Progress in Mathematical Physics

Volume 44

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Determining Spectra in Quantum Theory

Birkhäuser
Boston • Basel • Berlin

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Mathematics Subject Classicifications (2000): 31B15, 31C15, 35P05, 35P25, 35S30, 47A10, 47A11, 47A35, 47A40, 47A60, 60K40, 81Q10, 81Q15, 81U05, 81U10

Library of Congress Cataloging-in-Publication Data
Demuth, Michael, 1946-
Determining spectra in quantum theory / Michael Demuth, Maddaly Krisha.
p. cm. – (Progress in mathematical physics ; v. 44)
Includes bibliographical references and index.
ISBN-13: 978-0-8176-4366-9 (alk. paper)
ISBN-10: 0-8176-4366-4 (alk. paper)
1. Potential theory (Mathematics). 2. Scattering (Mathematics) 3. Spectral theory (Mathematics) 4. Operator theory. I. Krishna, M. (Maddaly) II. Title. III. Series.
QA404.7.D46 2005
515'.7222–dc22 2005048054

ISBN-10 0-8176-4366-4 eISBN 0-8176-4439-3 Printed on acid-free paper.
ISBN-13 978-0-8176-4366-9

 Birkhäuser

Printed in the United States of America. (TXQ/SB)

9 8 7 6 5 4 3 2 1 SPIN 11006008

Birkhäuser is a part of *Springer Science+Business Media*

www.birkhauser.com

Preface

The main objective of this book is to give a collection of criteria available in the spectral theory of selfadjoint operators, and to identify the spectrum and its components in the Lebesgue decomposition. Many of these criteria were published in several articles in different journals. We collected them, added some and gave some overview that can serve as a platform for further research activities.

Spectral theory of Schrödinger type operators has a long history; however the most widely used methods were limited in number. For any selfadjoint operator A on a separable Hilbert space the spectrum is identified by looking at the total spectral measure associated with it; often studying such a measure meant looking at some transform of the measure. The transforms were of the form $\langle f, \psi(A)f\rangle$ which is expressible, by the spectral theorem, as $\int \psi(x)d\mu(x)$ for some finite measure μ. The two most widely used functions ψ were the exponential function $\psi(x) = e^{sx}$ and the inverse function $\psi(x) = (x - z)^{-1}$. These functions are "usable" in the sense that they can be manipulated with respect to addition of operators, which is what one considers most often in the spectral theory of Schrödinger type operators.

Starting with this basic structure we look at the transforms of measures from which we can recover the measures and their components in Chapter 1.

In Chapter 2 we repeat the standard spectral theory of selfadjoint operators. The spectral theorem is given also in the Hahn–Hellinger form. Both Chapter 1 and Chapter 2 also serve to introduce a series of definitions and notations, as they prepare the background which is necessary for the criteria in Chapter 3.

Some criteria for spectral components are summarised in Chapter 3, which is the central part of this book. They are based on Borel and Fourier transform, on eigenfunction and commutator methods, and on results in scattering theory.

The criteria in Chapter 3 are very general and can be used for any semibounded selfadjoint operator. Nevertheless, we want to illustrate the power of the criteria. In doing so we investigate a series of applications. Hence we introduce in Chapter 4 a collection of operators that are of interest in this context. We do not intend to give the most general example or an exhaustive list of operators. We selected those examples of interest to us.

In Chapter 4 the unperturbed and perturbed operators are introduced. Here it is shown that for the chosen operators the criteria of Chapter 3 can be used. Finally in Chapter 5 we apply the criteria and give a series of spectral theroretic results for the perturbed operators introduced above, i.e., we ensure that the operators satisfy the assumptions stated in the criteria.

In order to maintain a fluent description, we present references and comments in the notes at the end of each chapter. If we do not prove a result in the text, we give a detailed reference for the reader to find a proof (or work out a proof on the lines given in the reference). We also give some additional information on some results in the notes. We try to give proofs in cases where we think that they improve the understanding of the text. While most of the results are from the literature, there are occasional new results not available in the literature and some of our proofs are improved or even new.

In this book there is a bias towards operators with absolutely continuous spectrum both for the deterministic and random cases; this bias is intentional and reflects our own interests.

Some theorems appear with the names of the authors and some may not. We tried as much as possible to attach a name; however it is possible that we unintentionally missed out in some cases.

There are a few books available on the subject of random Schrödinger operators in addition to several review articles. These are the books of Cycon et al. [54], Carmona–Lacroix [39], Figotin–Pastur [83], Stollmann [184] and the most recent one of Bourgain [30]. The reviews of Simon [177] and Bellissard [21] contain material not in the above books. There are more books dealing with spectral theory of deterministic Schrödinger operators that include the classical Achiezer–Glazman [1], Reed–Simon [159, 156, 158, 157] volumes, Weidmann [189] and the most recent ones of Amrein–Anne Boutet de Monvel–Georgescu [10] and Hislop–Sigal [92]. The mathematical theory of scattering has Amrein–Jauch-Sinha [12], Reed–Simon [158], Baumgartel–Wollenberg [20], Sigal [172], Pearson [154], Yafaev [192] and Dérezenskii–Gérard [72] to cite a few.

There seems to be a lot more material that is still not found in books. For example there seems to be scope for writing a book each on the density of states, the random magnetic fields, and one on transport in random media, and so on. Since the results are large in number, we decided to present just a few illustrative results and probably do some disservice to a large number of researchers by not including their work in this book; we apologise to all of them.

In the preparation of this book we greatly benefitted from discussions with Werner Kirsch, Peter Hislop, Walter Craig, Arne Jensen, Stefan Böcker, Ivan Vese'lic, Walter Renger, Andres Noll, Sven Eder, Eckhard Giere at various times. We thank Michael Baro who was kind enough to carefully read through the manuscript which helped us greatly in the preparation. We thank Anne Boutet de Monvel for readily agreeing to consider this book in the Progress in Mathematical Physics series.

We thank Ina Küster-Zippel and Ambika Vanchinathan for typesetting the manuscript and the staff at Birkhäuser especially Ann Kostant, Regina Gorenshteyn, and Elizabeth Loew, for their help in the publication process.

We acknowledge financial support from Deutsche Forschungsgemeinschaft and Indian National Science Academy. We are grateful to the kind hospitality provided by the Institutes of Mathematics at TU-Clausthal, Ruhr-Universität-Bochum and the Institute of Mathematical Sciences, Chennai at various times that made it possible for us to meet each other and work on this book.

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Measures and Transforms

1.1 Measures

In the following we consider a positive complete Borel measure on $\mathbb{R}$ to be a countably additive set function μ from the Borel subsets $\mathcal{B}_{\mathbb{R}}$ to $\mathbb{R}^+$, with the additional property that all subsets of measure zero sets are measurable. In addition the measure μ is said to be σ-finite if there exists a countable collection of sets $\{K_i\}$, $\mathbb{R} = \cup K_i$ with $\mu(K_i) < \infty$ for all i. It is said to be finite if $\mu(\mathbb{R}) < \infty$ and it is said to be a probability measure if $\mu(\mathbb{R}) = 1$. A finite complex measure is the sum $\mu_1 - \mu_2 + i(\mu_3 - \mu_4)$ of four finite positive measures $\mu_i, i = 1, 2, 3, 4$. When μ_3 and μ_4 are zero, such a complex measure is also called a signed measure. For any positive Borel measure, its topological support is defined as the smallest closed subset of $\mathbb{R}$ whose complement has measure zero. We use the following definitions concerning measures: In what follows by a measure it is always meant to be a positive measure in the above sense unless stated otherwise.

Definition 1.1.1. Let μ and ν be σ-finite Borel measures.

- μ is said to be *supported* on a set S if $\mu(\mathbb{R} \setminus S) = 0$.
- μ and ν are said to be *mutually singular* if there is a Borel set S such that $\mu(S) = 0$ and $\nu(\mathbb{R} \setminus S) = 0$.
- μ is *absolutely continuous* with respect to ν if $\nu(S) = 0 \implies \mu(S) = 0$ and they are said to be equivalent if $\mu(S) = 0 \iff \nu(S) = 0$.

Remark 1.1.2. The support of a measure should somehow capture the smallest and unique set on which the measure is concentrated. However, except for atomic measures it is not possible to find such sets. The above definition of support is so general that $\mathbb{R}$ itself is a support of any measure!

One of the ways to pin down a set where a measure is concentrated is to compare it with another measure, for example the Lebesgue measure. Such comparisons give a notion called minimal support. The minimal support of a measure μ can be defined in this sense to be the set S such that $\mu(\mathbb{R} \setminus S) = 0$

and for any $S_0 \subset S$ such that $\mu(\mathbb{R} \setminus S_0) = 0$, one has $|S \setminus S_0| = 0$. Another way to seek such set of concentration is in terms of smallest closed sets whose complement gets measure zero; this gives a unique notion of support called the topological support, but has the disadvantage that two mutually singular measures may have the same topological support. Therefore we will consider a support for a measure defined in the above sense in what follows.

A measure and its distribution function defined below go hand-in-hand in many applications.

Definition 1.1.3. Let μ be a measure such that $\mu((a,b]) < \infty$ for all $-\infty < a \leq b < \infty$, then its distribution function is defined as

$$\Phi_\mu(x) = \begin{cases} \mu((0,x]), & x > 0 \\ -\mu((x,0]), & x \leq 0. \end{cases}$$

Recall that:

1. The distribution function Φ_μ is a monotone increasing right continuous function.
2. A measure μ is regular, that is given any $\epsilon > 0$ and a Borel set E of finite measure, there exists a compact set C and an open set O such that $C \subset E \subset O$ and $\mu(E \setminus C) < \epsilon, \quad \mu(O \setminus E) < \epsilon$.
3. A measure μ is said to be G_δ outer regular, if given a Borel set E, there is a G_δ set $\tilde{E}$ with $E \subseteq \tilde{E}$ and $\mu(E) = \mu(\tilde{E})$ and it is said to be F_σ inner regular if given a Borel set E with $\mu(E) < \infty$, there is an F_σ set $\tilde{E}$ such that $\tilde{E} \subseteq E$ and $\mu(E) = \mu(\tilde{E})$.
4. The Lebesgue measure on $\mathbb{R}$ is the completion of the unique measure m on the Borel subsets of $\mathbb{R}$, which assigns for intervals a measure equal to their length.
5. The Lebesgue–Stieltjes measure m_G on $\mathbb{R}$ associated with a right continuous non-decreasing function G is the unique measure on the Borel sets such that $m_G((a,b)) = G(b) - G(a)$.
6. In the case of a probability measure μ, one can add a constant to the definition of the distribution function, so that $\Phi_\mu(-\infty) = 0$ and $\Phi_\mu(+\infty) = 1$. With this normalisation, a probability measure and its distribution function are uniquely associated to each other.
7. The Hausdorff measure h^α for an index $\alpha, 0 \leq \alpha \leq 1$ is
$$h^\alpha(S) = \lim_{\delta \to 0} \left(\inf_{\{A_i\}} \left\{ \sum_{i=1}^{\infty} |A_i|^\alpha, \quad |A_i| < \delta, \quad S \subset \cup A_i \right\} \right).$$
It may not be σ-finite on the Borel sets. It is only G_δ outer regular and F_σ inner regular when $0 < \alpha < 1$.
8. The Hausdorff dimension of a Borel set S, is defined as the number α such that $h^\beta(S) = \infty, \ \beta < \alpha$ and $h^\beta(S) = 0, \ \beta > \alpha$.

9. A measure μ is said to have exact dimension α if it is supported on a Borel set of Hausdorff dimension α and any set S with Hausdorff dimension smaller than α gets μ measure 0.
   The Hausdorff dimension of a measure μ is determined by

$$\alpha_\mu = \lim_{\epsilon \downarrow 0} \frac{\ln \mu((x-\epsilon, x+\epsilon))}{\ln \epsilon},$$

   if this is a constant for a.e. x with respect to μ.
10. A measure μ is said to be atomic if there is a countable set S on which it is supported and it is said to be continuous if μ gives measure zero to any countable set.

One splits a measure relative to the Lebesgue measure, mainly for its application in the spectral theory of Schrödinger operators.

Theorem 1.1.4 (Lebesgue decomposition). *Let μ be a measure on $\mathbb{R}$. Then there exist mutually singular measures μ_{ac}, μ_s, such that*

$$\mu = \mu_{ac} + \mu_s,$$

where μ_{ac} is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ and μ_s is singular with respect to the Lebesgue measure.

A decomposition with respect to the Hausdorff measures is also done.

Theorem 1.1.5 (Hausdorff decomposition). *Let μ be a measure on $\mathbb{R}$. Then there exist mutually singular measures $\mu_{\alpha,ac}, \mu_{\alpha s}$ such that*

$$\mu = \mu_{\alpha,ac} + \mu_{\alpha s},$$

where $\mu_{\alpha,ac}$ is absolutely continuous with respect to the Hausdorff measure h^α on $\mathbb{R}$ and $\mu_{\alpha s}$ is singular with respect to h^α.

Definition 1.1.6. A function G is said to be of *locally bounded variation* if for all $a, b \in \mathbb{R}$,

$$\sup \sum_{j=1}^{n} |G(x_j) - G(x_{j-1})| < \infty,$$

where the supremum ranges over all finite partitions, $a \le x_0 \le x_1 \le x_2 \cdots \le x_n = b$.

If f is a function of locally bounded variation, let us set

$$f(x-) = \lim_{y \uparrow x} f(y), \quad f_-(x) = f(x) - f(x-).$$

With this notation, the following integration by parts formula is valid.

Theorem 1.1.7. *Given two right continuous functions f, g of locally bounded variation, the following integration by parts formula is valid:*

$$f(b)g(b) - f(a)g(a) = \int_{(a,b]} f(x-)dm_g(x) + \int_{(a,b]} g(x-)dm_f(x) + \sum_{x \in (a,b]} (f_-(x))(g_-(x)),$$

where m_f, m_g denote the Lebesgue–Stieltjes measures associated with f and g, respectively.

In the above equation the sum on the right-hand side is only over a countable set, since the discontinuities can only be countable for f, g. The next is a statement on the measures, a part of which is a restatement of the theorem of De la Vallée Poussin, which we state for finite measures. We define

$$W_1 = \left\{ x : \overline{\lim}_{\epsilon \downarrow 0} \frac{\mu((x-\epsilon, x+\epsilon))}{2\epsilon} \neq \underline{\lim}_{\epsilon \downarrow 0} \frac{\mu((x-\epsilon, x+\epsilon))}{2\epsilon} \right\}.$$

Theorem 1.1.8. *Let μ be a finite measure. Then the distribution function Φ_μ is differentiable almost everywhere (with respect to the Lebesgue measure) and the derivative is zero almost everywhere if the measure is singular with respect to the Lebesgue measure. Further $\mu(W_1) = 0$.*

On the other hand there are measures μ such that they are supported on a set of the form

$$W_\alpha = \left\{ x : \overline{\lim}_{\epsilon \to 0} \frac{\mu((x-\epsilon, x+\epsilon))}{(2\epsilon)^\alpha} \neq \underline{\lim}_{\epsilon \to 0} \frac{\mu((x-\epsilon, x+\epsilon))}{(2\epsilon)^\alpha} \right\},$$

for some $0 < \alpha < 1$.

From the definition of the Lebesgue measure one can see the following theorem.

Theorem 1.1.9. *Let μ be a measure and let $I \subset \mathbb{R}$ be an interval. Suppose* $\sup_{\epsilon > 0} \frac{\mu((x-\epsilon, x+\epsilon))}{2\epsilon} < \infty$*, for all $x \in I$, then $\mu_s(I) = 0$.*

We finally state the Rogers–Taylor decomposition of measures with respect to Hausdorff measures. Let μ be a finite measure and define for $0 < \alpha < 1$,

$$(D_\mu^\alpha)(x) = \overline{\lim}_{\epsilon \to 0} \frac{\mu((x-\epsilon, x+\epsilon))}{(2\epsilon)^\alpha}.$$

Let

$$\begin{aligned} T_0(\alpha, \mu) &= \{x : D_\mu^\alpha(x) = 0\} \\ T(\alpha, \mu) &= \{x : 0 < D_\mu^\alpha(x) < \infty\} \\ T_\infty(\alpha, \mu) &= \{x : D_\mu^\alpha(x) = \infty\}. \end{aligned} \tag{1.1.1}$$

Then the following result is valid.

Theorem 1.1.10 (Rogers–Taylor). *Let μ be a finite positive Borel measure on $\mathbb{R}$. Then the sets $T_0(\alpha,\mu), T(\alpha,\mu)$ and $T_\infty(\alpha,\mu)$ are Borel sets and*

(i) $h^\alpha(T_\infty(\alpha,\mu)) = 0$.
(ii) $T(\alpha,\mu)$ *has finite* h^α *measure.*
(iii) $\mu(E \cap T(\alpha,\mu)) = 0$ *for any* E *with* $h^\alpha(E) = 0$.
(iv) $\mu(E \cap T_0(\alpha,\mu)) = 0$ *for any* E *of* σ*-finite* h^α *measure.*

1.2 Fourier Transform

The Fourier transform is one of the most widely used transforms in spectral theory, so we introduce the transform here.

Definition 1.2.1. The Schwartz class of functions is the set of functions defined as

$$\mathcal{S}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) : \lim_{|x|\to\infty} |x^m f^{(k)}(x)| = 0, \quad \text{for all} \quad k, m \in \mathbb{N}^d\},$$

where $x^m = \prod_{i=1}^d x_i^{m_i}$ and $f^{(k)} = \prod_{i=1}^d \frac{\partial^{k_i} f}{\partial x_i^{k_i}}$, for multi-indices m and k.

Then $\mathcal{S}(\mathbb{R}^d)$ is a locally convex vector space, with the family of seminorms

$$\|f\|_{j,k} = \sup_{x\in\mathbb{R}^d} |x^j f^{(k)}(x)|, \quad j,k \in \mathbb{N}^d.$$

It is a fact that $\mathcal{S}(\mathbb{R}^d)$ is a dense subset of every one of the Banach spaces $L^p(\mathbb{R}^d), 1 \le p < \infty$. Therefore it follows from standard facts in functional analysis that bounded linear operators defined on these spaces need only be defined on $\mathcal{S}(\mathbb{R}^d)$ and then they can be uniquely extended to the respective Banach space.

On $\mathcal{S}(\mathbb{R}^d)$ we define a linear mapping called Fourier transform.

Definition 1.2.2. The Fourier transform $\mathcal{F}$ is given by

$$(\mathcal{F}f)(\xi) = \frac{1}{\sqrt{(2\pi)^d}} \int_{\mathbb{R}^d} dx \ \ e^{-ix\cdot\xi} f(x), \quad f \in \mathcal{S}(\mathbb{R}^d),$$

where the integral is taken with respect to the Lebesgue measure.

We next collect a few theorems about the properties of the Fourier transform. We denote by $C_0(\mathbb{R}^d)$, the set of bounded continuous functions vanishing at infinity.

Theorem 1.2.3. *Consider the Fourier transform $\mathcal{F}$ on $\mathcal{S}(\mathbb{R}^d)$. Then the following statements are valid.*

1. *It is a bijection on $\mathcal{S}(\mathbb{R}^d)$.*

2. *Its extension to* $L^1(\mathbb{R}^d)$ *maps* $L^1(\mathbb{R}^d)$ *into the space* $C_0(\mathbb{R}^d)$.
3. *Its extension to* $L^2(\mathbb{R}^d)$ *is unitary as a map onto* $L^2(\mathbb{R}^d)$.
4. *Its extension to* $L^p(\mathbb{R}^d)$ *is bounded as a map into* $L^q(\mathbb{R}^d)$, *for* $1 < p < 2$, *with* $q = p/(p-1)$.

Remark 1.2.4. In the above theorem, we collected together, Riemann–Lebesgue Lemma in (2), the Fourier–Plancherel theorem in (3), and the Hausdorff–Young inequality in (4).

We note at this point the standard facts that the Fourier transform of $f(\cdot - y)$ is $e^{iy\xi}(\mathcal{F}f)(\xi)$ and that of $e^{ixk}f(x)$ is $(\mathcal{F}f)(\xi - k)$.

Definition 1.2.5. The convolution of two functions $f_1 \in L^p(\mathbb{R}^d), 1 \le p \le \infty$, $f_2 \in L^1(\mathbb{R}^d)$ is given by

$$f_1 * f_2(x) = \int dy \quad f_1(x-y) f_2(y).$$

In particular if $f, g \in \mathcal{S}(\mathbb{R}^d)$, then $f * g \in \mathcal{S}(\mathbb{R}^d)$. The convolution of a bounded continuous function f with a measure μ is defined similarly by

$$f * \mu(x) = \int_{\mathbb{R}} f(x-y) d\mu(y),$$

and gives a bounded continuous function.

Proposition 1.2.6. *We have for* $f_1 \in L^p(\mathbb{R}^d), f_2 \in L^1(\mathbb{R}^d)$,

$$\|f_1 * f_2\|_p \le \|f_1\|_p \|f_2\|_1, \quad 1 \le p \le \infty.$$

The following theorem is useful in many contexts. Let ϕ be a continuous function on $\mathbb{R}^d$ which is integrable with $\int_{\mathbb{R}^d} \phi(x) dx = 1$. Define for $\epsilon > 0$, $\phi_\epsilon(x) = \frac{1}{\epsilon^d} \phi(\frac{x}{\epsilon})$.

Theorem 1.2.7. *Let* ϕ_ϵ *be as above. Then, for any* $1 \le p < \infty$,

$$\phi_\epsilon * f \to f, \quad in \quad L^p(\mathbb{R}^d), \quad as \quad \epsilon \to 0.$$

In addition for $1 \le p \le \infty$,

$$\lim_{\epsilon \to 0} (\phi_\epsilon * f)(x) \to f(x), \quad for\ a.e. \quad x \in \mathbb{R}^d.$$

The Fourier transform converts convolutions to products as seen in the following theorem.

Theorem 1.2.8. *Let* $f, g \in \mathcal{S}(\mathbb{R}^d)$. *Then*

$$\mathcal{F}(f * g) = (2\pi)^{d/2} (\mathcal{F}f)(\mathcal{F}g).$$

We define the mapping $\mathcal{F}^*$ by $(\mathcal{F}^* f)(x) = (\mathcal{F}f)(-x)$, for $f \in \mathcal{S}(\mathbb{R}^d)$.

Theorem 1.2.9. *Then the following inversion formula is valid:*

$$\mathcal{F}^*\mathcal{F}f = f, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

The same inversion formula is also valid for all $f \in L^1(\mathbb{R}^d)$ such that $\mathcal{F}f \in L^1(\mathbb{R}^d)$.

Theorem 1.2.10. *Let μ be a finite complex measure. Then the Fourier transform $\widehat{\mu}$ of μ defined by*

$$\widehat{\mu}(t) = \int_{\mathbb{R}} e^{-itx} d\mu(x)$$

is a bounded continuous function on $\mathbb{R}$.

1.3 The Wavelet Transform

In this section we consider the continuous wavelet transform and show that it is possible to use this transform to recover probability measures from their transforms.

We recall that given a function ψ in $L^1(\mathbb{R})$ and any $g \in L^p(\mathbb{R}),\ 1 \le p \le \infty$, we can define the continuous wavelet transform of g associated with ψ by

$$(T_\psi g)(x_1, x_2) = \int_{\mathbb{R}} \psi\left(\frac{y - x_2}{x_1}\right) g(y)\, dy, \quad x_1 \in \mathbb{R}^+,\ x_2 \in \mathbb{R}.$$

Here $\mathbb{R}^+ = (0, \infty)$. This transform has interesting properties which we will not describe here.

It is interesting to note that there is a class of ψ for which the transform $T_\psi \mu$ is associated with a probability measure μ uniquely, and we can recover μ from $T_\psi \mu$.

To show this we consider a probability measure μ on $\mathbb{R}$. Let ψ be a bounded continuous function with $\psi(0) = 1, \quad \lim_{|t|\to\infty} \psi(t) = 0$ and define

$$\psi_a(x) = \psi\left(\frac{x}{a}\right), \quad a > 0.$$

Recall the convolution of a function f with a measure μ is defined as $f * \mu(x) = \int f(x - y) d\mu(y)$, when the integral converges. Therefore, if μ is a probability measure on $\mathbb{R}$ and ψ is a bounded continuous function on $\mathbb{R}$, the convolution $\psi_a * \mu$ is always defined and gives a bounded continuous function. We give below a theorem on how to recover μ given $\psi_a * \mu$. We define for $0 < \alpha \le 1$,

$$d_\mu^\alpha(x) = \lim_{\epsilon\to 0} \frac{\mu((x - \epsilon, x + \epsilon))}{(2\epsilon)^\alpha} = \lim_{\epsilon\to 0} \frac{\Phi_\mu(x + \epsilon) - \Phi_\mu(x - \epsilon)}{(2\epsilon)^\alpha}, \tag{1.3.2}$$

where Φ_μ denotes the distribution function of μ, see Definition 1.1.3.

Hypothesis 1.3.1. *Let ψ be a continuous function on $\mathbb{R}$ with $\psi(0) = 1$ and $A_\psi = \int \psi(x)dx \neq 0$. Further assume that*

1. *ψ is bounded.*
2. *ψ is differentiable, even and satisfies*

$$|\psi(x)| + |x\psi'(x)| \leq \langle x \rangle^{-\delta}, \quad \textit{for some} \quad \delta > 1,$$

where $\langle x \rangle = (1 + x^2)^{1/2}$.

Theorem 1.3.2 (Jensen–Krishna). *Let μ be a probability measure and let ψ be as in in Hypothesis 1.3.1(1). Then*

1. $\lim\limits_{a \to 0} \psi_a * \mu(x) = \mu(\{x\})$.
2. *Let ψ be as in Hypothesis 1.3.1(2). Then for every continuous function f of compact support, the following is valid:*

$$\lim_{a \to 0} \int \left(\frac{1}{a} \psi_a * \mu \right)(x) f(x) dx = A_\psi \int f(x) d\mu(x).$$

3. *Let ψ be as in Hypothesis 1.3.1(2) and suppose $d_\mu^\alpha(x)$ is finite, for some $0 < \alpha \leq 1$ and x; then*

$$\lim_{a \to 0} a^{-\alpha} \psi_a * \mu(x) = c_\alpha d_\mu^\alpha(x), \tag{1.3.3}$$

where $c_\alpha = \int_0^\infty \alpha 2^\alpha y^{\alpha - 1} \psi(y)\, dy$.

4. *Let S_μ denote the topological support of μ and let ψ be as in Hypothesis 1.3.1(2). Then for any $x \in \mathbb{R} \setminus S_\mu$ and any $0 \leq \alpha \leq 1$,*

$$\lim_{a \to 0} a^{-\alpha} \psi_a * \mu(x) = 0.$$

Remark 1.3.3. Equation (1.3.3) implies that if μ is purely singular, then the limit of $\psi_a * \mu(x)/a$ is zero almost everywhere with respect to the Lebesgue measure, since the derivative $d_\mu^1(x) = 0$ almost everywhere for purely singular μ.

Proof: (1) This part is a direct consequence of the definition of the integral noting that point-wise we have

$$\lim_{a \to 0} \psi_a(x) = \begin{cases} 0, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

We also need to use the dominated convergence theorem to interchange the limit and the integral.

(2) Since f is a continuous function of compact support and ψ_a is bounded for each $a > 0$, $f(x)\psi_a(x - y)$ is absolutely integrable and the integral is uniformly bounded in $y \in \mathbb{R}$. Therefore, by an application of Fubini, a change

of variable $x \to ax+y$ and the dominated convergence theorem, in that order, it follows that

$$\begin{aligned}
\lim_{a\to 0}\int dx\ f(x)\left(\frac{1}{a}\psi_a * \mu\right)(x) &= \lim_{a\to 0}\int dx f(x)\int \frac{1}{a}\psi_a(x-y)d\mu(y)\\
&= \lim_{a\to 0}\int d\mu(y)\int f(x)\frac{1}{a}\psi_a(x-y)dx\\
&= \lim_{a\to 0}\int d\mu(y)\int f(ax+y)\psi(x)dx\\
&= \int d\mu(y)\int (\lim_{a\to 0} f(ax+y))\psi(x)dx\\
&= \left(\int d\mu(y)f(y)\right)\left(\int \psi(x)dx\right).
\end{aligned}$$

(3) Consider a ψ as in Hypothesis 1.3.1; then,

$$\begin{aligned}
&\frac{1}{a^\alpha}\int_{\mathbb{R}} \psi_a(x-y)d\mu(y)\\
&\qquad = -\frac{1}{a^\alpha}\int_{\mathbb{R}} \frac{d\psi((x-y)/a)}{dy}\Phi_\mu(y)dy\\
&\qquad = -\frac{1}{a^\alpha}\int_{\mathbb{R}} \psi'(y)\Phi_\mu(x-ay)dy\\
&\qquad = -\int_0^\infty \psi'(y)(2y)^\alpha\frac{\Phi_\mu(x+ay)-\Phi_\mu(x-ay)}{(2ay)^\alpha}dy, \qquad (1.3.4)
\end{aligned}$$

where in the first step integration by parts is used, and in the next step the boundary terms are dropped, which can be done since ψ vanishes at ∞. The penultimate step involved a change of variables and the final step used the oddness of ψ' to split the integral into the positive and negative half lines.

Observe that

$$d_\mu^\alpha(x) = \lim_{a\to 0}\frac{\Phi_\mu(x+ay)-\Phi_\mu(x-ay)}{(2ay)^\alpha}$$

for each $y\in\mathbb{R}$, and is finite by assumption, so $\frac{\Phi_\mu(x+ay)-\Phi_\mu(x-ay)}{(2ay)^\alpha}$ is a bounded function. Finally the Hypothesis 1.3.1(2) implies that $\psi'|y|$ is integrable; therefore by dominated convergence theorem, the limit can be interchanged with the integral to obtain the right-hand side to be $(-\int_0^\infty \psi'(y)(2y)^\alpha dy)d_\mu^\alpha(x)$. Another integration by parts now yields the constant c_α, obtaining the theorem.

(4) By Hypothesis 1.3.1(2), $|\frac{1}{a^\alpha}\psi_a(x)| \le Ca^{\delta-\alpha}$. Therefore for any $0\le\alpha\le 1$, Lebesgue dominated convergence gives this result. □

The next theorem gives further properties of the the limits of $\psi_a * \mu$ and the first part is analogous to Wiener's theorem and its extension by Simon [177].

Theorem 1.3.4 (Jensen–Krishna). *Let μ be a probability measure on $\mathbb{R}$ and ψ be as in Hypothesis 1.3.1(2). Then for any bounded interval (c,d) the following are valid.*

1. *Let $C = \int_{\mathbb{R}} |\psi(x)|^2 \, dx$ (C is finite by the assumption on ψ). Then*

$$\lim_{a\to 0} \frac{1}{a} \int_c^d \|\psi_a * \mu\|^2(x) \;\; dx = C \left(\sum_{c<x<d} \mu(\{x\})^2 + \frac{\mu(\{c\})^2 + \mu(\{d\})^2}{2} \right).$$

2. *Suppose that for some $p > 1$,*

$$\sup_{a>0} \int_c^d \left| \frac{1}{a} (\psi_a * \mu) \right|^p (x) \; dx < \infty;$$

then μ is purely absolutely continuous in (c,d). In addition, for any compact subset S of (c,d)

$$\frac{1}{a} \psi_a * \mu \to A_\psi \, \frac{d\mu_{ac}}{dx}, \quad in \quad L^p(S), \quad as \quad a \to 0.$$

The converse that if μ is purely absolutely continuous with the density $\frac{d\mu_{ac}}{dx}$ in $L^p((c,d))$, then the supremum above is finite, is also valid.

3. *For $0 < p < 1$, we have*

$$\lim_{a\to 0} \int_c^d \left| \frac{1}{a} \psi_a * \mu \right|^p (x) \; dx = |A_\psi|^p \int_c^d \left| \frac{d\mu_{ac}}{dx} \right|^p \; dx.$$

Proof: (1) Expanding the integrals gives

$$\frac{1}{a} \int_c^d |\left(\psi_a * \mu\right)(x)|^2 dx = \iint d\mu(y_1) d\mu(y_2) \int_c^d dx \frac{1}{a} \overline{\psi_a(x-y_1)} \psi_a(x-y_2).$$

Since the function ψ_a is bounded, the interval (c,d) is bounded, and μ is a probability measure, the right-hand side integral converges absolutely, so Fubini can be used to interchange integrals to get the equality above. Let

$$h_a(y_1,y_2) = \int_c^d dx \; \frac{1}{a} \; \overline{\psi_a(x-y_1)} \psi_a(x-y_2).$$

Set $S_1 = \{x : |x| \leq |y_1 - y_2|/2\}$, $\;S_2 = \{x : |x| \geq |y_1 - y_2|/2\}$. Then on S_1 and S_2 the estimates

$$\begin{aligned} \langle x + y_2 - y_1 \rangle^{-\delta} &\leq c|y_2 - y_1|^{-\delta}, \quad x \in S_1, \\ \langle x \rangle^{-\delta} &\leq c|y_2 - y_1|^{-\delta}, \quad x \in S_2 \end{aligned}$$

are valid. Suppose $y_1 \neq y_2$; then using the bound $|\psi(x)| \leq C\langle x \rangle^{-\delta}$, we see that

$$
\begin{aligned}
|h_a(y_1,y_2)| &\le \frac{c}{a}\int_{-\infty}^{\infty}\langle (x+y_2-y_1)/a\rangle^{-\delta}\langle x/a\rangle^{-\delta}dx\\
&= \frac{c}{a}\left(\int_{S_1}+\int_{S_2}\right)(\langle (x+y_2-y_1)/a\rangle^{-\delta}\langle x/a\rangle^{-\delta})dx\\
&\le \frac{ca^{\delta}}{|y_1-y_2|^{\delta}}\int_{-\infty}^{\infty}\langle x/a\rangle^{-\delta}d(x/a)\\
&\le \frac{ca^{\delta}}{|y_1-y_2|^{\delta}}
\end{aligned}
$$

is valid. It follows that $\lim_{a\to 0} h_a(y_1,y_2) = 0$ for $y_1 \neq y_2$. We decompose $\mu = \mu_c + \mu_p$ into its continuous and atomic parts. Then

$$\int h_a(y_1,y_2)\ d\mu(y_2) = \int h_a(y_1,y_2)\ d\mu_c(y_2) + \int h_a(y_1,y_2)\ d\mu_p(y_2).$$

Therefore

$$
\begin{aligned}
\int h_a(y_1,y_2)\ d\mu(y_2)d\mu(y_1) = \int h_a(y_1,y_2)\ d\mu_c(y_2)d\mu(y_1)\\
+ \int h_a(y_1,y_2)\ d\mu_p(y_2)d\mu(y_1).
\end{aligned}
$$

The first term vanishes as $a \to 0$, by the dominated convergence theorem and the second term is

$$\sum_{y_1,y_2\in[c,d]} \mu(\{y_2\})\mu_p(\{y_1\})h_a(y_1,y_2) + \sum_{y_2\in[c,d]}\int h_a(y_1,y_2)\ d\mu_c(y_1).$$

The second term above also goes to zero as $a \to 0$ by the dominated convergence theorem again. Since the limit of $h_a(y_1,y_2)$ $a \to 0$ has a non-zero value only when $y_1 = y_2$, it only remains to compute the limit of $h_a(y_1,y_1)$ for $y_1 \in [c,d]$ as $a \to 0$. This value follows from the relations

$$h_a(y_1,y_1) = \int_c^d \frac{1}{a}|\psi_a(x-y_1)|^2dx = \int_{(c-y_1)/a}^{(d-y_1)/a} |\psi(x)|^2dx,$$

and the evenness of ψ.

(2) In view of Theorem 1.3.2(2), for f continuous and of compact support,

$$|A_\psi|\left|\int_c^d f(x)d\mu(x)\right| = \lim_{a\to 0}\left|\int_c^d f(x)\frac{1}{a}\left(\psi_a * \mu\right)(x)dx\right|.$$

Therefore

$$|A_\psi|\left|\int_c^d f(x)d\mu(x)\right| \le C\left(\int_c^d |f(x)|^q dx\right)^{\frac{1}{q}} \sup_{a>0}\left(\int_c^d \left|\frac{1}{a}\left(\psi_a * \mu\right)(x)\right|^p dx\right)^{\frac{1}{p}}.$$

Since $\sup_{a>0} \int_c^d |\frac{1}{a}\psi_a * \mu|^p(x)dx < \infty$, it follows that

$$\left|\int_c^d f(x)d\mu(x)\right| < C\left(\int_c^d |f(x)|^q \, dx\right)^{1/q}, \ q = p/(p-1).$$

Therefore the map $f \to \int_c^d f(x)d\mu(x)$ is a bounded linear functional on $L^q((c,d))$. Since the space of bounded linear functionals is precisely $L^p(c,d)$, q given above, the absolute continuity of $d\mu$ restricted to (c,d) follows. So we can set $d\mu = g(x)dx, \ g \in L^p((c,d))$, when restricted to (c,d). (Here restriction means taking the measure $\chi_{(c,d)}(\cdot)\mu$).

The converse part is easy by Proposition 1.2.6 and because $\psi \in L^1(\mathbb{R})$.

(3) First note that the estimate

$$\left|\int_{\mathbb{R}\setminus(c,d)} \frac{1}{a}(\psi_a * \mu)(x)\right| \leq Ca^{\delta-1}, \ \delta > 1,$$

is valid whenever $[\alpha,\beta] \subset (c,d)$ and $x \in [\alpha,\beta]$, in view of Hypothesis 1.3.1(2) on ψ. Hence, without loss of generality assume that μ is supported in $[c,d]$ and write $\mu = \mu_{ac} + \mu_s$, in terms of its absolutely continuous and singular parts using the Lebesgue decomposition. The singular part of the measure μ will not contribute to the limit as will be seen in a moment, so assume that μ is absolutely continuous, given by $d\mu(x) = g(x)dx$; then the assumption that μ is a finite measure implies $g \in L^1(\mathbb{R})$. In particular, for $0 < p < 1$, the reverse Hölder inequality

$$\int_c^d \left|\frac{1}{a}\psi_a * g - A_\psi g\right|^p dx \leq \left(\int_c^d \left|\frac{1}{a}\psi_a * g - A_\psi g\right| dx\right)^p (d-c)^{1-p}$$

implies that $\frac{1}{a}\psi_a * g \to A_\psi g$ in $L^p((c,d)), \quad 0 < p \leq 1$, using Theorem 1.2.7. Now the spaces $L^p((c,d))$, $0 < p < 1$, are metric spaces with the metric $d(f,g) = \|f-g\|_p^p$. It then follows from the triangle inequality for this metric that

$$\lim_{a\to 0} \int_c^d \left|\frac{1}{a}\psi_a * g\right|^p (x)dx = |A_\psi|^p \int_c^d |g(x)|^p dx.$$

Now we will show that the singular part of μ does not contribute to the limit. So assume that μ is purely singular and that its support S is contained in $[\alpha,\beta] \subset (c,d)$. Since μ is singular, by the definition of support, S satisfies $\mu(\mathbb{R}\setminus S) = 0$ and $|S| = 0$, with $|\cdot|$ denoting the Lebesgue measure. By the regularity of the Lebesgue measure, given an $\epsilon > 0$, there is an open O such that $S \subset O$, with $|O \setminus S| < \epsilon$. We also have $|O| \leq |O \setminus S| + |S| < \epsilon$. For the same ϵ, since the measure μ is regular we also have a compact $K \subset S$ such that $\mu(S \setminus K) < \epsilon$. In addition, since $K \subset S$, and S has Lebesgue measure zero K also has Lebesgue measure zero.

The above reverse Hölder inequality gives

$$\begin{aligned}
\int_c^d \left|\frac{1}{a}\psi_a * \mu\right|^p (x)dx \leq & \int_O \left|\frac{1}{a}\psi_a * \mu\right|^p (x)dx + \int_{(c,d)\setminus O} \left|\frac{1}{a}\psi_a * \mu\right|^p (x)dx \\
\leq & \ |O|^{1-p}\mu((c,d))^p \|\psi\|_1^p + |d-c|^{1-p}\left(\int_{(c,d)\setminus O} \left|\frac{1}{a}\psi_a * \mu\right| (x)dx\right)^p \\
\leq & \ C_1\epsilon^{1-p} + |d-c|^{1-p}\left(\int_{(c,d)\setminus O} \left|\frac{1}{a}\psi_a * \mu\right| (x)dx\right)^p .
\end{aligned}$$

Now consider a bounded continuous function h which is 1 on $(c,d) \setminus O$ and which is 0 on K.

Then using the Hypothesis 1.3.1(2) that $|\psi(x)| \leq C\langle x\rangle^{-\delta}$ and setting $\phi(x) = \langle x\rangle^{-\delta}$,

$$\begin{aligned}
\int_{(c,d)\setminus O} \left|\frac{1}{a}\psi_a * \mu\right| (x)dx &\leq \int_{(c,d)\setminus O} \frac{1}{a}\int_{\mathbb{R}} |\psi_a(x-y)|d\mu(y) \ \ dx \\
&\leq \int_{(c,d)\setminus O} \frac{1}{a}\int_{\mathbb{R}} \langle\frac{x-y}{a}\rangle^{-\delta} d\mu(y) \ \ dx \\
&\leq \int_{(c,d)\setminus O} h(x)\frac{1}{a}(\phi_a * \mu)(x) \ dx.
\end{aligned}$$

The function ϕ satisfies the Hypothesis 1.3.1(2), so the Theorem 1.3.2(2) is applicable with ψ replaced by ϕ. Therefore the last term, which has positive integrand, converges to $\int_{(c,d)\setminus O} h(x)d\mu(x)$ as a goes to zero, which is bounded by $\int_{(c,d)\setminus K} d\mu(x)$,

$$\int_{(c,d)\setminus O} h(x)d\mu(x) \leq \mu((c,d)\setminus K) \leq \mu((c,d)\setminus S) + \mu(S\setminus K) \leq C_2\epsilon,$$

using the facts that $\mu((c,d)\setminus S) = 0$ and $\mu(S\setminus K) < \epsilon$. Putting these estimates together one gets the theorem since ϵ is arbitrary. □

Corollary 1.3.5 *Let μ be a probability measure and let ψ be as in Hypothesis 1.3.1. Then we have that:*

1. *μ has no point part in (c,d) iff*

$$\varliminf_{a_n\to 0} \int_c^d \left|\frac{1}{a_n}\psi_{a_n} * \mu\right|^2 (x) \ dx = 0,$$

for some sequence a_n.

2. *μ has no absolutely continuous part in (c,d) iff for some $0 < p < 1$,*

$$\varliminf_{a_n\to 0} \int_c^d \left|\frac{1}{a_n}\psi_{a_n} * \mu\right|^p (x) \ dx = 0,$$

for some sequence a_n.

Proof: The proofs follow directly from the Theorem 1.3.4. □

Even when the quantity d_μ^α does not exist, it is possible to say something on the wavelet transforms, to cover the cases of measures which are not supported on the sets where such limits exist. Set, for $0 < \alpha \leq 1$,

$$C^\alpha_{\mu,\psi}(x) = \overline{\lim}_{a \to 0} \frac{\psi_a * \mu}{a^\alpha}(x) \quad \text{and} \quad D^\alpha_\mu(x) = \overline{\lim}_{\epsilon \to 0} \frac{\mu((x-\epsilon, x+\epsilon))}{(2\epsilon)^\alpha} \quad .$$

Then we have the following theorem where we set $\psi_\alpha(t) = \psi'(e^t)e^{(\alpha+1)t}$, $0 < \alpha < 1$.

Theorem 1.3.6 (Jensen–Krishna). *Let μ be a probability measure and let ψ satisfy the Hypothesis 1.3.1(2). Then for each $0 < \alpha \leq 1$, $C^\alpha_{\mu,\psi}(x)$ is finite for any x, whenever $D^\alpha_\mu(x)$ is finite for the same x, and if ψ is positive they are both finite or both infinite. In addition when ψ is positive and $\widehat{\psi_\alpha}$ is zero free then, the converse of Theorem 1.3.2(3) is valid, that is* $\lim_{a \to 0} \frac{1}{a^\alpha}(\psi_a * \mu)(x) < \infty$ *implies that $d^\alpha_\mu(x)$ is finite.*

Proof: Consider the case when $D^\alpha_\mu(x)$ is finite for some x and for some fixed α. Then for any $0 < y < 1$, $\mu((x-y, x+y)) \leq C|y|^\alpha$ for some finite constant C. So, using the last line in Equation (1.3.4) and estimating the right-hand side there, one has, by Hypothesis 1.3.1(2),

$$\left| \frac{1}{a^\alpha}(\psi_a * \mu)(x) \right| \leq C_1 \int_0^\infty |\psi'(y)|(2y)^\alpha dy \leq C_1 \int_0^\infty \langle y \rangle^{-\delta} |y|^{-1+\alpha} dy < \infty.$$

Now taking the limsup of the left-hand side the finiteness of $C^\alpha_{\mu,\psi}$ follows.

On the other hand since ψ is positive continuous with $\psi(0) = 1$, there is a $\beta > 0$ such that $\psi(y) > 1/2, \ -\beta < y < \beta$. Using this and the evenness of ψ,

$$\begin{aligned} \frac{1}{a^\alpha}(\psi_a * \mu)(x) &= \frac{1}{a^\alpha} \int \psi_a(x-y) d\mu(y) = \frac{1}{a^\alpha} \int \psi\left(\frac{y}{a}\right) d\mu(y+x) \\ &\geq \frac{1}{a^\alpha} \int_{-\beta a}^{\beta a} \frac{1}{2} d\mu(y+x) \\ &\geq \frac{1}{2a^\alpha}[\mu((x-a\beta, x+a\beta))], \end{aligned}$$

where the positivity of ψ is used to get the first inequality above. The above inequalities immediately imply, since β is fixed, that $D^\alpha_\mu(x) = \infty$ implies the same for $C^\alpha_{\mu,\psi}(x)$.

To see the last part suppose $\lim_{a \to 0} \frac{1}{a^\alpha}(\psi_a * \mu)(x) < \infty$; then $C^\alpha_\mu(x) < \infty$, hence by the earlier part of the theorem $D^\alpha_\mu(x) < \infty$. This implies that

$$\sup_{\delta > 0} \frac{\Phi_\mu(x+\delta) - \Phi_\mu(x-\delta)}{(2\delta)^\alpha} < \infty.$$

Using this fact we see that in the right-hand side of Equation 1.3.4, the quantity

$$\sup_{y>0}\sup_{a>0}\left|\frac{\Phi_\mu(x+ay)-\Phi_\mu(x-ay)}{(2ay)^\alpha}\right|<\infty.$$

This shows that we can write the integral on the right-hand side of Equation (1.3.4) as

$$\frac{1}{a^\alpha}\psi_a*\mu(x)=\int_{\mathbb{R}}K_{\alpha,\psi}(t)f_{x,\alpha,\mu}(t-s)dt,$$

after changing variables $y=e^t, a=e^{-s}$. Here we have taken

$$f_{x,\alpha,\mu}(t)=\frac{\Phi_\mu(x+e^t)-\Phi_\mu(x-e^t)}{2^\alpha e^{\alpha t}},\quad K_{\alpha,\psi}=-2^\alpha\psi_\alpha.$$

From the assumptions it is clear that $K_{\alpha,\psi}\in L^1(\mathbb{R})$ and $f_{x,\alpha,\mu}$ is a bounded function such that $e^{\alpha t}f_{x,\alpha,\mu}(t)$ non-decreasing. This gives

$$\lim_{a\to 0}\frac{1}{a^\alpha}\psi_a*\mu(x)=\lim_{s\to\infty}\int_{\mathbb{R}}K_{\alpha,\psi}(t)f_{x,\alpha,\mu}(t-s)dt.$$

By assumption the limit on the left side of the above equation exists, and so the one on the right-hand side exists. Further using the assumption on the Fourier transform of ψ_α we see that in particular $\widehat{\psi_\alpha}(0)\neq 0$ and hence $\int_{\mathbb{R}}K_{\alpha,\psi}(t)dt\neq 0$. Therefore we can write the limit as

$$\lim_{s\to\infty}\int_{\mathbb{R}}K_{\alpha,\psi}(t)f_{x,\alpha,\mu}(t-s)dt=C\int_{\mathbb{R}}K_{\alpha,\psi}(t)dt,\ C\neq 0.$$

Since $\widehat{K_{\alpha,\psi}}$ does not vanish anywhere, applying Wiener's Tauberian theorem it follows that for any $L^1(\mathbb{R})$ function $K(t)$ we have

$$\lim_{s\to\infty}\int_{\mathbb{R}}K(t)f_{x,\alpha,\mu}(t-s)dt=C\int_{\mathbb{R}}K(t)dt.$$

Choosing a special function $K_h(t)=\frac{1}{h}\chi_{[0,h]}(t)$ we see that for any $h>0$,

$$\lim_{s\to 0}\frac{1}{h}\int_0^h f_{x,\alpha,\mu}(t-s)dt=C. \tag{1.3.5}$$

This will imply that the limit $\lim_{s\to\infty}f_{x,\alpha,\mu}(-s)=C$, which is the required statement. To see this we use the fact that $e^t f_{x,\alpha,\mu}(t)$ is a non-decreasing function of t, since it is essentially the integral of a positive measure, and note the following inequalities for any positive t.

$$e^{s+t}f_{x,\alpha,\mu}(s+t)\geq e^t f_{x,\alpha,\mu}(s)$$
$$\frac{f_{x,\alpha,\mu}(s+t)-f_{x,\alpha,\mu}(s)}{t}\geq-\frac{1-e^{-t}}{t}f_{x,\alpha,\mu}(s)\geq -B$$

with B an upper bound for $f_{x,\alpha,\mu}$. These bounds imply that for any $\epsilon > 0$, if there is a sequence of points $s_n \to \infty$ such that $f_{x,\alpha,\mu}(-s_n) \geq C + \epsilon$, then we would get that

$$\frac{1}{h}\int_0^h f_{x,\alpha,\mu}(t - s_n)dt \geq -B\frac{h}{2} + C + \epsilon, \;\; h > 0.$$

This gives a contradiction to the convergence obtained in Equation (1.3.5). Similar calculation shows that we cannot have for any $\epsilon > 0$ a sequence $s_n \to \infty$ such that $f_{x,\alpha,\mu}(-s_n) \leq C - \epsilon$, proving the theorem. □

We note for later reference that in the case when $\psi(x) = \frac{1}{1+x^2}$, the Fourier transform of the function $\psi_\alpha(t) = \frac{4e^{(2+\alpha)t}}{(e^{2t}+1)^2}$ turns out to be

$$\widehat{\psi_\alpha}(\xi) = 8\pi(1 - e^{i\alpha\pi+\xi\pi})^{-1}(\alpha - i\xi)e^{\xi\frac{\pi}{2}}e^{i(\alpha-1)\frac{\pi}{2}}.$$

Therefore for $0 < \alpha \leq 1$, this Fourier transform is non zero for any $\xi \in \mathbb{R}$.

1.4 Borel Transform

Henceforth we denote the set of complex numbers by $\mathbb{C}$, the upper half plane by $\mathbb{C}^+$, the open unit disk by $\mathbb{D}$, and the unit circle by $\mathbb{T}$. In this setting the Borel transform is defined as:

Definition 1.4.1. Let μ be a measure on $\mathbb{R}$ satisfying the condition that

$$\int_{\mathbb{R}} d\mu(x)\;\frac{1}{1+x^2} < \infty. \tag{1.4.6}$$

Then the integral

$$\int_{\mathbb{R}} d\mu(x)\left\{\frac{1}{x-z} - \frac{x}{1+x^2}\right\}$$

defines the Borel transform F_μ of μ, $\quad z \in \mathbb{C}\setminus\mathbb{R}$.

The Borel transform given above is an analytic function in $\mathbb{C}^+ \cup \mathbb{C}^-$, and it maps these two components into themselves and is also called the Borel–Stieltjes transform of the measure μ. In the case when the measure is such that $\int d\mu(x)\frac{1}{1+|x|} < \infty$, then no regularization is needed.

Conversely we also have the well-known Herglotz representation theorem.

Theorem 1.4.2. *Let F be a function analytic in $\mathbb{C}^+$ such that its range is contained in $\mathbb{C}^+$. Then there exists a non-negative number a, a real number b and a measure μ satisfying the Inequality (1.4.6), such that*

$$F(z) = az + b + \int_{\mathbb{R}} d\mu(x)\left\{\frac{1}{x-z} - \frac{x}{1+x^2}\right\}.$$

The a, b, μ are uniquely associated with F.

Proof: We show that this theorem follows from Theorem 1.4.3 below. The functions $w(z) = (z-i)/(z+i)$ (respectively $z(w) = (w+1)/i(w-1)$) is a map from $\mathbb{C}^+$ onto $\mathbb{D}$ (respectively from $\mathbb{D}$ onto $\mathbb{C}^+$) and $w(\cdot)$ and $z(\cdot)$ are holomorphic on $\mathbb{C}^+$ and $\mathbb{D}$, respectively, and also they are the inverses of each other. Therefore given any function F mapping $\mathbb{C}^+$ to itself, we can take $G = -iF \circ z$ so that G is a function on $\mathbb{D}$ with positive real part. Therefore let G be a function, with positive real part, on $\mathbb{D}$ associated with the F. Then G satisfies the conclusion of Theorem 1.4.3, so that it is given by the Equation (1.4.7). Then we consider $iG(w)$, and rewrite the expression for it by separating the point mass of σ at $\theta = 0$, if any, to get

$$iG(w) = -\alpha + i\sigma(\{0\})\frac{1+w}{1-w} + i\int_0^{2\pi} d\sigma_0(\theta)\ \frac{e^{i\theta}+w}{e^{i\theta}-w},$$

where we have taken $\sigma_0 = \sigma - \sigma(\{0\})\delta_0$. We write w as the image of a point z in $\mathbb{C}^+$ using the map $w(.)$, we have, setting $F(z) = i(G \circ w)(z) = iG(w)$, and using the relation $z = (w+1)/i(w-1)$,

$$F(z) = -\alpha + \sigma(\{0\})z + \int_0^{2\pi} \frac{z\cos(\theta/2) - \sin(\theta/2)}{z\sin(\theta/2) + \cos(\theta/2)} d\sigma_0(\theta).$$

Changing variables now to $\lambda = -\cot(\theta/2)$, we get

$$F(z) = b + az + \int_{-\infty}^{\infty} \frac{z\lambda + 1}{\lambda - z} d\nu(\lambda),$$

where we have set $a = \sigma(\{0\})$, $b = -\alpha$ and $d\nu(\lambda) = d\sigma_0(2\cot^{-1}(\lambda))$. We note that the measure ν is also positive and finite. Now we set $d\mu(\lambda) = (1+\lambda^2)d\nu(\lambda)$ and note that $\frac{z\lambda+1}{(\lambda-z)(1+\lambda^2)} = \frac{1}{\lambda-z} - \frac{\lambda}{1+\lambda^2}$ gives the expression for F. $\square$

Theorem 1.4.3. *Let G be a function analytic in $\mathbb{D}$ with positive real part. Then there exists a unique real number α and a finite measure σ on the circle $\mathbb{T}$ such that*

$$G(w) = i\alpha + \int_0^{2\pi} d\sigma(\theta)\ \frac{e^{i\theta}+w}{e^{i\theta}-w}, \quad w \in \mathbb{D}. \tag{1.4.7}$$

Proof: We first note that since G is analytic in the disk we can write it as $G(w) = u(w) + iv(w)$, a sum of two harmonic functions. By the assumption on the real part of G, we see that u is a positive harmonic function in the disk. So if we show that u is given by

$$u(w) = \int_0^{2\pi} d\sigma(\theta)\mathrm{Re}\left(\frac{e^{i\theta}+w}{e^{i\theta}-w}\right),$$

with σ a positive finite measure, then the theorem follows, since the harmonic conjugate of u is fixed uniquely up to a constant, which we could determine at the point 0, as claimed in the theorem. We also note that the integrand is just the Poisson kernel

$$P_r(\theta-\phi) = \mathrm{Re}\left(\frac{e^{i\theta}+w}{e^{i\theta}-w}\right) = \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}, \quad w = re^{i\phi}.$$

Therefore the representation for u follows from the next lemma, which gives a representation for positive harmonic functions on the disk. □

Lemma 1.4.4. *Let h be a positive harmonic function in $\mathbb{D}$. Then there is a finite positive measure σ on the circle $\mathbb{T}$ such that*

$$h(re^{i\theta}) = (P_r * \sigma)(e^{i\theta}), \quad 0 \le r < 1, \quad \theta \in [0, 2\pi).$$

Proof: First assume that h is positive and harmonic in a disk of slightly large radius $\delta > 1$, that is in $\mathbb{D}_\delta = \{z : |z| < \delta\}$. Then h is bounded on $\overline{\mathbb{D}}$ and so in particular on $\mathbb{T}$. Let us determine (its harmonic conjugate) v so that $v(0) = 0$, and the function $f(z) = h(z) + iv(z)$ is analytic and has a power series representation

$$f(z) = h(z) + iv(z) = \sum_{n=0}^{\infty} a_n z^n$$

which converges absolutely and uniformly on the closed disk $\overline{\mathbb{D}}$. Using this, the real part of f, namely $h(z) = (f(z) + \overline{f(z)})/2$, can be written as

$$h(z) = \sum_{n=-\infty}^{\infty} A_n r^{|n|} e^{in\theta}, \quad z = re^{i\theta}.$$

This shows that on the circle $\mathbb{T}$, we have

$$h(e^{i\theta}) = \sum_{n=-\infty}^{\infty} A_n e^{in\theta},$$

so that A_n are the Fourier coefficients of the (bounded and hence in $L^1(\mathbb{T})$) function $h(e^{i\theta})$ on the circle. Writing this fact explicitly we have

$$h(re^{i\theta}) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} \frac{1}{2\pi} \int_0^{2\pi} d\phi \;\; h(e^{i\phi}) e^{-in\phi} = \int_0^{2\pi} P_r(\theta-\phi) d\sigma(\phi), \tag{1.4.8}$$

where we have set $d\sigma(\phi) = \frac{1}{2\pi} h(e^{i\phi}) d\phi$ which is a positive finite measure, in view of the fact that h is assumed to be positive. Now going to the case when h is positive harmonic in $\mathbb{D}$, we consider the sequence $h_\delta(z) = h(z/\delta), \delta > 1$, of positive functions harmonic in $\mathbb{D}_\delta$, and obtain a collection of finite measures σ_δ associated with each of these functions h_δ. We also note from the Equation (1.4.8) that $h(0) = h_\delta(0) = \sigma_\delta(\mathbb{T})$; therefore all the measures σ_δ are uniformly bounded. Using this fact and the fact that the unit ball of the dual of the space of continuous functions on $\mathbb{T}$ is weak*-compact, we can obtain a sequence σ_{δ_n} of measures on $\mathbb{T}$ that converge weakly (in the weak* sense) to a measure σ.

Using this limiting measure σ, the representation of Equation (1.4.8) is seen to be valid for h, even in the case when it is harmonic only in $\mathbb{D}$, proving the lemma. $\square$

As a corollary we have:

Corollary 1.4.5 *Let μ be a measure and let F_μ be its Borel transform. Let $d\mu_\epsilon = Im(F_\mu(x+i\epsilon))\ dx$ be a family of measures. Then $w - lim\ \ \mu_\epsilon = \mu$ as $\epsilon \to 0$.*

Proof: We first note that whenever μ is a finite measure, μ_ϵ is also a finite measure for any $\epsilon > 0$. This is because $\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\epsilon}{(x-y)^2+\epsilon^2} d\mu(y)dx < \infty$, by an application of Fubini's theorem.

Since $Im(F_\mu)$ is a positive harmonic function on $\mathbb{C}^+$, we will prove this statement in the setting of the disk $\mathbb{D}$, as done earlier in the proof of Theorem 1.4.2. We have to show that when h is a positive harmonic function on $\mathbb{D}$, the sequence of measures $\frac{1}{2\pi}h(re^{i\theta})d\theta$ converges to the measure σ, given in Lemma 1.4.4 (note that in the lemma we only showed that for some subsequence r_n the weak limit of $\frac{1}{2\pi}h(r_n e^{i\theta})d\theta$ is σ). Let f be a bounded continuous function on $\mathbb{T}$. Then we have to show that

$$\frac{1}{2\pi}\int_{\mathbb{T}} f(\phi)\ h(re^{i\phi})d\phi \to \int_{\mathbb{T}} f(\phi)\ d\sigma(\phi), \quad \text{as} \quad r \to 1.$$

We set, for $0 < \delta < 1$,

$$S_j(\delta) = \begin{cases} \{\phi \in [0, 2\pi) : \cos(\phi) \geq 1 - \delta\}, & j = 1, \\ \{\phi \in [0, 2\pi) : \cos(\phi) < 1 - \delta\} & j = 2. \end{cases}$$

From Equation (1.4.8), we get, by interchanging integrals and using the fact that $\frac{1}{2\pi}\int_0^{2\pi} P_r(\phi)d\phi = 1$,

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} f(\phi)h(re^{i\phi})\ d\phi - \int_0^{2\pi} f(\theta)\ d\sigma(\theta) & \\ &= \int_0^{2\pi} d\sigma(\theta)\left[\frac{1}{2\pi}\int_0^{2\pi} P_r(\phi)f(\phi+\theta)d\phi - f(\theta)\right] \\ &= \int_0^{2\pi} d\sigma(\phi)\left[\frac{1}{2\pi}\int_0^{2\pi} P_r(\phi)\left(f(\phi+\theta) - f(\theta)\right)d\phi\right] \\ &\leq \frac{1}{2\pi}\left(\int_{S_1(\delta)} + \int_{S_2(\delta)} |P_r(\phi)||f(\phi+\theta) - f(\theta)|\ d\phi\right) \\ &\leq \epsilon + \frac{2\|f\|_\infty}{2\pi}\int_{S_2(\delta)} |P_r(\phi)|d\phi. \end{aligned} \tag{1.4.9}$$

The first term was estimated using the fact that $P_r(\cdot)$ is non-negative function whose integral over $\mathbb{T}$ is 2π together with the uniform uniform continuity of f

on the compact set $\mathbb{T}$, which is valid by the assumption that f is continuous. As δ goes to zero the set $S_1(\delta)$ shrinks to the point $\{0\}$ and consequently $f(\theta+\phi)-f(\theta)$ goes to zero uniformly in θ; therefore our choice of δ can be based on a given $\epsilon > 0$ to get this bound. The boundedness of f and the fact that

$$\lim_{r\to 1}\int_{S_2(\delta)} |P_r(\phi)|\, d\phi = 0,$$

for any $\delta > 0$, are used to conclude that the second term goes to zero as r goes to one. The ϵ being arbitrary in the above argument, the result follows. □

Next we shall prove that functions with positive imaginary parts defined in the upper half plane have boundary values almost everywhere on the real axis.

Theorem 1.4.6. *Let $F : \mathbb{C}^+ \to \mathbb{C}^+$ be an analytic function. Then the boundary values $\lim_{\epsilon\downarrow 0} F(x+i\epsilon)$ exist and are finite for almost every x.*

Proof: Without loss of generality we assume that both the real and imaginary parts of F are positive harmonic. Otherwise we could consider a fractional linear transformation of $\mathbb{C}^+$, call it ψ, onto an open disk of unit radius contained in the positive quadrant and consider $\psi \circ F$ (whose real and imaginary parts are positive, by the choice of ψ. Since ψ and its inverse are homomorphic, existence of limits is preserved under composition with ψ and/or its inverse).

So we consider a positive harmonic F on $\mathbb{C}^+$ and suppose μ is the measure representing F, as per the Theorem 1.4.3 transported to the case of $\mathbb{C}^+$. We shall show the existence of the boundary values on $[-1,1]$, the argument is similar for any other interval. We then split the measure μ into two parts, $\mu = \mu_1 + \mu_2$, μ_1 supported on $[-2,2]$ and μ_2 supported on the complement. Then it is clear that $[-1,1]$ being away from the support of μ_2, the limits exist finitely almost everywhere on $[-1,1]$ for its Borel transform, so we have to consider only the case of the Borel transform of μ_1. Therefore without loss of generality assume that μ has compact support in $[-1,1]$. Denote the distribution function of μ by Φ, as per the Definition 1.1.3.

The derivative of Φ exists finitely almost everywhere in $(-1,1)$, by Theorem 1.1.8, with respect to the Lebesgue measure. Then the theorem follows from the following proposition. □

Proposition 1.4.7. *Let F be a positive harmonic function on $\mathbb{R}$ such that its representing measure μ is supported in $[-1,1]$. Then, for any x in $(-1,1)$, the limit $F(x+i0)$ exists and is finite whenever the derivative of Φ, the distribution function of μ, exists and is finite at x.*

Proof: Since F is positive harmonic, we can take it to be the imaginary part of the Borel transform F_μ of some μ which integrates $1/(1+x^2)$. Since this μ is supported on a compact set by assumption, it is finite. So we can get the proposition from Theorem 1.3.2 (2), by taking $\alpha = 1$ and $\psi(x) = \frac{1}{1+x^2}$. With this choice, $\mathrm{Im}F_\mu(x + i\epsilon) = \frac{1}{\epsilon}(\psi_\epsilon * \mu)(x)$. □

Corollary 1.4.8 *Suppose μ is a non-zero finite complex measure with finite total variation $|\mu|$ and let F_μ denote its Borel transform. Then the boundary values of $F_\mu(x + i0) = \lim_{\epsilon \to 0} F_\mu(x + i\epsilon)$ exist and $0 < |F_\mu(x + i0)| < \infty$, for a.e. x (with respect to Lebesgue measure).*

Proof: Since μ has finite total variation, we can write it as a sum of four finite positive measures $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$. Since the Borel transform F_μ is linear in μ, we have

$$F_\mu = F_{\mu_1} - F_{\mu_2} + iF_{\mu_3} - iF_{\mu_4},$$

so applying the above theorem to each of the right-hand sides we have the finiteness result.

To prove that the boundary values are non-zero, we first note that under the assumption on μ, $|F_\mu|$ is bounded on every half plane $\{z : \mathrm{Im}(z) \geq \epsilon > 0\}$. Consider the biholomorphic maps w and z defined in the proof of Theorem 1.4.2, we get

$$\int_0^{2\pi} \ln^+ |w \circ F_\mu \circ z|(re^{i\theta}) \ \ d\theta < \infty,$$

for any $r < 1$. Functions $g : \mathbb{D} \to \mathbb{D}$ satisfying $\int_0^{2\pi} \ln^+ |g|(re^{i\theta})d\theta < \infty$, are known as functions of bounded characteristic and by Theorem 2.2 of [79], it follows that

$$g(z) \neq 0, \quad \text{for a.e. } z \quad \text{on} \quad |z| = 1.$$

From this it follows that $F_\mu(x + i0)$ itself is non-zero almost everywhere. □

Corollary 1.4.9 *Let μ, ν be probability measures. If $F_\mu(x+i0) = F_\nu(x+i0)$, for x in a set of positive Lebesgue measure, then $F_\mu = F_\nu$ on $\mathbb{C}^+$ and hence $\mu = \nu$.*

Proof: In view of Corollary 1.4.8, the function $F_\mu - F_\nu$, which is the Borel transform of the signed measure $\mu - \nu$, cannot have vanishing boundary values on a set of positive Lebesgue measure unless $\mu - \nu = 0$. Hence the result is valid. □

We also have a converse of the Proposition 1.4.7, leading to the Lebesgue decomposition of μ, namely:

Theorem 1.4.10. *Let F_μ be the Borel transform of a positive measure μ such that the function $1/(1 + x^2)$ is integrable with respect to μ. Then, the distribution function Φ_μ of μ has a finite derivative at x whenever the limit $\lim_{\epsilon \downarrow 0} \mathrm{Im}(F_\mu(x + i\epsilon)$ exists finitely; further $\pi(D\Phi_\mu)(x) = \mathrm{Im}F_\mu(x + i0)$.*

Proof: We may assume that we are dealing with a probability measure μ, by first splitting the measure into two pieces $\mu_1 + \mu_2$, as we did in the proof of Theorem 1.4.6. We can choose μ_1 to have compact support containing x and μ_2 to have support at a finite distance from x. Then μ_2 does not contribute to the $\mathrm{Im}F_\mu(x + i0)$ and we may take μ_1 to be a probability by normalizing if necessary. The rest of the proof is a direct application of Theorem 1.3.2(3) and its converse given in Theorem 1.3.6 with $\psi(x) = (1 + x^2)$ for the case $\alpha = 1$. □

Since the derivative of the distribution function agrees with the density of the absolutely continuous part of a measure μ almost everywhere, one immediate corollary of the above theorem and Theorem 1.1.8 is:

Corollary 1.4.11 *Suppose μ is a measure. Then it is purely singular iff*

$$\lim_{\epsilon \longrightarrow 0} \mathrm{Im}F_\mu(x + i\epsilon) = 0,$$

for almost every x with respect to the Lebesgue measure.

An application of Theorem 1.3.2(3) and its converse given in Theorem 1.3.6 is the following proposition for Borel transforms, as in the case of Theorem 1.4.10, where we set

$$(c_\alpha H)(x) = \lim_{\epsilon \longrightarrow 0} \epsilon^{1-\alpha} H(x + i\epsilon),$$

when H is harmonic.

Proposition 1.4.12. *Let μ be a measure with $\int_{\mathbb{R}} \frac{1}{1+x^2} d\mu(x) < \infty$ and let F_μ and Φ_μ denote its Borel transform and its distribution function, respectively. Consider an $0 < \alpha \leq 1$ fixed; then the limits $(c_\alpha \mathrm{Im}F_\mu)(x + i0)$ exists finitely iff the limits $(d_\mu^\alpha)(x)$ exist finitely, in which case we have*

$$(c_\alpha \mathrm{Im}F_\mu)(x + i0) = d_\mu^\alpha(x) \int_0^\infty dy \frac{\alpha 2^\alpha y^{\alpha-1}}{1 + y^2},$$

with $d_\mu^\alpha(x)$ given in Equation (1.3.2).

The above proposition may not be of use in some cases, in view of the statement following Theorem 1.1.8. Therefore we present the following criteria to capture such measures. We start by defining some quantities. Let μ be a measure and F_μ its Borel transform. Then recall the definition of D_μ^α and $C_{\mu,\psi}^\alpha$ from Theorem 1.3.6. When $\psi(x) = 1/(1+x^2)$ we call $C_{\mu,\psi}^\alpha(x)$ as simply $C_\mu^\alpha(x)$ without reference to the function ψ. Note that if F_μ is the Borel transform of μ, then

$$C_\mu^\alpha(x) = \overline{\lim}_{\epsilon \downarrow 0} \epsilon^{1-\alpha} \mathrm{Im}F_\mu(x + i\epsilon).$$

In the following two theorems h^α is the α-dimensional Hausdorff measure (see Definition 1.1.3).

Theorem 1.4.13 (del Rio–Jitomirskaya–Last–Simon). *Let μ be a measure. We set $T_\alpha = \{x : C_\mu^\alpha(x) = \infty\}$, $0 \leq \alpha \leq 1$ and denote by χ_α the indicator function of T_α. We define $\mu_{\alpha s} = \chi_\alpha d\mu$ and $d\mu_{\alpha c} = (1-\chi_\alpha)d\mu$. Then $d\mu_{\alpha s}$ is singular with respect to the measure h^α and $\mu_{\alpha c}$ is continuous with respect to h^α. Further, μ restricted to $\{x : 0 < C_\mu^\alpha(x) < \infty\}$ is absolutely continuous with respect to h^α.*

As a corollary of this theorem one also has:

Corollary 1.4.14 *A measure has exact dimension $\alpha \in [0,1)$ if and only if*

1. *For any $\beta > \alpha$, $D_\mu^\beta(x) = \infty$ for a.e. x (w.r.t.μ).*
2. *For any $\beta < \alpha$, $D_\mu^\beta(x) = 0$ for a.e. x (w.r.t.μ).*

Theorem 1.4.15 (del Rio–Jitomirskaya–Last–Simon). *Consider a measure μ and let C_μ^α and D_μ^α be defined as above, for each $0 < \alpha < 1$. Then either $C_\mu^\alpha(x)$ and $D_\mu^\alpha(x)$ are both finite or both infinite for a.e. x.*

Proof: Consider the function $\psi(x) = \frac{1}{1+x^2}$ and apply the Theorem 1.3.6 to get the proof. □

The following theorem is a collection of well-known results.

Theorem 1.4.16. *Let μ be a measure that integrates the function $1/(1+x^2)$ and let F_μ be its Borel transform. Then,*

1. *$\frac{1}{\pi}\mathrm{Im}(F_\mu(x+i0)) = \frac{d\mu_{ac}}{dx}(x)$ for almost every x.*
2. *The singular part μ_s is supported on the set*

$$\{x : \lim_{\epsilon\downarrow 0} \mathrm{Im}(F_\mu(x+i\epsilon)) = \infty\}.$$

3. *The point mass of μ at x, if any, is obtained by*

$$\mu(\{x\}) = \lim_{\epsilon\downarrow 0} \epsilon\mathrm{Im}(F_\mu(x+i\epsilon)).$$

Proof: The first item follows immediately from Theorem 1.4.10. The last item is an application of Theorem 1.3.2(1) with $\psi(x) = 1/(1+x^2)$, by assuming μ to be a probability as done in the proof of Theorem 1.4.10. Now we turn to the proof of the second item. By Theorem 1.1.8, of De la Valée Poussin, μ is not supported on the set where the quantity $d_\mu^1(x)$ does not exist finite or infinite. On the other hand, $\{x : d_\mu^1(x) < \infty\}$, gets measure zero from the singular part μ_s of μ. Therefore it is enough to show that $\lim_{a\to 0}\mathrm{Im}F_\mu(x+i0) = \infty$ whenever $\lim_{\epsilon\to 0}\frac{\mu((x-\epsilon,x+\epsilon))}{\epsilon} = \infty$, to prove the result. We note that $\mathrm{Im}F_\mu(x+ia) = \frac{1}{a}(\psi_a * \mu)(x)$ for $\psi(x) = \frac{1}{1+x^2}$. By using Equation 1.3.4, we see the estimate,

$$\frac{1}{a}(\psi_a * \mu)(x) \geq \int_0^1 \frac{4y^2}{1+y^2}\left(\frac{\Phi_\mu(x+\epsilon y) - \Phi(x-\epsilon y)}{2y}\right).$$

This immediately implies the required statement. □

Based on the above theorem there are a few more criteria for identifying the components of a measure.

Theorem 1.4.17 (Simon). *Suppose μ is a measure that integrates $1/(1+x^2)$ and let F_μ be its Borel transform, and let (c,d) be a bounded interval.*

1. *Let $p > 1$, be given and suppose*
$$\sup_{\epsilon>0} \int_c^d dx \quad |\mathrm{Im}(F_\mu)(x+i\epsilon)|^p < C_p.$$
Then μ is purely absolutely continuous on (c,d) and $d\mu_{ac}/dx$ is in $L^p((c,d))$. Further for any compact subset $K \subset (c,d)$, $\frac{1}{\pi}\mathrm{Im}F_\mu(x+i\epsilon)dx$ converges to $d\mu_{ac}/dx$ in $L^p(K)$.
2. *For $p \in (0,1)$ we have $\lim_{\epsilon\downarrow 0} \int_c^d |\mathrm{Im}(F_\mu)(x+i\epsilon)|^p dx = \int_c^d |d\mu_{ac}/dx|^p dx$.*

Proof: This theorem is a corollary of Theorem 1.3.4 by taking $\psi(x) = 1/(1+x^2)$ and noting that $\mathrm{Im}F_\mu(x+i\epsilon) = \frac{1}{\epsilon}(\psi_\epsilon * \mu)(x)$. □

As a corollary of the above theorem we present the following criteria for the absence of some of the components of a measure.

Corollary 1.4.18 (Simon) *Suppose μ is a measure that integrates $1/(1+x^2)$. Then we have:*

1. *The pure point part of μ in (c,d) is absent whenever for some sequence $\epsilon_n \downarrow 0$,*
$$\underline{\lim}_{\epsilon_n\downarrow 0} \epsilon_n \int_c^d \left|\frac{1}{\pi}\mathrm{Im}F_\mu(x+i\epsilon_n)\right|^2 dx = 0.$$
2. *The absolutely continuous part of μ is absent in (c,d) if for some sequence $\epsilon_n \downarrow 0$, we have*
$$\underline{\lim}_{\epsilon_n\downarrow 0} \int_c^d \left|\frac{1}{\pi}\mathrm{Im}F_\mu(x+i\epsilon_n)\right|^p dx = 0, \quad \textit{for some } 0 < p < 1.$$

Proof: Since (c,d) is a bounded interval, μ is finite on (c,d). Now the proof follows from the Corollary 1.3.5, by setting $\psi(x) = \frac{1}{1+x^2}$. □

1.5 Gesztesy–Krein–Simon ξ Function

We now introduce the ξ function. It is intimately related to the spectral properties and will be used in some contexts. We note that whenever F is the Borel transform of a measure that integrates $1/1+x^2$, its argument takes values in $(0,\pi)$ in $\mathbb{C}^+$. Therefore the principle branch of the logarithm of F is well-defined and has well-defined boundary values almost everywhere on the real axis, being itself a function of the same type. Therefore its imaginary part is an essentially bounded function on the line by the maximum principle and gives rise to an absolutely continuous measure. This function plays a role in several areas of spectral and inverse spectral theories. We shall call it the Gesztesy–Krein–Simon ξ (GKS ξ) function.

Definition 1.5.1. Let μ be a measure that integrates $1/(1+x^2)$ and let F be its Borel transform. Then define the Gesztesy–Krein–Simon ξ function by

$$\xi(x) = \lim_{\epsilon \downarrow 0} \frac{1}{\pi} \mathrm{Im}(\ln F_\mu(x+i\epsilon)),$$

for almost every x in $\mathbb{R}$.

Theorem 1.5.2 (Gesztesy–Simon). *Let μ be a measure and consider the associated GKS ξ function. Then for any interval $[a,b]$,*

1. *The measure μ is purely singular iff $\xi(x)$ takes values in $\{0,1\}$ for almost every x in $[a,b]$.*
2. *μ has non-empty absolutely continuous component iff $\xi(x)$ takes values in $(0,1)$ for every x in some subset of positive Lebesgue measure.*

Proof: (1) First note that $\xi(x) = 0$ or 1 almost everywhere in $[a,b]$ iff the argument of $F(x+i0)$ is zero or π almost everywhere in $[a,b]$. This is valid iff almost everywhere $F(x+i0)$, is real. This in turn is valid, by Theorem 1.4.16)(1), iff the absolutely continuous part of μ is zero.

(2) As in the above argument $0 < \xi(x) < 1$ on a set S of positive Lebesgue measure iff $0 < \frac{1}{\pi}\mathrm{Im}(F)(x+i0)$ on S, and this is valid iff (again by Theorem 1.4.16(1)) $d\mu_{ac}(x)/dx > 0$ almost everywhere on S. $\square$

1.6 Notes

Section 1.1

In this section the statements on measures is available in standard books like those of Rudin [167], Saks [169], Chandrasekharan [41]. The facts regarding functions of locally bounded variation can be found in Chapters V.3 and in Chapter III of [41]. (These are stated in [41] only for functions of bounded variation, but the arguments go through for finite intervals (a,b)). The integration by parts formula, Equation (1.1.7), is obtained by using Fubini's theorem (see Section IV of [41]) for the product measure $M = dm_f dm_g$ on the rectangle $(a,b] \times (a,b]$ and using the fact that $M((a,b] \times (a,b]) = M(A_1) + M(A_2)$, where $A_1 = \{(x,y) : x \leq y\}$ and A_2 its complement in $(a,b] \times (a,b]$.

The properties of Hausdorff measures can be found in Rogers [164]. Decomposition of a measure with respect to Hausdorff measures was first used in the spectral theory of selfadjoint operators by Y. Last in [141]. This paper also has different types of decomposition of a finite positive measure, in analogy with the Lebesgue decomposition. After this insightful paper there was a lot of progress in the spectral theory of Schrödinger operators with singular continuous spectra. Especially noteworthy are explicit operators with various α-dimensional singular spectra and the results on stability of spectra under some classes of perturbations, similar to the stability of absolutely continuous spectra under trace class perturbations.

The proof of the theorem of de La Vallée Poussin (Theorem 1.1.8) is given in Saks ([169], Theorem 9.1 and Theorem 9.6 Chapter IV). Examples of measures whose supports have Hausdorff dimension smaller than one which are supported on the sets W_α stated after Theorem 1.1.8 are given in del Rio–Jitomirskaya–Last–Simon [60].

The statement in the Theorem 1.1.10 is the content of Theorem 67 of Rogers [164] (see also Last [141] Theorem 4.1). The procedure followed often for finding components of measures is to determine the set of points where $D_\mu^\alpha(x)$ is zero, finite positive, or infinite.

Section 1.2

The material in this section is based on Donoghue [75], Stein [182], Stein–Weiss [183]. In particular the Theorem 1.2.7 on the convergence is given in Theorems 1.18 and 1.25 of Stein–Weiss [183].

Section 1.3

This material is from the work of Jensen–Krishna [101], as yet unpublished. The proofs of the theorems here are adapted from those for the case of the Borel transforms given in the works of Simon [178] and del Rio–Jitomirskaya–Last–Simon [60].

The theorems in this section are generalisations of theorems for the Borel transform, since the wavelet transform associated with the function $\psi(x) = 1/(1+x^2)$ is the Borel transform, with $\mathbb{R}^+ \times \mathbb{R}$ identified with the upper half plane.

The Tauberian theorem of Wiener is the following. Suppose f is a bounded function on $\mathbb{R}$ and K is an $L^1(\mathbb{R})$ function such that $\widehat{K}$ does not vanish anywhere. Then whenever the limit

$$\lim_{s\to 0} \int_{\mathbb{R}} K(t)f(t-s)dt = C \int_{\mathbb{R}} K(x)$$

exists, the following limit exists and the is valid for every function $G \in L^1(\mathbb{R})$,

$$\lim_{s\to 0} \int_{\mathbb{R}} G(t)f(t-s)dt = C \int_{\mathbb{R}} G(x).$$

The proof of Theorem 1.3.6 is on the lines given by Donoghue, Theorem III, Chapter IV [74].

Section 1.4

Most of the material here is standard and well known in the literature; it is presented here for the sake of completeness, see for example the books of Donoghue [74], Katznelson [109], Nevanlinna [149] and Duren [79]. The functions that map the upper half plane to itself are called Herglotz functions and their collection is termed the Nevanlinna class in the literature. The material on the identification of the components of a measure via its Borel transform are from the works of Simon [178] and del Rio–Jitomirskaya–Last–Simon [60]. In addition to the Theorem 1.4.16 there are further criteria that could be used based on context as given in Simon [178]. Though some of the theorems here are presented as special cases of those on wavelet transforms, these theorems are the first to appear in the literature.

Section 1.5

The material in this section is from the work of Gesztesy–Simon [87]. Spectral shift functions were introduced by M.G. Krein [132], (an extended overview of this theory can be found in Yafaev [192]) to study perturbations of trace class type for Schrödinger operators in the context of scattering theory and spectral theory. In the context of inverse spectral theory this was further advanced and a general formulation of the spectral shift function in the form we have given here is given by Gesztesy–Simon [87], who introduced this function and gave its connection with the Krein spectral shift function in some cases. They also showed its usefulness in both the spectral theory and the inverse spectral theory of one-dimensional Schrödinger operators.

2

Selfadjointness and Spectrum

2.1 Selfadjointness

The theory of selfadjoint operators is very well explained in many textbooks. Therefore we restrict our summary to those facts which are used in what follows here. This section consists mainly of a series of definitions.

2.1.1 Linear Operators and Their Inverses

Definition 2.1.1. Let $\mathfrak{H}$ be a separable complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, which is always assumed to be linear in the second vector and conjugate linear in the first vector. A linear operator from $\mathfrak{H}$ to $\mathfrak{H}$ is a pair $\{\text{dom }(A), A\}$ where the domain of A, i.e., dom (A), is a linear subspace of $\mathfrak{H}$, and A is a linear map from $\mathfrak{H}$ into $\mathfrak{H}$, i.e., for any $f \in \text{dom }(A)$ $Af \in \mathfrak{H}$ and for $g \in \text{dom }(A), \alpha \in \mathbb{C}$:

$$A(f + \alpha\, g) \ = Af + \alpha\, Ag.$$

Very often the domain, dom (A), belonging to A is obvious. In that case we write only A instead of $\{\text{dom}\,(A), A\}$.

Definition 2.1.2. The range of A is the image of its domain, i.e.,

$$\text{ran }(A) \ = \{g \in \mathfrak{H}\,,\ g = Af,\ f \in \text{dom }(A)\}.$$

The null space or kernel of A is defined by

$$\text{null }(A) \ = \{f \in \text{dom }(A),\ Af = \theta\}.$$

θ denotes the zero vector in $\mathfrak{H}$.

Definition 2.1.3. A is called invertible if null $(A) = \{\theta\}$. In this case A^{-1}, the inverse of A, is defined by

$$\begin{aligned} &(i) \ \ \mathrm{dom}\ (A^{-1}) = \mathrm{ran}\ (A), \\ &(ii) \ \ A^{-1}(Af) = f, \quad \forall f \in \mathrm{dom}\ (A). \end{aligned}$$

Definition 2.1.4. Two linear operators $\{\mathrm{dom}(A_1), A_1\}, \{\mathrm{dom}(A_2), A_2\}$ in $\mathfrak{H}$ are equal if

$$\begin{aligned} &(i) \ \ \mathrm{dom}\ (A_1) = \mathrm{dom}\ (A_2), \\ &(ii) \ \ A_1 f = A_2 f, \quad \forall f \in \mathrm{dom}\ (A_1). \end{aligned}$$

In this case we write $A_1 = A_2$.

The operator $\{\mathrm{dom}\ (A_2), A_2\}$ is called an extension of $\{\mathrm{dom}\ (A_1), A_1\}$, or $\{\mathrm{dom}\ (A_1), A_1\}$ is called a restriction of $\{\mathrm{dom}\ (A_2), A_2\}$ if

$$\begin{aligned} &(i) \ \ \mathrm{dom}\ (A_1) \subseteq \mathrm{dom}\ (A_2) \\ &(ii) \ \ A_1 f = A_2 f, \quad \forall f \in \mathrm{dom}\ (A_1). \end{aligned}$$

In this case we write $A_1 \subseteq A_2$ or $A_2 \supseteq A_1$.

2.1.2 Closed Operators

Definition 2.1.5. A linear operator $\{\mathrm{dom}\ (A), A\}$ in $\mathfrak{H}$ is called closed if one of the following properties is satisfied:

(i) Assume a sequence $(f_n)_{n\in\mathbb{N}},\ f_n \in \mathrm{dom}\,(A)$ and assume that the limits

$$\begin{aligned} \lim_{n\to\infty} f_n &= f \in \mathfrak{H}, \\ \lim_{n\to\infty} Af_n &= g \in \mathfrak{H} \end{aligned}$$

exist. Then $f \in \mathrm{dom}\,(A)$ and $Af = g$.

(ii) The graph of A, a subset of $\mathfrak{H} \times \mathfrak{H}$ defined by

$$\mathrm{graph}\ (A) \ = \{(f, Af),\ f \in \mathrm{dom}\ (A)\},$$

is closed in $\mathfrak{H} \times \mathfrak{H}$ with respect to the usual norm in $\mathfrak{H} \times \mathfrak{H}$ (see the Remark 2.1.6).

Remark 2.1.6. (i) If $\{\mathrm{dom}\ (A), A\}$ is a closed operator, then dom (A) is a Banach space with respect to

$$\|f\|_A = \|f\| + \|Af\|\ , \quad f \in \mathrm{dom}\ (A).$$

(ii) If $(\lambda - A)$ is injective for some $\lambda \in \mathbb{C}$, then A is closed iff $(\lambda - A)^{-1}$ is closed.

(iii) The graph (A) is a Hilbert space with respect to the inner product

$$\langle (f_1, g_g),\ (f_2, g_2)\rangle = \langle f_1, f_2\rangle + \langle g_1, g_2\rangle.$$

The corresponding norm, sometimes called graph norm, is

$$\|(f,g)\| = \sqrt{\|f\|^2 + \|g\|^2}.$$

The topology induced by this norm is equivalent to the topology induced above by

$$\|(f,g)\|_{\mathfrak{H}\times\mathfrak{H}} = \|f\| + \|g\|.$$

Definition 2.1.7. An operator $\{\mathrm{dom}\,(A), A\}$ is called closable if there is a closed operator $\{\mathrm{dom}\,(B), B\}$ with $A \subseteq B$. The smallest closed extension of A, denoted by $\bar{A}$, is called the closure of A. An operator $\{\mathrm{dom}\,(A), A\}$ is closable if and only if $f_n \in \mathrm{dom}\,(A),\ f_n \to \theta,\ Af_n \to g$ implies $g = \theta$. The closure $\{\mathrm{dom}(\bar{A}), \bar{A}\}$ is then given by

$$\mathrm{dom}\ (\bar{A}) = \{f \in \mathfrak{H} : \exists (f_n) \in \mathrm{dom}(A),\quad \lim_{n\to\infty} f_n = f \quad \text{and} \quad \lim_{n\to\infty} Af_n \quad \text{exists}\}$$

and $\bar{A}f = \lim_{n\to\infty} Af_n$. Clearly we have

$$\mathrm{graph}\ (\bar{A}) = \overline{\mathrm{graph}\ (A)}.$$

If B is closed, a subset $\mathfrak{D} \subset \mathrm{dom}\,(B)$ is called a core of B if $\overline{B|_{\mathfrak{D}}} = B$.

Definition 2.1.8. A linear operator $\{\mathrm{dom}\,(B), B\}$ is bounded if there is a positive constant b such that

$$\|Bf\| \leq b\,\|f\|,\ \forall f \in \mathrm{dom}\,(B).$$

Its norm is given by

$$\|B\| \ = \sup\left\{\frac{\|Bf\|}{\|f\|},\ f \in \mathrm{dom}\,(B), f \neq \theta\right\}.$$

With $\mathfrak{B}(\mathfrak{H})$ we denote the set of all bounded operators defined on $\mathfrak{H}$, i.e., with $\mathrm{dom}\,(B) = \mathfrak{H}$.

Any bounded linear operator B has a unique extension $\bar{B}$ with $\mathrm{dom}\,(\bar{B}) = \overline{\mathrm{dom}\,(B)}$. In particular, if $\mathrm{dom}\,(B)$ is dense in $\mathfrak{H}$, $\bar{B}$ belongs to $\mathfrak{B}(\mathfrak{H})$.

In what follows we will not distinguish between a bounded operator B and its bounded extension $\bar{B} \in \mathfrak{B}(\mathfrak{H})$. If we consider bounded linear operators we mean operators in $\mathfrak{B}(\mathfrak{H})$. If an operator B is defined on the whole space $\mathfrak{H}$ and if B is closed, i.e., $\bar{B} = B$, then B is bounded (closed graph theorem).

Proposition 2.1.9. *Let* $\{\mathrm{dom}\,(A), A\}$ *be a closed operator and take* $B \in \mathfrak{B}(\mathfrak{H})$. *Then* AB *is a closed operator with* $\mathrm{dom}\,(AB) = \{f \in \mathfrak{H}, Bf \in \mathrm{dom}\,(A)\}$, *and* BA *with* $\mathrm{dom}\,(BA) = \mathrm{dom}\,(A)$ *is closed if* B^{-1} *exists and belongs to* $\mathfrak{B}(\mathfrak{H})$.

In spectral theory a special role is played by the ideal of compact operators and its subideals of Hilbert–Schmidt and trace class operators.

Definition 2.1.10. A bounded operator B in $\mathfrak{H}$ is called compact if for every bounded sequence $\{f_n\}$ in $\mathfrak{H}$, the sequence $\{Bf_n\}$ contains a Cauchy sequence.

In other words B is compact if and only if $\{Bf_n\}$ converges strongly whenever $\{f_n\}$ converges weakly.

Definition 2.1.11. A compact operator B is called a Hilbert–Schmidt operator if for a complete orthonormal family $\{\phi_i\}$ in $\mathfrak{H}$ the expression

$$\|B\|_{HS} = \left(\sum_{j=1}^{\infty} \|B\phi_j\|^2 \right)^{\frac{1}{2}}$$

is finite. The norm $\|\cdot\|_{HS}$ is called the Hilbert–Schmidt norm.

The set of Hilbert–Schmidt operators forms a new Hilbert space with the scalar product

$$\langle A, B\rangle = \mathrm{trace}(A^*B) = \sum_{j=1}^{\infty} \langle A\phi_j, B\phi_j\rangle.$$

(See also Lemma 3.6.19). The adjoint A^* of A is defined below.

Let $(E, \mathfrak{B}_E, m)$ be a σ-finite measure space and set $\mathfrak{H} = L^2(E, m)$. Let $b : E \times E \to \mathbb{C}$ be the measurable kernel of the integral operator

$$(Bf)(x) = \int_E b(x, y)f(y)dm(y).$$

Then B is a Hilbert–Schmidt operator if and only if

$$\|Bf\|_{HS}^2 = \int_E |b(x, y)|^2 dm(x)dm(y)$$

is finite.

Trace class operators will be defined in Section 2.7.

2.1.3 Adjoint and Selfadjoint Operators

Definition 2.1.12. An operator $\{\mathrm{dom}\,(A), A\}$ is called densely defined if $\mathrm{dom}\,(A)$ is dense in $\mathfrak{H}$. Let A be a densely defined operator in the Hilbert space $\mathfrak{H}$. Its adjoint operator $\{\mathrm{dom}\,(A^*), A^*\}$ is defined as follows: $\mathrm{dom}\,(A^*)$ is the set of all $g \in \mathfrak{H}$ for which a vector $h \in \mathfrak{H}$ exists such that

$$\langle g, Af\rangle = \langle h, f\rangle, \forall f \in \mathrm{dom}\,(A).$$

For each such g we define the operator A^* by

$$A^*g = h.$$

Remark 2.1.13. (a) Obviously, $g \in \mathrm{dom}\,(A^*)$ if $|\,\langle g, Af\rangle\,| \leq\; c_g\;\|f\|$.
(b) The graph of A^* is closed.
(c) The operator A is closable iff $\mathrm{dom}\,(A^*)$ is dense in $\mathfrak{H}$. In this case $\bar{A} = A^{**}$. Moreover, if A is closable, then $A^* = A^{***} = (\bar{A})^*$.
(d) A bounded inverse of A exists if and only if A^* has a bounded inverse. Then it follows that

$$(A^{-1})^* = (A^*)^{-1}.$$

Definition 2.1.14. A bounded linear operator $U \in \mathfrak{B}(\mathfrak{H})$ is called unitary if $U^* = U^{-1}$, i.e., $UU^* = U^*U = \mathbb{1}$

Definition 2.1.15. Let A be a closed, densely defined operator and let $A^*A = AA^*$. Then A is called normal.

Definition 2.1.16. A densely defined operator A is called symmetric if $A \subseteq A^*$. This is equivalent to

$$\langle f, Ag\rangle = \langle Af, g\rangle$$

for all $f, g \in \mathrm{dom}\,(A)$.

Because A^* has a closed graph a symmetric operator is always closable, its closure is A^{**}.

Definition 2.1.17. A densely defined operator A is called selfadjoint if $A = A^*$, i.e., if A is symmetric and $\mathrm{dom}\,(A) = \mathrm{dom}\,(A^*)$. A symmetric operator which has only one selfadjoint extension is called essentially selfadjoint.

Remark 2.1.18.

(a) A selfadjoint operator has only one selfadjoint extension, namely A itself.
(b) For any symmetric operator A, the operator A^{**} is defined and symmetric. If A^{**} is selfadjoint, A is essentially selfadjoint, and A^{**} is the only selfadjoint extension.

There are several useful criteria to decide whether an operator is selfadjoint or at least essentially selfadjoint.

Proposition 2.1.19.

1) Let A be a symmetric operator. Then A is selfadjoint if and only if $\mathrm{ran}(A + i\beta) = \mathrm{ran}\,(A - i\beta) = \mathfrak{H}$*, for $\beta \in \mathbb{R}$.*
2) Let A be a symmetric operator. A is selfadjoint if $\mathrm{ran}\,(A+\mu) = \mathfrak{H}$ *for some real $\mu \in \mathbb{R}$. (This is a very useful criterion for semibounded A defined in Definition 2.1.23).*
3) Let A be a symmetric operator. A is essentially selfadjoint if $\mathrm{ran}\,(A + i\beta)$ *and* $\mathrm{ran}\,(A - i\beta)$ *are dense in $\mathfrak{H}$ for some $\beta \in \mathbb{R}$.*

2.1.4 Sums of Linear Operators

The sum of two linear operators $\{\mathrm{dom}\,(A), A\}, \{\mathrm{dom}\,(B), B\}$ is defined as

$$\mathrm{dom}\,(A+B) = \mathrm{dom}\,(A) \cap \mathrm{dom}\,(B).$$
$$(A+B)f = Af + Bf$$

for all $f \in \mathrm{dom}\,(A+B)$. In this section we will explain the properties of $A+B$ if we know these properties for A. For instance if A is bounded, the sum $A+B$ is a well-defined operator on $\mathrm{dom}\,(B)$. If B is closed and A is bounded, then $A+B$ is also closed. For more general A the situation is somewhat more complicated.

Definition 2.1.20. Let $\{\mathrm{dom}\,(A), A\}$, $\{\mathrm{dom}\,(B), B\}$ be two linear densely defined, closed operators in $\mathfrak{H}$ with $\mathrm{dom}\,(A) \subseteq \mathrm{dom}\,(B)$. B is called relatively bounded with respect to A, or A-bounded, if there are positive constants α and β such that

$$\|Bf\| \leq \alpha\|Af\| + \beta\|f\|$$

$\forall f \in \mathrm{dom}\,A$. The infimum of all possible α is called the A-bound of B.

Remark 2.1.21. Obviously, a bounded operator B is A-bounded for any A with A-bound zero. If A is selfadjoint, then B is A-bounded iff $B(A-z)^{-1} \in \mathfrak{B}(\mathfrak{H})$ for some $z \in \mathbb{C}$. In this case the A-bound is smaller than 1 iff $\|B(A-z)^{-1}\| < 1$.

If the A-bound of B is smaller than 1, the sum $A+B$ is closable iff A is closable with $\mathrm{dom}\,(\bar{A}) = \mathrm{dom}\,(\overline{A+B})$; also $A+B$ is closed iff A is closed.

Concerning the selfadjointness of the operator sum the main theorem is

Theorem 2.1.22 (Kato–Rellich). *Let A be a selfadjoint operator in $\mathfrak{H}$, let B be symmetric and A-bounded with A-bound smaller than one. Then $A+B$ is selfadjoint on* $\mathrm{dom}\,(A)$.

This can be extended to the case where the A-bound is equal to 1. This means that if A is essentially selfadjoint, if B is symmetric and A-bounded with A-bound equal to one, then $A+B$ is essentially selfadjoint on $\mathrm{dom}\,(A)$ or on any core of A.

Definition 2.1.23. A symmetric operator A is called bounded from below, if its numerical range is bounded from below, i.e., if

$$\langle f, Af\rangle \geq \gamma\langle f, f\rangle, \ \forall f \in \mathrm{dom}\,(A), \ \gamma \in \mathbb{R}.$$

It is called non-negative if $\gamma = 0$. Often we write $A - \gamma \geq 0$ or $A \geq 0$ in the last case.

Let us collect some properties of semibounded operators:

(a) If A is symmetric and positive and if ran $(A + \mathbb{1})$ is dense in $\mathfrak{H}$, then A is essentially selfadjoint.
(b) If A is selfadjoint and bounded from below and if B is symmetric and A-bounded with A-bound smaller than 1, then $A + B$ is also bounded from below.

2.1.5 Sesquilinear Forms

Although the criteria above are elegant from the operator-theoretic point of view it is very often difficult to determine the domain on which a symmetric operator can be extended to a selfadjoint one. On the other hand, in quantum physics it is very often sufficient to study the expectation values of the operators. Hence a natural way to realize selfadjoint operators is to start with sesquilinear forms.

Definition 2.1.24. A (sesquilinear) form $\mathfrak{t}$ is a map

$$\mathrm{dom}\,(\mathfrak{t}) \times \ \mathrm{dom}\,(\mathfrak{t}) \to \mathbb{C},$$

where $\mathrm{dom}\,(\mathfrak{t})$ is a linear subspace of $\mathfrak{H}$. $\mathrm{dom}\,(\mathfrak{t})$ is called the form domain. $\mathfrak{t}$ is linear in the second argument and conjugate linear in the first. It is called symmetric if $\mathfrak{t}[f,g] = \overline{\mathfrak{t}[g,f]}$ for $f, g \in \mathrm{dom}\,(\mathfrak{t})$. A symmetric form is called bounded from below if there is a $\gamma \in \mathbb{R}$. such that

$$\mathfrak{t}[f,f] \geq \gamma\, \|f\|^2\,, \quad f \in \mathrm{dom}\,(\mathfrak{t})$$

and is called non-negative if $\gamma = 0$. For abbreviation we write $\mathfrak{t} \geq \gamma$ or $\mathfrak{t} \geq 0$, respectively.

Definition 2.1.25. Analogous to the closed operators where the domain of A is a Banach space with respect to the graph norm, the symmetric form $\mathfrak{t}$, bounded from below, i.e., $\mathfrak{t} \geq \ \gamma$, is called closed if $\mathrm{dom}\,(\mathfrak{t})$ is complete with respect to the form norm, induced by the inner product

$$\langle f, g\rangle_{\mathfrak{t}} = \mathfrak{t}[f,g] + (-\gamma + 1)\langle f, g\rangle.$$

Then $\{\mathrm{dom}\,(\mathfrak{t}), \langle .,.\rangle_{\mathfrak{t}}\}$ is a Hilbert space.

The first form representation theorem says the following:

Theorem 2.1.26. *Let $\mathfrak{a}$ be a densely defined, closed, symmetric semibounded form in $\mathfrak{H}$. Then there is a unique selfadjoint operator A in $\mathfrak{H}$, such that*

$$\begin{aligned} &\mathrm{dom}\,(A) \subseteqq \mathrm{dom}\,(\mathfrak{a}) \\ \textit{and} \qquad &\mathfrak{a}[k,f] = \langle k, Af\rangle \end{aligned}$$

for $f \in \operatorname{dom}(A)$ *and* $k \in \operatorname{dom}(\mathfrak{a})$. *Moreover, A is given by*

$$\begin{aligned}\operatorname{dom}(A) = \{\, f \in \operatorname{dom}(\mathfrak{a}), \exists\ g \in \mathfrak{H},\ &\textit{such that}\\ \mathfrak{a}[k,f] = \langle k, g\rangle\ &\textit{for all } k \in \operatorname{dom}(\mathfrak{a})\}\end{aligned}$$

and

$$Af = g.$$

A is called the operator associated with $\mathfrak{a}$.

Corollary 2.1.27 *If A_0 is a symmetric, semibounded operator in $\mathfrak{H}$, then*

$$\operatorname{dom}(\mathfrak{a}_0) = \operatorname{dom}(A_0) \quad \textit{and} \quad \mathfrak{a}_0[g,f] = \langle g, A_0\ f\rangle$$

establishes a densely defined, symmetric form bounded from below. Denoting its closure by $\{\operatorname{dom}(\mathfrak{a}), \mathfrak{a}\}$ this is a densely defined symmetric closed form bounded from below. It corresponds to a selfadjoint operator A. A is called the Friedrichs extension of A_0. A is given by

$$\begin{aligned}\operatorname{dom}(A) = &\ \operatorname{dom}(\mathfrak{a}) \cap \operatorname{dom}(A_0^*)\\ A = &\ A_0^*|_{\operatorname{dom}(A)}.\end{aligned}$$

If A_0 is essentially selfadjoint, the Friedrichs extension is the only selfadjoint extension of A.

For non-negative forms the domain of the associated operator is given by the second form representation theorem.

Theorem 2.1.28. *Let $\mathfrak{a}$ be a densely defined, closed symmetric form with $\mathfrak{a} \geq 0$. Let A be the associated selfadjoint operator. Then*

$$\operatorname{dom}(A^{1/2}) = \operatorname{dom}(\mathfrak{a})$$

and

$$\mathfrak{a}[g,f] = \langle A^{1/2}g, A^{1/2}f\rangle$$

for $f, g \in \operatorname{dom}(\mathfrak{a})$. (Hint: The square root $A^{1/2}$ exists and is defined in Corollary 2.3.6).

In the case of perturbations there is a form analogue of Theorem 2.1.22.

Theorem 2.1.29 (Kato–Lax–Lions–Milgram–Nelson). *Let A be a positive selfadjoint operator, associated to the form $\mathfrak{a}$. Let $\mathfrak{b}$ be symmetric form with* $\operatorname{dom}(\mathfrak{a}) \subseteq \operatorname{dom}(\mathfrak{b})$. *Assume*

$$|\ \mathfrak{b}[f,f]\ | \leq\ \alpha\ \|A^{1/2}f\|^2 + \beta\ \|f\|^2 = \alpha\,\mathfrak{a}[f,f] + \beta\ \|f\|^2$$

for $f \in \operatorname{dom}(\mathfrak{a})$ *with $\alpha < 1$ and $\beta \in \mathbb{R}$.*

Then there is a unique selfadjoint operator C such that

$$\langle f, Cg\rangle = \langle A^{1/2}f, A^{1/2}g\rangle +\ \mathfrak{b}[f,g] = \mathfrak{a}[f,g] + b[f,g]$$

for $f, g \in \operatorname{dom}(\mathfrak{a})$. C is bounded from below.

Remark 2.1.30. Theorem 2.1.29 will be called the KLMN-Theorem henceforth. If A and B are two positive selfadjoint operators, the KLMN-Theorem defines the meaning of the form sum of these operators, denoted by $A \dot{+} B$. Let B be relatively form bounded with respect to A, i.e.,

$$\begin{aligned} &(i) \quad \mathrm{dom}\,(\mathfrak{a}) \subseteq \mathrm{dom}\,(\mathfrak{b}) \\ &(ii) \quad \|B^{1/2}f\|^2 \leq \alpha\ \|A^{1/2}f\| + \beta\ \|f\|^2,\ \forall f \in \mathrm{dom}\,(\mathfrak{a}), \end{aligned}$$

with form bound $\alpha < 1$. Then $C = A \dot{+} B$. This may differ from the operator sum because $\mathrm{dom}\,(A) \cap \mathrm{dom}\,(B) = \{\theta\}$ is not excluded.

One can formulate the KLMN-Theorem also for more general selfadjoint operators B defining an appropriate domain for the associated form.

2.2 Spectrum and Resolvent Sets

Definition 2.2.1. Let $\{\mathrm{dom}\,(A), A\}$ be a closed linear operator in a Hilbert space $\mathfrak{H}$. A complex number z is called a regular point for A if $A - z$ is invertible with $\mathrm{dom}\,[(A-z)^{-1}] = \mathfrak{H}$ and if $(A-z)^{-1}$ is bounded, i.e., z is regular if $(A-z)^{-1}$ exists and is in $\mathfrak{B}(\mathfrak{H})$ (see Section 2.1.2).

We will denote

$$R(z, A) = (A - z)^{-1}.$$

The resolvent set of A is the set of all regular points, i.e.,

$$\mathrm{res}(A) = \{z \in \mathbb{C};\ (A-z)^{-1} \in \mathfrak{B}(\mathfrak{H})\}.$$

The spectrum of A is the complement of the resolvent set

$$\sigma(A) = \mathbb{C}\ \setminus\ \mathrm{res}(A).$$

The mapping

$$\begin{aligned} \mathrm{res}(A) &\mapsto \mathfrak{B}(\mathfrak{H}) \\ z &\mapsto (A-z)^{-1} = R(z, A) \end{aligned}$$

is called the resolvent of A.

Proposition 2.2.2. *Let $\{\mathrm{dom}(A), A\}$ be a closed linear operator. Then its resolvent $R(z, A)$ has the following properties:*

(i) res (A) *is an open set, hence the spectrum is a closed set.*
(ii) ran $(R(z, A)) = \mathrm{dom}\,(A)$ *and* $AR(z, A) = zR(z, A) + 1$.
(iii) The first resolvent equation holds:
$R(z_1, A) - R(z_2, A) = (z_1 - z_2)\ R(z_1, A)R(z_2, A)$,
for all $z_1, z_2 \in$ res (A).
$R(z_1, A)R(z_2, A) = R(z_2, A)R(z_1, A)$.

(iv) For $z_0 \in \operatorname{res}(A)$ *and for all* $z \in \operatorname{res}(A)$ *with* $|\, z - z_0 \,| < \|R(z_0, A)\|^{-1}$ *we have the representation*

$$\begin{aligned} R(z,A) &= \sum_{n=0}^{\infty} (z - z_0)^n R(z_0, A)^{n+1} \\ &= R(z_0, A) \sum_{n=0}^{\infty} (z - z_0)^n R(z_0, A)^n . \end{aligned}$$

(v) Locally the resolvent map $z \to R(z, A)^{n+1}$ *is analytic in res* (A) *and*

$$\frac{d^n}{dz^n} R(z, A) = (-1)^{n+1} n!\ R(z, A)^{n+1} .$$

(vi) If A *is a bounded operator, then*

$$R(z, A) = \frac{1}{z}\left(\frac{A}{z} - \mathbb{1}\right)^{-1} = \sum_{n=0}^{\infty} \frac{A^n}{z^{n+1}} \quad \textit{for} \ \ |\, z \,| > \|A\| .$$

Therefore

$$\sigma(A) \subseteqq \{z \in \mathbb{C}, |\, z \,| \leq \|A\|\}$$

and hence the spectrum of a bounded operator is compact and never empty.

Definition 2.2.3. The spectral radius of a bounded operator A is defined by

$$\operatorname{spr}\ (A) = \sup\{|\, z \,|\ ,\ z \in \sigma(A)\} .$$

Remark 2.2.4. Obviously we have $\operatorname{spr}(A) \leq \|A\|$. If A is normal $\operatorname{spr}(A) = \|A\|$.

Proposition 2.2.5. *If* A *is a closed linear operator with a non-empty resolvent set, then*

$$\sigma\big(R(z_0, A)\big) \setminus \{0\} = \left\{\frac{1}{z - z_0},\ z \in \sigma(A)\right\}$$

for each $z_0 \in \operatorname{res}(A)$. *Let* $\operatorname{dist}\ (z_0, \sigma(A))$ *denote the distance of* z_0 *from the spectrum of* A. *Then*

$$\operatorname{dist}\ \big(z_0, \sigma(A)\big) = \frac{1}{\operatorname{spr}\big(R(z_0, A)\big)} \ \geq \ \frac{1}{\|R(z_0, A)\|}$$

or

$$\|R(z_0, A)\| \ \geq \ \frac{1}{\operatorname{dist}\big(z_0, \sigma(A)\big)} .$$

Remark 2.2.6. Any $z \in \mathbb{C}$ belongs to $\sigma(A)$ whenever one of the following occurs:

1) $(A - z)$ may not be injective, so that $\text{null}\,(A - z) \neq \{\theta\}$. Then z is called an eigenvalue of A and $\text{null}\,(A - z)$ is the (geometric) eigenspace of z. The (geometric) multiplicity of z is given by dim null $(A - z)$. The vectors in $\text{null}\,(A - z)$ are called the eigenvectors of z.
2) $\text{null}\,(A - z) = \{\theta\}$ but $\text{ran}\,(A - z)$ is dense such that $(A - z)^{-1}$ is densely defined but unbounded.
3) $\text{null}\,(A - z) = \{\theta\}$ but $\text{ran}\,(A - z)$ is not dense. In this case $(A - z)^{-1}$ exists and can be bounded on ran $(A - z)$. However it can not be extended to an operator in $\mathfrak{B}(\mathfrak{H})$. This part of the spectrum is called the residual spectrum.

Definition 2.2.7. The values $z \in \mathbb{C}$ for which $(A - z)$ is not injective or ran $(A - z)$is not closed in $\mathfrak{H}$ are called the approximate point spectrum of A.

Lemma 2.2.8. *Let A be a closed operator and let z be in its approximate point spectrum. Then there is a sequence $(f_n)_{n\in\mathbb{N}}$, $f_n \in \text{dom}\,(A)$, $\|f_n\| = 1$ such that*

$$\lim_{n\to\infty} \|Af_n - zf_n\| = 0.$$

($\{f_n\}_{n\in\mathbb{N}}$ is called a Weyl sequence).

In what follows we will study selfadjoint linear operators $\{\text{dom}\,(A), A\}$ given in a separable Hilbert space $\mathfrak{H}$. This is a general assumption for the rest of this chapter and will not be repeated further.

Theorem 2.2.9. *Let $\{\text{dom}\,A, A\}$ be a selfadjoint operator in a separable Hilbert space $\mathfrak{H}$. Then we have*

(i) $\sigma(A) \subseteq \mathbb{R}$, *i.e., the spectrum is real.*
(ii) The residual spectrum is empty.
(iii) There is a sequence $(f_n)_{n\in\mathbb{N}}$, $f_n \in \text{dom}\,(A)$, $\|f_n\| = 1$ so that

$$\lim_{n\to\infty} \|(A - \lambda)f_n\| = 0$$

for any $\lambda \in \sigma(A)$.
(iv) Every eigenvalue is real. The eigensubspaces corresponding to different eigenvalues are orthogonal. There are at most countably many eigenvalues.
(v) For all $z \in \mathbb{C} \setminus \sigma(A)$ we have $R(z, A)^* = R(\bar{z}, A)$, such that $R(z, A)$ is normal. Hence

$$\text{spr}\,\big(R(z, A)\big) = \|R(z, A)\|\ .$$

Remark 2.2.10. Proposition 2.2.5 and (v) imply that

$$\|R(z, A)\| = \frac{1}{\text{dist}\,\big(z, \sigma(A)\big)}$$

and

$$\|R(z, A)\| \leq \frac{1}{|\ \text{Im}\ z\ |}\ , \ \text{for all}\ \ z \text{ with } \text{Im}\ z \neq 0.$$

Remark 2.2.11. For semibounded symmetric operators there is a basic relation between forms and resolvents. Let $\mathfrak{a}$ be a closed semibounded form and A the associated operator (see Section 2.1.4). Take $z \in \operatorname{res}(A)$ and $f \in \operatorname{dom}(A)$. Then there is an element $v \in \mathfrak{H}$, such that for all $g \in \operatorname{dom}(\mathfrak{a})$:

$$\mathfrak{a}(g,f) - z\langle g,f\rangle \ = \langle g,v\rangle,$$

which implies

$$(A-z)f = v \quad \text{or} \quad f = (A-z)^{-1}v.$$

2.3 Spectral Theorem

For selfadjoint operators the spectrum can be decomposed into several components. Moreover, the spectrum can be studied in some detail by introducing a family of projection operators which are uniquely associated to the selfadjoint operator A in the given Hilbert space $\mathfrak{H}$.

Definition 2.3.1. Let $E(\cdot)$ be an operator-valued mapping from $\mathbb{R}$ to $\mathfrak{B}(\mathfrak{H})$. Then $E(\cdot)$ is called a resolution of the identity or a spectral resolution if it satisfies the following properties:

(i) $E(\lambda)$ is an orthogonal projection operator for every $\lambda \in \mathbb{R}$.
(ii) $E(\cdot)$ is non-decreasing, i.e., $E(\lambda) \le E(\mu)$ for $\lambda < \mu$. This is equivalent to $E(\lambda)E(\mu) = E(\mu)E(\lambda) = E\big(\min(\mu,\lambda)\big)$.
(iii) $E(\cdot)$ is right-continuous, i.e., $\lim\limits_{\varepsilon\downarrow 0} E(\lambda+\varepsilon)f = E(\lambda)f$ for all $f \in \mathfrak{H}$ and all $\lambda \in \mathbb{R}$.
(iv) $\lim\limits_{\lambda\to-\infty} E(\lambda)f = 0$ and $\lim\limits_{\lambda\to\infty} E(\lambda)f = f$ for all $f \in \mathfrak{H}$.

For any $f \in \mathfrak{H}$ the function $\langle f, E(\lambda)f\rangle \ = \|E(\lambda)f\|^2$ is non-negative, nondecreasing. It tends to zero as $\lambda \to -\infty$ and to $\|f\|^2$ as $\lambda \to \infty$. Thus the mapping $\lambda \to \langle f, E(\lambda)f\rangle$ is a distribution function. Then from the facts stated after Definition 1.1.3, it can be seen that there is a measure uniquely associated to this distribution function. $\{E(\lambda), \lambda \in \mathbb{R}\}$ is called a resolution of the identity because

$$\int_{-\infty}^{\infty} d\langle E(\lambda)f,f\rangle \ = \|f\|^2.$$

The Lebesgue–Stieltjes integrals

$$\int_{-\infty}^{\infty} \varphi(\lambda)\ d\langle f, E(\lambda)f\rangle$$

exists for any bounded measurable function $\varphi(\cdot) : \mathbb{R} \to \mathbb{R}$.

Note that $\langle f, E(\lambda)f\rangle$ is also of bounded variation. These facts together with polarisation identity give a finite complex measure associated with the function $\langle g, E(\lambda)f\rangle, \forall f, g \in \mathfrak{H}$.

Any resolution of the identity can be used to define a selfadjoint operator.

Lemma 2.3.2. *Let $\{E(\lambda), \lambda \in \mathbb{R}\}$ be a resolution of the identity. Then the operator $\tilde{A}$ defined by*

$$\mathrm{dom}\,(\tilde{A}) \;=\left\{f \in \mathfrak{H}, \int_{-\infty}^{\infty} \lambda^2 \; d\langle f, E(\lambda)f\rangle \; < \infty\right\}$$

and

$$\langle f, \tilde{A}\; g\rangle \;= \int_{-\infty}^{\infty} \lambda \; d\langle f, E(\lambda)g\rangle$$

is selfadjoint, assuming $f, g \in \mathrm{dom}(\tilde{A})$.

Remark 2.3.3. The existence of the integral in the second equality above follows from

$$|\;\langle f, E(\lambda)g\rangle\;|^2 \quad\leq\quad \langle f, E(\lambda)f\rangle\;\langle g, E(\lambda)g\rangle.$$

Moreover, one has

$$\|\tilde{A}f\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\langle f, E(\lambda)f\rangle.$$

As a short abbreviation one usually writes

$$\tilde{A} = \int_{-\infty}^{\infty} \lambda \; dE(\lambda),$$

which means that $\tilde{A}$ is the selfadjoint operator associated to a given resolution of the identity.

The one-to-one correspondence between spectral families and selfadjoint operators is a consequence of the spectral theorem.

Theorem 2.3.4 (Spectral Theorem). *For every selfadjoint operator A in a separable Hilbert space $\mathfrak{H}$ there is a unique resolution of the identity $E_A(\cdot)$ such that*

$$A = \int_{-\infty}^{\infty} \lambda \; dE_A(\lambda).$$

Because of the one-to-one correspondence $E_A(\cdot)$ is said to be the spectral family associated to A.

The spectral theorem allows us to define functions of selfadjoint operators.

Proposition 2.3.5. *Let $\varphi(\cdot) : \mathbb{R} \to \mathbb{C}$ be a continuous function and A a selfadjoint operator in $\mathfrak{H}$. Then $\varphi(A)$ is defined by*

$$\operatorname{dom}\big(\varphi(A)\big) := \{f \in \mathfrak{H}, \int_{-\infty}^{\infty} \mid \varphi(\lambda) \mid^2 \; d\langle f, E_A(\lambda) f\rangle < \infty\},$$
$$\langle f, \varphi(A) g\rangle = \int_{-\infty}^{\infty} \varphi(\lambda)\; d\langle f, E_A(\lambda) g\rangle$$

for $f, g \in \operatorname{dom}\big(\varphi(A)\big)$, or briefly

$$\varphi(A) = \int_{-\infty}^{\infty} \varphi(\lambda)\; dE_A(\lambda).$$

If $\varphi(\cdot)$ is additionally bounded, then $\varphi(A)$ is a bounded operator (defined on $\mathfrak{H}$) and

$$\|\varphi(A)\| = \sup\; \{\mid (\varphi(\lambda) \mid,\; \lambda \in \operatorname{supp}\; \big(E_A(\lambda)\big)\}$$

where supp $\big(E_A(\lambda)\big)$ *denotes the support of the spectral measure associated to the spectral family.*

Corollary 2.3.6

(i) Let A be selfadjoint and $z \in$ res (A). Then we can represent its resolvent by

$$(A - z)^{-1} = \int_{-\infty}^{\infty} \frac{1}{\lambda - z}\; dE_A(\lambda).$$

(ii) Let A be non-negative, $A \geq 0$. Then one can define the square roof of A by

$$A^{1/2} = \int_{0}^{\infty} \lambda^{1/2}\; dE_A(\lambda).$$

(iii) The unitary group generated by A is given by

$$e^{itA} = \int_{-\infty}^{\infty} e^{it\lambda}\; dE_A(\lambda).$$

(iv) Let A be semibounded, $A \geq -c,\; c > 0$. The corresponding semigroup is given by

$$e^{-tA} = \int_{-c}^{\infty} e^{-t\lambda}\; dE_A(\lambda).$$

Theorem 2.3.7 (Spectral Mapping Theorem). *Let A be a selfadjoint operator. Let $\varphi(\cdot)$ be a bounded continuous function on $\sigma(A)$. Then*

$$\sigma\big(\varphi(A)\big) = \varphi\big(\sigma(A)\big).$$

2.4 Spectral Measures and Spectrum

Let A be a selfadjoint operator in the separable Hilbert space $\mathfrak{H}$ and $\{E_A(\lambda), \lambda \in \mathbb{R}\}$ its spectral family. There is a spectral measure associated to it.

Remark 2.4.1. (Construction of the Spectral Measure) Let $(a, b]$ be a semiclosed interval in $\mathbb{R}$. We define

$$P_A((a,b]) = E_A(b) - E_A(a).$$

Moreover, we define

$$\begin{aligned} P_A([a,b]) &= E_A(b) - E_A(a-), \\ P_A([a,b)) &= E_A(b-) - E_A(a-), \\ P_A((a,b)) &= E_A(b-) - E_A(a), \end{aligned}$$

where we used the abbreviation

$$E(a-) = s - \lim_{\varepsilon \downarrow 0} E(a - \varepsilon).$$

For any $f \in \mathfrak{H}$ we will define a projection valued measure on the Borel sets B in $\mathbb{R}$, i.e., we extend the definitions above to the Borel sets. For $U = \bigcup_{j=1}^{\infty} (a_j, b_j)$ with mutually disjoint intervals (a_j, b_j) we have

$$P_A(U) = s - \lim_{k \to \infty} \sum_{j=1}^{k} P_A\big((a_j, b_j)\big).$$

For any Borel set B of $\mathbb{R}$ and $f \in \mathfrak{H}$ we define

$$\langle f, P_A(B) f\rangle = \inf\{\langle f, P_A(U) f\rangle,\ U \supseteq B,\ U \text{open}\}.$$

It follows that

$$\langle f, P_A(B) f\rangle \ = \sup\{\langle f, P_A(C) f\rangle,\ C \subseteq B,\ C \text{ compact}\}.$$

Hence we obtain a projection valued mapping $B \to P_A(B), B$ Borel subset of $\mathbb{R}$, with the following properties:

$$P_A(B)^2 = P_A(B) = P_A(B)^*,$$
$$P_A(B_1 \cup B_2) = P_A(B_1) + P_A(B_2) \text{ if } \ B_1 \cap B_2 = \emptyset,$$
$$P_A\left(\bigcup_{j=1}^{\infty} B_j\right) = \sum_{j=1}^{\infty} P_A(Bj) \text{ in the strong sense, if } B_j \text{ are mutually disjoint.}$$

$P_A(\cdot)$ is a projection valued measure, called the spectral measure of A. For $f \in \mathfrak{H}$ the scalar product $\langle f, P_A(\cdot) f\rangle$ defines a measure on $\mathbb{R}$ which is used in the spectral theorem (Theorem 2.3.4). $\langle f, P_A(\cdot) f\rangle$ is also called a spectral measure.

The spectral measure is related to the resolvent of A.

Theorem 2.4.2 (Stone's Formula). *Let A be a selfadjoint operator and (a,b) an open bounded interval. Then*

$$P_A\big((a,b)\big) = s-\lim_{\delta\downarrow 0} \quad s-\lim_{\epsilon\downarrow 0} \frac{1}{2\pi i} \quad \int_{a+\delta}^{b-\delta} [(A-\lambda-i\varepsilon)^{-1} - (A-\lambda+i\varepsilon)^{-1}]\, d\lambda.$$

By using the spectral measure the spectrum of A can be investigated in much more detail.

Proposition 2.4.3. *λ is in the spectrum of A if and only if*

$$E_A(\lambda+\varepsilon) - E_A(\lambda-\varepsilon) = P_A((\lambda-\varepsilon, \lambda+\varepsilon]) \neq 0$$

for all $\varepsilon > 0$.

The spectral measure can be used to introduce a first and simple decomposition of the spectrum.

Decomposition 2.4.4. *The essential spectrum, $\sigma_{ess}(A)$, of a selfadjoint operator A is defined by*

$$\sigma_{ess}(A) \ = \{\lambda \in \mathbb{R},\ \text{dim ran}\ P_A\big((\lambda-\varepsilon, \lambda+\varepsilon)\big) = \infty \textit{ for all } \varepsilon > 0\}.$$

If this is not the case, i.e., if there is an $\varepsilon_0 > 0$ such that $\text{dim ran} P_A\big((\lambda-\varepsilon_0, \lambda+\varepsilon_0)\big) < \infty$, *then one says λ lies in the discrete spectrum of A, $\sigma_{disc}(A)$.*

By definition these spectral components are disjoint sets, and

$$\sigma(A) = \sigma_{ess}(A)\ \cup\ \sigma_{disc}(A). \tag{2.4.1}$$

$\sigma_{ess}(A)$ is always closed, whereas σ_{disc} is not necessarily closed. $\sigma_{disc}(A)$ consists of all isolated eigenvalues of finite multiplicity.

The essential spectrum can also be characterized by singular sequences (see Theorem 2.2.9 (iii)).

Theorem 2.4.5 (Weyl's Criterion). *Let A be a selfadjoint operator. Then $\lambda \in \sigma_{ess}(A)$ if and only if there is a sequence $(f_n)_{n\in\mathbb{N}}$, $f_n \in \text{dom}\,(A)$, $\|f_n\| = 1$, $\text{w}-\lim_{n\to\infty} f_n = 0$ and $\text{s}-\lim_{n\to\infty} (A-\lambda)\ f_n = 0$.*

2.5 Spectral Theorem in the Hahn–Hellinger Form

In this section we present a form of the spectral theorem coming out of a decomposition of a spectral measure, known as the Hahn–Hellinger theorem. In this section we write a selfadjoint operators $\{\mathrm{dom}(A),\ A\}$ as simply A.

Definition 2.5.1. A cyclic vector for a bounded selfadjoint operator A on a Hilbert space $\mathcal{H}$ is a vector $\phi \in \mathcal{H}$ such that the set

$$\{A^j\phi : 0 \le j < \infty\}$$

is total in $\mathcal{H}$ (that is the set of finite linear combination of elements from this set is dense in $\mathcal{H}$).

Cyclic vectors are used to obtain a typical form of a selfadjoint operator as given in Theorem 2.5.2 below. In the following $L^2(X, \mu, Y)$ means Y-valued L^2-functions on the measure space $(X, \mathfrak{B}_X, \mu)$.

To see how a typical selfadjoint operator looks, we consider a probability measure μ of compact support and the operator

$$(Mf)(x) = xf(x), \quad f \in L^2(\mathbb{R}, \mu, \mathbb{C}).$$

Then M is a selfadjoint operator and so is

$$(Mf_k)_i(x) = xf_{ki}(x), \quad 1 \le i \le k, \quad f_k \in L^2(\mathbb{R}, \mu, \mathbb{C}^k), \quad k \in \mathbb{N}.$$

In the case when μ does not have compact support, the operator M is defined only on its natural domain. More generally, the operator M given as

$$(Mf_j)_i(x) = xf_{ji}(x), \quad 1 \le i \le j \le \infty, \quad f \in \mathcal{H}, \tag{2.5.2}$$

is selfadjoint when $\{\mu_\infty, \mu_1, \dots, \mu_n, \dots\}$ are mutually singular (not necessarily non-zero) σ finite measures and

$$\mathcal{H} = L^2(\mathbb{R}, \mu_\infty, \ell^2(\mathbb{Z})) \bigoplus_{1\le j<\infty} L^2(\mathbb{R}, \mu_j, \mathbb{C}^j). \tag{2.5.3}$$

M has the domain

$$\left\{ f \in \mathcal{H} : \sum_{i=1}^{\infty} \int d\mu_\infty(x)|xf_{\infty i}|^2(x) + \sum_{j=1}^{\infty}\sum_{i=1}^{j} \int d\mu_j(x)|xf_{ji}|^2(x) < \infty \right\}.$$

The spectral theorem for selfadjoint operators is that up to unitary equivalence every selfadjoint operator is of the above form as stated in the following theorem. The proof is given after Lemma 2.5.5.

Theorem 2.5.2. *Let A be a selfadjoint operator on a separable Hilbert space $\mathcal{H}$. Then there exist a countable collection of mutually singular finite measures $\{\mu_\infty, \mu_1, \dots, \mu_n, \dots\}$, and an invertible isometry V from $\mathcal{H}$ to $L^2(\mathbb{R}, \mu_\infty, \ell^2(\mathbb{Z})) \oplus_{1 \le i < \infty} L^2(\mathbb{R}, \mu_i, \mathbb{C}^i)$, such that $A = V^{-1}MV$, where M is defined as in Equation (2.5.2). Further V maps the domain of A onto the domain of M.*

Remark 2.5.3. In the above theorem some of the measures may have empty support. The points in the support of μ_n are said to have spectral multiplicity n.

For the proof of Theorem 2.5.2 we need the following two lemmas.

Lemma 2.5.4. *Let A be a selfadjoint operator on $\mathcal{H}$ with a cyclic vector. Then A is unitarily equivalent to an operator M, given by $(Mf)(x) = xf(x)$, on $L^2(\mathbb{R}, \mu)$ for some probability measure μ.*

Proof: First let us assume that A is bounded so that the spectrum is compact. Let the cyclic vector be f, $\|f\| = 1$. Define the set

$$\mathcal{P} = \{g \in \mathcal{H} : g = p(A)f, \quad p \text{ a polynomial}\}.$$

We set the measure $\mu = \langle f, P_A(\cdot)f\rangle$. Then, using Proposition 2.3.5, Corollary 2.3.6, Remark 2.4.1, the map $U : \mathcal{H} \to L^2(\mathbb{R}, \mu)$, given by $Up(A)f = p$, for any polynomial p, is seen to be an isometry on $\mathcal{P}$ by an application of Proposition 2.3.5. Being a bounded operator U extends to the whole of $\mathcal{H}$ as an isometry, since $\mathcal{P}$ is dense in $\mathcal{H}$. Since $\mathcal{P}$ is total (by the assumption of cyclicity of f) in $\mathcal{H}$ and since the set of polynomials is total in the space $L^2(\mathbb{R}, \mu)$ (reason: The support K of μ is compact and $L^2(\mathbb{R}, \mu) = L^2(K, \mu)$ and polynomials in K are dense in $L^2(K, \mu)$), the map extends to an isometry to the whole of $\mathcal{H}$. Unitarity of the map U follows if we show that it is surjective. To see this, consider any $h \in L^2(\mathbb{R}, \mu)$. Then approximate h by polynomials p_n in $L^2(\mathbb{R}, \mu)$, Now the limit $g = s{-}\lim\ p_n(A)f$ exists because $p_n(A)f$ is a Cauchy sequence as can be seen by the estimate

$$\|p_n(A)f - p_m(A)f\|_{\mathcal{H}} = \|p_n - p_m\|_{L^2(\mathbb{R},\mu)} = \|p_n - p_m\|_{L^2(K,\mu)}.$$

The right-hand side goes to zero as $n, m \to \infty$ since p_n is Cauchy in $L^2(K, \mu)$. We claim that $Ug = h$. Indeed

$$\begin{aligned} \|Ug - h\| &= \|Ug - Up_n(A)f + Up_n(A)f - h\| \\ &\le \|Ug - Up_n(A)f\| + \|Up_n(A)f - h\| \\ &\le \|g - p_n(A)f\|_{\mathcal{H}} + \|p_n - h\|_{L^2(\mathbb{R},\mu)} \to 0, \quad n \to \infty, \end{aligned}$$

showing the claim. In the case when A is unbounded, the proof of isometry is the same as before. However for showing that U is onto, we start with $h \in L^2(\mathbb{R}, \mu)$ of compact support, in which case the proof of the existence of

$g \in \mathcal{H}$ with $Ug = h$ is as before. If h is a general element of $L^2(\mathbb{R}, \mu)$, then it is approximated by a sequence h_n of vectors each having compact support (K_n say) and each h_n is approximated by polynomials $p_{n,m}$ in the support K_n of h_n. We then pick a diagonal sequence p_{n_l,m_l} of polynomials such that the sequence of vectors $p_{n_l,m_l}\chi_{K_{n_l}}$ approximate h in $L^2(\mathbb{R}, \mu)$. Associated with these polynomials we have vectors g_l in $\mathcal{H}$ such that $Ug_l = p_{n_l,m_l}\chi_{K_{n_l}}$ on $L^2(\mathbb{R}, \mu_{n_l})$, where $\mu_{n_l}(S) = \frac{\mu(S \cap K_{n_l})}{\mu(K_{n_l})}$. As in the earlier part of the proof we can see that the sequence g_l is Cauchy in $\mathcal{H}$ and hence converges to some g there and we have $Ug = h$. Under the unitary equivalence established by U between $\mathcal{H}$ and $L^2(\mathbb{R}, \mu)$, Proposition 2.3.5 shows that $UAU^{-1} = M$. $\square$

Lemma 2.5.5. *Suppose μ and ν are mutually singular probability measures. Then $L^2(\mathbb{R}, \mu + \nu) = L^2(\mathbb{R}, \mu) \oplus L^2(\mathbb{R}, \nu)$.*

Proof: Let the supports of μ and ν be given by E and F, respectively, so $\text{supp}(\mu + \nu) = E \cup F$. Then by the assumption on the mutual singularity of μ and ν, $E \cap F = \emptyset$ up to sets of $\mu + \nu$ measure zero. We can write any function $f \in L^2(\mathbb{R}, \mu + \nu)$ as $f = f_1 + f_2$ with $f_1 = f|E$ and $f_2 = f|F$, so that f_1 and f_2 are mutually orthogonal. Given any $f, g \in L^2(\mathbb{R}, \mu + \nu)$, we see that f_i is orthogonal to g_j if $i \neq j$ in this decomposition. These imply that we have $L^2(\mathbb{R}, \mu + \nu) = L^2(E, \mu + \nu) \oplus L^2(F, \mu + \nu)$. But ν has no mass on E and μ has no mass on F so $\mu + \nu$ is the same as μ on E and is the same as ν on F respectively, from which the lemma follows. $\square$

Proof of Theorem 2.5.2: Consider any normalized vector $f_1 \in \mathcal{H}$ and consider the subspace $\mathcal{H}_1 = \text{clo span}\{A^j f_1 : j \in \mathbb{N} \cup \{0\}\}$. If $\mathcal{H}_1 = \mathcal{H}$, then f_1 is a cyclic vector for H and the use of Lemma 2.5.4 finishes the proof. Otherwise consider a normalized vector $f_2 \in \mathcal{H}_1^\perp$ and construct $\mathcal{H}_2 = \text{clo span}\{A^j f_2 : j \in \mathbb{N} \cup \{0\}\}$ and at the n^{th} stage we pick a normalized vector $f_n \in \mathcal{H}_{n-1}^\perp$. Continuing this procedure, in view of the separability of $\mathcal{H}$, using Zorn's lemma if necessary we see that there is an N (which could be finite or infinite) such that

$$\mathcal{H}_j \perp \mathcal{H}_k, \quad j \neq k, \qquad \mathcal{H} = \bigoplus_{j=1}^{N} \mathcal{H}_j,$$

and each $\mathcal{H}_j$ is a cyclic subspace for A. Then setting $\nu_j = \langle f_j, P_A(\cdot) f_j \rangle$ and using Lemma 2.5.4 we see that for each $j = 1, \ldots, N$, there is a unitary operator U_j mapping $\mathcal{H}_j$ onto $L^2(\mathbb{R}, \nu_j)$ such that $U_j A U_j^{-1} = M$ on $L^2(\mathbb{R}, \nu_j)$. Therefore, set

$$I = \begin{cases} \{1, \ldots, N\}, & \text{if } N < \infty, \\ \mathbb{N} \ \text{ if } N = \infty, & \text{then} \end{cases}$$

$$\mathcal{H} = \bigoplus_{j \in I} \mathcal{H}_j \equiv \bigoplus_{j \in I} L^2(\mathbb{R}, \nu_j). \tag{2.5.4}$$

In the above decomposition, however the different ν_j's need not have disjoint supports and need not be mutually singular. We would like to obtain a decomposition in terms of mutually singular measures, some of them possibly occurring with multiplicity. To this end let $\nu = \sum_{j=1}^{N} \epsilon_j \nu_j$, where $\epsilon_j > 0, \quad \sum_j \epsilon_j = 1$. Then we define the subsets

$$A_n = \mathrm{supp}(\nu_n), \quad E_k = \{x \in \mathbb{R} : \sum_{n=1}^{N} \chi_{A_n}(x) = k\},$$
$$E_\infty = \{x \in \mathbb{R} : \sum_{n=1}^{N} \chi_{A_n}(x) = \infty\}.$$

Then by construction E_k is a Borel subset for each k, since $\{A_n\}$ are Borel subsets of $\mathbb{R}$. Each x in the support of ν belongs to some E_k, so the $\{E_k\}$ form a partition of $\mathrm{supp}(\nu)$. Set $\mathbb{K} = \{k \in \mathbb{N} : E_k \neq \emptyset\} \cup \{\infty : E_\infty \neq \emptyset\}$. Then define, for each $n \in I, \ k \in \mathbb{K}, \ l \in \mathbb{N}, \ 1 \leq l \leq k$,

$$A_{n,k,l} = \{x \in E_k \cap A_n : \sum_{j=1}^{n} \chi_{A_j}(x) = l\}.$$

Thus the sets $A_{n,k,l}$ denote precisely the set of points in $E_k \cap A_n$ such that these points also belong to $l-1$ subsets A_j with $1 \leq j < n$. Then

$$A_n \cap E_k = \bigsqcup_{\substack{l \in \mathbb{N} \\ 1 \leq l \leq k}} A_{n,k,l}, \quad \forall n \in I, \ k \in \mathbb{K},$$

where $\bigsqcup$ denotes the disjoint union.

We now note that by construction $\{E_k\}, \ \{A_{n,k,l}\}$ are collections of mutually disjoint subsets of $\mathrm{supp}\,(\nu)$ (even though the A_n may not be disjoint, as n varies the $A_{n,k,l}$ are disjoint as n, k, l vary). Therefore

$$E_k = \bigcup_n A_n \cap E_k = \bigsqcup_{\substack{l \in \mathbb{N} \\ 1 \leq l \leq k}} \bigsqcup_{n \in I} A_{n,k,l}, \quad \forall k \in \mathbb{K}, \ 1 \leq l \leq k.$$

We now note that for each $n \in I$, the collections of measures $\{\nu_{|_{A_n \cap E_k}}\}$ and $\{\nu_{|_{A_{n,k,l}}}\}$ are collections of mutually singular measures (they are mutually singular because their supports are mutually disjoint). We also have

$$\sum_{k \in \mathbb{K}} \nu_{|_{A_n \cap E_k}} = \sum_{\substack{k \in \mathbb{K} \\ l \in \mathbb{N} \\ 1 \leq l \leq k}} \nu_{|_{A_{n,k,l}}}.$$

We now have

$$\begin{aligned}\oplus_{n\in I}L^2(\mathbb{R},\nu_n) &= \oplus_{n\in I}\oplus_{k\in\mathbb{K}} L^2(\mathbb{R},\nu_{|A_n\cap E_k})\\ &= \oplus_{n\in I}\oplus_{k\in\mathbb{K}}\oplus_{1\le l\le k}L^2(\mathbb{R},\nu_{|A_{n,k,l}})\\ &= \oplus_{k\in\mathbb{K}}\oplus_{1\le l\le k} L^2(\mathbb{R},\nu_{|E_k})\\ &= \oplus_{k\in\mathbb{K}}L^2(\mathbb{R},\nu_{|E_k},\mathbb{C}^k),\end{aligned}\tag{2.5.5}$$

where in the penultimate equality we used the fact that $\oplus_n L^2(\mathbb{R},\nu_{|A_{n,k,l}}) = L^2(\mathbb{R},\nu_{|E_k})$. This relation follows from Lemma 2.5.5 using the fact that $A_{n,k,l}$ are mutually disjoint and $\sqcup_n A_{n,k,l} = E_k$, $\nu|A_{n,k,l}$ are mutually singular and $\sum_n \nu_{|A_{n,k,l}} = \nu_{|E_k}$. We note that in the Equation (2.5.5), the element (g_{kl}), $k\in\mathbb{K}, 1\le l\le k$ on the right-hand side associated with the element (f_n), $\; n\in I$, is given by $g_{kl} = \sum_{n\in I} f_n\chi_{A_{n,k,l}}$. We now set $\mu_k = \nu_{|E_k}$, $\; k\in\mathbb{K}$, $\mu_\infty = \nu_{|E_\infty}$ and $\mu_k = 0, k\in\mathbb{N}\setminus\mathbb{K}$. With this definition of the μ_k's we have

$$\oplus_{n\in I}L^2(\mathbb{R},\nu_n) = L^2(\mathbb{R},\mu_\infty,\ell^2(\mathbb{Z}))\oplus_{k\in\mathbb{N}} L^2(\mathbb{R},\mu_k,\mathbb{C}^k).$$

We saw that $\mathcal{H} = \oplus_{n\in I}\mathcal{H}_n$ and there are unitary operators U_n that map each $\mathcal{H}_n$ onto $L^2(\mathbb{R},\nu_n)$ so that A goes to the operator M. Now using the unitary equivalences set up in equations (2.5.4) and (2.5.5), we see that A is unitarily equivalent to the direct sums of multiplication operators stated in the theorem. The theorem for the case of unbounded A is a routine after this. □

Definition 2.5.6. If we consider the measures μ_n, $\; n = \infty, 1, 2, \ldots$, in the above theorem, then the probability measure $\mu = \frac{1}{2}\mu_\infty + \sum_{n=1}^{\infty} 2^{-n-1}\mu_n$ is called a total spectral measure. The set of measures which are equivalent to this μ is called the measure class of the selfadjoint operator A.

2.6 Components of the Spectrum

Throughout this section A is assumed to be a selfadjoint operator. The separation of the spectrum in Section 2.4 into the essential and discrete spectrum is very rough. Nevertheless, in many problems one is mainly interested in the discrete spectrum or in the infimum of the essential spectrum where this Decomposition 2.4.4 is sufficient.

However, there are finer decompositions of spectra and it is possible to investigate their components separately.

A natural way is by decomposing the Hilbert space.

Definition 2.6.1. We define

$$\mathfrak{H}_p(A) = \text{clo span}\,\{f:\; f \text{ an eigenvector of } A\}.$$

$\mathfrak{H}_p(A)$ is the closed linear span of all eigenvectors of A, i.e., it is the closure of the linear manifold consisting of all finite linear combinations of eigenvectors of A.

Lemma 2.6.2. *If null* $(A - \lambda_i)$ *denotes the eigensubspace of the eigenvalue* λ_i*, then*

$$\mathfrak{H}_p(A) = \bigoplus_i \mathrm{null}\,(A - \lambda_i).$$

Definition 2.6.3. Since $\mathfrak{H}_p(A)$ is a subspace of $\mathfrak{H}$ we can define

$$\mathfrak{H}_c(A) = \mathfrak{H}_p(A)^{\perp},$$

so that

$$\mathfrak{H} = \mathfrak{H}_p(A) \ \oplus \ \mathfrak{H}_c(A).$$

$\mathfrak{H}_c(A)$ is called the continuous subspace of A. The name becomes clear in Proposition 2.6.7. In this context $\mathfrak{H}_p(A)$ is called the discontinuous subspace of A.

Thus we get the second decomposition of the spectrum.

Decomposition 2.6.4. *The restriction of* A *to* dom $(A) \cap \mathfrak{H}_p(A)$ *is denoted by* A_p*.* A_p *leaves* $\mathfrak{H}_p(A)$ *invariant, it is a selfadjoint operator in* $\mathfrak{H}_p(A)$*.*

On the other hand, let us denote by A_c *the restriction of* A *to* dom $(A) \cap \mathfrak{H}_c(A)$*.* A_c *is a selfadjoint operator in* $\mathfrak{H}_c$*. It leaves* $\mathfrak{H}_c(A)$ *invariant.*

This implies a decomposition of the operator

$$A = A_p \ \oplus \ A_c$$

and a further decomposition of the spectrum. The continuous spectrum of A *is given by*

$$\sigma_c(A) = \sigma(A_c).$$

The point spectrum of A *is given by*

$$\sigma_{pp}(A) = \sigma(A_p).$$

Remark 2.6.5. We will denote the set of eigenvalues of A by $\sigma_p(A)$. Note that

$$\sigma_{pp}(A) = \overline{\sigma_p(A)}.$$

Remark 2.6.6. Relations between $\sigma_{\mathrm{disc}}(A)$, $\sigma_{ess}(A)$, $\sigma_p(A)$, $\sigma_{pp}(A)$ and $\sigma_c(A)$ are the following:

(i) $\sigma_{\mathrm{disc}}(A) \subseteq \sigma_p(A) \subseteq \sigma_{pp}(A)$;
(ii) $\sigma_{\mathrm{ess}}(A)$ consists of $\sigma_c(A)$, all the accumulations points of $\sigma_{pp}(A)$, and all eigenvalues with infinite multiplicity.

The sets $\sigma_{pp}(A)$ and $\sigma_c(A)$ are not necessarily disjoint.

By means of the spectral measure we can characterize the spectral components so far and we can decompose the continuous subspace $\mathfrak{H}_c(A)$ in more detail (see Definition 2.6.8).

Proposition 2.6.7.

(i) λ *is an eigenvalue of* A *iff* $P_A(\{\lambda\}) = E_A(\lambda) - E_A(\lambda-) \neq 0$, *i.e.,* $E_A(\cdot)$ *has a jump at* λ.

(ii) λ *is in the spectrum of* A *iff* $E_A(\lambda+\varepsilon) - E_A(\lambda-\varepsilon) \neq 0$, *for any* $\varepsilon > 0$, *i.e.,* $E_A(\cdot)$ *is increasing at* λ.

(iii) $\lambda \in \sigma_{ess}(A)$ *iff* $\dim\big(E_A(\lambda+\varepsilon) - E_A(\lambda-\varepsilon)\big) = \infty$ *for all* $\varepsilon > 0$.

(iv) $\lambda \in \sigma_{disc}(A)$ *iff* $P_A(\{\lambda\}) \neq 0$ *is finite dimensional and if there is an* $\varepsilon_0 > 0$ *such that* $E_A(\lambda+\varepsilon_0) - E_A(\lambda-\varepsilon_0) = P_A(\{\lambda\})$.

(v) For $f \in \mathfrak{H}_c(A)$ *the scalar product* $\langle f, P_A(\{\lambda\})f\rangle = 0$ *for all* $\lambda \in \mathbb{R}$, *i.e.,* $\langle f, P_A\big((-\infty,\lambda)\big)f\rangle$ *is continuous on* $\mathbb{R}$, *or equivalently* $P_A\big((-\infty,\lambda)\big)f$ *is strongly continuous. Therefore* $\mathfrak{H}_c(A)$ *is called the spectrally continuous subspace.*

Moreover, the continuous subspace $\mathfrak{H}_c(A)$ can be decomposed into the absolutely continuous and the singularly continuous subspaces.

Definition 2.6.8. Let

$$\begin{aligned}\mathfrak{H}_{ac}(A) = \{&f \in \mathfrak{H}_c(A), \text{for which the measure } \langle f, P_A(\cdot)f\rangle \text{ is absolutely}\\ &\text{continuous with respect to the Lebesgue measure on } \mathbb{R}\},\\ = \{&f \in \mathfrak{H}_c(A),\ \langle f,\ P_A(B)f\rangle\ = 0 \text{ if } \mid B \mid = 0\},\\ \mathfrak{H}_{sc}(A) = \{&f \in \mathfrak{H}_c(A),\ \text{there is a Borel set } B_f, |B_f|\ = 0,\ \text{but } P_A(B_f)f = f\}.\end{aligned}$$

$\mathfrak{H}_{ac}(A)$, $\mathfrak{H}_{sc}(A)$ are subspaces of $\mathfrak{H}_c(A)$.

A restricted to $\operatorname{dom} A \cap \mathfrak{H}_{ac}(A)$ is denoted by A_{ac}.
A restricted to $\operatorname{dom} A \cap \mathfrak{H}_{sc}(A)$ is denoted by A_{sc}.

The absolutely continuous spectrum of A is defined by

$$\sigma_{ac}(A)\ = \sigma(A_{ac}),$$

its singularly continuous spectrum by

$$\sigma_{sc}(A)\ = \sigma(A_{sc}).$$

Remark 2.6.9. We have the following decompositions:

$$\begin{aligned}\mathfrak{H} &= \mathfrak{H}_p(A)\ \oplus\ \mathfrak{H}_{sc}(A)\ \oplus\ \mathfrak{H}_{ac}(A)\\ &= \mathfrak{H}_p(A)\ \oplus\ \mathfrak{H}_c(A)\\ &= \mathfrak{H}_{ac}(A)\ \oplus\ \mathfrak{H}_s(A).\end{aligned}$$

$\mathfrak{H}_s(A) = \mathfrak{H}_{ac}(A)^{\perp}$ is called the singular subspace of A. A restricted to $\operatorname{dom}(A) \cap \mathfrak{H}_s(A)$ is called A_s. $\sigma_s(A)$ is defined as $\sigma(A_s)$ and is called the singular spectrum of A.

For the spectrum we have

$$\begin{aligned}\sigma(A) &= \sigma_{pp}(A) \ \cup \ \sigma_c(A) \\ &= \sigma_{pp}(A) \ \cup \ \sigma_{ac}(A) \ \cup \ \sigma_{sc}(A) \\ &= \sigma_{\text{disc}} \ (A) \cup \ \sigma_{\text{ess}}(A) \\ &= \sigma_{ac}(A) \cup \sigma_s(A).\end{aligned}$$

In general, $\sigma_{pp}, \sigma_{ac}, \sigma_{sc}$ are not disjoint.

Remark 2.6.10. If μ is any total spectral measure associated with A, then it is decomposed as

$$\mu = \mu_{ac} + \mu_{sc} + \mu_p$$

where $\mu_{ac}, \mu_{sc}, \mu_p$ are respectively the absolutely continuous, singularly continuous and atomic parts of μ. Then

$$\begin{aligned}\sigma_{pp}(A) &= \overline{\text{supp } \mu_p} \\ \sigma_{sc}(A) &= \overline{\text{supp } \mu_{sc}} \\ \sigma_{ac}(A) &= \overline{\text{supp } \mu_{ac}}.\end{aligned}$$

A more detailed characteriztion for the spectrum can be given in the case of compact, Hilbert–Schmidt and trace class operators.

Proposition 2.6.11. *Let B be a compact selfadjoint operator on $\mathfrak{H}$ with* $\dim(\mathfrak{H}) = \infty$. *Then*

1. $\mathfrak{H}_c = \{\theta\}$, *which says that the spectrum of B is pure point.*
2. $\sigma_{ess}(B) = \{0\}$, *which means that 0 is in the spectrum and is either the accumulation point of the spectrum or it is an eigen value of infinite multiplicity.*
3. *Let $\{\mu_k\}$ be the non-zero eigenvalues. They are isolated and have finite multiplicities. Let E_k be the eigenprojection associated with μ_k; then B admits the spectral representation*

$$B = \sum_{k=1}^{\infty} \mu_k E_k. \tag{2.6.6}$$

4. *Let $\{\lambda_j\}$ be the non-zero eigenvalues of B, counting multiplicity, with the normalized eigenvectors $\{\phi_j\}$; then Equation (2.6.6) can be rewritten as*

$$B = \sum_{j=1}^{\infty} \lambda_j \langle \cdot, \phi_j \rangle \phi_j.$$

Proposition 2.6.12. *Let B be the selfadjoint Hilbert–Schmidt operator. Let $\{\lambda_j\}$ be the non-zero eigenvalues of B, counting multiplicity. Then*

$$\|B\|_{HS} = \sum_{j=1}^{\infty} |\lambda_j|^2.$$

Proof: This follows immediately from Definition 2.1.11, because $\sum_{j=1}^{\infty} \|B\phi_j\|^2$ is independent of the choice of the orthonormal basis $\{\phi_j\}$. $\square$

Definition 2.6.13. Let B be a compact selfadjoint operator in $\mathfrak{H}$. Let $\{\lambda_j\}$ be the non-zero eigenvalues of B, counting multiplicity. Then B is called a trace class operator if

$$\|B\|_{trace} = \sum_{j=1}^{\infty} |\lambda_j|$$

is finite. $\|\cdot\|_{trace}$ is the trace norm of B. In this case the trace defined by

$$trace(B) = \sum_{j=1}^{\infty} \langle \phi_j, B\phi_j \rangle$$

is finite and equals $\sum_{j=1}^{\infty} \lambda_j$.

Remark 2.6.14. Let B be a trace class integral operator in $L^2(\mathbb{R}^d)$. Denote by $b(\cdot,\cdot) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, its kernel. Let $C_+ = [-r, r]^d$ be the d-dimensional cube with sidelength $2r$, centered at the origin. Set

$$\tilde{b}(x, y) = \lim_{r \to 0} \frac{1}{|C_r|^2} \int_{C_r} \int_{C_r} b(x+u, y+v) du dv.$$

Then the trace of B is given by

$$trace(B) = \int_{\mathbb{R}^d} \tilde{b}(x, x) dx. \tag{2.6.7}$$

Of course, if the kernel is continuous, then

$$trace(B) = \int_{\mathbb{R}^d} b(x, x) dx.$$

2.7 Characterization of the States in Spectral Subspaces

There is a one-to-one correspondence between any selfadjoint operator A and its associated strongly continuous, unitary group $\{e^{itA},\ t \in \mathbb{R}\}$. e^{itA} maps $\mathrm{dom}\,(A)$ into itself, and

$$e^{itA} A f = A\ e^{itA} f$$

for all $f \in \mathrm{dom}\,(A)$ and all $t \in \mathbb{R}$.

In quantum mechanical systems the unitary group $\{e^{itA},\ t \in \mathbb{R}\}$ describes the dynamics of the system. The integral form of the Schrödinger equation is

$$e^{itA} f_0 = f_t$$

with the initial state $f_0 \in \mathfrak{H}$. Using the spectral theorem (Theorem 2.3.4) we have

$$\langle f, e^{itA} f\rangle = \int_{-\infty}^{\infty} e^{it\lambda}\, d\langle f, E_A(\lambda) f\rangle.$$

(see Corollary 2.3.6 (iii)).

The time evolution of eigenvectors is only a change of phase, which can be seen from the next proposition.

Proposition 2.7.1. *If f is an eigenvector of A with $Af = \lambda f$, then*

$$e^{itA}\, f = e^{it\lambda} f,$$

and

$$|\, \langle f, e^{itA} f\rangle \,| = \|f\|^2,$$

One can also characterize the time evolution of vectors in $\mathfrak{H}_{ac}(A)$ and in $\mathfrak{H}_c(A)$.

Proposition 2.7.2. *$\mathfrak{H}_{ac}(A)$ is the subspace spanned by*

$$\left\{ f \in \mathfrak{H},\ \int_{\mathbb{R}} |\, \langle f, e^{itA}\ f\rangle \,|^2\, dt < \infty \right\}.$$

Proposition 2.7.3. *Let $f \in \mathfrak{H}_c(A)$. Then*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^{\pm T} |\, \langle g,\ e^{itA} f\rangle \,|^2\ \ dt = 0, \quad \text{for any } g \in \mathfrak{H}.$$

It is natural to introduce the subspace $\mathfrak{H}_w(A)$.

Definition 2.7.4. Let

$$\mathfrak{H}_w(A) = \{ f \in \mathfrak{H} :\ \mathrm{w} - \lim_{t\to\ \infty} e^{itA} f = 0 \}.$$

Note that

$$\mathfrak{H}_w(A) = \{ f \in \mathfrak{H},\ \mathrm{w} - \lim_{t\to-\infty} e^{itA} f = 0 \}.$$

From the definition it follows that $\mathfrak{H}_w(A)$ is a subspace of $\mathfrak{H}$ and

$$\mathfrak{H}_w(A) \subseteq \mathfrak{H}_c(A).$$

Moreover,

$$\mathfrak{H}_{ac}(A) \subseteq \mathfrak{H}_w(A).$$

So far, we have studied the spectral theoretic description of the space $\mathfrak{H}$ and its vectors. However, for potential scattering in quantum mechanics there is a classification of states based on geometric criteria as bound and scattering states. The corresponding criteria are valid in the special Hilbert space $L^2(\mathbb{R}^d)$.

Definition 2.7.5. $f \in L^2(\mathbb{R}^d)$ is a bound state of the selfadjoint operator A if for any $\varepsilon > 0$ there is a compact set $B \subset \mathbb{R}^d$ such that

$$\|\chi_{_{B^c}}\ e^{itA}\ f\| < \varepsilon$$

for all $t \in \mathbb{R}$, where B^c denotes the complement of B. The corresponding closed subspace is denoted by $\mathfrak{H}_{\text{bound}}(A)$.

Scattering states are vectors for which

$$\lim_{t\to\pm\infty} \|\chi_{_B}\ e^{itA}\ f\| = 0$$

for all compact sets $B \subset \mathbb{R}^d$. That means they leave any bounded region. The corresponding closed subspace is denoted by $\mathfrak{H}_{\text{scatt}}(A)$.

Finally, we introduce the scattering states in mean by

$$\lim_{T\to\ \infty} \frac{1}{2T} \int_{-T}^{T} \|\chi_{_B}\ e^{itA}\ f\|^2 dt = 0,$$

and denote the corresponding closed subspace by $\mathfrak{H}_{\text{scatt, mean}}(A)$.

Between these geometrically characterized subspaces and the spectral subspaces defined above there are in general the following relations.

Proposition 2.7.6. *Let A be a selfadjoint operator in $L^2(\mathbb{R}^d)$. Then*

$$\begin{aligned}
\mathfrak{H}_p(A) &\subseteq \mathfrak{H}_{\text{bound}}(A),\\
\mathfrak{H}_{\text{scatt}}(A) &\subseteq \mathfrak{H}_{\text{scatt, mean}}(A),\\
\mathfrak{H}_{\text{bound}}(A) &\perp \mathfrak{H}_{\text{scatt, mean}}(A),\\
\mathfrak{H}_{\text{scatt, mean}}(A) &\subseteq \mathfrak{H}_c(A),\\
\mathfrak{H}_{\text{scatt}}(A) &\subseteq \mathfrak{H}_w(A).
\end{aligned}$$

Proposition 2.7.7. *Let A be a selfadjoint operator in $L^2(\mathbb{R}^d)$.*

(i) If $\chi_{_B}(A-z)^{-1}$ is a compact operator for some $z \in \operatorname{res}(A)$ and for all balls B in $\mathbb{R}^d$, then

$$\begin{aligned}
\mathfrak{H}_{\text{bound}}(A) &= \mathfrak{H}_p(A),\\
\mathfrak{H}_{\text{scatt}}(A) &= \mathfrak{H}_w(A),\\
\mathfrak{H}_{\text{scatt, mean}}(A) &= \mathfrak{H}_c(A).
\end{aligned}$$

(ii) If additionally $\mathfrak{H}_{sc}(A)$ is empty we get

$$\mathfrak{H}_{\text{scatt}}(A) = \mathfrak{H}_{ac}(A).$$

The proof of the last proposition is based on the RAGE-Theorem, which is important for many aspects in spectral theory.

Theorem 2.7.8 (Ruelle–Amrein–Georgescu–Enss). *Let A be a selfadjoint operator in $\mathfrak{H}$. Let C be a bounded operator such that $C(A+i)^{-1}$ is compact. Then for all $f \in \mathfrak{H}_c(A)$.*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|Ce^{-itA} f\|^2 dt = 0.$$

2.8 Notes

Section 2.1

§*2.1.1* and §*2.1.2* Standard textbooks for the theory of linear and selfadjoint operators are Kato [108], Reed–Simon [159] and [156]. Some further introductions are given e.g., by Amrein [9], Hislop–Sigal [92] or Stollmann [184].

§*2.1.3* Proofs of the selfadjointness criteria are given by Amrein [9, p. 26 ff], Kato [108, p.267 ff] or Reed–Simon [159, p. 255 ff].

§*2.1.4* Perturbations of selfadjoint operators are studied in the same textbooks mentioned for §2.1.3. The Kato–Rellich Theorem is in Amrein [9, p. 49], Kato [108, p. 287], Reed–Simon [156, p. 162]. The possibility to allow an $A-$ bound equal to 1 goes back to Wüst and is studied in Kato [108, p. 289], Reed–Simon [156, p. 164]. For the theory of sesquilinear forms a short and for our purposes sufficient summary is given in Stollmann [184, p. 115 ff]. In Theorem 2.1.28 the square root $A^{1/2}$ exists and is defined in Corollary 2.3.6. The KLMN (Kato–Lax–Lions–Milgram–Nelson)-theorem can be found in Reed-Simon [156, p.167 with notes on p. 323]. The correct definition of the form domain for semibounded operators is in Reed-Simon [156, p. 168] and also in Simon [173, p. 38].

Form sums are more general than operator sums. Sometimes operator sums may not make sense, because the intersection of the domains is zero, but the form sum may exist. The following easy example is due to Brasche [33].

Consider $-\frac{d^2}{dx^2}$ in $L^2(\mathbb{R})$ and assume a sequence of potentials

$$V_n(x) = \sum_{j=1}^{n} \frac{1}{2^j} \frac{1}{|x-x_j|^{1/2}} e^{-|x-x_j|}$$

with $x \in \mathbb{R}\setminus\{x_1,\dots,x_n\}$ and where $\{x_j\}$ are the enumerated rational numbers in $\mathbb{R}$. Then $V_n \in L^1(\mathbb{R})$ and $\{V_n(.)\}$ has an L^1-limit called V. V is a non-negative function in $L^1(\mathbb{R})$. Hence the form sum $-\frac{d^2}{dx^2}+V(.)$ defines a selfadjoint operator in $L^2(\mathbb{R})$.

On the other hand dom $\left(-\frac{d^2}{dx^2}\right) \subseteq C(\mathbb{R})$, continuous functions of $\mathbb{R}$. But

$$\int_{\mathcal{O}} V(x)^2 dx = \infty$$

for any non-empty open set $\mathcal{O}$ of $\mathbb{R}$. Hence $C(\mathbb{R})$ is not in $\operatorname{dom}(V)$ which implies

$$\operatorname{dom}\big(-\frac{d^2}{dx^2}\big) \cap \operatorname{dom}(V) = \{\theta\},$$

such that the operator sum does not define a selfadjoint operator in $L^2(\mathbb{R})$.

The operator sums does not define a selfadjoint operator in $L^2(\mathbb{R})$.

Section 2.2

The spectral theory is standard. For the non-selfadjoint part a good reference is Engel–Nagel [81, p. 238 ff]. In particular the estimate $\|R(z_0, A)\| \geq [\operatorname{dist}\big(z_0, \sigma(A)\big)]^{-1}$ should be emphasised (see Proposition 2.2.5). From that follows in Remark 2.2.10 for selfadjoint operators A the equality

$$\|R(z_0, A)\| = [\operatorname{dist}\big(z_0, \sigma(A)\big)]^{-1}.$$

In many textbooks one can find

$$\|R(z_0, A)\| \leq [\operatorname{dist}\big(z_0, \sigma(A)\big)]^{-1}$$

for selfadjoint A. However there is no selfadjoint operator where $\|R(z_0, A)\|$ is strictly smaller than $[\operatorname{dist}\big(z_0, \sigma(A)\big)]^{-1}$. Moreover, there are non-selfadjoint operators where the norm of the resolvent is strictly larger than the inverse of the distance between the resolvent value and the spectrum.

In general further good references for this section and the rest in this chapter are Reed–Simon [159], Amrein [9], Baumgärtel–Wollenberg [20], Hislop–Sigal [92], Weidmann [189].

Section 2.3

The resolution of the identity is described in Kato [108]. The proof of the spectral theorem (Theorem 2.3.4) is also in Kato [108, p. 360 ff]. The spectral mapping theorem is proved for resolvents in Engel–Nagel [81, p. 243] and for semigroups in Engel–Nagel [81, p. 270]. One can also find it in Reed–Simon[159, p. 222].

Section 2.4

A comprehensive overview of spectral measures and, spectral integrals etc. are given in Baumgärtel–Wollenberg [20, p. 45 ff]. Stone's formula (Theorem 2.4.2) is for instance in Reed–Simon [159, p.237]. The characterization of the spectrum in terms of the spectral measure is explained in Weidmann [189, p. 305 ff]. Theorem 2.4.5 (Weyl's Criteria) is taken from Reed–Simon [159, p. 237].

Section 2.5

The Hahn–Hellinger form of the spectral theorem is stated for C^* algebras by Sunder [186] and our proof is based on the proof of Theorem 3.5.12 given there.

Section 2.6

The definition of $\sigma_p, \sigma_{\text{ess}}, \sigma_{ac}, \sigma_{sc}$ and so on is standard and given in the textbooks mentioned in notes for Section 2.1 and Section 2.2. The trace formula in Equation (2.6.7) for integral operators is given by Brislawn [36].

Section 2.7

This section is taken from Amrein [9, p. 82 ff]. The proof of Proposition 2.7.2 can be found, e.g., in Akhiezer–Glazmann [1]. The proof of Proposition 2.7.3 is given in Amrein [9], page 110. The mean ergodic theorem is proved in Amrein [9, p. 106]. In Definition 2.7.5 the subspace properties of $\mathfrak{H}_{\text{bound}}, \mathfrak{H}_{\text{scatt}}, \mathfrak{H}_{\text{scatt, mean}}$ has to be proved, of course. For Proposition 2.7.6 and 2.7.7 see Amrein [9, p. 129 ff]. The formulation of the RAGE Theorem is given in Reed–Simon [158, p. 341]. It is due to Ruelle [168], Amrein–Georgescu [11], and Enss [82]. The simplest example of an operator without a singularly continuous spectrum is the Laplacian. Starting from this, one can find several conditions on potentials functions such that the perturbed operator has also no singularly continuous spectrum.

3

Criteria for Identifying the Spectrum

To identify the spectrum of a selfadjoint operator of interest there are several methods used. Almost all of them have their roots in analyzing the properties of measures and their transforms.

These properties are then translated to criteria on the operators or functions of them via the spectral theorem. In the case of differential and difference operators, however, criteria are also formulated by studying the associated solutions.

3.1 Borel Transform

In this section we present the Aronszajn–Donoghue theory of rank one perturbations of selfadjoint operators. This theory and its improvements were used to solve several spectral problems.

Consider a selfadjoint operator H on $\mathcal{H}$ and suppose ϕ is a normalized vector in $\mathcal{H}$ and consider the orthogonal projection P_ϕ onto the span of ϕ. We consider the operators

$$H_\lambda = H + \lambda P_\phi, \quad \lambda \in \mathbb{R} \tag{3.1.1}$$

which are rank one perturbations of H. We set $H_0 = H$.

We start with some preliminary lemmas.

Lemma 3.1.1. *Let ψ be a unit vector in $\mathcal{H}$ and let H_λ be as above. Then, for all $z \in \mathbb{C}^+$,*

$$\begin{aligned}
\langle \psi, (H_\lambda - z)^{-1}\phi\rangle &= \frac{\langle \psi, (H - z)^{-1}\phi\rangle}{\langle \phi, (H - z)^{-1}\phi\rangle} \cdot \frac{1}{\lambda + \langle \phi, (H - z)^{-1}\phi\rangle^{-1}}, \\
\langle \phi, (H_\lambda - z)^{-1}\phi\rangle &= \frac{1}{\lambda + \langle \phi, (H - z)^{-1}\phi\rangle^{-1}}.
\end{aligned}$$

Proof: The proof of the first relation is by using the second resolvent equation and collecting terms involving H_λ and simplifying. The second relation is obtained from the first by taking $\psi = \phi$. $\square$

Lemma 3.1.2. *Consider a separable Hilbert space $\mathcal{H}$ and H a selfadjoint operator on it. Suppose $\phi \in \mathcal{H}$ and $\|\phi\| = 1$ and consider H_λ as in Equation (3.1.1) for $\lambda \in \mathbb{R}$ and assume that $H_\lambda \neq 0$. Then*

1. *If ϕ is cyclic for H, then it is also cyclic for H_λ.*
2. *The cyclic subspaces $\mathcal{H}_\lambda$ and $\mathcal{H}_{\lambda'}$ generated by H_λ and $H_{\lambda'}$ on ϕ satisfy $\mathcal{H}_\lambda = \mathcal{H}_{\lambda'}$, for $\lambda, \lambda' \in \mathbb{R}$.*

Proof: We prove this lemma for bounded H only. (1) Since ϕ is cyclic for H, we can find, by the Gram–Schmidt procedure an orthonormal basis $\{\phi_n\}$ for $\mathcal{H}$, so that $\phi_0 = \phi$ and in this basis H is tridiagonal so there is no loss of generality to assume that H is tridiagonal to start with. That is, $(Hu_n) = a_n u_{n+1} + b_n u_n + a_{n-1} u_{n-1}$. Then ϕ_0 cyclic for H implies that $\langle \phi_n, H\phi_{n+1} \rangle \neq 0$ for any $n \geq 0$. The reason is that when H is tridiagonal $\langle \phi_k, H\phi_m \rangle = 0, \quad if \quad |k - m| \geq 2$ and so if $\langle \phi_n, H\phi_{n+1} \rangle = 0$ for some n, then $a_n = 0$. Using this fact we can see by induction that $\langle p(H)\phi, \phi_{n+1} \rangle = 0$ for any polynomial p of degree greater than n. This contradicts the assumption that ϕ is a cyclic vector of H. We have, by definition of H_λ and ϕ, that $\langle \phi_0, H_\lambda \phi_0 \rangle = \langle \phi_0, H\phi_0 \rangle + \lambda$ and for any pair $(n, m) \neq (0, 0)$,

$$\langle \phi_n, H_\lambda \phi_m \rangle = \langle \phi_n, H\phi_m \rangle + \lambda \langle \phi_n, P_\phi \phi_m \rangle = \langle \phi_n, H\phi_m \rangle.$$

This shows that ϕ_0 is also cyclic for any H_λ.

(2) We first note that if $\mathcal{H}_0$ is the cyclic subspace generated by H on ϕ, then the orthogonal complement $\mathcal{H}_1$ of $\mathcal{H}_0$ is left invariant by H and H_λ for any λ and on $\mathcal{H}_1$ they are both the same (since the term $\lambda P_\phi \mathcal{H}_1 = \{\theta\}$). Thus we can write $H = B \oplus C$ and $H_\lambda = B_\lambda \oplus C$. Now an argument as in (1) shows that the cyclic subspace generated by B_λ on ϕ agrees with $\mathcal{H}_0$ for any λ, hence the conclusion is valid. $\square$

The idea now is to determine the behaviour of the spectral measures $\mu_\lambda(\cdot) = \langle \phi, P_{H_\lambda}(\cdot)\phi \rangle, \ \ \lambda \neq 0$ associated with H_λ and ϕ in terms of the properties of the measure $\mu_0 = \langle \phi, P_H(\cdot)\phi \rangle$. Therefore consider the Borel transform,

$$F_\lambda(z) = \langle \phi, (H_\lambda - z)^{-1}\phi \rangle = \int_{\mathbb{R}} \frac{1}{x - z} \, d\mu_\lambda(x).$$

If we take ϕ with $\|\phi\| = 1$, then all the μ_λ will be probability measures.

We observe the relations

$$F_\lambda(z) = \frac{F_0(z)}{1 + \lambda F_0(z)}, \quad \mathrm{Im}(F_\lambda(z)) = \frac{\mathrm{Im}(F_0(z))}{|1 + \lambda F_0(z)|^2}, \tag{3.1.2}$$

which are derived using the second resolvent equation

$$(H_\lambda - z)^{-1} = (H - z)^{-1} - \lambda(H_\lambda - z)^{-1} P_\phi (H - z)^{-1}, \tag{3.1.3}$$

or directly using Lemma 3.1.1.

Then the following theorem gives some properties of the measure μ_λ in terms of the measure μ_0.

For this let us define the following sets, following Simon [177], for $\lambda \neq 0$.

$$\begin{aligned} S_{\lambda,0} &= \{x \in \mathbb{R} : (DF_0)(x) < \infty, F_0(x + i0) = -\lambda^{-1}\}. \\ S_{\lambda,\infty} &= \{x \in \mathbb{R} : (DF_0)(x) = \infty, F_0(x + i0) = -\lambda^{-1}\}, \\ L_0 &= \{x \in \mathbb{R} : 0 < \ \mathrm{Im}(F_0)(x + i0) < \infty\}, \end{aligned} \tag{3.1.4}$$

$$\text{where} \quad DF_0(x) = \lim_{\epsilon \to 0} \int_{\mathbb{R}} \frac{1}{(x-y)^2 + \epsilon^2} \ d\mu_0(y).$$

Then we have the following properties for the Lebesgue decomposition of μ_λ.

Theorem 3.1.3 (Aronszajn–Donoghue). *Let H_λ, ϕ be as in Equation (3.1.1). Then*

1. *The part $\mu_{\lambda,pp}$ is supported on the set $S_{\lambda,0}$ and is given by*

$$\mu_{\lambda,pp}(\{x\}) = \sum_{y \in S_{\lambda,0}} \frac{1}{\lambda^2 (DF_0)(y)} \delta(x - y).$$

2. *The part $\mu_{\lambda,sc}$ is supported in the set, $S_{\lambda,\infty}$.*
3. *The part $\mu_{\lambda,ac}$ is supported on the set L_0 for all λ.*

Proof: (1) We first note that when $(DF_0)(x) < \infty$ we have the relation,

$$F_0(x + i\epsilon) = F_0(x + i0) + i\epsilon(DF_0)(x) + i\epsilon\delta(x, \epsilon), \quad |\delta(x, \epsilon)| \to 0, \text{as} \ \ \epsilon \to 0,$$

which is shown by subtracting the first two terms on the right-hand side of the last equation from the left-hand side, and using the dominated convergence theorem. Therefore computing the imaginary part of $F_\lambda(x + i\epsilon)$ using the Equation (3.1.2), together with the fact that $\mathrm{R}e(\lambda F_0) = -1, \ \ x \in S_{\lambda,0}$ we find

$$\lim_{\epsilon \downarrow 0} \epsilon \mathrm{Im}(F_\lambda)(x + i\epsilon) = \frac{1}{\lambda^2 (DF_0)(x)}. \tag{3.1.5}$$

By Theorem 1.4.16(3), the above limit gives precisely the atom of μ_λ at the point x. On the other hand when $(DF_0)(x)$ is infinite, then there is a subsequence ϵ_n such that $\mathrm{Im}(F_0(x + i\epsilon_n)) \uparrow \infty$; therefore using equation (3.1.2) we see that $\underline{\lim}_{\epsilon_n \downarrow 0} \epsilon_n \mathrm{Im}(F_0(x + i\epsilon_n)) = 0$. So there is no atom at x.

(2) By Theorem 1.4.16(2), the singular part is supported on the set $\mathrm{Im}F_\lambda(x + i0) = \infty$ but not on the support of $(\mu_\lambda)_{pp}$. Hence, it is supported in $S_{\lambda,\infty}$.

(3) From the Equation (3.1.2) we see that $0 < \mathrm{Im}(F_\lambda)(x + i0) < \infty$ when $0 < \mathrm{Im}(F_0)(x + i0) < \infty$. Therefore the claim follows using Theorem 1.4.16(1). $\square$

We present a very useful criterion to exhibit the point spectrum for random operators. To state this a formula known as the spectral averaging formula that $\int \mu_\lambda \, d\lambda$ is the same as the Lebesgue measure is needed. In the following we denote m to be the Lebesgue measure on $\mathbb{R}$ and dx to be the infinitesimal element of the Lebesgue measure.

Proposition 3.1.4 (Simon). *Let $\mu_\lambda, \lambda \in \mathbb{R}$ be the family of probability measures, associated with the operators H_λ as in Equation (3.1.1). Then*

$$m = \int \mu_\lambda \;\; d\lambda.$$

Proof: For any λ, μ_λ is the spectral measure of H_λ associated with ϕ, we find that

$$\begin{aligned}\int d\lambda \int d\mu_\lambda(x) \;\; \frac{1}{1+x^2} &= \int d\lambda \;\; \int d\mu_\lambda(x) \;\; \frac{1}{2i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right) \\ &= \int d\lambda \;\; \frac{1}{2i}\left(\frac{1}{\lambda + F_0(i)^{-1}} - \frac{1}{\lambda + F_0(-i)^{-1}}\right) \\ &= \int d\lambda \;\; \frac{F_0(-i)^{-1} - F_0(i)^{-1}}{2i} \frac{1}{|\lambda + F_0(i)^{-1}|^2} < \infty,\end{aligned} \tag{3.1.6}$$

using the resolvent equation and Lemma 3.1.1. Here F_0 is the Borel transform of a positive measure μ_0 and hence satisfies $F_0(i) = \overline{F_0(-i)}$. Therefore using Fubini we conclude that the measure $\nu = \int \mu_\lambda \, d\lambda$ satisfies the condition $\int d\nu(x)/(1+x^2) < \infty$, so its Borel transform is defined and is unique. Hence, to show the equality of ν with the Lebesgue measure, it is enough to show by Theorem 1.4.2, the equality of their Borel transforms up to an addition of a real number. Further, since two Borel transforms F and G are the same, up to an addition of a real number, whenever $F(\cdot) - F(-i)$ and $G(\cdot) - G(-i)$ are the same since it is convenient for calculations.

To this end consider the integral, where we subtract the function $1/(i+x)$ for regularization,

$$\begin{aligned}\int_{\mathbb{R}} d\mu_\lambda(x)\left(\frac{1}{x-z} - \frac{1}{x+i}\right) &= F_\lambda(z) - F_\lambda(-i) \\ &= \frac{1}{\lambda - (-F_0(z)^{-1})} - \frac{1}{\lambda - (-F_0(-i)^{-1})},\end{aligned} \tag{3.1.7}$$

by the definition of Borel transform and Equations (3.1.2). Since a Borel transform F of a positive measure maps the upper and the lower half planes, respectively, to themselves, $-(F(z)^{-1})$ also does the same. Therefore a contour integration in the upper half plane to evaluate the integral gives the values

$$\int_{\mathbb{R}} d\lambda \;\; \left(\frac{1}{\lambda - (-F_0(z)^{-1})} - \frac{1}{\lambda - (-F_0(-i)^{-1})}\right) = 2\pi i, \quad z \in \mathbb{C}^+. \tag{3.1.8}$$

This shows that

$$\int d\nu(x) \left(\frac{1}{x-z} - \frac{1}{x+i} \right) = 2\pi i, \quad z \in \mathbb{C}^+. \tag{3.1.9}$$

On the other hand a similar computation along a contour in the upper half plane gives the values

$$\int_{\mathbb{R}} dx \left(\frac{1}{x-z} - \frac{1}{x+i} \right) = 2\pi i, \quad z \in \mathbb{C}^+, \tag{3.1.10}$$

showing that $\nu = m$. □

We show next the converse of the above proposition, that essentially the Lebesgue measure is the only one for which the above spectral averaging result holds.

Theorem 3.1.5. *Suppose σ is a measure on $\mathbb{R}$ such that $\int \frac{1}{1+x^2}\, d\sigma(x) < \infty$. Suppose H is any selfadjoint operator on any Hilbert space $\mathcal{H}$ and ϕ any normalized vector there. Let $\mu_\lambda, \lambda \in \mathbb{R}$ be probability measures associated with H and ϕ as in the Proposition 3.1.4 above. Suppose that the following relation is valid,* **independent** *of the triple $(\mathcal{H}, H, \phi)$,*

$$\sigma = \int \mu_\lambda \ \ d\sigma(\lambda),$$

as measures. Then σ is a multiple of the Lebesgue measure.

Proof: Since the assumption of the theorem is that the spectral averaging formula is independent of the the triple $(\mathcal{H}, H, \phi)$, it is sufficient to prove the theorem for one triple. Let $\mathcal{H} = \mathbb{C}$, $\phi = 1$ and let $H\psi = \eta\psi, \ \psi \in \mathcal{H}, \ \eta \in \mathbb{R}$. Then trivially $\mu_0 = \delta_\eta$ and $\mu_\lambda = \delta_{\lambda+\eta}$, the unit masses at η and $\eta + \lambda$, respectively. Using this and taking Borel transforms of the assumed equation, we get

$$F_\sigma(z) - F_\sigma(i) = F_\sigma(z-\eta) - F_\sigma(i-\eta)$$

for all $z \in \mathbb{C}^+$. Since this relation is true for all selfadjoint H, this relation is also valid as we change $\eta \in \mathbb{R}$. Since F_σ is analytic in the upper half plane, the derivative of $F_\sigma(z-\eta)$ is seen to go to a constant as η goes to ∞. Hence the derivative of F is a constant. From this it follows that $F(z) = a + bz$, where a is a complex constant. Writing this in the standard form $F(z) = a_1 + ia_2 + bz$, we conclude that the measure σ has the Borel transform ia_2, which shows that it must be a multiple of the Lebesgue measure. □

Corollary 3.1.6 *Suppose H is a selfadjoint operator and ϕ is a normalized vector. Let $H_\lambda = H + \lambda P_\phi$ and let μ_λ denote the spectral measure of H_λ with respect to the vector ϕ. Then*

- *for any fixed set B of Lebesgue measure zero $\mu_\lambda(B) = 0$ for almost every λ with respect to the Lebesgue measure and*

- *for almost all pairs (λ, λ') with respect to the Lebesgue measure, the singular parts $\mu_{\lambda,s}$ and $\mu_{\lambda',s}$ are mutually singular.*

Proof: By Proposition 3.1.4 and the fact that the Lebesgue measure (denote it by m) of B is zero, we have using Fubini

$$0 = m(B) = \left(\int d\lambda\ \mu_\lambda\right)(B) \geq \int_a^b \mu_\lambda(B)\ d\lambda > 0,$$

for any $-\infty < a < b < \infty$. This shows that $\mu_\lambda(B) = 0$ for almost all λ, showing the first part.

As for the second part, since each of the measures $\mu_{\lambda,s}$ is singular with respect to the Lebesgue measure, their supports A_λ are of zero Lebesgue measure. The corollary would be false if there is a fixed set A of zero Lebesgue measure such that

$$m(\{\lambda : \mu_{\lambda,s}(A) > 0\}) > 0.$$

But by Proposition 3.1.4 above we must have

$$0 = m(A) = \left(\int d\lambda\ \mu_{\lambda,s}\right)(A) \geq \left(\int_{\{\lambda:\mu_{\lambda,s}(A)>0\}} \mu_{\lambda,s}(A)\ \ d\lambda\right) > 0,$$

which is impossible, therefore the second part follows. □

Thus the spectral averaging result above says that any fixed set S of zero Lebesgue measure cannot be given non-zero measure by the measures μ_λ for a set of λ having positive Lebesgue measure. This fact together with the criterion established in identifying the measure associated with rank one perturbations leads to the following theorem of Simon–Wolff. Let F_0 denote the Borel transform of the measure μ_0 in the following theorem.

Theorem 3.1.7 (Simon–Wolff). *Let H_λ and ϕ be as in Equation (3.1.1) and consider the family of measures μ_λ, $\lambda \in \mathbb{R}$, and suppose that for almost every λ, $\mu_\lambda([a, b]) \neq 0$. Then the following are equivalent.*

1. *For almost all λ, μ_λ is pure point in $[a, b]$.*
2. *For almost every x in $[a, b]$ (with respect to Lebesgue measure) $(DF_0)(x) < \infty$.*

Proof: (2) $\implies$ (1): Let F_λ denote the Borel transform of μ_λ. First note that when $(DF_0)(x) < \infty$, the imaginary part of $F_0(x + i0)$ is zero, and so Im $F_\lambda(x+i0) = 0$ for any λ by Equation (3.1.2). Therefore (2) implies that the absolutely continuous parts of μ_λ give measure zero to $[a, b]$ for every λ, see Theorem 3.1.3(3). Let $S = \{x \in [a, b] : (DF_0)(x) = \infty\}$; then by assumption (2), $|S| = 0$. Therefore by the spectral averaging formula (Proposition 3.1.4), we find that,

$$\int \mu_\lambda(S) d\lambda = \int_S dx = 0, \quad \text{implies} \quad \mu_\lambda(S) = 0, \quad \text{for a.e. } \lambda.$$

We know by Theorem 3.1.3(2) that the singular continuous part of μ_λ is supported in $S_{\lambda,\infty} \cap [a,b]) \subset S \cap [a,b]$, for a.e. λ. Therefore $\mu_{\lambda,sc} = 0$, for a.e. λ, showing that μ_λ is pure point for almost every λ.

(1) $\implies$ (2): If μ_λ has only point masses in $[a,b]$ for almost all λ, then almost everywhere $F_0(x+i0)$ is real in $[a,b]$. By Theorem 3.1.3(1), therefore, μ_λ is supported in the complement of S for almost every λ, implying $\mu_\lambda(S) = 0$ for almost every λ. Hence by the spectral averaging formula $|S| = 0$. $\square$

In the next few theorems we look at the behaviour of rank two perturbations which have applications in establishing purity of absolutely continuous spectra, presented later in Chapter 5.

Let $\mathcal{H}_{\lambda,\phi}$, denote the cyclic subspace generated by a selfadjoint operator H_λ and vector ϕ. Also let

$$F_{\lambda,f}(z) = \langle f, (H_\lambda - z)^{-1} f\rangle, \quad F_{\lambda,f,g}(z) = \langle f, (H_\lambda - z)^{-1} g\rangle, \tag{3.1.11}$$

and let $\mu_{\lambda,f}$ and $\mu_{\lambda,f,g}$ denote the finite complex measures, representing the functions $F_{\lambda,f}$ and $F_{\lambda,f,g}$, respectively (via the spectral theorem and the Borel transform).

Then we start with a technical result.

Theorem 3.1.8. *Let H be a selfadjoint operator and let ϕ, ψ be normalized vectors. Let H_λ denote either $H + \lambda P_\phi$ or $H + \lambda P_\psi$. Suppose that $\mathcal{H}_{\lambda,\phi}$ is not orthogonal to $\mathcal{H}_{\lambda,\psi}$ for some λ. Then the limits*

$$F_{\lambda,\phi,\psi}(x+i0) = \lim_{\epsilon \to 0} F_{\lambda,\phi,\psi}(x+i\epsilon) \quad \textit{and} \quad F_{\lambda,\psi,\phi}(x+i0) = \lim_{\epsilon \to 0} F_{\lambda,\psi,\phi}(x+i\epsilon),$$

both exist almost everywhere with respect to Lebesgue measure.

Proof: We will prove this for $F_{\lambda,\phi,\psi}$, the other case is similar. We observe that by spectral theorem the measure $\mu_{\lambda,\phi,\psi}$ is a finite complex measure with finite total variation $|\mu_{\lambda,\phi,\psi}|$. Since $F_{\lambda,\phi,\psi}$ is the Borel transform of $\mu_{\lambda,\phi,\psi}$, the boundary values $F_{\lambda,\phi,\psi}(x+i0)$ exist finitely almost everywhere and are non-zero almost everywhere unless $\mu_{\lambda,\phi,\psi}$ vanishes identically, by Corollary 1.4.8. $\square$

We next observe, using the resolvent Equation (3.1.3), (in the case when $H_\lambda = H + \lambda P_\phi$), that

$$\begin{aligned} F_{\lambda,\phi}(z) &= \frac{1}{F_{0,\phi}(z)^{-1} + \lambda}, \\ F_{\lambda,\psi}(z) &= F_{0,\psi}(z) - \lambda \frac{F_{0,\phi,\psi}(z) F_{0,\psi,\phi}(z)}{1 + \lambda F_{0,\phi}(z)}. \end{aligned} \tag{3.1.12}$$

In studying the spectral properties of the operators H_λ for almost every λ (with respect to Lebesgue), fixed sets of zero Lebesgue measure do not matter by Proposition 3.1.4. Therefore we fix the set $S \subset \mathbb{R}$ of full measure such that for all points in S, each of the quantities $F_{0,\phi}(x+i0), F_{0,\psi}(x+i0), F_{0,\psi,\phi}(x+i0)$ exists finitely and is non-zero.

Theorem 3.1.9 (Jaksic–Last). *Consider a selfadjoint operator H and a pair of vectors $\phi, \psi \in \mathcal{H}$. Suppose $H_\lambda = H + \lambda P_\phi$ and suppose $\mathcal{H}_{\lambda,\phi}$ is not orthogonal to $\mathcal{H}_{\lambda,\psi}$ for some λ. Then for almost every λ, $\mu_{\lambda,\phi}$ is absolutely continuous with respect to $\mu_{\lambda,\psi}$.*

Proof: We note that if $\mathcal{H}_{\lambda,\phi}$ is not orthogonal to $\mathcal{H}_{\lambda,\psi}$ for one λ, then it is also not orthogonal for all λ, by a simple calculation.

We prove the theorem by showing that in the Lebesgue decomposition of $\mu_{\lambda,\phi} = \mu_{\lambda,\phi,ac} + \mu_{\lambda,\phi,s}$, $\mu_{\lambda,\psi} = \mu_{\lambda,\psi,ac} + \mu_{\lambda,\psi,s}$, the $\mu_{\lambda,\phi,ac}$ is absolutely continuous to $\mu_{\lambda,\psi,ac}$ and $\mu_{\lambda,\phi,s}$ is absolutely continuous to $\mu_{\lambda,\psi,s}$.

We first handle the absolutely continuous parts. The complement of the set S gets measure zero from both $\mu_{\lambda,\phi}$ and $\mu_{\lambda,\psi}$ for almost all λ so we can stick to the set S for all our considerations. For any $x \in S$ we have by taking imaginary parts in Equation (3.1.12),

$$\mathrm{Im}(F_{\lambda,\psi}(x+i0)) - \mathrm{Im}(F_{0,\psi}(x+i0)) = -\lambda \ \mathrm{Im}\frac{F_{0,\phi,\psi}(x+i0)F_{0,\psi,\phi}(x+i0)}{1+\lambda F_{0,\phi}(x+i0)}.$$

We then have

$$\mathrm{Im}(F_{\lambda,\psi}(x)) = \frac{|1+\lambda F_{0,\phi}(x)|^2 \mathrm{Im}(F_{0,\psi}(x)) - \lambda \ \mathrm{Im}\left(T_{\phi,\psi}(x)(1+\lambda\overline{F_{0,\phi}(x)})\right)}{|1+\lambda F_{0,\phi}(x)|^2}, \tag{3.1.13}$$

where we have set $T_{\phi,\psi}(x) = F_{0,\phi,\psi}(x)F_{0,\psi,\phi}(x)$, and wrote x instead of $x + i0$ in the arguments of functions appearing in Equation (3.1.13) for ease of writing. Since at the point $x \in S$, $\mathrm{Im}(F_{0,\phi}(x+i0))$, $F_{0,\phi,\psi}(x+i0)$, $F_{0,\psi,\phi}(x+i0)$ are all non-zero, the denominator on the right-hand side is non-zero for any λ and the numerator is a polynomial of degree 2 which vanishes identically if and only if the following set of equalities is valid:

$$\begin{aligned} &\mathrm{Im}(F_{0,\psi}(x+i0)) = 0, \qquad \mathrm{Im}(T_{\phi,\psi}(x)) = 0, \\ &\mathrm{Re}(T_{\phi,\psi})\mathrm{Im}(F_{0,\phi}(x+i0)) - \mathrm{Im}(T_{\phi,\psi})\mathrm{Re}(F_{0,\phi}(x+i0)) = 0. \end{aligned}$$

Since all the above cannot vanish when $T_{\phi,\psi} \neq 0$ and $\mathrm{Im}(F_{0,\phi}(x+i0)) \neq 0$, we see that the numerator on the right-hand side of Equation (3.1.13) can vanish at most for two values of λ for each $x \in S$. Therefore it follows that the left-hand side is non-zero for each $x \in S \cap \{x : \mathrm{Im}(F_{0,\phi}(x+i0)) \neq 0\}$, except perhaps for two values of λ. Then using Fubini we conclude that for almost all λ, $\mathrm{Im}(F_{\lambda,\psi}(x+i0) \neq 0$. We note that for each λ,

$$S \cap \{x : \mathrm{Im}(F_{0,\phi}(x+i0)) \neq 0\} = S \cap \{x : \mathrm{Im}(F_{\lambda,\phi}(x+i0)) \neq 0\}.$$

Therefore $\mathrm{Im}(F_{0,\phi}(x+i0)) = 0$ (and hence $\mathrm{Im}(F_{\lambda,\phi}(x+i0)) = 0$) whenever $\mathrm{Im}(F_{\lambda,\psi}(x+i0)) = 0$ for almost all pairs (λ, x). This statement shows the absolute continuity of $\mu_{\lambda,\phi,ac}$ with respect to $\mu_{\lambda,\psi,ac}$ for almost all λ.

We show next that the singular parts $\mu_{\lambda,\phi,s}$ and $\mu_{\lambda,\psi,s}$ satisfy the same property. We recall that for almost all λ, the supports of $\mu_{\lambda,\phi,s}, \mu_{\lambda,\psi,s}$ are contained in S and here for each λ the supports of the respective measures are in $\{x : \mathrm{Im}(F_{\lambda,\phi}(x+i0)) = \infty\}$ and $\{x : \mathrm{Im}(F_{\lambda,\psi}(x+i0)) = \infty\}$. We denote $\mu_{\lambda,\phi,s,\perp}$ to be the part of $\mu_{\lambda,\phi,s}$ that is singular with respect to $\mu_{\lambda,\psi,s}$. Then there is a set $S_{\lambda,\phi,s}$ with the property that $\mu_{\lambda,\psi}(S_{\lambda,\phi,s}) = 0$ and $\mu_{\lambda,\phi,s,\perp}$ is supported on $S_{\lambda,\phi,s}$ for almost all λ (by noting that $\mu_{\lambda,\phi,s,\perp}$ is singular with respect to $\mu_{\lambda,\psi}$ also). Therefore we have

$$\lim_{\epsilon\to 0} \frac{F_{\lambda,\psi}(x+i\epsilon)}{F_{\lambda,\phi}(x+i\epsilon)} = 0, \quad a.e. \quad x \quad \text{w.r.t.} \quad \mu_{\lambda,\phi,s,\perp}.$$

This is because for almost all $x \in S_{\lambda,\phi,s}$ w.r.t. $\mu_{\lambda,\phi,s,\perp}$, the absolute value of the numerator on the left-hand side has finite limits while the absolute value of the denominator has infinite limits. On the other hand using Equation (3.1.12) we see that

$$\begin{aligned}\lim_{\epsilon\to 0} \frac{F_{\lambda,\psi}(x+i\epsilon)}{F_{\lambda,\phi}(x+i\epsilon)} &= \lim_{\epsilon\to 0} \frac{F_{0,\psi}(x+i\epsilon)}{F_{\lambda,\phi}(x+i\epsilon)} - \lambda \frac{T_{\phi,\psi}(x)}{F_{0,\phi}(x+i0)} \\ &= -\lambda \frac{T_{\phi,\psi}(x)}{F_{0,\phi}(x+i0)} \neq 0.\end{aligned} \tag{3.1.14}$$

This gives a contradiction showing that $\mu_{\lambda,\phi,s,\perp}$ must be zero, proving the stated absolute continuity of $\mu_{\lambda,\phi}$ with respect to $\mu_{\lambda,\psi}$ for almost every λ. □

In the next theorem let $H_{\lambda,\eta} = H + \lambda P_\phi + \eta P_\psi$ and let $\mathcal{H}_{\lambda,\eta,f}$ denote the cyclic subspace generated by $H_{\lambda,\eta}$ and f.

Theorem 3.1.10 (Jaksić–Last). *Suppose H is a selfadjoint operator and suppose that $\mathcal{H}_{\lambda,\eta,\phi}$ is not orthogonal to $\mathcal{H}_{\lambda,\eta,\psi}$, for some (λ,η). Then for almost every (λ,η), $H_{\lambda,\eta}|\mathcal{H}_{\lambda,\eta,\phi}$ is unitarily equivalent to $H_{\lambda,\eta}|\mathcal{H}_{\lambda,\eta,\psi}$.*

Proof: The proof is an easy consequence of Theorem 3.1.9 and Fubini's theorem. To see this note that for a fixed λ, $H_{\lambda,\eta}$ is a rank one perturbation by ηP_ψ of $H_\lambda = H + \lambda P_\phi$; therefore by Theorem 3.1.9, the spectral measure $\mu_{\lambda,\eta,\psi}$ is absolutely continuous with respect to $\mu_{\lambda,\eta,\phi}$ for almost every η, that is for each fixed λ $\mu_{\lambda,\eta,\psi}(K) = 0$ whenever $\mu_{\lambda,\eta,\phi}(K) = 0$ for almost all η. Now integrating with respect to λ, we see that

$$\int_a^b d\lambda \quad \int_c^d d\eta \quad \mu_{\lambda,\eta,\psi}(K) = 0 \ \text{ if } \ \int_a^b d\lambda \quad \int_c^d d\eta \quad \mu_{\lambda,\eta,\phi}(K) = 0,$$

for any finite $a < b$ and $c < d$. This shows that for almost all pairs (λ,η) (with respect to Lebesgue measure) $\mu_{\lambda,\eta,\psi}$ is absolutely continuous with respect to $\mu_{\lambda,\eta,\phi}$. Now reversing the roles of ϕ and ψ we see the theorem. □

3.2 Fourier Transform

The next statement is the theorem of Wiener on identifying the point spectral part of a measure μ. Recall the definition of the Fourier transform of a finite complex measure from Theorem 1.2.10.

Theorem 3.2.1 (Wiener). *Let μ be a finite complex measure. Then*

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} dt\ |\widehat{\mu}(t)|^2 = \frac{1}{2\pi}\sum_{x\in\mathbb{R}} |\mu(x)|^2,$$

where $\widehat{\mu}$ denotes the Fourier transform of μ.

Proof: Consider the expression on the left-hand side of the above

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} dt|\widehat{\mu}(t)|^2 = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} dt\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\overline{d\mu(\lambda)}d\mu(\kappa)e^{it(\lambda-\kappa)}.$$

Interchanging the integrals by Fubini, since all the measures are finite or over finite sets, and integrating over the variable t, we get

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} dt|\widehat{\mu}(t)|^2 = \lim_{T\to\infty}\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\overline{d\mu(\lambda)}d\mu(\kappa)\frac{\sin(T(\lambda-\kappa))}{T(\lambda-\kappa)}.$$

The right-hand side of the above equals after a change of variable,

$$\lim_{T\to\infty}\frac{1}{2\pi}\int_{\mathbb{R}} d\mu(\kappa)\int_{\mathbb{R}}\overline{d\mu(\lambda+\kappa)}\frac{\sin(T\lambda)}{T\lambda}.$$

We now evaluate the integral

$$\begin{aligned}\lim_{T\to\infty}\int_{\mathbb{R}}\overline{d\mu(\lambda+\kappa)}\frac{\sin(T\lambda)}{T\lambda} &= \lim_{T\to\infty}\left\{\int_{|\lambda|<\epsilon}\overline{d\mu(\lambda+\kappa)}\frac{\sin(T\lambda)}{T\lambda}\right.\\ &\quad \left.+\int_{|\lambda|\geq\epsilon}\overline{d\mu(\lambda+\kappa)}\frac{\sin(T\lambda)}{T\lambda}\right\}.\end{aligned}\tag{3.2.15}$$

Since for $\lambda \neq 0$ the function $(\sin(T\lambda))/T\lambda$ is bounded and goes to zero as T goes to infinity, so the second term is zero for each ϵ. The first term converges to $\overline{\mu(\{x\})}$ since the function $(\sin(T\lambda))/T\lambda$ is bounded and has (the limiting) value 1 at $\lambda = 0$ for each fixed T. This proves the theorem. □

The theorem is also useful in identifying the absence of the point part of a measure, as seen from the following corollary whose proof is obvious from the statement of the previous theorem.

Corollary 3.2.2 *Suppose μ is a finite complex measure such that*

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T} dt\ |\widehat{\mu}(t)|^2 = 0;$$

then μ is a continuous measure.

We already saw the criteria formulated for identifying spectral types in the Propositions 2.7.1, 2.7.2, 2.7.3 and Theorem 2.7.8. Wiener's theorem is at the back of all these.

3.3 Wavelet Transform

In this section we briefly give an application of the wavelet transform method given in section 1.3 to identify spectra of selfadjoint operators. The theorems here are abstract and we have not given concrete and explicit operators to which to apply them.

Definition 3.3.1. Let A be a selfadjoint operator. We call a family S of orthonormal vectors a cyclic family for A if the set

$$\{p(A)f : f \in S, p \text{ a polynomial}\},$$

is a total set.

Theorem 3.3.2. *Suppose A is a selfadjoint operator on $\mathcal{H}$ and ψ a function satisfying Hypothesis 1.3.1. Then*

1. *λ is in the point spectrum of A if for some $f \in \mathcal{H}$, $\|f\| = 1$,*

$$\lim_{a \to 0} \langle f, \psi(\frac{A - \lambda}{a}) f\rangle \neq 0.$$

2. *Let $B \subset \mathbf{R}$ be a Borel set of positive Lebesgue measure. Then $B \cap \sigma_{ac}(A) \neq \emptyset$, if for some $f \in \mathcal{H}$, $\|f\| = 1$,*

$$\lim_{a \to 0} \frac{1}{a} \langle f, \psi(\frac{A - \lambda}{a}) f\rangle \neq 0, \quad \text{a.e. } \lambda \in B.$$

3. *The point spectrum of A in (c, d) is empty iff for some cyclic family $\{f_n\}$ of A, one has for every n,*

$$\underline{\lim}_{a \to 0} \frac{1}{a} \int_c^d \left| \langle f_n, \psi(\frac{A - \lambda}{a}) f_n\rangle \right|^2 \, d\lambda = 0.$$

5. *The absolutely continuous spectrum of A in (c, d) is empty iff for some cyclic family $\{f_n\}$ of A, one has for every n and some $0 < p < 1$,*

$$\underline{\lim}_{a \to 0} \int_c^d \left| \frac{1}{a} \langle f_n, \psi(\frac{A - \lambda}{a}) f_n\rangle \right|^p \, d\lambda = 0.$$

5. *Suppose ψ is positive and suppose there is some $f \in \mathcal{H}$ such that, for all $\lambda \in (a, b)$,*

$$\sup_{a > 0} \frac{1}{a} \langle f, \psi(\frac{A - \lambda}{a}) f\rangle < \infty.$$

Then there is no singular spectrum of A in (a, b).

Proof: (1) This follows from Theorem 1.3.2(1) since $\langle f, \psi(\frac{A-\lambda}{a} f\rangle = (\psi_a * \mu)(\lambda)$, where the measure $\mu = \langle f, P_A(\dot{)} f\rangle$.

(2) Suppose f satisfies the condition given here and the measure $\mu = \langle f, P_A(\dot{)} f\rangle$ and $\mu_{ac}(B) = 0$. Then by the definition of the absolutely continuous part of μ we should have $\frac{d\mu_{ac}}{dx}(x) = 0$ for almost every $x \in B$. This means that $d^1_\mu(x) = 0$ for almost every $x \in B$ for this μ, since we already know from Theorem 1.1.8 that for a singular measure ν, $d^1_\nu(x) = 0$ for almost every x with respect to the Lebesgue measure. Therefore we arrive at a contradiction to the Theorem 1.3.2(3) giving the result.

(3) and (4) are consequences of Corollary 1.3.5(1) and (2) respectively applied to the measures $\mu_n = \langle f_n, P_A(\dot{)} f_n\rangle$ for each n. Since f_n is a cyclic family the result also holds for the total spectral measure $\mu = \sum_{n=1}^\infty 2^{-n}\mu_n$.
(5) Let f satisfy the condition given here; then for the measure $\mu = \langle f, P_A(\dot{)} f\rangle$ we have $\lim_{a\to 0} \frac{1}{a^\alpha}\psi_a * \mu(\lambda) = 0$ for every $\lambda \in (a, b)$ for any $0 \leq \alpha < 1$, hence the result. □

Using these for random families of operators, which we will discuss in Chapters 4 and 5, we give further criteria. Consider H^ω to be a random family of selfadjoint operators such that for any bounded continuous function ψ, the family of operators $\psi(H^\omega)$ is weakly measurable. Then the above theorems extend to these families as follows.

Theorem 3.3.3. *Let ψ be a function as in Hypothesis 1.3.1 and let H^ω be a random family of selfadjoint operators such that $\psi(H^\omega)$ is weakly measurable. The absolutely continuous spectrum of H^ω in (c, d) is empty for almost every ω iff for some cyclic family $\{f_n\}$ of H^ω, one has for every n and some $0 < p < 1$,*

$$\varliminf_{a\to 0} \mathbb{E}\left(\int_c^d \left|\frac{1}{a}\langle f_n, \psi(\frac{H^\omega - \lambda}{a}) f_n\rangle\right|^p \, d\lambda\right) = 0.$$

Proof: By Fatou's lemma the condition in the theorem implies that

$$\varliminf_{a\to 0} \int_c^d \left|\frac{1}{a}\langle f_n, \psi(\frac{H^\omega - \lambda}{a}) f_n\rangle\right|^p \, d\lambda = 0, \quad \text{for a.e. } \ \omega,$$

which by Corollary 1.3.5(2) implies the absence of absolutely continuous spectrum in (a, b) for almost every ω. □

3.4 Eigenfunctions

We already saw the criterion of Weyl, Theorem 2.4.5, using vectors in the Hilbert space to identify a part of the spectrum of a selfadjoint operator. In this section we present a few more, but finer criteria that emerged recently. These criteria use the relative rate of decay of $\|(H - E)f_n\|$ for a selfadjoint operator H, for a sequence f_n of approximate eigenvectors.

We recall the definition of the α-dimensionality of the spectrum. We also recall the definition of D_μ^α stated after Theorem 1.3.6 for any positive finite measure μ.

Definition 3.4.1. Let H be a selfadjoint operator in a Hilbert space $\mathcal{H}$. Then a collection of vectors $\{\psi_n\}$ of norm 1 are said to be SOAEV (sequence of approximate eigenvectors) for H at a number $E \in \mathbb{R}$, if $\|(H-E)\psi_n\| \to 0$ as $n \to \infty$. The $\{\psi_n\}$ is said to be rooted at a vector $\phi \in \mathcal{H}$, if

$$\varliminf_{n\to\infty} \frac{|\langle \psi_n, \phi\rangle|^2}{\|(H-E)\psi_n\|} > 0.$$

The SOAEV is said to be optimally rooted at ϕ with respect to a probability measure μ if

$$\varliminf_{n\to\infty} \frac{|\langle \psi_n, \phi\rangle|^2}{\|(H-E)\psi_n\|(\operatorname{Im}\int_{\mathbb{R}} \frac{1}{x-E-i\|(H-E)\psi_n\|)} d\mu(x))} > 0.$$

In the next theorem we denote by μ^ϕ the spectral measure of the operator H associated with the vector $\phi \in \mathcal{H}$.

Theorem 3.4.2 (Christ–Kiselev–Last). *Let H be a selfadjoint operator acting on a Hilbert space $\mathcal{H}$ and let $\{\psi_n\}$ be a SOAEV for H at $E \in \mathbb{R}$. Then (i) the following three sets are the same:*

$$\left\{ E : \overline{\lim}_{\epsilon\to 0} \frac{\mu^\phi((E-\epsilon, E+\epsilon))}{2\epsilon} > 0 \right\},$$

$$\{E : \ \ \textit{there exists a} \ \ \textit{SOAEV for} \ \ H \ \ \textit{and} \ \ E \ \ \textit{rooted at} \ \ \phi\}$$

and

$$\{E : \textit{there exists a SOAEV for } H \textit{ and } E \textit{ which is optimally rooted at } \phi \textit{ w.r.t } \mu^\phi\}.$$

(ii) for $\mu = \mu^\phi$, and every $E \in \mathbb{R}$ and $\alpha \in [0,1]$, we have $D_\mu^\alpha(E) = \infty$ (respectively > 0) if and only if there exists a SOAEV $\{\psi_n\}$ for H and E, such that

$$\varliminf_{n\to\infty} \frac{|\langle \psi_n, \phi\rangle|^2}{\|(H-E)\psi_n\|^\alpha} = \infty \quad (\textit{respectively} \quad > 0).$$

(iii) Let ν be any finite positive Borel measure on $\mathbb{R}$, and let $S \subset \mathbb{R}$ be a Borel set. If for each $E \in S$ there exists a SOAEV for H and E that is optimally rooted at ϕ w.r.t. ν, then $\nu|S$ is absolutely continuous w.r.t. μ^ϕ.

The criteria given above are quite abstract, without depending upon a special form of the Hilbert space $\mathcal{H}$. In the case of L^2 spaces on domains and for differential operators some new criteria are available which go beyond the Weyl criterion and identify a point in a component of the spectrum via the behaviour of solutions to differential (or difference) equations. See the notes for more on this.

3.5 Commutators

It is possible to determine the spectral nature of a selfadjoint operator based on the behaviour of its commutator with some other given operator.

The use of commutator methods goes back to Kato and Putnam. Especially the positive commutator was exploited by Lavine for showing absolute continuity of the spectrum associated with repulsive potentials. Mourre proved a major abstract theorem on the local spectrum of an operator based on the local positivity of a commutator, a theorem that was a major leap in this area.

Theorem 3.5.1. *Suppose H is a selfadjoint operator in a Hilbert space $\mathcal{H}$. Let A be a densely defined selfadjoint operator with β in its resolvent set and let (a,b) be an interval.*

1. *If $\|(A-\overline{\beta})^{-1}(H-z)^{-1}(A-\beta)^{-1}\| \leq \frac{C}{|\mathrm{Im}(z)|^{\alpha}}$, $\forall z \in \mathbb{C}^{+}$, $\mathrm{Re}(z) \in [a,b]$, there is no singular spectrum of Hausdorff dimension less than $(1-\alpha)$ for H in $[a,b]$, $0 \leq \alpha < 1$.*
2. *In particular if $\alpha = 0$ in the above there is no singular spectrum for H in $[a,b]$.*

Proof: (1) Since A is a selfadjoint operator and β is in its resolvent set, the range of $(A-\beta)^{-1}$ is dense, which implies that if $\{\phi_k\}$ is an orthonormal basis for $\mathcal{H}$, then the set $\{\eta_k = (A-\beta)^{-1}\phi_k\}$ of vectors is total in $\mathcal{H}$. We choose and fix some orthonormal basis $\{\phi_k\}$ for $\mathcal{H}$ and set $\mu_k = \langle \eta_k, P_H(\cdot)\eta_k\rangle$. We then have

$$\begin{aligned}\langle (A-\beta)^{-1}\phi_k, (H-z)^{-1}(A-\beta)^{-1}\phi_k\rangle &= \langle \eta_k, (H-z)^{-1}\eta_k\rangle \\ &= \int_{\mathbb{R}} \frac{1}{x-z} d\mu_k(x).\end{aligned}$$

Using the assumption in (1) of the theorem we see that $|\int_{\mathbb{R}} \frac{1}{x-z} d\mu_k(x)| \leq C/|\mathrm{Im}(z)|^{\alpha}$, which implies that $C^{1-\alpha}_{\mu_k}(\mathrm{Re}(z)) < \infty$ which by Theorem 1.4.15 implies that $D^{1-\alpha}_{\mu_k}(\mathrm{Re}(z)) < \infty$, $\mathrm{Re}(z) \in [a,b]$. Denote $E = \mathrm{Re}(z)$. Since $D^{1-\alpha}_{\mu_k}(E) < \infty$, $E \in [a,b]$, for any $\sigma < 1-\alpha$, one has

$$\lim_{\epsilon \longrightarrow 0} \frac{\mu_k(E-\epsilon, E+\epsilon)}{\epsilon^{\sigma}} = \lim_{\epsilon \longrightarrow 0} \left(\frac{\mu_k(E-\epsilon, E+\epsilon)}{(2\epsilon)^{1-\alpha}} \frac{(2\epsilon)^{1-\alpha}}{\epsilon^{\sigma}} \right) = 0.$$

By Corollary 1.4.14, this estimate implies that the exact Hausdorff dimension of μ_k is at least $(1-\alpha)$.

The estimate is valid for each k, so any total spectral measure also has the same property from which the assertion follows.

(2) This part of the theorem follows by using the Theorem 1.4.16(2), and going through the argument as in (1) for the total spectral measure (see Definition 2.5.6). □

The next criterion involves a pair of operators for which we need a definition.

Definition 3.5.2. Let H be a selfadjoint operator and A a closed operator on a separable Hilbert space $\mathcal{H}$. Then A is said to be H-smooth if for almost every t, $e^{-itH}f \in dom(A), \quad \forall f \in \mathcal{H}$ and

$$\sup_{\|f\|=1} \int_{\mathbb{R}} \|Ae^{-itH}f\|^2 \quad dt \quad < \infty.$$

Theorem 3.5.3. *Let H be a selfadjoint operator on a separable Hilbert space $\mathcal{H}$ and let A be a closed H-smooth operator. Then* $\mathrm{ran}(A^*) \subset \mathcal{H}_{ac}(H)$.

The smoothness is a very strong assumption and positive commutators provide a way of verifying such a smoothness criterion. Given below are a few theorems that exploit this fact in various ways.

Theorem 3.5.4 (Kato–Putnam). *Suppose A, B are two bounded selfadjoint operators with $C = i[A, B] = i(AB - BA) > 0$. Then $C^{1/2}$ is A-smooth and B-smooth. If in addition* $\ker(C) = \{\theta\}$, *then A and B have purely absolutely continuous spectrum.*

We recall here that for any selfadjoint operator H we can define a collection of Hilbert spaces $\mathcal{H}_n, n \in \mathbb{Z}$, associated with H as those defined using the inner product

$$\langle f, g\rangle_n = \int_{\mathbb{R}} (1 + |\lambda|)^{n/2} d\langle f, P_H(\lambda)g\rangle. \tag{3.5.16}$$

We will put the conditions of the theorem in a definition.

Definition 3.5.5. Let H, A be selfadjoint operators in a separable Hilbert space $\mathcal{H}$ with domains $\mathrm{dom}(H), \mathrm{dom}(A)$ respectively. Suppose $S \subset \mathbb{R}$ is an interval and $P_H(S)$ the spectral projection of H associated with S moreover let $|S|$ denote its Lebesgue measure. Then we say A is locally conjugate to H on S if the following conditions hold:

1. dom(A) $\cap\, \mathrm{dom}(H)$ is a core for H.
2. e^{itA} leaves the domain of H invariant and for each $f \in \mathrm{dom}(H)$, $\sup_{|t|<1} \|He^{itA}f\| < \infty$.
3. The form $i[H, A]$ defined on $\mathrm{dom}(A) \cap \mathrm{dom}(H)$ is bounded below and is closable. Further the selfadjoint operator C associated with the closure of this form admits a domain containing $\mathrm{dom}(H)$.
4. There is a number $\alpha > 0$ such that

   $$P_H(S)CP_H(S) \geq \alpha P_H(S) + P_H(S)KP_H(S),$$

   for some compact selfadjoint operator K.
5. The form defined on $\mathrm{dom}(A) \cap \mathrm{dom}(H)$ by $[C, A]$ is bounded as a map from $\mathcal{H}_{+2}$ into $\mathcal{H}_{-2}$.

The existence of a local conjugate to H on S implies that the singular continuous part of the spectrum is absent in S, which is the content of the following theorem. The idea is to use the positivity estimate on the commutator to obtain an a priori bound for the norm of the resolvent operator $(H + i\epsilon P_H(S)CP_H(S) - z)^{-1}$ (independent of $\mathrm{Im}(z)$ but depending on ϵ) between two appropriate spaces and to use this bound in a differential inequality to show that it is indeed bounded independent of ϵ.

Theorem 3.5.6 (Mourre). *Suppose H, A are selfadjoint operators on a separable Hilbert space $\mathcal{H}$ and S an interval. Suppose A is locally conjugate to H on S. Then*

1. *there is only a finite number of eigenvalues of H in S.*
2. *$\sigma_{sc}(H) \cap S = \emptyset$.*

Proof: Part (1) of this theorem is proved in Corollary 3.5.9 and part (2) is proved in Lemma 3.5.11. □

We will prove a few propositions before proving this theorem.

Proposition 3.5.7. *Let H, A be selfadjoint operators satisfying the conditions (1)-(3) of Definition 3.5.5. Then*

1. *$(H - z)^{-1}$ leaves $\mathrm{dom}(A)$ invariant for all $z \in \rho(H)$.*
2. *$(A \pm i\lambda)^{-1}$ leaves $\mathrm{dom}(H)$ invariant for large λ. In addition $(H+i)i\lambda(A+i\lambda)^{-1}(H+i)^{-1}$ goes strongly to the identity as $|\lambda| \to \infty$.*

Proof: (1) Since A is a selfadjoint operator we consider the spectral projection $P_A((-r, r)) = E_A(r-) - E_A(-r)$, defined from the spectral family E_A associated with A, for any number r. This family of projections converges strongly to the identity as $r \to \infty$. We consider for $f \in \mathrm{dom}(A)$ and $z \in \rho(H)$,

$$\|P_A((-r, r))A(H - z)^{-1}f\|,$$

which is clearly bounded, since r is finite. Then commuting A to the right we get

$$\begin{aligned}\|P_A((-r, r))A(H - z)^{-1}f\| &\leq \|P_A((-r, r))(H - z)^{-1}Af\| \\ &+ \quad \|P_A((-r, r))(H - z)^{-1}[A, H](H - z)^{-1}f\| \\ &\leq \frac{1}{|\mathrm{dist}(z, \sigma(H))|}\left\{\|Af\| + \|[A, H](H - z)^{-1}f\|\right\}.\end{aligned} \tag{3.5.17}$$

We note that by Definition 3.5.5(3), whereby $\mathrm{dom}(H) \subset \mathrm{dom}(C)$, the right-hand side is finite for each $f \in \mathrm{dom}(A)$ and each $z \in \rho(A)$. Therefore the left-hand side is finite for each r and is an increasing bounded sequence in r. So its limit $\|A(H - z)^{-1}f\|$ is also finite. This proves the required statement.

(2) By (1) we have, in the operator sense

$$\begin{aligned}&(A+i\lambda)^{-1}(H+i)^{-1}-(H+i)^{-1}(A+i\lambda)^{-1}\\&=(A+i\lambda)^{-1}\left\{(H+i)^{-1}A-A(H+i)^{-1}\right\}(A+i\lambda)^{-1}\\&=(A+i\lambda)^{-1}\left\{(H+i)^{-1}[A,H](H+i)^{-1}\right\}(A+i\lambda)^{-1},\end{aligned}\tag{3.5.18}$$

where the last equality holds in the sense of quadratic forms on $\mathcal{H}$. By Definition 3.5.5(3), the operator $C(H+i)^{-1}$ is bounded. Using this fact together with the bound $\|(A+i\lambda)^{-1}\| \leq 1/|\lambda|$, we see that the operator $B(\lambda) = C(H+i)^{-1}(A+i\lambda)^{-1}$ is bounded by $\frac{c}{|\lambda|}$ and so $\|B(\lambda)\|$ goes to zero as $|\lambda|$ tends to ∞. Therefore

$$(A+i\lambda)^{-1}(H+i)^{-1}(1-B(\lambda))-(H+i)^{-1}(A+i\lambda)^{-1}\to 0$$

in $\mathcal{H}$. This proves the proposition since for sufficiently large λ, $(1-B(\lambda))$ is invertible and $i\lambda(A+i\lambda)^{-1}(1-B(\lambda))^{-1}$ converges strongly to the identity as $|\lambda|\to\infty$. □

Proposition 3.5.8. *Let H, A be as in the Proposition 3.5.7 and C as in (3) of Definition 3.5.5. Then*

1. for all $f \in \mathrm{dom}(H)$,

$$Cf = \lim_{|\lambda|\to\infty}[H, Ai\lambda(A+i\lambda)^{-1}]f.$$

2. If f is an eigenvector of H, then $\langle f, Cf\rangle = 0$.

Proof: (1) Let $f \in \mathrm{dom}(H)$ and $g \in \mathrm{dom}(A)\cap\mathrm{dom}(H)$. Then by Proposition 3.5.7(2), we find that for sufficiently large λ,

$$\begin{aligned}\langle f,[H,Ai\lambda(A+i\lambda)^{-1}]g\rangle &= \langle f,(HAi\lambda(A+i\lambda)^{-1}-Ai\lambda(A+i\lambda)^{-1}H)g\rangle\\&=\langle f,[HA-AH]i\lambda(A+i\lambda)^{-1}g\rangle\\&\quad+\langle Af, Hi\lambda(A+i\lambda)^{-1}-i\lambda(A+i\lambda)^{-1}Hg\rangle\\&=\langle f, Ci\lambda(A+i\lambda)^{-1}g\rangle\\&\quad+\langle f, A(A+i\lambda)^{-1}Ci\lambda(A+i\lambda)^{-1}g\rangle.\end{aligned}\tag{3.5.19}$$

Proposition 3.5.7 (2) implies that

$$\begin{aligned}Ci\lambda(A+i\lambda)^{-1}(H+i)^{-1}&=C(H+i)^{-1}(H+i)i\lambda(A+i\lambda)^{-1}(H+i)^{-1}\\&\to C(H+i)^{-1},\quad \text{as } |\lambda|\to\infty.\end{aligned}$$

This convergence together with the fact that $A(A+i\lambda)^{-1}\to\mathbb{1}$, implies the convergence

$$\lim_{|\lambda|\to\infty}[H, i\lambda(A+i\lambda)^{-1}]f = Cf,$$

for all $f\in\mathrm{dom}(H)$, which shows (1).

If f is an eigenvector of H it is in $\mathrm{dom}(H)$ and $Hf = Ef$, for some real E, so

$$\langle f, Cf\rangle = \lim_{|\lambda|\to\infty} \langle f, [H, i\lambda(A+i\lambda)^{-1}]f\rangle = 0.$$

This proves (2). □

Corollary 3.5.9 *Suppose H is a selfadjoint operator and A is locally conjugate to H on S. Then there are only finitely many eigenvalues of H in S.*

Proof: Suppose there are infinitely many eigenvalues of H in S and suppose the normalized eigenfunctions associated with these eigenvalues are denoted by $\{f_n\}$. Let $\alpha > 0$ and K compact, selfadjoint be such that

$$P_H(S)i[H, A]P_H(S) \geq \alpha P_H(S) + P_H(S)KP_H(S).$$

This inequality implies that for each n we have

$$\langle f_n, P_H(S)i[H, A]P_H(S)f_n\rangle \geq \alpha\langle f_n, P_H(S)f_n\rangle + \langle f_n, P_H(S)KP_H(S)f_n\rangle.$$

Then by Proposition 3.5.8(2), the left-hand side is zero, so we have for each n,

$$\alpha + \langle f_n, Kf_n\rangle = \alpha\langle f_n, f_n\rangle + \langle f_n, Kf_n\rangle \leq 0.$$

But f_n goes to zero weakly, hence $|\langle f_n, Kf_n\rangle|$ goes to zero, since K is compact, so this inequality is not possible for infinitely many n when $\alpha > 0$, hence the corollary follows. □

Proposition 3.5.10. *Let H be a selfadjoint operator in a separable Hilbert space $\mathcal{H}$ with domain* $\mathrm{dom}(H)$ *and B^*B a bounded positive operator on $\mathcal{H}$ and let $\epsilon\,\mathrm{Im}(z) > 0$. Then*

1. *$(H - z - i\epsilon B^*B)$ is invertible.*
2. *Let B' be an operator with $B'^*B' \leq B^*B$ and D any bounded operator on $\mathcal{H}$. Then*

$$\|B'(H - z - i\epsilon B^*B)^{-1}D\| \leq \frac{1}{\sqrt{\epsilon}}\|D^*(H - z - i\epsilon B^*B)^{-1}D\|^{\frac{1}{2}}.$$

Proof: (1) Since B^*B is bounded, $H - z - i\epsilon B^*B$ is closed on $\mathrm{dom}(H)$. When $f \in \mathrm{dom}(H)$ and $\epsilon\,\mathrm{Im}(z) > 0$ (so that $\|(\mathrm{Im}(z) + \epsilon B^*B)f\|^2 \geq \|\mathrm{Im}(z)f\|^2 + \|\epsilon B^*Bf\|^2$), we have

$$\begin{aligned}\|(H - z - i\epsilon B^*B)f\|^2 &= \|(H - \mathrm{Re}(z))f\|^2 + \|(\mathrm{Im}(z) + \epsilon B^*B)f\|^2 \\ &\qquad - 2\mathrm{Im}\langle(H - \mathrm{Re}(z))f, \epsilon B^*Bf\rangle \\ &= \|(H - \mathrm{Re}(z))f\|^2 + \|\mathrm{Im}(z)f\|^2 + \|\epsilon B^*B)f\|^2 \\ &\qquad - 2\mathrm{Im}\langle(H - \mathrm{Re}(z))f, \epsilon B^*Bf\rangle \\ &\geq (\mathrm{Im}(z))^2\|f\|^2.\end{aligned} \tag{3.5.20}$$

From this inequality and the fact that $(H - z - i\epsilon B^*B)$ is a closed operator, it follows that $(H - z - i\epsilon B^*B)$ is injective with closed range equal to $\mathcal{H}$, which is seen by noting that if $f \in \mathrm{dom}(H)$ and $(H - z - i\epsilon B^*B)f = 0$ then $f = 0$ by the above inequality. Therefore the open mapping theorem shows that $(H - z - i\epsilon B^*B)$ has a bounded inverse.

(2) We set $G_z(\epsilon) = (H - z - i\epsilon B^*B)^{-1}$ and note the inequalities,

$$\begin{aligned}\|B'G_z(\epsilon)D\|^2 &= \frac{1}{\epsilon}\|D^*G_z^*(\epsilon)\epsilon B'^*B'G_z(\epsilon)D\| \\ &\leq \frac{1}{2\epsilon}\|D^*G_z(\epsilon)^*\ 2(\mathrm{Im}(z) + \epsilon B^*B)G_z(\epsilon)D\| \\ &\leq \frac{1}{2\epsilon}\|D^*(G_z(\epsilon)^* - G_z(\epsilon))D\| \\ &\leq \frac{1}{2\epsilon}(\|D^*G_z(\epsilon)^*D\| + \|DG_z(\epsilon)D\|) \leq \frac{1}{\epsilon}\|D^*G_z(\epsilon)D\|,\end{aligned} \tag{3.5.21}$$

from which (2) follows. □

Lemma 3.5.11. *Let H be a selfadjoint operator with a local conjugate operator A on S. Then for each $x \in S$, which is not an eigenvalue of H, there is an interval S_x such that $\sigma_{sc}(H) \cap S_x = \emptyset$.*

Proof: Since A is locally conjugate to H on S, there exists an $\alpha > 0$ such that

$$P_H(S_x)i[H, A]P_H(S_x) \geq \alpha P_H(S_x) + P_H(S_x)KP_H(S_x),$$

for some compact, selfadjoint operator K. Let $S_0 \subset S$ denote the set S with the (finitely many) eigenvalues removed. Then for any $x \in S_0$, one has that $P_H((x-\delta, x+\delta)) \to 0$ strongly as $\delta \to 0$. Therefore for sufficiently small δ, replacing P_H by a smooth function vanishing outside $(x-\delta, x+\delta)$ and 1 on $(x - \delta/2, x + \delta/2)$ if necessary,

$$P_H((x-\delta, x+\delta))KP_H((x-\delta, x+\delta)) \leq \frac{\alpha}{2}P_H((x-\delta, x+\delta)).$$

Then on $S_x = (x - \delta, x + \delta)$,

$$P_H(S_x)i[H, A]P_H(S_x) \geq \frac{\alpha}{2}P_H(S_x). \tag{3.5.22}$$

We note that since the constant α is the same for the entire set S (hence the set S_0), by assumption, the above estimate is valid with the same α for each x but for a different set S_x as long as $x \in S_0$. Once we have this inequality, we will show that for any compact subset $[a, b] \subset S_x$,

$$\sup_{\lambda\in[a,b],\epsilon_1>0} \|\,|A+i|^{-1}\,(H - \lambda - i\epsilon_1)^{-1}\,|A+i|^{-1}\,\| < \infty. \tag{3.5.23}$$

This estimate implies, by Theorem 3.5.1, that there is no singular spectrum in $[a, b]$, which means the absence of singular continuous spectrum there, since there are no eigenvalues in $[a, b]$. This procedure holds for each $x \in S_0$, so there is no singular continuous spectrum in S_0 and hence in S.

Therefore we turn to establishing inequality (3.5.23) on a set $[a, b]$ on which inequality (3.5.22) is valid.

Let S_x be the set on which the estimate (3.5.22) is valid and let $[a, b] \subset S_x$ and set $z = \lambda + i\epsilon_1, \quad \lambda \in [a, b], \quad P_H = P_H(S_x)$. Let $D = |A + i|^{-1}, \quad G_z(\epsilon) = (H - z - i\epsilon B^* B)^{-1}$ and $F_z(\epsilon) = DG_z(\epsilon)D$. Finally let $B^* B = P_H C P_H$, where C is as in Definition 3.5.5. Then we have

$$\begin{aligned} \|P_H G_z(\epsilon) D\| &\leq \frac{c_1}{\sqrt{\epsilon}} \|F_z(\epsilon)\|^{\frac{1}{2}}, \\ \|(H + i) P_H G_z(\epsilon) D\| &\leq \frac{c_2}{\sqrt{\epsilon}} \|F_z(\epsilon)\|^{\frac{1}{2}}, \\ \|(1 - P_H) G_z(\epsilon) D\| &\leq \|(1 - P_H) G_z(0)\| \|(1 - i\epsilon B^* B G_z(\epsilon) D\| \\ &\leq c_2 \|(1 - P_H) G_z(0)\| \leq c_4, \\ \|(H + i)(1 - P_H) G_z(\epsilon) D\| &\leq c_2 \|(H + i)(1 - P_H) G_z(0)\| \leq c_5, \\ \|F_z(\epsilon)\| &\leq \frac{c_6}{\epsilon}, \end{aligned} \tag{3.5.24}$$

where the constants $c_1, \ldots, c_6$ may depend on x but do not depend on ϵ, ϵ_1. In the above, the first and the second estimates come from Proposition 3.5.10, by taking $B' = P_H, (H + i) P_H$ and noting that $D = |A + i|^{-1}$ is a bounded selfadjoint operator. The next two estimates come from using the fact that $\sigma = \text{dist}([a, b], \mathbb{R} \setminus S_x) > 0$, so $\|(I - P_H)(H - z)^{-1}\| \leq \frac{1}{\sigma}$. The last estimate comes from putting the first and the third inequalities in the collection of inequalities (3.5.24) together with the boundedness of D.

Further, the derivative of $F_z(\epsilon)$ is given by

$$\frac{d}{d\epsilon} F_z(\epsilon) = DG_z(\epsilon) P_H C P_H G_z(\epsilon) D. \tag{3.5.25}$$

This expression gives the differential inequality

$$\left\| \frac{d}{d\epsilon} F_z(\epsilon) \right\| \leq \|DG_z(\epsilon) C G_z(\epsilon) D\| + c_1 + \frac{c_2}{\sqrt{\epsilon}} \|F_z(\epsilon)\|^{\frac{1}{2}}, \tag{3.5.26}$$

using the identity,

$$P_H C P_H = C - (1 - P_H) C P_H - P_H C (1 - P_H) - (1 - P_H) C (1 - P_H),$$

and using the inequalities (3.5.24). By Definition 3.5.5(4) and the Proposition 3.5.12 below, we find that $G_z(\epsilon) : \text{dom}(A) \cap \text{dom}(H) \to \text{dom}(H)$ and $[B^* B, A]$ is bounded as a map from $\mathcal{H}_2 \to \mathcal{H}_{-2}$. Therefore the inequality (3.5.25) can be written as

$$\left\|\frac{d}{d\epsilon}F_z(\epsilon)\right\| \le \|DG_z(\epsilon)([H-z-i\epsilon B^*B,A]+i\epsilon[B^*B,A])G_z(\epsilon)D\| \\ + c_1 + \frac{c_2}{\sqrt{\epsilon}}\|F_z(\epsilon)\|^{\frac{1}{2}}. \tag{3.5.27}$$

Simplifying this inequality results in

$$\left\|\frac{d}{d\epsilon}F_z(\epsilon)\right\| \le c_1 + \frac{c_2}{\sqrt{\epsilon}}\|F_z(\epsilon)\|^{\frac{1}{2}} + c_3\|F_z(\epsilon)\|. \tag{3.5.28}$$

By integrating this differential inequality,

$$\|F_z(\epsilon)\| \le c, \quad \text{for } \mathrm{Re}(z) \in [a,b],$$

results as desired. □

Proposition 3.5.12. *Let H be selfadjoint and let A be its local conjugate on S. Then*

1. *any ψ with $t\widehat{\psi}(t) \in L^1(\mathbb{R})$, where $\widehat{\psi}$ denoting the Fourier transform of ψ, satisfies $\psi(H) : \mathrm{dom}(A) \cap \mathrm{dom}(H) \to \mathrm{dom}(A)$ and*

$$\|\{A\psi(H) - \psi(H)A\}f\| \le \|(H+i)f\| \int |t||\widehat{\psi}(t)|\ dt. \tag{3.5.29}$$

2. *Let B^*B be defined as in Lemma 3.5.11. Then $[B^*B, A]$ is a bounded map from $\mathcal{H}_2 \to \mathcal{H}_{-2}$.*
3. *The operator $G_z(\epsilon) = (H - z - i\epsilon B^*B)^{-1}$ maps $\mathrm{dom}(A) \cap \mathrm{dom}(H) \to \mathrm{dom}(H)$.*

Proof: (1) Let $f \in \mathrm{dom}(A) \cap \mathrm{dom}(H)$ and let $A(\lambda) = iA\lambda(A+i\lambda)^{-1}$.

$$\left\|\left\{A(\lambda)e^{-iHt} - e^{-iHt}A(\lambda)\right\}f\right\| \\ \le \sup_{\substack{g\in\mathrm{dom}(H)\cap\mathrm{dom}(A)\\ \|g\|=1}} \left|\int_0^t \langle g, e^{isH}[H,A(\lambda)]e^{-isH}f\rangle\ ds\right|. \tag{3.5.30}$$

Since e^{-isH} leaves $\mathrm{dom}(H)$ and $\mathrm{dom}(A)$ invariant,

$$\|\{A(\lambda)e^{-iHt} - e^{-iHt}A(\lambda)\}f\| \\ \le |t| \sup_{\substack{g\in\mathrm{dom}(A)\cap\mathrm{dom}(H)\\ \|g\|=1}} \sup_{|s|\le|t|} |\langle e^{-isH}g, [H,A(\lambda)]e^{-isH}f\rangle|. \tag{3.5.31}$$

The equation (3.5.19) and Propositions 3.5.7, 3.5.8 imply that

$$\|Ae^{-iHt}f\| \le \lim_{|\lambda|\to\infty} \|A(\lambda)e^{-itH}f\| \le c|t|\|(H+i)f\| + \|Af\|, \tag{3.5.32}$$

using the boundedness of $(H+i)^{-1}[H,A(\lambda)](H+i)^{-1}$.

Now the relation $\psi(H) = \int \widehat{\psi}(t)e^{-itH}dt$ implies that $\psi(H)$ maps the set $\mathrm{dom}(A) \cap \mathrm{dom}(H)$ into $D(A)$, if $|t|\widehat{\psi}(t) \in L^1(\mathbb{R})$ and so Equation 3.5.29 holds.

(2) Since the function with which the projector P_H is associated is smooth and vanishes outside S_x, its Fourier transform is rapidly decreasing. So by (1) P_H takes $\mathrm{dom}(A) \cap \mathrm{dom}(H)$ into $\mathrm{dom}(A) \cap \mathrm{dom}(H)$. Therefore $[B^*B, A]$ can be written on $\mathrm{dom}(A) \cap \mathrm{dom}(H)$ as

$$[B^*B, A] = [P_H, A]P_H + P_H[C, A]P_H + P_H C[P_H, A],$$

in the sense of quadratic forms. Given the condition (3) of Definition 3.5.5, and the inequality (3.5.29), we find that the form $[C, A]$ is bounded as a map from $\mathcal{H}_2$ to $\mathcal{H}_{-2}$ and in particular for $f \in \mathrm{dom}(H)$, setting $B(\lambda) = i\lambda(A + i\lambda)^{-1}$,

$$\begin{aligned} &\left\| [H - z - i\epsilon B^*B, A(\lambda)]f \right\|_{-2} \\ &\qquad \le \sup_{\substack{g \in \mathrm{dom}(A) \cap \mathrm{dom}(H) \\ \|g\|_2 = 1}} \Big\{ \big| \langle g, [H - z - i\epsilon B^*B, A]B(\lambda)f \rangle \\ &\qquad\qquad + \langle g, A(A + i\lambda)^{-1}[H - z - i\epsilon B^*B, A]B(\lambda)f \rangle \big| \Big\}. \end{aligned} \tag{3.5.33}$$

By Proposition 3.5.7 the operators $i\lambda(A + i\lambda)^{-1}$, $A(A + i\lambda)^{-1}$ are uniformly bounded from $\mathcal{H}_2$ to $\mathcal{H}_{-2}$ for λ large enough. It follows that $[H - z - i\epsilon B^*B, A(\lambda)]$ are uniformly bounded in λ from $\mathcal{H}_2$ to $\mathcal{H}_{-2}$. It follows from this that $G_z(\epsilon)$ preserves $\mathrm{dom}(A)$, hence it maps $\mathrm{dom}(A)$ into $\mathrm{dom}(A) \cap \mathrm{dom}(H)$. $\square$

3.6 Criteria Using Scattering Theory

For comparing the continuous spectra of two selfadjoint operators A and B in a separable Hilbertspace $\mathfrak{H}$, one can define an operator

$$\Omega_+(B, A) = \operatorname*{s-lim}_{t \to +\infty} e^{itB} e^{-itA} P_{ac}(A).$$

If this operator exists it establishes a unitary equivalence between A and B restricted to parts of their absolutely continuous subspaces.

The operator $\Omega_+(B, A)$ is known as the wave operator. Its notation comes from mathematical scattering theory, which has its origin in quantum mechanics. However, here we will study only the spectral theoretic point of view. Scattering theoretic details are not given.

Nevertheless we need some basic notions and some fundamental theorems from scattering theory. Therefore we start with the properties and features of wave operators.

3.6.1 Wave Operators

Definition and Properties

Definition 3.6.1. Let A and B be two selfadjoint operators in a separable Hilbert space $\mathfrak{H}$. The associated wave operators are defined as strong limits by

$$\Omega_+(B,A) = \mathrm{s} - \lim_{t\to+\infty} e^{itB}e^{-itA}P_{ac}(A) \tag{3.6.34}$$

and

$$\Omega_-(B,A) = \mathrm{s} - \lim_{t\to-\infty} e^{itB}e^{-itA}P_{ac}(A) \tag{3.6.35}$$

where $P_{ac}(A)\mathfrak{H} = \mathfrak{H}_{ac}(A)$ denotes the absolutely continuous subspace of A. By definition it is assumed that the strong limits in (3.6.34) and (3.6.35) exist.

In the following we will study only Ω_+. Ω_- has the corresponding features. Some of the basic properties of the wave operator Ω_+, if it exists, will be listed below. The proofs can be found in standard textbooks, some of them are referred to in the notes.

Proposition 3.6.2. *Let A, B be two selfadjoint operators in a separable Hilbert space $\mathfrak{H}$. Assume that $\Omega_+(B,A)$ exists. Then*

(i) *$\Omega_+(B,A)$ is a partial isometry with initial subspace $P_{ac}(A)\mathfrak{H}$. Its final subspace, denoted by $\mathfrak{H}_+$, is always contained in $P_{ac}(B)\mathfrak{H}$.*
(ii) *The singular subspace of A, $\mathfrak{H}_s(A) = \mathfrak{H} \ominus P_{ac}(A)\mathfrak{H}$ is always contained in the kernel of Ω_+.*
(iii) $\mathrm{ran}(\Omega_+)$ *is an invariant subspace of B. Moreover,*

$$\Omega_+[\mathrm{dom}\ (A)] \subset \mathrm{dom}\ (B) \tag{3.6.36}$$

and the intertwining relation

$$B\Omega_+(B,A) = \Omega_+(B,A)A \tag{3.6.37}$$

holds on $\mathrm{dom}\,(A)$.

Definition 3.6.3. Assume that the wave operator $\Omega_+(B,A)$ exists. It is called *complete* if

$$\mathfrak{H}_+ = \mathrm{ran}\,(\Omega_+) = P_{ac}\,(B)\mathfrak{H}.$$

Theorem 3.6.4. *Assume that the wave operator $\Omega_+(B,A)$ exists. Then it is complete if and only if $\Omega_+(A,B)$ exists.*

Only from the existence of the wave operator $\Omega_+(B,A)$ can one obtain a first consequence for the behaviour of the absolutely continuous spectrum.

Theorem 3.6.5. *Let A and B be two selfadjoint operators in a separable Hilbert space $\mathfrak{H}$ and assume that the wave operator $\Omega_+(B, A)$ exists.*
Then

$$\sigma_{ac}(A) \subseteq \sigma_{ac}(B). \tag{3.6.38}$$

Proof: By the intertwining relation we have for any real λ

$$e^{i\lambda B}\Omega_+ = \Omega_+ e^{i\lambda A}.$$

Let $\varphi(\cdot) : \mathbb{R} \to \mathbb{R}$, be a bounded function for which the Fourier inversion is valid. Using functional calculus we get

$$\varphi(A) = \int_{\mathbb{R}} (\mathcal{F}\varphi)(\lambda) e^{i\lambda A} d\lambda$$

where $\mathcal{F}\varphi$ denotes the usual Fourier transform (see Section 1.2). Therefore the intertwining relation extends to such functions. This implies for any $f \in \mathfrak{H}$

$$\begin{aligned} ||\varphi(B)\Omega_+ f||^2 &= ||\Omega_+ \varphi(A) f||^2 \\ &= ||P_{ac}(A)\varphi(A) f||^2 \\ &= ||\varphi(A) P_{ac}(A) f||^2. \end{aligned}$$

Again via functional calculus the last relation is extended to indicator functions of Borel subsets. Therefore $\Omega_+ f \in \mathfrak{H}_{ac}(B)$ for all $f \in \mathfrak{H}_{ac}(A)$ and

$$\begin{aligned} \sigma_{ac}(A) &= \sigma(A|P_{ac}(A)\mathfrak{H}) \\ &= \sigma(B|\mathfrak{H}_+) \\ &\subseteq \sigma(B|P_{ac}(B)\mathfrak{H}) \\ &= \sigma_{ac}(B). \end{aligned}$$

□

Corollary 3.6.6 *Suppose the wave operator $\Omega_+(B, A)$ is complete.*

(i) Then the absolutely continuous spectrum is invariant, i.e.,

$$\sigma_{ac}(A) = \sigma_{ac}(B).$$

(ii) If $\sigma_{ac}(A)$ is empty, then $\sigma_{ac}(B)$ is empty.
(iii) Assume additionally that $\mathrm{ran}(\Omega_+(A, B)) = P_c(H)\mathfrak{H}$, i.e., the range is equal to the continuous subspace of B. Then the singularly continuous spectrum of $B, \sigma_{sc}(B)$ is empty.

Existence by Cook's Method

The simplest way to show the existence of a strong limit in the definition of the wave operator is known as Cook's method derived from Duhamel's principle. Recall that a set $\mathfrak{D}$ in $\mathfrak{H}$ is a total if clospan$(\mathfrak{D}) = \mathfrak{H}$.

Theorem 3.6.7. *Let A and B be two selfadjoint operators in $\mathfrak{H}$. Let $\mathfrak{D}$ be a total subset of $\mathfrak{H}_{ac}(A)$. Assume for any $f \in \mathfrak{D}$ there is a real value $t_f > 0$, such that*

$$e^{-itA} f \in \operatorname{dom}(A) \cap \operatorname{dom}(B), \quad \text{for all } t \geq t_f.$$

Assume that

$$\int_{t_f}^{\infty} \left\| (B - A) e^{-itA} f \right\| dt < \infty. \tag{3.6.39}$$

Then $\Omega_+(B, A)$ exists.

The proof of Cook's criterion is based on the following simple fact: If g is a C^1-function and if g' is in $L^1((a, \infty))$, then the $\lim_{t \to \infty} g(t)$ exists because

$$|g(t) - g(s)| \leq \left| \int_s^t g'(u) du \right|$$

tends to zero as s goes to infinity.

Because A and B are unbounded operators one always has difficulties in handling domain questions. However in scattering theory very often such problems can be avoided by studying bounded functions of A and B. Cook's criterion for resolvents reads as follows.

Theorem 3.6.8. *Let $\widetilde{\mathfrak{D}}$ be a subset of $\mathfrak{H}_{ac}(A)$ such that $\mathfrak{D} = (z - A)^{-1} \widetilde{\mathfrak{D}}$ is total in $\mathfrak{H}_{ac}(A)$. Take a fixed $z \in \operatorname{res}(A) \cap \operatorname{res}(B)$. Assume*

$$\int_0^{\infty} \left\| \left[(z - B)^{-1} - (z - A)^{-1} \right] e^{-itA} u \right\| dt < \infty \tag{3.6.40}$$

for $u \in \widetilde{\mathfrak{D}}$.

Then the wave operator $\Omega_+(B, A)$ exists.

Proof: We have

$$\int_0^{\infty} \left\| \left[(z - B)^{-1} - (z - A)^{-1} \right] e^{-itA} u \right\| dt$$

$$= \int_0^{\infty} \left\| \left[B(z - B)^{-1} - (z - B)^{-1} A \right] e^{-itA} (z - A)^{-1} u \right\| dt.$$

By assumption the last integral is finite. From Cook's Criterion 3.6.7 follows the existence of

$$\lim_{t\to\infty} e^{itB}(z-B)^{-1}\, e^{-itA}(z-A)^{-1}u = \lim_{t\to\infty} e^{itB}(z-B)^{-1}\, e^{-itA}f$$

with $f=(z-A)^{-1}u \in \mathfrak{D}$. Because $\mathfrak{D}$ is total we get the existence of

$$s-\lim_{t\to\infty} (z-B)^{-1}\, e^{+itB}\, e^{-itA}P_{ac}(A). \tag{3.6.41}$$

On the other hand, let $g(\cdot):[\,0,\infty\,)\to[\,0,\infty\,)$ be a uniformly continuous function satisfying $\int_0^\infty g(t)dt<\infty$; then $\lim_{t\to\infty} g(t)=0$.

Using this fact and the condition in (3.6.40),

$$\lim_{t\to\infty} e^{itB}\left[(z-B)^{-1}-(z-A)^{-1}\right]\, e^{-itA}u=0$$

or

$$\lim_{t\to\infty} e^{itB}(z-B)^{-1}\, e^{-itA}u = \lim_{t\to\infty} e^{itB}\, e^{-itA}\,(z-A)^{-1}u.$$

Together with (3.6.41) $\lim_{t\to\infty} e^{itB}\, e^{-itA}(z-A)^{-1}u$ exists for all $u\in\widetilde{\mathfrak{D}}$, which is sufficient for the assertion. □

Cook's method is very useful if one has some information about the t-dependence of the unitary group e^{-itA}. In quantum mechanics this characterizes the evolution of a free system. In particular one knows this dependence if A is the Laplacian in $L^2(\mathbb{R}^d)$ or a function of the Laplacian. For arbitrary selfadjoint operators A it is difficult to study the time dependence of this evolution in general.

Existence and completeness by trace class conditions

One of the simplest cases for which existence of wave operators can be shown is when the pair of operators A,B differ by a trace class operator. In order to avoid domain problems we study differences of resolvents.

Theorem 3.6.9. *Let A and B be two selfadjoint operators in $\mathfrak{H}$. Let $z\in$ $\mathrm{res}(A)\cap\mathrm{res}(B)$. Assume that*

$$(z-B)^{-1}-(z-A)^{-1}\in\mathfrak{B}_{\mathrm{trace}}(\mathfrak{H}),$$

i.e., is a trace class operator.

Then the wave operators $\Omega_\pm(B,A)$ exist and are complete, implying the operators $A|\mathfrak{H}_{ac}(A)$ and $B|\mathfrak{H}_{ac}(B)$ are unitarily equivalent.

For the proof of Theorem 3.6.9 and for later use in considering the extended Pearson's estimate we introduce a useful dense subspace of $\mathfrak{H}_{ac}(A)$.

Definition 3.6.10. Let A be a selfadjoint operator in $\mathfrak{H}$ with the resolution of the identity $E_A(\cdot)$. In $\mathfrak{H}_{ac}(A)$ (see Definition 2.6.8) we define a linear submanifold

$$\mathcal{M}_\infty(A) \;=\; \{f \in \mathfrak{H}_{ac}(A);\; \frac{d\langle f, E_A(\lambda) f\rangle}{d\lambda} \;\in L^\infty(\mathbb{R})\}.$$

For $f \in \mathcal{M}_\infty(A)$ we define

$$\|f\|_{\mathcal{M}_\infty(A)} = \left\| \frac{d\langle f, E_A(\lambda) f\rangle}{d\lambda} \right\|^{1/2}_{L^\infty(\mathbb{R}).}.$$

Remark 3.6.11. $\|\cdot\|_{\mathcal{M}_\infty(A)}$ establishes a norm in $\mathcal{M}_\infty(A)$. $\mathcal{M}_\infty(A)$ is a dense subspace of $\mathfrak{H}_{ac}(A)$. For $f \in \mathcal{M}_\infty(A)$ and any $g \in \mathfrak{H}$ we have

$$\int_{-\infty}^{\infty} |\langle g,\; e^{-itA} f\rangle|^2 \; dt \;\leq\; 2\pi \; \|g\|^2_{\mathfrak{H}} \;\; \|f\|^2_{\mathcal{M}_\infty(A)}.$$

If K is a Hilbert–Schmidt operator with the Hilbert–Schmidt norm $\|K\|_{HS}$, one has

$$\int_{-\infty}^{\infty} \|K \; e^{-itA} f\|^2 \; dt \;\leq\; 2\pi \; \|K\|^2_{HS} \;\; \|f\|^2_{\mathcal{M}_\infty(A)} \; .$$

Proof of Theorem 3.6.9: Using Cook' s method we have

$$\begin{aligned}
&\big\|(z-B)^{-1}(e^{itB}e^{-itA} - e^{isB}e^{-isA})(z-A)^{-1} f\big\| \\
&\qquad = \left\| \int_s^t du \; e^{iuB}[(z-B)^{-1} - (z-A)^{-1}] e^{-iuA} f \right\| \\
&\qquad \leq \int_s^t du \left\| \sum_{i=1}^{\infty} \; \lambda_i \langle \varphi_i,\; e^{-iuA} f\rangle \; e^{iuB} \psi_i \right\|
\end{aligned}$$

where λ_i are absolutely summable, φ_i, ψ_i are unit vectors. Following the proof of Pearson's Theorem in Reed–Simon [158] (p. 24 ff) one obtains the existence of

$$\lim_{t\to\,\infty} (z-B)^{-1} \; e^{itB} e^{-itA} (z-A)^{-1} f$$

for all $f \in \mathcal{M}_\infty(A)$. The domain of A is dense in $\mathfrak{H}$ implying the existence of

$$\operatorname*{s-lim}_{t\to\,\infty} (z-B)^{-1} \; e^{itB} e^{-itA} P_{ac}(A). \tag{3.6.42}$$

By the use of the Riemann–Lebesgue Lemma and because $(z-B)^{-1} - (z-A)^{-1}$ is compact

$$\operatorname*{s-lim}_{t\to\infty} \; \left[(z-B)^{-1} - (z-A)^{-1}\right] \; e^{-itA} P_{ac}(A) = 0, \tag{3.6.43}$$

which together with Equation (3.6.42) implies the existence of

$$\mathrm{s} - \lim_{t\to\infty} e^{+itB} e^{-itA} P_{ac}(A)(z-A)^{-1}.$$

By a density argument we obtain the existence of $\Omega_+(B,A)$.

On the other hand we have not used any special property of A, so we can reverse the arguments and conclude the existence of $\Omega_+(A,B)$. This shows

$$\operatorname{ran} \Omega_+(B,A) = \mathfrak{H}_{ac}(B)$$
$$\operatorname{ran} \Omega_+(A,B) = \mathfrak{H}_{ac}(A)$$

and $A|\mathfrak{H}_{ac}(A)$ is unitarily equivalent to $B|\mathfrak{H}_{ac}(B)$. The operator $\Omega_+(B,A)$ is unitary from $\mathfrak{H}_{ac}(A)$ to $\mathfrak{H}_{ac}(B)$ and implements the unitary equivalence. □

Wave operators possess a stable behaviour if the operators A and B are replaced by a certain function of A and B. For instance, assume that

$$\Omega_+(B,A) = s - \lim_{t\to\infty} e^{itB} e^{-itA} P_{ac}(A)$$

exists. Take the linear function $\alpha : \mathbb{R} \to \mathbb{R}$

$$\alpha(\lambda) = a\lambda + b.$$

Then

$$\begin{aligned}\Omega_+(\alpha(B),\alpha(A)) &= \mathrm{s} - \lim_{t\to\infty} e^{it\alpha(B)} e^{-it\alpha(A)} P_{ac}(\alpha(A)) \\ &= \mathrm{s} - \lim_{t\to\infty} e^{itaB} e^{-itaA} P_{ac}(A).\end{aligned}$$

$\Omega_+(\alpha(B),\alpha(A))$ equals $\Omega_+(B,A)$ if $a = \alpha'(\lambda)$ is positive and $\Omega_-(B,A)$ if a is negative. This feature is called the *invariance principle*. There are two forms of the invariance principle.

Invariance principle 3.6.12. *Let $\alpha : \mathbb{R} \to \mathbb{R}$ be an admissible function (see Definition 3.6.13). Let A and B be two selfadjoint operators in a separable Hilbertspace $\mathfrak{H}$. Then we distinguish:*

(i) Weak invariance principle: Assume that $\Omega_+(B,A)$ and $\Omega_+(\alpha(B),\alpha(A))$ exist for a function α with positive derivative. Then

$$\Omega_+(B,A) = \Omega_+(\alpha(B),\alpha(A)).$$

(ii) Strong invariance principle: Assume that $\Omega_+(\alpha(B),\alpha(A))$ exists where α is a function with positive derivative. Then $\Omega_+(B,A)$ exists and

$$\Omega_+(B,A) = \Omega_+(\alpha(B),\alpha(A)).$$

Before considering trace class criteria we characterize the class of admissible functions.

Definition 3.6.13. Let $I_n, n \in \mathbb{N}$, be pairwise disjoint, open intervals in $\mathbb{R}$ such that $\bigcup_{n\in\mathbb{N}} \overline{I}_n = \mathbb{R}$. Here $\overline{I}_n$ denotes the closure of I_n. A real-valued function α on $\mathbb{R}$ is called admissible if on every I_n

1) α is continuously differentiable,
2) $\alpha' > 0$ or $\alpha' < 0$,
3) α' is locally of bounded variation, i.e., on each closed interval contained in I_n the function α' is of bounded variation.

Observe that for an admissible function α the operator $\alpha(A)$ is well defined via the spectral calculus. One has only to recognize that $\alpha(\cdot)$ is a Borel measurable function.

Some useful properties of admissible functions are:

Lemma 3.6.14.

(i) If α is finite a.e.with respect to the spectral measure $P_A(\cdot)$, then $P_{ac}(\alpha(A)) = P_{ac}(A)$.
(ii) α^{-1} exists on each set $\alpha(I_n)$ and is admissible there.
(iii) A composition of admissible functions is again admissible.
(iv) If α and β are bounded admissible functions, then the product $\alpha \cdot \beta$ is also admissible.

Remark 3.6.15. Having in mind the spectral theorem the admissible function is used only on the spectrum of A. If for instance A is positive, α has to be admissible in the sense above only on $(0, \infty)$. If necessary one can extend α appropriately to $(-\infty, 0)$.

Examples for admissible functions with positive derivatives on $(0, \infty)$ are $-e^{-\lambda}$, $-(1+\lambda)^{-1}$, $-\frac{1}{\lambda}+1$. Examples with negative derivatives are semigroups $e^{-\lambda}$, resolvents $(\lambda + 1)^{-1}$, or powers of resolvents $(\lambda + 1)^{-p}$, $p \geq 1$.

In case of trace class perturbations the invariance principle holds in its strong form which goes back to Birman.

Theorem 3.6.16. *Let A and B be two selfadjoint operators in $\mathfrak{H}$. Assume that α is a bounded admissible, real-valued function on $\mathbb{R}$ and that $\alpha(A) - \alpha(B)$ is a trace class operator such that $\Omega_\pm(\alpha(A), \alpha(B))$ exist and are complete (see Theorem 3.6.9). Then $\Omega_\pm(\beta(\alpha(A)), \beta(\alpha(B))$ exist and are complete for any other admissible function β. We get*

$$\Omega_+(\beta(\alpha(A)), \beta(\alpha(B))) = \Omega_+(\alpha(A), \alpha(B)) \tag{3.6.44}$$

if β has positive derivative and

$$\Omega_+(\beta(\alpha(A)), \beta(\alpha(B))) = \Omega_-(\alpha(A), \alpha(B)) \tag{3.6.45}$$

if β has negative derivative.

Remark 3.6.17. Of course β can be chosen to be α^{-1}. Theorem (3.6.16) is useful in many aspects. For instance assume that

$$e^{-B}(e^{-B} - e^{-A}) \in \mathfrak{B}_{\text{trace}}. \tag{3.6.46}$$

and

$$(e^{-B} - e^{-A})e^{A} \in \mathfrak{B}_{\text{trace}} \tag{3.6.47}$$

are trace class. Choose

$$\beta(\lambda) = \begin{cases} -\frac{1}{2} \ln\ \lambda & \text{in } (0, \infty) \\ -\lambda & \text{in } (-\infty, 0). \end{cases}$$

Then $\Omega_+(e^{-2B}, e^{-2A})$ exists and equals $\Omega_-(B, A)$. This kind of procedure will be used in more detail in Section 3.6.2.

On the other hand the conditions in (3.6.46), (3.6.47) are not the best possible. Following Birman's method they can be improved to sandwiched differences, enlarging the allowed set of possible A and B.

Theorem 3.6.18. *Let A and B be given as above and let α be an admissible bounded function with positive derivative. Let $\tau : \mathbb{R} \to \mathbb{R}$ be another bounded admissible function for which the range of $\tau(A)P_{ac}(A)$ is dense in $\mathfrak{H}_{ac}(A)$ and range of $\tau(B)P_{ac}(B)$ is dense in $\mathfrak{H}_{ac}(B)$. Let*

$$\tau(B)(\alpha(B) - \alpha(A))\tau(A) \in \mathfrak{B}_{\text{trace}}(\mathfrak{H}) \tag{3.6.48}$$

and

$$\tau(B) - \tau(A) \in \mathfrak{B}_{\text{comp}}(\mathfrak{H}). \tag{3.6.49}$$

Then

$$\sigma_{ac}(A) = \sigma_{ac}(B).$$

Proof: Define $\tilde{J} = \tau(B)\tau(A)$ such that (3.6.48) reads as

$$\alpha(B)\tilde{J} - \tilde{J}\alpha(A) \in \mathfrak{B}_{\text{trace}}.$$

Hence by Pearson's two-space version of the trace class criterion and by the invariance principle (Theorem 3.6.16), the following strong limits exist and are equal:

$$\begin{aligned} \operatorname*{s-lim}_{t \to \pm\infty} e^{it\alpha(B)}\, \tilde{J}\, e^{-it\alpha(A)} P_{ac}(\alpha(A)) &= \operatorname*{s-lim}_{t \to \pm\infty} e^{it\alpha(B)}\, \tilde{J} e^{-it\alpha(A)} P_{ac}(A) \\ &= \operatorname*{s-lim}_{t \to \pm\infty} e^{itB}\, \tilde{J}\, e^{-itA} P_{ac}(A) \\ &= \operatorname*{s-lim}_{t \to \infty} e^{itB}\, \tau(B)\tau(A)\, e^{-itA} P_{ac}(A). \end{aligned}$$

Using the Riemann–Lebesgue Lemma and (3.6.49) the last expression equals

$$\mathrm{s}-\lim_{t\to\pm\infty} e^{itB}\ e^{-itA}\ \tau^2(A)\ P_{ac}(A).$$

Now $\tau(A)\mathfrak{H}_{ac}(A)$ is assumed to be dense in $\mathfrak{H}_{ac}(A)$; therefore

$$\mathrm{s}-\lim_{t\to\pm\infty} e^{itB}\ e^{-itA}\ \tau(A)\ P_{ac}(A).$$

exist. This implies the existence of $\Omega_{\pm}(B,A)$. □

Sometimes the perturbed operator B is not given explicitly but can be determined as a limit of a sequence of operators B_n. In this case it is not easy to study directly the trace class property of

$$K=\tau(B)\ (\alpha(B)-\alpha(A))\ \tau(A),$$

which is central in the last theorem. However one can study trace class properties of

$$K_n=\tau(B_n)\ (\alpha(B_n)-\alpha(A))\ \tau(A).$$

The question arises as to whether this is sufficient for the completeness of $\Omega_{\pm}(B,A)$. The following result gives one answer.

Lemma 3.6.19. *Let $\{K_n\}$ be a sequence of trace class operators in the Hilbertspace $\mathfrak{H}$. Assume that K_n converges weakly to an operator K. Assume that the trace norms of K_n are bounded uniformly in n, i.e.,*

$$\|K_n\|_{\mathrm{trace}}\le M\ .$$

Then the operator K is also trace class.

Proof: $\mathfrak{B}_{\mathrm{trace}}(\mathfrak{H})$ is the dual of $\mathfrak{B}_{\mathrm{comp}}(\mathfrak{H})$. $\mathfrak{B}_{\mathrm{comp}}(\mathfrak{H})$ is separable. By Alaoglu's Theorem any bounded sequence in $\mathfrak{B}_{\mathrm{trace}}(\mathfrak{H})$ has a *weak*∗-convergent subsequence, so there is a subsequence K_{n_j} of K_n with

$$\lim_{j\to\infty}\mathrm{trace}(K_{n_j}C)=\mathrm{trace}(K'C)$$

for all $C\in\mathfrak{B}_{\mathrm{comp}}(\mathfrak{H})$ and $K'\in\mathfrak{B}_{\mathrm{trace}}(\mathfrak{H})$.

We chose the following C. Take arbitrary $f,g\in\mathfrak{H}$ with $\|f\|=1,\ \|g\|=1$ and set

$$C=\langle f,\cdot\rangle g.$$

For this rank one operator we have

$$\begin{aligned}\mathrm{trace}(K_{n_j}C)&=\sum_{i=1}^{\infty}\langle\ \varphi_i,K_{n_j}C\varphi_i\rangle\\&=\sum_{i=1}^{\infty}\langle\varphi_i,K_{n_j}\langle f,\varphi_i\rangle g\rangle\\&=\sum_{i=1}^{\infty}\langle\varphi_i,K_{n_j}\ g\rangle\langle f,\varphi_i\rangle,\end{aligned}$$

where $\{\varphi_i\}_{\mathbb{N}}$ is an orthonormal basis in $\mathfrak{H}$. Choosing this basis in such a way that $\varphi_1 = f$ gives

$$\operatorname{trace}(K_{n_j}C) = \langle f, K_{n_j} g\rangle.$$

Hence

$$\begin{aligned}\lim_{n_j \to \infty} \operatorname{trace}(K_{n_j}C) &= \langle f, K' g\rangle \\ &= \lim_{n_j \to \infty} \langle f, K_{n_j} g\rangle \qquad = \langle f, Kg\rangle.\end{aligned}$$

Because f and g are arbitrary $K' = K$, i.e., K is trace class. □

This lemma can be used to generalize the conclusion in Theorem 3.6.18.

Corollary 3.6.20 *Let A be a selfadjoint operator in $\mathfrak{H}$. Let B be a selfadjoint operator given as strong resolvent limit of a sequence of selfadjoint operators B_n.*

Assume two admissible function α, τ as in Theorem 3.6.18.

Let

$$\mathrm{s} - \lim_{n \to \infty} \alpha(B_n) = \alpha(B) \tag{3.6.50}$$

and

$$\mathrm{w} - \lim_{n \to \infty} \tau(B_n) = \tau(B). \tag{3.6.51}$$

Assume that

$$\tau(B_n)\ (\alpha(B_n) - \alpha(A))\ \tau(A) \in \mathfrak{B}_{\mathrm{trace}}(\mathfrak{H}) \tag{3.6.52}$$

with

$$\|\tau(B_n)\ (\alpha(B_n) - \alpha(A))\ \tau(A)\|_{\mathrm{trace}} \leq\ M, \tag{3.6.53}$$

uniformly in n. Moreover assume that

$$\tau(B_n) - \tau(A) \in \mathfrak{B}_{\mathrm{comp}}(\mathfrak{H}) \tag{3.6.54}$$

and that it is bounded uniformly in $n \in \mathbb{N}$.

Then

$$\sigma_{ac}(B) = \sigma_{ac}(A).$$

Proof: By assumption

$$\mathrm{w} - \lim_{n \to \infty} \tau(B_n)\ (\alpha(B_n) - \alpha(A))\ \tau(A) = \tau(B)\ (\alpha(B) - \alpha(A))\ \tau(A).$$

Thus Lemma 3.6.19 implies the last operator is trace class. Because of equations (3.6.51) and (3.6.54) the difference $\tau(B) - \tau(A)$ is compact. Hence Theorem 3.6.18 is applicable and gives the result. □

Pearson's Estimate

Before we apply the spectral theoretic criteria in Theorem 3.6.5, Corollaries 3.6.6, 3.6.8, Theorem 3.6.9 and Theorem 3.6.16 for the perturbation of the continuous spectrum we will explain Pearson's estimate and its generalization due to Theorem 3.6.18. Pearson's estimate is useful to compare quantitatively a scattered system with a free one.

Theorem 3.6.21 (Pearson's Estimate (PE)). *Let A and B be two selfadjoint operators in $\mathfrak{H}$. Let α be an admissible bounded function such that*

$$\alpha(B) - \alpha(A) \in \mathfrak{B}_{\text{trace}}.$$

Then $\Omega_+(B, A)$ exists and is complete. Moreover,

$$||\Omega_\pm(B, A) - \mathbb{1}||^2_{\mathfrak{B}(\mathcal{M}_\infty(A),\mathfrak{H})} \leq 16\pi \, ||\alpha(B) - \alpha(A)||_{\text{trace}}, \qquad (3.6.55)$$

where $||.||_{\text{trace}}$ denotes the trace norm and $\mathfrak{B}(\mathcal{M}_\infty(A), \mathfrak{H})$ denotes the set of bounded operators mapping $\mathcal{M}_\infty(A)$ to $\mathfrak{H}$.

The existence and completeness of $\Omega_\pm(B, A)$ follow from Theorem 3.6.16. In (PE) it is assumed that $\alpha(B) - \alpha(A)$ is a trace class operator. In Theorem 3.6.18 however, it is only assumed that

$$\tau(B)(\alpha(B) - \alpha(A))\tau(A) \in \mathfrak{B}_{\text{trace}}.$$

One can find examples, where the last condition is satisfied, but $\alpha(B) - \alpha(A)$ is not trace class. Hence the question arises as to whether we can find a Pearson's Estimate also under the weaker assumptions of Theorem 3.6.18. The answer is yes. We call the next estimate Extended Pearson's estimate (EPE).

Pearson's Estimate is true on $\mathcal{M}_\infty(A)$ which is a dense set in $\mathfrak{H}_{ac}(A)$. For the extended estimate we have to define another dense set in $\mathfrak{H}_{ac}(A)$, but first we need a lemma.

Lemma 3.6.22. *Let A be a selfadjoint operator in the Hilbert space $\mathfrak{H}$. Let $\tau(A) \in \mathfrak{B}(\mathfrak{H})$. Then $\tau(A) \in \mathfrak{B}(\mathcal{M}_\infty(A))$.*

Proof: $\tau(A)$ commutes with A. Thus

$$\langle \tau(A)f, \; P_A(S) \; \tau(A)f \rangle \quad \leq \quad \|\tau(A)\|^2_{\mathfrak{B}(H)} \quad \langle f, P_A(S)f \rangle$$

for all $f \in \mathfrak{H}$ and all Borel sets $S \subset \mathbb{R}$. Hence $f \in \mathfrak{H}_{ac}(A)$ implies $\tau(A)f \in \mathfrak{H}_{ac}$ (A). Moreover the derivative of $\lambda \to \; \langle \tau(A)f, \; E_A(\lambda)\tau(A)f \rangle$ exists for almost all λ. For $f \in \mathcal{M}_\infty(A)$ we obtain

$$\left\| \frac{d\langle \tau(A)f, \; E_A(\lambda)\tau(A)f \rangle}{d\lambda} \right\|_{L^\infty(\mathbb{R})} \leq \; \|\tau(A)\|^2_{\mathfrak{B}(\mathfrak{H})} \left\| \frac{d\langle f, \; E_A(\lambda)f \rangle}{d\lambda} \right\|_{L^\infty(\mathbb{R})}.$$

This means that $\tau(A)f \in \mathcal{M}_\infty(A)$ if $f \in \mathcal{M}_\infty(A)$, and

$$\|\tau(A)f\|_{\mathcal{M}_\infty(A)} \;\leq\; \|\tau(A)\|_{\mathfrak{B}(\mathfrak{H})}\; \|f\|_{\mathcal{M}_\infty(A)}.$$

Hence $\tau(A) \in \mathfrak{B}(\mathcal{M}_\infty(A))$ and $\|\tau(A)\|_{\mathfrak{B}(\mathcal{M}_\infty(A))} \;\leq\; \|\tau(A)\|_{\mathfrak{B}(\mathfrak{H})}$.

Definition 3.6.23. Let A be a selfadjoint operator in the Hilbertspace $\mathfrak{H}$. Let $\tau(A) \in \mathfrak{B}(\mathfrak{H})$. Define

$$\mathcal{M}_\infty(\tau(A)) = \tau(A)\; \mathcal{M}_\infty(A).$$

For $f \in \mathcal{M}_\infty(\tau(A))$ we introduce

$$\|f\|_{\mathcal{M}_\infty(\tau(A))} = \inf\{\|g\|_{\mathcal{M}_\infty(A)}; g \in \mathcal{M}_\infty(A),\; \tau(A)g = f\}.$$

Lemma 3.6.24. $\mathcal{M}_\infty(\tau(A))$ *has the following features:*

(i) $\|\cdot\|_{\mathcal{M}_\infty(\tau(A))}$ *is a norm,*
(ii) $(\mathcal{M}_\infty(\tau(A)),\; \|\cdot\|_{\mathcal{M}_\infty(\tau(A))})$ *is continuously embedded in* $(\mathcal{M}_\infty(A),\; \|\cdot\|_{\mathcal{M}_\infty(A)})$ *where*

$$\|f\|_{\mathcal{M}_\infty(A)} \;\leq\; \|\tau(A)\|_{\mathfrak{B}_{(\mathfrak{H})}}\; \|f\|_{\mathcal{M}_\infty(\tau(A))},$$

(iii) $\mathcal{M}_\infty(\tau(A))$ *is dense in* $\mathfrak{H}_{ac}(A)$ *if* $\operatorname{ran}\big(\tau(A)P_{ac}(A)\big)$ *is dense in* $\mathfrak{H}_{ac}(A)$.

Proof: (i), (ii): By Lemma 3.6.22, $\tau(A)g \in \mathcal{M}_\infty(A)$ if $g \in \mathcal{M}_\infty(A)$. For $f = \tau(A)g$ we have

$$\|f\|_{\mathcal{M}_\infty(A)} \leq \|\tau(A)\|\|g\|_{\mathcal{M}_\infty(A)}.$$

By the definition of $\|\cdot\|_{\mathcal{M}_\infty(\tau(A))}$ on has

$$\|f\|_{\mathcal{M}_\infty(A)} \leq \|\tau(A)\|\|f\|_{\mathcal{M}_\infty(\tau(A))}.$$

Hence if $\|f\|_{\mathcal{M}_\infty(\tau(A))} = 0$ then $\|f\|_{\mathcal{M}_\infty(A)} = 0$. But $\|\cdot\|_{\mathcal{M}_\infty(\tau(A))}$ is a norm, so $f = 0$.
(iii): $\tau(A)P_{ac}(A)$ is a subspace of $\mathfrak{H}_{ac}(A)$ and is assumed to be dense. Thus for $f \in \mathfrak{H}_{ac}(A)$, given an $\epsilon > 0$ we choose a $g \in \mathfrak{H}$ such that

$$\|f - \tau(A)P_{ac}(A)g\| \;\leq\; \frac{\epsilon}{1 + \|\tau(A)\|_{\mathfrak{B}(\mathfrak{H})}}\;.$$

Moreover $\mathcal{M}_\infty(A)$ is dense in $\mathfrak{H}_{ac}(A)$, so that there is a $g_0 \in \mathcal{M}_\infty(A)$ satisfying

$$\|g_0 - P_{ac}(A)g\| \;\leq\; \frac{\epsilon}{1 + \|\tau(A)\|_{\mathfrak{B}(\mathfrak{H})}}\;.$$

Therefore we have the estimate

$$\begin{aligned}\|f - \tau(A)g_0\| &\leq \|f - \tau(A)P_{ac}(A)g\| + \|\tau(A)P_{ac}(A)g - \tau(A)g_0\| \\ &\leq \epsilon\ .\end{aligned}$$

□

Now we are able to formulate the Extended Pearson's Estimate (EPE) in (3.6.56).

Theorem 3.6.25 (Extended Pearson's Estimate (EPE)). *Suppose*

- *A, B are selfadjoint operators in $\mathfrak{H}$.*
- *$\alpha(\cdot)$ is an admissible bounded function with positive derivative.*
- *$\tau(\cdot)$ is an admissible bounded function such that $\tau(A)\mathfrak{H}_{ac}(A)$ is dense in $\mathfrak{H}_{ac}(A)$ and*

$$\tau(B)(\alpha(B) - \alpha(A))\tau(A) \in \mathfrak{B}_{\text{trace}}(\mathfrak{H}),$$
$$\tau(B) - \tau(A) \in \mathfrak{B}_{\text{comp}}(\mathfrak{H}).$$

Then

$$\begin{aligned}&\|\Omega_\pm(B, A) - \mathbb{1}\|_{\mathfrak{B}(\mathcal{M}_\infty(\tau^2(A))} \qquad (3.6.56)\\ &\quad \leqq c(B, A)\left\{\|\tau(B)(\alpha(B) - \alpha(A))\tau(A)\|^{1/2}_{\text{trace}} + ||(\tau(B) - \tau(A))\tau(A)||_{\mathfrak{B}(\mathfrak{H})}\right\},\end{aligned}$$

where

$$c(B, A) \leq\ 16\,\pi \sup_{\lambda\in\mathbb{R}} |\tau(\lambda)|.$$

Proof: By the invariance principle ($\alpha'(\cdot) > 0$) we have for $f \in \mathfrak{H}_{ac}(A)$

$$\|\,[\Omega_\pm(B, A) - \mathbb{1}]\,f\|_{\mathfrak{H}} = \|\,[\Omega_\pm(\alpha(B), \alpha(A)) - \mathbb{1}]\,f\|_{\mathfrak{H}}.$$

Following Lemma 3.6.24 (iii) it suffices to consider $f \in \mathcal{M}_\infty(\tau^2(A))$ which is dense in $\mathfrak{H}_{ac}(A)$. Thus we take $f = \tau^2(A)g,\ g \in \mathcal{M}_\infty(\tau(A))$.

Then

$$\begin{aligned}\|\,[\Omega_\pm(B, A) - \mathbb{1}]\,f\|_{\mathfrak{H}} \leq &\left\|\left[\Omega_\pm(\alpha(B), \alpha(A))\tau^2(A) - \tau(B)\tau(A)\right]g\right\|_{\mathfrak{H}}\\ &+ \left\|\left[\tau(B)\tau(A) - \tau^2(A)\right]g\right\|_{\mathfrak{H}}.\end{aligned}$$

By Pearson's Estimate (3.6.55) the last expression is smaller than

$$\begin{aligned}16\pi\,\|\tau(B)\tau(A)\|^{1/2}_{B(\mathfrak{H})}\ \|g\|_{\mathcal{M}_\infty(\tau(A))}\ &\|\tau(B)(\alpha(B) - \alpha(A))\tau(A)\|^{1/2}_{\text{trace}}\\ &+ \|[\tau(B) - \tau(A)]\,\tau(A)\|_{\mathfrak{B}(\mathfrak{H})}\ \|g\|_{\mathfrak{H}}\ .\end{aligned}$$

Now $\|g\|_{\mathfrak{H}}$ can be estimated by $\|g\|_{\mathcal{M}_\infty(\tau(A))}$ because $g = P_{ac}(\tau(A))g$ and

$$\begin{aligned}\|g\|^2 &= \int_{\sigma(\tau(A))} d\langle g,\ E_{\tau(A)}(\lambda)g\rangle \\ &\leq \int_{-\|\tau(A)\|}^{\|\tau(A)\|} \frac{d\langle g,\ E_{\tau(A)}(\lambda)g\rangle}{d\lambda}\, d\lambda \\ &\leq 2\|\tau(A)\|\ \|g\|^2_{\mathcal{M}_\infty(A)}.\end{aligned}$$

The proof is completed by using the definition of $\|\cdot\|_{\mathcal{M}_\infty(\tau(A)^2)}$ (see Definition 3.6.23). □

At this stage the EPE looks complicated and not very applicable. But the estimate becomes easier when restricted to an L^2-space.

Let $(E, \mathfrak{B}_E, m)$ be a σ-finite measure space. We set $L^2 = L^2(E,m)$ and $L^q = L^q(E,m), 1 \leq q \leq \infty$. From the theory of p-summing operators on Banach spaces we can use the following lemma.

Lemma 3.6.26. *Suppose that $D \in \mathfrak{B}(L^\infty, L^1)$ and $G \in \mathfrak{B}(L^2, L^\infty)$.*

(1) If $DG \in \mathfrak{B}(L^2, L^\infty)$, then DG is a compact operator in L^2. Its operator norm can be estimated by

$$\|DG\|^2_{2,2} \leq \|D\|_{\infty,1}\, \|G\|_{2,\infty}\, \|DG\|_{2,\infty}\,. \tag{3.6.57}$$

(2) If $F \in \mathfrak{B}(L^1, L^2)$, then FDG is a trace class operator and its trace norm can be estimated by

$$||FDG||_{\text{trace}} \leq \kappa_G^2\, \|F\|_{1,2}\, \|G\|_{2,\infty}\, \|D\|_{\infty,1}\,. \tag{3.6.58}$$

κ_G is the complex Grothendieck constant, which is smaller than $\pi/2$.

Applying the last lemma to the Extended Pearson's Estimate (3.6.56) we obtain immediately:

Corollary 3.6.27 (Demuth–Eder) *Let A, B, α, τ be as in Theorem 3.6.25. Set $\mathfrak{H} = L^2(E,m)$. Assume further that*

$$\begin{aligned}\alpha(B) - \alpha(A) &\in \mathfrak{B}(L^\infty, L^1), \\ \tau(B) - \tau(A) &\in \mathfrak{B}(L^\infty, L^1), \\ [\tau(B) - \tau(A)]\cdot\tau(A) &\in \mathfrak{B}(L^2, L^\infty), \\ \tau(A) \in \mathfrak{B}(L^2, L^\infty), \tau(B) &\in \mathfrak{B}(L^1, L^2).\end{aligned}$$

Then

$$\begin{aligned}&\|\Omega_\pm(B,A) - \mathbb{1}\|^2_{\mathfrak{B}(\mathcal{M}_\infty(\tau^2(A))} \\ &\leq \text{const}\cdot\Big[\|\alpha(B)-\alpha(A)\|_{\infty,1} + \|\tau(B)-\tau(A)\|_{\infty,1}\Big]\,.\end{aligned} \tag{3.6.59}$$

Proof: The proof of Theorem 3.6.18 only needed the fact that $(\tau(B) - \tau(A))\tau(A)$ is compact. Then the result follows from Equations (3.6.48), (3.6.57) and (3.6.58). $\square$

This is much more general than the estimate in (3.6.55) from the original Pearson's Theorem. It becomes even more easier if we choose $\alpha(\cdot) \equiv \tau(\cdot)$.

Having in mind semigroups $\alpha(\lambda) = e^{-\lambda s}$, $s \geq 0$, on $L^2(E, m)$ which consist of integral operators, (3.6.59) gives an applicable tool to study quantitatively free and scattered systems (see also Corollary 3.6.43(iii)).

3.6.2 Stability of the Absolutely Continuous Spectra

General Trace Class Conditions

In the last section we gave a short summary of some results in mathematical theory of scattering to show their usefulness in spectral theory. Their use for the stability of the absolutely continuous spectrum is described further in what follows.

Corollary 3.6.28 *Let A and B be two selfadjoint operators in $\mathfrak{H}$. Let $\alpha(\cdot)$ be a bounded admissible function.*

(i) Assume that the wave operator $\Omega_+(\alpha(A), \alpha(B))$ exists; then

$$\sigma_{ac}(A) \subseteq \sigma_{ac}(B).$$

(ii) If

$$\alpha(A) - \alpha(B) \in \mathfrak{B}_{\mathrm{trace}}(\mathfrak{H}), \tag{3.6.60}$$

then

$$\sigma_{ac}(A) = \sigma_{ac}(B).$$

(iii) Let $\tau(\cdot)$ be a bounded admissible function such that$\tau(A)\mathfrak{H}_{ac}(A)$ is dense in $\mathfrak{H}_{ac}(A)$ and $\tau(B)\mathfrak{H}_{ac}(B)$ is dense in $\mathfrak{H}_{ac}(B)$ and let

$$\tau(B)(\alpha(B) - \alpha(A))\tau(a) \in \mathfrak{B}_{\mathrm{trace}}(\mathfrak{H})) \tag{3.6.61}$$

and

$$\tau(B) - \tau(A) \in \mathfrak{B}_{\mathrm{comp}}(\mathfrak{H}). \tag{3.6.62}$$

Then

$$\sigma_{ac}(A) = \sigma_{ac}(B).$$

(iv) Let A be a selfadjoint operator, B_n a sequence of selfadjoint operators converging to B in the strong resolvent sense. Let α be an admissible function such that

$$s - \lim_{n \to \infty} \alpha(B_n) = \alpha(B).$$

Let $\alpha(B_n)(\alpha(B_n) - \alpha(A))\alpha(A)$ be trace class and $\alpha(B_n) - \alpha(A)$ compact for any $n \in \mathbb{N}$. Assume

$$\|\alpha(B_n)(\alpha(B_n)-\alpha(A))\alpha(A)\|_{\text{trace}} \le M,$$
$$\|\alpha(B_n)-\alpha(A)\| \le M,$$

uniformly in n. Then
$$\sigma_{ac}(B) = \sigma_{ac}(A).$$

(v) If we take in (iii) $\mathfrak{H} = L^2(E,m), \tau(\cdot) = \alpha(\cdot), \alpha(A) \in \mathfrak{B}(L^2, L^\infty)$ and $\alpha(B) \in \mathfrak{B}(L^1, L^2)$, then the conditions in (3.6.61) and (3.6.62) are satisfied if
$$\alpha(B) - \alpha(A) \in \mathfrak{B}(L^\infty, L^1). \tag{3.6.63}$$

Remark 3.6.29. In the above (i) follows from Theorem 3.6.5, Theorem 3.6.7, Theorem 3.6.8. The assertion in (ii) is a consequence of Theorem 3.6.16 and the result in (iii) follows from Theorem 3.6.18. The stability of the absolutely continuous spectrum in (iv) is due to Corollary 3.6.20.

Here we see at least in (ii) and (iv) that there is no reference or assumption on wave or scattering operators. The results are purely spectral theoretic. The scattering theory is only a setting, in which we obtain consequences on the absolutely continuous spectra. We will emphasize and develop this point of view.

The trace class conditions in (3.6.60) and (3.6.61) are very abstract but more general than usual conditions coming from decomposing the differences as in (3.6.60) by

$$\alpha^2(A) - \alpha^2(B) = \alpha(A)(\alpha(A) - \alpha(B)) + (\alpha(A) - \alpha(B))\alpha(B). \tag{3.6.64}$$

The reader may have in mind operators bounded from below and admissible functions providing resolvents, $\alpha(\lambda) = \frac{1}{a+\lambda}, \lambda \ge -c$, $a < -c$, or semigroups where $\alpha(\lambda) = e^{-s\lambda}, s > 0$. Then one introduces in (3.6.64) another bounded operator X^{-1}, such that $X^{-1}\alpha(B)$ are Hilbert–Schmidt operators. Hence the remaining condition is

$$X(\alpha(A) - \alpha(B)) \in \mathfrak{B}_{HS}. \tag{3.6.65}$$

This trick is used very often in the literature. However the choice of X^{-1} has to be so strong such that the condition in (3.6.65) is very restrictive.

In (3.6.61) the condition is weaker because the sandwiched difference has an "energy cut off" or a smoothing property on both sides of the difference. Hence the sufficient condition for the difference itself becomes less restrictive. There are examples where the sandwiched difference $\alpha(B)(\alpha(B)-\alpha(A))\alpha(A)$ is trace class but $\alpha(B) - \alpha(A)$ is not trace class.

Integral Conditions

The theory above becomes more applicable if we restrict us to Hilbert spaces $L^2(E,m)$, where m denotes the Lebesgue measure. In Corollary 3.6.28 we have already seen that the trace class conditions can be satisfied if the operators $\alpha(A), \alpha(B)$ or $\tau(A), \tau(B)$ have some $L^p - L^q$ smoothing properties. Let us assume in the following that $\alpha = \tau$ and that $\alpha(A), \alpha(B)$ are integral operators in $L^2(E, m)$ having measurable kernels $\alpha_A(\cdot,\cdot), \alpha_B(\cdot,\cdot) : E \times E \to \mathbb{C}$. Then the expressions in (3.6.61) or (3.6.64) lead to the general question as to when products of integral operators are trace class. This will be answered in the next theorem.

Theorem 3.6.30 (Demuth–Stollmann–Stolz–van Casteren). *Let P, Q be integral operators in $L^2(E, m)$ with measurable kernels $P(\cdot,\cdot) : E \times E \to \mathbb{C}$ and $Q(\cdot,\cdot) : E \times E \to \mathbb{C}$. Assume that*

$$\begin{aligned} P(\cdot, x) &\in L^2(E, m) \qquad \textit{for} \quad \mathrm{m-a.e.}\ x \in E, \\ Q(x, \cdot) &\in L^2(E, m) \qquad \textit{for} \quad \mathrm{m-a.e.}\ x \in E, \end{aligned}$$

and that

$$\int_E ||P(\cdot, x)||_{L^2}\, ||Q(x, \cdot)||_{L^2}\, dm\,(x) < \infty. \tag{3.6.66}$$

Then there is a trace class operator $PQ : L^2 \to L^2$ with kernel

$$(PQ)(x, y) = \int_E P(x, u) Q(u, y)\, dm\,(u)$$

and its trace norm is estimated by

$$\|PQ\|_{\mathrm{trace}} \le \int_E \|P(\cdot, x)\|_{L^2}\, \|Q(x, \cdot)\|_{L^2}\ dm\,(x). \tag{3.6.67}$$

Remark 3.6.31. It is obvious that

$$\int_E ||P(\cdot, x)||_{L^2}\, ||Q(x, \cdot)||_{L^2}\ dm\,(x) \le ||P||_{HS}\, ||Q||_{HS}\,, \tag{3.6.68}$$

where $||.||_{HS}$ denotes the Hilbert–Schmidt norm. This means that the assumption in (3.6.66) is weaker than assuming that both P and Q are Hilbert–Schmidt operators. It is not even necessary for one of P or Q to be Hilbert–Schmidt. For instance, take $P(x, y) = p_1(x)p_2(y), Q(x, y) = q_1(x)q_2(y)$ where p_i, q_i are positive functions on E. Assume $p_1 \in L^2, q_2 \in L^2$, and $\langle p_2, q_1\rangle = \int_E p_2(x)q_1(x)\, dm\,(x) < \infty$.

Then PQ is a trace class operator and we obtain

$$\begin{aligned} ||PQ||_{\mathrm{trace}} &= ||p_1||_{L^2} ||q_2||_{L^2} \langle p_2, q_1\rangle \\ &\le ||p_1||_{L^2} ||p_2||_{L^2} ||q_1||_{L^2} ||q_2||_{L^2} \\ &= ||P||_{HS} ||Q||_{HS}. \end{aligned}$$

Proof of Theorem 3.6.30: Let $\varphi : E \to \mathbb{C}$ be a function determined below such that $\frac{1}{\varphi}$ exists. Then

$$\begin{aligned}
\|PQ\|^2_{\text{trace}} &\le \|P\phi\frac{Q}{\phi}\|^2_{\text{trace}} \\
&\le \|P\phi\|^2_{HS}\|\phi^{-1}Q\|^2_{HS} \\
&= \left[\int dm(x)\ dm(u)\,|P(x,u)|^2\,|\phi(u)|^2\right] \\
&\quad\times\left[\int dm(y)\ dm(w)\,|\phi(w)|^{-2}\,|Q(w,y)|^2\right] \\
&= \left[\int dm(u)|\phi(u)|^2\|P(\cdot,u)\|^2\right]\left[\int dm(u)|\phi(u)|^{-2}\|Q(u,\cdot)\|^2\right].
\end{aligned}$$

This becomes symmetric in P and Q if we choose

$$|\varphi(u)|^2\ \|P(\cdot,u)\|^2 = \frac{1}{|\varphi(u)|^2}\,\|\,Q(u,\cdot)\|^2\,,$$

i.e.,

$$|\varphi(u)|^4 = \frac{||Q(u,\cdot)||^2}{||P(\cdot,u)||^2}. \qquad \square$$

Remark 3.6.32. Assume that P is a bounded operator from L^1 to L^2. The Dunford–Pettis theorem implies that P is an integral operator the kernel of which satisfies

$$\operatorname*{ess\,sup}_{u\in E}\left(\int |P(y,u)|^2 dm(y)\right)^{1/2} < \infty.$$

Thus if Q has a kernel with the property

$$\int_E \sqrt{\int_E |Q(u,y)|^2\ dm(y)}\quad dm(u) < \infty, \tag{3.6.69}$$

the product PQ is trace class.

This last criterion will be applied to Theorems 3.6.16, 3.6.18 and to similar conditions ensuring the stability of the absolutely continuous spectrum σ_{ac} or the essential spectrum σ_{ess}. We will make the following assumption throughout this section.

Assumption 3.6.33. *Let A, B be two selfadjoint operators in $L^2(E)$. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a bounded admissible function.*

Let $\alpha(A)$ and $\alpha(B)$ be integral operators with measurable kernels $\alpha_A(\cdot,\cdot)$, $\alpha_B(\cdot,\cdot)$. $\alpha(A)$ and $\alpha(B)$ are also selfadjoint such that their kernels are symmetric. For them we formulate the following assumptions:

(A1) Let $\alpha(A)$, $\alpha(B)$ be $L^\infty - L^\infty$ smoothing, i.e.,

$$\operatorname*{ess\,sup}_{x \in E} \int_E |\alpha_A(x,y)|\, dm(y) = a_A < \infty,$$
$$\operatorname*{ess\,sup}_{x \in E} \int_E |\alpha_B(x,y)|\, dm(y) = a_B < \infty,$$

and set $a = \max\{a_A, a_B\}$.

(A2) Let $a_A(\cdot,\cdot), \alpha_B(\cdot,\cdot)$ *be Carleman kernels, i.e.,*

$$\operatorname*{ess\,sup}_{x \in E} \int_E |\alpha_A(x,y)|^2 dm(y) = b_A < \infty,$$
$$\operatorname*{ess\,sup}_{x \in E} \int_E |\alpha_B(x,y)|^2 dm(y) = b_B < \infty,$$

and set $b = \max\{b_A, b_B\}$.

(A3) Let $\alpha(A), \alpha(B)$ *be* $L^1 - L^\infty$ *smoothing, i.e.,*

$$\operatorname*{ess\,sup}_{x,y \in E} |\alpha_A(x,y)| = d_A < \infty,$$
$$\operatorname*{ess\,sup}_{x,y \in E} |\alpha_B(x,y)| = d_B < \infty,$$

and set $d = \max\{d_A, d_B\}$,

(Of course (A1) and (A3) imply (A2)).

Because α is assumed to be bounded, α^2 is also an admissible function (see Lemma 3.6.14 (iv)). We will formulate the results in terms of α^2 because the notation of the proofs is simpler.

Theorem 3.6.34. *Let A, B and α be as in Assumption 3.6.33 with α satisfying (A1) and (A3). Set*

$$D = \alpha^2(B) - \alpha^2(A).$$

D is an integral operator with the kernel $D(\cdot,\cdot)$.

Then $\sigma_{ac}(A) = \sigma_{ac}(B)$ *if*

$$\int_E \int_E |D(x,y)|\, dm(x) dm(y) < \infty. \tag{3.6.70}$$

Proof: Using Theorem 3.6.18 we will show

$$\alpha^2(B)(\alpha^2(B) - \alpha^2(A))\alpha^2(A) \in \mathfrak{B}_{\text{trace}}(L^2(E)) \tag{3.6.71}$$

and

$$\alpha^2(B) - \alpha^2(A) \in \mathfrak{B}_{\text{comp}}(L^2(E)). \tag{3.6.72}$$

For (3.6.71) we use the Theorem 3.6.30 in the form

$$\left|\left| \alpha^2(B)D\, \alpha^2(A)\right|\right|_{\text{trace}} \le ||\alpha(B)\alpha(B)D\alpha(A)||_{\text{trace}}\, ||\, \alpha(A)||\,.$$

Then

$$\begin{aligned}
||\alpha(B)\alpha(B)D\alpha(A)||_{\text{trace}} &\le a\int\limits_E dm(x)\sqrt{\int\limits_E dm(y)|\alpha_B(y,x)|^2} \\
&\qquad \times\sqrt{\int\limits_E dm(y)|\int\limits_E dm(u)\int\limits_E dm(v)\alpha_B(x,u)D(u,v)\alpha_A(v,y)|^2} \\
&\le a\cdot b^{1/2}\cdot \int\limits_E dm(x) \\
&\qquad \times \left[\int\limits_E dm(y)\int\limits_E dm(u_1)\int\limits_E dm(v_1)|\alpha_B(x,u_1)D(u_1,v_1)\alpha_A(v_1,y)|\right. \\
&\qquad\qquad \left.\times\int\limits_E dm(u_2)\int\limits_E dm(v_2)|\alpha_B(x,u_2)D(u_2,v_2)\alpha_A(v_2,y)|\right]^{1/2} \\
&\le a\cdot b\int\limits_E dm(x)\int\limits_E dm(u)\int dm(v)|\alpha_B(x,u)D(u,v)| \\
&\le a^2\cdot b\int dm(u)\int dm(v)\,|D(u,v)|.
\end{aligned}$$

The compactness in (3.6.72) is easy because D is Hilbert–Schmidt in view of the inequality (3.6.70) and the bound $|D(x,y)|\le 2ad$. □

Corollary 3.6.35 *Under the assumptions of Theorem 3.6.34 the Extended Pearson's Estimate (EPE) has the form*

$$||\Omega_\pm(B,A)-\mathbb{1}||_{\mathfrak{B}(\mathcal{M}_\infty(\alpha^4(A)))}\le \text{const}\cdot\int\limits_E dm(x)\int\limits_E dm(y)|D(x,y)|.$$

Remark 3.6.36.

(1) The trick explained in (3.6.64) and (3.6.65) would imply conditions of the following form

$$\alpha^4(B)-\alpha^4(A)=\alpha^2(B)X^{-1}XD+DX\;X^{-1}\alpha^2(A).$$

Take for X^{-1} the multiplication operator

$$(X^{-1}f)(x)=(1+|x|^2)^{-\frac{\varrho}{2}}f(x)$$

for an appropriate $\varrho>0$, such that $\alpha^2(B)X^{-1}$ and $X^{-1}\alpha^2(A)$ are Hilbert–Schmidt operators; then the remaining trace class condition is

$$\int_E dm(x) \int_E dm(y)\ (1+|x|^2)^\varrho\ |D(x,y)|^2$$
$$\leq 2ad \int dm(x) \int dm(y)\ (1+|x|^2)^\varrho\ |D(x,y)| < \infty. \quad (3.6.73)$$

This inequality is stronger than (3.6.70).

(2) If we use the CDS^2-theorem (see Theorem 3.6.30) for the decomposition as in (3.6.64), that is for

$$\alpha^4(B) - \alpha^4(A) = \alpha^2(B)D + D\alpha^2(A), \quad (3.6.74)$$

a sufficient trace class condition is

$$\int_E dm(x) \sqrt{\int_E |D(x,y)|^2\ dm(y)} < \infty \quad (3.6.75)$$

(see Remark 3.6.32). Also this condition is stronger than the condition in the inequality (3.6.70). Hence the inequality (3.6.70) seems to be the best possible condition to ensure that the absolutely continuous spectrum is stable in this context.

The bound in the inequality (3.6.70) establishes an L^1-condition for the stability of σ_{ac}. The next proposition is an L^2-condition for the stability of σ_{ess}.

Proposition 3.6.37. *Let A, B be two selfadjoint operators satisfying (A2) of Assumption 3.6.33. Then the difference $\alpha^4(B) - \alpha^4(A)$ is a Hilbert–Schmidt operator if*

$$\int dm(x) \left(\int dm(y) |D(x,y)| \right)^2 < \infty. \quad (3.6.76)$$

Proof: Using (3.6.64) the Hilbert–Schmidt norm of $\alpha^4(B) - \alpha^4(A)$ can be estimated by

$$\sqrt{\int dm(x) \left(\int dm(y)\ \ |D(x,y)| \right)^2},$$

giving the result. □

Corollary 3.6.38 *If α^4 is an admissible function, such that the condition in (3.6.76)is satisfied, then $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B)$.*

Finally, we show that $\int |D(x,y)| dm(y)$ also give a bound for the traces.

Proposition 3.6.39. *Let Assumption 3.6.33 be satisfied with (A2). Assume that $\alpha^4(B) - \alpha^4(A)$ is a trace class operator. Then its trace satisfies the estimate,*

$$\text{trace}(\alpha^4(B) - \alpha^4(A)) \leq 2b \int_E dm(x) \int_E dm(y)\, |D(x,y)|\, .$$

Proof: The trace is

$$\begin{aligned}\operatorname{trace}(\alpha^4(B)-\alpha^4(A)) &= \operatorname{trace}(\alpha(B)D\alpha(B))+\operatorname{trace}(\alpha(A)D\alpha(A))\\ &\le 2b\int_E dm(x)\int_E dm(y)|D(x,y)|.\end{aligned}$$

□

Semigroups and integral conditions

In the previous section arbitrary bounded admissible functions were allowed. However in the context of integral operators the most prominent example is $\alpha(\lambda)=e^{-t\lambda}$, $t>0$. This means that the operators $\alpha(A)=e^{-tA}$ are semigroups of operators generated by the semibounded selfadjoint operators A and B, respectively. Using the Laplace transform

$$(A+a)^{-1}f=\int_0^\infty e^{-at}e^{-tA}f\mathrm{dt}$$

$-a<\inf\sigma(A)$ one can transfer the condition to differences of resolvents or to powers of resolvents.

However for a large variety of applications it is sufficient to study the behaviour of the semigroups. Hence we will summarize and simplify the results of the last section to an admissible function of the form $\alpha(A)=e^{-tA}$.

Assumption 3.6.40. *Let A,B be two semibounded selfadjoint operators in $L^2(E)$. Let $\{e^{-tA},t\ge 0\}$, $\{e^{-tB},t\ge 0\}$ be the associated semigroups generated by A and B, respectively. Assume that the operators e^{-tA},e^{-tB} are ultracontractive, which means they induce operators in $\mathfrak{B}(L^2,L^\infty)$. Then they possess kernels which we denote by*

$$e^{-tA}(\cdot,\cdot),\ e^{-tB}(\cdot,\cdot).$$

Assume that the semigroups are $L^\infty-L^\infty$ smoothing and also L^1-L^∞ smoothing. Then the kernels satisfy $(A1)-(A3)$ of Assumption 3.6.33. Here we set

$$\operatorname*{ess\,sup}_{x\in E}\int_E|e^{-tA}(x,y)|dm(y)=a_A(t),$$

$$\operatorname*{ess\,sup}_{x\in E}\int_E|e^{-tB}(x,y)|dm(y)=a_B(t),$$

$$\operatorname*{ess\,sup}_{x\in E}\int_E|e^{-tA}(x,y)|^2\,dm(y)=b_A(t),$$

$$\operatorname*{ess\,sup}_{x\in E}\int_E|e^{-tB}(x,y)|^2\,dm(y)=b_B(t),$$

$$\operatorname*{ess\,sup}_{x,y\in E} \left|e^{-tA}(x,y)\right| = d_A(t),$$
$$\operatorname*{ess\,sup}_{x,y\in E} \left|e^{-tB}(x,y)\right| = d_B(t).$$

Set $a(t) = \max\{a_A(t), a_B(t)\}, b(t) = \max\{b_A(t), b_B(t)\}$ *and* $d(t) = \max\{d_A(t), d_B(t)\}$*, where* t *is arbitrary but fixed. The difference of the semigroups is denoted by*

$$D_t = e^{-tB} - e^{-tA}.$$

Its kernel is given by

$$D_t(x,y) = e^{-tB}(x,y) - e^{-tA}(x,y).$$

In Section 3.4.2.2. it turned out that the functions $\int dm(y)|D(x,y)|$ played a crucial role, where $D(x,y)$ was the kernel of $D = \alpha^2(B) - \alpha^2(A)$. Therefore we give this integral a special name if D is the semigroup difference.

Definition 3.6.41. Let $D_t = e^{-tB} - e^{-tA}$ with the kernel $D_t(\cdot,\cdot)$. We define

$$D_t(x) = \int_E |D_t(x,y)| dm(y),$$

and call it **comparison function**.

Now we are able to summarize the results of this chapter in terms of this comparison function.

Theorem 3.6.42 (Demuth). *Let* A, B *be semibounded selfadjoint operators and let the associated semigroups* e^{-tA}, e^{-tB} *satisfy the Assumptions 3.6.40. Then,*

(i) the semigroup difference

$$D_{2t} = e^{-2tB} - e^{-2tA}$$

is trace class if $\sqrt{D_t(\cdot)} \in L^1$*. Its trace norm can be estimated by*

$$\int_E \sqrt{D_t(x)}\, dm(x).$$

(ii) The sandwiched semigroup difference

$$e^{-tB}\, D_{2t}\, e^{-tA}$$

is trace class if $D_t(\cdot) \in L^1$*. Its trace norm can be estimated by*

$$\int D_t(x)\, dm(x).$$

(iii) The semigroup difference D_{2t} *is Hilbert–Schmidt if* $D_t(x) \in L^2$. *Its Hilbert–Schmidt norm is estimated by*

$$\left(\int_E |D_t(x)|^2 dm(x)\right)^{1/2}.$$

(iv) If D_{4t} *is a trace class operator and its trace has the estimate*

$$\text{trace } D_{4t} \leqq \int_E |D_{2t}(x)|\, dm(x).$$

Proof: The proofs are given in the last section (in Theorem 3.6.34, in Remark 3.6.36 (3.6.75), in Proposition 3.6.37 and Proposition 3.6.39). They will not be repeated here. □

Corollary 3.6.43 *Let A and B be the selfadjoint operators as given in the last theorem. Let* $D_1(\cdot)$ *be the comparison function for* $t = 1$. *Then*

(i) $\sigma_{ac}(A) = \sigma_{ac}(B)$, *if* $D_1(\cdot) \in L^1(E)$.
(ii) $\sigma_{\text{ess}}(a) = \sigma_{\text{ess}}(B)$, *if* $D_1(\cdot) \in L^2(E)$.
(iii) If $D_1(\cdot) \in L^1(E)$ *the wave operators* $\Omega_{\pm}(B, A)$ *exist and are complete. For* f *in a dense set of* $\mathfrak{H}_{ac}(A)$ *we have*
$$||(\Omega_{\pm}(B, A) - \mathbb{1})f|| \leq c_f \cdot \int D_1(x)\, dm(x).$$ □

3.7 Notes

Section 3.1

The theory of rank one perturbations was originally done in the work of Donoghue [76]. The revival of the theory in the context of random operators is recent and the works of Simon–Wolff [181]. Subsequently the theory was very effectively used to handle several problems in the random operators. The material presented in this section is based on Simon [177].

One of the components used in the spectral theory of random operators is spectral averaging formula, Proposition 3.1.4. We refer to Simon [177], Section I.3 for comments on the history of the formula. We present here in Theorem 3.1.5 a converse that the Lebesgue measure and its positive multiples are the only measures for which this integral formula is valid.

The Simon–Wolff Theorem 3.1.7 is from [181], in the form presented in [177].

Section 3.2

The theorem of Wiener presented here is used very much in scattering theory, especially in proving the RAGE theorem given earlier.

Section 3.4

Approximate eigenfunctions were used to determine spectra of selfadjoint operator in the theorem of Weyl, see Simon [159]. We refer to Chapter 4 for the definitions of operators used in the following.

Gilbert–Pearson [89] gave criteria for identifying components of the spectral measure in the case of one-dimensional Schrödinger operators and their theory is known as the subordinacy theory. Consider a Schrödinger operator $-\varDelta+V$ on $L^2(\mathbb{R}^+)$, with some boundary condition at 0 that gives us a selfadjoint operator when V is locally integrable. Then such an operator has simple spectrum and as such can be shown to be unitarily equivalent to a Jacobi matrix J given by

$$(Ju)(0) = a_0u(1)+b_0u(0), \;\; (Ju)(n) = a_nu(n)+b_nu(n)+a_{n+1}u(n-1), \; n \in \mathbb{N},$$

with $a_n > 0$ and $b_n \in \mathbb{R}$. Then the vector $\delta_0(n) = \delta_{0n}$ is a cyclic vector for J and therefore the measure $\mu = \langle \delta_0, P_J(\cdot)\delta_0\rangle$ is a total spectral measure for J. Further the Borel transform of μ, given by $m(z) = \int \frac{1}{x-z} d\mu(x)$, is the Green function $\langle \delta_0, (J-z)^{-1}\delta_0\rangle$ and it can be written in terms of the solutions of the difference equation $Ju - zu = 0$. For any infinite sequence $\{u(0), \dots, u(n), \dots\}$, let us denote for any $n \in \mathbb{N}$, Q_nu to be the n-dimensional vector $\{u(0), \dots, u(n-1)\}$ and let $\|Q_nu\|_2$ denote the ℓ^2 norm of this finite vector. With this notation let us call a solution u of the equation $Ju = zu$ subordinate if for any other solution v of this equation one has

$$\lim_{L\to\infty} \frac{\|Q_Lu\|_2}{\|Q_Lv\|_2} = 0.$$

(Note here that in view of the selfadjointness of J, the equation $Ju = zu$ cannot have an $\ell^2(\mathbb{Z}^+)$ solution for non-real z.)

Gilbert–Pearson use the notion of minimal support of a measure μ, which is its supported when compared to the Lebesgue measure. For a measure μ, their definition of minimal support of μ is the set S such that $\mu(\mathbb{R} \setminus S) = 0$ and for every subset $S_0 \subset S$ satisfying $\mu(\mathbb{R} \setminus S_0) = 0$ one has $|S \setminus S_0| = 0$. The set S is also called the essential support of μ.

So considering this setting a restatement of the result of Gilbert–Pearson ([89], Theorem 3) is:

Theorem 3.7.1 (Gilbert–Pearson). *Consider a Jacobi matrix J as above and let μ be the associated total spectral measure. Then,*

(i) the set

$$\{E : Ju = Eu \text{ has no subordinate solution}\},$$

is the minimal support of μ_{ac}.

(ii) The set

$$\{E : Ju = Eu \text{ has subordinate solutions}\},$$

is the minimal support of μ_s.

(iii) The set

$$\{E : Ju = Eu \text{ has subordinate solutions in } \ell^2(\mathbb{Z}^+)\},$$

is the minimal support of μ_p.

(iii) The set

$$\{E : Ju = Eu \text{ has subordinate solutions but not in } \ell^2(\mathbb{Z}^+)\},$$

is the minimal support of μ_{sc}.

This theorem is improved by Jitomirskaya–Last [103] to the following. Consider $A + B$ on $\ell^2(\mathbb{Z}^+)$ with

$$(Au)(n) = a_n u(n+1) + a_{n-1}u(n-1), \quad (Bu)(n) = b_n u(n), \ n \in \mathbb{Z}^+,$$

with $a_n > 0, \sum_{n=1}^{\infty} a_n^{-1} = \infty,\ b_n \in \mathbb{R}$. Let δ_1 denote the orthogonal projection onto the vector supported at 1. Consider the two solutions u_1, u_2 of the equation

$$H_\theta u = (A + B - \tan(\theta)\delta_1)u = Eu, \quad \theta \in (-\pi/2, \pi/2),$$

with the respective initial conditions

$$\begin{pmatrix} u_1(1) & u_2(1) \\ u_1(2) & u_2(2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{E+\tan(\theta)-b_1}{a_1} & -\frac{1}{a_1} \end{pmatrix}$$

(which amount to the orthogonal conditions "$u_1(0) = 0, u_1(1) = 1$ and $u_2(0) = 1, u_2(1) = 0$"). With these notations the theorem is

Theorem 3.7.2 (Jitomirskaya–Last). *Let $E \in \mathbb{R}$, $\alpha \in (0,1)$ and consider the solutions u_1, u_2 associated with E and the operator H_θ with the initial conditions given above. Then*

$$D_\mu^\alpha(E) = \infty \quad \Longleftrightarrow \quad \varliminf_{L\to\infty} \frac{\|Q_L u_1\|_2}{\|Q_L u_2\|_2^{\frac{\alpha}{2-\alpha}}} = 0.$$

Recent work of Christ–Kiselev–Last [47] gives criteria based on approximate eigenfunctions. The Theorem 3.4.2 is Theorem 1.5 of [47] and the proof can be found there. Although this seems like a powerful criterion, as of now this criterion has not been effectively used.

On the other hand a related criterion in terms of explicit generalized solutions for partial differential (difference) operators, the theorem of Kiselev–Last given below, is applicable for several concrete operators and is used to determine their spectral type.

Theorem 3.7.3 (Kiselev–Last). *Let H be a selfadjoint operator in $L^2(\mathbb{R}^d)$ given by $H = \Delta + V$ with $V(x) \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ (respectively the one given by* *equation $H = \Delta + V$ on $\ell^2(\mathbb{Z}^d)$). Suppose there are (generalized) solutions to the equation $Hf_{\scriptscriptstyle E} = Ef_{\scriptscriptstyle E}$, such that*

$$\varliminf_{R\to\infty} \frac{\|B_R f_E\|_2^2}{R^\alpha} < \infty,$$

where B_R denotes the operator of multiplication by the indicator function of the ball of radius R in $\mathbb{R}^d$ (respectively in $\mathbb{Z}^d$). Suppose, there is a vector $\phi \in L^2(\mathbb{R}^d)$ (respectively $\phi \in \ell^2(\mathbb{Z}^d)$), such that $\int_{\mathbb{R}^d} f_E(x)\phi(x) \neq 0$ (respectively $\sum_{\mathbb{Z}^d} f_E(n)\phi(n) \neq 0$). Then $D^\alpha_{\mu^\phi}(E) > 0$, where μ^ϕ is the spectral measure of H associated with the vector ϕ.

Section 3.5

The section on Mourre theory follows the original work of Mourre [148]. The method of finding a local conjugate to a given selfadjoint operator was widely used to obtain the absence of singular continuous spectrum in the case of N-particle Schrödinger operators [10]. The attempt to show the existence of a local conjugate seems to require some minimal smoothness of at least one of the summands when dealing with perturbations of two selfadjoint operators as it happens in the case of Schrödinger operators.

Section 3.6

Standard textbooks for mathematical scattering theory are those of Kato [108], Amrein–Jauch–Sinha [12] Reed–Simon [158], Baumgärtel–Wollenberg [20], Perry [155] or Yafaev [192].

Theorem 3.6.4

The proof of Theorem 3.6.4 is given e.g., in Reed–Simon [158] p. 19.

Theorem 3.6.7

The proof of Cook's criterion can be found in all of the standard textbook. The reader may consult Reed–Simon [158] page 20.

Theorem 3.6.8

The proof follows the arguments given on page 323 of Baumgärtel–Wollenberg [20].

Theorem 3.6.9

Trace class criteria go back to Rosenblum [165] and Kato [105], [106]. Then they were extended to trace class conditions for differences of operator-valued functions by Birman [27] and generalized to two-space scattering by Pearson [153].

Definition 3.6.10

The subspace $\mathcal{M}_\infty(A)$ was also introduced, for example, in Reed–Simon [158] page 23, and is studied in various aspects by Baumgärtel–Wollenberg [20] page 58 ff.

Lemma 3.6.14

Properties (i) and (ii) are taken from [20], page 158 ff. (iii) is new here; it can be proved in the same fashion as (i), (ii). Property (iv) is obvious from the definition.

Theorem 3.6.16

The proof can be taken from [20] pages 343 ff, or one can consult also [158] page 30.

New results on the invariance principle are given by Xia in [190] and [191].

Theorem 3.6.18

The possibility of the improvement of the trace class criteria to sandwiched differences goes back to an idea of Birman who studied wave operators for mutually subordinate operators (see e.g., [158] page 28). The crucial condition in case of subordination is that $P_B(S)(B-A)P_A(S)$ is a trace class operator for any bounded interval $S \subset \mathbb{R}$. That implies immediately the condition in Equation 3.6.48. This is more practicable, because there is only a rough knowledge about the spectral measures for general selfadjoint operators. A two-space version of the proof for Theorem 3.6.18 is given in [62] and in [64].

The two-space version of the usual trace class criterion due to Pearson is given in Reed–Simon [158] page 24. A proof for $\mathfrak{H}_{ac}(A) = \mathfrak{H}_{ac}\big(\alpha(A)\big)$ for admissible α is given by Baumgärtel–Wollenberg [20] page 158.

Lemma 3.6.19

The idea of extending the trace class conditions to operators which are limits of operator sequences and the proof idea of this lemma go back to T. Ichinose (private communication 2001). The motivation was to include Schrödinger operators with negative δ-like potentials.

Theorem 3.6.21

Pearson's estimate goes back to Rosenblum. It follows directly from the proof of Pearson's Theorem. A two-space version of Pearson's estimate is given in Reed–Simon [158], page 26.

Examples where $\tau(B)\big(\alpha(B)-\alpha(A)\big)\tau(A)$ is trace class, but $\alpha(B)-\alpha(A)$ is not trace class are given in van Casteren–Demuth–Stollmann–Stolz [70].

Definition 3.6.23

The idea to introduce the subspace $\mathcal{M}_\infty\big(\tau(A)\big)$ goes back to S. Eder (private communication 1999). The proof ideas are a refinement of the proofs for $\mathcal{M}_\infty(A)$.

Lemma 3.6.26

The estimates are consequences from the theory of p-summing operators. The interested reader may consult Diestel–Jarchow–Tonge [73]; Bergh–Löfström [26] or Eder [80]. The more general frame of lattice theory is given e.g., by Schäfer [170] or Meyer–Nieberg [144].

Remark 3.6.29

The trick to decompose $\alpha(B)\big(\alpha(A) - \alpha(B)\big)$ into $\alpha(B)X^{-1}$ and $X\big(\alpha(A) -\alpha(B)\big)$ was used by Arcenev [14]; Deift–Simon [59]; Baumgärtel–Demuth [19]. The sandwiched differences $\alpha(B)\big(\alpha(B) - \alpha(A)\big)\alpha(A)$ are studied by van Casteren–Demuth–Stollmann–Stolz in [70]. If A is the Laplacian in $L^2(\mathbb{R}^d)$ and B is the Laplacian perturbed by Dirichlet boundary conditions one can find examples where $\alpha(B)\big(\alpha(B) - \alpha(A)\big)\alpha(A)$ is trace class but $\alpha(B) - \alpha(A)$ is not even Hilbert–Schmidt (see also [70] and the notes for Chapter 5).

Theorem 3.6.30

An extended proof of this criterion is given by Demuth–Stollmann–Stolz–van Casteren [70] in Theorem (3.6.42). A two-space version of this theorem is given by Demuth in [62].

4

Operators of Interest

We will illustrate the usefulness of the spectral theoretic criteria developed in Chapter 3 for several physically motivated models described by different Schrödinger operators. These models will be classified with respect to different kinds of perturbations.

The first class of examples are deterministic perturbations. Here we restrict ourselves to the usual multiplication with a potential function. Further possible perturbations for instance by variable coefficients in the first and second order of the differential operator are not considered.

The second class of models describes the randomness of quantum mechanical systems. Here the potentials can vary randomly.

The third class are singular or domain perturbations. This class models the perturbations of potentials barriers of infinite height. In the context of Dirichlet forms these are domain perturbations. For local operators such perturbations are modelled by imposing Dirichlet boundary conditions.

4.1 Unperturbed Operators

The variety of interesting unperturbed operators is so large that we restrict to those operators which play an important role in quantum physics. It is natural to start with the Laplacian $-\Delta$ in $L^2(\mathbb{R}^d)$, because it describes the kinetic energy of a free non-relativistic particle, given by $\frac{p^2}{2m}$. Following Einstein the energy of a relativistic particle is

$$\sqrt{c^2p^2 + m^2c^4} - mc^2$$

where c is the speed of light. Note that this expression tends to $\frac{p^2}{2m}$ as $c \to \infty$.

Models for crystals doped with impurities are often given on lattices. Hence Δ on lattices is included.

However, as mentioned above, the variety of possible also interesting operators is huge. For diffusion problems operators in divergence form $-\nabla(a\nabla)$

are relevant, where a is a diffusion matrix with variable coefficients. It would be also of interest to study unperturbed operators of the form $(-i\nabla - \vec{A})^2$ where a magnetic potential is present. Moreover, from the mathematical point of view it is of interest to treat Laplace–Beltrami operators on locally finite Riemannian manifolds.

To find a selection of models illustrating the general theory of Chapter 3, we study only the Laplacian $-\Delta$ on $L^2(\mathbb{R}^d)$ and on $l^2(\mathbb{Z}^d)$, fractional powers of the Laplacian of the form $(-\Delta)^\alpha$ with $0 < \alpha < 1$, and the relativistic Hamiltonian $\sqrt{-\Delta + c^2} - c$.

4.1.1 Laplacians

For the sake of completeness we introduce the unique selfadjoint operators which correspond to the differential expressions mentioned above. We start with the Laplacian, because it is an important free operator. In what follows we use both $\widehat{\cdot}$ and $\mathcal{F}$ to denote the Fourier transform.

Definition 4.1.1. Let $L^2(\mathbb{R}^d, dk)$ be the image of $L^2(\mathbb{R}^d, dx)$ under Fourier transform. Let P_i be the multiplication operator in $L^2(\mathbb{R}^d, dk)$ given by

$$(\widehat{P_i\, f})(k) = k_i \widehat{f}(k), \quad i = 1, \dots, d.$$

P_i is called the momentum operator. It is selfadjoint on its domain

$$\mathrm{dom}\,(P_i) = \{f \in L^2(\mathbb{R}^d, dx);\ \int_{\mathbb{R}^d} |k_i|^2 |\widehat{f}(k)|^2 dk < \infty\}.$$

Let P^2 be the multiplication operator by $|k|^2 = \sum_{i=0}^{d} k_i^2$ in $L^2(\mathbb{R}^d, dk)$, i.e.,

$$(\widehat{\ P^2 f})(k) = |k|^2\ \widehat{f}(k) \tag{4.1.1}$$

with

$$\mathrm{dom}\,(P^2) = \{f \in L^2(\mathbb{R}^d, dx);\ \int_{\mathbb{R}^d} |k|^4\ |\widehat{f}(k)|^2\ dk < \infty\}.$$

Remark 4.1.2. According to the definitions above

$$P^2 = \sum_{i=0}^{d} P_i^2.$$

The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is contained in $\mathrm{dom}\,(P^2)$. For $f \in \mathcal{S}(\mathbb{R}^d)$ one has

$$(P^2 f)(x) = (-\Delta\, f)(x), \tag{4.1.2}$$

with $\Delta = \sum_{i=0}^{d} \frac{\partial^2}{\partial x_i^2}$. P^2 restricted to $S(\mathbb{R}^d)$ is essentially selfadjoint so that $\left(P^2|_{S(\mathbb{R}^d)}\right)^* = P^2$. Hence P^2 is the only selfadjoint realization of the Laplacian. This will be also denoted by $-\Delta$ or sometimes by H_0 (see below).

Remark 4.1.3. The definition of $-\Delta$ coincides with the Friedrichs extension (see Corollary 2.1.27). The associated sesquilinear form is

$$\mathfrak{a}[f,g] = \langle \sqrt{-\Delta}\, f,\ \sqrt{-\Delta}\, g\rangle \tag{4.1.3}$$

where $f, g \in \operatorname{dom}(\mathfrak{a})$ with

$$\begin{aligned}\operatorname{dom}(\mathfrak{a}) &= \operatorname{dom}(\sqrt{-\Delta})\\ &= \{f \in L^2(\mathbb{R}^d, dx);\ \int_{\mathbb{R}^d} |k|^2\ |\widehat{f}(k)|^2\ dk < \infty\}.\end{aligned}$$

Of course $\operatorname{dom}(-\Delta) \subset \operatorname{dom}(\sqrt{-\Delta})$ and $\operatorname{dom}(-\Delta)$ consist of all functions $g \in \operatorname{dom}(\sqrt{-\Delta})$ with $\sqrt{-\Delta}\ g \in \operatorname{dom}(\sqrt{-\Delta})$.

The previous concept of introducing the Laplacian in Definition 4.1.1 can be extended to functions of it.

Definition 4.1.4. Let g be a real-valued C^∞-function with the following properties:

(i) $g : \mathbb{R}_+ \to [b, \infty)$ for some $b \in \mathbb{R}_+$
(ii) $\lim\limits_{\lambda\to\infty} g(\lambda) = \infty$
(iii) The set $\{\lambda \in \mathbb{R}_+,\ g'(\lambda) = 0\}$ is countable and has an accumulation point at most at infinity.

Then we define

$$(H_0 f)(x) = (\mathcal{F}^* M_g\ \mathcal{F} f)(x) \tag{4.1.4}$$

or

$$(\mathcal{F}\ H_0 f)(k) = M_g\ (\mathcal{F} f)(k),$$

where M_g is the multiplication operator in $L^2(\mathbb{R}^d, dk)$ with the function $g(\cdot)$ given by

$$(M_g \widehat{f})(k) = g(|k|^2)\widehat{f}(k).$$

The domain of H_0 is

$$\begin{aligned}\operatorname{dom}(H_0) &= \mathcal{F}^* \operatorname{dom}(M_g) \\ &= \{f \in L^2(\mathbb{R}^d, dx) : \int_{\mathbb{R}^d} g(|k|^2)^2\ |\widehat{f}(k)|^2\ dk < \infty\}.\end{aligned} \tag{4.1.5}$$

Remark 4.1.5. The unperturbed operator H_0, defined above, is selfadjoint positive and has purely absolutely continuous spectrum.

Due to the assumptions (i)–(iii) on the function g we have included all the operators which we have in mind for the illustrating examples, i.e., we have included

$$\begin{aligned}&-\Delta, (-\Delta)^\alpha \text{ with } 0 < \alpha < 1,\\ &\sqrt{-\Delta + c^2} - c.\end{aligned} \tag{4.1.6}$$

As long as the theory is true for H_0 in general, we will not distinguish between these examples.

Moreover, it is possible to translate the Remark 4.1.3 to H_0. Its associated form is

$$\mathfrak{a}[f, g] = \langle H_0^{1/2} f,\ H_0^{1/2} g\rangle \tag{4.1.7}$$

with

$$\begin{aligned} \operatorname{dom}(\mathfrak{a}) &= \operatorname{dom}(H_0^{1/2}) \\ &= \left\{ f \in L^2(\mathbb{R}^d, dx) : \int_{\mathbb{R}^d} g(|k|^2)\ |\widehat{f}(k)|^2\ dk < \infty \right\}. \end{aligned}$$

We consider next the analog of the Laplacian on the lattice $\mathbb{Z}^d$, which we will continue to call the Laplacian.

Definition 4.1.6. Δ on $\ell^2(\mathbb{Z}^d)$ is defined as

$$(\Delta u)(n) = \sum_{|n-i|=1} u(i), \quad u \in \ell^2(\mathbb{Z}^d).$$

Remark 4.1.7. With this definition this operator is bounded selfadjoint and has purely absolutely continuous spectrum which is equal to $[-2d, 2d]$. Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and denote points in $\mathbb{T}^d$ by $(\theta_1, \ldots, \theta_d)$. Then Δ is unitarily equivalent to the operator of multiplication by $2\sum_{i=1}^{d} \cos(\theta_i)$. In view of the fact that the spectrum of Δ is symmetric about 0, we will not write $-\Delta$ in the case of lattice Laplacians, unlike the continuum case.

We give a few properties of two functions of Δ, namely $e^{it\Delta}$ and $(\Delta - z)^{-1}$. We use the notation $\{\delta_m\}$ to denote the standard basis for $\ell^2(\mathbb{Z}^d)$, that is δ_m for ℓ^2 sequences given by $\delta_m(m) = 1, \delta_m(n) = 0, m \neq n$.

Lemma 4.1.8. *Let $J_k(x)$ denote the Bessel function of the first kind with k an integer and x real. Then*

$$\langle \delta_m, e^{it\Delta}\delta_n\rangle = (2\pi)^d \prod_{j=1}^{d} i^{n_j - m_j} J_{n_j - m_j}(2t).$$

Proof: As stated earlier Δ is unitarily equivalent to the operator of multiplication by the function $2\sum_{j=1}^{d} \cos(\theta_j)$ on $L^2(\mathbb{T}^d)$ under the unitary equivalence implemented by the Fourier Series $u \to \widehat{u}$ given by $\widehat{u}(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}^d} e^{-in\theta} u(n)$, a priori defined on $\ell^1(\mathbb{Z}^d)$ sequences and extended to $\ell^2(\mathbb{Z}^d)$. Therefore writing $\langle \delta_m, e^{it\Delta}\delta_n\rangle$ in this representation and using the definition of the Bessel function of the first kind, namely,

$$J_k(x) = \frac{1}{i^k 2\pi} \int_0^{2\pi} e^{ix\cos(\theta)} e^{ik\theta} d\theta,$$

the lemma follows. □

Lemma 4.1.9. *Consider $J_k(x)$, $k \in \mathbb{Z}, x \in \mathbb{R}$. Then the asymptotic estimates*

$$|J_k(x)| \leq \begin{cases} \frac{C}{\sqrt{|k|}}, & |k| \to \infty, \\ \frac{C}{\sqrt{|x|}}, & |x| \to \infty, \end{cases}$$

are valid with C independent of x or k.

Proof: This is a direct application of the stationary phase method, to estimate the integral defining the Bessel function, and we refer to the notes for more on this. □

In the following proposition we will consider the properties of the matrix elements of the resolvent, $G_0(z,m,n) = \langle \delta_m, (\Delta - z)^{-1}\delta_n \rangle$.

Remark 4.1.10. It might be tempting to think, in analogy with the case of the continuous Laplacian, that the quantities $G_0(E+i0,m,n)$ have a singularity in E and that decay like $|n-m|^{-d+2}$ for large $|n-m|$ in dimensions $d \geq 3$. Both these expectations are incorrect as will be seen in the next proposition. In fact we cannot have $\sum_{m\in\mathbb{Z}^d} |G_0(E+i0,m,n)|^2 < \infty$ for a set E of positive Lebesgue measure in $[-2d, 2d]$, since such an estimate would imply absence of spectrum on that set, which is a contradiction. Therefore the best one can hope for is a decay like $|n-m|^{-\frac{d-1}{2}}$ for $G_0(E+i0,m,n)$ and we will prove something close to this in $d \geq 3$.

Proposition 4.1.11. *Consider the Laplacian Δ on $\ell^2(\mathbb{Z}^d)$. Then the following are valid.*

(i) For $d = 1$, $G_0(E+i0,m,n)$ is bounded continuous in the complement of any open set containing $\{-2,2\}$.
(ii) For $d = 2$, $G_0(E+i0,m,n)$ is bounded continuous in the complement of any open set containing $\{-4,0,4\}$.
(iii) For $d \geq 4$, $G_0(z,m,n)$ is bounded continuous in $\overline{\mathbb{C}^+}$ and satisfies,

$$|G_0(E+i0,m,n)| \leq C \left(\frac{1}{1+|(n-m)_{\pi(4)}|} \right)^{\frac{1}{3}-\epsilon} \prod_{j=5}^{d} \left(\frac{1}{1+|(n-m)_{\pi(j)}|} \right)^{\frac{1}{3}},$$

for any $\epsilon > 0$, where $\pi(\cdot)$ is a permutation of $\{1,\ldots,d\}$ such that $|(n-m)_{\pi(1)}| \leq |(n-m)_{\pi(2)}| \leq \cdots \leq |(n-m)_{\pi(d)}|$.

Proof: (i) Is an explicit calculation of

$$G_0(z,m,n) = \int_0^{2\pi} \frac{e^{i\theta(m-n)}}{2\cos(\theta) - z} d\theta = \frac{\operatorname{sgn}(m-n)}{2\pi i} \int\limits_{|w|=1} \frac{w^{|m-n|}}{w + w^{-1} - z} \frac{dw}{w},$$

using contour integration, where we have taken sgn(y) $= \frac{y}{|y|}$, $y \neq 0$. This gives the expression

$$G_0(z,m,n) = \operatorname{sgn}(m-n) \frac{\left(z \pm \sqrt{z^2 - 4}\right)^{|m-n|}}{\sqrt{z^2-4}},$$

where the choice of the square root is so that the numerator has exponential decay in $|n-m|$ when $z \in \mathbb{C}^+$. Taking boundary values now gives the result.

(ii) We set $m = 0$ for simplicity of writing (noting that everywhere below replacing n by $n - m$ will give the result for the case of non-zero m) and take $n = (n_1, n_2)$. Then,

$$G_0(z,0,n) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} d\theta_1 d\theta_2 \frac{e^{-in_1\theta_1 - in_2\theta_2}}{(2(\cos\theta_1 + \cos\theta_2) - z)} . \tag{4.1.8}$$

At this stage we note that since $Im(z)$ is non-zero, the function in the integrand is a bounded continuous function for all n_1, n_2. Therefore, we do the calculation below only at θ_1 with $\cos(\theta_1) \neq 0$, while we continue to write the integral as if it is valid for all θ_1. Since we are omitting only a set of measure zero (in θ_1) there is no problem with the definition of the integral. We change variables to $\theta = \frac{\theta_1+\theta_2}{2}, \phi = \frac{\theta_1-\theta_2}{2}$ and use a trigonometric identity involving sum of cosines and rewrite the expression as

$$G_0(z,0,n) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} d\theta d\phi \frac{e^{-i(n_1+n_2)\theta - i(n_1-n_2)\phi}}{(4(\cos\theta \cdot \cos\phi) - z)} . \tag{4.1.9}$$

For a further simplification we write the integral with respect to ϕ as a contour integral. Let

$$a = e^{-i\phi}, n_2 > 0 \quad \text{and} \quad a = e^{i\phi}, n_2 < 0; \tag{4.1.10}$$

then

$$\begin{aligned} G_0(z,0,n) = \operatorname{sgn}(n_1 - n_2) \frac{1}{2\pi} \int_0^{2\pi} d\theta \; e^{-i(n_1+n_2)\theta} \\ \times \frac{1}{2\pi i} \int_{|a|=1} da \frac{a^{|n_1-n_2|}}{(2\cos\theta \cdot a^2 + 2\cos\theta - z \cdot a)}. \end{aligned} \tag{4.1.11}$$

Denoting the roots in the denominator in the integrand as $z_\pm = (z \pm \sqrt{z^2 - 16(\cos\theta)^2})/4\cos\theta$, we have

$$\begin{aligned} G_0(z,0,n) &= \operatorname{sgn}(n_1-n_2)\frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-i(n_1+n_2)\theta} \\ &\quad \times \frac{1}{2\pi i}\int_{|a|=1} da \frac{a^{|n_1-n_2|}}{2\cos\theta(a-z_+)(a-z_-)}. \end{aligned} \tag{4.1.12}$$

Using the residue theorem and noting that when $\epsilon > 0$, the roots $z_\pm$ do not lie on the unit circle but fall one each in the interior and exterior of the unit disc (this can be seen by computing the product z_+z_- which equals 1 and noting that when z is non-real, neither z_+, z_- is on the unit circle). When $\mathrm{R}e(z) = E \in (-4,4)$, the integral is simplified to

$$\begin{aligned} G_0(z,0,n) &= \operatorname{sgn}(n_1-n_2)\frac{1}{2\pi}\int_0^{2\pi} d\theta\ e^{-i(n_1+n_2)\theta} \\ &\quad \times \frac{1}{2\pi i}\int_{|a|=1} da \frac{(z_+)^{|n_1-n_2|}}{2\cos\theta(z_+-z_-)}, \end{aligned} \tag{4.1.13}$$

where we made a choice of the square root so that $|z_+| < 1$, without actually specifying the choice. Since we are only interested in estimates of this integral, this ambiguity will not cause any problems. A further simplification of the integral yields

$$G_0(z,0,n) = \operatorname{sgn}(n_1-n_2)\frac{1}{2\pi}\int_0^{2\pi} d\theta\ \frac{e^{-i(n_1+n_2)\theta}(z_+)^{|n_1-n_2|}}{\sqrt{z^2-16\cos^2\theta}}. \tag{4.1.14}$$

Now a change of variables $x = \cos\theta$ gives

$$\begin{aligned} G_0(z,0,n) &= -\frac{\operatorname{sgn}(n_1-n_2)}{\pi} \\ &\quad \times \int_{-1}^{1} dx \frac{(x-\operatorname{sgn}(n_1+n_2)i\sqrt{1-x^2})^{|n_1+n_2|}(z_+(x,z))^{|n_1-n_2|}}{4\sqrt{1-x^2}\sqrt{\frac{z^2}{16}-x^2}}. \end{aligned} \tag{4.1.15}$$

We note here again that the above expression is valid only up to fixing a branch of the square root which we have left ambiguous. We have not computed z_+ explicitly, but note that it depends on z and x and satisfies $|z_+| \leq 1$ for $z \in \mathbb{C}^+$. We also note that $z_+(x,z)$ has no singularity at $x = 0$ event though this fact is not apparent from the expression for it; the reader who is not convinced can explicitly estimate and verify, by making a choice of the square root, that the bound $|z_+(x,z)| \leq 2|x|/|z|$ is valid for x in a neighbourhood of 0 for each fixed $z \in \mathbb{C}^+$.

It is clear from the expression above that when $\epsilon = 0$ and $E \in [-4,4]$, the integral diverges logarithmically when $E = 0, -4, 4$; otherwise the integrand has only a square root singularity at the points $-1, 1, -E/4, E/4$. Further the numerator in the integrand has modulus at most one. Let S denote the complement of any open set containing $\{-4,0,4\}$ and let $\delta = \operatorname{dist}(S, \{-4,0,4\})$. Then using the above facts we estimate the integral when $E \in S$:

$$|G_0(z,0,n)| \le \frac{1}{\pi}\int_{-1}^{1} dx \frac{1}{4\left|\sqrt{(1-x)(1+x)(\frac{E}{4}-x)(\frac{E}{4}+x)}\right|}. \tag{4.1.16}$$

Then a tedious calculation by splitting the integral appropriately into four parts according as $\{(E,x) : S \ni E < 0, x < E/4\}$, $\{(E,x) : S \ni E < 0, x > E/4\}$, $\{(E,x) : S \ni E > 0, x < E/4\}$ and $\{(E,x) : S \ni E > 0, x > E/4\}$, we obtain the estimate

$$|G_0(z,0,n)| \le C(\delta)^{-\frac{3}{2}}, \quad C \quad \text{independent of} \quad n, E, \epsilon. \tag{4.1.17}$$

From this the stated result follows.

(iii) We have the expression

$$G_0(z,m,n) = i\int_{-\infty}^{0} \langle \delta_n, e^{it\Delta}\delta_m\rangle e^{-itz} dt.$$

From this relation and Lemma 4.1.8 we see that

$$G_0(z,m,n) = i\ (2\pi)^d \int_{-\infty}^{0} e^{-itz} \prod_{j=1}^{d} i^{n_j - m_j} J_{n_j-m_j}(2t)\ dt.$$

Given the pair n, m let π be a permutation on $\{1,\dots,d\}$ such that $|n-m|_{\pi(j)}$ is non-decreasing as j increases (note that this permutation depends on the pair (n,m), but we will not show this dependence explicitly in π). By this choice we have

$$|(n-m)_{\pi(1)}| \le |(n-m)_{\pi(2)}| \le |(n-m)_{\pi(3)}| \le \cdots \le |(n-m)_{\pi(d)}|.$$

These choices are made to get as much decrease as possible for $G_0(z,m,n)$ with real z as $|(n-m)|$ increases. Then we use the estimates

$$\begin{aligned}
|J_{(n-m)_{\pi(1)}}(2t)\cdots J_{(n-m)_{\pi(3)}}(2t)| &\le \frac{C}{1+|t|}\\
|J_{(n-m)_{\pi(4)}}(2t)| &\le \left(\frac{C}{1+|t|}\right)^{\epsilon}\left(\frac{C}{1+|(n-m)_{\pi(4)}|}\right)^{\frac{1}{3}-\epsilon},\\
|J_{(n-m)_{\pi(j)}}(2t)| &\le \left(\frac{C}{1+|(n-m)_{\pi(j)}|}\right)^{\frac{1}{3}}, \quad j \ge 4,
\end{aligned}$$

together with the bound $|e^{-itz}| \le 1$, when $\mathrm{Im}\ (z) > 0,\ t \le 0$, to get the bound

$$\begin{aligned}
|G_0(z,m,n)| \le C &\left(\frac{1}{1+|(n-m)_{\pi(4)}|}\right)^{\frac{1}{3}-\epsilon} \prod_{j=5}^{d}\left(\frac{1}{1+|(n-m)_{\pi(j)}|}\right)^{\frac{1}{3}}.\\
&\times \int_0^{\infty}\left(\frac{1}{1+|t|}\right)^{1+\epsilon} dt.
\end{aligned}$$

The integrand in the integral defining $G_0(z, m, n)$ is uniformly bounded by an integrable function when Im $(z) \geq 0$. The integrand is continuous in $z \in \mathbb{C}$; therefore the boundedness and the continuity of the integral in $\overline{\mathbb{C}^+}$ follows by dominated convergence. $\square$

4.1.2 Unperturbed Semigroups and Their Kernels

Since all the operators H_0 in Definition 4.1.4 are selfadjoint and positive they generate strongly continuous, selfadjoint contraction semigroups in $L^2(\mathbb{R}^d, dx) = L^2(\mathbb{R}^d)$ denoted by $\{e^{-tH_0},\ t \geq 0\}$. These semigroups play a central role in spectral theory because they are related to the resolvents via the Laplace transform:

$$(H_0 + a)^{-1} f = \int_0^\infty e^{-a\lambda}\ e^{-\lambda H_0}\ f\ d\lambda,\ a > 0. \tag{4.1.18}$$

For all the generators H_0 that we have in mind here, the operator e^{-tH_0} for fixed $t > 0$ are integral operators in $L^2(\mathbb{R}^d)$. We denote their kernels by $e^{-tH_0}(\cdot, \cdot)$ mapping $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. In particular,

1) If $H_0 = -\Delta$, then

$$e^{-tH_0}(x, y) = \frac{1}{(4\pi t)^{d/2}}\ e^{-\frac{|x-y|^2}{4t}}. \tag{4.1.19}$$

2) If $H_0 = (-\Delta)^\alpha,\ 0 < \alpha < 1,$ then

$$e^{-tH_0}(x, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t|k|^{2\alpha} + ik\cdot(x-y)} dk, \tag{4.1.20}$$

which for $|x - y| \geq 1$ gives the estimates

$$\frac{c_1}{t^{d/2\alpha}\ |x - y|^{d+2\alpha}} \leq e^{-tH_0}(x, y) \leq \frac{c_2}{t^{d/2\alpha} |x - y|^{d+2\alpha}}\quad . \tag{4.1.21}$$

3) If $H_0 = (-\Delta)^{1/2}$, then

$$e^{-tH_0}(x, y) = \Gamma\left(\frac{d+1}{2}\right) \pi^{-(d+1)/2} \cdot \frac{t}{[t^2 + |x - y|^2]^{(d+1)/2}}. \tag{4.1.22}$$

4) If $H_0 = \sqrt{-\Delta + c^2} - c,$ then

$$\begin{aligned} e^{-tH_0}(x, y) = {} & \frac{1}{(2\pi)^d}\ \frac{t}{\sqrt{|\,x - y|^2 + t^2}} \\ & \times \int_{\mathbb{R}^d} \exp\ \left\{ct - \sqrt{(|x - y|^2 + t^2)(k^2 + c^2)}\right\}\ dk. \end{aligned} \tag{4.1.23}$$

4.1.3 Associated Processes

For the unperturbed operators H_0 considered here, i.e., for the Laplacian $-\Delta$, for the fractional powers $(-\Delta)^\alpha$, $0 < \alpha < 1$, and for the relativistic Hamiltonian $\sqrt{-\Delta + c^2} - c$, there is a powerful tool to handle the semigroup or resolvent differences by the theory of strong Markov processes associated to the generators H_0.

In this subsection we denote by H_0 one of the operators $-\Delta, (-\Delta)^\alpha$ with $0 < \alpha < 1$, $\sqrt{-\Delta + c^2} - c$. Since H_0 is selfadjoint the corresponding semigroup is symmetric and strongly continuous. The semigroup possesses the Feller property, i.e., $e^{-tH_0} C_0 \subset C_0$, where $C_0 = C_0(\mathbb{R}^d)$ are the continuous function vanishing at infinity. The kernels $e^{-tH_0}(x, y)$ are continuous in t, x, and y. Their total mass is one, i.e.,

$$\int_{\mathbb{R}^d} e^{-tH_0}(x, y)\, dy = 1, \tag{4.1.24}$$

$t > 0$, $x \in \mathbb{R}^d$, i.e., the semigroup is also Markovian. Hence the kernels $e^{-tH_0}(x, y)$ satisfy the so-called basic assumption of stochastic spectral analysis. The Kolmogorov construction can be used to determine a strong Markov process

$$\{(\Omega, \mathfrak{B}_\Omega, P_x),\ (X_t, t \geq 0),\ (\mathbb{R}^d, \mathfrak{B}_{\mathbb{R}^d})\} \tag{4.1.25}$$

which is associated to H_0, such that the one-dimensional distribution is given by

$$P_x\{X_t \in B\} = \int_B e^{-tH_0}(x, y)\, dy, \tag{4.1.26}$$

for any Borel set B in $\mathbb{R}^d$. In more detail: $(\Omega, \mathfrak{B}_\Omega, P_x)$ is a probability space, X_t are measurable functions from Ω to $\mathbb{R}^d$ for each t. $\mathfrak{B}_{\mathbb{R}^d}$ are the Borel sets in $\mathbb{R}^d$. The process is conservative because of Equation (4.1.24). In our case Ω consists of all paths $\omega : [0, \infty] \to \mathbb{R}^d \cup \{\infty\}$ with $\omega(0) = x$, which are right continuous with left hand limits. The state variables X_t can be identified with the paths, i.e., $X_t(\omega) = \omega(t)$. Because of (4.1.24) the lifetime of the process is infinite. The process has independent increments.

In general, the processes described above are called Lévy processes. For $(-\Delta)^\alpha$ the processes are called α-stable. For $-\Delta$ the process is the Wiener process. The Wiener paths are Hölder continuous (more details are given in the Notes 4.3).

For any Borel function f on $\mathbb{R}^d$ the relation between the process and the semigroup is given by

$$(e^{-tH_0} f) = E_x\{f(X_t)\} \tag{4.1.27}$$

$$= \int_{\mathbb{R}^d} e^{-tH_0}(x, y) f(y) dy \tag{4.1.28}$$

which is defined whenever the right-hand side makes sense. $E_x\{\cdot\}$ is the expectation with respect to P_x. The kernel $e^{-tH_0}(x, y)$ plays the role of the transition density function of the process. An essential advantage of the application of stochastic processes in spectral theory is that the kernels of the semigroups can be represented stochastically. For the Laplacian this is known as Brownian bridge. The same concept can be used for more general processes in particular for the Lévy processes here. We introduce a measure which pins the motion of the process to x at time 0 and to y at time t.

Definition 4.1.12. Let

$$\{(\Omega, \mathfrak{B}_\Omega, P_x), \{X_t, t \geq 0\}, (\mathbb{R}^d, \mathfrak{B}_{\mathbb{R}^d})\}$$

be the strong Markov process associated to the semigroup $\{e^{-tH_0},\ t \geq 0\}$. Let $\mathfrak{B}_{t-}$ be the σ-field in $\mathfrak{B}_\Omega$ which is generated by $\{X_s,\ 0 \leq s < t\}$. Let A be an event in $\mathfrak{B}_{t-}$. Then we define the pinned measure by

$$E_x^{y,t}\{A\} = E_x\{e^{-(t-s)H_0}(X_s, y)\mathbb{1}_A\},\ s < t. \tag{4.1.29}$$

Remark 4.1.13. The pinned measure makes sense since it is a martingale on $[0, t)$ (see the Notes 4.3). Because the process is conservative satisfying Equation (4.1.24),

$$\int_{\mathbb{R}^d} E_x^{y,t}\{A\}\ dy = E_x\{1_A\},\ \text{for } A \in \mathfrak{B}_{t-}. \tag{4.1.30}$$

In this context the unperturbed semigroup has a stochastic representation of the form

$$\begin{aligned}(e^{-tH_0}f)(x) &= \int_{\mathbb{R}^d} e^{-tH_0}(x, y)\ f(y)\ dy \\ &= \int_{\mathbb{R}^d} E_x^{y,t}\{f(X_t)\}\ dy \qquad (4.1.31)\\ &= \int_{\mathbb{R}^d} E_x\{e^{-(t-s)H_0}(X_s, y)\ f(X_t)\}\ dy \qquad (4.1.32)\end{aligned}$$

with some $0 \leq s < t$.

4.1.4 Regular Dirichlet Forms, Capacities and Equilibrium Potentials

There are at least three reasons to introduce and to explain here the notion of Dirichlet forms and in particular of regular Dirichlet forms which are Dirichlet forms in regular Dirichlet spaces. The first reason is the correspondence between such forms and Hunt processes. This links selfadjoint operators with the theory of stochastic processes. The second reason is that regular Dirichlet forms model Dirichlet boundary conditions in a natural way. This association will be used in Section 4.2 and also in the applications of Chapter 5. The third reason is that any regular Dirichlet form admits the notion of a set function called capacity. This notion is intimately related to singular perturbations studied in Chapter 5.

Definition 4.1.14. Let $\big(\mathfrak{a}, \mathrm{dom}\,(\mathfrak{a})\big)$ be the non-negative closed form on $L^2(\mathbb{R}^d)$ associated to the selfadjoint operator H_0 (see Definition 2.1.25 and Theorem 2.1.26). Such a form is called a Dirichlet form if it satisfies the Markov property, i.e., if

$$0 \le (e^{-tH_0} f)(x) \le 1 \tag{4.1.33}$$

for a.e. $x \in \mathbb{R}^d$ as long as $0 \le f(x) \le 1$ for a.e. $x \in \mathbb{R}^d$. In this case $\mathrm{dom}\,(\mathfrak{a})$ is also called a Dirichlet space.

The Dirichlet space is called regular, and correspondingly the Dirichlet form is called regular if $\mathrm{dom}\,(\mathfrak{a}) \cap C_c(\mathbb{R}^d)$ is not only dense in the Hilbertspace $\{\mathrm{dom}\,(\mathfrak{a}),\ \langle\cdot,\cdot\rangle_{\mathfrak{a}}\}$ (see Definition 2.1.25) but also dense in $C_c(\mathbb{R}^d)$ with respect to the L^∞-norm.

It is relatively straightforward to go from a conservative Hunt process to the corresponding Dirichlet form. From the transition function $P_x\{X_t \in B\} = \int\limits_B p_t(x, dy),\ B \in \mathfrak{B}_{\mathbb{R}^d}$, one gets the semigroup

$$\int_{\mathbb{R}^d} f(y)\ p_t(x, dy) = E_x\{f(X_t)\}. \tag{4.1.34}$$

This semigroup fulfills the Markov property. Moreover, it is strongly continuous implying finally the existence of an associated regular Dirichlet form.

It is not so easy to prove the converse. However we have the following theorem.

Theorem 4.1.15. *Let $\big(\mathfrak{a}, \mathrm{dom}\,(\mathfrak{a})\big)$ be a given regular Dirichlet form on $L^2(\mathbb{R}^d)$. Then there is a Hunt process whose Dirichlet form is the given one.*

Remark 4.1.16. For the examples we have in mind in this chapter, i.e., for $-\Delta,\ (-\Delta)^\alpha,\ \sqrt{-\Delta + c^2}\ -\ c$, all these connections between the operators. The processes and the regular Dirichlet forms are satisfied. Let H_0 be one of these selfadjoint operators. Then the semigroup $\{e^{-tH_0}, t\ \ge 0\}$ is symmetric, strongly continuous, Markovian, the form $\big(\langle H_0^{1/2} f,\ H_0^{1/2} g\rangle,\ \mathrm{dom}\,(H_0^{1/2})\big)$ is a regular Dirichlet form in $L^2(\mathbb{R}^d)$, the associated process is a Lévy process, and their transition density functions are the kernels $e^{-tH_0}(x, y)$ given in Section 4.1.2.

Later we will study singular or domain perturbations of H_0. For this it is useful to introduce the notion of capacity and of the equilibrium potential.

Definition 4.1.17.

1) Let $\big(\mathfrak{a}, \mathrm{dom}\,(\mathfrak{a})\big)$ be a regular Dirichlet form in $L^2(\mathbb{R}^d)$. Set

$$\begin{aligned} \mathfrak{a}_1[f] &= \mathfrak{a}[f] + ||f||^2 \\ &= ||H_0^{1/2} f||^2 + ||f||^2 \end{aligned}$$

for all $f \in \mathrm{dom}\,(\mathfrak{a}) = \mathrm{dom}\,(H_0^{1/2})$.
If O is an open set in $\mathbb{R}^d$, the capacity is defined as

$$\mathrm{cap}\ (O) = \inf\{\mathfrak{a}_1[f],\ f \in \mathrm{dom}\,(\mathfrak{a}),\ f \geq 1 \text{ a.e. on } O\}.$$

If there is no such f we write cap $(O) = \infty$.
If Γ is an arbitrary set of $\mathbb{R}^d$ its capacity is

$$\mathrm{cap}(\Gamma) = \inf\{\mathrm{cap}(O),\ O \supset \Gamma,\ O \text{ open}\}.$$

2) A statement is said to hold quasi-everywhere (q.e.) if its holds outside of sets with capacity zero.
3) A function f is called quasi-continuous (q.c.) if for each $\epsilon > 0$ there is an open set $O \subset \mathbb{R}^d$ with $\mathrm{cap}(O) < \epsilon$ such that f is continuous on $\mathbb{R}^d \setminus O$.

Theorem 4.1.18. *Let $\big(\mathfrak{a}, \mathrm{dom}\,(\mathfrak{a})\big)$ be a regular Dirichlet form. Then for each $f \in \mathrm{dom}\,(\mathfrak{a})$ there is a function $\tilde{f}$, which is quasi continuous and coincides with f almost everywhere with respect to the Lebesgue measure.*

$\tilde{f}$ is called a quasi-continuous version of f.

It turns out that the capacity can be computed, i.e., the infinium in its definition is attained.

Theorem 4.1.19. *Let $\big(\mathfrak{a}, \mathrm{dom}\,(\mathfrak{a})\big)$ be a regular Dirichlet form in $L^2(\mathbb{R}^d)$ and let Γ be an arbitrary set in $\mathbb{R}^d$. Then*

$$\mathrm{cap}(\Gamma) = \inf\ \{\mathfrak{a}_1[f],\ f \in \mathrm{dom}\,(\mathfrak{a}),\ \tilde{f} \geq 1 \quad q.e.\ on\ \ \Gamma\}. \tag{4.1.35}$$

If the set $\{\ f \in \mathrm{dom}\,(\mathfrak{a}),\ \tilde{f} \geq 1 \quad q.e.\ on\ \ \Gamma\}$ is not empty there is a minimizing element $v_\Gamma \in \mathrm{dom}\,(\mathfrak{a})$ such that

$$\mathrm{cap}(\Gamma) = \mathfrak{a}_1[v_\Gamma]. \tag{4.1.36}$$

Definition 4.1.20. The unique element $v_\Gamma \in \mathrm{dom}\,(\mathfrak{a})$ which realizes the infimum in the notion of the capacity is called the equilibrium potential. (In the literature this is also called the one-equilibrium potential.)

For the equilibrium potential v_Γ we have

$$\begin{array}{rrl} & 0 \leq v_\Gamma(x) \leq 1 & \text{almost everywhere} \\ \textit{and} & \tilde{v}_\Gamma(x) = 1 & \text{q.e. on } \Gamma, \end{array}$$

where $\tilde{v}_\Gamma$ is a quasi-continuous version of v_Γ. If Γ is open $v_\Gamma(x) = 1$ a.e. on Γ.

For the spectral theory in the presence of Dirichlet boundary conditions the equilibrium potential plays a role similar to that of the potential function for regular perturbations due to the fact that there is a stochastic representation for the equilibrium potential. The proof of the next theorem is given in the notes.

Theorem 4.1.21. *Assume* $\big(\mathfrak{a}, \mathrm{dom}\,(\mathfrak{a})\big)$ *to be a regular Dirichlet form with the associated process* $\{(\Omega, \mathfrak{B}_\Omega, P_x),\ \{X_t, t \geq 0\},\ (\mathbb{R}^d, \mathfrak{B}_{\mathbb{R}^d})\}$. *Let* Γ *be a Borel set of finite capacity. Define by*

$$\tau_{_\Gamma} = \ \inf\,\{s,\ s > 0,\ X_s \in \Gamma\},$$

the first hitting time of Γ.

Then the equilibrium potential can be represented stochastically by

$$v_\Gamma(x) = E_x\{e^{-\tau_\Gamma},\ \tau_{_\Gamma} < \infty\} \tag{4.1.37}$$

which holds almost everywhere. Here, as usual, $E_x\{\cdot\}$ *denotes the expectation with respect to* $P_x(\cdot)$.

There is a close connection between equilibrium potentials and Radon measures which we will outline in the following. A positive Radon measure μ on $\mathbb{R}^d$ is said to be of finite energy integral if there is a positive constant c such that (recall $\tilde{f}$ to be the quasicontinuous version of f),

$$\int_{\mathbb{R}^d} |\tilde{f}(x)|\ d\mu(x) \leq\ c\sqrt{\mathfrak{a}_1[f]}$$

for all $f \in \mathrm{dom}\,(\mathfrak{a}) \cap C_c(\mathbb{R}^d)$. The collection of such measures is denoted by S_0. For a Radon measure of finite energy integral the map $f \to \int |\tilde{f}(x)|\ d\mu(x)$ defines a continuous linear functional on the Hilbert space $\{\mathrm{dom}\,(\mathfrak{a}), \langle\cdot,\cdot\rangle_{\mathfrak{a}}\}$. Hence by the theorem of Riesz there is a function $u_\mu \in \mathrm{dom}\,(\mathfrak{a})$ such that

$$\mathfrak{a}_1(u_\mu, f) = \int_{\mathbb{R}^d} \tilde{f}\ d\mu \tag{4.1.38}$$

for all $f \in \mathrm{dom}\,(\mathfrak{a}) \cap C_c(\mathbb{R}^d)$. u_μ is called the (one-) potential of the measure μ.

For a given equilibrium potential $v_{_\Gamma}$ there is a unique Radon measure, called $\mu_{_\Gamma}$ in S_0, such that $u_{\mu_{_\Gamma}} = v_{_\Gamma}$. $\mu_{_\Gamma}$ is called the equilibrium measure of Γ. It does not charge sets of capacity zero.

For compact Γ we get

$$\mathrm{cap}(\Gamma) = \mathfrak{a}_1[v_{_\Gamma}] = \mu_{_\Gamma}(\Gamma). \tag{4.1.39}$$

For the examples of this section the equilibrium measure is

$$\mu_{_\Gamma}(B) = \int_{\mathbb{R}^d} E_x\{e^{-\tau_\Gamma},\ X_{\tau_{_\Gamma}} \in B,\ \tau_{_\Gamma} < \infty\}\ dx. \tag{4.1.40}$$

Using Definition 4.1.17 we see the following.

Corollary 4.1.22 *Let* Γ *be a compact set in* $\mathbb{R}^d$. *Take the regular Dirichlet form* $\big(\mathfrak{a}, \mathrm{dom}\,(\mathfrak{a})\big)$. *Then the capacity of* Γ *is given by*

$$\begin{aligned}\operatorname{cap}(\Gamma) &= \|H_0^{1/2} v_\Gamma\|^2 + \|v_\Gamma\|^2 \\ &= \mu_\Gamma(\Gamma) \\ &= \int_{\mathbb{R}^d} E_x\{e^{-\tau_\Gamma},\ \tau_\Gamma < \infty\}\ dx \\ &= \int_{\mathbb{R}^d} v_\Gamma(x)\ dx. \end{aligned} \tag{4.1.41}$$

4.2 Perturbed Operators

The final objective in the next chapter is to study the spectra of operators which arise as perturbations of the free H_0, introduced in Section 4.1. We have in mind different kinds of perturbations: deterministic and random potentials, singular or domain perturbations.

In this section we intend to ensure that the perturbed operators are selfadjoint. Hence they generate strongly continuous semigroups which are quasi-bounded. We collect some characteristic features of the perturbed operators and the associated semigroups.

4.2.1 Deterministic Potentials

Let H_0 be one of the operators $-\Delta,\ (-\Delta)^\alpha,\ \alpha \in (0,1),\ \sqrt{-\Delta + c^2} - c$, although more general free Feller generators could be considered.

H_0 is assumed to be perturbed by a deterministic potential operator M_V determined by a potential function $V : \mathbb{R}^d \to \mathbb{R}$, i.e.,

$$(M_V f)(x) = V(x) f(x)$$

for f in the natural domain of M_V in $L^2(\mathbb{R}^d)$. Note that M_V is selfadjoint if V is Lebesgue measurable.

Here we restrict the class of potentials in such a way that the form sum $H_0 \dot{+} M_V$ becomes selfadjoint. This class of potentials is a straightforward generalization of the Kato class which is connected with the Laplacian. This generalized class is called a Kato–Feller class because it is related to the free Feller semigroup $\{e^{-tH_0}, t \geq 0\}$.

In the sequel we follow the convention that if T is an operator from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$, then its norm is

$$\|T\|_{p,q} = \sup\{\|Tf\|_q,\ f \in L^p(\mathbb{R}^d),\ \|f\|_p = 1\} \tag{4.2.42}$$

for $1 \leq p \leq \infty,\ 1 \leq q \leq \infty$.

Definition 4.2.1. Kato–Feller class. Let $W : \mathbb{R}^d \to [0, \infty]$ be a measurable function on $\mathbb{R}^d$ with non-negative values. Let H_0 be one of the operators $-\Delta,\ (-\Delta)^\alpha,\ \alpha \in (0,1)$, or $\sqrt{-\Delta + c^2} - c$. Denote by $e^{-tH_0}(x,y)$ the kernels of the corresponding semigroups.

A function W is said to belong to the class $K = K(H_0)$ if

$$\limsup_{t\downarrow 0} \sup_{x\in\mathbb{R}^d} \int_0^t (e^{-sH_0}W)(x)ds = 0. \tag{4.2.43}$$

W is said to belong to the class $K_{\mathrm{loc}} = K_{\mathrm{loc}}(H_0)$ if $1_B W$ is in $K(H_0)$ for any compact set B in $\mathbb{R}^d$.

Set $V_+ = \max(V, 0)$ and $V_- = \max(-V, 0)$.

Then the potential $V = V_+ - V_-$ is said to belong to the Kato–Feller class if $V_+ \in K_{\mathrm{loc}}(H_0)$ and $V_- \in K(H_0)$. Correspondingly, M_V is called the Kato–Feller (multiplication) operator if V is in the Kato–Feller class.

Before discussing some properties of Kato–Feller potentials we introduce the Kato–Feller norm.

Definition 4.2.2. Let $V : \mathbb{R}^d \to \mathbb{R}$. Its Kato–Feller norm is defined by

$$||V||_{KF} = \sup_{x\in\mathbb{R}^d} \int_0^1 (e^{-sH_0}|V|)(x)ds,$$

whenever this expression is finite.

Remark 4.2.3.

(i) $\|V\|_{KF}$ is finite for $V \in K(H_0)$.

(ii) At first glance the definition of Kato–Feller potentials looks very technical. However it is natural in the following sense. Any $V \in K(H_0)$ is relatively form bounded with respect to H_0. The form bound is given by $||(H_0 + a)^{-1}M_V||_{\infty,\infty}$. The class $K(H_0)$ can even be characterized by

$$V \in K(H_0) \quad \text{iff} \quad \lim_{a\to\infty} ||(H_0 + a)^{-1}M_V||_{\infty,\infty} = 0\ . \tag{4.2.44}$$

These facts will be proved in the next lemma. Hence the KLMN theorem is applicable.

(iii) Moreover, let $\{(\Omega, \mathfrak{B}_\Omega, P_x),\ (X_t, t \geq 0),\ (\mathbb{R}^d, \mathfrak{B}_{\mathbb{R}^d})\}$ be the process associated to H_0 (see(4.1.25)). $E_x\{.\}$ denotes the expectation for the process. Let $V = V_+ - V_-$ be a Kato–Feller potential. Then V_- is in $K(H_0)$ and we have

$$\sup_{x\in\mathbb{R}^d} E_x\left\{\int_0^{t_0} V_-\big(X_s\big)ds\right\} \tag{4.2.45}$$
$$= \sup_{x\in\mathbb{R}^d} \int_0^{t_0} \left(e^{-sH_0}V_-\right)(x)ds = \alpha.$$

If t_0 is small enough α becomes smaller than one. Then the lemma of Kashminskii implies

$$\sup_{x\in\mathbb{R}^d} E_x\left\{\exp\int_0^{t_0} V_-\big(X_s\big)ds\right\} \le \frac{1}{1-\alpha}. \tag{4.2.46}$$

Hence for Kato–Feller potentials we have the estimate

$$\sup_{x\in\mathbb{R}^d} E_x\left\{\exp\left\{-\int_0^{t} V\big(X_s\big)ds\right\}\right\} \le c\, e^{ct} \tag{4.2.47}$$

for any $t \ge 0$.

(iv) If V_+ is in $K_{\mathrm{loc}}(H_0)$ it turns out that $V_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ (see Lemma 4.2.5), such that $C_c^\infty(\mathbb{R}^d) \subseteq \mathrm{dom}\,(M_{V_+}^{1/2})$.

On the other hand, obviously $\mathrm{dom}\,(-\Delta) \subseteq \mathrm{dom}\,(H_0) \subseteq \mathrm{dom}\,(H_0^{1/2})$ for all the H_0 considered here. Therefore the form

$$\langle H_0^{1/2}f,\ H_0^{1/2}g\rangle + \langle M_{V_+}^{1/2}f,\ M_{V_+}^{1/2}g\rangle$$

on $\mathrm{dom}\,(H_0^{1/2}) \cap\ \mathrm{dom}\,(M_{V_+}^{1/2})$ is densely defined, symmetric, nonnegative. Following Theorem 2.1.26 we denote the associated selfadjoint operator by $H_0 \dotplus M_{V_+}$.

Next we prove the assertions of the last remark.

Lemma 4.2.4.

(i) V belongs to $K(H_0)$ if and only if

$$\lim_{a\to\infty} ||(H_0+a)^{-1}M_V||_{\infty,\infty} = 0.$$

(ii) If $V \in K(H_0)$, then the domain of $M_V^{1/2}$ contains $\mathrm{dom}\,(H_0^{1/2})$.

(iii) $||(H_0+a)^{-1}M_V||_{\infty,\infty}$ is the relative form bound of M_V with respect to H_0 for $V \in K(H_0)$. Moreover we have an estimate of the form

$$||(H_0+a)^{-1}M_V||_{\infty,\infty} \le \frac{1}{1-e^{-a}}\ ||V||_{KF}. \tag{4.2.48}$$

Proof: (i) By the definition of $K(H_0)$ the potential $V(\cdot)$ is non-negative. Thus we have

$$\begin{aligned} &\big((H_0+a)^{-1}M_V\big)(x) \\ &= \int_0^\infty e^{-as}(e^{-sH_0}V)(x)ds \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \int_{k\eta}^{(k+1)\eta} e^{-as}(e^{-sH_0}V)(x)ds, \\
&\qquad\qquad \text{(for some } \eta > 0) \\
&= \sum_{k=0}^{\infty} e^{-k\eta a} \int_0^{\eta} e^{-as}(e^{-(s+k\eta)H_0}V)(x)ds \\
&= \sum_{k=0}^{\infty} e^{-k\eta a} \int_0^{\eta} e^{-as}(e^{-k\eta H_0}e^{-sH_0}V)(x)ds \\
&\qquad\qquad \text{(semigroup property)} \\
&= \sum_{k=0}^{\infty} e^{-k\eta a} \int_0^{\eta} e^{-as} \int_{\mathbb{R}^d} du\ (e^{-k\eta H_0})(x,u)(e^{-sH_0}V)(u) \\
&= \sum_{k=0}^{\infty} e^{-k\eta a} \int_{\mathbb{R}^d} du\ e^{-k\eta H_0}(x,u) \int_0^{\eta} ds\ e^{-as}(e^{-sH_0}V)(u) \\
&\leq \sup_{u\in\mathbb{R}^d} \int_0^{\eta} e^{-as}(e^{-sH_0}V)(u)ds \cdot \sum_{k=0}^{\infty} e^{-k\eta a} \int_{\mathbb{R}^d} du\ (e^{-k\eta H_0})(x,u) \\
&= \frac{1}{1-e^{-a\eta}} \sup_{u\in\mathbb{R}^d} \int_0^{\eta} e^{-as}(e^{-sH_0}V)(u)\ ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{1-e^{-a\eta}} \sup_{u\in\mathbb{R}^d} \left((H_0+a)^{-1}V\right)(u) \\
&\leq \frac{1}{1-e^{-a\eta}} \ ||(H_0+a)^{-1}M_V||_{\infty,\infty} \quad .
\end{aligned}
$$

Hence

$$
(1-e^{-a\eta})\ ||(H_0+a)^{-1}M_V||_{\infty,\infty} \leq \sup_{u\in\mathbb{R}^d} \int_0^{\eta} e^{-as}(e^{-sH_0}V)(u)\ ds \tag{4.2.49}
$$

$$
\leq ||(H_0+a)^{-1}M_V||_{\infty,\infty}. \tag{4.2.50}
$$

Using (4.2.50)

$$
\begin{aligned}
&\limsup_{\eta\downarrow 0} \quad \sup_{x\in\mathbb{R}^d} \int_0^{\eta} (e^{-sH_0}V)(x)\ ds \\
&\qquad \leq ||(H_0+a)^{-1}V||_{\infty} = ||(H_0+a)^{-1}M_V||_{\infty,\infty}.
\end{aligned}
$$

i.e., $V \in K(H_0)$ if $\lim\limits_{a \to \infty} ||(H_0+a)^{-1}M_V||_{\infty,\infty} = 0$.

On the other hand, using (4.2.49)

$$\lim_{a \to \infty} ||(H_0+a)^{-1}M_V||_{\infty,\infty} \leq \sup_{x\in\mathbb{R}^d} \int_0^\eta (e^{-sH_0}V)(x)\ ds,$$

which becomes arbitrarily small as η tends to zero.

(ii) and (iii): Let $f \in \operatorname{dom}(H_0^{1/2})$ with $f = (H_0+a)^{-1/2}g$.

Then

$$\begin{aligned}
\langle M_V^{1/2}f,\ M_V^{1/2}f\rangle &= \langle M_V^{1/2}(H_0+a)^{-1/2}g,\ M_V^{1/2}(H_0+a)^{-1/2}g\rangle \\
&\leq ||M_V^{1/2}(H_0+a)^{-1/2}||_{2,2}^2 \quad ||g||^2 \\
&= ||(H_0+a)^{-1/2}M_V^{1/2}||_{2,2}^2 \quad ||g||^2 \\
&= ||M_V^{1/2}(H_0+a)^{-1}M_V^{1/2}||_{2,2} \quad ||g||^2.
\end{aligned}$$

Now, we claim

$$||M_V^{1/2}(H_0+a)^{-1}M_V^{1/2}||_{2,2}\ \leq\ ||(H_0+a)^{-1}M_V||_{\infty,\infty}, \tag{4.2.51}$$

which will be shown at the end of this proof.

Using the claim we get

$$\begin{aligned}
\langle M_V^{1/2}f,\ M_V^{1/2}f\rangle &\leq ||(H_0+a)^{-1}M_V||_{\infty,\infty} \quad ||(H_0+a)^{1/2}f||^2 \qquad (4.2.52)\\
&= \varepsilon(a)\ (a\langle f,f\rangle + \langle H_0^{1/2}f, H_0^{1/2}f\rangle)
\end{aligned}$$

with $\varepsilon(a) = ||(H_0+a)^{-1}M_V||_{\infty,\infty}$.

This proves (ii) and (iii). It remains to show (4.2.51) and (4.2.48). (4.2.48) follows from (4.2.49) for $\eta = 1$. For (4.2.51) we consider

$$\begin{aligned}
&\left|\ \langle M_V^{1/2}(H_0+a)^{-1}M_V^{1/2}f, g\rangle\right| \\
&= \left|\int_{\mathbb{R}^d} dx\ V(x)^{1/2}[(H_0+a)^{-1}M_V^{1/2}\overline{f}](x)g(x)\right| \\
&= \left|\int_{\mathbb{R}^d} dx \int_0^\infty ds\ e^{-as} \int_{\mathbb{R}^d} du\ e^{-sH_0}(x,u)V(x)^{1/2}\overline{f}(u)V(u)^{1/2}g(x)\right|.
\end{aligned}$$

Continuing the estimate we have

$$\begin{aligned}
\leq \quad &\left(\int_{\mathbb{R}^d} dx \int_0^\infty ds\ e^{-as} \int_{\mathbb{R}^d} du\ (e^{-sH_0})(x,u)V(x)|f(u)|^2\right)^{1/2} \\
&\times \left(\int_{\mathbb{R}^d} dx \int_0^\infty ds\ e^{-as} \int_{\mathbb{R}^d} du\ (e^{-sH_0})(x,u)V(u)|g(x)|^2\right)^{1/2}
\end{aligned}$$

$$= \left(\int_{\mathbb{R}^d} dx\ V(x)[(H_0+a)^{-1}|f|^2](x) \right)^{1/2}$$

$$\times \left(\int_{\mathbb{R}^d} dx[(H_0+a)^{-1}V](x)|g(x)|^2 \right)^{1/2}$$

$$\leq \left\| M_V(H_0+a)^{-1} \right\|_{1,1}^{1/2} \ \left\| (H_0+a)^{-1}M_V \right\|_{\infty,\infty}^{1/2} \ \|f\|\ \|g\|$$

$$= \left\| (H_0+a)^{-1}M_V \right\|_{\infty,\infty} \ \|f\|\ \|g\|.$$

$\square$

Lemma 4.2.5. *If a potential function V is in $K_{loc}(H_0)$, then it is in $L^1_{loc}(\mathbb{R}^d)$.*

Proof: Let $|K_n|$ be the volume of a ball K_n with radius n. Then obviously

$$\sup_n \inf_{y\in B} \frac{a}{|K_n|} \int_{K_n} dx \int_0^\infty ds \int_B dy\ e^{-as}(e^{-sH_0})(x,y)\cdot\ |V(y)|$$

$$\leq \quad \sup_{x\in\mathbb{R}^d} a \int_0^\infty ds\ e^{-as} \int_{\mathbb{R}^d} dy\ (e^{-sH_0})(x,y)\cdot\ V(y)1_B(y)$$

$$\leq \quad \sup_{x\in\mathbb{R}^d} a \int_0^\infty ds\ e^{-as} \int_{\mathbb{R}^d} (e^{-sH_0}1_B V)(x).$$

The right-hand side is finite. It suffices to ensure that

$$\sup_n \inf_{y\in B} \frac{1}{|K_n|}\ h_n(y)$$

is strictly positive with

$$h_n(y) = a \int_{K_n} dx \int_0^\infty ds\ e^{-as}(e^{-sH_0})(x,y).$$

This follows because $h_n(y) \to 1$, as $n \to \infty$ for all y. The sequence $1 - h_n(y)$ is decreasing for all $y \in \mathbb{R}^d$. Thus by Dini's Lemma $\lim_{n\to\infty} \sup_{y\in B}(1 - h_n(y)) = 0$ for all compact subsets B. Hence $\inf_{y\in B} h_n(y) > 0$. $\square$

Remark 4.2.6. Let $V = V_+ - V_-$ be a Kato–Feller potential, i.e., $V_+ \in K_{\mathrm{loc}}(H_0)$, $V_- \in K(H_0)$. By Lemma 4.2.4 (equations (4.2.48) and (4.2.52), M_{V_-} is relatively form bounded with respect to $H_0 \dot{+} M_{V_+}$ on $\mathrm{dom}\,(H_0^{1/2}) \cap \mathrm{dom}\,(M_{V_+}^{1/2})$ with the form bound $(1-e^{-a})^{-1}||V_-||_{KF}$. This becomes smaller than 1 for a large enough. By the KLMN-Theorem (see Theorem 2.1.29) the semibounded selfadjoint operator associated to the form

$$\langle H_0^{1/2} f,\ H_0^{1/2} g\rangle + \langle M_{V_+}^{1/2} f,\ M_{V_+}^{1/2} g\rangle\ - \langle M_{V_-}^{1/2} f,\ M_{V_-}^{1/2} g\rangle$$

is defined on the domain $\operatorname{dom}(H_0^{1/2}) \cap\ \operatorname{dom}(M_{V_+}^{1/2})$. This operator will be denoted by $H_0 \dot{+} M_V$.

Remark 4.2.7. The operator $H_0 \dot{+} M_V$ generates a strongly continuous positivity preserving semigroup $\{e^{-t(H_0 \dot{+} M_V)},\ t \geq 0\}$ on $C_0(\mathbb{R}^d)$. For any $t > 0$ the operator $e^{-t(H_0 \dot{+} M_V)}$ is an integral operator. We denote the kernels by $(e^{-t(H_0 \dot{+} V)})(x, y)$, i.e.,

$$(e^{-t(H_0 \dot{+} M_V)} f)(x) = \int\limits_{\mathbb{R}^d} \left(e^{-t(H_0 \dot{+} V)}\right)(x, y)\ f(y)\ dy, \quad f \in C_0(\mathbb{R}^d).$$

The functions $(t, x, y) \to e^{-t(H_0 \dot{+} V)}(x, y)$ are continuous on the set $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

The semigroup $\{e^{-t(H_0 \dot{+} M_V)},\ t \geq 0\}$ can be extended to $L^p(\mathbb{R}^d)$ and acts as a strongly continuous semigroup in $L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$. In particular, it maps $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. Hence there is a selfadjoint positivity preserving strongly continuous semigroup in $L^2(\mathbb{R}^d)$ with the selfadjoint generator $H_0 \dot{+} M_V$.

The semigroup can be represented stochastically by the Feynman–Kac formula. Let $\{(\Omega, \mathfrak{B}_\Omega, P_x),\ (X_t, t \geq 0),\ (\mathbb{R}^d, \mathfrak{B}_{\mathbb{R}^d})\}$ be the process associated to H_0 (see Section 4.1.3). Then

$$(e^{-t(H_0 \dot{+} M_V)} f)(x) \quad = \quad E_x \left\{ e^{-\int\limits_0^t V(X_u) du} f(X_t) \right\}, \tag{4.2.53}$$

$f \in L^2(\mathbb{R}^d)$. And for the kernels we have (see Remark 4.1.13)

$$(e^{-t(H_0 \dot{+} V)})(x, y) = \lim_{s \uparrow t}\ E_x \left\{ e^{-\int\limits_0^s V(X_u) du} (e^{-(t-s)H_0})(X_s, y) \right\} \tag{4.2.54}$$

$$= E_x^{y,t} \left\{ e^{-\int\limits_0^t V(X_u) du} \right\}, \tag{4.2.55}$$

with the pinned measure from Definition 4.1.12.

For the perturbed kernels there is a series of interesting estimates, which are useful for the spectral theoretic criteria.

Proposition 4.2.8. *Let M_V be a Kato–Feller operator. Note that all the free semigroups $\{e^{-tH_0},\ t \leq 0\}$ considered here are $L^1 - L^\infty$ smoothing. Then we obtain the following estimate:*

$$\left(e^{-t(H_0 \dot{+} V)}\right)(x,y) \le ||e^{-\frac{t}{2}(H_0 \dot{+} 4M_V)}||_{\infty,\infty}^{1/2} \; ||e^{-\frac{t}{2}H_0}||_{1,\infty}^{1/2} \cdot \left(e^{-tH_0}(x,y)\right)^{1/2}. \tag{4.2.56}$$

and

$$\left(e^{-t(H_0 \dot{+} V)}\right)(x,y) \le ce^{ct} \left[\sup_{x\in\mathbb{R}^d} \left(e^{-\frac{t}{2}H_0}(x,x)\right)^{1/2}\right] \cdot \left(e^{-tH_0}(x,y)\right)^{1/2}. \tag{4.2.57}$$

The inequalities 4.2.56 or 4.2.57 are important for many applications. For the sake of completeness we give a proof below.

Proof: By (4.2.55)

$$\begin{aligned}
\left[e^{-t(H_0 \dot{+} V)}(x,y)\right]^2 &= \left[E_x^{y,t}\left\{e^{-\int_0^t V(X_u)du}\right\}\right]^2 \\
&\le E_x^{y,t}\left\{e^{-2\int_0^t V(X_u)du}\right\} e^{-tH_0}(x,y) \\
&\le ||e^{-t(H_0 \dot{+} 2M_V)}||_{1,\infty} \; e^{-tH_0}(x,y) \\
&= ||e^{-\frac{t}{2}(H_0 \dot{+} 2M_V)} \; e^{-\frac{t}{2}(H_0 \dot{+} 2M_V)}||_{1,\infty} \; e^{-tH_0}(x,y) \\
&\le ||e^{-\frac{t}{2}(H_0 \dot{+} 2M_V)}||_{2,\infty} \; || \; e^{-\frac{t}{2}(H_0 \dot{+} 2M_V)}||_{1,2} \; e^{-tH_0}(x,y) \\
&= ||e^{-\frac{t}{2}(H_0 \dot{+} 2M_V)}||_{2,\infty}^2 \; e^{-tH_0}(x,y),
\end{aligned}$$

where the last equality is because they are selfadjoint operators. For estimating $||e^{-\frac{t}{2}(H_0 \dot{+} 2M_V)}||_{2,\infty}$ we consider

$$\begin{aligned}
\left|\left[e^{-\frac{t}{2}(H_0 \dot{+} 2M_V)}f\right](x)\right|^2 &\le E_x\left\{e^{-4\int_0^{\frac{t}{2}} V(X_u)du}\right\} \cdot E_x\left\{\left|f\left(X_{\frac{t}{2}}\right)\right|^2\right\} \\
&\le \left|\left|e^{-\frac{t}{2}(K_0 \dot{+} 4M_V)}\right|\right|_{\infty,\infty} \cdot \sup_{x,y\in\mathbb{R}^d} \left(e^{-\frac{t}{2}H_0}\right)(x,y) \; \|f\|^2;
\end{aligned}$$

therefore

$$\left|\left|e^{-\frac{t}{2}(H_0 \dot{+} 2M_V)}\right|\right|_{2,\infty} \le \left|\left|e^{-\frac{t}{2}(H_0 \dot{+} 4M_V)}\right|\right|_{\infty,\infty} \cdot \left|\left|e^{-\frac{t}{2}H_0}\right|\right|_{1,\infty}$$

which proves (4.2.56).

For (4.2.57) we use (4.2.47) to get

$$\left|\left|e^{-\frac{t}{2}(H_0 \dot{+} 4M_V)}\right|\right|_{\infty,\infty}^{1/2} \le c\, e^{ct}.$$

Moreover,

$$
\begin{aligned}
\left|\left|e^{-\frac{t}{2}H_0}\right|\right|_{1,\infty} &= \sup_{x,y}\left(e^{-\frac{t}{2}H_0}\right)(x,y) \\
&= \sup_{x,y}\int du\; e^{-\frac{t}{4}H_0}(x,u)\; e^{-\frac{t}{4}H_0}(u,y) \\
&\le \sup_{x,y}\left(\int du\;\left(e^{-\frac{t}{4}H_0}(x,u)\right)^2\right)^{1/2} \\
&\qquad\times\left(\int du\;\left(e^{-\frac{t}{4}H_0}(u,y)\right)^2\right)^{1/2} \\
&= \sup_{x,y}\left(e^{-\frac{t}{2}H_0}(x,x)\right)^{1/2}\;\left(e^{-\frac{t}{2}H_0}(y,y)\right)^{1/2} \\
&\le \sup_{x}\; e^{-\frac{t}{2}H_0}(x,x),
\end{aligned}
$$

which justifies (4.2.57). □

Remark 4.2.9. Note that in our case

$$e^{-tH_0}(x,x) \;=\; \frac{1}{(2\pi)^d}\int\limits_{\mathbb{R}^d} e^{-tg(k^2)}dk \tag{4.2.58}$$

(see Definition (4.1.4).

This implies

$$e^{-t(-\Delta)}(x,x) \le c\; t^{-\frac{d}{2}}, \tag{4.2.59}$$

$$e^{-t(-\Delta)^\alpha}(x,x) \le c\; t^{-\frac{d}{2\alpha}}, \quad \alpha\in(0,1), \tag{4.2.60}$$

$$e^{-t(\sqrt{(-\Delta)+c^2}-c)}(x,x) \le c\; t^{-d}\; . \tag{4.2.61}$$

4.2.2 Random Potentials

In this section we discuss some random selfadjoint operators and discuss their spectral properties.

To begin, we consider an ergodic dynamical system, namely a topological space Ω, its Borel sigma algebra $\mathfrak{B}_\Omega$, a probability measure $\mathbb{P}$ on it such that there is an action of a group G as automorphisms on Ω so that $\mathbb{P}$ is an invariant and ergodic measure with respect to the action of the group G. We denote the automorphism associated with a group element g as T_g. We also assume that $\mathcal{H}$ is a separable Hilbert space on which the group G has a unitary representation and this family of unitary operators is denoted by U_g. We make the following assumption on the group G.

Hypothesis 4.2.10. *There is a total set S of vectors in $\mathcal{H}$ such that for each $f\in S$, there is a sequence $g_n(f)$ of elements of G with the property $U_{g_n}f$ is a collection of orthonormal vectors.*

We then consider a family of selfadjoint operators $H^\omega, \omega \in \Omega$, on $\mathcal{H}$ such that they have a common core, which happens to be the whole space in case they are all bounded. To say anything reasonable about these operators we impose further conditions. The first of which is

Hypothesis 4.2.11. *The spectral measures $P_{H^\omega}(\cdot)$ of the operators H^ω are weakly measurable, and satisfy the compatibility condition*

$$U_g P_{H^\omega}(B) U_g^{-1} = P_{H^{T_g \omega}}(B), \quad \forall B \in \mathfrak{B}_{\mathbb{R}}.$$

Definition 4.2.12. We say that a family of selfadjoint operators $H^\omega, \omega \in \Omega$ is affiliated to the ergodic dynamical system $(\Omega, \mathfrak{B}_\Omega, \mathbb{P}, G)$, if it satisfies the Hypothesis 4.2.11.

Under the above conditions, we have some elementary theorems.

Theorem 4.2.13 (Pastur). *Let $(\Omega, \mathfrak{B}_\Omega, \mathbb{P}, G)$ be an ergodic dynamical system as above and let H^ω be a family of selfadjoint operators on $\mathcal{H}$ affiliated to it. Then for almost all ω,* $\operatorname{dimran}((P_{H^\omega}(B)))$ *is either* 0 *or* ∞ *for every interval B.*

Theorem 4.2.14 (Pastur). *Consider an ergodic dynamical system and a family of selfadjoint operators H^ω on $\mathcal{H}$ associated to it. Then there is a constant set Σ such that*

$$\sigma(H^\omega) = \Sigma, \quad \text{for a.e. } \omega.$$

Moreover the discrete spectrum of H^ω is empty for almost every ω.

Theorem 4.2.15 (Pastur, Kunz–Souillard, Kirsch–Martinelli). *Consider an ergodic dynamical system $(\Omega, \mathfrak{B}_\Omega, \mathbb{P}, G)$ and let H^ω be a family of selfadjoint operators on $\mathcal{H}$ associated to it. Then there are constant sets Σ_{ac}, Σ_{sc} and Σ_{pp} such that*

$$\sigma_{ac}(H^\omega) = \Sigma_{ac}, \quad \sigma_{sc}(H^\omega) = \Sigma_{sc}, \quad \sigma_{pp}(H^\omega) = \Sigma_{pp}, \quad \text{for a.e. } \omega.$$

Consider $\ell^2(\mathbb{Z}^d)$ and the discrete Laplacian Δ defined in Definition 4.1.6. To this we add a diagonal random operator given by taking the dynamical system $\Omega = [a, b]^{\mathbb{Z}^d}$, with product topology, its Borel sigma algebra and the measure $\mathbb{P} = \times_{i \in \mathbb{Z}^d} \mu_i$, $\mu_i \equiv \mu$, where μ is a probability measure with support $[a, b]$. Then the group $G = \mathbb{Z}^d$ acts on Ω by translation, i.e., $T_k \omega(n) = \omega(n-k)$. Also the group G has a unitary representation in $\ell^2(\mathbb{Z}^\nu)$ given by $(U_k u)(n) = u(n-k)$. We take the set $S = \{\delta_i : i \in \mathbb{Z}^d\}$, where $\delta_i(k) = 1,$ for $i = k$ and 0 otherwise. Then the representation is total, since the δ_i themselves form an orthonormal basis for $\ell^2(\mathbb{Z}^d)$. In this setting the space $(\Omega, \mathfrak{B}_\Omega, \mathbb{P}, G)$ gives an ergodic dynamical system and the collection of operators $H^\omega = \Delta + V^\omega$ with $V^\omega u(n) = \omega(n) u(n), \; u \in \ell^2(\mathbb{Z}^d)$ is associated to this ergodic dynamical system. This model is called the Anderson model.

One can also consider the operators $V^\omega u(n) = \sum_{m\in\mathbb{Z}^d} a_{m,n}\omega(m)u(n)$ on $\ell^2(\mathbb{Z}^d)$ where a_{mn} is a real-valued sequence. When $a_{m,n} = \delta_{mn}$ we recover the ergodic operators, or when $a_{nm} = a_n\delta_{nm}$ with a_n going to 0 at ∞, we get models with decaying randomness. (See Section 5.2.1).

4.2.3 Singular Perturbations

The notion of a singular perturbation is somewhat ambiguous. One can find several definitions of this kind of perturbations in the literature. In this section we take a physical point of view. As singular perturbations we have in mind impenetrable potential barriers, i.e., models where the positive potentials function becomes infinite on some closed set $\Gamma \subset \mathbb{R}^d$. This set will be called an obstacle or the obstacle region.

There are several possibilities to formulate such a problem mathematically. Nevertheless, the principle is always the same. At first one restricts the mathematical consideration to the complement of Γ, we call it $\Sigma = \mathbb{R}^d \setminus \Gamma$. Then one studies the system on Σ or in $L^2(\Sigma)$ and tries to find a selfadjoint Hamiltonian which is a representative for this singular perturbation considered. We will mention four procedures: restriction of operators, form domain restrictions, restriction of processes to a killing process, and restriction of semigroups as will be explained in the following remarks. The restriction of semigroups are considered in more detail.

Remark 4.2.16 (Restriction of Operators). Starting with an operator in $L^2(\mathbb{R}^d)$ one restricts it to $L^2(\Sigma)$ and tries to find a unique selfadjoint extension of it. This is a possible procedure in particular for local operators which means if

$$\text{supp }(H_0 f) \subset \text{supp }(f),$$

if $f \in \text{dom}\, H_0$. (In our examples here only the Laplacian is local). Then H_0 will be restricted to

$$\mathfrak{D} = \{f,\ f \in \text{dom}\,(H_0),\ \text{supp }(f) \subset \Sigma\}.$$

If the set of infinitely differentiable functions with compact support in Σ is contained in $\text{dom}\, H_0$, then $H_0|_{\mathfrak{D}}$ is symmetric in $L^2(\Sigma)$. However in general the domain of H_0 may be too small to provide essential selfadjointness. For this one has to find suitable boundary conditions.

For local and non-local operators there are simpler description of such obstacle perturbations and the corresponding operators.

Remark 4.2.17 (Restriction of Form Domains). In the framework of regular Dirichlet forms such perturbations can be considered as a variation of the domain. We start with the regular Dirichlet form

$$\mathfrak{a}[f,g] = \langle H_0^{\frac{1}{2}} f, H_0^{\frac{1}{2}} g\rangle,\ \text{dom}\,(\mathfrak{a}) = \text{dom }\ (H_0^{\frac{1}{2}}),$$

see Definition 4.1.17. Recall that

$$\mathfrak{a}_1[f,g] = \mathfrak{a}[f,g] + \langle f, g\rangle.$$

Then $\{\mathrm{dom}\,(\mathfrak{a}),\ \mathfrak{a}_1[.,.]\}$ forms a Hilbert space. Now one defines a closed subspace of $\{\mathrm{dom}\,(\mathfrak{a}),\ \mathfrak{a}_1[.,.]\}$ by restricting the form domain. Set

$$(\mathrm{dom}\,(\mathfrak{a}))_\Sigma \ = \{f, f \in \mathrm{dom}\,(H_0^{\frac{1}{2}}),\ \tilde{f} = 0 \text{ q. e. } on\ \Gamma\}.$$

$\tilde{f}$ is the quasi-continuous version of f (see Theorem 4.1.18).

The form $(\mathfrak{a}, (\mathrm{dom}\,(\mathfrak{a}))_\Sigma)$ is again a regular Dirichlet form but now defined in $L^2(\Sigma)$. That form corresponds uniquely to a selfadjoint operator in $L^2(\Sigma)$, which will be denoted by $H_{(\mathrm{dom}\,(\mathfrak{a}))_\Sigma}$.

Remark 4.2.18 (Restriction of Processes). Following Theorem 4.1.15 and its preceding remark there is a one-to-one correspondence between regular Dirichlet forms and Hunt processes. Therefore it is also possible to start with the process associated to H_0 and modify it in such a way that the corresponding Dirichlet form coincides with the one mentioned above in Remark 4.2.17.

Let $(\mathfrak{a}, \mathrm{dom}\,(\mathfrak{a}))$ be the regular Dirichlet form in $L^2(\mathbb{R}^d)$ associated to H_0. Let

$$\{\left(\Omega, \mathfrak{B}_\Omega, (P_x)_{x\in\mathbb{R}^d\cup\{\infty\}}\right),\ (X_t,\ t \geq 0),\ (\mathbb{R}^d, \mathfrak{B}_{\mathbb{R}^d})\}$$

be the corresponding process where $\mathbb{R}^d \cup \{\infty\}$ means the one-point compactification of $\mathbb{R}^d$, such that the path maps $[0,\infty] \to \mathbb{R}^d \cup \{\infty\}$. Consider a closed set Γ. Then one constructs a subprocess by killing the trajectories on Γ. Let $\Sigma = \mathbb{R}^d \setminus \Gamma$ and let $\Sigma \cup \{\infty\}$ be the one-point compactification of Σ. Define a new set of paths by

$$X_\Sigma(t) \ = \begin{cases} X_t & \text{if} \quad 0 \leq t \leq \tau_\Gamma \\ \{\infty\} & \text{if} \quad t > \tau_\Gamma, \end{cases}$$

where τ_Γ is the first hitting time of Γ, that is,

$$\tau_\Gamma = \inf\{s,\ s > 0,\ X_s \in \Gamma\}.$$

The triple

$$\left\{\left(\Omega, \mathfrak{B}_\Omega, (P_x)_{x\in\Sigma\cup\{\infty\}}\right),\ (X_\Sigma\,(t);\ t \geq 0),\ (\Sigma, \mathfrak{B}_\Sigma)\right\}$$

is then a Hunt process acting in Σ. It is called the killed process because the trajectories are "killed" upon entering the obstacle region. The transition function of the process is

$$p_t^\Sigma(x,B) = p_x\{X_t \in B,\ \tau_\Gamma > t\}, \tag{4.2.62}$$

$B \in \mathfrak{B}_\Sigma$. The density process determines a new regular Dirichlet form in $L^2(\Sigma)$. This form corresponds to a selfadjoint operator which coincides with $H_{(\mathrm{dom}\,(\mathfrak{a}))_\Sigma}$ given above for the form domain restrictions.

So far we have described only possible procedures to find a selfadjoint Hamiltonian which models the singular perturbation by an obstacle. Next we will explain a further possibility in some more detail.

Remark 4.2.19 (Restriction of Semigroups). Let us introduce a family of operators given by

$$[\widetilde{T}_{H_0,\Sigma}(t)f] = E_x\{f\left(X_t\right),\ \tau_\Gamma > t\} \tag{4.2.63}$$

for $t > 0$, where τ_Γ is again the first hitting time,

$$\tau_\Gamma = \inf\{s,\ s > 0,\ X_s \in \Gamma\},$$

and for the moment we take $f \in L^2(\mathbb{R}^d)$. Finally, we will argue that an appropriate restriction of $\widetilde{T}_{H_0,\Sigma}(t)$ is a selfadjoint strongly continuous semigroup on $L^2(\Sigma)$. For that we need a geometric condition for Γ.

Definition 4.2.20. An element $x \in \mathbb{R}^d$ is called a τ_Γ-regular point of Γ if and only if

$$P_x\{\tau_\Gamma = 0\} = 1.$$

The set of all τ_Γ − regular points of Γ will be denoted by Γ^r.

Throughout the rest of this chapter we assume that $\Gamma = \Gamma^r$. Then $[\widetilde{T}_{H_0,\Sigma}(t)f](x) = 0$ if $x \in \Gamma$. Hence it is reasonable to restrict $\widetilde{T}_{H_0,\Sigma}(t)$ to $L^2(\Sigma)$, i.e., we set

$$T_{H_0,\Sigma}(t) = \widetilde{T}_{H_0,\Sigma}(t)\mid_{L^2(\Sigma)}. \tag{4.2.64}$$

Obviously, $T_{H_0,\Sigma}(t)$ are bounded operators in $L^2(\Sigma)$ for any t. Using Definition 4.1.12 $T_{H_0,\Sigma}(t)$ are integral operators for $t > 0$ with the kernels

$$\left(T_{H_0,\Sigma}(t)\right)(x,y) = E_x^{y,t}\{\tau_\Gamma < t\}. \tag{4.2.65}$$

These kernels are symmetric and $T_{H_0,\Sigma}(t)$ are selfadjoint.

Proposition 4.2.21. *The family $\{T_{H_0,\Sigma}(t),\ t \geq 0\}$ is a strongly continuous semigroup in $L^2(\Sigma)$.*

Proof: We write τ_Γ as $\tau_\Gamma(X)$ to denote the dependence on the paths X.

The semigroup property follows because

$$\begin{aligned}
[T_{H_0,\Sigma}(t)\ T_{H_0,\Sigma}(s)f](x) = &\ E_x\left\{[T_{H_0,\Sigma}(s)f]\left(X_t\right),\ \tau_\Gamma(X) > t\right\} \\
= &\ E_x\left\{E_{X_t}\{f\left(\tilde{X}_s\right),\ \tau_\Gamma(X) > t,\ \tau_\Gamma(\tilde{X}) > s\}\right\}
\end{aligned}$$

where $\tilde{X}(.)$ are the trajectories starting at X_t. Let ϑ_ρ be the shift operator given by

$$\vartheta_\rho(X)(\sigma) = X_{\sigma+\rho} \qquad \text{for } \rho, \sigma > 0.$$

Then the last expression can be rewritten as

$$E_x\left\{E_{X_t}\{f\left(\vartheta_t(X)(s)\right),\ \tau_\Gamma(X)>t,\ \tau_\Gamma\circ\vartheta_t(X)>s\}\right\}.$$

τ_Γ is a terminal stopping time which means that on the event $\tau_\Gamma>t$ we have

$$\tau_\Gamma+\tau_\Gamma\circ\vartheta_t=\tau_\Gamma.$$

Hence

$$\begin{aligned}[T_{H_0,\Sigma}(t)\ T_{H_0,\Sigma}(s)f](x) &= E_x\left\{E_{X_t}\{f\left(X_{t+s}\right),\ \tau_\Gamma>t+s\}\right\}\\ &= E_x\{f\left(X_{t+s}\right),\ \tau_\Gamma>t+s\}\\ &= [T_{H_0,\Sigma}(t+s)f](x).\end{aligned}$$

The strong continuity follows finally by the dominated convergence and is based on the following point wise consideration. Noting that the trajectories are right continuous we get for continuous functions f

$$\begin{aligned}\lim_{t\downarrow 0}|E_x\{f\left(X_t\right),\ \tau_\Gamma>t\}-f(x)| &\le \lim_{t\downarrow 0}\ E_x\left\{|f\left(X_t\right)-f(x)|,\ \tau_\Gamma>t\right\}\\ &\qquad+\lim_{t\downarrow 0}\ E_x\left\{|f(x)|,\ \tau_\Gamma\le t\right\}\\ &\le \lim_{t\downarrow 0}\ E_x\{|f\left(X(t)\right)-f(x)|\}\\ &\qquad+|f(x)|\ P_x\{\tau_\Gamma=0\}\\ &=0,\end{aligned}$$

where we used $P_x\{\tau_\Gamma=0\}=0$ for $x\in\Sigma$. □

Hence $\{T_{H_0,\Sigma}(t),\ t\ge 0\}$ forms a selfadjoint strongly continuous semigroup in $L^2(\Sigma)$. It has a selfadjoint generator denoted by $(H_0)_\Sigma$. Summarizing that we obtained:

Corollary 4.2.22 *If $\Gamma=\Gamma^r$, we have*

$$\left(e^{-t(H_0)_\Sigma}f\right)(x)=E_x\{f\left(X_t\right),\ \tau_\Gamma>t\} \tag{4.2.66}$$

is a strongly continuous semigroup in $L^2(\Sigma)$ with the kernel

$$\left(e^{-t(H_0)_\Sigma}\right)(x,y)=E_x^{y,t}\{\tau_\Gamma<t\}. \tag{4.2.67}$$

The semigroup is contractive implying $(H_0)_\Sigma\ge 0$.

The process corresponding to $e^{-t(H_0)_\Sigma}$ has the same transition function as the killed process considered in Remark 4.2.18 (see (4.2.62)). The associated quadratic form of $\{e^{-t(H_0)_\Sigma};\ t\ge 0\}$ is given by

$$\tilde{\mathfrak{a}}[f,f]=\lim_{t\downarrow 0}\ \frac{1}{t}\ |\langle f,f\rangle-\langle f,\ e^{-t(H_0)_\Sigma}f\rangle|$$

for $f \in \operatorname{dom}((H_0)_\Sigma^{\frac{1}{2}})$. Thus $(H_0)_\Sigma$ coincides with $(H_0)_{(\operatorname{dom}(\mathfrak{a}))_\Sigma}$ (see Remark 4.2.17) and with the operator associated to the killed process from Remark 4.2.18.

Let us come back to the beginning of this section, where we mentioned the physical point of view that the singular perturbations here are impenetrable potential barriers. Assume a function

$$W_\Gamma(x) = \begin{cases} \{\infty\} & \text{for} \quad x \in \Gamma \\ 0 & \text{for} \quad x \notin \Gamma \ . \end{cases}$$

Consider the formal Feynman–Kac expression

$$E_x\left\{ e^{-\int\limits_0^t W_\Gamma(X_s)ds} f(X_t) \right\} . \tag{4.2.68}$$

Then it is clear that

$$e^{-\int\limits_0^t W_\Gamma(X_s)ds} = \begin{cases} 1 & \text{if} \quad X_s \in \Sigma \quad \text{for all} \quad 0 \le s \le t \\ 0 & \text{if } \operatorname{meas}\{s : X_s \in \Gamma\} > 0 \ . \end{cases}$$

We set $T_{\Gamma,t} = \operatorname{meas}\{s : s \le t, X_s \in \Gamma\}$. This is called the spending time of the trajectory in Γ.

This description fits the killing of the process in Γ as well as the introduction of the semigroup $\{T_{H_0,\Sigma}(t);\ t \ge 0\}$ in Proposition 4.2.21, because (4.2.68) is then equal to

$$E_x\{f(X_t)\,,\ T_{\Gamma,t} > 0\} \ . \tag{4.2.69}$$

Although the last three methods are sufficient to introduce a selfadjoint operator which models the obstacle system one wants to clarify the relation between $(H_0)_\Sigma$ and the free operator H_0, or better its restriction to $L^2(\Sigma)$. In doing so we introduce the harmonic extension operator.

Definition 4.2.23. Assume a free operator H_0 with its associated process (see Section 4.1.3). Let $\Gamma = \Gamma^r$ and τ_Γ the first hitting time of Γ.

Then the harmonic extension operator V_Γ^a is defined by

$$\begin{aligned} \operatorname{dom}(V_\Gamma^a) &= \operatorname{dom}(H_0), \\ [V_\Gamma^a f](x) &= E_x\left\{e^{-a\tau_\Gamma} f\left(X_{\tau_\Gamma}\right),\ \tau_\Gamma < \infty\right\} , \end{aligned}$$

with $a > 0$. (In the following we set $a = 1$.)

Note that the definition makes sense also for bounded continuous functions. Hence

$$\begin{aligned} (V_\Gamma^1 1)(x) &= E_x\{e^{-\tau_\Gamma},\ \tau_\Gamma < \infty\} \\ &= v_\Gamma(x) , \end{aligned} \tag{4.2.70}$$

where v_Γ is the equilibrium potential defined in (4.1.37).

Because we assumed $\Gamma = \Gamma^r$ we get for any function g which is zero on Γ, i.e., for functions g with $M_{\chi_\Sigma} g = g$, that

$$[V_\Gamma^1 g](x) = 0 \ . \tag{4.2.71}$$

The harmonic extension operator is the correct operator to express the resolvent difference of the free and the singularly perturbed operator. This formula is known as Dynkin's formula.

Proposition 4.2.24 (Dynkin's formula). *Let Γ be an obstacle region with $\Gamma = \Gamma^r$. Denote by $J : L^2(\mathbb{R}^d) \to L^2(\Sigma)$ the restriction operator of any $f \in L^2(\mathbb{R}^d)$ to $L^2(\Sigma)$, where $\Sigma = \mathbb{R}^d \backslash \Gamma$ as before. Then*

$$(1 + H_0)^{-1} - J^* \left(1_{L^2(\Sigma)} + (H_0)_\Sigma\right)^{-1} J = V_\Gamma^1 (1 + H_0)^{-1} \tag{4.2.72}$$

holds on $L^2(\mathbb{R}^d)$.

Proof: Let $g \in \operatorname{dom} H_0$. Then

$$\begin{aligned}
&\Big\{\big[(1 + H_0)^{-1} - J^* \left(1_{L^2(\Sigma)} + (H_0)_\Sigma\right)^{-1} J\big](1 + H_0) g\Big\}(x) \\
&= \int_0^\infty ds\ e^{-s} E_x \Big\{\big[(1 + H_0) g\big]\big(X_s\big),\ \tau_\Gamma \le s\Big\} \\
&= E_x \left\{ \int_{\tau_\Gamma}^\infty ds\ e^{-s} \big[(1 + H_0) g\big]\big(X_s\big) \right\} \\
&= E_x \left\{ e^{-\tau_\Gamma} \int_0^\infty ds\ e^{-s} \big[(1 + H_0) g\big]\big(X_{s+\tau_\Gamma}\big) \right\} \\
&= E_x \left\{ e^{-\tau_\Gamma} \int_0^\infty ds\ e^{-s} E_{X_{\tau_\Gamma}} \Big\{\big[(1 + H_0) g\big]\big(\tilde{X}_s\big)\Big\} \right\} . \\
&\qquad \text{(strong Markov property)} \\
&= E_x \left\{ e^{-\tau_\Gamma} \int_0^\infty ds\ e^{-s} \big[e^{-s H_0} (1 + H_0) g\big]\big(X_{\tau_\Gamma}\big) \right\} \\
&= E_x \Big\{ e^{-\tau_\Gamma} g\big(X_{\tau_\Gamma}\big)\Big\}.
\end{aligned}$$

□

Equation (4.2.72) is called Dynkin's formula. By the definition of J we have $(Jf)(x) = f(x)$ for $x \in \Sigma$. J^* is the extension operator, extending

$g \in L^2(\Sigma)$ to $L^2(\mathbb{R}^d)$ by setting it zero on the orthogonal complement of $L^2(\Sigma)$.

Note that

$$JJ^*g = \quad 1_{L^2(\Sigma)}g, \quad g \in L^2(\Sigma),$$

and

$$J^*Jf = \quad M_{\chi(\Sigma)}f, \quad f \in L^2(\mathbb{R}^d).$$

(4.2.72) is the analogue of the second resolvent equation for regular potentials, i.e., of

$$(a+H_0)^{-1} - (a+H_0 \dot{+} M_V)^{-1} = (a+H_0 \dot{+} M_V)^{-1}M_V(a+H_0)^{-1}, \quad (4.2.73)$$

where V is, for example, a Kato–Feller potential and with a large enough. In other words, the harmonic extension operator V_Γ^1 is the counterpart of $(a + H_0 \dot{+} M_V)^{-1}M_V$. Dynkin's formula can be extended to more general perturbations and free operators. This aspect is explained in the notes to some extent.

Using Dykin's formula one finds at least a general relation between JH_0J^* and $(H_0)_\Sigma$.

Proposition 4.2.25. *Let H_0 and Γ be given as described above. Then $(H_0)_\Sigma$ (see Corollary 4.2.22) is a selfadjoint extension of the operator JH_0J^*, that is if $f \in L^2(\Sigma)$ and $f \in \operatorname{dom}(H_0)$, then $f \in \operatorname{dom}(H_0)_\Sigma$ and*

$$JH_0J^*f = (H_0)_\Sigma\, f.$$

Proof: By the definition of $(H_0)_\Sigma$ and J we obtain

$$\begin{aligned} (H_0)_\Sigma &= (H_0)_\Sigma 1_{L^2(\Sigma)} \\ &= \big(1_{L^2(\Sigma)} + (H_0)_\Sigma\big)JJ^* - JJ^* \qquad (4.2.74) \\ &= \big(1_{L^2(\Sigma)} + (H_0)_\Sigma\big)J(1 - V_\Gamma^1)J^* - JJ^*\ , \end{aligned}$$

because of (4.2.71). Using Dynkin's formula

$$J^*\left(1_{L^2(\Sigma)} + (H_0)_\Sigma\right)^{-1} J = (1 - V_\Gamma^1)(1 + H_0)^{-1}$$

or

$$\left(1_{L^2(\Sigma)} + (H_0)_\Sigma\right)^{-1} J = J(1 - V_\Gamma^1)(1 + H_0)^{-1}.$$

The right-hand side is contained in $\operatorname{dom}(H_0)_\Sigma$. In particular any $J^*f \in \operatorname{dom}(H_0)$ is contained in $\operatorname{dom}((H))_\Sigma)$ as long as $f \in L^2(\Sigma)$. For $g \in \operatorname{dom}(H_0)$ with $g = (1 + H_0)^{-1}h$ we get

$$\left(1_{L^2(\Sigma)} + (H_0)_\Sigma\right)^{-1} J(1 + H_0)g = J(1 - V_\Gamma^1)g$$

or

$$J(1 + H_0)g = \left(1_{L^2(\Sigma)} + (H_0)_\Sigma\right) J(1 - V_\Gamma^1)g\ .$$

This will be used in (4.2.74). If $J^*f \in \operatorname{dom}(H_0)$ we have $JV_\Gamma^1 J^* f = 0$ such that

$$(H_0)_\Sigma f = \quad JH_0J^*f,$$

which gives the proposition. □

To prove that JH_0J^* is essentially selfadjoint on a certain domain such that $(H_0)_\Sigma$ is its unique selfadjoint extension is a separate problem. It has to be considered separately for every free operator H_0. For local H_0 the literature is large. For non-local free Feller generators holds at least that

$$e^{-t(H_0)_\Sigma} C_0(\Sigma) \subseteq C_0(\Sigma) \ .$$

$C_0(\Sigma)$ denotes the functions f which are continuous on Σ and for which

$$\lim_{x \to x_0} f(x) = 0$$

if $x_0 \in \partial\Sigma$ (boundary of Σ) or if $x_0 = \{\infty\}$.

This form of restricted Feller property implies that $\operatorname{dom}((H_0)_\Sigma) \cap C_0(\Sigma)$ is a core for $\operatorname{dom}((H_0)_\Sigma)$.

Further extension problems will not be studied here. We use the generator $(H_0)_\Sigma$ to model the singular perturbations by such obstacles, which we have in mind in the following.

4.3 Notes

Section 4.1.1

The introduction of the selfadjoint realization of the Laplacian is standard. One can find it in many textbooks. A clear and complete proof is given by Amrein [9] p. 38 ff. Note that a multiplication operator $(M_g f)(k) = g(k)f(k)$ is selfadjoint if $g(\cdot)$ is real-valued and $|g(k)|$ is infinite at most on a set of Lebesgue measure zero. This is a weak condition which allows very general selfadjoint functions of the Laplacian. Note that the domain of M_g is dense in L^2 but $C_c^\infty(\mathbb{R}^d)$ is not necessarily contained in $\operatorname{dom}(M_g)$.

For (4.1.2) we have mentioned the Schwartz space on which the Laplacian is essentially selfadjoint. Note that there are further sets where the same is valid. One can choose $C_c^\infty(\mathbb{R}^d)$, infinitely differentiable functions of compact support, or one can take $\widehat{C}_c^\infty(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d, dx);\ \widehat{f} \in C_c^\infty(\mathbb{R}^d)\}$.

That all the selfadjoint operators $g(-\Delta)$ have purely absolutely continuous spectrum follows (see Remark 4.1.5) for instance from Kato [108] p. 520 Example 19.

Section 4.1.2

The Gaussian kernel in (4.1.19) is well known. In this context we recommend the book by Davies [56] on heat kernels and their relation to spectral theory. For α-stable processes with the generator $(-\Delta)^\alpha, 0 < \alpha < 1$, we refer to a series of articles by Chen–Song [42], [43], [44] and to the books of Chung [48] or Chung–Zhao [49].

The kernel of $e^{-t(-\Delta)^{1/2}}$ (see(4.1.22)) can be found for example in Stein–Weiss [183], Theorem 1.14.

For relativistic Schrödinger operators to get the expression for the kernel as given in Equation (4.1.23) we refer to Carmona–Masters–Simon [40]. However there are other ways of constructing Feller processes from the kernels. The reader may consult the first chapter of the book by Demuth–van Casteren [65] (see also the notes for Section 4.1.3).

Section 4.1.3

Standard textbooks for this subsection are the books by Fukushima–Oshima–Takeda [86], Ma–Röckner [143] and Demuth-van Casteren [65]. The basic assumptions on stochastic spectral analysis (BASSA) were introduced in [65] p. 5. They will be repeated here for the Euclidean space $\mathbb{R}^d$:

Let $(t, x, y) \to p(t, x, y)$ be a continuous function of all the variables mapping $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$. For $p(\cdot, \cdot, \cdot)$ we assume the following:

A1 p satisfies the Chapman–Kolmogorov identity, i.e.,

$$\int p(s, x, z)\, p(t, z, y)\, dz = p(s + t, x, y),$$

$s, t > 0;\ x, y \in \mathbb{R}^d$. Its total mass is smaller or equal to one, i.e.,

$$\int p(t, x, y)\, dy \leq 1 \qquad t > 0,\ x \in \mathbb{R}^d.$$

A2 p satisfies the Feller property, i.e., for $f \in C_0(\mathbb{R}^d)$ the function $x \to \int f(y)\, p(t, x, y)\, dy$ is also in $C_0(\mathbb{R}^d)$.

A3 Let $f \in C_0(\mathbb{R}^d)$. Then we assume for every $x \in \mathbb{R}^d$ that

$$\lim_{t \downarrow 0} \int f(y)\, p(t, x, y)\, dy = f(x).$$

A4 p is assumed to be symmetric, i.e.,

$$p(t, x, y) = p(t, y, x)$$

for all $t > 0$ and all $x, y \in \mathbb{R}^d$.

If these assumptions are satisfied p establishes a Markovian selfadjoint semigroup and a strong Markov process

$$\{(\Omega, \mathfrak{B}_\Omega, P_x),\ (X_t, t \geq 0),\ (\mathbb{R}^d, \mathfrak{B}_{\mathbb{R}^d})\}.$$

A proof of this fact can be found in Blumenthal–Getoor [28]. X_t are the state variables, defined on Ω with values in $\mathbb{R}^d \cup \{\infty\}$. The one-dimensional distribution is

$$P_x(X_t \in B) = \int_B p(t,x,y)\ dy$$

$t > 0,\ B \in \mathfrak{B}$. The sample paths are cadlag.

Because of the Feller property the corresponding process is called a Feller process which is a subclass of the Hunt processes. This is proved in Fukushima–Oshima–Takeda [86] Theorem A.2.2 p. 315. For a definition of a Hunt process see also [86] p. 310 ff. Here we will not go into further details about the theory of Hunt processes.

However, it is obvious that the kernels of Section 4.1.2 for $-\Delta$, $(-\Delta)^\alpha,\ \alpha \in (0,1)$, $\sqrt{-\Delta + c^2} - c$ satisfy these basic assumptions (A1–A4). They establish processes, which are even more regular, that means they establish Levý processes (for a definition see [86] p. 140). They are spatial homogeneous and have independent increments. The α-stable processes then form a subclass of the Levý processes. They are studied by Chen and Song [43], [44] and also by Chung [48] or Chung–Zhao [49]. Finally the Wiener process associated to the Laplacian is the most regular one. Their paths are Hölder continuous.

The pinned measure $E_x^{y,t}\{\cdot\}$ in (4.1.29) is studied in some detail and under several aspects by Demuth–van Casteren in [65], e.g., see p. 63 and p. 77. The definition in (4.1.29) makes sense, because the new process $\{e^{-(t-s)H_0}(X_s, y),\ 0 \leq s < t\}$ is a martingale on $[0,t)$. This can be seen easily by

$$\begin{aligned} E_x\{e^{-tH_0}(X_{s_2}, y)\ |\mathfrak{F}_{s_1}\} &= E_{X_{s_1}}\{e^{-tH_0}(X_{s_2 - s_1}, y)\} \\ &= \int_{\mathbb{R}^d} e^{-(s_2-s_1)H_0}(X_{s_1}, z)\ e^{-(t-s_2)H_0}(z,y)\ dz \\ &= e^{-(t-s_1)H_0}(X_{s_1}, y), \end{aligned}$$

for all s_1 with $0 \leq s_1 < s_2$ and $s_2 < t$. The total mass of this measure is $e^{-tH_0}(x,y)$.

Section 4.1.4

The theory of regular Dirichlet forms and their relation to Hunt processes is given for instance in the textbooks by Fukushima–Oshima–Takeda [86], by Ma–Röckner [143], or by Fukushima [85]. For the definition of the regular Dirichlet form we refer to [86] p. 6 and Lemma 1.4.2 on p. 27. For the definition of the Hunt process we refer again to [86] p. 310 ff as in the notes for Section 4.1.3. The very interesting result in Theorem 4.1.15, the one-to-one correspondence between Hunt processes and regular Dirichlet form, can be found in [86] Theorem 7.2.1 p. 302. In Remark 4.1.16 we stated that all the H_0 studied here gives rise to a regular Dirichlet form. This is obvious because $\mathrm{dom}(H_0^{\frac{1}{2}}) \supseteq \mathrm{dom}\,(H_0) \supseteq \mathrm{dom}\,(-\Delta) \supseteq C_c^\infty(\mathbb{R}^d)$ which is dense in $C_c(\mathbb{R}^d)$ with respect to the supremum norm. The potential theory of Dirichlet forms and the notion of capacity is given in [86] Chapter 2, p. 64 ff. Theorem 4.1.18 is a copy of Theorem 2.1.3 in [86] p. 68. Theorem 4.1.19 is also given in [86] as Theorem 2.1.5 p. 70 or by Fukushima [85] Lemma 3.1.1 p. 62.

For the stochastic representation of the equilibrium potential stated in Theorem 4.1.19 one can find different proofs. Here we mention [86] Lemma 4.2.1 p. 141 and Theorem 4.2.5 p. 145. Also Demuth–van Casteren studied this in a more general setting in [65] p. 101. The proof of $v_A(x) = E_x\{e^{-\tau_A}\}$ is sometimes spread over different parts in the books. A connected and clear proof is given by Noll [151] Theorem 4.1. Because this is the central relation for the study of singular perturbations we will repeat his proof here.

Proof of Theorem 4.1.21

Let $\big(\mathfrak{a}, \mathrm{dom}\,(\mathfrak{a})\big)$ be a regular Dirichlet form in $L^2(\mathbb{R}^d)$ with the generator H_0. A function $u \in L^2$ is called 1-excessive if $e^{-tH_0}\ u \leq e^t\ u$. For instance the equilibrium potential v_Γ is 1-excessive.

Assume now $v \in \mathrm{dom}\,(\mathfrak{a})$ and an arbitrary function $u \in L^2(\mathbb{R}^d)$. Without proof we state a criterion for u being also in $\mathrm{dom}\,(\mathfrak{a})$:

Let v and u be 1-excessive and assume $u \leq v$. Let $v \in \mathrm{dom}\,(\mathfrak{a})$. Then $u \in \mathrm{dom}\,(\mathfrak{a})$ and $\mathfrak{a}_1[u] \leq \mathfrak{a}_1[v]$.

That fact can be used in the present situation. v_Γ is already 1-exessive. Set $u_\Gamma(x) = E_x\{e^{-\tau_\Gamma},\ \tau_\Gamma < \infty\}$. We will show that $u_\Gamma \in \mathrm{dom}\,(\mathfrak{a})$ and equals v_Γ. At first it turns out that u_Γ is 1-excessive, because

$$
\begin{aligned}
(e^{-tH_0} u_\Gamma)(x) &= E_x\{u_\Gamma(X_t)\} \\
&= E_x\{u_\Gamma(X_t);\ \tau_\Gamma \leq t\} + E_x\{u_\Gamma(X_t);\ t < \tau_\Gamma < \infty\} \\
&\leq P_x\{\tau_\Gamma \leq t\} + E_x\{u_\Gamma(X_t);\ t < \tau_\Gamma < \infty\} \\
&\leq e^t\ E_x\{e^{-\tau_\Gamma}\ ;\ \tau_\Gamma \leq t\} + E_x\{e^{-(\tau_\Gamma - t)};\ \ t < \tau_\Gamma < \infty\} \\
&\leq e^t\ E_x\{e^{-\tau_\Gamma}\ ;\ \tau_\Gamma < \infty\} \\
&= e^t\ u_\Gamma(x).
\end{aligned}
$$

Moreover $u_\Gamma = 1$ q.e. on Γ. Indeed if there is some $x_0 \in \Gamma$ with $u_\Gamma(x_0) \neq 1$, then x_0 is a irregular point with respect to the first hitting time τ_Γ because $P_{x_0}\{\tau_\Gamma > 0\} > 0$. But sets of irregular point have capacity zero.

Hence it suffices to show $u_\Gamma \leq v_\Gamma$. Then we obtain $u_\Gamma \in \mathrm{dom}\,(\mathfrak{a})$ and $\mathfrak{a}_1[u_\Gamma] \leq \mathfrak{a}_1[v_\Gamma]$. Because v_Γ is unique, then $u_\Gamma = v_\Gamma$ a.e. follows.

For the estimate $u_\Gamma \leq v_\Gamma$ we consider first an open set $\mathcal{O} \subset \mathbb{R}^d$ and define the random variable

$$Y_t = e^{-t}\ v_{\mathcal{O}}(X_t),\ t \geq 0.$$

For fixed $x \in \mathbb{R}^d$ this is a super martingale, because for $s \geq t \geq 0$

$$\begin{aligned} E_x\{Y_s \mid \mathfrak{F}_t\} &= e^{-s}\ E_x\{v_{\mathcal{O}}(X_s) \mid \mathfrak{F}_t\} \\ &= e^{-s}\ E_{X_t}\{v_{\mathcal{O}}(X_{s-t})\} \\ &= e^{-s}\ (e^{-(s-t)H_0}\ v_{\mathcal{O}})(X_t) \\ &\leqq e^{-s}\ e^{(s-t)} v_{\mathcal{O}}(X_t) \\ &= Y_t, \end{aligned}$$

where we used that the equilibrium potential is 1-excessive. Using Doob's optional sampling theorem (for a proof see e.g., Demuth–van Casteren [65] p. 361) it follows that

$$E_x\{Y_{\min\,(s,\tau_{\mathcal{O}})}\} \leq v_{\mathcal{O}}(x).$$

When $s \to \infty$ one gets using $v_{\mathcal{O}}(X_{\tau_{\mathcal{O}}}) = 1$ and $u_{\mathcal{O}}(x) = E_x e^{-\tau_{\mathcal{O}}}$,

$$u_{\mathcal{O}}(x) \leq\ v_{\mathcal{O}}(x)$$

with $v_{\mathcal{O}}(X_{\tau_{\mathcal{O}}})$ and $u_{\mathcal{O}}(x) = E_x\{e^{-\tau_{\mathcal{O}}} v_{\mathcal{O}}(X_{\tau_{\mathcal{O}}})\}$, for any open $\mathcal{O} \subset \mathbb{R}^d$. For general Γ we take a sequence of open sets $\mathcal{O}_n \supset \Gamma$ such that cap $(\mathcal{O}_n) \to$ cap (Γ). This implies $v_{\mathcal{O}_n} \to v_\Gamma$ in $\{\mathrm{dom}\,(\mathfrak{a}),\ \langle\cdot,\cdot\rangle_{\mathfrak{a}}\}$.

Moreover,

$$u_\Gamma(x) \leq\ u_{\mathcal{O}_n}(x) \leq\ v_{\mathcal{O}_n}(x),$$

such that for $n \to \infty$ we get the desired estimate $u_\Gamma(x) \leq\ v_\Gamma(x)$ a.e. □

In [86] Theorem 4.2.5 p. 145 it is shown that $E.\{e^{-\tau_\Gamma},\ \tau_\Gamma < \infty\}$ is a quasi-continuous version of v_Γ for nearly Borel sets Γ.

For an introduction into the potential theory we recommend Fukushima [85] chapter 3 or Fukushima–Oshima–Takeda [86]. Equation (4.1.38) is given in [86] Theorem 2.2.2 p. 77 ff, Corollary 4.1.22 is studied also there [86] Lemma 2.2.6 p. 79 or by Demuth–vanCasteren [65] p. 406 ff.

Section 4.2.1

This section is mainly taken from the book by Demuth–vanCasteren [65] Chapter 2 and Chapter 3. It is also useful to look at the book by van Casteren [188].

Here we have restricted the consideration to the unperturbed operators $-\Delta, (-\Delta)^\alpha,\ \alpha \in (0,1),\ \sqrt{-\Delta + c^2} - c$. For the Laplacian one can consult the review articles by Simon [175] and [176]and also by Aizenman–Simon [7]. They studied the Kato class potentials in some detail and gave a lot of spectral applications. Different areas of mathematical physics are investigated using functional integration. A good reference for this is the book of Simon [174].

Kashminskii's Lemma is proved in [65] p. 127. The proof of Lemma 4.2.4 is simplified here in comparing with [65] p. 58 ff. Lemma 4.2.5 is given in [65] p. 60. Central is Remark 4.2.7, which is Theorem 2.5 in [65]. The proofs are omitted here. The proof of the continuity of the perturbed kernels

$\left(e^{-t(H_0\dot{+}V)}\right)(x,y)$ is difficult if we only assume $V_+ \in K_{\mathrm{loc}}(H_0)$. It is given in Chapter 3 of [65]. This proof becomes much easier if we would assume also $V_+ \in K(H_0)$. For that one can use the full Dyson expansion (see van Casteren [188]).

The estimate in Proposition 4.2.8 is very useful, because the perturbed kernels are estimated by the free one. Note that the singularity at $t = 0$ is the same for the free or perturbed kernels (see Remark 4.2.9). The proof given here is a special version of the proof of Theorem 2.9 [65] p. 69.

Section 4.2.2

Random Schrödinger operators indexed by parameters from ergodic probability spaces are used to describe disordered materials and transport, or the lack of it, in them. Some of the general theory needed here can be found in Carmona–Lacroix [39] and Figotin–Pastur [83].

One of the basic theorems in this area is the stability of the spectra of such ergodic families of operators. Pastur [152] proved the constancy of spectra and the spectral types with respect to the Lebesgue decomposition. The proof of results on the constancy of spectra require proving the measurability of spectral projections of the associated operators as functions of the random parameter. These are worked out in detail and the proofs of Theorems 4.2.13, 4.2.14 and 4.2.15 can be found in the sections V.1–V.3 of Carmona–Lacroix [39].

The selfadjointness of the random operators is not obvious when the potentials V^ω are unbounded and one of the earlier results in this direction is proved by Kirsch–Martinelli [116].

The constancy of singular spectra of Hausdorff dimension smaller than one is also valid for these random operators as shown by Last in [141].

Section 4.2.3 Singular perturbations

A more detailed overview of all the possibilities to find a selfadjoint operator modelling obstacle perturbations is given by Noll [151] in Section 3. He also gives a simple example (Example 3.1 in [151]) where the domain of the restricted operator becomes too small to provide essential selfadjointness.

To find suitable boundary condition for higher order differential operators is a difficult task. The reader may consult Agronovich [3] or Rosenblum–Shubin–Solomyak [166].

The restriction of form domains considered in Remark 4.2.17 is an elegant and natural method to model such singular perturbations. The simplest case is the Laplacian. For that case, Simon [174] used it in Theorem 21.1 and showed in Section 7 that $(H_0)_\Sigma$ is the Friedrichs extension of $-\Delta$ restricted to $C_c^\infty(\Sigma)$, which holds for an arbitrary closed set in Γ. An analogous result was given by Baumgärtel–Demuth [19]. Elliptic differential operators of second order with variable coefficients were studied for instance by Arendt–Batty [13] in Example 4.9 and Example 5.6 and in Section 7. Further useful references are given there. Another worthwhile source is the book of Davies [56], Chapter

2. In all these cases of local operators the singular perturbation is finally modelled by imposing Dirichlet boundary conditions to the free operator H_0.

The restriction of form domains and the associated construction of killing processes are investigated extensively in the book of Fukushima–Oshima–Takeda [86]. Concerning Remark 4.2.17 the reader may consult Theorem 4.4.3 in [86] and concerning Remark 4.2.18 to Theorem A.2.10 in [86].

It is more natural, from a physical point of view to consider a form, since it specifies expectation values of observables. Therefore the selfadjoint operator $H_{(\mathrm{dom}\,(\mathfrak{a}))_\Sigma}$, associated to the form $\mathfrak{a}$ seems to be a more natural object to consider than a selfadjoint operator obtained as an extension of a symmetric operator.

Concerning Remark 4.2.19 one can construct two kinds of semigroups. One is based on 4.2.63, i.e., on

$$E_x\left\{f\big(X_t\big),\ \tau_\Gamma > t\right\}.$$

The corresponding semigroup $\{T_{H_0,\Sigma}(t),\ t \geq 0\}$ was called a pseudo-Dirichlet semigroup by Arend–Batty in [13].

The other is based on

$$E_x\left\{f\big(X_t\big),\ \sigma_\Gamma > t\right\}.$$

where $\sigma_\Gamma = \inf\left\{s > 0,\ \int\limits_0^s 1_\Gamma\big(X_u\big)du > 0\right\}$.

This semigroup is related to the family of operators given in (4.2.68). σ_Γ is called the penetration time of Γ. Of course $\tau_\Gamma \leq \sigma_\Gamma$. If Γ has an interior such that $\Gamma = \Gamma^r = (\inf \Gamma)^r$ then $\tau_\Gamma = \sigma_\Gamma$. That was proved by Demuth–van Casteren [65] in Proposition 2.24. This problem is related to the Kac regularity studied e.g., by Herbst–Zhao [91].

Here we have considered only $T_{H_0,\Sigma}(t) = e^{-t(H_0)_\Sigma}$ because it is the more general one. The symmetry of the kernels $e^{-t(H_0)}(x,y)$ in (4.2.65) was shown by Demuth–van Casteren [65] in Lemma D10 for Theorem D4. Potentials of the form $W_\Gamma(x)$ were also considered by Arendt–Batty [13] Section 5 and the corresponding semigroups were called barred semigroups.

The harmonic extension operator and Dynkin's formula is given by Fukushima–Oshima–Takeda [86] (see page 136) for nearly Borel sets. The proofs of Proposition 4.2.24 and Proposition 4.2.25 are taken from [65] Proposition 2.31 and Corollary 2.32, respectively.

The proof of the Feller property

$$e^{-t(H_0)_\Sigma}\ C_0(\Sigma) \subseteq C_0(\Sigma)$$

is given in [65] Theorem D21. For the Brownian motion one can have a look at Doob [77], Part 3, Chapter II, §2, pp. 720–722.

It would be possible to combine the considerations of perturbations by deterministic potentials (Section 4.2.1) and the singular perturbations discussed here. First one ensures that $H_0 \dot{+} M_V$ is a selfadjoint semibounded operator for which the Feynman–Kac formula holds (see Remark 4.2.7). Then one studies

$$E_x \left\{ e^{-\int_0^t V(X_u)\,du} f(X_t),\ \tau_\Gamma > t \right\}$$

in an analogous way as $E_x\left\{f(X_t),\ \tau_\Gamma > t\right\}$ in the present section. This kind of generalization is studied throughout the book of Demuth–van Casteren [65].

Another more general possibility is to study perturbations by measures. Formally one resets the measure $V(x)dx$ given by the potential function $V(.)$ by a measure μ. μ is a signed measure of the Borel σ-algebra in $\mathbb{R}^d$. For this we recommend the articles by Albeverio–Brasche–Röckner [8] and by Brasche [34] and the references therein.

Dynkin's formula in Proposition 4.2.24 can be extended to very general perturbations and is not restricted to differential operators of second order.

Let L be any non-negative selfadjoint operator in a separable Hilbertspace $\mathfrak{H}$. Set $H_0 = L + 1$ and define a new Hilbertspace $\mathfrak{H}_1$, consisting of all $f \in \operatorname{dom}(L^{1/2}) = \operatorname{dom}(H_0^{\frac{1}{2}})$ with the scalar product

$$\langle f, g\rangle_1 = \langle L^{1/2} f,\ L^{1/2} g\rangle + \langle f, g\rangle\ .$$

Let M be a closed linear operator mapping $\mathfrak{H}_1$ into an auxiliary Hilbertspace $\mathfrak{H}_{\mathrm{aux}}$. Assume that M is compact and has dense range in $\mathfrak{H}_{\mathrm{aux}}$. Let H_β be defined by

$$||H_\beta^{1/2} f||^2 = ||H_0^{1/2} f||^2 + \beta\ ||Mf||^2_{\mathrm{aux}}.$$

Then there is a non-negative selfadjoint operator H_∞ in $\overline{\ker(M)}$ (closure of $\ker(M)$ in $\mathfrak{H}$) such that

$$\begin{aligned} \operatorname{dom}(H_\infty^{1/2}) &= \ker(M)\ , \\ ||H_\infty^{1/2} f|| &= ||H_0^{1/2} f||, \qquad f \in \operatorname{dom}(H_\infty^{1/2}). \end{aligned}$$

This means

$$\operatorname*{s-lim}_{\beta \to \infty} H_\beta^{-1} g = 0 \quad \text{if } g \in \left(\overline{\ker(M)}\right)^\perp$$

and

$$\operatorname*{s-lim}_{\beta \to \infty} H_\beta^{-1} f = H_\infty^{-1} f \quad \text{if } f \in \overline{\ker(M)}\,.$$

Decompose $\mathfrak{H} = \overline{\ker(M)} \oplus \left(\overline{\ker(M)}\right)^\perp$ the extended Dynkin's formula is a representation of the resolvent difference

$$H_0^{-1} - \left(H_\infty^{-1} \oplus 0\right).$$

This is based on the resolvent differences with finite β, i.e., on

$$H_\beta^{-1} - H_0^{-1} = -H_0^{-1/2} B^* \left(\frac{1}{\beta} + B\ B^* \right)^{-1} B\ H_0^{-1/2} \tag{4.3.75}$$

with $B = M\ H_0^{-1/2}$. Let $\{e_n\}$ be an orthonormal system in $\mathfrak{H}$ and $\{g_n\}$ an orthonormal system in $\mathfrak{H}_{\text{aux}}$. Because M is a compact operator from $\mathfrak{H}_1$ into $\mathfrak{H}_{\text{aux}}$, B is compact as operator from $\mathfrak{H}$ into $\mathfrak{H}_{\text{aux}}$. Let us write

$$Bf = \sum_{k=1}^{\infty} \lambda_k\ \langle e_k,\ f\rangle\ g_k$$

and

$$B^* h = \sum_{k=1}^{\infty} \lambda_k\ \langle g_k,\ h\rangle_{\text{aux}}\ e_h$$

with $\lambda_k > 0$ and $\lambda_k \to 0$ as $k \to \infty$. It turns out that λ_k^2 are the eigenvalues of BB^* with eigenvectors g_k. Therefore for $h \in \mathfrak{H}_{\text{aux}}$, one has

$$\left(\frac{1}{\beta} + BB^* \right)^{-1} h = \sum_{k=1}^{\infty} \frac{1}{\frac{1}{\beta} + \lambda_k^2}\ \langle g_k,\ h\rangle\ g_k\ .$$

Using Equation (4.3.75) we get

$$(H_0^{-1} - H_\beta^{-1})f = \sum_{k=1}^{\infty} \frac{\lambda_k^2}{\frac{1}{\beta} + \lambda_k^2}\ \langle H_0^{-1/2} e_k,\ H_0^{-1} f\rangle\ H_0^{-1/2} e_k\ .$$

When β tends to infinity the resolvent difference of H_0^{-1} and H_∞^{-1} is given by

$$\left[H_0^{-1} - (H_\infty^{-1} \oplus 0) \right] f = \sum_{k=1}^{\infty} \langle H_0^{-1/2} e_k,\ H_0^{-1} f\rangle\ H_0^{-1/2} e_k\ . \tag{4.3.76}$$

$\{H_0^{-1/2} e_k\}$ is an orthonormal system in $\mathfrak{H}_1$. It spans the orthogonal complement of $\ker(M)$ in $\mathfrak{H}_1$. This means that

$$P = \sum_k \langle H_0^{-1/2} e_k,\ .\ \rangle_1\ H_0^{-1/2} e_k$$

is the projector onto $\ker(M)^{\perp_{\mathfrak{H}_1}}$. Thus the extended Dynkin's formula is

$$\left[H_0^{-1} - (H_\infty^{-1} \oplus 0) \right] f = P\ H_0^{-1} f. \tag{4.3.77}$$

$\mathfrak{H}_{\text{aux}}$ can be for instance the Hilbertspace $L^2(\mathbb{R}^d,\ \mu)$ where μ is some non-negative measure on $\mathfrak{B}_{\mathbb{R}^d}$. Thus M or better M_μ models perturbations by measures. If μ is absolutely continuous with respect to the Lebesgue measure, then $d\mu = V(x)dx$ and $V(.)$ is the regular perturbation by multiplication with potential functions, i.e., if M equals M_{1_Γ}. Equation (4.3.77) is a generalization of Dynkin's formula in Proposition 4.2.24 as can be seen below.

Let Γ be an obstacle region and $\mathfrak{H}_{\mathbb{R}^d\setminus\Gamma}$ be the subspace of $\mathfrak{H}_1$ of all $f \in \mathrm{dom}\,(L^{1/2})$ such their quasi-continuous version is zero quasi-everywhere on Γ. Then the operator V_Γ^1 in 4.2.74 is the orthogonal projection in the Hilbertspace $\mathfrak{H}_1$ onto the orthogonal complement of $\mathfrak{H}_{\mathbb{R}^d\setminus\Gamma}$. Moreover for every closed subset Γ there is a measure μ such that $\mathfrak{H}_{\mathbb{R}^d\setminus\Gamma} = \ker(M_\mu)$. More details are given in [35].

5

Applications

5.1 Borel Transforms

In this chapter a selection of results are presented for the purpose of illustrating the application of various techniques outlined earlier in Section 5.1. The choice involves a general theorem in one-dimensional random Jacobi operators where the determinacy of the randomness is linked to the existence of an absolutely continuous component in the spectrum. The second example presented involves several dimensions, a fully non-deterministic case (independent randomness at different sites) in the high disorder regime where the spectral type turns out to be pure point. The third example involves random operators on a tree (which does not degenerate to the one-dimensional lattice), where in the low disorder regime an absolutely continuous spectrum is exhibited. All the above methods ultimately involve controlling the boundary behaviour of Borel transforms of some spectral measures of the associated operators.

In Section 5.1 we use the matrix elements $\langle f, (A-z)^{-1}g\rangle$ of selfadjoint operators A with f, g elements in a Hilbert space $\mathcal{H}$. Then these matrix elements are Borel transforms of finite complex measures $\mu^A_{f,g} = \langle f, P_A(\cdot)g\rangle$, which turn out to be probability measures when $f = g$ and $\|f\| = 1$. So the theorems on Borel transforms from Chapter 1 are applicable for such functions and are used here.

In Section 5.2 we apply the criteria for the stability of absolutely continuous spectra to random and deterministic potentials and singular perturbations.

5.1.1 Kotani Theory

In this section Kotani's theory is presented in the one-dimensional discrete case. We consider the space $\ell^2(\mathbb{Z})$ and work with the discrete Schrödinger operator (see Section 4.2.)

$$(\Delta u)(n) = u(n+1) + u(n-1), \quad u \in \ell^2(\mathbb{Z}),$$

which is perturbed with a random potential $(V^\omega u)(n) = q^\omega(n)u(n)$, where $q^\omega(n)$, $n \in \mathbb{Z}$, is a real-valued discrete time process on some probability space $(\Omega, \mathfrak{B}_\Omega, \mathbb{P})$. In the discussion below we consider the case when $q^\omega(n) = v(n)$ a non-random potential (in which case we denote the operator V^ω just by V), when the theorem is valid in that context.

Before we proceed further we start with an abstract definition, for a self-adjoint operator on $\ell^2(\mathbb{Z})$. We take the standard basis $\{\delta_m\}$ for $\ell^2(\mathbb{Z})$, that is $\delta_m(m) = 1, \delta_m(n) = 0, m \neq n$. For convenience we define $G(z,n,m) = \langle \delta_n, (A-z)^{-1}\delta_m\rangle$, for z in $\mathrm{res}(A)$, the resolvent set of A and set $G(E + i0, n, m) = \lim_{\epsilon \downarrow 0} G(E + i\epsilon, n, m)$ whenever the right-hand side limit exists.

Let us recollect here some basic facts about the discrete Schrödinger operator here. Let $H = \Delta + V$; then consider the operators $H_k^\pm$ on $\ell^2(\mathbb{Z}_k^\pm)$, ($\mathbb{Z}_k^+ = \{n \in \mathbb{Z} : n \geq k+1\}$, $\mathbb{Z}_k^- = \{n \in \mathbb{Z} : n \leq k-1\}$) given by

$$\begin{aligned} (H_k^+ u)(n) &= (Hu)(n), \qquad n > k+1, \\ (H_k^+ u)(k+1) &= u(k+2) + v(k+1)u(k+1) \\ (H_k^- u)(n) &= (Hu)(n), \qquad n < k-1, \\ (H_k^- u)(k-1) &= u(k-2) + v(k-1)u(k-1). \end{aligned} \tag{5.1.1}$$

Then the following relation is valid for the associated resolvent kernels,

$$\begin{aligned} G(z,n,m) &= \quad \langle \delta_n, (H-z)^{-1}\delta_m\rangle, \\ G_k^+(z,n,m) &= \begin{cases} \langle \delta_n, (H_k^+ - z)^{-1}\delta_m\rangle, n, m \geq k+1 \\ 0, \quad \text{otherwise}, \end{cases} \\ G_k^-(z,n,m) &= \begin{cases} \langle \delta_n, (H_k^- - z)^{-1}\delta_m\rangle, n, m \leq k-1 \\ 0, \quad \text{otherwise}. \end{cases} \end{aligned} \tag{5.1.2}$$

We give special names to the following two values of the resolvent kernels,

$$m^+(z,k) = G_k^+(z, k+1, k+1), \quad m^-(z,k) = G_k^-(z, k-1, k-1),$$

since these quantities are related as follows.

Lemma 5.1.1. *Let H be a discrete Schrödinger operator as above. Then the integral kernels of H, $H_k^\pm$ are related by*

$$G(z,k,k) = \frac{1}{(m^+(z,k-1))^{-1} - m^-(z,k)}, \quad k \in \mathbb{Z}, \tag{5.1.3}$$

which can be rewritten as

$$G(z,k,k) = \frac{-1}{m^+(z,k) + m^-(z,k) - v(k) + z}, \quad k \in \mathbb{Z}. \tag{5.1.4}$$

Further $G(z, k, k)$, $m^\pm(z,k)$ are functions analytic in $\mathbb{C}^+$ mapping it to itself.

Proof: Let us denote by $< \phi, \cdot > \psi$, the rank one operator associated with the vectors ψ, ϕ in $\ell^2(\mathbb{Z})$. Then using the standard basis $\{\delta_n\}$, we can write

$$H = H_k + \langle \delta_k, \cdot \rangle \delta_{k-1} + \langle \delta_{k-1}, \cdot \rangle \delta_k, \quad H_k = H_k^- \oplus H_{k-1}^+.$$

We use the second resolvent equation,

$$(H-z)^{-1} = (H_k - z)^{-1} - (H-z)^{-1}(\langle \delta_k, \cdot \rangle \delta_{k-1} + \langle \delta_{k-1}, \cdot \rangle \delta_k)(H_k - z)^{-1}.$$

Now apply both sides to the vector δ_k and δ_{k-1}, respectively, and take the inner product of the result with the vector δ_k to get the equalities

$$\begin{aligned} G(z,k,k) &= m^+(z,k-1)\left(1 - G(z,k,k-1)\right), \\ G(z,k,k-1) &= -G(z,k,k)m^-(z,k), \end{aligned}$$

respectively. Eliminating $G(z,k,k-1)$ from the above two relations gives the the Equation (5.1.3).

We use the resolvent equations above for the pair $H_{k-1}^+, V_k \oplus H_k^+$, to compute $\langle \delta_k, (H_{k-1}^+ - z)^{-1}\delta_k \rangle$ and $\langle \delta_k, (H_{k-1}^+ - z)^{-1}\delta_{k+1} \rangle$ and simplify to obtain the following recurrence relation for m^+ for any $k \in \mathbb{Z}$ (and use a similar procedure for m^-).

$$\begin{aligned} \big(v(k) - z - m^+(z,k+1)\big)\, m^+(z,k) &= 1, \\ \big(v(k) - z - m^-(z,k)\big)\, m^-(z,k+1) &= 1. \end{aligned} \tag{5.1.5}$$

Using the last relation it is possible to rewrite the Equation (5.1.3) as (5.1.4). The second part of the assertion of the lemma is by direct verification, using the spectral theorem (which is used to write each of the $G(z,k,k)$ or $m^{\pm}(z,k)$ as $\int_{\mathbb{R}} d\mu(x)/(x-z)$ for an appropriate positive finite measure μ). $\square$

Definition 5.1.2. We say that a selfadjoint operator A defined on $\ell^2(\mathbb{Z})$ is locally reflectionless if there is a set $\Sigma_{rl} \subset \sigma(A)$, of positive Lebesgue measure, such that

$$Re(G(E + i0, n, n)) = 0, \quad \forall \ \ E \in \Sigma_{rl}, \quad n \in \mathbb{Z}.$$

Proposition 5.1.3. *Let H be a discrete Schrödinger operator. Suppose for some $E \in \mathbb{R}$ and for some $k \in \mathbb{Z}$ the limits $G(E+i0,n,n), m^{\pm}(E+i0,n)$ exist for two consecutive integers $n = k, k+1$ and* $\mathrm{Re}(G(E+i0,n,n) = 0, \quad n = k, k+1$. *Then*

$$\mathrm{Re}(m^-(E+i0,n) - v(n) + E) = \mathrm{Re}(m^+(E+i0,n)), \qquad n = k, k+1$$

and

$$\mathrm{Im}(m^+(E+i0,n) = \mathrm{Im}(m^-(E+i0,n)), \qquad n = k+1.$$

Proof: From Equation (5.1.4), taking real parts and setting them to be zero using the condition $\mathrm{Re}(G(E+i0,n,n)) = 0, \quad n = k, k+1$, we obtain the equality of real parts of the $m^{\pm}$ stated in the first equation.

For the second relation we proceed as follows. First take real parts in Equation (5.1.3) and set it to zero to obtain the relation

$$\frac{\mathrm{Re}(m^-(E+i0,k+1)}{\mathrm{Re}(m^+(E+i0,k))} = (\mathrm{Re}(m^-(E+i0,k+1)))^2 + (\mathrm{Im}(m^-(E+i0,k+1)))^2. \tag{5.1.6}$$

We then rewrite the first equality in equations 5.1.5 as

$$m^+(E+i0,k) = \frac{1}{v(k) - E - m^+(E+i0,k+1)},$$

take real parts on both sides and substitute the value $\mathrm{Re}(m^-(E+i0,k+1))$ for $\mathrm{Re}(v(k) - E - m^+(E+i0,k+1))$, obtained from Equation (5.1.4) and the condition $\mathrm{Re}(G(E+i0,k+1,k+1)) = 0$, to get

$$\frac{\mathrm{Re}(m^-(E+i0,k+1))}{\mathrm{Re}(m^+(E+i0,k+1))} = \mathrm{Re}(m^-(E+i0,k+1))^2 + \mathrm{Im}(m^+(E+i0,k+1))^2. \tag{5.1.7}$$

Using equations (5.1.6) and (5.1.7), and noting that $\mathrm{Im}(m^{\pm}(E+i0,k+1) \geq 0$, we obtain the second relation of the proposition. $\square$

Corollary 5.1.4 *Let H be a locally reflectionless discrete Schrödinger operator. Then for every E in the set Σ_{rl} we have* $\mathrm{Im}(m^+(E+i0,k)) = \mathrm{Im}(m^-(E+i0,k))$ *and* $\mathrm{Re}(m^+(E+i0,k)) = -\mathrm{Re}(m^-(E+i0,k)) + v(k) - E$.

Proof: We first note that the boundary values of $G(E+i0,k,k), m^{\pm}(E+i0,k)$ exist for almost every E with respect to Lebesgue measure since each of these is a Borel transform of a positive measure, by definition. Therefore we can assume without loss of generality that Σ_{rl} itself is the set of positive measure such that $\mathrm{Re}(G(E+i0,k,k)) = 0$ and where $m^{\pm}(E+i0,k)$ exist for all k and all $E \in \Sigma_{rl}$. Now the corollary is immediate from the previous Proposition 5.1.3. $\square$

The following deterministic theorem is valid for discrete Schrödinger operators, in that the locally reflectionless ones among them are deterministic in the sense given below.

Theorem 5.1.5. *Let H be a locally reflectionless discrete Schrödinger operator. Then the knowledge of $\{v(n), \quad n \leq n_0\}$ determines $\{v(n), \quad n > n_0\}$ for any finite n_0.*

Proof: First note that the knowledge of $v(n)$ for $n \leq k$ determines the operator H_k^- on $\ell^2(\{n : n \leq k-1\})$ completely and hence its resolvent kernel, $m^-(z,k)$ completely for any $z \in \mathbb{C}^+$. The knowledge of $m^-(z,k), z \in \mathbb{C}^+$, determines $m^-(E+i0,k)$ on Σ_{rl}, via taking boundary values and hence determines also $m^+(E+i0,k+1)$ on Σ_{rl} by using the above Corollary 5.1.4.

Since the function $m^+(z, k+1)$ (constructed as the resolvent kernel of the operator H_k^+) is analytic in the upper half plane with positive imaginary part, its boundary values on a set of positive measure determine the function completely. Thus we know the function for any $z \in \mathbb{C}^+$. Now the values of $v(n), n \geq k+1$ can be read from the Taylor expansion at infinity of $m^+(z, k)$, since H is bounded and δ_k is a cyclic vector for H_k^+. $\square$

Before we proceed further we need to make a connection between the behaviour of solutions near ∞ and the spectrum.

It is a fact that from the Weyl–Titchmarsh theory of selfadjointness of difference operators, we can get an expression for the functions $m^{\pm}(z, n)$ in terms of the solutions $u_{\pm}(z, n)$ which are square summable near $\pm\infty$ (in n) for each z in $\mathbb{C}$; it is part of the theory that they always exist and either decay exponentially or grow exponential at ∞ for z in the resolvent set of H. We state this fact in a lemma in the random setting.

Lemma 5.1.6. *Let $H^\omega, \omega \in \Omega$ be a discrete ergodic random Schrödinger operator with $(\Omega, \mathfrak{B}_\Omega, \mathbb{P}, \mathbb{Z})$ an ergodic dynamical system, where $\mathbb{E}(\ln(1 + |q^\omega(0)|)) < \infty$. Then the limits*

$$-\gamma(z) = \lim_{n\to\infty} \frac{1}{n} \ln |\frac{u_+(n)}{u_+(0)}|,$$

exist for almost every ω are independent of ω. γ is a non-negative sub-harmonic function in $\mathbb{C}$. Further

$$-\gamma(z) = \mathbb{E}\{\ln |m^+(z, 0)|\},$$

where

$$m^{\pm}(z, k) = -u_{\pm}(z, k \pm 1)/u_{\pm}(z, k), \quad k \in \mathbb{Z}^+.$$

We then have a proposition on the values of γ.

Proposition 5.1.7. *Let H^ω be as in Lemma 5.1.6; then,*

$$\mathbb{E}\{\ln(1 + \frac{\operatorname{Im}(z)}{\operatorname{Im}(m^+(z, 0))})\} = -2\mathbb{E}\{\ln |m^+(z, 0)|\} = -2\mathbb{E}\{\ln |m^+(z, k)|\}.$$

Proof: We take the imaginary parts in the relation (5.1.5), for m^+ divide by $\operatorname{Im}(m^+)$ to get, for any $k \in \mathbb{Z}$,

$$\begin{aligned} \ln(1 + \frac{\operatorname{Im}(z)}{Im(m^+(z, k+1))}) &= \ln(-\operatorname{Im}(m^+(z, k))^{-1}) - \ln(\operatorname{Im}(m^+(z, k+1)) \\ &= -\ln \operatorname{Im}(m^+(z, k+1)) + \ln(\operatorname{Im}(m^+(z, k))) \\ &\quad - \ln |m^+(z, k+1)|^2, \end{aligned} \tag{5.1.8}$$

using the identity $-\operatorname{Im}(a^{-1}) = \operatorname{Im}(a)/|a|^2$, for any complex number a. From the definition of m^+ we see that it is the Borel transform of a positive measure, so when z is in the upper half plane the function is bounded both above and below uniformly, as a function of ω, hence integrable with respect to $\mathbb{P}$. The same is true of the imaginary part of m^+. Therefore the quantities appearing in the above equality are integrable, so taking integrals with respect to $\mathbb{P}$ and using the invariance of the $\mathbb{P}$ under shifts, we get the proposition. $\square$

For the following we set $M^{\pm}(z,k) = \mathrm{Im}(m^{\pm}(z,k)) + \frac{1}{2}\mathrm{Im}(z)$.

Proposition 5.1.8. *The following inequality is valid with $z = E + i\epsilon$:*

$$\mathbb{E}\left[\frac{1}{M^+(z,k)} + \frac{1}{M^-(z,k)} - 4\mathrm{Im}(G(z,k,k))\right] \leq 4\left[\frac{\gamma(z)}{\epsilon} - \frac{\partial\gamma}{\partial y}(z)\right].$$

Proof: The proof of this theorem relies on the Thouless formula relating the density of states measure dn to the Lyapunov exponent, namely,

$$\gamma(z) = \int \ln|(x-z)|\, dn(x).$$

We define the density of states measure via its Borel transform as

$$\int \frac{1}{x-z} dn(x) = \mathbb{E}\left\{\langle\delta_o, (H^\omega - z)^{-1}\delta_o\rangle\right\} = \mathbb{E}\{G(z,0,0)\} = \mathbb{E}\{G(z,k,k)\}.$$

(In the literature the density of states is defined differently and the above relation is actually a theorem, but for our purposes here we can take it as a definition.) We delegate to the notes for references to proof of Thouless formula. Given these facts and the Relation (5.1.4), we have, with $z = E + i\epsilon$,

$$\frac{\partial\gamma}{\partial y}(E + i\epsilon) = \int \frac{\epsilon}{(x-E)^2 + \epsilon^2} dn(x);$$

therefore

$$\frac{\partial\gamma}{\partial y}(E + i\epsilon) = \mathbb{E}\{\mathrm{Im}(G(E + i\epsilon, k, k))\}.$$

The Proposition 5.1.7 together with the inequality $\ln(1+x) \geq x/(1+x)$ gives the bound for the terms

$$\mathbb{E}\left\{\frac{1}{M^{\pm}(z,k)}\right\} \leq 2\gamma(E + i\epsilon)/\epsilon, \tag{5.1.9}$$

completing the proof. $\square$

Finally we prove the theorem of Kotani which says that for ergodic discrete random Schrödinger operators the existence of absolutely continuous spectrum implies that they are reflectionless and hence deterministic.

Theorem 5.1.9. *Consider an ergodic discrete random Schrödinger operator $H^\omega, \omega \in \Omega$, with $(\Omega, \mathfrak{B}_\Omega, \mathbb{P}, \mathbb{Z})$ an ergodic dynamical system satisfying the conditions of Lemma 5.1.6. If the set $\{E : \gamma(E) = 0\}$ has positive Lebesgue measure, then the potential V^ω is deterministic.*

Proof: For the following proof we note that $G(z,k,k), m^{\pm}(z,k)$ are functions of ω for each fixed z,k and write their evaluations at the point ω as $G(z,k,k)(\omega), m^{\pm}(z,k)(\omega)$, etc.

Let S denote the set where $\gamma(E)$ is zero and also the boundary values $\frac{\partial\gamma}{\partial y}(E+i0)$ exist. These boundary values exist almost everywhere on $\mathbb{R}$ because $\frac{\partial\gamma}{\partial y}$ is the imaginary part of a Borel transform. Then for $E \in S$, we have, as ϵ goes to zero,

$$\lim_{\epsilon\to 0}\frac{\gamma(E+i\epsilon)-\gamma(E+i0)}{\epsilon} = \lim_{\epsilon\to 0}\frac{\gamma(E+i\epsilon)}{\epsilon} = \lim_{\epsilon\to 0}\frac{\partial}{\partial y}\gamma(E+i\epsilon).$$

the right-hand side of the inequality in Proposition 5.1.8 goes to zero. From this observation, using the Inequality (5.1.9), the fact that $m^{\pm}(z,\omega,k)$ are Borel transforms and Fatou's lemma, we see that for almost all ω the boundary values of $M^{\pm}(E+i0,\omega,k)$ exist and are finite. Together with the bound from the inequality in Proposition 5.1.8 it follows that

$$\mathbb{E}\left[\frac{1}{M^{+}(E+i0,k)}+\frac{1}{M^{-}(E+i0,k)}-4\mathrm{Im}(G(E+i0,k,k)^{-1})\right]=0.$$

Since $-\mathrm{Im}(G(E+i0,k,k)^{-1}) = M^{+}(E+i0,k)+M^{-}(E+i0,k)$, from the definition of these quantities and Relation (5.1.4), we have

$$\left[\frac{1}{M^{+}}+\frac{1}{M^{-}}-4(M^{+}+M^{-})|G|^2\right]=\left[\frac{1}{M^{+}}+\frac{1}{M^{-}}\right][1-4(M^{+}M^{-})|G|^2],$$

where we suppressed the arguments $E+i0$ and k for simplicity. This relation is rewritten as

$$\left[\frac{1}{M^{+}}+\frac{1}{M^{-}}\right][(\mathrm{Im}(G^{-1}))^2+(\mathrm{Re}(G^{-1}))^2-4(M^{+}M^{-})][|G|^2],$$

which simplifies to the product, by using the identity $(a+b)^2=(a-b)^2-4ab$,

$$\left(\frac{1}{M^{+}}+\frac{1}{M^{-}}\right)|G(E+i0,k,k)|^2\left[(M^{+}-M^{-})^2+(\mathrm{Re}(-G^{-1}))^2\right].$$

Therefore,

$$\mathbb{E}\left\{\left(\frac{1}{M^{+}}+\frac{1}{M^{-}}\right)|G(E+i0,k,k)|^2\left[(M^{+}-M^{-})^2+(\mathrm{Re}(-G^{-1}))^2\right]\right\}=0.$$

This immediately shows that for all $E \in S$, $\mathrm{Re}(G(E+i0,\omega,k,k)) = 0$ for almost every ω and every $k \in \mathbb{Z}$. Now Theorem 5.1.5 gives the stated determinacy of V^{ω}. $\square$

5.1.2 Aizenman–Molchanov Method

In this section we present the proof of localization using the Aizenman–Molchanov method. In this method one shows that for operators of the form $H^\omega = \Delta + V^\omega$ on $\ell^2(\mathbb{Z}^d)$, the resolvent kernels $\langle \delta_n, (H^\omega - E - i\epsilon)^{-1}\delta_m\rangle$ have a good decay in $|n - m|$ by showing that the averages $\mathbb{E}\{\|\langle \delta_n, (H^\omega - E - i\epsilon)^{-1}\delta_m\rangle\|^s\}$ have good decay in $|n-m|$ for some $0 < s < 1$.

The idea of controlling the above averages is the following observation. Suppose that A is a bounded selfadjoint operator on some separable Hilbert space $\mathcal{H}$ and suppose $\{\phi_k, k \in I\}$ is an orthonormal basis there, with I some countable set indexing the basis. Suppose E is a real number such that for some $0 < s < 1$,

$$|E| > \left(\sup_{k\in I} \sum_{l\in I} | < \phi_k, A\phi_l > |^s\right)^{1/s} ;$$

then we also have

$$\sum_{l\in I} |\langle \phi_k, (A - E)^{-1}\phi_l\rangle|^s < \infty.$$

The second idea is that if instead of a number E we have $\lambda q^\omega - E$, with q^ω an operator diagonal in the basis ϕ_k such that the diagonal entries $q^\omega(k)$ are real valued i.i.d. random variables. Then the same procedure can be adopted for controlling the sums of $|\langle \phi_k, (A + \lambda q^\omega - E)^{-1}\phi_l\rangle|^s$, by making use of Lemma 3.1.1.

Let us first state an abstract theorem to illustrate the idea.

Theorem 5.1.10. *Consider a bounded selfadjoint operator A on a separable Hilbert space $\mathcal{H}$ such that for some $0 < s < 1$,*

$$\|A\|_s = (\sup_{k\in I} \sum_{l\in I} |\langle \phi_k, A\phi_l\rangle|^s)^{1/s} < \infty,$$

for some orthonormal basis $\{\phi_k\}$. Then whenever E is in the resolvent of A satisfying $|E| > \|A\|_s$, for each k, we have

$$\sum_{l\in I} |\langle \phi_k, (A - E)^{-1}\phi_l\rangle|^s < \infty.$$

Proof: Let us for simplicity denote

$$G(k, l) = \langle \phi_k, (A - E)^{-1}\phi_l\rangle$$

and consider the identity,

$$EG(k, l) = -\delta_{k,l} + \sum_{m\in I} A(k, m)G(m, l),$$

where $A(k,m) = \langle \phi_k, A\phi_m \rangle$. Now we take the modulus of both sides raised to the power s take the factor E to the other side and get the inequality

$$|G(k,l)|^s \leq \frac{|\delta_{k,l}|^s}{|E|^s} + \sum_{m \in I} \frac{|A(k,m)|}{|E|^s}^s |G(m,l)|^s.$$

We set

$$K(k,l,E) = |A(k,l)/E|^s, K^{*0}(k,l) = \frac{\delta_{k,l}}{|E|^s}$$
$$K^{*m}(k,l,E) = \sum_{l_1,\cdots,l_m} K(k,l_1,E)K(l_1,l_2,E)\cdots K(l_m,l,E)$$

and repeat the above estimate $|k-l|/2$ times to get

$$|G(k,l)|^s \leq \sum_{r=0}^{\frac{|k-l|}{2}-1} K^{*r}(k,l) + \sum_{m \in I} K^{*\frac{|k-l|}{2}}(k,m)|G(m,l)|^s. \tag{5.1.10}$$

By assumption E is in the resolvent set of A, so $|G(m,l)|$ is uniformly bounded in m,l. Further the assumption that $\|A\|_s/E < 1$ implies that the right-hand side is bounded by

$$\sum_{r=0}^{\frac{|k-l|}{2}-1} K^{*r}(k,l) + C\alpha^{\frac{|k-l|}{2}},$$

where we set $\alpha = \|A\|_s/|E| < 1$. Using this inequality in the Inequality (5.1.10) and taking sums over l now gives the result. $\square$

To deal with random operators we denote by P_k the orthogonal projection onto the range of ϕ_k. Then for any real λ, consider

$$A^\omega_\lambda = A + \sum_{k \in I} \lambda q^\omega(k) P_k, \tag{5.1.11}$$

where $q^\omega(k), k \in I$ are independent random variables with distribution μ. We assume that μ is absolutely continuous with respect to Lebesgue measure in the following.

First we give a definition that picks out a class of μ to be considered in this theory.

Definition 5.1.11. A probability measure μ is said to be τ-regular for $0 < \tau \leq 1$ if for any $x \in \mathbb{R}$,

$$\mu\left((x-a, x+a)\right) \leq C|a|^\tau, \quad \text{any} \quad a > 0,$$

with C independent of x. The smallest number C with this property is denoted by $C_{\mu,\tau}$.

Remark 5.1.12. In the case when μ has a density p in $L^p(\mathbb{R})$, $p > 1$, by Hölder's inequality it follows that it is $\frac{p-1}{p}$-regular with $C_{\mu,\tau} \leq \|f\|_p$.

Then to study the measure class of the operators A_λ^ω for almost all ω, it is enough to study the measure class to which the spectral measure of A_λ^ω associated with each of the vectors ϕ_k belongs. We do this for the vector ϕ_0, the proof for the others is similar.

Theorem 5.1.13 (Aizenman–Molchanov). *Consider the family of operators as in Equation (5.1.11), with $q^\omega(k)$ distributed according to a τ-regular probability measure μ, $0 < \tau \leq 1$. Then there is a λ_0 such that for all $\lambda > \lambda_0$, the spectrum of A_λ^ω is pure point for almost every ω.*

Proof: Set $G^\omega(k,l,E) =< \phi_k, (A_\lambda^\omega - E - i0)^{-1}\phi_l >$, suppressing the indices λ, for this proof.

We show that for each $\lambda > \lambda_0$,

$$\sum_{l \in I} \mathbb{E}(|G^\omega(k,l,E)|^s) < \infty, \quad \text{for any} \quad E \in [a,b], \tag{5.1.12}$$

for some $0 < s < 1$. Since we are dealing with a sum of positive terms and the interval $[a,b]$ is finite, the above estimate together with Fubini implies that

$$\int_a^b dE \;\; \mathbb{E}\left(\sum_{l \in I} |G^\omega(k,l,E)|^s\right) < \infty. \tag{5.1.13}$$

This inequality immediately implies that for all $k \in I$ and for almost all ω,

$$\sum_{l \in I} |G^\omega(k,l,E)|^2 < \infty, \quad \text{for a.e.} \quad E \in [a,b]. \tag{5.1.14}$$

Such an estimate implies that for almost all ω we have

$$(DF_k^\omega)(E) = \int \frac{1}{|x - E + i0|^2} d\nu_k^\omega(x) < \infty, \quad \text{for a.e.} \quad E \in [a,b], \tag{5.1.15}$$

where

$$F_k^\omega(E) = \langle \phi_k, (A_\lambda^\omega - z)^{-1}\phi_k \rangle = \int \frac{1}{x - E + i0} d\nu_k^\omega(x). \tag{5.1.16}$$

Therefore if we consider A_λ^ω, for typical ω we would satisfy the condition of Theorem 3.1.7 and then the spectral measures of the rank one perturbations $A_\lambda^\omega + \delta P_k$ associated with the vector ϕ_k, will have only pure point part in $[a,b]$, if any, for almost all δ. This means that for almost all ω the spectral measure of A_λ^ω with respect to ϕ_k has only a pure point component, if any, in $[a,b]$. Since the arguments did not depend on specific $k \in I$, it follows that each of the spectral measures ν_k^ω and hence any total spectral measure of A_λ^ω

has only a pure point component, if any, in $[a, b]$ for almost every ω. Therefore proving the Inequality (5.1.12), which is shown in Lemma 5.1.15, completes the proof. $\square$

We shall need a crucial lemma that decouples randomness at one site from the others in the proof of Lemma 5.1.15. For this reason the Lemma 5.1.14 given next is often called the "decoupling lemma."

Lemma 5.1.14. *Suppose μ is an absolutely continuous probability measure that is also τ-regular. Then for any $0 < s < \tau$ and any $\beta \in \mathbb{C}$ we have*

$$\inf_{\alpha \in \mathbb{C}} \int d\mu(x) \quad \frac{|x-\alpha|^s}{|x-\beta|^s} \geq C_{s,\tau}(\mu) \int d\mu(x) \quad \frac{1}{|x-\beta|^s},$$

where $C_{s,\tau}(\mu) \to 0, s \to \tau$.

Proof: Suppose we prove the inequality

$$\begin{aligned} |x-\beta|^{-s} + |y-\beta|^{-s} \leq & \frac{|x-\alpha|^s}{|x-\beta|^s}\{|y-\alpha|^{-s} + |y-\beta|^{-s}\} \\ & + \frac{|y-\alpha|^s}{|y-\beta|^s}\{|x-\alpha|^{-s} + |x-\beta|^{-s}\}. \end{aligned} \tag{5.1.17}$$

Then by integration over the variables x, y using the symmetry of the expressions over x, y, we would get

$$\int d\mu(x) \quad \frac{1}{|x-\beta|^s} \leq \int d\mu(x) \frac{|x-\alpha|^s}{|x-\beta|^s} \int d\mu(y) \quad \left(\frac{1}{|y-\alpha|^s} + \frac{1}{|y-\beta|^s}\right).$$

This inequality now implies the lemma since

$$\begin{aligned} \frac{\int d\mu(x) \quad \frac{1}{|x-\beta|^s}}{\int d\mu(x) \frac{|x-\alpha|^s}{|x-\beta|^s}} & \leq \int d\mu(y) \quad \left(\frac{1}{|y-\alpha|^s} + \frac{1}{|y-\beta|^s}\right) \\ & \leq 2\left(t \quad \mu(\mathbb{R}) + \int_t^\infty dt_1 \quad \mu\{|x-\beta|^{-s} \geq t_1\}\right). \\ & \leq D_{s,\tau}. \end{aligned} \tag{5.1.18}$$

Here we used the fact that μ is τ-regular to estimate the second term as $\int_t^\infty C|t_1|^{-\tau/s} dt_1$ and then minimize over t. The lemma is then valid with $C_{s,\tau}(\mu) = 1/D_{s,\tau}$. Therefore we need to prove the inequality (5.1.17), which we multiply by $|(x-\beta)(y-\beta)|^s$, to get an inequality

$$(\frac{|x-\alpha|^s}{|y-\alpha|^s} - 1)|y-\beta|^s + (\frac{|y-\alpha|^s}{|x-\alpha|^s} - 1)|x-\beta|^s + |x-\alpha|^s + |y-\alpha|^s \geq 0,$$

which is equivalent to inequality (5.1.17). Since the above expression is symmetric in x and y, we may take $|y-\beta| \geq |x-\beta|$. We use the triangle inequality along with the estimate $(a+b+c)^s \leq a^s + b^s + c^s$ valid for positive number a, b, c and $0 < s < 1$ to get $|y-\beta|^s \leq |x-\beta|^s + |x-\alpha|^s + |y-\alpha|^s$. Then we obtain an equivalent version of the inequality (5.1.17) given by

$$\begin{aligned} &\frac{|x-\alpha|^s}{|y-\alpha|^s}|y-\beta|^s + \left(\frac{|y-\alpha|^s}{|x-\alpha|^s} - 2\right)|x-\beta|^s \\ &\geq \left(\frac{|x-\alpha|^s}{|y-\alpha|^s} + \frac{|y-\alpha|^s}{|x-\alpha|^s} - 2\right)|x-\beta|^s \\ &\geq 0. \end{aligned} \tag{5.1.19}$$

This form of the inequality is obvious since $t + t^{-1} - 2 \geq 0$ always for any positive t. □

Using the decoupling lemma we show that in the model considered in Equation (5.1.11) has pure point spectrum for large enough coupling constant.

Lemma 5.1.15. *Consider the family of operators as in Equation (5.1.11), with $q^\omega(k)$ distributed according to a τ-regular probability measure μ, $0 < \tau \leq 1$. Then there is a λ_0 such that for all $\lambda > \lambda_0$, $0 < s < \tau$,*

$$\sum_{n \in I} \mathbb{E}(|G^\omega(k, n, E + i0)|^s) < \infty,$$

for any $E \in [a, b]$.

Proof: Consider an interval $[a, b]$ with non-empty intersection with the spectrum of A_λ^ω. Then we shall show that the estimate in Equation (5.1.12) is valid. For this lemma we shall drop the index ω in $G^\omega(k, l, z)$. Then we consider the simple identity

$$(\lambda q(l) - E - i\epsilon)G(0, l, E + i\epsilon) = \delta_{0,l} - \sum_{k \in I} A(k, l)G(0, k, E + i\epsilon).$$

Set $G_l(k, l, z) = \langle \phi_k, (A_\lambda^{\cdot} - \lambda q(l)P_l - z)^{-1}\phi_l \rangle$. Then $G_l(k, l, z)$ and $q(l)$ are independent random variables.

The rank one perturbation formula given in Lemma 3.1.1 gives us the relation

$$G(0, l, E + i\epsilon) = \left(\frac{G_l(0, l, E + i\epsilon)}{G_l(l, l, E + i\epsilon)}\right) \frac{1}{\lambda q(l) + (G_l(l, l, E + i\epsilon))^{-1}}.$$

Set $B_{0,l}(E + i\epsilon) = \frac{G_l(0,l,E+i\epsilon)}{G_l(l,l,E+i\epsilon)}$ and $D_l(E + i\epsilon) = G_l(l, l, E + i\epsilon)^{-1}$. Using this in the equation above we see that

$$(\lambda q^\omega(l) - E - i\epsilon)\frac{B_{0,l}(E + i\epsilon)}{\lambda q(l) + D_l(E + i\epsilon)} = \delta_{0,l} - \sum_{k \in I} A(k, l)G(0, k, E + i\epsilon).$$

Then taking absolute values on both sides, raising to power s taking averages over ω and simplifying we get

$$\mathbb{E}(\frac{|\lambda q(l) - E - i\epsilon|^s |B_{0,l}(E+i\epsilon)|^s}{|\lambda q(l) + D_l(E+i\epsilon)|^s}) \leq \delta_{0,l} + \sum_{k\in I} |A(k,l)|^s \mathbb{E}(|G(0,k,E+i\epsilon)|^s).$$

We denote $\mathbb{E}(|G(m,k,E+i\epsilon)|^s) = K(m,k)(E+i\epsilon)$; then using the independence of $D_l(E+i\epsilon)$ and $q(l)$, the above inequality becomes

$$\mathbb{E}\int d\mu(x) \frac{|\lambda x - E - i\epsilon|^s |B_{0,l}(E+i\epsilon)|^s}{|\lambda x + D_l(E+i\epsilon)|^s} \leq \delta_{0,l} + \sum_{k\in I} |A(k,l)|^s K(0,k)(E+i\epsilon),$$

which using the Lemma (5.1.14) and some algebra is seen to be

$$|\lambda|^s C_{s,\tau}(\mu) K(0,l)(E+i\epsilon) \leq \delta_{0,l} + \sum_{k\in I} |A(k,l)|^s K(0,k)(E+i\epsilon).$$

Therefore we take λ_0 so that $\|A\|_s/|\lambda_0| C_{s,\tau}(\mu)^{1/s} < 1$, in which case it is also true that $\|A\|_s/|\lambda| C_{s,\tau}(\mu)^{1/s} <$ for any $\lambda \geq \lambda_0$. Therefore Theorem 5.1.10 is valid for all such λ, provided $K(m,n)(E+i\epsilon)$ is uniformly bounded in m and n and ϵ point wise in E. This we show in the lemma below. □

Lemma 5.1.16. *Consider the operators A_λ^ω with $q(n)$ and its distribution μ satisfying the conditions in Lemma 5.1.15. Then there is a constant C(λ, s), independent of E and $\epsilon > 0$, such that. for all $n, m \in I$, $\epsilon > 0$ and $E \in \mathbb{R}$ the estimate*

$$\mathbb{E}(|G^\omega(n,m,E+i\epsilon)|^s) \leq C(\lambda, s) < \infty,$$

is valid.

Proof: In this lemma also we drop the superscript ω in $G^\omega(n,m,z)$. We take $n = 0$ and prove the lemma, the case of general n is similar. Using the rank one perturbation formula of Lemma 3.1.1 we see that

$$G(0,0,E+i\epsilon) = \frac{1}{\lambda(q(0) - (\lambda^{-1} D_0(E+i\epsilon))},$$

using the notations of the previous theorem. Therefore taking averages, since $D_0(E+i\epsilon)$ is independent of the random variable $q(0)$, we obtain

$$\begin{aligned} \mathbb{E}(|G(0,0,E+i\epsilon)|^s) &= \int_{\mathbb{R}} d\mu(x) \frac{1}{|\lambda|^s |x - (\frac{D_0(E+i\epsilon)}{\lambda})|^s} \\ &\leq \frac{1}{|\lambda|^s} \sup_{a\in\mathbb{R}} \int d\mu(x) \ \ |x-a|^{-s} < \infty. \end{aligned} \tag{5.1.20}$$

The integral was estimated as in the inequality 5.1.18.

The proof of the estimate for $G(0,l,E+i\epsilon)$ when l is not equal to 0 is given below under the further assumption that the density of μ is bounded and

$\int d\mu(x)|x|^\tau < \infty$. The proof without this assumption is found in [6], Theorem II.1.

Consider distinct sites $m \neq n$ fixed in the rest of the proof and consider $G(m,n,z)$. Let $H_1 = H - \lambda q(m) - \lambda q(n)$ in the following proof. Let R denote the orthogonal projection onto the two-dimensional subspace generated by ϕ_m, ϕ_n. Then one has the rank two formula

$$\left[R(H-z)^{-1}R\right]^{-1} = \left(R(H_1-z)^{-1}R\right)^{-1} + \begin{pmatrix} \lambda q(n) & 0 \\ 0 & \lambda q(m) \end{pmatrix},$$

where the two-dimensional matrix $(R(H_1-z)^{-1}R)^{-1}$ has the form

$$\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \gamma \end{pmatrix}, \quad \mathrm{Im}\ (\alpha),\ \mathrm{Im}\ (\gamma) \geq 0 \ \text{ when } \mathrm{Im}\ (z) > 0.$$

Using these facts and the formula

$$\left(A^{-1}\right)_{ij} = \frac{-A_{ij}}{\det(A)},$$

valid for an invertible two-dimensional matrix we see that

$$(H-z)^{-1}(m,n) = \frac{-\overline{\beta}}{(\lambda q(m)+\alpha)(\lambda q(n)+\gamma) - |\beta|^2}, \quad \mathrm{Im}\ (\alpha), \mathrm{Im}\ (\gamma) \geq 0. \tag{5.1.21}$$

Here the quantities α, β, γ are random variables independent of $q(m)$ and $q(n)$ and hence will be treated as constants while integrating with respect to $q(m)$ and $q(n)$ which we call x and y, respectively. Then we have to estimate the quantity

$$\mathbb{E}|G(m,n,z)|^s = \mathbb{E}\int d\mu(x)\int d\mu(y)\left|\frac{-\overline{\beta}}{\lambda^2(x+\frac{\alpha}{\lambda})(y+\frac{\gamma}{\lambda}) - |\beta|^2}\right|^s. \tag{5.1.22}$$

The idea of estimating this integral is the following: the factor β from the numerator is controlled by a factor of $|\beta|$ occurring in the denominator, while the rest of the terms are integrated in taking the average in view of the absolute continuity of the measure μ. But we need to take care of some technical details for this. Let $\alpha/\lambda = \alpha_1 + i\alpha_2, \gamma/\lambda = \gamma_1 + i\gamma_2$. Then we have to estimate the integral

$$\begin{aligned} I &= \int d\mu(x)\int d\mu(y)\left|\frac{-\overline{\beta}}{\lambda^2(x+\alpha_1+i\alpha_2)(y+\gamma_1+i\gamma_2) - |\beta|^2}\right|^s \\ &\leq \int d\mu(x)\int d\mu(y)\left|\frac{|\beta|}{\lambda^2(x+\alpha_1)(y+\gamma_1) - |\beta|^2 - \alpha_2\gamma_2}\right|^s. \end{aligned} \tag{5.1.23}$$

In the above we split the region of integration into two pieces:

$$\begin{aligned} S_1 &= \{(x,y) : 2|\lambda^2(x+\alpha_1)(y+\gamma_1)| \leq \beta^2 + \alpha_2\gamma_2\}, \\ S_2 &= \{(x,y) : 2|\lambda^2(x+\alpha_1)(y+\gamma_1)| > \beta^2 + \alpha_2\gamma_2\}. \end{aligned}$$

Using the fact that $\alpha_2\gamma_2 > 0$, the integrand in the above integral is seen to be bounded on S_1 by the function

$$\frac{2^s}{|\lambda|^s |x+\alpha_1|^{\frac{s}{2}} |y+\gamma_1|^{\frac{s}{2}}}$$

and on S_2 by the function

$$\frac{\sqrt{2}^s |y+\gamma_1|^{\frac{s}{2}}}{|\lambda|^s |x+\alpha_1|^{\frac{s}{2}} \left|\left(y+\gamma_1 - \frac{\beta^2+\alpha_2\gamma_2}{\lambda^2(x+\alpha_2)}\right)\right|^s}.$$

Therefore writing the integral on these two sets and using these bounds for the integrands we obtain

$$\begin{aligned}
&\int d\mu(x) \int d\mu(y) \left|\frac{|\beta|}{\lambda^2(x+\alpha_1)(y+\gamma_1) - |\beta|^2 - \alpha_2\gamma_2}\right|^s \\
&\leq \int d\mu(x) d\mu(y) \frac{2^s}{|\lambda|^s |x+\alpha_1|^{\frac{s}{2}} |y+\gamma_1|^{\frac{s}{2}}} \\
&+ \quad \sqrt{2}^s |\lambda|^{-s} \int d\mu(x) \frac{1}{|x+\alpha_1|^{\frac{s}{2}}} \int d\mu(y) \frac{|y+\gamma_1|^{\frac{s}{2}}}{\left|y+\gamma_1 - \frac{\beta^2+\alpha_2\gamma_2}{\lambda^2(x+\alpha_2)}\right|^s}.
\end{aligned} \tag{5.1.24}$$

Now using the properties of μ, the first term on the right-hand side is bounded by $C|\lambda|^{-s}$ as in the inequality (5.1.18). As for the second term, we first use the assumptions on the measure μ that it has bounded density and a τ moment together with the Lemma 5.1.17 to get rid of the factor $|y+\gamma|$ in the numerator of the y integral. Then we see the boundedness of the resulting integral as before, for example as in the inequality (5.1.18). □

We usually take the indexing set to be one of $\mathbb{Z}^d$ so it makes sense to talk about $|n|$ in such cases. The above procedure is valid if λ is a function of $n \in I$ with $\lambda(n) \to \infty$ as $|n| \to \infty$ also. In the cases when λ is small or when $\lambda(n) \to 0$ as $|n| \to \infty$ there is an alternative procedure which works at the edges of the spectrum.

We start with a lemma to present this procedure.

Lemma 5.1.17 (Aizenman). *Let μ be an absolutely continuous probability measure whose density f satisfies $\int_{\mathbb{R}} dx |f(x)|^{1+q} = Q < \infty$ for some $q > 0$. Let $0 < \tau \leq 1$ and suppose $B = \int_{\mathbb{R}} d\mu(x)\ |x|^\tau < \infty$. Then for any*

$$\kappa < \left[1 + \frac{2}{\tau} + \frac{1}{q}\right]^{-1}$$

we have

$$\int_{\mathbb{R}} d\mu(x) \ \frac{|x|^\kappa}{|x-\alpha|^\kappa} < K_\kappa \int_{\mathbb{R}} d\mu(x) \ \frac{1}{|x-\alpha|^\kappa}, \quad \textit{for all} \quad \alpha \in \mathbb{C},$$

with K_κ given by

$$K_\kappa = B^{\frac{\kappa}{\tau}}(2^{1+2\kappa}+4)\left[B^{1-\frac{\kappa}{\tau}} + B^{\frac{\kappa}{\tau}}\ C(Q,\kappa,\tau,q)^{\frac{\tau-2\kappa}{\tau}}\right] < \infty.$$

Remark 5.1.18. (i) We see from the explicit form of the constant K_κ that the moment B can be made sufficiently small by the choice of μ even when its support is large. This will ensure that in some models of random operators, the Simon–Wolff criterion is valid in a part of the spectrum. This is the reason for our writing K_κ in this form.

(ii) In the above we can set $q = \infty$ in which case $Q = \|f\|_\infty$.

Proof: The strategy is to consider the ratio

$$\frac{\int_{\mathbb{R}} d\mu(x)\ \frac{|x|^\kappa}{|x-\alpha|^\kappa}}{\int_{\mathbb{R}} d\mu(x)\ \frac{1}{|x-\alpha|^\kappa}}$$

and to find a uniform bound in α. This is done by obtaining an upper bound on the numerator in terms of α and a lower bound for the denominator with the same asymptotic α, such that the quotient is estimated by K_κ, which is independent of α.

Note first that B finite and $\kappa < \tau$ implies that $|x-\alpha|^\kappa$ is integrable even if α is purely real and we have

$$\int_a^b f(x)dx \le Q^{\frac{1}{1+q}}|b-a|^{\frac{q}{1+q}} \tag{5.1.25}$$

by the Hölder inequality. Hence

$$\begin{aligned}\int d\mu(x)\ \frac{1}{|x-\alpha|^\kappa} &\le 1 + \int_1^\infty dt\ \mu(\{x : \frac{1}{|x-\alpha|^\kappa} \ge t\}) \\ &\le 1 + \frac{\kappa(2^q Q)^{\frac{1}{1+q}}}{\frac{q}{1+q} - \kappa} \\ &\equiv C(Q,\kappa,\tau,q),\end{aligned} \tag{5.1.26}$$

where the integral is estimated using the estimate in equation (5.1.25). Consider the region $|\alpha| > (2B)^{\frac{1}{\tau}}$:

We then estimate for fixed α the contributions from the regions $|x| \le |\alpha|/2$ and $|x| > |\alpha|/2$ to obtain

$$\begin{aligned}\int d\mu(x)\ \frac{|x|^\kappa}{|x-\alpha|^\kappa} &\le \frac{2^\kappa}{|\alpha|^\kappa}(\int d\mu(x)\ |x|^\kappa + \int d\mu(x)\ \frac{|x|^{2\kappa}}{|x-\alpha|^\kappa}) \\ &\le \frac{2^\kappa}{|\alpha|^\kappa}(B + B^{\frac{2\kappa}{\tau}}\ C(Q,\kappa,\tau,q)^{\frac{\tau-2\kappa}{\tau}}),\end{aligned} \tag{5.1.27}$$

with κ chosen so that $\kappa/(1-2\kappa/\tau) < q/(1+q)$. (Here we have explicitly calculated the p occurring in the lemma of Aizenman in terms of κ and τ).

For a fixed τ and q this condition is satisfied whenever κ satisfies the inequality stated in the lemma.

The lower bounds on $\int d\mu(x) 1/|x-\alpha|^\kappa$ is obtained first by noting that $B < \infty$ implies

$$\mu(\{x : |x|^\tau > (2B)\}) \leq \frac{1}{2}.$$

Since $|\alpha| > (2B)^{\frac{1}{\tau}}$, we have the trivial estimate

$$\begin{aligned}\int d\mu(x)\frac{1}{|x-\alpha|^\kappa} &\geq \int_{|x|>(2B)^{\frac{1}{\tau}}} d\mu(x)\frac{1}{|x-\alpha|^\kappa} + \int_{|x|\leq(2B)^{\frac{1}{\tau}}} d\mu(x)\frac{1}{|x-\alpha|^\kappa} \\ &\geq \int_{|x|\leq(2B)^{\frac{1}{\tau}}} d\mu(x)\frac{1}{|x-\alpha|^\kappa} \\ &\geq \frac{1}{2(|\alpha| + (2B)^{\frac{1}{\tau}})^\kappa}.\end{aligned} \tag{5.1.28}$$

Putting the inequalities in (5.1.27) and (5.1.28) together we obtain (remembering that $|\alpha| > (2B)^{\frac{1}{\tau}}$)

$$\frac{\int_{\mathbb{R}} d\mu(x) \;\; \frac{|x|^\kappa}{|x-\alpha|^\kappa}}{\int_{\mathbb{R}} d\mu(x) \;\; \frac{1}{|x-\alpha|^\kappa}} \leq 2^{1+2\kappa} B^{\frac{\kappa}{\tau}} \left[B^{1-\frac{\kappa}{\tau}} + B^{\frac{\kappa}{\tau}} \; C(Q,\kappa,\tau,q)^{\frac{\tau-2\kappa}{\tau}}\right]. \tag{5.1.29}$$

Consider the region $|\alpha| < (2B)^{\frac{1}{\tau}}$.

Estimating as in Equation (5.1.27) but now splitting the region as $|x| \leq (2B)^{\frac{1}{\tau}}$ and $|x| > (2B)^{\frac{1}{\tau}}$, we obtain the analogue of the estimate in Equation (5.1.27), in this region of α as

$$\begin{aligned}\int d\mu(x) \; \frac{|x|^\kappa}{|x-\alpha|^\kappa} &\leq \frac{1}{(2B)^{\frac{1}{\tau}}}\left(\int d\mu(x) \; |x|^\kappa + \int d\mu(x) \; \frac{|x|^{2\kappa}}{|x-\alpha|^\kappa}\right) \\ &\leq \frac{1}{(2B)^{\frac{1}{\tau}}}(B + B^{\frac{2\kappa}{\tau}} \; C(Q,\kappa,\tau,q)^{\frac{\tau-2\kappa}{\tau}}).\end{aligned} \tag{5.1.30}$$

Similarly the estimate for the denominator term is done as in Equation (5.1.28):

$$\begin{aligned}\int d\mu(x)\frac{1}{|x-\alpha|^\kappa} &\geq \int_{|x|>(2B)^{\frac{1}{\tau}}} d\mu(x)\frac{1}{|x-\alpha|^\kappa} + \int_{|x|\leq(2B)^{\frac{1}{\tau}}} d\mu(x)\frac{1}{|x-\alpha|^\kappa} \\ &\geq \int_{|x|\leq(2B)^{\frac{1}{\tau}}} d\mu(x)\frac{1}{|x-\alpha|^\kappa} \\ &\geq \frac{1}{2((2B)^{\frac{1}{\tau}} + (2B)^{\frac{1}{\tau}})} \\ &= \frac{1}{4(2B)^{\frac{1}{\tau}}}.\end{aligned} \tag{5.1.31}$$

Using the above two inequalities we obtain the estimate,

$$\frac{\int_{\mathbb{R}} d\mu(x) \ \frac{|x|^\kappa}{|x-\alpha|^\kappa}}{\int_{\mathbb{R}} d\mu(x) \ \frac{1}{|x-\alpha|^\kappa}} \leq 4\left[B^{1-\frac{\kappa}{\tau}} + B^{\frac{\kappa}{\tau}} \ C(Q,\kappa,\tau,q)^{\frac{\tau-2\kappa}{\tau}}\right], \tag{5.1.32}$$

when $|\alpha| \leq (2B)^{\frac{1}{\tau}}$. Using the inequalities (5.1.29) and (5.1.31) obtained for these two regions of values of α we finally get

$$\frac{\int_{\mathbb{R}} d\mu(x) \ \frac{|x|^\kappa}{|x-\alpha|^\kappa}}{\int_{\mathbb{R}} d\mu(x) \ \frac{1}{|x-\alpha|^\kappa}} \leq B^{\frac{\kappa}{\tau}}(2^{1+2\kappa}+4)\left[B^{1-\frac{\kappa}{\tau}} + B^{\frac{\kappa}{\tau}} \ C(Q,\kappa,\tau,q)^{\frac{\tau-2\kappa}{\tau}}\right], \tag{5.1.33}$$

for any $\alpha \in \mathbb{R}$. □

In the following we consider $H^\omega = \Delta + V^\omega$ on $\ell^2(\mathbb{Z}^d)$, with $V^\omega(n) = a_n q^\omega(n)$. We assume that $q(n)$ are i.i.d. random variables while a_n is a non-negative sequence going to zero as $n \to \infty$. We take the constants K_κ given in Lemma 5.1.17; then the theorem is

Theorem 5.1.19. *Suppose H^ω is as above with a probability measure μ such that $\int_{\mathbb{R}} |x|^\tau d\mu(x) < \infty$. Suppose further that μ satisfies the assumptions of Lemma 5.1.17, for some $p > 1$ and $\tau = 1$. Consider a κ such that*

$$0 < \kappa < \left[1 + \frac{2}{\tau} + \frac{1}{p-1}\right]^{-1}$$

and suppose that for all $E \in (a,b)$,

$$K_\kappa \ (\sup_{l\in\mathbb{Z}^d} |a_l|)\left(\sum_{n\in\mathbb{Z}^d} |G^0(k,n,E+i0)|^\kappa\right) < 1.$$

Then for almost all ω,

$$\sigma_{ac}(H^\omega) \cap (a,b) = \emptyset.$$

Proof: By the second resolvent equation we have

$$G^\omega(n,m,z) = G^0(n,m,z) - \sum_{l\in\mathbb{Z}^d} G^\omega(n,l,z)V^\omega(l)G^0(l,m,z). \tag{5.1.34}$$

We denote by

$$G_l^\omega(n,m,z) =< \delta_n, (H^\omega - V^\omega(l)P_l - z)^{-1}\delta_m >,$$

where P_l is the orthogonal projection onto the subspace generated by δ_l. Then using the rank one formula of Lemma 3.1.1

$$G^\omega(n,l,z) = \frac{\frac{G_l^\omega(n,l,z)}{G_l^\omega(l,l,z)}}{V^\omega(l) + G_l^\omega(l,l,z)^{-1}}$$

whose proof is again using the resolvent equation. We see that equation (5.1.34) can be rewritten as

$$G^{\omega}(n,m,z) = G^0(n,m,z) - \sum_{l\in\mathbb{Z}^d} \frac{\frac{G_l^{\omega}(n,l,z)}{G_l^{\omega}(l,l,z)}}{V^{\omega}(l) + G_l^{\omega}(l,l,z)^{-1}} V^{\omega}(l) G^0(l,m,z). \tag{5.1.35}$$

Raising both the sides to power s where $0 < s < 1$, we get

$$|G^{\omega}(n,m,z)|^s \leq |G^0(n,m,z)|^s + \sum_{l\in\mathbb{Z}^d} \left| \frac{\frac{G_l^{\omega}(n,l,z)}{G_l^{\omega}(l,l,z)}}{V^{\omega}(l) + (G_l^{\omega}(l,l,z)^{-1}} \right|^s |V^{\omega}(l)|^s \left|G^0(l,m,z)\right|^s. \tag{5.1.36}$$

Now observing that G_l is independent of the random variable $V^{\omega}(l)$, we see that

$$\mathbb{E}(|G^{\omega}(n,m,z)|^s) \leq |G^0(n,m,z)|^s + \sum_{l\in\mathbb{Z}^d} \mathbb{E}\left(\left| \frac{\frac{G_l^{\omega}(n,l,z)}{G_l^{\omega}(l,l,z)}}{V^{\omega}(l) + (G_l^{\omega}(l,l,z)^{-1}} \right|^s |V^{\omega}(l)|^s \right) \left|G^0(l,m,z)\right|^s. \tag{5.1.37}$$

This becomes, integrating with respect to the variable $q^{\omega}(l)$ and remembering that $V^{\omega}(l) = a_l q^{\omega}(l)$,

$$\begin{aligned} \mathbb{E}\left(|G^{\omega}(n,m,z)|^s\right) = |G^0(n,m,z)|^s &+ \sum_{l\in\mathbb{Z}^d} \mathbb{E}\left(\left| \frac{G_l^{\omega}(n,l,z)}{G_l^{\omega}(l,l,z)} \right|^s \right. \\ &\times \left. \left(\int d\mu(x) \; \frac{|x|^s}{|x + a_l^{-1} G_l^{\omega}(l,l,z)^{-1}|^s} \right) \right) \left|G^0(l,m,z)\right|^s \end{aligned} \tag{5.1.38}$$

which, using the Lemma 5.1.17, yields

$$\begin{aligned} \mathbb{E}(|G^{\omega}(n,m,z)|^s) \leq |G^0(n,m,z)|^s &+ \sum_{l\in\mathbb{Z}^d} K_s \mathbb{E}\left(\left| \frac{Gl(n,l,z)}{G_l^{\omega}(l,l,z)} \right|^s \right) \\ &\times \left(\int d\mu(x) \; \frac{1}{|x + a_l^{-1} G_l^{\omega}(l,l,z)^{-1}|^s} \right) \left|G^0(l,m,z)\right|^s, \end{aligned} \tag{5.1.39}$$

where K_s is the constant appearing in Lemma 5.1.17 with κ set equal to s. We take $K = (\sup_n |a_n|^s)K_s$, and rewrite the above equation to obtain

$$\mathbb{E}\left(|G^{\omega}(n,m,z)|^s\right) \leq |G^0(n,m,z)|^s + \sum_{l\in\mathbb{Z}^d} K\mathbb{E}(|(G^{\omega}(n,l,z)|^s|G^0(l,m,z)|^s. \tag{5.1.40}$$

We now sum both sides over m, set

$$I = \sum_{m\in\mathbb{Z}^d} \mathbb{E}(|G^{\omega}(n,m,z)|^s)$$

and obtain the inequality

$$I \leq \sum_{m\in\mathbb{Z}^d} \left|G^0(n,m,z)\right|^s + \sup_{l\in\mathbb{Z}^d} \sum_{m\in\mathbb{Z}^d} KI \left|G^0(l,m,z)\right|^s.$$

Therefore, using the assumption that

$$K \sup_{l\in\mathbb{Z}^d} \sum_{m\in\mathbb{Z}^d} \left|G^0(l,m,z)\right|^s < 1, \quad E \in (a,b), \tag{5.1.41}$$

we obtain

$$\int_a^b dE \sum_{m\in\mathbb{Z}^d} \mathbb{E}\left(|G^{\omega}(n,m,E+i0)|^s\right) < \infty,$$

by an application of Fatou's lemma. This implies that for almost all $E \in [a,b]$ and almost all ω, we have the finiteness of

$$\sum_{m\in\mathbb{Z}^d} |G^{\omega}(n,m,E+i0)|^2 < \infty,$$

satisfying the condition of Theorem 3.1.7.

Therefore arguing as in the proof of Theorem 5.1.13 we see that

$$\nu_n^{\omega}(\cdot) = \langle \delta_n, E_{H^{\omega}}(\cdot)\delta_n \rangle$$

are pure point in $[a,b]$ almost every ω. This happens for all n; hence the total spectral measure of H^{ω} itself is pure point in $[a,b]$ for almost all ω. □

5.1.3 Bethe Lattice

In this section we consider a model with a random potential on the Bethe lattice Γ_K of connectivity $K+1$ to illustrate a technique that was used to show that a part of the spectrum is purely absolutely continuous for almost every realization of the random potential. In this model the random potential is stationary (with respect to some automorphisms of Γ_K to itself). The Bethe lattice is a tree (infinite connected graph with no loops such that any two vertices are connected by a unique path) with $K+1$ edges incident on each vertex. The distance $d(i,j)$ between two vertices i,j is taken to be the length of the shortest path joining them. (For this purpose, the length of an edge is taken to be 1). Therefore in $\mathit{\Gamma}_K$, each vertex has $K+1$ neighbours (vertices at a distance 1).

The vertices in $\mathit{\Gamma}_K$ can be labelled as follows. An arbitrary vertex is declared as the origin and it is given the coordinate (0). Every other point is uniquely identified by its distance to the origin together with the vertices that fall on the unique path connecting it to the origin. Therefore a point at a distance l from the origin has the coordinate $(a_1, a_2, \ldots, a_l)$ with $1 \leq a_1 \leq K+1$ and $1 \leq a_i \leq K, \quad i > 1$. Thus, the points at a distance 1 are labelled $(1), (2), (3), \ldots, (K+1)$, while those at a distance 2 from the origin going through a_1 are labelled $(a_1, 1), \ldots, (a_1, K)$, since there are exactly K of them. Given this labelling, the following transformations τ_1, τ_2 give automorphisms of $\mathit{\Gamma}_K$.

$$\begin{aligned} \tau_1(0) = (1), \quad \tau_1(a_1, \ldots, a_l) &= (1, a_1, \ldots, a_l), \quad 1 \leq a_1 \leq K \\ \tau_1(K+1, a_2, \ldots, a_l) &= (a_2 + 1, a_3, \ldots, a_l), \\ \tau_2(a_1, \ldots, a_l) &= (b_1, \ldots, b_l), \quad b_i = a_i + 1 \mod (K+1), \ \forall \ i. \end{aligned} \tag{5.1.42}$$

The advantage of these automorphisms is that given any vertex $(a_1, \ldots, a_l)$ in $\mathit{\Gamma}_K$ at a distance l from the origin, there is a unique non-negative integer m such that $(a_1, \ldots, a_l) = \tau_2^m \tau_1^l(0)$. (Indeed one can verify that $0 \leq m < (K+1)K^{(l-1)}$).

In view of this we can relabel the points of $\mathit{\Gamma}_K$ as

$$\mathit{\Gamma}_K = \{(0,0)\} \cup \{(j, m) : j \in \mathbb{N}, \quad 0 \leq m < (K+1)K^{(j-1)}\}.$$

We need to have a bit of notation here for later use. Given any site $\alpha \in \mathit{\Gamma}_K$, denote by $\mathit{\Gamma}_{K,\alpha}$ the graph obtained by removing the vertex α and all the edges connecting α to its nearest neighbours (i.e., all β with $d(\alpha, \beta) = 1$). Then the resulting set is

$$\{\alpha\} \bigcup_{\beta : d(\alpha,\beta)=1} \mathit{\Gamma}_{K,\alpha,\beta},$$

where $\mathit{\Gamma}_{K,\alpha,\beta}$ is the branch of the graph passing through β obtained by removing the edge connecting β to α. This branch has all the vertices with connectivity $K+1$ except the vertex β which has connectivity K.

Consider the space $\ell^2(\mathit{\Gamma}_K)$. Note that since the number of points at a distance j grows exponentially in j, the ℓ^2-sequences should decay exponentially fast to zero as a function of j.

Consider the Laplacians on this Hilbert space given by

$$(\mathit{\Delta}_\mathit{\Gamma} u)(\beta') = \sum_{\gamma \in \mathit{\Gamma} : d(\beta', \gamma) = 1} u(\gamma),$$

where the set $\mathit{\Gamma}$ is one of $\mathit{\Gamma}_K, \mathit{\Gamma}_{K,\alpha}$ or $\mathit{\Gamma}_{K,\alpha,\beta}$. The operator $\mathit{\Delta}_\mathit{\Gamma}$ for one of the above sets is evidently bounded and symmetric and hence also selfadjoint.

It is clear from their definition that for any α, the $\Gamma_{K,\alpha,\beta}$ for different β are isomorphic as graphs.

This isomorphism implies that $\Delta_{\Gamma_{K,\alpha,\beta}}$ and $\Delta_{\Gamma_{K,\alpha,\beta'}}$ are unitarily equivalent as operators on the respective ℓ^2-spaces..

Let δ_α denote the element in $\ell^2(\Gamma_K) = \ell^2(\{\alpha\}) \bigoplus_{\beta:d(\alpha,\beta)=1} \ell^2(\Gamma_{K,\alpha,\beta})$ which is $\delta_\alpha(\beta) = \delta_{\alpha\beta}$, the latter being the Kronecker delta. Then the collection $\{\delta_\alpha, \alpha \in \Gamma_K\}$ forms an orthonormal basis for $\ell^2(\Gamma_K)$.

In the following we identify the spectrum and the structure of the matrix elements of the resolvent of Δ_Γ.

We suppress the dependence on K in the notation for the resolvent kernels and denote for any $z \in \mathbb{C}^+$,

$$\begin{aligned} R_{\alpha,\beta}(\gamma,\gamma',z) &= \langle \delta_\gamma, (\Delta_{\Gamma_{K,\alpha,\beta}} - z)^{-1}\delta_{\gamma'}\rangle, \\ R_\alpha(\gamma,\gamma',z) &= \langle \delta_\gamma, (\Delta_{\Gamma_{K,\alpha}} - z)^{-1}\delta_{\gamma'}\rangle, \\ R(\gamma,\gamma',z) &= \langle \delta_w, (\Delta_{\Gamma_K} - z)^{-1}\delta_{\gamma'}\rangle. \end{aligned} \tag{5.1.43}$$

The above relations are understood to be in the sense of boundary values from $\mathbb{C}^+$ when z is replaced by a real parameter E.

Given these notations the following is used to identify the spectrum and the spectral type of Δ_{Γ_K}.

Lemma 5.1.20. *Consider an $\alpha \in \Gamma_K$ and β such that $d(\alpha,\beta) = 1$. Then the resolvent kernels $R(\alpha,\alpha,z)$ are given by the*

1. *$R(\alpha,\alpha,z) = -1/(z + (K+1)R_{\alpha,\beta}(\beta,\beta,z))$ and*
2. *$R_{\alpha,\beta}(\beta,\beta,z) = -\frac{z}{2K} + \frac{1}{K}\sqrt{(\frac{z}{2})^2 - K}$,*

where the square root in (2) is chosen so that $R_{\alpha,\beta}(\beta,\beta,z) \in \mathbb{C}^+$ when $z \in \mathbb{C}^+$.

Proof: (1) We already noticed that for a given $\alpha \in \Gamma_K$, the graphs $\Gamma_{K,\alpha,\beta}$ are isomorphic for different β with $d(\beta,\alpha) = 1$ and also for different pairs (α,β) and (γ,γ') with $d(\alpha,\beta) = 1, d(\gamma,\gamma') = 1$, the corresponding $\Gamma_{K,\alpha,\beta}$ and $\Gamma_{K,\gamma,\gamma'}$ are isomorphic. This isomorphism induces an isometric isomorphism of the different $\ell^2(\Gamma_{K,\alpha,\beta})$, with the bases $\{\delta_\gamma,\ \gamma \in \Gamma_{K,\alpha,\beta}\}$, getting mapped bijectively onto each other. Therefore the resolvent kernels satisfy

$$R_{\alpha,\beta}(\beta,\beta,z) = R_{\gamma,\gamma'}(\gamma',\gamma',z) \tag{5.1.44}$$

for any β, γ' at unit distance from α, γ, respectively. This observation together with the second resolvent equation implies that

$$\begin{aligned} \langle \delta_\alpha, (\Delta_{\Gamma_K} - z)^{-1}\delta_\beta\rangle &= \langle \delta_\alpha, (\Delta_{\Gamma_{K,\alpha}} - z)^{-1}\delta_\beta\rangle \\ &\quad - \langle \delta_\alpha, (\Delta_{\Gamma_K} - z)^{-1}\delta_\alpha\rangle\langle \delta_\beta, (\Delta_{\Gamma_{K,\alpha}} - z)^{-1}\delta_\beta\rangle \\ &= R(\alpha,\beta,z) - R(\alpha,\alpha,z)R_{\alpha,\beta}(\beta,\beta,z). \end{aligned}$$

The same resolvent equation also gives

$$\begin{aligned}R(\alpha,\alpha,z) &= \langle \delta_\alpha, (\Delta_{\Gamma_K} - z)^{-1}\delta_\alpha\rangle \\ &= \langle \delta_\alpha, (\Delta_{\Gamma_{K,\alpha}} - z)^{-1}\delta_\alpha\rangle \\ &\quad - \sum_{\beta:d(\beta,\alpha)=1} \langle \delta_\alpha, (\Delta_{\Gamma_K} - z)^{-1}\delta_\beta\rangle\langle \delta_\alpha, (\Delta_{\Gamma_{K,\alpha}} - z)^{-1}\delta_\alpha\rangle \\ &= \frac{-1}{z} - \frac{-1}{z}\sum_{\beta:d(\alpha,\beta)=1} R(\alpha,\beta,z).\end{aligned}$$

These two equations together imply that

$$R(\alpha,\alpha,z) = \frac{-1}{z}[1 + (K+1)R(\alpha,\alpha,z)R_{\alpha,y}(\beta,\beta,z))],$$

or

$$R(\alpha,\alpha,z) = \frac{-1}{z + (K+1)R_{\alpha,\beta}(\beta,\beta,z)},$$

for some β such that $d(\alpha,\beta) = 1$.

(2) A similar calculation repeated with the pair of operators $\Delta_{\Gamma_{K,\alpha,\beta}}$ and $\Delta_{\Gamma_{\beta,w,z}}$, where β is chosen with $d(\alpha,\beta) = 1$ and $\gamma \neq \alpha$ is chosen so that $d(\beta,\gamma) = 1$, gives the relation

$$\begin{aligned}R_{\alpha,\beta}(\beta,\gamma,z) &= R_{\beta,\gamma}(\beta,\gamma,z) - R_{\alpha,\beta}(\beta,\beta,z)R_{\beta,\gamma}(\gamma,\gamma,z), \\ &\qquad\qquad\qquad\qquad \gamma \neq \alpha, \quad d(\gamma,\beta) = 1, \\ R_{\alpha,\beta}(\beta,\beta,z) &= R_{\beta,\gamma}(\beta,\beta,z) - \sum_{\substack{\gamma:\gamma\neq\alpha \\ d(\beta,\gamma)=1}} R_{\alpha,\beta}(\beta,\gamma,z)R_{\beta,\gamma}(\beta,\beta,z).\end{aligned}$$

Simplifying the above two equations using the Relation (5.1.44), yields

$$R_{\alpha,\beta}(\beta,\beta,z) = -\frac{z}{2K} + \frac{1}{K}\sqrt{\frac{z^2}{4} - K},$$

where we can fix the signature of the square root to be positive on the positive axis, so that the function maps the upper half plane to itself as it is a property of $R_{\alpha,\beta}(\beta,\beta,z)$. □

Theorem 5.1.21. *The spectrum of Δ_{Γ_K} is purely absolutely continuous and is the interval $[-2\sqrt{K}, 2\sqrt{K}]$.*

Proof: We will show that the boundary values $R(\alpha,\alpha,E+i0)$ have finite non-zero imaginary part for all $E \in (-2\sqrt{K}, 2\sqrt{K})$ and have real and finite boundary values elsewhere in $\mathbb{R}$ for all $\alpha \in \Gamma_K$. Such a statement will show, by an application of Theorem 1.4.16, that any total spectral measure associated with the operator Δ_{Γ_K} is purely absolutely continuous and has support equal to $[-2\sqrt{K}, 2\sqrt{K}]$, showing the assertion in the theorem.

First observe that the boundary values $R_{\alpha,\beta}(\beta,\beta,E+i0) = \frac{-E}{2K} + \frac{1}{K}\sqrt{\frac{E^2}{4} - K}$ exist finitely for any $E \in \mathbb{R}$ and being independent of x, y. Further when $|E| < 2\sqrt{K}$, the imaginary part of this quantity is non-zero while for $|E| \geq 2\sqrt{K}$ it is purely real. Using this fact we see that $R(\alpha,\alpha,E+i0)$, which is given by

$$R(\alpha,\alpha,E+i0) = \frac{-1}{E + (1+\frac{1}{K})(-\frac{E}{2} + \sqrt{\frac{E^2}{4} - K})},$$

also has finite and non-zero imaginary part for $E \in (-2\sqrt{K}, 2\sqrt{K})$ and has zero imaginary part for $|E| \geq 2\sqrt{K}$. This shows, by an application of Theorem 1.4.16, that the spectral measure of Δ_{Γ_K} associated with δ_x is purely absolutely continuous and has support equal to $[-2\sqrt{K}, 2\sqrt{K}]$. Since $R(\alpha,\alpha,E+i0)$ is independent of α, this happens for any total spectral measure of Δ_{Γ_K}, proving the claim. $\square$

We next consider the operators

$$H_\lambda^\omega = \Delta_{\Gamma_K} + \lambda V^\omega,$$

where $V^\omega(\alpha),\ \ \alpha \in \Gamma_K$, are real-valued i.i.d. random variables with common distribution μ.

Hypothesis 5.1.22. *We assume that $V^\omega(\alpha), \alpha \in \Gamma_K$, are independent random variables with identical distributions given by a probability measure μ such that $h(t) = \int_{\mathbb{R}} e^{-itx} d\mu(x)$ satisfies:*

1. *$h(t)$ is bounded and differentiable,*
2. *h' is bounded and absolutely continuous and*
3. *h'' is bounded on $(0,\infty)$.*

Then the spectrum of H^ω is $[-2\sqrt{K}, 2\sqrt{K}] + \mathrm{supp}(\mu)$. We denote the resolvent kernels associated with the operators $\Delta_\Gamma + \lambda V^\omega$ for $\Gamma \in \{\Gamma_K, \Gamma_{K,\alpha}, \Gamma_{K,\alpha,\beta}\}$ by $R^\omega(\cdot,\cdot,z,\lambda), R_\alpha^\omega(\cdot,\cdot,z,\lambda)$ and $R_{\alpha,\beta}^\omega(\cdot,\cdot,z,\lambda)$, respectively. Then the proof of the following lemma is analogous to the proof of Lemma 5.1.20.

Lemma 5.1.23. *Consider the operators H_λ^ω with V^ω satisfying the Hypothesis 5.1.22. Let $\alpha \in \Gamma_K$ and β satisfy $d(\alpha,\beta) = 1$. Then*

1. *$R^\omega(\alpha,\alpha,z,\lambda) = -[1/(z - \lambda V^\omega(\alpha))] + \sum_{\beta : d(\alpha,\beta)=1} R_{\alpha,\beta}^\omega(\beta,\beta,z,\lambda))$ and*
2. *$R_{\alpha,\beta}^\omega(\beta,\beta,z,\lambda) = -[1/(z - \lambda V^\omega(\beta))] + \sum_{\substack{\gamma : d(\gamma,\beta)=1 \\ \gamma \neq \alpha}} R_{\beta,\gamma}^\omega(\gamma,\gamma,z,\lambda))$.*

Theorem 5.1.24 (Klein). *Consider $\ell^2(\Gamma_K)$ and H_λ^ω given above with the random variables $V^\omega(\alpha)$ distributed with an absolutely continuous distribution*

μ. Then there is a $\lambda_0 > 0$ such that for all $0 \le \lambda < \lambda_0$, there is an interval $(E_-(\lambda), E_+(\lambda)) \subset \sigma(H^\omega_\lambda)$ such that

$$\sigma_s(H^\omega_\lambda) \cap (E_-(\lambda), E_+(\lambda)) = \emptyset, \quad a.e. \quad \omega.$$

The aim of the rest of the section is to prove this theorem for which we need some auxiliary results. We first need some simple relations involving Gaussian integrals. We state these relations as a proposition, the proof of which is obtained by integration. We set $\langle x, y\rangle = x_1y_1 + x_2y_2$ for $x, y \in \mathbb{R}^2$.

Proposition 5.1.25. *Let Z be a complex number, whose imaginary part is positive. Then the following relations are valid:*

1. $\frac{i}{\pi}\int_{\mathbb{R}^2} e^{iZ\langle x,x\rangle}dx = i\int_0^\infty e^{iZr^2}2rdr = \frac{1}{Z}$, *where* $r = (\langle x, x\rangle)^{1/2}$.
2. $\frac{1}{\pi}\int_{\mathbb{R}^2} e^{-i\langle x,y\rangle}\frac{\partial}{\partial s^2}\left(e^{iZs^2}\right)dy = e^{iZ\langle x,x\rangle}$, *where* $s = (\langle y, y\rangle)^{1/2}$.

Let us define for $\alpha \in \Gamma_K$ with $d(\alpha, 0) = 1$,

$$\begin{aligned}\zeta_{\lambda,z}(s^2) &= \mathbb{E}\left\{e^{\frac{i}{4}R^\omega_{o,\alpha}(\alpha,\alpha,z,\lambda)s^2}\right\},\\ \xi_{\lambda,z}(s^2, r^2) &= \mathbb{E}\left\{e^{\frac{i}{4}\mathrm{Re}(R^\omega_{o,\alpha}(\alpha,\alpha,z,\lambda))(s^2-r^2)-\frac{1}{4}\mathrm{Im}(R^\omega_{o,\alpha}(\alpha,\alpha,z,\lambda))(s^2+r^2)}\right\}.\end{aligned} \tag{5.1.45}$$

Then the following proposition sets up a relation valid for ξ, ζ. We set $z = E + i\epsilon, r^2 = \langle x, x\rangle, s^2 = \langle y, y\rangle$ and $\partial_s \equiv \frac{\partial}{\partial s^2}$.

Proposition 5.1.26. *Let H^ω_λ be operators with V^ω satisfying the Hypothesis 5.1.22. Consider any $\lambda, E \in \mathbb{R}$ and $\epsilon > 0$; then*

$$\begin{aligned}\mathbb{E}(R^\omega(0, 0, z, \lambda)) &= \frac{i}{\pi}\int_{\mathbb{R}^2} e^{izx^2}h(\lambda x^2)[\zeta_{\lambda,z}(x^2)]^{K+1}dx,\\ \zeta_{\lambda,z}(r^2) &= \frac{1}{\pi}\int_{\mathbb{R}^2} e^{-i\langle x,y\rangle}\partial_s\{e^{izs^2}h(\lambda s^2)[\zeta_{\lambda,z}(s^2)]^K\}dy,\end{aligned} \tag{5.1.46}$$

and

$$\begin{aligned}\mathbb{E}(|R^\omega(0, 0, z, \lambda)|^2) &= \frac{1}{\pi^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} dy_+dy_- \quad e^{iE(s_+^2-s_-^2)-\epsilon(s_+^2+s_-^2)}\\ &\quad\times h(\lambda(s_+^2 - s_-^2))[\xi_{\lambda,z}(s_+^2, s_-^2)]^{K+1},\\ \xi_{\lambda,z}(r_+^2, r_-^2) &= \frac{1}{\pi^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} dy_+dy_- \quad e^{-i(\langle x_+,y_+\rangle-\langle x_-,y_-\rangle)}\partial_{s_+}\partial_{s_-}\\ &\quad\times\{e^{iE(s_+^2-s_-^2)-\epsilon(s_+^2+s_-^2)}h(\lambda(s_+^2 - s_-^2))[\xi_{\lambda,z}(s_+^2, s_-^2)]^K\},\\ &= \frac{1}{\pi^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} dy_+dy_- \quad e^{-i(<x_+,y_+>-<x_-,y_->)}\partial_{s_+}\partial_{s_-}\\ &\quad\times\{e^{iE(s_+^2-s_-^2)-\epsilon(s_+^2+s_-^2)}h(\lambda(s_+^2 - s_-^2))[\xi_{\lambda,z}(s_+^2, s_-^2)]^K\}.\end{aligned} \tag{5.1.47}$$

Proof: We first note that the graphs $\Gamma_{K,\alpha,\beta}$ are mutually disjoint for any fixed $\alpha, \beta \in \{\gamma : d(\alpha,\gamma) = 1\}$. Therefore, by the independence assumption on $V^\omega(\beta)$ for different β, it follows that $R^\omega_{\alpha,\beta}(\beta,\beta,z,\lambda)$ are independent functions of ω as β varies with $d(\alpha,\beta) = 1$.

The proof of Equation (5.1.46) is to take $Z = R^\omega(0,0,z,\lambda)$, use Relations (1), (2) of Lemma 5.1.23 in Proposition 5.1.25 (1) and (2), respectively. The resulting integrals are simplified using the independence of the functions $R^\omega_{0,\alpha}(\alpha,\alpha,z,\lambda)$ for different α to convert the average of the exponential into a product. The interchange of the order of integration required is justified for $\lambda \neq 0$, $\epsilon > 0$ or for $\lambda = 0, |E| < 2\sqrt{K}$, since in these cases the integrals converge absolutely. The equation (5.1.47) is similarly obtained by taking absolute value squares taking expectations with respect to ω and using similar reasoning as above. $\square$

Let us define the Banach spaces

$$\begin{aligned} \mathcal{H}_p &= \overline{\{f \in C^1(\mathbb{R}^+) : \|f\|_p + \|f'\|_p < \infty\}}, \quad 1 \leq p \leq \infty, \\ \mathcal{K}_p &= \mathcal{H}_p \otimes \mathcal{H}_p, \end{aligned} \tag{5.1.48}$$

where as usual $\|\cdot\|_\infty$ is the sup norm and $\|f\|_p^p = \int_{\mathbb{R}^+} |f|^p(x)\, dx, \;\; 1 \leq p < \infty$. Again following standard notation $\overline{S}$ denotes the completion of the set S with respect to the indicated norm.

The aim of defining these spaces is to use a fixed point theorem and show that the function $\xi_{\lambda,E+i0}$ is in $\mathcal{K}_\infty$ for a set of λ when E varies in a subinterval of $(-2\sqrt{K}, 2\sqrt{K})$.

To this end we define

$$\tilde{f}(x) = f(x^2), \quad (Tf)(x) = -2\mathcal{F}(\tilde{f}')(x) = -\frac{1}{\pi}\int_{\mathbb{R}^2} e^{-i<x,y>} f'(y^2) dy,$$

and set $\mathcal{T} = T \otimes T$. These operators will help in formulating the fixed point problem we plan to solve. We denote the operator of multiplication by the function g on any of the spaces $\mathcal{H}_p$ or $\mathcal{K}_p$ by M_g. We also set

$$\begin{aligned} B_1(\lambda,z)(x) &= e^{iEx^2} e^{-\epsilon x^2} h(\lambda x^2), \\ B_2(\lambda,z)(x,y) &= e^{iE(x^2-y^2)} e^{-\epsilon(x^2+y^2)} h(\lambda(x^2+y^2)), \end{aligned}$$

where $z = E + i\epsilon$ and h is the function given in the Hypothesis 5.1.22.

Lemma 5.1.27. *Let $g \in \mathcal{S}(\mathbb{R}^+)$, the space of smooth rapidly decreasing (at ∞) functions on $\mathbb{R}^+$.*

(i) Then the operator $M_g T M_g$ is compact as an operator from $\mathcal{H}_\infty$ to $\mathcal{H}_1$.
(ii) The operator $M_{g\times g}\mathcal{T} M_{g\times g}$, as an operator from $\mathcal{K}_\infty$ to $\mathcal{K}_1$, is compact.

Proof: (i) Since $g \in \mathcal{S}(\mathbb{R}^+)$, all its derivatives are also there. Therefore g is in $\mathcal{H}_p$ for all p and so is the function $g_1(s) = (1+s)^N g(s)$, for any positive integer N. Fix $N = 2$ and write $g = g_1 g_2$, where $g_2(s) = (1+s)^{-2}, \quad x \in \mathbb{R}^2$. Then

$$M_g T M_g = M_{g_1} M_{g_2} T M_{g_2} M_{g_1}.$$

Now Cauchy–Schwarz implies that M_{g_1} is a bounded linear operator from $\mathcal{H}_2$ to $\mathcal{H}_1$ and also from $\mathcal{H}_\infty$ to $\mathcal{H}_2$, since g_1 and its derivative are in $\mathcal{H}_2$, as a direct calculation shows. The relation $M_{g_2} T M_{g_2} = M_{g_2} \mathcal{F} M_{g_2'} + M_{g_2} \mathcal{F} M_{g_2} \partial$, is valid with $\partial f = f'$ being a bounded linear operator from $\mathcal{H}_2$ to itself. Therefore the compactness of $M_{g_2} T M_{g_2}$ as an operator from $\mathcal{H}_2$ to itself follows if we show that $M_{\tilde{g}} \mathcal{F} M_{\tilde{g}}$ is a compact operator from $L^2(\mathbb{R}^2)$ to itself for any $g \in \mathcal{S}(\mathbb{R}^+)$. It is a fact that this operator is Hilbert–Schmidt from $L^2(\mathbb{R}^2)$ to itself. (This is because $M_g \mathcal{F} M_g$ is an integral operator with kernel $K(x,y) = \frac{1}{4\pi} g(x) e^{-i<x,y>} g(y)$, which satisfies $\int_{\mathbb{R}^2 \times \mathbb{R}^2} |K(x,y)|^2 dx dy < \infty$ by Cauchy–Schwarz whenever $g \in L^2(\mathbb{R}^2)$. This finiteness is precisely the condition for the linear operator given by K to be Hilbert–Schmidt.)

(ii) The proof of this part is similar to that of (i). $\square$

Lemma 5.1.28. *Let H^ω_λ be as in Theorem 5.1.24 and consider the functions ζ, ξ, B_1, B_2 and the operator $T, \mathcal{T}$ defined earlier. Then we have*

(i) $\zeta_{\lambda,z} \in \mathcal{H}_\infty, \quad \xi_{\lambda,z} \in \mathcal{K}_\infty$ for all $\lambda \in \mathbb{R}$ and $z = E + i\epsilon, \quad \epsilon > 0$. The maps $(\lambda, E, \epsilon) \to \zeta_{\lambda, E+i\epsilon}$ and $(\lambda, E, \epsilon) \to \xi_{\lambda, E+i\epsilon}$ as $\epsilon \to 0$ are continuous from $\mathbb{R} \times \mathbb{R} \times (0,\infty)$ to $\mathcal{H}_\infty$ and $\mathcal{K}_\infty$, respectively.

(ii) If $|E| < 2\sqrt{K}$, then $\zeta_{0,E} \in \mathcal{H}_\infty, \quad \xi_{0,E} \in \mathcal{K}_\infty$ and

$$\zeta_{0,E+i\epsilon} \to \zeta_{0,E} \quad \textit{in} \quad \mathcal{H}_\infty, \quad \xi_{0,E+i\epsilon} \to \xi_{0,E} \quad \textit{in} \quad \mathcal{K}_\infty, \text{ as } \epsilon \to 0,$$

respectively.

(iii) The Equations (5.1.46) and (5.1.47) become the fixed point equations in $\mathcal{H}_\infty$ and $\mathcal{K}_\infty$, respectively given by

$$\zeta_{\lambda,z} = T M_{B_1(\lambda,z)} \zeta_{\lambda,z}, \quad \xi_{\lambda,z} = \mathcal{T} M_{B_2(\lambda,z)} \xi_{\lambda,z},$$

valid for $(\lambda, E, \epsilon) \in \mathbb{R} \times \mathbb{R} \times (0,\infty) \cup \{(0,E,0) : |E| < 2\sqrt{K}\}$.

Proof: (i) Note that when $\epsilon > 0$, the imaginary parts $\mathrm{Im}(R^\omega_\alpha(\alpha,\alpha,E+i\epsilon,\lambda)$ are strictly positive, and the first resolvent equation shows that for each ω, $\|(H^\omega_\lambda - E - i\epsilon)^{-1} - (H^\omega_{\tilde{\lambda}} - \tilde{E} - i\tilde{\epsilon})^{-1}\| \to 0$ as $(E,\epsilon,\lambda) \to (\tilde{E},\tilde{\epsilon},\tilde{\lambda})$ in $(E,\epsilon,\lambda) \in \mathbb{R} \times (0,\infty) \times \mathbb{R}$. Therefore for each ω the function $R^\omega_\alpha(\alpha,\alpha,z,\lambda)$ is a strictly positive jointly continuous function of (E,ϵ,λ) in the above set. Hence the function $e^{-\mathrm{Im}(R^\omega_\alpha(\alpha,\alpha,z,\lambda))x^2}$ is a rapidly decreasing function of x^2 which is also continuous in (E,ϵ,λ). Therefore the function $\zeta_{\lambda,z}$ is continuous in the variables (E,ϵ,λ) with respect to the topology of $\mathcal{H}_\infty$. A similar argument asserts the statement for ξ.

(ii) Since for $\lambda = 0$ the quantities involved are the matrix elements of resolvents of the Laplacians, using the explicit expressions from Lemma 5.1.20 the continuity assertions for $\zeta_{0,z}, \xi_{0,z}$ follow.

(iii) This item is a restatement of Equations (5.1.46) and (5.1.47). $\square$

Lemma 5.1.29. *(i) The map $F : \mathbb{R} \times \mathbb{R} \times [0, \infty) \times \mathcal{H}_\infty \to \mathcal{H}_\infty$, defined by*

$$F(\lambda, E, \epsilon, f) = TM_{B_1(\lambda, E+i\epsilon)} f^K - f,$$

is continuous. F is continuously Frechet differentiable with respect to f, with the derivative given by

$$F_f(\lambda, E, \epsilon, f_0) = KTM_{B_1(\lambda, E+i\epsilon) f_0^{K-1}} - I.$$

Further, for $|E| < 2\sqrt{K}$,

$$F(0, E, 0, \zeta_{0,E+i0}) = 0, \quad \text{and} \quad 0 \notin \sigma(F_f(\lambda, E, \epsilon, \zeta_{0,E+i0})).$$

(ii) The map $Q : \mathbb{R} \times \mathbb{R} \times [0, \infty) \times \mathcal{K}_\infty \to \mathcal{K}_\infty$, defined by

$$Q(\lambda, E, \epsilon, f) = \mathcal{T}M_{B_1(\lambda, E+i\epsilon)} f^K - f,$$

is continuous. Q is continuously Frechet differentiable with respect to f, with the derivative given by

$$Q_f(\lambda, E, \epsilon, f_0) = K\mathcal{T}M_{B_1(\lambda, E+i\epsilon) f_0^{K-1}} - I.$$

Further, for $|E| < 2\sqrt{K}$,

$$Q(0, E, 0, \zeta_{0,E+i0}) = 0, \quad \text{and} \quad 0 \notin \sigma(Q_f(\lambda, E, \epsilon, \xi_{0,E+i0})).$$

Proof: All the statements here except for the one on the spectrum of the operators F_f and Q_f are obtained from the previous lemma. The proof that 0 is not in the spectrum of the operators $F_f(\lambda, E, \epsilon, \zeta_{0,E+i0})$ and $Q_f(\lambda, E, \epsilon, \xi_{0,E+i0})$ is as in the proof of Theorem 3.5 of [2]. □

Theorem 5.1.30. *Let X be a complete metric space, Y a Banach space, and f a continuous function from an open set $U \subset X \times Y \to Y$ which has a Frechet derivative, $f_y(x, y)$, with respect to $y \in Y$. This derivative is continuous in U. Suppose $f(x_0, y_0) = 0$, for some point $(x_0, y_0) \in U$ and suppose that $f_y(x, y)$ is a Banach space isomorphism of Y. Then*

1. *there exist $r, \delta > 0$ such that for each $x \in \{(w, y_0) : d(w, x_0) < r\}$ there is a unique $u(x) \in \{y \in Y : \|y - y_0\| < \delta\}$, such that $f(x, u(x)) = 0$,*
2. *the map $x \to u(x)$ from $\{w \in X : d(w, x_0) < r\}$ to Y is continuous.*

Proof: See Theorem 2.7.2. of [150]. □

Proposition 5.1.31. *For any $E : |E| < 2\sqrt{K}$, there are $\lambda_E > 0, \delta_E > 0$ such that the maps*

$$(-\lambda_E, \lambda_E) \times (E - \delta_E, E + \delta_E) \times (0, \infty) \ni (\lambda, E', \epsilon) \to \xi_{\lambda, E'+i\epsilon} \in \mathcal{K}_\infty,$$

and

$$(-\lambda_E, \lambda_E) \times (E - \delta_E, E + \delta_E) \times (0, \infty) \ni (\lambda, E', \epsilon) \to \zeta_{\lambda, E'+i\epsilon} \in \mathcal{K}_\infty,$$

have continuous extensions to $((-\lambda_E, \lambda_E) \times (E - \delta_E, E + \delta_E) \times [0, \infty)$ satisfying the Relations (5.1.46) and (5.1.47), respectively.

Proof: The Lemma 5.1.29(i) and (ii) show that the functions F and Q satisfy the hypotheses of the Theorem 5.1.30 at $(0, E, 0, \zeta_{0,E})$ and $(0, E, 0, \xi_{0,E})$, respectively. Therefore for each E with $|E| < 2\sqrt{K}$, there exist positive numbers $\lambda_E, \delta_E, \epsilon_E$ and σ_E such that for each

$$(\lambda, E', \epsilon) \in ((-\lambda_E, \lambda_E) \times (E - \delta_E, E + \delta_E) \times [0, \epsilon_E),$$

there is a unique $\alpha_{(\lambda,E',\epsilon)} \in \mathcal{K}_\infty$ with $\|\alpha_{(\lambda,E',\epsilon)} - \xi_{0,E}\|_{\mathcal{K}_\infty} < \sigma_E$, such that $Q(\lambda, E', \epsilon, \alpha_{(\lambda,E',\epsilon)}) = 0$. Further the map

$$((-\lambda_E, \lambda_E) \times (E - \delta_E, E + \delta_E) \times [0, \epsilon_E) \ni (\lambda, E', \epsilon) \to \alpha_{(\lambda,E',\epsilon)} \in \mathcal{K}_\infty$$

is continuous. A similar statement is valid for the function F. □

Proof of Theorem 5.1.24: For the purpose of this proof we shall denote the points of Γ_K by α, β, γ, to avoid confusion with the integration variables x, y etc. We take $z = E + i\epsilon, \quad \epsilon > 0$, and prove that there is a λ_0 such that for $0 \leq \lambda < \lambda_0$ there is an interval $(E_-(\lambda), E_+(\lambda))$ such that $\mathbb{E}|R^\omega(\alpha, \alpha, E + i\epsilon, \lambda)|^2 < \infty$, with the bound uniform in $\epsilon > 0$, for all $\alpha \in \Gamma_K$. This proves the theorem. In the following we prove the statement only for $\alpha = 0$; for all other α the proof is the same.

Proposition 5.1.26, Relation (5.1.46) implies that if $\xi_{\lambda,E+i0} \in \mathcal{K}_\infty$, for an interval $(E_(\lambda), E_+(\lambda))$ for any given λ, then the average $\int_{E_(\lambda)}^{E_+(\lambda)} \mathbb{E}(|R^\omega(0, 0, E + i0, \lambda)|^2) < \infty$, from which it follows that for almost every ω, the spectral measure of H^ω_λ with respect to the vector δ_0 is purely absolutely continuous. The stated finiteness is the content of the previous proposition. □

5.1.4 Jaksić–Last Theorem

In this section we consider the question of purity of absolutely continuous spectra, which uses the theory of unitary equivalence of a family of selfadjoint operators restricted to different subspaces of a given Hilbert space, presented in Chapter 3.

Theorem 5.1.32 (Jaksić–Last). *Suppose H is a selfadjoint operator and $\{\phi_n\}$ is an orthonormal basis. Suppose $\{q(n), n \in I\}$ are independent real-valued random variables with absolutely continuous distribution. Consider the operators $H^\omega = H + \sum_{n \in I} q^\omega(n) P_{\phi_n}$. Then H^ω restricted to the cyclic subspaces $\mathcal{H}_{\omega,\phi_n}$ and $\mathcal{H}_{\omega,\phi_m}$ are unitarily equivalent for almost every ω, whenever $\mathcal{H}_{\omega,\phi_n}$ and $\mathcal{H}_{\omega,\phi_m}$ are non-orthogonal.*

Proof: Under the assumptions on $\{q(n)\}$ it follows that the conditional probability distribution of the pair $(q(n), q(m))$ fixing the values of others $\{q^\omega(k), k \neq n, m\}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$. Therefore, from Theorem 3.1.10 (using $\lambda = q^\omega(n), \quad \eta = q^\omega(m)$) and using Fubini for the variables $\{q^\omega(k) : k \neq n, m\}$, it follows that the spectral measures μ_{ω,ϕ_n} and μ_{ω,ϕ_m} are equivalent for almost every ω. This implies the conclusion of the theorem. □

Corollary 5.1.33 *Suppose H^ω, $\{\phi_n\}$ are as in Theorem 5.1.32. Suppose further that*

1. *$\{\phi_n : n \in I\}$ is a cyclic family for H^ω.*
2. *For each pair $n, m \in I$, the cyclic subspaces $\mathcal{H}_{\omega,\phi_n}$ and $\mathcal{H}_{\omega,\phi_m}$ are not orthogonal.*

Then for almost every ω and for each $n \in I$, the spectral measure μ_{ω,ϕ_n} is a total spectral measure for H^ω.

Proof: Condition (1) implies that the measure $\mu_\omega = \sum_{k\in I} \alpha(k)\mu_{\omega,\phi_k}$, where $\alpha(k)$ is a strictly positive function on I with $\sum_{k\in k} \alpha(k) = 1$, is a total spectral measure. But under the conditions of the corollary Theorem 5.1.32 implies that for each $n \in I$, μ_ω is equivalent to μ_{ω,ϕ_n} for almost every ω, which proves the corollary. $\square$

In this section we use the notation $\text{ess} - \text{supp}(\mu)$ to mean support of a measure up to sets of Lebesgue measure zero.

Corollary 5.1.34 *Let H^ω and $\{\phi_n\}$ satisfy the conditions of Corollary 5.1.33 and let μ_ω denote any total spectral measure. Then*

1. *the singular parts of μ_ω and $\mu_{\omega'}$ are mutually singular for almost every pair (ω, ω').*
2. *There exists a fixed Borel set $A \subset \mathbb{R}$ such that for almost every ω,*
   a) *$A = \text{ess} - \text{supp}\ \mu_{\omega,ac}$.*
   b) *$\mu_{\omega,s}(B) = 0, \qquad \text{whenever} \quad |B \setminus A| = 0.$*

Proof: (1) If the result is not true, then for some set S of zero Lebesgue measure, $\mu_{\omega,s}(S) > 0$ for a set of ω of positive measure. By the equivalence of μ_ω to μ_{ω,ϕ_n} for any n, this implies that $\mu_{\omega,\phi_1}(S) > 0$ for a set of ω of positive measure. Since the conditional probability distribution of $q(1)$ given $\{q^\omega(n), n \neq 1\}$ is absolutely continuous, it follows that for fixed ω_0, the self-adjoint operator $H_\lambda = H^{\omega_0} + \lambda P_{\phi_1}$ satisfies the property that $\mu_{\lambda,\phi_1,s}(S) > 0$ for a set of λ of positive Lebesgue measure. But by Corollary 3.1.6 this is impossible for a fixed set S of positive Lebesgue measure, hence S must have Lebesgue measure zero.

(2)(a) The first part of the proof is by the Kolmogorov zero-one law. We first note that the absolutely continuous spectrum of H^ω does not change under perturbation by finite rank operators, by an application of Theorem 3.6.9, hence the essential support of $\mu_{\omega,ac}$ does not depend on the random variables $\{q(n)\}$ for any finite collection of indices $n \in I$. Let A_ω denote an essential support of $\mu_{\omega,ac}$. Then the function

$$F(x, \omega) = \int_{-\infty}^{x} \chi_{A_\omega}(y)h(y)dy$$

for any fixed positive integrable function h, is measurable in ω and continuous in x, since by the general theory (see Section 4.2.2) the spectral projections

χ_{A_ω} are weakly measurable as a function of ω. Further these functions $F(x,\cdot)$ are also independent of $\{q(n)\}$ for any finite collection of indices n. Then by the Kolmogorov zero-one law it follows that the sets $B_{r,s} = F(r,\cdot)^{-1}((-\infty, s))$ have probability 0 or 1 for any pair of numbers r, s. For each $r \in \mathbb{Q}$ define the functions $\alpha(r) = \inf\{s \in \mathbb{Q} : \text{prob}(B_{r,s}) = 1\}$. Then we see that $\{\omega : F(r,\omega) = \alpha(r)\}$ gets probability 1 and we can define a function G on $\mathbb{R}$ such that $G(r) = \alpha(r)$ which is seen to be continuous (using the continuity of $F(x,\omega)$ for any fixed ω). Further $\mathbb{Q}$ being a countable set, we see that $\Omega_0 = \cap_{r\in\mathbb{Q}}\{\omega : F(r,\omega) = G(r)\}$ has probability 1. Therefore by continuity for each $\omega \in \Omega_0$, we have $F(\cdot,\omega) = G(\cdot)$, which also shows that G is indeed the distribution function of a positive absolutely continuous measure μ and its support, which we denote A, agrees with the support of $\mu_{\omega,ac}$ for almost every $\omega \in \Omega_0$.

(2)(b) By the first part of Corollary 3.1.6 it is enough to show the result for the set A. We note that by the equivalence of μ_{ω,ϕ_1} to μ_ω, we have

$$A = \text{ess} - \text{supp} \ \ \mu_{\omega,\phi_1,ac}, \quad \text{for a.e. } \omega.$$

Let Ω_0 denote the set of measure 1 on which the above is valid. Fix any $\omega \in \Omega_0$ and consider

$$H^\omega_\lambda = H^\omega + \lambda P_{\phi_1},$$

and let $\mu_{\omega,\lambda,\phi_1}$ be the spectral measure of H^ω_λ with respect to ϕ_1. Then by the invariance of the absolutely continuous spectrum under trace class perturbations, it follows that $A = \text{ess} - \text{supp} \ \ \mu_{\omega,\lambda,\phi_1,ac}$ also, see Theorems 3.6.9 and 3.6.18. Therefore for almost every λ with respect to Lebesgue measure $\mu_{\omega,\lambda,\phi_1,s}(A) = 0$. This is valid for all $\omega \in \Omega_0$. Since the conditional distribution of $q(1)$ given $q^\omega(n), n \neq 1$, is absolutely continuous with respect to the Lebesgue measure, this implies using Fubini that for almost every $\omega \in \Omega$ we have $\mu_{\omega,\phi_1,s}(A) = 0$ and by the equivalence of μ_{ω,ϕ_1} to μ_ω it is also valid for $\mu_{\omega,s}$. □

Remark: For a family of operators, H^ω, as in the above corollary showing the existence of absolutely continuous spectrum in an interval, amounts to showing its purity too for almost every ω.

5.2 Scattering

5.2.1 Decaying Random Potentials

In this section we consider models of operators of the form

$$H^\omega = \Delta + V^\omega, \quad \text{on} \quad \ell^2(\mathbb{Z}^d), \tag{5.2.49}$$

where V^ω is an operator of multiplication by a function of the form $V(n) = a_n q_n$, with the q_n's i.i.d. random variables with distribution μ.

These models provide examples of some random operators, though non-stationary, where the expected Anderson–Mott transition from dense pure point spectrum to purely absolutely continuous spectrum occurs.

Theorem 5.2.1. *Suppose $q_n, \quad n \in \mathbb{Z}^d$ are i.i.d. random variables distributed according to μ, a measure of finite variance and whose density $d\mu/dx$ is in $L^p(\mathbb{R})$ for some $p > 1$. Suppose the real-valued sequence a_n is chosen so that $|a_n|(1+|n|)^\alpha$ is bounded for some $\alpha > 1$. Let H^ω be as in Equation (5.2.49). Then for almost every ω, we have*

$$\sigma_{ac}(H^\omega) = \sigma(\Delta).$$

Proof: The theorem is proved by first showing that for almost every ω, the wave operators (see Definition 3.6.1) for the pair $\{\Delta, H^\omega\}$ exist, which shows that $\sigma_{ac}(H^\omega) \supset \sigma_{ac}(\Delta)$ and then use the Corollary 5.1.34 to show that the absolutely continuous spectrum is pure. The absolutely continuous spectrum in the complement of $\sigma(\Delta)$ is ruled out explicitly by showing that there is only pure point spectrum there.

So we show that the wave operators

$$W_+^\omega = \lim_{t \to \infty} e^{itH^\omega} e^{-it\Delta}$$

exist in the strong sense. To show this limit exists, it is enough to consider the set D of vectors of finite support, and for each $f \in D$, we show that the limits

$$\lim_{t \to \infty} e^{itH^\omega} e^{-it\Delta} f$$

exists. Since f is a vector of finite support, it is sufficient to show that the above limit exists for $f = \delta_n$ for any $n \in \mathbb{Z}^d$ (where $\{\delta_n\}$ is the standard basis for $\ell^2(\mathbb{Z}^d)$), from which it follows for any finite linear combinations of these δ_n's and hence for any $f \in D$. Therefore we show that the sequence

$$W_{t,n,\omega} = e^{itH^\omega} e^{-it\Delta} \delta_n$$

is Cauchy, for almost every ω and for every n. We again approximate the vector δ_n by a sequence $\phi_k(\Delta)\delta_n$, where the $\{\widehat{\phi_k}\}$ are a sequence of smooth functions with compact support in $\mathbb{T}^d \setminus \{\theta_i : \sin\theta_i = 0, \quad i = 1, \ldots, d\}$. We recall that Δ is precisely the operator of multiplication by the function $\sum_{i=1}^d 2\cos(\theta_i)$ in the space $L^2(\mathbb{T}^d)$, which is the range of $\ell^2(\mathbb{Z}^d)$ under the Fourier series map $\widehat{\ }$. The function $-\sin(\theta_i)$ is the derivative of $\cos(\theta_i)$ and the above condition on the support of ϕ_k is needed in the stationary phase estimate of the lemma below. We now set $n = 0$ and fix a ϕ_k but we see in the proof that the proof works for any $n \in \mathbb{Z}^d$ and any k. Denote $W_{t,o,k,\omega} = e^{itH^\omega} e^{-it\Delta}\phi_k(\Delta)\delta_0$ and consider

$$
\begin{aligned}
[\mathbb{E}\|W_{s,0,k,\omega} - W_{t,0,k,\omega}\|]^2 &\le \mathbb{E}\left(\|W_{s,0,k,\omega} - W_{t,0,k,\omega}\|^2\right) \\
&\le \mathbb{E}\Big\langle (W_{s,0,k,\omega} - W_{t,0,k,\omega}), \\
&\qquad\qquad \int_t^s d\tau \quad e^{i\tau H^\omega} iV^\omega e^{-i\tau\varDelta}\phi_k(\varDelta)\delta_0 \Big\rangle \\
&\le \int_t^s d\tau \quad \left(\mathbb{E}\|V^\omega e^{-i\tau\varDelta}\phi_k(\varDelta)\delta_0\|^2\right)^{1/2},
\end{aligned}
\tag{5.2.50}
$$

where we interchanged the τ integral and the expectation with respect to randomness by using Fubini. If we now show that

$$
\int_1^\infty d\tau \quad \left(\mathbb{E}\|V^\omega e^{-i\tau\varDelta}\phi_k(\varDelta)\delta_0\|^2\right)^{1/2} < \infty,
$$

then $\mathbb{E}\|W_{s,0,k,\omega} - W_{t,0,k,\omega}\|$ goes to zero as s, t go to infinity. Consequently by the dominated convergence theorem $W_{s,0,k,\omega}$ is Cauchy for almost every ω, which is the required result. We note that $\mathbb{E}|V^\omega(n)|^2 = \sigma^2|a_n|^2$, where σ^2 is the second moment of μ. Therefore we have for some $\beta > 0$,

$$
\begin{aligned}
\mathbb{E}\|V^\omega e^{-i\tau\varDelta}\phi_k(\varDelta)\delta_0\|^2 &\le \sum_{n\in\mathbb{Z}^d} |a_n|^2\sigma^2|\langle\delta_n, e^{-i\tau\varDelta}\phi_k(\varDelta)\delta_0\rangle|^2 \\
&\le \sum_{|n|\le\beta\tau} |a_n|^2\sigma^2|\langle\delta_n, e^{-i\tau\varDelta}\phi_k(\varDelta)\delta_0\rangle|^2 \\
&\quad + \sum_{|n|\ge\beta\tau} |a_n|^2\sigma^2|\langle\delta_n, e^{-i\tau\varDelta}\phi_k(\varDelta)\delta_0\rangle|^2 \\
&\le \frac{C}{(1+|\tau|)^N} + \frac{C}{(1+|\tau|)^{2\alpha}}, \quad \text{for any} \quad N \in \mathbb{N},
\end{aligned}
$$

where the third estimate uses the condition on a_n and the first estimate is given in Lemma 5.2.2. Since $\alpha > 1$ by assumption, the integrability in τ follows, completing the proof of the existence of wave operators. The existence of wave operators implies, Proposition 3.6.2 (i), that

$$
[-2d, 2d] = \sigma(\varDelta) = \sigma_{ac}(\varDelta) \subset \sigma_{ac}(H^\omega)
$$

for almost every ω.

Now we note that the collection of vectors δ_n, $n \in \mathbb{Z}^d$, is a cyclic family for the operators H^ω, since it is an orthonormal basis. Further an explicit computation shows that for any $n, m \in \mathbb{Z}^d$, there is a k such that

$$
\langle\delta_n, \varDelta^k\delta_m\rangle \neq 0,
$$

which shows that

$$
\langle\delta_n, (H^\omega)^k\delta_m\rangle \neq 0, \quad \forall \;\; \omega.
$$

This implies that the cyclic subspaces $\mathcal{H}_{\omega,\delta_n}$ are mutually non-orthogonal for almost every ω. We can take the total spectral measure $\mu_\omega = \sum_{n\in\mathbb{Z}^d} \alpha_n \mu_{\omega,\delta_n}$, with $\alpha_n > 0, \ \ \forall\, n \ \ \sum_n \alpha_n = 1$ and use Corollary 5.1.34, to find that there is a fixed Borel set A with $\sigma_{ac}(H^\omega) = A$, up to a set of Lebesgue measure zero, and $\sigma_s(H^\omega) \cap B = \emptyset$, for any Borel set B with $|B \setminus A| = \emptyset$. We already showed that $\sigma_{ac}(H^\omega) \supset [-2d, 2d]$. Therefore $|A \cap [-2d, 2d] \setminus [-2d, 2d]| = \emptyset$, showing that $\sigma_s(H^\omega) \cap (-2d, 2d) = \emptyset$, for almost every ω.

It now remains to show that there is no absolutely continuous spectrum outside the set $(-2d, 2d)$. This follows from Lemma 5.2.3. □

Lemma 5.2.2. *Let ϕ be a function such that $\widehat{\phi}$ is smooth and has compact support in $\mathbb{T}^d \setminus \{\theta_i : \sin(\theta_i) = 0, \quad i = 1, \ldots, d\}$. Then there exists a $\beta_0 > 0$ depending on ϕ such that for any $0 < \beta < \beta_0$,*

$$\sum_{|n|\le\beta\tau} |\langle \delta_n, e^{-i\tau\Delta}\phi(\Delta)\delta_0\rangle|^2 \le \frac{C}{(1+\tau)^N},$$

for arbitrary positive integer N.

Proof: First we show that for any n with $|n| \le \beta\tau$,

$$|\langle \delta_n, e^{-i\tau\Delta}\phi(\Delta)\delta_0\rangle| \le \frac{C}{(1+\tau)^N},$$

for any positive integer N, the statement in the lemma follows immediately, since the volume of the region $\{n : |n| \le \beta\tau\}$ grows at most as $|\tau|^d$ as τ goes to ∞. We go to the spectral representation of Δ and write the expression on the left as

$$\left| \int_{\mathbb{T}^d} \prod_{i=1}^d \, d\theta_i \ \ e^{-i\sum_{i=1}^d (\tau\cos(\theta_j) - n_j\theta_j)} \widehat{\phi}(\theta_1, \ldots, \theta_d) \right|.$$

We note that for any index j, $\frac{|n_j|}{\tau} \le \beta$. So we pick some j and integrate by parts with respect to the variable θ_j, N times. We use the identity $e^{if(\theta)} = \frac{1}{if'(\theta)} \frac{de^{if(\theta)}}{d\theta}$, and the integration by parts formula

$$\int e^{if(x)} g(x) dx = -\int e^{if(x)} \frac{d}{dx}\left(\frac{1}{if'(x)} g(x)\right) dx,$$

which is valid when the boundary terms are zero, to obtain the above expression as

$$\left| (-1)^N \int_{\mathbb{T}^d} \prod_{i=1}^d \, d\theta_i \ \ e^{-i\sum_{k=1}^d (\tau\cos(\theta_k) - n_k\theta_k)} \frac{d^N}{d\theta_j^N}\left(\frac{1}{J(\theta_j, \tau, n_j)} \widehat{\phi}(\theta_1, \ldots, \theta_d)\right) \right|.$$

Here $J(\theta_j, \tau, n_j) = \frac{d}{d\theta_j}(\tau\cos(\theta_j) - n_j\theta_j) = -(\tau\sin(\theta_j) + n_j)$. Now the condition on the support of $\widehat{\phi}$ (and the fact that its derivatives have their supports inside it) ensure that in the region of integration we always have $\inf_{\theta_j} |\sin(\theta_j)| > 0$, so the choice $\beta_0 = \frac{1}{2}\inf|\sin(\theta_j)|$, ensures the estimate

$$|\tau(\sin(\theta_j) + \frac{n_j}{\tau})| \geq \frac{1}{2}|\tau|,$$

for every $\beta < \beta_0$. Since the factor $\frac{1}{J}$ occurs to the power N, we get the bound stated in the lemma. All the other factors occurring in the integrals are bounded and their bound can be absorbed into the constant C (which does depend on N but not on τ). □

Lemma 5.2.3. *Consider H^ω as in Theorem 5.2.1. Then for almost every ω, $\sigma_{ac}(H^\omega) \cap [(-\infty, -2d) \cup (2d, \infty)] = \emptyset$.*

Proof: We first note that the variance of distribution μ is finite, $\int |x| d\mu(x) < \infty$. Therefore the conclusions of the Lemma 5.1.17 are valid with $q = p - 1$ and $\tau = 1$. In the following we fix an $s, 0 < s < (p-1)/(3p-2)$, and use the constant K_s occurring in that lemma. Given any $E_0 > 2d$, consider the set $S = (-\infty, -E_0) \cup (E_0, \infty)$. Then consider an $\epsilon > 0$ such that

$$K_s \sup_{\{z \in \mathbb{C} : |\mathrm{Re}(z)| > E_0\}} \sum_{n \in \mathbb{Z}^d} |G^0(0, n, z)|^s \epsilon^s < 1. \tag{5.2.51}$$

Given this ϵ, we consider the set of sites

$$\Lambda_\epsilon = \{k \in \mathbb{Z}^d : |a_k| < \epsilon\}.$$

We then denote the operator $H^\omega_{\Lambda_\epsilon} = \Delta + \sum_{k \in \Lambda_\epsilon} a_k q^\omega(k) P_k$, where P_k is the one-dimensional orthogonal projection onto the ray generated by δ_k. This operator $H^\omega_{\Lambda_\epsilon}$ satisfies the conditions of Theorem 5.1.19. Therefore the spectrum of $H^\omega_{\Lambda_\epsilon}$ is pure point in S for almost every ω. On the other hand since H^ω and $H^\omega_{\Lambda_\epsilon}$ differ by a finite rank operator their absolutely continuous spectra are the same by Corollary 3.6.28. Therefore there is no absolutely continuous spectrum in S for H^ω for almost every ω. Let $E_n > 2d$ be a sequence of numbers converging to $2d$. Then for any E_n there is an $\epsilon(n)$ and a set Λ_{ϵ_n} whose complement is a finite set, with which the inequality 5.2.51 is satisfied for $|\mathrm{Re}(z)| > E_n$. Therefore for almost all ω the set $(-\infty, -E_n) \cup (E_n, \infty)$ has no absolutely continuous spectrum. This being valid for any $E_n > 2d$, the theorem follows. □

Remark 5.2.4. In the above proof we only showed that there is no absolutely continuous spectrum outside $[-2d, 2d]$ while one can show that there is no continuous spectrum there. See notes for more on this.

5.2.2 Obstacles and Potentials

In the following section we discuss two classes of problems, perturbations of free operators by potentials and by obstacles. The theory is seen to be similar for both cases.

For the free operator H_0 one can take any generator of a strongly continuous semigroup such that Assumption 3.6.40 can be verified. We restrict ourselves to the physically interesting operators $-\Delta$, $(-\Delta)^\alpha$, $\alpha \in (0,1)$, $\sqrt{-\Delta + c^2} - c$ because the ideas of proof are the same for all of them. For potential perturbations one could also consider higher order differential operators; however in the case of obstacles the use of stochastic methods restricts the choice to at most second order partial differential operators.

Let H_0 be one of the operators mentioned above. Fix $\rho > 0$ by

$$\operatorname*{ess\,sup}_{x \in \mathbb{R}^d} e^{-tH_0}(x,x) \;\le\; c\, t^{-\rho} \tag{5.2.52}$$

i.e. $\rho = \frac{d}{2}$ for $-\Delta$, $\rho = \frac{d}{2\alpha}$ for $(-\Delta)^\alpha$, $\alpha \in (0,1)$, $\rho = d$ for $\sqrt{-\Delta + c^2} - c$.

Let M_V be a Kato–Feller operator with respect to H_0. Then we define the resolvent difference

$$R_p^{V,a} = (H_0 \dotplus M_V + a)^{-p} - (H_0 + a)^{-p}, \; p \in \mathbb{N},$$

a large enough and for the semigroup difference

$$S_{V,t} = e^{-t(H_0 \dotplus V)} - e^{-tH_0}$$

$t \ge 0$.

Let Γ be a closed region in $\mathbb{R}^d$ and set $\Sigma = \mathbb{R}^d \setminus \Gamma$. Take $(H_0)_\Sigma$ as in Corollary 4.2.22. Then we denote the resolvent difference by

$$R_p^{\Sigma,a} = (H_0 + a)^{-p} - J^*\big((H_0)_\Sigma + a\big)^{-p} J, \quad p \in \mathbb{N},$$

and the semigroup difference by

$$S_{\Sigma,t} = e^{-tH_0} - J^* e^{-t(H_0)_\Sigma} J \; .$$

All these differences establish integral operators the kernels of which are denoted by $R_p^{V,a}(.\,,.)$, $S_{V,t}(.\,,.)$, $R_p^{\Sigma,a}(.\,,.)$, $S_{\Sigma,t}(.\,,.)$.

Proposition 5.2.5. *The kernels $R_p^{V,a}(.,.), S_{V,t}(.,.), R_\rho^{\Sigma,a}(.,.), S_{\Sigma,t}(.,.)$ satisfy Assumption 3.6.33 if $p > \rho$.*

Proof: For $R_p^{\Sigma,a}(.\,,.)$ and $S_{\Sigma,t}(.\,,.)$ this is obvious and follows directly from the properties of $e^{-tH_0}(.\,,.)$. For $R_p^{V,a}$ and $S_{V,t}$ we notice that due to the Laplace transform,

$$R_p^{V,a} f = \frac{1}{\Gamma(p)} \int_0^\infty d\lambda\; \lambda^{p-1} e^{-a\lambda}\; S_{V,\lambda} f,$$

it suffices to study $S_{V,\lambda}$. The kernel of $S_{V,\lambda}$ is

$$S_{V,\lambda}(x,y) = E_x^{y,\lambda}\Big\{e^{-\int_0^\lambda V(X_u)du}\Big\} - E_x^{y,\lambda}\{1\},$$

where we used the notation of Chapter 4. Now

$$\begin{aligned}
&\operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} E_x^{y,\lambda}\Big\{e^{-\int_0^\lambda V(X_u)du}\Big\}dy \\
&= \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} E_x\Big\{e^{-\int_0^\lambda V(X_u)du}\Big\} \\
&\leq \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} E_x\Big\{e^{\int_0^\lambda V_-(X_u)du}\Big\} \\
&\leq c\, e^{c\lambda},
\end{aligned}$$

using (4.2.47). This estimate implies (A1) in Assumption 3.6.33. For (A2) we use the semigroup property

$$\begin{aligned}
&\operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \int |e^{-\lambda(H_0\dot{+}V)}(x,y)|^2 dy \\
&= \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} e^{-2\lambda(H_0\dot{+}V)}(x,x) \\
&\leq c\, e^{c\lambda}\Big[\operatorname*{ess\,sup}_{x\in\mathbb{R}^d} (e^{-\lambda H_0})(x,x)\Big]^{\frac{1}{2}}\Big[\operatorname*{ess\,sup}_{x\in\mathbb{R}^d} (e^{-2\lambda H_0})(x,x)\Big]^{\frac{1}{2}} \\
&\leq c\, \lambda^{-\rho}\, e^{c\lambda},
\end{aligned}$$

where we used (4.2.57) and Remark 4.2.9. ρ depends on the order of H_0 and of the dimension d (see (5.2.52)). Similarly, we get the estimate

$$\operatorname*{ess\,sup}_{x,y} |e^{-\lambda(H_0\dot{+}V)}(x,y)| \leq\ c\,\lambda^{-\rho}\, e^{c\lambda}. \qquad \square$$

Hence the theory of Section 3.2.3 is applicable. The comparison functions for the respective resolvent semigroup differences are denoted by

$$\begin{aligned}
R_p^{V,a}(.) &= \int_{\mathbb{R}^d} |R_p^{V,a}(.\,,y)|dy\ , \\
S_{V,t}(.) &= \int_{\mathbb{R}^d} |S_{V,t}(.\,,y)|dy\ , \\
R_p^{\Sigma,a}(.) &= \int_{\mathbb{R}^d} |R_p^{\Sigma,a}(.\,,y)|dy\ , \\
S_{\Sigma,t}(.) &= \int_{\mathbb{R}^d} |S_{\Sigma,t}(.\,,y)|dy\ .
\end{aligned}$$

Estimating the comparison functions the parallel structure in the case of obstacle and potential perturbations becomes clear. For obstacles the comparison functions are determined by the equilibrium potential v_Γ, introduced in Definition 4.1.20. For potentials the corresponding role is taken by $[(H_0 + M_V + a)^{-1}|V|](.)$. This becomes obvious in the next proposition.

Proposition 5.2.6. *The comparison functions can be estimated as follows:*

$$R_p^{V,2a}(x) \leq \frac{c}{a}\ [(H_0 \dot{+} M_V + a)^{-1}|V|](x),$$

and in particular

$$R_1^{V,a}(x) \leq \frac{1}{a}\ [(H_0 \dot{+} M_V + a)^{-1}|V|](x).$$

Moreover

$$S_{V,t}(x)\ \leq e^{at}\ [(H_0 \dot{+} M_V + a)^{-1}|V|](x).$$

On the other hand

$$\begin{aligned} R_p^{\Sigma,2}(x) &\leq c\ v_\Gamma(x), \\ R_1^{\Sigma,1} &= v_\Gamma(x), \\ S_{\Sigma,t}(x) &\leqq e^t\ v_\Gamma(x). \end{aligned}$$

As usual c are different positive constants.

Proof: For proving these inequalities we integrate the kernels in Proposition 5.2.5 with respect to one of the variables. The first inequality follows from

$$\begin{aligned}
&R_p^{V,2a}(x) \\
&= \frac{1}{\Gamma(p)} \int_{\mathbb{R}^d} dy \Big| \int_0^\infty d\lambda\ e^{-2a\lambda}\ \lambda^{p-1} \\
&\qquad\qquad \times \Big[e^{-\lambda(H_0 \dot{+} V)}(x,y) - e^{-\lambda H_0}(x,y)\Big] \Big| \\
&= \frac{1}{\Gamma(p)} \int_{\mathbb{R}^d} dy \Big| \int_{\mathbb{R}^d} du \int_0^\infty d\lambda\ e^{-2a\lambda}\ \lambda^{p-1} \\
&\qquad \times \int_0^\lambda ds\ e^{-s(H_0 \dot{+} V)}(x,u)\ V(u)\ e^{-(\lambda-s)H_0}(u,y) \Big| \\
&\leqq \frac{1}{\Gamma(p)} \int_0^\infty d\lambda \int_0^\infty ds\ e^{-2a\lambda}\ \lambda^{p-1} \int_{\mathbb{R}^d} du\ e^{-s(H_0 \dot{+} V)}(x,u)|V(u)| \\
&= \frac{1}{\Gamma(p)} \int_0^\infty ds \int_s^\lambda d\lambda\ e^{-2a\lambda}\ \lambda^{p-1} \big(e^{-s(H_0 \dot{+} V)}|V|\big)(x) \\
&\leq \frac{c}{a} \int_0^\infty ds\ e^{-as} \big(e^{-s(H_0 \dot{+} V)}|V|\big)(x) \\
&= \frac{c}{a} \big[(H_0 \dot{+} M_V + a)^{-1}|V|\big](x)\ .
\end{aligned}$$

The second resolvent equation yields

$$
\begin{aligned}
R_1^{V,a}(x) &= \int_{\mathbb{R}^d} dy \left| \int_{\mathbb{R}^d} du \ (H_0 \dot{+} M_V + a)^{-1}(x,u) V(u) (H_0 + a)^{-1}(u,y) \right| \\
&\le \frac{1}{a} \big((H_0 \dot{+} M_V + a)^{-1} |V| \big)(x) \ ,
\end{aligned}
$$

because

$$
\int_{\mathbb{R}^d} dy \ (H_0 + a)^{-1}(u,y) = \int_{\mathbb{R}^d} dy \int_0^\infty d\lambda \ e^{-a\lambda} \ e^{-\lambda H_0}(x,y) = \frac{1}{a} \ .
$$

For the semigroup difference Duhamel's formula gives

$$
\begin{aligned}
S_{V,t}(x) &= \int_{\mathbb{R}^d} dy \left| \int_0^t ds \int_{\mathbb{R}^d} du \ e^{-s(H_0 \dot{+} V)}(x,u) V(u) \ e^{-(t-s)H_0}(u,y) \right| \\
&\le \int_0^t ds \ \big(e^{-s(H_0 \dot{+} V)} |V| \big)(x) \\
&\le e^{at} \int_0^t ds \ e^{-as} \big(e^{-s(H_0 \dot{+} V)} |V| \big)(x) \\
&= e^{at} \big((H_0 \dot{+} M_V + a)^{-1} |V| \big)(x) \ .
\end{aligned}
$$

For the obstacle differences we use Dynkin's formula (see Proposition 4.2.24):

$$
\begin{aligned}
R_p^{\Sigma,2}(x) &= \int_{\mathbb{R}^d} dy \left[(H_0 + 2)^{-p}(x,y) - \big(J^* \left[(H_0)_\Sigma + 2 \right]^{-p} J \big)(x,y) \right] \\
&= \frac{1}{\Gamma(p)} \int_0^\infty d\lambda \ \lambda^{p-1} \ e^{-2\lambda} \ E_x\{\tau_\Gamma < \lambda\} \\
&\le c \ E_x \left\{ \int_{\tau_\Gamma}^\infty e^{-\lambda} \ d\lambda \right\} \\
&= c \ E_x \left\{ e^{-\tau_\Gamma} \right\} \\
&= c \ v_\Gamma(x), \qquad (c \text{ depends on } \ p).
\end{aligned}
$$

$R_1^{\Sigma,1}(x) = v_\Gamma(x)$ is obvious. Finally,

$$
\begin{aligned}
S_{\Sigma,t}(x) &= E_x\{\tau_\Gamma > t\} \\
&\leqq e^t E_x\{e^{-\tau_\Gamma}\} \\
&= e^t v_\Gamma(x).
\end{aligned}
$$

□

In Section 3.2.2 and 3.2.3 we emphasised that the L^1-norm of the comparison function plays the essential role for the stability of the absolutely continuous spectrum and for the continuity of the wave operators, whereas for the essential spectrum the L^2-norm is responsible.

For potential perturbations the L^1-norm of all comparison functions are determined by

$$\int_{\mathbb{R}^d} \big((H_0 \dot{+} M_V + a)^{-1}|V|\big)(x)\ dx\ .$$

If a is large enough this can be estimated by the L^1-norm of V because

$$\begin{aligned}
&\int_{\mathbb{R}^d} du \int_0^\infty d\lambda\ e^{-a\lambda} \int_{\mathbb{R}^d} dx\ e^{-\lambda(H_0 \dot{+} V)}(x,u)|V(u)| \\
&= \int_{\mathbb{R}^d} du \int_0^\infty d\lambda\ e^{-a\lambda}\ E_u\Big\{e^{-\int_0^\lambda V\left(X_s\right)ds}\Big\}|V(u)| \\
&\le\ c \int_0^\infty d\lambda\ e^{-(a-c)\lambda}||V||_{L^1}\ ,
\end{aligned}$$

where we used (4.2.47) because V is a Kato–Feller potential.

The L^2-norm of the comparison function can be estimated by the L^2-norm of V.

$$\begin{aligned}
&\int_{\mathbb{R}^d} dx \left|\int_0^\infty d\lambda\ e^{-a\lambda} \int_{\mathbb{R}^d} du\ e^{-\lambda(H_0 \dot{+} V)}(x,u)V(u)\right|^2 \\
&\qquad \le \int dx\ \frac{1}{a}\int_0^\infty d\lambda\ e^{-a\lambda} \int_{\mathbb{R}^d} du\ e^{-\lambda(H_0 \dot{+} V)}(x,u) \\
&\qquad\qquad \times \int_{\mathbb{R}^d} dv\ e^{-\lambda(H_0 \dot{+} V)}(x,v)|V(v)|^2 \\
&\qquad \le\ \frac{c}{a}\int_0^\infty d\lambda\ e^{-a\lambda}\ e^{c\lambda}||V||^2_{L^2}\ .
\end{aligned}$$

Here we use Equation (4.2.47) twice. For the obstacles we know that

$$\int v_{\Gamma}(x)\ dx =\ \mathrm{cap}\ (\Gamma)$$

(see Corollary 4.1.22).

Of course the L^2-norm of v_{Γ} can also be estimated by $\mathrm{cap}\ (\Gamma)$.

Thus we have obtained the following result.

Corollary 5.2.7 *Let H_0 be one of the operators $-\Delta, (-\Delta)^\alpha, \alpha \in (0,1)$, $\sqrt{-\Delta + c^2} - c$. Let V be a Kato–Feller potential. Let Γ be a closed set in $\mathbb{R}^d$. Set $\Sigma = \mathbb{R}^d \setminus \Gamma$. Let $(H_0)_\Sigma$ be defined as in Corollary 4.2.22.*

a) Then we know in the potential case
- $\sigma_{ac}(H_0 \dot{+} M_V) = \sigma_{ac}(H_0)$ *if* $V \in L^1(\mathbb{R}^d)$.
- $\sigma_{ess}(H_0 \dot{+} M_V) = \sigma_{ess}(H_0)$ *if* $V \in L^2(\mathbb{R}^d) + L^1(\mathbb{R}^d)$.
- *The wave operators $\Omega_\pm(H_0 \dot{+} M_V, H_0)$ exist and are complete. They are continuous with respect to the potential in the sense*

$$||\big(\Omega_\pm(H_0 \dot{+} M_V, H_0) - \mathbb{1}\big)f|| \le\ c_f\ ||V||_{L^1},$$

for all f in dense set of $\mathfrak{H}_{ac}(H_0)$.

b) For the obstacle perturbations we have

$$\sigma_{ac}\big((H_0)_\Sigma\big) = \sigma_{ac}(H_0)$$

and

$$\sigma_{ess}\big((H_0)_\Sigma\big) = \sigma_{ess}(H_0)$$

if cap (Γ) is finite.
The wave operators $\Omega_\pm\big((H_0)_\Sigma,\ J, H_0\big)$ exist and are continuous in terms of the capacity, i.e.,

$$\Big\| \Big(\Omega_\pm\big((H_0)_\Sigma,\ J, H_0\big) - \mathbb{1}\Big) f \Big\| \ \leq\ c_f \ \mathrm{cap}\ (\Gamma)$$

for f in a dense set of $\mathfrak{H}_{ac}(H_0)$.

Remark 5.2.8. The results for Kato–Feller potentials are not surprising. They can be proved by several methods. Only the continuity of wave operators in this generality seems to be of some interest.

More interesting are the obstacle perturbations. Note that also unbounded sets Γ can have finite capacities. The continuity of the wave operators in terms of cap (Γ) can be used also for further considerations in scattering theory, for instance for estimating the scattering amplitudes or the scattering phases.

One can construct examples where $S_{\Sigma,t}$ is a Hilbert–Schmidt operator with "finite trace" i.e., for which

$$\int_{\mathbb{R}^d} S_{\Sigma,t}(x,x)\ dx < \infty\ ,$$

but $S_{\Sigma,t}$ is not a trace class operator.

As long as the free operators H_0 consist of functions of $-\Delta$, which is true in our examples, one can show the absence of singularly continuous spectrum. The singularly continuous spectrum of $(H_0)_\Sigma$ or $H_0 \dotplus M_V$ is empty, if

$$\int_{\mathbb{R}^d} S_{\Sigma,t}(x)(1+|x|^2)^{s/2} dx < \infty\ ,$$

or if

$$\int_{\mathbb{R}^d} S_{V,t}(x)(1+|x|^2)^{s/2} dx < \infty\ ,$$

respectively, for some $s > 1$. In terms of the potentials these condition are satisfied if

$$\int_{\mathbb{R}^d} v_\Gamma(x)(1+|x|^2)^{s/2} dx < \infty\ ,$$

or if

$$\int_{\mathbb{R}^d} |V(x)|(1+|x|^2)^{s/2} dx < \infty\ ,$$

$s > 1$, respectively. The condition with the equilibrium potential v_Γ allows also unbounded star shaped Γ.

Finally, we will derive two examples for illustrating Corollary 3.6.20. The obstacle perturbation can be approached by increasing potential barriers on the obstacle region Γ, which means the following.

Let H_0 be given as before. Take a bounded potential function $\beta \mathbb{1}_\Gamma(x)$, where $\mathbb{1}_\Gamma(.)$ is the indicator function of Γ and $\beta > 0$. Assume that Γ has an interior and suppose $\Gamma = \Gamma^r = (\mathrm{int}(\Gamma))^r$, where $(\mathrm{int}\ (\Gamma))^r$ are the τ_Γ −regular points of int (Γ) (see Definition 4.2.20).

The operator $H_0 + \beta M_{\mathbb{1}_\Gamma}$ is well defined by the Kato–Rellich Theorem (Theorem 2.1.22), it is positive and selfadjoint. Setting $H_\beta = H_0 + \beta M_{\mathbb{1}_\Gamma}$ the Feynman–Kac formula reads

$$\big(e^{-tH_\beta} f\big)(x) = E_x\Big\{e^{-\beta \int_0^t \mathbb{1}_\Gamma\left(X_s\right) ds} f\big(X_t\big)\Big\} \ ,$$

$f \in L^2(\mathbb{R}^d)$. In Section 4.2.3 we have already introduced

$$\begin{aligned} T_{\Gamma,t} &= \text{meas } \{s,\ s \le t,\ X_s \in \Gamma\} \\ &= \int_0^t \mathbb{1}_\Gamma\big(X_s\big) ds \ . \end{aligned}$$

$T_{\Gamma,t}$ is called the spending time of the trajectory in Γ. Compare this with (4.2.68), (4.2.69).

Under the present assumptions on Γ we have $T_{\Gamma,t} > 0$ iff $\tau_\Gamma < t$, or $T_{\Gamma,t} = 0$ iff $\tau_\Gamma \ge t$. Considering the convergence in β one has, recalling that $\Sigma = \mathbb{R}^d \setminus \Gamma$,

$$\begin{aligned} \lim_{\beta\to\infty} (e^{-tH_\beta} f)(x) &= \lim_{\beta\to\infty} (e^{-tH_\beta} \mathbb{1}_\Gamma f)(x) + \lim_{\beta\to\infty} (e^{-tH_\beta} \mathbb{1}_\Sigma f)(x) \\ &= \lim_{\beta\to\infty} E_x\Big\{(e^{-\beta T_{\Gamma,t}} \mathbb{1}_\Gamma\big(X_t\big) f\big(X_t\big),\ T_{\Gamma,t} > 0\Big\} \\ &\quad + E_x\Big\{\mathbb{1}_\Sigma\big(X_t\big) f\big(X_t\big),\ T_{\Gamma,t} = 0\Big\} \\ &\quad + \lim_{\beta\to\infty} \Big\{\mathbb{1}_\Sigma\big(X_t\big) f\big(X_t\big) e^{-\beta T_{\Gamma,t}},\ T_{\Gamma,t} > 0\Big\} \\ &= E_x\Big\{\mathbb{1}_\Sigma\big(X_t\big) f\big(X_t\big);\ T_{\Gamma,t} = 0\Big\} \\ &= E_x\Big\{\mathbb{1}_\Sigma\big(X_t\big) f\big(X_t\big),\ \tau_\Gamma < t\Big\}. \end{aligned}$$

Hence

$$\begin{aligned} \text{s} - \lim_{\beta\to\infty}\ e^{-tH_\beta} \mathbb{1}_\Gamma &= 0, \\ \text{s} - \lim_{\beta\to\infty} \mathbb{1}_\Gamma\ e^{-tH_\beta} &= 0, \end{aligned}$$

and

$$\text{s} - \lim_{\beta\to\infty} \mathbb{1}_\Sigma\ e^{-tH_\beta} \mathbb{1}_\Sigma = e^{-t(H_0)_\Sigma} \ . \tag{5.2.53}$$

Thus Corollary 3.6.20 and Corollary 3.6.28 open the possibility to study the absolutely continuous spectrum of $(H_0)_\Sigma$ by clarifying when the trace norm of $e^{-tH_0}(e^{-tH_0} - e^{-tH_\beta})e^{-tH_\beta} = K_\beta$ is uniformly bounded in β (see (3.6.52)). Note that K_β does not contain $(H_0)_\Sigma$. The semigroups in K_β satisfy all the conditions of Assumption 3.6.40. Hence $||K_\beta||_{\mathrm{tr}}$ can be estimated by the L^1−norm of the associated comparison function, i.e.,

$$
\begin{aligned}
||K_\beta||_{\mathrm{tr}} \;&\leq\; c \int_{\mathbb{R}^d} dx\; E_x\{\mathbb{1} - e^{-\beta T_{\Gamma,t}}\} \\
&=\; c \int_{\mathbb{R}^d} dx\; E_x\{\mathbb{1} - e^{-\beta T_{\Gamma,t}},\; T_{\Gamma,t} > 0\} \\
&\leq c \int_{\mathbb{R}^d} dx\; E_x\{T_{\Gamma,t} > 0\} \\
&\leq c \int_{\mathbb{R}^d} dx\; E_x\{\tau_\Gamma < t\} \\
&\leq\; c\, e^t \int_{\mathbb{R}^d} v_\Gamma(x) dx \\
&=\; c\, e^t \;\mathrm{cap}\;(\Gamma)\;.
\end{aligned}
$$

Using this approximation we get the same result as in Proposition 5.2.6. However notice we have not used the definition of $(H_0)_\Sigma$ directly. Here $(H_0)_\Sigma$ plays only the role of the strong resolvent limit of H_β.

The same procedure is possible for instance for negative δ-like potentials. Consider $L^2(\mathbb{R})$ and assume that the formal operator $H_\delta = H_0 - \delta(a)$ is defined as the strong resolvent limit of $H_\varepsilon = H_0 - \frac{1}{\varepsilon}\mathbb{1}_{[a-\varepsilon,a+\varepsilon]}$ as $\varepsilon \downarrow 0$. The Feynman–Kac formula given in (4.2.53), in Remark 4.2.7 does not make sense for H_δ. But it can be applied to H_ε as long as $\varepsilon > 0$. Therefore we use again Corollary 3.6.28 and obtain $\sigma_{ac}(H_\delta) = [0,\infty)$ if

$$||e^{-H_\varepsilon}\big(e^{-H_\varepsilon} - e^{-H_0}\big)e^{-H_0}||_{\mathrm{tr}} < M_1\;,$$

with a constant M_1, which has to be independent of ε. Using Corollary 5.2.7 we need

$$\frac{1}{\varepsilon} \int_{\mathbb{R}^1} \chi_{[a-\varepsilon,a+\varepsilon]}(x)\; dx = 2$$

and

$$\sup_x\; E_x\Big\{e^{\frac{1}{\varepsilon}\;\int_0^t \chi_{[a-\varepsilon,a+\varepsilon]}\big(X_s\big)\; ds}\;\Big\} <\; M_2$$

with $M_2 \neq M_2(\varepsilon)$. By the lemma of Kashminskii (see Remark 4.2.3) it suffices to have

$$\sup_x \int_0^{t_0}\; ds\; E_x\Big\{\frac{1}{\varepsilon}\chi_{[a-\varepsilon,a+\varepsilon]}\big(X_s\big)\Big\} \leq\; \alpha < 1,$$

for t_0 small enough and with $\alpha \neq \alpha(\varepsilon)$.

To satisfy the last condition we consider

$$\begin{aligned}
&\frac{1}{\varepsilon} \sup_x \ E_x\Big\{ \int_0^{t_0} \ ds \ \chi_{[a-\varepsilon,a+\varepsilon]}\big(X_s\big)\Big\} \\
&= \frac{1}{\varepsilon} \sup_x \int_0^{t_0} \ ds \int\limits_{[a-\varepsilon,a+\varepsilon]} dy \quad e^{-sH_0}(x,y) \\
&= \frac{1}{\varepsilon} \int_0^{t_0} \ ds \int_{a-\varepsilon}^{a+\varepsilon} \ dy \ s^{-\rho} \\
&= 2 \int_0^{t_0} \ s^{-\rho} \ ds \ .
\end{aligned}$$

That is independent of ε and becomes small enough for small t_0 if $\rho < 1$. Hence the result is applicable for $d = 1$ and $\rho = \frac{1}{2\alpha}$ for $H_0 = (-\Delta)^\alpha$ and $\rho = \frac{1}{2}$ for $H_0 = -\Delta$.

5.3 Notes

Section 5.1.1

In the one-dimensional spectral theory of random Schrödinger operators, most of the results use a combination of the behaviour of solutions of the difference of differential equations together with the properties of Borel transforms. The absence of absolutely continuous spectrum on the set $\{E : \gamma(E) = 0\}$ (when this set has positive Lebesgue measure) is a result of Pastur and the other direction that the absolutely continuous spectrum is the essential closure of this set is due to Kotani.

In this section the existence of Lyapunov exponent, Lemma 5.1.2, can be found in Theorem 11.3, discrete case of Equation (11.4), Figotin–Pastur [83]. The Thouless formula mentioned in the proof of Proposition 5.1.8 is proved in Theorem 11.6 of Figotin–Pastur [83]. In the literature the density of states are defined as follows. Consider a cube Λ_L of side length L centered at the origin and consider H^ω restricted to Λ_L with Dirichlet boundary conditions at the boundaries of Λ_L, call the resulting matrix H^ω_L. It is shown that the quantity $\lim_{L\to\infty} \frac{1}{L^d}\#\{x \in \sigma(H^\omega_L) : x \leq E\}$ exists, is independent of ω for almost every ω and is the distribution function of a measure which is the same as dn we defined as the density of states measure in the text. We refer to Figotin–Pastur [83] for details.

The reflectionless property of ergodic potentials in one dimension was first observed by Craig [53].

In the case of finite-valued potentials Kotani identified the case when there can be absolutely continuous spectrum. Consider a finite subset S of $\mathbb{R}$ and consider the set of potentials $\Omega = S^{\mathbb{Z}}$ equipped with the metric $d(x,y) = \sum_{n\in\mathbb{Z}} 2^{-|n|}|x_n - y_n|$. Then Ω is a metric space and consider potentials q

coming from the support of some probability measure $\mathbb{P}$ invariant and ergodic with respect to the action of $\mathbb{Z}$ on $S^{\mathbb{Z}}$ by shifts. The potential is periodic if and only if the support of $\mathbb{P}$ is a finite set. Kotani [128] proved that if there is any absolutely continuous spectrum for such finite-valued random potentials the potential has to be periodic. The precise statement is the following.

Theorem 5.3.1 (Kotani). *Let S, Ω and $\mathbb{P}$ be as above and consider the random operators $H_\omega = \Delta + q^\omega$ on $\ell^2(\mathbb{Z})$, $\omega \in \operatorname{supp}(\mathbb{P})$. If $\sigma_{ac}(H_\omega) \neq \emptyset$, then $\mathbb{P}$ is supported on a finite set in $S^{\mathbb{Z}}$.*

One of the operators which was widely studied in one dimension is the almost Mathieu operator which is given by

$$H_{\lambda,\beta,\omega} = \Delta + \lambda \cos(\beta \cdot + \omega).$$

It is clear that when β is a rational multiple of 2π, the above operator is periodic and hence its spectrum is purely absolutely continuous for all λ, ω. On the other hand when β is an irrational multiple of 2π, then there is rich spectral structure and the most general theorem for this operator is by Jitomirskaya [102] which is:

Theorem 5.3.2. *Consider $H_{\lambda,\beta,\omega}$ as above. Then for almost all $\beta \in \mathbb{R}, \omega \in \mathbb{R}$, the spectrum $\sigma(H_{\lambda,\beta,\omega})$ is*

(i) pure point with exponentially decaying eigenfunctions for $\lambda > 2$.
(ii) It is purely singular continuous for $\lambda = 2$ and
(iii) it is purely absolutely continuous for $0 \leq \lambda < 2$.

There is a vast literature on various almost periodic operators in one dimension which we have not even touched upon. See for example Last–Simon [142], del Rio–Jitomirskaya–Last–Simon [60], Schlag [171] and Bourgain [30] for getting an overview of the many developments in one- and two-dimensional random operators subsequent to the books of Carmona–Lacroix [39] and Figotin–Pastur [83].

Section 5.1.2

Rigorous proof of localization in the Anderson model was a long standing open problem subsequent to the work of P. Anderson on the transport properties of disordered systems. In the early 1980s Fröhlich and Spencer obtained a fundamental estimate, showing exponential decay of the Green's function with distance, for the Anderson model in the large disorder and large energy regime valid on a set of large positive measure. This estimate, which used an analysis Green's function at "multiple scales" (multi-scale analysis), quickly led to proofs of localization in these models. There were many refinements of this technique, which required as a part the so-called Wegner estimate on the density of states. A good book for this theory is that of Stollmann [184]. The proof was simplified substantially by Aizenman–Molchanov [6]. Localization is also exhibited recently for a class of almost periodic operators in two dimensions, see [30].

The material presented here follows Aizenman–Molchanov [6], Aizenman–Graf [5] and Aizenman [4]. Our proof of Lemma 5.1.14 follows the one given in Aizenman–Graf [5].

Section 5.1.3

The Bethe lattice is a strange object; it disconnects into several equivalent pieces on removing a vertex, similar to a one-dimensional lattice while having exponentially many points at a fixed distance from the origin in some sense behaving like an infinite-dimensional lattice far away from the origin. For the Anderson model on the Bethe lattice both localizations, in the high disorder regime and extended states in the low disorder regime, are known. We present here only the proof of the existence of extended states by Abel Klein [124]. The determination of the spectrum of the Laplacian is simplified in our presentation.

Section 5.1.4

One of the major open problems in the spectral theory of the Anderson model is to exhibit extended states in the low disorder regime for high dimension. The theorem of Jaksić–Last presented here from the work [94] shows that the existence of extended states also automatically guarantees their purity, under some very general conditions. This theorem was applied to the case of decaying randomness where only existence can be shown by some methods.

Section 5.2.1

In one dimension the case of slowly decreasing potentials both in the random and deterministic settings have been considered. In the case of random potentials with decay, in one dimension, Kotani–Ushiroya [131], Delyon–Simon–Souillard [61], Remling [137, 163, 162, 160, 161], Christ–Kiselev [46, 45], Kiselev [120], Last–Simon [142], Molchanov–Novitskii–Vainberg [146] have considered such potentials and the general rule has been that if the decay of the random potential is faster than the inverse square root of the distance, then there is absolutely continuous spectrum and if the decay is slower than the inverse square root, then there is pure point spectrum and in the borderline case there is even some singular continuous spectrum. In this direction Deift–Killip [58] proved a result, for the presence of absolutely continuous spectrum, with decaying randomness in one dimension using a relation between the transmission coefficient and the potential. To state their result in the case of half-line operators consider the half-line Schrödinger operator on $L^2(\mathbb{R}^+)$ with Dirichlet boundary condition at 0,

$$H_D = -\Delta + V, \quad \text{on} \quad \{u \in \operatorname{dom}(\Delta) : u(0) = 0\},$$

or the half-line discrete Schrödinger operator

$$h_D = \Delta_D + q, \quad (\Delta_D u)(n) = (\Delta u)(n), \quad n > 1, \quad (\Delta_D u)(1) = u(2).$$

on $\ell^2(\mathbb{Z}^+)$. Then Deift–Killip [58] show that perturbations by square integrable potentials in the above cases do not alter the absolutely continuous spectrum. Their theorem is the following.

Theorem 5.3.3. *Consider the operators H_D with $V \in L^2(\mathbb{R}^+)$ or h_D with $q \in \ell^2(\mathbb{Z}^+)$. Then*

(i) any total spectral measure μ of H_D satisfies $\overline{\mathrm{supp}(\mu_{ac})} = [0, \infty)$ and
(ii) any total spectral measure μ of h_D satisfies $\mathrm{supp}(\mu_{ac}) = [-2, 2]$.

Since the full line Schrödinger operators differ from the direct sum of two half-line operators by a rank one perturbation, this theorem also easily extends to the full line case.

In the multi-dimensional case an absolutely continuous spectrum for random Anderson type models with decaying randomness is shown by Krishna [134, 133] using scattering theory and subsequently by Molchanov [145], Jaksić–Molchanov [97, 99, 98, 100], Molchanov–Vainberg [147], Jaksić–Last [94, 95, 96], Anne Boutet de Monvel–Sahbani [31, 32], Hundertmark–Kirsch [93], Kirsch [114], Krutikov [136] and Bourgain [29] for various models. Kirsch–Krishna–Obermeit [115] showed the existence of a mobility edge in some multi-dimensional models of the Anderson type with decaying randomness and Jaksić–Last [141] showed the purity of the absolutely continuous part for these models. The theorem of [115] involve a condition relating the sequence a_n and the measure μ, In this context we note that if $\sum |a_n|^p < \infty$ and μ has a finite p-th moment, then the associated operator V^ω is compact for almost every ω, hence there is no essential spectrum for H^ω outside the spectrum of $\varDelta$, as observed by Delyon–Simon–Souillard [61].

In the early years of the study of random Schrödinger operators in one dimension, there was a model known as the "Maryland model" which involved studying random potentials of the form $\tan(\alpha n + \omega)$ and these gave unbounded random potentials. Simon–Spencer [180] proved that in general any unbounded perturbation of the lattice Laplacian in one dimension cannot have any absolutely continuous spectrum. Their method of proof involves showing that such an operator and a direct sum of matrices have resolvents which differ by a trace class operator. Hence by the trace class theory of scattering (Theorem 3.6.9) there cannot be any absolutely continuous spectrum for the original operator. The "Maryland model" is studied in higher dimension also by Jaksić–Molchanov [100]. Both decaying and growing randomness is also considered in the paper [135].

Section 5.2.2

The applications follow more or less directly from the theory in Section 3.2. Instead of the conditions in Theorem 3.6.18 for the stability of σ_{ac} it would be also sufficient if $e^{-t(H_0 \dot{+} V)} - e^{-tH_0}$ or if $e^{-tH_0} - J^* e^{-t(H_0)^{\Sigma}} J$ is trace class. Using the CDS^2-trace class criterion (Theorem 3.6.30) these semigroup differences are trace class if $\int_{\mathbb{R}^d} \sqrt{|V(x)|} dx < \infty$, or if $\int_{\mathbb{R}^d} \sqrt{v_\Gamma(x)} dx < \infty$, respectively (compare it with Theorem 3.6.42 (i)). These is more restrictive than $V \in L^1, \; v_\Gamma \in L^1$.

The results in Corollary 5.2.7 are given in [62]. The continuity of the wave operators in terms of cap (Γ) is new. The continuity of wave operators is studied in some detail by Brüning–Gesztesy [37]. One can also find some results in the book of Reed–Simon [158] p. 74. The result is not restricted to the wave operators. At least for the Laplacian it can be extended to scattering phases proved by Demuth–McGillivray in [66]. In Remark 5.2.8 we mentioned examples of unbounded Γ with finite capacities. For $(-\Delta)^\alpha$ such examples are given by Giere [88] or by Baro–Demuth–Giere [18]. Examples of "non-trace class" operators with finite trace were considered by Demuth–Stollmann–van Casteren [71]. They study $H_0 = -\Delta$ and $\Gamma = \underset{n}{\cup} B(a_n, r_n)$, a union of balls centered in a_n with decreasing radii. Then the capacity of Γ is finite if $\sum\limits_{n=1}^{\infty} r_n^{d-2} < \infty$. For sets of finite capacity we know that $\int_{\mathbb{R}^d} \; S_{\Sigma,t}(x,x)dx < \infty$. On the other hand, a necessary condition for $S_{\Sigma,t}$ to be a trace class operator is that $\mathbb{1}_\Gamma e^{t\Delta}$ is trace class. Then necessarily $\sum\limits_{n=1}^{\infty} r_n^{d/2} < \infty$. Assume $r_n = \left(\frac{1}{n}\right)^{2/5}$ and $d = 5$; then $\sum\limits_{n=1}^{\infty} \left(\frac{1}{n}\right)^{6/5} < \infty$ but $\sum\limits_{n=1}^{\infty} \left(\frac{1}{n}\right) = \infty$.

The conditions for empty singularly continuous spectra goes back to the theory of Enss [82]. Further details for the Laplacian are given by Demuth-Sinha [69] and for $(-\Delta)^\alpha$ and $\sqrt{-\Delta + c^2} - c$ in [18].

Increasing potential barriers, i.e., convergence of $H_\beta = H_0 + \beta M_{\mathbb{1}_\Gamma}$ are considered in the book by Demuth-van Casteren [65] in Chapter 7. If we assume that $\Gamma = \Gamma^r = (\text{int }(\Gamma))^r$, then the first hitting time τ_Γ and the penetration time σ_Γ (see also the notes for Section 4.2.3), $\sigma_\Gamma = \inf\{s > 0, \; T_{\Gamma,s} > 0\}$ coincide. A proof is given in [65] Proposition 2.24. Hence $E_x\{\mathbb{1}_\Sigma(X_t) f(X_t), \; T_{\Gamma,t} = 0\} = E_x\{\mathbb{1}_\Sigma(X_t) f(X_t), \; \tau_\Gamma < 0\}$, which is used in the proof for Equation (5.2.53). To consider the sequences of operators for studying the spectral behaviour of its limit seems to be new. The examples where H_Σ models negative δ-like potentials open the extension of Feynman-Kac methods to such perturbations. For the Laplacian, examples in $\mathbb{R}^2$ are possible with δ-like potentials on a finite curve in $\mathbb{R}^2$.

If $H_0 = -\Delta$, then $(H_0)_\Sigma$ is the Laplacian with Dirichlet boundary condition on $\delta\Gamma$. Therefore the considerations seems to restricted to imposing Dirichlet boundary condition. However there is a trick to include also Neumann boundary conditions. We write $-\Delta_\Sigma^N$ for the Neumann Laplacian. Then it holds, in general,

$$e^{t\Delta_\Sigma^D} \;\le\; e^{t\Delta}$$

and

$$e^{t\Delta_\Sigma^D} \;\le\; e^{t\Delta_\Sigma^N} \; .$$

But there is no relation between $e^{t\Delta_\Sigma^N}$ and $e^{t\Delta}$. However, for the comparison function we get an estimate, namely

$$\left(e^{t\Delta} - e^{t\Delta_\Sigma^N}\right)(x) \le \left(e^{t\Delta} - e^{t\Delta_\Sigma^D}\right)(x) \le e^t v_\Gamma(x) \; .$$

This means that $\sigma_{ac}(-\Delta_\Sigma^N) = [0, \infty]$ if cap (Γ) is finite.

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