1988), 91–102.
208. M. Miyanishi, *Recent developments in affine algebraic geometry: (From the personal viewpoints of the author)*, preprint, 67 pages.
209. ______, *A remark on an iterative infinite higher derivation*, J. Math. Kyoto Univ. **8** (1968), 411–415.
210. ______, *$\mathbb{G}_a$-action of the affine plane*, Nagoya Math. J. **41** (1971), 97–100.
211. ______, *Some remarks on polynomial rings*, Osaka J. Math. **10** (1973), 617–624.
212. ______, *Algebraic characterization of the affine plane*, J. Math. Kyoto Univ. **15** (1975), 169–184.

213. ______, *Lectures on Curves on Rational and Unirational Surfaces*, Springer-Verlag (Berlin, Heidelberg, New York), 1978, Published for Tata Inst. Fund. Res. , Bombay.
214. ______, *Regular subring of a polynomial ring*, Osaka J. Math. **17** (1980), 329–338.
215. ______, *Non-complete algebraic surfaces*, Springer-Verlag (Berlin,Heidelberg, New York), 1981.
216. ______, *An algebro-topological characterization of the affine space of dimension three*, Amer. J. Math. **106** (1984), 1469–1486.
217. ______, *Normal affine subalgebras of a polynomial ring*, Algebraic and Topological Theories—to the memory of Dr. Takehiko Miyata (Tokyo), Kinokuniya, 1985, pp. 37–51.
218. ______, *Algebraic characterizations of the affine 3-space*, Proc. Algebraic Geom. Seminar, Singapore, World Scientific, 1987, pp. 1469–1486.
219. ______, *Algebraic geometry*, Translations of Math. Monographs, vol. 136, American Mathematical Society, Providence, 1990.
220. ______, *Vector fields on factorial schemes*, J. Algebra **173** (1995), 144–165.
221. ______, *Open algebraic surfaces*, CRM Monograph Series, vol. 12, American Mathematical Society, 2000.
222. M. Miyanishi and T. Sugie, *Affine surfaces containing cylinderlike open sets*, J. Math. Kyoto U. **20** (1980), 11–42.
223. ______, *On a projective plane curve whose complement has logarithmic Kodaira dimension* $-\infty$, Osaka J. Math. **18** (1981), 1–11.
224. ______, *Homology planes with quotient singularities*, J. Math. Kyoto U. **31** (1991), 755–788.
225. M. Miyanishi and S. Tsunoda, *Non-complete algebraic surfaces with logarithmic Kodaira dimension* $-\infty$ *and with non-connected boundaries at infinity*, Japan J. Math. **10** (1984), 195–242.
226. ______, *Open algebraic surfaces with logarithmic Kodaira dimension* $-\infty$ *and logarithmic del Pezzo surfaces of rank 1*, Proc. Symp. Pure Math. **46** (1987), 435–450.
227. S. Mukai, *Counterexample to Hilbert's fourteenth problem for the 3-dimensional additive group*, RIMS Preprint 1343, Kyoto, 2001.
228. ______, *Finite and infinite generation of Nagata invariant ring*, Talk abstract, Oberwolfach, 2004. Avail. at `www.kurims.kyoto-u.ac.jp/~mukai/paper/Oberwolfach04.pdf`.
229. ______, *Finite generation of the Nagata invariant rings in A-D-E cases*, preprint 2005, to appear.
230. ______, *Geometric realization of T-shaped root systems and counterexamples to Hilbert's fourteenth problem*, Algebraic Transformation Groups and Algebraic Varieties, 123-129, Springer-Verlag, Berlin, 2004, Encyclopaedia Math. Sci. 132.
231. S. Mukai and H. Naito, *On some invariant rings for the two dimensional additive group action*, avail. at `http://www.eprints.math.sci.hokudai.ac.jp`.
232. D. Mumford, *Abelian Varieties*, Oxford Univ. Press (Oxford, UK), 1970.
233. ______, *Hilbert's fourteenth problem–the finite generation of subrings such as rings of invariants*, Proc. Symp. Pure Math. 28 (Providence), Amer. Math. Soc., 1976, pp. 431–444.

234. D. Mumford and J. Fogarty, *Geometric Invariant Theory (Third enlarged edition)*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 34, Springer-Verlag, 1994.
235. M. Nagata, *On the 14-th Problem of Hilbert*, Amer. J. Math. **81** (1959), 766–772.
236. ______, *On the Fourteenth Problem of Hilbert*, Proc. I.C.M. 1958, Cambridge University Press, 1960, pp. 459–462.
237. ______, *Note on orbit spaces*, Osaka J. Math. **14** (1962), 21–31.
238. ______, *Lectures on the Fourteenth Problem of Hilbert*, Lecture Notes, vol. 31, Tata Inst., Bombay, 1965.
239. ______, *On Automorphism Group of* $k[x, y]$, Lectures in Math. Kyoto Univ., vol. 5, Kinokuniya Bookstore, Tokyo, 1972.
240. ______, *Polynomial Rings and Affine Spaces*, CBMS Regional Conference Series in Mathematics, vol. 37, American Mathematical Society, Providence, Rhode Island, 1978.
241. M. Nagata and A. Nowicki, *Rings of constants for k-derivations in* $k[x_1, ..., x_n]$, J. Math. Kyoto Univ. **28** (1988), 111–118.
242. P. E. Newstead, *Introduction to Moduli Problems and Orbit Spaces*, Tata Institute, Bombay, 1978.
243. E. Noether, *Der Endlichkeitssatz der Invarianten enlicher Gruppen*, Math. Ann. **77** (1916), 89–92.
244. ______, *Der Endlichkeitssatz der Invarianten enlicher linearer Gruppen der Charakteristik p*, Nachr. Ges. Wiss. Göttingen (1926), 28–35.
245. D. G. Northcott, *Affine Sets and Afine Groups*, London Math. Society Lecture Note Series, vol. 39, Cambridge University Press, Cambridge, UK, 1980.
246. P. Nousiainen and M. Sweedler, *Automorphisms of polynomial and power series rings*, J. Pure Appl. Algebra **29** (1983), 93–97.
247. A. Nowicki, *Polynomial Derivations and their Rings of Constants*, Uniwersytet Mikolaja Kopernika, Toruń, 1994.
248. ______, *Rings and fields of constants for derivations in characteristic zero*, J. Pure Appl. Algebra **96** (1994), 47–55.
249. N. Onoda, *Linear* $\mathbb{G}_a$*-actions on polynomial rings*, Proceedings of the 25th Symposium on Ring Theory (Okayama, Japan) (Y. Tsushima and Y. Watanabe, eds.), 1992, pp. 11–16.
250. K. Pommerening, *Invariants of unipotent groups: A survey*, Invariant Theory (New York), Lectures Notes in Math., vol. 1278, Springer-Verlag, 1987, pp. 8–17.
251. V. L. Popov, *Hilbert's theorem on invariants*, Soviet Math. Dokl. **20** (1979), 1318–1322.
252. ______, *Contraction of the actions of reductive algebraic groups*, Math. USSR-Sb. **58** (1987), 311–335.
253. ______, *On actions of* $\mathbb{G}_a$ *on* $\mathbb{A}^n$, Algebraic Groups, Utrecht 1986 (New York), Lectures Notes in Math., vol. 1271, Springer-Verlag, 1987, pp. 237–242.
254. ______, *Groups, Generators, Syzygies, and Orbits in Invariant Theory*, Translations of Math. Monographs, vol. 100, Amer. Math. Soc., Providence, 1992.
255. ______, *On polynomial automorphisms of affine spaces*, Izv. Math. **65** (2001), 569–587.
256. D. Quillen, *Projective modules over polynomial rings*, Invent. Math. **36** (1976), 167–171.

257. G. Kemper R. Bryant, *Global degree bounds and the transfer principle for invariants*, J. Algebra **284** (2005), 80–90.
258. C. P. Ramanujam, *A topological characterization of the affine plane as an algebraic variety*, Ann. of Math. **94** (1971), 69–88.
259. D. Rees, *On a problem of Zariski*, Illinois J. Math. **2** (1958), 145–149.
260. R. Rentschler, *Opérations du groupe additif sur le plan affine*, C. R. Acad. Sc. Paris **267** (1968), 384–387.
261. M. Roberts, *On the covariants of a binary quantic of the nth degree*, Quart. J. Pure Appl. Math. **4** (1861), 168–178.
262. P. Roberts, *A prime ideal in a polynomial ring whose symbolic blow-up is not noetherian*, Proc. Amer. Math. Soc. **94** (1985), 589–592.
263. ———, *An infinitely generated symbolic blow-up in a power series ring and a new counterexample to Hilbert's fourteenth problem*, J. Algebra **132** (1990), 461–473.
264. J. Roé, *On the existence of plane curves with imposed multiple points*, J. Pure Appl. Algebra **156** (2001), 115–126.
265. M. Rosenlicht, *On quotient varieties and the affine embedding of certain homogeneous spaces*, Trans. Amer. Math. Soc. **101** (1961), 211–223.
266. K.P. Russell, *Simple birational extensions of two dimensional affine rational domains*, Compositio Math. **33** (1976), 197–208.
267. ———, *On Affine-Ruled Rational Surfaces*, Math. Annalen **255** (1981), 287–302.
268. C. S. de Salas, *Invariant theory for unipotent groups and an algorithm for computing invariants*, Proc. London Math. Soc. **81** (1999), 387–404.
269. A. Sathaye, *An application of generalized Newton Puiseux expansions to a conjecture of D. Daigle and G. Freudenburg*, Algebra, Arithmetic and Geometry with Applications (West Lafayette, IN, 2000), Springer Verlag, 2004, pp. 687–701.
270. J. Schröer, *Varieties of pairs of nilpotent matrices annihilating each other*, Comment. Math. Helv. **79** (2004), 396–426.
271. G. Schwarz, *Book review: Groups, generators, syzygies, and orbits in invariant theory, by V.L. Popov*, Bull. Amer. Math. Soc. **29** (1993), 299–304.
272. A. Seidenberg, *Derivations and integral closure*, Pacific J. Math. **16** (1966), 167–173.
273. J. P. Serre, *A course in arithmetic*, Springer-Verlag (Berlin, Heidelberg, New York), 1973.
274. ———, *Trees*, Springer-Verlag (Berlin, Heidelberg, New York), 1980.
275. C. S. Seshadri, *On a theorem of Weitzenböck in invariant theory*, J. Math. Kyoto Univ. **1** (1962), 403–409.
276. I. R. Shafarevich, *On some infinite dimensional groups*, Rend. Mat. Appl. (5) **25** (1966), 208–212.
277. A. R. Shastri, *Polynomial representations of knots*, Tôhoku Math. J. **44** (1992), 11–17.
278. I. P. Shestakov and U. U. Umirbaev, *Poisson brackets and two-generated subalgebras of ring of polynomials*, J. Amer. Math. Soc. **17** (2004), 181–196.
279. ———, *The tame and the wild automorphisms of polynomial rings in three variables*, J. Amer. Math. Soc. **17** (2004), 197–227.
280. T. Siebert, *Lie algebras of derivations and affine algebraic geometry over fields of characteristic 0*, Math. Ann. **305** (1996), 271–286.

281. M. K. Smith, *Stably tame automorphisms*, J. Pure Appl. Algebra **58** (1989), 209–212.
282. D. M. Snow, *Triangular actions on* $\mathbb{C}^3$, Manuscripta Mathematica **60** (1988), 407–415.
283. ______, *Unipotent actions on affine space*, Topological Methods in Algebraic Transformation Groups, Progress in Mathematics, vol. 80, Birkhäuser, 1989, pp. 165–176.
284. V. Srinivas, *On the embedding dimension of an affine variety*, Math. Ann. **289** (1991), 125–132.
285. Y. Stein, *On the density of image of differential operators generated by polynomials*, J. Analyse Math. **52** (1989), 291–300.
286. R. Steinberg, *Nagata's example*, Algebraic Groups and Lie Groups, Cambridge University Press, 1997, pp. 375–384.
287. T. Sugie, *Algebraic characterization of the affine plane and the affine 3-space*, Topological Methods in Algebraic Transformation Groups, Progress in Mathematics, vol. 80, Birkhäuser, 1989, pp. 177–190.
288. A. Suslin, *Projective modules over a polynomial ring*, Soviet Math. Doklady **17** (1976), 1160–1164.
289. M. Suzuki, *Propriétés topologiques des polynômes de deux variables complexes, et automorphismes algébriques de l'espace* $\mathbb{C}^2$, J. Math. Soc. Japan **26** (1974), 241–257.
290. R. G. Swan, Algebraic Geoemetry Seminar lecture notes, University of Chicago, 1979 (unpublished).
291. L. Tan, *An algorithm for explicit generators of the invariants of the basic* $\mathbb{G}_a$*-actions*, Comm. Algebra **17** (1989), 565–572.
292. R. Tanimoto, *A note on Hilbert's Fourteenth Problem for monomial derivations*, to appear in "Affine Algebraic Geometry", Proceedings of the 2004 Conference to honor the retirement of M. Miyanishi.
293. ______, *On Freudenburg's counterexample to the Fourteenth Problem of Hilbert*, preprint 2004, 32 pages.
294. ______, *Rings of invariants of* $\mathbb{G}_a$ *acting linearly on polynomial rings*, preprint 2005.
295. ______, *Linear counterexamples to the fourteenth problem of Hilbert*, J. Algebra **275** (2004), 331–338.
296. G. M. Tuynman, *The derivation of the exponential map of matrices*, Amer. Math. Monthly **102** (1995), 818–820.
297. Andrzej Tyc, *An elementary proof of the Weitzenböck theorem*, Colloq. Math. **78** (1998), 123–132.
298. W. van der Kulk, *On polynomial rings in two variables*, Nieuw Arch. Wisk. **1** (1953), 33–41.
299. P. van Rossum, *Tackling problems on affine space with locally nilpotent derivations on polynomial rings*, Ph.D. thesis, Univ. Nijmegen, The Netherlands, 2001.
300. W. V. Vasconcelos, *Derivations of Commutative Noetherian Rings*, Math Z. **112** (1969), 229–233.
301. S. Vénéreau, *Automorphismes et variables de l'anneau de polynômes* $A[y_1, ..., y_n]$, Ph.D. thesis, Institut Fourier des mathématiques, Grenoble, 2001.
302. Z. Wang, *Locally Nilpotent Derivations of Polynomial Rings*, Ph.D. thesis, Univ. Ottawa, 1999.

303. ———, *Homogeneization of locally nilpotent derivations and an application to* $k[x, y, z]$, J. Pure Appl. Algebra **196** (2005), 323–337.
304. R. Weitzenböck, *Über die Invarianten von linearen Gruppen*, Acta Math. **58** (1932), 231–293.
305. J. Wilkens, *On the cancellation problem for surfaces*, C. R. Acad. Sci. Paris Sér. I Math. **326** (1998), 1111–1116.
306. J. Winkelmann, *On free holomorphic* $\mathbb{C}$*-actions on* $\mathbb{C}^n$ *and homogeneous Stein manifolds*, Math. Ann. **286** (1990), 593–612.
307. ———, *Invariant rings and quasiaffine quotients*, Math. Z. **244** (2003), 163–174.
308. D. L. Wright, *Algebras which resemble symmetric algebras*, Ph.D. thesis, Columbia University, New York, 1975.
309. ———, *The amalgamated free product structure of* $GL_2([k[X_1, ..., X_n])$ *and the weak jacobian theorem for two variables*, J. Pure Appl. Algebra **12** (1978), 235–251.
310. ———, *Abelian subgroups of* $Aut_k(k[x, y])$ *and applications to actions on the affine plane*, Illinois J. Math. **23** (1979), 579–634.
311. ———, *On the Jacobian Conjecture*, Illinois J. of Math. **25** (1981), 423–440.
312. ———, *Two-dimensional Cremona groups acting on simplicial complexes*, Trans. Amer. Math. Soc. **331** (1992), 281–300.
313. H. Yoshihara, *On plane rational curves*, Proc. Japan Acad. (Ser. A) **55** (1979), 152–155.
314. O. Zariski, *Interpretations algebrico-geometriques du quatorzieme problem de Hilbert*, Bull. Sci. Math. **78** (1954), 155–168.
315. V.D. Zurkowski, *Locally finite derivations*, preprint, 26 pages.
316. ———, *Locally finite derivations in dimension three*, preprint, 76 pages.

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