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Integrable Hamiltonian Hierarchies

Spectral and Geometric Methods

Springer

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Gerdjikov, V.S. et al., *Integrable Hamiltonian Hierarchies: Spectral and Geometric Methods*, Lect. Notes Phys. 748 (Springer, Berlin Heidelberg 2008), DOI 10.1007/978-3-540-77054-1

ISBN: 978-3-540-77053-4 e-ISBN: 978-3-540-77054-1

DOI 10.1007/978-3-540-77054-1

Lecture Notes in Physics ISSN: 0075-8450

Library of Congress Control Number: 2008923362

Cover design: eStudio Calamar S.L., F. Steinen-Broo, Pau/Girona, Spain

Printed on acid-free paper

9 8 7 6 5 4 3 2 1

springer.com

Preface

In the past decades now a famous class of evolution equations has been discovered and intensively studied, a class including the nowadays celebrated *Korteweg-de Vries* equation, *sine-Gordon* equation, *nonlinear Schrödinger* equation, etc. The equations from this class are known also as *the soliton equations* or equations solvable by the so- called *Inverse Scattering Transform Method*. They possess a number of interesting properties, probably the most interesting from the geometric point of view of being that most of them are *Liouville integrable* Hamiltonian systems. Because of the importance of the soliton equations, a dozen monographs have been devoted to them. However, the great variety of approaches to the soliton equations has led to the paradoxical situation that specialists in the same field sometimes understand each other with difficulties. We discovered it ourselves several years ago during a number of discussions the three of us had. Even though by friendship binds us, we could not collaborate as well as we wanted to, since our individual approach to the field of integrable systems (finite and infinite dimensional) is quite different. We have become aware that things natural in one approach are difficult to understand for people using other approaches, though the objects are the same, in our case – *the Recursion (generating) Operators* and their applications to finite and infinite dimensional (not necessarily integrable) Hamiltonian systems. Since even between us, in order to overcome our differences, we needed some serious efforts, we decided that it was time to bring together the analytic and geometric aspects, if not of the theory of the soliton equations (this would be too ambitious) but at least the analytic and the geometric aspects of the so-called Recursion Operators, which are among the powerful tools for the study of soliton equations. We had to do it in such a way, that a specialist in one of the approaches can read and understand the value of the other approach. However, the material we started to collect soon began growing rapidly, and we realized that a book should be written on this topic. The realization of the book project took longer than we expected – more than six years. But now we are happy that we are able to present a text which in our opinion reflects our original ideas.

The book has two parts, the first is dedicated to the analytic approach to the Recursion operators, the second, to the geometric nature of these operators, that is, to their interpretation as mixed tensor fields with special geometric properties over the manifold of potentials.

As we mentioned, we expect that the book will be useful to specialists in the Recursion Operator approach to the soliton equations. However, with an intent to target a larger audience, we have included some other important topics, such as the construction of the soliton solutions, for example. We have tried to develop the material in such a way that the book proves useful for graduate students who want to enter this interesting field of research.

The present book is based on some material that has become already classical, as well as on some of our works. The last few have been written in collaboration with many other friends and colleagues, namely:

Sergio De Filippo, Giuseppe Marmo, Mario Salerno, Giovanni Landi, Yanus Grabowski, Andrei Borowiz, Giovanni Sparano, Alexandre Vinogradov, Patrizia Vitale, Fabrizio Canfora, Luca Parisi, Boris Florko, Ljudmila Bordag, Peter Kulish, Evgenii Khristov, David Kaup, Evgenii Doktorov, Mikhail Ivanov, Yordan Vaklev, Marco Boiti, Flora Pempinelli, Nikolay Kostov, Ivan Uzunov, Evstati Evstatiev, Georgi Diankov, Rossen Ivanov, Rossen Dandoloff, Georgi Grahovski, Assen Kyuldjiev, Viktor Enol'skii, Bakhtiyor Baizakov, Vladimir Konotop, Jianke Yang, Adrian Constantin, Tihomir Valchev, Victor Atanasov.

We would like to extend our thanks to all of them.

We are grateful to Evgenii Doktorov, Nikolay Kostov, David Kaup, Georgi Grahovski, Rossen Ivanov for careful reading of the manuscript and for the many useful discussions we had with them.

A crucial factor which helped us complete the book was the financial support of Istituto Nazionale di fisica Nucleare, Gruppo Collegato di Salerno, and of Dipartimento di Fisica "E.R.Caianiello" at the University of Salerno, Italy. We would also like to thank the Institute for Nuclear Research and Nuclear energy, Sofia and the University of Cape Town for their organizational and financial support, as well as the National science foundation of Bulgaria (contract No. 1410).

Sofia, Salerno, Cape Town
2000–2007

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Part I Integrable Hamiltonian Hierarchies: Spectral Methods

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Introduction

1.1 Brief Historical overview

Several of the nonlinear evolution equations to which this book is dedicated, the so-called soliton equations, are so important and so well known that they have become a paradigm. Three of them are

(i) The Korteweg-de Vries (KdV) equation [1, 2, 3, 4, 5, 6]:

$$v_t + v_{xxx} + 6v_x v(x,t) = 0 \; ; \tag{1.1}$$

(ii) The sine-Gordon (s-G) equation [7, 8]:

$$w_{xt} + \sin w(x,t) = 0 \; ; \tag{1.2}$$

(iii) The nonlinear Schrödinger (NLS) equation [9, 10]:

$$iu_t + u_{xx} + 2|u|^2 u(x,t) = 0 \; . \tag{1.3}$$

In fact, the first two were known for more than a century. The sine-Gordon (s-G) equation has been introduced in the middle of the 19th century in relation to the study of the surfaces with constant negative curvature [11]. The special properties of s-G, namely, the fact that one can obtain new non-trivial solutions starting from a trivial one was discovered also long ago by Bäcklund and Darboux [12, 13]. The transformations doing it are known today as Bäcklund and Darboux transformations.

The KdV equation was also discovered in the 19th century [14]; it describes hydrodynamic waves in shallow and narrow channels.

For rather a long time (more than a half century), the importance of these two equations was not fully realized, though there have been made some significant steps. For example, in 1971 Lamb Jr. [17] used the Bäcklund transformation to derive the n-soliton solutions of s-G. Studying them he predicted a new phenomenon: the self-induced transparency which soon was

Gerdjikov, V.S. et al.: *Introduction.* Lect. Notes Phys. **748**, 3–36 (2008)
DOI 10.1007/978-3-540-77054-1_1

discovered experimentally. A number of other important applications of the soliton equations to physical problems can be found in [9, 10, 16, 15, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67].

The special properties of the KdV equation started to be unveiled in the 1950s with the Fermi-Pasta-Ulam experiment [68]. These authors studied a chain of 64 anharmonic oscillators of mass m, whose evolution is described by the equations:

$$\begin{gathered} \frac{m}{K}\frac{d^2 y_i}{dt^2} = (y_{i+1} + y_{i-1} - 2y_i) + \alpha\left((y_{i+1} - y_i)^2 - (y_i - y_{i-1})^2\right) \ , \\ i = 1, 2, \dots, N-1, \qquad y_0 = y_N = 0 \ , \end{gathered} \tag{1.4}$$

N designates the number of the corresponding oscillators and K and α are constants. As a result, they discovered that the stochastization does not occur; in other words, there is no energy exchange between the different modes. Later, Kruskal and Zabusky used this results for the analysis of the KdV equation [69], which can be considered as the continuous limit of (1.4).

It has been discovered that this property is related to the fact that the KdV equation has (an infinite number of) higher conservation laws; see [1] and the references therein. These results helped to reveal that the evolution of the KdV equation is an isospectral deformation of the Sturm-Liouville equation:

$$-\frac{d^2\psi}{dx^2} + (v(x,t) - k^2)\psi(x,k) = 0 \ , \tag{1.5}$$

and finally lead Lax to the discovery of the inverse scattering method (ISM) [2, 5, 6].

This was the first case when for a *nonlinear* partial differential equation general Cauchy-type theorem of existence and uniqueness of solution was proved.

The intensive study of KdV continued, and soon another curios fact was discovered: the purely elastic interactions of the solitons, each preserving its velocity after the interaction. But for about three years since the Lax paper was published, the KdV equation was believed to be the only NLEE integrable by the ISM, possessing an infinite number of integrals of motion and N-soliton solutions.

However, in 1971, there appeared two pioneer papers [9, 70] that started the “blow-up instability” in the theory of solitons. They stimulated the interest of many scientists from various branches of Mathematics and Physics, and the special properties of KdV equation were fully understood.

In the first of these papers, Zakharov and Shabat [9] discovered a second nonlinear evolution equation (NLEE) integrable by the ISM: the now famous NLS equation (1.3). They showed that the NLS, just like the KdV, has an infinite number of integrals of motion and possesses N-soliton solutions, whose interaction is again purely elastic. Due to the large number of physical applications of the NLS equation, these results attracted the attention of a number of physicists working in plasma physics, nonlinear optics, superconductivity, etc. The soliton solutions available in explicit form allowed to understand a number of purely nonlinear phenomena that could not be explained by the old perturbative methods.

The third NLEE integrable by the ISM, the MKdV equation

$$w_t + w_{xxx} + 6\kappa_1 w_x w^2(x,t) = 0, \qquad \kappa_1 = \pm 1 . \tag{1.6}$$

was found by Wadati [71] in 1972.

It turned out that both the NLS [9] and the mKdV equations [71] allow Lax representation:

$$[L(\lambda), M(\lambda)] = 0 , \tag{1.7}$$

Their Lax representations can be handled more conveniently, if we choose as a Lax operator $L(\lambda)$ the so-called Zakharov-Shabat (ZS) system:

$$\begin{gathered} L(\lambda)\psi(x,t,\lambda) \equiv i\frac{d\psi}{dx} + (q(x,t) - \lambda\sigma_3)\,\psi(x,t,\lambda) = 0 , \\ q(x,t) = \begin{pmatrix} 0 & q^+ \\ q^- & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \end{gathered} \tag{1.8}$$

where $q^+(x,t) = u(x,t)$, and $q^-(x,t) = u^*(x,t)$, and the operator $M(\lambda)$ is polynomial of second degree in λ.

The above result stimulated both mathematicians and theoretical physicists to look for other, more general, Lax operators and new equations of soliton type. Such were found soon. In [7], Ablowitz, Kaup, Newell and Segur (AKNS for short) discovered that the Zakharov-Shabat system (1.8) can be used to solve also the s-G equation, provided one chooses M in the form $M_0(x,t) + M_{-1}(x,t)/\lambda$. Nearly at the same time, Faddeev and Takhtadjan [8] found another Lax representation for the s-G equation, with M being an integral operator.

The Heisenberg ferromagnet equation (HF):

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx}, \qquad \mathbf{S}\cdot\mathbf{S} = 1 , \tag{1.9}$$

which is gauge equivalent to the NLS equation [22, 73]. Here $\mathbf{S}$ is a three-component vector, and $\times$ is the cross product of these vectors. In matrix form, (1.9) is written as:

$$iS_t = \frac{1}{2}\,[S, S_{xx}]\,, \qquad S^2(x,t) = \mathbb{1} , \tag{1.10}$$

where the 2×2 matrix is obtained from the vector $\mathbf{S}$ by:

$$S(x,t) = \sum_{k=1}^{3} \mathbf{S}_k(x,t)\sigma_k , \tag{1.11}$$

and σ_k are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{1.12}$$

As we shall see, the pair of equations NLS and HF provides the first example of the so-called gauge equivalent NLEE.

The vector NLS equation:

$$i\mathbf{u}_t + \mathbf{u}_{xx} + (\mathbf{u}^\dagger, \mathbf{u})\mathbf{u}(x,t) = 0, \qquad \mathbf{u} = \begin{pmatrix} u_1(x,t) \\ \vdots \\ u_n(x,t) \end{pmatrix} , \tag{1.13}$$

known also as the Manakov model [24, 74], and the N-wave system [25]:

$$i[J, Q_t] + i[I, Q_x] + [[I,Q],[J,Q(x,t)]] = 0, \qquad Q_{jj} = 0 , \tag{1.14}$$

where $Q(x,t)$ is $n \times n$ matrix whose diagonal elements vanish, were shown to be integrable with the help of a generalized Zakharov–Shabat system of the form:

$$i\frac{d\psi}{dx} + (q(x,t) - \lambda J)\,\psi(x,t,\lambda) = 0 . \tag{1.15}$$

For the Manakov, model one can use $(n+1) \times (n+1)$-component system of the form (1.15) with

$$q(x,t) = \begin{pmatrix} 0 & \mathbf{u}(x,t) \\ \mathbf{u}^\dagger(x,t) & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{1}_n \end{pmatrix} , \tag{1.16}$$

where $\mathbf{u} = (u_1, \dots, u_n)^T$, while for the N-wave system we must choose an $n \times n$ system of the form (1.15) with

$$q(x,t) = [J, Q(x,t)], \qquad J = \operatorname{diag}(a_1, \dots, a_n) . \tag{1.17}$$

where $a_1 > a_2 > \cdots > a_n$ are real constants.

Along with the continuous models, soon there appeared also discrete models, described by systems of ordinary differential equations, which can be treated in a similar way. The first among them were two versions of the Toda chain [75, 76, 77, 78, 79, 80, 81, 82]:

$$\frac{d^2 q_k}{dt^2} = e^{q_{k+1}-q_k} - e^{q_k - q_{k-1}}, \qquad k = 1, \dots, N , \tag{1.18}$$

The first version, for which one must put $e^{-q_0} \equiv e^{q_{N+1}} \equiv 0$, is known as the Toda molecule system; the second version assumes $q_{N+1} \equiv q_1$ and is known as the affine Toda chain [83, 84].

Due to its success in dealing with nonlinear models, numerous attempts to apply the new ideas followed, and the list of nonlinear integrable equations constantly increased during the last decades; see e.g. the list of integrable systems in [40, 66, 85].

The second pioneer result was obtained by Zakharov and Faddeev in [70]. It has been proved that the KdV equation is an infinite dimensional completely integrable Hamiltonian system. The paper [70] started another important trend. This was the first case when for an infinite dimensional Hamiltonian system the action-angle variables were obtained. It stimulated the study of the Hamiltonian structures and Hamiltonian hierarchies of the soliton equations, which is one of the main aims of this book. Doing this in the first part we apply the spectral methods.

The geometric methods treating the recursion operators will be introduced in the second part. It turns out, however, that the mathematical ideas and techniques are so different from those introduced in the first part that they cannot be treated simultaneously. Because of that, we preferred to separate the ideas and to put in the first part, essentially, the "spectral" ideas and in the second one – the geometric ones. From one side, the reader who has read the first part will easily compare the results, and from the other, both parts are relatively independent and can be read separately.

1.2 Fundamental Properties of the Soliton Equations

Let us outline the main idea of the ISM and the fundamental properties of the relevant soliton equations. A beginner can use also one of the numerous review papers and monographs on ISM [18, 31, 37, 38, 40, 42, 47, 49, 53, 56, 66, 86, 87, 88, 89, 90, 91, 92, 93].

1.2.1 Solving Nonlinear Cauchy Problems

The basic idea underlying the ISM consists in "changing the variables" passing from $q(x,t)$ to the scattering matrix $T(\lambda)$ of L, which is defined in Chap. 2. Let us outline it in the example of the nonlinear Cauchy problem for the NLS equation with the initial condition:

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + 2|u|^2 u(x,t) = 0 \,, \tag{1.19a}$$

$$u(x,t)|_{t=0} = u_0(x) \,. \tag{1.19b}$$

Consider the operator L_0 be of the form (1.8) with the potential $q_0(x) = q(x, t = 0)$ defined by

$$q_0(x) = \begin{pmatrix} 0 & u_0(x) \\ u_0^*(x) & 0 \end{pmatrix} .$$

The first step toward the solution of (1.19) consists in solving the direct scattering problem for L_0 and constructing the scattering matrix $T(0, \lambda)$.

The next step is to find $T(t,\lambda)$ for any moment $t>0$. Rather surprisingly, one gets *a linear evolution* equation for $T(t,\lambda)$ of the form

$$i\frac{dT}{dt} + [f(\lambda)\sigma_3, T(t,\lambda)] = 0\,, \tag{1.20}$$

where the function $f(\lambda)$ is known as the dispersion law of the corresponding NLEE. For the NLS equation $f(\lambda) = -2\lambda^2$. The equation (1.20) is trivially solved:

$$T(t,\lambda) = \exp(if(\lambda)\sigma_3 t)T(0,\lambda)\exp(-if(\lambda)\sigma_3 t)\,, \tag{1.21}$$

i.e. one can easily find the scattering matrix $T(t,\lambda)$ for any time $t>0$.

The last step consists in recovering the potential $q(x,t)$ from the scattering matrix $T(t,\lambda)$. Thus, the solution of the nonlinear Cauchy problem (1.19) can be done along the scheme:

$$\begin{tikzcd} q_0(x) \arrow[r] & L_0 \arrow[d, "\mathrm{I}"'] & L|_{t>0} \arrow[r] & q(x,t) \\ & T(0,\lambda) \arrow[r, "\mathrm{II}"] & T(t,\lambda) \arrow[u, "\mathrm{III}"'] & \end{tikzcd} \tag{1.22}$$

We shall explain how to perform the transitions along arrows I and III in (1.22) in Chaps. 3 and 4. We shall show that both steps I and III are reduced to solving linear integral equations, for which one can prove theorems of existence and uniqueness. Solving the problem corresponding to step III is known as the inverse scattering problem for L which has given the method its name.

Even at this stage, we see that along with the NLS equation we can solve a number of other more complicated NLEE, which have different dispersion laws $f(\lambda)$. This shows that we should look at the NLS equation as just one member of an infinite set (a hierarchy) of NLEE, which can be integrated quite analogously.

Returning to the scheme (1.22), the ISM allows one to reduce the *Cauchy problem for a nonlinear equation* to a sequence of *three linear problems*, each having unique solution. Therefore, for each of the NLEE in the hierarchy, we can prove the analog of *Cauchy theorem of existence and uniqueness*. We also conclude that the solutions of the soliton equations are parametrized by the scattering data of the Lax operator L. Since generically L may have both continuous and discrete spectrum, we can expect *two qualitatively different* types of solutions: (a) solutions parametrized by data on the continuous spectrum only and (b) solutions parametrized by the data on the discrete spectrum of L. While type (a) solutions are rather close to the usual linear waves, type (b) solutions are known as soliton solutions and possess a number of exceptional properties, which are outlined below. Of course, one may have also "mixed" solutions, parametrized by data on both the continuous and discrete spectra. For those, one can prove that their properties are mostly characterized by their "solitonic" ingredient.

1.2.2 Hierarchies of Soliton Equations

As mentioned in the previous subsection, to each dispersion law $f(\lambda)$ there corresponds an NLEE. In fact, applying the ISM to the Zakharov–Shabat system (1.8), one can solve each of the systems of NLEE of the form:

$$i\sigma_3 \frac{dq}{dt} + 2f(\Lambda)q(x,t) = 0 \; , \tag{1.23}$$

where $f(\lambda)$ is the dispersion law, and Λ is one of the so-called recursion operators Λ_+, Λ_- or $\Lambda = (\Lambda_+ + \Lambda_-)/2$:

$$\Lambda_\pm X = \frac{i}{4}\left[\sigma_3, \frac{dX}{dx}\right] + \frac{i}{2} q(x,t) \int_{\pm\infty}^{x} \mathrm{tr}\,\left(q(y,t)[\sigma_3, X(y,t)]\right) dy \; , \tag{1.24}$$

Choosing in (1.23) the dispersion law $f(\lambda) = -2\lambda^2$ and $q^+ = (q^-)^*$, we get the NLS equation (1.3). Taking $f(\lambda) = -8\lambda^3$ and $q^+ = \kappa_1(q^-)$ and real we get the mKdV equation (1.6).

The class of polynomial dispersion laws has a special property where the corresponding systems (1.23) are local in $q(x,t)$, i.e. depend only on q and its x-derivatives: q_x, q_{xx} etc. In addition, the explicit form of the NLEE does not depend on the choice of the recursion operator (Λ_+, Λ_- or Λ) we have used. One of our aims in Chap. 4 will be to prove that the NLEE (1.23) is equivalent to the set of linear equation (1.20) for the scattering matrix $T(\lambda,t)$.

1.2.3 Linearized NLEE and Fourier Transform

The equivalence between (1.23) and (1.20) stimulated the development of the important idea proposed in [94], to interpret the ISM as a generalized Fourier transform. An indication showing that this indeed may be so is the fact that in the limit of small potentials:

$$q(x,t) = \epsilon q^{(0)}(x,t) + \mathcal{O}(\epsilon^2) \; , \tag{1.25}$$

the NLEE (1.23) becomes linear and the ISM reduces to the usual Fourier transform.

Indeed, let us consider $\epsilon \ll 1$ and keep only the first-order terms with respect to ϵ. The scattering matrix of the Zakharov–Shabat system in this approximation (known as the Born approximation) takes the form:

$$T(\lambda,t) = \mathbb{1} + i\epsilon R^{(0)}(\lambda,t) + \mathcal{O}(\epsilon^2) \; , \tag{1.26}$$

where

$$R^{(0)}(\lambda,t) \equiv \begin{pmatrix} 0 & \rho_+^{(0)}(\lambda,t) \\ \rho_-^{(0)}(\lambda,t) & 0 \end{pmatrix} = \int_{-\infty}^{\infty} dy\, e^{i\lambda\sigma_3 y} q^{(0)}(y,t) e^{-i\lambda\sigma_3 y} \; . \tag{1.27}$$

Thus $q^{(0)}(\lambda,t)$ and $R^{(0)}(\lambda,t)$ are related by the Fourier transform:

$$q^{(0)}(\lambda,t) \equiv \begin{pmatrix} 0 & q^{+}_{(0)}(x,t) \\ q^{-}_{(0)}(x,t) & 0 \end{pmatrix} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy\, e^{-i\lambda\sigma_3 y} R^{(0)}(y,t) e^{i\lambda\sigma_3 y} \,. \tag{1.28}$$

In this limit the recursion operators go into:

$$\Lambda_{\pm} X \underset{\epsilon\to 0}{\rightarrow} D_0 X + \mathcal{O}(\epsilon^2), \qquad D_0 X = \frac{i}{4}\left[\sigma_3, \frac{dX}{dx}\right] \,. \tag{1.29}$$

and the NLEE (1.23) takes the form:

$$i\sigma_3 \frac{dq^{(0)}}{dt} + 2f(D_0) q^{(0)}(x,t) = 0 \,, \tag{1.30}$$

which is a linear partial differential equation with constant coefficients. From (1.28) and (1.29) we get:

$$D_0 q^{(0)}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, \lambda e^{-i\lambda\sigma_3 x} R^{(0)}(\lambda,t) e^{i\lambda\sigma_3 x} \,. \tag{1.31}$$

Therefore the left-hand side of (1.30) goes into:

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda\sigma_3 x}\left(\frac{i}{2}\left[\sigma_3, \frac{dR^{(0)}}{dt}\right] + 2f(\lambda) R^{(0)}(\lambda,t) e^{i\lambda\sigma_3 x}\right) \\ &= \left[\sigma_3, \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda\sigma_3 x}\left(\frac{i}{2}\frac{dR^{(0)}}{dt} + f(\lambda)[\sigma_3, R^{(0)}(\lambda,t)]\right) e^{i\lambda\sigma_3 x}\right] \end{aligned} \tag{1.32}$$

and vanishes if and only if $R^{(0)}(\lambda,t)$ satisfies the linear evolution equation:

$$i\frac{dR^{(0)}}{dt} + f(\lambda)[\sigma_3, R^{(0)}(\lambda,t)] = 0 \,. \tag{1.33}$$

It is easy to check that if we insert (1.26) into (1.20) and keep only the terms linear in ϵ we get again (1.33).

The conclusion here is that in the limit of small potentials (Born approximation) the NLEE (1.23) becomes linear, and the ISM simplifies to a standard Fourier transform, which is closely related to the spectral decomposition of the linear operator D_0. Of course, D_0 has no discrete eigenvalues, so all soliton solutions in this limit disappear.

The deep and important idea proposed in [94], and extended in [95, 96, 97, 98] is that this analogy survives also for potentials that are not small. Instead of the usual Fourier transform one has to use its generalization, which is related to the expansions over the "squared solutions" of L and to the spectral decompositions of $\Lambda_{\pm}$ and Λ. The proofs outlined in Chap. 5 allow one to conclude that L and Λ have the same set of discrete eigenvalues, which are responsible for the soliton solutions.

1.2.4 Integrals of Motion

If we denote the matrix elements of $T(\lambda, t)$ by:

$$T(\lambda, t) = \begin{pmatrix} a^+(\lambda) & -b^-(\lambda, t) \\ b^+(\lambda, t) & a^-(\lambda) \end{pmatrix} \tag{1.34}$$

from (1.20) we get:

$$\frac{da^\pm(\lambda)}{dt} = 0, \qquad i\frac{db^\pm(\lambda)}{dt} \mp 2f(\lambda)b^\pm(\lambda) = 0\,. \tag{1.35}$$

The first two equations mean that the functions $a^\pm(\lambda)$ are t-independent. One can prove that they are analytic functions of λ for $\lambda \in \mathbb{C}_\pm$. Expanding them, or rather $\ln a^\pm(\lambda)$ in Taylor series, e.g.:

$$\pm \ln a^\pm(\lambda) = \sum_{k=1}^{\infty} C_k \lambda^{-k}\,, \tag{1.36}$$

one can generate an infinite number of integrals of motion C_k. Thus, the functions $\pm \ln a^\pm(\lambda)$ can be viewed as generating functionals of the integrals of motion for the hierarchy of NLEE.

The C_k can be expressed as functionals of the potential $q(x,t)$; the special choice of the Taylor series in (1.36) has the advantage of the densities of C_k being local in q, i.e. they depend only on q and their x-derivatives.

1.2.5 Soliton Solutions and Soliton Interactions

Though some of the soliton solutions of the sine-Gordon and KdV equations were already known in 19th century, the real value of their important properties was discovered after the ISM was developed. Indeed, after the discovery of the ISM, it was realized that the GLM equation [99, 100, 101] can be solved in closed form in the reflectionless case, i.e. if $\rho^+(\lambda) = \rho^-(\lambda) = 0$ for all real values of λ. The corresponding solutions are parametrized only by the data related to the discrete spectrum of the Lax operator and are known as N-soliton solutions. For the N-soliton solution of the NLS equation, the corresponding ZS system has N pairs of discrete eigenvalues $\lambda_j^+ = (\lambda_j^-)^* \in \mathbb{C}_\pm$, $j = 1, \dots, N$.

So, in the reflectionless case, the GLM has a degenerate kernel, and as a result solving it leads to a set of algebraic equations, which is solvable in closed form. Thus, for the N-soliton solution of the NLS equation, we have [9]:

$$u_{\mathrm{Ns}}(x,t) = \frac{\det M_1(x,t)}{\det M_0(x,t)}\,, \tag{1.37}$$

where

$$M_{ij}(x,t) = \frac{1 + \gamma_j^*(x,t)\gamma_k(x,t)}{\lambda_j^- - \lambda_k^+} \qquad M_1(x,t) = \begin{pmatrix} M(x,t) & \boldsymbol{\gamma}(x,t) \\ \mathbf{c}_0^T & 0 \end{pmatrix}\,, \tag{1.38}$$

$$\boldsymbol{\gamma}(x,t) = (\gamma_1, \dots, \gamma_N)^T, \quad \mathbf{c}_0^T = (1, 1, \dots, 1), \quad \gamma_k(x,t) = e^{i\lambda_k^+(x-\xi_{0,k}+\lambda_k^+ t)} . \tag{1.39}$$

Taking the limits $t \to \infty$ and $t \to -\infty$ in these formulas, one is able to analyze the N-soliton interaction in the case when the solitons have different velocities.

The soliton solutions of the NLEE turned out to be very important due to their exceptional stability properties and due to their special type of interaction. Indeed, let for $t \to -\infty$ the solution of the NLEE is represented as a sum of N one-soliton solutions, which move with different velocities. In this limit, they are well separated between themselves so that they practically do not interact. We will write this in the form:

$$u(x,t) \underset{t\to-\infty}{\longrightarrow} \sum_{k=1}^{N} u_{1\mathrm{s}}(x,t;\mu_k,\nu_k,\xi_k^-,\delta_k^-) \tag{1.40}$$

where the parameters ξ_k^-, μ_k, ν_k and δ_k^- characterize the asymptotic values of the “center” of mass position, the velocity, the amplitude, and the phase of the soliton. We also assume that the solitons are ordered in such a way that the leftmost one is the fastest, while the rightmost one is the slowest.

For $t \to \infty$ the asymptotics of the solution $u(x,t)$ becomes:

$$u(x,t) \underset{t\to\infty}{\longrightarrow} \sum_{k=1}^{N} u_{1\mathrm{s}}(x,t;\mu_k,\nu_k,\xi_k^+,\delta_k^+) \tag{1.41}$$

This means that again $u(x,t)$ reduces to a sum of one-soliton solutions, which have the same velocities and amplitudes but are ordered in inverse order: i.e. now the leftmost soliton is the slowest while the rightmost one is the fastest. The other difference between (1.40) and (1.41) consists in the change of the relative center of mass positions, ξ_k^- to ξ_k^+, and the relative phases from δ_k^- to δ_k^+. The pure elastic nature of the N-soliton interactions is due to the infinite set of integrals of motion.

Thus, the soliton interaction provides us with examples of nonlinear interactions, in which the nonlinearity exactly compensates for the linear dissipation. In fact, this is the fundamental physical property of these equations. Mathematically, it is reflected in the fact that the corresponding NLEE is solvable by the ISM. The other properties discussed below are all consequence of it.

1.2.6 Hamiltonian Hierarchies

The importance of the Hamiltonian properties of the soliton equations, along with the fact that the KdV equation is an infinite dimensional completely integrable Hamiltonian system, was realized by Zakharov and Faddeev [70]. This paper stimulated the studies of the Hamiltonian properties for the other soliton equations. Soon, it was proved the NLS and the s-G equations also are infinite dimensional completely integrable Hamiltonian system [102, 103]. It is easy to check that the NLS equation can be cast naturally into Hamiltonian form:

$$i\frac{dq^{\pm}}{dt} = \{H_{(0)}, q^{\pm}\}_{(0)} \ , \tag{1.42}$$

$$\Omega_{(0)}(X_{H_{(0)}}, \cdot) = \delta H_{(0)}, \tag{1.43}$$

where $X_{H_{(0)}}$ is the vector field related to the Hamiltonian $H_{(0)}$. The Poisson bracket $\{\cdot, \cdot\}_{(0)}$, the symplectic 2-form $\Omega_{(0)}$ and $H_{(0)}$ are given by:

$$\begin{aligned} H_{(0)} &= -8iC_3 \ , \\ \{F, G\}_{(0)} &= \int_{-\infty}^{\infty} dx \left(\frac{\delta F}{\delta q^+(x)} \frac{\delta G}{\delta q^-(x)} - \frac{\delta F}{\delta q^-(x)} \frac{\delta G}{\delta q^+(x)} \right) \ , \qquad (1.44) \\ \Omega_{(0)} &= \int_{-\infty}^{\infty} dx \, \delta q^+(x) \wedge \delta q^-(x) \ . \qquad (1.45) \end{aligned}$$

In fact, all NLS type equations allow such Hamiltonian formulation with the same canonical symplectic form and with Hamiltonian being linear combinations of the integrals of motion C_k.

The next important step in the theory of NLEEs considered as Hamiltonian systems has been made by Magri [104], who discovered that the KdV equation allows a second Hamiltonian formulation. Soon after that it was realized that in fact each soliton equation possesses a hierarchy of Hamiltonian structures, namely, there exist an infinite sequence of choices for the Hamiltonian $H^{(m)}$ and symplectic forms $\Omega^{(m)}$ and Poisson brackets $\{\cdot, \cdot\}_{(m)}$ such that the corresponding equations of motion

$$i\frac{dq^{\pm}}{dt} = \{H_{(m)}, q^{\pm}\}_{(m)} \ , \tag{1.46}$$

$$\Omega_{(m)}(X_{H_{(m)}}, \cdot) = \delta H_{(m)}(\cdot), \tag{1.47}$$

provide the same soliton equation for all values of $m = 1, 2, \ldots$. It was also shown that these hierarchies, i.e. both $H^{(m)}$ and $\Omega^{(m)}$, are generated again by the recursion operator Λ. In particular, one has the so-called Lenard relation stating that the gradients of the integrals of motion are related by Λ [105, 106]:

$$\frac{\delta C_{m+1}}{\delta q^T(x)} = \Lambda \frac{\delta C_m}{\delta q^T(x)} \ . \tag{1.48}$$

In a number of important cases, the study of the spectral properties of the operator L leads to the explicit construction of the action–angle variables. In other words, there is an explicit procedure that allows to prove that generically the soliton equations describe infinite–dimensional completely integrable Hamiltonian systems. Naturally, each such system possesses an infinite number of integrals of motion.

1.2.7 Exact Integrability and "Action-Angle" Variables

We have already mentioned that the NLEE (1.23) are equivalent to the linear equations (1.20) for the scattering matrix. If one introduces the variables on the continuous spectrum,

$$\eta(\lambda) = -\frac{1}{\pi} \ln(a^+ a^-(\lambda)), \qquad \kappa(\lambda) = \frac{1}{2} \ln \left(\frac{b^+(\lambda)}{b^-(\lambda)} \right) , \tag{1.49}$$

then one can prove that

$$\frac{d\eta(\lambda)}{dt} = 0, \qquad i\frac{d\kappa(\lambda)}{dt} - 2f(\lambda) = 0 , \tag{1.50}$$

that is, they satisfy equations generalizing the ones for the action-angle variables for the finite-dimensional case.

In Chap. 5, we introduce a special set of "squared solutions" – the so-called *symplectic basis*, which relate the potential $q(x,t)$ directly to the corresponding "action-angle" variables. Next, in Chap. 7, we prove the equivalence of the NLEE (1.23) and the set of the (1.49) augmented with a set of similar equations for the discrete spectrum. These results generalize the Liouville theorem for complete integrability [107, 108] to the infinite-dimensional Hamiltonian systems.

1.2.8 The Hierarchy of Bäcklund and Darboux Transformations

The Bäcklund transformations [12], along with the closely related Darboux transformations [13], were discovered in the 19th century in studying surfaces with constant negative curvature. When in the 70s it was realized that these surfaces are described by NLEE, integrable by the ISM, the interest toward these transformations was revived. Here we refer to only some of the numerous monographs and reviews from that time[17, 15, 42, 97, 98, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137].

The Bäcklund transformation allows starting from a given solution of the NLEE $u(x,t)$ to construct a new solution $\tilde{u}(x,t)$, which depends on *additional parameters*. For example, if we choose $u(x,t)$ to be the trivial solution $u = 0$ and apply to it the Bäcklund transformation, we shall get the one-soliton solution of the corresponding NLEE. The additional parameters on which it depends characterize the location x_1 of the soliton mass center and its velocity v_1. If we again apply the Bäcklund transformation to the one-soliton solution, we obtain the two-soliton solution etc.

Since the equations we are dealing with are nonlinear, it is clear that they do not allow a superposition principle, namely the sum of two solutions *is not* a solution of the NLEE. The Bäcklund transformation, however, allows us to construct solutions that satisfy a generalized superposition principle, which is expressed by the following commutative diagram:

$$\begin{array}{ccc} u_1(x,t) & \xrightarrow{x_2,v_2} & u_{12}(x,t) \; = u_{21}(x,t) \\ {\scriptstyle x_1}\big\uparrow{\scriptstyle v_1} & & {\scriptstyle x_1}\big\uparrow{\scriptstyle v_1} \\ u(x,t) & \xrightarrow{x_2,v_2} & u_2(x,t) \end{array} \tag{1.51}$$

where each arrow means a Bäcklund transformation (BT) with fixed values of the additional parameters. The final result of two subsequent applications of Bäcklund transformations *does not depend on the order* in which they are applied.

Equation (1.51) provides the scheme of the BT for those soliton equations, whose soliton solutions are parametrized only by their velocities v_j and the centers of the masses x_j; such are the KdV and mKdV equations, the s-G equation. The BT can be understood also as a mapping of the Lax operator L with potential $u(x,t)$ to a Lax operator L_j of the same form but with different potential equal to $u_j(x,t)$. Comparing the spectra of the operators L and L_j, one finds that L_j has an additional pair of discrete eigenvalues $\pm i\zeta_j$ determined by the velocities v_j.

The same scheme applies also to more complicated cases. The solitons of the NLS equation are parametrized by four parameters μ_k, ν_k, ξ_k, δ_k, but their BT again can be represented by the diagram (1.51), adding of course some additional parameters. Again, one concludes that the BT maps the Lax operator L into L_j, having an additional pair of discrete eigenvalues $\zeta_j^{\pm} = \mu_j \pm i\nu_j$.

Repeating the BT two or more times allows one to get multisolution solutions. Then the commutativity of the diagram (1.51) means that the spectrum of the Lax operator L_{12} with potential $u_{12}(x,t)$ is independent of the order in which we have added the two additional pairs of discrete eigenvalues.

The BT also forms hierarchies which can be described in a way analogous to the hierarchy of soliton equations. Using the generalized Wronskian relations [111, 112], one can derive the equivalence between the class of BT:

$$ig(\boldsymbol{\Lambda}_+)Q_-(x,t) + f(\boldsymbol{\Lambda}_+)Q_+(x,t) = 0 \,, \tag{1.52}$$
$$Q_+(x,t) = q^{(2)}(x,t) + q^{(1)}(x,t), \qquad Q_-(x,t) = \sigma_3(q^{(2)}(x,t) - q^{(1)}(x,t)),$$

and the following linear relations between the reflection coefficients of the two Lax operators:

$$ig(\lambda)(\rho_2^{\pm}(\lambda) - \rho_1^{\pm}(\lambda)) \mp f(\lambda)(\rho_2^{\pm}(\lambda) + \rho_1^{\pm}(\lambda)) = 0 \,, \tag{1.53}$$

and similar set of relations for the discrete spectra of L_1 and L_2. Here $\boldsymbol{\Lambda}$ is a generalization of the recursion operators $\boldsymbol{\Lambda}_\pm$:

$$\begin{aligned} \boldsymbol{\Lambda}_\pm X = \frac{i}{4}\left[\sigma_3, \frac{dX}{dx}\right] &+ \frac{i}{8}Q_-(x,t)\int_{\pm\infty}^{x} dy\, \mathrm{tr}\, \left(Q_-(y,t), [\sigma_3, X(y,t)]\right) \\ &+ \frac{i}{8}Q_+(x,t)\int_{\pm\infty}^{x} dy\, \mathrm{tr}\, \left(Q_+(y,t), [\sigma_3, X(y,t)]\right) \,, \end{aligned} \tag{1.54}$$

and $f(\lambda)$ and $g(\lambda)$ are some polynomials. It is easy to check that for $q^{(1)}(x,t) = q^{(2)}(x,t)$ one finds $Q_- = 0$, $Q_+(x,t) = 2q(x,t)$, and the operators $\boldsymbol{\Lambda}_\pm$ go into the recursion operators $\Lambda_\pm$ (1.24). The spectral properties of the operators $\boldsymbol{\Lambda}_\pm$ are directly related to the completeness relations of the "products of solutions" of the two Lax operators L_j with potentials $q^{(j)}(x,t)$, $j = 1,2$, respectively. These relations can be derived in complete analogy to the "squared solutions" [97, 98, 138].

1.2.9 Gauge Equivalent Hierarchies

The Lax representation (1.7) is invariant under the action of the so–called group of gauge transformations:

$$L \to \widetilde{L} = g^{-1}Lg(x,t), \qquad M \to \widetilde{M} = g^{-1}Mg(x,t) \ , \tag{1.55}$$

where $g = g(x,t)$ is a smooth function of x and t taking values in the group $SL(2)$.

This freedom may be used in two ways. First, it can be used to remove some additional degrees of freedom by fixing the gauge of the Lax operator L. As a bonus, after fixing the gauge, one gets a nondegenerate Hamiltonian formulation for the remaining degrees of freedom.

Generically, there exist more than one natural way of fixing the gauge of a given Lax pair. This can be used in order to find gauge equivalence between different soliton equations. The famous example which we mention here is the one between the NLS equation (1.3) and the HF equation (1.10). According to [49]:

$$S(x,t) = g^{-1}\sigma_3 g(x,t) \ . \tag{1.56}$$

and the function $g(x,t)$ is fixed uniquely by the condition:

$$i\frac{dg}{dx} + q(x,t)g(x,t) = 0, \qquad \lim_{x\to\infty} g(x,t) = \mathbb{1} \ , \tag{1.57}$$

i.e. $g(x,t) = \psi(x,t,\lambda = 0)$ is the Jost solution of the Zakharov–Shabat system at $\lambda = 0$. The relevant Lax operator for the HF equation is given by:

$$\widetilde{L}\widetilde{\psi} \equiv i\frac{d\widetilde{\psi}}{dx} - \lambda S(x,t)\widetilde{\psi}(x,t,\lambda) = 0, \tag{1.58}$$

The gauge equivalence can be used to analyze the HF hierarchy much in the same way as that of NLS hierarchy. It is also possible to reformulate all fundamental properties of the soliton equations from one gauge to another; see Chap. 8.

The HF equation can be put into canonical Hamiltonian form with

$$\widetilde{H}_{(0)} = \int_{-\infty}^{\infty} dx \, \mathrm{tr}\, S^2(x,t) \ ,$$

$$\{S_a(x,t), S_b(x,t)\} = i\epsilon_{abc} S_c(x)\delta(x-y) , \qquad (1.59)$$

$$\widetilde{\Omega}_{(0)} = \int_{-\infty}^{\infty} dx \operatorname{tr} \left(\delta S \underset{'}{\wedge} [S(x,t), \delta S] \right) ,$$

and, of course, we have a Hamiltonian hierarchy generated by the recursion operator $\widetilde{\Lambda} = g^{-1}\Lambda g(x,t)$ gauge equivalent to Λ [139, 140, 141].

We shall mention also the nontrivial relation between the Hamiltonian hierarchies of NLS and HF [142]:

$$\Omega_{(0)} = \widetilde{\Omega}_{(2)} + \delta C_2 \wedge \delta C_1 , \qquad (1.60)$$

where C_1 and C_2 are the first two integrals of motion; see (1.36). From this relation it follows that the two hierarchies are dynamically equivalent.

We intend to present the theory of the generating (recursion) operators in a form that can easily be reformulated from one gauge into another. This has been achieved in the works [139, 140, 143] but has never been presented in a full form as we shall do in the present monograph.

1.3 The ISM as the Generalized Fourier Transform

Let us outline the main ideas developed in the first part of this book. They are based on the famous AKNS method [94], which naturally introduces the main objects in this book – the recursion operators and the expansions over their eigenfunctions: the squared solutions. We introduce also some related topics, which shall be treated in the first part of the book.

In Chap. 2, we outline the AKNS approach [94]. More precisely, we write down the Lax representation (1.7) with $L(\lambda)$ being the Zakharov–Shabat system (1.8) and take $M(\lambda)$ to be polynomial of order N in λ:

$$M\chi \equiv \left(i\frac{d}{dt} + \sum_{k=0}^{N} \lambda^{N-k} V_k(x,t) \right) \chi(x,t,\lambda) . \qquad (1.61)$$

Of course, since the Lax representation must hold identically with respect to λ, we obtain a set of $N+1$ equations relating $V_k(x,t)$ and $q(x,t)$. These are treated as recursion relations that have to be solved. In doing so, we need to split each matrix $V_k(x,t) = V_k^{\mathrm{d}}(x,t) + V_k^{\mathrm{f}}(x,t)$ into diagonal $V_k^{\mathrm{d}}(x,t)$ and off-diagonal $V_k^{\mathrm{f}}(x,t)$ parts and treat them separately. Thus, we naturally obtain that $V_{k+1}^{\mathrm{f}}(x,t)$ is obtained from $V_k^{\mathrm{f}}(x,t)$ by acting with the recursion operators $\Lambda_\pm$; see (2.38).

As a result, we find an explicit description of the hierarchy of M-operators in terms of the recursion operators $\Lambda_\pm$. We stress that the splitting of a 2×2 matrix into diagonal and off-diagonal parts is a grading of the algebra $sl(2)$. So we slightly modify the AKNS approach by keeping $V_k^{\mathrm{f}}(x,t)$ as 2×2 matrices with zeroes on the diagonal, rather taking the zeroes out and "squashing" $V_k^{\mathrm{f}}(x,t)$ into two-component columns:

$$V_k^{\mathrm{f}}(x,t) = \begin{pmatrix} 0 & V_k^{+,\mathrm{f}} \\ V_k^{-,\mathrm{f}} & 0 \end{pmatrix} \to \begin{pmatrix} V_k^{+,\mathrm{f}} \\ V_k^{-,\mathrm{f}} \end{pmatrix} . \tag{1.62}$$

Doing such an operation at the time looked like a simplification; in fact, using it nothing is lost, and one obtains the correct expressions for the components $V_k^{\pm,\mathrm{f}}$. However, keeping the matrix structure along with the grading property allows one to transfer AKNS results from one gauge into another; we demonstrate this later in Chap. 8.

Next, in Sect. 2.3, we derive the time evolution of the scattering matrix $T(t,\lambda)$ in terms of the dispersion law $f(\lambda)$ of the NLEE. Thus, it becomes clear that given the dispersion law $f(\lambda)$ one can determine not only the time dependence of $T(t,\lambda)$ but also the corresponding M operator in the Lax representation and the relevant NLEE; see (1.23). Here, we also give several specific choices for $f(\lambda)$ and the corresponding important examples of NLEE, such as NLS, mKdV, and sine-Gordon.

In the last two Sects. 2.4 and 2.5, we outline two of the natural generalization of the AKNS approach. The first one is treating the block-matrix Zakharov–Shabat system, thus allowing to construct the Lax representations for the vector and matrix NLS equations (1.13). The second one deals with the generalized Zakharov–Shabat system relevant for solving the N-wave equations (1.14). Both cases are dealing with more complicated algebraic structures, but the idea of using particular gradings of these algebras for solving the corresponding more complex recursion relations goes through. Thus, we are able to construct the corresponding generalizations of the recursion operators and to express with them the hierarchies of M-operators and multicomponent NLEE. These results are also given in explicit gauge covariant form, so they can be used to treat also the gauge equivalent NLEE.

Chapter 3 is devoted to the direct scattering problem for the Zakharov–Shabat system. In order to avoid unnecessary mathematical difficulties, we do our considerations under two simplifying conditions **C1** and **C2**. The first one requires that the potential be a Schwartz type function. Condition **C2** is an unexplicit restriction on $q(x)$ requesting that the discrete spectrum of $L(\lambda)$ is finite and simple. In Sect. 3.1, we introduce the Jost solutions and derive the analyticity properties of their columns. Next, we introduce the important notion of fundamental analytic solutions (FAS) $\chi^{\pm}(x,\lambda)$. The next Sect. 3.2 uses the FAS to construct the kernel $R^{\pm}(x,y,\lambda)$ of the resolvent of $L(\lambda)$:

$$\begin{aligned} R(x,y,\lambda) &= R^{\pm}(x,y,\lambda), \qquad \text{for } \lambda \in \mathbb{C}_{\pm} , \\ R^{\pm}(x,y,\lambda) &= \frac{1}{i}\chi^{\pm}(x,\lambda)\Theta^{\pm}(x-y)\hat{\chi}^{\pm}(y,\lambda) , \\ \Theta^{+}(x) = \operatorname{diag}&(-\theta(-x),\theta(x)), \qquad \Theta^{-}(x) = \operatorname{diag}(\theta(x),-\theta(-x)) . \end{aligned} \tag{1.63}$$

Equation (1.63) provides a kernel of bounded integral operator, analytic in λ for $\operatorname{Im}\lambda \neq 0$, which has poles at the zeroes of $a^{\pm}(\lambda) = \det \chi^{\pm}(x,\lambda)$. This determines the spectrum of L: Its continuous part fills in the real axis and the discrete one consists of the sets of zeroes of $a^{\pm}(\lambda) = \det \chi^{\pm}(x,\lambda)$.

In Sect. 3.3, we derive the asymptotic of FAS for $\lambda \to \infty$, which is used in Sect. 3.4 to derive the dispersion relations for $\ln a^{\pm}(\lambda)$. This relation makes it obvious that the analyticity properties of $a^{\pm}(\lambda)$ allow one to determine them using just one of the two minimal sets of scattering data $\mathcal{T}_i$, $i = 1, 2$ introduced in Sect. 3.5. On the continuous spectrum, $\mathcal{T}_i$ are determined by the reflection coefficients

$$\mathcal{T}_1 \equiv \{\rho^+(\lambda), \quad \rho^-(\lambda)\}, \qquad \mathcal{T}_2 \equiv \{\tau^+(\lambda), \quad \tau^-(\lambda)\} , \tag{1.64}$$

where $\rho^{\pm}(\lambda) = b^{\pm}(\lambda)/a^{\pm}(\lambda)$, and $\tau^{\pm}(\lambda) = b^{\mp}(\lambda)a^{\pm}(\lambda)$.

In Sect. 3.6, using the analyticity properties of the Jost solution, we derive their spectral representations. Section 3.7, the last in this chapter, is devoted to the completeness property of the Jost solutions. It is derived applying the contour integration method to the kernel of the resolvent $R^{\pm}(x, y, \lambda)$ and is directly related to the spectral decomposition of $L(\lambda)$.

In Chap. 4 we outline several approaches to the solution of the inverse scattering problem (ISP) for the ZS system. In the first two sections, we explain the classical approach to the problem based on the Gelfand–Levitan-Marchenko (GLM) equation. There, we explain two different ways of deriving this equation.

In Sect. 4.3, we show that the ISP for the ZS system is equivalent to a (possibly singular) Riemann–Hilbert problem. Sections 4.4 and 4.5 are devoted to two different versions of the Zakharov–Shabat dressing method, which is the most effective method for deriving the reflectionless potentials and the soliton solutions of the corresponding NLEE.

The main idea in Chap. 5 is to show that the mapping of the potential $q(x)$ of $L(\lambda)$ onto the minimal sets of scattering data $\mathcal{T}_i$, $i = 1, 2$ is one-to-one. Similarly, we analyze the mappings from the variations $\delta q(x)$ onto $\delta\mathcal{T}_i$:

$$\rho^{\pm}(\lambda) = \frac{i}{(a^{\pm})^2} [\![q, \boldsymbol{\Phi}^{\pm}(x,\lambda)]\!], \qquad \delta\rho^{\pm}(\lambda) = \frac{\mp i}{(a^{\pm})^2} [\![\sigma_3 \delta q, \boldsymbol{\Phi}^{\pm}(x,\lambda)]\!] , \tag{1.65}$$

$$\tau^{\pm}(\lambda) = \frac{i}{(a^{\pm})^2} [\![q, \boldsymbol{\Psi}^{\pm}(x,\lambda)]\!], \qquad \delta\tau^{\pm}(\lambda) = \frac{\pm i}{(a^{\pm})^2} [\![\sigma_3 \delta q, \boldsymbol{\Psi}^{\pm}(x,\lambda)]\!] . \tag{1.66}$$

Here by $[\![\cdot \, , \, \cdot]\!]$ we denote the following skew-scalar product on the phase space $\mathcal{M}$:

$$[\![X(x), Y(x)]\!] = -[\![Y(x), X(x)]\!] = \frac{1}{2} \int_{-\infty}^{\infty} dy \, \mathrm{tr}\, \left(X(y)[\sigma_3, Y(y)] \right) , \tag{1.67}$$

The "squared solutions" $\boldsymbol{\Psi}^{\pm}(x,\lambda)$ and $\boldsymbol{\Phi}^{\pm}(x,\lambda)$ are defined by the FAS as follows:

$$\boldsymbol{\Psi}^{\pm}(x,\lambda) = (\chi^{\pm}\sigma_{\mp}\hat{\chi}^{\pm})^{\mathrm{f}}(x,\lambda), \qquad \boldsymbol{\Phi}^{\pm}(x,\lambda) = (\chi^{\pm}\sigma_{\mp}\hat{\chi}^{\pm})^{\mathrm{f}}(x,\lambda) , \tag{1.68}$$

where $\sigma_{\pm} = \sigma_1 \pm i\sigma_2$.

The basic tools for deriving (1.65) and (1.66) are the Wronskian relations introduced in Sect. 5.1. Thus, we see that the elements of $\mathcal{T}_i$ and $\delta\mathcal{T}_i$ can be viewed as Fourier-like integrals, whose integrands are products of $q(x)$ (or $\delta q(x)$) with the squared solutions. In Sect. 5.2, we introduce three sets of squared solutions: $\{\boldsymbol{\Psi}^{\pm}(x,\lambda)\}$, $\{\boldsymbol{\Phi}^{\pm}(x,\lambda)\}$, and the symplectic basis $\{\mathbf{P}(x,\lambda), \mathbf{Q}(x,\lambda)\}$. Equation (1.68) shows that the squared solutions are constructed explicitly through the FAS $\chi^{\pm}(x,\lambda)$ which ensures their analyticity properties.

Next, we introduce the Green function $G(x,y,\lambda) = G^{\pm}(x,y,\lambda)$ for $\lambda \in \mathbb{C}_{\pm}$:

$$G^{\pm}(x,y,\lambda) = G_1^{\pm}(x,y,\lambda)\theta(x-y) - G_2^{\pm}(x,y,\lambda)\theta(y-x)\ , \tag{1.69a}$$

$$G_1^{\pm}(x,y,\lambda) = \frac{1}{(a^{\pm}(\lambda))^2}\boldsymbol{\Psi}^{\pm}(x,\lambda) \otimes \boldsymbol{\Phi}^{\pm}(y,\lambda)\ , \tag{1.69b}$$

$$G_2^{\pm}(x,y,\lambda) = \frac{1}{(a^{\pm}(\lambda))^2}\left(\boldsymbol{\Phi}^{\pm}(x,\lambda) \otimes \boldsymbol{\Psi}^{\pm}(y,\lambda) + \frac{1}{2}\boldsymbol{\Theta}^{\pm}(x,\lambda) \otimes \boldsymbol{\Theta}^{\pm}(y,\lambda)\right)\ , \tag{1.69c}$$

where $\boldsymbol{\Theta}^{\pm}(x,\lambda) = (\chi^{\pm}\sigma_3\hat{\chi}^{\pm})^{\mathrm{f}}(x,\lambda)$. This makes it possible, applying the contour integration method to $G(x,y,\lambda)$, to prove the completeness of the sets of "squared solutions" in $\mathcal{M}$.

Thus, we can expand any function in $\mathcal{M}$, including $q(x)$ and $\sigma_3\delta q(x)$ over each of the sets $\{\boldsymbol{\Psi}^{\pm}(x,\lambda)\}$, $\{\boldsymbol{\Phi}^{\pm}(x,\lambda)\}$ and $\{\mathbf{P}(x,\lambda), \mathbf{Q}(x,\lambda)\}$. Doing this in Sect. 5.3, we find that the elements of $\mathcal{T}_i$ and $\delta\mathcal{T}_i$ can be viewed as expansion coefficients of these expansions. In addition, the completeness relation allows one to prove Proposition 5.3, stating that there is one-to-one correspondence between the function $X(x)$ and its expansion coefficients. As a consequence, it follows that there is one-to-one correspondence between $q(x)$ (resp. $\delta q(x)$) and each of the minimal sets of scattering data $\mathcal{T}_i$ (resp. $\delta\mathcal{T}_i$).

In fact, the completeness relations can be viewed as spectral decompositions of the operators $\Lambda_{\pm}$, Λ, for which $\{\boldsymbol{\Psi}^{\pm}(x,\lambda)\}$, $\{\boldsymbol{\Phi}^{\pm}(x,\lambda)\}$ and $\{\mathbf{P}(x,\lambda), \mathbf{Q}(x,\lambda)\}$ are sets of eigenfunctions. In Sects. 5.4 and 5.5, we show that $\Lambda_{\pm}$ in fact coincide with the recursion operators Λ derived by the AKNS approach, while the elements of the symplectic basis are eigenfunctions of $\Lambda = 1/2(\Lambda_+ + \Lambda_-)$.

Note that the Green functions $G^{\pm}(x,y,\lambda)$ (1.69) have as denominators $(a^{\pm}(\lambda))^2$. Therefore, if we assume that the discrete eigenvalues $\lambda_j^{\pm}$ of L are simple then $G^{\pm}(x,y,\lambda)$ will have poles of second order. As a result, the discrete spectrum of $\Lambda_{\pm}$ will contain the same eigenvalues $\lambda_j^{\pm}$ but will be doubly degenerated.

In Sect. 5.5, we also derive the biorthogonality relations between the squared solutions. As a result, we prove that the elements of the set $\{\boldsymbol{\Psi}^{\pm}(x,\lambda)\}$ are biorthogonal to those of $\{\boldsymbol{\Phi}^{\pm}(x,\lambda)\}$ with respect to the skew-scalar product (1.67). As to the symplectic basis, it is an orthogonal one with respect to $[\![\cdot,\cdot]\!]$. These relations allow us also to obtain an integral representation for the Green functions of the recursion operators.

In the last section of this Chapter, 5.6, we derive the generalized Wronskian relations that interrelate the sets of scattering data of two ZS systems L_j with two different potentials $q^{(j)}(x)$, $j = 1, 2$ as follows [144, 145]:

$$\rho_2^{\pm}(\lambda) + \rho_1^{\pm}(\lambda) = \frac{i}{a_2^{\pm} a_1^{\pm}} \left[\!\left[Q_+(x), \boldsymbol{\Phi}'^{,\pm}(x,\lambda) \right]\!\right], \tag{1.70}$$

$$\rho_2^{\pm}(\lambda) - \rho_1^{\pm}(\lambda) = \frac{\mp i}{a_2^{\pm} a_1^{\pm}} \left[\!\left[Q_-(x), \boldsymbol{\Phi}'^{,\pm}(x,\lambda) \right]\!\right] , \tag{1.71}$$

where $Q_+(x) = q^{(1)}(x) + q^{(2)}(x)$ and $Q_-(x) = \sigma_3(q^{(1)}(x) - q^{(2)}(x))$. The "products of solutions" $\boldsymbol{\Psi}'^{,\pm}(x,\lambda)$ and $\boldsymbol{\Phi}'^{,\pm}(x,\lambda)$ of L_2 and L_1 are defined by:

$$\boldsymbol{\Psi}'^{,\pm}(x,\lambda) = (\chi^{(1),\pm}\sigma_{\mp}\hat{\chi}^{(2),\pm})^{\mathrm{f}}(x,\lambda), \qquad {}'\boldsymbol{\Phi}^{\pm}(x,\lambda) = (\chi^{(2),\pm}\sigma_{\pm}\hat{\chi}^{(1),\pm})^{\mathrm{f}}(x,\lambda) . \tag{1.72}$$

Here, $\chi^{(j),\pm}(x,\lambda)$ are the FAS of L_j, $j = 1, 2$. These generalized Wronskian relations are fundamental in analyzing the hierarchy of BT. Crucial in this analysis is to prove the completeness of the "products of solutions" rigorously done in [97, 98] and to derive the recursion operatorss $\boldsymbol{\Lambda}_{\pm}$ (1.54) for which the "products of solutions" are eigenfunctions.

Thus, we find two sets $\{\boldsymbol{\Psi}'^{,\pm}(x,\lambda)\}$ and $\{{}'\boldsymbol{\Phi}^{\pm}(x,\lambda)\}$ of "products of solutions" of the two ZS system. Using a natural generalization of the Green function $G(x,y,\lambda)$ (1.69), we prove that they are also complete sets of functions in $\mathcal{M}$. The expansions of $Q_+(x)$ and $Q_-(x)$ are derived. At the end, we derive the explicit form of the operators $\boldsymbol{\Lambda}_{\pm}$ for which the "products of solutions" are eigenfunctions are constructed. They coincide with the operators (1.54) generating the class of BT.

In Chap. 6, we shall show how the expansions over the "squared solutions," derived in Chap. 5, can be used for the analysis of the solvable NLEEs related to the ZS system. Thus, we shall demonstrate the important role the recursion operators $\Lambda_{\pm}$ and $\boldsymbol{\Lambda}_{\pm}$ play in deriving all fundamental properties of the soliton equations. We specially underlie the importance of the expansions over the symplectic basis, which allow one an easy proof of the complete integrability of the NLEEs and explicit derivation of their action-angle variables.

In Sect. 6.1, we prove a theorem that shows the equivalence between the NLEE (1.23) and the corresponding set of *linear evolution equations* for the elements of $\mathcal{T}_i$, $i = 1, 2$. Sect. 6.2 contains several important examples of NLEE, which are two-component generalizations of NLS and mKdV, as well as the mixed GNLS-GmKdV equation. Special attention is focused on the NLEE with singular dispersion laws, such as the generalized Maxwell-Bloch.

Most of the physically important NLEE such as NLS, mKdV, sine-Gordon etc., require special involutions to be imposed on the ZS system. One of them is

$$q(x,t) = \varepsilon_0 q^{\dagger}(x,t), \qquad \varepsilon_0 = \pm 1 , \tag{1.73}$$

and the other one is

$$q^{-}(x,t) = \eta_0 q^{+}(x,t), \qquad \eta_0 = \pm 1 . \tag{1.74}$$

These involutions result in corresponding symmetries on the spectral properties and on sets $\mathcal{T}_i$ of the ZS system. For example, the involution (1.73) with $\varepsilon_0 = 1$ results in symmetry of the discrete eigenvalues, namely, they must come in mutually complex conjugated pairs $\lambda_j^+ = (\lambda_j^-)^*$. For $\varepsilon_0 = -1$, the corresponding ZS system becomes equivalent to a self-adjoint eigenvalue problem, whose continuous spectrum fills in the real axis. As a result, no discrete eigenvalues are possible, and the relevant NLEE can have no soliton solutions.

The other involution (1.74) with $\eta_0 = 1$ results in different symmetry of the discrete eigenvalues, namely, they must come in mutually opposite pairs $\lambda_j^+ = -\lambda_j^-$. It is also possible to impose both involutions at the same time. Then, of course, both symmetries should be applied, which means that we may have two types of eigenvalues: (i) purely imaginary ones, coming in pairs and (ii) quadruplets $\pm\lambda_j^+$ and $\pm(\lambda_j^+)^*$. As a result, all equations enjoying both symmetries such as sine-Gordon, mKdV etc. have two types of solitons: topological and breathers. The consequences of all these involutions are derived in Sect. 6.3.

Section 6.4 is devoted to the derivation of the general properties of the NLEE using the spectral theory of the recursion operators. Here, we display the hierarchy of the M-operators and the hierarchy of integrals of motion C_k. We derive the recurrent relations for the densities of C_k and show how they can be expressed in terms of the recursion operators $\Lambda_\pm$. The section ends by deriving the Lenard relation (1.48).

Section 6.5, the text in Chap. 6, uses the expansions over the products of solutions of two ZS systems to describe the hierarchy of BT of the NLEE. We prove a theorem, which establishes the equivalence between the BT and a certain interrelation between the spectral properties of the two ZS systems. The section ends with two explicit examples of BT for the NLS and the sine-Gordon equations.

In Chap. 7, we explain how the NLEEs analyzed above can be viewed as infinite dimensional Hamiltonian systems. We start with several basic examples. Next, we go to the generic NLEE, whose phase space $\mathcal{M}^{\mathbb{C}}$ is equivalent to the space of pairs of smooth complex-valued functions $\{q^+(x), q^-(x)\}$. This phase space and the Hamiltonian dynamics on it can be viewed as a complexification of the standard Hamiltonian dynamics and the well-known Hamiltonian systems come up as different real forms of them. In Sect. 7.3, we show how the generic NLEE, related to the ZS system, can be cast in Hamiltonian form. We show that the expansion coefficients of $\sigma_3 \delta q(x)$ over the symplectic basis can be viewed as action-angle variables of the NLEE. It is shown that the orthogonality properties of the symplectic basis with respect to the skew-scalar product is equivalent to the fact that the action-angle variables satisfy canonical Poisson brackets. A simple consequence of this fact is the involution of the integrals of motion C_k.

In Sect. 7.4, we show that the complete integrability of the generic NLEE allows one to introduce, for each of these equations, a hierarchy of Hamiltonian structures. This means that there exist a sequence of Hamiltonians $H_{(m)}^{\mathbb{C}}$ and symplectic forms $\Omega^{\mathbb{C}}_{(m)}$, $m = 1, 2, \dots$, such that the corresponding Hamiltonian equations of motion generate the same NLEE. This hierarchy is generated by the recursion operator Λ in a natural way. Using the fact that its spectral decompositions use the symplectic basis, we prove that each of the symplectic forms $\Omega^{\mathbb{C}}_{(m)}$ is closed, and any two of symplectic forms $\Omega^{\mathbb{C}}_{(m)}$ and $\Omega^{\mathbb{C}}_{(p)}$ are compatible. Likewise, we prove that the corresponding hierarchy of Poisson brackets $\{ \cdot, \cdot\}^{\mathbb{C}}_{(m)}$ satisfy Jacobi identity.

In Sect. 7.5, we analyze the effects of involutions (1.73) and (1.74) on the hierarchies of Hamiltonian structures and on the action-angle variables. Special attention is paid to the fact that involution (1.74) results in the degeneracy of all symplectic forms with even index Ω_{2p}; also "half" of the integrals of motion vanish, $C_{2p} = 0$.

Chapter 8 explains how all the results obtained up to now for the NLEEs (6.7) can be reformulated in a natural way for the gauge-equivalent NLEEs. In fact, this was the reason for what we called explicitly gauge-covariant formulation of the results in the Chaps. 5, 6, and 7. Here, we mention the most famous examples of gauge-equivalent NLEEs. The first one is the equivalence between the KdV and the mKdV equations; the corresponding relation is provided by the so-called Miura transformation. The second such example, relates the NLS equation with the Heisenberg ferromagnet (HF) equation in the semiclassical approximation. In terms of the 2×2 matrix-valued function $S(x,t)$ this equation reads:

$$i\frac{\partial S}{\partial t} + \left[S(x,t), \frac{\partial^2 S}{\partial x^2} \right] = 0 \,. \tag{1.75}$$

where $S(x,t)$ satisfies: $\operatorname{tr} S(x,t) = 0$, $S^2(x,t) = \mathbb{1}$, and $\lim_{x\to\pm\infty} S(x,t) = \sigma_3$. It is also solvable by the ISM applied to the gauge-transformed Lax pair.

In Sect. 8.1, we introduce the group of gauge transformations of the Lax representations. We also explain how one can take out the auxiliary gauge degrees of freedom by properly fixing up the gauge. Thus the ZS system provides us a good example of such fixing. Another important example is known as the pole gauge; the corresponding Lax operator is

$$\tilde{L}\widetilde{\psi}(x,t,\lambda) \equiv i\frac{d\widetilde{\psi}}{dx} - \lambda S(x,t)\widetilde{\psi}(x,t,\lambda) = 0 \,. \tag{1.76}$$

In Sect. 8.2, we outline how the AKNS approach should be modified in order to handle the gauge-equivalent systems. To this end, one should apply a different grading of the corresponding algebra, which is compatible with the gauge transformation. As a result, we derive the gauge-equivalent recursion operator in terms of the new potential $S(x)$:

$$\widetilde{\Lambda}_{\pm}\widetilde{X} = \frac{i}{4}\left(\left[S(x,t), \frac{d\widetilde{X}}{dx} + \frac{1}{2}S_x \int_{\pm\infty}^{x} dy \operatorname{tr}\left(S_y \widetilde{X}(y,t)\right) \right] \right) \,. \tag{1.77}$$

In fact, due to our explicitly gauge-covariant approach, the operators $\Lambda_\pm$ and $\widetilde{\Lambda}_\pm$ are also gauge equivalent, i.e. $\widetilde{\Lambda}_\pm = g^{-1}(x)\Lambda_\pm g(x)$.

In Sect. 8.3, we analyze the direct and the inverse scattering problems for $\widetilde{L}$. They are closely related to the direct and inverse scattering problems for L, analyzed in Chaps. 3 and 4. We construct the FAS $\widetilde{\chi}^\pm(x,\lambda)$ of $\widetilde{L}$ and use them to construct the kernel of the resolvent for $\widetilde{L}$. As a result, we derive the completeness relation for the Jost solutions, which can be viewed also as the spectral decomposition of $\widetilde{L}$. The Zakharov–Shabat dressing method can also be adjusted in order to treat gauge-equivalent equations. This is explained in Sect. 8.4, where we also outline the derivation of the soliton solutions.

Section 8.5 contains reformulation of the Wronskian relations for the gauge-equivalent system $\widetilde{L}$. Thus we are able to analyze the mapping between $S(x)$ (resp. $\delta S(x)$) and the minimal sets of scattering data $\widetilde{\mathcal{T}}_i$, $i=1,2$ of $\widetilde{L}$. Denoting their elements on the continuous spectrum in analogy with (1.64) by $\widetilde{\rho}^\pm(\lambda)$ and $\widetilde{\tau}^\pm(\lambda)$ we establish that

$$\widetilde{\rho}^\pm(\lambda) = \frac{i\lambda}{(\widetilde{a}^\pm(\lambda))^2}\,[\![\pi_S\sigma_3, \widetilde{\boldsymbol{\Phi}}^\pm(x,\lambda)]\!]_{\sim}, \quad \widetilde{\tau}^\pm(\lambda) = \frac{i\lambda}{(\widetilde{a}^\pm(\lambda))^2}\,[\![\pi_S\sigma_3, \widetilde{\boldsymbol{\Psi}}^\pm(x,\lambda)]\!]_{\sim}\,, \tag{1.78}$$

$$\begin{aligned} \delta\widetilde{\rho}^\pm(\lambda) &= \mp\frac{i\lambda}{2(\widetilde{a}^\pm(\lambda))^2}\,[\![[S,\delta S(x)], \widetilde{\boldsymbol{\Phi}}^\pm(x,\lambda)]\!]_{\sim}\,,\\ \delta\widetilde{\tau}^\pm(\lambda) &= \pm\frac{i\lambda}{2(\widetilde{a}^\pm(\lambda))^2}\,[\![[S(x),\delta S(x)], \widetilde{\boldsymbol{\Psi}}^\pm(x,\lambda)]\!], \end{aligned} \tag{1.79}$$

where $\pi_S\sigma_3 = 1/4[S(x),[S(x),\sigma_3]]$. Here use a gauge-modified skew-scalar product $[\![\cdot\,,\,\cdot]\!]_{\sim}$:

$$[\![\widetilde{X}, \widetilde{Y}]\!]_{\sim} = \int_{-\infty}^{\infty} \langle \widetilde{X}(x), [S(x), \widetilde{Y}(x)]\rangle\,. \tag{1.80}$$

The "squared solutions" entering in the right-hand sides of (1.78) and (1.79) are related to the "squared solutions" of L by the gauge transformation:

$$\widetilde{\boldsymbol{\Psi}}^\pm(x,\lambda) = g^{-1}(x)\boldsymbol{\Psi}^\pm(x,\lambda)g(x), \qquad \widetilde{\boldsymbol{\Phi}}^\pm(x,\lambda) = (a_0^\pm)^2 g^{-1}(x)\boldsymbol{\Phi}^\pm(x,\lambda)g(x)\,, \tag{1.81}$$

where $a_0^\pm = a^\pm(0)$. Note the additional factor λ in the right-hand side of (1.78) compared to (1.65) and (1.66).

In view of (1.81), it is only natural to expect that the sets of gauge-equivalent "squared solutions" will also satisfy the completeness relation in $\widetilde{\mathcal{M}}$, whose elements $\widetilde{X}$ have the form $\widetilde{X}(x) = g^{-1}(x)X(x)g(x)$, $X(x)\in\mathcal{M}$. Therefore, one is able to reformulate all results relevant for the NLS hierarchy into the corresponding ones for the HF hierarchy. This is done consistently in Sects. 8.7, 8.8, 8.9 and 8.10.

In particular, we prove that a generic NLEE from the HF hierarchy of the form

$$iS(x)\frac{\partial S}{\partial t} - \frac{i}{4}\widetilde{f}(\widetilde{\Lambda}_+)\left[S(x,t), \frac{\partial S}{\partial x}\right] = 0\ , \tag{1.82}$$

where $\lambda\widetilde{f}(\widetilde{\lambda}) = f(\lambda)$. The result can be obtained applying the gauge transformation to (1.23) and using the fact that (see Lemma 8.5)

$$\left[S(x,t), \widetilde{\frac{\partial q}{\partial t}}\right] = \widetilde{\Lambda}_+ \left[S(x,t), \frac{\partial S}{\partial t}\right], \qquad \widetilde{q}(x,t) = -\frac{1}{4}\left[S(x,t), \frac{\partial S}{\partial x}\right]\ . \tag{1.83}$$

Next, using the expansions over the sets of "squared solutions" $\widetilde{\boldsymbol{\Psi}}^{\pm}(x,\lambda)$ and $\widetilde{\boldsymbol{\Phi}}^{\pm}(x,\lambda)$, we prove that the ISM applied to $\widetilde{L}$ can also be viewed as a generalized Fourier transform (GFT). It is natural to treat these expansions as spectral decompositions over the eigenfunctions of the recursion operators $\widetilde{\Lambda}_\pm$. This allows us also to derive all fundamental properties of the HF hierarchy in a uniform way in terms of $\widetilde{\Lambda}_\pm$.

The HF type equations also allow a hierarchy of Hamiltonian structures $\widetilde{\Omega}^{\mathbb{C}}_{(m)}, \widetilde{H}_{(m)}$, $m = 1, 2, \ldots$, where the Hamiltonians $\widetilde{H}_{(m)}$ are also linear combinations of the integrals of motion C_k. In Sect. 8.9, we analyze the interrelations between the two Hamiltonian hierarchies. Here, it is very effective to use the symplectic basis $\{\widetilde{\mathbf{P}}(x,\lambda), \widetilde{\mathbf{Q}}(x,\lambda)\}$, which maps directly $[S(x,t), \delta S]$ into the action-angle variables of the corresponding HF type NLEE. The important and nontrivial result consists in the relation:

$$\widetilde{\Omega}^{\mathbb{C}}_{(m)} = \Omega^{\mathbb{C}}_{(m-2)} - \delta \ln a_0^+ \wedge \delta C_{m-1}\ . \tag{1.84}$$

Note that the additional term in the right-hand side of (1.84) is a wedge product of the variations of two of the integrals of motion: $\ln a_0^+$ and C_{m-2}, which are common for all the NLEE. Therefore, one can conclude that both Hamiltonian hierarchies are dynamically equivalent.

Chapter 9 deals with the modern approach to the Hamiltonian systems based on the method of the classical r-matrix. In Sect. 9.1, we introduce the notion of the classical r-matrix and show how it can be applied to the hierarchy of NLS type equations. In Sect. 9.2, we apply similar considerations for the NLEE from the HF hierarchy. In both cases, the classical r-matrix provides an effective way to establish the involutivity of the integrals of motion C_k.

In Sect. 9.3, following [146, 147, 148], we derive the classical Yang-Baxter equation. It ensures that the Poisson brackets introduced via r satisfy Jacobi identity. We also show that applying appropriate averaging procedure to a given solution of the classical Yang-Baxter equation, one is able to obtain another solution of the same equation. In this way, we derive [49] the so-called trigonometric and the elliptic classical r-matrices.

Section 9.4 explains a "trick" of how it is possible to relate to each r-matrix a compatible Lax operator. Applied to the simplest classical r-matrix, this "trick" produces the Lax operator $\widetilde{L}$ relevant for the HF hierarchy. Applied to the elliptic classical r-matrices, we obtain the Lax operator relevant for the Landau-Lifshitz eq.

In Sect. 9.5, we show that proper use of the classical r-matrix approach allows one to prove the involutivity of the integrals of motion also for the generalized ZS systems, related to any simple Lie algebra. Section 9.6, the last, is a brief review on the possibility of extending the classical r-matrix approach for other types of Lax operators.

1.4 Related Developments, Bibliography and Comments

The ISM and its development have generated a number of fruitful ideas both in physics and mathematics.

It is impossible, in a single monograph, to embrace all the variety of methods and beautiful properties that are inherent in the soliton theory. Therefore, we try to list the important references, where these methods and properties can be found.

This monograph can be used also as an introduction to the inverse scattering method. Though there are a number of monographs on the topic which we refer to [37, 40, 49, 87, 149], the way we cover the material and the combination with the geometric aspects seem to be absent.

1. Important results were obtained also in analyzing integrable NLEE in 2+1 dimensions. Most effective here are the methods based on the nonlocal Riemann-Halbert problem and on the $\overline{\partial}$ and the nonlocal $\overline{\partial}$ problems [39, 87, 150, 151, 152, 153, 154, 155, 156].
2. The soliton equations can be considered important in differential geometry [13, 110, 117, 121, 123, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194]. More details of these aspects are contained in the second part of the monograph.
3. Here, we list just several of the reviews and monographs, which the interested reader may find useful [87, 149, 155, 195, 196, 197, 198, 199, 200, 201].

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2

The Lax Representation and the AKNS Approach

In the present Chapter, we outline the famous AKNS approach [1] to the integrable equations. This approach soon became popular, because it provided a simple and effective tool for deriving NLEE allowing Lax representation.

In the first three sections of this Chapter, we show how the AKNS method allows one to derive the family of integrable NLEE related to the Zakharov-Shabat system $L(\lambda)$ and their Lax representations $[L(\lambda), M(\lambda)] = 0$. Assuming $M(\lambda)$ to be polynomial in λ, we derive the recursion procedure for calculating the coefficients of $M(\lambda)$ in terms of $q(x)$ and its derivatives. These relations are solved in compact form using the recursion (generating) operator Λ, which plays a fundamental role in the theory of NLEE. Our derivation is slightly different from that of AKNS in the sense that it is gauge covariant. The advantage of such formulation will become clear in Chap. 8, where we treat the gauge-equivalent NLEE. In the last two sections of this Chapter, we outline two of the natural generalizations of the AKNS approach.

2.1 The Lax Representation in the AKNS Approach

By definition, one can apply the ISM to a given NLEE only if it allows the so-called Lax representation:

$$iL_t = [L, A] \ . \tag{2.1}$$

In his original paper [2] Lax has chosen L to be the Sturm-Liouville operator:

$$-\frac{d^2\psi}{dx^2} + (v(x,t) - k^2)\psi(x,t,k) = 0 \ . \tag{2.2}$$

Taking A as the third-order ordinary differential operator:

$$A\psi \equiv 4\frac{d^3\psi}{dx^3} - 6v(x,t)\frac{d\psi}{dx} - 3\frac{d(v(x,t))}{dx}\psi(x,t,k) \tag{2.3}$$

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DOI 10.1007/978-3-540-77054-1_2

Lax proved that (2.1) is satisfied if and only if $v(x,t)$ satisfies the KdV equation:

$$v_t + v_{xxx} + 6v_x v(x,t) = 0 \; . \tag{2.4}$$

Zakharov and Shabat were the first to realize that it is useful to consider Lax representations (2.1) with L-operators more general than (2.2). In [3], they considered as Lax operator the system:

$$L\chi \equiv \left(i\frac{d}{dx} + U(x,t,\lambda) \right) \chi(x,t,\lambda) = 0 \; , \tag{2.5a}$$

$$U(x,t,\lambda) = q(x,t) - \lambda\sigma_3 \; , \tag{2.5b}$$

$$q(x,t) = \begin{pmatrix} 0 & q^+ \\ q^- & 0 \end{pmatrix} \tag{2.5c}$$

with $q^+ = (q^-)^* = u(x,t)$ which is the ZS system. Then they constructed explicitly a 2×2 matrix operator A such that the Lax representation (2.1) became equivalent to the NLS equation for $u(x,t)$.

Below, we shall use the AKNS approach [1, 4], which is technically more convenient. In it, we rewrite (2.1) as the compatibility condition:

$$[L(\lambda), M(\lambda)] = 0 \; . \tag{2.6}$$

of two linear operators, whose potentials depend nontrivially on the spectral parameter λ. The λ-dependence is chosen explicitly, and as a rule it is taken to be polynomial or rational in λ. Then the M–operator takes the form:

$$M \equiv i\frac{d}{dt} + V(x,t,\lambda) \; , \tag{2.7}$$

where $V(x,t,\lambda)$ has a prescribed dependence on λ (say, a polynomial one). We also require that the condition (2.6) holds identically with respect to λ. As we shall explain below, this gives us the possibility to express the coefficients of $V(x,t,\lambda)$ in terms of the potential $q(x,t)$ of L.

The compatibility condition (2.6) can be understood also as the zero curvature condition for some connection defined on a conveniently chosen fiber bundle.

Let $\chi(x,t,\lambda)$ be a fundamental solution of L, i.e. this is a matrix-valued function whose determinant does not vanish:

$$L(\lambda)\chi(x,t,\lambda) = 0, \qquad \det\chi(x,t,\lambda) \neq 0 \; . \tag{2.8}$$

From the compatibility condition (2.6) there follows:

$$\begin{aligned} [L(\lambda), M(\lambda)]\chi(x,t,\lambda) &\equiv L(\lambda)M(\lambda)\chi(x,t,\lambda) - M(\lambda)L(\lambda)\chi(x,t,\lambda) \\ &= L(\lambda)M(\lambda)\chi(x,t,\lambda) = 0 \; . \end{aligned} \tag{2.9}$$

i.e. if $\chi(x,t,\lambda)$ is a fundamental solution of $L(\lambda)$, then $M(\lambda)\chi(x,t,\lambda)$ is also a fundamental solution of $L(\lambda)$. From the general theory of ordinary differential operators, it is known that every two fundamental solutions of a given

ODE must be linearly related. Therefore, there exist an x-independent matrix $C(\lambda, t)$ such that

$$M(\lambda)\chi(x,t,\lambda) = \chi(x,t,\lambda)C(\lambda,t) \; . \tag{2.10}$$

In Sect. 2.3 below, we shall analyze in greater detail the convenient choices for $C(\lambda, t)$; as a rule we shall assume it is t-independent. Here, we just remark that the compatibility condition (2.6) holds true for any $C(\lambda, t)$.

Thus, as M operator we choose in agreement with (2.7) and (2.10):

$$M\chi \equiv \left(i\frac{d}{dt} + V(x,t,\lambda) \right) \chi(x,t,\lambda) = \chi(x,t,\lambda)C(\lambda) \tag{2.11}$$

where $V(x, t, \lambda)$ is a polynomial of order N in λ

$$V(x,t,\lambda) = \sum_{k=0}^{N} \lambda^{N-k} V_k(x,t) \; . \tag{2.12}$$

Let us outline the AKNS approach. To this end, we insert the expression (2.11) into (2.6) and equate to zero the coefficients in front of the positive powers of λ. This gives:

$$[V_0(x,t), \sigma_3] = 0 \; , \tag{2.13a}$$

$$i\frac{dV_k}{dx} + [q, V_k(x,t)] - [\sigma_3, V_{k+1}(x,t)] = 0 \; , \tag{2.13b}$$

for $k = 0, 1, \ldots, N-1$ and the λ–independent term gives:

$$-i\frac{\partial q}{\partial t} + i\frac{\partial V_N}{\partial x} + [q(x,t), V_N(x,t)] = 0 \; . \tag{2.13c}$$

The (2.13) with $k = 1$ will be treated as the initial condition for the recurrent relations, which allow one to express subsequently the coefficients $V_k(x, t)$ in (2.12) through $q(x, t)$ and its x–derivatives. Thus, (2.13c) finally turns into an NLEE for the off–diagonal matrix $q(x, t)$ or into a system of NLEE for the coefficient functions $q^{\pm}(x, t)$.

Let us list some of the specific choices for the M-operator, which lead to integrable equations.

If we choose:

$$V(x,t,\lambda) = -i\sigma_3 q_x - q^+ q^- \sigma_3 - 2\lambda q(x,t) + 2\lambda^2 \sigma_3 \; , \tag{2.14}$$

we easily find that

(1) The coefficients in front of the positive powers of λ in the compatibility condition (2.6) vanish identically;
(2) The term independent of λ in (2.6) leads to:

$$-iq_t + \sigma_3 q_{xx} + 2q^+ q^- \sigma_3 q(x,t) = 0 \; . \tag{2.15}$$

Thus, it becomes obvious that the choice of L (2.5) and M (2.11), (2.14) in the Lax representation (2.6) is equivalent to the system (2.15), which generalizes the NLS equation.

The next example is related to the KdV and mKdV equations. In both cases $V(x,t,\lambda)$ is a cubic polynomial of λ:

$$\begin{aligned} &V_0 = -4\sigma_3, \qquad V_1 = 4q(x,t), \qquad V_2 = 2q^+q^-\sigma_3 + 2i\sigma_3 q_x \ , \\ &V_3 = -i(q^+q^-_x - q^-q^+_x)\sigma_3 - q_{xx} - 2q^+q^-q(x,t) \ , \end{aligned} \tag{2.16}$$

The compatibility condition (2.6) in this case leads to the following system of NLEE for $q^\pm(x,t)$:

$$\begin{aligned} &\frac{\partial q^+}{\partial t} + \frac{\partial^3 q^+}{\partial x^3} + 6q^+q^-(x,t)\frac{\partial q^+}{\partial x} = 0 \ , \\ &\frac{\partial q^-}{\partial t} + \frac{\partial^3 q^-}{\partial x^3} + 6q^-q^+(x,t)\frac{\partial q^-}{\partial x} = 0 \ , \end{aligned} \tag{2.17}$$

One can obtain two important soliton equations by imposing proper constraints (involutions) on $q^\pm(x,t)$. Indeed, choosing $q^+ = v(x,t)$, $q^- = 1$, we see that the system (2.17) reduces to the KdV equation:

$$\frac{\partial v}{\partial t} + \frac{\partial^3 v}{\partial x^3} + 6\frac{\partial v}{\partial x}v(x,t) = 0 \ , \tag{2.18}$$

Similarly, imposing the involution $q^+ = \kappa q^- = p(x,t)$, where $p(x,t)$ can be viewed also as a real-valued function, we obtain the modified KdV (mKdV) equation

$$\frac{\partial p}{\partial t} + \frac{\partial^3 p}{\partial x^3} + 6\frac{\partial p}{\partial x}p^2(x,t) = 0 \ . \tag{2.19}$$

The last example is connected with the s-G equation:

$$w_{xt} + \gamma \sin 2w(x,t) = 0 \ . \tag{2.20}$$

In this case $V(x,t,\lambda)$ has the form:

$$V(x,t,\lambda) = \frac{\gamma}{2\lambda}\left(\cos 2w(x,t)\,\sigma_3 - \sin 2w(x,t)\,\sigma_1\right) \ , \tag{2.21}$$

where $q^\pm$ are expressed through the real valued-function $w(x,t)$ as follows:

$$q^+(x,t) = -q^-(x,t) = -iw_x(x,t) \ . \tag{2.22}$$

If instead of (2.21) we use:

$$V(x,t,\lambda) = \frac{\gamma}{2\lambda}\left(\cosh 2w(x,t)\,\sigma_3 + \sinh 2w(x,t)\,\sigma_2\right) \ , \tag{2.23}$$

the compatibility condition (2.6) leads to the so-called sinh–Gordon equation:

$$\frac{\partial^2 w}{\partial x \partial t} + \gamma \sinh 2w(x,t) = 0 \ . \tag{2.24}$$

2.2 The Recursion Operators and the NLEE

Following the ideas of AKNS, we shall solve the recursion relations (2.13b) with generic initial conditions, i.e. for arbitrary choice of N and

$$V_0 = c_0\sigma_3, \qquad c_0 = \text{const} . \tag{2.25}$$

The analysis of these relations involves the splitting off of each $V_k(x,t)$ into diagonal and off-diagonal parts. This corresponds to splitting of the algebra $\mathfrak{g} = sl(2)$ into a direct sum $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$ of linear subspaces, corresponding to the kernel and the image of ad_{σ_3} considered as operator on $sl(2)$. Therefore, $\mathfrak{g}^{(0)}$ consists of all diagonal 2×2 matrices with vanishing trace, while $\mathfrak{g}^{(1)}$ contains all off-diagonal matrices. Such splitting has the grading property:

$$[X^{(0)}, Y^{(0)}] = 0, \qquad [X^{(0)}, Y^{(1)}] \in \mathfrak{g}^{(1)}, \qquad [X^{(1)}, Y^{(1)}] \in \mathfrak{g}^{(0)} , \tag{2.26}$$

where $X^{(0)}$, $Y^{(0)}$ and $X^{(1)}$, $Y^{(1)}$ are arbitrary elements of $\mathfrak{g}^{(0)}$ and $\mathfrak{g}^{(1)}$, respectively. We shall make use of the projectors onto $\mathfrak{g}^{(1)}$ defined by the above splitting:

$$\pi_0 \cdot \equiv \frac{1}{4}[\sigma_3, [\sigma_3, \cdot\,]] . \tag{2.27}$$

Applied to any traceless 2×2 matrix X, π_0 projects out its diagonal part:

$$\pi_0 X \equiv X - X^{\mathrm{d}} = \begin{pmatrix} 0 & X_{12} \\ X_{21} & 0 \end{pmatrix} \in \mathfrak{g}^{(1)} , \tag{2.28}$$

and

$$(\mathbb{1} - \pi_0)X = X^{\mathrm{d}} = X_{11}\sigma_3 , \tag{2.29}$$

Each $V_k(x,t)$ can be split into:

$$V_k(x,t) = w_k(x,t)\sigma_3 + V_k^{\mathrm{f}}(x,t) , \tag{2.30}$$

where

$$V_k^{\mathrm{f}}(x,t) = \pi_0 V_k(x,t) , \tag{2.31a}$$

$$w_k(x,t) = \frac{1}{2}\mathrm{tr}\,(V_k(x,t)\sigma_3) . \tag{2.31b}$$

We start by the relation (2.13b) with $k=0$:

$$i\frac{dc_0}{dx}\sigma_3 + [q(x,t), \sigma_3] - [\sigma_3, V_1(x,t)] = 0 . \tag{2.32}$$

The diagonal term here is the one proportional to dc_0/dx. It vanishes with c_0 as a constant. The two off-diagonal terms in (2.32) give us:

$$V_1^{\mathrm{f}}(x,t) = -c_0 q(x,t) \;. \tag{2.33}$$

For generic k, we extract first the diagonal part by multiplying (2.13b) by σ_3 and taking the trace. Using (2.31) we find:

$$i\frac{dw_k}{dx} + \frac{1}{2}\mathrm{tr}\,(\sigma_3[q(x,t), V_k(x,t)]) = 0 \;. \tag{2.34}$$

Note that in the second term of (2.34) only the off-diagonal part of V_k contributes. Thus (2.34) relates w_k and V_k^{f}. Integrating it we get:

$$w_k(x,t) = c_k + \frac{i}{2}\int_{\pm\infty}^{x} dy\,\mathrm{tr}\,(\sigma_3[q(y,t), V_k^{\mathrm{f}}(y,t)]) \;, \tag{2.35}$$

where c_k is an integration constant. Next the off-diagonal part of (2.13b) gives:

$$i\frac{dV_k^{\mathrm{f}}}{dx} + [q(x,t),\sigma_3]w_k(x,t) = [\sigma_3, V_{k+1}^{\mathrm{f}}(x,t)] \;. \tag{2.36}$$

It remains to apply $\frac{1}{4}[\sigma_3,\cdot]$ to both sides of (2.36) and to make use of (2.27) and (2.35) to find:

$$\begin{aligned} V_{k+1}^{\mathrm{f}}(x,t) &= \frac{i}{4}\left[\sigma_3, \frac{dV_k^{\mathrm{f}}}{dx}\right] - \frac{1}{4}\left[\sigma_3,[\sigma_3, q(x,t)]\right] w_k(x,t) \\ &= \frac{i}{4}\left[\sigma_3, \frac{dV_k^{\mathrm{f}}}{dx}\right] - \frac{i}{2}q(x,t)\int_{\pm\infty}^{x} dy\,\mathrm{tr}\,(\sigma_3[q(y,t), V_k^{\mathrm{f}}(y,t)]) \\ &\quad - c_k q(x,t) \;. \end{aligned} \tag{2.37}$$

Therefore the recurrent relation (2.13) now can be rewritten in the following compact form:

$$V_{k+1}^{\mathrm{f}}(x,t) = \Lambda_\pm V_k^{\mathrm{f}}(x,t) - c_k q(x,t) \;, \tag{2.38a}$$

$$V_1(x,t) = -c_0 q(x,t) \;, \tag{2.38b}$$

where by $\Lambda_\pm$ we have denoted the recursion operators:

$$\Lambda_\pm X \equiv \frac{i}{4}\left[\sigma_3, \frac{dX}{dx}\right] - \frac{i}{2}q(x,t)\int_{\pm\infty}^{x} dy\,\mathrm{tr}\,(\sigma_3[q(y,t), X(y,t)]) \;. \tag{2.39}$$

As we shall see in the next chapters, these operators play an important role in the theory of the NLEE. Here, we shall use them to write down the solution of the recurrent relations in the following compact form:

$$V_k^{\mathrm{f}}(x,t) = -\sum_{p=0}^{k-1} c_p \Lambda_\pm^{k-p-1} q(x,t) \;, \tag{2.40a}$$

$$w_k(x,t) = c_k - \frac{i}{2}\sum_{p=0}^{k-1} c_p \times \int_{\pm\infty}^{x} dy\, \mathrm{tr}\left(\sigma_3\left[q(y,t), \Lambda_\pm^{k-p-1} q(y,t)\right]\right) . \qquad (2.40b)$$

We shall show below that, although the operators $\Lambda_\pm$ are integro-differential applying their positive powers to $q(x,t)$, we always get expressions, which are local in $q(x,t)$, i.e. depend only on q and its x-derivatives.

The explicit solution of the recursion relation (2.13b) allows us now to describe the class of all NLEE, which can be solved applying the ISM to the ZS system. To do this, we have to insert the expression for $V_N(x,t)$ from (2.40) into (2.13) and to separate again the diagonal and the off-diagonal parts in it. The diagonal part gives us the necessary expression for $w_N(x,t)$ as an integral containing $q(x,t)$ and $V_N^{\mathrm{f}}(x,t)$, i.e. we get (2.35) with $k=N$. The off-diagonal part leads to the following NLEE:

$$-i\frac{\partial q}{\partial t} + i\frac{\partial V_N}{\partial x} + [q(x,t), \sigma_3] w_N(x,t) = 0 . \qquad (2.41)$$

Now, we apply to both sides $-\frac{1}{4}[\sigma_3, \cdot]$ and using (2.35) find:

$$\frac{i}{4}\left[\sigma_3, \frac{\partial q}{\partial t}\right] - \Lambda_\pm V_N^{\mathrm{f}}(x,t) + c_N q(x,t) = 0 , \qquad (2.42)$$

or

$$\frac{i}{4}\left[\sigma_3, \frac{\partial q}{\partial t}\right] + f(\Lambda_\pm) q(x,t) = 0 , \qquad (2.43)$$

where $f(\lambda)$ is the polynomial:

$$f(\lambda) = \sum_{p=0}^{N} c_p \lambda^{N-p} . \qquad (2.44)$$

In the form (2.43), the NLEE is quite analogous to the generic partial differential equation with constant coefficients (1.30) in Chap. 1. Indeed, since $q(x,t)$ is an off-diagonal matrix, then $[\sigma_3, q_t] = 2\sigma_3 q_t$ and that makes the (1.30) and (2.43) quite analogous; one just has instead of $f(D_0)$, $f(\Lambda_\pm)$.

Our main aim will be to prove that this analogy is not coincidental, and its roots are in the spectral decompositions of the recursion operators.

2.3 Evolution of the Scattering Data

We introduced already some NLEE having Lax representation. In this subsection, we shall explain the idea of what earlier was called a type of "change of variables", which linearizes the NLEE. To this end, we shall use the ZS system (2.5) with a complex-valued potential $q(x,t)$.

We shall suppose also that the potential $q(x,t)$ depends on the additional parameter t in such a way that its coefficients $q^{\pm}(x,t)$ satisfy one of the above mentioned NLEE. Another important choice consists in fixing up the class of functions, to which the potential belongs. Here, and in what follows, we assume that $q(x,t)$ belongs to the space $\mathcal{M}$ of off-diagonal 2×2 matrix-valued complex functions of Schwartz-type; i.e. it is an infinitely differentiable function tending to 0 for $|x| \to \infty$ faster than any negative power of x. We also assume that these properties are fulfilled for all values of t.

Note that the ZS system can be viewed formally as a quantum mechanical problem for the scattering of a "plane wave" on the "potential" $q(x,t)$. This "scattering" will be used, however, as a technical tool and will not be assigned any real physical meaning. Nevertheless, we shall make use of the well-developed theory for solving the direct and inverse scattering problems in quantum mechanics, which can easily be generalized to complex-valued "potentials." Thus, we shall omit the quotation marks as we use the standard terminology.

We recall some well-known facts from the theory of the linear differential equations. By $\chi(x,t,\lambda)$, we shall denote a matrix-valued solution of (2.5). Since $\operatorname{tr} U(x,t,\lambda) = 0$, then $\det \chi(x,t,\lambda)$ does not depend on x. $\chi(x,t,\lambda)$ is called a *fundamental solution* if its determinant does not vanish, i.e. $\det \chi(x,t,\lambda) \neq 0$.

Any fundamental solution of (2.5) can be fixed up uniquely by specifying its value at a given point $x = x_0$. Another important property of the linear systems in general and of the ZS system in particular is that any two fundamental solutions must be linearly related; see (2.47) below.

A special role in the direct and inverse scattering theory for the ZS system is played by the so-called Jost solutions $\psi(x,t,\lambda)$ and $\phi(x,t,\lambda)$. They are special fundamental solutions of (2.5) introduced by fixing up their asymptotics for $x \to \infty$ (or to $x \to -\infty$) to be plane waves:

$$\lim_{x\to\infty} \exp(i\lambda\sigma_3 x)\psi(x,t,\lambda) = \mathbb{1}, \qquad \lambda \in \mathbb{R} \tag{2.45a}$$

$$\lim_{x\to-\infty} \exp(i\lambda\sigma_3 x)\phi(x,t,\lambda) = \mathbb{1}, \qquad \lambda \in \mathbb{R}\,. \tag{2.45b}$$

By plane wave above, we mean the matrix-valued function $\exp(-i\lambda x\sigma_3)$ for real values of the spectral parameter λ; obviously it is a solution of (2.5) for the asymptotic value of the potential $q(x,t) = 0$.

In the special cases in (2.45), x_0 is taken to be ∞ and $-\infty$ correspondingly. Both solutions have determinants equal to 1:

$$\det \psi(x,t,\lambda) = \det \phi(x,t,\lambda) = 1, \qquad \lambda \in \mathbb{R}\,. \tag{2.46}$$

so they are fundamental, and they must be linearly related. This means that there exist the so-called scattering matrix $T(t,\lambda)$ such that

$$\phi(x,t,\lambda) = \psi(x,t,\lambda)T(t,\lambda), \qquad \lambda \in \mathbb{R}\,. \tag{2.47}$$

Let us denote the entries of the scattering matrix $T(t,\lambda)$ by:

$$T(t,\lambda) = \begin{pmatrix} a^+(\lambda) & -b^-(t,\lambda) \\ b^+(t,\lambda) & a^-(\lambda) \end{pmatrix} . \tag{2.48}$$

From (2.46) and (2.47) it follows that

$$\det T(t,\lambda) \equiv a^+(\lambda)a^-(\lambda) + b^+(t,\lambda)b^-(t,\lambda) = 1, \qquad \lambda \in \mathbb{R} . \tag{2.49}$$

This is known as the "unitarity" condition for the scattering matrix $T(t,\lambda)$.

Next, we derive the corresponding evolution of the scattering matrix $T(t,\lambda)$. To this end, we make use of the explicit form of the M-operator (2.11) derived in the previous section with conveniently chosen $C(\lambda)$. Consider (2.11) with $\chi = \phi(x,t,\lambda)$:

$$M\phi \equiv \left(i\frac{d}{dt} + V_N(x,t,\lambda) \right) \phi(x,t,\lambda) = \phi(x,t,\lambda)C(\lambda) , \tag{2.50}$$

multiply it on the left by $\exp(i\lambda\sigma_3 x)$ and take the limit $x \to -\infty$. Assuming that the asymptotics of the Jost solution $\phi(x,t,\lambda)$ for $x \to -\infty$ in (2.45a) is valid for all t, we get:

$$\begin{aligned} \lim_{x\to-\infty} e^{i\lambda\sigma_3 x} V_-(x,t,\lambda)\phi(x,t,\lambda) &\equiv \lim_{x\to-\infty} V_N(x,t,\lambda) \\ &= f(\lambda)\sigma_3 \\ &= C(\lambda) \end{aligned} \tag{2.51}$$

Thus, we find that $C(\lambda)$ can be directly related to the dispersion law of the NLEE:

$$C(\lambda) = f(\lambda)\sigma_3 . \tag{2.52}$$

In the limit $x \to \infty$, in view of (2.47) we get:

$$\left(i\frac{dT}{dt} + \lim_{x\to\infty} V_N(x,t,\lambda)T(t,\lambda) \right) = T(t,\lambda)C(\lambda) . \tag{2.53}$$

With (2.52), (2.56), we find that the scattering matrix $T(t,\lambda)$ satisfies the following *linear* evolution equation:

$$i\frac{dT}{dt} + f(\lambda)[\sigma_3, T(t,\lambda)] = 0 . \tag{2.54}$$

Written in terms of the entries of $T(\lambda)$ (2.54), the evolution takes the form of linear equations:

$$i\frac{da^\pm}{dt} = 0, \qquad i\frac{db^\pm}{dt} \mp 2f(\lambda)b^\pm(t,\lambda) = 0 \tag{2.55}$$

that can be easily solved for any choice of the *dispersion law* $f(\lambda)$.

The same results can be derived by taking $\chi = \psi(x,t,\lambda)$ and considering the limits $x\pm \to \infty$. Thus we established that

$$V_+(\lambda) = V_-(\lambda) = C(\lambda) = f(\lambda)\sigma_3, \qquad V_\pm(\lambda) = \lim_{x\to\pm\infty} V_N(x,t,\lambda) \quad (2.56)$$

But (2.6) means also that $q(x,t)$ satisfies the NLEE (2.43). Therefore, we outlined the proof of the following

Theorem 2.1 ([1]). *If $q(x,t) \in \mathcal{M}$ and satisfies the NLEE (2.43), then the scattering matrix $T(t,\lambda)$ satisfies the linear evolution equation (2.54).*

Thus the dispersion law $f(\lambda)$ of the corresponding NLEE determines both the NLEE itself through (2.43) and the evolution of the scattering data through (2.54) or (2.55).

Calculating the limits $V_\pm(\lambda)$ from the explicit expressions for $V(x,t,\lambda)$ corresponding to the NLS, KdV and s-G equations we get:

$$f_{\rm NLS}(\lambda) = -2\lambda^2, \qquad f_{\rm KdV}(\lambda) = -4\lambda^3, \qquad f_{\text{s-G}}(\lambda) = \frac{\gamma}{2\lambda}\,. \quad (2.57)$$

The two functions $a^\pm(\lambda)$ are in fact t–independent. This means that if we expand them in asymptotic series in λ their expansion coefficients also will be t–independent, i.e. they will be integrals of motion for the corresponding NLEE. In what follows, we treat $a^\pm(\lambda)$ as generating functionals of the integrals of motion of the NLEE.

2.4 Generalizations of the AKNS Method I

The AKNS method can be applied also to special multicomponent generalizations of the NLS type equations. One way to do this is to apply it to the block-matrix generalization of the Zakharov-Shabat system.

$$\boldsymbol{L}\boldsymbol{\chi} \equiv \left(i\frac{d}{dx} + \boldsymbol{U}(x,t,\lambda)\right)\boldsymbol{\chi}(x,t,\lambda) = 0\,, \quad (2.58a)$$

$$\boldsymbol{U}(x,t,\lambda) = \boldsymbol{q}(x,t) - \lambda\boldsymbol{\sigma}\,, \quad (2.58b)$$

$$\boldsymbol{q}(x,t) = \begin{pmatrix} 0 & \boldsymbol{q}^+ \\ \boldsymbol{q}^- & 0 \end{pmatrix}, \qquad \boldsymbol{\sigma} = \frac{2}{s+p}\begin{pmatrix} p\mathbb{1}_s & 0 \\ 0 & -s\mathbb{1}_p \end{pmatrix}, \quad (2.58c)$$

where $\boldsymbol{q}^+(x,t)$ and $(\boldsymbol{q}^-)^T(x,t)$ are rectangular $s\times p$ matrix-valued functions, $\mathbb{1}_s$ and $\mathbb{1}_p$ are the unit matrices of dimension s and p, $s+p=n$.

As M operator we choose:

$$\boldsymbol{M}\boldsymbol{\chi} \equiv \left(i\frac{d}{dt} + \boldsymbol{V}(x,t,\lambda)\right)\boldsymbol{\chi}(x,t,\lambda) = \boldsymbol{\chi}(x,t,\lambda)\boldsymbol{C}(\lambda) \quad (2.59)$$

where $\boldsymbol{V}(x,t,\lambda)$ is a polynomial of order N in λ

$$\boldsymbol{V}(x,t,\lambda) = \sum_{k=0}^{N} \lambda^{N-k} \boldsymbol{V}_k(x,t) \ , \tag{2.60}$$

$$\boldsymbol{C}(\lambda) = \lim_{x\to\infty} \boldsymbol{V}(x,t,\lambda) = \lim_{x\to-\infty} \boldsymbol{V}(x,t,\lambda) \ . \tag{2.61}$$

The compatibility condition $[\boldsymbol{L},\boldsymbol{M}] = 0$ holds true for any choice of the matrix $\boldsymbol{C}(\lambda)$. Now $\boldsymbol{U}(x,t,\lambda)$ and $\boldsymbol{V}(x,t,\lambda)$ are elements (of special form) of the algebra $sl(n)$. Since this condition must hold identically with respect to λ, we equate to zero the coefficients in front of all powers of λ with the result:

$$[\boldsymbol{V}_0(x,t), \boldsymbol{\sigma}] = 0 \ , \tag{2.62a}$$

$$i\frac{d\boldsymbol{V}_k}{dx} + [\boldsymbol{q}(x,t), \boldsymbol{V}_k(x,t)] - [\boldsymbol{\sigma}, \boldsymbol{V}_{k+1}(x,t)] = 0 \ , \tag{2.62b}$$

for $k = 0, 1, \ldots, N-1$. The λ–independent term provides the corresponding multicomponent NLEE:

$$-i\frac{\partial \boldsymbol{q}}{\partial t} + i\frac{\partial \boldsymbol{V}_N}{\partial x} + [\boldsymbol{q}(x,t), \boldsymbol{V}_N(x,t)] = 0 \ . \tag{2.62c}$$

These relations again can be viewed as recursion relations, allowing to determine $\boldsymbol{V}_k(x,t)$ in terms of $\boldsymbol{q}(x,t)$ and its derivatives. Generalizing the AKNS approach, we split each $\boldsymbol{V}_k(x,t)$ into block-diagonal and block-off-diagonal parts. This corresponds to splitting of the algebra $\mathfrak{g} = sl(n)$ into a direct sum $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$ of linear subspaces, corresponding to the kernel and the image of the operator $\operatorname{ad}_{\boldsymbol{\sigma}}$ on $sl(n)$. Since the nonvanishing eigenvalues of $\operatorname{ad}_{\boldsymbol{\sigma}}$ are equal to ± 2, the projector $\boldsymbol{\pi}_0$ onto $\mathfrak{g}^{(1)}$ takes the form:

$$\boldsymbol{\pi}_0 \cdot \equiv \frac{1}{4}[\boldsymbol{\sigma}, [\boldsymbol{\sigma}, \cdot\,]] \ . \tag{2.63}$$

Applied to any $n \times n$ matrix $\boldsymbol{X}$ it projects out its block-diagonal part:

$$\boldsymbol{\pi}_0 \boldsymbol{X} = \boldsymbol{X} - \boldsymbol{X}^{(0)} = \begin{pmatrix} 0 & \boldsymbol{X}_{12} \\ \boldsymbol{X}_{21} & 0 \end{pmatrix} , \tag{2.64}$$

The projector onto $\mathfrak{g}^{(0)}$ is given by:

$$(\mathbb{1} - \boldsymbol{\pi}_0)\boldsymbol{X} = \boldsymbol{X}^{(0)} = \begin{pmatrix} \boldsymbol{X}_{11} & 0 \\ 0 & \boldsymbol{X}_{22} \end{pmatrix}, \qquad \operatorname{tr} \boldsymbol{X}^{(0)} = 0 \ . \tag{2.65}$$

Therefore, $\mathfrak{g}^{(0)}$ consists of all block-diagonal matrices (2.65) with vanishing trace $\operatorname{tr} \boldsymbol{X}_{11} + \operatorname{tr} \boldsymbol{X}_{22} = 0$, while $\mathfrak{g}^{(1)}$ contains all block-off-diagonal matrices. Such splitting also has the grading property:

$$[\boldsymbol{X}^{(0)}, \boldsymbol{Y}^{(0)}] = 0, \qquad [\boldsymbol{X}^{(0)}, \boldsymbol{Y}^{(1)}] \in \mathfrak{g}^{(1)}, \qquad [\boldsymbol{X}^{(1)}, \boldsymbol{Y}^{(1)}] \in \mathfrak{g}^{(0)} \ , \tag{2.66}$$

where $\boldsymbol{X}^{(i)}$, $\boldsymbol{Y}^{(i)}$ are arbitrary elements of $\mathfrak{g}^{(i)}$, $i = 1, 2$. Each $\boldsymbol{V}_k(x, t)$ can be split into:

$$\boldsymbol{V}_k(x,t) = \boldsymbol{w}_k(x,t) + \boldsymbol{V}_k^{\mathrm{f}}(x,t) \;, \tag{2.67}$$

where

$$\boldsymbol{V}_k^{\mathrm{f}}(x,t) = \boldsymbol{\pi}_0 \boldsymbol{V}_k(x,t) \;, \tag{2.68a}$$

$$\boldsymbol{w}_k(x,t) = (\mathbb{1}_n - \boldsymbol{\pi}_0)\boldsymbol{V}_k(x,t) \;. \tag{2.68b}$$

Then (2.62a) means that $\boldsymbol{V}_0^{\mathrm{f}}(x,t) = 0$, i.e. $\boldsymbol{V}_0(x,t) = \boldsymbol{w}_0(x,t)$. From the block-diagonal part of (2.62b) with $k = 1$ we conclude that

$$\frac{d\boldsymbol{w}_0}{dx} = 0 \;, \tag{2.69}$$

i.e. we assume that $\boldsymbol{w}_0 = \mathrm{const} \in \mathfrak{g}^{(0)}$. The block-off-diagonal part of (2.62b) with $k = 1$ is equivalent to:

$$\boldsymbol{V}_1^{\mathrm{f}}(x,t) = \mathrm{ad}_{\boldsymbol{\sigma}}^{-1}[\boldsymbol{q}(x,t), \boldsymbol{w}_0] \;. \tag{2.70}$$

For $k > 1$, we again use the same splitting with the results:

$$i\frac{d\boldsymbol{w}_k}{dx} + [\boldsymbol{q}(x,t), \boldsymbol{V}_k^{\mathrm{f}}(x,t)] = 0 \;. \tag{2.71}$$

Thus, (2.71) establishes a relation between $\boldsymbol{w}_k$ and $\boldsymbol{V}_k^{\mathrm{f}}$. Integrating it we get:

$$\boldsymbol{w}_k(x,t) = \boldsymbol{w}_k^0 + i\int_{\pm\infty}^{x} dy\, [\boldsymbol{q}(y,t), \boldsymbol{V}_k^{\mathrm{f}}(y,t)] \;, \tag{2.72}$$

where $\boldsymbol{w}_k^0 \in \mathfrak{g}^{(0)}$ is a matrix-valued integration constant.

Next, the block-off-diagonal part of (2.62b) gives:

$$i\frac{d\boldsymbol{V}_k^{\mathrm{f}}}{dx} + [\boldsymbol{q}(x,t), \boldsymbol{w}_k(x,t)] = [\boldsymbol{\sigma}, \boldsymbol{V}_{k+1}^{\mathrm{f}}(x,t)] \;. \tag{2.73}$$

It remains to apply $[\boldsymbol{\sigma}, \cdot]$ to both sides of (2.73) and to make use of (2.63) and (2.72) to find:

$$\begin{aligned}
\boldsymbol{V}_{k+1}^{\mathrm{f}}(x,t) &= \frac{i}{4}\left[\boldsymbol{\sigma}, \frac{d\boldsymbol{V}_k^{\mathrm{f}}}{dx}\right] - \frac{1}{4}\left[\boldsymbol{\sigma}, [\boldsymbol{w}_k(x,t), \boldsymbol{q}(x,t)]\right] \;, \\
&= \frac{i}{4}\left[\boldsymbol{\sigma}, \frac{d\boldsymbol{V}_k^{\mathrm{f}}}{dx}\right] + \frac{i}{4}\left[\boldsymbol{\sigma}, \left[\boldsymbol{q}(x,t)\int_{\pm\infty}^{x} dy\, [\boldsymbol{q}(y,t), \boldsymbol{V}_k^{\mathrm{f}}(y,t)]\right]\right] \\
&\quad - \frac{1}{4}\left[\boldsymbol{\sigma}, [\boldsymbol{w}_k^0, \boldsymbol{q}(x,t)]\right] \;.
\end{aligned} \tag{2.74}$$

Thus, the recurrent relation (2.62) acquires the following compact form:

$$\boldsymbol{V}^{\mathrm{f}}_{k+1}(x,t) = \boldsymbol{\Lambda}_{\pm}\boldsymbol{V}^{\mathrm{f}}_{k}(x,t) - \frac{1}{4}\left[\boldsymbol{\sigma}, [\boldsymbol{w}^0_k, \boldsymbol{q}(x,t)]\right] , \tag{2.75a}$$

$$\boldsymbol{V}_1(x,t) = -\mathrm{ad}_{\boldsymbol{\sigma}}^{-1}[\boldsymbol{w}^0_k, \boldsymbol{q}(x,t)] , \tag{2.75b}$$

where by $\boldsymbol{\Lambda}_{\pm}$ we have denoted the recursion operators:

$$\boldsymbol{\Lambda}_{\pm}\boldsymbol{X} \equiv \frac{i}{4}\left[\boldsymbol{\sigma}, \frac{d\boldsymbol{X}}{dx}\right] + \frac{i}{4}\left[\boldsymbol{\sigma}, \left[\boldsymbol{q}(x,t), \int_{\pm\infty}^{x} dy\, [\boldsymbol{q}(y,t), \boldsymbol{X}(y,t)]\right]\right] . \tag{2.76}$$

The formal solution to this recurrent relations is given by:

$$\boldsymbol{V}^{\mathrm{f}}_{k+1}(x,t) = -\frac{1}{4}\sum_{p=0}^{k} \boldsymbol{\Lambda}_{\pm}^{k-p}\left[\boldsymbol{\sigma}, \left[\boldsymbol{w}^0_p, \boldsymbol{q}(x,t)\right]\right] . \tag{2.77}$$

Applying the same reasoning to the λ-independent term in the compatibility condition, we get the explicit form for the multicomponent NLS-type (MNLS-type) equations:

$$\frac{i}{4}\left[\boldsymbol{\sigma}, \frac{\partial \boldsymbol{q}}{\partial t}\right] - \boldsymbol{\Lambda}_{\pm}\boldsymbol{V}^{\mathrm{f}}_{N}(x,t) + \frac{1}{4}\left[\boldsymbol{\sigma}, \left[\boldsymbol{w}^0_N, \boldsymbol{q}(x,t)\right]\right] = 0 . \tag{2.78}$$

It remains to insert the solution (2.77) for $\boldsymbol{V}^{\mathrm{f}}_{N}(x,t)$ into (2.78) to get these NLEE in terms of the recursion operators $\boldsymbol{\Lambda}_{\pm}$:

$$\frac{i}{4}\left[\boldsymbol{\sigma}, \frac{\partial \boldsymbol{q}}{\partial t}\right] + \frac{1}{4}\sum_{p=0}^{N} \boldsymbol{\Lambda}_{\pm}^{N-p}\left[\boldsymbol{\sigma}, \left[\boldsymbol{w}^0_p, \boldsymbol{q}(x,t)\right]\right] = 0 . \tag{2.79}$$

Obviously the multicomponent analog of the dispersion law for the NLEE (2.79) is provided by the matrix-valued polynomial $\boldsymbol{f}(\lambda)$:

$$\boldsymbol{f}(\lambda) = \sum_{p=0}^{N} \lambda^{N-p}\boldsymbol{w}^0_p \in \mathfrak{g}^{(0)} . \tag{2.80}$$

Let us list several important examples of MNLS-type equations.

The Manakov model [5] originally was obtained by taking $N = 2$, $\boldsymbol{f}_{\mathrm{Man}}(\lambda) = -2\lambda^2\boldsymbol{\sigma}$ with $s = 1$, $p = 2$; then $\boldsymbol{q}^+ = (\boldsymbol{q}^-)^\dagger$ is a two-component vector $\mathbf{u}(x,t)$ satisfying:

$$i\mathbf{u}_t + \mathbf{u}_{xx} + (\mathbf{u}^\dagger, \mathbf{u})\mathbf{u}(x,t) = 0, \qquad \mathbf{u} = \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix} . \tag{2.81}$$

It became famous due to its numerous applications in nonlinear optics [6, 7, 8, 9, 10].

Of course, one can consider a generalization of the Manakov model with p-component vectors, $p > 2$. Also the use of an involution of the form $\boldsymbol{q}^+ = B_0(\boldsymbol{q}^-)^\dagger$, where $B_0 = \operatorname{diag}(\epsilon_1, \dots, \epsilon_p)$ with $\epsilon_j = \pm 1$ leads to another version of the Manakov model:

$$i\mathbf{u}_t + \mathbf{u}_{xx} + (\mathbf{u}^\dagger, B_0\mathbf{u})\mathbf{u}(x,t) = 0, \qquad \mathbf{u} = \begin{pmatrix} u_1(x,t) \\ \vdots \\ u_p(x,t) \end{pmatrix} . \tag{2.82}$$

Matrix NLS models. The above two models and all other multicomponent generalizations of the MNLS equation are particular cases of the system:

$$i\frac{\partial \boldsymbol{q}^+}{\partial t} + \frac{\partial^2 \boldsymbol{q}^+}{\partial x^2} + 2\boldsymbol{q}^+\boldsymbol{q}^-\boldsymbol{q}^+(x,t) = 0 \, , \tag{2.83a}$$

$$-i\frac{\partial \boldsymbol{q}^-}{\partial t} + \frac{\partial^2 \boldsymbol{q}^-}{\partial x^2} + 2\boldsymbol{q}^-\boldsymbol{q}^+\boldsymbol{q}^-(x,t) = 0 \, , \tag{2.83b}$$

or in matrix form:

$$\frac{i}{2}\left[\boldsymbol{\sigma}, \frac{\partial \boldsymbol{q}}{\partial t}\right] + \frac{\partial^2 \boldsymbol{q}}{\partial x^2} + 2\boldsymbol{q}^3(x,t) = 0 \, . \tag{2.84}$$

The dispersion law of this equation is

$$\boldsymbol{f}_{\mathrm{MNLS}}(\lambda) = -2\lambda^2\boldsymbol{\sigma} \, . \tag{2.85}$$

Let us impose on $\boldsymbol{q}(x,t)$ the condition:

$$\boldsymbol{q}(x,t) = B\boldsymbol{q}^\dagger(x,t)B^{-1}, \qquad B = \begin{pmatrix} B_0 & 0 \\ 0 & B_1 \end{pmatrix} \in \mathfrak{g}^{(0)} \, , \tag{2.86}$$
$$B_0 = \operatorname{diag}(\epsilon_1, \dots, \epsilon_s), \qquad B_1 = \operatorname{diag}(\eta_1, \dots, \eta_p) \, ,$$

where $\epsilon_j = \pm 1$ and $\eta_s = \pm 1$. Each specific choice of the sets of ϵ_j and η_s provides an allowed involution of the system (2.83). The involution (2.86) means that the block matrices $\boldsymbol{q}^\pm(x,t)$ are related by:

$$\boldsymbol{q}^+(x,t) = \boldsymbol{r}(x,t), \qquad \boldsymbol{q}^-(x,t) = B_0\boldsymbol{r}^\dagger(x,t)B_1 \tag{2.87}$$

Inserting (2.87) into (2.83), we easily find that the second equation (2.83b) can be obtained from the first one (2.83a) with hermitian conjugation. As a result, we get the following matrix NLS equation:

$$i\frac{\partial \boldsymbol{r}}{\partial t} + \frac{\partial^2 \boldsymbol{r}}{\partial x^2} + 2\boldsymbol{r}B_0\boldsymbol{r}^\dagger\boldsymbol{r}(x,t) = 0 \, . \tag{2.88}$$

Vector and matrix mKdV models. The well-known mKdV equation (2.18) is characterized by dispersion law, which is cubic in λ; see (2.16). We also choose here $s = p$, i.e. $n = 2p$.

Choosing in (2.79) $\boldsymbol{f}_{\mathrm{mKdV}} = -4\lambda^3\boldsymbol{\sigma}$, we obtain the following multicomponent generalization of the system (2.17):

$$\frac{\partial \boldsymbol{q}^+}{\partial t} + \frac{\partial^3 \boldsymbol{q}^+}{\partial x^3} + 3\boldsymbol{q}^+\boldsymbol{q}^-(x,t)\frac{\partial \boldsymbol{q}^+}{\partial x} + 3\frac{\partial \boldsymbol{q}^+}{\partial x}\boldsymbol{q}^-\boldsymbol{q}^+(x,t) = 0 \; , \tag{2.89a}$$

$$\frac{\partial \boldsymbol{q}^-}{\partial t} + \frac{\partial^3 \boldsymbol{q}^-}{\partial x^3} + 3\boldsymbol{q}^-\boldsymbol{q}^+(x,t)\frac{\partial \boldsymbol{q}^-}{\partial x} + 3\frac{\partial \boldsymbol{q}^-}{\partial x}\boldsymbol{q}^+\boldsymbol{q}^-(x,t) = 0 \; , \tag{2.89b}$$

The multicomponent mKdV equation is obtained from the system (2.89) imposing the involution:

$$B\boldsymbol{q}^*(x,t)B^{-1} = -\boldsymbol{q}(x,t), \qquad B = \begin{pmatrix} 0 & B_2 \\ B_2^{-1} & 0 \end{pmatrix} \; , \tag{2.90}$$

This choice of B satisfies $B^2 = \mathbb{1}$, i.e. the constraint (2.90) is an involution. If we denote $\boldsymbol{q}^+(x,t) = \boldsymbol{r}(x,t)$ then we have:

$$\boldsymbol{q}^+(x,t) = \boldsymbol{r}(x,t), \qquad \boldsymbol{q}^-(x,t) = -B_2^{-1}\boldsymbol{r}^*(x,t)B_2^{-1} \; . \tag{2.91}$$

Then the system (2.89) becomes equivalent to:

$$\frac{\partial \boldsymbol{r}}{\partial t} + \frac{\partial^3 \boldsymbol{r}}{\partial x^3} - 3\boldsymbol{r}B_2\boldsymbol{r}^*B_2\frac{\partial \boldsymbol{r}}{\partial x} - 3\frac{\partial \boldsymbol{r}}{\partial x}B_2\boldsymbol{r}^*B_2\boldsymbol{r}(x,t) = 0 \; , \tag{2.92}$$

for the complex-valued $p\times p$-matrix function $\boldsymbol{r}(x,t)$. If we choose $B_2 = \mathbb{1}_p$, we get another version of the multicomponent mKdV equation:

$$\frac{\partial \boldsymbol{r}}{\partial t} + \frac{\partial^3 \boldsymbol{r}}{\partial x^3} - 3\frac{\partial \boldsymbol{r}}{\partial x}\boldsymbol{r}^*\boldsymbol{r}(x,t) - 3\boldsymbol{r}\boldsymbol{r}^*(x,t)\frac{\partial \boldsymbol{r}}{\partial x} = 0 \; . \tag{2.93}$$

Imposing additional involution, we can make $\boldsymbol{r}(x,t)$ either real-valued $p\times p$ matrix or purely imaginary one.

In order to solve these multicomponent generalizations of the NLS and mKdV equations, we need to develop the direct and inverse scattering theory for the block-matrix Zakharov-Shabat system (2.58a). Its Jost solutions are also introduced by fixing up their asymptotics for $x \to \infty$ (or to $x \to -\infty$) to be plane waves, that is, we require that $\boldsymbol{\psi}(x,t,\lambda)$ and $\boldsymbol{\phi}(x,t,\lambda)$ be fundamental solution of $\boldsymbol{L}$ satisfying:

$$\lim_{x\to\infty} \exp(i\lambda\boldsymbol{\sigma}x)\boldsymbol{\psi}(x,t,\lambda) = \mathbb{1}, \qquad \lambda \in \mathbb{R} \tag{2.94a}$$

$$\lim_{x\to-\infty} \exp(i\lambda\boldsymbol{\sigma}x)\boldsymbol{\phi}(x,t,\lambda) = \mathbb{1}, \qquad \lambda \in \mathbb{R} \; . \tag{2.94b}$$

Note that these definitions of the Jost solutions are compatible with the $\boldsymbol{M}$-operator in the form (2.59) with the special choice (2.58a) for $\boldsymbol{C}(\lambda)$.

One can check that

$$\det \psi(x,t,\lambda) = \det \phi(x,t,\lambda) = 1, \qquad \lambda \in \mathbb{R} \; . \tag{2.95}$$

so they are fundamental, and they must be linearly related by the scattering matrix $\boldsymbol{T}(t,\lambda)$:

$$\boldsymbol{\phi}(x,t,\lambda) = \boldsymbol{\psi}(x,t,\lambda)\boldsymbol{T}(t,\lambda), \qquad \lambda \in \mathbb{R}\,. \tag{2.96}$$

It is natural that the scattering matrix $\boldsymbol{T}(t,\lambda)$ will have the same type of block-matrix structure as $\boldsymbol{U}(x,t,\lambda)$:

$$\boldsymbol{T}(t,\lambda) = \begin{pmatrix} \boldsymbol{a}^{+}(t,\lambda) & -\boldsymbol{b}^{-}(t,\lambda) \\ \boldsymbol{b}^{+}(t,\lambda) & \boldsymbol{a}^{-}(t,\lambda) \end{pmatrix}\,. \tag{2.97}$$

From (2.95) and (2.96), it follows that the generalization of the "unitarity" condition (2.49) is:

$$\det \boldsymbol{T}(t,\lambda) = 1, \qquad \lambda \in \mathbb{R}\,. \tag{2.98}$$

Next we conclude that

$$\boldsymbol{V}_{+}(\lambda) = \boldsymbol{V}_{-}(\lambda) = \boldsymbol{C}(\lambda) = \boldsymbol{f}(\lambda), \qquad V_{\pm}(\lambda) = \lim_{x\to\pm\infty} V(x,t,\lambda) \tag{2.99}$$

so $\boldsymbol{T}(t,\lambda)$ must satisfy the following linear evolution equation:

$$i\frac{d\boldsymbol{T}}{dt} + [\boldsymbol{f}(\lambda), \boldsymbol{T}(t,\lambda)] = 0\,. \tag{2.100}$$

In the special case, when $\boldsymbol{f}(\lambda) = f(\lambda)\boldsymbol{\sigma}$ from (2.97) and (2.100) we find:

$$i\frac{d\boldsymbol{a}^{\pm}}{dt} = 0, \qquad i\frac{d\boldsymbol{b}^{\pm}}{dt} \mp 2f(\lambda)\boldsymbol{b}^{\pm}(t,\lambda) = 0 \tag{2.101}$$

that can be easily solved for any $f(\lambda)$.

Thus, we outlined the proof of the following generalization of Theorem 2.1:

Theorem 2.2 ([11, 12]). *If $\boldsymbol{q}(x,t)$ satisfies the NLEE (2.78), then the scattering matrix $\boldsymbol{T}(t,\lambda)$ satisfies the linear evolution equation (2.100).*

Remark 2.3. Not all MNLS equations are local. Only equations, whose Hamiltonians are from the principal series, i.e. ones whose dispersion laws are of the form $\boldsymbol{f}(\lambda) = f(\lambda)\boldsymbol{\sigma}$ are local. Such equations are superintegrable: They have more generating functionals of integrals of motion than are necessary for integrability. These functionals are not all in involutions. Due to this, boomerons and trappons are possible [13, 14].

2.5 Generalizations of the AKNS Method II

The AKNS method can be applied also to Lax operators generalizing the Zakharov-Shabat system to the following first-order $n \times n$ system:

$$L_{\mathrm{g}}\chi_{\mathrm{g}} \equiv \left(i\frac{d}{dx} + U_{\mathrm{g}}(x,t,\lambda) \right) \chi_{\mathrm{g}}(x,t,\lambda) = 0 \,, \tag{2.102a}$$

$$U_{\mathrm{g}}(x,t,\lambda) = q(x,t) - \lambda J \,, \tag{2.102b}$$

$$q(x,t) = \begin{pmatrix} 0 & q_{12} & \dots & q_{1n-1} & q_{1n} \\ q_{12} & 0 & \dots & q_{2n-1} & q_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{n1} & q_{n2} & \dots & q_{n-1n} & 0 \end{pmatrix} , \tag{2.102c}$$

$$J = \mathrm{diag}\,(a_1, a_2, \dots, a_{n-1}, a_n), \qquad \mathrm{tr}\, J = 0 \,. \tag{2.102d}$$

The second operator in the Lax representation is also a first-order $n \times n$ matrix-valued operator:

$$M_{\mathrm{g}}\chi_{\mathrm{g}} \equiv \left(i\frac{d}{dt} + V_{\mathrm{g}}(x,t,\lambda) \right) \chi_{\mathrm{g}}(x,t,\lambda) = \chi_{\mathrm{g}}(x,t,\lambda) C_{\mathrm{g}}(\lambda) \tag{2.103}$$

where $V_{\mathrm{g}}(x,t,\lambda)$ is a polynomial of order N in λ

$$V_{\mathrm{g}}(x,t,\lambda) = \sum_{k=0}^{N} \lambda^{N-k} V_k(x,t) \,. \tag{2.104}$$

Here and below, we shall use the same letter for the potential $q(x,t)$ and for the coefficients $V_k(x,t)$, remembering that now they are $n \times n$ matrices.

The recurrent relations (2.13) now are modified into:

$$[V_0(x,t), J] = 0 \,, \tag{2.105a}$$

$$i\frac{dV_k}{dx} + [q, V_k(x,t)] - [J, V_{k+1}(x,t)] = 0 \,, \tag{2.105b}$$

for $k = 0, 1, \dots, N-1$ and the λ–independent term gives the corresponding NLEEs:

$$-i\frac{\partial q}{\partial t} + i\frac{\partial V_N}{\partial x} + [q(x,t), V_N(x,t)] = 0 \,. \tag{2.105c}$$

Before proceeding with solving the recurrent relations (2.105), we shall fix up the gauge of the Lax operator L_{g}, taking J to be a constant diagonal matrix. We assume that J has n different real eigenvalues. Without loss of generality we can consider them ordered:

$$J = \mathrm{diag}\,(a_1, a_2, \dots, a_n), \qquad a_1 > a_2 > \dots > a_n \,, \tag{2.106}$$

and $\mathrm{tr}\, J = 0$. Applying a convenient gauge transformation commuting with J we can always achieve that

$$q(x,t) = [J, \widetilde{q}(x,t)], \qquad \text{i.e.} \qquad q_{jj} = 0 \,. \tag{2.107}$$

In analogy with the analysis of the previous sections, we again will need to split off each $V_k(x,t)$ into diagonal and off-diagonal parts. Now the corresponding algebra $\mathfrak{g} = sl(n)$ is split into a direct sum $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(\mathrm{f})}$ of linear subspaces, corresponding to the kernel and the image of ad_J.

Obviously, if the eigenvalues of J are all different then the kernel $\mathfrak{g}^{(0)}$ of ad_J will consist of the diagonal matrices, or more precisely, of the Cartan subalgebra of $sl(n)$. In contrast, with the $sl(2)$-case such splitting satisfies only two of the properties in (2.26), namely,

$$[X^{(0)}, Y^{(0)}] = 0, \qquad [X^{(0)}, Y^{(1)}] \in \mathfrak{g}^{(1)} , \tag{2.108}$$

whereas $[X^{(1)}, Y^{(1)}] \not\in \mathfrak{g}^{(0)}$; such commutators contain both diagonal and off-diagonal parts. The operator ad_J is well defined on the whole algebra $\mathfrak{g}$, while its inverse is well defined only on $\mathfrak{g}^{(1)}$. In components we have:

$$([J, X])_{jk} = (a_j - a_k) X_{jk}, \qquad \left(\mathrm{ad}_J^{-1} Y^{(\mathrm{f})}\right)_{jk} = \frac{Y^{(\mathrm{f})}_{jk}}{a_j - a_k} , \tag{2.109}$$

where by definition $Y^{(\mathrm{f})}$ is off-diagonal, $Y^{(\mathrm{f})}_{jj} = 0$. The analog of the projector π_J is given by:

$$\pi_J \cdot \equiv \mathrm{ad}_J^{-1} [J, \cdot\,] . \tag{2.110}$$

Applied to any $n \times n$ matrix X, it projects it out onto its off-diagonal part:

$$\pi_J X = X - X^{(0)} = \begin{pmatrix} 0 & X_{12} & \dots & X_{1n-1} & X_{1n} \\ X_{21} & 0 & \dots & X_{2n-1} & X_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ X_{n-1,1} & X_{n-1,2} & \dots & 0 & X_{n-1,n} \\ X_{n,1} & X_{n,2} & \dots & X_{n,n-1} & 0 \end{pmatrix} , \tag{2.111}$$

Then the projector onto $\mathfrak{g}^{(0)}$ is:

$$(\mathbb{1} - \pi_J) X = X^{(0)} = \begin{pmatrix} X_{11} & 0 & \dots & 0 & 0 \\ 0 & X_{22} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & X_{n-1,n-1} & 0 \\ 0 & 0 & \dots & 0 & X_{n,n} \end{pmatrix} . \tag{2.112}$$

Below, in this and the next subsections, we shall use the following generalization of the Condition **C1** (see page 71 below): $q(x,t)$ belongs to the space $\mathcal{M}_J$ of $n \times n$ if $q(x,t) \equiv \pi_J q(x,t)$, and its matrix elements are complex Schwartz–type functions. Each $V_k(x,t)$ can be split into:

$$V_k(x,t) = V_k^{(0)}(x,t) + V_k^{\mathrm{f}}(x,t) , \tag{2.113}$$

where

$$V_k^{(0)}(x,t, \equiv (\mathbb{1} - \pi_J) V_k(x,t) = w_k(x,t) \in \mathfrak{g}^{(0)} \ , \tag{2.114a}$$

$$V_k^{\mathrm{f}}(x,t) = \pi_J V_k(x,t) \ . \tag{2.114b}$$

From (2.105a), we immediately get that $V_0^{\mathrm{f}} = 0$, i.e.:

$$V_0(x,t) = w_0(x,t) \in \mathfrak{g}^{(0)} \ . \tag{2.115a}$$

Next, we consider (2.105b) with $k = 0$. Projecting it onto $\mathfrak{g}^{(0)}$ we obtain:

$$\frac{\partial w_0}{\partial x} = 0 \ . \tag{2.115b}$$

Therefore, in what follows we can assume that

$$V_0(x,t) \equiv w_0^0 \in \mathfrak{g}^{(0)} \tag{2.115c}$$

is a constant diagonal matrix. The off-diagonal part of (2.105b) with $k = 0$

$$[q(x,t), w_0^0] - [J, V_1(x,t)] = 0 \ , \tag{2.116a}$$

allows one to determine only the off-diagonal part of $V_1(x,t)$:

$$V_1^{\mathrm{f}}(x,t) = -\mathrm{ad}\,_J^{-1} \left[w_0^0, q(x,t) \right] \ . \tag{2.116b}$$

Analogously, for $k = 1$ (2.105b) gives:

$$i \frac{\partial w_1}{\partial x} + (\mathbb{1} - \pi_J)[q(x,t), V_1^{\mathrm{f}}(x,t)] = 0 \ , \tag{2.117a}$$

$$i \frac{\partial V_1^{\mathrm{f}}}{\partial x} + \pi_J \left([q(x,t), V_1^{\mathrm{f}}(x,t)] \right) = [J, V_2^{\mathrm{f}}(x,t)] \ . \tag{2.117b}$$

Inserting (2.116b) into (2.117a), we find

$$\frac{\partial w_1}{\partial x} = 0 \ , \tag{2.118}$$

i.e. we can assume that $w_1 = w_1^0 = \mathrm{const}$. Equation (2.117b) leads to:

$$V_2^{\mathrm{f}}(x,t) = \mathrm{ad}\,_J^{-1} \left(i \frac{\partial v_1^{\mathrm{f}}}{\partial x} + \pi_J \left([q(x,t), V_1^{\mathrm{f}}] \right) \right) \ . \tag{2.119}$$

For $k > 1$ we get in a similar way:

$$i \frac{\partial w_k}{\partial x} + (\mathbb{1} - \pi_J)[q(x,t), V_k^{\mathrm{f}}(x,t)] = 0 \ , \tag{2.120a}$$

$$i \frac{\partial V_k^{\mathrm{f}}}{\partial x} + \pi_J \left([q(x,t), V_k^{\mathrm{f}}(x,t)] \right) + [q(x,t), w_k(x,t)] = [J, V_{k+1}^{\mathrm{f}}(x,t)] \ . \tag{2.120b}$$

Formally integrating (2.120a), we can express $w_k(x,t)$ through $V_k^{\mathrm{f}}(x,t)$ by:

$$w_k(x,t) = w_k^0 + i\int_{\pm\infty}^{x} dy\ (\mathbb{1} - \pi_J)[q(y,t), V_k^{\mathrm{f}}(y,t)]\ , \tag{2.121}$$

where $w_k^0 \in \mathfrak{g}^{(0)}$ are constant diagonal matrices. We insert it into (2.120b) with the result:

$$\begin{aligned} &i\frac{\partial V_k^{\mathrm{f}}}{\partial x} + \pi_J[q(x,t), V_k^{\mathrm{f}}] + [q(x,t), w_k^0] \\ &\qquad + i\pi_J\left[q(x,t), \int_{\pm\infty}^{x} dy\ (\mathbb{1} - \pi_J)[q(y,t), V_k^{\mathrm{f}}(y,t)]\right] = [J, V_{k+1}^{\mathrm{f}}]. \end{aligned} \tag{2.122}$$

Here, and below, we shall use the fact that $\pi_J[q(x,t), w_k^0] \equiv [q(x,t), w_k^0]$.

Applying to both sides of (2.122) ad_J^{-1} we get:

$$V_{k+1}^{\mathrm{f}} = \Lambda_\pm V_k^{\mathrm{f}} + \mathrm{ad}_J^{-1}[q(x,t), w_k^0]\ , \tag{2.123}$$

where the recursion operators $\Lambda_\pm$ are defined by:

$$\begin{aligned} \Lambda_\pm X \equiv \mathrm{ad}_J^{-1}\Bigg\{ & i\frac{\partial X}{\partial x} + \pi_J[q(x), X(x)] \\ & + i\pi_J\left[q(x), \int_{\pm\infty}^{x} dy\ (\mathbb{1} - \pi_J)[q(y), X(y)]\right]\Bigg\}\ . \end{aligned} \tag{2.124}$$

Note that the structure of $\Lambda_\pm$ ensures that if $X \in \mathfrak{g}^{(1)}$ then also $\Lambda_\pm X \in \mathfrak{g}^{(1)}$.

Thus, we have cast the recursion relations (2.105) in the form (2.123); (2.116b) must be viewed as the initial condition for them. Its formal solution can be written down in compact form as:

$$V_{k+1}^{\mathrm{f}} = -\sum_{s=0}^{k} \Lambda_\pm^s \mathrm{ad}_J^{-1}[w_{k-s}^0, q(x)]\ . \tag{2.125}$$

It remains to repeat the “splitting” procedure also to the NLEEs (2.105c):

$$i\frac{\partial w_N}{\partial x} + (\mathbb{1} - \pi_J)[q(x,t), V_N^{\mathrm{f}}(x,t)] = 0\ , \tag{2.126a}$$

$$-i\frac{\partial q}{\partial t} + i\frac{\partial V_N^{\mathrm{f}}}{\partial x} + \pi_J\left([q(x,t), V_N^{\mathrm{f}}(x,t)]\right) + [q(x,t), w_N(x,t)] = 0\ , \tag{2.126b}$$

with the result

$$w_N(x,t) = w_N^0 + i\int_{\pm\infty}^{x} dy\ (\mathbb{1} - \pi_J)[q(y,t), V_N^{\mathrm{f}}(y,t)]\ , \tag{2.127}$$

and applying the operator ad_J^{-1} to both sides of (2.127) we get:

$$-i\operatorname{ad}_J^{-1}\frac{\partial q}{\partial t} + \Lambda_\pm V_N^{\mathrm{f}} - \operatorname{ad}_J^{-1}[w_N^0, q(x,t)] = 0 . \qquad (2.128)$$

This is the generic form of the NLEE solvable by the ISM applied to L_{g} (2.102). Using (2.125), we can write it down in compact form:

$$i\operatorname{ad}_J^{-1}\frac{\partial q}{\partial t} + \sum_{s=0}^{N} \Lambda_\pm^s \operatorname{ad}_J^{-1}[w_{N-s}^0, q(x,t)] = 0 . \qquad (2.129)$$

Note that in solving the recurrent relations we obtained $N+1$ integration constants w_k^0. These constant diagonal matrices determine the function:

$$f_{\mathrm{g}}(\lambda) = \sum_{s=0}^{N} w_{N-s}^0 \lambda^s , \qquad (2.130)$$

which is the proper generalization of the dispersion law $f(\lambda)$. Indeed, one can check that $f_{\mathrm{g}}(\lambda)$ determine the evolution of the scattering matrix of L_{g}.

The simplest of these NLEE is obtained already for $N=1$. This is the famous N-wave equation:

$$\begin{aligned} & i\left[J, \frac{\partial Q}{\partial t}\right] - i\left[I, \frac{\partial Q}{\partial x}\right] + [[I, Q(x,t)], [J, Q(x,t)]] = 0 , \qquad (2.131) \\ & Q(x,t) = \operatorname{ad}_J^{-1} q(x,t) \in \mathfrak{g}^{(1)}, \end{aligned}$$

where $I = w_0^0 \in \mathfrak{g}^{(0)}$. Its dispersion law is linear in λ:

$$f_{\mathrm{Nw}}(\lambda) = \lambda I . \qquad (2.132)$$

The scattering matrix $T_{\mathrm{g}}(\lambda,t)$ and the Jost solutions of L_{g} are natural generalizations of the ones for the Zakharov-Shabat system L. They are defined by:

$$\lim_{x\to\infty} \psi_{\mathrm{g}}(x,t,\lambda)e^{i\lambda Jx} = \mathbb{1}, \qquad \lim_{x\to-\infty} \phi_{\mathrm{g}}(x,t,\lambda)e^{i\lambda Jx} = \mathbb{1} , \qquad (2.133a)$$

$$T_{\mathrm{g}}(\lambda,t) = \hat{\psi}_{\mathrm{g}}(x,t,\lambda)\phi_{\mathrm{g}}(x,t,\lambda) . \qquad (2.133b)$$

The detailed investigation of the direct and inverse scattering problems for L_{g} comes out of the scope of the present Chapter. Here, we shall just derive the t-dependence of $T_{\mathrm{g}}(\lambda,t)$ using the Lax representation (2.102). We fix up $C_{\mathrm{g}}(\lambda)$ in the right hand-side of (2.103) in such a way that the definitions of the Jost solutions (2.133a) are valid for all time t, i.e.:

$$\begin{aligned} C_{\mathrm{g}}(\lambda) &\equiv \lim_{x\to\pm\infty} V_{\mathrm{g}}(x,t,\lambda) \\ &= \sum_{s=0}^{N} w_{N-s}^0 \lambda^s = f_{\mathrm{g}}(\lambda) . \qquad (2.134) \end{aligned}$$

Then, we recall that the Jost solution $\phi_{\rm g}(x,t\lambda)$ must satisfy (2.103) and consider its limits for $x \to -\infty$ and $x \to \infty$. In view of our choice for $C_{\rm g}(\lambda)$ (2.134), the first limit becomes the identity $0 = 0$. Doing the second limit, we make use of (2.133b): $\phi_{\rm g}(x,t\lambda) = \psi_{\rm g}(x,t\lambda)T_{\rm g}(t,\lambda)$ and get:

$$i\frac{dT_{\rm g}}{dt} + [f_{\rm g}(\lambda), T_{\rm g}(t,\lambda)] = 0 \, . \tag{2.135}$$

This result can be formulated as the following generalization of theorem

Theorem 2.3 ([15]). *Let $q(x,t) \in \mathcal{M}_J$ and satisfies the NLEE (2.129) then the scattering matrix $T_{\rm g}(t,\lambda)$ satisfies the linear evolution equation (2.135).*

Thus, we demonstrated the analogy between the NLEE related to the ZS system (2.43), and their generalizations (2.79) and (2.129), and the generic partial differential equation with constant coefficients (1.30) in Chap. 1. The polynomials $f(\lambda)$ determine the dispersion laws of these equations. In the case of the NLEE, the derivative operator $\frac{1}{i}\frac{\partial}{\partial x}$ has been replaced by the corresponding recursion operator $\Lambda_\pm$. This analogy is not coincidental, and its roots are in the spectral decompositions of the recursion operators.

2.6 Comments and Bibliographical Review

1. The KdV, NLS, MKdV, s-G, N-wave equations are only several of the NLEE that are integrable and have a wide range of applications in physics. In fact, they describe different regimes of wave-wave interactions, which do not depend on the physical origin of the waves. This explains their universality [16, 17]. Here, we give a short list of monographs and review papers [4, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37] in which these problems are analyzed and which contain the necessary references.
2. The fact that to each Lax operator L, one can relate a hierarchy of solvable NLEE became obvious in 1974 after the AKNS Chapter [1]. In it, they proposed a modification of the Lax approach, which simplified substantially the derivation of the relevant NLEE. The AKNS scheme, formulated initially for the ZS system, substantially simplified finding new “higher” NLEE related to a given Lax operator L and reduced it to the solving of a set of recurrent relations. They constructed also the recursion operators $\Lambda_\pm$, which solves the recurrent relations and plays fundamental role in deriving the properties of the NLEE. Another important fact discovered by AKNS [1] was the importance of the Wronskian relations and the squared solutions of L in studying the mapping between the potential $q(x,t)$ of L and the scattering data of L. They revealed that the squared solutions are eigenfunctions of the recursion operators $\Lambda_\pm$ and may be viewed as

natural generalizations of the usual exponentials. As a consequence, the ISM can be viewed as a generalized Fourier transform. In order to establish this fact rigorously, one needs to prove that the squared solutions are complete sets of functions in the space of allowed potentials $\mathcal{M}$ of L. In 1976, Kaup [38] formulated the completeness relation for the squared solutions of the ZS system. Later in 1979, Kaup and Newell [39] derived the fundamental properties of the NLS hierarchy through the recursion operators $\Lambda_\pm$ using the completeness property of its eigenfunctions – the squared solutions.
At about the same time, Khristov and one of the authors of the present monograph (VSG) [40, 41], independently of Kaup and Newell, proposed a rigorous proof of Kaup's compeleteness relation and applied it to the theory of the NLS-type equations. It was shown also that the completeness relation of the squared solutions can be viewed as the spectral decomposition of the recursion operator Λ.
Besides, in [40, 41] it was proved that the "products" of solutions of two different ZS systems also satisfy a completeness relation. These products of solutions are eigenfunctions of the operators $\boldsymbol{\Lambda}_\pm$ generalizing $\Lambda_\pm$ and generating the Bäcklund transformations of the NLEE. These results extend the results of Calogero and Degasperis [11, 12, 42]. The same type of results have been derived also for the Sturm-Liouville problem [43, 44, 45], for the ZS system with periodic boundary conditions [45, 46, 47, 48] and for the Sturm-Liouville problem on the semiaxis [49, 50].
3. The AKNS paper stimulated a number of other scientists [11, 12, 13, 14, 27, 39, 40, 41, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118]. In 1976, Calogero and Degasperis [11, 12] proposed generalized Wronskian identities to describe the class of Bäcklund transformations for the NLS-type NLEE.
4. The necessity to consider Lax operators generalizing the ZS system naturally called for generalizations of the AKNS approach and for the explicit derivation of the corresponding recursion operators Λ. Here, we list some of the best known ones:
   - the ZS system in the pole gauge [71, 81, 82, 118];
   - $n \times n$ ZS system [27, 119, 120, 121];
   - ZS system related to symmetric spaces [7, 9, 72, 122, 123];
   - The natural generalizations of the Zakharov-Shabat system to simple Lie algebras of rank higher than one:
$$i\frac{d\psi}{dx} + (Q(x,t) - \lambda J)\,\psi(x,t,\lambda) = 0, \qquad Q(x,t) = [J, Q'(x,t)]\,, \tag{2.136}$$
where $Q'(x,t)$ takes values in the simple Lie algebra $\mathfrak{g}$, and J is a constant element of some fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ see [9, 124, 125]

and [51, 74, 75, 84, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151];

- Polynomial generalizations of the Zakharov-Shabat system to simple Lie algebras of higher rank:

$$i\frac{d\psi}{dx} + \left(\sum_{k=1}^{n} \lambda^{n-k} Q_k(x,t) - \lambda^n J \right) \psi(x,t,\lambda) = 0 \ , \tag{2.137}$$

  where $Q_{n-1}(x,t) = [J, Q'(x,t)]$ and $Q_k(x,t)$ take values in the simple Lie algebra $\mathfrak{g}$, and J is a constant element of some fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The best known examples of this form are related to the $sl(2)$ algebra [70, 78, 114, 115, 121, 152, 153, 154, 155, 156, 157, 158]; for the bundles (pencils) with qubic and higher powers in λ see [70, 152, 156, 157, 159, 160]). They can be extended also to algebras of higher ranks, as well as to nonvanishing boundary conditions case [161];
- Gelfand–Dickey–Zakharov–Shabat problem [162];
- to the difference version of ZS system known as the Ablowitz-Ladik system [79, 163, 164, 165], their gauge equivalent ones [166], and their multicomponent generalizations [37, 76];
- for ZS system with periodic boundary conditions, see [47, 167, 168, 169, 170] and the numerous references therein. Another important class of boundary conditions, whose treatment requires a number of additional constructions are the constant boundary conditions; see [161, 171, 172].
- for ZS system with elliptic dependence on λ see [107, 173].

5. For a number of important choices of $L(\lambda)$, to the best of our knowledge, the derivation of the AKNS scheme has not yet been done, and the corresponding recursion operators Λ are not yet known. This refers to the cases in which $L(\lambda)$ is rational function of λ [174, 175].
6. The formal approach to the recursion operators and NLEE is outlined in a series of papers [127, 176]; see also [177];
7. The so-called U-V-systems were introduced by Zakharov and Mikhailov [174, 175] where

$$L(\lambda) = i\frac{d}{dx} + U(x,t,\lambda), \qquad M(\lambda) = i\frac{d}{dt} + V(x,t,\lambda) \ , \tag{2.138}$$

   and $U(x,t,\lambda)$ and $V(x,t,\lambda)$ are rational functions of λ taking values in $\mathfrak{g}$. Such Lax pairs allow to solve the principal chiral field equation in $1+1$ dimensions, as well as a number of fermionic models in field theory;
8. U-V-systems with elliptic λ-dependence were used to solve the Landau-Lifshitz equations and its generalizations related to the $sl(n)$-algebras [173, 178].

For the last two items, the recursion operator is known only for the simplest case of rational U-V-system relevant for the principal chiral field [61] and for the $sl(2)$-Landau-Lifshitz equation [179, 180].

9. Quite different from the operators in [179, 180] is the recursion operator found for the Landau-Lifshitz equation with Lax pairs using some deformations of the algebra $so(4)$, see [181].
10. Discrete systems such as the Ablowitz-Ladik system [182] and its multicomponent generalizations have been treated along the same lines in [76, 79, 183, 166].

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3

The Direct Scattering Problem for the Zakharov–Shabat System

In the first section of this chapter, we derive the analyticity properties of the Jost solutions of $L(\lambda)$ and construct its fundamental analytic solutions (FAS) $\chi^{\pm}(x,\lambda)$. In Chap. 2, the FAS are used to construct the kernel of the resolvent $R^{\pm}(x,y,\lambda)$ of $L(\lambda)$, whose properties determine the spectrum of $L(\lambda)$.

The analyticity properties of $\chi^{\pm}(x,\lambda)$ ensure also the analyticity of $a^{\pm}(\lambda) = \det \chi^{\pm}(x,\lambda)$, whose zeroes determine the poles of $R^{\pm}(x,y,\lambda)$ and the discrete eigenvalues of $L(\lambda)$. In Sects. 3.3, 3.4, 3.5 and 3.6, we derive the asymptotic behavior of $a^{\pm}(\lambda)$ and $\chi^{\pm}(x,\lambda)$ for $\lambda \to \infty$.

The contour integration method and the analyticity properties of $\chi^{\pm}(x,\lambda)$ allow one to derive integral representations for the Jost solutions and $a^{\pm}(\lambda)$, thus extracting two minimal sets of scattering data $\mathcal{T}_i$, $i = 1, 2$ for $L(\lambda)$. Each of these two sets determines uniquely $\chi^{\pm}(x,\lambda)$ and $a^{\pm}(\lambda)$ in their whole regions of analyticity. In Sect. 3.7, we apply the contour integration method and derive the completeness relation for the Jost solutions of $L(\lambda)$. This relation can be viewed as the spectral decomposition of $L(\lambda)$.

In this, and in the next chapters, t plays the role of an auxiliary parameter but for the sake of brevity we shall omit it.

3.1 Analytic Properties of the Jost Solutions

We begin by a detailed study of the direct scattering problem for the ZS system:

$$L\chi \equiv \left(i\frac{d}{dx} + q(x) - \lambda\sigma_3 \right) \chi(x,\lambda) = 0 \; , \tag{3.1}$$

where the potential $q(x)$ satisfies two conditions, **C1** and **C2**:

Condition C1: $q(x)$ belongs to the space $\mathcal{M}$ of off-diagonal 2×2 matrix-valued functions, whose matrix elements are complex Schwartz-type functions.

This condition is made for simplification. It can be weakened substantially, but this would require deeper mathematical analysis, which is out of the scope

Gerdjikov, V.S. et al.: *The Direct Scattering Problem for the Zakharov–Shabat System*. Lect. Notes Phys. **748**, 71–95 (2008)
DOI 10.1007/978-3-540-77054-1_3

of the present monograph. We use **C1** in order to outline better the main ideas of the ISM. For reasons that will become clear, condition **C2** will be formulated at the end of this section.

We recall the definitions of the Jost solutions (2.45) and the scattering matrix (2.48):

$$\lim_{x\to\infty} \exp(i\lambda\sigma_3 x)\psi(x,\lambda) = \mathbb{1}, \tag{3.2a}$$

$$\lim_{x\to-\infty} \exp(i\lambda\sigma_3 x)\phi(x,\lambda) = \mathbb{1}\ , \tag{3.2b}$$

$$T(\lambda) = \psi^{-1}(x,\lambda)\phi(x,\lambda) = \begin{pmatrix} a^+(\lambda), & -b^-(\lambda) \\ b^+(\lambda), & a^-(\lambda) \end{pmatrix}\ . \tag{3.2c}$$

for $\lambda \in \mathbb{R}$. Together with (3.1), we consider the associated systems, whose solutions are related to $\psi(x,\lambda)$ and $\phi(x,\lambda)$ by:

$$\xi(x,\lambda) = \psi(x,\lambda)\exp(i\lambda\sigma_3 x), \tag{3.3a}$$

$$\hat{\psi}(x,\lambda) \equiv \psi^{-1}(x,\lambda) \tag{3.3b}$$

$$\varphi(x,\lambda) = \phi(x,\lambda)\exp(i\lambda\sigma_3 x)\ . \tag{3.3c}$$

Obviously, $\xi(x,\lambda)$ and $\varphi(x,\lambda)$ satisfy the associated to (3.1) linear systems:

$$i\frac{d\xi}{dx} + q(x)\xi(x,\lambda) - \lambda[\sigma_3, \xi(x,\lambda)] = 0\ , \tag{3.4}$$

but with different boundary conditions, which according to (3.2) are:

$$\lim_{x\to\infty} \xi(x,\lambda) = \mathbb{1}, \qquad \lim_{x\to-\infty} \varphi(x,\lambda) = \mathbb{1}\ . \tag{3.5}$$

As regards $\hat{\psi}(x,\lambda)$, it satisfies another associated to (3.1) system:

$$i\frac{d\hat{\psi}}{dx} - \hat{\psi}(x,\lambda)(q(x) - \lambda\sigma_3) = 0\ . \tag{3.6}$$

Each of the above systems of differential equations, together with the corresponding boundary conditions, can be rewritten as a system of integral equations. More specifically, for $\xi(x,\lambda)$ and $\varphi(x,\lambda)$ these equations are:

$$\xi(x,\lambda) = \mathbb{1} + i\int_{\infty}^{x} dy\, e^{-i\lambda\sigma_3(x-y)} q(y)\xi(y,\lambda) e^{i\lambda\sigma_3(x-y)}\ , \tag{3.7}$$

and

$$\varphi(x,\lambda) = \mathbb{1} + i\int_{-\infty}^{x} dy\, e^{-i\lambda\sigma_3(x-y)} q(y)\varphi(y,\lambda) e^{i\lambda\sigma_3(x-y)}\ , \tag{3.8}$$

More explicitly, the equation for $\xi(x,\lambda)$ is:

$$\xi(x,\lambda) = \mathbb{1} \tag{3.9}$$

$$+\, i \int_{\infty}^{x} dy \begin{pmatrix} q^{+}(y)\xi_{21}(y,\lambda), & q^{+}(y)\xi_{22}(y,\lambda)e^{-2i\lambda(x-y)} \\ q^{-}(y)\xi_{11}(y,\lambda)e^{2i\lambda(x-y)}, & q^{-}(y)\xi_{12}(y,\lambda) \end{pmatrix} ,$$

and we have an analogous one for $\varphi(x,\lambda)$; obviously the lower limit in the integral for $\varphi(x,\lambda)$ will be $-\infty$.

For later convenience, we shall split the matrix-valued solutions $\xi(x,\lambda)$ and $\phi(x,\lambda)$ into pairs of columns:

$$\xi(x,\lambda) = \left(\xi^{-},\, \xi^{+}\right)(x,\lambda), \qquad \varphi(x,\lambda) = \left(\varphi^{+}\,, \varphi^{-}\right)(x,\lambda)\,, \tag{3.10}$$

where, as we shall show below, the superscript $+$ (resp. $-$) means that the corresponding column is analytic for $\lambda \in \mathbb{C}_{+}$ (resp. for $\lambda \in \mathbb{C}_{-}$), where by $\mathbb{C}_{+}$ (resp. by $\mathbb{C}_{-}$) we denote the upper (resp., the lower) half-plane . These columns satisfy the integrals equations:

$$\xi^{-}(x,\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \int_{\infty}^{x} dy\, G_2(x-y,\lambda) q(y) \xi^{-}(y,\lambda)\,, \tag{3.11}$$

$$\xi^{+}(x,\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \int_{\infty}^{x} dy\, G_1(x-y,\lambda) q(y) \xi^{-}(y,\lambda)\,, \tag{3.12}$$

$$\phi^{+}(x,\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \int_{-\infty}^{x} dy\, G_2(x-y,\lambda) q(y) \phi^{-}(y,\lambda)\,, \tag{3.13}$$

$$\phi^{-}(x,\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \int_{-\infty}^{x} dy\, G_1(x-y,\lambda) q(y) \phi^{-}(y,\lambda)\,, \tag{3.14}$$

where the Green functions $G_a(x-y,\lambda)$, $a = 1,2$, are given by:

$$G_1(x-y,\lambda) = \begin{pmatrix} e^{-2i\lambda(x-y)} & 0 \\ 0 & 1 \end{pmatrix} , \tag{3.15}$$

$$G_2(x-y,\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & e^{2i\lambda(x-y)} \end{pmatrix} . \tag{3.16}$$

The equations (3.11), (3.12), (3.13) and (3.14) are integral equations of Volterra type. Their solutions are given by Neumann series which for (3.11) has the form:

$$\xi^{-}(x,\lambda) = \sum_{j=0}^{\infty} \xi_j^{-}(x,\lambda), \qquad \xi_0^{-}(x,\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \tag{3.17}$$
$$\xi_{j+1}^{-}(x,\lambda) = i \int_{\infty}^{x} dy\, G_2(x-y,\lambda) q(y) \xi_j^{-}(y,\lambda) \,.$$

The existence of the solution $\xi^{-}(x,\lambda)$ depends on the convergence of the series (3.17), and one needs estimates for the components of $\xi_j^{-}(x,\lambda) = \begin{pmatrix} \xi_j^{(1),-} \\ \xi_j^{(2),-} \end{pmatrix}(x,\lambda)$. Such estimates have the form (see [1, 2, 3, 4]):

$$|\xi_{2j}^{(1),-}(x,\lambda)| \le \frac{1}{(j!)^2}\left(\int_\infty^x dx'\,|q^+(x')|\right)^j \left(\int_\infty^x dx'\,|e^{2i\lambda(x-x')}q^-(x')|\right)^j , \quad (3.18)$$

$$|\xi_{2j+1}^{(2),-}(x,\lambda)| \le \frac{1}{j!(j+1)!}\left(\int_\infty^x dx'\,|q^+(x')|\right)^j \left(\int_\infty^x dx'\,|e^{2i\lambda(x-x')}q^-(x')|\right)^{j+1} .$$

Thus one gets:

$$|\xi^{(1),-}(x,\lambda)| \le I_0(2\sqrt{s(x)}), \qquad |\xi^{(2),-}(x,\lambda)| \le I_1(2\sqrt{s(x)}) , \quad (3.19)$$

$$s(x) = \int_\infty^x dx'\,|q^+(x')| \int_\infty^x dx'\,|e^{2i\lambda(x-x')}q^-(x')| , \quad (3.20)$$

where $I_0(s)$ and $I_1(s)$ are the Bessel functions of the first kind.

Similar estimates hold true also for the other columns of the Jost solutions. If λ is real, then the exponentials in the Green functions will be bounded for all x and y, and the only condition for the existence of the Jost solutions will be the integrability of the potential, i.e. the existence of $s(x)$ for $x \to \pm\infty$. This class of potentials is very general, which means that the elements of the scattering matrix $T(\lambda)$ for real-valued λ may not be smooth.

Lemma 3.1. *The solutions $\phi^+(x,\lambda)$, $\xi^+(x,\lambda)$ (resp. $\phi^+(x,\lambda)$, $\xi^-(x,\lambda)$) are analytic functions of λ for $\lambda \in \mathbb{C}_+$ (resp. $\lambda \in \mathbb{C}_-$).*

Proof. The idea of the proof is based on the fact that for potentials of **C1** type one can prove the existence not only of the Jost solutions $\xi^\pm(x,\lambda)$ and $\phi^\pm(x,\lambda)$ but also of their derivatives

$$\frac{d^p\xi^\pm(x,\lambda)}{d\lambda^p}, \qquad \frac{d^p\phi^\pm(x,\lambda)}{d\lambda^p} , \quad (3.21)$$

for all integer $p = 1, 2, \dots$. Indeed, the terms in the corresponding Neumann series for $\frac{d^p\xi^\pm(x,\lambda)}{d\lambda^p}$ will contain integrals of the form:

$$\int_\infty^x dx'\,|(x-x')^m q^\pm(x')| \int_\infty^x dx'\,|e^{2i\lambda(x-x')}q^-(x')|, \qquad 0 \le m \le p , \quad (3.22)$$

which are convergent if $q^\pm(x)$ are Schwartz-type functions (condition **C1**).

If λ becomes complex, then the exponential factors in (3.18) have to be taken into account. If for some values of λ they happen to be decreasing, one again will be able to prove the existence of the corresponding Jost solutions. For example, in the equation for $\xi^-(x,\lambda)$, the factor in the integrand for $s(x)$ is $e^{2i\lambda(x-x')}$ which is decreasing for $\operatorname{Im}\lambda < 0$, since we integrate over $x < x' < \infty$. Thus, the Neumann series for $\xi^-(x,\lambda)$ is convergent for all $\operatorname{Im}\lambda \le 0$. Besides each of the terms in the series is an analytic function of λ, which means that $\xi^-(x,\lambda)$ is analytic function for $\lambda \in \mathbb{C}_-$.

Lemma 3.2. *Let the potential $q(x)$ satisfy the condition* **C1**. *Then the scattering matrix elements $T(\lambda)$ are Schwartz-type functions on the real λ-axis.*

Proof. The idea of the proof is based on evaluating the limits of $e^{i\lambda x\sigma_3}\xi^{\pm}(x,\lambda)$ and their derivatives $e^{i\lambda x\sigma_3}(d^p\xi^{\pm}/dx^p)(x,\lambda)$ for $x \to \pm\infty$ and for real λ.

Remark 3.3. As we mentioned in the beginning of this chapter, the condition **C1** can be substantially weakened. For example, if we require that $q^{\pm}(y)$ are smooth functions such that

$$\lim_{y\to\pm\infty} |y^p q^{\pm}(y)| < \infty \,, \tag{3.23}$$

then $d^n\xi^-/d\lambda^n$ will allow continuation onto the real λ-axis only for $n < p$.

As a basic tool in what follows, we shall use the *fundamental analytic solutions* (FAS) of the ZS system (3.1), which are obtained by combining the pairs of columns of the Jost solutions with the same analyticity properties:

$$\begin{aligned} \chi^+(x,\lambda) &= \left(\varphi^+,\xi^+\right)\exp(-i\lambda\sigma_3 x) = \left(\phi^+,\psi^+\right) \\ \chi^-(x,\lambda) &= \left(\xi^-,\varphi^-\right)\exp(-i\lambda\sigma_3 x) = \left(\psi^-,\phi^-\right). \end{aligned} \tag{3.24}$$

We mentioned already that any two fundamental solutions of (3.1) are linearly related. Comparing (3.24) with (3.2) we obtain:

$$\chi^+(x,\lambda) = \psi(x,\lambda)\begin{pmatrix} a^+, & 0 \\ b^+, & 1 \end{pmatrix} = \phi(x,\lambda)\begin{pmatrix} 1, & b^- \\ 0, & a^+ \end{pmatrix} \,, \tag{3.25a}$$

$$\chi^-(x,\lambda) = \psi(x,\lambda)\begin{pmatrix} 1, & -b^- \\ 0, & a^- \end{pmatrix} = \phi(x,\lambda)\begin{pmatrix} a^-, & 0 \\ -b^+, & 1 \end{pmatrix} \,. \tag{3.25b}$$

Since $\det\psi = \det\phi = 1$ from (3.2) we find:

$$\det T(\lambda) = 1, \qquad \text{i.e.,} \qquad a^+a^- + b^+b^- = 1, \qquad \lambda \in \mathbb{R}\,. \tag{3.26}$$

and from (3.25) we get:

$$\det\chi^+(x,\lambda) = a^+(\lambda), \qquad \det\chi^-(x,\lambda) = a^-(\lambda)\,. \tag{3.27}$$

From (3.27), it follows that $a^+(\lambda)$ and $a^-(\lambda)$ are analytic functions of λ for $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\lambda < 0$, respectively.

Let us make now analogous considerations for the system (3.6) associated to (3.1). We can do this in two different ways. The first one is to write down the corresponding set of integral equations and to analyze them as above. The second, more "economic" approach, consists in using (3.3b) and the already known results about the solutions of the ZS system. It is not difficult to evaluate $\hat\psi(x,\lambda)$ – the inverse of the Jost solution $\psi(x,\lambda)$ with the result:

$$\hat\psi(x,\lambda) = \begin{pmatrix} \psi_2^+ & -\psi_1^+ \\ -\psi_2^- & \psi_1^- \end{pmatrix}(x,\lambda) = \begin{pmatrix} \tilde\psi^+(x,\lambda) \\ -\tilde\psi^-(x,\lambda) \end{pmatrix} \,, \tag{3.28}$$

where by "hat" we denote the inverse matrix: $\hat{\psi} \equiv \psi^{-1}$. We have also introduced the "tilde"-operation, which transforms any two-component vector $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ into:

$$\tilde{X} = (X_2, -X_1) = X^T(-i\sigma_2) \; . \tag{3.29}$$

If we insert (3.28) into (3.3c) we have:

$$\begin{pmatrix} a^+(\lambda), & -b^-(\lambda) \\ b^+(\lambda), & a^-(\lambda) \end{pmatrix} = \begin{pmatrix} \tilde{\psi}^+(x,\lambda) \\ -\tilde{\psi}^-(x,\lambda) \end{pmatrix} \left(\phi^+, \phi^-\right)(x,\lambda) \; , \tag{3.30}$$

where we must use the standard rules for multiplying block matrices. For example, for the $(1,1)$-element in the right-hand side of (3.30), we get:

$$\begin{aligned} a^+(\lambda) &= \tilde{\psi}^+(x,\lambda)\phi^+(x,\lambda) \\ &= (\psi_2^+\phi_1^+ - \psi_1^+\phi_2^+)(x,\lambda) = W(\psi^+,\phi^+) \; , \end{aligned} \tag{3.31}$$

i.e. we reproduce the well-known expression for $a^+(\lambda)$ as the Wronskian of the two Jost solutions $\psi^+(x,\lambda)$ and $\phi^+(x,\lambda)$.

Analogously, we can evaluate the inverses of $\tilde{\chi}^+(x,\lambda)$ and $\tilde{\chi}^-(x,\lambda)$ to find:

$$\hat{\chi}^+(x,\lambda) = \frac{1}{a^+(\lambda)} \begin{pmatrix} \tilde{\psi}^+(x,\lambda) \\ -\tilde{\phi}^+(x,\lambda) \end{pmatrix} \; , \tag{3.32a}$$

$$\hat{\chi}^-(x,\lambda) = \frac{1}{a^-(\lambda)} \begin{pmatrix} \tilde{\phi}^-(x,\lambda) \\ -\tilde{\psi}^-(x,\lambda) \end{pmatrix} \; . \tag{3.32b}$$

In particular, for the ratio of these two solutions on the real λ-axis, we get:

$$a^+(\lambda)\hat{\chi}^+(x,\lambda)\chi^-(x,\lambda) = \begin{pmatrix} 1 & -b^-(\lambda) \\ -b^+(\lambda) & 1 \end{pmatrix} \; . \tag{3.33}$$

In the next two chapters, we shall see that this relation plays an important role.

We conclude that $\hat{\chi}^\pm(x,\lambda) \equiv \left(\chi^\pm(x,\lambda)\right)^{-1}$ are fundamental analytic solutions of (3.6).

It should be noted that $\chi^\pm(x,\lambda)$ and $\hat{\chi}^\pm(x,\lambda)$ are fundamental solutions of (3.1) and (3.6), respectively, only if their determinants *do not vanish,* i.e. if $a^\pm(\lambda) \neq 0$. This cannot be ensured apriory for all λ with $\mathrm{Im} \gtrless 0$. However, the number of zeroes of $a^\pm(\lambda)$ and their location is of crucial importance for future constructions. In order to reasonably simplify our considerations, we shall impose one more implicit condition on the potential $q(x)$ of (3.1), imposing requirements for the zeroes of $a^\pm(\lambda)$:

Condition C2. The potential $q(x)$ of the system (3.1) is such that the corresponding transition coefficients $a^+(\lambda)$ and $a^-(\lambda)$ have a finite number of simple zeroes in their regions of analyticity located at $\lambda_k^\pm$:

$$\left\{\lambda_k^\pm : a^\pm(\lambda_k^\pm) = 0, \quad \mathrm{Im}\,\lambda_k^\pm \gtrless 0, \quad k = 1, 2, \ldots, N\right\} \; . \tag{3.34}$$

Remark 3.4. It can be shown that the set of potentials satisfying conditions **C1** and **C2** is dense in the manifold of all potentials.

3.2 The Spectrum of *L*

In order to find out what is the spectrum of the operator L, we shall use the fundamental analytic solutions (3.25) and construct with their help the resolvent of L. Let us introduce the functions:

$$R^+(x,y,\lambda) = \frac{1}{i}\chi^+(x,\lambda)\begin{pmatrix} -\theta(y-x), & 0 \\ 0, & \theta(x-y) \end{pmatrix}\hat{\chi}^+(y,\lambda)\ , \tag{3.35a}$$

$$R^-(x,y,\lambda) = \frac{1}{i}\chi^-(x,\lambda)\begin{pmatrix} \theta(x-y), & 0 \\ 0, & -\theta(y-x) \end{pmatrix}\hat{\chi}^-(y,\lambda)\ , \tag{3.35b}$$

where $\theta(x)$ is the step function

$$\theta(x) = \begin{cases} 1 & \text{for } x > 0\ , \\ 1/2 & \text{for } x = 0\ , \\ 0 & \text{for } x < 0. \end{cases} \tag{3.36}$$

As known, $d\theta/dx = \delta(x)$. Then:

1. $R^{\pm}(x,y,\lambda)$ satisfy the equation:

$$i\frac{dR^{\pm}}{dx} + (q(x) - \lambda\sigma_3)R^{\pm}(x,y,\lambda) = \delta(x-y)1\!\!1\ ; \tag{3.37}$$

2. $R^{\pm}(x,y,\lambda)$ are analytic functions of λ for $\operatorname{Im}\lambda \gtrless 0$, respectively, at all the points $\lambda \neq \lambda_k^{\pm}$. At the points $\lambda_k^{\pm}$ $R^{\pm}(x,y,\lambda)$ have poles.

These are usual conditions, which the kernel of the resolvent of the operator L (3.1) must satisfy. Now we define the resolvent by:

$$R_\lambda f \equiv \int_{-\infty}^{\infty} dy\, R(x,y,\lambda)f(y) \tag{3.38}$$

where the kernel is defined by:

$$R(x,y,\lambda) = \begin{cases} R^+(x,y,\lambda),\ \operatorname{Im}\lambda > 0\ , \\ R^-(x,y,\lambda),\ \operatorname{Im}\lambda < 0. \end{cases} \tag{3.39}$$

In order to ensure that the integral operator R_λ is well defined, we must verify that $R^{\pm}(x,y,\lambda)$ fall off fast enough for $x,\ y \to \pm\infty$. But for $x,\ y \to \infty$ we have:

$$\begin{aligned} \chi^+(x,\lambda) &\underset{x\to\infty}{\longrightarrow} \exp(-i\lambda\sigma_3 x)\begin{pmatrix} a^+, & 0 \\ b^+, & 1 \end{pmatrix}\ , \\ \hat{\chi}^+(y,\lambda) &\underset{y\to\infty}{\longrightarrow} \begin{pmatrix} 1/a^+, & 0 \\ -b^+/a^+, & 1 \end{pmatrix}\exp(i\lambda\sigma_3 y), \end{aligned} \tag{3.40}$$

and inserting this into (3.35) we get:

$$R^{+}(x,y,\lambda) \underset{x,y,\to\infty}{\longrightarrow} \frac{1}{i}\begin{pmatrix} -e^{-i\lambda(x-y)}\theta(y-x), & 0 \\ -e^{i\lambda(x+y)}b^{+}/a^{+}, & e^{i\lambda(x-y)}\theta(x-y) \end{pmatrix} . \qquad (3.41)$$

So, we conclude that for $\operatorname{Im}\lambda > 0$ all the matrix elements of R^{+} fall off exponentially when x and y tend independently to ∞. Analogously, we can check for all the other combinations, e.g. for $x \to \infty$, $y \to -\infty$ etc. Thus, we find that the kernel of the integral operator (3.38) falls off exponentially when $|x|$ and $|y|$ tend to ∞ and $\operatorname{Im}\lambda \neq 0$.

Defining the spectrum of L as usual as the complement to the set of the points where R_λ is bounded, we see that all points $\lambda \neq \lambda_k^{\pm}$ with $\operatorname{Im}\lambda \neq 0$ are regular, i.e. do not belong to the spectrum of L. The points λ from the real axis are not regular. Indeed for $\operatorname{Im}\lambda = 0$ $R^{\pm}(x,y,\lambda)$ are only bounded functions for $|x|$ and $|y|$ tending to ∞. Since in this case

$$R(x,y,\lambda) = \frac{1}{2}\big(R^{+}(x,y,\lambda) + R^{-}(x,y,\lambda)\big), \qquad \operatorname{Im}\lambda = 0 , \qquad (3.42)$$

the corresponding integral operator (3.38) is not bounded. In other words, the continuous spectrum of L fills up the whole real axis.

The resolvent R_λ has also other types of singularities in λ, which are localized at the zeroes of $a^{\pm}(\lambda)$. Indeed from (3.27), it follows that $\hat{\chi}^{\pm}(y,\lambda)$ have pole singularities at $\lambda_k^{\pm}$ (3.34), whose locations do not depend on y. In view of condition **C2** and (3.35), we conclude that R_λ has simple poles at $\lambda_k^{\pm}$. This means that $\lambda_k^{\pm}$ are simple discrete eigenvalues of the operator L.

Conclusion 3.5 *The continuous spectrum of L fills up the real axis $\mathbb{R}$ in the complex λ-plane; since for each value of $\lambda \in \mathbb{R}$ (3.1) has two linearly independent solutions (e.g. ψ^{+} and ψ^{-} then the continuous spectrum of L is doubly degenerate. The discrete spectrum of L is located at the zeroes of $a^{+}(\lambda)$ in the upper half-plane and at the zeroes of $a^{-}(\lambda)$ in the lower half-plane.*

Remark 3.6. Note that there are cases when the functions $a^{\pm}(\lambda)$ do not have zeroes, and then the operator L has no discrete eigenvalues. As an example, we give here the case when $q^{-} = -{q^{+}}^{*}$. Then the linear system (3.1) reduces to the eigenvalue problem for a self-adjoint operator, whose spectrum can be located only on the real axis; at the same time the continuous spectrum fills up the whole axis.

Condition **C2** we have imposed states that $a^{+}(\lambda)$ and $a^{-}(\lambda)$ have the same number of zeroes. Although we do not know of a theorem stating this fact for generic potentials $q(x)$ (i.e. potentials without additional involutions), we also do not know counterexamples. All explicit procedures for adding new discrete eigenvalues to the spectrum of L (such as the Bäcklund transformation method, the dressing method etc., the Gel'fand–Levitan equation) always add pairs of eigenvalues λ_k^{+} and λ_k^{-}, $\operatorname{Im}\lambda_k^{\pm} \gtrless 0$.

Let us now consider the behavior of the solutions $\chi^\pm$ in the neighborhood of the points $\lambda_k^\pm$. Since $\det \chi^\pm(x, \lambda_k^\pm) = a^\pm(\lambda_k^\pm) = 0$, the columns of these solutions must be linearly dependent. Therefore:

$$\phi^+(x, \lambda_k^+) = b_k^+ \psi^+(x, \lambda_k^+), \qquad \phi^-(x, \lambda_k^-) = -b_k^- \psi^-(x, \lambda_k^-)\,, \tag{3.43}$$

or briefly:

$$\phi_k^\pm(x) = \pm b_k^\pm \psi_k^\pm(x), \tag{3.44}$$

where

$$\phi_k^\pm(x) = \phi^\pm(x, \lambda_k^\pm), \qquad \psi_k^\pm(x) = \psi^\pm(x, \lambda_k^\pm)\,. \tag{3.45}$$

At the points $\lambda_k^\pm$, the system (3.1) has just one solution. We remind that $\chi^\pm$ for $x \to \pm\infty$ have the following asymptotic behavior (see (3.25)):

$$\begin{aligned} \chi^+(x,\lambda) &\underset{x\to\infty}{\longrightarrow} \exp(-i\lambda\sigma_3 x)\begin{pmatrix} a^+ & 0 \\ b^+ & 1 \end{pmatrix}, \\ \chi^+(x,\lambda) &\underset{x\to-\infty}{\longrightarrow} \exp(-i\lambda\sigma_3 x)\begin{pmatrix} 1 & b^- \\ 0 & a^+ \end{pmatrix}, \end{aligned} \qquad \operatorname{Im}\lambda \gtrsim 0 \tag{3.46}$$

$$\begin{aligned} \chi^-(x,\lambda) &\underset{x\to\infty}{\longrightarrow} \exp(-i\lambda\sigma_3 x)\begin{pmatrix} 1 & -b^- \\ 0 & a^- \end{pmatrix}, \\ \chi^-(x,\lambda) &\underset{x\to-\infty}{\longrightarrow} \exp(-i\lambda\sigma_3 x)\begin{pmatrix} a^- & 0 \\ -b^+ & 1 \end{pmatrix}, \end{aligned} \qquad \operatorname{Im}\lambda \lesssim 0 \tag{3.47}$$

Strictly speaking, these limits hold true for all $\lambda \in \mathbb{C}_\pm$ only in the case of potentials on compact support. If the potential $q(x)$ falls off exponentially with $|x| \to \infty$, then (3.46), (3.47) hold only in a strip around the real axis. For generic potentials these limits hold only on the real axis.

Remark 3.7. In the study of the properties of the ZS system we use the following tactics: (i) first, we derive the given property for potentials on compact support; (ii) next, we consider the limit when the support of $q(x)$ expands to cover the whole x-axis. The last step in most cases will be skipped; to perform it rigorously, a study of the corresponding system of integral equations (3.7), (3.8) and (3.9) is required. In detail, the methods for investigating the asymptotic behavior of $\chi^\pm$ both for $|x| \to \infty$ and $|\lambda| \to \infty$ are explained in [5, 3].

The above approach can be generalized in order to include the case of discrete eigenvalues with multiplicities greater than 1. This does not cause any additional difficulties, but the corresponding formulae become much more involved. Besides, this case can be obtained as the limit when two simple eigenvalues approach each other. Another reason to limit ourselves with the potentials satisfying **C2** is that the manifold of **C2**-potentials is dense in the manifold of all potentials (smooth and tending fast enough to 0 for $|x| \to \infty$) of L.

3.3 Asymptotic Behavior for $\lambda \to \infty$

One of the basic methods used in this part is the so-called contour integration method. To do this, we shall need not only the analytic properties of the solutions $\chi^{\pm}(x,\lambda)$, but also their behavior for $\lambda \to \infty$.

Let us denote by $\eta^{\pm}(x,\lambda) = \chi^{\pm}(x,\lambda)\exp(i\lambda\sigma_3 x)$; obviously the functions satisfy (3.4):

$$i\frac{d\eta^{\pm}}{dx} + q(x)\eta^{\pm} - \lambda[\sigma_3, \eta^{\pm}(x,\lambda)] = 0 \,. \tag{3.48}$$

In the simplest case, when $q(x) \equiv 0$ the Jost solutions of this equation are $\eta^{\pm}(x,\lambda) = \mathbb{1}$. In the generic case, $\eta^{\pm}(x,\lambda)$ are analytic functions of λ for $\lambda \in \mathbb{C}_{\pm}$. For $\lambda = \infty$, they allow asymptotic expansions over the inverse powers of λ of the form:

$$\eta^{\pm}(x,\lambda) = \mathbb{1} + \sum_{k=1}^{\infty} \eta_k^{\pm}(x)\lambda^{-k}, \qquad \lambda \in \mathbb{C}_{\pm} \,. \tag{3.49}$$

If we insert (3.49) into (3.48), we get the following relations:

$$q(x) - [\sigma_3, \eta_1^{\pm}(x)] = 0 \,, \tag{3.50a}$$

for $\eta_1^{\pm}$, which can be considered as the initial condition, and

$$i\frac{d\eta_k^{\pm}}{dx} + q(x)\eta_k^{\pm}(x) - [\sigma_3, \eta_{k+1}^{\pm}(x)] = 0 \,. \tag{3.50b}$$

The form of these recurrent relations is close to the one for $V_k(x,t)$; see (3.7). So, they are solved in an analogous way by splitting $\eta_k^{\pm}(x)$:

$$\eta_k^{\pm}(x) = \left(\eta_k^{\pm}(x)\right)^{\mathrm{d}} + \left(\eta_k^{\pm}(x)\right)^{\mathrm{f}} \tag{3.51}$$

into diagonal and off-diagonal part. There is, however, an important difference: $\eta^{\pm}(x,\lambda)$ are elements of the Lie group $SL(2)$, not of the Lie algebra $sl(2)$, so $(\eta^{\pm})^{\mathrm{d}}$ are not proportional to σ_3. The initial condition (3.50) gives:

$$\left(\eta_1^{+}(x)\right)^{\mathrm{f}} = \left(\eta_1^{-}(x)\right)^{\mathrm{f}} = \frac{1}{4}[\sigma_3, q(x)] \tag{3.52}$$

and the diagonal part of (3.50b) with $k = 1$ gives:

$$i\frac{d}{dx}\left(\eta_1^{\pm}(x)\right)^{\mathrm{d}} + q(x)\left(\eta_1^{\pm}(x)\right)^{\mathrm{f}} = 0 \,, \tag{3.53}$$

which can easily be integrated, since $\left(\eta_1^{\pm}(x)\right)^{\mathrm{f}}$ is already known from (3.52). In the integration, we have to take into account the behavior of $\eta^{\pm}(x,\lambda)$ for $x \to \pm\infty$, see (3.46) and (3.47). Thus we obtain:

$$\eta_1^+(x) = \frac{1}{2}\begin{pmatrix} -i\int_{-\infty}^{x} dy\, q^+q^-(y), & q^+(x) \\ -q^-(x), & i\int_{\infty}^{x} dy\, q^-q^+(y) \end{pmatrix}, \tag{3.54}$$

and

$$\eta_1^-(x) = \frac{1}{2}\begin{pmatrix} -i\int_{\infty}^{x} dy\, q^+q^-(y), & q^+(x) \\ -q^-(x), & i\int_{-\infty}^{x} dy\, q^+q^-(y) \end{pmatrix}. \tag{3.55}$$

Continuing this procedure, we subsequently find $\eta_2^{\pm}(x)$, $\eta_3^{\pm}(x)$, …etc. As a bonus, knowing $\eta_1^{\pm}(x)$ from (3.46), (3.47) and (3.54), and (3.55), we can find the first nontrivial coefficients in the expansions of $a^{\pm}(\lambda)$ and $b^{\pm}(\lambda)$ over the inverse powers of λ:

$$\begin{aligned} a^+(\lambda) &= \lim_{x\to\infty}\left(\eta^+(x,\lambda)\right)_{11} = 1 - \frac{i}{2\lambda}\int_{-\infty}^{\infty} dy\, q^+q^-(y) + \mathcal{O}\left(\lambda^{-2}\right), \\ a^-(\lambda) &= \lim_{x\to\infty}\left(\eta^-(x,\lambda)\right)_{22} = 1 + \frac{i}{2\lambda}\int_{-\infty}^{\infty} dy\, q^+q^-(y) + \mathcal{O}\left(\lambda^{-2}\right), \end{aligned} \tag{3.56}$$

and

$$\begin{aligned} b^+(\lambda) &= \lim_{x\to\infty}\left(\eta^+(x,\lambda)\right)_{21} e^{2i\lambda x} = \mathcal{O}\left(\frac{1}{\lambda}\right), \\ b^-(\lambda) &= -\lim_{x\to\infty}\left(\eta^-(x,\lambda)\right)_{12} e^{-2i\lambda x} = \mathcal{O}\left(\frac{1}{\lambda}\right), \end{aligned} \tag{3.57}$$

3.4 The Dispersion Relation for $a^{\pm}(\lambda)$

Here, we shall apply the contour integration method to the diagonal elements of $T(\lambda)$. As a result, we shall show that $a^{\pm}(\lambda)$ are not independent but can be determined by the locations of their zeroes $\lambda_k^{\pm}$ and by the reflection coefficients $\rho^{\pm}(\lambda) = b^{\pm}/a^{\pm}(\lambda)$, $\lambda \in \mathbb{R}$ of the system (3.1).

We already required that $a^{\pm}(\lambda)$ have only simple zeroes (condition **C2**) in their regions of analyticity. From (3.56), we see that $a^{\pm}(\lambda)$ tend to 1 for $\lambda \to \infty$. Then we can introduce the functions:

$$f^+(\lambda) = \prod_{k=1}^{N} \frac{\lambda - \lambda_k^-}{\lambda - \lambda_k^+}\, a^+(\lambda), \qquad f^-(\lambda) = \prod_{k=1}^{N} \frac{\lambda - \lambda_k^+}{\lambda - \lambda_k^-}\, a^-(\lambda), \tag{3.58}$$

which have the same asymptotic behavior for $\lambda \to \infty$ as $a^{\pm}(\lambda)$, i.e. $f^{\pm}(\lambda) = 1 + \mathcal{O}(1/\lambda)$ and have no zeroes in their regions of analyticity. Then the functions $\ln f^{\pm}(\lambda)$, where ln is the principal branch of the logarithm, are well defined and analytic for $\operatorname{Im}\lambda \gtrless 0$, respectively, and tend to 0 for $\lambda \to \infty$ fast enough. Let us now apply the Cauchy theorem to the integrals of these functions along the contours $C_{\pm}$; see Fig. 3.1.

The contours $C_{\pm,R}$ are defined as follows: (a) $C_{+,R}$ consists of the segment $[-R+i0, R+i0]$ and the semicircle D_R^+ of radius R and centered at the origin

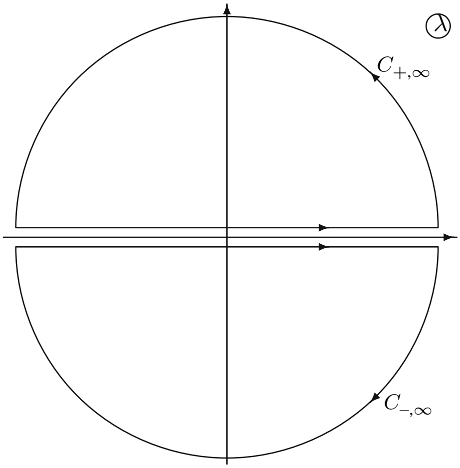

Fig. 3.1. The contours $C_\pm = \mathbb{R} \cup C_{\pm,\infty}$ of integrations

oriented anti-clockwise. (b) $C_{-,R}$ consists of the segment $[-R-i0, R-i0]$ and the semicircle D_R^- of radius R and centered at the origin oriented clockwise. When R tends to ∞ the segments fill in the real axis and the semicircles $D_R^\pm$ go to the infinite arcs $C_{\pm,\infty}$. Then $C_\pm = \mathbb{R} \cup C_{\pm,\infty}$ and

$$\frac{1}{2\pi i} \oint_{C_+} \frac{d\mu}{\mu - \lambda} \ln f^+(\mu) = \ln f^+(\lambda), \qquad \text{for } \operatorname{Im} \lambda > 0 \tag{3.59a}$$

$$\frac{1}{2\pi i} \oint_{C_-} \frac{d\mu}{\mu - \lambda} \ln f^-(\mu) = 0, \qquad \text{for } \operatorname{Im} \lambda > 0\ . \tag{3.59b}$$

The integrals in the left-hand sides can be calculated directly by integrating along the contours. Note that the integration along the infinite semiarcs gives no contribution, since $\ln f^\pm(\lambda)$ tends to zero fast enough for $\lambda \to \infty$. Summing up the two integrals we get for $\lambda \in \mathbb{C}_+$:

$$\ln f^+(\lambda) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\mu}{\mu - \lambda} \big(\ln f^+(\mu) + \ln f^-(\mu) \big)\ . \tag{3.60}$$

It remains to replace $f^\pm(\lambda)$ by their definitions (3.58) to obtain:

$$\ln a^+(\lambda) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\mu \ln \big(a^+(\mu) a^-(\mu) \big)}{\mu - \lambda} + \sum_{k=1}^{N} \ln \frac{\lambda - \lambda_k^+}{\lambda - \lambda_k^-}\ . \tag{3.61}$$

Analogous calculations can be done also for $\lambda \in \mathbb{C}_-$. Taking into account that the contour C_- is negatively orientated, we have:

$$-\ln a^-(\lambda) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\mu\ \ln \big(a^+(\mu) a^-(\mu) \big)}{\mu - \lambda} + \sum_{k=1}^{N} \ln \frac{\lambda - \lambda_k^+}{\lambda - \lambda_k^-}\ . \tag{3.62}$$

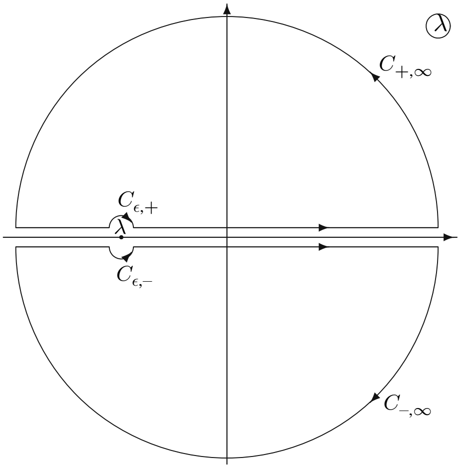

Fig. 3.2. The contours $C_{\pm} = \mathbb{R} \cup C_{\varepsilon,\pm} \cup C_{\pm,\infty}$ of integrations in the case when λ is on the real axis

The case when $\lambda \in \mathbb{R}$ requires a bit more attention. The contours C_+ and C_- now have to be chosen slightly deformed in the neighborhood of λ, as shown in Fig. 3.2. Then both integrals:

$$\oint_{C_{\pm}} \frac{d\mu}{\mu - \lambda} \ln f^{\pm}(\mu) = 0 \;, \tag{3.63}$$

vanish, since the integrands have no singularities lying inside the contours. The integration along the real axis, however, gives additional contributions coming from the integrations along the small semi-circles surrounding λ:

$$0 = \frac{1}{2\pi i} \oint_{C_+} \frac{d\mu \, \ln f^+(\mu)}{\mu - \lambda} = \frac{1}{2\pi i} \fint_{-\infty}^{\infty} \frac{d\mu \, \ln f^+(\mu)}{\mu - \lambda} - \frac{1}{2} \ln f^+(\lambda) \;, \tag{3.64a}$$

$$0 = \frac{1}{2\pi i} \oint_{C_-} \frac{d\mu \, \ln f^-(\mu)}{\mu - \lambda} = \frac{1}{2\pi i} \fint_{-\infty}^{\infty} \frac{d\mu \, \ln f^-(\mu)}{\mu - \lambda} + \frac{1}{2} \ln f^-(\lambda), \tag{3.64b}$$

where by $\fint_{-\infty}^{\infty}$ we denote the principal value integral. Summing up (3.64a) and (3.64b) and replacing $f^{\pm}(\lambda)$ by the right-hand sides of (3.58) we get:

$$\frac{1}{2} \ln \frac{a^+(\lambda)}{a^-(\lambda)} = \frac{1}{2\pi i} \fint_{-\infty}^{\infty} \frac{d\mu \, \ln(a^+(\mu)a^-(\mu))}{\mu - \lambda} + \sum_{k=1}^{N} \ln \frac{\lambda - \lambda_k^+}{\lambda - \lambda_k^-} \;. \tag{3.65}$$

In short, (3.61), (3.62), and (3.65) can be written as:

$$\mathcal{A}(\lambda) = \frac{1}{2\pi i} \fint_{-\infty}^{\infty} \frac{d\mu \, \ln(a^+(\mu)a^-(\mu))}{\mu - \lambda} + \sum_{k=1}^{N} \ln \frac{\lambda - \lambda_k^+}{\lambda - \lambda_k^-} \;, \tag{3.66}$$

where $\mathcal{A}(\lambda)$ is a piecewise analytic function of λ defined by:

$$\mathcal{A}(\lambda) = \begin{cases} \ln a^+(\lambda), & \operatorname{Im}\lambda > 0\ , \\ \frac{1}{2}\ln\left(a^+(\lambda)/a^-(\lambda)\right), & \operatorname{Im}\lambda = 0\ , \\ -\ln a^-(\lambda), & \operatorname{Im}\lambda < 0. \end{cases} \tag{3.67}$$

These formulae which allow one to recover the functions $a^\pm(\lambda)$ in their whole regions of analyticity are known as the dispersion relations.[1]

3.5 Minimal Sets of Scattering Data

The derivation of the dispersion relations for $a^\pm(\lambda)$ demonstrates the advantage of the analytic functions – the possibility to reconstruct them in their *whole domain of analyticity* knowing just their values on the boundaries of this domain and the residues at their pole singularities. In this section, we shall make use of the analyticity of $a^\pm(\lambda)$ and will show that in fact all matrix elements of $T(\lambda)$ can be determined uniquely from each of the following *minimal sets of scattering data*:

$$\mathcal{T}_1 \equiv \left\{\rho^+(\lambda), \rho^-(\lambda), \quad \lambda \in \mathbb{R}, \quad \lambda_k^\pm, C_k^\pm, \quad k = 1, \dots, N\right\}\ , \tag{3.68a}$$

$$\mathcal{T}_2 \equiv \left\{\tau^+(\lambda), \tau^-(\lambda), \quad \lambda \in \mathbb{R}, \quad \lambda_k^\pm, M_k^\pm, \quad k = 1, \dots, N\right\}\ , \tag{3.68b}$$

Here, we have used the following notations:

$$\rho^\pm(\lambda) = \frac{b^\pm(\lambda)}{a^\pm(\lambda)}, \qquad \tau^\pm(\lambda) = \frac{b^\mp(\lambda)}{a^\pm(\lambda)}, \qquad \lambda \in \mathbb{R}\ . \tag{3.69}$$

The functions $\rho^\pm(\lambda)$ are known in the literature as the reflection coefficients corresponding to the potential $q(x)$, $\lambda_k^\pm$ are the discrete eigenvalues of L and

$$C_k^\pm = \frac{b_k^\pm}{\dot{a}_k^\pm}, \qquad M_k^\pm = \frac{1}{b_k^\pm \dot{a}_k^\pm}, \qquad \dot{a}_k^\pm = \left.\frac{da^\pm}{d\lambda}\right|_{\lambda=\lambda_k^\pm}\ , \tag{3.70}$$

for $k = 1, \dots, N$, characterize the normalization of the corresponding Jost solutions $\psi_k^\pm(x)$ and $\phi_k^\pm(x)$.

In order to demonstrate how $a^+(\lambda)$ and $a^-(\lambda)$ can be recovered from $\mathcal{T}_1$, we divide both parts of the "unitarity" condition (3.26) by a^+a^-; this immediately produces:

$$1 + \rho^+(\lambda)\rho^-(\lambda) = \frac{1}{a^+(\lambda)a^-(\lambda)}\ . \tag{3.71}$$

Next, we insert (3.71) into (3.66) and rewrite the dispersion relation in the form:

[1] This is not to be mixed up with the dispersion law $f(\lambda)$ in (2.54), (2.55) which determines the evolution of the scattering data.

$$\mathcal{A}(\lambda) = \frac{i}{2} \int_{-\infty}^{\infty} \frac{d\mu\, \eta(\mu)}{\mu - \lambda} + \sum_{k=1}^{N} \ln \frac{\lambda - \lambda_k^+}{\lambda - \lambda_k^-} \,, \tag{3.72a}$$

where

$$\eta(\mu) = \frac{1}{\pi} \ln \left(1 + \rho^+(\mu)\rho^-(\mu) \right) . \tag{3.72b}$$

If we take subsequently $\operatorname{Im} \lambda > 0$ and $\operatorname{Im} \lambda < 0$ in (3.72) and exponentiate both sides we find:

$$a^+(\lambda) = \prod_{k=1}^{N} \frac{\lambda - \lambda_k^+}{\lambda - \lambda_k^-} \exp \left(\frac{i}{2} \int_{-\infty}^{\infty} \frac{d\mu\, \eta(\mu)}{\mu - \lambda} \right) , \qquad \operatorname{Im} \lambda > 0 \,, \tag{3.73}$$

$$a^-(\lambda) = \prod_{k=1}^{N} \frac{\lambda - \lambda_k^-}{\lambda - \lambda_k^+} \exp \left(-\frac{i}{2} \int_{-\infty}^{\infty} \frac{d\mu\, \eta(\mu)}{\mu - \lambda} \right) , \qquad \operatorname{Im} \lambda < 0 \,. \tag{3.74}$$

As for real λ from the dispersion relation we conclude that:

$$\sqrt{\frac{a^+(\lambda)}{a^-(\lambda)}} = \prod_{k=1}^{N} \frac{\lambda - \lambda_k^+}{\lambda - \lambda_k^-} \exp \left(\frac{i}{2} \int_{-\infty}^{\infty} \frac{d\mu\, \eta(\mu)}{\mu - \lambda} \right) . \tag{3.75}$$

Using (3.71) we get:

$$a^+(\lambda) = \frac{1}{\sqrt{1 + \rho^+(\lambda)\rho^-(\lambda)}} \prod_{k=1}^{N} \frac{\lambda - \lambda_k^+}{\lambda - \lambda_k^-} \exp \left(\frac{i}{2} \int_{-\infty}^{\infty} \frac{d\mu\, \eta(\mu)}{\mu - \lambda} \right) ,$$

and

$$a^-(\lambda) = \frac{1}{\sqrt{1 + \rho^+(\lambda)\rho^-(\lambda)}} \prod_{k=1}^{N} \frac{\lambda - \lambda_k^-}{\lambda - \lambda_k^+} \exp \left(-\frac{i}{2} \int_{-\infty}^{\infty} \frac{d\mu\, \eta(\mu)}{\mu - \lambda} \right) ,$$

Next, from the definition of the reflection coefficients (3.69), we also recover $b^\pm(\lambda)$, since:

$$\begin{aligned} b^\pm(\lambda) &= \rho^\pm(\lambda) a^\pm(\lambda) \\ &= \frac{\rho^\pm(\lambda)}{\sqrt{1 + \rho^+(\lambda)\rho^-(\lambda)}} \prod_{k=1}^{N} \frac{\lambda - \lambda_k^\pm}{\lambda - \lambda_k^\pm} \exp \left(\pm\frac{i}{2} \int_{-\infty}^{\infty} \frac{d\mu\, \eta(\mu)}{\mu - \lambda} \right) . \end{aligned} \tag{3.76}$$

Thus, we established that the all matrix elements of the scattering matrix $T(\lambda)$ of the ZS system L (3.1) can be reconstructed from $\mathcal{T}_1$ (3.68a).

Finally, we remark that from the unitarity condition (3.26) and from (3.69) there follows:

$$1 + \tau^+(\lambda)\tau^-(\lambda) = 1 + \rho^+(\lambda)\rho^-(\lambda) = \frac{1}{a^+(\lambda)a^-(\lambda)} \,, \tag{3.77}$$

for $\lambda \in \mathbb{R}$. Then following similar arguments as above, we can reconstruct $T(\lambda)$ also from $\mathcal{T}_2$.

3.6 Spectral Representations of the Jost Solutions

It is natural to expect that from the analyticity properties of the Jost solutions one should be able to derive in analogous way dispersion relations like the ones for $a^{\pm}(x,\lambda)$. This indeed can be done by using the same methods.

We start from the linear relations between the Jost solutions (3.2c), which we rewrite in an equivalent form:

$$\frac{1}{a^-(\lambda)}\phi^-(x,\lambda) = \psi^+(x,\lambda) - \rho^-(\lambda)\psi^-(x,\lambda)\ , \tag{3.78a}$$

$$\frac{1}{a^+(\lambda)}\phi^+(x,\lambda) = \psi^-(x,\lambda) + \rho^+(\lambda)\psi^+(x,\lambda)\ . \tag{3.78b}$$

Next, we multiply (3.78a) by $e^{-i\lambda x}$ and (3.78b) by $e^{i\lambda x}$ to obtain:

$$\frac{1}{a^-(\lambda)}\varphi^-(x,\lambda) = \xi^+(x,\lambda) - \rho^-(\lambda)e^{-2i\lambda x}\xi^-(x,\lambda)\ , \tag{3.79a}$$

$$\frac{1}{a^+(\lambda)}\varphi^+(x,\lambda) = \xi^-(x,\lambda) + \rho^+(\lambda)e^{2i\lambda x}\xi^+(x,\lambda)\ . \tag{3.79b}$$

We now apply the contour integration method to the following integrals:

$$\mathcal{J}_1(x,\lambda) = \frac{1}{2\pi i}\left(\oint_{C_+}\frac{d\mu\,\varphi^+(x,\mu)}{(\mu-\lambda)a^+(\mu)} - \oint_{C_-}\frac{d\mu\,\xi^-(x,\mu)}{\mu-\lambda}\right)\ , \tag{3.80a}$$

$$\mathcal{J}_2(x,\lambda) = \frac{1}{2\pi i}\left(\oint_{C_+}\frac{d\mu\,\xi^+(x,\mu)}{\mu-\lambda} - \oint_{C_-}\frac{d\mu\,\varphi^-(x,\mu)}{(\mu-\lambda)a^-(\mu)}\right)\ , \tag{3.80b}$$

We shall outline some of the details only for the case when we consider $\mathcal{J}_2(x,\lambda)$ with $\operatorname{Im}\lambda > 0$. Applying Cauchy theorem, we have to keep in mind that (i) $1/a^+(\lambda)$ has simple poles at $\lambda = \lambda_k^+$; (ii) $1/a^-(\lambda)$ has simple poles at $\lambda = \lambda_k^-$; (iii) $\phi^-(x,\lambda)$ and $\xi^+(x,\lambda)$ have no poles, and therefore the integrand of the first integral in $\mathcal{J}_2(x,\lambda)$ has only a simple pole at $\lambda = \mu$, and iv) the contour C^+ is positively oriented, while C^- is a negatively orientated contour. Thus we get:

$$\mathcal{J}_2(x,\lambda) = \xi^+(x,\lambda) + \sum_{k=1}^{N}\frac{\varphi_k^-(x)}{(\lambda_k^- - \lambda)\dot{a}_k^-}\ . \tag{3.81}$$

Remark 3.8. Here and below, by $\varphi_k^{\pm}(x)$, $\xi_k^{\pm}(x)$ etc., we denote the values of the functions $\varphi^{\pm}(x,\lambda)$, $\xi^{\pm}(x,\lambda)$ etc. for $\lambda = \lambda_k^{\pm}$, i.e.:

$$\varphi_k^{\pm}(x) = \varphi^{\pm}(x,\lambda_k^{\pm}), \qquad \xi_k^{\pm}(x) = \xi^{\pm}(x,\lambda_k^{\pm}).$$

Integrating over the infinite arcs, we make use of the asymptotics of $\xi^+(x,\lambda)$ and $\varphi^-(x,\lambda)$ to get:

$$
\begin{aligned}
&\frac{1}{2\pi i}\oint_{C_\infty^+}\frac{d\lambda}{\lambda-\mu}\xi^+(x,\lambda)-\frac{1}{2\pi i}\oint_{C_\infty^-}\frac{d\lambda}{\lambda-\mu}\frac{\varphi^-(x,\lambda)}{a^-(\lambda)}\\
&=\frac{1}{2\pi i}\oint_{C_\infty^+}\frac{d\lambda}{\lambda-\mu}\begin{pmatrix}0\\1\end{pmatrix}-\frac{1}{2\pi i}\oint_{C_\infty^-}\frac{d\lambda}{\lambda-\mu}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}
\end{aligned}
\tag{3.82}
$$

Finally, the integral along the real axis is evaluated with the help of (3.79a) to be equal to:

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\lambda}{\lambda-\mu}\left(\xi^+(x,\lambda)-\frac{\varphi^-(x,\lambda)}{a^-(\lambda)}\right)\\
&=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\lambda}{\lambda-\mu}\rho^-(\lambda)e^{-2i\lambda x}\xi^-(x,\lambda)\ ,
\end{aligned}
\tag{3.83}
$$

which leads to the following integral representation for $\xi^+(x,\lambda)$:

$$
\begin{aligned}
\xi^+(x,\lambda)=&\begin{pmatrix}0\\1\end{pmatrix}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\mu}{\mu-\lambda}\rho^-(\mu)e^{-2i\mu x}\xi^-(x,\mu)\\
&+\sum_{k=1}^{N}\frac{\varphi_k^-(x)}{\dot a_k^-(\lambda-\lambda_k^-)}\ .
\end{aligned}
\tag{3.84}
$$

It remains to use the relations (3.43), which after multiplying by the corresponding exponential factors take the form:

$$
\varphi_k^+(x)=b_k^+e^{2i\lambda_k^+x}\xi_k^+(x),\qquad \varphi_k^-(x)=-b_k^-e^{-2i\lambda_k^-x}\xi_k^-(x)\ . \tag{3.85}
$$

Thus, we find:

$$
\begin{aligned}
\xi^+(x,\lambda)=&\begin{pmatrix}0\\1\end{pmatrix}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\mu}{\mu-\lambda}\rho^-(\mu)e^{-2i\mu x}\xi^-(x,\mu)\\
&-\sum_{k=1}^{N}\frac{C_k^-e^{-2i\lambda_k^-x}\xi_k^-(x)}{\lambda-\lambda_k^-},\qquad \lambda\in\mathbb{C}_+\ ,
\end{aligned}
\tag{3.86}
$$

where the constants $C_k^\pm$ are introduced in (3.70).

Skipping the details, we just give the result which is obtained from working out the integral $\mathcal{J}_2(x,\lambda)$ (3.80b) with $\lambda\in\mathbb{C}_-$:

$$
\begin{aligned}
\xi^-(x,\lambda)=&\begin{pmatrix}1\\0\end{pmatrix}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\mu}{\mu-\lambda}\rho^+(\mu)e^{2i\mu x}\xi^+(x,\mu)\\
&+\sum_{k=1}^{N}\frac{C_k^+e^{2i\lambda_k^+x}\xi_k^+(x)}{\lambda-\lambda_k^+},\qquad \lambda\in\mathbb{C}_-\ .
\end{aligned}
\tag{3.87}
$$

Now, it is clear why the manifold $\mathcal{T}_1$ (3.68a) can be considered as the minimal set of scattering data. Indeed, $\mathcal{T}_1$ is nothing else but the set of coefficients, determining the system (3.86), (3.87), which is a system of singular integral equations for the Jost solutions $\xi^+(x,\lambda)$ and $\xi^-(x,\lambda)$. This system admits unique solution, so $\mathcal{T}_1$ determines uniquely the Jost solutions $\xi^\pm(x,\lambda)$. We skip the details of the proof; below we shall prove it by other means.

Quite analogously, one can derive the integral representation for the other pair of Jost solutions $\varphi^\pm(x,\lambda)$:

$$\begin{aligned}\varphi^-(x,\lambda) &= \begin{pmatrix}0\\1\end{pmatrix} + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\mu}{\mu-\lambda}\varphi^+(x,\lambda)\tau^+(\mu)e^{-2i\mu x}\\ &\quad -\sum_{k=1}^{N}\frac{M_k^+}{\lambda_k^+-\lambda}e^{-2i\lambda_k^+ x}\varphi_k^+(x), \qquad \lambda\in\mathbb{C}_-\ ,\end{aligned} \tag{3.88}$$

$$\begin{aligned}\varphi^+(x,\lambda) &= \begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\mu}{\mu-\lambda}\varphi^-(x,\lambda)\tau^-(\mu)e^{2i\mu x}\\ &\quad +\sum_{k=1}^{N}\frac{M_k^-}{\lambda_k^--\lambda}e^{2i\lambda_k^- x}\varphi_k^-(x), \qquad \lambda\in\mathbb{C}_+\ ,\end{aligned} \tag{3.89}$$

and the constants $M_k^\pm$ are given in (3.70).

Thus, we find that $\mathcal{T}_2$ (3.68b) is actually the set of coefficients determining the system (3.89), (3.88), which again can be viewed as a system of singular integral equations for the Jost solutions $\varphi^\pm(x,\lambda)$. It has unique solution and therefore $\mathcal{T}_2$ determines uniquely the Jost solutions $\varphi^\pm(x,\lambda)$.

3.7 Completeness of the Jost Solutions

It is well known that for any self-adjoint operator A acting on the Hilbert space $\mathcal{H}$, one can introduce a spectral measure and then prove the spectral theorem. This is done with the help of a family of self-adjoint commuting projectors E_u, $u\in\mathbb{R}$, such that:

$$E_uE_v = E_s, \qquad s=\min\{u,v\}\ . \tag{3.90}$$

Roughly speaking, the subspace $E_u\mathcal{H}\subset\mathcal{H}$ is a direct sum of all eigenspaces of A corresponding to the eigenvalues $v\le u$, which in this case must are real. The set of projectors E_s is called a decomposition of the unity. This means that any element $g\in\mathcal{H}$ can be represented as

$$g=\int_{-\infty}^{\infty} dE_u\cdot g\ , \tag{3.91}$$

where dE_u is the spectral measure. Then the operator A can be written down as the following Lebesque–Stieltjes integral:

$$A\cdot g=\int_{-\infty}^{\infty} u\, dE_u\cdot g \tag{3.92}$$

for any element $g \in \mathcal{H}$; This provides the spectral decomposition of A.

As a simple and well-known example of such decomposition, one can view the usual Fourier integral decomposition:

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda\, e^{i\lambda x} \widetilde{g}(\lambda) \,, \tag{3.93}$$

In this case

$$dE_u f = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iu(x-y)} f(y) dy \tag{3.94}$$

$$\frac{1}{i} \frac{dg(x)}{dx} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda\, \lambda e^{i\lambda x} \widetilde{g}(\lambda) \,. \tag{3.95}$$

Thus, the Fourier transform is the spectral decomposition of the operator $-id/dx$.

Unfortunately, such decompositions are usually proved only for self-adjoint and unitary operators. As it is the ZS system is neither unitary nor self-adjoint, so we cannot use general methods developed for proving the spectral theorem.

Below we prove the completeness relation of the Jost solution of L, which is the analog of (3.91). Our proof, although not that rigorous, has, however, other advantages. The first is that it can be applied to a wider class of operators, not necessarily self-adjoint or unitary. The second advantage is that we can obtain explicit expressions for the corresponding projection operators E_u in terms of the Jost solutions of L.

Our derivation uses again the contour integration method. Now, we apply it to the integral:

$$\begin{aligned} \mathcal{J}_R(x,y) &= \frac{1}{2\pi i} \left(\oint_{C_+} d\lambda\, R^+(x,y,\lambda) - \oint_{C_-} d\lambda\, R^-(x,y,\lambda) \right) \\ &= \sum_{k=1}^{N} \left(\operatorname*{Res}_{\lambda=\lambda_k^+} R^+(x,y,\lambda) + \operatorname*{Res}_{\lambda=\lambda_k^-} R^-(x,y,\lambda) \right) \,, \end{aligned} \tag{3.96}$$

where $R^{\pm}(x,y,\lambda)$ provided by (3.35) determine the kernel of the resolvent of L.

At the points of the discrete spectrum, i.e. in the neighborhood of $\lambda_k^{\pm}$, we have:

$$a^{\pm}(\lambda) = (\lambda - \lambda_k^{\pm}) \dot{a}_k^{\pm} + \frac{1}{2} (\lambda - \lambda_k^{\pm})^2 \ddot{a}_k^{\pm} + \cdots \,, \tag{3.97}$$

$$\chi^+(x,\lambda_k^+) \equiv \chi_k^+(x) = \psi_k^+(b_k^+,\, 1) = \phi_k^+(x)(1,\, 1/b_k^+) \,, \tag{3.98}$$

$$\hat{\chi}^+(x,\lambda_k^+) \simeq \begin{pmatrix} 1 \\ -b_k^+ \end{pmatrix} \frac{\tilde{\psi}_k^+(x)}{(\lambda - \lambda_k^+)\dot{a}_k^+} = \begin{pmatrix} 1/b_k^+ \\ -1 \end{pmatrix} \frac{\tilde{\phi}_k^+(x)}{(\lambda - \lambda_k^+)\dot{a}_k^+} \,, \tag{3.99}$$

$$\chi^-(x,\lambda_k^-) \equiv \chi_k^-(x) = \psi_k^-(1,\, -b_k^-) = \phi_k^-(x)(-1/b_k^-,\, 1) \,, \tag{3.100}$$

$$\hat{\chi}^-(x, \lambda_k^-) \simeq -\begin{pmatrix} b_k^- \\ 1 \end{pmatrix} \frac{\tilde{\psi}_k^-(x)}{(\lambda - \lambda_k^-)\dot{a}_k^-} = \begin{pmatrix} 1 \\ 1/b_k^- \end{pmatrix} \frac{\tilde{\phi}_k^-(x)}{(\lambda - \lambda_k^-)\dot{a}_k^-}, \quad (3.101)$$

Here, like in Remark 3.8 $\tilde{\phi}_k^-(x)$, ... denote the values of the corresponding functions for $\lambda = \lambda_k^\pm$.

In (3.97), we just used the analyticity properties of $a^\pm(\lambda)$ and the **C2** condition. To derive (3.98), (3.99), (3.100) and (3.101), we used (3.43) and (3.44).

Then, we can find the residues of the resolvent kernel:

$$\begin{aligned} \operatorname*{Res}_{\lambda=\lambda_k^\pm} R^\pm(x, y, \lambda) &= \lim_{\lambda \to \lambda_k^\pm} R^\pm(x, y, \lambda) \\ &= \pm i \frac{\phi_k^\pm(x)\tilde{\psi}_k^\pm(y)}{\dot{a}_k^\pm}. \end{aligned} \quad (3.102)$$

Now, we calculate the jump of $R(x, y, \lambda)$ on the real λ-axis. From (3.35) we have:

$$R^\pm(x, y, \lambda) = \frac{i}{2}\epsilon(x - y)\chi^\pm(x)\hat{\chi}^\pm(y) \mp \frac{i}{2}\chi^\pm(x)\sigma_3\hat{\chi}^\pm(y), \quad (3.103)$$

where $\epsilon(x - y) = \theta(x - y) - \theta(y - x)$, i.e.

$$\begin{aligned} R^+(x, y, \lambda) - R^-(x, y, \lambda) &= \frac{i}{2}\epsilon(x - y)\left(\chi^+(x)\hat{\chi}^+(y) - \chi^-(x)\hat{\chi}^-(y)\right) \\ &\quad - \frac{i}{2}\left(\chi^+(x)\sigma_3\hat{\chi}^+(y) + \chi^-(x)\sigma_3\hat{\chi}^-(y)\right). \end{aligned} \quad (3.104)$$

But

$$\chi^+(x)\hat{\chi}^+(y) = \psi(x)\hat{\psi}(y) = \chi^-(x)\hat{\chi}^-(y) \quad (3.105)$$

so

$$\begin{aligned} &R^+(x, y, \lambda) - R^-(x, y, \lambda) = -\frac{i}{2}\left(\chi^+(x)\sigma_3\hat{\chi}^+(y) - \chi^-(x)\sigma_3\hat{\chi}^-(y)\right) \\ &= -\frac{i}{2}\left(\chi^+(x)(\mathbb{1} + \sigma_3)\hat{\chi}^+(y) + \chi^-(x)(\mathbb{1} - \sigma_3)\hat{\chi}^-(y)\right) \\ &= -i\left(\frac{\phi^+(x, \lambda)\tilde{\psi}^+(y, \lambda)}{a^+(\lambda)} + \frac{\phi^-(x, \lambda)\tilde{\psi}^-(y, \lambda)}{a^-(\lambda)}\right). \end{aligned} \quad (3.106)$$

Finally, the integrals over the arcs of the infinite circle are calculated explicitly by using the asymptotics of the FAS for $\lambda \to \infty$:

$$\chi^\pm(x, \lambda) = e^{-i\lambda\sigma_3 x}\left(\mathbb{1} + \mathcal{O}(1/\lambda)\right) \quad (3.107)$$

i.e.

$$R^+_{\rm as}(x,y,\lambda) = \frac{1}{i} e^{-i\lambda\sigma_3 x} \begin{pmatrix} -\theta(y-x) & 0 \\ 0 & \theta(x-y) \end{pmatrix} e^{i\lambda\sigma_3 y} \ , \tag{3.108a}$$

$$R^-_{\rm as}(x,y,\lambda) = \frac{1}{i} e^{-i\lambda\sigma_3 x} \begin{pmatrix} \theta(x-y) & 0 \\ 0 & -\theta(y-x) \end{pmatrix} e^{i\lambda\sigma_3 y} \ , \tag{3.108b}$$

since the terms of order $1/\lambda$ and higher do not contribute to the integrals. Consequently

$$\begin{aligned}
&\frac{1}{2\pi i}\left(\oint_{C_{+,\infty}} d\lambda\, R^+_{\rm as}(x,y,\lambda) - \oint_{C_{-,\infty}} d\lambda\, R^-_{\rm as}(x,y,\lambda)\right) \\
&= \frac{1}{2\pi}\sigma_3 \int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda\sigma_3(x-y)} \\
&= \delta(x-y)\sigma_3 \ .
\end{aligned} \tag{3.109}$$

Combining the relations (3.96), (3.102), (3.106), and (3.109) leads to the following expressions for $\mathcal{J}_R(x,y)$:

$$\begin{aligned}
&\mathcal{J}_R(x,y) \\
&= \delta(x-y)\sigma_3 - \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \left(\frac{\phi^+(x,\lambda)\tilde{\psi}^+(y,\lambda)}{a^+(\lambda)} + \frac{\phi^-(x,\lambda)\tilde{\psi}^-(y,\lambda)}{a^-(\lambda)} \right) \\
&= i\sum_{k=1}^{N} \left(\frac{\phi^+_k(x)\tilde{\psi}^+_k(y)}{\dot{a}^+_k} - \frac{\phi^-_k(x)\tilde{\psi}^-_k(y)}{\dot{a}^-_k} \right)
\end{aligned} \tag{3.110}$$

and, hence, to the following completeness relation:

$$\begin{aligned}
\delta(x-y)\mathbb{1} = {}& \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \left(\frac{\phi^+(x,\lambda)\tilde{\psi}^+(y,\lambda)}{a^+(\lambda)} + \frac{\phi^-(x,\lambda)\tilde{\psi}^-(y,\lambda)}{a^-(\lambda)} \right)\sigma_3 \\
&+ i\sum_{k=1}^{N} \left(\frac{\phi^+_k(x)\tilde{\psi}^+_k(y)}{\dot{a}^+_k} - \frac{\phi^-_k(x)\tilde{\psi}^-_k(y)}{\dot{a}^-_k} \right)\sigma_3 \ .
\end{aligned} \tag{3.111}$$

Then, every vector function $Y(x) = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$, whose components are smooth and fall off fast enough for $x \to \pm\infty$, can be expanded over the Jost solutions of L in the form:

$$\begin{aligned}
Y(x) = {}& \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, \left(\phi^+(x,\lambda)y^+(\lambda) + \phi^-(x,\lambda)y^-(\lambda)\right) \\
&+ i\sum_{k=1}^{N} \left(\phi^+_k(x)y^+_k - \phi^-_k(x)y^-_k\right) \ ,
\end{aligned} \tag{3.112}$$

where the expansion coefficients are given by:

$$y^{\pm}(\lambda) = \frac{1}{a^{\pm}(\lambda)} \int_{-\infty}^{\infty} dy\, \tilde{\psi}^{\pm}(y,\lambda)\sigma_3 Y(y) \,, \tag{3.113a}$$

$$y_k^{\pm} = \frac{1}{\dot{a}_k^{\pm}} \int_{-\infty}^{\infty} dy\, \tilde{\psi}_k^{\pm}(y)\sigma_3 Y(y) \,. \tag{3.113b}$$

From (3.112) we see that

$$\begin{aligned} LY(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda\, \lambda \left(\phi^{+}(x,\lambda)y^{+}(\lambda) + \phi^{-}(x,\lambda)y^{-}(\lambda)\right) \\ + i \sum_{k=1}^{N} \left(\lambda_k^{+}\phi_k^{+}(x)y_k^{+} - \lambda_k^{-}\phi_k^{-}(x)y_k^{-}\right) \,. \end{aligned} \tag{3.114}$$

The completeness relation (3.111) is the analog of (3.91) while (3.114) generalizes (3.92).

3.8 Comments and Bibliographical Review

1. The direct and the inverse scattering problems for the Sturm-Liouville equation (1.5) have appeared as important physical problems in the context of quantum mechanics in the 1950s. Note that for real-valued potentials the operator (1.5) is self-adjoint, which allows for rigorous spectral analysis along the lines of the monographs [6, 7]; see also [8, 9, 10].

   A fundamental property of the Jost solutions lies in the fact that each of the columns $\psi^{\pm}(x,\lambda)$ and $\phi^{\pm}(x,\lambda)$ allow analytic continuation for $\lambda \in \mathbb{C}_{\pm}$. Detailed proof of this can be found in the monographs [2, 4, 5, 11, 12, 13, 14] and in many of the review papers.

   The spectral theory of self-adjoint and unitary operators is a well-developed area of mathematical analysis; see [15, 16]. However, as a rule, the Lax operators that are important for the NLEE are *neither* self-adjoint *nor* unitary. It does not seem realistic to us to generalize the rigorous methods of the spectral analysis to generic non-self-adjoint operators. Indeed, there are well-known examples showing that there exist non-self-adjoint operators with smooth potentials on *finite support*, whose discrete spectrum contains *infinitely* many discrete eigenvalues.

   In order to simplify matters, we introduced the two rather restrictive conditions **C.1** and **C.2**. We are aware that condition **C.2** is a strong *implicit* condition on potential $q(x)$. Nevertheless, we proceed with it for three reasons:

   R1. The manifold of potentials satisfying both **C.1** and **C.2** is not empty; in particular we are able to construct *explicitly* an infinite number of such potentials.

   R2. One can establish the analyticity properties of the Jost solutions for these potentials and then use them for solving both the direct and the inverse scattering problems for L.

R3. This approach allows us to analyze and derive the fundamental properties of the corresponding class of NLEE. In a large number of cases, the derived solutions of these NLEE have physical significance, which was our motivation to continue with the research.

For this special class of ordinary differential operators, whose potential satisfies both conditions **C.1** and **C.2**, one is able to construct the spectral theory also in cases when they are neither self-adjoint nor unitary. The basic tool for this is the notion of the fundamental analytic solution (FAS). For the ZS system, it is easily constructed by combining the Jost solutions with similar analyticity properties. For the generalized $sl(n)$ ZS system (1.15) with real $J = \mathrm{diag}\,(a_1, \dots, a_n)$, $a_k > a_j$ for $k < j$, this important result was obtained by Shabat [17, 18]. It was generalized for the case of $\mathfrak{g} \simeq sl(n)$ and complex-valued a_j by Mikhailov [19], Caudrey [20, 21], and Beals and Coifman [22, 23, 24, 25, 26]; for any simple Lie algebra, see [27]. In all these cases, it is possible to follow [28] to construct the kernel of the resolvent of L in terms of FAS using the contour integration method to derive the completeness relation for the eigenfunctions of L. In fact, this relation provides the spectral decomposition for L.

2. Both the direct and the inverse scattering problems for the Sturm-Liouville equation (1.5) have appeared as important physical problems in the context of quantum mechanics in the 1950s when most of the fundamental results, including the famous Gel'fand–Levitan–Marchenko (GLM) equation, were obtained; see monographs [6, 7]. For a concise explanation and derivation of the direct scattering problem, see [5, 6, 7, 15, 29, 30, 31, 32, 33, 34, 35, 36]. The derivation and the analysis of the GLM equation can be found in [2, 3, 4, 5, 12, 13, 14, 37, 38].
3. The spectral theory of linear differential operators is well developed for two important classes of operators: self-adjoint and unitary; see [16, 34]. Here, we demonstrate how this theory can be extended for ordinary differential operators that are neither self-adjoint nor unitary. The corresponding spectral densities in our case are expressed explicitly in terms of the FAS of $L(\lambda)$. These results can be generalized also for the generalized Zakharov–Shabat system [28], for the polynomial bundles [39, 40, 41, 42], and others.

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4

The Inverse Scattering Problem for the Zakharov–Shabat System

In this chapter, we outline several approaches to the solution of the inverse scattering problem (ISP) for the ZS system. In the first two sections, we explain the classical approach to the problem based on the Gel'fand–Levitan–Marchenko equation. There we explain two different ways of deriving this equation.

In Sect. 4.3, we show that the ISP for the ZS system is equivalent to a (possibly singular) Riemann–Hilbert problem.

The next two Sects. 4.4 and 4.5, are devoted to two different versions of the Zakharov–Shabat dressing method, which is the most effective method for deriving the reflectionless potentials and the soliton solutions of the corresponding NLEE.

4.1 Derivation of the GLM Equation

A basic notion in the classical approach to the inverse scattering problem is the notion of the transformation operators, which are introduced by:

$$\begin{aligned}
\psi(x,\lambda) &\equiv (\mathbb{1} + \mathbf{K}_+)e^{-i\lambda\sigma_3 x} \\
&= e^{-i\lambda\sigma_3 x} + \int_x^\infty dy\, K_+(x,y)e^{-i\lambda\sigma_3 y}\ , \qquad (4.1a) \\
\phi(x,\lambda) &\equiv (\mathbb{1} + \mathbf{K}_-)e^{-i\lambda\sigma_3 x} \\
&= e^{-i\lambda\sigma_3 x} + \int_{-\infty}^x dy\, K_-(x,y)e^{-i\lambda\sigma_3 y}\ , \qquad (4.1b)
\end{aligned}$$

In other words, the transformation operators $\mathbf{K}_\pm$ are Volterra-type integral operators, which transform the "plane wave" $e^{-i\lambda\sigma_3 x}$ into the corresponding Jost solution of L. Their kernels $K_\pm(x,y)$ are 2×2 matrix functions. The existence of such operators will become clear in the sequel.

Gerdjikov, V.S. et al.: *The Inverse Scattering Problem for the Zakharov–Shabat System*. Lect. Notes Phys. **748**, 97–132 (2008)
DOI 10.1007/978-3-540-77054-1_4

The representations (4.1) for the Jost solutions are valid for any real value of the spectral parameter λ. The relations (4.1), which are in matrix form can be rewritten for each of the two columns $\psi^{\pm}(x,\lambda)$ and $\phi^{\pm}(x,\lambda)$ separately. After multiplying with the factors $e^{\mp i\lambda x}$ we cast them in equivalent form:

$$\xi^{+}(x,\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_x^\infty dy\, K_+^{(2)}(x,y) e^{i\lambda(y-x)}\,, \tag{4.2a}$$

$$\xi^{-}(x,\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_x^\infty dy\, K_+^{(1)}(x,y) e^{-i\lambda(y-x)}\,, \tag{4.2b}$$

where $K_+^{(i)}(x,y)$, $i=1,2$ are the first and the second columns of the matrix $K_+(x,y)$. While (4.1a) in matrix form is valid only for real values of λ, we assume that the (4.2a) and (4.2b) hold also for complex λ and extend to the domains of analyticity of their left-hand sides, i.e. to the upper and lower half-planes of the complex λ-plane correspondingly. In particular, putting $\lambda = \lambda_k^+$ in (4.2a) and $\lambda = \lambda_k^-$ in (4.2b) we obtain:

$$\xi_k^{+}(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_x^\infty dy\, K_+^{(2)}(x,y) e^{i\lambda_k^+(y-x)}\,, \tag{4.3a}$$

$$\xi_k^{-}(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_x^\infty dy\, K_+^{(1)}(x,y) e^{-i\lambda_k^-(y-x)}\,, \tag{4.3b}$$

for $k=1,\ldots,N$.

Analogical relations can also be derived for $\varphi^{\pm}(x,\lambda) = \phi^{\pm}(x,\lambda)e^{\pm i\lambda x}$ and $\varphi_k^{\pm}(x)$. Of course, in their right-hand side, there will appear the columns $K_-^{(i)}$, $i=1,2$ of $K_-(x,y)$, and the integration extends over the interval $(-\infty,x)$.

The first method for deriving the GLM equation is based on the analyticity properties of the Jost solutions. More specifically, it uses the integral representations (3.86) and (3.87). Inserting in them (4.2) and (4.3), we find that $K_\pm(x,y)$ must satisfy:

$$\begin{aligned}\int_x^\infty dy\, K_+^{(2)}(x,y) e^{i\mu(y-x)} &= \begin{pmatrix} \tilde{\mathcal{F}}_+^{+}(x,\mu) \\ 0 \end{pmatrix} \\ &\quad + \int_x^\infty dy\, K_+^{(1)}(x,y) \tilde{\mathcal{F}}_+^{+}\left(\frac{x+y}{2},\mu\right)\,,\end{aligned} \tag{4.4a}$$

$$\begin{aligned}\int_x^\infty dy\, K_+^{(1)}(x,y) e^{-i\mu(y-x)} &= \begin{pmatrix} 0 \\ \tilde{\mathcal{F}}_+^{-}(x,\mu) \end{pmatrix} \\ &\quad + \int_x^\infty dy\, K_+^{(2)}(x,y) \tilde{\mathcal{F}}_+^{-}\left(\frac{x+y}{2},\mu\right)\,,\end{aligned} \tag{4.4b}$$

where

$$\tilde{\mathcal{F}}_+^{+}(x,\mu) = \frac{1}{2\pi i}\int_{-\infty}^\infty \frac{d\lambda\, \rho^-(\lambda) e^{-2i\lambda x}}{\lambda-\mu} - \sum_{k=1}^N \frac{C_k^- e^{-2i\lambda_k^- x}}{\mu - \lambda_k^-}\,, \tag{4.5a}$$

$$\tilde{\mathcal{F}}_+^-(x,\mu) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\lambda \rho^+(\lambda) e^{2i\lambda x}}{\lambda-\mu} + \sum_{k=1}^{N} \frac{C_k^+ e^{2i\lambda_k^+ x}}{\mu-\lambda_k^+} \ . \tag{4.5b}$$

The next step is to multiply (4.4a) by $e^{i\mu(x-z)}$ and (4.4b) by $e^{-i\mu(x-z)}$ and integrate over $d\mu$ along the real axis. If $z \geq x$ in the left-hand side of (4.4) we obtain $K_+^{(i)}(x,z)$. The integration over $d\mu$ in the right-hand sides is performed by using the formulae:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d\mu}{\mu-\lambda} e^{-i\mu(x-z)\sigma_3} = \frac{i}{2} e^{-i\lambda\sigma_3(x-z)}\sigma_3 \ , \tag{4.6a}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d\mu}{\mu-\lambda_k^-} e^{-i\mu(x-z)\sigma_3} = -ie^{-i\lambda_k^-(x-z)}\sigma_3 \ , \tag{4.6b}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d\mu}{\mu-\lambda_k^+} e^{-i\mu(x-z)\sigma_3} = -ie^{-i\lambda_k^+(x-z)}\sigma_3 \ . \tag{4.6c}$$

In this way from (4.4) we get:

$$K_+^{(2)}(x,z) = \begin{pmatrix} F_+^+(x+z) \\ 0 \end{pmatrix} + \int_x^{\infty} dy\, K_+^{(1)}(x,y) F_+^+(y+z) \ , \tag{4.7a}$$

$$K_+^{(1)}(x,z) = \begin{pmatrix} 0 \\ F_+^-(x+z) \end{pmatrix} + \int_x^{\infty} dy\, K_+^{(2)}(x,y) F_+^-(y+z) \ , \tag{4.7b}$$

where

$$F_+^+(x) = \frac{1}{4\pi}\int_{-\infty}^{\infty} d\lambda\, \rho^-(\lambda) e^{i\lambda x} + i\sum_{k=1}^{N} C_k^- e^{-i\lambda_k^- x} \ , \tag{4.8a}$$

$$F_+^-(x) = \frac{1}{4\pi}\int_{-\infty}^{\infty} d\lambda\, \rho^+(\lambda) e^{-i\lambda x} + i\sum_{k=1}^{N} C_k^+ e^{i\lambda_k^+ x} \ . \tag{4.8b}$$

It is convenient to rewrite (4.7) in matrix form:

$$K_+(x,z) = F_+(x+z) + \int_x^{\infty} dy\, K_+(x,y) F_+(y+z) \ , \tag{4.9}$$

where

$$F_+(x) = \begin{pmatrix} 0 & F_+^+(x) \\ F_+^-(x) & 0 \end{pmatrix} \ . \tag{4.10}$$

The equation (4.9) is the famous Gel'fand–Levitan–Marchenko equation. Given the scattering data $\mathcal{T}_1$ (see (3.68a)), one determines the kernel $F_+(x)$ of the GLM equation.

From the general theory of the Volterra type integral equations, one concludes that the GLM equation (4.9) with the kernel $F_+(x)$ (4.10) has unique solution $K_+(x,y)$. Given the solution $K_+(x,y)$, we easily obtain the Jost solutions $\psi^{\pm}(x,\lambda)$ through (4.1a).

The last step of the solution of the ISP is the determination of the potential $q(x)$, corresponding to $\mathcal{T}_1$. In order to do this, we replace the right-hand side of (4.1a) into the ZS system, multiply on the left with $e^{i\lambda\sigma_3 x}$, and require that $\psi(x,\lambda)$ from (4.1a) is a solution for each λ, i.e.

$$\begin{aligned} q(x) + \int_x^{\infty} dy \left(i\frac{\partial K_+}{\partial x} + q(x)K_+(x,y) - \lambda\sigma_3 K_+(x,y) \right) e^{i\lambda\sigma_3(x-y)} \\ = iK_+(x,x) \, . \end{aligned} \tag{4.11}$$

The term containing λ in the integrand can be integrated by parts and since $\lim_{y\to\infty} K_+(x,y) = 0$ we obtain:

$$\begin{aligned} q(x) + i\int_x^{\infty} dy \left(i\frac{\partial K_+}{\partial x} + i\sigma_3\frac{\partial K_+}{\partial y}\sigma_3 + q(x)K_+(x,y) \right) e^{i\lambda\sigma_3(x-y)} \\ = iK_+(x,x) - i\sigma_3 K_+(x,x)\sigma_3 \, . \end{aligned} \tag{4.12}$$

The equation (4.12) is satisfied identically with respect to λ if and only if the following two relations hold:

$$q(x) = i\sigma_3[\sigma_3, K_+(x,x)] \, , \tag{4.13a}$$

$$i\frac{\partial K_+}{\partial x} + i\sigma_3\frac{\partial K_+}{\partial y}\sigma_3 + q(x)K_+(x,y) = 0 \, . \tag{4.13b}$$

More specifically, (4.13a) allows knowing the solution $K_+(x,y)$ of the GLM equation to recover the corresponding potential $q(x)$, which automatically comes in an off-diagonal form.

4.2 Derivation of GLM Equation from the Completeness Relation (3.111)

In the next approach to the GLM equation, we use as a starting point the completeness relation (3.111) of the Jost solutions and the linear relations (3.79) between them. First we insert (3.78) into the integrand of the completeness relation (3.111) and obtain:

$$\frac{\phi^+(x,\lambda)\tilde{\psi}^+(y,\lambda)}{a^+(\lambda)} + \frac{\phi^-(x,\lambda)\tilde{\psi}^-(y,\lambda)}{a^-(\lambda)} = \psi(x,\lambda)(\sigma_3 + R(\lambda))\hat{\psi}(y,\lambda) \tag{4.14a}$$

$$= \phi(x,\lambda)(\sigma_3 - \tilde{R}(\lambda))\hat{\phi}(y,\lambda) \tag{4.14b}$$

$$R(\lambda) = \begin{pmatrix} 0 & \rho^-(\lambda) \\ \rho^+(\lambda) & 0 \end{pmatrix}, \qquad \tilde{R}(\lambda) = \begin{pmatrix} 0 & \tau^+(\lambda) \\ \tau^-(\lambda) & 0 \end{pmatrix} . \tag{4.14c}$$

Recall that:

$$\rho^{\pm}(\lambda) = \frac{b^{\pm}(\lambda)}{a^{\pm}(\lambda)}, \qquad \tau^{\pm}(\lambda) = \frac{b^{\mp}(\lambda)}{a^{\pm}(\lambda)} \, . \tag{4.14d}$$

As a next step, we need the transformation operators also for the inverse of $\psi(x,\lambda)$ and $\phi(x,\lambda)$. In order to do this, we shall use the following formula for inverting a 2×2 matrix with unit determinant, which are easily verified:

$$\hat{\psi}(x,\lambda) = \sigma^{-1}\psi^T(x,\lambda)\sigma, \qquad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \, . \tag{4.15}$$

Then from (4.1a) we get:

$$\begin{aligned} \hat{\psi}(x,\lambda) &= e^{i\lambda\sigma_3 x} + \int_x^\infty dy \, e^{i\lambda\sigma_3 y}\sigma^{-1}K_+^T(x,y)\sigma \, , \\ &= e^{i\lambda\sigma_3 x} + \int_x^\infty dy \, e^{i\lambda\sigma_3 y}K_+^\tau(x,y) \, , \end{aligned} \tag{4.16}$$

where

$$K_+^\tau(x,y) = \begin{pmatrix} K_{+,22} & -K_{+,12} \\ -K_{+,21} & K_{+,11} \end{pmatrix} = \begin{pmatrix} \tilde{K}_+^{(2)} \\ -\tilde{K}_+^{(1)} \end{pmatrix} \, . \tag{4.17}$$

Further, we insert (4.1) and (4.16) into the integral of the right-hand side of (3.111). After somewhat long but elementary calculations we get:

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^\infty d\lambda \left(\frac{\phi^+(x,\lambda)\tilde{\psi}^+(y,\lambda)}{a^+(\lambda)} + \frac{\phi^-(x,\lambda)\tilde{\psi}^-(y,\lambda)}{a^-(\lambda)} \right) \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty d\lambda \, \psi(x,\lambda)(\sigma_3 + R(\lambda))\hat{\psi}(y,\lambda) && (4.18) \\ &= \sigma_3\delta(x-y) + H_c(x,y) + \int_y^\infty dy'' \, H_c(x,y'')K_+^\tau(y,y'') \, . && (4.19) \end{aligned}$$

where we have assumed that $x < y$ and

$$H_c(x,y) = F_c(x,y) + K_+(x,y)\sigma_3 + \int_x^\infty dy' \, K_+(x,y')F_c(y',y) \, , \tag{4.20a}$$

$$F_c(x,y) = \frac{1}{2\pi}\int_{-\infty}^\infty d\lambda \, e^{-i\lambda\sigma_3 x}R(\lambda)e^{i\lambda\sigma_3 y} \, . \tag{4.20b}$$

The terms in the discrete spectrum in (3.111) can be transformed in a similar way. Using (4.3) we have:

$$i\left(\frac{\phi_k^+(x)\tilde{\psi}_k^+(y)}{\dot{a}_k^+} - \frac{\phi_k^-(x)\tilde{\psi}_k^-(y)}{\dot{a}_k^-} \right)$$

$$= i\left(C_k^+ e^{i\lambda_k^+(x+y)}\xi_k^+(x)\tilde{\xi}_k^+(y) + C_k^- e^{-i\lambda_k^-(x+y)}\xi_k^-(x)\tilde{\xi}_k^-(y)\right)$$

$$= H_k(x,y) + \int_y^\infty dy''\, H_k(x,y'')K_+^\tau(y,y'') \,, \tag{4.21a}$$

$$H_k(x,y) = F_k^+(x+y) + \int_x^\infty dz\, K_+(x,z)F_k^+(z+y) \,, \tag{4.21b}$$

$$F_k^+(x) = \begin{pmatrix} 0 & -C_k^- e^{-i\lambda_k^- x} \\ C_k^+ e^{i\lambda_k^+ x} & 0 \end{pmatrix} . \tag{4.21c}$$

Inserting (4.18), (4.21a), (4.21b) and (4.21c) into the completeness relation (3.111), we find:

$$\sigma_3\delta(x-y) = \sigma_3\delta(x-y) + H(x,y) + \int_y^\infty dy''\, H(x,y'')K_+^\tau(y,y'') \,, \tag{4.22}$$

where

$$H(x,y) = H_c(x,y) + \sum_{k=1}^N H_k(x,y) \,. \tag{4.23}$$

Thus, from (4.22) we get:

$$H(x,y) + \int_y^\infty dy''\, H(x,y'')K_+^\tau(y,y'') = 0 \,. \tag{4.24}$$

Let us briefly recall some well-known facts about the Volterra type integral equations. Let us consider a generic integral equation of Volterra type:

$$H(x,y) + \int_y^\infty dy''\, H(x,y'')K_+^\tau(y,y'') = V(x,y) \,. \tag{4.25}$$

whose kernel $K_+^\tau(y,y'')$ and right-hand side $V(x,y)$ are smooth and fall off fast enough for y, y'' tending to ∞. More precisely, these conditions ensure that $K_+^\tau(y,y'')$ is a kernel of a bounded integral operator.

Then the solution of (4.25) exists and is given by the following series:

$$H(x,y) = V(x,y) + \sum_{p=1}^\infty \frac{(-1)^p}{p!} V^{(p)}(x,y) \,, \tag{4.26}$$

with

$$\begin{aligned} V^{(p)}(x,y) &= \int_y^\infty dy'' V^{(p-1)}(x,y'')K_+^\tau(y,y''), \qquad p = 1,2,\dots \,, \\ V^{(0)}(x,y) &= V(x,y) \,. \end{aligned} \tag{4.27}$$

The conditions on $K_+^\tau(x,y)$ and $V(x,y)$ ensure that the integrals in $V^{(p)}(x,y)$ are convergent for any p and that moreover the series (4.26) is convergent. The equation (4.24) above is obtained from (4.25) with $V(x,y)=0$. Thus, the only solution of (4.24) is the trivial one:

$$H(x,y) = 0 \, , \tag{4.28}$$

which due to the definition of $H(x,y)$ gives the GLM equation.

4.3 The Riemann–Hilbert Problem

Solving the Riemann–Hilbert problem allows to recover a section-analytic function of λ knowing its values on the boundaries of its analyticity regions.

4.3.1 The Additive RHP for Scalar and Matrix Functions

We start with the additive version of the RHP. In fact, in Sects. 3.5, 3.6, and 3.7, we have derived the solutions of particular additive RHP both for scalar and vector-valued functions. Here, we give a somewhat more general treatment of the problem.

As a basic tool for solving the RHP, we shall use the well-known Plemelji–Sokhotsky formula. Let Γ be a smooth closed contour, which divides the complex λ-plane into two regions $\Gamma_+\backslash\Gamma$ and $\Gamma_-\backslash\Gamma$. Let for definiteness Γ be positively orientated and $\Gamma_-\backslash\Gamma$ be the inside of Γ, see Fig. 4.1.

Let also $F(\lambda)$ be a smooth bounded function on Γ and let $F^+(\lambda)$ and $F^-(\lambda)$ be analytic functions of λ for $\lambda \in \Gamma_+\backslash\Gamma$ and $\lambda \in \Gamma_+\backslash\Gamma$, respectively, such that

$$F^+(\lambda) - F^-(\lambda) = F(\lambda), \qquad \text{for } \lambda \in \Gamma \, , \tag{4.29}$$

where of course we assume that $F^+(\lambda)$ and $F^-(\lambda)$ allow continuous extensions to Γ.

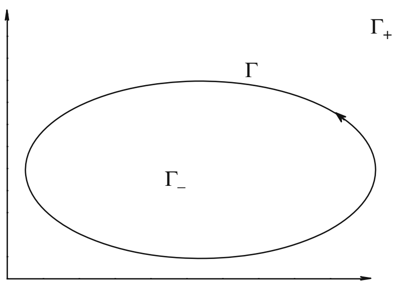

Fig. 4.1. The contour Γ and the regions $\Gamma_\pm\backslash\Gamma$ for a generic RHP

The problem of reconstructing $F^{\pm}(\lambda)$ for all $\lambda \in \Gamma_{\pm}$ and satisfying (4.29) for $\lambda \in \Gamma$ is known as additive RHP. The RHP (4.29) allows a class of solutions of the form:

$$F^{+}(\lambda) = H(\lambda) + \frac{1}{2\pi i} \oint_{\Gamma} \frac{F(\mu)}{\mu - \lambda} d\mu, \qquad \lambda \in \Gamma_{+} \backslash \Gamma\,, \tag{4.30a}$$

$$F^{-}(\lambda) = H(\lambda) + \frac{1}{2\pi i} \oint_{\Gamma} \frac{F(\mu)}{\mu - \lambda} d\mu, \qquad \lambda \in \Gamma_{-} \backslash \Gamma\,, \tag{4.30b}$$

where $H(\lambda)$ is arbitrary entire function of λ.

The fact that (4.30a) and (4.30b) provide a solution to the RHP (4.29) follows from the formulae giving the limit values of $F^{\pm}(\lambda)$ for $\lambda \to \Gamma$:

$$F^{+}(\lambda) = H(\lambda) + \frac{1}{2} F(\lambda) + \frac{1}{2\pi i} \text{V.P.} \oint_{\Gamma} \frac{F(\mu)}{\mu - \lambda} d\mu\,, \tag{4.31a}$$

$$F^{-}(\lambda) = H(\lambda) - \frac{1}{2} F(\lambda) + \frac{1}{2\pi i} \text{V.P.} \oint_{\Gamma} \frac{F(\mu)}{\mu - \lambda} d\mu\,; \tag{4.31b}$$

they are known as the Plemelji–Sokhotsky formulae.

In order to pick up a unique solution of the RHP, one has to normalize it by fixing up the value of, say $F^{+}(\lambda)$ at $\lambda = \lambda_0 \in \Gamma_{+}$, that is, to require $F^{+}(\lambda) = F_0^{+}$. This fixes up the value of $H(\lambda_0)$ to be:

$$H(\lambda_0) \equiv H_0 = F_0^{+} - \frac{1}{2\pi i} \oint_{\Gamma} \frac{d\mu\, F(\mu)}{\mu - \lambda_0}\,. \tag{4.32}$$

Then introducing $\widetilde{H}(\lambda) = H(\lambda) - H(\lambda_0)$ we get:

$$F^{+}(\lambda) = \widetilde{H}(\lambda) + F_0^{+} + \frac{1}{2\pi i} \oint_{\Gamma} d\mu\, F(\mu) \left(\frac{1}{\mu - \lambda} - \frac{1}{\mu - \lambda_0} \right)\,, \tag{4.33a}$$

$$F^{-}(\lambda) = \widetilde{H}(\lambda) + F_0^{+} + \frac{1}{2\pi i} \oint_{\Gamma} d\mu\, F(\mu) \left(\frac{1}{\mu - \lambda} - \frac{1}{\mu - \lambda_0} \right)\,. \tag{4.33b}$$

However, the entire function $\widetilde{H}(\lambda)$ changes the behavior of $F^{+}(\lambda)$ when $\lambda \to \infty$. The condition that $F^{+}(\lambda)$ is bounded when $\lambda \to \infty$ means that $\widetilde{H}(\lambda)$ must be bounded when $\lambda \to \infty$. By Liouville's theorem this means that $\widetilde{H}(\lambda) = \text{const}$. But $\widetilde{H}(\lambda) = 0$ for $\lambda = \lambda_0$, so the constant equals 0. Therefore, the solution of the normalized RHP is provided by:

$$F^{+}(\lambda) = F_0^{+} + \frac{1}{2\pi i} \oint_{\Gamma} d\mu\, F(\mu) \left(\frac{1}{\mu - \lambda} - \frac{1}{\mu - \lambda_0} \right), \quad \lambda \in \Gamma_{+} \backslash \Gamma\,, \tag{4.34a}$$

$$F^{-}(\lambda) = F_0^{+} + \frac{1}{2\pi i} \oint_{\Gamma} d\mu\, F(\mu) \left(\frac{1}{\mu - \lambda} - \frac{1}{\mu - \lambda_0} \right), \quad \lambda \in \Gamma_{-} \backslash \Gamma\,. \tag{4.34b}$$

The additive RHP can be effectively solved also for matrix-valued functions $F^{\pm}(x, \lambda)$, which may depend also on additional parameters (such as x).

Indeed, the spectral representations for the Jost solutions obtained in Sect. 3.6 are in fact solutions of an additive RHP for functions that are two-component vector functions depending analytically on λ and also on the real parameter x. Let us choose Γ to be the real axis, $\Gamma_\pm \equiv \mathbb{C}_\pm$, $\lambda_0 = \infty$ and replace in (4.29) $F^+(\lambda)$, $F^-(\lambda)$, $F(\lambda)$, F_0^+ by

$$\xi^+(x,\lambda), \qquad \frac{\varphi^-(x,\lambda)}{a^-(\lambda)}, \qquad -\rho^-(\lambda)e^{-2i\lambda x}\xi^-(x,\lambda), \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

respectively. Then (3.79a) can be considered as a generalization of the additive RHP (4.29) and the spectral representation (3.86) provides the solution for $\xi^+(x,\lambda)$ of this RHP.

Likewise, taking $F^+(\lambda)$, $F^-(\lambda)$, $F(\lambda)$, F_0^+ to be

$$\frac{\varphi^+(x,\lambda)}{a^+(\lambda)}, \qquad \xi^-(x,\lambda), \qquad \rho^+(\lambda)e^{2i\lambda x}\xi^+(x,\lambda), \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

respectively, we can treat (3.79b) as a generalization of the RHP (4.29), and the spectral representation (3.87) provides the solution for $\xi^-(x,\lambda)$ of this RHP.

An additional difficulty here is to take into account the possible pole singularities of $\varphi^\pm(x,\lambda)/a^\pm(\lambda)$ (i.e. possible zeroes of $a^\pm(\lambda)$), which correspond to the discrete eigenvalues of L. In the terminology of the RHP $\lambda_j^\pm$ are the singular points of the RHP.

Remark 4.1. The contour integration method [46] outlined also in the previous Chapter is an effective tool to solve additive RHP.

4.3.2 The Multiplicative RHP for Scalar Functions

By multiplicative RHP, we mean the problem of constructing two functions $F^+(\lambda)$ and $F^-(\lambda)$ analytic for $\lambda \in \Gamma_+$ and $\lambda \in \Gamma_-$, respectively, such that:

$$F^+(\lambda) = F^-(\lambda)F(\lambda), \qquad \text{for } \lambda \in \Gamma\ . \tag{4.35}$$

If the functions $F^\pm(\lambda)$ are scalar and have no zeroes in their regions of analyticity, we can reduce the multiplicative RHP to an additive one by just taking the log of both sides of (4.35):

$$\ln F^+(\lambda) - \ln F^-(\lambda) = \ln F(\lambda), \qquad \text{for } \lambda \in \Gamma\ . \tag{4.36}$$

Such solution we shall call regular. The solution will be unique if we normalize it by $F^+_{\rm reg}(\lambda_0) = F_0^+$ for $\lambda_0 \in \Gamma_+ \backslash \Gamma$. Then the regular solution of (4.35) is given by:

$$F^+_{\rm reg}(\lambda) = F_0^+ \exp\left(\mathcal{F}(\lambda) - \mathcal{F}(\lambda_0)\right), \qquad \lambda \in \Gamma_+ \backslash \Gamma, \tag{4.37a}$$

$$F^-_{\rm reg}(\lambda) = F_0^+ \exp\left(\mathcal{F}(\lambda) - \mathcal{F}(\lambda_0)\right), \qquad \lambda \in \Gamma_- \backslash \Gamma, \tag{4.37b}$$

$$\mathcal{F}(\lambda) = \frac{1}{2\pi i} \text{V.P.} \int_{\Gamma} d\mu \, \frac{\ln F(\mu)}{\mu - \lambda}, \qquad \lambda \in \mathbb{C} \backslash \Gamma, \ . \tag{4.37c}$$

We can also describe a special class of singular solutions $F^{\pm}_{\text{sing}}(\lambda)$ of the RHP (4.35), which have simple zeroes at a set of points $\lambda_j^{\pm} \in \Gamma_{\pm} \backslash \Gamma$, $j = 1, \dots, N$. Such singular solutions of (4.35) can be reduced to a regular one for the functions $F^{\prime,\pm}(\lambda)$

$$F^{\prime,+}(\lambda) = \frac{F_N(\lambda_0)}{F_N(\lambda)} F^+_{\text{sing}}(\lambda), \qquad F^{\prime,-}(\lambda) = \frac{F_N(\lambda)}{F_N(\lambda_0)} F^-_{\text{sing}}(\lambda) \ , \tag{4.38}$$

$$F_N(\lambda) = \prod_{j=1}^{N} \frac{\lambda - \lambda_j^+}{\lambda - \lambda_j^-} \ .$$

Indeed, the additional factors expressed through $F_N(\lambda)$ are such that $F^{\prime,\pm}(\lambda)$ have no zeroes for $\lambda \in \Gamma_{\pm}$ and satisfy the multiplicative RHP (4.35) with the same sewing function $F(\lambda)$. Thus, using (4.37) and (4.38) we can write down the singular solutions of the RHP as:

$$F^+_{\text{sing}}(\lambda) = F_0^+ \prod_{j=1}^{N} \frac{\lambda - \lambda_j^+}{\lambda - \lambda_j^-} \prod_{j=1}^{N} \frac{\lambda_0 - \lambda_j^-}{\lambda_0 - \lambda_j^+} \exp\left(\mathcal{F}(\lambda) - \mathcal{F}(\lambda_0)\right), \qquad \lambda \in \Gamma_+ \backslash \Gamma \ , \tag{4.39a}$$

$$F^-_{\text{sing}}(\lambda) = F_0^+ \prod_{j=1}^{N} \frac{\lambda - \lambda_j^-}{\lambda - \lambda_j^+} \prod_{j=1}^{N} \frac{\lambda_0 - \lambda_j^+}{\lambda_0 - \lambda_j^-} \exp\left(\mathcal{F}(\lambda) - \mathcal{F}(\lambda_0)\right), \qquad \lambda \in \Gamma_- \backslash \Gamma \ , \tag{4.39b}$$

Note that the factor $F_N(\lambda)/F_N(\lambda_0)$ equals to 1 for $\lambda = \lambda_0$, which makes it compatible with the normalization condition at $\lambda = \lambda_0$. For $\lambda \to \infty$, this factor tends to a constant $1/F_N(\lambda_0)$, thus ensuring the boundness of $F^+_{\text{sing}}(\lambda)$ for $\lambda \to \infty$. This explains also why we took equal number of zeroes in $\Gamma_+ \backslash \Gamma$ and $\Gamma_- \backslash \Gamma$.

The RHP we considered above were of generic form. However, one can think of such problems with additional involution properties, e.g.

$$F^+(\lambda) = (F^-(\lambda^*))^* \ . \tag{4.40}$$

Such involution requires that the contour Γ be symmetric, i.e. if $\lambda \in \Gamma$ then also $\lambda^* \in \Gamma$. In the singular case, this involution will relate the sets of zeroes by $\lambda_j^+ = (\lambda_j^-)^*$.

In what follows, we shall mostly use the so-called canonical normalization, i.e. we shall take $\lambda_0 = \infty$. This simplifies the form of the RHP solution above.

Following Remark 4.1, we note that deriving the dispersion relations for $a^{\pm}(\lambda)$ in Sect. 3.4 we solved a singular RHP of the form (4.35). To see this, it is necessary to choose the contour Γ to be the real axis; then $\Gamma_+ \backslash \Gamma \equiv \mathbb{C}_+$ and $\Gamma_- \backslash \Gamma \equiv \mathbb{C}_-$ are the upper and lower half-plane, respectively. Next we put:

$$F^+_{\text{sing}}(\lambda) = a^+(\lambda) \ , \tag{4.41a}$$

$$F^+_{\mathrm{sing}}(\lambda) = \frac{1}{a^-(\lambda)} \,, \tag{4.41b}$$

and

$$F(\lambda) = \frac{1}{1+\rho^+(\lambda)\rho^-(\lambda)}, \qquad \lambda \in \mathbb{R} \,. \tag{4.41c}$$

Then $a^+(\lambda)$ and $a^-(\lambda)$ have a set of N simple zeroes located at $\lambda_k^\pm \in \mathbb{C}_\pm$. The rest of the details are given in Sect. 3.4.

Taking the contour Γ to be the real axis requires some additional restrictions on $F(\lambda)$, namely it must tend to 1 for $\lambda \to \infty$, i.e. $\rho^\pm(\lambda)$ fast enough for $\lambda \to \infty$. This property is ensured by imposing condition **C1** on the potential $q(x)$.

4.3.3 The Multiplicative RHP for Matrix-Valued Functions

The formulation of the problem is similar to the one in the preceding subsection with the substantial difference that now the functions $F^\pm(\lambda)$ and $F(\lambda)$ are $n \times n$ matrices.

Let us consider, like above, the positively orientated contour Γ splitting the complex λ-plane into two regions $\Gamma_+\backslash\Gamma$ and $\Gamma_-\backslash\Gamma$. Let also $F(\lambda)$ be smooth enough function defined for $\lambda \in \Gamma$.

We shall say that the ($n \times n$-matrix valued) functions $F^\pm(\lambda)$ are regular solution of the multiplicative RHP if:

1. on the contour Γ they satisfy the equation:

$$F^+(\lambda) = F^-(\lambda)F(\lambda), \qquad \lambda \in \Gamma \,; \tag{4.42}$$

2. all matrix elements of $F^\pm(\lambda)$ have no singularities for $\lambda \in \Gamma_\pm\backslash\Gamma$;
3. $\det F^+(\lambda)$ (resp. $\det F^-(\lambda)$) has no zeroes for $\lambda \in \Gamma_+\backslash\Gamma$ (resp. $\lambda \in \Gamma_-\backslash\Gamma$).

One can expect that the RHP (4.42) will have unique solution only after imposing a normalization condition:

$$F^+(\lambda_0) = F_0^+ \,, \tag{4.43}$$

where F_0^+ is a nondegenerate matrix. In analogy with the scalar case, we may look for the solutions $F^\pm(\lambda)$ for $\lambda \in \Gamma_\pm$ in the form:

$$F^+(\lambda) = F_0^+ + \frac{1}{2\pi i} \oint_\Gamma \left(\frac{\mathfrak{f}(\mu)}{\mu - \lambda} - \frac{\mathfrak{f}(\mu)}{\mu - \lambda_0} \right) d\mu \,, \tag{4.44}$$

$$F^-(\lambda) = F_0^+ - \frac{1}{2\pi i} \oint_\Gamma \left(\frac{\mathfrak{f}(\mu)}{\mu - \lambda} - \frac{\mathfrak{f}(\mu)}{\mu - \lambda_0} \right) d\mu \,. \tag{4.45}$$

The next step is to find a relation between the function $\mathfrak{f}(\mu)$ in the integrands in (4.44), (4.45) and the sewing function $F(\lambda)$. To do this, we insert the (4.44), (4.45) into (4.42), and after some calculations we obtain:

$$\mathfrak{f}(\lambda)\mathfrak{k}(\lambda) + F_0^+ + \frac{1}{2\pi i}\oint_\Gamma \left(\frac{\mathfrak{f}(\mu)}{\mu - \lambda} - \frac{\mathfrak{f}(\mu)}{\mu - \lambda_0} \right) d\mu = 0 \,, \tag{4.46}$$

where $\mathfrak{k}(\lambda)$ is the Caley transform $\mathfrak{k}(\lambda) = (\mathbb{1} + F(\lambda))(\mathbb{1} - F(\lambda))^{-1}$ of the sewing function $F(\lambda)$. Given $F(\lambda)$, we can interpret (4.46) as singular integral equation for $\mathfrak{k}(\lambda)$.

This equation simplifies for the so-called canonical normalization:

$$\lim_{\lambda\to\infty} F^+(\lambda) = \mathbb{1} \,, \tag{4.47}$$

i.e. $\lambda_0 = \infty$ and $F_0^+ = \mathbb{1}$. Then (4.46) takes the form:

$$\mathfrak{f}(\lambda)\mathfrak{k}(\lambda) + \mathbb{1} + \frac{1}{2\pi i}\oint_\Gamma \frac{\mathfrak{f}(\mu)}{\mu - \lambda} = 0 \,, \tag{4.48}$$

which is easier (though not always easy!) to analyze.

Along with the regular solutions, the RHP (4.42) has also singular solutions for which $\det g^+(\lambda)$ and $\det g^-(\lambda)$ may have zeroes or poles in $\Gamma_\pm \backslash \Gamma$.

The singular solutions to (4.42) are ones for which condition (3) does not hold. That means that for some value of $\lambda = \lambda_1^+$ the matrix $F(\lambda_1^+)$ has vanishing eigenvalue; in what follows, we shall assume that this eigenvalue is of multiplicity 1. The corresponding eigenspace will be characterized by a projector P_1 of rank 1.

We shall construct the corresponding singular solution $F_1^\pm(\lambda)$ of the RHP using the so-called dressing factors:

$$F_1^\pm(\lambda) = u_1(\lambda)F^\pm(\lambda) \,, \tag{4.49a}$$

$$u_1(\lambda) = \mathbb{1} + (c_1(\lambda) - 1)P_1 \,, \tag{4.49b}$$

$$c_1(\lambda) = \frac{\lambda - \lambda_1^+}{\lambda - \lambda_1^-} \,, \tag{4.49c}$$

where P_1 is a projector of rank one. Therefore, it can be written down as:

$$P_1 = \frac{|n^{(1)}\rangle\langle m^{(1)}|}{\langle m^{(1)}|n^{(1)}\rangle} \,, \tag{4.49d}$$

where $|n^{(1)}\rangle$ is a two-component column vector and $\langle m^{(1)}|$ is a two-component row vector

$$|n_1\rangle = F(\lambda_1^+)|n_{0,1}\rangle, \qquad \langle m_1| = \langle m_{0,1}|\hat{F}_1^-(\lambda_1^-) \,, \tag{4.49e}$$

and $|n_{0,1}\rangle$ and $\langle m_{0,1}|$ are two arbitrary constant vectors. Obviously (4.49d) satisfies the characteristic equation $P_1^2 = P_1$ identically.

This singular solution is such that $F_1^+(\lambda)$ has a vanishing eigenvalue at $\lambda = \lambda_1^+$, while $F_1^-(\lambda)$ has a pole at $\lambda = \lambda_1^-$. The projector P_1 determines the structure of both singularities through the dressing factor $u_1(\lambda)$. Note that the singular RHP will be characterized by the same sewing function as the regular RHP.

The dressing procedure described above can be applied also to a singular solution. Indeed, we can introduce

$$F_2^{\pm}(\lambda) = u_2(\lambda) F_1^{\pm}(\lambda) \ , \tag{4.50a}$$

$$u_2(\lambda) = \mathbb{1} - (c_2(\lambda) - 1) P_2 \ , \tag{4.50b}$$

$$c_2(\lambda) = \frac{\lambda - \lambda_2^+}{\lambda - \lambda_2^-} \ , \tag{4.50c}$$

which will give new singular solution provided P_2 is a projector

$$P_2 = \frac{|n^{(2)}\rangle\langle m^{(2)}|}{\langle m^{(2)}|n^{(2)}\rangle} \ , \tag{4.51}$$

where

$$|n^{(2)}\rangle = F_1(\lambda_2^+)|n_{0,2}\rangle, \qquad \langle m^{(2)}1| = \langle m_{0,2}|\hat{F}_1^-(\lambda_2^-) \ , \tag{4.52}$$

and $\lambda_2^+ \neq \lambda_1^+$ and $\lambda_2^- \neq \lambda_1^-$.

These formulae show why we required $\lambda_2^{\pm} \neq \lambda_1^{\pm}$; the reason is that $\lambda_2^{\pm}$ are regular points for $g_1^{\pm}(\lambda)$ and consequently $g_1^+(\lambda_2^+)$ and $g_1^-(\lambda_2^-)$ are invertible matrices.

Repeating this procedure $N-1$ times, we can construct singular solutions with singularities located at the prescribed points $\lambda_k^{\pm}$ (4.42).

4.4 The Zakharov–Shabat Dressing Method

In this section, we shall show that there is a direct relation between the matrix RHP and the inverse scattering problem for the ZS system. This is the main idea of the Zakharov–Shabat dressing method.

We know that the RHP involves analytic functions of λ. On the other hand, we constructed in Sect. 3.1 the fundamental solutions $\chi^+(x,\lambda)$ and $\chi^-(x,\lambda)$, which are analytic functions of λ for $\lambda \in \mathbb{C}_+$ and $\lambda \in \mathbb{C}_-$, respectively. From the definitions of $\chi^{\pm}(x,\lambda)$ (3.25), we find that on the real axis they are related by:

$$\chi^+(x,\lambda) = \chi^-(x,\lambda) G_0(\lambda), \qquad \lambda \in \mathbb{R} \ , \tag{4.53}$$

where

$$G_0(\lambda) = \begin{pmatrix} 1 & b^-/a^- \\ 0 & 1/a^- \end{pmatrix} \begin{pmatrix} a^+ & 0 \\ b^+ & 1 \end{pmatrix} = \frac{1}{a^-(\lambda)} \begin{pmatrix} 1 & b^-(\lambda) \\ b^+(\lambda) & 1 \end{pmatrix} \ . \tag{4.54}$$

Strictly speaking, $\chi^{\pm}(x,\lambda)$ have a defect which prevents us from treating (4.53) as RHP. They do not allow canonical normalization, because for $\lambda \to \infty$

they behave like $e^{i\lambda\sigma_3 x}$. This defect can be avoided by going over from $\chi^{\pm}(x,\lambda)$ to

$$\eta^{\pm}(x,\lambda) = \chi^{\pm}(x,\lambda)\exp(i\lambda\sigma_3 x)\ . \tag{4.55}$$

Multiplying both sides of (4.53) by $\exp(i\lambda\sigma_3 x)$ we get:

$$\eta^{+}(x,\lambda) = \eta^{-}(x,\lambda)G(x,\lambda), \qquad \lambda \in \mathbb{R}\ , \tag{4.56}$$

where the sewing function $G(x,\lambda)$ depends on x according to:

$$G(x,\lambda) = \exp(-i\lambda\sigma_3 x)\, G_0(\lambda)\, \exp(i\lambda\sigma_3 x), \qquad \lambda \in \mathbb{R}\ . \tag{4.57}$$

From the construction of the solutions $\eta^{\pm}(x,\lambda)$, we know that (see (3.49)):

$$\lim_{\lambda\to\infty} \eta^{+}(x,\lambda) = \lim_{\lambda\to\infty} \eta^{-}(x,\lambda) = \mathbb{1}\ . \tag{4.58}$$

i.e. they indeed can be treated as solutions to the RHP (4.53) with a canonical normalization (4.58).

The contour Γ in our case is the real axis $\Gamma \equiv \mathbb{R}$ and $\Gamma_{\pm} \equiv \mathbb{C}_{\pm}$.

Note that x enters in the RHP as an external parameter. The x-dependence of the sewing function $G(x,\lambda)$ (4.57) is of a "plane-wave" type. It has the advantage that $G(x,\lambda)$ is bounded for all values of x and $\lambda \in \mathbb{R}$.

In addition, condition **C1** on $q(x)$ ensures that $b^{\pm}(\lambda)$ are Schwartz-type functions of λ on the real axis; so is $a^{-}(\lambda) - 1$. Besides, from **C2** it follows that $a^{-}(\lambda) - 1$ has no zeroes on the real λ-axis – no discrete eigenvalues on $\mathbb{R}$. Thus, we check that $G(x,\lambda)$ satisfies all the necessary conditions for being sewing function of an RHP uniformly with respect to x.

We showed up to here that the FAS of the ZS system (or rather $\eta^{\pm}(x,\lambda)$, which are directly related to $\chi^{\pm}(x,\lambda)$), satisfy the RHP (4.56) with canonical normalization. Generically, they provide a singular solution to this RHP, because

$$\det \eta^{\pm}(x,\lambda) = a^{\pm}(\lambda)\ , \tag{4.59}$$

and according to the condition **C2** (3.34) we have

$$\det \eta^{\pm}(x,\lambda_k^{\pm}) = a^{\pm}(\lambda_k^{\pm}) = 0, \qquad \lambda_k^{\pm} \in \mathbb{C}_{\pm}\ , \tag{4.60}$$

for $k = 1, \ldots, N$. This means that the singularities of the RHP coincide with the discrete eigenvalues of the ZS system.

Now, we shall prove that any solution of the RHP (4.56) with a sewing function $G(x,\lambda)$ (4.57) is a solution to (3.4), which is equivalent to the ZS system. To this end we put:

$$X^{\pm}(x,\lambda) = \left(i\frac{d\eta^{\pm}}{dx} - \lambda[\sigma_3, \eta^{\pm}(x,\lambda)] \right) \hat{\eta}^{\pm}(x,\lambda)\ . \tag{4.61}$$

Using (4.56) and (4.57) for λ on the real line, we subsequently get:

$$
\begin{aligned}
X^+(x,\lambda) &= \left(i\frac{d\eta^+}{dx} - \lambda[\sigma_3, \eta^+(x,\lambda)]\right)\hat{\eta}^+(x,\lambda) \\
&= \left(i\frac{d\eta^-}{dx}G - \lambda[\sigma_3, \eta^-(x,\lambda)G] + i\eta^-\frac{dG}{dx}\right)\hat{G}\hat{\eta}^-(x,\lambda) \\
&= \left[\left(i\frac{d\eta^-}{dx} - \lambda[\sigma_3, \eta^-]\right)G + \eta^-\left(i\frac{dG}{dx} - \lambda[\sigma_3, G]\right)\right]\hat{G}\hat{\eta}^-(x,\lambda) \\
&= \left(i\frac{d\eta^-}{dx} - \lambda[\sigma_3, \eta^-]\right)\hat{\eta}^-(x,\lambda) \\
&= X^-(x,\lambda)\ .
\end{aligned} \tag{4.62}
$$

But $X^+(x,\lambda)$ is analytic for $\operatorname{Im}\lambda > 0$, $X^-(x,\lambda)$ is analytic for $\operatorname{Im}\lambda < 0$, and the equality $X^+(x,\lambda) = X^-(x,\lambda)$ (4.62) shows that $X^+(x,\lambda)$, $X^-(x,\lambda)$ extend to a function analytic everywhere in the complex λ-plane. Their limits for $\lambda \to \infty$ are given by (see (4.58) and (3.49)):

$$
\begin{aligned}
\lim_{\lambda\to\infty} g^\pm(x,\lambda) &= \lim_{\lambda\to\infty}\left(\frac{id\eta^\pm}{dx} - \lambda[\sigma_3, \eta^\pm(x,\lambda)]\right)\hat{\eta}^\pm(x,\lambda) \\
&= -[\sigma_3, \eta_1^\pm(x)]\ .
\end{aligned} \tag{4.63}
$$

In view of (3.50), both functions $g^+(x,\lambda)$ and $g^-(x,\lambda)$ have the same limit, which we denote by $-q(x)$. Then the functions:

$$
g^+(x,\lambda) + q(x), \qquad g^-(x,\lambda) + q(x)\ , \tag{4.64}
$$

extend to a function analytic everywhere in the complex λ-plane and tending to 0 for $\lambda \to \infty$. Therefore, they both vanish identically. Multiplying (4.61) by η^+ (or by η^-) on the right and taking into account (4.64) we get:

$$
\left(i\frac{d\eta^\pm}{dx} + q(x)\eta^\pm(x,\lambda) - \lambda[\sigma_3, \eta^\pm(x,\lambda)]\right) = 0\ , \tag{4.65}
$$

which coincides with (3.4). Thus, we proved that each regular solution $\eta^\pm(x,\lambda)$ of the RH problem satisfies (4.65), and consequently $\chi^\pm(x,\lambda) = \eta^\pm(x,\lambda)\exp(-i\lambda\sigma_3 x)$ are solutions of the Zakharov–Shabat system (3.1).

The cases when $\eta^\pm(x,\lambda)$ are solutions of a singular RH problem is treated analogously. In this case, the relations (4.62) hold true for all $\lambda \in \mathbb{C}\backslash\{\lambda_k^\pm\}$, i.e. for all λ except a finite number of points. This does not influence the final result.

We need also to recover the potential $q(x)$ from the solutions $\eta^\pm(x,\lambda)$ of the RHP. In fact, the answer can be obtained from (4.65). Taking in it the limit $\lambda \to \infty$ and making use of (3.49) we find:

$$
\begin{aligned}
q(x) &= -\lim_{\lambda\to\infty}\lambda[\sigma_3, \eta^\pm(x,\lambda)]\hat{\eta}^\pm(x,\lambda) \\
&= \lim_{\lambda\to\infty}\lambda(\eta^\pm(x,\lambda)\sigma_3\hat{\eta}^\pm(x,\lambda) - \sigma_3)\ .
\end{aligned} \tag{4.66}
$$

4.4.1 The RHP and the Singular Integral Equations

In Sect. 4.3, we mentioned that the RHP can be reduced to a set of singular integral equations. Their derivation is based on the analytic properties of $\eta^{\pm}(x,\lambda)$. They satisfy an RHP with canonical normalization, which can be written in the form:

$$\eta^{+}(x,\lambda) = \frac{\eta^{-}(x,\lambda)}{a^{-}(\lambda)} \left(\mathbb{1} + K(x,\lambda)\right) , \tag{4.67}$$

or

$$\eta^{-}(x,\lambda) = \frac{\eta^{+}(x,\lambda)}{a^{+}(\lambda)} \left(\mathbb{1} - K(x,\lambda)\right) , \tag{4.68}$$

where

$$K(x,\lambda) = \begin{pmatrix} 0 & b^{-}(\lambda)e^{-2i\lambda x} \\ b^{+}(\lambda)e^{2i\lambda x} & 0 \end{pmatrix} . \tag{4.69}$$

We can apply again the contour integration method to the integrals:

$$\mathcal{J}_1(x,\lambda) = \frac{1}{2\pi i} \oint_{C_+} \frac{d\mu \eta^{+}(x,\mu)}{\mu - \lambda} - \frac{1}{2\pi i} \oint_{C_-} \frac{d\mu}{\mu - \lambda} \frac{\eta^{-}(x,\mu)}{a^{-}(\mu)} , \tag{4.70}$$

$$\mathcal{J}_2(x,\lambda) = \frac{1}{2\pi i} \oint_{C_+} \frac{d\mu}{\mu - \lambda} \frac{\eta^{+}(x,\mu)}{a^{+}(\mu)} - \frac{1}{2\pi i} \oint_{C_-} \frac{d\mu \eta^{-}(x,\mu)}{\mu - \lambda} . \tag{4.71}$$

Skipping the details, we only give the corresponding spectral representations for $\eta^{+}(x,\lambda)$ $(\lambda \in \mathbb{C}_+)$:

$$\eta^{+}(x,\lambda) = \mathbb{1} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\mu}{\mu - \lambda} \frac{\eta^{-}(x,\mu)}{a^{-}(\mu)} K(x,\mu) - \sum_{k=1}^{N} \frac{\eta_k^{-}(x)}{\dot{a}_k^{-}(\lambda_k^{-} - \lambda)} . \tag{4.72}$$

and for $\eta^{-}(x,\lambda)$ $(\lambda \in \mathbb{C}_-)$:

$$\eta^{-}(x,\lambda) = \mathbb{1} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\mu}{\mu - \lambda} \frac{\eta^{+}(x,\mu)}{a^{+}(\mu)} K(x,\mu) - \sum_{k=1}^{N} \frac{\eta_k^{+}(x)}{\dot{a}_k^{+}(\lambda_k^{+} - \lambda)} . \tag{4.73}$$

The systems of (4.73) and (4.72) can be viewed as systems of linear integral equations for the solutions of the RH problem (4.67). Now, keeping in mind that the quantities $\eta_k^{\pm}(x)$ are the values of $\eta^{\pm}(x,\lambda)$ at the points $\lambda_k^{\pm}$, we use (3.97)–(3.101) and find that $\eta_k^{\pm}(x)$ have the following structure:

$$\eta_k^{+}(x) = \xi_k^{+}(x)\big(b_k^{+} e^{2i\lambda_k^{+} x}, 1\big), \qquad \eta_k^{-}(x) = \xi_k^{-}(x)\big(1, -b_k^{-} e^{-2i\lambda_k^{-} x}\big) . \tag{4.74}$$

It is easy to see also that, if we take the second column of (4.72) and the first column of (4.73), we obtain (3.86) and (3.87).

4.4.2 Reflectionless Potentials

The singular integral equations that we derived in the preceding subsection cannot be solved explicitly in the generic case. However, there are particular cases, when it is possible to solve them explicitly. In this way, the so-called reflectionless potentials of L (3.1) are constructed. Their name suggests that for these solutions the reflection coefficients vanish, that is,

$$\rho^+(\lambda) = \rho^-(\lambda) = 0, \qquad \lambda \in \mathbb{R}\,, \tag{4.75}$$

or equivalently:

$$\tau^+(\lambda) = \tau^-(\lambda) = 0, \qquad \lambda \in \mathbb{R}\,. \tag{4.76}$$

Then the systems of equations simplify and (3.87), (3.86) are transformed into:

$$\xi^-(x,\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sum_{k=1}^{N} \frac{C_k^+}{\lambda_k^+ - \lambda} e^{2i\lambda_k^+ x} \xi_k^+, \qquad \lambda \in \mathbb{C}_-\,, \tag{4.77}$$

$$\xi^+(x,\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{k=1}^{N} \frac{C_k^-}{\lambda_k^- - \lambda} e^{-2i\lambda_k^- x} \xi_k^-, \qquad \lambda \in \mathbb{C}_+\,, \tag{4.78}$$

and (3.89), (3.88) – into:

$$\varphi^+(x,\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{k=1}^{N} \frac{M_k^-}{\lambda_k^- - \lambda} e^{2i\lambda_k^- x} \varphi_k^-, \qquad \lambda \in \mathbb{C}_+\,, \tag{4.79}$$

$$\varphi^-(x,\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sum_{k=1}^{N} \frac{M_k^+}{\lambda_k^+ - \lambda} e^{-2i\lambda_k^+ x} \varphi_k^+, \qquad \lambda \in \mathbb{C}_-\,, \tag{4.80}$$

In order to solve the systems (4.77), (4.78) (or (4.80), (4.79)), it is enough to calculate $\xi_k^\pm(x)$ (or $\varphi_k^\pm(x)$). This can be done by putting $\lambda = \lambda_p^-$ in (4.77) (resp. (4.80)) and $\lambda = \lambda_p^+$ in (4.78) (resp. (4.79)). Thus, we obtain the following linear algebraic equations for $\xi_k^\pm(x)$ and $\varphi_k^\pm(x)$:

$$\left(\boldsymbol{\xi}^-(x), \boldsymbol{\xi}^+(x) \right) \begin{pmatrix} \mathbb{1} & D^- \\ D^+ & \mathbb{1} \end{pmatrix} = \left(\mathbf{e}^+, \mathbf{e}^- \right)\,, \tag{4.81a}$$

$$D_{kp}^+(x) = \frac{b_k^+(x)}{l_{kp}\dot{a}_k^+}, \qquad D_{kp}^-(x) = \frac{b_k^-(x)}{l_{pk}\dot{a}_k^-}\,, \tag{4.81b}$$

and

$$\left(\boldsymbol{\varphi}^+(x), \boldsymbol{\varphi}^-(x) \right) \begin{pmatrix} \mathbb{1} & F^+ \\ F^- & \mathbb{1} \end{pmatrix} = \left(\mathbf{e}^+, \mathbf{e}^- \right)\,, \tag{4.82a}$$

$$F_{kp}^+(x) = \frac{1}{l_{kp} b_k^-(x)\dot{a}_k^-}, \qquad F_{kp}^-(x) = \frac{1}{l_{pk} b_k^+(x)\dot{a}_k^+}\,, \tag{4.82b}$$

where we used the notations:

$$\boldsymbol{\xi}^{\pm}(x) = \left(\xi_1^{\pm}(x), \dots, \xi_N^{\pm}(x)\right), \quad \boldsymbol{\varphi}^{\pm}(x) = \left(\varphi_1^{\pm}(x), \dots, \varphi_N^{\pm}(x)\right), \tag{4.83a}$$

$$\mathbf{e}^{\pm} = \underbrace{\left(e^{\pm}, \dots, e^{\pm}\right)}_{N-\text{times}}, \qquad e^{+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad e^{-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{4.83b}$$

$$l_{kp} = \lambda_k^{+} - \lambda_p^{-}, \qquad b_k^{\pm}(x) = b_k^{\pm} e^{\pm 2i\lambda_k^{\pm} x} = e^{-z_k \pm i\phi_k}, \tag{4.83c}$$

$$z_k = 2\nu_k (x - \xi_{0,k}), \qquad \phi_k = \frac{\mu_k}{\nu_k} z_k + \delta_k, \tag{4.83d}$$

The relations (4.74) show that the systems of (4.81) and (4.82) are equivalent. Therefore, it will be enough to consider only one of them. For example, the solution of (4.81) can be calculated by inverting the block matrix in the left-hand side of (4.81). This inverse equals:

$$\begin{pmatrix} \mathbb{1} & D^{-} \\ D^{+} & \mathbb{1} \end{pmatrix}^{-1} = \begin{pmatrix} (\mathbb{1} - D^{-}D^{+})^{-1} & -D^{-}(\mathbb{1} - D^{+}D^{-})^{-1} \\ -D^{+}(\mathbb{1} - D^{-}D^{+})^{-1} & (\mathbb{1} - D^{+}D^{-})^{-1} \end{pmatrix}. \tag{4.84}$$

Thus, we can obtain explicit expressions for $\xi_k^{\pm}(x)$ and $\varphi_k^{\pm}(x)$. In order to recover the Jost solutions, it is enough to insert these expressions into the left-hand sides of (4.77)–(4.79). The corresponding reflectionless potential is given by:

$$q_{\mathrm{Ns}}(x) = \left[\sigma_3, \sum_{k=1}^{N} \left(\frac{b_k^{+}(x)}{\dot{a}_k^{+}} \xi_k^{+}(x), -\frac{b_k^{-}(x)}{\dot{a}_k^{-}} \xi_k^{-}(x) \right)\right]. \tag{4.85}$$

Also, in view of (4.75), (4.76), the scattering data, i.e. the scattering matrix, has the form:

$$T_{\mathrm{Ns}}(\lambda) = \begin{pmatrix} a_{\mathrm{Ns}}^{+}(\lambda) & 0 \\ 0 & a_{\mathrm{Ns}}^{-}(\lambda) \end{pmatrix}, \qquad a_{\mathrm{Ns}}^{+}(\lambda) a_{\mathrm{Ns}}^{-}(\lambda) = 1, \tag{4.86}$$

where

$$a_{\mathrm{Ns}}^{+}(\lambda) = \frac{1}{a_{\mathrm{Ns}}^{-}(\lambda)} = \prod_{k=1}^{N} \frac{\lambda - \lambda_k^{+}}{\lambda - \lambda_k^{-}}. \tag{4.87}$$

The next step would be to insert the results for $\xi_k^{\pm}(x)$ and $\varphi_k^{\pm}(x)$ into the right-hand side of (4.77)–(4.80). This immediately gives us the explicit formulae for the Jost solutions and for the FAS. They are meromorphic functions of λ and so is the corresponding scattering matrix. Note that due to (4.75) and (4.76) it will be a diagonal matrix. More specifically, we get:

$$\phi_{\mathrm{Ns}}^{+}(x,\lambda) = \psi_{\mathrm{Ns}}^{-}(x,\lambda) a_{\mathrm{Ns}}^{+}(\lambda), \qquad \phi_{\mathrm{Ns}}^{-}(x,\lambda) = \psi_{\mathrm{Ns}}^{+}(x,\lambda) a_{\mathrm{Ns}}^{-}(\lambda), \tag{4.88}$$

and

$$\chi_{\mathrm{Ns}}^{+}(x,\lambda) = a_{\mathrm{Ns}}^{+}(\lambda) \chi_{\mathrm{Ns}}^{-}(x,\lambda). \tag{4.89}$$

4.4.3 The 1-Soliton Case

The solution of (4.77), (4.78), (4.80) and (4.79) is simple and is given by:

$$\xi_1^-(x) = \frac{1}{A_1(x)} \begin{pmatrix} 1 \\ -b_1^+(x) \end{pmatrix}, \qquad \xi_1^+(x) = \frac{1}{A_1(x)} \begin{pmatrix} b_1^-(x) \\ 1 \end{pmatrix}, \tag{4.90a}$$

$$\varphi_1^+(x) = \frac{b_1^+(x)}{A_1(x)} \begin{pmatrix} b_1^-(x) \\ 1 \end{pmatrix}, \qquad \varphi_1^-(x) = \frac{b_1^-(x)}{A_1(x)} \begin{pmatrix} -1 \\ b_1^+(x) \end{pmatrix}, \tag{4.90b}$$

where

$$A_1(x) = 1 + b_1^+(x) b_1^-(x) , \tag{4.90c}$$

$$b_1^\pm(x) = b_1^\pm e^{\pm 2i\lambda_1^\pm x} , \tag{4.90d}$$

These formulae can be written in compact form as follows:

$$\eta_1^+(x) = \mathbb{1} - P_1(x), \qquad \eta_1^-(x) = P_1(x) , \tag{4.91a}$$

$$\begin{aligned} P_1(x) &= \frac{|n_1(x)\rangle\langle m_1(x)|}{\langle m_1(x)|n_1(x)\rangle} \\ &= \frac{1}{1 + b^+(x)b^-(x)} \begin{pmatrix} 1 & -b_1^-(x) \\ -b_1^+(x) & b_1^+(x)b_1^-(x) \end{pmatrix} , \end{aligned} \tag{4.91b}$$

$$|n_1(x)\rangle = e^{-i\lambda_k^+ \sigma_3 x} |n_{10}\rangle, \qquad |n_{10}\rangle = \begin{pmatrix} 1 \\ -b_{10}^+ \end{pmatrix} , \tag{4.91c}$$

$$\langle m_1(x)| = \langle m_{10}| e^{i\lambda_k^- \sigma_3 x}, \qquad \langle m_{10}| = \left(1, -b_{10}^-\right) . \tag{4.91d}$$

Inserting (4.90) into (4.72) and (4.73) we get the following explicit expressions for the FAS:

$$\eta_{1s}^+(x,\lambda) = \mathbb{1} + (c_1(\lambda) - 1) P_1(x) , \tag{4.92a}$$

$$\eta_{1s}^-(x,\lambda) = \mathbb{1} + \left(\frac{1}{c_1(\lambda)} - 1 \right) (\mathbb{1} - P_1(x)) , \tag{4.92b}$$

or

$$\eta_{1s}^+(x,\lambda) = \eta_{1s}^-(x,\lambda) c_1(\lambda) . \tag{4.93}$$

The corresponding potential is given by:

$$\begin{aligned} q_{1s}(x) &= -(\lambda_1^+ - \lambda_1^-) [\sigma_3, P_1(x)] \\ &= \frac{2(\lambda_1^+ - \lambda_1^-)}{1 + b_1^+ b_1^-} \begin{pmatrix} 0 & b_1^- \\ -b_1^+ & 0 \end{pmatrix} . \end{aligned} \tag{4.94}$$

Let us note that in the 1-soliton case we have

$$a_{1s}^+(\lambda) = \frac{1}{a_{1s}^-(\lambda)} = c_1(\lambda) ; \tag{4.95}$$

see formula (4.87) with $\nu = 1$. From (4.67), we find that in the reflectionless case $G(x, \lambda)$ is proportional to the unit matrix; consequently, it is x-independent. In the special case $N = 1$, comparing (4.67) with (4.93), we can interpret (4.93) as an RHP.

Another important fact illustrated by (4.92) and generic for the N-soliton case is the following. Both $\eta^+(x, \lambda)$ and $\eta^-(x, \lambda)$ are meromorphic functions of λ (fraction-linear in our case). In fact, they can be extended to the whole λ-plane with the exception of their pole-singularities, located at $\lambda_1^\pm$.

In fact, what we just derived is the form of the simplest reflectionless potential of the ZS system. In order to get the 1-soliton solution of one of the NLEE with a dispersion law $f(\lambda)$, we need to recall that $b_1^\pm$ must depend also on the time t; see (2.56). Therefore, instead of $b_1^\pm(x)$ we must use:

$$\begin{aligned}
b_1^\pm(x, t) &\equiv b_1^\pm(x) e^{\mp 2if(\lambda_k^\pm)t} = \pm i e^{-z_1 \pm i\phi_1}, \qquad z_1(x, t) = 2\nu_1(x - \xi_1(t)) \,, \\
\phi_1(x, t) &= \frac{\mu_1}{\nu_1} z_1(x, t) + \delta_1(t), \qquad \delta_1(t) = \frac{2}{\nu_1} \left(\mu_1 f_{1,1} - \nu_1 f_{0,1}\right) t + \delta_1(0) \,, \\
\xi_1(t) &= \frac{1}{2\nu_1} \left(2f_{1,1} t + \ln |b_{01}^+|\right), \qquad f(\lambda_1^\pm) = f_{0,1} \pm i f_{1,1}.
\end{aligned} \tag{4.96}$$

Inserting (4.90d) and (4.96) into the right-hand side of (4.94) we get

$$q_{1s}^{-,*}(x, t) = q_{1s}^+(x, t) = u_{1s}(x, t), \qquad u_{1s} = \frac{2i\nu_1 e^{\mp i\phi_1(x,t)}}{\cosh(z_1(x, t))} \,. \tag{4.97}$$

In this parametrization, $\xi_1(t)$ must be interpreted as the center of mass of the soliton, μ_1 as its velocity, ν_1 as its amplitude, and δ_1 as its phase. Obviously, the 1-soliton solution is a (traveling) wave moving with constant velocity.

4.5 The Dressing Method: The Singular Solutions

The dressing method supposes that we already know explicitly (at least) one regular solution $\eta_0^\pm(x, \lambda)$ of (4.67). By "regular", here, we mean that $\eta_0^+(x, \lambda)$ and $\eta_0^-(x, \lambda)$ have no singularities or zeroes in their regions of analyticity. In particular, this means that

$$a_0^\pm(\lambda) = \det \eta_0^\pm(x, \lambda) \neq 0 \qquad \text{for} \qquad \lambda \in \mathbb{C}_\pm \,. \tag{4.98}$$

As we already know, the condition (4.98) means that the corresponding ZS system has no discrete eigenvalues.

It is only natural that we apply the same idea, like in Sect. 4.3.3, only now the dressing factor should depend parametrically on x. Indeed, we rewrite the formulae (4.49) in the form:

$$\eta^+(x, \lambda) = u_1(x, \lambda)\eta_0^+(x, \lambda) \,, \tag{4.99a}$$

$$u_1(x,\lambda) = \mathbb{1} + (c_1(\lambda) - 1)P_1(x), \qquad c_1(\lambda) = \frac{\lambda - \lambda_1^+}{\lambda - \lambda_1^-}\,, \tag{4.99b}$$

$$\eta^-(x,\lambda) = \frac{1}{c_1(\lambda)} u_1(x,\lambda)\eta_0^-(x,\lambda)\,, \tag{4.99c}$$

where $P_1(x)$ is a projector, like in (4.91a), with conveniently chosen right and left eigenvectors $|n_1(x)\rangle$ and $\langle m_1(x)|$. Our aim here will be to find out how these eigenvectors depend on x and how they are determined by the scattering data of L.

First of all, let us take the determinant of both sides of (4.99a) and (4.99c) remembering that $\det \eta^\pm(x,\lambda) = a^\pm(\lambda)$ and $\det \eta_0^\pm(x,\lambda) = a_0^\pm(\lambda)$, where $a_0^\pm(\lambda)$ are related to the regular RHP. Thus we find:

$$a^+(\lambda) = c_1(\lambda)a_0^+(\lambda), \qquad a^-(\lambda) = \frac{a_0^-(\lambda)}{c_1(\lambda)}\,. \tag{4.100}$$

Since $a_0^\pm(\lambda)$ are regular (i.e. have no zeroes in $\lambda \in \mathbb{C}_\pm$, respectively) then from (4.100) we get that $a^+(\lambda)$ and $a^-(\lambda)$ have simple zeroes at $\lambda = \lambda_1^+$ and $\lambda = \lambda_1^-$, respectively. This explains why we introduced the additional factor $\frac{1}{c_1(\lambda)}$ in (4.99c). With it, the functions $\eta^\pm(x,\lambda)$ satisfy an RHP

$$\eta^+(x,\lambda) = \eta^-(x,\lambda)G_1(x,\lambda), \qquad G_1(x,\lambda) = G_0(x,\lambda)c_1(\lambda)\,, \tag{4.101}$$

with slightly different sewing function. The new sewing function $G_1(x,\lambda)$ is like the old one $G_0(x,\lambda)$, due to the fact that $\operatorname{Im}\lambda_1^\pm \neq 0$ has no singularities on the real axis.

In order to determine $|n_1(x)\rangle$ and $\langle m_1(x)|$, we remind that $\eta^\pm(x,\lambda)$ have singularities at the points $\lambda = \lambda_1^\pm$. Indeed from the definition of the FAS (3.24) and their inverse (3.32), and from the structure of their degeneracies at the points of the discrete spectrum (3.43) and (3.44), we find:

$$\chi^+(x,\lambda_1^+) = \psi_1^+(x)(b_1^+,1)\,, \tag{4.102a}$$

$$\operatorname*{Res}_{\lambda=\lambda_1^-} \hat{\chi}^-(x,\lambda) = -\frac{1}{\dot{a}_1^-}\begin{pmatrix} b_1^- \\ 1 \end{pmatrix}\tilde{\psi}_1^-(x)\,, \tag{4.102b}$$

which means that

$$\chi^+(x,\lambda_1^+)|n_1\rangle = 0, \qquad |n_1\rangle = \begin{pmatrix} 1 \\ -b_1^+ \end{pmatrix} \tag{4.103a}$$

$$\operatorname*{Res}_{\lambda=\lambda_1^-} \langle m_1|\hat{\chi}^-(x,\lambda) = 0, \qquad \langle m_1| = (1, -b_1^-)\,, \tag{4.103b}$$

where the constants $b_1^\pm = C_1^\pm \dot{a}_1^\pm$ are related to the data characterizing the discrete eigenvalues $\lambda_1^\pm$.

Let us now insert (4.99) into (4.103):

$$(\mathbb{1} - P_1(x))\chi_0^+(x,\lambda_1^+)|n_1\rangle = 0 \,, \tag{4.104a}$$
$$\langle m_1|\hat{\chi}_0^-(x,\lambda_1^-)(\mathbb{1} - P_1(x)) = 0 \,, \tag{4.104b}$$
$$\chi_0^\pm(x,\lambda) = \eta_0^\pm(x,\lambda)e^{i\lambda\sigma_3 x} \tag{4.104c}$$

and compare with the general definition of $P_1(x)$ (4.91a). This gives

$$|n_1(x)\rangle = \chi_0^+(x,\lambda_1^+)|n_1\rangle = \eta_0^=(x,\lambda_1^+)e^{i\lambda_1^+\sigma_3 x}\begin{pmatrix} 1 \\ -b_1^+ \end{pmatrix} \,, \tag{4.105a}$$
$$\langle m_1(x)| = \langle m_1|\hat{\chi}_0^-(x,\lambda_1^-) = (1, -b_1^-)e^{-i\lambda_1^-\sigma_3 x}\hat{\eta}_0^-(x,\lambda_1^-) \,. \tag{4.105b}$$

These formulae make the dressing procedure effective. Indeed, $|n_1\rangle$ and $\langle m_1|$ are determined by the scattering data of L and the x-dependence of $|n_1(x)\rangle$, and $\langle m_1(x)|$ is determined by the solutions of the regular RHP, which are supposed to be known. Then (4.105) and (4.99) provide us with the new singular solutions $\eta^+(x,\lambda)$ and $\eta^-(x,\lambda)$ of the RHP, whose singularities are located at the points $\lambda_1^\pm$.

Let us display also another way of determining the x-dependence of $|n_1(x)\rangle$ and $\langle m_1(x)|$. Now, we shall use the fact that $\eta_0^\pm(x,\lambda)$ are related to the ZS with a known potential $q_0(x)$:

$$i\frac{d\eta_0^\pm}{dx} + q_0(x)\eta_0^\pm(x,\lambda) - \lambda[\sigma_3, \eta_0^\pm(x,\lambda)] = 0 \,, \tag{4.106a}$$

while $\eta^\pm(x,\lambda)$ is related to the ZS:

$$i\frac{d\eta^\pm}{dx} + q(x)\eta^\pm(x,\lambda) - \lambda[\sigma_3, \eta^\pm(x,\lambda)] = 0 \,, \tag{4.106b}$$

with a potential $q(x)$ which is to be found. Then, from (4.99a) and (4.106), we find that the dressing factor $u(x,\lambda)$ must satisfy the equation:

$$i\frac{du}{dx} + q(x)u(x,\lambda) - u(x,\lambda)q_0(x) - \lambda[\sigma_3, u(x,\lambda)] = 0 \,. \tag{4.107}$$

The anzats for $u(x,\lambda)$ (4.99b) must be compatible with (4.107), i.e. (4.107) must hold identically with respect to λ. This can be achieved by requiring the limit for $\lambda \to \infty$ of the left-hand side of (4.107) to vanish; its residue at $\lambda = \lambda_1^-$ vanishes too. Easy calculation shows that the first condition leads to the following relation between $q(x)$ and $q_0(x)$:

$$q(x) - q_0(x) = -(\lambda_1^+ - \lambda_1^-)[\sigma_3, P_1(x)] \,. \tag{4.108}$$

The second condition gives us the following nonlinear equation for $P_1(x)$:

$$\begin{aligned} i\frac{dP_1}{dx} &+ q(x)P_1(x) - P_1(x)q_0(x) - \lambda_1^+\sigma_3 P_1(x) + \lambda_1^- P_1(x)\sigma_3 \\ &+ (\lambda_1^+ - \lambda_1^-)P_1(x)\sigma_3 P_1(x) = 0, \end{aligned} \tag{4.109}$$

Note that $P_1(x)$ satisfies (4.109) identically, provided the eigenvectors $|n_1(x)\rangle$ and $\langle m_1(x)|$ are solutions of the equations:

$$L_1^+|n_1(x)\rangle \equiv i\frac{d|n_1\rangle}{dx} + (q_0(x) - \lambda_1^+\sigma_3)|n_1(x)\rangle = 0 \; , \qquad (4.110a)$$

$$\langle m_1(x)|\hat{L}_1^- \equiv i\frac{d\langle m_1|}{dx} - \langle m_1(x)|(q_0(x) - \lambda_1^-\sigma_3) = 0 \; . \qquad (4.110b)$$

The solutions to (4.110) are given by (4.105) so, as should be expected, both methods give compatible results. Thus, we can construct singular solutions to the RHP, provided we know explicitly the regular solutions $\eta_0^\pm(x,\lambda)$ to the RHP. The simplest possible choice for $\eta_0^\pm(x,\lambda)$ is the trivial one

$$\eta_0^\pm(x,\lambda) = \mathbb{1} \; , \qquad (4.111)$$

which corresponds to $q_0(x) = 0$. Simple comparison between (4.91) and (4.99), (4.105) shows that the dressing method in this case leads to the same reflectionless potential as the GLM equation.

The dressing procedure can be repeated subsequently, which allows one to construct solutions of the RH problem with arbitrary number of pairs of singularities. Besides this “limitation” the anzats (4.99b) requires that $\lambda_k^\pm \neq \lambda_j^\pm$ for $k \neq j$. This will become evident below. Here, we note that all known examples of reflectionless potentials, tending sufficiently fast to zero when $|x| \to \infty$ satisfy this condition. The attempt to construct a solution with odd number of singularities using this scheme, or with singularities lying on the real axis, lead to potentials, violating the boundary condition (i.e., not tending fast enough to 0 for $|x| \to \infty$); see [1].

Now, let us show how we can add N pairs of eigenvalues to the spectrum of L. The corresponding singular solutions to the RHP are related to the regular ones $\eta_0^\pm(x,\lambda)$ by:

$$\eta^+(x,\lambda) = u_N(x,\lambda)\cdots u_2(x,\lambda)u_1(x,\lambda)\eta_0^+(x,\lambda) \; , \qquad (4.112a)$$

$$\eta^-(x,\lambda) = c(\lambda)u_N(x,\lambda)\cdots u_2(x,\lambda)u_1(x,\lambda)\eta_0^-(x,\lambda) \; , \qquad (4.112b)$$

$$u_k(x,\lambda) = \mathbb{1} + (c_k(\lambda) - 1)P_k(x) \; , \qquad (4.112c)$$

$$c_k(\lambda) = \frac{\lambda - \lambda_k^+}{\lambda - \lambda_k^-}, \qquad c(\lambda) = \prod_{k=1}^N c_k(\lambda) \; , \qquad (4.112d)$$

$$P_k(x) = \frac{|n_k(x)\rangle\langle m_k(x)|}{\langle m_k(x)|n_k(x)\rangle} \; . \qquad (4.112e)$$

The x-dependence of the vectors $|n_k(x)\rangle$ and $\langle m_k(x)|$:

$$|n_k(x)\rangle = u_{k-1}(x,\lambda_k^+)\cdots u_1(x,\lambda_k^+)\eta_0^+(x,\lambda_k^+)|n_k\rangle \; , \qquad (4.112f)$$

$$\langle m_k(x)| = \langle m_k|\hat{\eta}_0^-(x,\lambda_k^-)\hat{u}_1(x,\lambda_k^-)\cdots \hat{u}_{k-1}(x,\lambda_k^-) \; , \qquad (4.112g)$$

$$|n_k\rangle = \begin{pmatrix} 1 \\ -b_k^+ \end{pmatrix}, \qquad \langle m_0| = (1, -b_k^-) \; , \qquad (4.112h)$$

where the constants $b_k^\pm = C_k^\pm \dot{a}_k^\pm$ specify the data characterizing the discrete spectrum of L. The formulae (4.112f), (4.112g) explain why we need the condition $\lambda_k^\pm \neq \lambda_j^\pm$ for $k \neq j$. Indeed, it is important that the factors $u_j(x, \lambda_k^+)$ and $\hat{u}_j(x, \lambda_k^-)$ with

$$\hat{u}_j(x, \lambda_k^-) = \mathbb{1} + \left(\frac{1}{c_j(\lambda_k^-)} - 1 \right) P_j(x) \tag{4.113}$$

must be regular (invertible) matrices.

This version of the dressing method can be applied to any regular solutions $\eta_0^\pm(x, \lambda)$ and not just to the trivial one (4.111). A drawback is that the dressing factors must be constructed consecutively. In order to write down $u_k(x, \lambda)$ or $P_k(x)$, one needs to know all $u_j(x, \lambda)$ with $j = 1, 2, \ldots, k-1$. Note also that $u_k(x, \lambda)$ and $u_j(x, \lambda)$ do not commute for $k \neq j$. This fact may lead to the wrong conclusion that the final results for $\eta^\pm(x, \lambda)$ depend on the order in which the different pairs of singularities are added.

It is a technical, although cumbersome calculation, which allows us to see that the recurrent procedure in (4.112) for the trivial choice of $\eta_0^\pm(x, \lambda)$ (4.111) gives the same answer as the solution to the system of (4.77), (4.78) (or (4.79), (4.80)). We end this section by the relation:

$$q(x) - q_0(x) = -\sum_{k=1}^{N} (\lambda_k^+ - \lambda_k^-)[\sigma_3, P_k(x)] \ , \tag{4.114}$$

which naturally generalizes (4.108) to $N > 1$ and coincides with (4.85). Since the proof of this fact is highly technical we shall omit it.

4.6 Soliton Interactions

Among the first important results which demonstrated the special properties of the NLEE was the fact that the solitons interact in a purely elastic manner. This was established by calculating the limits of the N-soliton solutions for $t \to -\infty$ and then comparing it with the limit for $t \to \infty$. Assuming that all solitons move with different velocities, one finds that in this limit they become well separated between themselves and do not interact:

$$u(x, t) \underset{t \to -\infty}{\longrightarrow} \sum_{k=1}^{N} u_{1\mathrm{s},k}(x, t; \mu_k, \nu_k, \xi_k^-, \delta_k^-) \ . \tag{4.115}$$

where $u_{1\mathrm{s},k}$ is the 1-soliton solution (4.97) of the NLEE with the soliton parameters μ_k, ν_k, ξ_k^- and δ_k^-. The last two characterize the asymptotic values of the "center" of mass position and of the soliton phase. We also assume that the solitons are ordered in such a way that the leftmost one is the fastest, while the rightmost one is the slowest. In the simplest case, when we have just two solitons the following picture appears.

With time evolution, the faster soliton will overtake the slower one which is in front of it. Since the soliton equations are nonlinear, the sum of two one-soliton solutions *is not a solution* of the NLEE. Due to the nonlinearity the two solitons interact, which means that they may substantially be deformed and overlapped. After the interaction, however, they separate again and recover their initial shape and velocities. Thus, the soliton interaction is a purely elastic process, during which no dissipation of energy occurs. It is more complicated to study the N-soliton solution, but the result for it is analogous. The interaction is again purely elastic and is reduced to a sequence of two-soliton interactions. Thus for $t \to \infty$, one gets the following asymptotics for the solution $u(x,t)$:

$$u(x,t) \underset{t\to\infty}{\longrightarrow} \sum_{k=1}^{N} u_{1\mathrm{s}}(x,t;\mu_k,\nu_k,\xi_k^+,\delta_k^+) \tag{4.116}$$

The above means that again $u(x,t)$ asymptotically becomes a sum of one-soliton solutions, which have the same velocities and amplitudes but are ordered in inverse order: i.e. now the leftmost soliton is the slowest, while the rightmost is the fastest. The other difference between (4.115) and (4.116) consists in the change of the relative center of mass positions, ξ_k^- to ξ_k^+, and the relative phases from δ_k^- to δ_k^+. The pure elastic nature of the N-soliton interactions is due to the infinite set of integrals of motion. Thus, the soliton interaction is such that the nonlinear interactions exactly compensate for the dissipation coming from the linear terms.

This method of studying the N-soliton interactions has the advantage of being exact. Unfortunately, it does not allow one to handle cases when two or more of the solitons have equal velocities. Such solitons do not separate asymptotically, and the limits of such N-soliton solutions for $t \to \pm\infty$ remain very complicated and do not allow to draw any definite conclusions.

This called for an alternative method for analyzing the N-soliton interactions and such was proposed by Karpman, Maslov and Solov'ev [2, 3]. It is not exact, because it is based on the so-called adiabatic approximation, in which the small parameter ϵ is the overlap between the neighboring solitons, and only terms that are of the order of ϵ are taken into account. This "drawback" becomes an advantage, because one is also able to treat nonintegrable NLEE. Indeed, in many physical applications described by soliton equations, one often needs to take into account additional small (perturbational) effects that violate the integrability.

Here, we briefly outline this approach, which is important for many physical applications.

The N-soliton train for the NLS equation (i.e. $f(\lambda) = -2\lambda^2$) is defined as the solution to the NLS equation with the initial condition:

$$u(x,0) = \sum_{k=1}^{N} u_{1\mathrm{s},k}(x,0), \qquad u_{1\mathrm{s},k}(x,t) = \frac{2i\nu_k e^{i\phi_k}}{\cosh z_k}\,, \tag{4.117}$$

$$
\begin{aligned}
z_k &= 2\nu_k(x - \xi_k(t)), \qquad \phi_k(z,t) = \frac{\mu}{\nu} z_k + \delta_k(t) \,, \\
\xi_k(t) &= \mu_k t + \xi_{0,k}, \qquad \delta_k(t) = (\mu_k^2 + \nu_k^2)t + \delta_{k,0},
\end{aligned} \tag{4.118}
$$

In order that the adiabatic approximation holds, they must satisfy a set of constraints:

$$
|\nu_k - \nu_0| \ll \nu_0, \quad |\mu_k - \mu_0| \ll \mu_0, \quad |\nu_k - \nu_0||\xi_{k+1,0} - \xi_{k,0}| \gg 1 \,, \tag{4.119}
$$

where $\nu_0 = \frac{1}{N}\sum_{k=1}^N \nu_k$, and $\mu_0 = \frac{1}{N}\sum_{k=1}^N \mu_k$ are the average amplitude and velocity, respectively. The idea of Karpman, Solov'ev, Maslov's was to derive a dynamical system for the $4N$ parameters that would describe the dynamics of the N-soliton train. They did this for two solitons and derived a dynamical systems for the 8 parameters, which they were able to solve exactly.

Later, in [4, 5, 6], these results were extended for the general case when the number of solitons is $N > 2$. Skipping the details of derivation, we formulate the final result. Let us introduce the complex variables:

$$
q_k(t) = 2i\lambda_0\xi_k(t) + 2k\ln(2\nu_0) + i(k\pi - \delta_k(t) - \delta_0) \,, \tag{4.120}
$$

where $\delta_0 = 1/N\sum_{k=1}^N \delta_k$ and $\lambda_0 = \mu_0 + i\nu_0$. Then the dynamical system mentioned above becomes the so-called complex Toda chain (CTC) [4, 5]:

$$
\frac{\mathrm{d}^2 q_j}{\mathrm{d}t^2} = 4\nu_0^2 \left(e^{q_{j+1} - q_j} - e^{q_j - q_{j-1}} \right), \qquad j = 1, \ldots, N \,. \tag{4.121}
$$

with free-ends conditions, i.e. $e^{-q_0} \equiv e^{q_{N+1}} \equiv 0$.

Thus the problem of determining the evolution of an NLS N-soliton train in this approximation is reduced to solving the CTC for N sites, which like RTC [7], is integrable; see [8, 9, 10, 11].

One should keep in mind that the N-soliton train *is not* an exact N-soliton solution taken at $t = 0$; normally a small portion of its energy ($\sim 1 \div 2\%$) is due to "radiation" coming from the continuous spectrum of L. Another important remark is that the values of the soliton parameters *are not* directly related to the spectral data of L; we will discuss this below. In fact ξ_k, μ_k, ν_k and δ_k taken at $t = 0$ show the center of mass positions, velocities, amplitudes, and phases of the initial N-soliton train

There are obvious similarities between the RTC [8, 9, 10, 11] and CTC [4, 5]:

S1 The CTC Lax representation is the same as for the RTC: $\dot{L} = [B, L]$, where

$$
L = \sum_{k=1}^{N} \left(b_k E_{kk} + a_k (E_{k,k+1} + E_{k+1,k}) \right) \,, \tag{4.122}
$$

Here $a_k = \frac{1}{2} e^{(q_{k+1} - q_k)/2}$ and $b_k = \frac{1}{2}(\mu_k + i\nu_k)$. The matrices $(E_{kn})_{pq} = \delta_{kp}\delta_{nq}$, and $(E_{kn})_{pq} = 0$, whenever p or q becomes 0 or $N+1$.

S2 The integrals of motion in involution are provided by the eigenvalues, ζ_k, of $L_0 = L(t = 0)$ and can be calculated from the initial conditions.
S3 The solutions of both the CTC and the RTC are determined by the scattering data $S_{L_0} = \{\zeta_k, r_k\}_{k=1}^N$ of L_0, where r_k are the first components of the properly normalized eigenvectors of L_0 [10, 11].
S4 The set of eigenvalues ζ_k determine the asymptotic behavior of the solutions of (4.121). This fact will be used to classify the regimes of asymptotic behavior.

Along with them there are also important differences, namely:

D1 While for RTC, q_k, r_k and ζ_k are all real, for CTC they generically take complex values;
D2 While for RTC $\zeta_k \neq \zeta_j$ for $k \neq j$, for CTC no such restriction holds.

As a consequence of (S2), (S4) and (D2), the only possible asymptotic behavior in the RTC is an asymptotically separating, free motion of the solitons. For CTC it is κ_k that determines the asymptotic velocity of the k-th soliton. For simplicity and without loss of generality we assume that: $\operatorname{tr} L_0 = 0$; $\zeta_k \neq \zeta_j$ for $k \neq j$; and $\operatorname{Re}\zeta_1 \leq \operatorname{Re}\zeta_2 \leq \cdots \leq \operatorname{Re}\zeta_N$. Then we have:

- Regime (i): $\operatorname{Re}\zeta_k \neq \operatorname{Re}\zeta_j$ for $k \neq j$, i.e. the asymptotic velocities are all different. Then we have asymptotically separating, free solitons; see also [4, 5]. This is the only dynamical regime possible for RTC;
- Regime (ii): $\operatorname{Re}\zeta_1 = \operatorname{Re}\zeta_2 = \cdots = \operatorname{Re}\zeta_N = 0$, i.e. all N solitons move with the same mean asymptotic velocity and form a "bound state." The key question now will be the nature of the internal motions in such a bound state: Is it quasi-equidistant or not? In [6], the CTC model is used to determine the sets of $4N$ soliton parameters that ensure quasi-equidistant motion; this is very important for soliton-based fiber optics communications.
- Regimes (iii): a variety of intermediate situations when one group (or several groups) of particles move with the same mean asymptotic velocity; then they would form one (or several) bound state(s), and the rest of the particles will have free asymptotic motion.
- Regimes (iv): degenerate regimes when two or more of the eigenvalues ζ_k become equal (e.g., $\zeta_1 = \zeta_2$).

Obviously the regimes (ii) and (iii), as well as the degenerate and singular cases, which we do not consider here (see e.g., [12, 13]), have no analogies in the RTC and physically are qualitatively different from (i).

We have compared the predictions from the CTC model with the numerical solutions of the NLS and verified excellent match for the asymptotic regimes (i), (ii), and (iii). Particularly, the bound states regimes indeed take place and are very well described. Our analytic approach allows us to predict the set of initial parameters, for which each of these asymptotic regimes takes place. We put special stress on the bound state and the quasi-equidistant regimes, since such behavior is desirable in long-range fiber optics communications [4, 6, 14].

Another important issue is the possibility to treat the effects of possible perturbations. Many physical effects in nonlinear optics such as birefringence, or nonlinear gain lead to the necessity to add additional terms to the NLS equations that violate its integrability. The effects of such terms on the N-soliton train dynamics can be calculated; the result is a perturbed CTC which again can be useful [15, 16].

These ideas were applied also to the higher NLS equations [17, 18, 19] and modified NLS equation [20], and again it was shown that a CTC model describes very well the N-soliton train dynamics. In [17, 18], the interrelation between the $4N$ soliton parameters and the scattering data of the corresponding ZS system L was analyzed. In particular, it was shown that the discrete eigenvalues of L are clustered in a circle of radius $\sqrt{\epsilon}$ around λ_0 and are well approximated by the eigenvalues ζ_k of the CTC Lax matrix.

A natural generalization of the CTC describes the N-soliton train interaction also for the Manakov [21, 22] model and other multicomponent NLS equations, which proves the hypothesis proposed in [21] in 1998.

4.7 Comments and Bibliographical Review

1. The inverse scattering problems for the Sturm–Liouville equation (1.5) became important with the development of quantum mechanics in the 1950s. At that time it was important to determine the potentials of the atoms and nuclei from the experimental data of electron scattering. The problem was made simpler by the fact that the Sturm-Liouville operator is a self-adjoint operator and therefore it allows rigorous spectral analysis. As a result the GLM equation has been discovered and analyzed [6, 7], see also [25]. This equation reflects the fact that the Jost solutions of (1.5) are analytic functions of the spectral parameter λ. The Jost solutions of the ZS system also possess analyticity properties, namely their columns $\psi^{\pm}(x,\lambda)$ and $\phi^{\pm}(x,\lambda)$ allow analytic continuation for $\lambda \in \mathbb{C}_{\pm}$, for detailed proofs see [1, 26, 27, 28, 29, 30, 31].

   However, the spectral theory of self-adjoint and unitary operators [32, 33] is not very useful for the analysis of Lax operators. The reason is that as a rule the Lax operators are *neither* self-adjoint *nor* unitary. Therefore our approach, as we mentioned in Sect. 3.8, is not so rigorous and is based on the two conditions **C.1** and **C.2**. These conditions allow us to construct the spectral theory for several classes of Lax operators that are neither self-adjoint nor unitary. The basic tool for this is the notion of the fundamental analytic solution (FAS). While for the ZS system it is easily constructed, for the generalized $sl(n)$ ZS system (1.15) with real $J = \operatorname{diag}(a_1, \ldots, a_n)$, $a_k > a_j$ for $k < j$ the construction is more complicated and involves the Gauss decomposition of the scattering matrix $T(\lambda)$ [34, 35]. The GZS system with complex-valued a_j was considered by Mikhailov [36], Caudrey [37, 38] and Beals and Coifman [39, 40, 41, 42, 43]; the generalization for any simple Lie algebra and complex-valued J is constructed in [44]. For all these cases, one can construct the kernel of the resolvent of L in terms of FAS and then use the contour integration method to derive the completeness relation for the eigenfunctions of L [45]. In fact this amounts to the explicit construction of the spectral decomposition for L.

2. The Riemann–Hilbert problem reduces to a system of singular integral equations. In most cases it has been well studied and shown that RHP with canonical normalization has unique regular solution, see [46, 47, 48, 49].
3. The GLM equation is directly related to the spectral decomposition of L because it can be derived from the completeness relation of the Jost solutions (3.111). The GLM equation is comparatively easy to derive and analyze for systems of second order such as the Sturm–Liouville problem, for the ZS system. The derivation of the GLM equation for the $n \times n$ generalization of the ZS system was done by Zakharov and Manakov in their preprint [50]. The substantial difficulties encountered in the derivation to our opinion are related to the difficulties in constructing the FAS for gZS system.

   The fundamental differences between the well-known GLM for the ZS system and the generalized GLM (gGLM) for the generalized ZS system are as follows. While the GLM is an integral equation of Volterra type, the gGLM is an integral equation of Fredholm type. As a consequence, one can prove that the GLM has unique solution, while for the gGLM one needs to analyze and solve first for the Fredholm alternative. Therefore, even the derivation of the soliton solutions from the gGLM encounters problems.

   A way out of these difficulties was proposed by Shabat [34, 35] who succeeded to reduce the inverse scattering problem for the gZS system to a Riemann–Hilbert for the matrix-valued FAS. The next important step done by Zakharov and Shabat consisted in devising an elegant algebraic method which allowed one starting from a simple (or even trivial) regular solution of the RHP to construct new *singular* solutions of the RHP. This method [27, 51, 52] has several equivalent formulations now, see [53, 54] and allowed one to derive the soliton solutions of a large number of classes of NLEE, related to different types of Lax operators.
4. The dressing method is based on an ansatz for the dressing factor which is natural and well known for the basic examples of gZS system related to the algebra $sl(n)$ with potentials $q(x)$ vanishing fast enough for $x \to \pm\infty$. Imposing on $q(x)$ algebraic reductions (e.g., requesting that $q(x)$ belongs to the simple algebra $\mathfrak{g}$, or imposing on it reduction in the sense of Mikhailov [55]) may substantially change the dressing factors. Such analysis has been done for the N-wave type NLEE related to the simple Lie algebras of low rank with imposed $\mathbb{Z}_2$-reductions on them, see [56, 57, 58] where all inequivalent $\mathbb{Z}_2$-reductions of N-wave NLEE were classified. Recently in [59]

we outlined how the dressing factor can be constructed for the N-wave equations with $\mathbb{Z}_2 \times \mathbb{Z}_2$-reductions. The additional $\mathbb{Z}_2$ reductions makes possible the existence of two types of discrete eigenvalues: doublets which pairs of $\lambda_i^{\pm} = \pm i\rho_i$, ρ_i – real and quadruplets which consist of $\pm\lambda_k^{+}$, $\pm(\lambda_k^{+})^*$ where λ_k^{+} is a generic complex number. As a result, we have to consider two types of solitons here, in analogy with the situation with the s-G equation.

5. The special properties of the soliton interactions attracted the attention of many scientists since the discovery of the ISM. The first classical results in that direction were obtained by studying and comparing the asymptotics of the exact N-soliton solutions for $t \to -\infty$ and $t \to \infty$. These studies revealed the purely elastic interactions of the solitons for the KdV, NLS and s-G equations, for detailed derivation and explanation see the monograph [28]. These results, however, are valid only for the generic case when the solitons move with different velocities; if two or more of the solitons have the same velocity, the asymptotics acquire rather complex form and become uninformative.

   One way out of this situation was to apply numeric methods. However, the number of initial parameters grows quickly with the number of the solitons[1] which makes it very difficult to determine the regions in the soliton parameter space corresponding to a given dynamical regime.

   The method proposed by Karpman, Maslov and Soloviev [2,3] and developed further in [4, 5, 6, 17] became more effective. Using the adiabatic approximation it allowed one to derive a system of dynamical equations for the soliton parameters, which for the NLS equation turned out to be the CTC. The fact that CTC is also completely integrable dynamical system allowed one knowing the initial soliton parameters to predict the asymptotical dynamics of the N-soliton train [4]. It also allowed one to describe the regions in the soliton parameter space responsible for the different dynamical regimes [6].

   By taking into account the effects of different perturbations, one naturally obtains a CTC with perturbation terms. Many scientists have combined this method with numerical tools in order to study the effects of perturbations on specific physical problems; the list of references here necessarily shows only some of them: [4, 6, 18, 28, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83].

6. All the soliton equations mentioned up to now: KdV, NLS, N-wave, mKdV, etc. have wide applications to various physical problems. This is due to the fact that they describe wave–wave interactions which are not sensitive to physical characteristics of the waves. The following references are just a representative sample of the numerous relevant publications: [7, 26, 27, 28, 30, 64, 65, 66, 67, 72, 75, 79, 84, 85, 86, 87, 88, 89, 90, 91,

[1] Ror NLS each soliton is characterized by four parameters.

92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121].

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5

The Generalized Fourier Transforms

The main idea here is to show that the mapping of the potential $q(x)$ of $L(\lambda)$ onto the minimal sets of scattering data $\mathcal{T}_i$, $i = 1, 2$ is one-to-one. Similarly, we analyze the mappings from the variations $\delta q(x)$ onto $\delta\mathcal{T}_i$. The basic tools for doing that are the Wronskian relations (Sect. 5.1), which allow one to express the elements of $\mathcal{T}_i$ and $\delta\mathcal{T}_i$ as Fourier-like integrals, whose integrands are products of $q(x)$ (or $\delta q(x)$) with the squared solutions. In Sect. 5.2, we introduce three sets of squared solutions: $\{\boldsymbol{\Psi}^{\pm}(x,\lambda)\}$, $\{\boldsymbol{\Phi}^{\pm}(x,\lambda)\}$ and the symplectic basis $\{\mathbf{P}(x,\lambda), \mathbf{Q}(x,\lambda)\}$. The squared solutions are constructed explicitly through the FAS $\chi^{\pm}(x,\lambda)$, which ensures their analyticity properties. This makes it possible, by applying the contour integration method to a properly chosen Green function, to prove that they are complete sets of functions in the space of allowed potentials.

Since the sets of squared solutions are complete, we can expand any function, including $q(x)$ and $\sigma_3\delta q(x)$ over each of the sets $\{\boldsymbol{\Psi}^{\pm}(x,\lambda)\}$, $\{\boldsymbol{\Phi}^{\pm}(x,\lambda)\}$ and $\{\mathbf{P}(x,\lambda), \mathbf{Q}(x,\lambda)\}$. Doing this in Sect. 5.3, we find that the elements of $\mathcal{T}_i$ and $\delta\mathcal{T}_i$ can be viewed as expansion coefficients of these expansions.

In the next two Sects., 5.4, 5.5, we introduce the recursion operators $\Lambda_{\pm}, \Lambda$ for which $\{\boldsymbol{\Psi}^{\pm}(x,\lambda)\}$, $\{\boldsymbol{\Phi}^{\pm}(x,\lambda)\}$ and $\{\mathbf{P}(x,\lambda), \mathbf{Q}(x,\lambda)\}$ are sets of eigenfunctions. Therefore, the completeness relations of the squared solutions can be viewed as spectral decompositions of the recursion operators. We also derive the biorthogonality relations between the squared solutions which allows us to obtain an integral representation for the Green functions of the recursion operators.

In the last section of this chapter, we derive the generalized Wronskian relations that relate the scattering data and the potentials of two Zakharov–Shabat systems with potentials $q(x)$ and $q'(x)$. Thus, we find two sets $\{\boldsymbol{\Psi}'^{,\pm}(x,\lambda)\}$ and $\{'\boldsymbol{\Phi}^{\pm}(x,\lambda)\}$ of "products of solutions" of the two ZS systems and prove that they are also complete sets of functions. The expansions of $q(x) + q'(x)$ and $\sigma_3(q'(x) - q(x))$ are derived. The explicit form of the operators $\Lambda'_{\pm}$, for which the "products of solutions" are eigenfunctions are constructed.

Gerdjikov, V.S. et al.: *The Generalized Fourier Transforms*. Lect. Notes Phys. **748**, 133–173 (2008)
DOI 10.1007/978-3-540-77054-1_5

5.1 The Wronskian Relations

The analysis of the mapping $\mathcal{F}\colon \mathcal{M} \to \mathcal{T}$ between the class of the allowed potentials $\mathcal{M}$ and the scattering data of L starts with the so-called Wronskian relations. As we shall see, they would allow us to

1. formulate the idea that the ISM is a generalized Fourier transforms (GFT);
2. determine explicitly the proper generalizations of the usual exponents;
3. introduce the skew-scalar product on $\mathcal{M}$, which endows it with a symplectic structure.

All these ideas will be worked out for the Zakharov–Shabat system:

$$L\psi \equiv i\frac{d\psi}{dx} + U(x,t,\lambda)\psi(x,t,\lambda) = 0 \tag{5.1a}$$

for which

$$U(x,t,\lambda) = q(x,t) - \lambda\sigma_3 \tag{5.1b}$$

At the end, we shall discuss various generalizations of this system.

First of all, we remark that with (5.1) one can associate the following systems:

$$i\frac{d\hat{\psi}}{dx} - \hat{\psi}(x,t,\lambda)U(x,t,\lambda) = 0 \tag{5.2}$$

$$i\frac{d\delta\psi}{dx} + \delta U(x,t,\lambda)\psi(x,t,\lambda) + U(x,t,\lambda)\delta\psi(x,t,\lambda) = 0 \tag{5.3}$$

$$i\frac{d\dot{\psi}}{dx} + \dot{U}(x,t,\lambda)\psi(x,t,\lambda) + U(x,t,\lambda)\dot{\psi}(x,t,\lambda) = 0\ , \tag{5.4}$$

where $\delta\psi$ corresponds to a given variation $\delta q(x,t)$ of the potential, while by dot, we denote the derivative with respect to the spectral parameter; for example, $\dot{U}(x,t,\lambda) = -\sigma_3$, and as before by $\hat{\psi}$ we denote ψ^{-1}.

5.1.1 The Mapping from q to $\mathcal{T}$

We start with the identity:

$$\begin{aligned}
(\hat{\chi}\sigma_3\chi(x,\lambda) - \sigma_3)\big|_{-\infty}^{\infty} &= -i\int_{-\infty}^{\infty} dx\, \frac{d}{dx}\left(i\hat{\chi}\sigma_3\chi\right)(x,\lambda) \\
&= -i\int_{-\infty}^{\infty} dx\, \hat{\chi}[U(x,\lambda),\sigma_3]\chi(x,\lambda)\ , \\
&= -i\int_{-\infty}^{\infty} dx\, \hat{\chi}[q(x),\sigma_3]\chi(x,\lambda)\ ,
\end{aligned} \tag{5.5}$$

where $\chi(x,\lambda)$ can be any fundamental solution of L. For convenience, we choose them to be the FAS introduced earlier.

The left-hand side of (5.5) can be calculated explicitly by using the asymptotics of $\chi^{\pm}(x,\lambda)$ for $x \to \pm\infty$. It would be expressed by the matrix elements of the scattering matrix $T(\lambda)$, i.e. by the scattering data of L as follows:

$$\left(\hat{\chi}^{+}\sigma_3\chi^{+}(x,\lambda) - \sigma_3\right)\Big|_{-\infty}^{\infty} = -2\begin{pmatrix} 0 & b^{-}(\lambda) \\ b^{+}(\lambda) & 0 \end{pmatrix} \tag{5.6}$$

$$\left(\hat{\chi}^{-}\sigma_3\chi^{-}(x,\lambda) - \sigma_3\right)\Big|_{-\infty}^{\infty} = -2\begin{pmatrix} 0 & b^{-}(\lambda) \\ b^{+}(\lambda) & 0 \end{pmatrix} \tag{5.7}$$

We have seen that there exist two independent sets of scattering data $\mathcal{T}_1$ and $\mathcal{T}_2$, which contain the two sets of reflection coefficients:

$$\rho^{\pm}(\lambda) = \frac{b^{\pm}(\lambda)}{a^{\pm}(\lambda)}, \qquad \tau^{\pm}(\lambda) = \frac{b^{\mp}(\lambda)}{a^{\pm}(\lambda)}, \qquad \lambda \in \mathbb{R} \tag{5.8}$$

We shall show that these Wronskian relations allow us to express each of the reflection coefficients $\rho^{\pm}(\lambda)$ and $\tau^{\pm}(\lambda)$ as integrals of the potential $q(x)$ multiplied by the some bilinear combination of eigenfunctions of L. Indeed, let us multiply both sides of (5.5) by σ_+, take the trace, and divide by $a^{+}(\lambda)$. Fixing up $\chi \equiv \chi^{+}(x,\lambda)$ we find:

$$\begin{aligned} \rho^{+}(\lambda) &= -\frac{i}{2a^{+}(\lambda)}\int_{-\infty}^{\infty} \operatorname{tr}\left(\hat{\chi}^{+}[q(x),\sigma_3]\chi^{+}(x,\lambda)\sigma_+\right) \\ &= -\frac{i}{2a^{+}(\lambda)}\int_{-\infty}^{\infty} \operatorname{tr}\left([q(x),\sigma_3]\chi^{+}(x,\lambda)\sigma_+\hat{\chi}^{+}(x,\lambda)\right) \\ &= -\frac{i}{(a^{+}(\lambda))^2}\left[\!\left[q(x), \mathcal{E}_{+}^{+}(x,\lambda)\right]\!\right], \end{aligned} \tag{5.9}$$

where

$$\mathcal{E}_{+}^{+}(x,\lambda) = \chi^{+}(x,\lambda)\sigma_+\hat{\chi}^{+}(x,\lambda), \tag{5.10}$$

and

$$[\![X,Y]\!] \equiv \frac{1}{2}\int_{-\infty}^{\infty} dx \operatorname{tr}\left(X(x),[\sigma_3,Y(x)]\right) = -[\![Y,X]\!]. \tag{5.11}$$

As we see, the skew-symmetric scalar product $[\![\cdot,\cdot]\!]$ appeared in a natural way. It is immediately checked that it depends only on the off-diagonal parts of the matrices X and Y. Therefore, in the right-hand side of (5.9) only the off-diagonal part of $\mathcal{E}^{+}(x,\lambda)$ will contribute. Using the explicit form of the FAS $\chi^{+}(x,\lambda)$ and its inverse, we find:

$$\mathcal{E}^{+}(x,\lambda) = \frac{1}{a^{+}(\lambda)}\begin{pmatrix} -\phi_1^{+}(x,\lambda)\phi_2^{+}(x,\lambda) & (\phi_1^{+}(x,\lambda))^2 \\ -(\phi_2^{+}(x,\lambda))^2 & \phi_1^{+}(x,\lambda)\phi_2^{+}(x,\lambda) \end{pmatrix}. \tag{5.12}$$

Thus we can rewrite one of the Wronskian relations (5.5) in the form:

$$\begin{aligned} \rho^+(\lambda) &= \frac{i}{2a^+(\lambda)} \int_{-\infty}^{\infty} dx \operatorname{tr}\left(q(x)[\sigma_3, \mathit{\Phi}^+(x,\lambda)]\right) \\ &= \frac{i}{(a^+(\lambda))^2} [\![q(x), \boldsymbol{\Phi}^+(x,\lambda)]\!] \ , \end{aligned} \tag{5.13}$$

where we introduced the "squared" solutions of L:

$$\boldsymbol{\Phi}^+(x,\lambda) = a^+(\lambda)\mathcal{E}_+^+(x,\lambda)^{\mathrm{f}} = \begin{pmatrix} 0 & (\phi_1^+(x,\lambda))^2 \\ -(\phi_2^+(x,\lambda))^2 & 0 \end{pmatrix} . \tag{5.14}$$

Now (5.13) really shows us that the mapping $\mathcal{F}$ has the sense of GFT. Indeed, the reflection coefficient $\rho^+(\lambda)$ is represented as an integral, whose integrand contains the potential $q(x)$ and the "squared" solution $\mathit{\Phi}^+(x,\lambda)$ or rather $\boldsymbol{\Phi}^+(x,\lambda)$. In the limit, when $q(x) \simeq 0$ the FAS $\chi^+(x,\lambda) \simeq e^{-i\lambda\sigma_3 x}$ and $\boldsymbol{\Phi}^+(x,\lambda) \simeq e^{-2i\lambda x}\sigma_+$. As a result, in this limit we recover from (5.13) the well-known Born approximation, namely that

$$\begin{aligned} \rho^+_{\mathrm{Born}} &= i \int_{-\infty}^{\infty} dx \operatorname{tr}\left(q(x)[\sigma_3, \sigma_+]\right) e^{-2i\lambda x} \\ &= 2i \int_{-\infty}^{\infty} dx\, q^-(x) e^{-2i\lambda x} \ . \end{aligned} \tag{5.15}$$

In this limit, the mapping $\mathcal{F}$ goes into the standard Fourier transform. Our aim is to show that this interpretation holds true for any potential $q(x) \in \mathcal{M}$. We shall return once again to this point at the end of this chapter.

To achieve our aim, we need additional formulae, which can be derived from (5.5). Acting in an analogous way as before we find:

$$\rho^\pm(\lambda) = \frac{i}{(a^\pm(\lambda))^2} [\![q(x), \boldsymbol{\Phi}^\pm(x,\lambda)]\!] \ , \tag{5.16}$$

and

$$\tau^\pm(\lambda) = \frac{i}{(a^\pm(\lambda))^2} [\![q(x), \boldsymbol{\Psi}^\pm(x,\lambda)]\!] \ , \tag{5.17}$$

where the "squared" solutions $\boldsymbol{\Phi}^\pm(x,\lambda)$ and $\boldsymbol{\Psi}^\pm(x,\lambda)$ are defined by:

$$\boldsymbol{\Phi}^\pm(x,\lambda) = a^\pm(\mathcal{E}_\pm^\pm(x,\lambda))^{\mathrm{f}} = \begin{pmatrix} 0 & \pm(\phi_1^\pm(x,\lambda))^2 \\ \mp(\phi_2^\pm(x,\lambda))^2 & 0 \end{pmatrix} \tag{5.18}$$

$$\boldsymbol{\Psi}^\pm(x,\lambda) = a^\pm(\mathcal{E}_\mp^\pm(x,\lambda))^{\mathrm{f}} = \begin{pmatrix} 0 & \mp(\psi_1^\pm(x,\lambda))^2 \\ \pm(\psi_2^\pm(x,\lambda))^2 & 0 \end{pmatrix} \tag{5.19}$$

These "squared" solutions effectively coincide with the ones that appeared originally in [1]. We keep this form with the zeroes on the diagonal for later purposes, when we pose analogous problems for the gauge-equivalent system $\tilde{L}$.

5.1.2 The Mapping from δq to $\delta\mathcal{T}$

The second type of Wronskian relations relates the variation of the potential $\delta q(x)$ to the corresponding variations of the scattering data. To this purpose, we start with the identity:

$$\hat{\chi}\delta\chi(x,\lambda)\big|_{-\infty}^{\infty} = -i\int_{-\infty}^{\infty} dx\, \frac{d}{dx}\left(i\hat{\chi}\delta\chi\right)(x,\lambda) \tag{5.20}$$

To calculate the integrand in (5.20), we need to use the (5.3), satisfied by $\frac{d}{dx}\delta\chi(x,\lambda)$. This, we find by taking the variation of the Zakharov–Shabat system. Inserting (5.1b) into (5.3) we find:

$$i\frac{d}{dx}\delta\chi(x,\lambda) + (q(x) - \lambda\sigma_3)\delta\chi(x,\lambda) + \delta q(x)\chi(x,\lambda) = 0\ . \tag{5.21}$$

Using this we easily find:

$$\hat{\chi}\delta\chi(x,\lambda)\big|_{-\infty}^{\infty} = i\int_{-\infty}^{\infty} dx\, \hat{\chi}\delta q(x)\chi(x,\lambda)\ . \tag{5.22}$$

We apply again the same ideas as in the previous subsection. Evaluating the left-hand side of (5.22) with $\chi(x,\lambda) \equiv \chi^{+}(x,\lambda)$ and $\chi(x,\lambda) \equiv \chi^{-}(x,\lambda)$ we find:

$$\hat{\chi}^{+}\delta\chi^{+}(x,\lambda)\big|_{-\infty}^{\infty} = \begin{pmatrix} \delta\ln a^{+}(\lambda) & -a^{+}(\lambda)\delta\tau^{+}(\lambda) \\ a^{+}(\lambda)\delta\rho^{+}(\lambda) & -\delta\ln a^{+}(\lambda) \end{pmatrix}, \tag{5.23}$$

and

$$\hat{\chi}^{-}\delta\chi^{-}(x,\lambda)\big|_{-\infty}^{\infty} = \begin{pmatrix} -\delta\ln a^{-}(\lambda) & -a^{-}(\lambda)\delta\rho^{-}(\lambda) \\ a^{-}(\lambda)\delta\tau^{-}(\lambda) & \delta\ln a^{-}(\lambda) \end{pmatrix}. \tag{5.24}$$

Next, multiplying by $\sigma_{\pm}$ and taking the trace we arrive to:

$$\delta\rho^{\pm}(\lambda) = \mp\frac{i}{2(a^{\pm}(\lambda))^2}\left[\!\left[[\sigma_3, \delta q(x)], \boldsymbol{\Phi}^{\pm}(x,\lambda)\right]\!\right] \tag{5.25}$$

$$\delta\tau^{\pm}(\lambda) = \pm\frac{i}{2(a^{\pm}(\lambda))^2}\left[\!\left[[\sigma_3, \delta q(x)], \boldsymbol{\Psi}^{\pm}(x,\lambda)\right]\!\right] \tag{5.26}$$

and

$$\delta\mathcal{A}(\lambda) = -\frac{i}{4a^{\pm}(\lambda)}\left[\!\left[[\sigma_3, \delta q(x)], \boldsymbol{\Theta}^{\pm}(x,\lambda)\right]\!\right]\ . \tag{5.27}$$

Here $\boldsymbol{\Psi}^{\pm}(x,\lambda)$ and $\boldsymbol{\Phi}^{\pm}(x,\lambda)$ are the same "squared" solutions as in (5.18), (5.19),

$$\boldsymbol{\Theta}^{\pm}(x,\lambda) = a^{+}(\lambda)(\chi^{\pm}(x,\lambda)\sigma_3\hat{\chi}^{\pm}(x,\lambda))^{\mathrm{f}}$$

$$= \begin{pmatrix} 0 & -2(\phi_1^{\pm}\psi_1^{\pm})(x,\lambda) \\ 2(\phi_2^{\pm}\psi_2^{\pm})(x,\lambda) & 0 \end{pmatrix} , \tag{5.28}$$

and $\mathcal{A}(\lambda)$ is introduced in (3.67).

These relations are basic in the analysis of the NLEE related to the Zakharov–Shabat system and their Hamiltonian structures. We shall use them later, assuming that the variation of $q(x)$ is due to its time evolution. In this case, $q(x,t)$ depends on t in such a way that it satisfies certain NLEE. Then we consider variations of the type:

$$\delta q(x,t) = \frac{\partial q}{\partial t}\delta t + \mathcal{O}((\delta t)^2) . \tag{5.29}$$

Keeping only the first-order terms with respect to δt we find:

$$\rho_t^{\pm}(\lambda) = \mp \frac{i}{2(a^{\pm}(\lambda))^2} [\![[\sigma_3, q_t(x)], \boldsymbol{\Phi}^{\pm}(x,\lambda)]\!] \tag{5.30}$$

$$\tau_t^{\pm}(\lambda) = \pm \frac{i}{2(a^{\pm}(\lambda))^2} [\![[\sigma_3, q_t(x)], \boldsymbol{\Psi}^{\pm}(x,\lambda)]\!] . \tag{5.31}$$

We postpone the application of these relations until later.

5.1.3 Still More Wronskian Relations

The third type of Wronskian relations, which allows us to treat the conserved densities of the integrals of motion are of the form:

$$\begin{aligned} (\hat{\chi}\dot{\chi}(x,\lambda) + ix\sigma_3)|_{-\infty}^{\infty} &= i\int_{-\infty}^{\infty} dx\, (\hat{\chi}(x,\lambda)\dot{U}(x,\lambda)\chi(x,\lambda) + \sigma_3) \\ &= -i\int_{-\infty}^{\infty} dx\, (\hat{\chi}\sigma_3\chi(x,\lambda) - \sigma_3) , \end{aligned} \tag{5.32}$$

where we remind that “dot” means derivative with respect to λ.

The left-hand sides of (5.32) for $\chi(x,\lambda) = \chi^{\pm}(x,\lambda)$ are given by:

$$\left(\frac{1}{2}\operatorname{tr} \hat{\chi}\dot{\chi}(x,\lambda)\sigma_3 + ix \right)\Big|_{-\infty}^{\infty} = \pm \frac{\dot{a}^{\pm}}{a^{\pm}(\lambda)} = \frac{d\mathcal{A}}{d\lambda} , \tag{5.33}$$

where $\mathcal{A}(\lambda) = \pm \ln a^{\pm}(\lambda)$ for $\lambda \in \mathbb{C}_{\pm}$ was introduced in (3.67).

Next, we multiply the integrand in the right-hand side of (5.32) by $\frac{1}{2}\sigma_3$ and take the trace. It can be rearranged as follows:

$$\begin{aligned} \frac{1}{2}\operatorname{tr}\left(\hat{\chi}^{\pm}\sigma_3\chi^{\pm}(x,\lambda)\sigma_3\right) - 1 &= \frac{1}{2}\int_{\pm\infty}^{x} dy\, \frac{d}{dy}\left(\operatorname{tr}\left(\hat{\chi}^{\pm}\sigma_3\chi^{\pm}(y,\lambda)\sigma_3\right)\right) \\ &= -\frac{i}{2}\int_{\pm\infty}^{x} dy\, \operatorname{tr}\left(\hat{\chi}^{\pm}[U(y,\lambda),\sigma_3]\chi^{\pm}(y,\lambda)\sigma_3\right) \end{aligned}$$

$$= -\frac{i}{2}\int_{\pm\infty}^{x} dy \,\mathrm{tr}\,\left([q(y),\sigma_3]\chi^{\pm}\sigma_3\hat{\chi}^{\pm}(y,\lambda)\right)$$
$$= -\frac{i}{2a^{\pm}(\lambda)}\int_{\pm\infty}^{x} dy \,\mathrm{tr}\,\left([q(y),\sigma_3]\boldsymbol{\Theta}^{\pm}(y,\lambda)\right) \ , \tag{5.34}$$

which means that

$$\frac{d\mathcal{A}(\lambda)}{d\lambda} = -i\int_{-\infty}^{\infty} dx \left(\frac{1}{2}\mathrm{tr}\,\left(\hat{\chi}^{\pm}\sigma_3\chi^{\pm}(x,t,\lambda)\sigma_3\right) - 1\right)$$
$$= -\frac{1}{2a^{\pm}(\lambda)}\int_{-\infty}^{\infty} dx \int_{\pm\infty}^{x} dy \,\mathrm{tr}\,\left([q(y),\sigma_3]\boldsymbol{\Theta}^{\pm}(y,\lambda)\right) \ . \tag{5.35}$$

5.2 Completeness of the "Squared" Solutions

The important fact, which follows from the Wronskian relations is that the transition from the potential $q(x)$ to the scattering data $\mathcal{T}_k$, $k = 1, 2$ is closely related to the expansions over the "squared" solutions.

In this section, we shall prove their basic property – their completeness, which underlies the invertibility of the maps $q(x) \to \mathcal{T}_k$, $k = 1, 2$, and the uniqueness of the solution of the inverse scattering problem.

The proof is based again on the contour integration method, this time applied to the following Green function:

$$G(x,y,\lambda) = \begin{cases} G^{+}(x,y,\lambda), & \text{for } \lambda \in \mathbb{C}_+ \ , \\ 1/2(G^{+}(x,y,\lambda) + G^{-}(x,y,\lambda)), & \text{for } \lambda \in \mathbb{R} \ , \\ G^{-}(x,y,\lambda), & \text{for } \lambda \in \mathbb{C}_- \ , \end{cases} \tag{5.36}$$

where

$$G^{\pm}(x,y,\lambda) = G_1^{\pm}(x,y,\lambda)\theta(x-y) - G_2^{\pm}(x,y,\lambda)\theta(y-x) \ , \tag{5.37}$$

$$G_1^{\pm}(x,y,\lambda) = \frac{1}{(a^{\pm}(\lambda))^2}\boldsymbol{\Psi}^{\pm}(x,\lambda)\otimes\boldsymbol{\Phi}^{\pm}(y,\lambda) \ , \tag{5.38}$$

$$G_2^{\pm}(x,y,\lambda) = \frac{1}{(a^{\pm}(\lambda))^2}\left(\boldsymbol{\Phi}^{\pm}(x,\lambda)\otimes\boldsymbol{\Psi}^{\pm}(y,\lambda) + \frac{1}{2}\boldsymbol{\Theta}^{\pm}(x,\lambda)\otimes\boldsymbol{\Theta}^{\pm}(y,\lambda)\right) \ . \tag{5.39}$$

Now we consider the integral

$$\mathcal{J}_G(x,y) = \frac{1}{2\pi i}\left(\oint_{C_+} d\lambda\, G^{+}(x,y,\lambda) - \oint_{C_-} d\lambda\, G^{-}(x,y,\lambda)\right)$$
$$= \sum_{k=1}^{N}\left(\operatorname*{Res}_{\lambda=\lambda_k^+} G^{+}(x,y,\lambda) + \operatorname*{Res}_{\lambda=\lambda_k^-} G^{-}(x,y,\lambda)\right) \ . \tag{5.40}$$

Obviously the poles of $G^{\pm}$ coincide with $\lambda_k^{\pm}$; if $a^{\pm}(\lambda)$ have first-order zeroes at $\lambda_k^{\pm}$, then $G^{\pm}$ would have second-order poles at these points.

Theorem 5.1. *For real* λ

$$G_1^{+}(x,y,\lambda) + G_2^{+}(x,y,\lambda) = G_1^{-}(x,y,\lambda) + G_2^{-}(x,y,\lambda)\ . \tag{5.41}$$

Proof. From (5.37) and (5.38), (5.39) we find:

$$\begin{aligned} &G_1^{\pm}(x,y,\lambda) + G_2^{\pm}(x,y,\lambda) + \frac{1}{2}\mathbb{1}\otimes\mathbb{1} \\ &\quad = \left(\chi^{\pm}(x,\lambda)\otimes\chi^{\pm}(y,\lambda)\right) \varPi \left(\hat{\chi}^{\pm}(x,\lambda)\otimes\hat{\chi}^{\pm}(y,\lambda)\right)\ , \end{aligned} \tag{5.42}$$

where

$$\varPi = \sigma_+\otimes\sigma_- + \sigma_-\otimes\sigma_+ + \frac{1}{2}(\sigma_3\otimes\sigma_3 + \mathbb{1}\otimes\mathbb{1})\ , \tag{5.43}$$

which is known as the second Casimir endomorphism of the algebra $sl(2)$.[1] Its important property can be formulated as follows:

$$\varPi(X\otimes Y) = (Y\otimes X)\varPi\ . \tag{5.44}$$

Then from (5.42) we have:

$$\begin{aligned} &G_1^{\pm}(x,y,\lambda) + G_2^{\pm}(x,y,\lambda) + \frac{1}{2}\mathbb{1}\otimes\mathbb{1} \\ &\quad = \left(\chi^{\pm}(x,\lambda)\hat{\chi}^{\pm}(y,\lambda)\right)\otimes\left(\chi^{\pm}(y,\lambda)\hat{\chi}^{\pm}(x,\lambda)\right)\varPi\ , \end{aligned} \tag{5.45}$$

But from the linear relations between the FAS and the Jost solutions we have:

$$\chi^{+}(x,\lambda)\hat{\chi}^{+}(y,\lambda) = \chi^{-}(x,\lambda)\hat{\chi}^{-}(y,\lambda) = \psi(x,\lambda)\hat{\psi}(y,\lambda)\ . \tag{5.46}$$

This concludes the proof.

The expression in (5.46) can be viewed as a solution of the Zakharov–Shabat system, which is normalized to $\mathbb{1}$ for $x=y$. It is well known that such solutions are meromorphic functions of λ.

Next, we calculate the residues of the Green functions at $\lambda_k^{\pm}$. Here and below by $\boldsymbol{\varPsi}_k^{\pm}(x)$, $\boldsymbol{\varPhi}_k^{\pm}(x)$ etc., we denote $\boldsymbol{\varPsi}^{\pm}(x,\lambda_k^{\pm})$, $\boldsymbol{\varPhi}^{\pm}(x,\lambda_k^{\pm})$ etc. With these notations we have:

$$\operatorname*{Res}_{\lambda=\lambda_k^{\pm}} G^{\pm}(x,y,\lambda) = \operatorname*{Res}_{\lambda=\lambda_k^{\pm}} G_1^{\pm}(x,y,\lambda) = X_k^{\pm}(x,y)\ , \tag{5.47}$$

$$\begin{aligned} X_k^{\pm}(x,y) = \frac{1}{(\dot{a}_k^{\pm})^2}\Bigg(&\boldsymbol{\varPsi}_k^{\pm}(x)\otimes\dot{\boldsymbol{\varPhi}}_k^{\pm}(y) + \dot{\boldsymbol{\varPsi}}_k^{\pm}(x)\otimes\boldsymbol{\varPhi}_k^{\pm}(y) \\ &-\frac{\ddot{a}_k^{\pm}}{\dot{a}_k^{\pm}}\boldsymbol{\varPsi}_k^{\pm}(x)\otimes\boldsymbol{\varPhi}_k^{\pm}(y)\Bigg)\ . \end{aligned} \tag{5.48}$$

[1] There, we identify the endomorphisms of $sl(2)$ and the space $sl(2)\otimes sl(2)$.

Working out the residues, we shall make use of the relations (3.98), (3.99), (3.100) and (3.101). Thus, we find that $\boldsymbol{\Psi}_k^{\pm}(x)$ and $\boldsymbol{\Phi}_k^{\pm}(x)$ have the form:

$$\boldsymbol{\Psi}_k^{\pm}(x) = \pm\pi_0(\psi_k^{\pm}(x)\tilde{\psi}_k^{\pm}(x))) \ , \tag{5.49a}$$

$$\boldsymbol{\Phi}_k^{\pm}(x) = \mp(b_k^{\pm})^2\pi_0\psi_k^{\pm}(x)\tilde{\psi}_k^{\pm}(x))) \tag{5.49b}$$

$$= -(b_k^{\pm})^2\boldsymbol{\Psi}_k^{\pm}(x) \ , \tag{5.49c}$$

Let us now evaluate the integrals along the infinite arcs of the contours. We have:

$$\begin{aligned}\mathcal{J}_{G,\infty} &= \frac{1}{2\pi i}\left(\oint_{C_{+,\infty}} d\lambda\, G^+(x,y,\lambda) - \oint_{C_{-,\infty}} d\lambda\, G^-(x,y,\lambda)\right) \\ &= \frac{1}{2\pi i}\left(\oint_{C_{+,\infty}} d\lambda\, G^+_{\rm as}(x,y,\lambda) - \oint_{C_{-,\infty}} d\lambda\, G^-_{\rm as}(x,y,\lambda)\right) \ . \end{aligned} \tag{5.50}$$

Using (3.107), we evaluate the asymptotic values of $G^{\pm}(x,y,\lambda)$ for $\lambda \in \mathbb{C}_{\pm}$:

$$G^+_{1,\rm as}(x,y,\lambda) = \sigma_- \otimes \sigma_+ e^{2i\lambda(x-y)}\left(1 + \mathcal{O}(1/\lambda)\right) \ , \tag{5.51a}$$

$$G^-_{1,\rm as}(x,y,\lambda) = \sigma_+ \otimes \sigma_- e^{-2i\lambda(x-y)}\left(1 + \mathcal{O}(1/\lambda)\right) \ , \tag{5.51b}$$

$$G^+_{2,\rm as}(x,y,\lambda) = \sigma_+ \otimes \sigma_- e^{-2i\lambda(x-y)}\left(1 + \mathcal{O}(1/\lambda)\right) \ , \tag{5.52a}$$

$$G^-_{2,\rm as}(x,y,\lambda) = \sigma_- \otimes \sigma_+ e^{2i\lambda(x-y)}\left(1 + \mathcal{O}(1/\lambda)\right) \ , \tag{5.52b}$$

i.e.

$$\begin{aligned}G^+_{\rm as}(x,y,\lambda) = \Big(&\sigma_- \otimes \sigma_+ e^{2i\lambda(x-y)}\theta(x-y) \\ &- \sigma_+ \otimes \sigma_- e^{-2i\lambda(x-y)}\theta(y-x)\Big)\left(1 + \mathcal{O}(1/\lambda)\right) \ , \end{aligned} \tag{5.53a}$$

for $\lambda \in \mathbb{C}_+$ and

$$\begin{aligned}G^-_{\rm as}(x,y,\lambda) = \Big(&\sigma_+ \otimes \sigma_- e^{-2i\lambda(x-y)}\theta(x-y) \\ &- \sigma_- \otimes \sigma_+ e^{2i\lambda(x-y)}\theta(y-x)\Big)\left(1 + \mathcal{O}(1/\lambda)\right) \ , \end{aligned} \tag{5.53b}$$

for $\lambda \in \mathbb{C}_-$.

Due to Jordan lemma, the terms of order $1/\lambda$ do not contribute to the integral, so we can drop them. Then the integrands in $\mathcal{J}_{G,\infty}$, due to (5.53a) and (5.53b), are entire functions of λ, so we may freely deform the contours $C_{\pm,\infty}$ till they coincide with the real axis. Then we subtract $G^+_{\rm as} - G^-_{\rm as}$ and find out that the θ-functions conveniently add up to give 1. Thus:

$$\begin{aligned}\mathcal{J}_{G,\infty} &= \frac{i}{2\pi}\int_{-\infty}^{\infty} d\lambda\, \left(\sigma_- \otimes \sigma_+ e^{2i\lambda(x-y)} - \sigma_+ \otimes \sigma_- e^{-2i\lambda(x-y)}\right) \\ &= -\frac{i}{2}\delta(x-y)\Pi_0 \ , \end{aligned} \tag{5.54}$$

$$\Pi_0 = \sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_+ \ . \tag{5.55}$$

Finally, we have to evaluate the jump of the Green function across the real axis. Due to (5.41) we find:

$$\begin{aligned} & G^+(x,y,\lambda) - G^-(x,y,\lambda) \\ &= G_1^+(x,y,\lambda) - G_1^-(x,y,\lambda) \qquad (5.56) \\ &= \frac{\boldsymbol{\Psi}^+(x,\lambda)\otimes\boldsymbol{\Phi}^+(y,\lambda)}{(a^+(\lambda))^2} - \frac{\Psi^-(x,\lambda)\otimes\boldsymbol{\Phi}^-(y,\lambda)}{(a^-(\lambda))^2} \ . \qquad (5.57) \end{aligned}$$

Now we equate both answers for the integral $\mathcal{J}_G(x,y)$:

$$\begin{aligned}\mathcal{J}_G(x,y) &= -\frac{i}{2}\delta(x-y)\Pi_0 \\ &\quad -\frac{i}{2\pi}\int_{-\infty}^{\infty} d\lambda\, \left(\frac{\boldsymbol{\Psi}^+(x,\lambda)\otimes\boldsymbol{\Phi}^+(y,\lambda)}{(a^+(\lambda))^2} - \frac{\boldsymbol{\Psi}^-(x,\lambda)\otimes\boldsymbol{\Phi}^-(y,\lambda)}{(a^-(\lambda))^2}\right) \\ &= \sum_{k=1}^{N}\left(X_k^+(x,y) + X_k^-(x,y)\right) \ , \end{aligned} \tag{5.58}$$

where $X_k^{\pm}(x,y)$ are defined by (5.48). Thus, the completeness relation for the "squared" solutions acquires the form:

$$\begin{aligned}\delta(x-y)\Pi_0 &= -\frac{1}{\pi}\int_{-\infty}^{\infty} d\lambda\, \left(\frac{\boldsymbol{\Psi}^+(x,\lambda)\otimes\boldsymbol{\Phi}^+(y,\lambda)}{(a^+(\lambda))^2} - \frac{\boldsymbol{\Psi}^-(x,\lambda)\otimes\boldsymbol{\Phi}^-(y,\lambda)}{(a^-(\lambda))^2}\right) \\ &\quad + 2i\sum_{k=1}^{N}\left(X_k^+(x,y) + X_k^-(x,y)\right) \ , \end{aligned} \tag{5.59a}$$

$$\Pi_0 = \sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_+ \ . \tag{5.59b}$$

As we will see in the next Section, this relation is compatible with the one derived for the first time by Kaup [2].

Remark 5.2. The derivation of the above completeness relation can be made quite rigorous. First, we remind that we identify $sl(2)\otimes sl(2)$ by $sl(2)\otimes sl^*(2)$ using the Killing form of $sl(2)$. Next, $sl(2)\otimes sl^*(2)$ is naturally isomorphic with the space of endomorphisms of $sl(2)$ (as linear space). Finally, we take a function $f(y)$ from the class $L^1(\mathbb{R})$ with values in $sl(2)$ and act by the

Green function (now understood as endomorphism) on it. Then performing essentially the same steps as in the proof, we obtain a completeness relation (5.68). The formula which we used in (5.54), namely,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda(x-y)}d\lambda = \delta(x-y)\ , \tag{5.60}$$

simply means that we apply our construction to functions for which $\mathcal{F}_0^{-1}\mathcal{F}_0 f = f$, where $\mathcal{F}_0$ is the usual Fourier transform. Thus, for the functions of Schwartz class, we have our expansions, and for the smooth functions we have them in distributional sense.

Also, since $sl(2)\otimes sl(2)$ can be identified with $sl^*(2)\otimes sl(2)$, and this again can be identified with the endomorphisms of $sl(2)$ the completeness relation obtained here can be used in two different ways, getting two different expansions; see the next section. This was one of the main reasons for presenting them in the above tensor form.

The constant tensor Π_0 is closely related to the skew-scalar product $[\![\,\cdot,\cdot\,]\!]$ we had earlier; see (5.11). Indeed, if we interpret Π_0 as 4×4 matrix, we can write:

$$[\![X,Y]\!] = -\int_{-\infty}^{\infty} dx\,\mathrm{tr}\left(\frac{1}{2}[\sigma_3, X(x)]\otimes \mathbb{1}\right)\Pi_0\left(\frac{1}{2}\mathbb{1}\otimes[\sigma_3, Y(x)]\right).$$

5.2.1 The Symplectic Basis

Here, we introduce the so-called symplectic basis, which will be extensively used in the analysis of the Hamiltonian structures of the NLEE. The elements of this basis are certain linear combinations of the “squared” solutions, namely:

$$\begin{aligned}\mathbf{P}(x,\lambda) &= \frac{1}{\pi}\left(\tau^+(\lambda)\boldsymbol{\Phi}^+(x,\lambda) - \tau^-(\lambda)\boldsymbol{\Phi}^-(x,\lambda)\right)\\ &= -\frac{1}{\pi}\left(\rho^+(\lambda)\boldsymbol{\Psi}^+(x,\lambda) - \rho^-(\lambda)\boldsymbol{\Psi}^-(x,\lambda)\right)\ ,\end{aligned} \tag{5.61a}$$

$$\mathbf{P}_k^{\pm}(x) = 2iC_k^{\pm}\boldsymbol{\Psi}_k^{\pm}(x) = -2iM_k^{\pm}\boldsymbol{\Phi}_k^{\pm}(x)\ , \tag{5.61b}$$

$$\begin{aligned}\mathbf{Q}(x,\lambda) &= \frac{\tau^+(\lambda)\boldsymbol{\Phi}^+(x,\lambda) + \rho^+(\lambda)\boldsymbol{\Psi}^+(x,\lambda)}{2b^+(\lambda)b^-(\lambda)}\\ &= \frac{\rho^-(\lambda)\boldsymbol{\Psi}^-(x,\lambda) + \tau^-(\lambda)\boldsymbol{\Phi}^-(x,\lambda)}{2b^+(\lambda)b^-(\lambda)}\ ,\end{aligned} \tag{5.61c}$$

$$\mathbf{Q}_k^{\pm}(x) = \frac{1}{2}\left(C_k^{\pm}\dot{\boldsymbol{\Psi}}_k^{\pm}(x) + M_k^{\pm}\dot{\boldsymbol{\Phi}}_k^{\pm}(x)\right)\ , \tag{5.61d}$$

where we must recall that

$$C_k^{\pm} = \frac{b_k^{\pm}}{\dot{a}_k^{\pm}}, \qquad M_k^{\pm} = \frac{1}{b_k^{\pm}\dot{a}_k^{\pm}} \; . \tag{5.62}$$

Like the other systems of squared solutions, the symplectic basis also satisfies completeness relation. It is derived by adding (5.59) and the relation (5.59)', where the primed relation is obtained from (5.59) by exchanging the factors to the left of the tensor product with the ones to the right of it.

The relation itself has the form:

$$\begin{aligned} \delta(x-y)\Pi_0 = &\int_{-\infty}^{\infty} d\lambda \; (\mathbf{P}(x,\lambda)\otimes\mathbf{Q}(y,\lambda) - \mathbf{Q}(x,\lambda)\otimes\mathbf{P}(y,\lambda)) \\ &+ \sum_{k=1}^{N} \left(Z_k^{+}(x,y) + Z_k^{-}(x,y)\right) \; , \end{aligned} \tag{5.63a}$$

$$Z_k^{\pm}(x,y) = \left(\mathbf{P}_k^{\pm}(x)\otimes\mathbf{Q}_k^{\pm}(y) - \mathbf{Q}_k^{\pm}(x)\otimes\mathbf{P}_k^{\pm}(y)\right) \; . \tag{5.63b}$$

5.3 Expansions Over the "Squared" Solutions

Using the completeness relations, one can expand any generic element $X(x)$ of the phase space $\mathcal{M}$ over each of the three complete sets of "squared solutions." In this section, we explain how this can be done. We remind that $X(x)$ is a generic element of $\mathcal{M}$ if it is an off-diagonal matrix-valued function, which falls off fast enough for $|x| \to \infty$. Obviously $X(x)$ can be written down in terms of its matrix elements $X_{\pm}(x)$ as:

$$X(x) = X_+(x)\sigma_+ + X_-(x)\sigma_- \; . \tag{5.64}$$

From (5.59b) we get:

$$\begin{aligned} \frac{1}{2}\mathrm{tr}_1\left([\sigma_3, X(x)]\otimes \mathbb{1}\right)\Pi_0 &= -\frac{1}{2}\mathrm{tr}_2 \Pi_0\left(\mathbb{1}\otimes[\sigma_3, X(x)]\right)\Pi_0 \\ &= -X(x) \; , \end{aligned} \tag{5.65}$$

where tr_1 (and tr_2) mean that we are taking the trace of the elements in the first (or the second) position of the tensor product.

Now, we multiply (5.63) on the right by $\frac{1}{2}[\sigma_3, X(x)]\otimes\mathbb{1}$, take tr_1, and integrate over dx. This leads to the expansion of $X(x)$ over the system $\boldsymbol{\Phi}^{\pm}$.

$$\begin{aligned} X(x) = &\frac{1}{\pi}\int_{-\infty}^{\infty} d\lambda \; \left(\psi_X^{+}(\lambda)\boldsymbol{\Phi}^{+}(x,\lambda) - \psi_X^{-}(\lambda)\boldsymbol{\Phi}^{-}(x,\lambda)\right) \\ &- 2i\sum_{k=1}^{N}{}^{\pm} \left(\psi_{X,k}^{\pm}\dot{\boldsymbol{\Phi}}_k^{\pm} + \dot{\psi}_{X,k}^{\pm}\boldsymbol{\Phi}_k^{\pm}\right) \; , \end{aligned} \tag{5.66}$$

where $\sum_{k=1}^{\pm,N} X_k^{\pm} \equiv \sum_{k=1}^{N}(Z_k^+ + Z_k^-)$ and

$$\psi_X^{\pm}(\lambda) = \frac{[\![\boldsymbol{\Psi}^{\pm}(x,\lambda), X(x)]\!]}{(a^{\pm}(\lambda))^2}, \qquad \psi_{X,k}^{\pm} = \frac{[\![\boldsymbol{\Psi}_k^{\pm}(x), X(x)]\!]}{(\dot{a}_k^{\pm})^2}, \tag{5.67a}$$

$$\dot{\psi}_{X,k}^{\pm} = \frac{1}{(\dot{a}_k^{\pm})^2}\left[\!\!\left[\dot{\boldsymbol{\Psi}}_k^{\pm}(x) - \frac{\ddot{a}_k^{\pm}}{\dot{a}_k^{\pm}}\boldsymbol{\Psi}_k^{\pm}(x), X(x)\right]\!\!\right]. \tag{5.67b}$$

Analogously, we can multiply (5.59) on the left by $\frac{1}{2}\mathbb{1}\otimes[\sigma_3, X(x)]$, take tr_2, and integrate over dx. This leads to the expansion of $X(x)$ over the system $\boldsymbol{\Psi}^{\pm}$, which runs as follows:

$$\begin{aligned} X(x) &= -\frac{1}{\pi}\int_{-\infty}^{\infty} d\lambda\, \left(\phi_X^+(\lambda)\boldsymbol{\Psi}^+(x,\lambda) - \phi_X^-(\lambda)\boldsymbol{\Psi}^-(x,\lambda)\right) \\ &\quad + 2i\sum_{k=1}^{N}{}^{\pm} \left(\phi_{X,k}^{\pm}\dot{\boldsymbol{\Psi}}_k^{\pm} + \dot{\phi}_{X,k}^{\pm}\boldsymbol{\Psi}_k^{\pm}\right), \end{aligned} \tag{5.68}$$

$$\phi_X^{\pm}(\lambda) = \frac{[\![\boldsymbol{\Phi}^{\pm}(x,\lambda), X(x)]\!]}{(a^{\pm}(\lambda))^2}, \qquad \phi_{X,k}^{\pm} = \frac{[\![\boldsymbol{\Phi}_k^{\pm}(x), X(x)]\!]}{(\dot{a}_k^{\pm})^2}, \tag{5.69a}$$

$$\dot{\phi}_{X,k}^{\pm} = \frac{1}{(\dot{a}_k^{\pm})^2}\left([\![\dot{\boldsymbol{\Phi}}_k^{\pm}(x), X(x)]\!] - \frac{\ddot{a}_k^{\pm}}{\dot{a}_k^{\pm}}[\![\boldsymbol{\Phi}_k^{\pm}(x), X(x)]\!]\right). \tag{5.69b}$$

The same procedure applied to the completeness relation (5.63) for the symplectic basis leads to:

$$\begin{aligned} X(x) &= \int_{-\infty}^{\infty} d\lambda\, (\kappa_X(\lambda)\mathbf{P}(x,\lambda) - \eta_X(\lambda)\mathbf{Q}(x,\lambda)) \\ &\quad + \sum_{k=1}^{N}{}^{\pm} \left(\kappa_{X,k}^{\pm}\mathbf{P}_k^{\pm} - \eta_{X,k}^{\pm}\mathbf{Q}_k^{\pm}\right), \end{aligned} \tag{5.70}$$

$$\kappa_X(\lambda) = [\![\mathbf{Q}(x,\lambda), X(x)]\!], \qquad \eta_X(\lambda) = [\![\mathbf{P}(x,\lambda), X(x)]\!] \tag{5.71a}$$

$$\kappa_{X,k}^{\pm} = [\![\mathbf{Q}_k^{\pm}(x), X(x)]\!], \qquad \eta_{X,k}^{\pm} = [\![\mathbf{P}_k^{\pm}(x), X(x)]\!] \tag{5.71b}$$

The completeness relations derived above allow us to establish a one-to-one correspondence between the element $X(x) \in \mathcal{M}$ and its expansion coefficients. Indeed, from (5.59) and (5.63), we derived the expansions (5.66), (5.68), and (5.70) with the inversion formulae (5.67), (5.69), and (5.71), respectively. Using them we prove the following:

Proposition 5.3. *The function $X(x) \equiv 0$ if and only if one of the following sets of relations holds:*

$$\psi_X^+(\lambda) = \psi_X^-(\lambda) \equiv 0, \qquad \lambda \in \mathbb{R} , \tag{5.72a}$$

$$\psi_{X,k}^\pm = \dot{\psi}_{X,k}^\pm = 0, \qquad k = 1, \dots, N ; \tag{5.72b}$$

$$\phi_X^+(\lambda) = \phi_X^-(\lambda) \equiv 0, \qquad \lambda \in \mathbb{R} , \tag{5.73a}$$

$$\phi_{X,k}^\pm = \dot{\phi}_{X,k}^\pm = 0, \qquad k = 1, \dots, N ; \tag{5.73b}$$

$$\kappa_X(\lambda) = \eta_X(\lambda) \equiv 0, \qquad \lambda \in \mathbb{R} , \tag{5.74a}$$

$$\kappa_{X,k}^\pm = \eta_{X,k}^\pm = 0, \qquad k = 1, \dots, N ; \tag{5.74b}$$

Proof. Let us show that from $X(x) \equiv 0$ there follows (5.72). To this end, we insert $X(x) \equiv 0$ into the right-hand sides of the inversion formulae (5.67) and immediately get (5.72). The fact that from (5.72) there follows $X(x) \equiv 0$ is readily obtained by inserting (5.72) into the right-hand side of (5.66).

The equivalence of $X(x) \equiv 0$ to (5.73) and (5.74) is proved analogously using the inversion formulae (5.69), (5.71) and the expansions (5.68) and (5.70). The proposition is proved.

5.3.1 Expansions of $q(x)$

Here we calculate the expansion coefficients for $X(x) \equiv q(x)$. As the readers have guessed already, their evaluation will be based on the Wronskian relations (5.25), (5.26), which we derived above. From them we have:

$$\psi_q^\pm(\lambda) = \frac{1}{(a^\pm(\lambda))^2} [\![\boldsymbol{\Psi}^\pm(x,\lambda), q(x)]\!] = i\tau^\pm(\lambda) , \tag{5.75a}$$

$$\psi_{q,k}^\pm = \frac{1}{(\dot{a}_k^\pm)^2} [\![\boldsymbol{\Psi}_k^\pm(x), q(x)]\!] = 0 , \tag{5.75b}$$

$$\begin{aligned} \dot{\psi}_{q,k}^\pm &= \frac{1}{(\dot{a}_k^\pm)^2} \left([\![\dot{\boldsymbol{\Psi}}_k^\pm(x), q(x)]\!] - \frac{\ddot{a}_k^\pm}{\dot{a}_k^\pm} [\![\boldsymbol{\Psi}_k^\pm(x), q(x)]\!] \right) \\ &= iM_k^\pm , \end{aligned} \tag{5.75c}$$

As a result, we get the following expansion of $q(x)$ over the system $\boldsymbol{\Phi}^\pm$:

$$\begin{aligned} q(x) = \frac{i}{\pi} \int_{-\infty}^{\infty} d\lambda \, \left(\tau^+(\lambda) \boldsymbol{\Phi}^+(x,\lambda) - \tau^-(\lambda) \boldsymbol{\Phi}^-(x,\lambda) \right) \\ + 2 \sum_{k=1}^{N} \left(M_k^+ \boldsymbol{\Phi}_k^+(x) + M_k^- \boldsymbol{\Phi}_k^-(x) \right) . \end{aligned} \tag{5.76}$$

Using the other Wronskian relations for $\boldsymbol{\Phi}^{\pm}(x,\lambda)$ we have:

$$\phi_q^{\pm}(\lambda) = \frac{1}{(a^{\pm}(\lambda))^2} [\![\boldsymbol{\Phi}^{\pm}(x,\lambda), q(x)]\!] = i\rho^{\pm}(\lambda) \,, \tag{5.77a}$$

$$\phi_{q,k}^{\pm} = \frac{1}{(\dot{a}_k^{\pm})^2} [\![\boldsymbol{\Phi}_k^{\pm}(x), q(x)]\!] = 0 \,, \tag{5.77b}$$

$$\begin{aligned} \dot{\phi}_{q,k}^{\pm} &= \frac{1}{(\dot{a}_k^{\pm})^2} \left([\![\dot{\boldsymbol{\Phi}}_k^{\pm}(x), q(x)]\!] - \frac{\ddot{a}_k^{\pm}}{\dot{a}_k^{\pm}} [\![\boldsymbol{\Phi}_k^{\pm}(x), q(x)]\!] \right) \\ &= iC_k^{\pm} \,, \end{aligned} \tag{5.77c}$$

and as a consequence get the expansion for $q(x)$ over the system $\boldsymbol{\Psi}^{\pm}$:

$$\begin{aligned} q(x) = &-\frac{i}{\pi} \int_{-\infty}^{\infty} d\lambda \, \left(\rho^{+}(\lambda) \boldsymbol{\Psi}^{+}(x,\lambda) - \rho^{-}(\lambda) \boldsymbol{\Psi}^{-}(x,\lambda) \right) \\ &- 2 \sum_{k=1}^{N} \left(C_k^{+} \boldsymbol{\Psi}_k^{+}(x) + C_k^{-} \boldsymbol{\Psi}_k^{-}(x) \right) \,. \end{aligned} \tag{5.78}$$

The expansion coefficients over the symplectic basis are given by:

$$\eta_q(\lambda) = [\![\mathbf{P}(x,\lambda), q(x)]\!] = 0, \qquad \kappa_q(\lambda) = [\![\mathbf{Q}(x,\lambda), q(x)]\!] = i \,, \tag{5.79a}$$

$$\eta_{q,k}^{\pm} = [\![\mathbf{P}_k^{\pm}(x), q(x)]\!] = 0, \qquad \kappa_{q,k}^{\pm} = [\![\mathbf{Q}_k^{\pm}(x), q(x)]\!] = i \,, \tag{5.79b}$$

and therefore:

$$q(x) = i \int_{-\infty}^{\infty} d\lambda \, \mathbf{P}(x,\lambda) + i \sum_{k=1}^{N} \left(\mathbf{P}_k^{+}(x) + \mathbf{P}_k^{-}(x) \right) \,. \tag{5.80}$$

Note that only half of the elements in the symplectic basis contribute to the right-hand side of (5.80). We shall see that this makes the above basis quite special.

5.3.2 Expansions of $\sigma_3 \delta q(x)$

Here, we evaluate the expansion coefficients for $X(x) \equiv \sigma_3 q(x)$. Their calculation is again based on the Wronskian relations (5.25), (5.26), which we introduced above. We have:

$$\psi_{\sigma_3 \delta q}^{\pm}(\lambda) = \frac{1}{(a^{\pm}(\lambda))^2} [\![\boldsymbol{\Psi}^{\pm}(x,\lambda), \sigma_3 \delta q(x)]\!] = \pm i \delta \tau^{\pm}(\lambda) \,, \tag{5.81a}$$

$$\psi_{\sigma_3 \delta q,k}^{\pm} = \frac{1}{(\dot{a}_k^{\pm})^2} [\![\boldsymbol{\Psi}_k^{\pm}(x), \sigma_3 \delta q(x)]\!] = \pm i \delta \lambda_k^{\pm} M_k^{\pm} \,, \tag{5.81b}$$

$$\dot{\psi}^{\pm}_{\sigma_3\delta q,k} = \frac{1}{(\dot{a}_k^{\pm})^2}\left([\![\dot{\boldsymbol{\Psi}}_k^{\pm}(x), \sigma_3\delta q(x)]\!] - \frac{\ddot{a}_k^{\pm}}{\dot{a}_k^{\pm}} [\![\boldsymbol{\Psi}_k^{\pm}(x), \sigma_3\delta q(x)]\!] \right)$$
$$= \pm i\delta M_k^{\pm} \; . \tag{5.81c}$$

As a result, we get the following expansion of $\sigma_3\delta q(x)$ over the system $\boldsymbol{\Phi}^{\pm}$:

$$\sigma_3\delta q(x) = \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda \, \left(\delta\tau^{+}(\lambda)\boldsymbol{\Phi}^{+}(x,\lambda) + \delta\tau^{-}(\lambda)\boldsymbol{\Phi}^{-}(x,\lambda)\right) \tag{5.82}$$
$$+ 2\sum_{k=1}^{N}\Big(M_k^{+}\delta\lambda_k^{+}\dot{\boldsymbol{\Phi}}_k^{+}(x) + \delta M_k^{+}\boldsymbol{\Phi}_k^{+}(x) - M_k^{-}\delta\lambda_k^{-}\dot{\boldsymbol{\Phi}}_k^{-}(x)$$
$$- \delta M_k^{-}\boldsymbol{\Phi}_k^{-}(x)\Big) \; .$$

The other two expansions for $\sigma_3\delta q(x)$ over the system $\boldsymbol{\Psi}^{\pm}$ can be found in a similar way. Indeed, as:

$$\phi^{\pm}_{\sigma_3\delta q}(\lambda) = \frac{1}{(a^{\pm}(\lambda))^2}[\![\boldsymbol{\Phi}^{\pm}(x,\lambda), \sigma_3\delta q(x)]\!] = \mp i\delta\rho^{\pm}(\lambda) \; , \tag{5.83a}$$

$$\phi^{\pm}_{\sigma_3\delta q,k} = \frac{1}{(\dot{a}_k^{\pm})^2}[\![\boldsymbol{\Phi}_k^{\pm}(x), \sigma_3\delta q(x)]\!] = \mp i\delta\lambda_k^{\pm}C_k^{\pm} \; , \tag{5.83b}$$

$$\dot{\phi}^{\pm}_{\sigma_3\delta q,k} = \frac{1}{(\dot{a}_k^{\pm})^2}\left([\![\dot{\boldsymbol{\Phi}}_k^{\pm}(x), \sigma_3\delta q(x)]\!] - \frac{\ddot{a}_k^{\pm}}{\dot{a}_k^{\pm}} [\![\boldsymbol{\Phi}_k^{\pm}(x), \sigma_3\delta q(x)]\!] \right)$$
$$= \mp i\delta C_k^{\pm} \; , \tag{5.83c}$$

we get:

$$\sigma_3\delta q(x) = \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda \, \left(\delta\rho^{+}(\lambda)\boldsymbol{\Psi}^{+}(x,\lambda) + \delta\rho^{-}(\lambda)\boldsymbol{\Psi}^{-}(x,\lambda)\right) \tag{5.84}$$
$$+ 2\sum_{k=1}^{N}\Big(C_k^{+}\delta\lambda_k^{+}\dot{\boldsymbol{\Psi}}_k^{+}(x) + \delta C_k^{+}\boldsymbol{\Psi}_k^{+}(x) - C_k^{-}\delta\lambda_k^{-}\dot{\boldsymbol{\Psi}}_k^{-}(x)$$
$$- \delta C_k^{-}\boldsymbol{\Psi}_k^{-}(x)\Big) \; .$$

Finally, with the expansion coefficients over the symplectic basis being given by:

$$\eta_{\sigma_3\delta q}(\lambda) = [\![\mathbf{P}(x,\lambda), \sigma_3\delta q(x)]\!] = -i\delta\eta(\lambda) \; , \tag{5.85a}$$

$$\kappa_{\sigma_3\delta q}(\lambda) = [\![\mathbf{Q}(x,\lambda), \sigma_3\delta q(x)]\!] = -i\delta\kappa(\lambda) \; , \tag{5.85b}$$

$$\eta^{\pm}_{\sigma_3\delta q,k} = [\![\mathbf{P}_k^{\pm}(x), \sigma_3\delta q(x)]\!] = \mp 2\delta\lambda_k^{\pm} \; , \tag{5.85c}$$

$$\kappa^{\pm}_{\sigma_3\delta q,k} = [\![\mathbf{Q}_k^{\pm}(x), \sigma_3\delta q(x)]\!] = \mp i\delta \ln b_k^{\pm} \; , \tag{5.85d}$$

we obtain:

$$\sigma_3 \delta q(x) = i \int_{-\infty}^{\infty} d\lambda \, (\delta\kappa(\lambda) \mathbf{P}(x,\lambda) - \delta\eta(\lambda) \mathbf{Q}(x,\lambda)) \\ + i \sum_{k=1}^{N} \left(\delta\eta_k^+ \mathbf{Q}_k^+(x) - \delta\kappa_k^+ \mathbf{P}_k^+(x) + \delta\eta_k^- \mathbf{Q}_k^-(x) - \delta\kappa_k^- \mathbf{P}_k^-(x) \right), \tag{5.86}$$

where

$$\eta(\lambda) = \frac{1}{\pi} \ln \left(1 + \rho^+(\lambda)\rho^-(\lambda)\right), \qquad \eta_k^\pm = \mp 2i\lambda_k^\pm , \tag{5.87a}$$

$$\kappa(\lambda) = \frac{1}{2} \ln \frac{b^+(\lambda)}{b^-(\lambda)}, \qquad \kappa_k^\pm = \pm \ln b_k^\pm . \tag{5.87b}$$

In the next chapter, we shall see how this set of variables is related to the action-angle variables of the corresponding NLEE.

The expansion (5.86) allows us to introduce one more minimal set of scattering data:

$$\mathcal{T} \equiv \left\{ \eta(\lambda), \kappa(\lambda), \quad \lambda \in \mathbb{R}, \quad \eta_k^\pm, \quad \kappa_k^\pm, \quad k = 1, \ldots, N \right\} , \tag{5.88}$$

which, like $\mathcal{T}_1$ and $\mathcal{T}_2$ in (3.68), allows to recover uniquely both the scattering matrix $T(\lambda)$ and the corresponding potential. Indeed, to determine $T(\lambda)$ from (5.88), we make use of the dispersion relations (3.66), (3.67), which allow us to find $a^\pm(\lambda)$ in their whole domains of analyticity, knowing $\eta(\lambda)$ and $\lambda_k^\pm$. Then, knowing $a^\pm(\lambda)$ and $b^+(\lambda)/b^-(\lambda) = \exp(2\kappa(\lambda))$, it is easy to determine $b^\pm(\lambda)$ as functions on the real λ-axis. The coefficients $C_k^\pm = b_k^\pm/\dot{a}_k^\pm$ and $M_k^\pm = 1/(b_k^\pm \dot{a}_k^\pm)$ are obtained through $b_k^\pm = \exp(\pm\kappa_k^\pm)$ (5.87b) and $\dot{a}_k^\pm = da^\pm/d\lambda|_{\lambda=\lambda_k^\pm}$ (3.97).

5.4 Generating Operators Revisited

We introduced the generating (or the recursion) operators in Chap. 2 by solving the set of recurrent relations derived by AKNS. Here, we shall obtain them using another idea as the operators, for which the squared solutions are eigenfunctions.

First of all, note that the squared solutions $\Psi^\pm(x,\lambda)$ and $\Phi^\pm(x,\lambda)$ satisfy the equation:

$$i\frac{d\mathcal{E}_\alpha}{dx} + [q(x) - \lambda\sigma_3, \mathcal{E}_\alpha(x,\lambda)] = 0 . \tag{5.89}$$

which actually is satisfied by any function which is a linear combination of functions of the form (compare with (5.10)):

$$\mathcal{E}_\alpha(x,\lambda) = a(\lambda)\chi\sigma_\alpha\hat{\chi}(x,\lambda) , \tag{5.90}$$

where $\chi(x,\lambda)$ is any fundamental solution of the Zakharov–Shabat system and $a(\lambda) = \det \chi(x,\lambda)$ is an x-independent function. In what follows, we shall specify $\chi(x,\lambda) = \chi^{\pm}(x,\lambda)$ and $a(\lambda) = a^{\pm}(\lambda)$, respectively.

Next, omitting for brevity the subscript α, we introduce the splitting:

$$\mathcal{E}(x,\lambda) = e(x,\lambda)\sigma_3 + \boldsymbol{\Phi}(x,\lambda) \ , \tag{5.91a}$$

$$e(x,\lambda) = \frac{1}{2}\operatorname{tr}\left(\mathcal{E}(x,\lambda)\sigma_3\right) \ , \tag{5.91b}$$

$$\boldsymbol{\Phi}(x,\lambda) = \pi_0(\mathcal{E}(x,\lambda)) \ . \tag{5.91c}$$

Let us multiply (5.89) by $\frac{1}{2}\sigma_3$ and take the trace. We obtain:

$$i\frac{de}{dx} + \langle \sigma_3[q(x), \boldsymbol{\Phi}(x,\lambda)]\rangle = 0, \qquad \langle X, Y\rangle = \frac{1}{2}\operatorname{tr} XY \ . \tag{5.92}$$

Then

$$e(x,\lambda) = i\int_{\pm\infty}^{x} dy\, \langle \sigma_3[q(y), \boldsymbol{\Phi}(y,\lambda)]\rangle + \lim_{y\to\pm\infty} e(y,\lambda) \ . \tag{5.93}$$

Next, we apply to (5.89) the projector π_0 (i.e. take its off-diagonal part):

$$i\frac{d\boldsymbol{\Phi}}{dx} + \pi_0[q(x), e(x,\lambda)\sigma_3 + \boldsymbol{\Phi}(x,\lambda)] = \lambda[\sigma_3, \boldsymbol{\Phi}(x,\lambda)] \tag{5.94}$$

and insert into it (5.93). Thus we obtain

$$\begin{aligned} &\frac{i}{4}\left[\sigma_3, \frac{d\boldsymbol{\Phi}}{dx}\right] - iq(x)\int_{\pm\infty}^{x} dy\, \langle \sigma_3[q(y), \boldsymbol{\Phi}(y,\lambda)]\rangle - \lambda\boldsymbol{\Phi}(y,\lambda) \\ &\quad = q(x)\lim_{y\to\pm\infty} e(y,\lambda) \ , \end{aligned} \tag{5.95}$$

or in compact form:

$$(\Lambda_{\pm} - \lambda)\boldsymbol{\Phi}(y,\lambda) = q(x)\lim_{y\to\pm\infty} e(y,\lambda) \ , \tag{5.96}$$

where the integro-differential operators $\Lambda_{\pm}$ are defined by:

$$\Lambda_{\pm}X = \frac{i}{4}\left[\sigma_3, \frac{dX}{dx}\right] - iq(x)\int_{\pm\infty}^{x} dy\, \langle \sigma_3[q(y), X(y)]\rangle \ , \tag{5.97}$$

If it were for the nonzero terms in the right-hand side of (5.95) we could say that any of the operators $\Lambda_{\pm}$ will do the job. Since it is not so we have to go further and evaluate the right-hand side of (5.95) and find for what specific choices of $\boldsymbol{\Phi}(x,\lambda)$ it vanishes. To this end, we need the asymptotics of the “squared” solutions for $x \to \pm\infty$, which are collected in Table 5.1.

Table 5.1. The limits of $\mathcal{E}^{\pm}(x,\lambda)$ for $x \to \pm\infty$.

	$\sigma_\alpha = \sigma_+$	$\sigma_\alpha = \sigma_-$	$\sigma_\alpha = \sigma_3$
		$\mathcal{E}^+_\alpha(x,\lambda)$	
$+\infty$	$E_1^+(x,\lambda)$	$e^{2i\lambda x}\sigma_-$	$a^+(\lambda)\sigma_3 + 2b^+(\lambda)e^{2i\lambda x}\sigma_-$
$-\infty$	$e^{-2i\lambda x}\sigma_+$	$E_2^+(x,\lambda)$	$a^+(\lambda)\sigma_3 - 2b^-(\lambda)e^{-2i\lambda x}\sigma_+$
		$\mathcal{E}^-_\alpha(x,\lambda)$	
$+\infty$	$e^{-2i\lambda x}\sigma_+$	$E_1^-(x,\lambda)$	$a^-(\lambda)\sigma_3 + 2b^-(\lambda)e^{-2i\lambda x}\sigma_+$
$-\infty$	$E_2^-(x,\lambda)$	$e^{2i\lambda x}\sigma_-$	$a^-(\lambda)\sigma_3 - 2b^-(\lambda)e^{2i\lambda x}\sigma_-$

Here we have used the notations:

$$E_1^+ = \begin{pmatrix} -a^+(\lambda)b^+(\lambda) & (a^+(\lambda))^2 e^{-2i\lambda x} \\ -(b^+(\lambda))^2 e^{2i\lambda x} & a^+(\lambda)b^+(\lambda) \end{pmatrix} , \tag{5.98a}$$

$$E_1^- = \begin{pmatrix} -a^-(\lambda)b^-(\lambda) & -(b^-(\lambda))^2 e^{-2i\lambda x} \\ (a^-(\lambda))^2 e^{2i\lambda x} & a^-(\lambda)b^-(\lambda) \end{pmatrix} , \tag{5.98b}$$

$$E_2^+ = \begin{pmatrix} a^+(\lambda)b^-(\lambda) & -(b^-(\lambda))^2 e^{-2i\lambda x} \\ (a^+(\lambda))^2 e^{2i\lambda x} & -a^+(\lambda)b^-(\lambda) \end{pmatrix} , \tag{5.98c}$$

$$E_2^- = \begin{pmatrix} a^-(\lambda)b^+(\lambda) & (a^-(\lambda))^2 e^{-2i\lambda x} \\ -(b^+(\lambda))^2 e^{2i\lambda x} & -a^-(\lambda)b^+(\lambda) \end{pmatrix} . \tag{5.98d}$$

Thus we find the following relations, which in particular show that $\boldsymbol{\Psi}^{\pm}(x,\lambda)$ and $\boldsymbol{\Phi}^{\pm}(x,\lambda)$ are eigenfunctions of the operators Λ_+ and Λ_-, respectively:

$$(\Lambda_+ - \lambda)\boldsymbol{\Psi}^{\pm}(x,\lambda) = 0 , \tag{5.99a}$$

$$(\Lambda_- - \lambda)\boldsymbol{\Psi}^{\pm}(x,\lambda) = q(x)a^{\pm}(\lambda)b^{\mp}(\lambda) , \tag{5.99b}$$

and

$$(\Lambda_- - \lambda)\boldsymbol{\Phi}^{\pm}(x,\lambda) = 0 , \tag{5.100a}$$

$$(\Lambda_+ - \lambda)\boldsymbol{\Phi}^{\pm}(x,\lambda) = -q(x)a^{\pm}(\lambda)b^{\pm}(\lambda) , \tag{5.100b}$$

The "squared" solutions $\boldsymbol{\Theta}^{\pm}(x,\lambda)$, which are related to the diagonal of the resolvent, and which were used to obtain the densities of the integrals of motion, are *not* eigenfunctions of $\Lambda_{\pm}$. For them we get:

$$(\Lambda_+ - \lambda)\boldsymbol{\Theta}^{\pm}(x,\lambda) = q(x)a^{\pm}(\lambda) , \tag{5.101a}$$

$$(\Lambda_- - \lambda)\boldsymbol{\Theta}^{\pm}(x,\lambda) = q(x)a^{\pm}(\lambda) \, . \tag{5.101b}$$

It is also natural to evaluate the action of $\Lambda_\pm$ on the elements of the symplectic basis:

$$(\Lambda_\pm - \lambda)\mathbf{P}(x,\lambda) = 0 \, , \tag{5.102a}$$

$$(\Lambda_\pm - \lambda)\mathbf{Q}(x,\lambda) = \mp\frac{1}{2}q(x) \, . \tag{5.102b}$$

From these relations, it is easy to see that the symplectic basis consists of the eigenfunctions of the operator:

$$\Lambda = \frac{1}{2}(\Lambda_+ + \Lambda_-) \, , \tag{5.103}$$

that is

$$(\Lambda - \lambda)\mathbf{P}(x,\lambda) = 0, \qquad (\Lambda - \lambda)\mathbf{Q}(x,\lambda) = 0 \, , \tag{5.104}$$

Below, we shall need also the action of the recursion operators on the eigenfunctions of the discrete spectrum. The calculations are performed in an analogous way and the results are:

$$(\Lambda_+ - \lambda_k^{\pm})\boldsymbol{\Psi}_k^{\pm}(x) = 0, \qquad (\Lambda_+ - \lambda_k^{\pm})\dot{\boldsymbol{\Psi}}_k^{\pm}(x) = \boldsymbol{\Psi}_k^{\pm}(x) \, , \tag{5.105}$$

$$(\Lambda_- - \lambda_k^{\pm})\boldsymbol{\Phi}_k^{\pm}(x) = 0, \qquad (\Lambda_- - \lambda_k^{\pm})\dot{\boldsymbol{\Phi}}_k^{\pm}(x) = \boldsymbol{\Phi}_k^{\pm}(x) \, , \tag{5.106}$$

and

$$(\Lambda - \lambda_k^{\pm})\mathbf{P}_k^{\pm}(x) = 0, \qquad (\Lambda - \lambda_k^{\pm})\mathbf{Q}_k^{\pm}(x) = 0 \, . \tag{5.107}$$

5.5 Spectral Properties of the Λ-Operators

In the preceding section, we showed that the sets of "squared solutions"

$$\begin{aligned}\{\Psi\} \equiv \{&\boldsymbol{\Psi}^{\pm}(x,\lambda), \quad \lambda \in \mathbb{R}\, ,\\ &\boldsymbol{\Psi}_k^{\pm}(x), \quad \dot{\boldsymbol{\Psi}}_k^{\pm}(x), \quad k = 1,\dots,N\} \, ,\end{aligned} \tag{5.108a}$$

$$\begin{aligned}\{\Phi\} \equiv \{&\boldsymbol{\Phi}^{\pm}(x,\lambda), \quad \lambda \in \mathbb{R}\, ,\\ &\boldsymbol{\Phi}_k^{\pm}(x), \quad \dot{\boldsymbol{\Phi}}_k^{\pm}(x), \quad k = 1,\dots,N\} \, ,\end{aligned} \tag{5.108b}$$

$$\begin{aligned}\{\mathbf{P},\mathbf{Q}\} \equiv \{&\mathbf{P}(x,\lambda), \mathbf{Q}(x,\lambda), \quad \lambda \in \mathbb{R}\, ,\\ &\mathbf{P}_k^{\pm}(x), \mathbf{Q}_k^{\pm}(x), \quad k = 1,\dots,N\} \, ,\end{aligned} \tag{5.108c}$$

are complete sets of functions in the phase space $\mathcal{M}$. As a consequence, any function $X(x) \in \mathcal{M}$ can be expanded over any of the sets (5.108).

On the other hand, the formulae (5.99), (5.100), and (3.3) allow us to interpret $\boldsymbol{\Psi}^{\pm}(x,\lambda)$, $\boldsymbol{\Phi}^{\pm}(x,\lambda)$ and $\mathbf{P}(x,\lambda)$, $\mathbf{Q}(x,\lambda)$ as eigenfunctions of the operators Λ_+, Λ_- and $\Lambda = \frac{1}{2}(\Lambda_+ + \Lambda_-)$, respectively. These are the eigenfunctions corresponding to the continuous spectrum of the Λ-operators, which fills up the real axis in the complex λ-plane and is therefore doubly degenerate. Note that the L operator has the same doubly degenerate continuous spectrum.

The discrete spectrum of each of these operators Λ_+, Λ_-, Λ, and L is located at the same sets of points $\lambda_k^{\pm} \in \mathbb{C}_{\pm}$, $k = 1, \ldots, N$. However, there is a substantial difference in the structure of the corresponding eigenspaces. In Sect. 3.1, we imposed condition **C2**, which means that the eigenvalues of L are simple, i.e. the corresponding eigenspaces of L are one-dimensional.

As can be seen from the completeness relations (5.59) and (5.63), the eigenspaces of Λ_+, Λ_-, and Λ are two-dimensional, so each eigenvalue is doubly degenerate. For the operator Λ_+, the eigenspaces corresponding to $\lambda_k^{\pm}$ are spanned by $\Psi_k^{\pm}(x)$ and $\dot{\Psi}_k^{\pm}(x)$. While the former are eigenfunctions the latter are not; $\dot{\Psi}_k^{\pm}(x)$ are known as the adjoint eigenfunctions of Λ_+.

Quite analogous is the interpretation of $\Phi_k^{\pm}(x)$ and $\dot{\Phi}_k^{\pm}(x)$ as eigenfunctions and adjoint eigenfunctions of Λ_-. Finally each of the eigenspaces of Λ is spanned by two linearly independent eigenfunctions; see (3.3).

As we shall see in Part II, the degeneracy of the eigenspaces of the Λ-operators plays a fundamental role in their geometric properties.

5.5.1 The Biquadratic Relations

Let us analyze the properties of the squared solutions and the Λ-operators with respect to the skew-symmetric scalar product $[\![\cdot,\cdot]\!]$. We start with the relation:

$$[\![\Lambda_+ X, Y]\!] = [\![X, \Lambda_- Y]\!] \ , \tag{5.109a}$$

which means that the operator Λ_- is the “adjoint” to Λ_+ with respect to the skew-symmetric scalar product. The relation (5.109a) follows from the explicit expressions of Λ_+ and Λ_- and integration by parts. An easy consequence of (5.109a) is

$$[\![\Lambda X, Y]\!] = [\![X, \Lambda Y]\!] \ , \tag{5.109b}$$

which means that Λ is “self-adjoint” with respect to the skew-symmetric scalar product.

The skew-symmetric scalar products of the squared solutions can be evaluated with the help of the so-called “biquadratic” relations for the Jost solutions. In order to derive them, we shall use the definition (5.90) of $\mathcal{E}_\alpha(x,\lambda)$ and remind that $\mathcal{E}_\alpha(x,\lambda)$ and $\mathcal{E}_\beta(x,\mu)$ are solutions of the following equations:

$$i\frac{d\mathcal{E}_\alpha}{dx} + [q(x) - \lambda\sigma_3, \mathcal{E}_\alpha(x,\lambda)] = 0 \ , \tag{5.110}$$

$$i\frac{d\mathcal{E}_\alpha}{dx} + [q(x) - \mu\sigma_3, \mathcal{E}_\beta(x,\mu)] = 0 \,. \tag{5.111}$$

Now, we multiply (5.110) and (5.111) by $\mathcal{E}_\beta(x,\mu)$ and $\mathcal{E}_\alpha(x,\lambda)$, respectively, add them together and take the trace. After some standard rearrangements using the properties of the trace we get:

$$i\frac{d}{dx}\frac{\operatorname{tr}(\mathcal{E}_\alpha(x,\lambda)\mathcal{E}_\beta(x,\mu))}{\lambda-\mu} + \operatorname{tr}\,(\mathcal{E}_\alpha(x,\lambda),[\sigma_3,\mathcal{E}_\beta(x,\mu)) = 0 \,. \tag{5.112}$$

Now, we integrate with respect to dx to get:

$$\begin{aligned}[\![\mathcal{E}_\alpha(x,\lambda),\mathcal{E}_\beta(x,\mu)]\!] &\equiv [\![\boldsymbol{\Phi}_\alpha(x,\lambda),\boldsymbol{\Phi}_\beta(x,\mu)]\!] \\ &= \left.\frac{i\operatorname{tr}(\mathcal{E}_\alpha(x,\lambda)\mathcal{E}_\beta(x,\mu))}{2(\mu-\lambda)}\right|_{x=-\infty}^{\infty} \,. \end{aligned} \tag{5.113}$$

Here, we used the fact that only the off-diagonal parts $\boldsymbol{\Phi}_\alpha$, $\boldsymbol{\Phi}_\beta$ of $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ contribute to the skew-symmetric scalar product. Setting $\chi(x,\lambda) \equiv \chi^+(x,\lambda)$ or $\chi(x,\lambda) \equiv \chi^-(x,\lambda)$, and taking σ_α, σ_β to be σ_+, σ_- or σ_3, we obtain in the left-hand side of (5.113) all the possible relations of the type (5.113) for the "squared solutions." The right-hand side of (5.113) can be evaluated explicitly in terms of the scattering data with the help of the well-known asymptotics of $\chi^\pm(x,\lambda)$ for $x \to \pm\infty$.

Obviously, the knowledge of the skew-symmetric scalar products (5.113) allows one to calculate the expansion coefficients and thus to expand any of the "squared" solutions (e.g. $\boldsymbol{\Psi}^+(x,\lambda)$ over any of the other sets of "squared" solutions (e.g. over $\boldsymbol{\Phi}^+(x,\lambda)$. Here, we have to note that since $\boldsymbol{\Psi}^+(x,\lambda)$ do not tend to 0 for $x \to \pm\infty$, some of these expansion coefficients will contain singularities.

We already encountered the necessity of these limits in the calculation of the generating operators. But now, in addition to the data in Table 5.1, for real λ, we need also the formulae:

$$\lim_{y\to-\infty} \text{V.P.}\,\frac{e^{\pm i\lambda y}}{\lambda} = \mp i\pi\delta(\lambda), \qquad \lim_{y\to\infty} \text{V.P.}\,\frac{e^{\pm i\lambda y}}{\lambda} = \pm i\pi\delta(\lambda) \,, \tag{5.114}$$

Exactly, the terms of this type are responsible for the appearance of the above mentioned singularities.

The results for all possible "biquadratic" relations are collected in Table 5.2, where

$$V^\pm(\lambda-\mu) = \frac{1}{\lambda-\mu} \pm i\pi\delta(\lambda-\mu) \,, \tag{5.115a}$$

$$U^\pm(\lambda,\mu) = \frac{iu^\pm(\lambda)u^\pm(\mu)}{\mu-\lambda} \,, \tag{5.115b}$$

Table 5.2. The skew-symmetric scalar products between the "squared solutions"; the notations are given in formulae (5.115)

	$\boldsymbol{\Phi}^+(x,\mu)$	$\boldsymbol{\Phi}^-(x,\mu)$	$\boldsymbol{\Psi}^+(x,\mu)$	$\boldsymbol{\Psi}^-(x,\mu)$
$\boldsymbol{\Phi}^+(x,\lambda)$	$V^+(\lambda-\mu)$	$U(\lambda,\mu)$	$-D^+(\lambda,\mu)$	0
$\boldsymbol{\Phi}^-(x,\lambda)$	$-U(\mu,\lambda)$	$V^-(\lambda,\mu)$	0	$D^-(\lambda,\mu)$
$\boldsymbol{\Psi}^+(x,\lambda)$	$D^+(\lambda,\mu)$	0	$-U^+(\lambda,\mu)$	$W(\lambda,\mu)$
$\boldsymbol{\Psi}^-(x,\lambda)$	0	$-D^-(\lambda,\mu)$	$-W(\mu,\lambda)$	$-U^-(\lambda,\mu)$
$\frac{\boldsymbol{\Theta}^+(x,\lambda)}{a^+(\lambda)}$	$v^+(\mu)Y^+$	$v^-(\mu)Y^+$	$u^+(\mu)Y^+$	$u^-(\mu)Y^+$
$\frac{\boldsymbol{\Theta}^-(x,\lambda)}{a^-(\lambda)}$	$v^+(\mu)Y^-$	$v^-(\mu)Y^-$	$u^+(\mu)Y^-$	$u^-(\mu)Y^-$

$$W(\lambda,\mu) = 2\pi\delta(\lambda-\mu)a^+(\lambda)a^-(\lambda) - \frac{iu^+(\lambda)u^-(\mu)}{\mu-\lambda}\,, \tag{5.115c}$$

$$U(\lambda,\mu) = -2\pi\delta(\lambda-\mu)a^+(\lambda)a^-(\lambda) + \frac{iv^+(\lambda)v^-(\mu)}{\mu-\lambda}\,, \tag{5.115d}$$

$$D^\pm(\lambda,\mu) = 2\pi(a^\pm(\lambda))^2\delta(\lambda-\mu), \tag{5.115e}$$

$$v^\pm(\lambda) = (a^\pm(\lambda))^2\rho^\pm(\lambda), \qquad u^\pm(\lambda) = (a^\pm(\lambda))^2\tau^\pm(\lambda)\,, \tag{5.115f}$$

$$Y^\pm \equiv Y^\pm(\lambda-\mu) = -\frac{i}{\lambda-\mu} \pm 2\pi\delta(\lambda-\mu)\,. \tag{5.115g}$$

From formulae (5.66) and (5.67) we see that in order to expand the function $X(x)$ over the systems of "squared solutions", one needs to evaluate the skew-symmetric scalar products $[\![\boldsymbol{\Phi}^\pm(x,\lambda), X(x)]\!]$, etc. Therefore, the results in Table 5.2 allow us to expand the "squared" solutions over themselves, say $\boldsymbol{\Theta}^\pm(x,\lambda)$ over any of the systems of "squared" solutions. Skipping the details, we just list the results. For $\lambda \in \mathbb{C}_+$, the expansion over $\boldsymbol{\Psi}^\pm$ reads:

$$\begin{aligned}
\frac{\boldsymbol{\Theta}^{+,f}(x,\lambda)}{a^+(\lambda)} &= 2\rho^+(\lambda)\boldsymbol{\Psi}^+(x,\lambda) \\
&\quad - \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{d\mu}{\mu-\lambda}\left(\rho^+(\mu)\boldsymbol{\Psi}^+(x,\mu) - \rho^-(\mu)\boldsymbol{\Psi}^-(x,\mu)\right) \\
&\quad - 2\sum_{k=1}^{N}\left(\frac{C_k^+}{\lambda_k^+-\lambda}\boldsymbol{\Psi}_k^+(x) + \frac{C_k^-}{\lambda_k^--\lambda}\boldsymbol{\Psi}_k^-(x)\right) \qquad (5.116a) \\
&= 2\rho^+(\lambda)\boldsymbol{\Psi}^+(x,\lambda) + (\Lambda_+-\lambda)^{-1}q(x)\,, \qquad (5.116b)
\end{aligned}$$

while the one over $\boldsymbol{\Phi}^{\pm}$ has the form:

$$
\begin{aligned}
\frac{\boldsymbol{\Theta}^{+,f}(x,\lambda)}{a^{+}(\lambda)} &= -2\tau^{+}(\lambda)\boldsymbol{\Phi}^{+}(x,\lambda) \\
&\quad + \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{d\mu}{\mu-\lambda}\left(\tau^{+}(\mu)\boldsymbol{\Phi}^{+}(x,\mu) - \tau^{-}(\mu)\boldsymbol{\Phi}^{-}(x,\mu)\right) \\
&\quad + 2\sum_{k=1}^{N}\left(\frac{M_k^{+}}{\lambda_k^{+}-\lambda}\boldsymbol{\Phi}_k^{+}(x) + \frac{M_k^{-}}{\lambda_k^{-}-\lambda}\boldsymbol{\Phi}_k^{-}(x)\right) \qquad (5.117a) \\
&= -2\tau^{+}(\lambda)\boldsymbol{\Phi}^{+}(x,\lambda) + (\Lambda_{-}-\lambda)^{-1}q(x)\ . \qquad (5.117b)
\end{aligned}
$$

Analogously, for $\lambda \in \mathbb{C}_{-}$ we find:

$$
\frac{\boldsymbol{\Theta}^{-,f}(x,\lambda)}{a^{-}(\lambda)} = -2\rho^{-}(\lambda)\boldsymbol{\Psi}^{+}(x,\lambda) + (\Lambda_{+}-\lambda)^{-1}q(x)\ , \qquad (5.118a)
$$

and

$$
\frac{\boldsymbol{\Theta}^{-,f}(x,\lambda)}{a^{-}(\lambda)} = -2\tau^{-}(\lambda)\boldsymbol{\Phi}^{-}(x,\lambda) + (\Lambda_{-}-\lambda)^{-1}q(x)\ . \qquad (5.118b)
$$

Obviously relations (5.101) can be considered as consequences of (5.116), (5.117a), (5.117b), (5.118a) and (5.118b).

5.5.2 Biorthogonality of the Squared Solutions

The important consequence of the results in the previous section is the biorthogonality property of the "squared" solutions with respect to the skew-symmetric scalar product. Indeed, from Table 5.2 and from (7.50) there follows:

$$
\begin{aligned}
[\![\boldsymbol{\Phi}^{+}(x,\lambda), \boldsymbol{\Psi}^{+}(x,\mu)]\!] &= -\pi(a^{+}(\lambda))^2\delta(\lambda-\mu)\ , \\
[\![\boldsymbol{\Phi}^{-}(x,\lambda), \boldsymbol{\Psi}^{-}(x,\mu)]\!] &= \pi(a^{-}(\lambda))^2\delta(\lambda-\mu)\ . \qquad (5.119)
\end{aligned}
$$

Using these relations, we can expand $\boldsymbol{\Phi}^{+}(x,\mu)$ and $\boldsymbol{\Phi}^{-}(x,\mu)$ over "themselves". The result is trivial: $\boldsymbol{\Phi}^{+}(x,\lambda) = \boldsymbol{\Phi}^{+}(x,\lambda)$. Analogously with (5.119), we can expand $\boldsymbol{\Psi}^{+}(x,\mu)$ and $\boldsymbol{\Psi}^{-}(x,\mu)$ over themselves with the same trivial result.

In order to derive the corresponding relations for the "squared" solutions related to the discrete spectrum, we would need the limits in Table 5.2 evaluated for $\lambda = \lambda_k^{\pm}$. In fact we already used them implicitly in the evaluation of the expansion coefficients of $\boldsymbol{\Theta}^{\pm}(x,\mu)/a^{\pm}(\mu)$. Skipping the technical details we list the results:

$$[\![\boldsymbol{\Phi}_k^{\pm}(x), \boldsymbol{\Psi}_m^{\pm}(x)]\!] = 0, \qquad [\![\dot{\boldsymbol{\Phi}}_k^{\pm}(x), \boldsymbol{\Psi}_m^{\pm}(x)]\!] = -\frac{i}{2}(\dot{a}_k^{\pm})^2\delta_{km} , \tag{5.120}$$

$$[\![\boldsymbol{\Phi}_k^{\pm}(x), \dot{\boldsymbol{\Psi}}_m^{\pm}(x)]\!] = -\frac{i}{2}(\dot{a}_k^{\pm})^2\delta_{km}, \qquad [\![\dot{\boldsymbol{\Phi}}_k^{\pm}(x), \dot{\boldsymbol{\Psi}}_m^{\pm}(x)]\!] = -\frac{i}{2}\dot{a}_k^{\pm}\ddot{a}_k^{\pm}\delta_{km} .$$

All the other skew-symmetric scalar products vanish.

Next, we can use the above results to derive also the skew-symmetric scalar products between the elements of the symplectic basis:

$$[\![\mathbf{P}(x,\lambda), \mathbf{P}(x,\mu)]\!] = 0, \qquad [\![\mathbf{Q}(x,\lambda), \mathbf{Q}(x,\mu)]\!] = 0 , \tag{5.121a}$$

$$[\![\mathbf{P}(x,\lambda), \mathbf{Q}(x,\mu)]\!] = -i\delta(\lambda-\mu), \qquad [\![\mathbf{P}_k^{\pm}(x), \mathbf{Q}_m^{\pm}(x)]\!] = -i\delta_{km} , \tag{5.121b}$$

In other words, we can say the the symplectic basis is the analog of the orthogonal basis with respect to the skew-symmetric scalar product.

5.5.3 The Green Functions of $\Lambda_{\pm}$

Let us now briefly outline the relation between $G(x,y,\lambda)$ (5.64) and the Green functions of the operators $\Lambda_{\pm}$. This can be done by applying the contour integration method to the following integral:

$$\mathcal{J}_4(x,y,\lambda) = \frac{1}{2\pi i}\left(\oint_{C_+} \frac{d\mu\, G^+(x,y,\mu)}{\mu-\lambda} - \oint_{C_-} \frac{d\mu\, G^-(x,y,\mu)}{\mu-\lambda}\right) . \tag{5.122}$$

Due to the additional factor $\mu-\lambda$ in the denominator, we conclude now that the integrals along the infinite semi-arcs vanish. However, the same factor for $\lambda\in\mathbb{C}_+$ adds one more singular point to the integrand of the first integral in the right-hand side of (5.122). The residue at that point is equal to $G^+(x,y,\lambda)$.

Skipping the rest of the details, we formulate the analog of the spectral decomposition of the Green function $G(x,y,\lambda)$ (5.36), which follows from (5.122):

$$G(x,y,\lambda) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\mu}{\mu-\lambda}\left(\frac{\boldsymbol{\Psi}^+(x,\mu)\otimes\boldsymbol{\Phi}^+(y,\mu)}{(a^+(\mu))^2}\right. \tag{5.123a}$$

$$\left. - \frac{\boldsymbol{\Psi}^-(x,\mu)\otimes\Phi^-(y,\mu)}{(a^-(\mu))^2}\right) + 2i\sum_{k=1}^{N}\left(Y_k^+ + Y_k^-\right)(x,y) ,$$

$$Y_k^{\pm}(x,y) = \frac{1}{(\lambda-\lambda_k^{\pm})(\dot{a}_k^{\pm})^2}\left\{\dot{\boldsymbol{\Psi}}_k^{\pm}(x)\otimes\boldsymbol{\Phi}_k^{\pm}(y) + \boldsymbol{\Psi}_k^{\pm}(x)\otimes\dot{\boldsymbol{\Phi}}_k^{\pm}(y)\right.$$

$$\left. - \left(\frac{\ddot{a}_k^{\pm}}{\dot{a}_k^{\pm}} + \frac{1}{\lambda-\lambda_k^{\pm}}\right)\boldsymbol{\Psi}_k^{\pm}(x)\otimes\boldsymbol{\Phi}_k^{\pm}(y)\right\} . \tag{5.123b}$$

Let us apply $(\Lambda_+ - \lambda)\otimes\mathbb{1}$ on (5.123) on the left and make use of (5.105), (5.106) and the completeness relation (5.89). This gives the result:

$$\begin{aligned}((\Lambda_+ - \lambda) \otimes \mathbb{1})\, G(x, y, \lambda) &= \Pi_0 \delta(x - y) \\ &= (\sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_+)\delta(x - y) \,. \quad (5.124)\end{aligned}$$

We can now apply to both sides of (5.124) the contraction $\mathcal{C}_1$, which is defined by:

$$\mathcal{C}_1(X \otimes Y) = XY \,, \tag{5.125}$$

i.e. it reduces the tensor product to the standard matrix multiplication. Then (5.124) leads to:

$$(\Lambda_+ - \lambda)\, \mathbf{G}^{(1)}(x, y, \lambda) = \mathbb{1}\delta(x - y) \,, \tag{5.126}$$

where $\mathbf{G}^{(1)} = \mathcal{C}_1(G(x, y, \lambda)(\mathbb{1} \otimes \sigma_3))$.

We remind that by construction $G(x, y, \lambda)$ is section-analytic function of λ. We recall also that in Sect. 5.2 we established that it falls off exponentially for $|x|$, $|y| \to \infty$. These properties, combined with (5.126), show us that $\mathbf{G}^{(1)}(x, y, \lambda)$ is the resolvent kernel of the operator Λ_+.

Quite analogously, we can construct the resolvent kernel of the operator Λ_-. This time we shall need another contraction:

$$\mathcal{C}_2(X \otimes Y) = YX = \mathcal{C}_1(Y \otimes X) \,, \tag{5.127}$$

which first exchanges the positions of the elements in the tensor product and then applies the contraction $\mathcal{C}_1$. We apply $\mathcal{C}_2$ to:

$$(\mathbb{1} \otimes (\Lambda_- - \lambda))\, G(y, x, \lambda) = \Pi_0 \delta(x - y) \,, \tag{5.128}$$

which also follows from (5.123) and (5.99), (5.100). As a result we find that

$$(\Lambda_- - \lambda)\mathbf{G}^{(2)}(x, y, \lambda) = \delta(x - y) \,, \tag{5.129}$$

where $\mathbf{G}^{(2)}(x, y, \lambda) = \mathcal{C}_2\left(G(x, y, \lambda)(\mathbb{1} \otimes \sigma_3)\right)$. Thus $\mathbf{G}^{(2)}(x, y, \lambda)$ is the kernel of the Green function of Λ_-.

The relations (5.126) and (5.129) can be derived also by a rather tedious direct calculation using the explicit form of $\Lambda_\pm$ and $G(x, y, \lambda)$.

We end this section with the following proposition.

Proposition 5.4. *For any smooth function $f(\lambda)$, which has no singularities on the spectrum of L the following relations hold:*

$$f(\Lambda_+)q(x) = f(\Lambda_-)q(x) = f(\Lambda)q(x) \,. \tag{5.130}$$

Proof. From the relations (5.105), (5.106) and (5.107) we find:

$$(f(\Lambda_+) - f(\lambda))\boldsymbol{\Psi}^{\pm}(x, \lambda) = 0 \,, \tag{5.131a}$$

$$(f(\Lambda_+) - f(\lambda_k^{\pm}))\boldsymbol{\Psi}_k^{\pm}(x) = 0 \,, \tag{5.131b}$$

$$(f(\Lambda_+) - f(\lambda_k^\pm))\dot{\boldsymbol{\Psi}}_k^\pm(x) = \dot{f}(\lambda_k^\pm)\boldsymbol{\Psi}_k^\pm(x) \;, \tag{5.131c}$$

$$(f(\Lambda_-) - f(\lambda))\boldsymbol{\Phi}^\pm(x,\lambda) = 0 \;, \tag{5.132a}$$

$$(f(\Lambda_-) - f(\lambda_k^\pm))\boldsymbol{\Phi}_k^\pm(x) = 0 \;, \tag{5.132b}$$

$$(f(\Lambda_-) - f(\lambda_k^\pm))\dot{\boldsymbol{\Phi}}_k^\pm(x) = \dot{f}(\lambda_k^\pm)\boldsymbol{\Phi}_k^\pm(x) \;, \tag{5.132c}$$

$$(f(\Lambda) - f(\lambda))\mathbf{P}(x,\lambda) = 0, \quad (f(\Lambda) - f(\lambda))\mathbf{Q}(x,\lambda) = 0 \;, \tag{5.133a}$$

$$(f(\Lambda) - f(\lambda_k^\pm))\mathbf{P}_k^\pm(x) = 0, \quad (f(\Lambda) - f(\lambda_k^\pm))\mathbf{Q}_k^\pm(x) = 0 \;. \tag{5.133b}$$

On the other hand, from (5.76), (5.78) and (5.80), we have:

$$\begin{aligned} f(\Lambda_+)q(x) = {} & \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda\, f(\lambda)\left(\tau^+(\lambda)\boldsymbol{\Phi}^+(x,\lambda) - \tau^-(\lambda)\boldsymbol{\Phi}^-(x,\lambda)\right) \\ & + 2\sum_{k=1}^{N}\left(f_k^+ M_k^+ \boldsymbol{\Phi}_k^+(x) + f_k^- M_k^- \boldsymbol{\Phi}_k^-(x)\right) \;, \end{aligned} \tag{5.134}$$

$$\begin{aligned} f(\Lambda_-)q(x) = {} & -\frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda\, f(\lambda)\left(\rho^+(\lambda)\boldsymbol{\Phi}^+(x,\lambda) - \rho^-(\lambda)\boldsymbol{\Phi}^-(x,\lambda)\right) \\ & - 2\sum_{k=1}^{N}\left(f_k^+ C_k^+ \boldsymbol{\Phi}_k^+(x) + f_k^- C_k^- \boldsymbol{\Phi}_k^-(x)\right) \;, \end{aligned} \tag{5.135}$$

$$f(\Lambda)q(x) = i\int_{-\infty}^{\infty} d\lambda\, f(\lambda)\mathbf{P}(x,\lambda) + i\sum_{k=1}^{N}\left(f_k^+\mathbf{P}_k^+(x) + f_k^-\mathbf{P}_k^-(x)\right) \;. \tag{5.136}$$

where $f_k^\pm = f(\lambda_k^\pm)$.

Now, due to the definitions of the functions of the symplectic basis (5.61a) and (5.61b), we see that the right-hand sides of (5.134) and (5.135) coincide with the right-hand side of (5.136). The proposition is proved.

5.6 Expansions Over the "Products of Solutions"

5.6.1 Generalized Wronskian Relations

The Wronskian relations derived in Sect. 5.1 can be generalized and used to relate the scattering data and the potentials of two different Zakharov–Shabat systems.

Let us consider two ZS systems: Equation (5.1) with potential $U(x,\lambda)$ and (5.1') with potential $U'(x,\lambda)$:

$$U(x,\lambda) = q'(x,t) - \lambda\sigma_3 \ . \tag{5.137}$$

All quantities such as Jost solutions, scattering matrix and its elements, FAS etc. will be denoted by the same letter with additional "prime".

We start with the generalized Wronskian identities [3, 4]:

$$\begin{aligned}
(\hat{\chi}'\sigma_3\chi(x,\lambda) - \sigma_3)|_{-\infty}^{\infty} &= -i\int_{-\infty}^{\infty} dx\, \hat{\chi}'[U'(x,\lambda)\sigma_3 - U(x,\lambda)\sigma_3]\chi(x,\lambda)\ ,\\
&= i\int_{-\infty}^{\infty} dx\, \hat{\chi}\sigma_3(q'(x)+q(x))\chi(x,\lambda),
\end{aligned} \tag{5.138}$$

$$\begin{aligned}
(\hat{\chi}'\chi(x,\lambda) - \mathbb{1})|_{-\infty}^{\infty} &= -i\int_{-\infty}^{\infty} dx\, \hat{\chi}'[U'(x,\lambda) - U(x,\lambda))\chi(x,\lambda),\\
&= -\int_{-\infty}^{\infty} dx\, \hat{\chi}'(q'(x)-q(x))\chi(x,\lambda)\ ,
\end{aligned} \tag{5.139}$$

where $\chi(x,\lambda)$ and $\chi'(x,\lambda)$ can be any fundamental solution of L and L', respectively; for convenience, we choose them to be the FAS introduced earlier.

The left-hand sides of (5.138) and (5.139) can be calculated explicitly by using the asymptotics of $\chi^{\pm}(x,\lambda)$ and $\chi'^{,\pm}(x,\lambda)$ for $x \to \pm\infty$. They are expressed by the matrix elements of the scattering matrix $T(\lambda)$ and $T'(\lambda)$ as follows:

$$\left(\hat{\chi}'^{,\pm}\sigma_3\chi^{\pm}(x,\lambda) - \sigma_3\right)\Big|_{-\infty}^{\infty} = \frac{1}{a'^{,\pm}}\begin{pmatrix} a^{\pm} - a'^{,\pm} & \mp(b'^{,-}a^{\pm} + b^{-}a'^{,\pm}) \\ \mp(b'^{,+}a^{\pm} + b^{+}a'^{,\pm}) & a^{\pm} - a'^{,\pm} \end{pmatrix}, \tag{5.140}$$

$$\left(\hat{\chi}'^{,\pm}\chi^{\pm}(x,\lambda) - \mathbb{1}\right)\Big|_{-\infty}^{\infty} = \frac{1}{a'^{,\pm}}\begin{pmatrix} \pm(a^{\pm} - a'^{,\pm}) & (b'^{,-}a^{\pm} - b^{-}a'^{,\pm}) \\ -(b'^{,+}a^{\pm} - b^{+}a'^{,\pm}) & \mp(a^{\pm} - a'^{,\pm}) \end{pmatrix} \tag{5.141}$$

The individual matrix elements are obtained by multiplying both sides of the above equations by the Pauli matrices σ_α and taking the trace. After such operation the integrands in the right-hand sides of (5.138), (5.139) are expressed through the skew-scalar product (5.12) like in Sect. 5.1 but now instead of $\mathcal{E}(x,\lambda)$ (5.11) there appear

$$\mathcal{E}_{\alpha}'^{;\pm}(x,\lambda) = \chi^{\pm}(x,\lambda)\sigma_\alpha\hat{\chi}'^{,\pm}(x,\lambda)\ , \tag{5.142}$$

and instead of the "squared solutions" we naturally get the "products of the solutions"

$$\boldsymbol{\Phi}'^{,\pm}(x,\lambda) = a'^{,\pm}(\mathcal{E}_{\pm}'^{;\pm}(x,\lambda))^{\mathrm{f}} = \begin{pmatrix} 0 & \pm\phi_1'^{,\pm}\phi_1^{\pm}(x,\lambda) \\ \mp\phi_2'^{;\pm}\phi_2^{\pm}(x,\lambda) & 0 \end{pmatrix} \tag{5.143}$$

$$\boldsymbol{\Psi}'^{,\pm}(x,\lambda) = a'^{,\pm}(\mathcal{E}'^{,\pm}_{\mp}(x,\lambda))^{\mathrm{f}} = \begin{pmatrix} 0 & \mp\psi_1'^{,\pm}\psi_1^{\pm}(x,\lambda) \\ \pm\psi_2'^{,\pm}\psi_2^{\pm}(x,\lambda) & 0 \end{pmatrix}, \quad (5.144)$$

$$\boldsymbol{\Theta}'^{,\pm}(x,\lambda) = a'^{,\pm}(\mathcal{E}'^{,\pm}_{3}(x,\lambda))^{\mathrm{f}} = \begin{pmatrix} 0 & -\phi_1'^{,\pm}\psi_1^{\pm} - \phi_1^{\pm}\psi_1'^{,\pm} \\ \phi_2'^{,\pm}\psi_2^{\pm} + \phi_2^{\pm}\psi_2'^{,\pm} & 0 \end{pmatrix}, \quad (5.145)$$

$$\boldsymbol{\Xi}'^{,\pm}(x,\lambda) = a'^{,\pm}(\mathcal{E}'^{,\pm}_{0}(x,\lambda))^{\mathrm{f}} = \begin{pmatrix} 0 & \pm\phi_1'^{,\pm}\psi_1^{\pm} \mp \phi_1^{\pm}\psi_1'^{,\pm} \\ \mp\phi_2'^{,\pm}\psi_2^{\pm} \pm \phi_2^{\pm}\psi_2'^{,\pm} & 0 \end{pmatrix}, \quad (5.146)$$

One can repeat the same arguments with $\chi'(x,\lambda)$ and $\chi(x,\lambda)$ interchanged. Thus, we get a second set of Wronskian relations which involve a dual form of $\mathcal{E}'^{,\pm}_{\alpha}$:

$$'\mathcal{E}^{\pm}_{\alpha}(x,\lambda) = \chi'^{,\pm}(x,\lambda)\sigma_\alpha\hat{\chi}^{\pm}(x,\lambda)\,. \quad (5.147)$$

The dual set of the “products of solutions”:

$$'\boldsymbol{\Phi}^{\pm}(x,\lambda) = a^{\pm}(\,'\mathcal{E}^{\pm}_{\pm}(x,\lambda))^{\mathrm{f}}, \qquad '\boldsymbol{\Psi}^{\pm}(x,\lambda) = a^{\pm}(\,'\mathcal{E}^{\pm}_{\mp}(x,\lambda))^{\mathrm{f}}\,, \quad (5.148)$$

$$'\boldsymbol{\Theta}^{\pm}(x,\lambda) = a^{\pm}(\,'\mathcal{E}^{\pm}_{3}(x,\lambda))^{\mathrm{f}}, \qquad '\boldsymbol{\Xi}^{\pm}(x,\lambda) = a^{\pm}(\,'\mathcal{E}^{\pm}_{0}(x,\lambda))^{\mathrm{f}} \quad (5.149)$$

in fact coincide with $\boldsymbol{\Phi}'^{,\pm}(x,\lambda)$, $\boldsymbol{\Psi}'^{,\pm}(x,\lambda)$, $\boldsymbol{\Theta}'^{,\pm}(x,\lambda)$ and $\boldsymbol{\Xi}'^{,\pm}(x,\lambda)$, respectively. Thus we find:

$$\begin{aligned} \rho'^{,\pm}(\lambda) + \rho^{\pm}(\lambda) &= \frac{i}{a^{+}a'^{,+}}\int_{-\infty}^{\infty} \mathrm{tr}\,\left(\hat{\chi}^{+}[q(x),\sigma_3]\chi^{+}(x,\lambda)\sigma_{+}\right) \\ &= \frac{i}{a^{+}a'^{,+}}\left[\!\left[q(x)+q'(x), \boldsymbol{\Phi}'^{,\pm}(x,\lambda)\right]\!\right]\,, \end{aligned} \quad (5.150)$$

$$\tau'^{,\pm}(\lambda) + \tau^{\pm}(\lambda) = \frac{i}{a^{+}a'^{,+}}\left[\!\left[q(x)+q'(x), \boldsymbol{\Psi}'^{,\pm}(x,\lambda)\right]\!\right]\,, \quad (5.151)$$

and

$$\mp\left(\rho'^{,\pm}(\lambda) - \rho^{\pm}(\lambda)\right) = \frac{i}{a^{+}a'^{,+}}\left[\!\left[\sigma_3(q'(x)-q(x)), \boldsymbol{\Phi}'^{,\pm}(x,\lambda)\right]\!\right]\,, \quad (5.152)$$

$$\pm(\tau'^{,\pm}(\lambda) + \tau^{\pm}(\lambda)) = \frac{i}{a^{+}a'^{,+}}\left[\!\left[\sigma_3(q'(x)-q(x)), \boldsymbol{\Psi}'^{,\pm}(x,\lambda)\right]\!\right]\,, \quad (5.153)$$

These “products of solutions” effectively coincide with the ones that appeared originally in [3, 4]. We keep this form with the zeroes on the diagonal for later purposes, when we pose analogous problems for the gauge-equivalent system $\tilde{L}$.

The generalized Wronskian relations derived above allow one to analyze the mappings between the pairs of potentials $\{q(x), q'(x)\}$ and the pairs of minimal sets of scattering data $\{\mathcal{T}', \mathcal{T}\}$. To this end, we will show that the "products of solutions" (5.143), (5.144), just like the "squared" solutions, are complete sets of functions in the space of allowed potentials.

The proof is based again on the contour integration method, this time applied to the Green function:

$$G'(x,y,\lambda) = \begin{cases} G'^{,+}(x,y,\lambda), & \text{for } \lambda \in \mathbb{C}_+ \,, \\ 1/2(G'^{,+}(x,y,\lambda) + G'^{,-}(x,y,\lambda)), & \text{for } \lambda \in \mathbb{R} \,, \\ G'^{,-}(x,y,\lambda), & \text{for } \lambda \in \mathbb{C}_- \,, \end{cases} \tag{5.154}$$

where

$$G'^{,\pm}(x,y,\lambda) = G_1'^{,\pm}(x,y,\lambda)\theta(x-y) - G_2'^{,\pm}(x,y,\lambda)\theta(y-x) \,, \tag{5.155}$$

$$G_1'^{,\pm}(x,y,\lambda) = \frac{1}{a'^{,\pm}(\lambda)a^{\pm}(\lambda)} \boldsymbol{\Psi}'^{,\pm}(x,\lambda) \otimes {}'\boldsymbol{\Phi}^{\pm}(y,\lambda) \,, \tag{5.156}$$

$$\begin{aligned} G_2'^{,\pm}(x,y,\lambda) = \frac{1}{a'^{,\pm}(\lambda)a^{\pm}(\lambda)} &\left(\boldsymbol{\Phi}'^{,\pm}(x,\lambda) \otimes {}'\boldsymbol{\Psi}^{\pm}(y,\lambda) \right. \\ &\left. + \frac{1}{2}\boldsymbol{\Theta}'^{,\pm}(x,\lambda) \otimes {}'\boldsymbol{\Theta}^{\pm}(y,\lambda) + \frac{1}{2}\boldsymbol{\Xi}'^{,\pm}(x,\lambda) \otimes {}'\boldsymbol{\Xi}^{\pm}(y,\lambda) \right) . \end{aligned} \tag{5.157}$$

Now, we consider the integral

$$\begin{aligned} \mathcal{J}_G'(x,y) &= \frac{1}{2\pi i} \left(\oint_{C_+} d\lambda\, G'^{,+}(x,y,\lambda) - \oint_{C_-} d\lambda\, G'^{,-}(x,y,\lambda) \right) \\ &= \sum_{k=1}^{N} \left(\operatorname*{Res}_{\lambda=\lambda_k^+} G'^{,+}(x,y,\lambda) + \operatorname*{Res}_{\lambda=\lambda_k^-} G'^{,-}(x,y,\lambda) \right) . \end{aligned} \tag{5.158}$$

Obviously $G'^{,\pm}$ has poles whenever $a^{\pm}(\lambda)$ or $a'^{,\pm}(\lambda)$ have zeroes. In what follows, we will denote the sets of zeroes of $a^{\pm}(\lambda)$ and $a'^{,\pm}(\lambda)$ by $\mathcal{Z}^{\pm}$ and $\mathcal{Z}'^{,\pm}$, respectively:

$$\mathcal{Z}^{\pm} = \left\{ \lambda_j^{\pm}, \quad a^{\pm}(\lambda_j^{\pm}) = 0, \quad j = 1, \dots, N \right\}, \qquad \lambda_j^{\pm} \in \mathbb{C}_{\pm} \tag{5.159}$$

$$\mathcal{Z}'^{,\pm} = \left\{ \lambda_j'^{,\pm}, \quad a'^{,\pm}(\lambda_j'^{,\pm}) = 0, \quad j = 1, \dots, N' \right\}, \qquad \lambda_j'^{,\pm} \in \mathbb{C}_{\pm} \tag{5.160}$$

To be more specific, we assume that all zeroes are simple. Therefore, $G'^{,\pm}(x,y,\lambda)$ would have second-order poles at those $\lambda_j^{\pm}$, which happen to be equal to some $\lambda_k'^{,\pm}$. Without loss of generality, we split the set of indices $\mathcal{N} \equiv \{1, \dots, N\}$ into three subsets: $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_1 \cup \mathcal{N}_2$, where

$$
\begin{aligned}
\mathcal{N}_0 &= \{j \in \mathcal{N}_0 \colon a^{\pm}(\lambda_j^{\pm}) = 0, \quad a'^{,\pm}(\lambda_j'^{,\pm}) = 0\} \,, \\
\mathcal{N}_1 &= \{j \in \mathcal{N}_1 \colon a^{\pm}(\lambda_j^{\pm}) = 0, \quad a'^{,\pm}(\lambda_j'^{,\pm}) \neq 0\} \,, \\
\mathcal{N}_2 &= \{j \in \mathcal{N}_2 \colon a^{\pm}(\lambda_j^{\pm}) \neq 0, \quad a'^{,\pm}(\lambda_j'^{,\pm}) = 0\}.
\end{aligned}
\tag{5.161}
$$

With these notations, the Green function $G'^{,\pm}(x, y, \lambda)$ will have poles of second order at $\lambda_j^{\pm}$ if $j \in \mathcal{N}_0$ and first-order poles if $j \in \mathcal{N}_1 \cup \mathcal{N}_2$. Applying Cauchy residue theorem, we have:

$$
\mathcal{J}'_G(x, y) = \sum_{j\in\mathcal{N}_0} (X_j'^{,+} + X_j'^{,-}) + \sum_{j\in\mathcal{N}_1} (Y_j'^{,+} + Y_j'^{,-}) + \sum_{j\in\mathcal{N}_2} (Z_j'^{,+} + Z_j'^{,-}) \,, \tag{5.162}
$$

where

$$
\begin{aligned}
X_j'^{,\pm}(x, y) = \frac{1}{\dot{a}_j^{\pm} \dot{a}_j'^{,\pm}} \Big\{ & \boldsymbol{\Psi}_j'^{,\pm}(x) \otimes {}'\dot{\boldsymbol{\Phi}}_j^{\pm}(y) + \dot{\boldsymbol{\Psi}}_j'^{,\pm}(x) \otimes {}'\boldsymbol{\Phi}_j^{\pm}(y) \\
& - \left(\frac{\dot{a}_j^{\pm} \ddot{a}_j'^{,\pm} + \ddot{a}_j^{\pm} \dot{a}_j'^{,\pm}}{2\dot{a}_j^{\pm} \dot{a}_j'^{,\pm}} \right) \boldsymbol{\Psi}_j'^{,\pm}(x) \otimes {}'\boldsymbol{\Phi}_j^{\pm}(y) \Big) \,, \qquad j \in \mathcal{N}_0;
\end{aligned}
\tag{5.163}
$$

$$
Y_j'^{,\pm}(x, y) = \frac{1}{\dot{a}_j^{\pm} a'^{,\pm}(\lambda_j^{\pm})} \boldsymbol{\Psi}_j'^{,\pm}(x) \otimes {}'\boldsymbol{\Phi}_j^{\pm}(y), \qquad j \in \mathcal{N}_1; \tag{5.164}
$$

$$
Z_j'^{,\pm}(x, y) = \frac{1}{a_j^{\pm}(\lambda_j'^{,\pm}) \dot{a}'^{,\pm}} \boldsymbol{\Psi}_j'^{,\pm}(x) \otimes {}'\boldsymbol{\Phi}_j^{\pm}(y), \qquad j \in \mathcal{N}_2 \,. \tag{5.165}
$$

Working out the residues we shall make use of the relations (5.49).

Theorem 5.5. *For real* λ

$$
G_1'^{,+}(x, y, \lambda) + G_2'^{,+}(x, y, \lambda) = G_1'^{,-}(x, y, \lambda) + G_2'^{,-}(x, y, \lambda) \,. \tag{5.166}
$$

Proof. From (5.37) and (5.38), (5.39) we find:

$$
\begin{aligned}
& G_1'^{,\pm}(x, y, \lambda) + G_2'^{,\pm}(x, y, \lambda) \\
&= \left(\chi^{\pm}(x, \lambda) \otimes \chi'^{,\pm}(y, \lambda) \right) \mathit{\Pi} \left(\hat{\chi}'^{,\pm}(x, \lambda) \otimes \hat{\chi}^{\pm}(y, \lambda) \right) \,,
\end{aligned}
\tag{5.167}
$$

where $\mathit{\Pi}$ is the second Casimir endomorphism of the algebra $sl(2)$ and has the property (5.44). It remains to use (5.46) to conclude the proof.

The integrals along the infinite arcs of the contours are evaluated in a way similar to the one in Sect. 5.2. We have:

$$
\mathcal{J}'_{G,\infty} = -\frac{i}{2} \delta(x - y) \mathit{\Pi}_0 \,, \tag{5.168}
$$

where $\mathit{\Pi}_0$ is the same as in (5.59b).

Finally, we have to evaluate the jump of the Green function across the real axis. Due to (5.167) we find:

$$
\begin{aligned}
G^{\prime,+}(x,y,\lambda) - G^{\prime,-}(x,y,\lambda) &= G_1^{\prime,+}(x,y,\lambda) - G_1^{\prime,-}(x,y,\lambda) \\
&= \frac{\boldsymbol{\Psi}^{\prime,+}(x,\lambda) \otimes {}'\boldsymbol{\Phi}^{+}(y,\lambda)}{a^{+}(\lambda)a^{\prime,+}(\lambda)} - \frac{\Psi^{\prime,-}(x,\lambda) \otimes {}'\boldsymbol{\Phi}^{-}(y,\lambda)}{a^{-}(\lambda)a^{\prime,-}(\lambda)} \, .
\end{aligned} \tag{5.169}
$$

Equating both answers for the integral $\mathcal{J}_G(x,y)$, we get the completeness relation for the "products" of solutions in the form:

$$
\begin{aligned}
\delta(x-y)\mathit{\Pi}_0 = &-\frac{1}{\pi}\int_{-\infty}^{\infty} d\lambda \left(\frac{\boldsymbol{\Psi}^{\prime,+}(x,\lambda) \otimes {}'\boldsymbol{\Phi}^{+}(y,\lambda)}{a^{+}(\lambda)a^{\prime,+}(\lambda)} - \frac{\Psi^{\prime,-}(x,\lambda) \otimes {}'\boldsymbol{\Phi}^{-}(y,\lambda)}{a^{-}(\lambda)a^{\prime,-}(\lambda)} \right) \\
&+ 2i \left(\sum_{j\in\mathcal{N}_0}^{\pm} X_j^{\prime,\pm} + \sum_{j\in\mathcal{N}_1}^{\pm} Y_j^{\prime,\pm} + \sum_{j\in\mathcal{N}_2}^{\pm} Z_j^{\prime,\pm} \right) \, ,
\end{aligned} \tag{5.170}
$$

This relation is compatible with the one derived for the first time in [5, 6, 7].

5.6.2 Expanding $\boldsymbol{X(x)}$ Over the "Products of Solutions"

The next step is to use the completeness relation (5.170) to obtain the expansions over the products of solutions. Using the same technique as in Sect. 5.2, we derive the following expansion of the function $X(x)$ in (5.64):

$$
\begin{aligned}
X(x) = &\frac{1}{\pi}\int_{-\infty}^{\infty} d\lambda \, \left(\psi_X^{\prime,+}(\lambda)' \boldsymbol{\Phi}^{+}(x,\lambda) - \psi_X^{\prime,-}(\lambda)' \boldsymbol{\Phi}^{-}(x,\lambda) \right) \\
&- 2i \sum_{k\in\mathcal{N}_0}^{\pm} \left(\psi_{X,k}^{\pm}\, {}'\dot{\boldsymbol{\Phi}}_k^{\pm} - \dot{\psi}_{X,k}^{\pm}\, {}'\boldsymbol{\Phi}_k^{\pm} \right) - 2i \sum_{k\in\mathcal{N}_1\cup\mathcal{N}_2}^{\pm} \psi_{X,k}^{\pm}\, {}'\boldsymbol{\Phi}_k^{\pm} \, ,
\end{aligned} \tag{5.171}
$$

$$
\begin{aligned}
X(x) = &-\frac{1}{\pi}\int_{-\infty}^{\infty} d\lambda \, \left(\phi_X^{\prime,+}(\lambda) \boldsymbol{\Psi}^{\prime,+}(x,\lambda) - \phi_X^{\prime,-}(\lambda) \boldsymbol{\Psi}^{\prime,-}(x,\lambda) \right) \\
&+ 2i \sum_{k\in\mathcal{N}_0}^{\pm} \left(\phi_{X,k}^{\pm} \dot{\boldsymbol{\Psi}}_k^{\prime,\pm} + \dot{\phi}_{X,k}^{\pm} \boldsymbol{\Psi}_k^{\prime,\pm} \right) + 2i \sum_{k\in\mathcal{N}_1\cup\mathcal{N}_2}^{\pm} \phi_{X,k}^{\pm} \boldsymbol{\Psi}_k^{\prime,\pm} \, ,
\end{aligned} \tag{5.172}
$$

where

$$
\psi_X^{\prime,\pm}(\lambda) = \frac{[\![\boldsymbol{\Psi}^{\prime,\pm}(x,\lambda), X(x)]\!]}{a^{\pm}(\lambda)a^{\prime,\pm}(\lambda)}, \qquad \psi_{X,k}^{\prime,\pm} = \frac{[\![\boldsymbol{\Psi}_k^{\prime,\pm}(x), X(x)]\!]}{A_k^{\pm}} \, , \tag{5.173a}
$$

$$
\dot{\psi}_{X,k}^{\pm} = \frac{1}{A_k^{\pm}} \left[\!\!\left[\dot{\boldsymbol{\Psi}}_k^{\pm}(x) - \frac{\ddot{a}_k^{\pm}}{\dot{a}_k^{\pm}} \boldsymbol{\Psi}_k^{\pm}(x), X(x) \right]\!\!\right], \qquad k \in \mathcal{N}_0 \, , \tag{5.173b}
$$

$$\phi_X^{\prime,\pm}(\lambda) = \frac{[\![{}'\boldsymbol{\Phi}^{\pm}(x,\lambda), X(x)]\!]}{a^{\pm}(\lambda)a^{\prime,\pm}(\lambda)}, \qquad \phi_{X,k}^{\prime,\pm} = \frac{[\![{}'\boldsymbol{\Phi}_k^{\prime,\pm}(x), X(x)]\!]}{A_k^{\pm}}, \quad (5.173c)$$

$$\dot{\phi}_{X,k}^{\pm} = \frac{1}{A_k^{\pm}} \left[\!\!\left[\dot{\boldsymbol{\Phi}}_k^{\pm}(x) - \frac{\ddot{a}_k^{\pm}}{\dot{a}_k^{\pm}} \boldsymbol{\Phi}_k^{\pm}(x), X(x) \right]\!\!\right], \qquad k \in \mathcal{N}_0 \,, \quad (5.173d)$$

$$A_k^{\pm} = \begin{cases} \dot{a}_k^{\pm}\dot{a}_k^{\prime,\pm}, \text{ for } k \in \mathcal{N}_0 \,, \\ \dot{a}_k^{\pm} a_k^{\prime,\pm}, \text{ for } k \in \mathcal{N}_1 \,, \\ \dot{a}_k^{\pm} a_k^{\prime,\pm}, \text{ for } k \in \mathcal{N}_2 . \end{cases} \quad (5.173e)$$

Proposition 5.6. *The function $X(x) \equiv 0$ if and only if one of the following sets of relations holds:*

$$\psi_X^{\prime,+}(\lambda) = \psi_X^{\prime,-}(\lambda) \equiv 0, \qquad \lambda \in \mathbb{R} \,, \quad (5.174a)$$

$$\psi_{X,k}^{\prime,\pm} = 0, \qquad k = 1, \dots, N; \qquad \dot{\psi}_{X,k}^{\prime,\pm} = 0, \quad k \in \mathcal{N}_0 \,, \quad (5.174b)$$

$$\phi_X^{\prime,+}(\lambda) = \phi_X^{\prime,-}(\lambda) \equiv 0, \qquad \lambda \in \mathbb{R} \,, \quad (5.175a)$$

$$\phi_{X,k}^{\prime,\pm} = 0, \quad k = 1, \dots, N; \qquad \dot{\phi}_{X,k}^{\prime,\pm} = 0, \quad k \in \mathcal{N}_0 \quad (5.175b)$$

Proof. Let $X(x) \equiv 0$. Then all coefficients in the right-hand sides of (5.174) and (5.175) vanish due to (5.173). Let now all coefficients in the left-hand sides of (5.174) and (5.175) vanish. Then inserting them in the right hand sides of the expansions (5.171) and (5.172), we get that $X(x) \equiv 0$.

It is easy to check that the expansion coefficients $\phi_X^{\pm}(\lambda)$ and $\psi_X^{\pm}(\lambda)$ can be explicitly evaluated for two particularly important choices of $X(x)$. Indeed, using (5.142) and (5.147), it is easy to express the skew-scalar products $[\![\boldsymbol{\Psi}^{\prime,\pm}(x,\lambda), X(x)]\!]$ and $[\![{}'\boldsymbol{\Phi}^{\pm}(x,\lambda), X(x)]\!]$ for $X(x) = q(x) + q'(x)$ and $X(x) = \sigma_3(q'(x) - q(x))$ through the reflection coefficients $\rho^{\pm}(\lambda), \rho^{\prime,\pm}(\lambda)$ and $\tau^{\pm}(\lambda), \tau^{\prime,\pm}(\lambda)$. Skipping the details one gets the following expansions:

$$\begin{aligned} &\sigma_3(q'(x) - q(x)) \\ &= \frac{i}{\pi} \int_{-\infty}^{\infty} d\lambda \, \left((\rho^{\prime,+}(\lambda) - \rho^{+}(\lambda)) \boldsymbol{\Psi}^{\prime,+}(x,\lambda) + (\rho^{\prime,-}(\lambda) - \rho^{-}(\lambda)) \boldsymbol{\Psi}^{\prime,-}(x,\lambda) \right) \\ &\quad + 2 \sum_{k \in \mathcal{N}_0}^{\pm} \pm (C_k^{\prime,\pm} - C_k^{\pm}) \boldsymbol{\Psi}_k^{\prime,\pm}(x) - 2 \sum_{k \in \mathcal{N}_1}^{\pm} \pm C_k^{\pm} \boldsymbol{\Psi}_k^{\prime,\pm}(x) \\ &\quad + 2 \sum_{k \in \mathcal{N}_2}^{\pm} \pm C_k^{\prime,\pm} \boldsymbol{\Psi}_k^{\prime,\pm}(x), \end{aligned} \quad (5.176)$$

$$= \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda\, \left((\tau^{\prime,+}(\lambda) - \tau^{+}(\lambda))' \boldsymbol{\Phi}^{+}(x,\lambda) + (\tau^{\prime,-}(\lambda) - \tau^{-}(\lambda))' \boldsymbol{\Phi}^{-}(x,\lambda) \right)$$

$$+ 2\sum_{k\in\mathcal{N}_0}^{\pm} \pm (M_k^{\prime,\pm} - M_k^{\pm})' \boldsymbol{\Phi}_k^{\pm}(x) - 2\sum_{k\in\mathcal{N}_1}^{\pm} \pm M_k^{\pm\prime} \boldsymbol{\Phi}_k^{\pm}(x)$$

$$+ 2\sum_{k\in\mathcal{N}_2}^{\pm} \pm M_k^{\prime,\pm\prime} \boldsymbol{\Phi}_k^{\pm}(x)\,, \qquad (5.177)$$

and

$$q'(x) + q(x)$$

$$= -\frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda\, \left((\rho^{\prime,+}(\lambda) + \rho^{+}(\lambda)) \boldsymbol{\Psi}^{\prime,+}(x,\lambda) - (\rho^{\prime,-}(\lambda) + \rho^{-}(\lambda)) \boldsymbol{\Psi}^{\prime,-}(x,\lambda) \right)$$

$$- 2\sum_{k\in\mathcal{N}_0}^{\pm} (C_k^{\prime,\pm} + C_k^{\pm}) \boldsymbol{\Psi}_k^{\prime,\pm}(x) - 2\sum_{k\in\mathcal{N}_1}^{\pm} C_k^{\pm} \boldsymbol{\Psi}_k^{\prime,\pm}(x) - 2\sum_{k\in\mathcal{N}_2}^{\pm} C_k^{\prime,\pm} \boldsymbol{\Psi}_k^{\prime,\pm}(x)\,, \qquad (5.178)$$

$$= \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda\, \left((\tau^{\prime,+}(\lambda) + \tau^{+}(\lambda))' \boldsymbol{\Phi}^{+}(x,\lambda) - (\tau^{\prime,-}(\lambda) + \tau^{-}(\lambda))' \boldsymbol{\Phi}^{-}(x,\lambda) \right)$$

$$+ 2\sum_{k\in\mathcal{N}_0}^{\pm} (M_k^{\prime,\pm} + M_k^{\pm})' \boldsymbol{\Phi}_k^{\pm}(x) + 2\sum_{k\in\mathcal{N}_1}^{\pm} M_k^{\pm\prime} \boldsymbol{\Phi}_k^{\pm}(x) + 2\sum_{k\in\mathcal{N}_2}^{\pm} M_k^{\prime,\pm\prime} \boldsymbol{\Phi}_k^{\pm}(x)\,, \qquad (5.179)$$

In other words, the elements of the two sets of scattering data $\mathcal{T}$ and $\mathcal{T}'$ determine the expansion coefficients of $\sigma_3(q'(x) - q(x))$ and $q'(x) + q(x)$ over the products of solutions.

5.6.3 Generalized Recursion Operators

The completeness relations obtained above can be treated as spectral decompositions of the operators $\Lambda'_{\pm}$, for which the products of solutions are eigenfunctions:

$$\begin{aligned} \Lambda'_{+} \boldsymbol{\Psi}^{\prime,\pm}(x,\lambda) &= \lambda \boldsymbol{\Psi}^{\prime,\pm}(x,\lambda), & \lambda &\in \mathcal{Z}^{\pm} \cup \mathcal{Z}^{\prime,\pm}\,, \\ \Lambda'_{+} \dot{\boldsymbol{\Psi}}_j^{\prime,\pm}(x) &= \lambda_j^{\pm} \dot{\boldsymbol{\Psi}}_j^{\prime,\pm}(x) + \boldsymbol{\Psi}_j^{\prime,\pm}(x), & j &\in \mathcal{N}_0\,, \end{aligned} \qquad (5.180)$$

$$\begin{aligned} \Lambda'_{-}{}',\boldsymbol{\Phi}^{\pm}(x,\lambda) &= \lambda', \boldsymbol{\Phi}^{\pm}(x,\lambda), & \lambda &\in \mathcal{Z}^{\pm} \cup \mathcal{Z}^{\prime,\pm}\,, \\ \Lambda'_{-}{}', \dot{\boldsymbol{\Phi}}_j^{\pm}(x) &= \lambda_j^{\pm\prime}, \dot{\boldsymbol{\Phi}}_j^{\pm}(x) + {}', \boldsymbol{\Phi}_j^{\pm}(x), & j &\in \mathcal{N}_0\,, \end{aligned} \qquad (5.181)$$

The integro-differential operators $\Lambda'_{\pm}$ generalize the recursion operators $\Lambda_{\pm}$. In order to derive their explicit form, we make use of the equation:

$$i\frac{d\mathcal{E}_\alpha^{\prime;\pm}}{dx} + q(x)\mathcal{E}_\alpha^{\prime;\pm}(x,\lambda) - \mathcal{E}_\alpha^{\prime;\pm}(x,\lambda)q'(x) - \lambda[\sigma_3, \mathcal{E}_\alpha^{\prime;\pm}(x,\lambda)] = 0 \ , \qquad (5.182)$$

which is satisfied by all $\mathcal{E}_\alpha^{\prime;\pm}(x,\lambda)$ in (5.142). It is only natural to introduce in (5.182) the splitting

$$\mathcal{E}'(x,\lambda) = \mathcal{E}'^{,\mathrm{d}}(x,\lambda) + \mathcal{E}'^{,\mathrm{f}}(x,\lambda) \qquad (5.183)$$

of $\mathcal{E}_\alpha^{\prime;\pm}(x,\lambda)$ into diagonal and off-diagonal parts. Here, for simplicity of notations we dropped some of the upper and the lower index of $\mathcal{E}'$. Now, we must keep in mind that the trace of $\mathcal{E}'(x,\lambda)$ is nonvanishing, so $\mathcal{E}'(x,\lambda)$ is an element of the algebra $gl(2)$ rather than $sl(2)$. Therefore, its diagonal part consists of two summands:

$$\mathcal{E}'(x,\lambda) = \frac{1}{2}\mathbb{1}\mathrm{tr}\left(\mathcal{E}'(x,\lambda)\right) + \frac{1}{2}\sigma_3\mathrm{tr}\left(\mathcal{E}'(x,\lambda)\sigma_3\right) . \qquad (5.184)$$

Taking the diagonal part of the left-hand side of (5.182) we find:

$$i\frac{d\mathcal{E}'^{,\mathrm{d}}}{dx} + q(x)\mathcal{E}'^{,\mathrm{f}}(x,\lambda) - \mathcal{E}'^{,\mathrm{f}}(x,\lambda)q'(x) = 0 \ , \qquad (5.185)$$

which can be formally integrated with the result:

$$\begin{aligned}\mathcal{E}'^{,\mathrm{d}}(x,\lambda)\Big|_{x=\pm\infty}^{x} &= -\frac{i}{2}\mathbb{1}\int_{\pm\infty}^{x} dy\ \mathrm{tr}\ \left((q'(y) - q(y))\mathcal{E}'^{,\mathrm{f}}(y,\lambda)\right) \qquad (5.186)\\ &+ \frac{i}{4}\sigma_3\int_{\pm\infty}^{x} dy\ \mathrm{tr}\ \left(\sigma_3\left[q'(y) + q(y), \mathcal{E}'^{,\mathrm{f}}(y,\lambda)\right]\right) .\end{aligned}$$

It remains to consider the off-diagonal part of (5.182) and to insert in it the expression for $\mathcal{E}'^{,\mathrm{d}}(x,\lambda)$ from (5.186). The result is:

$$\Lambda'_\pm\mathcal{E}'^{,\mathrm{f}}(x,\lambda) = = \lambda\mathcal{E}'^{,\mathrm{f}}(x,\lambda) + \frac{1}{4}\left[\sigma_3, q(x)e_\pm^{\prime;\mathrm{d}}(\lambda) - e_\pm^{\prime;\mathrm{d}}(\lambda)q'(x)\right], \qquad (5.187)$$

$$e_\pm^{\prime;\mathrm{d}}(\lambda) = \lim_{x\to\pm\infty}\mathcal{E}'^{,\mathrm{d}}(x,\lambda) \ , \qquad (5.188)$$

where $\Lambda'_\pm$ acts on any off-diagonal 2×2 matrix-valued function $X(x)$ by:

$$\begin{aligned}\Lambda'_\pm X &= \frac{i}{4}\left[\sigma_3, \frac{dX}{dx}\right] - \frac{i}{8}(q'(x) + q(x))\int_{\pm\infty}^{x} dy\ \mathrm{tr}\ \left(\sigma_3\left[q'(y) + q(y), X(y)\right]\right)\\ &+ \frac{i}{4}\sigma_3(q'(x) - q(x))\int_{\pm\infty}^{x} dy\ \mathrm{tr}\ \left((q'(y) - q(y))X(y)\right) . \qquad (5.189)\end{aligned}$$

In order to establish the relations in (5.180), (5.181), it remains to use the proper definitions of the "products of solutions" in (5.143) and (5.144) and to evaluate the limits of the corresponding diagonal parts $e_\pm^{\prime;\mathrm{d}}(\lambda)$.

Remark 5.7. We leave it to the reader to derive the spectral decompositions of the Green functions $G^{\prime,\pm}(x,y,\lambda)$, following the same ideas as in Sect. 5.5.3.

Remark 5.8. One can check that the recursion operators $\Lambda'_{\pm}$ evaluated for $q(x) = q'(x)$ coincide with the recursion operators $\Lambda_{\pm}$. The expansions (5.178) and (5.179) for $q(x) = q'(x)$ go into the expansion for $q(x)$; see (5.76) and (5.77a). Taken in the limit $q'(x) \to q(x) + \delta q(x)$, where $\delta q(x)$ is small, the expansions (5.176) and (5.177) go into the expansions (5.82), (5.84). The proof is based on the fact that if $\delta q(x)$ is uniformly small enough, then the Zakharov–Shabat systems L and L' have the same sets of discrete eigenvalues. Besides, the differences $\rho^{\prime,\pm}(\lambda) - \rho^{\pm}(\lambda)$, $\tau^{\prime,\pm}(\lambda) - \tau^{\pm}(\lambda)$ will also be small. Then, up to terms of higher order in δq we have:

$$\begin{aligned} (\rho^{\prime,\pm}(\lambda) - \rho^{\pm}(\lambda))\boldsymbol{\Psi}^{\prime,\pm}(x,\lambda) &\to \delta\rho^{\pm}(\lambda)\boldsymbol{\Psi}^{\pm}(x,\lambda)\,, \\ (\tau^{\prime,\pm}(\lambda) - \tau^{\pm}(\lambda))'\boldsymbol{\Phi}^{\prime,\pm}(x,\lambda) &\to \delta\tau^{\pm}(\lambda)\boldsymbol{\Psi}^{\pm}(x,\lambda)\,, \end{aligned} \tag{5.190}$$

$$\begin{aligned} C_k^{\prime,\pm}\boldsymbol{\Psi}^{\prime,\pm}(x) - C_k^{\pm}\boldsymbol{\Psi}^{\pm}(x) &\to \delta\lambda_k^{\pm} C_k^{\pm}\dot{\boldsymbol{\Psi}}_k^{\pm}(x) + \delta C_k^{\pm}\boldsymbol{\Psi}_k^{\pm}(x)\,, \\ M_k^{\prime,\pm}\boldsymbol{\Phi}^{\prime,\pm}(x) - M_k^{\pm}\boldsymbol{\Phi}^{\pm}(x) &\to \delta\lambda_k^{\pm} M_k^{\pm}\dot{\boldsymbol{\Phi}}_k^{\pm}(x) + \delta M_k^{\pm}\boldsymbol{\Phi}_k^{\pm}(x), \end{aligned} \tag{5.191}$$

thus reproducing the expansions of $\sigma_3\delta q(x)$ over the "squared solutions".

Remark 5.9. To the best of our knowledge no symplectic basis for the "products of solutions" is known.

5.7 Comments and Bibliographical Review

1. The Wronskian relations are a well-known method for analyzing spectral problems for ordinary differential operators; see [8, 9]. Their importance for solving NLEE was realized at the end of the 60s [10, 11], which led to their applications in solving NLEE [1] and to a number of their generalizations in the 70s [3, 4, 5, 6, 7, 12, 13, 14, 15]. They allow one to derive the explicit form of the "squared solutions" [1, 12, 16, 17], which play the role of generalized exponentials.
2. The first paper in which the Green function for the squared solutions of the Sturm-Liouville problem was proposed in [18]. However, his result did not receive much attention.

   Next important result in this direction was Kaup's formulation of the completeness (or the closure) relation for the "squared solutions" of the ZS system [2]. The Green function for the squared solutions of the ZS system was proposed in [5, 7] and used to prove in a more rigorous way their completeness relation [2]. In [5, 7], a new complete set of functions – the symplectic basis – was introduced by taking proper linear combinations of the "squared solutions". The advantage of using the symplectic basis is in the fact that it directly relates the potential $q(x)$ of ZS system to the action-angle variables of the hierarchy of NLS-type equations.

3. A direct consequence of the completeness relation for the "squared solutions" is the possibility to expand any function $X(x) \in \mathcal{M}$ over the "squared solutions" or over the elements of the symplectic basis. In the process one naturally encounters the skew-scalar product (5.11). The expansion coefficients of $X(x)$ are expressed through the skew-scalar products of $X(x)$ with the "squared solutions".
4. These skew-scalar products can be expressed in terms of the scattering data of L for three important particular choices for $X(x)$: $q(x)$, $\sigma_3 q_t$, $\sigma_3 \delta q(x)$. The expansions for these three functions over the "squared solutions", along with the fact that the "squared solutions" are eigenfunctions of the recursion operator Λ, are basic for deriving the fundamental properties of the NLEE and their Hamiltonian hierarchy. These facts have been realized independently by Kaup and Newell [19] and Gerdjikov and Khristov [5, 7]. The symplectic basis introduced in the latter papers simplified very much the derivation of the action-angle variables of the corresponding NLEE and the proof of compatibility of their Hamiltonian hierarchies.
5. Using a natural generalization of the Green function for the "squared solutions", one is able to prove the completeness of the products of solutions for two ZS system with potentials $q(x)$ and $q'(x)$ [5, 6, 7, 20]. One can use these relation and expand over the "products of solutions" the sum $q(x) + q'(x)$ and the difference $\sigma_3(q(x) - q'(x))$ of these potentials. The expansion coefficients are expressed in terms of the scattering data of the two ZS system. As a result one is able to describe the class of Bäcklund transformations [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31] for the corresponding NLEE. The results in [5, 6, 7] make more precise the results of Calogero and Degasperis, based on the generalized Wronskian relations [3, 4].
6. The Green function for the "squared solutions" was also used to outline the construction of the spectral theory of the corresponding recursion operators $\Lambda_\pm$. They have been extended to a number of generalizations of the ZS system, including:
   - the $n \times n$ ZS system related to the $sl(n)$ algebras, allowing to solve the N-wave-type equations [32, 33, 34, 35]. Using the possibility to express the resolvent of L through the FAS, one can also derive the spectral decomposition of the Lax operator [34].
   - to quadratic and higher order polynomial in λ bundles [36, 37, 38, 39, 40, 41]
   - to ZS system related to the simple Lie algebra $\mathfrak{g}$ [42, 43] of two different ZS system.
   - to discrete analogs of the ZS system [44, 45, 46, 47] and their multicomponenet generalizations [48]
   - to ZS system with nonvanishing boundary conditions [49, 50];

- to Lax operators gauge equivalent to the ZS system [13, 14, 51] and to the generalized ZS system [52];

7. Generalizations of the construction of the recursion operator for 2 + 1-dimensional NLEE was proposed by Konopelchenko and Zakharov in [53, 54, 55] and by Fokas and Santini in [56]. Unfortunately, the construction of their eigenfunctions and spectral decomposition is still an unsolved problem.
8. For all these generalized ZS systems, it was possible to introduce the Green function for the recursion operators $\Lambda_{\pm}$ in terms of FAS. Again using the contour integration method one can prove the completeness relation for the "squared solutions" of L, which are eigenfunctions of $\Lambda_{\pm}$. Thus, we are able to construct the spectral decompositions also for $\Lambda_{\pm}$, which are integro-differential non-self-adjoint operators.

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6

Fundamental Properties of the Solvable NLEEs

In this chapter, we shall show how the expansions over the "squared solutions," derived in Chap. 5, can be used for the analysis of the solvable NLEEs related to the ZS system. First, we shall describe the class of these NLEEs and show that the expansions over the "squared solutions" linearize them. After that the NLEEs can be solved trivially. Next, we shall demonstrate the important role of the recursion (generating) operators in the theory of the NLEEs. They allow us to describe the class of integrable NLEE and construct their integrals of motion and hierarchy of Hamiltonian structures. An easy consequence of the expansions over the symplectic basis is the complete integrability of the NLEEs and the explicit form of their action-angle variables.

In the last section of this Chapter, we demonstrate how the expansions over the "products of solutions" and the generalized recursion operators can be used to describe the class of Bäcklund transformations for the NLEE.

6.1 Description of the Class of NLEEs

The main tool we shall use now are the expansions over the "squared" solutions of the potential $q(x,t)$ and its time derivative $\sigma_3 q_t$. The latter is obtained by considering a special type of variations $\delta q(x,t)$, namely:

$$\sigma_3 \delta q(x,t) = \sigma_3 q_t \delta t + \mathcal{O}((\delta t)^2) . \tag{6.1}$$

Keeping only the terms of order δt, from (5.82), (5.84), and (5.86) we find:

$$\begin{aligned} \sigma_3 q_t(x,t) &= \frac{i}{\pi} \int_{-\infty}^{\infty} d\lambda \, \left(\rho_t^+(t,\lambda) \boldsymbol{\Psi}^+(x,t,\lambda) + \rho_t^-(t,\lambda) \boldsymbol{\Psi}^-(x,t,\lambda) \right) \\ &\quad + 2 \sum_{k=1}^{N} \Big(C_k^+(t) \lambda_{k,t}^+ \dot{\boldsymbol{\Psi}}_k^+(x,t) + C_{k,t}^+ \boldsymbol{\Psi}_k^+(x,t) \\ &\quad - \; C_k^-(t) \lambda_{k,t}^- \dot{\boldsymbol{\Psi}}_k^-(x,t) - C_{k,t}^- \boldsymbol{\Psi}_k^-(x,t) \Big) , \end{aligned} \tag{6.2}$$

Gerdjikov, V.S. et al.: *Fundamental Properties of the Solvable NLEEs*. Lect. Notes Phys. **748**, 175–209 (2008)
DOI 10.1007/978-3-540-77054-1_6

$$\begin{aligned}\sigma_3 q_t(x,t) &= -\frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda\, \left(\tau_t^+(t,\lambda)\boldsymbol{\Phi}^+(x,t,\lambda) + \tau_t^-(t,\lambda)\boldsymbol{\Phi}^-(x,t,\lambda)\right) \\ &+ 2\sum_{k=1}^{N} \Big(M_k^+(t)\lambda_{k,t}^+\dot{\boldsymbol{\Phi}}_k^+(x,t) + M_{k,t}^+\boldsymbol{\Phi}_k^+(x,t) \\ &- M_k^-(t)\lambda_{k,t}^-\dot{\boldsymbol{\Phi}}_k^-(x,t) - M_{k,t}^-\boldsymbol{\Phi}_k^-(x,t)\Big)\,,\end{aligned} \tag{6.3}$$

$$\begin{aligned}\sigma_3 q_t(x,t) &= i\int_{-\infty}^{\infty} d\lambda\, (\kappa_t(\lambda)\mathbf{P}(x,t,\lambda) - \eta_t(\lambda)\mathbf{Q}(x,t,\lambda)) \\ &+ i\sum_{k=1}^{N}\left(\eta_{k,t}^+\mathbf{Q}_k^+(x,t) - \kappa_{k,t}^+\mathbf{P}_k^+(x,t) + +\eta_{k,t}^-\mathbf{Q}_k^-(x,t) - \kappa_{k,t}^-\mathbf{P}_k^-(x,t)\right),\end{aligned} \tag{6.4}$$

where

$$\begin{aligned}\eta(\lambda) &= \frac{1}{\pi}\ln\left(1+\rho^+(t,\lambda)\rho^-(t,\lambda)\right), \qquad \eta_k^\pm = \mp 2i\lambda_k^\pm\,, \\ \kappa(t,\lambda) &= \frac{1}{2}\ln\frac{b^+(t,\lambda)}{b^-(t,\lambda)}, \qquad \kappa_k^\pm(t) = \pm\ln b_k^\pm(t)\,.\end{aligned} \tag{6.5}$$

As already mentioned, the dispersion law $f(\lambda)$

$$f(\lambda) = \sum_k f_k\lambda^k\,, \tag{6.6}$$

determines the evolution of the scattering data of the particular NLEEs. In most of the examples of NLEEs given below the corresponding dispersion laws are polynomial in λ. However, the theorem proved below holds true for a much larger class of dispersion laws.

Theorem 6.1. *Let the potential $q(x,t)$ satisfy conditions* **C1** *and* **C2** *and let the function $f(\lambda)$ be meromorphic for $\lambda \in \mathbb{C}$ and has no singularities on the spectrum of L. Then the NLEEs:*

$$i\sigma_3 q_t + 2f(\Lambda_+)q(x,t) = 0\,, \tag{6.7a}$$

$$i\sigma_3 q_t + 2f(\Lambda_-)q(x,t) = 0\,, \tag{6.7b}$$

$$i\sigma_3 q_t + 2f(\Lambda)q(x,t) = 0\,, \tag{6.7c}$$

are pairwise equivalent to the following linear evolution equations for the scattering data:

$$i\rho_t^\pm \mp 2f(\lambda)\rho^\pm(\lambda,t) = 0\,, \tag{6.8a}$$

$$iC_{k,t}^\pm \mp 2f_k^\pm C_k^\pm(t) = 0\,, \tag{6.8b}$$

$$\lambda_{k,t}^\pm C_k^\pm(t) = 0\,, \tag{6.8c}$$

$$i\tau_t^\pm \pm 2f(\lambda)\tau^\pm(\lambda,t) = 0\,, \tag{6.9a}$$

$$iM_{k,t}^\pm \pm 2f_k^\pm M_k^\pm(t) = 0\,, \tag{6.9b}$$

$$\lambda^{\pm}_{k,t} M^{\pm}_k(t) = 0 \,, \tag{6.9c}$$

$$i\eta_t = 0, \qquad i\kappa_t - 2f(\lambda) = 0 \,, \tag{6.10a}$$

$$i\eta^{\pm}_{k,t} = 0, \qquad i\kappa^{\pm}_{k,t} - 2f^{\pm}_k = 0 \,, \tag{6.10b}$$

Proof. Inserting (6.2) and (5.134) into the left-hand side of the NLEEs (6.7a), we obtain that the following expansion over the "squared" solutions $\mathbf{\Psi}^{\pm}$:

$$\begin{aligned}
&\frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda\, \left[\left(i\rho^+_t - 2f(\lambda)\rho^+(t,\lambda)\right)\mathbf{\Psi}^+(x,t,\lambda)\right.\\
&+\left.\left(i\rho^-_t + 2f(\lambda)\rho^-(t,\lambda)\right)\mathbf{\Psi}^-(x,t,\lambda)\right]\\
&+2\sum_{k=1}^{N}\left[\left(iC^+_{k,t} - 2f^+_k C^+_k(t)\right)\mathbf{\Psi}^+_k(x,t) - \left(iC^-_{k,t} + 2f^-_k C^-_k(t)\right)\mathbf{\Psi}^-_k(x,t)\right.\\
&+\left. i\lambda^+_{k,t} C^+_k(t)\dot{\mathbf{\Psi}}^+_k(x,t) - i\lambda^-_{k,t} C^-_k(t)\dot{\mathbf{\Psi}}^-_k(x,t)\right] = 0
\end{aligned} \tag{6.11}$$

must vanish. In order to prove the equivalence between (6.7a) and (6.8), it remains to use proposition 5.3.

Analogously, inserting the expansions (6.2) and (5.135) into the left-hand side of (6.7a), we find that the following expansion over $\mathbf{\Phi}^{\pm}$:

$$\begin{aligned}
&\frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda\, \left[\left(i\tau^+_t + 2f(\lambda)\tau^+(t,\lambda)\right)\mathbf{\Phi}^+(x,t,\lambda)\right.\\
&+\left.\left(i\tau^-_t - 2f(\lambda)\tau^-(t,\lambda)\right)\mathbf{\Phi}^-(x,t,\lambda)\right]\\
&+2\sum_{k=1}^{N}\left[\left(iM^+_{k,t} + 2f^+_k M^+_k(t)\right)\mathbf{\Phi}^+_k(x,t) - \left(iM^-_{k,t} - 2f^-_k M^-_k(t)\right)\mathbf{\Phi}^-_k(x,t)\right.\\
&+\left. i\lambda^+_{k,t} M^+_k(t)\dot{\mathbf{\Phi}}^+_k(x,t) - i\lambda^-_{k,t} M^-_k(t)\dot{\mathbf{\Phi}}^-_k(x,t)\right] = 0
\end{aligned} \tag{6.12}$$

must vanish. Using again the Proposition 5.3, one immediately proves the equivalence between (6.7b) and (6.9).

Finally, inserting the expansions over the symplectic basis (see (6.4) and (5.136)) into the left-hand side of (6.7c) we get:

$$\begin{aligned}
&i\int_{-\infty}^{\infty} d\lambda\, \left\{i\eta_t \mathbf{Q}(x,t,\lambda) - (i\kappa_t - 2f(\lambda))\,\mathbf{P}(x,t,\lambda)\right\}\\
&+i\sum_{k=1}^{N}\left\{i\eta^+_{k,t}\mathbf{Q}^+_k(x,t) - \left(i\kappa^+_{k,t} - 2f^+_k\right)\mathbf{P}^+_k(x,t)\right.\\
&+\left. i\eta^-_{k,t}\mathbf{Q}^-_k(x,t) - \left(i\kappa^-_{k,t} - 2f^-_k\right)\mathbf{P}^-_k(x,t)\right\} = 0\,.
\end{aligned} \tag{6.13}$$

Using again Proposition 5.3, we establish the equivalence of (6.7c) and (6.10).

To complete the proof of the theorem, it is necessary to invoke Proposition 5.4 and (5.130), from which it follows that the left-hand sides of the NLEEs (6.7a)–(6.7c) coincide. The theorem is proved.

Remark 6.2. Note the special role of the symplectic basis and the related scattering data (5.88). From (6.10), we see that half of these data, namely, $\eta(\lambda)$, $\eta_k^\pm$ are time-independent, while the other half, $\kappa(\lambda)$, $\kappa_k^\pm$, depends linearly on time. This implies that $\mathcal{T}$ in fact provides us with global action–angle variables for the NLEEs (6.7). We shall return to this question in Sect. 6.2 below.

Remark 6.3. The terms in the NLEEs, corresponding to f_0 and f_1 in the dispersion law can be removed by the following simple change of variables:

$$q(x,t) \to \tilde{q}(x,t) = e^{-i\alpha\sigma_3 t} q(x+\beta t, t) \ .$$

Indeed, if we denote by $\tilde{\Lambda}$ the operator, obtained from Λ, by changing $q(x,t)$ to $\tilde{q}(x,t)$ we have:

$$\begin{aligned}
\tilde{\Lambda}^n \tilde{q}(x,t) &= e^{-i\alpha\sigma_3 t}(\Lambda^n q)(x+\beta t, t) \ , \\
i\sigma_3 \tilde{q}_t &= e^{-i\alpha\sigma_3 t}\left(i\sigma_3 q_t + \alpha q(x+\beta t, t) + i\beta\sigma_3 q_x\right) \\
&= e^{-i\alpha\sigma_3 t} i\sigma_3 q_t + (\alpha + 2\beta\tilde{\Lambda}\tilde{q}(x,t) \\
&= e^{-i\alpha\sigma_3 t} i\sigma_3 q_t + (\alpha + 2\beta\tilde{\Lambda})\tilde{q} \ .
\end{aligned}$$

After this change of variables the NLEEs (6.7) will go into:

$$i\sigma_3 \tilde{q}_t + \tilde{f}(\tilde{\Lambda})\tilde{q}(x,t) = 0 \ ,$$

where $\tilde{f}(\lambda) = f(\lambda) - \alpha/2 - \beta\lambda$.

6.2 Examples of NLEEs

Here, we list several examples of physically important NLEEs, which fall into the above scheme. The Theorem 6.1 shows that each NLEE is specified by the corresponding function $f(\lambda)$. In physics, this function is known as the dispersion law of the NLEEs; clearly $f(\lambda)$ fixes up uniquely both the explicit form of the NLEEs and the evolution of the scattering data. Below, we list examples of two types of dispersion laws: (a) regular, i.e. polynomial in λ and (b) singular, including negative powers of λ.

6.2.1 Polynomial Dispersion Laws

In order to find the explicit form of the NLEEs, we shall need to calculate $\Lambda_\pm^p q(x,t)$ for $p = 1,2,3$. Recall that

$$\Lambda_\pm X = \frac{i}{4}\left[\sigma_3, \frac{dX}{dx}\right] + \frac{i}{2} q(x,t) \int_{\pm\infty}^{x} dy \, \mathrm{tr}\left(q(y,t), [\sigma_3, X(y,t)]\right) . \quad (6.14)$$

The calculation shows that

$$\Lambda_{\pm} q(x,t) = \frac{i}{4}[\sigma_3, q_x] \ , \tag{6.15}$$

$$(\Lambda_{\pm})^2 q(x,t) = -\frac{1}{4}\left(q_{xx} + 2q^+q^-q(x,t)\right) \ , \tag{6.16}$$

$$(\Lambda_{\pm})^3 q(x,t) = -\frac{i}{16}\left[\sigma_3, q_{xxx} + 6q^+q^-q_x\right] \ , \tag{6.17}$$

These expressions illustrate two important facts. The first one was actually introduced by (5.130); the second one, which will be analyzed below, is that $\Lambda^k q(x,t)$ are local in $q(x,t)$ for positive k, i.e. $\Lambda^k q(x,t)$ depend only on q and its x-derivatives.

The generic NLEEs will be systems of equations for the two complex-valued functions $q^+(x,t)$ and $q^-(x,t)$, which parametrize the potential $q(x,t)$. Next, we shall show how these systems of NLEEs can be simplified by additional algebraic restrictions on $q(x,t)$. Let us consider some examples.

Example 6.4 (The GNLS equation). This generalization of the NLS equation is obtained by choosing $f(\lambda) = c_2\lambda^2$. Then (6.7a) and (6.16) lead to the following system:

$$iq_t^+ - \frac{c_2}{2}(q_{xx}^+ + 2(q^+)^2q^-(x,t)) = 0 \ , \tag{6.18}$$

$$-iq_t^- - \frac{c_2}{2}(q_{xx}^- + 2(q^-)^2q^+(x,t)) = 0 \ . \tag{6.19}$$

We can put for simplicity $c_2 = -2$ and require that $q^+ = \varepsilon_0(q^-)^* = u(x,t)$ with $\epsilon_0 = \pm 1$. Then the second equation (6.19) is obtained from the first one (6.18) by complex conjugation, and the system reduces to the standard NLS equation:

$$iu_t + u_{xx} + 2\epsilon_0|u|^2u(x,t) = 0, \qquad \epsilon_0 = \pm 1 \ . \tag{6.20}$$

Example 6.5 (The GmKdV equation). The dispersion law for the generalized mKdV equation is given by $f(\lambda) = -8\lambda^3$. Then (6.7a) and (6.17) lead to:

$$\begin{aligned} q_t^+ + q_{xxx}^+ + 6q^+q_x^+q^-(x,t) = 0 \ , \\ q_t^- + q_{xxx}^- + 6q^-q_x^-q^+(x,t) = 0 \ , \end{aligned} \tag{6.21}$$

In order to get the famous mKdV equation, we need to impose the relation $q^+ = \eta_0 q^- = v(x,t)$, with $\eta_0 = \pm 1$ and we obtain:

$$v_t + v_{xxx} + 6\eta_0 v_x v^2(x,t) = 0, \qquad \eta_0 = \pm 1 \ . \tag{6.22}$$

Here, $v(x,t)$ is a complex–valued function. It is possible, by imposing another restriction on q, to make $v(x,t)$ either real or purely imaginary function. This equation finds applications in the physics of fluids, in plasma physics [1, 2], and in differential geometry [3].

Example 6.6 (Mixed GNLS–GmKdV equation). The last example here is a generalization of the NLEEs with a dispersion law $f(\lambda) = -2\lambda^2 - 8c_3\lambda^3$, where c_3 is some real constant. The corresponding system of NLEEs is:

$$iq_t^+ + q_{xx}^+ + 2(q^+)^2 q^-(x,t) + ic_3(q_{xxx}^+ + 6q^+q^-q_x^+) = 0 \, ,$$

$$-iq_t^- + q_{xx}^- + 2(q^-)^2 q^+(x,t) + ic_3(q_{xxx}^- + 6q^+q^-q_x^-) = 0 \, , \qquad (6.23)$$

Here, like in the NLS case, we can impose the restriction $q^+ = \varepsilon_0(q^-)^* = w(x,t)$ which leads to:

$$iw_t + w_{xx} + 2\epsilon_0|w|^2 w(x,t) + ic_3(w_{xxx} + 6\epsilon_0|w|^2 w_x) = 0 \, . \qquad (6.24)$$

This equation has applications in nonlinear optics [4].

6.2.2 Singular Dispersion Laws

There are also important examples of NLEEs, characterized by singular dispersion relations; an example of such is provided by $f(\lambda) = c_4/\lambda$. To treat such cases, we shall need also explicit expression for $(\Lambda_\pm)^{-1}$. Generically, it can be obtained as follows. Let us introduce $g(x,t)$ as the solution of the Zakharov–Shabat system at $\lambda = 0$:

$$i\frac{dg}{dx} + q(x,t)g(x,t) = 0, \qquad \lim_{x\to-\infty} g(x,t) = \mathbb{1} \, . \qquad (6.25)$$

Note that for the proof of Theorem 6.1 we needed the condition that the dispersion law is regular on the real axis. Now we shall need modifications in order to treat singular dispersion laws. We need to impose additional implicit restriction on the class of potentials $q(x,t)$, which would ensure the convergence of the integrals in (6.11)–(6.13) and the validity of the theorem, namely, we require that $q(x,t)$ be such that the reflection coefficients $\rho^\pm(t,\lambda)$ are smooth functions of λ and vanish fast enough for $\lambda \to 0$:

$$\lim_{\lambda\to 0} \lambda^{-k}\rho^+(t,\lambda) = \lim_{\lambda\to 0} \lambda^{-k}\rho^-(t,\lambda) = \text{const} \, , \qquad (6.26)$$

for $k = 1, \dots, N_1$. This condition with conveniently chosen N_1 will ensure the convergence of the integrals in (6.11)–(6.13). Clearly, N_1 is determined by the order of the singularity of $f(\lambda)$; it must be such that $\lim_{\lambda\to 0} \lambda^{N_1} f(\lambda) = \text{const}$.

The condition (6.26) means that:

$$T(t,0) = \begin{pmatrix} a^+(0) & 0 \\ 0 & a^-(0) \end{pmatrix}, \qquad a^+(0)a^-(0) = 1 \, . \qquad (6.27)$$

Recall also that the asymptotic of $g(x,t)$ for $x \to \infty$ is:

$$\lim_{x\to-\infty} g(x,t) = T(t,0)\ . \tag{6.28}$$

Let us now apply a similarity transformation $\mathcal{E} \to \tilde{\mathcal{E}} = \hat{g}\mathcal{E}g$ to the solution $\mathcal{E}(x,t,\lambda)$ of the equation (5.89). We get:

$$i\frac{d\tilde{\mathcal{E}}}{dx} = \lambda \hat{g}(x,t)[\sigma_3, \mathcal{E}(x,t,\lambda)]g(x,t)\ . \tag{6.29}$$

Next, we divide by λ and integrate over x, which yields:

$$\begin{aligned} &\frac{1}{\lambda}\tilde{\mathcal{E}}(x,t,\lambda) - \frac{1}{\lambda}\lim_{y\to\pm\infty}\tilde{\mathcal{E}}(y,t,\lambda) \\ &\qquad = -i\int_{\pm\infty}^{x} dy\, \hat{g}(y,t)[\sigma_3, \mathcal{E}(y,t,\lambda)]g(y,t)\ . \end{aligned} \tag{6.30}$$

In order to get $(\Lambda_\pm)^{-1}$, it remains to apply the inverse similarity transformation to (6.30) and to take off the diagonal part of $\mathcal{E}$ with the result:

$$\begin{aligned} &\frac{1}{\lambda}\pi_0\mathcal{E}(x,t,\lambda) - \frac{1}{\lambda}\lim_{y\to\pm\infty}\pi_0\left(h(x,y,t)\mathcal{E}(y,t,\lambda)\hat{h}(x,y,t)\right) \\ &\qquad = (\Lambda_\pm)^{-1}\pi_0\mathcal{E}(x,t,\lambda)\ , \end{aligned} \tag{6.31}$$

where $h(x,y,t) = g(x,t)\hat{g}(y,t)$ and

$$(\Lambda_\pm)^{-1}X(x,t) = -i\pi_0 g(x,t)\int_{\pm\infty}^{x} dy\, \hat{g}(y,t)[\sigma_3, X(y,t)]g(y,t)\hat{g}(x,t)\ . \tag{6.32}$$

If we put $X = q(x,t)$ in (6.32) and make use of (6.25) we get:

$$\begin{aligned} \Lambda_\pm^{-1}q(x,t) &= -i\Lambda_\pm^{-1}\left(g_x\hat{g}(x,t)\right) \\ &= -\pi_0\left(g(x,t)\left(\hat{g}(y,t)\sigma_3 g(y,t)\right)\big|_{y=\pm\infty}^{x}\hat{g}(x,t)\right) \\ &= \pi_0\left(g(x,t)\sigma_3\hat{g}(x,t)\right)\ . \end{aligned} \tag{6.33}$$

Here, we find again that both Λ_+^{-1} and Λ_-^{-1} produce the same effect on $q(x,t)$, which is a consequence of the regularity condition (6.26) and (6.27). Indeed, the contribution to the right-hand side of (6.33) from the limit $x \to +\infty$ or $x \to -\infty$, turns out to be the same.[1]

Example 6.7 (The generalized Maxwell–Bloch (GMB) equation).
For the NLEEs with dispersion law $f_4(\lambda) = c_4/\lambda$ we get:

$$i\sigma_3 q_t + 2\pi_0 c_4\left(g(x,t)\sigma_3\hat{g}(x,t)\right) = 0\ . \tag{6.34}$$

[1] There is no reason to expect that $\Lambda_+^{-k}q(x,t)$ will be equal to $\Lambda_-^{-k}q(x,t)$ for $k > 1$.

Of course generically we do not have explicit expression for $g(x,t)$ in terms of $q(x,t)$. Nevertheless, we can consider the system, consisting of (6.34) and (6.25), with the boundary condition:

$$\lim_{x\to\infty} g(x,t) = \mathbb{1}, \qquad g(x,t) = \begin{pmatrix} g_1^- & g_1^+ \\ g_2^- & g_2^+ \end{pmatrix} . \tag{6.35}$$

The resulting system of equations written in components has the form:

$$i\frac{dg_1^\pm}{dx} + q^+(x,t)g_2^\pm(x,t) = 0 , \tag{6.36a}$$

$$i\frac{dg_2^\pm}{dx} + q^-(x,t)g_1^\pm(x,t) = 0 , \tag{6.36b}$$

$$i\frac{dq^+}{dt} - 4c_4 g_1^- g_1^+(x,t) = 0 , \tag{6.36c}$$

$$i\frac{dq^-}{dt} - 4c_4 g_2^- g_2^+(x,t) = 0 , \tag{6.36d}$$

Imposing the relation $q^+(x,t) = (q^-(x,t))^*$, we find that the matrix elements of $g(x,t)$ are related by:

$$g_1^-(x,t) = (g_2^+(x,t))^*, \qquad g_2^-(x,t) = -(g_1^+(x,t))^* . \tag{6.37}$$

Then the GMB equation

$$\frac{dE_p}{dx} + Q(x,t)E_s(x,t) = 0 , \tag{6.38a}$$

$$\frac{dE_s}{dx} - Q^*(x,t)E_p(x,t) = 0 , \tag{6.38b}$$

$$\frac{dQ}{dt} - E_s^*(x,t)E_p(x,t) = 0 , \tag{6.38c}$$

follows from (6.36) and (6.37) if we put

$$g_1^+(x,t) = E_p(x,t), \qquad g_2^+(x,t) = -E_s(x,t), \qquad c_4 = -\frac{1}{4} . \tag{6.39}$$

The GMB equation (6.38) describes the propagation of light in a two-level media; see [5] and the references therein. In the physical applications $E_p(x,t)$, $E_s(x,t)$, and $Q(x,t)$ have the meaning of a pump wave, the Stockes wave, and the normalized effective polarization of the medium, respectively. Of course, here we are using normalized variables and dimensionless units.

Example 6.8 (The sine-Gordon equation).

The NLEEs (6.34) greatly simplifies for two specific choices of the potential $q(x,t)$, namely:

$$q^+(x,t) = -q^-(x,t), \qquad \text{i.e.,} \qquad q(x,t) = w_x\sigma_2 , \tag{6.40a}$$

$$q^{+}(x,t) = q^{-}(x,t), \qquad \text{i.e.,} \qquad q(x,t) = -iw_x\sigma_1 \; . \tag{6.40b}$$

It happens because if (6.40) holds, then $[q(x,t), q(y,t)] = 0$ for any values of x, t and y; as a consequence for $g(x,t)$ we find:

$$g(x,t) = \cos w(x,t)\mathbb{1} + i\sin w(x,t)\sigma_2 \; , \tag{6.41}$$

$$g(x,t) = \cosh w(x,t)\mathbb{1} + \sinh w(x,t)\sigma_1 \; . \tag{6.42}$$

Then (6.36a), (6.36b), and (6.25) are satisfied identically. The explicit form of $\Lambda_{\pm}^{-1}q(x,t)$ also greatly simplifies, giving:

$$(\Lambda_{\pm})^{-1}q(x,t) = -\sin 2w(x,t)\sigma_1, \qquad \eta_0 = -1 \; , \tag{6.43}$$

$$(\Lambda_{\pm})^{-1}q(x,t) = -i\sin 2w(x,t)\sigma_2, \qquad \eta_0 = 1 \; , \tag{6.44}$$

Thus fixing up $c_4 = -1/4$ (6.34) transforms into one of the well-known versions

$$2w_{xt} + \sin 2w(x,t) = 0 \; , \tag{6.45}$$

$$2w_{xt} + \sinh 2w(x,t) = 0 \; , \tag{6.46}$$

of the s-G equation.

Obviously, for these specific forms of $q(x,t)$ we are able to evaluate the explicit form of the NLEEs, whose dispersion laws contain any finite number of inverse power of λ. However, due to the form of $(\Lambda_{\pm})^{-1}$ (6.32) these NLEEs are in general nonlocal. It seems that the s-G equation is the only local NLEE in the class of equations having singular dispersion laws.

6.3 Involutions of the Zakharov–Shabat System

Up to now, we considered the operator L with a generic potential, i.e. we supposed q^{+} and q^{-} to be independent complex-valued functions. However, the attentive reader has noticed already that the most important of the above examples – the NLS equation, the mKdV equation, and the s-G equation are obtained after imposing special constraints on the potential $q(x,t)$. These constraints lead to symmetries of second order of the ZS system, which are known as involutions. In this section, we shall formulate them in an algebraic way and will show what is their implication on the spectral data of L.

The first involution we shall consider has the form:

$$q^{-}(x,t) = \varepsilon_0 (q^{+}(x,t))^{*}, \qquad \varepsilon_0 = \pm 1 \; , \tag{6.47}$$

and the second one is

$$q^{-}(x,t) = \eta_0 q^{+}(x,t), \qquad \eta_0 = \pm 1 \; . \tag{6.48}$$

Each restriction on $q(x,t)$ like (6.47) or (6.48) imposes constraints on

1. the Jost solutions, the FAS, and the scattering matrix;
2. the minimal sets of scattering data $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}$;
3. the corresponding NLEE;
4. the possible dispersion laws $f(\lambda)$.

We start with (1). The relation (6.47) means that for $U(x,t,\lambda)$ the following $\mathbb{Z}_2$ symmetry holds:

$$U^*(x,t,\lambda^*) = -\epsilon^{-1}U(x,t,\lambda)\epsilon, \qquad \epsilon = \begin{pmatrix} 0 & 1 \\ -\varepsilon_0 & 0 \end{pmatrix} . \tag{6.49}$$

This implies that FAS and the scattering matrix must satisfy:

$$(\chi^{\pm}(x,t,\lambda^*))^* = \epsilon^{-1}\chi^{\mp}(x,t,\lambda)\epsilon \;, \tag{6.50}$$
$$(T(t,\lambda^*))^* = \epsilon^{-1}T(t,\lambda)\epsilon \;. \tag{6.51}$$

Written in components, the relation (6.51) has the form:

$$a^-(\lambda) = (a^+(\lambda^*))^* \;, \tag{6.52a}$$
$$b^-(t,\lambda) = \varepsilon_0(b^+(t,\lambda^*))^* \;, \tag{6.52b}$$

i.e. for the reflection coefficients we obtain:

$$\rho^-(t,\lambda) = \varepsilon_0(\rho^+(t,\lambda^*))^* \;, \tag{6.53a}$$
$$\tau^-(t,\lambda) = \varepsilon_0(\tau^+(t,\lambda^*))^* \;. \tag{6.53b}$$

and as a consequence

$$\eta(\lambda) = \eta^*(\lambda), \qquad \kappa(\lambda) = -\kappa^*(\lambda), \qquad \lambda \in \mathbb{R} \;. \tag{6.53c}$$

In fact, (6.52a) is valid for any complex value of λ, while (6.52b), (6.53) hold only for $\lambda \in \mathbb{R}$.

These formulae show the effect of the involutions on the scattering data, related to the continuous spectrum of L. Analyzing their analogs for the discrete spectrum, we have to consider separately the case of (6.47) with $\varepsilon_0 = -1$. Such involution allows to reformulate the Zakharov–Shabat system into:

$$\mathcal{L}\psi(x,t,\lambda) \equiv \left(i\sigma_3 \frac{d\psi}{dx} + U_0(x,t)\psi(x,t,\lambda) \right) = \lambda\psi(x,t,\lambda) \;, \tag{6.54}$$

where $U_0(x,t) = \sigma_3 q(x,t) \equiv U_0^\dagger(x,t)$ is a Hermitian matrix. The linear system (6.54) on the line then is an eigenvalue problem for the self-adjoint operator $\mathcal{L}$. From the general theory of such operators, the spectrum of $\mathcal{L}$ must be located on the real λ-axis. This is indeed the case, because the continuous spectrum of $\mathcal{L}$ fills up the whole real λ-axis. In addition, for real λ, the unitarity condition for the scattering matrix reads:

$$a^+(\lambda)a^-(\lambda) + b^+(t,\lambda)b^-(t,\lambda) = 1, \qquad \lambda \in \mathbb{R} \;. \tag{6.55a}$$

Taking into account (6.52) with $\varepsilon_0 = -1$ we get:

$$|a^+(\lambda)|^2 = 1 + |b^+(t,\lambda)|^2, \qquad \lambda \in \mathbb{R}\,, \tag{6.55b}$$

which means that for all $\lambda \in \mathbb{R}$ $|a^+(\lambda)|^2 \geq 1$, and therefore $\mathcal{L}$ (or, equivalently, L) does not have discrete eigenvalues on the real λ-axis.

In short, the involution (6.52) with $\varepsilon_0 = -1$ ensures that L does not have discrete eigenvalues; then the minimal sets of scattering data consist of:

$$\mathcal{T}_1 \equiv \{\rho^\pm(t,\lambda), \quad \lambda \in \mathbb{R}\}\,, \tag{6.56a}$$
$$\mathcal{T}_2 \equiv \{\tau^\pm(t,\lambda), \quad \lambda \in \mathbb{R}\}\,, \tag{6.56b}$$
$$\mathcal{T} \equiv \{\eta(\lambda), \kappa(t,\lambda), \quad \lambda \in \mathbb{R}\}\,. \tag{6.56c}$$

The absence of discrete spectrum means that

1. The corresponding NLEEs do not have soliton solutions. In particular, this holds true for the NLS and the mixed NLS–mKdV equations with the "wrong" sign $\varepsilon_0 = -1$ and also for the sinh-Gordon equation (6.46).
2. The corresponding completeness relations remain true, if the terms corresponding to the discrete spectrum are dropped.

Let us now consider the involution (6.47) with $\varepsilon_0 = 1$ and (6.48) with $\eta_0 = \pm 1$. Each of these choices, or combinations of them, allows the existence of discrete spectrum of L, of course, restricted by the involution. Let us derive these restrictions.

Recall that the eigenvalues $\lambda_k^\pm$ are located at the zeroes of $a^\pm(\lambda)$, which are analytic in $\mathbb{C}_\pm$, respectively. In the neighborhood of some discrete eigenvalues $\lambda_k^\pm$, according to condition **C2**, $a^\pm(\lambda)$ have simple zeroes, namely:

$$a^+(\lambda) = (\lambda - \lambda_k^+)\left(\dot{a}_k^+ + \frac{1}{2}(\lambda - \lambda_k^+)\ddot{a}_k^+ + \cdots\right)\,, \tag{6.57a}$$
$$a^-(\lambda) = (\lambda - \lambda_k^-)\left(\dot{a}_k^- + \frac{1}{2}(\lambda - \lambda_k^-)\ddot{a}_k^- + \cdots\right)\,. \tag{6.57b}$$

If we impose now (6.47) with $\varepsilon_0 = 1$ (or equivalently, (6.52a)) from (6.57), we find that (6.57b) after complex conjugation goes into (6.57a) provided:

$$\lambda_k^- = (\lambda_k^+)^*, \qquad \dot{a}_k^- = (\dot{a}_k^+)^*, \qquad \ddot{a}_k^- = (\ddot{a}_k^+)^*\,. \tag{6.58}$$

Thus we get that the data on the discrete spectrum must satisfy:

$$b_k^- = (b_k^+)^*, \qquad C_k^- = (C_k^+)^*, \qquad M_k^- = (M_k^+)^*, \tag{6.59a}$$
$$\eta_k^+ = (\eta_k^-)^*, \qquad \kappa_k^+ = -(\kappa_k^-)^*\,. \tag{6.59b}$$

We deduce that the minimal sets of scattering data consist of twice less elements (compare with (3.68)):

$$\mathcal{T}_1 \equiv \left\{\rho^+(t,\lambda), \quad \lambda \in \mathbb{R}, \quad \lambda_k^+,\ C_k^+(t), \quad k = 1,\ldots,N\right\}\,, \tag{6.60a}$$

$$\mathcal{T}_2 \equiv \left\{ \tau^+(t,\lambda), \quad \lambda \in \mathbb{R}, \quad \lambda_k^+, \; M_k^+(t), \quad k = 1,\dots,N \right\} , \tag{6.60b}$$
$$\mathcal{T} \equiv \left\{ \eta(\lambda), \kappa(t,\lambda), \quad \lambda \in \mathbb{R}, \quad \eta_k^\pm, \; \kappa_k^\pm, \quad k = 1,\dots,N \right\} , \tag{6.60c}$$

where $\eta(\lambda)$, $\eta_k^\pm$ take real values, while $\kappa(\lambda)$, $\kappa_k^\pm$ are purely imaginary.

As mentioned above, the reduction has an effect also on the dispersion laws. Indeed, combining (6.51) and the generic evolution equation for $T(\lambda)$:

$$i\frac{dT}{dt} + f(\lambda)[\sigma_3, T(t,\lambda)] = 0 , \tag{6.61}$$

we easily find that both are consistent only if $f(\lambda)$ satisfies:

$$\sum_p f_p \lambda^p = f(\lambda) = (f(\lambda^*))^* = \sum_p f_p^* \lambda^p , \tag{6.62}$$

that is, if the coefficients f_p are real.

Let us now consider the second involution (6.48). Then $U(x,t,\lambda)$ satisfies:

$$U(x,t,\lambda) = \sigma U(x,t,-\lambda)\sigma^{-1}, \qquad \sigma = \begin{pmatrix} 0 & 1 \\ \eta_0 & 0 \end{pmatrix} . \tag{6.63}$$

Consequently, the FAS and $T(t,\lambda)$ satisfy:

$$\chi^\pm(x,t,-\lambda) = \sigma^{-1}\chi^\mp(x,t,\lambda)\sigma , \tag{6.64}$$
$$T(t,-\lambda) = \sigma^{-1}T(t,\lambda)\sigma , \tag{6.65}$$

or by components:

$$a^-(\lambda) = a^+(-\lambda) , \tag{6.66a}$$
$$b^+(t,\lambda) = -\eta_0 b^-(t,-\lambda) , \tag{6.66b}$$

$$\rho^+(t,\lambda) = -\eta_0\rho^-(t,-\lambda) , \tag{6.67a}$$
$$\tau^+(t,\lambda) = -\eta_0\tau^-(t,-\lambda) , \tag{6.67b}$$
$$\eta(\lambda) = \eta(-\lambda), \qquad \kappa(\lambda) = -\kappa(-\lambda) . \tag{6.67c}$$

From (6.66) it also follows that

$$\lambda_k^+ = -\lambda_k^-, \qquad \dot{a}_k^- = -\dot{a}_k^+, \qquad \ddot{a}_k^- = \ddot{a}_k^+ , \tag{6.68a}$$
$$b_k^+(t) = -\eta_0 b_k^-(t) , \tag{6.68b}$$
$$C_k^+(t) = \eta_0 C_k^-(t), \qquad M_k^+(t) = \eta_0 M_k^-(t) , \tag{6.68c}$$
$$\eta_k^+ = \eta_k^-, \qquad \kappa_k^+ = -\kappa_k^- . \tag{6.68d}$$

Therefore, the minimal sets of scattering data formally look precisely like (6.60a) and (6.60b).

The involution (6.48), just like (6.47), restricts the possible dispersion laws by requiring them to be odd functions of λ:

$$f(\lambda) = -f(-\lambda) \tag{6.69}$$

i.e., $f_{2p} = 0$.

6.4 Fundamental Properties of the Soliton Solutions

The properties of the scattering data related to the discrete spectrum of L allow to describe the general structure of the soliton solutions of the corresponding NLEEs. Note that generically each soliton solution of the NLEEs (6.7) is parametrized by four complex parameters: λ_k^+, λ_k^-, C_k^+ and C_k^-. Considered as dynamical system such soliton has four degrees of freedom.

When an involution is imposed the situation changes. First, we consider the involution (6.47) with $\varepsilon_0 = 1$. Then due to (6.59), we find that the soliton solution is parametrized by two complex parameters λ_k^+ and C_k^+; this means that these solitons have two degrees of freedom. The first degree of freedom corresponds to the overall motion of the soliton, while the second describes the internal motion. The same holds true when only (6.48) is imposed.

Note that the involutions restrict also the possible dispersion laws. As a consequence of (6.62) only real f_p are possible, while (6.69) allows only odd functions as dispersion laws. Indeed, we see that (6.18) and (6.19) are not compatible with the involution (6.47) if c_2 is a complex constant. Analogously (6.23) is not compatible with the involution (6.48).

The most interesting situation is when the involutions (6.47) with $\varepsilon_0 = 1$ and (6.48) are imposed simultaneously. The NLEEs, compatible with both involutions, must have dispersion laws which are odd functions of λ with real coefficients f_p. The most famous examples of such equations are the s-G equation (6.45) and the mKdV equation (6.22). Let us now take a generic soliton solution and see what will happen to its parameters when both involutions are present. Due to (6.58) and (6.68a) each soliton is characterized by a quadruplet of eigenvalues:

$$\lambda_k^+, \quad -(\lambda_k^+)^*, \quad -\lambda_k^+, \quad (\lambda_k^+)^*, \qquad \lambda_k^\pm = \lambda_{0,k} \pm i\lambda_{1,k}, \tag{6.70a}$$
$$C_k^+, \quad \eta_0(C_k^+)^*, \quad \eta_0 C_k^+, \quad (C_k^+)^* , \tag{6.70b}$$

or equivalently

$$M_k^+, \quad \eta_0(M_k^+)^*, \quad \eta_0 M_k^+, \quad (M_k^+)^*, \qquad M_k^+ = e^{\xi_0 + i\varphi_0} , \tag{6.70c}$$

where

$$C_k^\pm = \frac{\pm i}{\dot{a}_k^\pm} e^{-z_k \pm i\phi_k}, \qquad M_k^\pm = \frac{\mp i}{\dot{a}_k^\pm} e^{z_k \mp i\phi_k}, \qquad \dot{a}_k^\pm \simeq \frac{\pm 1}{2i\lambda_{1,k}} \prod_{j\neq k} \frac{\lambda_k^\pm - \lambda_j^\pm}{\lambda_k^\mp - \lambda_j^\mp} . \tag{6.70d}$$

Here, we have used the convenient parametrization of the one-soliton solutions, which allows one to identify the center of mass position, the amplitude, the velocity, and the phase of the soliton; see (4.96). Such soliton is described by two complex parameters and again has two degrees of freedom. In the literature, this type of soliton solutions for the s-G equation are known as the breather solutions.

There is also a special case, when λ_k^+ is taken to be purely imaginary: $\lambda_k^+ = is_k$. Such soliton is parametrized only by a pair of eigenvalues:

$$\lambda_k^+ = is_k \qquad \lambda_k^- = -is_k \;, \tag{6.71a}$$
$$C_k^+ = \eta_0 (C_k^+)^* \;, \tag{6.71b}$$

or

$$M_k^+ = \eta_0 (M_k^+)^* \;. \tag{6.71c}$$

In other words, the eigenvalues are determined by one real parameter; another real parameter is needed for C_k^+ (or M_k^+), which must be either real (for $\eta_0 = 1$) or purely imaginary (for $\eta_0 = -1$). Such soliton has only one degree of freedom, which describes the overall motion.

The two types of soliton solutions for the s-G equation differ also by their topological properties: The soliton with one degree of freedom has a nontrivial topological charge (equal to 1), while the breather has vanishing topological charge and may be viewed as a bound state of two solitons with opposite topological charges. These otherwise intriguing properties of the s-G solitons come out of the context of the present monograph; the interested reader is referred to the literature, e.g. [6] and the references therein.

6.4.1 The M-Operators in Terms of $\Lambda_\pm$.

Since the dispersion law determines uniquely the NLEEs, it must determine also the M-operator. In Sect. (2.2), we showed how the M-operators of the NLEEs can be evaluated explicitly. In this subsection, we derive a compact expression for the M-operators in terms of $\Lambda_\pm$ and establish its relation to the diagonal of the resolvent of the ZS system.

Let us first consider the case when the dispersion law is $f^{(N)}(\lambda) = c_N \lambda^N$. Using formulae (2.40), after some calculations we get:

$$\begin{aligned} V^{(N)}(x,t,\lambda) &= -c_N \sum_{k=0}^{N} \lambda^{N-k} \left(V_k^{\mathrm{f}}(x,t) + w_k(x,t)\sigma_3 \right) \\ &= c_N \lambda^N \sigma_3 - c_N \sum_{k=1}^{N} \lambda^{N-k} \Bigg(\Lambda_\pm^{k-1} q(x,t) \\ &\quad + \frac{i}{2}\sigma_3 \int_{\pm\infty}^{x} dy \, \mathrm{tr} \left(\sigma_3 \left[q(y,t), \Lambda_\pm^{k-1} q(y,t) \right] \right) \Bigg) \;. \end{aligned} \tag{6.72}$$

It is easy to check that formally:

$$\sum_{k=1}^{N} \lambda^{N-k} \Lambda_\pm^{k-1} = \lambda^{N-1} \sum_{k=1}^{N} (\Lambda_\pm/\lambda)^{k-1} = \frac{\Lambda_\pm^N - \lambda^N}{\Lambda_\pm - \lambda} \;. \tag{6.73}$$

Therefore from (6.72) follows:

$$V^{(N)}(x,t,\lambda) = c_n \lambda^N \sigma_3 - c_N \Big(F^{(N)}(\Lambda_\pm, \lambda) q(x,t) + \frac{i}{2}\sigma_3 \int_{\pm\infty}^{x} dy \, \mathrm{tr}\, \Big(\sigma_3 \Big[q(y,t), F^{(N)}(\Lambda_\pm, \lambda) q(y,t) \Big] \Big) \Big) \tag{6.74a}$$

where

$$F^{(N)}(\Lambda, \lambda) = \frac{\Lambda^N - \lambda^N}{\Lambda - \lambda} \ . \tag{6.74b}$$

For generic dispersion law $f(\lambda) = \sum_{p\geq 0} c_p \lambda^p$ we have:

$$V(x,t,\lambda) = f(\lambda)\sigma_3 - \Big(F(\Lambda_\pm, \lambda) q(x,t) + \frac{i}{2}\sigma_3 \int_{\pm\infty}^{x} dy \, \mathrm{tr}\, \left(\sigma_3 \left[q(y,t), F(\Lambda_\pm, \lambda) q(y,t) \right] \right) \Big) \ , \tag{6.75a}$$

where

$$F(\Lambda, \lambda) = \sum_{p\geq 0} c_p F^{(p)}(\Lambda, \lambda) = \frac{f(\Lambda) - f(\lambda)}{\Lambda - \lambda} \ . \tag{6.75b}$$

Note that for polynomial dispersion laws the functions $F(\Lambda, \lambda)$ are also polynomial in Λ. From Proposition 5.4, we conclude that for polynomial dispersion laws the M-operators obtained through Λ_+ coincide with the ones obtained through Λ_- or Λ.

Another method to obtain the explicit form of the M-operators is based on the diagonal of the resolvent $R(x,y,t,\lambda)$ (3.35). From (3.35), one can see that the diagonal of the resolvent $R(x,x,t,\lambda)$ involves the function $\theta(x-y)$ which is discontinuous for $x = y$. Assuming that $\theta(x)$ is defined, as in (3.36), the diagonal of the resolvent is given by the following regular expression:

$$R(x,t,\lambda) = \frac{i}{2}\chi^\pm(x,t,\lambda)\sigma_3 \hat{\chi}^\pm(x,t,\lambda), \qquad \lambda \in \mathbb{C}_\pm \ , \tag{6.76}$$

which obviously satisfies:

$$i\frac{dR}{dx} + [q(x,t) - \lambda J, R(x,t,\lambda)] = 0 \ . \tag{6.77}$$

Besides, $R(x,t,\lambda)$ is piecewise analytic function of λ, bounded for $\lambda \to \infty$. Using the asymptotics of $\chi^\pm(x,t,\lambda)$ for $\lambda \to \infty$ we get the following expansion for $R(x,t,\lambda)$ over the inverse powers of λ:

$$R(x,t,\lambda) = \frac{i}{2}\left(\sigma_3 + \sum_{k=1}^{\infty} R_k(x,t)\lambda^{-k} \right) \ . \tag{6.78}$$

Following Gel'fand, we shall demonstrate that the coefficients $V_k(x,t)$ of the M-operators are simply expressed through $R_k(x,t)$. Let us first consider the case with dispersion law $f^{(N)}(\lambda) = c_N\lambda^N$ and introduce the splitting of $\lambda^N R(x,t,\lambda)$ into "positive" and "negative" parts:

$$-2i\lambda^N R(x,t,\lambda) = \tilde{R}_+^{(N)}(x,t,\lambda) + \tilde{R}_-^{(N)}(x,t,\lambda) \, , \tag{6.79}$$

where

$$\tilde{R}_+^{(N)}(x,t,\lambda) = \sigma_3 + \sum_{k=1}^{N} \lambda^{N-k} R_k(x,t) \, , \tag{6.80a}$$

$$\tilde{R}_-^{(N)}(x,t,\lambda) = \sum_{k=N+1}^{\infty} \lambda^{N-k} R_k(x,t) \, . \tag{6.80b}$$

Let us now insert (6.79) into (6.77) and split the left-hand side into "positive" and "negative" parts which must vanish independently. Thus we find:

$$i\frac{\tilde{R}_+^{(N)}}{dx} + \left[q(x,t) - \lambda\sigma_3, \tilde{R}_+^{(N)}(x,t,\lambda)\right] = \left[\sigma_3, \widetilde{R}_{N+1}(x,t)\right] \, , \tag{6.81}$$

which provides us with the following expression for $V^{(N)}(x,t,\lambda)$:

$$\begin{aligned} V^{(N)}(x,t,\lambda) &= c_N \tilde{R}_+^{(N)}(x,t,\lambda) \\ &= c_N \left(\sigma_3\lambda^N + \sum_{k=1}^{N} \lambda^{N-k} R_k(x,t)\right) \, , \end{aligned} \tag{6.82}$$

and for generic dispersion laws we get:

$$V(x,t,\lambda) = \sum_{p\geq 0} c_p \tilde{R}_+^{(p)}(x,t,\lambda) \, . \tag{6.83}$$

From this equation and from the fact that $\lim_{x\to\pm\infty} R_k(x,t) = 0$ we derive:

$$\lim_{x\to\pm\infty} V(x,t,\lambda) = f(\lambda)\sigma_3 \, , \tag{6.84}$$

i.e. the asymptotics of $V(x,t,\lambda)$ are determined by the dispersion law. The inverse is also true: Knowing the dispersion law we are able to recover $V(x,t,\lambda)$ by the formulae (6.75) above.

6.4.2 The Trace Identities and Λ–Operators

In Sect. 3.4, we have obtained the dispersion relation (3.66):

$$\mathcal{A}(\lambda) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\mu}{\mu-\lambda} \ln(a^+(\mu)a^-(\mu)) + \sum_{k=1}^{N} \ln\frac{\lambda-\lambda_k^+}{\lambda-\lambda_k^-}$$

$$= \frac{i}{2} \int_{-\infty}^{\infty} \frac{d\mu}{\mu - \lambda} \eta(\mu) + \sum_{k=1}^{N} \ln \frac{2i\lambda + \eta_k^+}{2i\lambda - \eta_k^-} \ , \tag{6.85}$$

where

$$\mathcal{A}(\lambda) = \begin{cases} \ln a^+(\lambda), & \lambda \in \mathbb{C}_+ \ , \\ \frac{1}{2} \ln(a^-(\lambda)/a^-(\lambda)), & \lambda \in \mathbb{R} \ , \\ -\ln a^-(\lambda), & \lambda \in \mathbb{C}_- \ , \end{cases} \tag{6.86}$$

and for $\lambda \in \mathbb{R}$ the integrals are understood in the principal value sense.

We showed also that $\mathcal{A}(\lambda)$ is the generating functional of the integrals of motion; the integrals themselves are obtained expanding $\mathcal{A}(\lambda)$ in Taylor series, in the neighborhood of any point $\lambda \in \mathbb{C}_\pm$. Let us assume that we have (at least asymptotically) the expansions over the negative (positive) powers of λ:

$$\mathcal{A}(\lambda) = i \sum_{p=1}^{\infty} C_p \lambda^{-p}, \qquad \mathcal{A}(\lambda) = -i \sum_{p=0}^{\infty} C_{-p} \lambda^p \tag{6.87}$$

As we shall see below, the densities of the integrals C_p depend only on q and its derivatives with respect to x. This usually is referred saying that C_p, $p = 1, 2, \ldots$ are local integrals of motion. We shall also see that C_{-p}, $p = 1, 2, \ldots$ are the nonlocal integrals of motion.

In order to express the integrals C_p through the scattering data, we expand the right-hand side of (6.85) over the negative and positive powers of λ:

$$\frac{1}{\mu - \lambda} = -\sum_{p=1}^{\infty} \frac{\mu^{p-1}}{\lambda^p}, \qquad \frac{1}{\mu - \lambda} = \sum_{p=0}^{\infty} \frac{\lambda^p}{\mu^{p+1}} \ . \tag{6.88}$$

We shall need also the expansions of

$$\begin{aligned} \ln \frac{\lambda - \lambda_k^+}{\lambda - \lambda_k^-} &= \ln \frac{\lambda_k^+}{\lambda_k^-} - \sum_{p=1}^{\infty} \frac{\lambda^p}{p} \left((\lambda_k^+)^{-p} - (\lambda_k^-)^{-p} \right) \\ &= -\sum_{p=1}^{\infty} \frac{1}{p\lambda^p} \left((\lambda_k^+)^p - (\lambda_k^-)^p \right) . \end{aligned} \tag{6.89}$$

As a result for C_p we find:

$$C_p = -\frac{1}{2} \int_{-\infty}^{\infty} d\mu \, \mu^{p-1} \eta(\mu) + \frac{i}{p} \sum_{k=1}^{N} \left(\left(\frac{i\eta_k^+}{2} \right)^p - \left(\frac{\eta_k^-}{2i} \right)^p \right) \ , \tag{6.90}$$

$$C_0 = -\frac{1}{2} \int_{-\infty}^{\infty} \frac{d\mu}{\mu} \eta(\mu) + i \sum_{k=1}^{N} \ln \left(-\frac{\eta_k^+}{\eta_k^-} \right) \ , \tag{6.91}$$

$$C_{-p} = -\frac{1}{2} \int_{-\infty}^{\infty} \frac{d\mu}{\mu^{p+1}} \eta(\mu) - \frac{i}{p} \sum_{k=1}^{N} \left(\left(\frac{i\eta_k^+}{2} \right)^{-p} - \left(\frac{\eta_k^-}{2i} \right)^{-p} \right) \ , \tag{6.92}$$

where $p = 1, 2, \ldots$. The convergence of the integrals defining C_p, $p > 0$ is a consequence of condition **C1** (see p. 71), which ensures that the reflection coefficients $\rho^{\pm}(t, \lambda)$, $\tau^{\pm}(t, \lambda)$ and the action variables $\eta(\lambda)$ (6.5) are Schwartz–type functions of λ for $\lambda \in \mathbb{R}$. The integrals defining C_0 and C_{-p}, $p = 1, \ldots, N_1$ are convergent due to the constraint (6.26), characteristic for singular dispersion laws.

6.4.3 Recurrent Relations for the Densities of C_p

There are several ways of deriving the recurrent relations for the densities of C_p. Here, we shall display one of them, proposed in [6]. The idea is to start with a special solution of the L–operator:

$$Y^{+}(x, y, t, \lambda) = \chi^{+}(x, t, \lambda)\hat{\chi}^{+}(y, t, \lambda), \qquad x \geq y \ , \tag{6.93}$$

which is obviously normalized by the condition $Y^{+}(x, t, \lambda)|_{x=y} = \mathbb{1}$, and to write it down in the form:

$$Y^{+}(x, y, t, \lambda) = (\mathbb{1} + W(x, t, \lambda))e^{Z(x,y,t,\lambda)}(\mathbb{1} + W(y, t, \lambda))^{-1} \ , \tag{6.94}$$

where $Z(x, y, t, \lambda)$ is a diagonal matrix and

$$W(x, t, \lambda) = \begin{pmatrix} 0 & W^{+}(x, t, \lambda) \\ W^{-}(x, t, \lambda) & 0 \end{pmatrix} . \tag{6.95}$$

The solution (6.93) is an analytic function of $\lambda \in \mathbb{C}_{+}$. This follows[2] immediately from the analytic properties of $\chi^{+}(x, t, \lambda)$. For $W(x, t, \lambda)$ and $Z(x, y, t, \lambda)$, there exist the following expansions over the negative powers of λ:

$$\begin{aligned} W(x, t, \lambda) &= \sum_{n=1}^{\infty} W_n(x, t)\lambda^{-n} \ , \\ Z(x, y, t, \lambda) &= -i\lambda(x - y)\sigma_3 + \sum_{p=1}^{\infty} Z_n(x, y)\lambda^{-p} \ . \end{aligned} \tag{6.96}$$

Next, we insert (6.94), (6.95) into (5.1a). The common factor $(1 + W(y, t, \lambda))^{-1}$ can be dropped, and the rest can be split into diagonal

$$i\frac{dZ}{dx} + q(x, t)W(x, t, \lambda) - \lambda\sigma_3 = 0 \ , \tag{6.97}$$

and off-diagonal parts:

$$i\frac{dW}{dx} + iW(x, t, \lambda)\frac{dZ}{dx} + q(x, t) - \lambda\sigma_3 W(x, t, \lambda) = 0 \ . \tag{6.98}$$

[2] Due to the special normalization condition there follows that $Y^{+}(x, y, t, \lambda)$ is a meromorphic function of λ; in what follows we shall not need these properties.

Taking into account (6.97), we rewrite (6.98) as:

$$
\begin{aligned}
i\frac{dW}{dx} &- W(x,t,\lambda)q(x,t)W(x,t,\lambda) \\
&+ q(x,t) - \lambda[\sigma_3, W(x,t,\lambda)] = 0 \;,
\end{aligned}
\tag{6.99}
$$

Then the expansion (6.96) leads to the recurrent relations:

$$
[\sigma_3, W_{n+1}] = i\frac{dW_n}{dx} - \sum_{\substack{k+p=n \\ k,p\geq 1}} W_k(x,t)q(x,t)W_p(x,t) \;, \tag{6.100}
$$

with the initial condition:

$$
W_1(x,t) = \frac{1}{4}[\sigma_3, q(x,t)] \;. \tag{6.101}
$$

Since $W(x,t,\lambda)$ is off-diagonal $W_{n+1}(x,t)$ is obtained by applying to the right-hand side of (6.100) $\mathrm{ad}_{\sigma_3}^{-1}$. Then the fact that $W_{n+1}(x,t)$ is local in $q(x,t)$ follows from the fact that $W_1(x,t)$, ..., $W_n(x,t)$ are.

Knowing $W_n(x,t)$ from (6.97) and (6.96), we can recover the expansion coefficients $Z_n(x,y)$ through:

$$
\frac{dZ_n}{dx} = iq(x,t)W_n(x,t) \;, \tag{6.102}
$$

or

$$
Z_n(x,y,t) = i\int_y^x dx'\, q(x')W_n(x') \;. \tag{6.103}
$$

The next step will be to show how $Z_n(x,y)$ and $W_n(x,t)$ are related to the densities of C_n. To this end, we shall take the limits $x \to \infty$, and $y \to -\infty$, and making use of the asymptotics of $\chi^+(x,t,\lambda)$ (3.25), we find that:

$$
\lim_{\substack{x\to\infty \\ y\to-\infty}} \chi^+(x,t,\lambda)\hat{\chi}^+(y,t,\lambda) = \lim_{\substack{x\to\infty \\ y\to-\infty}} e^{-i\lambda\sigma_3 x}T(t,\lambda)e^{i\lambda\sigma_3 y} \;, \tag{6.104}
$$

$$
\lim_{\substack{x\to-\infty \\ y\to\infty}} \chi^+(x,t,\lambda)\hat{\chi}^+(y,t,\lambda) = \lim_{\substack{x\to-\infty \\ y\to\infty}} e^{-i\lambda\sigma_3 x}\hat{T}(t,\lambda)e^{i\lambda\sigma_3 y} \;, \tag{6.105}
$$

From the other side, the anzats (6.94) means that we may look for $\chi^+(x,t,\lambda)$ in the form:

$$
\chi^+(x,t,\lambda) = (\mathbb{1} + W(x,t,\lambda))e^{\tilde{Z}(x,t,\lambda)} \;, \tag{6.106}
$$

where the diagonal matrix $\tilde{Z}(x,t,\lambda)$ is related to $Z(x,y,t,\lambda)$ through:

$$Z(x,y,t,\lambda) = \tilde{Z}(x,t,\lambda) - \tilde{Z}(y,t,\lambda) \; . \tag{6.107}$$

We can plug (6.106) into (3.25) and take the limits $x \to \infty$ and $x \to -\infty$ to find:

$$\lim_{x\to -\infty} (\tilde{Z}(x,t,\lambda) + i\lambda\sigma_3 x) = \begin{pmatrix} 0 & 0 \\ 0 & \ln a^+(\lambda) \end{pmatrix} \; , \tag{6.108}$$

$$\lim_{x\to \infty} (\tilde{Z}(x,t,\lambda) + i\lambda\sigma_3 x) = \begin{pmatrix} \ln a^+(\lambda) & 0 \\ 0 & 0 \end{pmatrix} \; , \tag{6.109}$$

for $\lambda \in \mathbb{C}_+$. Thus from (6.108), (6.109) and (6.107) we find:

$$\lim_{\substack{x\to\infty \\ y\to -\infty}} (\tilde{Z}(x,t,\lambda) - \tilde{Z}(y,t,\lambda) + i\lambda\sigma_3(x-y)) = \ln a^+(\lambda)\sigma_3 \; . \tag{6.110}$$

Analogously, starting from $Y^-(x,y,t,\lambda) = \chi^-(x,t,\lambda)\hat{\chi}^-(y,t,\lambda)$ for $\lambda \in \mathbb{C}_-$ we find that:

$$\lim_{\substack{x\to\infty \\ y\to -\infty}} (Z(x,y,t,\lambda) + i\lambda\sigma_3(x-y)) = -\ln a^-(\lambda)\sigma_3 \; . \tag{6.111}$$

The main conclusion from (6.110), (6.111) and (6.96) is that

$$\begin{aligned} \mathcal{A}(\lambda) &= \lim_{\substack{x\to\infty \\ y\to -\infty}} \left(i\lambda(x-y) + \frac{1}{2}\,\mathrm{tr}\,(Z(x,y,t,\lambda)\sigma_3 \right) \\ &= \frac{1}{2} \lim_{\substack{x\to\infty \\ y\to -\infty}} \sum_{p=1}^{\infty} \lambda^{-p} \mathrm{tr}\,(Z_p(x,t,y)\sigma_3) \; , \end{aligned} \tag{6.112}$$

i.e.

$$C_p = \frac{1}{2i} \lim_{\substack{x\to\infty \\ y\to -\infty}} \mathrm{tr}\,(Z_p(x,t,y)\sigma_3) \; . \tag{6.113}$$

Finally, taking into account (6.103) we get:

$$\mathcal{A}(\lambda) = \frac{i}{2}\int_{-\infty}^{\infty} dx\,\mathrm{tr}\,(q(x,t)W(x,t,\lambda)\sigma_3) \; , \tag{6.114}$$

or

$$C_p = \frac{1}{2}\int_{-\infty}^{\infty} dx\,\mathrm{tr}\,\left(q(x,t)W_p(x,t)\sigma_3\right) \; . \tag{6.115}$$

The first four of them have the form:

$$C_1 = -\frac{1}{4}\int_{-\infty}^{\infty} dx\,\mathrm{tr}\,(q^2) = -\frac{1}{2}\int_{-\infty}^{\infty} dx\, q^+ q^-(x)\;, \tag{6.116}$$

$$C_2 = \frac{i}{8}\int_{-\infty}^{\infty} dx\,\mathrm{tr}\,(q q_x) = \frac{i}{8}\int_{-\infty}^{\infty} dx\,(q^+ q_x^- - q_x^+ q^-)\;, \tag{6.117}$$

$$\begin{aligned} C_3 &= \frac{1}{16}\int_{-\infty}^{\infty} dx\,\mathrm{tr}\,(-q_x q_x + q^4) \\ &= \frac{1}{8}\int_{-\infty}^{\infty} dx\,(-q_x^+ q_x^- + (q^+ q^-)^2)\;, \end{aligned} \tag{6.118}$$

$$\begin{aligned} C_4 &= \frac{i}{32}\int_{-\infty}^{\infty} dx\,\mathrm{tr}\,(q_x q_{xx}\sigma_3 - 3q^3 q_x \sigma_3) \\ &= \frac{i}{32}\int_{-\infty}^{\infty} dx\,(q_x^+ q_{xx}^- - q_{xx}^+ q_x^- - 3q^+ q^-(q^+ q_x^- - q_x^+ q^-))\;. \end{aligned} \tag{6.119}$$

The so-called trace identities are obtained by equating the two expressions for C_p: those expressed in terms of the scattering data (6.90) with the ones expressed in terms of the potential $q(x)$ (6.116). For example, the first three trace identities are:

$$\int_{-\infty}^{\infty} d\mu\,\eta(\mu) + \sum_{k=1}^{N}(\eta_k^+ + \eta_k^-) = \int_{-\infty}^{\infty} dx\, q^+ q^-(x)\;, \tag{6.120}$$

$$\int_{-\infty}^{\infty} d\mu\,\mu\eta(\mu) + \frac{i}{4}\sum_{k=1}^{N}((\eta_k^+)^2 - (\eta_k^-)^2) = -\frac{i}{4}\int_{-\infty}^{\infty} dx\,(q^+ q_x^- - q_x^+ q^-)\;, \tag{6.121}$$

$$\int_{-\infty}^{\infty} d\mu\,\mu^2\eta(\mu) - \frac{1}{12}\sum_{k=1}^{N}((\eta_k^+)^3 + (\eta_k^-)^3) = \frac{1}{4}\int_{-\infty}^{\infty} dx\,(q_x^+ q_x^- - (q^+ q^-)^2)\;, \tag{6.122}$$

$$\begin{aligned} &\int_{-\infty}^{\infty} d\mu\,\mu^3\eta(\mu) - \frac{i}{32}\sum_{k=1}^{N}((\eta_k^+)^4 - (\eta_k^-)^4) \\ &= \frac{i}{16}\int_{-\infty}^{\infty} dx\,(-q_x^+ q_{xx}^- + q_{xx}^+ q_x^- + 3q^+ q^-(q^+ q_x^- - q_x^+ q^-)). \end{aligned} \tag{6.123}$$

Now, we see that the recurrent relations (6.100), (6.115) allow one to evaluate the densities of the integrals C_p, considered as functionals of the potential $q(x,t)$. Obviously from (6.100) there follows that all the densities C_p with $p \geq 1$ are local, i.e. depend only on $q(x,t)$ and its x-derivatives; the ones with $p \leq -1$ are generically nonlocal. Sometimes, after a change of variables C_{-1} may become local, as in the s-G case.

6.4.4 Generating Operators and the Integrals of Motion

Now, we demonstrate that there is close relation between the densities of C_p and the diagonal of the resolvent of L. As a result, we shall derive a compact expressions for C_p through the generating operators $\Lambda_\pm$.

To this end, we remind the Wronskian relations (5.34), (5.35) and use also the diagonal of the resolvent (6.76):

$$\begin{aligned}\frac{d\mathcal{A}}{d\lambda} &= -\frac{i}{2}\int_{-\infty}^{\infty} dx\, \left(\operatorname{tr}\left(\chi^\pm\sigma_3\hat{\chi}^\pm(x,t,\lambda)\sigma_3\right) - 2\right) \\ &= \mp\int_{-\infty}^{\infty} dx\, \left(\operatorname{tr}\left(R^\pm(x,x,t,\lambda)\sigma_3\right) - 1\right) \\ &= -\frac{1}{2a^\pm(\lambda)}\int_{-\infty}^{\infty} dx \int_{\pm\infty}^{x} dy \operatorname{tr}\left([q(x,t),\sigma_3]\Theta^\pm(y,t,\lambda)\right)\ ,\end{aligned} \tag{6.124}$$

Thus, we see the relation between the diagonal of the resolvent of the ZS system and the generating functional of integrals of motion of the NLEEs.

Let us now make use of (5.116)–(5.118b) to express the densities of C_p through the generating operators $\Lambda_\pm$, Λ. To this end, we have to investigate the asymptotic expansions of the right-hand sides of, say (5.116) and (5.118a):

$$\frac{\Theta^\pm(x,t,\mu)}{a^\pm(\mu)} = \pm 2\rho^\pm(t,\mu)\boldsymbol{\Psi}^\pm(x,t,\mu) + (\Lambda_+ - \mu)^{-1}q(x,t)\ , \tag{6.125}$$

where $\mu \in \mathbb{C}_\pm$, respectively. We recall the definition of $\boldsymbol{\Psi}^\pm(x,\mu)$:

$$\begin{aligned}\boldsymbol{\Psi}^\pm(x,t,\mu) &= a^\pm(\mu)\left(\chi^\pm\sigma_\mp\hat{\chi}^\pm(x,t,\mu)\right)^{\mathrm{f}} \\ &= e^{\pm 2i\mu x}a^\pm(\mu)\left(\eta^\pm\sigma_\mp\hat{\eta}^\pm(x,t,\mu)\right)^{\mathrm{f}}\ .\end{aligned} \tag{6.126}$$

Here, $\eta^\pm(x,t,\mu)$ and its inverse $\hat{\eta}^\pm(x,t,\mu)$ are analytic functions of μ for $\mu \in \mathbb{C}_\pm$ that allow expansions of the form:

$$\eta^\pm(x,t,\mu) = \chi^\pm(x,t,\mu)e^{i\mu\sigma_3 x} = \mathbb{1} + \sum_{k=1}^{\infty}\eta_k^\pm(x,t)\mu^{-k}\ , \tag{6.127a}$$

$$\hat{\eta}^\pm(x,t,\mu) = e^{-i\mu\sigma_3 x}\hat{\chi}^\pm(x,t,\mu) = \mathbb{1} + \sum_{k=1}^{\infty}\hat{\eta}_k^\pm(x,t)\mu^{-k}\ . \tag{6.127b}$$

Thus, for the expansion of $\boldsymbol{\Psi}^\pm(x,t,\mu)/a^\pm(\mu)$ we find:

$$\frac{\boldsymbol{\Psi}^\pm(x,t,\mu)}{a^\pm(\mu)} = e^{\pm 2i\mu x}\left(\sigma_\mp + \sum_{s=1}^{\infty}\mu^{-s}\sum_{\substack{k+p=s\\ k,p\geq 1}}\left(\eta_k^\pm(x,t)\sigma_\mp\hat{\eta}_p^\pm(x,t)\right)^{\mathrm{f}}\right)\ . \tag{6.128}$$

We derive the coefficient in front of μ^s in the expansion of $\boldsymbol{\Psi}^{\pm}(x,t,\mu)/a^{\pm}(\mu)$ using the well-known formula:

$$\frac{1}{2\pi i}\oint_{C_{\pm,\infty}} d\mu\, \mu^{-s+1}\frac{\boldsymbol{\Psi}^{\pm}(x,t,\mu)}{a^{\pm}(\mu)} = 0\ , \qquad (6.129)$$

for all s; this result is due to the presence of the exponential factor in the right-hand side of (6.128). In other words, we find that the first term in the right-hand side of (6.125) does not contribute to the expansion of $\boldsymbol{\Theta}^{\pm}(x,t,\mu)/a^{\pm}(\mu)$.

Analogously, we can treat also (5.117) and (5.118b) to prove that the first term in the right-hand side of

$$\frac{\Theta^{\pm}(x,t,\mu)}{a^{\pm}(\mu)} = \mp 2\tau^{\pm}(t,\mu)\boldsymbol{\Phi}^{\pm}(x,t,\mu) + (\Lambda_{+}-\mu)^{-1}q(x,t)\ , \qquad (6.130)$$

is irrelevant for the expansion of $\boldsymbol{\Theta}^{\pm}(x,t,\mu)/a^{\pm}(\mu)$.

As a result, we see that it is enough to expand formally $(\Lambda_{\pm}-\mu)^{-1}q(x,t)$ in power series in μ; the result is given by the right hand side of (6.88) with λ replaced by Λ_{+} and Λ_{-}, respectively. Comparing this with (6.87) and (6.124) we get:

$$C_p = \frac{1}{ip}\int_{-\infty}^{\infty} dx \int_{\pm\infty}^{x} dy\, \mathrm{tr}\,\big([\sigma_3, q(y,t)], \Lambda_{\pm}^{p} q(y,t)\big) \qquad (6.131a)$$

for $p = 1, 2, \ldots$ and

$$C_{-p} = \frac{i}{p}\int_{-\infty}^{\infty} dx \int_{\pm\infty}^{x} dy\, \mathrm{tr}\,\big([\sigma_3, q(y,t)], \Lambda_{\pm}^{-p} q(y,t)\big) \qquad (6.131b)$$

for $p = 1, 2, \ldots$.

6.4.5 The Lenard Relation

Analogous expressions exist also for δC_p. To derive them, one must insert (6.125) and (6.88) into the Wronskian relation (5.27)) with the result:

$$\begin{aligned}\delta C_p &= \frac{1}{4}\left[\!\left[\sigma_3 \delta q(x,t), \Lambda_{\pm}^{p-1} q(x,t)\right]\!\right] \\ &= -\frac{1}{2}\int_{-\infty}^{\infty} dx \mathrm{tr}\,\Big(\delta q(x,t), \Lambda_{\pm}^{p-1} q(x,t)\Big)\ . \end{aligned} \qquad (6.132)$$

This means that their variational derivatives have the form:

$$\frac{\delta C_p}{\delta q(x,t)} = -\frac{1}{2}\Lambda_{\pm}^{p-1} q(x,t), \qquad p = 1, 2, \ldots \qquad (6.133)$$

Note that the right-hand side of (6.133) does not depend on the choice of the generating operator that we use; it will be the same for Λ_+, Λ_-, and Λ. The Lenard relation is obtained easily from (6.133):

$$\frac{\delta C_p}{\delta q(x,t)} = \Lambda_\pm \frac{\delta C_{p-1}}{\delta q(x,t)} = \Lambda \frac{\delta C_{p-1}}{\delta q(x,t)} \ . \tag{6.134}$$

Three important facts must be noted here:

1. The variational derivatives of C_p and C_{p-1} are related by an Λ-operator, which is *p-independent*;
2. As a consequence of (6.133) and (6.134), the nonlinear parts of the NLEEs (6.7) are in fact variational derivatives of conveniently chosen linear combinations of C_ps;

They underly the Hamiltonian formulation of the NLEEs, which we shall display in the next Chapter.

6.5 The Class of Bäcklund Transformations

The proper Bäcklund transformations (BT) is known also as the auto-BT transform given solution $q(x,t)$ of a NLEE to another solution $q'(x,t)$ of the same NLEE. The NLEEs we are studying are solvable by the ISM applied to the Zakharov-Shabat system. Therefore, the BT can be viewed also as an automorphism of the classes of allowed potentials of L. Note that the BT *are not isospectral;* we will come back to this point below.

Remark 6.9. In this section, for notational simplicity, the t-dependence will be omitted if it does not lead to ambiguity.

We shall describe a large class of BTs using the expansions over the products of solutions of two ZS system: L and L' (see Sect. 5.6) and the properties of the generalized recursion operators $\Lambda_\pm$. Our considerations will involve also functions (generically, polynomials) of $\Lambda'_\pm$ which act naturally on the "products of solutions":

$$\begin{aligned} g(\Lambda'_+)\boldsymbol{\Psi}'^{,\pm}_j(x,\lambda) &= g(\lambda)\boldsymbol{\Psi}'^{,\pm}_j(x,\lambda) = 0, \quad \lambda \in \mathbb{R} \cup \mathcal{Z}^\pm \cup \mathcal{Z}'^{,\pm} \ , \\ g(\Lambda'_+)\dot{\boldsymbol{\Psi}}'^{,\pm}_j(x) &= g(\lambda_j^\pm)\dot{\boldsymbol{\Psi}}'^{,\pm}_j(x) + \dot{g}(\lambda_j^\pm)\boldsymbol{\Psi}'^{,\pm}_j(x), \qquad j \in \mathcal{N}_0, \end{aligned} \tag{6.135}$$

$$\begin{aligned} g(\Lambda'_-)\boldsymbol{\Phi}'^{,\pm}_j(x,\lambda) &= g(\lambda)\boldsymbol{\Phi}'^{,\pm}_j(x,\lambda) = 0, \quad \lambda \in \mathbb{R} \cup \mathcal{Z}^\pm \cup \mathcal{Z}'^{,\pm} \ , \\ g(\Lambda'_-)\dot{\boldsymbol{\Phi}}'^{,\pm}_j(x) &= g(\lambda_j^\pm)\dot{\boldsymbol{\Phi}}'^{,\pm}_j(x) + \dot{g}(\lambda_j^\pm)\boldsymbol{\Phi}'^{,\pm}_j(x), \qquad j \in \mathcal{N}_0, \end{aligned} \tag{6.136}$$

where the sets of discrete eigenvalues were introduced in (5.159). Using these relations, we obtain the following expansions over the "products of solutions":

$$
\begin{aligned}
& g(\Lambda'_+)\sigma_3(q'(x) - q(x)) \\
&= \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda\, g(\lambda)\left((\rho'^{,+} - \rho^+)\boldsymbol{\Psi}'^{,+}(x,\lambda) + (\rho'^{,-} - \rho^-)\boldsymbol{\Psi}'^{,-}(x,\lambda)\right) \\
&\quad + 2\sum_{k\in\mathcal{N}_0}^{\pm} \pm g(\lambda_k^\pm)(C_k'^{,\pm} - C_k^\pm)\boldsymbol{\Psi}_k'^{,\pm}(x) + 2\sum_{k\in\mathcal{N}_1}^{\pm} \pm g(\lambda_k^\pm)C_k^\pm \boldsymbol{\Psi}_k'^{,\pm}(x) \\
&\quad + 2\sum_{k\in\mathcal{N}_2}^{\pm} \pm g(\lambda_k'^{,\pm})C_k'^{,\pm}\boldsymbol{\Psi}_k'^{,\pm}(x)\,, \qquad (6.137) \\
&= \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda\, g(\lambda)\left((\tau'^{,+} - \tau^+)'\boldsymbol{\Phi}^+(x,\lambda) + (\tau'^{,-} - \tau^-)'\boldsymbol{\Phi}^-(x,\lambda)\right) \\
&\quad + 2\sum_{k\in\mathcal{N}_0}^{\pm} \pm g(\lambda_k^\pm)(M_k'^{,\pm} - M_k^\pm)'\boldsymbol{\Phi}_k^\pm(x) + 2\sum_{k\in\mathcal{N}_1}^{\pm} \pm g(\lambda_k^\pm)M_k'^{,\pm\prime}\boldsymbol{\Phi}_k^\pm(x) \\
&\quad + 2\sum_{k\in\mathcal{N}_2}^{\pm} \pm g(\lambda_k'^{,\pm})M_k'^{,\pm\prime}\boldsymbol{\Phi}_k^\pm(x)\,, \qquad (6.138)
\end{aligned}
$$

and

$$
\begin{aligned}
& h(\Lambda'_+)(q'(x) + q(x)) = \\
&= -\frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda\, h(\lambda)\left((\rho'^{,+} + \rho^+)\boldsymbol{\Psi}'^{,+}(x,\lambda) - (\rho'^{,-} + \rho^-)\boldsymbol{\Psi}'^{,-}(x,\lambda)\right) \\
&\quad - 2\sum_{k\in\mathcal{N}_0}^{\pm} h(\lambda_k^\pm)(C_k'^{,\pm} + C_k^\pm)\boldsymbol{\Psi}_k'^{,\pm}(x) - 2\sum_{k\in\mathcal{N}_1}^{\pm} h(\lambda_k^\pm)C_k^\pm \boldsymbol{\Psi}_k'^{,\pm}(x) \\
&\quad - 2\sum_{k\in\mathcal{N}_2}^{\pm} h(\lambda_k'^{,\pm})C_k'^{,\pm}\boldsymbol{\Psi}_k'^{,\pm}(x)\,, \qquad (6.139) \\
&= \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda\, h(\lambda)\left((\tau'^{,+} + \tau^+)'\boldsymbol{\Phi}^+(x,\lambda) - (\tau'^{,-} + \tau^-)'\boldsymbol{\Phi}^-(x,\lambda)\right) \\
&\quad + 2\sum_{k\in\mathcal{N}_0}^{\pm} h(\lambda_k^\pm)(M_k'^{,\pm} + M_k^\pm)'\boldsymbol{\Phi}_k^\pm(x) + 2\sum_{k\in\mathcal{N}_1}^{\pm} h(\lambda_k^\pm)M_k'^{,\pm\prime}\boldsymbol{\Phi}_k^\pm(x) \\
&\quad + 2\sum_{k\in\mathcal{N}_2}^{\pm} h(\lambda_k'^{,\pm})M_k'^{,\pm\prime}\boldsymbol{\Phi}_k^\pm(x)\,, \qquad (6.140)
\end{aligned}
$$

In other words, the elements of the two sets of scattering data $\mathcal{T}$ and $\mathcal{T}'$ determine the expansion coefficients of $\sigma_3(q'(x) - q(x))$ and $q'(x) + q(x)$ over the products of solutions.

Theorem 6.10. *Let the potentials $q(x)$ and $q'(x)$ satisfy conditions* **C1** *and* **C2** *and let the the functions $g(\lambda)$ and $h(\lambda)$ be meromorphic for $\lambda \in \mathbb{C}$ with no singularities on the spectrums of L and L'. Then the BTs:*

$$
g(\Lambda'_+)\sigma_3(q'(x) - q(x)) + h(\Lambda'_+)(q'(x) + q(x)) = 0\,, \qquad (6.141a)
$$

$$
g(\Lambda'_-)\sigma_3(q'(x) - q(x)) - h(\Lambda'_-)(q'(x) + q(x)) = 0\,, \qquad (6.141b)
$$

are pairwise equivalent to the following linear relations between the scattering data $\mathcal{T}$ and $\mathcal{T}'$:

$$\rho^{\prime,\pm}(\lambda) = \frac{H^{\pm}(\lambda)}{H^{\mp}(\lambda)}\rho^{\pm}(\lambda), \qquad H^{\pm}(\lambda) = g(\lambda) \pm h(\lambda)\ , \tag{6.142a}$$

$$C_k^{\prime,\pm} = \frac{H^{\pm}(\lambda_k^{\pm})}{H^{\mp}(\lambda_k^{\pm})} C_k^{\pm}, \quad k \in \mathcal{N}_0\ , \tag{6.142b}$$

$$H^{\mp}(\lambda_k^{\prime,\pm}) C_k^{\prime,\pm} = 0, \quad k \in \mathcal{N}_2, \qquad H^{\pm}(\lambda_k^{\mp}) C_k^{\pm} = 0, \quad k \in \mathcal{N}_1\ . \tag{6.142c}$$

$$\tau^{\prime,\pm}(\lambda) = \frac{H^{\pm}(\lambda)}{H^{\mp}(\lambda)}\tau^{\pm}(\lambda), \qquad H^{\pm}(\lambda) = g(\lambda) \pm h(\lambda)\ , \tag{6.143a}$$

$$M_k^{\prime,\pm} = \frac{H^{\pm}(\lambda_k^{\pm})}{H^{\mp}(\lambda_k^{\pm})} M_k^{\pm}, \quad k \in \mathcal{N}_0\ , \tag{6.143b}$$

$$H^{\mp}(\lambda_k^{\prime,\pm}) M_k^{\prime,\pm} = 0, \quad k \in \mathcal{N}_2, \qquad H^{\pm}(\lambda_k^{\pm}) M_k^{\pm} = 0, \quad k \in \mathcal{N}_1\ . \tag{6.143c}$$

Proof. Conditions **C1** and **C2** ensure the existence of the expansions over the products of solutions (6.137)–(6.140). Summing the expansions (6.137) and (6.139), we get the following expansion for the left-hand side of (6.141a):

$$\begin{aligned}
& g(\Lambda'_+)\sigma_3(q'(x) - q(x)) + h(\Lambda'_+)(q'(x) + q(x)) \\
&= \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda\, \left(g(\lambda)\left((\rho^{\prime,+}(\lambda) - \rho^{+}(\lambda))\right) - h(\lambda)\left((\rho^{\prime,+}(\lambda) + \rho^{+}(\lambda))\right)\right) \boldsymbol{\Psi}^{\prime,+}(x,\lambda) \\
&+ \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda\, \left(g(\lambda)\left((\rho^{\prime,-}(\lambda) - \rho^{-}(\lambda))\right) + h(\lambda)\left((\rho^{\prime,-}(\lambda) + \rho^{-}(\lambda))\right)\right) \boldsymbol{\Psi}^{\prime,-}(x,\lambda) \\
&+ 2\sum_{k\in\mathcal{N}_0}^{\pm} \left(\pm H^{\mp}(\lambda_k^{\pm}) C_k^{\prime,\pm} - H^{\pm}(\lambda_k^{\pm}) C_k^{\pm}\right) \boldsymbol{\Psi}_k^{\prime,\pm}(x) \\
&+ 2\sum_{k\in\mathcal{N}_1}^{\pm} \mp H^{\pm}(\lambda_k^{\pm}) C_k^{\pm} \boldsymbol{\Psi}_k^{\prime,\pm}(x) + 2\sum_{k\in\mathcal{N}_2}^{\pm} \pm H^{\mp}(\lambda_k^{\prime,\pm}) C_k^{\prime,\pm} \boldsymbol{\Psi}_k^{\prime,\pm}(x)\ .
\end{aligned} \tag{6.144}$$

In order to obtain the equivalence between the BT (6.141a) and the set of eqs. (6.142), it remains to apply Proposition 5.6. The equivalence between the BT (6.141b) follows analogously from the expansions (6.138) and (6.140) and Proposition 5.6.

The next lemma clears the relation between the scattering data $\mathcal{T}$ and $\mathcal{T}'$

Lemma 6.11. *Let $q(x)$ and $q'(x)$ be the potentials of two ZS system L and L' with minimal sets of scattering data $\mathcal{T}$ and $\mathcal{T}'$, respectively. Let also $q(x)$ and $q'(x)$ satisfy the BT (6.141a) with some given functions $g(\lambda)$ and $h(\lambda)$. Then:*

1. The potentials $q(x)$ and $q'(x)$ are such that

$$\begin{aligned} H^+(\lambda) &= \prod_{k\in\mathcal{N}_1} (\lambda-\lambda_k^-) \prod_{k\in\mathcal{N}_2} (\lambda-\lambda_k^{',+}) H_0^+ \, , \\ H^-(\lambda) &= \prod_{k\in\mathcal{N}_1} (\lambda-\lambda_k^+) \prod_{k\in\mathcal{N}_2} (\lambda-\lambda_k^{',-}) H_0^- , \end{aligned} \tag{6.145}$$

where $H_0^\pm$ are some constants;

2. The matrix elements of $T(\lambda)$ (2.48) and $T'(\lambda)$ are related by:

$$\begin{aligned} a^{',+}(\lambda) &= \prod_{k\in\mathcal{N}_1} \frac{\lambda-\lambda_k^-}{\lambda-\lambda_k^+} \prod_{k\in\mathcal{N}_2} \frac{\lambda-\lambda_k^{',+}}{\lambda-\lambda_k^{',-}} a^+(\lambda) \, , \\ a^{',-}(\lambda) &= \prod_{k\in\mathcal{N}_1} \frac{\lambda-\lambda_k^+}{\lambda-\lambda_k^-} \prod_{k\in\mathcal{N}_2} \frac{\lambda-\lambda_k^{',-}}{\lambda-\lambda_k^{',+}} a^-(\lambda) \, , \\ b^{',+}(\lambda) &= \frac{H_0^+}{H_0^-} b^+(\lambda), \qquad b^{',-}(\lambda) = \frac{H_0^-}{H_0^+} b^-(\lambda), \end{aligned} \tag{6.146}$$

Proof. 1. We use (6.142c) and the fact that the normalization constants $C_k^\pm$, $C_k^{',\pm}$ and $M_k^\pm$, $M_k^{',\pm}$ do not vanish. Therefore, (6.142c) and (6.143c) in fact mean that

$$H^\mp(\lambda_k^{',\pm}) = 0, \quad k\in\mathcal{N}_2, \qquad H^\pm(\lambda_k^\pm) = 0, \quad k\in\mathcal{N}_1 \, . \tag{6.147}$$

Therefore, $H^\pm(\lambda)$ must be of the form (6.145), with $H_0^\pm(\lambda)$ being functions of λ such that they are analytic in the whole complex λ plane and tending to a constant for $\lambda\to\infty$. According to Liouville theorem such functions are in fact constants.

2. From (6.142a), we have that $\rho^{',+}\rho^{',-} = \rho^+\rho^-$. Inserting that in the dispersion relations for $a^{',\pm}(\lambda)$ and $a^\pm(\lambda)$ (see (3.61) and (3.62)) we recover the first two lines of (6.146). The last line of (6.146) follows from $\rho^\pm(\lambda) = b^\pm(\lambda)/a^\pm(\lambda)$ and $\rho^{',\pm}(\lambda) = b^{',\pm}(\lambda)/a^{',\pm}(\lambda)$.

Obviously, acting by generic polynomial expressions of $\Lambda'_\pm$ on either $\sigma_3(q'(x)-q(x))$ or $(q'(x)+q(x))$, we get highly nonlinear and nonlocal expressions. Below, we demonstrate some simple cases [7, 8], when these nonlocal expressions can be turned into local. In both examples, the functions $g(\lambda)$ and $h(\lambda)$ will be linear

$$g(\lambda) = g_1\lambda + g_0, \qquad h(\lambda) = h_1\lambda + h_0 \, , \tag{6.148}$$

where g_a, h_a, $a = 1,2$ are complex constants such that the roots of $H^\pm(\lambda)$ are $\lambda_k^{',\pm} = \mu_k \pm i\nu_k$ with μ_k real and $\nu_k > 0$. Using the explicit form of Λ'_+ we get a BT of the form:

$$g(\Lambda'_+)\sigma_3(q'(x)-q(x)) + h(\Lambda'_+)(q'(x)+q(x))$$

$$
\begin{aligned}
&= \frac{i}{2}\frac{d}{dx}\left(g_1(q'-q) + h_1\sigma_3(q'+q)\right) + \frac{i}{2}\left(g_1(q'+q) + h_1\sigma_3(q'-q)\right)Z(x) \\
&\quad + g_0\sigma_3(q'-q) + h_0(q'+q) = 0\,.
\end{aligned} \tag{6.149}
$$

where

$$
Z(x) = \frac{1}{2}\int_{\infty}^{x} dy\,\mathrm{tr}\,\left(q'^2 - q^2\right)(y) = \int_{\infty}^{x} dy\,(q'_+q'_- - q_+q_-)\,. \tag{6.150}
$$

Let us show that $Z(x)$ can be expressed locally through $q'_\pm$ and $q_\pm$. Multiply (6.149) from the left by $p_1(x) = g_1(q'-q) + h_1\sigma_3(q'+q)$, take the trace, and integrate with respect to dx. Thus, we obtain for $Z(x)$ the quadratic equation:

$$
\frac{1}{2}(g_1^2 - h_1^2)Z^2 - 4i(h_0g_1 - h_1g_0)Z + \mathrm{tr}\,p_1^2 - C_1 = 0\,, \tag{6.151}
$$

$$
p_1(x) = g_1(q'-q) + h_1\sigma_3(q'+q) \tag{6.152}
$$

where C_1 is an integration constant which can be fixed up by the condition $\lim_{x\to\infty} Z(x) = 0$. This gives $C_1 = 0$ and the following solution for $Z(x)$:

$$
Z_{1,2}(x) = 2i\alpha_0 \pm \sqrt{\frac{\mathrm{tr}\,p_1^2(x)}{2(h_1^2 - g_1^2)} - 4\alpha_0^2}, \qquad \alpha_0 = \frac{g_1h_0 - g_0h_1}{g_1^2 - h_1^2}\,, \tag{6.153}
$$

$$
\begin{aligned}
\mathrm{tr}\,p_1^2(x) = 2g_1^2(q'_+ - q_+)(q'_- - q_-) - 2h_1^2)(q'_+ + q_+)(q'_- + q_-) \\
- 4g_1h_1(q'_+q_- - q'_-q_+).
\end{aligned} \tag{6.154}
$$

With (6.153) the BT (6.141a) becomes local. From the equations (5.94), we see that this BT adds two new eigenvalues to the spectrum of the ZS system $\lambda_1^{\prime,\pm} = -(f_0 \mp g_0)/(f_1 \mp g_1)$. In the examples below, we shall show that the BT goes into the well-known BT [9, 10]. Before we go into the examples, we note that in the cases when additional involutions like (6.47) and/or (6.48) are imposed, one needs to impose the corresponding restrictions on the functions $g(\lambda)$ and $h(\lambda)$. In the case when $q_- = \pm q_-^{\prime,*}$, the compatibility condition for the system (16.150) requires that

$$
H^+(\lambda) = (H^-(\lambda^*))^*\,. \tag{6.155}
$$

If $q^+ = \eta_0 q^-$ then the functions $g(\lambda)$ and $h(\lambda)$ must satisfy:

$$
H^+(\lambda) = H^-(-\lambda)\,. \tag{6.156}
$$

In case both involutions (6.47) and (6.48) are imposed, the functions $g(\lambda)$ and $h(\lambda)$ must satisfy both restrictions (6.155) and (6.156) simultaneously.

Example 6.12. The BT for the nonlinear Schrödinger equation is obtained using the involution (6.47) with $\epsilon = -1$:

$$
g(\lambda) = -i\nu_1, \qquad h(\lambda) = \lambda - \mu_1, \qquad q_- = -q_+^*, \quad q'_- = -q_+^{\prime,*}\,, \tag{6.157}
$$

where μ_1 and ν_1 are real parameters and $\nu_1 > 0$. In that case, $Z(x) = 4\nu_1 + 2\sqrt{4\nu_1^2 - |v_2 - v_1|^2}$ and (16.150) goes into the well-known BT [9]:

$$\frac{d(q'_+ + q_+)}{dx} = -2i\mu_1(q'_+ + q_+)(x) + (q'_+ - q_+)\sqrt{4\nu_1^2 - |q'_+ - q_+|^2} \,. \quad (6.158)$$

It adds to the spectrum of the ZS system two eigenvalues $\tilde{\lambda}_2^{\pm} = \mu_1 \pm i\nu_1$.

Example 6.13. The BT for the sine-Gordon equation is obtained if both involutions (6.47) and (6.48) are imposed and

$$g(\lambda) = -i\nu_1, \qquad h(\lambda) = \lambda, \qquad q_+ = -q_- = -iw_x, \qquad q'_+ = -q'_- = -iw'_x \,, \quad (6.159)$$

with $u(x)$ and $u'(x)$ being real functions and $\nu_1 > 0$. This gives $Z(x) = 2\nu_1 + \sqrt{4\nu_1^2 - (w'_x + w_x)^2}$ and (6.149) goes into

$$\frac{d}{dx}(w'_x + w_x) + (w'_x - w_x)\sqrt{4\nu_1^2 - (w'_x + w_x)^2} = 0 \,. \quad (6.160)$$

or, equivalently:

$$\frac{d(w'_x + w_x)}{\sqrt{4\nu_1^2 - (w'_x + w_x)^2}} + (w'_x - w_x)dx = 0 \,. \quad (6.161)$$

Integrating once we get:

$$\arcsin\frac{w'_x + w_x}{2\nu_1} + w'(x,t) - w(x,t) = c_2 \,, \quad (6.162)$$

The integration constant c_2 is determined by the asymptotic behavior of w and w' as $c_2 = 2\pi k$, where k is an integer. Finally we get:

$$w'x + w_x = -2\nu_1 \sin(w' - w) \,, \quad (6.163)$$

which is the x-part of the BT for the sine-Gordon equation.

6.6 Comments and Bibliographical Review

1. A number of papers have approached the complete integrability of the infinite-dimensional Hamiltonian systems [1, 2, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87]. They are integrable by the ISM applied to different Lax operators.
2. By now tens of specific NLEEs are known for their applications in physics. The list of such NLEEs along with their applications can be found in [6, 20, 48, 54, 72, 87, 88, 89, 90].

3. A lot of studies were aimed at finding the integrability criterium, i.e. to discover a criteria that would ensure the integrability of a given nonlinear partial differential equation. One of these criteria based on establishing the existence of a higher integral of motion was proposed in [91]. Thus Mikhailov, Shabat, and Yamilov [91] were able to classify all integrable versions of the NLS-type equations. This approach is rigorous and fundamental and meets serious difficulties of technical nature when the number of the fields exceeds 2.
4. Another method is the so-called Painlevé test [57, 70, 92]. If given NLEE passes the Painlevé test, then there are chances that it may have Lax representation and so might be integrable through the ISM.
5. The hierarchy of M-operators in the Lax representation are naturally generated by the recursion operators $\Lambda_{\pm}$. For the ZS system, this has been shown in [6, 7, 8, 14, 93] and also generalized to several classes of Lax operators [19, 36, 48, 70, 72, 87, 94, 95]. We also specially note the relation between the M-operators and the classical r-matrix found in [6].
6. It is well known that the integrals of motion C_k for the ZS system have local densities, i.e. they are expressed only in terms of $q(x,t)$ and its x-derivatives [6, 14, 96].
7. The trace identities for the ZS system play an important role in the analysis of the mapping between the potential and the scattering data [6, 7, 8, 93, 97].
8. The involutions of the ZS system analyzed above can be viewed as simple particular realizations of the reduction group $\mathbb{Z}_2$ proposed by Mikhailov [98]. This method is important for deriving new integrable equations from generic multicomponent NLEE. One of its important consequences was the discovery of the two-dimensional Toda field theories [54, 98, 99, 100, 101, 102].

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7

Hierarchies of Hamiltonian Structures

In this chapter, we explain how the NLEEs analyzed above can be viewed as infinite dimensional Hamiltonian systems. We start by several basic examples. Next, we go to the generic NLEE, whose phase space $\mathcal{M}^{\mathbb{C}}$ is equivalent to the space of pairs of smooth complex-valued functions $\{q^+(x), q^-(x)\}$. This phase space and the Hamiltonian dynamics on it can be viewed as a complexification of the standard Hamiltonian dynamics, and the well-known Hamiltonian systems come up as different real forms of them. In Sect. 7.3, using the expansion of $\sigma_3 \delta q(x)$ over the symplectic basis, we derive the action-angle variable for the generic NLEE. In the next section we demonstrate that each generic NLEE allows a hierarchy of compatible Hamiltonian structures generated by the recursion operator Λ. The different real forms of these Hamiltonian structures are derived in Sect. 7.5. These are obtained from the generic ones by imposing additional involutions. Using the additional symmetry properties that the scattering data acquire due to the involution we demonstrate that in some cases "half" of the Hamiltonian structures become degenerated.

7.1 Hamiltonian Properties: Basic Examples

The Hamiltonian properties of some particular equations have been well known for a long time. For example, the NLS equation

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + 2|u|^2 u(x,t) = 0 , \tag{7.1}$$

can be written down as

$$\frac{\partial u}{\partial t} = \{H_{\rm NLS}, u\}_{(0)}, \qquad \frac{\partial u^*}{\partial t} = \{H_{\rm NLS}, u^*\}_{(0)} , \tag{7.2}$$

where

$$H_{\rm NLS} = \int_{-\infty}^{\infty} dx\, (|u_x|^2 - |u(x,t)|^4) . \tag{7.3}$$

Gerdjikov, V.S. et al.: *Hierarchies of Hamiltonian Structures.* Lect. Notes Phys. **748**, 211–245 (2008)
DOI 10.1007/978-3-540-77054-1_7

The phase space $\mathcal{M}_{\mathrm{NLS}}$ is the infinite dimensional manifold of all complex-valued functions $u(x,t)$, which are smooth and fall off fast enough for $|x| \to \infty$. The Poisson brackets on $\mathcal{M}_{\mathrm{NLS}}$ are introduced by:

$$\begin{aligned} \{u(x), u(y)\}_{(0)} &= 0, \qquad \{u^*(x), u^*(y)\}_{(0)} = 0 \,, \\ \{u(x), u^*(y)\}_{(0)} &= i\delta(x-y) \,. \end{aligned} \tag{7.4}$$

Then it easy to check that the Hamiltonian equations (7.2) coincide with the NLS (7.1) and its complex conjugate.

The phase space $\mathcal{M}_{\mathrm{NLS}}$ can be viewed equivalently as the space of pairs of real-valued functions $u_0(x)$ and $u_1(x)$ satisfying $u(x) = u_0(x) + iu_1(x)$ and

$$\begin{aligned} \{u_0(x), u_0(y)\}_{(0)} &= 0, \qquad \{u_1(x), u_1(y)\}_{(0)} = 0 \,, \\ \{u_0(x), u_1(y)\}_{(0)} &= -\frac{1}{2}\delta(x-y) \,. \end{aligned} \tag{7.5}$$

The Hamiltonian formulation of the mKdV and sine-Gordon equations

$$\frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} + 6\kappa_1 \frac{\partial w}{\partial x} w^2(x,t) = 0, \qquad \kappa_1 = \pm 1; \tag{7.6}$$

$$\frac{\partial^2 w}{\partial x \partial t} + \sin(w(x,t)) = 0 \,, \tag{7.7}$$

require different construction.[1] The corresponding phase space $\mathcal{M}_{\mathrm{mKdV}}$ is the space of real-valued functions $w(x,t)$, which are smooth and fall off fast enough for $|x| \to \infty$. The Poisson brackets on $\mathcal{M}_{\mathrm{mKdV}}$ are introduced by:

$$\{w(x), w(y)\}_{(1)} = \frac{\partial}{\partial x}\delta(x-y) \,, \tag{7.8}$$

and the Hamiltonian is given by:

$$H_{\mathrm{mKdV}} = \int_{-\infty}^{\infty} dx \, \left(\frac{1}{2} w w_{xx} + \frac{\kappa_1}{2} w^4 \right) \,, \tag{7.9}$$

The phase space for the s-G equation is the same as $\mathcal{M}_{\mathrm{mKdV}}$, but the Poisson brackets on it are introduced by:

$$\{w(x), w(y)\}_{(1)} = \partial_x^{-1}\delta(x-y) \,, \tag{7.10}$$

where $\partial_x^{-1} \cdot = 1/2 \left(\int_{-\infty}^{x} dy + \int_{\infty}^{x} dy \right) \cdot$ and the Hamiltonian is given by:

$$H_{\mathrm{sG}} = \int_{-\infty}^{\infty} dx \, (1 - \cos w(x,t)) \,. \tag{7.11}$$

[1] For the moment, we leave aside the problems related to the topological charge of the sine-Gordon equation.

One can also use an alternative way to describe the Hamiltonian dynamics which makes use of symplectic forms rather than of Poisson brackets. The symplectic form corresponding to the Poisson brackets (7.4), (7.5):

$$\Omega_0 = \frac{1}{i} \int_{-\infty}^{\infty} dx\, \delta u(x) \wedge \delta u^*(x) = 2 \int_{-\infty}^{\infty} dx\, \delta u_1(x) \wedge \delta u_0(x) \ , \tag{7.12}$$

is called canonical symplectic form.

Remark 7.1. In this Chapter, in order to avoid unnecessary complications of the notations by $\delta u(x)$, we shall denote the infinite-dimensional analog of the 1-form; then $\delta u(x) \wedge \delta u^*(x)$ will be the infinite-dimensional analog of the canonical 2-form on $\mathcal{M}_{\mathrm{NLS}}$. For more precise definitions, notations, and explanations see the Second part.

To each pair of symplectic form Ω and a Hamiltonian H, one can put into correspondence a Hamiltonian vector-field X_H. By definition it satisfies the relation:

$$i_{X_H}\Omega_0 + \delta H \equiv \Omega_0(X_H, \cdot) + \delta H = 0 \ . \tag{7.13}$$

The infinite-dimensional analog of the Hamiltonian vector-field X_{NLS} corresponding to Ω_0 (7.12) and H_{NLS} (7.3) is given by:

$$X_H \cdot \equiv X_{\mathrm{NLS}} \cdot = i \int_{-\infty}^{\infty} dx \left(\frac{\delta H}{\delta u^*(x)} \frac{\delta}{\delta u(x)} \cdot - \frac{\delta H}{\delta u(x)} \frac{\delta}{\delta u^*(x)} \cdot \right) \ , \tag{7.14}$$

Then, the Hamiltonian equation of motion (7.2) can be rewritten as:

$$\frac{\partial u}{\partial t} + (X_{\mathrm{NLS}}, \delta u(x)) = 0 \ , \tag{7.15}$$

where by $(X_{\mathrm{NLS}}, \delta u(x))$ we mean the result of evaluation of the vector field X_{NLS} on the 1-form $\delta u(x)$. Obviously, with X_{NLS} as in (7.14) and H as in (7.3), we find that (7.15) coincides with the NLS equation (7.1).

The symplectic form corresponding to the Poisson brackets (7.8) for mKdV equals:

$$\Omega_{\mathrm{mKdV}} = \int_{-\infty}^{\infty} dx\, \partial_x^{-1} \delta w(x) \wedge \delta w(x) \ , \tag{7.16}$$

where $\partial_x^{-1} \cdot$ was introduced above. The relevant Hamiltonian vector field for Ω_{mKdV} and H_{mKdV} is:

$$X_{\mathrm{mKdV}} = - \int_{-\infty}^{\infty} dx\, \frac{\partial}{\partial x} \left(\frac{\delta H_{\mathrm{mKdV}}}{\delta w(x)} \right) \frac{\delta}{\delta w(x)} \cdot \tag{7.17}$$

Again, one can check that the Hamiltonian equation of motion

$$\frac{\partial w}{\partial t} + (X_{\mathrm{mKdV}}, \delta w(x)) = 0 \ , \tag{7.18}$$

provides the mKdV equation (7.6).

The third example is the symplectic form related to the sine-Gordon equation. The corresponding phase space is $\mathcal{M}_{\mathrm{mKdV}}$, but the Poisson brackets (7.10) are different and so are the symplectic forms:

$$\Omega_{\mathrm{sG}} = \int_{-\infty}^{\infty} dx\, \partial_x \delta w(x) \wedge \delta w(x)\ . \tag{7.19}$$

The Hamiltonian vector field X_{sG} corresponding to Ω_{sG} and H_{sG} (7.7) is:

$$X_{\mathrm{sG}} = \int_{-\infty}^{\infty} dx\, \partial_x^{-1} \left(\frac{\delta H_{\mathrm{sG}}}{\delta w(x)} \right) \frac{\delta}{\delta w(x)}. \tag{7.20}$$

so that the equation of motion

$$\frac{\partial w}{\partial t} + (X_{\mathrm{sG}}, \delta w(x)) \equiv \frac{\partial w}{\partial t} + \partial_x^{-1} \sin w(x,t) = 0\ , \tag{7.21}$$

after differentiating both sides with respect to x, provides the sine-Gordon equation (7.7).

The phase spaces $\mathcal{M}_{\mathrm{sG}}$ and $\mathcal{M}_{\mathrm{mKdV}}$ are isomorphic, but the symplectic structures introduced above are substantially different. One can view $\mathcal{M}_{\mathrm{mKdV}}$ as a subspace of $\mathcal{M}_{\mathrm{NLS}}$ obtained with the reduction $u = u^* = w$. However, on $\mathcal{M}_{\mathrm{NLS}}$ one can use the canonical Poisson brackets (7.5) and the relevant canonical symplectic form Ω_{NLS}, but these become degenerate under the reduction $u = u^* = w$. So, it is not accidental that the symplectic forms Ω_{mKdV} and Ω_{sg} are both noncanonical.

At the same time, the NLS, mKdV, and sG-equations are particular members of the class of generic NLEEs:

$$\frac{i}{2}\left[\sigma_3, \frac{\partial q}{\partial t}\right] + 2f(\Lambda) q(x,t) = 0, \qquad q(x,t) = \begin{pmatrix} 0 & q^+ \\ q^- & 0 \end{pmatrix}, \tag{7.22}$$

with specifically chosen dispersion laws $f(\lambda)$ and reductions (or involution) chosen among:

(α) $q^-(x) = \epsilon_0 (q^+(x))^*$, where $\epsilon_0 = \pm 1$;
(β) $q^-(x) = \epsilon_1 q^+(x)$ where $\epsilon_1 = \pm 1$; and
(γ) both (α) and (β) hold.

The examples above illustrate the general fact that on the same phase space, one can introduce different Poisson brackets and symplectic forms. It also raises the questions:

- Do the generic NLEEs allow Hamiltonian formulation and in what sense;
- Can one view the Hamiltonian formulations of NLS, sG and mKdV equations as particular cases of the generic ones obtained with the proper involution;
- Are all these Hamiltonian formulations compatible and in what sense.

The answer to these questions will be given later in this Chapter. It is based on the fact that the generic NLEEs allow not just one but a hierarchy of Hamiltonian structures generated by the recursion operator Λ.

To describe the simplest of them, one needs properly chosen phase space $\mathcal{M}^{\mathbb{C}}$ and Hamiltonian $H^{\mathbb{C}}$. A natural candidate for $\mathcal{M}^{\mathbb{C}}$ is the manifold of pairs of complex-valued functions $\{q^+(x), q^-(x), -\infty \leq x \leq \infty\}$ with the following Poisson brackets:

$$\begin{aligned} &\{q^+(x), q^+(y)\}^{\mathbb{C}}_{(0)} = 0, \qquad \{q^-(x), q^-(y)\}^{\mathbb{C}}_{(0)} = 0\,, \\ &\{q^-(x), q^+(y)\}^{\mathbb{C}}_{(0)} = i\delta(x-y)\,. \end{aligned} \tag{7.23}$$

They can be viewed as generalizations of the canonical Poisson brackets (7.4). Obviously, if we impose the constraint $q^-(x) = (q^+(x))^*$, the phase space $\mathcal{M}^{\mathbb{C}}$ will reduce to $\mathcal{M}_{\rm NLS}$. Using the additional involution (β), one can consider also $\mathcal{M}_{\rm mKdV}$ as a subspace of $\mathcal{M}^{\mathbb{C}}$. As it is, $\mathcal{M}^{\mathbb{C}}$ can be viewed as complexification of $\mathcal{M}_{\rm NLS}$.

What we shall do now is the following: taking as Hamiltonians $H^{\mathbb{C}}$ properly chosen linear combination of the integrals of motion C_k, we shall describe the Hamiltonian properties of the generic NLEEs (7.22). Note that such $H^{\mathbb{C}}$ are analytic functionals of $q^{\pm}(x)$.

We shall see that the Hamiltonian formulations of NLS, mKdV, and sG equations can be understood as real Hamiltonian forms obtained from the complexified ones with one of the above mentioned involutions on $\mathcal{M}^{\mathbb{C}}$. Special attention will be paid to the degeneracy of some of the Hamiltonian structures which the involution (β) induces.

Similar constructions are possible also for finite-dimensional systems. One can complexify also a dynamical systems with n degrees of freedom. As a result, one obtains dynamical systems with $2n$ degrees of freedom. After that, using involutive automorphisms, one can extract the corresponding real Hamiltonian forms, which are new dynamical systems with n degrees of freedom; for details see Chap. 12 of the second part.

7.2 Complexified Phase Spaces and Hamiltonians

The canonical Poisson brackets between any two functionals on $\mathcal{M}^{\mathbb{C}}$ are given by:

$$\{F, G\}^{\mathbb{C}}_{(0)} = i \int_{-\infty}^{\infty} dx \left(\frac{\delta F}{\delta q^-(x)} \frac{\delta G}{\delta q^+(x)} - \frac{\delta F}{\delta q^+(x)} \frac{\delta G}{\delta q^-(x)} \right)\,. \tag{7.24}$$

where both F and G are complex-valued functionals on $\mathcal{M}^{\mathbb{C}}$ depending analytically on $q^{\pm}(x)$.

Then, the corresponding canonical symplectic form can be written as:

$$\Omega^{\mathbb{C}}_{(0)} = \frac{1}{i} \int_{-\infty}^{\infty} dx\, \delta q^-(x) \wedge \delta q^+(x)\,. \tag{7.25}$$

Next, we need to specify the Hamiltonian as a functional over $\mathcal{M}^{\mathbb{C}}$. A generic functional on $\mathcal{M}^{\mathbb{C}}$, analytic with respect to both q^+ and q^-, has the form:

$$\begin{aligned} H^{\mathbb{C}} = c + \sum_{k+m\geq 1} \int_{-\infty}^{\infty} dx_1 \dots dx_k dy_1 \dots dy_m H^{(k,m)}(x_1,\dots,x_k|y_1,\dots,y_k) \\ q^+(x_1)\dots q^+(x_k) q^-(y_1)\dots q^-(y_m) \, , \end{aligned} \tag{7.26}$$

where for simplicity we assume that the kernels $H^{(k,m)}(x_1,\dots,x_k|y_1,\dots,y_k)$ are distributions symmetric with respect to each of the sets of arguments $\{x_1,\dots,x_k\}$ and $\{y_1,\dots,y_m\}$. In our case, these kernels will be products of Dirac δ-functions and their derivatives. Condition **C1** on p.73 ensures that $H^{\mathbb{C}}$ is properly defined along with all functionals of the type (7.26).

Let us illustrate this construction. For example, choosing:

$$\begin{aligned} H^{(1,1)}(x_1|y_1) = c_0\delta(x_1-y_1) + c_1\left(\frac{\partial\delta(x_1-y_1)}{\partial x_1} - \frac{\partial\delta(x_1-y_1)}{\partial y_1}\right) \\ + c_2\frac{\partial^2\delta(x_1-y_1)}{\partial x_1^2} \, , \end{aligned} \tag{7.27}$$

$$H^{(2,2)}(x_1,x_2|y_1,y_2) = h_2\delta(x_1-y_1)\delta(x_2-y_2)\delta(x_1-x_2) \, , \tag{7.28}$$

and setting all other kernel functions equal to zero, we see that the functionals defined by $H^{(1,1)}(x_1|y_1)$ and $H^{(2,2)}(x_1,x_2|y_1,y_2)$ are equal to:

$$\begin{aligned} H^{(1,1)} = c_0\int_{-\infty}^{\infty} dx\, q^+(x)q^-(x) - c_1\int_{-\infty}^{\infty} dx\left(\frac{\partial q^+}{\partial x}q^-(x) - q^+\frac{\partial q^-(x)}{\partial x}\right) \\ + c_2\int_{-\infty}^{\infty} dx\, \frac{\partial^2 q^+(x)}{\partial x^2}q^-(x) \, , \end{aligned} \tag{7.29}$$

$$H^{(2,2)} = h_2\int_{-\infty}^{\infty} dx\, (q^+(x)q^-(x))^2 \, , \tag{7.30}$$

where c_0, c_1, c_2 and h_2 are some constants. Functionals whose densities depend only on $q^\pm(x)$ and their x-derivatives are called local functionals.

In what follows, we shall need also the variational derivatives of H with respect to $q^\pm(x)$, which are evaluated using the standard rules:

$$\begin{aligned} \frac{\delta q^+(x)}{\delta q^+(y)} = \delta(x-y), \qquad \frac{\delta q^-(x)}{\delta q^-(y)} = \delta(x-y) \, , \\ \frac{\delta q^+(x)}{\delta q^-(y)} = 0, \qquad \frac{\delta q^-(x)}{\delta q^+(y)} = 0 \, , \end{aligned} \tag{7.31}$$

where $\delta(x-y)$ is the Dirac delta-function.

In order to treat adequately the complexified Hamiltonian structures, we have to be more specific in defining the Poisson brackets (7.23), namely, we shall use:

$$\begin{aligned}
&\{q^+_\alpha(x), q^+_\beta(y)\}^{\mathbb{C}}_{(0)} = 0, \qquad \{q^-_\alpha(x), q^-_\beta(y)\}^{\mathbb{C}}_{(0)} = 0, \qquad \alpha, \beta = 0, 1;\\
&\{q^-_0(x), q^+_0(y)\}^{\mathbb{C}}_{(0)} = 0; \qquad \{q^-_1(x), q^+_1(y)\}^{\mathbb{C}}_{(0)} = 0; \qquad (7.32)\\
&\{q^-_0(x), q^+_1(y)\}^{\mathbb{C}}_{(0)} = \frac{1}{2}\delta(x-y), \qquad \{q^-_1(x), q^+_0(y)\}^{\mathbb{C}}_{(0)} = \frac{1}{2}\delta(x-y).
\end{aligned}$$

The generic Hamiltonian equations of motion generated by $H^{\mathbb{C}}$ and the Poisson brackets (7.24) are the following:

$$\frac{\partial q^+}{\partial t} = \{H^{\mathbb{C}}, q^+(x,t)\}^{\mathbb{C}}_{(0)} = i\frac{\delta H^{\mathbb{C}}}{\delta q^-(x)}\,, \tag{7.33a}$$

$$-\frac{\partial q^-}{\partial t} = -\{H^{\mathbb{C}}, q^-(x)\}^{\mathbb{C}}_{(0)} = i\frac{\delta H^{\mathbb{C}}}{\delta q^+(x)}\,, \tag{7.33b}$$

They are equivalent to a standard Hamiltonian system, provided $H^{\mathbb{C}}$ is analytic with respect to q^+ and q^-. The analyticity of $H^{\mathbb{C}}$ means that its real and imaginary parts $H^{\mathbb{C}}_0$ and $H^{\mathbb{C}}_1$ satisfy the analog of Cauchy–Riemann equations:

$$\frac{\delta H^{\mathbb{C}}_0}{\delta q^{\pm}_0(x)} = \frac{\delta H^{\mathbb{C}}_1}{\delta q^{\pm}_1(x)}, \qquad \frac{\delta H^{\mathbb{C}}_1}{\delta q^{\pm}_0(x)} = -\frac{\delta H^{\mathbb{C}}_0}{\delta q^{\pm}_1(x)}\,, \tag{7.34}$$

where $q^{\pm}_\alpha$, $\alpha = 0, 1$ are the real and imaginary parts of $q^{\pm}(x)$:

$$q^{\pm}(x) = q^{\pm}_0(x) + iq^{\pm}_1(x)\,. \tag{7.35}$$

Inserting (7.34), (7.32) into (7.33) we get the system:

$$\frac{\partial q^+_0}{\partial t} = \frac{\delta H^{\mathbb{C}}_{(0)}}{\delta q^-_1(x)}\,, \tag{7.36a}$$

$$\frac{\partial q^+_1}{\partial t} = \frac{\delta H^{\mathbb{C}}_{(0)}}{\delta q^-_0(x)}\,, \tag{7.36b}$$

$$\frac{\partial q^-_0}{\partial t} = -\frac{\delta H^{\mathbb{C}}_{(0)}}{\delta q^+_1(x)}\,, \tag{7.36c}$$

$$\frac{\partial q^-_1}{\partial t} = -\frac{\delta H^{\mathbb{C}}_{(0)}}{\delta q^+_0(x)}\,, \tag{7.36d}$$

which can be viewed as the equation of motion of an infinite-dimensional Hamiltonian system with real-valued Hamiltonian $H^{\mathbb{C}}_{(0)}$. The elements of the phase space $\mathcal{M}^{\mathbb{C}}$ can be viewed also as the 4-tuples of real functions $\{q^+_0, q^+_1, q^-_0, q^-_1\}$ vanishing fast enough for $x \to \pm\infty$. We designate the space of such 4-tuples by $\mathcal{M}_{\mathbb{R}}$.

The Poisson brackets on $\mathcal{M}_{\mathbb{R}}$ are introduced by (7.32). We shall use also the symplectic form on $\mathcal{M}_{\mathbb{R}}$, which is defined by the real part of $\Omega^{\mathbb{C}}_{(0)}$:

$$\operatorname{Re}\Omega^{\mathbb{C}}_{(0)} = \int_{-\infty}^{\infty} dx\, \left(\delta q^+_0 \wedge \delta q^-_1 + \delta q^+_1 \wedge \delta q^-_0\right)\,. \tag{7.37}$$

Remark 7.2. If we impose on $\Omega^{\mathbb{C}}_{(0)}$ the involution α), then $q_0^+ = \epsilon_0 q_0^-$, $q_1^+ = -\epsilon_0 q_1^-$, which means that $\operatorname{Im} \Omega^{\mathbb{C}}_{(0)} = 0$ and $\operatorname{Re} \Omega^{\mathbb{C}}_{(0)}$ becomes proportional Ω_0 from (7.12).

In what follows, we shall use the formal Hamiltonian formulation of the generic NLEEs with complex-valued Hamiltonian $H^{\mathbb{C}}$ and complex valued dynamical fields $q^{\pm}(x)$ as in (7.33). Those who prefer the standard Hamiltonian formulations using real-valued Hamiltonians and dynamical fields can always rewrite the generic NLEEs in its equivalent form (7.36).

7.3 The Generic NLEEs as Completely Integrable Complex Hamiltonian System

For the sake of compactness, we shall use the gauge-covariant formulation of the generic NLEEs. The usefulness of this formulation will become clear in the next Chapter, dedicated to the gauge-equivalent NLEE and their Hamiltonian properties.

So, below we shall view the phase space $\mathcal{M}^{\mathbb{C}}$ as the space of 2×2 off-diagonal matrices $q(x) = \begin{pmatrix} 0 & q^+ \\ q^- & 0 \end{pmatrix}$. The variational derivatives (or the "gradients") of the functional $H^{\mathbb{C}}$ then will be written as:

$$\nabla_q H^{\mathbb{C}} \equiv \frac{\delta H^{\mathbb{C}}}{\delta q^T(x)} = \begin{pmatrix} 0 & \frac{\delta H^{\mathbb{C}}}{\delta q^-(x)} \\ \frac{\delta H^{\mathbb{C}}}{\delta q^+(x)} & 0 \end{pmatrix} . \tag{7.38}$$

It remains to recall the definition of the skew-scalar product $[\![\cdot \, , \, \cdot]\!]$, and after simple calculation one is able to write down the canonical Poisson brackets (7.24) as follows:

$$\begin{aligned} \{F, G\}^{\mathbb{C}}_{(0)} &= \frac{i}{2} \int_{-\infty}^{\infty} \operatorname{tr} \left(\nabla_q F, [\sigma_3, \nabla_q G] \right) dx \\ &= i [\![\nabla_q F, \nabla_q G]\!] . \end{aligned} \tag{7.39}$$

The corresponding canonical symplectic form $\Omega_0^{\mathbb{C}}$ and Hamiltonian vector field $X_{H^{\mathbb{C}}}$ become:

$$\begin{aligned} \Omega_0^{\mathbb{C}} &= i \int_{-\infty}^{\infty} dx \, \left(\delta q^+(x) \wedge \delta q^-(x) \right) \equiv \frac{i}{4} \int_{-\infty}^{\infty} \operatorname{tr} \left(\sigma_3 \delta q(x) \underset{'}{\wedge} [\sigma_3, \sigma_3 \delta q(x)] \right) \\ &= \frac{i}{2} [\![\sigma_3 \delta q \underset{'}{\wedge} \sigma_3 \delta q]\!] . , \end{aligned} \tag{7.40}$$

$$X_{H^{\mathbb{C}}} \cdot = -i \left[\!\!\left[\nabla_q H^{\mathbb{C}}, \nabla_q \cdot \right]\!\!\right] = -\{H^{\mathbb{C}}, \cdot\}^{\mathbb{C}}_{(0)} . \tag{7.41}$$

Above, by the symbol $\underset{'}{\wedge}$, we mean that we first perform the matrix multiplication keeping the order of the factors and then replace the standard multiplication by an exterior product $\wedge$.

With all these notations we can write down (7.33) in the form:

$$i\sigma_3 \frac{\partial q}{\partial t} + \nabla_q H^{\mathbb{C}} = 0 \,. \tag{7.42}$$

The system (6.18), (6.19) generalizing the NLSE can be written down as complex Hamiltonian system (7.42) with $H^{\mathbb{C}}$ chosen to be:

$$H^{\mathbb{C}} = \int_{-\infty}^{\infty} dx \, \left[q_x^- q_x^+ - (q^+ q^-)^2 \right] = -8C_3 \,. \tag{7.43}$$

Quite analogously, one may check that all the other examples of NLEEs also allow complex Hamiltonian structures with the symplectic structure introduced on $\mathcal{M}_{\mathbb{C}}$ by (7.23).

Each of the generic NLEE (7.22) with dispersion law $f(\lambda)$ can be written down in the form (7.42). It is only natural to expect that the corresponding Hamiltonian H should be expressed in terms of the integrals of motion C_p. Indeed, (6.133) can be written down as:

$$\nabla_q C_p = -\frac{1}{2} \Lambda^{p-1} q(x) \,. \tag{7.44}$$

Then if we choose

$$H^{\mathbb{C}} = \sum_k 4 f_k C_{k+1} \,, \tag{7.45}$$

we get:

$$\nabla_q H^{\mathbb{C}} = -2 f(\Lambda) q(x) \,. \tag{7.46}$$

Thus (7.42) coincides with the NLEEs (7.22) with dispersion law $f(\lambda) = \sum_k f_k \lambda^k$.

The next step in studying the Hamiltonian properties of (6.7) is to check whether the integrals of motion C_p are in involution. A bit more general is the problem to evaluate the Poisson brackets between the entries in the minimal sets of scattering data $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}$. To do this, we use the compact expression (7.39) for the Poisson brackets through the skew-scalar product on $\mathcal{M}_{\mathbb{C}}$ and the inversion formulae (5.81), (5.83), (5.85). Thus:

$$\frac{\delta \tau^{\pm}(t,\lambda)}{\delta q(x)} \equiv \nabla_q \tau^{\pm}(t,\lambda) = \frac{\mp i}{(a^{\pm}(\lambda))^2} \boldsymbol{\Psi}^{\pm}(x,\lambda) \,, \tag{7.47a}$$

$$\nabla_q \lambda_k^{\pm} = \mp i C_k^{\pm} \boldsymbol{\Psi}_k^{\pm}(x) \,, \tag{7.47b}$$

$$\nabla_q M_k^{\pm} = \frac{\mp i}{(\dot{a}_k^{\pm})^2} \left(\dot{\boldsymbol{\Psi}}_k^{\pm}(x) - \frac{\ddot{a}_k^{\pm}}{\dot{a}_k^{\pm}} \boldsymbol{\Psi}_k^{\pm}(x) \right) \,, \tag{7.47c}$$

Analogously:

$$\nabla_q \rho^{\pm}(t,\lambda) = \frac{\pm i}{(a^{\pm}(\lambda))^2} \boldsymbol{\Phi}^{\pm}(x,\lambda) \,, \tag{7.48a}$$

$$\nabla_q \lambda_k^{\pm} = \pm i M_k^{\pm} \boldsymbol{\Phi}_k^{\pm}(x) \,, \tag{7.48b}$$

$$\nabla_q C_k^{\pm} = \frac{\pm i}{(\dot{a}_k^{\pm})^2} \left(\dot{\boldsymbol{\Phi}}_k^{\pm}(x) - \frac{\ddot{a}_k^{\pm}}{\dot{a}_k^{\pm}} \boldsymbol{\Phi}_k^{\pm}(x) \right) \tag{7.48c}$$

and

$$\nabla_q \eta(\lambda) = i\boldsymbol{P}(x,\lambda), \qquad \nabla_q \eta_k^{\pm} = i\boldsymbol{P}_k^{\pm}(x) \,, \tag{7.49a}$$

$$\nabla_q \kappa(t,\lambda) = i\boldsymbol{Q}(x,\lambda), \qquad \nabla_q \kappa_k^{\pm} = i\boldsymbol{Q}_k^{\pm}(x) \,, \tag{7.49b}$$

where $\boldsymbol{P}(x,\lambda)$, $\boldsymbol{Q}(x,\lambda)$ etc. are the elements of the symplectic basis (5.61) introduced in Sect. 5.2.1.

From the above relations, there follows that the Poisson brackets between the scattering data are expressed through the skew-scalar products of the corresponding "squared" solutions. So, in order to evaluate them, we need to recall the results from Sect. 5.5.1 and Table 5.2. Indeed:

$$\begin{aligned} \{\rho^+(t,\lambda), \tau^+(t,\mu)\}_{(0)} &= -\frac{i[\![\boldsymbol{\Phi}^+(x,\lambda), \boldsymbol{\Psi}^+(x,\mu)]\!]}{(a^+(\lambda))^2 (a^+(\mu))^2} \,, \\ &= -i\pi\delta(\lambda-\mu) \,, \end{aligned} \tag{7.50a}$$

$$\begin{aligned} \{\rho^-(t,\lambda), \tau^-(t,\mu)\}_{(0)} &= -\frac{i[\![\boldsymbol{\Phi}^-(x,\lambda), \boldsymbol{\Psi}^-(x,\mu)]\!]}{(a^-(\lambda))^2 (a^-(\mu))^2} . \\ &= i\pi\delta(\lambda-\mu) \,. \end{aligned} \tag{7.50b}$$

and

$$\begin{aligned} \{\eta(\lambda), \kappa(\mu)\}_{(0)}^{\mathbb{C}} &= i\delta(\lambda-\mu), \qquad \{\eta(\lambda), \kappa_k^{\pm}\}_{(0)}^{\mathbb{C}} = 0 \,, \\ \{\kappa(\lambda), \eta_k^{\pm}\}_{(0)}^{\mathbb{C}} &= 0, \qquad \{\eta_k^{\pm}, \kappa_m^{\pm}\}_{(0)}^{\mathbb{C}} = i\delta_{km}, \end{aligned} \tag{7.51a}$$

$$\{\eta(\lambda), \eta(\mu)\}_{(0)}^{\mathbb{C}} = 0, \qquad \{\eta_k^{\pm}, \eta_m^{\pm}\}_{(0)}^{\mathbb{C}} = 0, \qquad \{\eta(\lambda), \eta_k^{\pm}\}_{(0)}^{\mathbb{C}} = 0 \,, \tag{7.51b}$$

$$\{\kappa(\lambda), \kappa(\mu)\}_{(0)}^{\mathbb{C}} = 0, \qquad \{\kappa_k^{\pm}, \kappa_m^{\pm}\}_{(0)}^{\mathbb{C}} = 0, \qquad \{\kappa(\lambda), \kappa_k^{\pm}\}_{(0)}^{\mathbb{C}} = 0 \,. \tag{7.51c}$$

So, the orthogonality properties of the symplectic basis with respect to the skew-scalar product are directly related to the fact that the set of variables $\{\eta(\lambda), \kappa(\lambda), \eta_k^{\pm}, \kappa_k^{\pm}\}$ satisfy canonical Poisson brackets.

We just proved that $\{\eta(\lambda), \eta_k^{\pm}\}$ are in involution (see (7.51b) above). They are also time independent due to (7.22). The variables $\{\kappa(t,\lambda), \kappa_k^{\pm}(t)\}$ are also in involution but depend on time linearly; see (6.10). Thus, these two sets of variables have all the necessary properties to be global "action-angle" variables for the LEES (6.7).

Here our notion of complete integrability is a bit broader than usual. By action-angle variables we understand a complete canonical basis in $\mathcal{M}^{\mathbb{C}}$ which is such that the Hamiltonian depends only on the action variables. We do not necessarily require that the range of the angle variables is $[0, 2\pi]$; for us it is enough that they depend linearly on time, thus allowing to solve exactly the dynamical equations.

From the trace identities (3.72), we know that the integrals of motion C_k are expressed in terms of $\eta(\lambda)$ and $\eta_k^{\pm}$ only; therefore, they are in involution.

Another way to prove that any two of the integrals of motion C_p are in involution is based on the relation:

$$\frac{[\![\boldsymbol{\Theta}^+(x,\lambda), \boldsymbol{\Theta}^+(x,\mu)]\!]}{a^+(\lambda)a^+(\mu)} = 0 \,, \tag{7.52}$$

which is derived in complete analogy with relations (5.121). Since from (5.27) we get $\boldsymbol{\Theta}(x,\lambda) = \nabla_q \mathcal{A}(\lambda)$ and $\mathcal{A}(\lambda)$ is analytic with respect to $\lambda \in \mathbb{C}_\pm$, we can rewrite (7.52) into the form:

$$\begin{aligned} [\![\nabla_q \mathcal{A}(\lambda), \nabla_q \mathcal{A}(\mu)]\!] &= \{\mathcal{A}(\lambda), \mathcal{A}(\mu)\}^{\mathbb{C}}_{(0)} \\ &= \sum_{n,m=1}^{\infty} \frac{\{C_n, C_m\}^{\mathbb{C}}_{(0)}}{\lambda^m \mu^n} = 0 \,. \end{aligned} \tag{7.53}$$

for all complex λ and μ. As this relation must hold identically with respect to λ and μ, one concludes that

$$\{C_n, C_m\}^{\mathbb{C}}_{(0)} = 0 \,, \tag{7.54}$$

for $n, m = 1, 2, \ldots,$.

Thus we conclude that the NLEE (6.7) are infinite dimensional completely integrable complex Hamiltonian systems with respect to the canonical Poisson brackets (7.24) defined on $\mathcal{M}^{\mathbb{C}}$.

When treating the complete integrability of an infinite-dimensional system, the most difficult point is to ensure that the action-angle variables really span the whole phase space $\mathcal{M}^{\mathbb{C}}$. We are now going to present a more rigorous proof of this fact based on the completeness relation for the symplectic basis.

The most straightforward way to derive the action-angle variables of the NLEE (6.7) is to insert into the right-hand side of (7.40) the expansion (5.86) for $\sigma_3 \delta q(x)$. This gives:

$$\begin{aligned} \Omega^{\mathbb{C}}_{(0)} &= \frac{i}{2} \left[\!\!\left[\sigma_3 \delta q(x) \mathop{\wedge}_{,} \left(i \int_{-\infty}^{\infty} d\lambda\ (\delta\eta(\lambda) \boldsymbol{Q}(x,\lambda) - \delta\kappa(t,\lambda) \boldsymbol{P}(x,\lambda)) \right.\right.\right. \\ &\quad \left.\left.\left. + i \sum_{k=1}^{N} (\delta\eta_k^+ \boldsymbol{Q}_k^+(x) - \delta\kappa_k^+ \boldsymbol{P}_k^+(x) + \delta\eta_k^- \boldsymbol{Q}_k^-(x) - \delta\kappa_k^- \boldsymbol{P}_k^-(x)) \right) \right]\!\!\right] \\ &= -\frac{1}{2} \int_{-\infty}^{\infty} d\lambda\ \big(\delta\eta(\lambda) \wedge [\![\sigma_3 \delta q(x), \boldsymbol{Q}(x,\lambda)]\!] \\ &\quad - \delta\kappa(t,\lambda) \wedge [\![\sigma_3 \delta q(x), \boldsymbol{P}(x,\lambda)]\!] \big) \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2}\sum_{k=1}^{N}(\delta\eta_k^+ \wedge [\![\sigma_3\delta q(x), \boldsymbol{Q}_k^+(x)]\!] - \delta\kappa_k^+ \wedge [\![\sigma_3\delta q(x), \boldsymbol{P}_k^+(x)]\!] \\
&+ \frac{1}{2}\delta\eta_k^- \wedge [\![\sigma_3\delta q(x), \boldsymbol{Q}_k^-(x)]\!] - \delta\kappa_k^- \wedge [\![\sigma_3\delta q(x), \boldsymbol{P}_k^-(x))]\!] \\
&= i\int_{-\infty}^{\infty} d\lambda\, \delta\kappa(t,\lambda) \wedge \delta\eta(\lambda) + i\sum_{k=1}(\delta\kappa_k^+ \wedge \delta\eta_k^+ + \delta\kappa_k^- \wedge \delta\eta_k^-) \, .
\end{aligned} \tag{7.55}
$$

In the above calculation, we made use of the inversion formulae for the symplectic basis (5.85) and identified the skew-symmetric scalar products of $\sigma_3\delta q(x)$ with the elements of the symplectic basis with the variations of $\delta\eta(\lambda)$, $\delta\kappa(t,\lambda)$, etc.

From (7.55), we see also that the 2-form $\Omega_0^{\mathbb{C}}$ expressed by the variables $\eta(\lambda)$, $\kappa(\lambda)$, $\eta_k^\pm$ and $\kappa_k^\pm$ has a canonical form. Recall now the trace identities (6.131). From them it follows that the Hamiltonian $H^{\mathbb{C}}$ of the NLEE depends only on the variables $\eta(\lambda)$, $\eta_k^\pm$:

$$
H^{\mathbb{C}} = -2\int_{-\infty}^{\infty} d\mu\, f(\mu)\eta(\mu) + 4i\sum_{k=1}^{N}\left(F_k^+ - F_k^-\right) \, , \tag{7.56}
$$

where

$$
F_k^\pm = F(\lambda_k^\pm), \qquad F(\lambda) = \int^{\lambda} d\lambda' f(\lambda') \, . \tag{7.57}
$$

Remark 7.3. The variables $\eta(\lambda)$, $\kappa(\lambda)$, $\eta_k^\pm$ and $\kappa_k^\pm$ are indeed the analogs of the action-angle variables for the complex Hamiltonian systems (6.7) in the sense that they form a canonical basis in $\mathcal{M}^{\mathbb{C}}$, and the Hamiltonian depends only on "half" of them. They satisfy the following equations of motion:

$$
\frac{d\eta}{dt} = 0, \qquad \frac{d\eta_k^\pm}{dt} = 0, \qquad i\frac{d\kappa}{dt} - 2f(\lambda) = 0, \qquad i\frac{d\kappa_k^\pm}{dt} - 2f(\lambda_k^\pm) = 0 \, , \tag{7.58}
$$

from which we see that the "action" variables are time-independent, while the "angle" variables are linear functions of t. However, here both types of variables are complex-valued. In addition, the "angle" variables $\kappa(\lambda)$, $\kappa_k^\pm$ (see eq. (5.87b)) can be written as:

$$
\kappa(\lambda) = \frac{1}{2}\ln\left|\frac{b^+(\lambda)}{b^-(\lambda)}\right| + \frac{i}{2}\arg\frac{b^+(\lambda)}{b^-(\lambda)}, \qquad \kappa_k^\pm = \pm\ln|b_k^\pm| \pm i\arg b_k^\pm \, , \tag{7.59}
$$

from which we find that only their imaginary parts can be viewed as angles taking values in the range $[0, 2\pi]$.

Note that the derivation of this result is based on the completeness relation of the symplectic basis. This ensures: (i) the uniqueness and the invertibility of the mapping from $\{q^{\pm}(x)\}$ to $\mathcal{T}$; (ii) the nondegeneracy of the 2-form $\Omega^{\mathbb{C}}_{(0)}$ on $\mathcal{M}^{\mathbb{C}}$.

One can view $q^{\pm}(x)$ as local coordinates on $\mathcal{M}^{\mathbb{C}}$; any functional F or G on $\mathcal{M}^{\mathbb{C}}$ can be expressed in terms of $q^{\pm}(x)$. The variations δF and δG of the functionals F and G are the analogs of 1-forms over $\mathcal{M}^{\mathbb{C}}$. They can be expressed in terms of the "gradients" by:

$$\delta F = [\![\nabla_q F, \sigma_3 \delta q]\!], \qquad \delta G = [\![\nabla_q G, \sigma_3 \delta q]\!] \;, \tag{7.60}$$

The "gradients" $\nabla_q F$ and $\nabla_q G$ are elements of the space $T_q\mathcal{M}^{\mathbb{C}}$ tangential to $\mathcal{M}^{\mathbb{C}}$.

At the same time, the mapping to $\mathcal{T}$ is one-to-one, therefore it is possible to express F and G in terms of the scattering data. To this end, we consider the expansions of $\nabla_q F$ and $\nabla_q G$ over the symplectic basis:

$$\begin{aligned}\nabla_q F = i \int_{-\infty}^{\infty} (\eta_F(\lambda)\boldsymbol{Q}(x,\lambda) - \kappa_F(\lambda)\boldsymbol{P}(x,\lambda))\, d\lambda \\ + i \sum_{k=1}^{N}{}^{\pm} \left(\eta^{\pm}_{F,k}\boldsymbol{Q}^{\pm}_k(x) - \kappa^{\pm}_{F,k}\boldsymbol{P}^{\pm}_k(x) \right) \;,\end{aligned} \tag{7.61}$$

$$\begin{aligned}\eta_F(\lambda) &= i[\![\boldsymbol{P}(x,\lambda), \nabla_q F]\!], \qquad \kappa_F(\lambda) = i[\![\boldsymbol{Q}(x,\lambda), \nabla_q F]\!] \;, \\ \eta^{\pm}_{F,k} &= i[\![\boldsymbol{P}^{\pm}_k(x), \nabla_q F]\!], \qquad \kappa^{\pm}_{F,k} = i[\![\boldsymbol{Q}^{\pm}_k(x), \nabla_q F]\!] \;.\end{aligned} \tag{7.62}$$

Similar expansion for $\nabla_q G$ is obtained from (7.61) by changing F to G. Such expansions will hold true provided F and G are restricted in such a way that the expansion coefficients $\eta_F(\lambda)$ and $\kappa_F(\lambda)$ are smooth and fall off fast enough for $\lambda \to \pm\infty$. In what follows, we shall assume that the functionals F and G satisfy the following

Condition C4. The functionals F and G are restricted by the following implicit condition: the expansion coefficients $\eta_F(\lambda)$ and $\kappa_F(\lambda)$ and $\eta_G(\lambda)$ and $\kappa_G(\lambda)$ are Schwartz-type functions of λ for real λ.

Using the biquadratic relations satisfied by the elements of the symplectic basis (5.121), we can express the Poisson brackets between F and G in terms of their expansion coefficients as follows:

$$\begin{aligned}\{F,G\}^{\mathbb{C}}_{(0)} &= -i[\![\nabla_q F, \nabla_q G]\!] \\ &= \int_{-\infty}^{\infty} d\lambda\, (\eta_F\kappa_G - \kappa_F\eta_G)(\lambda) + \sum_{k=1}^{N}{}^{\pm} \left(\eta^{\pm}_{F,k}\kappa^{\pm}_{G,k} - \kappa^{\pm}_{F,k}\eta^{\pm}_{G,k} \right) \;.\end{aligned} \tag{7.63}$$

In particular, if we choose $F = H^{\mathbb{C}}$, then from (5.136) we find that $\eta_{H^{\mathbb{C}}}(\lambda) = 0$, $\eta^{\pm}_{H^{\mathbb{C}},k} = 0$ and

$$\kappa_{H^{\mathbb{C}}}(\lambda) = -2f(\lambda), \qquad \kappa^{\pm}_{H^{\mathbb{C}},k} = -2f(\lambda^{\pm}_k),$$

which gives:

$$\{H,G\}^{\mathbb{C}}_{(0)} = 2\int_{-\infty}^{\infty} d\lambda\, f(\lambda)\eta_G(\lambda) + 2\sum_{k=1}^{N}(f(\lambda^+_k)\eta^+_{G,k} + f(\lambda^-_k)\eta^-_{G,k}) \,. \quad (7.64)$$

Equation (7.64) allows us to describe inexplicitly the set of functionals G that are in involution with all integrals of motion of the generic NLEE. Indeed, the right-hand side of (7.64) will vanish identically for all choices of the dispersion law $f(\lambda)$ only if the expansion coefficients of $\nabla_q G$ satisfy:

$$\eta_g(\lambda) = 0, \qquad \lambda \in \mathbb{R}; \qquad \eta^{\pm}_{G,k} = 0, \qquad \forall k = 1,\dots,N \,. \quad (7.65)$$

One can also describe the Hamiltonian vector fields in terms of the symplectic basis:

$$\begin{aligned} X_{H^{\mathbb{C}}}\cdot &\equiv -\{H^{\mathbb{C}},\cdot\}^{\mathbb{C}}_{(0)} \quad (7.66)\\ &= -2i\int_{-\infty}^{\infty} d\lambda\, f(\lambda)\,[\![\boldsymbol{P}(x,\lambda),\nabla_q\cdot\,]\!] - 2i\sum_{k=1}^{N}{}^{\pm} f(\lambda^{\pm}_k)\,[\![P^{\pm}_k(x),\nabla_q\cdot\,]\!] \,, \end{aligned}$$

where $f(\lambda)$ is the dispersion law of the generic NLEE with Hamiltonian $H^{\mathbb{C}}$.

We end this subsection by noting the special role of the subspace $\mathcal{L}^{\mathbb{C}} \subset \mathcal{M}^{\mathbb{C}}$ spanned by $\boldsymbol{P}(x,\lambda)$ and $\boldsymbol{P}^{\pm}_k(x)$, $k = 1,\dots,N$, i.e. by "half" of the elements of the symplectic basis. Obviously, all Hamiltonian vector fields with Hamiltonians of the form (7.45) induce dynamics which is tangent to $\mathcal{L}^{\mathbb{C}}$. This is a maximal subspace of $\mathcal{M}^{\mathbb{C}}$ on which the symplectic form $\Omega^{\mathbb{C}}_{(0)}$ is degenerate. Therefore, $\mathcal{L}^{\mathbb{C}}$ is the Lagrange submanifold of $\mathcal{M}^{\mathbb{C}}$. Dynamics defined by Hamiltonian vector field that are not tangential violates the complete integrability [1].

7.4 The Hierarchy of Hamiltonian Structures

The complete integrability of the generic NLEE makes them rather special. They have an infinite number of integrals of motion C_n, which are in involution and moreover satisfy the relation:

$$\nabla_q C_{n+m} = \Lambda^m \nabla_q C_n \,, \quad (7.67)$$

which generalizes the Lenard relation (6.133). The important fact here is that the recursion operator Λ is universal one and does not depend on either n or m. This has far-reaching consequences which we outline below.

The first one consists of the possibility of introducing a hierarchy of Poisson brackets:

$$\{F, G\}^{\mathbb{C}}_{(m)} = \frac{1}{i} \left[\!\left[\nabla_q F, \Lambda^m \nabla_q G \right]\!\right] . \tag{7.68}$$

Below, we shall show that these Poisson brackets satisfy all the necessary properties.

First, using the fact that Λ is "self-adjoint" with respect to the skew-symmetric scalar product (5.109b), we easily check that the Poisson bracket defined by (7.68) is skew-symmetric. Indeed:

$$\begin{aligned}\{F, G\}^{\mathbb{C}}_{(m)} &= \frac{1}{i} \left[\!\left[\nabla_q F, \Lambda^m \nabla_q G \right]\!\right] \\ &= -\frac{1}{i} \left[\!\left[\Lambda^m \nabla_q G, \nabla_q F \right]\!\right] \\ &= -\frac{1}{i} \left[\!\left[\nabla_q G, \Lambda^m \nabla_q F \right]\!\right] = -\{G, F\}^{\mathbb{C}}_{(m)} .\end{aligned} \tag{7.69}$$

We have also the Leibnitz rule:

$$\begin{aligned}\{FG, H\}^{\mathbb{C}}_{(m)} &= \frac{1}{i} \left[\!\left[\nabla_q (FG), \Lambda^m \nabla_q H \right]\!\right] \\ &= \frac{1}{i} F \left[\!\left[\nabla_q G, \Lambda^m \nabla_q H \right]\!\right] + \frac{1}{i} \left[\!\left[\nabla_q F, \Lambda^m \nabla_q H \right]\!\right] G \\ &= F\{G, H\}^{\mathbb{C}}_{(m)} + \{F, H\}^{\mathbb{C}}_{(m)} G ,\end{aligned} \tag{7.70}$$

since $\nabla_q(FG) = F\nabla_q G + (\nabla_q F)G$.

Using the expansion (7.61) of $\nabla_q F$, an analogous one for $\nabla_q G$ and the fact that the elements $\boldsymbol{P}(x, \lambda)$ and $\boldsymbol{Q}(x, \lambda)$ are eigenfunctions of Λ (see (5.104)), we find:

$$\begin{aligned}&\{F, G\}^{\mathbb{C}}_{(m)} = -i \left[\!\left[\nabla_q F, \Lambda^m \nabla_q G \right]\!\right] \qquad (7.71)\\ &= \int_{-\infty}^{\infty} d\lambda\, \lambda^m \left(\eta_F \kappa_G - \kappa_F \eta_G \right)(\lambda) + \sum_{k=1}^{N}{}^{\pm} (\lambda_k^{\pm})^m \left(\eta^{\pm}_{F,k} \kappa^{\pm}_{G,k} - \kappa^{\pm}_{F,k} \eta^{\pm}_{G,k} \right) .\end{aligned}$$

The Jacobi identity is far from trivial to check in these terms. This will be done using the corresponding symplectic form and for that reason we postpone the proof until later.

The existence of a hierarchy of Poisson brackets entails that there must exist also hierarchy of vector fields, symplectic forms etc.

Let us define:

$$\Omega^{\mathbb{C}}_{(m)} = \frac{i}{2} \left[\!\left[\sigma_3 \delta q(x) \mathbin{\overset{\wedge}{,}} \Lambda^m \sigma_3 \delta q(x) \right]\!\right] . \tag{7.72}$$

These 2-forms are not canonical. The proof of the fact that $\delta \Omega^{\mathbb{C}}_{(m)} = 0$ is performed by recalculating them in terms of the "action-angle" variables. For this, we follow the same idea as in the calculation of $\Omega^{\mathbb{C}}_{(0)}$; see (7.55). We insert the expansion for $\sigma_3 \delta q(x)$ over the symplectic basis and then act on this expansion by Λ^m. This is easy to do because of (5.133) and the result is:

$$\begin{aligned}\Lambda^m \sigma_3 \delta q(x) = i \int_{-\infty}^{\infty} & d\lambda\, \lambda^m \left(\delta\eta(\lambda) \boldsymbol{Q}(x,\lambda) - \delta\kappa(t,\lambda) \boldsymbol{P}(x,\lambda) \right) \\ & + i \sum_{k=1}^{N} \left((\lambda_k^+)^m \left(\delta\eta_k^+ \boldsymbol{Q}_k^+(x) - \delta\kappa_k^+ \boldsymbol{P}_k^+(x) \right) \right. \\ & + \left. (\lambda_k^-)^m \left(\delta\eta_k^- \boldsymbol{Q}_k^-(x) - \delta\kappa_k^- \boldsymbol{P}_k^-(x) \right) \right) . \end{aligned} \tag{7.73}$$

Calculating the skew-symmetric scalar products of $\sigma_3 \delta q(x)$ with the right-hand side of (7.73), we again obtain the variations of the η and κ-variables. Finally we get:

$$\begin{aligned}\Omega_{(m)}^{\mathbb{C}} = i \int_{-\infty}^{\infty} & d\lambda\, \lambda^m \delta\kappa(t,\lambda) \wedge \delta\eta(\lambda) \\ & + i \sum_{k=1} ((\lambda_k^+)^m \delta\kappa_k^+ \wedge \delta\eta_k^+ + (\lambda_k^-)^m \delta\kappa_k^- \wedge \delta\eta_k^-) . \end{aligned} \tag{7.74}$$

Remark 7.4. The right-hand sides of (7.74) are well defined for all $m \geq 0$ for potentials $q(x)$ satisfying condition **C1**. This condition ensures that $\kappa(t,\lambda)$ and $\eta(\lambda)$ are Schwartz-type functions of λ.

Remark 7.5. For negative values of m, the existence of the integrals in (7.74) is ensured only provided we put additional restrictions on $q(x)$, which would ensure that $\lim_{\lambda\to 0} \lambda^m \delta\kappa(t,\lambda) \wedge \delta\eta(\lambda)$ exist for all $m < 0$.

Now it is easy to prove.

Proposition 7.6 *Let the potential $q(x)$ satisfy condition* **C1** *on p. 73. Then each of the symplectic forms $\Omega_{(m)}^{\mathbb{C}}$ for $m \geq 0$ is closed, i.e.*

$$\delta \Omega_{(m)}^{\mathbb{C}} = 0, \qquad m = 0, 1, 2, \dots . \tag{7.75}$$

If in addition $q(x)$ satisfies the condition in Remark 7.5, then each of the symplectic forms $\Omega_{(m)}^{\mathbb{C}}$ is closed also for $m < 0$.

Proof. Indeed, the conditions in Proposition 7.6 is such that the integral in the right-hand side of (7.74) is well defined, so we can interchange the integration with the operation of taking the external differential δ. Therefore, we have:

$$\begin{aligned}\delta\Omega_{(m)}^{\mathbb{C}} &= i \int_{-\infty}^{\infty} d\lambda\, \lambda^m \delta \left(\delta\kappa(t,\lambda) \wedge \delta\eta(\lambda) \right) \\ & \quad + i \sum_{k=1} \delta \left((\lambda_k^+)^m \delta\kappa_k^+ \wedge \delta\eta_k^+ + (\lambda_k^-)^m \delta\kappa_k^- \wedge \delta\eta_k^- \right) \\ &= 0 , \end{aligned} \tag{7.76}$$

where we used the simple fact that $\delta(\delta g(\lambda)) \equiv 0$ for any $g(\lambda)$.

Corollary 7.7 *Direct consequence of Proposition 7.6 is that the Poisson brackets* $\{\cdot, \cdot\}^{\mathbb{C}}_{(m)}$ *satisfy Jacobi identity.*

Now with respect to the symplectic form $\Omega^{\mathbb{C}}_{(m)}$ to $H^{\mathbb{C}}$ there corresponds the following Hamiltonian vector field:

$$
\begin{aligned}
&X^{(m)}_{H^{\mathbb{C}}} \cdot \equiv -\{H^{\mathbb{C}}, \cdot\}^{\mathbb{C}}_{(m)} \qquad (7.77)\\
&= -2i \int_{-\infty}^{\infty} d\lambda\, \lambda^m f(\lambda) [\![\boldsymbol{P}(x,\lambda), \nabla_q \cdot]\!] - 2i \sum_{k=1}^{N} {}^{\pm} (\lambda_k^{\pm})^m f(\lambda_k^{\pm}) [\![P_k^{\pm}(x), \nabla_q \cdot]\!] \,,
\end{aligned}
$$

The corresponding equation of motion is one of the higher generic NLEEs, namely:

$$
i\frac{\partial q}{\partial t} + 2\Lambda^m f(\Lambda) q(x,t) = 0 \,. \qquad (7.78)
$$

The same NLEE (7.78) can be obtained also using the canonical Poisson brackets $\{\cdot, \cdot\}^{\mathbb{C}}_{(0)}$ with the Hamiltonian $H^{\mathbb{C}}_{(m)}$ given by:

$$
H^{\mathbb{C}}_{(m)} = \sum_k 4 f_k C_{k+m+1} \,. \qquad (7.79)
$$

Indeed, from the Lenard relation and from (7.46) there follows that

$$
\nabla_q H^{\mathbb{C}}_{(m)} = 2\Lambda^m f(\Lambda) q(x) \,. \qquad (7.80)
$$

It is also easy to check using (7.67) that we have the infinite chain of relations:

$$
\begin{aligned}
\cdots &= \Lambda^{-1} \nabla_q H^{\mathbb{C}}_{(m+1)} = \nabla_q H^{\mathbb{C}}_{(m)} = \Lambda \nabla_q H^{\mathbb{C}}_{(m-1)} \\
&= \cdots = \Lambda^m \nabla_q H^{\mathbb{C}}_{(0)} = \Lambda^{m+1} \nabla_q H^{\mathbb{C}}_{(-1)} = \cdots \,, \qquad (7.81)
\end{aligned}
$$

where we have put $H^{\mathbb{C}}_{(0)} = H^{\mathbb{C}}$. The elements of this infinite chain are well defined only if the potential $q(x)$ satisfies conditions **C1** and **C4**.

Using the "self-adjoint" properties of Λ with respect to the skew-symmetric scalar product, we can write down formally:

$$
\begin{aligned}
X^{(m)}_{H^{\mathbb{C}}} \cdot &\equiv -i [\![\nabla_q H^{\mathbb{C}}_{(0)}, \Lambda^m \nabla_q \cdot]\!] \\
&= -i [\![\Lambda^p \nabla_q H^{\mathbb{C}}_{(0)}, \Lambda^{m-p} \nabla_q \cdot]\!] = \{\nabla_q H^{\mathbb{C}}_{(p)}, \cdot\}^{\mathbb{C}}_{(m-p)} \,, \qquad (7.82)
\end{aligned}
$$

for all $p = \pm 1, \pm 2, \ldots$. But each Hamiltonian vector field determines uniquely the corresponding equations of motion. Therefore, the chain of relations (7.81) shows that each generic NLEE allows a hierarchy of Hamiltonian formulations:

$$
\frac{\partial q^+}{\partial t} = \{H^{\mathbb{C}}_{(p)}, q^+(x)\}^{\mathbb{C}}_{(-p)} \,, \qquad (7.83)
$$

$$-\frac{dq^-}{dt} = -\{H^{\mathbb{C}}_{(p)}, q^-(x)\}^{\mathbb{C}}_{(-p)} , \tag{7.84}$$

One can also explain that the Hamiltonian vector field obtained through $\Omega^{\mathbb{C}}_{(m)}$ and $H^{\mathbb{C}}_{(m)}$ is in fact independent of m, provided of course $\Omega^{\mathbb{C}}_{(m)}$ is well defined. It is given by (7.41), which means (as one could expect) that the NLEE generated by the pair $\Omega^{\mathbb{C}}_{(m)}$ and $H^{\mathbb{C}}_{(m)}$ does not depend on m and coincides with (7.22).

One can relate the existence of the hierarchy of Hamiltonian structures to the simple fact that the generic NLEE:

$$\Lambda^m \left(i\sigma_3 \frac{\partial q}{\partial t} + 2f(\Lambda) q(x,t) \right) = 0 , \tag{7.85}$$

are equivalent to the NLEE (7.22). In terms of the "action–angle" variables we get

$$\begin{aligned} &i\lambda^m \eta_t = 0, \qquad \lambda^m (i\kappa_t - 2f(\lambda)) = 0 , \\ &i(\lambda_k^{\pm})^m \eta_k^{\pm} = 0, \qquad (\lambda_k^{\pm})^m (i\kappa_{k,t}^{\pm} - 2f(\lambda_k^{\pm})) = 0 . \end{aligned} \tag{7.86}$$

For $m = 0$, we recover the equations from Chap. 7. For $m \neq 0$ see Remarks 7.4 and 7.5.

7.5 Involutions and Hierarchies

7.5.1 The Involution $q^- = (q^+)^* = u(x)$

This involution applied to the system (6.18), (6.19) allows one to get the NLS equation. Applied to the system (6.21), it gives us the complex mKdV equation.

The involution $q^+ = (q^-)^* = u$ is a consequence of (6.47) with $\varepsilon_0 = 1$. Its effects on the action–angle variables are as follows. From (5.87) and (6.52)–(6.59) we find:

$$\begin{aligned} &\eta(\lambda) = \tfrac{1}{\pi} \ln(1 + |\rho^+(t,\lambda)|^2), \qquad \kappa(\lambda) = i \arg b^+(t,\lambda), \qquad \lambda \in \mathbb{R}, \\ &\eta_k^{\pm} = 2\lambda_{1,k} \mp 2i\lambda_{(0),k}, \qquad \kappa_k^{\pm} = \ln |b_k^+| \pm i \arg b_k^+ , \end{aligned} \tag{7.87}$$

where $k = 1, \ldots, N$, and

$$\lambda_k^{\pm} = \lambda_{0,k} \pm i\lambda_{1,k}, \qquad \lambda_{1,k} > 0 . \tag{7.88}$$

Thus, we conclude that $\Omega_{(0)}$ becomes purely real and has the form:

$$\Omega_{(0)} = \int_{-\infty}^{\infty} d\lambda \, \delta\eta(\lambda) \wedge \delta \arg b^+(t,\lambda)$$

$$+ 4\sum_{k=1}^{N} \left(\delta\lambda_{0,k} \wedge \delta \ln |b_k^+(t)| - \delta\lambda_{1,k} \wedge \delta \arg b_k^+(t)\right) . \tag{7.89}$$

As regards the integrals of motion, they become real valued. (see (6.90)):

$$C_p = -\frac{1}{2}\int_{-\infty}^{\infty} d\lambda\, \lambda^{p-1}\eta(\lambda) - \frac{2}{p}\sum_{k=1}^{N} \mathrm{Im}\,(\lambda_k^+)^p . \tag{7.90}$$

The Hamiltonian $H^{(0)}$ also becomes real:

$$\begin{aligned} H_{(0)} &= 4\sum_p f_p C_{p+1} \\ &= -2\int_{-\infty}^{\infty} d\mu\, f(\mu)\eta(\mu) - 8\sum_{k=1}^{N} \mathrm{Im}\,(F_k^+) , \end{aligned} \tag{7.91}$$

where $f(\lambda)$ is the dispersion law and $F(\lambda)$ and F_k^+ are introduced in (7.57).

Analogously, for the symplectic forms of the hierarchy $\Omega_{(m)}$ we get:

$$\begin{aligned} \Omega_{(m)} &= \int_{-\infty}^{\infty} d\lambda\, \lambda^m \delta\eta(\lambda) \wedge \delta \arg b^+(t,\lambda) \\ &+ \frac{4}{m+1}\sum_{k=1}^{N} \left(\delta \ln |b_k^+(t)| \wedge \delta\mathrm{Re}\,(\lambda_k^+)^{m+1} - \delta \arg b_k^+(t) \wedge \delta\mathrm{Im}\,(\lambda_k^+)^{m+1}\right) . \end{aligned} \tag{7.92}$$

The hierarchy of Hamiltonians is provided by:

$$\begin{aligned} H_{(m)} &= 4\sum_p f_p C_{p+m+1} \\ &= -2\int_{-\infty}^{\infty} d\mu\, \mu^m f(\mu)\eta(\mu) - 8\sum_{k=1}^{N} \mathrm{Im}\,(F_k^{(m),+}) . \end{aligned} \tag{7.93}$$

where

$$F^{(m),+}(\lambda) = \int^{\lambda} d\lambda'\, f(\lambda')\lambda'^{,m}, \qquad F_k^{(m),+} = F^{(m),+}(\lambda_k^+) . \tag{7.94}$$

Thus, we see that the overall effect of the reduction is to decrease “twice” the number of the dynamical variables both on the continuous and discrete spectrum. Now two of the three types of action variables, $\arg b^+(t,\lambda)$ and $\arg b_k^+$, are real and take values in the interval $[0, 2\pi]$; the third type $\ln |b_k^+|$ are also real but may take arbitrary values.

The reduction imposes also restrictions on the dispersion law of the NLEE, see (6.62). In other words, the reduction (6.47) admits only dispersion laws whose expansion coefficients are real:

$$f(\lambda) = \sum_p f_p \lambda^p, \qquad f_p = f_p^* \,. \tag{7.95}$$

Some of the most important examples of NLEE obtained by this reduction are the NLS equation, the complex mKdV equation, and a combination of both NLS-cmKdV ($u = q^+(x,t)$):

$$\text{NLS} \qquad iu_t + u_{xx} + 2|u|^2 u(x,t) = 0 \,, \tag{7.96}$$

$$\text{cmKdV:} \qquad u_t + u_{xxx} + 6|u|^2 u_x(x,t) = 0 \tag{7.97}$$

$$\begin{aligned} \text{NLS-cmKdV:} \qquad & iu_t + u_{xx} + 2|u|^2 u(x,t) \\ & \qquad + ic_0(u_{xxx} + 6|u^2|u_x) = 0 \,. \end{aligned} \tag{7.98}$$

Their dispersion laws are given by:

$$\begin{aligned} f_{\mathrm{NLS}}(\lambda) = -2\lambda^2, \qquad f_{\mathrm{cmKdV}}(\lambda) = -4\lambda^3 \,, \\ f_{\mathrm{NLS-cmKdV}}(\lambda) = -2\lambda^2 - 4c_0\lambda^3 \,. \end{aligned} \tag{7.99}$$

7.5.2 The Involution $q^- = -(q^+)^* = u(x)$

This involution is obtained from (6.47) when $\varepsilon_0 = -1$. Applied to systems (6.18), (6.19) and (6.21), it allows one to get different versions of the NLS and cmKdV equations:

$$iu_t + u_{xx} - 2|u|^2 u(x,t) = 0 \,, \tag{7.100}$$

$$u_t + u_{xxx} - 2|u|^2 u_x(x,t) = 0 \,, \tag{7.101}$$

with “wrong” signs of the nonlinearity; compare with (7.96) and (7.97). However, this involution changes drastically the class of all solutions of these NLEE, in particular it makes impossible the existence of soliton solutions.

Indeed, as was explained in Sect. 6.3 above, this involution makes the Zakharov–Shabat system equivalent to a self-adjoint eigenvalue problem (6.54) which has no discrete eigenvalues.

Therefore, we have scattering data only on the continuous spectrum given by (6.52) with $\varepsilon_0 = -1$, and the action-angle variables are provided by:

$$\eta(\lambda) = \tfrac{1}{\pi} \ln(1 - |\rho^+(t,\lambda)|^2), \qquad \kappa(\lambda) = i \arg b^+(t,\lambda), \qquad \lambda \in \mathbb{R} \tag{7.102}$$

From (6.55b), it follows that $|\rho^+(t,\lambda)| < 1$ and, therefore, the action variables are well defined.

As in the previous Subsection, we conclude that $\Omega_{(0)}$ becomes purely real and has the form:

$$\Omega_{(0)} = \int_{-\infty}^{\infty} d\lambda \, \delta\eta(\lambda) \wedge \delta \arg b^+(t,\lambda) \,, \tag{7.103}$$

Analogously, for Ω_m we get:

$$\Omega_{(m)} = \int_{-\infty}^{\infty} d\lambda\, \lambda^m \delta\eta(\lambda) \wedge \delta \arg b^+(t,\lambda) \; . \tag{7.104}$$

The integrals of motion remain purely real, (see (6.90)):

$$C_p = -\frac{1}{2} \int_{-\infty}^{\infty} d\lambda\, \lambda^{p-1} \eta(\lambda) \; , \tag{7.105}$$

and the Hamiltonian becomes:

$$H_{(0)} = 4 \sum_p f_p C_{p+1} = -2 \int_{-\infty}^{\infty} d\mu\, f(\mu) \eta(\mu) \; , \tag{7.106}$$

where $f(\lambda)$ is the dispersion law.

The overall effect of the reduction is to decrease twice the number of the dynamical variables. Now, we have truly action-angle variables: $\eta(\mu)$ and $\arg b^+(t,\lambda)$ taking values in the interval $[0, 2\pi]$. Therefore, in some sense the phase space $\mathcal{M}$ must be isomorphic to an infinite-dimensional torus.

7.5.3 The Involution $q^+ = \pm q^- = u(x)$

This involution is obtained from (6.48) with $\varepsilon_1 = \pm 1$. Its consequences on the Hamiltonian structures and on the conservation laws are more serious now. Indeed, from (6.48) it is obvious that the canonical symplectic form $\Omega_{(0)}$ becomes identically zero: $\Omega_0 \equiv 0$. In fact, from (7.74) and (6.67), (6.68), we find that for all symplectic forms with even indices:

$$\Omega_{(2p)} \equiv 0 \; . \tag{7.107}$$

As for the symplectic forms with odd indices we get:

$$\begin{aligned} \Omega_{(2p+1)} &= 2i \int_0^{\infty} d\lambda\, \lambda^{2p+1} \delta \ln \frac{b^+(t,\lambda)}{b^+(t,-\lambda)} \wedge \delta\eta(\lambda) \\ &\quad + \frac{2}{p+1} \sum_{k=1}^{N} \delta\kappa_k^+ \wedge \delta(\lambda_k^+)^{2p+2} \; . \end{aligned} \tag{7.108}$$

The reason for (7.107) is that due to (6.67) for $m = 2p$ the integrand becomes an odd function of λ, and the terms under the summation sign pairwise cancel each other. For odd values of $m = 2p+1$ these terms add up.

We have a similar situation for the integrals of motion. From (6.66), (6.67) and (6.68) it follows that:

$$C_{2p} = 0 \; , \tag{7.109}$$

$$C_{2p+1} = -\int_0^\infty d\lambda\, \lambda^{2p} \eta(\lambda) - \frac{2}{2p+1} \sum_{k=1}^N \mathrm{Im}\, (\lambda_k^+)^{2p+1} . \tag{7.110}$$

Like before, the reduction imposes restriction on $f(\lambda)$, which now must be odd function of λ:

$$f(\lambda) = -f(-\lambda), \qquad \text{or} \qquad f(\lambda) = \sum_p f_{2p-1} \lambda^{2p-1} ; \tag{7.111}$$

however, the coefficients f_{2p-1} may take complex values. As a result, equations of NLS type (i.e. containing second derivatives with respect to x) are not allowed.

As examples of interesting NLEE related to this reduction, we point out the mKdV and sG. For instance, the dispersion law of mKdV equation (see (7.99)) is $f_{\mathrm{mKdV}} = -4\lambda^3$, and the equation itself reads ($q^+ = q^- = v(x)$):

$$v_t + v_{xxx} + 6v^2 v_x = 0 . \tag{7.112}$$

As mentioned above, $\Omega_{(0)} \equiv 0$ and $H_{(0)} = \sum_p f_{2p-1} C_{2p} \equiv 0$, i.e. the canonical Hamiltonian formulation is degenerate. Luckily, we have a whole hierarchy of Hamiltonian structures, e.g.:

$$\Omega_{(1)}, \qquad H_{(1)} = \sum_p f_{2p-1} C_{2p+1} , \tag{7.113}$$

$$\Omega_{(-1)}, \qquad H_{(-1)} = \sum_p f_{2p-1} C_{2p-1} , \tag{7.114}$$

which provide us with a Hamiltonian formulation of these equations.

It takes some additional effort to see that the s-G equation is also a member of this hierarchy. Indeed, it is known that its dispersion law is:

$$f_{\mathrm{sG}}(\lambda) = -\frac{1}{2\lambda} . \tag{7.115}$$

The corresponding NLEE (6.7a) is of the form:

$$i\sigma_3 q_t - \Lambda_\pm^{-1} q(x,t) = 0 , \tag{7.116}$$

which requires the calculation of the operators $(\Lambda_\pm)^{-1}$ inverse to $\Lambda_\pm$. Generically the evaluation of $(\Lambda_\pm)^{-1}$ explicitly through $q(x)$ is not possible. But under the reduction (6.48), the matter is simplified and this becomes possible. If we denote $Z_\pm(x) = \Lambda_\pm^{-1} q(x)$ then obviously:

$$q(x) \equiv \Lambda_\pm Z_\pm(x) = \frac{i}{4} \left[\sigma_3, \frac{dZ_\pm}{dx} \right] - iq(x) \int_{\pm\infty}^x dy\, \langle \sigma_3, [q(y), Z_\pm(y)] \rangle , \tag{7.117}$$

and due to the reduction, the matrix $q(x)$ has only one independent entry: $q(x) = q^+(x)\sigma$ where

$$\sigma = \begin{pmatrix} 0 & 1 \\ \varepsilon_1 & 0 \end{pmatrix} . \tag{7.118}$$

From (7.117), there follows that $Z_\pm(x)$ also has just one independent matrix element and $Z_\pm(x) = z_\pm(x)\sigma_3\sigma$. Inserting these expressions into (7.117) we get the following relation:

$$q^+(x) = \frac{i}{2}\frac{dz_\pm(x)}{dx} + 2i\varepsilon_1 q^+(x)\int_{\pm\infty}^{x} dy\, q^+(y)z_\pm(y) , \tag{7.119}$$

which must be treated as an integral equation for $z_\pm(x)$. Now we choose $q^+(x)$ in the form:

$$q^+(x) = R(z_\pm)\frac{dz_\pm(x)}{dx} , \tag{7.120}$$

which leads to the following integral equation for the function $R(z_\pm)$:

$$R(z_\pm) = \frac{i}{2} + 2i\varepsilon_1 R(z_\pm)\int_{\pm\infty}^{z_\pm} dz z R(z) . \tag{7.121}$$

If we divide by $R(z_\pm)$ and differentiate with respect to $z_\pm$ we get:

$$\frac{dR(z_\pm)}{R^3(z_\pm)} = 2\varepsilon_1 d(z_\pm)^2 , \tag{7.122}$$

which has the solution:

$$R^2(z_\pm) = \frac{1}{c_0^2 - 4\varepsilon_1(z_\pm)^2}, \qquad \varepsilon_1 = \pm 1 , \tag{7.123}$$

where c_0 is an integration constant. Next, we insert (7.123) into (7.120) and integrate over $z_\pm$. The result depends on the choice of the sign ε_1, namely:

$$q^+(x) = \frac{d}{dx}\left(\frac{1}{2}\arcsin\frac{2z_\pm}{c_0}\right), \qquad \varepsilon_1 = 1 , \tag{7.124}$$

$$q^+(x) = \frac{d}{dx}\left(\frac{1}{2}\operatorname{arcsinh}\frac{2z_\pm}{c_0}\right), \qquad \varepsilon_1 = -1 , \tag{7.125}$$

or equivalently

$$z_\pm(x) = \frac{c_0}{2}\sin\left(2\int_{\pm\infty}^{x} dy\, q^+(y) + c_1\right), \qquad \varepsilon_1 = 1 , \tag{7.126}$$

$$z_\pm(x) = \frac{c_0}{2}\sinh\left(2\int_{\pm\infty}^{x} dy\, q^+(y) + c_1\right), \qquad \varepsilon_1 = -1 . \tag{7.127}$$

Therefore, we get:

$$\Lambda_\pm^{-1} q(x) \equiv Z_\pm(x) = \frac{c_0}{2}\sigma_3\sigma\left(\sin(2w(x) + c_0')\right), \qquad \varepsilon_1 = 1 , \tag{7.128}$$

$$\Lambda_{\pm}^{-1} q(x) \equiv Z_{\pm}(x) = \frac{c_0}{2} \sigma_3 \sigma \left(\sinh(2w(x) + c_0') \right), \qquad \varepsilon_1 = -1 \; , \quad (7.129)$$

where we introduced the notation

$$w(x) = \int_{\pm\infty}^{x} dy \, q^{+}(y) + c_1 \; , \qquad (7.130)$$

which means that

$$q^{+}(x) = w_x \; . \qquad (7.131)$$

Finally, the NLEE (7.116) with the dispersion law (7.115) is cast into one of the forms:

$$\frac{\partial^2 w}{\partial x \partial t} + \frac{c_0}{2} \sin(2w(x,t)) = 0, \qquad \varepsilon_1 = 1 \; , \qquad (7.132)$$

$$\frac{\partial^2 w}{\partial x \partial t} + \frac{c_0}{2} \sinh(2w(x,t)) = 0, \qquad \varepsilon_1 = -1 \; , \qquad (7.133)$$

which are known as the sine-Gordon and the sinh-Gordon equations, respectively.

7.5.4 Applying Both Involutions (6.47) and (6.48) with $\varepsilon_0 = 1$

It is natural to ask whether it is possible to impose both involutions we considered simultaneously. The answer is positive only if they commute. Indeed, we can calculate the interrelation between $U(x, -\lambda^*)$ and $U(x, \lambda)$ in two ways: (i) first applying involution (6.47) and then (6.48) and (ii) first applying (6.48) then (6.47). The two results will be identical, if the matrices ϵ and σ commute, i.e. the involutions are compatible provided:

$$\varepsilon_0 = -\varepsilon_1 \; . \qquad (7.134)$$

Again, we can consider two possibilities. The first one:

$$\varepsilon_0 = 1, \qquad \varepsilon_1 = -1 \; , \qquad (7.135)$$

means that

$$(q^{+}(x))^* = q^{-}(x) = -q^{+}(x) \; , \qquad (7.136)$$

or equivalently

$$q^{\pm}(x) = \pm i u(x) \; , \qquad (7.137)$$

where $u(x)$ is a real-valued function.

The second possibility is

$$\varepsilon_0 = -1, \qquad \varepsilon_1 = 1 \; , \qquad (7.138)$$

which means that

$$(q^+(x))^* = -q^-(x) = q^+(x) , \tag{7.139}$$

or equivalently

$$q^\pm(x) = \pm v(x) , \tag{7.140}$$

where $v(x)$ is a real-valued function.

Let us start with the first case and impose on the scattering data both sets of restrictions (6.52), (6.58), (6.59) and (6.66), (6.68) at the same time. This has to be done with special care for the discrete spectrum.

Indeed, the discrete spectrum of L may contain two types of eigenvalues:

1. pairs of purely imaginary eigenvalues:

$$\lambda_k^\pm = \pm i s_k, \qquad k = 1, \dots, N_0 , \tag{7.141}$$

2. quadruplets of complex eigenvalues, lying on the vertices of a rectangle in the complex λ-plane:

$$\lambda_k^\pm = \lambda_{0,k} \pm i\lambda_{1,k}, \qquad \lambda_{k+N_1}^\pm = -\lambda_{0,k} \pm i\lambda_{1,k}, \qquad k = N_0+1, \dots, N_0+N_1 . \tag{7.142}$$

The functions $\eta(\lambda)$ and $\kappa(t,\lambda)$ retain the properties:

$$\kappa(t,\lambda) = \kappa(-\lambda), \qquad \eta(\lambda) = -\eta(-\lambda) , \tag{7.143}$$

(compare (6.66) and (5.87)) so that only the 2-forms $\Omega_{(2p+1)}$ are nondegenerate. A bit more care is needed to calculate the contribution of the discrete spectrum to $\Omega_{(2p-1)}$. Skipping the details we present only the final result:

$$\begin{aligned}
&\Omega_{(2p-1)} \\
&= -2\int_0^\infty d\lambda\, \lambda^{2p-1} \delta \arg b^+(t,\lambda) \wedge \delta\eta(\lambda) + \frac{2(-1)^p}{p} \sum_{k=1}^{N_0} \delta \ln |b_k^+| \wedge \delta s_k^{2p} \\
&\quad + \frac{4}{p} \sum_{k=N_0+1}^{N_0+N_1} \left(\delta(\ln |b_k^+|) \wedge \delta(\mathrm{Re}\,(\lambda_k^+)^{2p} - \delta \arg b_k^+ \wedge \delta(\mathrm{Im}\,(\lambda_k^+)^{2p} \right) ,
\end{aligned} \tag{7.144}$$

for $p = 1, 2, \dots$. Of course, these formulae are not valid for $p = 0$. In this case we have:

$$\begin{aligned}
\Omega_{(-1)} &= -2\int_0^\infty \frac{d\lambda}{\lambda} \delta \arg b^+(t,\lambda) \wedge \delta\eta(\lambda) + 2\sum_{k=1}^{N_0} \delta \ln |b_k^+| \wedge \delta \ln s_k \\
&\quad + 4 \sum_{k=N_0+1}^{N_0+N_1} \left(\delta(\ln |b_k^+|) \wedge \delta \ln |\lambda_k^+| - \delta \arg b_k^+ \wedge \delta \arg \lambda_k^+ \right) .
\end{aligned} \tag{7.145}$$

The 2-form $\Omega_{(-1)}$ is well defined only for the class of potentials for which the reflection coefficients $\rho^{\pm}(0) = 0$ and $\tau^{\pm}(0) = 0$.

Analogously, for the integrals of motion we get:

$$\begin{aligned} C_{2p+1} = &- \int_0^\infty d\lambda\, \lambda^{2p} \eta(\lambda) - \frac{4}{2p+1} \sum_{k=1}^{N_0} (-1)^p s_k^{2p+1} \\ &- \frac{8}{2p+1} \sum_{k=N_0+1}^{N_0+N_1} \operatorname{Im} (\lambda_k^+)^{2p+1} . \end{aligned} \tag{7.146}$$

The constraints on the dispersion laws are provided by both (7.95) and (7.111), i.e. $f(\lambda)$ must be odd function of λ with real coefficients. As already shown above, the simplest nontrivial choices for dispersion laws compatible with both reductions are $1/\lambda$ and λ^3; they lead to the sine-Gordon and mKdV equations, respectively. The choice $f(\lambda) = \lambda$ leads to trivial linear equation.

Analyzing the second possibility (7.138) and (7.139) does not involve any difficulties. As mentioned above, the Zakharov–Shabat system in this case does not have discrete eigenvalues. So, we can apply the considerations from Sect. 7.5.2 taking into account that the additional involution forces $\eta(\lambda)$ to be even function and $\kappa(\lambda)$ to be odd function of λ. Therefore, the action-angle variables are uniquely specified by:

$$\eta(\lambda) = \frac{1}{\pi} \ln(1 - |\rho^+(t,\lambda)|^2), \qquad \kappa(\lambda) = i \arg b^+(t,\lambda), \quad 0 \leq \lambda . \tag{7.147}$$

Then the symplectic forms $\Omega_{(2p)}$ vanish identically and $\Omega_{(2p-1)}$ reduce to

$$\Omega_{(2p-1)} = 2 \int_0^\infty d\lambda\, \lambda^{2p-1} \delta\eta(\lambda) \wedge \delta \arg b^+(t,\lambda) . \tag{7.148}$$

The integrals of motion C_{2p} become identically zero while C_{2p+1} remain nontrivial and purely real (see (7.105)):

$$C_{2p-1} = -\frac{1}{2} \int_{-\infty}^\infty d\lambda\, \lambda^{2p-2} \eta(\lambda) . \tag{7.149}$$

The relevant Hamiltonians $H_{(2p-1)}$ must be linear combinations of C_{2p-1} only. From (7.106) we see that they are real:

$$H_{(2p-1)} = 4 \sum_p f_{2p} C_{2p-1} = -2 \int_0^\infty d\mu\, \mu^{2p-2} f(\mu) \eta(\mu) , \tag{7.150}$$

where $f(\lambda)$ is the dispersion law which must be odd function of λ.

7.6 Comments and Bibliographical Review

1. Generically the NLEE of soliton type are infinite-dimensional Hamiltonian systems. These facts have been extensively studied [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57]. A new important trend that started with the development of the ISM consists in constructing the complete integrability and explicitly constructing their action-angle variables. In fact, this amounts to generalizing the Liouville theorem for infinite-dimensional systems. The rigorous proof uses the completeness of the squared solutions, in order to prove that the number of integrals of motion is sufficient to ensure integrability.
2. There is a natural way to complexify the Hamiltonian systems [58, 59, 60, 61]. Generalizing the dynamical variables to take complex values results in a new Hamiltonian system having twice more degrees of freedom. Using involutions (i.e. automorphisms second order) of the corresponding symplectic structure, it is possible also to extract different real Hamiltonian forms of the complexified systems [60, 62].
3. An important step in the Hamiltonian theory of NLEEs has been started by Magri [63], who discovered that the KdV equation allows a second Hamiltonian formulation. Soon after that it was realized that in fact all soliton equation possess a hierarchy of Hamiltonian structures generated by the recursion operators [64, 65, 66].
4. A number of papers have approached the complete integrability of the infinite-dimensional Hamiltonian systems[3, 4, 10, 26, 27, 31, 33, 34, 37, 47, 49, 50, 57, 59, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130]
5. More refined methods for deriving new soliton equations are based on the use of the so-called reduction group. The method was proposed by Mikhailov [131, 132] and allowed to prove the integrability of the two-dimensional affine Toda field theories related to the simple Lie algebras [110, 131, 133, 134, 135, 136]. Mikhailov's method was generalized also for the class of N-wave equations and their gauge-equivalent ones [137, 138]. The study of the ISP for the relevant Lax operator (2.136) with complex-valued J is technically rather complicated (see [139, 140, 141, 142]) and requires good knowledge of graded and Kac-Moody algebras, which are out of the scope of the present monograph. These studies have been developed in [32, 48, 56, 92, 123, 124, 131, 132, 136, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159]. More recently, an attempt to classify all inequivalent reductions of the N-wave-type equations related to low rank Lie algebras was done in [62, 160, 161]. Another important trend here is to construct new classes of infinite-dimensional Lie algebras [159, 162].

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8

The NLEEs and the Gauge Transformations

In this Chapter, we show that all the results obtained up to now for the NLEEs (6.7) can be reformulated in a natural way for the gauge-equivalent NLEEs. In fact, this was the reason for what we called explicitly gauge-covariant formulation of the results in the Chaps. 5, 6, and 7.

In order to demonstrate that this is not a problem of pure academic interest, we mention the most famous examples of gauge-equivalent NLEEs. The first one is the equivalence between the KdV and the mKdV equations; the corresponding relation is provided by the so-called Miura transformation. The second such example relates the NLS equation with the Heisenberg ferromagnet (HF) equation in the semiclassical approximation. In terms of the 2×2 matrix-valued function $S(x,t)$ this equation reads:

$$i\frac{\partial S}{\partial t} + \left[S(x,t), \frac{\partial^2 S}{\partial x^2} \right] = 0 \, . \tag{8.1}$$

where $S(x,t)$ satisfies:

$$\operatorname{tr} S(x,t) = 0, \qquad S^2(x,t) = \mathbb{1} \, , \tag{8.2a}$$

and

$$\lim_{x\to\pm\infty} S(x,t) = \sigma_3 \, . \tag{8.2b}$$

One convenient way to parametrize $S(x,t)$ is to use the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \, , \tag{8.3}$$

and put:

$$S(x,t) = \sum_{a=1}^{3} S_a(x,t)\sigma_a = \begin{pmatrix} S_3(x,t) & S^+(x,t) \\ S^-(x,t) & -S_3(x,t) \end{pmatrix} \, . \tag{8.4}$$

where $S^\pm(x,t) = S_1(x,t) \mp iS_2(x,t)$. If $S_a(x,t)$ are real-valued then $S^\dagger(x,t) = S(x,t)$ is hermitian. Equation (8.4) shows that instead of 2×2 matrices $S(x,t)$

Gerdjikov, V.S. et al.: *The NLEEs and the Gauge Transformations*. Lect. Notes Phys. **748**, 247–314 (2008)
DOI 10.1007/978-3-540-77054-1_8

one may use the real vector $\mathbf{S}(x,t) = (S_1(x,t), S_2(x,t), S_3(x,t))^T$. We shall need the commutation relations:

$$[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c \ , \tag{8.5}$$

where a, b, and c take the values $1, 2, 3$, and ϵ_{abc} is the completely antisymmetric tensor. Taking into account (8.5), one can rewrite the HF equation (8.1) in the form:

$$\frac{\partial \mathbf{S}}{\partial t} + 2\mathbf{S}(x,t) \times \frac{\partial^2 \mathbf{S}}{\partial x^2} = 0 \ , \tag{8.6}$$

where $\times$ means the vector product of two vectors. The conditions (8.2) in terms of the vector $\mathbf{S}(x,t)$ mean:

$$(\mathbf{S}(x,t), \mathbf{S}(x,t)) = 1, \qquad \lim_{x\to\pm\infty} \mathbf{S}(x,t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \tag{8.7}$$

Equation (8.6) with real-valued vector-function $\mathbf{S}(x,t)$ is the best known form of the HF equation in the continuous limit.

Along with it, we shall consider also the complexified HF (CHF) equation, which has the same form (8.6) constrained by (8.7), but the components of $\mathbf{S}(x,t)$ are complex-valued. Then the corresponding matrix $S(x,t)$ is not hermitian.

In what follows, we shall prefer the formulation in terms of matrix $S(x,t)$. All our results can be immediately reformulated in terms of $\mathbf{S}(x,t)$ using simple rules. If $A(x,t)$ and $B(x,t)$ are 2×2 matrices satisfying (8.2) and $\mathbf{A}(x,t)$, $\mathbf{B}(x,t)$ are the corresponding vectors (see (8.4)), then

$$\begin{gathered} [A(x,t), B(x,t)] \leftrightarrow 2i\mathbf{A}(x,t) \times \mathbf{B}(x,t) \ , \\ \langle A(x,t), B(x,t)\rangle \equiv \frac{1}{2}\mathrm{tr}\,(A(x,t), B(x,t)) \leftrightarrow (\mathbf{A}(x,t), \mathbf{B}(x,t)) \ . \end{gathered} \tag{8.8}$$

In this Chapter, we analyze also the group of gauge transformations of the Lax equation. It is well known that the gauge degrees of freedom have no physical meaning. A natural way to avoid them is to fix the gauge, and we shall demonstrate different possible ways of fixing it. The first one takes us back to the Zakharov–Shabat system and to the class of NLS-type equations. More attention will be paid to the second possibility known as the pole gauge, which allows one to solve the class of Heisenberg ferromagnet type of NLEEs and their complexified versions.

We shall show the complexified NLS-type equations are gauge equivalent to the complexified HF-type equations. The latter are solved by the ISM for the operator $\tilde{L}$ (8.34). As a consequence of these results, we shall recover the well-known equivalence between the NLS equation and the HF equation. The gauge-covariant formulation of the generalized Fourier transforms developed earlier will allow transfer of all fundamental properties of the NLS-type equations into the corresponding properties for the HF-type equations. We establish also the interrelations between their hierarchies of Hamiltonian structures.

8.1 The Group of the Gauge Transformations

The Lax representation written in the form:

$$[L, M] = 0 \,, \tag{8.9}$$

provides the compatibility condition of the linear operators:

$$L\psi(x,t,\lambda) \equiv i\frac{\partial \psi}{\partial x} + U(x,t,\lambda)\psi(x,t,\lambda) = 0 \,, \tag{8.10a}$$

$$M\psi(x,t,\lambda) \equiv i\frac{\partial \psi}{\partial t} + V(x,t,\lambda)\psi(x,t,\lambda) = \psi(x,t,\lambda)C(\lambda) \,, \tag{8.10b}$$

Here, $U(x,t,\lambda)$, $V(x,t,\lambda)$ and $C(\lambda)$ take values in the simple Lie algebra $\mathfrak{g}$. The compatibility condition for L and M in (8.10) is purely algebraic in nature:

$$-i\frac{\partial U}{\partial t} + i\frac{\partial V}{\partial x} + [U(x,t,\lambda), V(x,t,\lambda)] = 0 \,, \tag{8.11}$$

and holds true for any x-independent $C(\lambda)$. As explained in Chap. 2 for further convenience we specify

$$C(\lambda) = \lim_{x\to\pm\infty} V(x,t,\lambda) \,. \tag{8.12}$$

If given L and M satisfy (8.9), then $\widetilde{L}$ and $\widetilde{M}$ defined by:

$$L \to \tilde{L} = g^{-1}Lg(x,t), \qquad M \to \tilde{M} = g^{-1}Mg(x,t) \,, \tag{8.13}$$

are also compatible, i.e. $[\widetilde{L}, \widetilde{M}] = 0$. Here, $g(x,t)$ is a nondegenerate matrix-valued function of x and t. The transformed Lax pair is:

$$\widetilde{L}\widetilde{\psi}(x,t,\lambda) \equiv i\frac{\partial \widetilde{\psi}}{\partial x} + \widetilde{U}(x,t,\lambda)\widetilde{\psi}(x,t,\lambda) = 0 \,, \tag{8.14a}$$

$$\widetilde{M}\widetilde{\psi}(x,t,\lambda) \equiv i\frac{\partial \widetilde{\psi}}{\partial t} + \widetilde{V}(x,t,\lambda)\widetilde{\psi}(x,t,\lambda) = \widetilde{\psi}(x,t,\lambda)C(\lambda) \,, \tag{8.14b}$$

where $\widetilde{\psi}(x,t,\lambda) = g^{-1}(x,t)\psi(x,t,\lambda)$ and

$$\widetilde{U}(x,t,\lambda) = g^{-1}(x,t)U(x,t,\lambda)g(x,t) + ig^{-1}(x,t)\frac{\partial g}{\partial x} \,, \tag{8.15a}$$

$$\widetilde{V}(x,t,\lambda) = g^{-1}(x,t)V(x,t,\lambda)g(x,t) + ig^{-1}(x,t)\frac{\partial g}{\partial t} \,. \tag{8.15b}$$

Equation (8.15) is known as the gauge transformation that leaves invariant the compatibility condition (8.11). The function $g(x,t)$ specifies the gauge degrees of freedom. We shall see that there are more than one natural ways to fix it up. To each of these choices of $g(x,t)$ there corresponds a specific form of the Lax operator L (or, equivalently, of $U(x,t,\lambda)$) and a class of NLEE written in terms of $U_a(x,t)$, $a=0,1$:

$$U(x,t,\lambda) = U_0(x,t) + \lambda U_1(x,t) \ , \tag{8.16}$$

In order to be more specific in what follows, we shall consider L to be of Zakharov–Shabat type, where both $U_0(x,t)$ and $U_1(x,t)$ are 2×2 matrix-valued functions. The space $\mathcal{U}$ of pairs of matrix-valued functions $U_0(x,t)$, $U_1(x,t)$ splits into subspaces invariant with respect to the dynamics of the NLEE. Fixing the gauge $g(x,t)$ corresponds to a specific choice of one of these subspaces, as well as to a convenient choice of the local coordinates on it.

For example, $\operatorname{tr} U_0(x,t)$ and $\operatorname{tr} U_1(x,t)$ can be used to define one such invariant subspace. Indeed, taking the trace of the compatibility condition (8.11) we get:

$$-i\frac{\partial}{\partial t}\operatorname{tr} U(x,t,\lambda) + i\frac{\partial}{\partial t}\operatorname{tr} V(x,t,\lambda) = 0 \ . \tag{8.17}$$

If

$$V(x,t,\lambda) = \sum_{k=0}^{N} \lambda^{N-k} V_k(x,t) \ , \tag{8.18}$$

then from (8.17) we find:

$$\frac{\partial}{\partial x}\operatorname{tr} V_k(x,t) = 0 \ , \tag{8.19a}$$

for $k=0,1,\ldots,N-2$ and

$$-\frac{\partial}{\partial t}\operatorname{tr} U_1(x,t) + \frac{\partial}{\partial x}\operatorname{tr} V_1(x,t) = 0 \ , \tag{8.19b}$$

$$-\frac{\partial}{\partial t}\operatorname{tr} U_0(x,t) + \frac{\partial}{\partial x}\operatorname{tr} V_0(x,t) = 0 \ . \tag{8.19c}$$

The last two equations allow one to conclude that both $\operatorname{tr} U_0(x,t)$ and $\operatorname{tr} U_1(x,t)$ are densities of conserved quantities of all the NLEE related to L. Fixing up their values:

$$\int_{-\infty}^{\infty} dx \operatorname{tr} U_0(x,t) = I_{(0)}, \qquad \int_{-\infty}^{\infty} dx \operatorname{tr} U_1(x,t) = I_{(1)} \ , \tag{8.20}$$

means that we have determined the subspace of $\mathcal{U}$ on which these two integrals are constants. Since the traces satisfy the linear Equation (8.17), they can be easily "separated" from the other degrees of freedom. Putting

$$\operatorname{tr} U_a(x,t) = 0, \qquad \operatorname{tr} V_k(x,t) = 0 \;, \tag{8.21}$$

for $a = 0, 1$ and $k = 0, \dots, N$ means that our Lax pair is restricted to the simple Lie algebra $\mathfrak{g} \simeq sl(2)$.

The next supposition that simplifies our task is to assume that the eigenvalues of $U_1(x,t)$ are constants, i.e. do not depend on x and t. Then, rescaling the spectral parameter λ with this constant, we can consider $U_1(x,t)$ to have as eigenvalues ± 1, that is, U_1 is of the form:

$$U_1(x,t) = -g_1(x,t)\sigma_3 g_1^{-1}(x,t) \;. \tag{8.22}$$

where $g_1(x,t)$ is an unknown 2×2 matrix function. The compatibility condition (8.11) must hold identically with respect to λ. In particular, this means that the coefficient at λ^{N+1} must vanish, i.e.:

$$[U_1(x,t), V_0(x,t)] = 0 \;. \tag{8.23}$$

Therefore, $V_0(x,t)$ must also be of the form

$$V_0(x,t) = f_0 g_1(x,t)\sigma_3 g_1^{-1}(x,t) \;. \tag{8.24}$$

Let us assume in addition that the eigenvalues $\pm f_0$ of $V_0(x,t)$ are constants.

Finally, let us apply the gauge transformation (8.13) with $g = g_1(x,t)$ to our linear problems. Then the Lax pair simplifies to:

$$L^{(1)}\psi^{(1)} \equiv i\frac{\partial \psi^{(1)}}{\partial x} + \left(U_0^{(1)}(x,t) - \lambda\sigma_3\right)\psi^{(1)}(x,t,\lambda) = 0 \;, \tag{8.25a}$$

$$\begin{aligned} M^{(1)}\psi^{(1)} &\equiv i\frac{\partial \psi^{(1)}}{\partial t} + \left(\sum_{k=1}^{N} \lambda^{N-k} V_k^{(1)}(x,t) + f_0\lambda^N\sigma_3\right)\psi^{(1)}(x,t,\lambda) \\ &= \psi^{(1)}(x,t,\lambda)C(\lambda) \;, \end{aligned} \tag{8.25b}$$

where

$$U_0^{(1)}(x,t) = g_1^{-1}U_0(x,t)g_1(x,t) + i g_1^{-1}(x,t)\frac{\partial g_1}{\partial x} \;, \tag{8.25c}$$

$$V_N^{(1)}(x,t) = g_1^{-1}V_N(x,t)g_1(x,t) + i g_1^{-1}(x,t)\frac{\partial g_1}{\partial t} \;, \tag{8.25d}$$

$$V_k^{(1)}(x,t) = g_1^{-1}V_k(x,t)g_1(x,t), \qquad k = 1, \dots, N-1 \;. \tag{8.25e}$$

One possible way to fix the gauge is to choose the leading terms of $L^{(1)}$ and $M^{(1)}$ as constant diagonal traceless matrices. The gauge transformation (8.25) that allowed us to do this "hides" the gauge variables $g_1(x,t)$ in $U_0^{(1)}(x,t)$. Also by "eliminating" the x and t-dependence of $U_1(x,t)$, we reduced the space $\mathcal{U}$ to $\mathcal{U}^{(1)}$, which is parametrized by $U_0^{(1)}(x,t)$ only. Since $\operatorname{tr} U_0^{(1)}(x,t) = 0$, it contains only three independent complex-valued functions.

However, there are still gauge degrees of freedom that can be fixed. Indeed, one can use the gauge transformations that preserve the leading terms of $L^{(1)}$ and $M^{(1)}$, i.e. one that commutes with σ_3, in order to eliminate the diagonal terms in $U_0^{(1)}(x,t)$ and $V_0^{(1)}(x,t)$. Let us make gauge transformation with $g = g_2(x,t)$, where

$$g_2(x,t) = e^{ig_{02}(x,t)\sigma_3} \; . \tag{8.26}$$

and $g_{02}(x,t)$ is a scalar function. It transforms the Lax pair (8.25) into:

$$L^{(2)}\psi^{(2)} \equiv i\frac{\partial \psi^{(2)}}{\partial x} + \left(U_0^{(2)}(x,t) - \lambda\sigma_3\right)\psi^{(2)}(x,t,\lambda) = 0 \; , \tag{8.27a}$$

$$\begin{aligned} M^{(2)}\psi^{(2)} &\equiv i\frac{\partial \psi^{(2)}}{\partial t} + \left(\sum_{k=1}^{N} \lambda^{N-k} V_k^{(2)}(x,t) + f_0\lambda^N\sigma_3\right)\psi^{(2)}(x,t,\lambda) \\ &= \psi^{(2)}(x,t,\lambda)C(\lambda) \; , \end{aligned} \tag{8.27b}$$

where

$$U_0^{(2)}(x,t) = g_2^{-1}U_0^{(1)}(x,t)g_2(x,t) + ig_2^{-1}(x,t)\frac{\partial g_2}{\partial x} \; , \tag{8.27c}$$

$$V_N^{(2)}(x,t) = g_2^{-1}V_N^{(1)}(x,t)g_2(x,t) + ig_2^{-1}(x,t)\frac{\partial g_2}{\partial t} \; , \tag{8.27d}$$

$$V_k^{(2)}(x,t) = g_2^{-1}V_k^{(1)}(x,t)g_2(x,t), \qquad k = 1,\ldots,N-1 \; . \tag{8.27e}$$

Since $g_2(x,t)$ is diagonal:

$$ig_2^{-1}(x,t)\frac{\partial g_2}{\partial x} = -\sigma_3\frac{\partial g_{02}}{\partial x}, \qquad ig_2^{-1}(x,t)\frac{\partial g_2}{\partial t} = -\sigma_3\frac{\partial g_{02}}{\partial t} \; , \tag{8.28}$$

We can choose $g_{02}(x,t)$ to satisfy:

$$\frac{\partial g_{02}}{\partial x} = \frac{1}{2}\operatorname{tr}\left(\sigma_3 U_0^{(1)}(x,t)\right), \tag{8.29}$$

which means that the diagonal elements of $U_0^{(2)}(x,t)$ vanish. The constraints (8.29) together with the assumptions we made already eliminate all gauge degrees of freedom. Imposing them on the Lax pair $L^{(2)}$ and $M^{(2)}$ give us the Zakharov–Shabat system:

$$L\psi = i\frac{\partial \psi}{\partial x} + \left(q(x,t) - \lambda\sigma_3\right)\psi(x,t,\lambda) = 0 \; , \tag{8.30a}$$

$$\begin{aligned} M\psi &= i\frac{\partial \psi}{\partial t} + \left(\sum_{k=1}^{N} \lambda^{N-k} V_k(x,t) + f_0\lambda^N\sigma_3\right)\psi(x,t,\lambda) \\ &= \psi(x,t,\lambda)C(\lambda) \; , \end{aligned} \tag{8.30b}$$

where

$$q(x,t) = q^{+}(x,t)\sigma_{+} + q^{-}(x,t)\sigma_{-} = \begin{pmatrix} 0 & q^{+}(x,t) \\ q^{-}(x,t) & 0 \end{pmatrix} , \qquad (8.30c)$$

$$V_1(x,t) = -f_0 q(x,t) . \qquad (8.30d)$$

The functions $V_k(x,t)$ cannot be simplified further; they are generic functions satisfying the condition **C1** and taking values in the algebra $sl(2)$. As explained in Chap. 2, they can be expressed through $q(x,t)$ and its x-derivatives using the recurrent relations.

The arguments above demonstrate that the Zakharov–Shabat system is in fact one of the generic systems related to the $sl(2)$ algebra. It also demonstrates that the space of functions $\mathcal{U}$ parametrizing the Lax operator L after eliminating the gauge variables shrinks into the phase space $\mathcal{M}$ studied in the previous Chapter. This way of fixing the gauge is called canonical.

The second way to fix up the gauge for the system (8.25) is known in the literature as the pole gauge. Using the same arguments as above, we impose conditions (8.22) and (8.24) to get:

$$L^{(3)}\psi^{(3)} \equiv i\frac{\partial \psi^{(3)}}{\partial x} + \left(U_0^{(3)}(x,t) - \lambda g^{-1}(x,t)\sigma_3 g(x,t)\right)\psi^{(3)}(x,t,\lambda) = 0 , \qquad (8.31a)$$

$$\begin{aligned} M^{(3)}\psi^{(3)} &\equiv i\frac{\partial \psi^{(3)}}{\partial t} + \left(\sum_{k=1}^{N} \lambda^{N-k} V_k^{(3)}(x,t) + f_0 \lambda^N g^{-1}(x,t)\sigma_3 g(x,t)\right)\psi^{(3)}(x,t,\lambda) \\ &= \psi^{(3)}(x,t,\lambda) C(\lambda) . \end{aligned} \qquad (8.31b)$$

The alternative way to eliminate the gauge degrees of freedom from (8.31) consists in restricting $g(x,t)$ in such a way that $U_0(x,t)$ and $V_N(x,t)$ given by (8.25c), (8.25d) vanish, i.e.:

$$U_0^{(3)} \equiv i\frac{\partial g}{\partial x} + U_0(x,t)g(x,t) = 0 , \qquad (8.32a)$$

$$V_0^{(3)} \equiv i\frac{\partial g}{\partial t} + V_N(x,t)g(x,t) = 0 , \qquad (8.32b)$$

In what follows, we shall denote by $g(x,t)$ the special solution of the system (8.32) satisfying:

$$\lim_{x\to\infty} g(x,t) = \mathbb{1} . \qquad (8.33)$$

In fact $g(x,t)$ is the Jost solution taken at $\lambda = 0$. Then the corresponding Lax operators L and M take the form:

$$\tilde{L}\widetilde{\psi}(x,t,\lambda) \equiv i\frac{\partial \widetilde{\psi}}{\partial x} - \lambda S(x,t)\widetilde{\psi}(x,t,\lambda) = 0 , \qquad (8.34a)$$

$$\tilde{M}\widetilde{\psi}(x,t,\lambda) \equiv i\frac{\partial \widetilde{\psi}}{\partial t} + \left(\sum_{k=1}^{N-1} \lambda^{N-k} \widetilde{V}_k(x,t) + f_0 \lambda^N S(x,t) \right) \widetilde{\psi}(x,t,\lambda)$$
$$= \widetilde{\psi}(x,t,\lambda) C(\lambda) \, , \qquad (8.34b)$$

where

$$S(x,t) = g^{-1}(x,t)\sigma_3 g(x,t), \qquad \widetilde{\psi}(x,t,\lambda) = g^{-1}(x,t)\psi(x,t,\lambda) g_0 \, . \quad (8.34c)$$

Obviously the Zakharov–Shabat system (8.30) and the pole-gauge system (8.34) are related by a gauge transformation (8.13), with $g(x,t)$ fixed up by conditions (8.32), (8.33), i.e. $g(x,t)$ is the Jost solution of L for $\lambda = 0$:

$$g(x,t) = \psi(x,t,\lambda = 0) \, . \qquad (8.35)$$

The gauge transformation taking the ZS system from the canonical to the pole gauges can be viewed as an implicit change of variables for the NLEEs. Indeed, given the potential $q(x,t)$ of the Zakharov–Shabat system one determines uniquely the Jost solution $\psi(x,t,\lambda)$, which in turn provides $g(x,t)$ (8.35) and $S(x,t)$ (8.34c). If, on the other hand, we are given $S(x,t)$ satisfying $S^2(x,t) = \mathbb{1}$, then we have to find the matrix $g(x,t)$ that diagonalizes it. This procedure is not unique; it will rather give $h(x,t)g(x,t)$ where $h(x,t)$ is a diagonal matrix-valued function. This function is determined from the condition that

$$i\mathrm{tr}\,(\sigma_3 \frac{\partial g}{\partial x} g^{-1}(x,t)) = -\mathrm{tr}\,(\sigma_3 q(x,t)) = 0 \qquad (8.36)$$

i.e., $q(x,t)$ is off-diagonal.

8.2 Gauge-Equivalent NLEE

It is natural to apply the AKNS method, in order to obtain the Lax pairs for the generic NLEEs of Heisenberg ferromagnet (HF) type. Below, we show how it can be done. If we insert the explicit form of $\widetilde{L}$ and $\widetilde{M}$ (8.34) into the compatibility condition $\left[\widetilde{L}, \widetilde{M} \right] = 0$ we get:

$$i\lambda \frac{\partial S}{\partial t} + i\frac{\partial \widetilde{V}}{\partial x} - \lambda \left[S(x,t), \widetilde{V}(x,t,\lambda) \right] = 0 \, . \qquad (8.37)$$

This equation holds identically with respect to λ which leads to the set of relations:

$$\lambda^N \quad : \quad i f_0 \frac{\partial S}{\partial x} - [S(x,t), \widetilde{V}_1(x,t)] = 0 \, , \qquad (8.38a)$$

$$\lambda^{N-k} \quad : \quad i\frac{\partial \widetilde{V}_k}{\partial x} - [S(x,t), \widetilde{V}_{k+1}(x,t)] = 0 \, , \qquad (8.38b)$$

$$\lambda \quad : \qquad i\frac{\partial S}{\partial t} + i\frac{\partial \widetilde{V}_{N-1}}{\partial x} = 0 \; . \tag{8.38c}$$

We see that the HF-type NLEE come up as the coefficient proportional to λ in the Lax representation and take the form of a conservation law.

In order to derive the explicit form of these NLEEs in terms of $S(x,t)$, we have to view (8.38a) and (8.38b) for $k = 1, \dots, N-2$ as recurrent relations and solve them to find $\widetilde{V}_k(x,t)$ in terms of $S(x,t)$ and its derivatives.

The method of solving is similar to the AKNS method explained in Chap. 2. In fact, there we introduced a grading in the algebra $sl(2)$ compatible with the eigenspaces of $\mathrm{ad}_{\sigma_3}\cdot$. Now, as the linear in λ term in the Lax operator instead of the constant element σ_3, we encounter its image after the gauge transformation $S(x,t)$. Therefore, now we need a different grading of $sl(2)$, one that is according to the eigenspaces of $\mathrm{ad}_S\cdot$.

In what follows, we shall introduce also the projector $\pi_S\cdot = \mathrm{ad}_S^{-1}\mathrm{ad}_S\cdot$. It splits the algebra $sl(2)$ considered as linear space into $\widetilde{\mathfrak{g}}^{(0)} \oplus \widetilde{\mathfrak{g}}^{(1)}$. The first space $\widetilde{\mathfrak{g}}^{(0)}$ is the eigenspace of π_S corresponding to the eigenvalue 0, and $\widetilde{\mathfrak{g}}^{(1)}$ is the eigenspace corresponding to the eigenvalue 1. Then, π_S projects on $\widetilde{\mathfrak{g}}^{(1)}$ parallel to $\widetilde{\mathfrak{g}}^{(0)}$.

The subspace of elements $\widetilde{X} = \pi_S \widetilde{X}$ is called the orbit of $g \simeq sl(2)$ passing through $S(x,t)$. This splitting has the grading property, namely:

$$[\widetilde{\mathfrak{g}}^{(0)}, \widetilde{\mathfrak{g}}^{(0)}] = 0, \qquad [\widetilde{\mathfrak{g}}^{(0)}, \widetilde{\mathfrak{g}}^{(1)}] \in \widetilde{\mathfrak{g}}^{(1)}, \qquad [\widetilde{\mathfrak{g}}^{(1)}, \widetilde{\mathfrak{g}}^{(1)}] \in \widetilde{\mathfrak{g}}^{(0)} \; . \tag{8.39}$$

In fact, $\widetilde{\mathfrak{g}}^{(0)}$ is isomorphic to the Cartan subalgebra of $sl(2)$, which with our choice of basis is one-dimensional and spanned by $S(x,t)$.

In order to apply effectively the AKNS method, it is convenient to introduce a special x and t-dependent basis in the algebra $sl(2)$:

$$\begin{aligned} \widetilde{\sigma}_3(x,t) &= g^{-1}(x,t)\sigma_3 g(x,t) \equiv S(x,t) \; , \\ \widetilde{\sigma}_\pm(x,t) &= g^{-1}(x,t)\sigma_\pm g(x,t) \; , \end{aligned} \tag{8.40}$$

and the covariant derivatives:

$$\nabla_x \cdot \equiv g^{-1}(x,t)\left(\frac{\partial}{\partial x}\cdot\right) g(x,t) = \frac{\partial}{\partial x} \cdot - \left[g^{-1} g_x, \cdot\right] \; , \tag{8.41}$$

$$\nabla_t \cdot \equiv g^{-1}(x,t)\left(\frac{\partial}{\partial t}\cdot\right) g(x,t) = \frac{\partial}{\partial t} \cdot - \left[g^{-1} g_t, \cdot\right] \; . \tag{8.42}$$

Obviously the covariant derivatives and $\widetilde{\sigma}_\alpha$ satisfy:

$$\nabla_x \widetilde{\sigma}_\alpha = 0, \qquad \nabla_t \widetilde{\sigma}_\alpha = 0, \qquad \alpha = 3, \pm \; . \tag{8.43}$$

After these preliminaries the first step is to split $\widetilde{V}_k(x,t)$ into "diagonal" and "off-diagonal" parts:

$$\widetilde{V}_k(x,t) = \widetilde{V}_k^{\mathrm{d}}(x,t) + \widetilde{V}_k^{\mathrm{f}}(x,t) \; , \tag{8.44}$$

where

$$\widetilde{V}_k^{\rm d}(x,t) = (\mathbb{1} - \pi_S)\widetilde{V}_k(x,t) \in \widetilde{\mathfrak{g}}^{(0)}, \qquad \widetilde{V}_k^{\rm f}(x,t) = \pi_S \widetilde{V}_k(x,t) \in \widetilde{\mathfrak{g}}^{(1)} \ . \quad (8.45)$$

Using the invariance properties of the trace, we derive the following compact expression for $\widetilde{V}_k^{\rm d}(x,t)$:

$$\widetilde{V}_k^{\rm d}(x,t) = S(x,t)\langle \widetilde{V}_k(x,t) S(x,t)\rangle \ , \quad (8.46)$$

where we remind that $\langle X, Y\rangle = \frac{1}{2}\mathrm{tr}\,(XY)$. Now the recurrent relations (8.38) can be written as follows:

$$\widetilde{V}_0(x,t) \equiv \widetilde{V}_0^{\rm d}(x,t) = f_0 S(x,t) \ , \quad (8.47a)$$

$$i\nabla_x \widetilde{V}_k^{\rm d}(x,t) + \left[g^{-1}g_x, \widetilde{V}_k^{\rm f}(x,t)\right] = 0 \ , \quad (8.47b)$$

$$i\nabla_x \widetilde{V}_k^{\rm f}(x,t) + \left[g^{-1}g_x, \widetilde{V}_k^{\rm d}(x,t)\right] = [S(x,t), \widetilde{V}_{k+1}^{\rm f}(x,t)] \ . \quad (8.47c)$$

In what follows, we need to express $g^{-1}g_x$ and $g^{-1}g_t$ in terms of the dynamical variables $S(x,t)$ and its derivatives. This is achieved using the relations:

$$S_x = [S, g^{-1}g_x] \in \widetilde{\mathfrak{g}}^{(1)}, \qquad S_t = [S, g^{-1}g_t] \in \widetilde{\mathfrak{g}}^{(1)} \ . \quad (8.48)$$

Since $\mathrm{ad}_S\cdot$ is invertible on $\widetilde{\mathfrak{g}}^{(1)}$ we find:

$$i\widetilde{q}(x) \equiv g^{-1}g_x = \mathrm{ad}_S^{-1}\frac{\partial S}{\partial x} = \frac{1}{4}\left[S(x,t), \frac{\partial S}{\partial x}\right], \qquad g^{-1}g_t = \frac{1}{4}\left[S(x,t), \frac{\partial S}{\partial t}\right] \ . \quad (8.49)$$

Next, we integrate (8.47b) using the properties of the "moving frame" (8.40) with the result:

$$\widetilde{V}_k^{\rm d}(x,t) = iS(x,t)\int_{\pm\infty}^{x} dy\, \langle S_y, \widetilde{V}_k^{\rm f}(y,t)\rangle + S(x,t)\lim_{y\to\pm\infty}\langle \widetilde{V}_k^{\rm d}(y,t), S(y)\rangle \ . \quad (8.50)$$

Inserting this result into the recurrent relation (8.47c), we can rewrite them in the following compact form:

$$\widetilde{V}_{k+1}^{\rm f}(x,t,\lambda) = \widetilde{\Lambda}_{\pm}\widetilde{V}_k^{\rm f}(x,t,\lambda) + \frac{i}{4}f_k[S, S_x] \ , \quad (8.51)$$

where the recursion operator $\widetilde{\Lambda}_{\pm}$ is defined by:

$$\widetilde{\Lambda}_{\pm}\widetilde{X} = \frac{i}{4}\left(\left[S(x,t), \frac{d\widetilde{X}}{dx} + S_x\int_{\pm\infty}^{x} dy \langle S_y \widetilde{X}(y,t)\rangle\right]\right) \ . \quad (8.52)$$

and $f_k = \lim_{y\to\pm\infty}\langle \widetilde{V}_k^{\rm d}(y,t)S(y)\rangle$ is an integration constant. The solution of this recursion is given by:

$$\widetilde{V}^{\mathrm{f}}_{k+1} = \frac{i}{4} \sum_{p=0}^{k} f_p \widetilde{\Lambda}_{\pm}^{k-p} [S, S_x] \; . \tag{8.53}$$

It remains to insert into (8.38c) the expression for $V_{N-1}(x,t)$ following from (8.53) to get the HF-type NLEEs in terms of $S(x,t)$ and the recursion operator $\widetilde{\Lambda}_{\pm}$:

$$i \frac{\partial S}{\partial t} + i \frac{\partial}{\partial x} \widetilde{f}^{(1)}(\widetilde{\Lambda}_{\pm})[S, S_x] = 0 \; , \tag{8.54}$$

where $\widetilde{f}^{(1)}(\lambda) = \sum_{p=0}^{N-2} f_p \lambda^{N-p-2}$ determines the dispersion law of the NLEE. In this form, the NLEE takes a form of a conservation law.

Another equivalent form of the generic NLEE of HF type can be obtained by applying the operator $\mathrm{ad}_S \cdot$ to both sides of (8.38c) and then apply the splitting procedure into "diagonal" and "off-diagonal" parts to $\frac{\partial V_{N-1}}{\partial x}$, which we used in solving the recurrent relations. As a result, we obtain:

$$i \frac{\partial S}{\partial t} + i \left[S(x,t), f(\widetilde{\Lambda}_{\pm})[S, S_x] \right] = 0 \; , \tag{8.55}$$

where $f(\lambda) = \sum_{p=0}^{N-1} f_p \lambda^{N-p-1}$ describes the dispersion law of the NLEE.

In order to derive examples of HF-type generic NLEE, we need to evaluate the first few powers of $\widetilde{\Lambda}_{\pm}$ on $[S, S_x]$. It is easy to see that:

$$\begin{aligned} \widetilde{\Lambda}_{\pm}[S, S_x] &= \frac{i}{4} \left[S, [S, S_{xx}] \right] \\ &= \frac{i}{4} \left(S_{xx} - S(x,t) \langle S_{xx} S(x,t) \rangle \right) \; . \end{aligned} \tag{8.56}$$

In deriving the second line of (8.56), we used the fact that $1/4[S,[S,S_{xx}]] \equiv (S_{xx})^{\mathrm{f}} = S_{xx} - (S_{xx})^{\mathrm{d}}$, and (8.46) for $(S_{xx})^{\mathrm{d}}$. Applying $\widetilde{\Lambda}_{\pm}$ to the second line of (8.56), we get:

$$\widetilde{\Lambda}_{\pm}^2 [S, S_x] = -\frac{1}{16} \left\{ [S, S_{xxx}] - S_x \langle S_{xx}, S \rangle + 2[S, S_x] \langle S_x, S_x \rangle \right\} \; . \tag{8.57}$$

Therefore, if we choose the dispersion law $f(\lambda) = f_0 \lambda^2 + f_1 \lambda$, we get the following generic NLEE:

$$\begin{aligned} & i \frac{\partial S}{\partial t} - f_1 \left[S(x,t), \frac{\partial^2 S}{\partial x^2} \right] \\ & - \frac{i f_0}{8} \left\{ \left[S, \frac{\partial^3 S}{\partial x^3} \right] - \frac{\partial S}{\partial x} \langle S_{xx}, S(x,t) \rangle + 2 \left[S(x,t), \frac{\partial S}{\partial x} \right] \langle S_x, S_x \rangle \right\} = 0. \end{aligned} \tag{8.58}$$

Choosing $f_1 = c_2$, $f_0 = 0$, we get that the complexified HF equation is gauge equivalent to the GNLS system (6.18), (6.19).

The NLEE (8.58) with $f_1 = 0$, $f_0 = -8$ gives a complexification of one of the higher HF-type NLEE; it is gauge equivalent to the GmKdV system (6.21).

The generic NLEE with $f_0 f_1 \neq 0$ is gauge equivalent to the mixed GNLS-G mKdV (6.23).

We have already outlined several possible reductions, which can be applied to the Zakharov–Shabat system L. Using them, one is able to study also the important reductions of the above-mentioned systems such as NLS equation, mKdV equation, s-G equation etc. Therefore, one of our tasks will be to find out what would be the corresponding reductions on $\widetilde{L}$ (i.e. on $S(x,t)$) and how one can derive the properties of the corresponding HF-type NLEEs.

The compatibility condition of $\widetilde{L}$ and $\widetilde{M}$ means that these linear systems have common set of eigenfunctions, or a common fundamental solution. As such one can use the Jost solutions $\tilde{\psi}(x,t,\lambda)$ and $\tilde{\phi}(x,t,\lambda)$

$$\tilde{L}\tilde{\psi}(x,t,\lambda) \equiv i\frac{d\widetilde{\psi}}{dx} - \lambda S(x,t)\tilde{\psi}(x,t,\lambda) = 0, \qquad \tilde{L}\tilde{\phi}(x,t,\lambda) = 0 \text{ ,} \tag{8.59a}$$

$$\tilde{M}\tilde{\psi}(x,t,\lambda) \equiv i\frac{d\widetilde{\psi}}{dt} + \widetilde{V}(x,t,\lambda)\tilde{\psi}(x,t,\lambda) = \tilde{\psi}(x,t,\lambda)C(\lambda) \text{ ,} \tag{8.59b}$$

$$\tilde{M}\tilde{\phi}(x,t,\lambda) \equiv i\frac{d\widetilde{\phi}}{dt} + \widetilde{V}(x,t,\lambda)\tilde{\phi}(x,t,\lambda) = \tilde{\phi}(x,t,\lambda)C(\lambda), \tag{8.59c}$$

$$\widetilde{V}(x,t,\lambda) = \sum_{k=1}^{N-1} \lambda^{N-k}\widetilde{V}_k(x,t) + f_0\lambda^N S(x,t) \text{ ,} \tag{8.59d}$$

defined by:

$$\lim_{x\to\infty} \tilde{\psi}(x,t,\lambda)e^{i\lambda\sigma_3 x} = \mathbb{1}, \qquad \lim_{x\to-\infty} \tilde{\phi}(x,t,\lambda)e^{i\lambda\sigma_3 x} = \mathbb{1} \text{ ,} \tag{8.60}$$

The asymptotic behavior of the Jost solutions for $x \to \pm\infty$ is determined by (8.2b). Using the freedom to choose $C(\lambda)$, we can define it by:

$$C(\lambda) = \lim_{x\to\pm\infty} \tilde{V}(x,t,\lambda) = f(\lambda)\sigma_3 \text{ ;} \tag{8.61}$$

then the definitions (8.60) are valid for any t.

The next step is to introduce the scattering matrix $\widetilde{T}(\lambda,t)$:

$$\tilde{T}(t,\lambda) = \tilde{\psi}^{-1}(x,t,\lambda)\tilde{\phi}(x,t,\lambda) = \begin{pmatrix} \tilde{a}^+(\lambda) & -\tilde{b}^-(t,\lambda) \\ \tilde{b}^+(t,\lambda) & \tilde{a}^-(\lambda) \end{pmatrix} \text{ .} \tag{8.62}$$

and determine its t-dependence. To do this, we evaluate the limit $x \to \infty$ in (8.59c) and making use of (8.60), (8.61), and (8.62) we get:

$$i\frac{d\widetilde{T}}{dt} + C(\lambda)\widetilde{T}(\lambda,t) = \widetilde{T}(\lambda,t)C(\lambda) \text{ ,} \tag{8.63a}$$

or

$$i\frac{d\widetilde{T}}{dt} + f(\lambda)\left[\sigma_3, \widetilde{T}(\lambda,t)\right] = 0 \text{ .} \tag{8.63b}$$

In particular, from (8.59c), there follows that $\widetilde{V}(x,t,\lambda=0)=0$ and, therefore, $f(0)=0$ (see (8.61)) and

$$i\left.\frac{d\widetilde{T}}{dt}\right|_{\lambda=0}=0\,. \tag{8.64}$$

Thus the problem of solving the nonlinear Cauchy problem reduces to a sequence of three linear problems, each of which have unique solution. First, given the initial condition $S^{(0)}(x)=S(x,t=0)$ of the NLEE, one solves the direct scattering problem for $\widetilde{L}$ and determines $\widetilde{T}(\lambda,0)$. Next, one solves the evolution equation (8.63) and finds $\widetilde{T}(\lambda,t)$. The third step consists in solving the ISP for $\widetilde{L}$, thus reconstructing the solution $S(x,t)$.

8.3 Direct and Inverse Scattering Problem for $\widetilde{L}$

Before going into the discussion of the scattering problem for $\tilde{L}$, we show the close relation between the solutions of the inverse scattering problems for the gauge-equivalent systems L and $\tilde{L}$. From now on, most of the notations concerning the system $\tilde{L}$, such like the scattering matrix and its matrix elements, "squared" solutions etc. will be denoted by the same letters as for L but with additional "tilde".

We already introduced the Jost solutions $\tilde{\psi}(x,t,\lambda)$ and $\tilde{\phi}(x,t,\lambda)$, as well as the scattering matrix $\tilde{T}(t,\lambda)$ of $\tilde{L}$. Their properties depend very much on the class of functions to which the potential $S(x,t)$ belongs. For the sake of simplicity, we determine this class by a condition analogous to **C1** on p. 71:

Condition $\widetilde{\text{C}}$1. The matrix-valued function $S(x,t)-\sigma_3$ is complex-valued Schwartz-type function of x for all t.

An important consequence of (8.2b) and (8.34c) is:

$$\lim_{x\to\pm\infty}[g(x,t),\sigma_3]=0\,, \tag{8.65}$$

or, since $\lim_{x\to-\infty} g(x,t)=\widehat{T}(0)$, we also find:

$$\left[\widehat{T}(0),\sigma_3\right]=0, \qquad \text{i.e.,} \qquad T(0)=\begin{pmatrix} a_0^+ & 0 \\ 0 & a_0^- \end{pmatrix}, \qquad a_0^\pm=a^\pm(0)\,, \tag{8.66}$$

and $b^\pm(0)=0$.

In order to avoid technicalities in analyzing the discrete spectrum of $\widetilde{L}$ and possible singularities at $\lambda=0$, we adopt the following two conditions, which may be considered as implicit constraints on the potential $S(x)$.

Condition $\widetilde{\text{C}}$2. The functions $\widetilde{a}^\pm(\lambda)$ have at most finite number of simple zeroes at $\lambda=\lambda_j^\pm\in\mathbb{C}_\pm$.

Condition $\widetilde{\text{C}}$3. The functions $\widetilde{b}^\pm(\lambda)$ are smooth functions of λ in the neighborhood of $\lambda\simeq 0$, more specifically:

$$\left.\frac{d^k b^+(t,\lambda)}{d\lambda^k}\right|_{\lambda=0} = \left.\frac{d^k b^-(t,\lambda)}{d\lambda^k}\right|_{\lambda=0} = 0 \; , \tag{8.67}$$

for $k = 1, \ldots, N_3$. For most of our purposes, we shall need only $N_3 = 2$.

The Jost solutions of $\tilde{L}$ are related to those of L by:

$$\tilde{\psi}(x,t,\lambda) = g^{-1}(x,t)\psi(x,t,\lambda) \; , \tag{8.68a}$$

$$\tilde{\phi}(x,t,\lambda) = g^{-1}(x,t)\phi(x,t,\lambda)\widehat{T}_0 \; , \tag{8.68b}$$

where $T_0 = T(t, \lambda = 0)$. To show this, it is enough to use the definitions of the Jost solutions and the scattering matrix (8.62), keeping in mind that (8.68) taken for $\lambda = 0$ gives:

$$\lim_{x\to-\infty} g(x,t) = \widehat{T}_0 \; . \tag{8.69}$$

Then the relation between $\tilde{T}(t,\lambda)$ and $T(t,\lambda)$ is

$$\tilde{T}(t,\lambda) = T(t,\lambda)\widehat{T}(0) \; , \tag{8.70}$$

where we also took into account that $a_0^{\pm}$ (and consequently, $T(0)$) are t-independent.

With the condition $\widetilde{\mathrm{C}}2$ (8.70) in components reads:

$$\tilde{a}^{\pm}(\lambda) = \frac{a^{\pm}(\lambda)}{a_0^{\pm}}, \qquad \tilde{b}^{\pm}(t,\lambda) = \frac{b^{\pm}(t,\lambda)}{a_0^{\pm}} \; , \tag{8.71}$$

The approach developed up to now had its deep roots in the possibility to introduce the fundamental analytic solutions of L. In order that the gauge transformation preserves these properties, we need condition $\widetilde{\mathrm{C}}3$. Writing the Jost solutions into pairs of columns:

$$\widetilde{\psi}(x,t,\lambda) = |\widetilde{\psi}^-(x,t,\lambda), \widetilde{\psi}^+(x,t,\lambda)|, \qquad \widetilde{\phi}(x,t,\lambda) = |\widetilde{\phi}^+(x,t,\lambda), \widetilde{\phi}^-(x,t,\lambda)| \; , \tag{8.72}$$

one is able to prove that each of the columns is analytic in the corresponding half-plane $\lambda \in \mathbb{C}_{\pm}$. The simplest way to demonstrate this fact is to establish the relations between the Jost solutions of L and $\widetilde{L}$:

$$\widetilde{\psi}^{\pm}(x,t,\lambda) = g^{-1}(x,t)\psi^{\pm}(x,t,\lambda), \qquad \widetilde{\phi}^{\pm}(x,t,\lambda) = g^{-1}(x,t)\phi^{\pm}(x,t,\lambda)/a_0^{\pm} \; , \tag{8.73}$$

Thus, the FAS of both systems are related through:

$$\widetilde{\chi}^+(x,t,\lambda) \equiv |\widetilde{\phi}^+(x,t,\lambda), \widetilde{\psi}^+(x,t,\lambda)| = g^{-1}(x,t)\chi^+(x,t,\lambda)\widehat{A}^+(0), \tag{8.74a}$$

$$\widetilde{\chi}^-(x,t,\lambda) \equiv |\widetilde{\psi}^-(x,t,\lambda), \widetilde{\phi}^-(x,t,\lambda)| = g^{-1}(x,t)\chi^-(x,t,\lambda)\widehat{A}^-(0), \tag{8.74b}$$

$$A^+(0) = \begin{pmatrix} a_0^+ & 0 \\ 0 & 1 \end{pmatrix}, \qquad A^-(0) = \begin{pmatrix} 1 & 0 \\ 0 & a_0^- \end{pmatrix}.$$

Thus, we find that $\widetilde{\chi}^{\pm}(x,t,\lambda)$ satisfy the RHP

$$\widetilde{\chi}^{+}(x,t,\lambda) = \widetilde{\chi}^{-}(x,t,\lambda)\widetilde{G}_0(t,\lambda), \qquad \widetilde{G}_0(t,\lambda) = A^{-}(0)G_0(t,\lambda)\widehat{A}^{+}(0) . \tag{8.75}$$

In order to impose correctly the normalization condition on this RHP, we rewrite it in terms of $\widetilde{\eta}^{\pm}(x,t,\lambda)$, defined as:

$$\widetilde{\eta}^{\pm}(x,t,\lambda) = \widetilde{\chi}^{\pm}(x,t,\lambda)e^{i\lambda\sigma_3 x} \tag{8.76}$$

and we obtain

$$\widetilde{\eta}^{+}(x,t,\lambda) = \widetilde{\eta}^{-}(x,t,\lambda)\widetilde{G}(x,t,\lambda) , \tag{8.77}$$

$$\widetilde{G}(x,t,\lambda) = e^{-i\lambda\sigma_3 x}\widetilde{G}_0(t,\lambda)e^{i\lambda\sigma_3 x} ,$$

$$\lim_{\lambda\to 0}\widetilde{\eta}^{\pm}(x,t,\lambda) = \mathbb{1} . \tag{8.78}$$

Remark 8.1. We shall call the FAS $\widetilde{\eta}^{\pm}(x,t,\lambda)$ regular, if the functions

$$\widetilde{a}^{\pm}(\lambda) = \det\widetilde{\eta}^{\pm}(x,t,\lambda) , \tag{8.79}$$

have no zeroes for $\lambda \in \mathbb{C}_{\pm}$.

Remark 8.2. Note that one of the effects of the gauge transformation consists of the shift of the normalization point of the RHP. Here, we have it at $\lambda = 0$ while for the Zakharov–Shabat system it was at $\lambda = \infty$.

The method of RHP is very effective both for solving the ISP and for constructing the soliton solutions. Therefore, our next step will be to show that if $\widetilde{\eta}^{\pm}(x,t,\lambda)$ is a solution of the RHP (8.77) with sewing function $\widetilde{G}(x,t,\lambda)$, then $\widetilde{\eta}^{\pm}(x,t,\lambda)$ are FAS of $\widetilde{L}$. To this end, we consider the functions:

$$\widetilde{h}^{\pm}(x,t,\lambda) = i\frac{\partial\widetilde{\eta}^{\pm}}{\partial x}\widehat{\widetilde{\eta}}^{\pm}(x,t,\lambda) + \lambda\widetilde{\eta}^{\pm}(x,t,\lambda)\sigma_3\widehat{\widetilde{\eta}}^{\pm}(x,t,\lambda) . \tag{8.80}$$

Assuming that $\widetilde{\eta}^{\pm}(x,t,\lambda)$ are regular solutions, then $\widetilde{h}^{\pm}(x,t,\lambda)$ are analytic for $\lambda \in \mathbb{C}_{\pm}$. Next, we use the (8.77) to show that:

$$\begin{aligned}
\widetilde{h}^{+}(x,t,\lambda) &= i\frac{\partial(\widetilde{\eta}^{-}\widetilde{G})}{\partial x}\widehat{\widetilde{G}}\widehat{\widetilde{\eta}}^{-}(x,t,\lambda) + \lambda\widetilde{\eta}^{-}(x,t,\lambda)\widetilde{G}\sigma_3\widehat{\widetilde{G}}\widehat{\widetilde{\eta}}^{-}(x,t,\lambda) \\
&= i\frac{\partial\widetilde{\eta}^{-}}{\partial x}\widehat{\widetilde{\eta}}^{-}(x,t,\lambda) + \lambda\widetilde{\eta}^{-}(x,t,\lambda)\sigma_3\widehat{\widetilde{\eta}}^{-}(x,t,\lambda) \\
&= \widetilde{h}^{-}(x,t,\lambda) .
\end{aligned} \tag{8.81}$$

Therefore, $\widetilde{h}^{+}(x,t,\lambda)$ and $\widetilde{h}^{-}(x,t,\lambda)$ are the two “halfs” of one functions which is analytic in the whole complex λ-plane. In order to find out more about it, we evaluate its limit for $\lambda \to \infty$. To do this, we use the asymptotic expansion of $\chi^{\pm}(x,t,\lambda)$ and (8.74a), (8.75) to get:

$$\begin{aligned}\widetilde{\eta}^{\pm}(x,t,\lambda) &= g^{-1}(x,t)\chi^{\pm}(x,t,\lambda)e^{i\lambda\sigma_3 x}\widehat{A}^{\pm}(0) \\ &= g^{-1}(x,t)\left(\mathbb{1} + \frac{1}{\lambda}\chi^{(1)}(x,t) + \mathcal{O}(\lambda^{-2})\right)\widehat{A}^{\pm}(0)\,,\end{aligned} \tag{8.82}$$

where we recall that both $g(x,t)$ and $\chi^{(1)}(x,t)$ are expressed by the potential $q(x,t)$ of the Zakharov–Shabat system as follows:

$$\chi^{(1)}(x,t) = \frac{1}{4}[\sigma_3, q(x,t)], \qquad i\frac{\partial g}{\partial x} + q(x,t)g(x,t) = 0\,, \tag{8.83}$$

and, therefore,

$$\begin{aligned}&\lim_{\lambda\to\infty}\widetilde{h}^{\pm}(x,t,\lambda) = \lim_{\lambda\to\infty}\left\{i\frac{\partial g^{-1}}{\partial x}g(x,t)\right. \\ &\left.+\,\lambda g^{-1}(x,t)(\mathbb{1}+\lambda^{-1}\chi^{(1)}(x,t)+\cdots)\sigma_3(\mathbb{1}-\lambda^{-1}\chi^{(1)}(x,t)+\cdots)g(x,t)\right\} \\ &= i\frac{\partial g^{-1}}{\partial x}g(x,t) + g^{-1}(x,t)\left(\lambda\sigma_3 + [\chi^{(1)}(x,t),\sigma_3]\right)g(x,t) \\ &= -ig^{-1}(x,t)\frac{\partial g}{\partial x} + \lambda S(x,t) - g^{-1}(x,t)q(x,t)g(x,t) \\ &= \lambda S(x,t)\,.\end{aligned} \tag{8.84}$$

Thus, we have proved that the function $\widetilde{h}^{\pm}(x,t,\lambda) - \lambda S(x,t)$ is analytic for all λ and tends to 0 for $\lambda \to \infty$. Applying the Liouville's theorem, we conclude that this function is identically equal to 0, i.e.

$$\widetilde{h}^{\pm}(x,t,\lambda) - \lambda S(x,t) = 0\,, \tag{8.85}$$

i.e.

$$i\frac{\partial\widetilde{\eta}^{\pm}}{\partial x} - \lambda S(x,t)\widetilde{\eta}^{\pm}(x,t,\lambda) + \lambda\widetilde{\eta}^{\pm}(x,t,\lambda)\sigma_3 = 0\,. \tag{8.86}$$

Combining this with (8.76), we deduce that $\widetilde{\chi}^{\pm}(x,t,\lambda)$ are FAS of the Lax operator $\widetilde{L}$.

Let us assume now that we know the regular FAS $\widetilde{\eta}^{\pm}(x,t,\lambda)$ satisfying the linear system (8.86). Taking into account the normalization condition (8.78), we find the following Taylor series expansion for it:

$$\widetilde{\eta}^{\pm}(x,t,\lambda) = \eta_0^{\pm} + \sum_{p=1}^{\infty}\lambda^p\widetilde{\eta}_p^{\pm}(x,t)\,, \tag{8.87}$$

Inserting (8.87) into (8.86) and taking the coefficient linear in λ we get:

$$i\frac{\partial\widetilde{\eta}_1^{\pm}(x,t)}{\partial x} - S(x,t)\eta_0^{\pm} + \eta_0^{\pm}\sigma_3 = 0\,, \tag{8.88a}$$

i.e.

$$S(x,t) = \eta_0^{\pm}\sigma_3\hat{\eta}_0^{\pm} + i\frac{\partial\widetilde{\eta}_1^{\pm}(x,t)}{\partial x}\hat{\eta}_0^{\pm} \ . \qquad (8.88b)$$

This formula allows us to recover the potential $S(x,t)$ of $\widetilde{L}$ from the FAS of the RHP (8.77).

The FAS of $\widetilde{L}$ can be used to construct the kernel of the resolvent of $\widetilde{L}$. Skipping the details which are the same as for the ZS system L, we just give the result:

$$\widetilde{R}^{+}(x,y,\lambda) = \frac{1}{i}\widetilde{\chi}^{+}(x,\lambda)\begin{pmatrix} -\theta(y-x), & 0 \\ 0, & \theta(x-y) \end{pmatrix}\hat{\widetilde{\chi}}^{+}(y,\lambda)\ , \qquad (8.89a)$$

$$\widetilde{R}^{-}(x,y,\lambda) = \frac{1}{i}\widetilde{\chi}^{-}(x,\lambda)\begin{pmatrix} \theta(x-y), & 0 \\ 0, & -\theta(y-x) \end{pmatrix}\hat{\widetilde{\chi}}^{-}(y,\lambda)\ , \qquad (8.89b)$$

where $\theta(x-y)$ is the step-function. The kernels $\widetilde{R}^{\pm}(x,y,\lambda)$ define the integral operator $\widetilde{\boldsymbol{R}}$ which acts on two-component vector functions $\widetilde{\boldsymbol{f}}(x)$ as follows:

$$\widetilde{\boldsymbol{R}}_{\lambda}\widetilde{\boldsymbol{f}} = \int_{-\infty}^{\infty} dy\,\widetilde{R}^{\pm}(x,y,\lambda)\widetilde{\boldsymbol{f}}(y), \qquad \text{for } \lambda\in\mathbb{C}_{\pm}\ . \qquad (8.90)$$

Theorem 8.3. *Let $S(x)$ satisfy conditions $\widetilde{\mathrm{C}}.1$ and $\widetilde{\mathrm{C}}.2$ and let $\lambda_j^{\pm}$ be the simple zeroes of $\widetilde{a}^{\pm}(\lambda)$. Then*

1. *$\widetilde{R}^{\pm}(x,y,\lambda)$ is an analytic function of λ for $\lambda\in\mathbb{C}_{\pm}$ having pole singularities at $\lambda_j^{\pm}\in\mathbb{C}_{\pm}$;*
2. *$\widetilde{R}^{\pm}(x,y,\lambda)$ is a kernel of a bounded integral operator for $\mathrm{Im}\,\lambda\neq 0$;*
3. *$\widetilde{R}(x,y,\lambda)$ is uniformly bounded function for $\lambda\in\mathbb{R}$ and is a kernel of an unbounded integral operator;*
4. *$\widetilde{R}^{\pm}(x,y,\lambda)$ satisfy the equation:*

$$\widetilde{L}(\lambda)\widetilde{R}^{\pm}(x,y,\lambda) = \mathbb{1}\delta(x-y)\ . \qquad (8.91)$$

Proof (Idea of the proof).

1. is obvious from the fact that $\widetilde{\chi}^{\pm}(x,\lambda)$ are the FAS of $\widetilde{L}(\lambda)$;
2. Assume that $\mathrm{Im}\,\lambda>0$ and consider the asymptotic behavior of $\widetilde{R}^{+}(x,y,\lambda)$ for $x,y\to\infty$. From (8.74), we find that

$$\widetilde{R}^{+}(x,y,\lambda) = g^{-1}(x)e^{-\lambda\sigma_3 x}T^{-}(\lambda)\Theta^{+}(x-y)\hat{T}^{-}(\lambda)e^{i\lambda\sigma_3 y}, \qquad (8.92)$$
$$\Theta^{+}(x-y) = \mathrm{diag}\,(-\theta(y-x),\theta(x-y))\ .$$

Due to the fact that

$$\widetilde{\chi}^{+}(x,\lambda) = \widetilde{\psi}(x,\lambda)\widetilde{T}^{-}(\lambda) \underset{x\to\infty}{\simeq} e^{-i\lambda\sigma_3 x}\widetilde{T}^{-}(\lambda)\ , \qquad (8.93)$$

$$\widetilde{T}^{-}(\lambda) = \begin{pmatrix} \widetilde{a}^{+}(\lambda) & 0 \\ \widetilde{b}^{+}(\lambda) & 1 \end{pmatrix} , \qquad (8.94)$$

with the lower triangular matrix $\widetilde{T}^{-}(\lambda)$ (compare with (3.40)), and due to the choice of $\Theta^{+}(x-y)$, one can check that for $\lambda \in \mathbb{C}_{+}$ the right-hand side of (8.92) falls off exponentially for $x \to \infty$ and arbitrary choice of y. All other possibilities are treated analogously.

3. For $\lambda \in \mathbb{R}$ the arguments of (2) cannot be applied because the exponentials in the right-hand side of (8.92) $\operatorname{Im}\lambda = 0$ only oscillate. Thus, we conclude that the kernel $\widetilde{R}^{\pm}(x, y, \lambda)$ for $\lambda \in \mathbb{R}$ is only a bounded function, and the corresponding integral operator $\widetilde{R}_{\lambda}$ is unbounded. This is consistent with the fact that the continuous spectrum of $\widetilde{L}$ fills up the real λ-axis.
4. The proof of (8.91) follows from the fact that $\widetilde{L}(\lambda)\widetilde{\chi}^{+}(x, \lambda) = 0$ and

$$\frac{d\Theta^{\pm}(x-y)}{dx} = \pm \mathbb{1}\delta(x-y) . \qquad (8.95)$$

Corollary 8.4. *Let $\widetilde{\eta}_0^{\pm}(x, \lambda)$ be regular solutions of the RHP (8.78) and let $S^{(0)}(x)$ be the corresponding potential of the operator $\widetilde{L}_0$. Then, the operator $\widetilde{L}_0$ has no discrete eigenvalues.*

Proof. If $\widetilde{\eta}_0^{\pm}(x, \lambda)$ are regular solutions then $\det \widetilde{\eta}_0^{\pm}(x, \lambda) = \det \widetilde{\chi}_0^{\pm}(x, \lambda) = \widetilde{a}^{\pm}(\lambda)$ have no zeroes and therefore the kernels of the resolvent $\widetilde{R}^{\pm}(x, y, \lambda)$ have no poles for $\lambda \in \mathbb{C}_{+}$.

The explicit form of the resolvent allows to derive the completeness relation of the Jost solutions. It can be obtained by applying the gauge transformation to (3.111). We get:

$$\begin{aligned} \delta(x-y)\mathbb{1} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \left(\frac{\widetilde{\phi}^{+}(x, \lambda)\hat{\widetilde{\psi}}^{+}(y, \lambda)}{\widetilde{a}^{+}(\lambda)} + \frac{\widetilde{\phi}^{-}(x, \lambda)\hat{\widetilde{\psi}}^{-}(y, \lambda)}{\widetilde{a}^{-}(\lambda)} \right) \\ &\quad + i \sum_{k=1}^{N} \left(\frac{\widetilde{\phi}_k^{+}(x)\hat{\widetilde{\psi}}_k^{+}(y)}{\dot{\widetilde{a}}_k^{+}} - \frac{\widetilde{\phi}_k^{-}(x)\hat{\widetilde{\psi}}_k^{-}(y)}{\dot{\widetilde{a}}_k^{-}} \right) . \end{aligned} \qquad (8.96)$$

The last remark is that from the above results it follows that the gauge transformations are isospectral.

8.4 The Dressing Method and Soliton Solutions

The contour integration method applied in Chap. 4 to the FAS of the Zakharov–Shabat system can be extended also to the gauge-equivalent system $\widetilde{L}$. Indeed, the gauge transformation is λ-independent. This allows us to apply it directly to the final result, namely, to the spectral expansions of $\eta^+(x,t,\lambda)$ (4.72) and $\eta^-(x,t,\lambda)$ (4.73). Making use of (8.74) and (8.71), after some calculations, we get for $\eta^+(x,\lambda)$ with $\lambda \in \mathbb{C}_+$:

$$\widetilde{\eta}^+(x,\lambda) = g^{-1}(x) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\mu}{\mu-\lambda}\frac{\widetilde{\eta}^-(x,\mu)}{\widetilde{a}^-(\mu)}\widetilde{K}(x,\mu) - \sum_{k=1}^{N}\frac{\widetilde{\eta}_k^-(x)}{\lambda_k^- - \lambda}\widetilde{K}_k^-(x)\,, \tag{8.97a}$$

and for $\eta^-(x,\lambda)$ with $\lambda \in \mathbb{C}_-$:

$$\widetilde{\eta}^-(x,\lambda) = g^{-1}(x) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\mu}{\mu-\lambda}\frac{\widetilde{\eta}^+(x,\mu)}{\widetilde{a}^+(\mu)}\widetilde{K}(x,\mu) - \sum_{k=1}^{N}\frac{\widetilde{\eta}_k^+(x)}{\lambda_k^+ - \lambda}\widetilde{K}_k^+(x)\,, \tag{8.97b}$$

where we have used the notations:

$$\widetilde{K}(x,\lambda) = \begin{pmatrix} 0 & \widetilde{b}^-(\lambda)e^{-2i\lambda x} \\ \widetilde{b}^+(\lambda)e^{2i\lambda x} & 0 \end{pmatrix}, \tag{8.98a}$$

$$\widetilde{\eta}_k^+(x) = \widetilde{\xi}_k^+(x)(\widetilde{b}_k^+ e^{2i\lambda_k^+ x}, 1), \qquad \widetilde{\eta}_k^-(x) = \widetilde{\xi}_k^-(x)(1, -\widetilde{b}_k^- e^{-2i\lambda_k^- x})\,, \tag{8.98b}$$

$$\widetilde{\eta}_k^\pm(x) = \widetilde{\eta}^\pm(x,\lambda_k^\pm), \qquad \widetilde{C}_k^\pm = \frac{\widetilde{b}_k^\pm}{\dot{\widetilde{a}}_k^\pm}, \qquad \widetilde{M}_k^\pm = \frac{1}{\widetilde{b}_k^\pm \dot{\widetilde{a}}_k^\pm}\,, \tag{8.98c}$$

$$\widetilde{K}_k^+(x) = \begin{pmatrix} 0 & \widetilde{M}_k^+ e^{-2i\lambda_k^+ x} \\ \widetilde{C}_k^+ e^{2i\lambda_k^+ x} & 0 \end{pmatrix}, \qquad \widetilde{K}_k^-(x) = \begin{pmatrix} 0 & \widetilde{C}_k^- e^{-2i\lambda_k^- x} \\ \widetilde{M}_k^- e^{2i\lambda_k^- x} & 0 \end{pmatrix}. \tag{8.98d}$$

We also used the simple fact that the zeroes of the functions $\widetilde{a}^\pm(\lambda)$ due to (8.71) coincide with $\lambda_k^\pm$, the zeroes of $a^\pm(\lambda)$.

The system of (8.97) cannot be treated directly as system of singular linear integral equations allowing one to solve the RH problem (8.77) and recover $\widetilde{\eta}^\pm(x,\lambda)$ in their whole regions of analyticity. The reason for this is that they contain $g^{-1}(x)$, which is not explicitly given. However, we also know that the solutions $\widetilde{\eta}^\pm(x,\lambda)$ are such that for $\lambda = 0$ they become equal to $\mathbb{1}$. Therefore, we can put $\lambda = 0$ in (8.97), which will provide us with the spectral decompositions of $g^{-1}(x)$:

$$g^{-1}(x) = \mathbb{1} - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\mu}{\mu}\frac{\widetilde{\eta}^-(x,\mu)}{\widetilde{a}^-(\mu)}\widetilde{K}(x,\mu) + \sum_{k=1}^{N}\frac{\widetilde{\eta}_k^-(x)}{\lambda_k^-}\widetilde{K}_k^-(x)\,, \tag{8.99a}$$

$$g^{-1}(x) = \mathbb{1} - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\mu}{\mu}\frac{\widetilde{\eta}^+(x,\mu)}{\widetilde{a}^+(\mu)}\widetilde{K}(x,\mu) + \sum_{k=1}^{N}\frac{\widetilde{\eta}_k^+(x)}{\lambda_k^+}\widetilde{K}_k^+(x)\,. \tag{8.99b}$$

It remains to insert these expressions for $g^{-1}(x)$ back into (8.97) to get the system:

$$\widetilde{\eta}^+(x,\lambda) = \mathbb{1} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\mu\,\lambda}{\mu(\mu-\lambda)}\frac{\widetilde{\eta}^-(x,\mu)}{\widetilde{a}^-(\mu)}\widetilde{K}(x,\mu) - \sum_{k=1}^{N}\frac{\lambda\widetilde{\eta}_k^-(x)}{\lambda_k^-(\lambda_k^- - \lambda)}\widetilde{K}_k^-(x)\,, \tag{8.100a}$$

$$\widetilde{\eta}^-(x,\lambda) = \mathbb{1} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\mu\,\lambda}{\mu(\mu-\lambda)}\frac{\widetilde{\eta}^+(x,\mu)}{\widetilde{a}^+(\mu)}\widetilde{K}(x,\mu) - \sum_{k=1}^{N}\frac{\lambda\widetilde{\eta}_k^+(x)}{\lambda_k^+(\lambda_k^+ - \lambda)}\widetilde{K}_k^+(x)\,, \tag{8.100b}$$

These system of singular integral equations can be treated in exactly the same way like (4.72), (4.73) in Chap. 4.

Again, of special interest here is the reflectionless case: $\widetilde{K}(x,\lambda)=0$. Then the integral terms in the right-hand sides of (8.100) vanish, and we find that in this case the FAS $\widetilde{\eta}^\pm(x,\lambda)$ are rational functions of λ. An important feature of (8.100) is that the poles of $\widetilde{\eta}^+(x,\lambda)$ (resp. $\widetilde{\eta}^-(x,\lambda)$) lie in the lower half-plane $\mathbb{C}_-$ (resp. upper half-plane $\mathbb{C}_+$). Then the systems of (8.100) simplify and are transformed into:

$$\widetilde{\xi}_{\rm Ns}^-(x,\lambda) = \begin{pmatrix}1\\0\end{pmatrix} - \sum_{k=1}^{N}\frac{\lambda\widetilde{C}_k^+}{\lambda_k^+(\lambda_k^+-\lambda)}e^{2i\lambda_k^+ x}\widetilde{\xi}_k^+, \qquad \lambda\in\mathbb{C}_-\,, \tag{8.101}$$

$$\widetilde{\xi}_{\rm Ns}^+(x,\lambda) = \begin{pmatrix}0\\1\end{pmatrix} + \sum_{k=1}^{N}\frac{\lambda\widetilde{C}_k^-}{\lambda_k^-(\lambda_k^--\lambda)}e^{-2i\lambda_k^- x}\widetilde{\xi}_k^-, \qquad \lambda\in\mathbb{C}_+\,, \tag{8.102}$$

and

$$\widetilde{\varphi}_{\rm Ns}^+(x,\lambda) = \begin{pmatrix}1\\0\end{pmatrix} + \sum_{k=1}^{N}\frac{\lambda\widetilde{M}_k^-}{\lambda_k^-(\lambda_k^--\lambda)}e^{2i\lambda_k^- x}\widetilde{\varphi}_k^-, \qquad \lambda\in\mathbb{C}_+\,, \tag{8.103}$$

$$\widetilde{\varphi}_{\rm Ns}^-(x,\lambda) = \begin{pmatrix}0\\1\end{pmatrix} - \sum_{k=1}^{N}\frac{\lambda\widetilde{M}_k^+}{\lambda_k^+(\lambda_k^+-\lambda)}e^{-2i\lambda_k^+ x}\widetilde{\varphi}_k^+, \qquad \lambda\in\mathbb{C}_-\,. \tag{8.104}$$

In order to solve the system (8.101), (8.102) (or (8.103), (8.104)), it is enough to calculate $\widetilde{\xi}_k^\pm(x)$ (or $\widetilde{\varphi}_k^\pm(x)$). This can be done by putting $\lambda=\lambda_p^-$ in (8.101) (resp. (8.104)) and $\lambda=\lambda_p^+$ in (8.102) (resp. (8.103)).

$$\widetilde{\xi}_p^-(x) = \begin{pmatrix}1\\0\end{pmatrix} - \sum_{k=1}^{N}\frac{\lambda_p^-\widetilde{C}_k^+}{\lambda_k^+(\lambda_k^+-\lambda_p^-)}e^{2i\lambda_k^+ x}\widetilde{\xi}_k^+, \qquad \lambda\in\mathbb{C}_-\,, \tag{8.105}$$

$$\widetilde{\xi}_p^+(x) = \begin{pmatrix}0\\1\end{pmatrix} + \sum_{k=1}^{N}\frac{\lambda_p^+\widetilde{C}_k^-}{\lambda_k^-(\lambda_k^--\lambda_p^+)}e^{-2i\lambda_k^- x}\widetilde{\xi}_k^-, \qquad \lambda\in\mathbb{C}_+\,, \tag{8.106}$$

and

$$\widetilde{\varphi}_p^+(x) = \begin{pmatrix}1\\0\end{pmatrix} + \sum_{k=1}^{N}\frac{\lambda_p^+\widetilde{M}_k^-}{\lambda_k^-(\lambda_k^--\lambda_p^+)}e^{2i\lambda_k^- x}\widetilde{\varphi}_k^-, \qquad \lambda\in\mathbb{C}_+\,, \tag{8.107}$$

$$\widetilde{\varphi}_p^-(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sum_{k=1}^N \frac{\lambda_p^- \widetilde{M}_k^+}{\lambda_k^+(\lambda_k^+ - \lambda_p^-)} e^{-2i\lambda_k^+ x} \widetilde{\varphi}_k^+, \qquad \lambda \in \mathbb{C}_- \ . \tag{8.108}$$

Solving this system using, e.g., Kramer's rules, we find $\widetilde{\eta}_k^\pm$ as rational functions of the exponentials $\exp(\pm 2i\lambda_j^\pm x)$. Then, we insert the results back into the spectral decompositions (8.101)–(8.104), and we obtain explicitly the reflectionless FAS.

Thus, we have the following linear algebraic equations for $\widetilde{\xi}_k^\pm(x)$ and $\widetilde{\varphi}_k^\pm(x)$:

$$\left(\widetilde{\boldsymbol{\xi}}^-(x), \widetilde{\boldsymbol{\xi}}^+(x) \right) \begin{pmatrix} \mathbb{1} & \widetilde{D}^-(x) \\ \widetilde{D}^+(x) & \mathbb{1} \end{pmatrix} = \left(\mathbf{e}^+, \mathbf{e}^- \right) \ , \tag{8.109a}$$

$$\widetilde{D}_{kp}^+(x) = \frac{\widetilde{l}_{kp}^+ \widetilde{b}_k^+(x)}{l_{kp} \dot{\widetilde{a}}_k^+}, \qquad \widetilde{D}_{kp}^-(x) = \frac{\widetilde{l}_{kp}^- \widetilde{b}_k^-(x)}{l_{pk} \dot{\widetilde{a}}_k^-} \ , \tag{8.109b}$$

$$\widetilde{l}_{kp}^+ = \frac{\lambda_k^+(\lambda_k^+ - \lambda_p^-)}{\lambda_k^+}, \qquad \widetilde{l}_{kp}^- = \frac{\lambda_k^-(\lambda_p^+ - \lambda_k^-)}{\lambda_p^+} \ ,$$

and

$$\left(\widetilde{\boldsymbol{\varphi}}^+(x), \widetilde{\boldsymbol{\varphi}}^-(x) \right) \begin{pmatrix} \mathbb{1} & \widetilde{F}^+ \\ \widetilde{F}^- & \mathbb{1} \end{pmatrix} = \left(\mathbf{e}^+, \mathbf{e}^- \right) \ , \tag{8.110a}$$

$$\widetilde{F}_{kp}^+(x) = \frac{1}{l_{kp}^+ \widetilde{b}_k^-(x) \dot{\widetilde{a}}_k^-}, \qquad \widetilde{F}_{kp}^-(x) = \frac{1}{l_{pk}^- \widetilde{b}_k^+(x) \dot{\widetilde{a}}_k^+} \ , \tag{8.110b}$$

where we used the notations:

$$\widetilde{\boldsymbol{\xi}}^\pm(x) = \left(\widetilde{\xi}_1^\pm(x), \dots, \widetilde{\xi}_N^\pm(x) \right), \quad \widetilde{\boldsymbol{\varphi}}^\pm(x) = \left(\widetilde{\varphi}_1^\pm(x), \dots, \widetilde{\varphi}_N^\pm(x) \right) \ ,$$

$$\mathbf{e}^\pm = \underbrace{\left(e^\pm, \dots, e^\pm \right)}_{N-\text{times}}, \qquad e^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad e^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ , \tag{8.111a}$$

$$\widetilde{b}_k^\pm(x) = \widetilde{b}_k^\pm e^{\pm 2i\lambda_k^\pm x} = e^{-z_k \pm i\phi_k} \ , \tag{8.111b}$$

$$z_k = 2\nu_k(x - \xi_{0,k}), \qquad \phi_k = \frac{\mu_k}{\nu_k} z_k + \delta_k \ . \tag{8.111c}$$

The relations (8.38) show that the systems of (8.109) and (8.110) are equivalent. Therefore, it will be enough to consider only one of them. For example, the solution of (8.109) can be calculated by inverting the block matrix in the left-hand side of (8.109). Since

$$\begin{pmatrix} \mathbb{1} & \widetilde{D}^- \\ \widetilde{D}^+ & \mathbb{1} \end{pmatrix}^{-1} = \begin{pmatrix} (\mathbb{1} - \widetilde{D}^-\widetilde{D}^+)^{-1} & -\widetilde{D}^-(\mathbb{1} - \widetilde{D}^+\widetilde{D}^-)^{-1} \\ -\widetilde{D}^+(\mathbb{1} - \widetilde{D}^-\widetilde{D}^+)^{-1} & (\mathbb{1} - \widetilde{D}^+\widetilde{D}^-)^{-1} \end{pmatrix} \ , \tag{8.112}$$

we can obtain explicit expressions for $\widetilde{\xi}_k^\pm(x)$ and $\widetilde{\varphi}_k^\pm(x)$.

The next step is to insert the results for $\widetilde{\xi}_k^{\pm}(x)$ and $\widetilde{\varphi}_k^{\pm}(x)$ into the right-hand side of (8.101)–(8.104). This gives us immediately the explicit formulae for the Jost solutions and for the FAS. They are meromorphic functions of λ, and so is the corresponding scattering matrix:

$$\widetilde{T}_{\rm Ns}(\lambda) = \begin{pmatrix} \widetilde{a}^+_{\rm Ns}(\lambda) & 0 \\ 0 & \widetilde{a}^-_{\rm Ns}(\lambda) \end{pmatrix}, \qquad \widetilde{a}^+_{\rm Ns}(\lambda)\widetilde{a}^-_{\rm Ns}(\lambda) = 1 \;, \tag{8.113}$$

where

$$\widetilde{a}^+_{\rm Ns}(\lambda) = \frac{1}{\widetilde{a}^-_{\rm Ns}(\lambda)} = \prod_{k=1}^{N} \frac{(\lambda-\lambda_k^+)\lambda_k^-}{(\lambda-\lambda_k^-)\lambda_k^+} \;. \tag{8.114}$$

More specifically, we get:

$$\widetilde{\phi}^+_{\rm Ns}(x,\lambda) = \widetilde{\psi}^-_{\rm Ns}(x,\lambda)\widetilde{a}^+_{\rm Ns}(\lambda), \qquad \widetilde{\phi}^-_{\rm Ns}(x,\lambda) = \widetilde{\psi}^+_{\rm Ns}(x,\lambda)\widetilde{a}^-_{\rm Ns}(\lambda) \;, \tag{8.115}$$

and

$$\widetilde{\chi}^+_{\rm Ns}(x,\lambda) = \widetilde{a}^+_{\rm Ns}(\lambda)\widetilde{\chi}^-_{\rm Ns}(x,\lambda) \;. \tag{8.116}$$

The corresponding reflectionless potential can be recovered using (8.34c), together with the fact that in the reflectionless case the decompositions for $g^{-1}(x,t)$ (8.99) is given by any of the expressions:

$$g^{-1}_{\rm Ns}(x) = \mathbb{1} + \sum_{k=1}^{N} \frac{\widetilde{\eta}_k^-(x)}{\lambda_k^-}\widetilde{K}_k^-(x) \;, \tag{8.117a}$$

$$g^{-1}_{\rm Ns}(x) = \mathbb{1} + \sum_{k=1}^{N} \frac{\widetilde{\eta}_k^+(x)}{\lambda_k^+}\widetilde{K}_k^+(x) \;. \tag{8.117b}$$

Since we already know the explicit formulae for $\widetilde{\eta}_k^{\pm}(x)$, all we need to do is to insert them into the right-hand sides of (8.117), thus getting $g^{-1}(x,t)$ explicitly. If we denote it by:

$$g^{-1}_{\rm Ns}(x,t) = \begin{pmatrix} \widehat{g}_{{\rm Ns},11}(x,t) & \widehat{g}_{{\rm Ns},12}(x,t) \\ \widehat{g}_{{\rm Ns},21}(x,t) & \widehat{g}_{{\rm Ns},22}(x,t) \end{pmatrix} \;, \tag{8.118}$$

then its inverse $g(x,t)$ will be equal to:

$$g_{\rm Ns}(x,t) = \begin{pmatrix} \widehat{g}_{{\rm Ns},22}(x,t) & -\widehat{g}_{{\rm Ns},12}(x,t) \\ -\widehat{g}_{{\rm Ns},21}(x,t) & \widehat{g}_{{\rm Ns},11}(x,t) \end{pmatrix} \;. \tag{8.119}$$

In deriving (8.119), we made use of the fact that $\det g_{\rm Ns}(x,t) = 1$. Then,

$$\begin{aligned} S_{\rm Ns}(x) &= g^{-1}_{\rm Ns}\sigma_3 g_{\rm Ns}(s,t) \qquad (8.120) \\ &= \sigma_3\left(\widehat{g}_{{\rm Ns},11}\widehat{g}_{{\rm Ns},22} + \widehat{g}_{{\rm Ns},12}\widehat{g}_{{\rm Ns},21}\right) + 2\begin{pmatrix} 0 & -\widehat{g}_{{\rm Ns},11}\widehat{g}_{{\rm Ns},12} \\ \widehat{g}_{{\rm Ns},21}\widehat{g}_{{\rm Ns},22} & 0 \end{pmatrix} \;. \end{aligned}$$

From (8.120), one can also easily derive the N-soliton solutions of the HF-type NLEE. To this end, it is enough to insert the explicit t-dependence into the scattering data, i.e. we have to change $\widetilde{b}^{\pm}(x)$ to $\widetilde{b}^{\pm}(x,t)$:

$$\widetilde{b}^{\pm}(x,t) = b_k^{\pm} \exp(\pm 2i\lambda_k^{\pm} x \mp 2i\lambda_k^{\pm} \widetilde{f}_k^{\pm} t), \qquad \widetilde{f}_k^{\pm} = \widetilde{f}(\lambda_k^{\pm}) \, , \tag{8.121}$$

where $\widetilde{f}(\lambda)$ is the dispersion law of the corresponding NLEE.

In the simplest one-soliton case, the solution of the (8.101), (8.102), (8.104), (8.103) is given by:

$$\widetilde{\xi}_1^{-}(x) = \frac{1}{\widetilde{A}_1(x,t)} \begin{pmatrix} 1 \\ -\widetilde{b}_1^{+}(x,t) \end{pmatrix}, \qquad \widetilde{\xi}_1^{+}(x) = \frac{1}{\widetilde{A}_1(x,t)} \begin{pmatrix} \widetilde{b}_1^{-}(x,t) \\ 1 \end{pmatrix} , \tag{8.122a}$$

$$\widetilde{\varphi}_1^{+}(x,t) = \frac{\widetilde{b}_1^{+}(x,t)}{\widetilde{A}_1(x,t)} \begin{pmatrix} \widetilde{b}_1^{-}(x,t) \\ 1 \end{pmatrix}, \qquad \widetilde{\varphi}_1^{-}(x,t) = \frac{\widetilde{b}_1^{-}(x,t)}{\widetilde{A}_1(x,t)} \begin{pmatrix} -1 \\ \widetilde{b}_1^{+}(x,t) \end{pmatrix} , \tag{8.122b}$$

where

$$\widetilde{A}_1(x,t) = 1 + \widetilde{b}_1^{+}(x,t)\widetilde{b}_1^{-}(x,t) \, , \tag{8.122c}$$

These formulae can be written in compact form as follows:

$$\widetilde{\eta}_1^{+}(x,t) = \mathbb{1} - \widetilde{P}_1(x,t), \qquad \widetilde{\eta}_1^{-}(x,t) = \widetilde{P}_1(x,t) \, , \tag{8.123a}$$

$$\begin{aligned} \widetilde{P}_1(x,t) &= \frac{|\widetilde{n}_1(x,t)\rangle\langle\widetilde{m}_1(x,t)|}{\langle\widetilde{m}_1(x,t)|\widetilde{n}_1(x,t)\rangle} \\ &= \frac{1}{1+\widetilde{b}^{+}(x,t)\widetilde{b}^{-}(x,t)} \begin{pmatrix} 1 & -\widetilde{b}_1^{-}(x,t) \\ -\widetilde{b}_1^{+}(x,t) & \widetilde{b}_1^{+}(x,t)\widetilde{b}_1^{-}(x,t) \end{pmatrix} , \end{aligned} \tag{8.123b}$$

$$|\widetilde{n}_1(x)\rangle = e^{-i\lambda_k^{+}\sigma_3 x + i\widetilde{f}_1^{+}\sigma_3 t}|\widetilde{n}_{10}\rangle, \qquad |\widetilde{n}_{10}\rangle = \begin{pmatrix} 1 \\ -\widetilde{b}_{10}^{+} \end{pmatrix} , \tag{8.123c}$$

$$\langle\widetilde{m}_1(x)| = \langle\widetilde{m}_{10}|e^{i\lambda_k^{-}\sigma_3 x - i\widetilde{f}_1^{-}\sigma_3 t}, \qquad \langle\widetilde{m}_{10}| = \begin{pmatrix} 1, -\widetilde{b}_{10}^{-} \end{pmatrix} , \tag{8.123d}$$

Inserting (8.122) into (8.97) and (8.97b), we get the following explicit expressions for the FAS:

$$\widetilde{\eta}_{1\mathrm{s}}^{+}(x,t,\lambda) = \mathbb{1} + (\widetilde{c}_1(\lambda)-1)\widetilde{P}_1(x,t), \quad \widetilde{c}_1(\lambda) = \frac{\lambda_1^{-}}{\lambda_1^{+}} \frac{\lambda-\lambda_1^{+}}{\lambda-\lambda_1^{-}} , \tag{8.124a}$$

$$\widetilde{\eta}_{1\mathrm{s}}^{-}(x,t,\lambda) = \mathbb{1} + \left(\frac{1}{\widetilde{c}_1(\lambda)} - 1 \right) (\mathbb{1} - \widetilde{P}_1(x,t)) \, , \tag{8.124b}$$

or

$$\widetilde{\eta}_{1\mathrm{s}}^{+}(x,t,\lambda) = \widetilde{\eta}_{1\mathrm{s}}^{-}(x,t,\lambda)\widetilde{c}_1(\lambda) \, . \tag{8.125}$$

The new potential $S_{1\mathrm{s}}(x,t)$ is related to the old one $S(x,t)$ by:

$$S_{1s}(x) = \left(\mathbb{1} + \left(\frac{\lambda_1^-}{\lambda_1^+} - 1\right) \widetilde{P}_1(x,t)\right) S(x,t) \left(\mathbb{1} + \left(\frac{\lambda_1^+}{\lambda_1^-} - 1\right) \widetilde{P}_1(x,t)\right) . \tag{8.126}$$

In the 1-soliton case we have

$$\widetilde{a}_{1s}^+(\lambda) = \frac{1}{\widetilde{a}_{1s}^-(\lambda)} = \widetilde{c}_1(\lambda) ; \tag{8.127}$$

see formula (8.114) with $\nu = 1$. From (8.116), we find that in the reflectionless case $G(x,\lambda)$ is proportional to the unit matrix; consequently it is x-independent. Another important fact illustrated by (8.124) and generic for the N-soliton case is the following. Both $\widetilde{\eta}^+(x,\lambda)$ and $\widetilde{\eta}^-(x,\lambda)$ are meromorphic functions of λ (fraction-linear in our case). In fact, they can be extended to the whole λ-plane with the exception of their pole-singularities, located at $\lambda_1^\pm$.

8.5 The Wronskian Relations and the Gauge Equivalence

We can derive the Wronskian relations for the system $\widetilde{L}$ in two independent but equivalent ways.

First of all, we remark that with $\widetilde{L}$ one can associate the following systems:

$$i\frac{d\hat{\widetilde{\psi}}}{dx} + \lambda\hat{\widetilde{\psi}}(x,t,\lambda)S(x,t) = 0 , \tag{8.128}$$

$$i\frac{d\delta\widetilde{\psi}}{dx} - \lambda\delta S(x,t)\widetilde{\psi}(x,t,\lambda) - \lambda S(x,t)\delta\widetilde{\psi}(x,t,\lambda) = 0 , \tag{8.129}$$

$$i\frac{d\dot{\widetilde{\psi}}}{dx} - \lambda S(x,t)\widetilde{\psi}(x,t,\lambda) - S(x,t)\dot{\widetilde{\psi}}(x,t,\lambda) = 0 , \tag{8.130}$$

where $\delta\widetilde{\psi}$ corresponds to a given variation $\delta S(x,t)$ of the potential, while by dot we denote the derivative with respect to the spectral parameter.

We start with the identity:

$$\begin{aligned}\left(\hat{\widetilde{\chi}}\sigma_3\widetilde{\chi}(x,\lambda) - \sigma_3\right)\Big|_{-\infty}^{\infty} &= i\int_{-\infty}^{\infty} dx\, \frac{d}{dx}\left(i\hat{\widetilde{\chi}}\sigma_3\widetilde{\chi}\right)(x,\lambda) \\ &= i\lambda\int_{-\infty}^{\infty} dx\, \hat{\chi}[S(x,t),\sigma_3]\widetilde{\chi}(x,\lambda) ,\end{aligned} \tag{8.131}$$

where $\widetilde{\chi}(x,\lambda)$ can be any fundamental solution of $\widetilde{L}$. For convenience, we choose them to be the FAS introduced above.

The left-hand side of (8.131) can be calculated explicitly by using the asymptotics of $\widetilde{\chi}^\pm(x,\lambda)$ for $x \to \pm\infty$. It would be expressed by the matrix elements of the scattering matrix $\widetilde{T}(\lambda)$, i.e. by the scattering data of $\widetilde{L}$ as follows:

$$\left(\hat{\widetilde{\chi}}^{+}\sigma_3\widetilde{\chi}^{+}(x,\lambda)-\sigma_3\right)\Big|_{-\infty}^{\infty}=-2\begin{pmatrix}0 & \widetilde{b}^{-}(\lambda)\\ \widetilde{b}^{+}(\lambda) & 0\end{pmatrix} \tag{8.132}$$

$$\left(\hat{\widetilde{\chi}}^{-}\sigma_3\widetilde{\chi}^{-}(x,\lambda)-\sigma_3\right)\Big|_{-\infty}^{\infty}=-2\begin{pmatrix}0 & \widetilde{b}^{-}(\lambda)\\ \widetilde{b}^{+}(\lambda) & 0\end{pmatrix}. \tag{8.133}$$

It is well known that there exist two independent sets of scattering data $\widetilde{\mathcal{T}}_1$ and $\widetilde{\mathcal{T}}_2$, which contain the two sets of reflection coefficients:

$$\widetilde{\mathcal{T}}_1\equiv\{\widetilde{\rho}^{\pm}(\lambda),\quad \lambda\in\mathbb{R};\lambda_k^{\pm},\ \widetilde{C}_k^{\pm}\},\qquad \widetilde{\mathcal{T}}_2\equiv\{\widetilde{\tau}^{\pm}(\lambda),\quad \lambda\in\mathbb{R};\lambda_k^{\pm},\ \widetilde{M}_k^{\pm}\}, \tag{8.134}$$

where $k=1,\dots,N$. Here and below:

$$\begin{aligned}\widetilde{\rho}^{\pm}(\lambda)&\equiv\frac{\widetilde{b}^{\pm}(\lambda)}{\widetilde{a}^{\pm}(\lambda)}=\rho^{\pm}(\lambda),\qquad \widetilde{C}_k^{\pm}=C_k^{\pm},\\ \widetilde{\tau}^{\pm}(\lambda)&\equiv\frac{\widetilde{b}^{\mp}(\lambda)}{\widetilde{a}^{\pm}(\lambda)}=(a_0^{\pm})^2\tau^{\pm}(\lambda),\qquad \widetilde{M}_k^{\pm}=(a_0^{\pm})^2M_k^{\pm}.\end{aligned} \tag{8.135}$$

We now use the invariance of the Killing form under the group of gauge transformations. Then the analogs of (5.16)–(5.19) acquire the form:

$$\widetilde{\rho}^{\pm}(\lambda)=\frac{i\lambda}{(\widetilde{a}^{\pm}(\lambda))^2}[\![\pi_S\sigma_3,\widetilde{\boldsymbol{\Phi}}^{\pm}(x,\lambda)]\!]_{\sim}, \tag{8.136}$$

$$\widetilde{\tau}^{\pm}(\lambda)=\frac{i\lambda}{(\widetilde{a}^{\pm}(\lambda))^2}[\![\pi_S\sigma_3,\widetilde{\boldsymbol{\Psi}}^{\pm}(x,\lambda)]\!]_{\sim}, \tag{8.137}$$

where we recall that $\pi_S\sigma_3=1/4[S(x),[S(x),\sigma_3]]$. The skew-symmetric scalar product $[\![\cdot\,,\,\cdot]\!]_{\sim}$ is given by:

$$[\![\widetilde{X},\widetilde{Y}]\!]_{\sim}=\int_{-\infty}^{\infty}\langle\widetilde{X}(x),[S(x),\widetilde{Y}(x)]\rangle\,dx. \tag{8.138}$$

Note the additional factor λ in the right-hand sides of (8.136), (8.137) compared to (5.16) and (5.17).

The "squared" solutions $\widetilde{\boldsymbol{\Phi}}^{\pm}(x,\lambda)$ and $\widetilde{\boldsymbol{\Psi}}^{\pm}(x,\lambda)$ are

$$\widetilde{\mathcal{E}}_{\alpha}^{\pm}(x,\lambda)=\widetilde{a}^{\pm}(\lambda)\widetilde{\chi}^{\pm}(x,\lambda)\sigma_{\alpha}\hat{\widetilde{\chi}}^{\pm}(x,\lambda),\qquad \alpha=3,\pm; \tag{8.139a}$$

$$\widetilde{\boldsymbol{\Psi}}^{\pm}(x,\lambda)=\left(\widetilde{\mathcal{E}}_{\mp}^{\pm}(x,\lambda)\right)^{\mathrm{f}}=g^{-1}(x)\boldsymbol{\Psi}^{\pm}(x,\lambda)g(x), \tag{8.139b}$$

$$\widetilde{\boldsymbol{\Phi}}^{\pm}(x,\lambda)=\left(\widetilde{\mathcal{E}}_{\pm}^{\pm}(x,\lambda)\right)^{\mathrm{f}}=(a_0^{\pm})^2g^{-1}(x)\boldsymbol{\Phi}^{\pm}(x,\lambda)g(x), \tag{8.139c}$$

$$\widetilde{\boldsymbol{\Theta}}^{\pm}(x,\lambda)=\left(\widetilde{\mathcal{E}}_{3}^{\pm}(x,\lambda)\right)^{\mathrm{f}}=a_0^{\pm}g^{-1}(x)\boldsymbol{\Theta}^{\pm}(x,\lambda)g(x). \tag{8.139d}$$

Here and below, the superscript "f" is understood in the sense of the splitting (8.44). Indeed, due to the definition of $[\![\cdot,\cdot]\!]_{\sim}$ the "diagonal" parts of $\widetilde{\mathcal{E}}_{\alpha}^{\pm}(x,\lambda)^{\rm d}$ do not contribute to the right-hand sides of the Wronskian relations.

One can introduce also the symplectic basis by the following linear combinations of the "squared" solutions (compare with (5.61)):

$$\begin{aligned} \widetilde{\boldsymbol{P}}(x,\lambda) &= \frac{1}{\pi}\left(\widetilde{\tau}^{+}(\lambda)\widetilde{\boldsymbol{\Phi}}^{+}(x,\lambda) - \widetilde{\tau}^{-}(\lambda)\widetilde{\boldsymbol{\Phi}}^{-}(x,\lambda)\right) \\ &= -\frac{1}{\pi}\left(\widetilde{\rho}^{+}(\lambda)\widetilde{\boldsymbol{\Psi}}^{+}(x,\lambda) - \widetilde{\rho}^{-}(\lambda)\widetilde{\boldsymbol{\Psi}}^{-}(x,\lambda)\right) , \qquad (8.140a) \end{aligned}$$

$$\widetilde{\boldsymbol{P}}_{k}^{\pm}(x) = 2i\widetilde{C}_{k}^{\pm}\widetilde{\boldsymbol{\Psi}}_{k}^{\pm}(x) = -2i\widetilde{M}_{k}^{\pm}\widetilde{\boldsymbol{\Phi}}_{k}^{\pm}(x) , \qquad (8.140b)$$

$$\begin{aligned} \widetilde{\boldsymbol{Q}}(x,\lambda) &= \frac{\widetilde{\tau}^{+}(\lambda)\widetilde{\boldsymbol{\Phi}}^{+}(x,\lambda) + \widetilde{\rho}^{+}(\lambda)\widetilde{\boldsymbol{\Psi}}^{+}(x,\lambda)}{2\widetilde{b}^{+}(\lambda)\widetilde{b}^{-}(\lambda)} \\ &= \frac{\widetilde{\rho}^{-}(\lambda)\widetilde{\boldsymbol{\Psi}}^{-}(x,\lambda) + \widetilde{\tau}^{-}(\lambda)\widetilde{\boldsymbol{\Phi}}^{-}(x,\lambda)}{2\widetilde{b}^{+}(\lambda)\widetilde{b}^{-}(\lambda)} , \qquad (8.140c) \end{aligned}$$

$$\widetilde{\boldsymbol{Q}}_{k}^{\pm}(x) = \frac{1}{2}\left(\widetilde{C}_{k}^{\pm}\dot{\widetilde{\boldsymbol{\Psi}}}_{k}^{\pm}(x) + \widetilde{M}_{k}^{\pm}\dot{\widetilde{\boldsymbol{\Phi}}}_{k}^{\pm}(x)\right) . \qquad (8.140d)$$

Taking into account (8.139) we get that

$$\widetilde{\boldsymbol{P}}(x,\lambda) = g^{-1}(x)\boldsymbol{P}(x,\lambda)g(x), \quad \widetilde{\boldsymbol{P}}_{k}^{\pm}(x) = g^{-1}(x)\boldsymbol{P}_{k}^{\pm}(x)g(x) , \qquad (8.141a)$$

$$\widetilde{\boldsymbol{Q}}(x,\lambda) = g^{-1}(x)\boldsymbol{Q}(x,\lambda)g(x), \quad \widetilde{\boldsymbol{Q}}_{k}^{\pm}(x) = g^{-1}(x)\boldsymbol{Q}_{k}^{\pm}(x)g(x) . \qquad (8.141b)$$

for $k = 1, \ldots, N$ and the gauge analogs of (5.79)

$$[\![\pi_{S}\sigma_{3}, \widetilde{\boldsymbol{P}}(x,\lambda)]\!]_{\sim} = 0, \qquad [\![\pi_{S}\sigma_{3}, \widetilde{\boldsymbol{Q}}(x,\lambda)]\!]_{\sim} = -\frac{i}{\lambda} , \qquad (8.142a)$$

$$[\![\pi_{S}\sigma_{3}, \widetilde{\boldsymbol{P}}_{k}^{\pm}(x)]\!]_{\sim} = 0, \qquad [\![\pi_{S}\sigma_{3}, \widetilde{\boldsymbol{Q}}_{k}^{\pm}(x)]\!]_{\sim} = -\frac{i}{\lambda_{k}^{\pm}} , \qquad (8.142b)$$

The second type of Wronskian relations, which we shall consider, relate the variation of the potential $\delta S(x)$ to the corresponding variations of the scattering data. To this purpose, we start with the identity:

$$\hat{\widetilde{\chi}}\delta\widetilde{\chi}(x,\lambda)\Big|_{-\infty}^{\infty} = -i\lambda\int_{-\infty}^{\infty} dx\, \hat{\widetilde{\chi}}\delta S(x)\widetilde{\chi}(x,\lambda) . \qquad (8.143)$$

For the left-hand side of (8.143) we find:

$$\hat{\widetilde{\chi}}^{+}\delta\widetilde{\chi}^{+}(x,\lambda)\Big|_{-\infty}^{\infty} = \begin{pmatrix} \delta\ln\widetilde{a}^{+}(\lambda) & -\widetilde{a}^{+}(\lambda)\delta\widetilde{\tau}^{+}(\lambda) \\ \widetilde{a}^{+}(\lambda)\delta\widetilde{\rho}^{+}(\lambda) & -\delta\ln\widetilde{a}^{+}(\lambda) \end{pmatrix} , \qquad (8.144)$$

and

$$\hat{\widetilde{\chi}}^{-}\delta\widetilde{\chi}^{-}(x,\lambda)\Big|_{-\infty}^{\infty} = \begin{pmatrix} -\delta\ln\widetilde{a}^{-}(\lambda) & -\widetilde{a}^{-}(\lambda)\delta\widetilde{\rho}^{-}(\lambda) \\ \widetilde{a}^{-}(\lambda)\delta\widetilde{\tau}^{-}(\lambda) & \delta\ln\widetilde{a}^{-}(\lambda) \end{pmatrix} , \qquad (8.145)$$

Next, multiplying by $\sigma_{\pm}$ and taking the trace we arrive at:

$$\delta\widetilde{\rho}^{\pm}(\lambda) = \mp\frac{i\lambda}{2(\widetilde{a}^{\pm}(\lambda))^2}\,[\![[S,\delta S(x)], \widetilde{\boldsymbol{\Phi}}^{\pm}(x,\lambda)]\!]_{\sim} , \qquad (8.146)$$

$$\delta\widetilde{\tau}^{\pm}(\lambda) = \pm\frac{i\lambda}{2(\widetilde{a}^{\pm}(\lambda))^2}\,[\![[S(x),\delta S(x)], \widetilde{\boldsymbol{\Psi}}^{\pm}(x,\lambda)]\!] , \qquad (8.147)$$

and

$$\delta\widetilde{\mathcal{A}}(\lambda) = -\frac{i\lambda}{4\widetilde{a}^{\pm}(\lambda)}\,[\![[S(x),\delta S(x)], \widetilde{\boldsymbol{\Theta}}^{\pm}(x,\lambda)]\!]_{\sim} . \qquad (8.148)$$

In addition to $\widetilde{\boldsymbol{\Psi}}^{\pm}(x,\lambda)$ and $\widetilde{\boldsymbol{\Phi}}^{\pm}(x,\lambda)$ introduced above, here, we use also the "squared" solutions:

$$\widetilde{\boldsymbol{\Theta}}^{\pm}(x,\lambda) = \widetilde{a}^{+}(\lambda)(\widetilde{\chi}^{\pm}(x,\lambda)\sigma_3\hat{\widetilde{\chi}}^{\pm}(x,\lambda))^{\mathrm{f}} \qquad (8.149)$$

and $\widetilde{\mathcal{A}}(\lambda)$ is introduced by:

$$\widetilde{\mathcal{A}}(\lambda) = \begin{cases} \ln\widetilde{a}^{+}(\lambda), & \operatorname{Im}\lambda > 0 , \\ \frac{1}{2}\ln\left(\widetilde{a}^{+}(\lambda)/\widetilde{a}^{-}(\lambda)\right), & \operatorname{Im}\lambda = 0 , \\ -\ln\widetilde{a}^{-}(\lambda), & \operatorname{Im}\lambda < 0. \end{cases} \qquad (8.150)$$

Note that again we get an additional factor of λ in the right-hand sides of (8.146), (8.147), and (8.148) compared to (5.25), (5.26), and (5.27).

The third type of Wronskian relations is the analog of (5.32):

$$\left(\hat{\widetilde{\chi}}\dot{\widetilde{\chi}}(x,\lambda) + ix\sigma_3\right)\Big|_{-\infty}^{\infty} = -i\int_{-\infty}^{\infty} dx\,(\hat{\widetilde{\chi}}(x,\lambda)S(x)\widetilde{\chi}(x,\lambda) - \sigma_3) , \qquad (8.151)$$

where we remind that "dot" means derivative with respect to λ. Taking the trace with σ_3 of the left-hand sides and evaluating the limits gives:

$$\left(\frac{1}{2}\operatorname{tr}(\hat{\widetilde{\chi}}\dot{\widetilde{\chi}}(x,\lambda)\sigma_3) + ix\right)\Bigg|_{-\infty}^{\infty} = \pm\frac{\dot{\widetilde{a}}^{\pm}}{\widetilde{a}^{\pm}(\lambda)} = \frac{d\widetilde{\mathcal{A}}}{d\lambda} , \qquad (8.152)$$

where $\widetilde{\mathcal{A}}(\lambda)$ was introduced in (8.150).

In the analysis of the NLEE related to the system $\widetilde{L}$ and their Hamiltonian structures basic role plays a particular class of variations of $S(x)$ resulting from its time evolution. In this case $S(x,t)$, depends on t in such a way that it satisfies certain NLEE. Then, we consider variations of the type:

$$\delta S(x) \simeq \frac{\partial S}{\partial t}\delta t + \mathcal{O}((\delta t)^2) . \qquad (8.153)$$

Keeping only the first-order terms with respect to δt we find:

$$\widetilde{\rho}_t^{\pm}(\lambda) = \mp\frac{i\lambda}{2(\widetilde{a}^{\pm}(\lambda))^2} \left[\!\left[[S(x,t), S_t], \widetilde{\boldsymbol{\Phi}}^{\pm}(x,t,\lambda) \right]\!\right]_{\sim} , \tag{8.154}$$

$$\widetilde{\tau}_t^{\pm}(\lambda) = \pm\frac{i\lambda}{2(\widetilde{a}^{\pm}(\lambda))^2} \left[\!\left[[S(x,t), S_t], \widetilde{\boldsymbol{\Psi}}^{\pm}(x,t,\lambda) \right]\!\right]_{\sim} . \tag{8.155}$$

We postpone the application of these relations until later, and now we consider the second method for the derivation of the Wronskian relations. It consists in applying successively the gauge transformations to the Wronskian relations for the Zakharov–Shabat system derived in Chap. 5. Doing this, we shall use (8.71), from which one finds that the "squared" solutions $\widetilde{\boldsymbol{\Phi}}^{\pm}(x,\lambda)$, and $\widetilde{\boldsymbol{\Psi}}^{\pm}(x,\lambda)$ are related to $\boldsymbol{\Phi}^{\pm}(x,\lambda)$ and $\boldsymbol{\Psi}^{\pm}(x,\lambda)$ by (8.139). Then, (5.16)–(5.19) acquire the form:

$$\widetilde{\rho}^{\,\pm}(\lambda) = \frac{i}{(\widetilde{a}^{\pm}(\lambda))^2} \left[\!\left[\widetilde{q}(x), \widetilde{\boldsymbol{\Phi}}^{\pm}(x,\lambda) \right]\!\right]_{\sim} , \tag{8.156}$$

$$\widetilde{\tau}^{\pm}(\lambda) = \frac{i}{(\widetilde{a}^{\,\pm}(\lambda))^2} \left[\!\left[\widetilde{q}(x), \widetilde{\boldsymbol{\Psi}}^{\pm}(x,\lambda) \right]\!\right]_{\sim} . \tag{8.157}$$

which means that

$$\left[\!\left[\widetilde{q}(x), \widetilde{\boldsymbol{P}}(x,\lambda) \right]\!\right]_{\sim} = 0, \qquad \left[\!\left[\widetilde{q}(x), \widetilde{\boldsymbol{Q}}(x,\lambda) \right]\!\right]_{\sim} = -i , \tag{8.158a}$$

$$\left[\!\left[\widetilde{q}(x), \widetilde{\boldsymbol{P}}_k^{\pm}(x) \right]\!\right]_{\sim} = 0, \qquad \left[\!\left[\widetilde{q}(x), \widetilde{\boldsymbol{Q}}_k^{\pm}(x) \right]\!\right]_{\sim} = -i . \tag{8.158b}$$

Analogously, applying the gauge transformation to the second type of Wronskian relations see (5.25), (5.26) we get:

$$\delta\widetilde{\rho}^{\pm}(\lambda) = \mp\frac{i}{2(\widetilde{a}^{\pm}(\lambda))^2} \left[\!\left[[S(x), \widetilde{\delta q(x)}], \widetilde{\boldsymbol{\Phi}}^{\pm}(x,\lambda) \right]\!\right]_{\sim} , \tag{8.159}$$

$$\delta\widetilde{\tau}^{\pm}(\lambda) = \pm\frac{i}{2(\widetilde{a}^{\pm}(\lambda))^2} \left[\!\left[[S(x), \widetilde{\delta q(x)}], \widetilde{\boldsymbol{\Psi}}^{\pm}(x,\lambda) \right]\!\right]_{\sim} , \tag{8.160}$$

where

$$\widetilde{\delta} q(x) = g^{-1}(x)\delta q(x) g(x) . \tag{8.161}$$

Now, we need to express $\widetilde{\delta} q$ in terms of $S(x)$ and $\delta S(x)$. Using (8.49) and (8.52) $\widetilde{q}(x)$ equals:

$$\begin{aligned}\widetilde{q}(x) \equiv g^{-1}(x) q(x) g(x) &= -\frac{i}{4}\left[S(x), \frac{\partial S}{\partial x}\right] \\ &= \widetilde{\Lambda}_{\pm}\pi_S \sigma_3 .\end{aligned} \tag{8.162}$$

Next, we prove:

Lemma 8.5. *The following relations hold:*

$$\widetilde{\delta q(x)} \equiv g^{-1}(x)\delta q(x) g(x) = \widetilde{\Lambda}_-^*(\delta S(x)) , \tag{8.163}$$

$$\widetilde{\delta q(x)} = \widetilde{\Lambda}_+^*(\delta S(x)) - i\frac{\partial S}{\partial x}\delta\left(\ln a_0^+\right) . \tag{8.164}$$

Proof. From (8.34c) there follows the relation:

$$\delta S(x) = [S(x), g^{-1}(x)\delta g(x)] , \tag{8.165}$$

which means that

$$\left(g^{-1}(x)\delta g(x)\right)^{\mathrm{f}} = \frac{1}{4}[S(x), \delta S(x)] . \tag{8.166}$$

Let us consider the matrix function $B(x,\lambda) = \delta\psi\psi^{-1}(x,\lambda)$, where $\psi(x,\lambda)$ is the Jost solution introduced in (8.60). Note that $B(x,0) = \delta g(x) g^{-1}(x)$. Using (8.59a) and (8.128), we can see that $B(x,\lambda)$ satisfies the following equation:

$$i\frac{\partial B}{\partial x} + [q(x), B(x,\lambda)] - \lambda[\sigma_3, B(x,\lambda)] = \delta q(x) . \tag{8.167}$$

Separating the diagonal and off-diagonal parts of (8.167) and taking proper care of the integration constants we find that

$$\delta q(x) = [\sigma_3, (\Lambda_\pm - \lambda)B^{\mathrm{f}}(x,\lambda)] + [\sigma_3, q(x)] \lim_{x\to\pm\infty}\langle\sigma_3, B(x,\lambda)\rangle . \tag{8.168}$$

In the limit $\lambda \to 0$ we easily get:

$$\lim_{x\to\infty}\langle\sigma_3, B(x,0)\rangle = 0, \qquad \lim_{x\to-\infty}\langle\sigma_3, B(x,0)\rangle = -\delta\ln a_0^+ . \tag{8.169}$$

Next, we apply the similarity transformation with $g(x)$ to both sides of (8.168), put $\lambda = 0$ and use the fact that

$$\widetilde{\Lambda}_\pm \widetilde{X} = g^{-1}(x)(\Lambda_\pm X) g(x) . \tag{8.170}$$

This gives:

$$\begin{aligned} g^{-1}(x)(\delta q(x)) g(x) &= [S(x), \widetilde{\Lambda}_+ \widetilde{B(x)^{\mathrm{f}}}] \\ &= [S(x), \widetilde{\Lambda}_- \widetilde{B(x)^{\mathrm{f}}}] - [S, \widetilde{q}(x)]\delta(\ln a_0^+) . \end{aligned} \tag{8.171}$$

It remains only to use the relation (8.162) to complete the proof.

It is useful to reformulate the above lemma in a somewhat different way, namely, that the Gateau derivative $q'(S)$ of the mapping $q \to S$ is given by:

$$g^{-1}(x) q'(S) g(x) = \widetilde{\Lambda}_-^* = \widetilde{\Lambda}_+^* - i\frac{\partial S}{\partial x}\delta(\ln a_0^+)$$

$$= \widetilde{\Lambda}^* - \frac{i}{2}\frac{\partial S}{\partial x}\delta(\ln a_0^+) , \tag{8.172}$$

where we recall that $\widetilde{\Lambda} = (\widetilde{\Lambda}_+ + \widetilde{\Lambda}_-)/2$.

The operators $\widetilde{\Lambda}_\pm$ and $\widetilde{\Lambda}$ satisfy "conjugation"-type relations with respect to the skew-symmetric scalar product $[\![\cdot\,,\,\cdot]\!]_\sim$:

$$[\![\widetilde{\Lambda}_+\widetilde{X}, \widetilde{Y}\,]\!]_\sim = [\![\widetilde{X}, \widetilde{\Lambda}_-\widetilde{Y}\,]\!]_\sim, \qquad [\![\widetilde{\Lambda}\widetilde{X}, \widetilde{Y}\,]\!]_\sim = [\![\widetilde{X}, \widetilde{\Lambda}\widetilde{Y}\,]\!]_\sim , \tag{8.173}$$

for any choice of the functions $\widetilde{X}(x)$ and $\widetilde{Y}(x)$ such that

$$\widetilde{X}(x) \equiv \pi_S \widetilde{X}(x), \qquad \widetilde{Y}(x) \equiv \pi_S \widetilde{Y}(x) . \tag{8.174}$$

Another fact that singles out these recursion operators as important objects in the theory of HF-type NLEE is that they have the "squared" solutions $\widetilde{\boldsymbol{\Psi}}^{\pm}(x,\lambda)$ and $\widetilde{\boldsymbol{\Phi}}^{\pm}(x,\lambda)$ as eigenfunctions, namely:

$$\widetilde{\Lambda}_+ \widetilde{\boldsymbol{\Psi}}^{\pm}(x,\lambda) = \lambda \widetilde{\boldsymbol{\Psi}}^{\pm}(x,\lambda), \qquad \widetilde{\Lambda}_+ \widetilde{\boldsymbol{\Psi}}^{\pm}_k(x) = \lambda_k^{\pm} \widetilde{\boldsymbol{\Psi}}^{\pm}_k(x) , \tag{8.175a}$$

$$\widetilde{\Lambda}_- \widetilde{\boldsymbol{\Phi}}^{\pm}(x,\lambda) = \lambda \widetilde{\boldsymbol{\Phi}}^{\pm}(x,\lambda), \qquad \widetilde{\Lambda}_- \widetilde{\boldsymbol{\Phi}}^{\pm}_k(x) = \lambda_k^{\pm} \widetilde{\boldsymbol{\Phi}}^{\pm}_k(x) , \tag{8.175b}$$

$$\widetilde{\Lambda} \widetilde{\boldsymbol{P}}^{\pm}(x,\lambda) = \lambda \widetilde{\boldsymbol{P}}^{\pm}(x,\lambda), \qquad \widetilde{\Lambda} \widetilde{\boldsymbol{P}}^{\pm}_k(x) = \lambda_k^{\pm} \widetilde{\boldsymbol{P}}^{\pm}_k(x) , \tag{8.175c}$$

$$\widetilde{\Lambda} \widetilde{\boldsymbol{Q}}^{\pm}(x,\lambda) = \lambda \widetilde{\boldsymbol{Q}}^{\pm}(x,\lambda), \qquad \widetilde{\Lambda} \widetilde{\boldsymbol{Q}}^{\pm}_k(x) = \lambda_k^{\pm} \widetilde{\boldsymbol{Q}}^{\pm}_k(x) . \tag{8.175d}$$

In addition, applying the gauge transformation to the (5.99b), (5.100b) and (5.101) we get:

$$(\widetilde{\Lambda}_- - \lambda)\widetilde{\boldsymbol{\Psi}}^{\pm}(x,\lambda) = -\frac{i}{4}\left[S(x), \frac{\partial S}{\partial x}\right]\widetilde{a}^{\pm}(\lambda)\widetilde{b}^{\mp}(\lambda) , \tag{8.176a}$$

$$(\widetilde{\Lambda}_+ - \lambda)\widetilde{\boldsymbol{\Phi}}^{\pm}(x,\lambda) = -\frac{i}{4}\left[S(x), \frac{\partial S}{\partial x}\right]\widetilde{a}^{\pm}(\lambda)\widetilde{b}^{\pm}(\lambda) , \tag{8.176b}$$

$$(\widetilde{\Lambda}_+ - \lambda)\widetilde{\boldsymbol{\Theta}}^{\pm}(x,\lambda) = -\frac{i}{4}\left[S(x), \frac{\partial S}{\partial x}\right]\widetilde{a}^{\pm}(\lambda) , \tag{8.176c}$$

$$(\widetilde{\Lambda}_- - \lambda)\widetilde{\boldsymbol{\Theta}}^{\pm}(x,\lambda) = -\frac{i}{4}\left[S(x), \frac{\partial S}{\partial x}\right]\widetilde{a}^{\pm}(\lambda) , \tag{8.176d}$$

These important relations allow us to give an alternative proof of the lemma 8.5. It consists in comparing the two expressions for $\delta\rho^{\pm}(\lambda)$ given by (8.146) and (8.159) and using the following chain of relations:

$$[\![[S(x), \widetilde{\delta q(x)}], \widetilde{\boldsymbol{\Phi}}^{\pm}(x,\lambda)\,]\!]_\sim = \lambda[\![[S(x), \delta S(x)], \widetilde{\boldsymbol{\Phi}}^{\pm}(x,\lambda)\,]\!]_\sim$$

$$= [\![[S(x), \delta S(x)], \widetilde{\Lambda}_- \widetilde{\boldsymbol{\Phi}}^{\pm}(x,\lambda)]\!]_{\sim}$$

$$= [\![\widetilde{\Lambda}_+ [S(x), \delta S(x)], \widetilde{\boldsymbol{\Phi}}^{\pm}(x,\lambda)]\!]_{\sim} \; . \quad (8.177)$$

From them we conclude that the function

$$\widetilde{H}(x) = [S(x), \widetilde{\delta q(x)}] - \widetilde{\Lambda}_+ [S(x), \delta S(x)] \quad (8.178)$$

has vanishing skew-symmetric scalar products with the squared solutions $\widetilde{\boldsymbol{\Phi}}^{\pm}(x,\lambda)$ for all real values of λ. It is not difficult to check that also $[\![\widetilde{H}(x), \widetilde{\boldsymbol{\Phi}}_k^{\pm}(x)]\!]_{\sim} = 0$ for all $k = 1, \ldots, N$. In order to conclude that $\widetilde{H}(x) = 0$ we need to use the completeness of the family $\widetilde{\boldsymbol{\Phi}}^{\pm}(x,\lambda)$, $\widetilde{\boldsymbol{\Phi}}_k^{\pm}(x)$, a fact that we shall prove in the next section.

8.6 Generalized Fourier Transform and Gauge Transformations

We have already established that the gauge transformations preserve the analytic properties of the Jost solutions. An immediate consequence of this fact are the analyticity properties of the "squared" solutions $\widetilde{\boldsymbol{\Psi}}^{\pm}(x,\lambda)$ and $\widetilde{\boldsymbol{\Phi}}^{\pm}(x,\lambda)$ recall that the superscript + (resp. −) means analyticity in the upper (resp. lower) complex half-plane. Then in complete analogy with the Zakharov–Shabat system, we can introduce the Green function:

$$\widetilde{G}^{\pm}(x,y,\lambda) = \widetilde{G}_1^{\pm}(x,y,\lambda)\theta(x-y) - \widetilde{G}_2^{\pm}(x,y,\lambda)\theta(y-x) \; , \quad (8.179)$$

$$\widetilde{G}_1^{\pm}(x,y,\lambda) = \frac{1}{(\widetilde{a}^{\pm}(\lambda))^2} \widetilde{\boldsymbol{\Psi}}^{\pm}(x,\lambda) \otimes \widetilde{\boldsymbol{\Phi}}^{\pm}(y,\lambda) \; , \quad (8.180)$$

$$\widetilde{G}_2^{\pm}(x,y,\lambda) = \frac{1}{(\widetilde{a}^{\pm}(\lambda))^2} \Big(\widetilde{\boldsymbol{\Phi}}^{\pm}(x,\lambda) \otimes \widetilde{\boldsymbol{\Psi}}^{\pm}(y,\lambda)$$
$$+ \frac{1}{2} \widetilde{\boldsymbol{\Theta}}^{\pm}(x,\lambda) \otimes \widetilde{\boldsymbol{\Theta}}^{\pm}(y,\lambda) \Big) \; . \quad (8.181)$$

and apply the contour integration method for the integral

$$\widetilde{\mathcal{J}}_{\widetilde{G}}(x,y) = \frac{1}{2\pi i} \left(\oint_{C_+} d\lambda \, \widetilde{G}^{+}(x,y,\lambda) - \oint_{C_-} d\lambda \, \widetilde{G}^{-}(x,y,\lambda) \right) \; . \quad (8.182)$$

One can evaluate $\widetilde{\mathcal{J}}_{\widetilde{G}}(x,y)$ by the residue theorem having in mind that the poles of $\widetilde{G}^{\pm}$ coincide with $\lambda_k^{\pm}$; if $\widetilde{a}^{\pm}(\lambda)$ have first-order zeroes at $\lambda_k^{\pm}$, then $\widetilde{G}^{\pm}$ will have second-order poles at these points. Next, we integrate directly along the contours. The integration along the real λ-axis gives us the contribution from the continuous spectrum, while integrating along the infinite semicircles of the contours $C_\pm$ results in the terms with $\delta(x-y)$. Equating both answers for $\widetilde{\mathcal{J}}_{\widetilde{G}}(x,y)$ leads to the completeness relation.

Another way to derive it consists in applying the gauge transformation to (5.59). In this way, we get

$$\begin{aligned}
&\delta(x-y)\widetilde{\Pi}_0(x,y) \\
&= -\frac{1}{\pi}\int_{-\infty}^{\infty} d\lambda \left(\frac{\widetilde{\boldsymbol{\Psi}}^+(x,\lambda)\otimes\widetilde{\boldsymbol{\Phi}}^+(y,\lambda)}{(\widetilde{a}^+(\lambda))^2} - \frac{\widetilde{\boldsymbol{\Psi}}^-(x,\lambda)\otimes\widetilde{\boldsymbol{\Phi}}^-(y,\lambda)}{(\widetilde{a}^-(\lambda))^2}\right) \\
&\quad + 2i\sum_{k=1}^{N}\left(\widetilde{X}_k^+(x,y) + \widetilde{X}_k^-(x,y)\right) ,
\end{aligned} \tag{8.183a}$$

$$\widetilde{\Pi}_0 = \widetilde{\sigma}_+(x)\otimes\widetilde{\sigma}_-(y) - \widetilde{\sigma}_-(x)\otimes\widetilde{\sigma}_+(y) , \tag{8.183b}$$

where $\widetilde{\sigma}_\pm(x)$ are defined in (8.40) and

$$\begin{aligned}
\widetilde{X}_k^\pm(x,y) = \frac{1}{(\dot{\widetilde{a}}_k^\pm)^2}&\left(\widetilde{\boldsymbol{\Psi}}_k^\pm(x)\otimes\dot{\widetilde{\boldsymbol{\Phi}}}_k^\pm(y) + \dot{\widetilde{\boldsymbol{\Psi}}}_k^\pm(x)\otimes\widetilde{\boldsymbol{\Phi}}_k^\pm(y)\right. \\
&\left. -\frac{\ddot{\widetilde{a}}_k^\pm}{\dot{\widetilde{a}}_k^\pm}\widetilde{\boldsymbol{\Psi}}_k^\pm(x)\otimes\widetilde{\boldsymbol{\Phi}}_k^\pm(y)\right) .
\end{aligned} \tag{8.183c}$$

The completeness relation for the symplectic basis has the form (compare with (5.63)):

$$\begin{aligned}
\delta(x-y)\widetilde{\Pi}_0(x,y) &= \int_{-\infty}^{\infty} d\lambda \left(\widetilde{\boldsymbol{P}}(x,\lambda)\otimes\widetilde{\boldsymbol{Q}}(y,\lambda) - \widetilde{\boldsymbol{Q}}(x,\lambda)\otimes\widetilde{\boldsymbol{P}}(y,\lambda)\right) \\
&\quad + \sum_{k=1}^{N}\left(\widetilde{Z}_k^+(x,y) + \widetilde{Z}_k^-(x,y)\right) ,
\end{aligned} \tag{8.184a}$$

$$\widetilde{Z}_k^\pm(x,y) = \left(\widetilde{\boldsymbol{P}}_k^\pm(x)\otimes\widetilde{\boldsymbol{Q}}_k^\pm(y) - \widetilde{\boldsymbol{Q}}_k^\pm(x)\otimes\widetilde{\boldsymbol{P}}_k^\pm(y)\right) . \tag{8.184b}$$

As we see below $\widetilde{\Pi}_0(x,y)$ is compatible with the splitting (8.44).

Using (8.184), one can expand any function $\widetilde{X}(x) \equiv \pi_S\widetilde{X}(x)$ satisfying condition $\widetilde{\mathrm{C}}.1$ over the "squared" solutions of $\widetilde{L}$. To this end, we use

$$\widetilde{X}(x) = \widetilde{X}_+(x)\widetilde{\sigma}_+ + \widetilde{X}_-(x)\widetilde{\sigma}_- , \tag{8.185}$$

and (8.183b) to get:

$$\begin{aligned}
\frac{1}{2}\mathrm{tr}_1\left([S(x),\widetilde{X}(x)]\otimes\mathbb{1}\right)\widetilde{\Pi}_0 &= -\frac{1}{2}\mathrm{tr}_2\widetilde{\Pi}_0\left(\mathbb{1}\otimes[S(x),\widetilde{X}(x)]\right)\widetilde{\Pi}_0 \\
&= -\widetilde{X}(x) .
\end{aligned} \tag{8.186}$$

where tr_1 (and tr_2) mean that we are taking the trace of the elements in the first (or the second) position of the tensor product. In this way, we get:

$$
\begin{aligned}
\widetilde{X}(x) = & -\frac{1}{\pi}\int_{-\infty}^{\infty} d\lambda\, \left(\widetilde{\phi}_X^{+}(\lambda)\widetilde{\boldsymbol{\Psi}}^{+}(x,\lambda) - \widetilde{\phi}_X^{-}(\lambda)\widetilde{\boldsymbol{\Psi}}^{-}(x,\lambda)\right) \\
& + 2i \sum_{k=1}^{N}{}^{\pm} \left(\widetilde{\phi}_{X,k}^{\pm}\dot{\widetilde{\boldsymbol{\Psi}}}{}_k^{\pm} + \dot{\widetilde{\phi}}{}_{X,k}^{\pm}\widetilde{\boldsymbol{\Psi}}_k^{\pm}\right) ,
\end{aligned}
\tag{8.187}
$$

$$
\widetilde{\phi}_X^{\pm}(\lambda) = \frac{\underset{\sim}{[\![}\widetilde{\boldsymbol{\Phi}}^{\pm}(x,\lambda), \widetilde{X}(x)\underset{\sim}{]\!]}}{(\widetilde{a}^{\pm}(\lambda))^2}, \qquad \widetilde{\phi}_{X,k}^{\pm} = \frac{\underset{\sim}{[\![}\widetilde{\boldsymbol{\Phi}}_k^{\pm}(x), \widetilde{X}(x)\underset{\sim}{]\!]}}{(\dot{\widetilde{a}}{}_k^{\pm})^2} , \tag{8.188a}
$$

$$
\dot{\widetilde{\phi}}{}_{X,k}^{\pm} = \frac{1}{(\dot{\widetilde{a}}{}_k^{\pm})^2}\left(\underset{\sim}{[\![}\dot{\widetilde{\boldsymbol{\Phi}}}{}_k^{\pm}(x), \widetilde{X}(x)\underset{\sim}{]\!]} - \frac{\ddot{\widetilde{a}}{}_k^{\pm}}{\dot{\widetilde{a}}{}_k^{\pm}} \underset{\sim}{[\![}\widetilde{\boldsymbol{\Phi}}_k^{\pm}(x), \widetilde{X}(x)\underset{\sim}{]\!]} \right) . \tag{8.188b}
$$

The same procedure, applied to the completeness relation (8.184) for the symplectic basis leads to:

$$
\begin{aligned}
\widetilde{X}(x) = & \int_{-\infty}^{\infty} d\lambda\, \left(\widetilde{\kappa}_X(\lambda)\widetilde{\boldsymbol{P}}(x,\lambda) - \widetilde{\eta}_X(\lambda)\widetilde{\boldsymbol{Q}}(x,\lambda)\right) \\
& + \sum_{k=1}^{N}{}^{\pm} \left(\widetilde{\kappa}_{X,k}^{\pm}\widetilde{\boldsymbol{P}}_k^{\pm} - \widetilde{\eta}_{X,k}^{\pm}\widetilde{\boldsymbol{Q}}_k^{\pm}\right) ,
\end{aligned}
\tag{8.189}
$$

$$
\widetilde{\kappa}_X(\lambda) = \underset{\sim}{[\![}\widetilde{\boldsymbol{Q}}(x,\lambda), \widetilde{X}(x)\underset{\sim}{]\!]}, \qquad \widetilde{\eta}_X(\lambda) = \underset{\sim}{[\![}\widetilde{\boldsymbol{P}}(x,\lambda), \widetilde{X}(x)\underset{\sim}{]\!]} \tag{8.190a}
$$

$$
\widetilde{\kappa}_{X,k}^{\pm} = [\![\widetilde{\boldsymbol{Q}}_k^{\pm}(x), \widetilde{X}(x)]\!], \qquad \widetilde{\eta}_{X,k}^{\pm} = \underset{\sim}{[\![}\widetilde{\boldsymbol{P}}_k^{\pm}(x), \widetilde{X}(x)\underset{\sim}{]\!]} \tag{8.190b}
$$

The completeness relations we obtained in the above allow us to establish a one-to-one correspondence between the element $\widetilde{X}(x) \in \widetilde{\mathcal{M}}$ and its expansion coefficients. Indeed, from (8.183) and (8.184), we derived the expansions (8.187) and (8.189) with the inversion formulae (8.188) and (8.190), respectively. Using them we can prove the following:

Proposition 8.1 *The function $\widetilde{X}(x) \equiv 0$ if and only if one of the following sets of relations holds:*

$$
\widetilde{\phi}_X^{+}(\lambda) = \widetilde{\phi}_X^{-}(\lambda) \equiv 0, \qquad \lambda \in \mathbb{R} , \tag{8.191a}
$$

$$
\widetilde{\phi}_{X,k}^{\pm} = \dot{\widetilde{\phi}}{}_{X,k}^{\pm} = 0, \qquad k = 1, \dots, N ; \tag{8.191b}
$$

or

$$
\widetilde{\kappa}_X(\lambda) = \widetilde{\eta}_X(\lambda) \equiv 0, \qquad \lambda \in \mathbb{R} , \tag{8.192a}
$$

$$
\widetilde{\kappa}_{X,k}^{\pm} = \widetilde{\eta}_{X,k}^{\pm} = 0, \qquad k = 1, \dots, N ; \tag{8.192b}
$$

Proof. Let us show that from $\widetilde{X}(x) \equiv 0$ there follows (8.191). To this end, we insert $\widetilde{X}(x) \equiv 0$ into the right-hand sides of the inversion formulae (8.188) and immediately get (8.191). The fact that from (8.191) there follows $\widetilde{X}(x) \equiv 0$ is readily obtained by inserting it into the right-hand side of (8.187).

The equivalence of $\widetilde{X}(x) \equiv 0$ and (8.191) or (8.192) is proved analogously using the inversion formulae (8.188), (8.190) and the expansions (8.187) and (8.189). The proposition is proved.

Of special importance are the expansions of $\widetilde{q}(x)$ and $\widetilde{\delta q(x)}$. The Wronskian relations that we derived above allow to derive their expansion coefficients. Thus, we find (compare with (5.78) and (5.80)):

$$\begin{aligned} \widetilde{q}(x) &= -\frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda \left(\widetilde{\rho}^{+}(\lambda)\widetilde{\boldsymbol{\Psi}}^{+}(x,\lambda) - \widetilde{\rho}^{-}(\lambda)\widetilde{\boldsymbol{\Psi}}^{-}(x,\lambda)\right) \\ &\quad - 2\sum_{k=1}^{N}\left(\widetilde{C}_k^{+}\widetilde{\boldsymbol{\Psi}}_k^{+}(x) + \widetilde{C}_k^{-}\widetilde{\boldsymbol{\Psi}}_k^{-}(x)\right) . \end{aligned} \tag{8.193a}$$

where $\widetilde{C}_k^{\pm} = C_k^{\pm}$.

$$\widetilde{q}(x) = i\int_{-\infty}^{\infty} d\lambda\, \widetilde{\boldsymbol{P}}(x,\lambda) + i\sum_{k=1}^{N}\left(\widetilde{\boldsymbol{P}}_k^{+}(x) + \widetilde{\boldsymbol{P}}_k^{-}(x)\right) . \tag{8.193b}$$

Analogously, for $\widetilde{\delta q(x)}$ we have:

$$S(x)\widetilde{\delta q(x)} = \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda \left(\delta\widetilde{\rho}^{+}(\lambda)\widetilde{\boldsymbol{\Psi}}^{+}(x,\lambda) + \delta\widetilde{\rho}^{-}(\lambda)\widetilde{\boldsymbol{\Psi}}^{-}(x,\lambda)\right) \tag{8.194a}$$

$$+ 2\sum_{k=1}^{N}\left(\widetilde{C}_k^{+}\delta\lambda_k^{+}\dot{\widetilde{\boldsymbol{\Psi}}}_k^{+}(x) + \delta\widetilde{C}_k^{+}\widetilde{\boldsymbol{\Psi}}_k^{+}(x) - \widetilde{C}_k^{-}\delta\lambda_k^{-}\dot{\widetilde{\boldsymbol{\Psi}}}_k^{-}(x) - \delta\widetilde{C}_k^{-}\widetilde{\boldsymbol{\Psi}}_k^{-}(x)\right) .$$

and

$$\begin{aligned} S(x)\widetilde{\delta q(x)} &= i\int_{-\infty}^{\infty} d\lambda \left(\delta\widetilde{\kappa}(\lambda)\widetilde{\boldsymbol{P}}(x,\lambda) - \delta\widetilde{\eta}(\lambda)\widetilde{\boldsymbol{Q}}(x,\lambda)\right) \\ &\quad + i\sum_{k=1}^{N}\left(\delta\widetilde{\eta}_k^{+}\widetilde{\boldsymbol{Q}}_k^{+}(x) - \delta\widetilde{\kappa}_k^{+}\widetilde{\boldsymbol{P}}_k^{+}(x) + \delta\widetilde{\eta}_k^{-}\widetilde{\boldsymbol{Q}}_k^{-}(x) - \delta\widetilde{\kappa}_k^{-}\widetilde{\boldsymbol{P}}_k^{-}(x)\right) , \end{aligned} \tag{8.194b}$$

where

$$\widetilde{\eta}(\lambda) = \frac{1}{\pi}\ln\left(1 + \widetilde{\rho}^{+}(\lambda)\widetilde{\rho}^{-}(\lambda)\right) , \qquad \widetilde{\eta}_k^{\pm} = \mp 2i\lambda_k^{\pm} , \tag{8.195a}$$

$$\widetilde{\kappa}(\lambda) = \frac{1}{2}\ln\frac{\widetilde{b}^{+}(\lambda)}{\widetilde{b}^{-}(\lambda)}, \qquad \widetilde{\kappa}_k^{\pm} = \pm\ln\widetilde{b}_k^{\pm} . \tag{8.195b}$$

In the next chapter, we shall see how this set of variables is related to the action–angle variables of the corresponding NLEE.

Now we apply the expansions to variations of the type:

$$\delta\widetilde{q} = \frac{\partial \widetilde{q}}{\partial t}\delta t \,. \tag{8.196}$$

We easily get:

$$\begin{aligned} S(x)\frac{\partial \widetilde{q}}{\partial t} &= \frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda\, \left(\widetilde{\rho}_t^{+}(\lambda)\widetilde{\boldsymbol{\Psi}}^{+}(x,\lambda) + \widetilde{\rho}_t^{-}(\lambda)\widetilde{\boldsymbol{\Psi}}^{-}(x,\lambda)\right) \\ &+ 2\sum_{k=1}^{N}\left(\widetilde{C}_k^{+}\lambda_{k,t}^{+}\dot{\widetilde{\boldsymbol{\Psi}}}_k^{+}(x) + \widetilde{C}_{k,t}^{+}\widetilde{\boldsymbol{\Psi}}_k^{+}(x) - \widetilde{C}_{k,t}^{-}\lambda_{k,t}^{-}\dot{\widetilde{\boldsymbol{\Psi}}}_k^{-}(x) - \widetilde{C}_{k,t}^{-}\widetilde{\boldsymbol{\Psi}}_k^{-}(x)\right). \end{aligned} \tag{8.197a}$$

and

$$\begin{aligned} S(x)\frac{\partial \widetilde{q}}{\partial t} &= i\int_{-\infty}^{\infty} d\lambda\, \left(\widetilde{\kappa}_t(\lambda)\widetilde{\boldsymbol{P}}(x,\lambda) - \widetilde{\eta}_t(\lambda)\widetilde{\boldsymbol{Q}}(x,\lambda)\right) \\ &+ i\sum_{k=1}^{N}\left(\widetilde{\eta}_{k,t}^{+}\widetilde{\boldsymbol{Q}}_k^{+}(x) - \widetilde{\kappa}_{k,t}^{+}\widetilde{\boldsymbol{P}}_k^{+}(x) + \widetilde{\eta}_{k,t}^{-}\widetilde{\boldsymbol{Q}}_k^{-}(x) - \widetilde{\kappa}_{k,t}^{-}\widetilde{\boldsymbol{P}}_k^{-}(x)\right), \end{aligned} \tag{8.197b}$$

Using the first set of Wronskian relations (8.136), (8.137) (resp. (8.146), (8.147)), we can evaluate the expansion coefficients of $\pi_S\sigma_3$ (resp. $[S(x,t),\delta S]$) with the result:

$$\begin{aligned} \pi_S\sigma_3 &= -\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{d\lambda}{\lambda}\, \left(\widetilde{\rho}^{+}(\lambda)\widetilde{\boldsymbol{\Psi}}^{+}(x,\lambda) - \widetilde{\rho}^{-}(\lambda)\widetilde{\boldsymbol{\Psi}}^{-}(x,\lambda)\right) \\ &\quad -2\sum_{k=1}^{N}\left(\frac{\widetilde{C}_k^{+}}{\lambda_k^{+}}\widetilde{\boldsymbol{\Psi}}_k^{+}(x) + \frac{\widetilde{C}_k^{-}}{\lambda_k^{-}}\widetilde{\boldsymbol{\Psi}}_k^{-}(x)\right). \end{aligned} \tag{8.198a}$$

$$\pi_S\sigma_3 = i\int_{-\infty}^{\infty}\frac{d\lambda}{\lambda}\,\widetilde{\boldsymbol{P}}(x,\lambda) + i\sum_{k=1}^{N}\left(\frac{1}{\lambda_k^{+}}\widetilde{\boldsymbol{P}}_k^{+}(x) + \frac{1}{\lambda_k^{-}}\widetilde{\boldsymbol{P}}_k^{-}(x)\right)\,. \tag{8.198b}$$

Analogously for $[S(x),\delta S]$ we have:

$$\begin{aligned} [S(x),\delta S(x)] &= \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{d\lambda}{\lambda}\,\left(\delta\widetilde{\rho}^{+}(\lambda)\widetilde{\boldsymbol{\Psi}}^{+}(x,\lambda) + \delta\widetilde{\rho}^{-}(\lambda)\widetilde{\boldsymbol{\Psi}}^{-}(x,\lambda)\right) \\ &+ 2\sum_{k=1}^{N}\left(\widetilde{C}_k^{+}\delta\ln\lambda_k^{+}\dot{\widetilde{\boldsymbol{\Psi}}}_k^{+}(x) + \frac{\delta\widetilde{C}_k^{+}}{\lambda_k^{+}}\widetilde{\boldsymbol{\Psi}}_k^{+}(x) - \widetilde{C}_k^{-}\delta\ln\lambda_k^{-}\dot{\widetilde{\boldsymbol{\Psi}}}_k^{-}(x) - \frac{\delta\widetilde{C}_k^{-}}{\lambda_k^{-}}\widetilde{\boldsymbol{\Psi}}_k^{-}(x)\right). \end{aligned} \tag{8.199a}$$

$$\begin{aligned} [S(x),\delta S(x)] &= i\int_{-\infty}^{\infty}\frac{d\lambda}{\lambda}\,\left(\delta\widetilde{\kappa}(\lambda)\widetilde{\boldsymbol{P}}(x,\lambda) - \delta\widetilde{\eta}(\lambda)\widetilde{\boldsymbol{Q}}(x,\lambda)\right) \\ &+ i\sum_{k=1}^{N}\left(\frac{\delta\widetilde{\eta}_k^{+}}{\lambda_k^{+}}\widetilde{\boldsymbol{Q}}_k^{+}(x) - \frac{\delta\widetilde{\kappa}_k^{+}}{\lambda_k^{+}}\widetilde{\boldsymbol{P}}_k^{+}(x) + \frac{\delta\widetilde{\eta}_k^{-}}{\lambda_k^{-}}\widetilde{\boldsymbol{Q}}_k^{-}(x) - \frac{\delta\widetilde{\kappa}_k^{-}}{\lambda_k^{-}}\widetilde{\boldsymbol{P}}_k^{-}(x)\right), \end{aligned} \tag{8.199b}$$

For $\delta S(x)$ of the type $\frac{\partial S}{\partial t}\delta t$ we get:

$$\left[S(x), \frac{\partial S}{\partial t}\right] = \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{d\lambda}{\lambda}\left(\widetilde{\rho}_t^{+}(\lambda)\widetilde{\boldsymbol{\Psi}}^{+}(x,\lambda) + \widetilde{\rho}_t^{-}(\lambda)\widetilde{\boldsymbol{\Psi}}^{-}(x,\lambda)\right) \tag{8.200a}$$

$$+ 2\sum_{k=1}^{N}\left(\widetilde{C}_k^{+}\frac{\lambda_{k,t}^{+}}{\lambda_k^{+}}\dot{\widetilde{\boldsymbol{\Psi}}}_k^{+}(x) + \frac{\widetilde{C}_{k,t}^{+}}{\lambda_k^{+}}\widetilde{\boldsymbol{\Psi}}_k^{+}(x) - \widetilde{C}_{k,t}^{-}\frac{\lambda_{k,t}^{-}}{\lambda_k^{-}}\dot{\widetilde{\boldsymbol{\Psi}}}_k^{-}(x) - \frac{\widetilde{C}_{k,t}^{-}}{\lambda_k^{-}}\widetilde{\boldsymbol{\Psi}}_k^{-}(x)\right).$$

$$\left[S(x), \frac{\partial S}{\partial t}\right] = i\int_{-\infty}^{\infty} \frac{d\lambda}{\lambda}\left(\widetilde{\kappa}_t(\lambda)\widetilde{\boldsymbol{P}}(x,\lambda) - \widetilde{\eta}_t(\lambda)\widetilde{\boldsymbol{Q}}(x,\lambda)\right) \tag{8.200b}$$

$$+ i\sum_{k=1}^{N}\left(\frac{\widetilde{\eta}_{k,t}^{+}}{\lambda_k^{+}}\widetilde{\boldsymbol{Q}}_k^{+}(x) - \frac{\widetilde{\kappa}_{k,t}^{+}}{\lambda_k^{+}}\widetilde{\boldsymbol{P}}_k^{+}(x) + \frac{\widetilde{\eta}_{k,t}^{-}}{\lambda_k^{-}}\widetilde{\boldsymbol{Q}}_k^{-}(x) - \frac{\widetilde{\kappa}_{k,t}^{-}}{\lambda_k^{-}}\widetilde{\boldsymbol{P}}_k^{-}(x)\right),$$

The expansions (8.198) are compatible with (8.193). To demonstrate this, we apply the operator $\widetilde{\Lambda}_+$ (resp. $\widetilde{\Lambda}$) to the right-hand side of (8.193a) (resp. to (8.193b)), take into account the relation (8.162), and use the fact that $\widetilde{\boldsymbol{\Psi}}^{\pm}(x,\lambda)$ (resp. $\boldsymbol{P}(x,\lambda)$, $\boldsymbol{Q}(x,\lambda)$) are eigenfunctions of $\widetilde{\Lambda}_+$ (resp. $\widetilde{\Lambda}$); see (8.175). In the same manner, using lemma 8.5, we demonstrate the compatibility of the (8.194), (8.197) to the (8.199), (8.200).

8.7 Fundamental Properties of the Gauge-Equivalent NLEEs

Now, we are ready to describe the fundamental properties of the HF-type NLEEs. Again, we have the choice to either use the expansions over the "squared" solutions derived above or to apply the gauge transformation to the results about the NLS type equations we had in Chap. 6. The latter approach applied to theorem 6.1 directly produces the following.

Theorem 8.6. *Let the function $f(\lambda)$ be meromorphic for $\lambda \in \mathbb{C}$ and has no singularities on the spectrum of $\widetilde{L}$. Then, the NLEE:*

$$iS(x)\frac{\partial \widetilde{q}}{\partial t} + 2f(\widetilde{\Lambda}_+)\widetilde{q}(x,t) = 0\,, \tag{8.201a}$$

$$iS(x)\frac{\partial \widetilde{q}}{\partial t} + 2f(\widetilde{\Lambda}_-)\widetilde{q}(x,t) = 0\,, \tag{8.201b}$$

$$iS(x)\frac{\partial \widetilde{q}}{\partial t} + 2f(\widetilde{\Lambda})\widetilde{q}(x,t) = 0\,, \tag{8.201c}$$

are pairwise equivalent to the following linear evolution equations for the scattering data:

$$i\widetilde{\rho}_t^{\pm} \mp 2f(\lambda)\widetilde{\rho}^{\pm}(\lambda,t) = 0\,,$$

$$
\begin{aligned}
i\widetilde{C}^{\pm}_{k,t} \mp 2f^{\pm}_{k}\widetilde{C}^{\pm}_{k}(t) &= 0\,, \\
\lambda^{\pm}_{k,t}\widetilde{C}^{\pm}_{k}(t) &= 0\,,
\end{aligned}
\tag{8.202a}
$$

$$
\begin{aligned}
i\widetilde{\tau}^{\pm}_{t} \pm 2f(\lambda)\widetilde{\tau}^{\pm}(\lambda,t) &= 0\,, \\
i\widetilde{M}^{\pm}_{k,t} \pm 2f^{\pm}_{k}\widetilde{M}^{\pm}_{k}(t) &= 0\,, \\
\lambda^{\pm}_{k,t}\widetilde{M}^{\pm}_{k}(t) &= 0,
\end{aligned}
\tag{8.202b}
$$

$$
\begin{aligned}
i\widetilde{\eta}_{t} = 0, &\qquad i\widetilde{\kappa}_{t} - 2f(\lambda) = 0\,, \\
i\widetilde{\eta}^{\pm}_{k,t} = 0, &\qquad i\widetilde{\kappa}^{\pm}_{k,t} - 2f^{\pm}_{k} = 0,
\end{aligned}
\tag{8.202c}
$$

Theorem 8.7. *Let the function $\widetilde{f}(\lambda)$ be smooth for real λ and such that it has no singularities on the spectrum of $\widetilde{L}$ and $\widetilde{f}(0) = 0$. Then the NLEE:*

$$
iS(x,t)\frac{\partial S}{\partial t} - \frac{i}{4}\widetilde{f}(\widetilde{\Lambda}_{+})\left[S(x,t), \frac{\partial S}{\partial x}\right] = 0\,, \tag{8.203a}
$$

$$
iS(x,t)\frac{\partial S}{\partial t} - \frac{i}{4}\widetilde{f}(\widetilde{\Lambda}_{-})\left[S(x,t), \frac{\partial S}{\partial x}\right] = 0\,, \tag{8.203b}
$$

$$
iS(x,t)\frac{\partial S}{\partial t} - \frac{i}{4}\widetilde{f}(\widetilde{\Lambda})\left[S(x,t), \frac{\partial S}{\partial x}\right] = 0\,, \tag{8.203c}
$$

are pairwise equivalent: (i) to the linear evolution equations (8.202a), (8.202b) and (8.202c) for the scattering data with $f(\lambda) = \lambda\widetilde{f}(\lambda)$ and (ii) to the NLEE (8.201a), (8.201b) and (8.201c) with $f(\lambda) = \lambda\widetilde{f}(\lambda)$.

It is not difficult to find that the class of the solvable NLEEs can be written down also in the form:

$$
2F(\widetilde{\Lambda}_{\pm})\widetilde{\Lambda}_{\pm}[S, S_t] + iG(\widetilde{\Lambda}_{\pm})[S, S_x] = 0\,. \tag{8.204}
$$

Needless to say, these equations are equivalent to the linear equations (8.202) for the scattering data, provided the functions $F(\lambda)$ and $G(\lambda)$ are such that $G(\lambda) = f(\lambda)F(\lambda)$. As before, one can write in (8.204) either the recursion operators $\widetilde{\Lambda}_{\pm}$ or the operator $\widetilde{\Lambda}$ without changing anything.

Sometimes one prefers to describe the same set of equations by means of the adjoint operators $\widetilde{\Lambda}^{*}_{\mp}$. If we introduce $F_1(\lambda) = 2\lambda F(\lambda)$ and $G_1(\lambda) = iG(\lambda)$, then with the help of Lemma 8.5 we have [2]

$$
F_1(\widetilde{\Lambda}^{*}_{\mp})S_t + G_1(\widetilde{\Lambda}^{*}_{\mp})S_x = 0 \tag{8.205}
$$

Written like that the equations are invariant under the permutation $x \leftrightarrow t$, $F_1 \leftrightarrow G_1$.

Remark 8.8. A simple argument applied to (8.204) shows that they may be written in the symmetric form as well:

$$
F_2(\widetilde{\Lambda}^{*}_{\mp})q_t + G_2(\widetilde{\Lambda}^{*}_{\mp})q_x = 0\,, \tag{8.206}
$$

where $F_2 = \frac{1}{2}F$ and $G_2(\lambda) = \frac{i}{4}\lambda^{-1}G(\lambda)$.

Note that for $F(\lambda) = 1$, $G(\lambda) = 4i\lambda^2$, one gets the HFE (8.1), which is gauge equivalent to the NLSE. The choice $F = 1/(8i)$ and $G(\lambda) = f_0\lambda^3 + f_1\lambda^2$ leads to the (8.58), gauge equivalent to the NLS-mKdV type (6.23).

These NLEEs have an infinite number of integrals of motion. They are generated by $\widetilde{\mathcal{A}}(\lambda)$ and can be expressed through the scattering data of $\widetilde{L}$ using the dispersion expansions:

$$\begin{aligned}
\widetilde{\mathcal{A}}(\lambda) &= \mathcal{A}(\lambda) - \mathcal{A}(0) \\
&= \frac{\lambda}{2\pi i} \int_{-\infty}^{\infty} \frac{d\mu}{\mu(\mu-\lambda)} \ln(\widetilde{a}^+(\mu)\widetilde{a}^-(\mu)) + \sum_{k=1}^{N} \ln \frac{(\lambda-\lambda_k^+)\lambda_k^-}{(\lambda-\lambda_k^-)\lambda_k^-} \\
&= \frac{\lambda}{2} \int_{-\infty}^{\infty} \frac{d\mu}{\mu(\mu-\lambda)} \widetilde{\eta}(\mu) + \sum_{k=1}^{N} \ln \frac{2\lambda/\eta_k^+ + 1}{-2\lambda/\eta_k^- + 1} \,, \qquad (8.207)
\end{aligned}$$

If we consider the asymptotic expansion of $\widetilde{\mathcal{A}}(\lambda)$ over the positive and negative powers of λ:

$$\widetilde{\mathcal{A}}(\lambda) = i \sum_{p=1}^{\infty} \widetilde{C}_p \lambda^{-p}, \qquad \widetilde{\mathcal{A}}(\lambda) = -i \sum_{p=1}^{\infty} \widetilde{C}_{-p} \lambda^{p} \,, \qquad (8.208)$$

then from (8.207) we find:

$$\widetilde{C}_p = -\frac{1}{2} \int_{-\infty}^{\infty} d\mu\, \mu^{p-1} \widetilde{\eta}(\mu) + \frac{i}{p} \sum_{k=1}^{N} \left(\left(\frac{i\widetilde{\eta}_k^+}{2} \right)^p - \left(\frac{\widetilde{\eta}_k^-}{2i} \right)^p \right) \,, \qquad (8.209)$$

$$\widetilde{C}_{-p} = -\frac{1}{2} \int_{-\infty}^{\infty} \frac{d\mu}{\mu^{p+1}} \widetilde{\eta}(\mu) - \frac{i}{p} \sum_{k=1}^{N} \left(\left(\frac{i\widetilde{\eta}_k^+}{2} \right)^{-p} - \left(\frac{\widetilde{\eta}_k^-}{2i} \right)^{-p} \right) \,, \qquad (8.210)$$

where $p = 1, 2, \dots$. The convergence of the integrals defining $\widetilde{C}_p$, $p > 0$ is a consequence of conditions $\widetilde{\mathrm{C}}.1$ and $\widetilde{\mathrm{C}}.3$, which ensure that $\widetilde{\rho}^{\pm}(t,\lambda)$, $\widetilde{\tau}^{\pm}(t,\lambda)$ and $\widetilde{\eta}(\lambda)$ are Schwartz–type functions of λ for $\lambda \in \mathbb{R}$ and vanish for $\lambda = 0$ along with their first few derivatives.

There exist a close relation between the densities of $\widetilde{C}_p$ and the diagonal of the resolvent

$$\widetilde{R}^{\pm}(x,\lambda) = \pm \frac{i}{2} \widetilde{\chi}^{\pm}(x,\lambda) \sigma_3 \hat{\widetilde{\chi}}^{\pm}(x,\lambda) = \pm \frac{i}{2\widetilde{a}^{\pm}(\lambda)} \widetilde{\Theta}^{\pm}(x,\lambda) \,, \qquad (8.211)$$

of $\widetilde{L}$. Considering it, we are able to obtain compact expressions for $\widetilde{C}_p$ through the generating operators $\widetilde{\Lambda}_{\pm}$. To this end, we recall the Wronskian relations (8.152)

$$\frac{d\widetilde{\mathcal{A}}}{d\lambda} = -i \int_{-\infty}^{\infty} dx \left(\langle \hat{\widetilde{\chi}}^{\pm} S(x) \widetilde{\chi}^{\pm}(x,\lambda) \sigma_3 \rangle - 1 \right)$$

$$
\begin{aligned}
&= \mp i \int_{-\infty}^{\infty} dx \, \left(\langle (S(x) \widetilde{R}^{\pm}(x,x,\lambda) \rangle) - 1 \right) \\
&= -\frac{i}{2\widetilde{a}^{\pm}(\lambda)} \int_{-\infty}^{\infty} dx \int_{\pm\infty}^{x} dy \, \left\langle S_y, \widetilde{\boldsymbol{\Theta}}^{\pm}(y,\lambda) \right\rangle \, . \qquad (8.212)
\end{aligned}
$$

This is the relation between the diagonal of the resolvent of $\widetilde{L}$ and the generating functional of integrals of motion of the HF-type NLEE.

In deriving the last line of (8.212), we used the fact that both $\widetilde{R}^{\pm}(x,\lambda)$ and $\widetilde{\Theta}^{\pm}(x,\lambda)$ satisfy the equation:

$$
i\frac{d\widetilde{\Theta}^{\pm}}{dx} - \lambda [S(x), \widetilde{\Theta}^{\pm}(x,\lambda)] \equiv [\widetilde{L}(\lambda), \widetilde{\Theta}^{\pm}(x,\lambda)] = 0 \, . \qquad (8.213)
$$

This equation shows that $\widetilde{\Theta}^{\pm}(x,\lambda)$ generates matrix-valued functions commuting with $\widetilde{L}(\lambda)$. Note that $\widetilde{\Theta}^{\pm}(x,\lambda)$ in the right-hand side of (8.211) has both "diagonal" and "off-diagonal" parts, while in the right-hand side of (8.212) only its "off-diagonal" part $\widetilde{\boldsymbol{\Theta}}^{\pm}(x,\lambda)$ contributes.

The "squared solution" $\widetilde{\boldsymbol{\Theta}}^{\pm}(x,\lambda)$, as noted above, satisfies also (8.176c), (8.176d) which using (8.162) provides:

$$
(\widetilde{\Lambda}_{\pm} - \lambda) \frac{\widetilde{\boldsymbol{\Theta}}^{\pm}(x,\lambda)}{\widetilde{a}^{\pm}(\lambda)} \equiv (\widetilde{\Lambda} - \lambda) \frac{\widetilde{\boldsymbol{\Theta}}^{\pm}(x,\lambda)}{\widetilde{a}^{\pm}(\lambda)} = \widetilde{\Lambda} \pi_S \sigma_3 \, . \qquad (8.214)
$$

If we apply now the operator $(\widetilde{\Lambda} - \lambda)^{-1}$ to both sides of (8.214) and expand formally its right-hand side over the powers of λ^{-1} we get:

$$
\begin{aligned}
\frac{\widetilde{\boldsymbol{\Theta}}^{\pm}(x,\lambda)}{\widetilde{a}^{\pm}(\lambda)} &= (\widetilde{\Lambda} - \lambda)^{-1} \widetilde{\Lambda} \pi_S \sigma_3 \\
&= -\sum_{s=0}^{\infty} \lambda^{-s-1} \widetilde{\Lambda}^{s+1} \pi_S \sigma_3 \, . \qquad (8.215)
\end{aligned}
$$

Let us denote by $\widetilde{V}_s(x)$ and $\widetilde{V}_s^{\mathrm{f}}(x)$ the expansion coefficients of $\widetilde{\Theta}(x,\lambda)$ and $\widetilde{\boldsymbol{\Theta}}(x,\lambda)$, respectively:

$$
\frac{\widetilde{\Theta}^{\pm}(x,\lambda)}{\widetilde{a}^{\pm}(\lambda)} = \sum_{s=1}^{\infty} \lambda^{-s} \widetilde{V}_s(x), \qquad \frac{\widetilde{\boldsymbol{\Theta}}^{\pm}(x,\lambda)}{\widetilde{a}^{\pm}(\lambda)} = \sum_{s=1}^{\infty} \lambda^{-s} \widetilde{V}_s^{\mathrm{f}}(x) \, . \qquad (8.216)
$$

Next, if we recall the splitting (8.44) $\widetilde{V}_s(x) = \widetilde{V}_s^{\mathrm{d}}(x) + \widetilde{V}_s^{\mathrm{f}}(x)$, we shall get:

$$
\widetilde{V}_s^{\mathrm{f}}(x) = -\widetilde{\Lambda}^s \pi_S \sigma_3 \equiv \frac{i}{4} \widetilde{\Lambda}^{s-1} [S(x), S_x] \, , \qquad (8.217)
$$

$$
\widetilde{V}_k^{\mathrm{d}}(x,t) = iS(x,t) \int_{\pm\infty}^{x} dy \, \langle S_y, \widetilde{V}_k^{\mathrm{f}}(y,t) \rangle + S(x,t) \lim_{y \to \pm\infty} \langle \widetilde{V}_k^{\mathrm{d}}(y,t), S(y) \rangle \, .
$$

Therefore, if we choose the dispersion law to be $f(\lambda) = \lambda^N$ the corresponding operator $\widetilde{M}(\lambda)$ will have as potential

$$\widetilde{V}^{(N)}(x,\lambda) = \sum_{s=0}^{N} \lambda^{N-s} \widetilde{V}_s^{\mathrm{f}}(x) = \mathcal{P}_+ \left(\lambda^N \frac{\widetilde{\Theta}^{\pm}(x,\lambda)}{\widetilde{a}^{\pm}(\lambda)} \right) , \tag{8.218}$$

where the operator $\mathcal{P}_+$ applied to the series picks up only the coefficients with non-negative power of λ.

Thus, we have demonstrated that the diagonal of the resolvent is a generating functional of the Lax representations for the HF-type NLEE.

Analogous arguments applied to (8.148) allow to write that

$$\delta\widetilde{\mathcal{A}} = i\lambda \int_{-\infty}^{\infty} dx \left\langle \delta S(x) \frac{\Theta^{\pm}(x,\lambda)}{\widetilde{a}^{\pm}(\lambda)} \right\rangle . \tag{8.219}$$

If we combine this with the expansions (8.208) and (8.215), we get:

$$\begin{aligned} \delta\widetilde{C}_p &= \frac{1}{4} [\![[S(x), \widetilde{\delta q(x)}], \widetilde{\Lambda}_{\pm}^{p-1} \widetilde{q}(x)]\!]_{\sim} \\ &= \frac{1}{4} [\![\widetilde{\Lambda}[S(x), \delta S(x)], \widetilde{\Lambda}^{p} \pi_S \sigma_3]\!]_{\sim} \\ &= \frac{1}{4} [\![[S(x), \delta S(x)], \widetilde{\Lambda}^{p+1} \pi_S \sigma_3]\!]_{\sim} \\ &= -\frac{1}{2} \int_{-\infty}^{\infty} dx \, \mathrm{tr} \left(\delta S(x), \widetilde{\Lambda}^{p+1} \pi_S \sigma_3 \right) . \end{aligned} \tag{8.220}$$

This means that the variational derivatives of $\delta\widetilde{C}_p$ have the form:

$$\frac{\delta\widetilde{C}_p}{\delta\widetilde{S}^T(x)} = -\frac{1}{2} \widetilde{\Lambda}^{p+1} \pi_S \sigma_3, \qquad p = 1, 2, \ldots \tag{8.221}$$

Note that the right-hand side of (8.221) does not depend on the choice of the generating operator that we use; it will be the same for $\widetilde{\Lambda}_+$, $\widetilde{\Lambda}_-$, and $\widetilde{\Lambda}$. The Lenard relation is obtained easily from (8.221):

$$\frac{\delta\widetilde{C}_p}{\delta\widetilde{S}(x)^T} = \widetilde{\Lambda}_{\pm} \frac{\delta\widetilde{C}_{p-1}}{\delta\widetilde{S}(x)} = \widetilde{\Lambda} \frac{\delta\widetilde{C}_{p-1}}{\delta\widetilde{S}(x)} . \tag{8.222}$$

Two important facts must be noted here:

1. The variational derivatives of $\widetilde{C}_p$ and $\widetilde{C}_{p-1}$ are related by a $\widetilde{\Lambda}$-operator, which is *p-independent*;
2. As a consequence of (8.221) and (8.222), the nonlinear parts of the NLEE (8.201) are in fact variational derivatives of conveniently chosen linear combinations of $\widetilde{C}_p$s;

The compact formulae (8.212) are the starting point for obtaining the series of conservation laws with a local densities for the HF-type equations. From (8.150) and (8.71) it follows that $\widetilde{\mathcal{A}}(\lambda) = \mathcal{A}(\lambda) - \ln a^+(0)$. Therefore, the generating functionals for the NLS-type equations and HF-type equations differ only by the quantity $\ln a^+(0)$, which is a conservation law by itself. Applying to the integrand of (6.131) the gauge transformation we obtain:

$$\widetilde{C}_m = \frac{1}{4im} \int_{-\infty}^{+\infty} dx \int_{\infty}^{x} \langle S_y, \widetilde{\Lambda}_+^m [S, S_y] \rangle \, dy \equiv \int_{-\infty}^{+\infty} \rho_m(x) dx \ . \tag{8.223}$$

The integrals with $m > 0$ have local densities in terms of $S(x)$ and its x-derivatives. This result follows easily from the locality of the related integrals C_m in terms of $q(x)$ and from (8.162).

One can also check that $\widetilde{C}_0 = 0$. All other integrals $\widetilde{C}_m$ with $m < 0$ are expected to be nonlocal in terms of $S(x)$. The only exception is:

$$\begin{aligned} \widetilde{C}_{-1} &= -\frac{i}{4} \int_{-\infty}^{+\infty} dx \int_{\infty}^{x} \langle S_y, \widetilde{\Lambda}_+^{-1} [S, S_y] \rangle \, dy \\ &= \int_{-\infty}^{+\infty} dx \int_{\infty}^{x} \langle S_y, \pi_S \sigma_3 \rangle \, dy \\ &= \int_{-\infty}^{+\infty} dx (\langle S(x), \sigma_3 \rangle - 1) \\ &= \int_{-\infty}^{+\infty} dx (\mathbf{S}_3(x) - 1) \ . \end{aligned} \tag{8.224}$$

In fact, these NLEE have two more conserved quantities:

$$\widetilde{C}_{-1;a} = \int_{-\infty}^{+\infty} dx \mathbf{S}_a(x), \qquad a = 1, 2 \ , \tag{8.225}$$

For the HF equation, the existence of the integrals $\widetilde{C}_{-1}$ and $\widetilde{C}_{-1;a}$ means that all three components of the total spin are conserved. We shall discuss these integrals in the next Section.

Utilizing the explicit form of $\widetilde{\Lambda}_+$, one is able to obtain (up to numerical factors and x-derivative terms) the first three conserved densities:

$$\begin{aligned} &\rho_1 = \langle S_x, S_x \rangle, \qquad \rho_2 = \langle S, [S_x, S_{xx}] \rangle \ , \\ &\rho_3 = 5 \langle S_x, S_x \rangle^2 - 4 \langle S_{xx}, S_{xx} \rangle \ . \end{aligned} \tag{8.226}$$

At first sight ρ_3 differs from the expression obtained in [1], which contains rational dependence on S_x, i.e. a factor $\langle S_x, S_x \rangle^{-1}$. But, if $S_x \neq 0$, then S, S_x and $[S, S_x]$ are linearly independent and form a basis in $sl(2, \mathbb{C})$. Expanding S_{xx} over this basis one can check that

$$\rho_3 = \langle S_x, S_x \rangle^2 - \left(\frac{d}{dx} \langle S_x, S_x \rangle^2 - \langle S_{xx}, [S, S_x] \rangle \right) \langle S_x, S_x \rangle^{-1} \ , \tag{8.227}$$

which coincides with the quantity in [1]. Besides, since the operators $\widetilde{\Lambda}_{\pm}$ depend polynomially on S, it is obvious that all the densities $\rho_m(x)$ are also local functions in S and its x-derivatives.

8.8 The Generic HF-Type NLEE as Completely Integrable Complex Hamiltonian System

Again, we have two ways to approach the Hamiltonian properties of the HF-type equations. The first one is to do it directly, the second, to make use of the gauge transformation. We shall first use the direct approach, and next we shall analyze the relationships between the two hierarchies of Hamiltonian structures.

It is only natural to consider the phase space of the HF-type equations as the space of allowed matrix-valued functions $S(x,t)$ (resp. vector-valued functions $\mathbf{S}(x,t)$) that satisfy the corresponding equation. One may also say that the phase space $\widetilde{\mathcal{M}}^{\mathbb{C}}$ consists of the class of allowed potentials of the linear problem $\widetilde{L}$ (8.34a). By "allowed" potential, we understand the space of functions

$$\widetilde{\mathcal{M}} = \left\{ S(x) : S^2 = \mathbb{1}, \quad \lim_{x\to\pm\infty} S(x) = \sigma_3, \quad S(x) \in sl(2,\mathbb{C}) \right\} .$$

satisfying conditions $\widetilde{\mathrm{C}}.1$, $\widetilde{\mathrm{C}}.2$, $\widetilde{\mathrm{C}}.3$.

$\widetilde{\mathcal{M}}^{\mathbb{C}}$ is not a linear space due to the nonlinear constraints on $S(x,t)$ in (8.2a). This constraint can be formulated also as

$$(\mathbf{S},\mathbf{S}) \equiv \mathbf{S}_1^2 + \mathbf{S}_2^2 + \mathbf{S}_3^2 = 1 \;, \tag{8.228}$$

which means that only two of the three complex-valued functions $\mathbf{S}_a(x,t)$ are independent. On this phase space, there exist a canonical way to introduce Poisson and symplectic structures:

$$\{\mathbf{S}_a(x), \mathbf{S}_b(y)\}_{\widetilde{(0)}} = \epsilon_{abc}\mathbf{S}_c(x)\delta(x-y) \;. \tag{8.229}$$

It is easy to check that these Poisson brackets are compatible with the constraint (8.228) due to the fact that

$$\{\mathbf{S}^2(x), \mathbf{S}_a(y)\}_{\widetilde{(0)}} \equiv \{\mathbf{S}_1^2 + \mathbf{S}_2^2 + \mathbf{S}_3^2, \mathbf{S}_a(y)\}_{\widetilde{(0)}} = 0 \;. \tag{8.230}$$

Of course, the choice $\mathbf{S}_1^2 + \mathbf{S}_2^2 + \mathbf{S}_3^2 = 1$ reflects specific normalization of the spin vector. Choosing

$$H_{\mathrm{HF}} = \int_{-\infty}^{\infty} dx\, \langle S_x, S_x \rangle \equiv \int_{-\infty}^{\infty} dx\, (\mathbf{S}_x, \mathbf{S}_x) \;, \tag{8.231}$$

one easily finds that the corresponding Hamiltonian equations of motion:

$$\frac{\partial S}{\partial t} = \{H_{\mathrm{HF}}, S\}_{\widetilde{(0)}} \,, \tag{8.232a}$$

or in components:

$$\frac{\partial \mathbf{S}_a}{\partial t} = \{H_{\mathrm{HF}}, \mathbf{S}_a\}_{\widetilde{(0)}} \,, \tag{8.232b}$$

gives the Heisenberg ferromagnet equations of motion (8.1). Of course, the HF model has physical applications for real-valued components $\mathbf{S}_a(x,t)$, or equivalently, for hermitian matrices $S(x,t) = S^\dagger(x,t)$. The models that we were studying are more general in the sense that we are dealing with complex-valued components $\mathbf{S}_a(x,t)$, or with nonhermitian matrices $S(x,t)$. These models can be obtained from the physical ones by the complexification procedure described in the previous chapter. That is why we shall first analyze the Hamiltonian properties of the complexified HF-type equations and then we shall pass to their real Hamiltonian forms.

We can introduce local coordinates on the phase space $\widetilde{\mathcal{M}}^{\mathbb{C}}$ in two different but equivalent ways by using : (i) 2×2 complex matrix-valued functions $S(x,t)$ as in (8.4) satisfying (8.2); (ii) vector-valued functions $\mathbf{S}(x)$ satisfying (8.228). Obviously, all necessary quantities can be expressed either through $S(x)$ or $\mathbf{S}$. For example, the variational derivatives (or the "gradients") of the functional $H^{\mathbb{C}}$ can be written as:

$$\nabla_S H^{\mathbb{C}} \equiv \frac{\delta H^{\mathbb{C}}}{\delta S^T(x)} = \begin{pmatrix} \frac{\delta H^{\mathbb{C}}}{\delta S_3(x)} & \frac{\delta H^{\mathbb{C}}}{\delta S^-(x)} \\ \frac{\delta H^{\mathbb{C}}}{\delta S^+(x)} & -\frac{\delta H^{\mathbb{C}}}{\delta S_3(x)} \end{pmatrix} \equiv \sum_{a=1}^{3} \frac{\delta H^{\mathbb{C}}}{\delta S_a(x)} \sigma_a \,, \tag{8.233a}$$

with $S^\pm(x,t) = S_1(x,t) \mp iS_2(x,t)$, or

$$\nabla_{\mathbf{S}} H^{\mathbb{C}} \equiv \left(\frac{\delta H^{\mathbb{C}}}{\delta S_1(x)}, \frac{\delta H^{\mathbb{C}}}{\delta S_2(x)}, \frac{\delta H^{\mathbb{C}}}{\delta S_3(x)} \right)^T \,, \tag{8.233b}$$

The first important consequence of the fact is that $\widetilde{\mathcal{M}}^{\mathbb{C}}$ is not a linear space and that the tangent space $T_S\widetilde{\mathcal{M}}^{\mathbb{C}}$ *is not* isomorphic to $\widetilde{\mathcal{M}}^{\mathbb{C}}$. Along with the tangent space, one should consider also the co-tangent space $T_S^*\widetilde{\mathcal{M}}^{\mathbb{C}}$.

The symplectic structure can of course be introduced not only by the Poisson brackets (8.228) but also through a symplectic form, which is given by:

$$\widetilde{\Omega}_{(0)} = \int_{-\infty}^{\infty} \langle \delta S(x) \mathop{\wedge}_{,} [S(x), \delta S(x)] \rangle dx \,. \tag{8.234}$$

Such symplectic form can be interpreted also as the natural symplectic form on the co-adjoint orbit of $sl(2)$ passing through the element $S(x)$. The corresponding Hamiltonian equations of motion can be written out in equivalent form as:

$$\widetilde{\Omega}(X_H, \cdot) + \delta H = 0 \,, \tag{8.235}$$

where H is the Hamiltonian of the corresponding NLEE.

We shall limit ourselves to the class of Hamiltonians that are analytic functions of $S(x,t)$.

Recalling the definition of the skew-symmetric scalar product $[\![\cdot\ ,\ \cdot]\!]_{\sim}$, (8.138) one is able to cast the canonical Poisson brackets (8.229) as follows:

$$\begin{aligned}\{F,G\}_{\widetilde{0}}^{\mathbb{C}} &= i\int_{-\infty}^{\infty}\langle\nabla_S F,[S(x),\nabla_S G]\rangle dx\\ &= i[\![\nabla_S F,\nabla_q G]\!]_{\sim}\ .\end{aligned} \tag{8.236}$$

The corresponding canonical symplectic form and Hamiltonian vector field become:

$$\begin{aligned}\widetilde{\Omega}_{(0)}^{\mathbb{C}} &= \frac{i}{2}\int_{-\infty}^{\infty} dx\,\langle\delta S(x)\underset{'}{\wedge}[S(x),\delta S(x)])\rangle\\ &= -\frac{i}{8}[\![[S,\delta S(x)]\underset{'}{\wedge}[S,\delta S(x)]\,]\!]_{\sim}\ ,\end{aligned} \tag{8.237a}$$

$$\widetilde{\Omega}_{(0)}^{\mathbb{C}} = \int_{-\infty}^{\infty} dx\,\Big(\mathbf{S}(x),\delta\mathbf{S}(x)\underset{'}{\wedge}\delta\mathbf{S}(x)\Big)\ . \tag{8.237b}$$

and

$$\widetilde{X}_{H^{\mathbb{C}}}\cdot = -i\Big[\!\!\Big[\nabla_S H^{\mathbb{C}},\nabla_S\,\cdot\Big]\!\!\Big]_{\sim} = -\{H^{\mathbb{C}},\,\cdot\}_{\widetilde{(0)}}^{\mathbb{C}}\ , \tag{8.238a}$$

$$\widetilde{X}_{H^{\mathbb{C}}}\cdot = -\{H^{\mathbb{C}},\,\cdot\}_{\widetilde{(0)}}^{\mathbb{C}} = 2\int_{-\infty}^{\infty}\left(\nabla_{\mathbf{S}}H^{\mathbb{C}}\cdot\mathbf{S}(x)\times\nabla_{\mathbf{S}}\cdot\right)\ , \tag{8.238b}$$

In what follows, we shall use mostly the first realization of $\widetilde{\mathcal{M}}^{\mathbb{C}}$ and the $sl(2)$-type notations. The advantage is that they can easily be generalized to any semisimple Lie algebra. Using the formula above, the reader can cast all results in vector notations.

Now, we can write down the complexified version of the HF system (8.1) in the form:

$$i\frac{\partial S}{\partial t} + \nabla_S H_{\mathrm{HF}}^{\mathbb{C}} = 0\ , \tag{8.239}$$

where

$$H^{\mathbb{C}} = H_{\mathrm{HF}}^{\mathbb{C}} = \frac{1}{2}\int_{-\infty}^{\infty}\langle S_x,S_x\rangle dx = \frac{1}{2}\widetilde{C}_1\ . \tag{8.240}$$

It generalizes the HF in the sense that $S(x,t)$ is generic (nonhermitian) complex-valued matrix. We shall not repeat here the ideas of complexifying Hamiltonian systems; they are in complete analogy with the ones already discussed in the previous Chapter.

Returning to the generic NLEE (8.201) with dispersion law $f(\lambda)$, each of them can be written down in the form (8.239). It is only natural to expect that the corresponding Hamiltonian H should be expressed in terms of the integrals of motion $\widetilde{C}_p$. Indeed, if we make use of (8.221) and choose

$$H^{\mathbb{C}} = \sum_k 4 f_k \widetilde{C}_{k+1} \tag{8.241}$$

we get:

$$\nabla_S H^{\mathbb{C}} = 2f(\widetilde{\Lambda})\widetilde{\Lambda}\pi_S\sigma_3 \; . \tag{8.242}$$

Thus Equation (8.239) coincides with the NLEE (8.201).

Next, we evaluate the Poisson brackets between the entries in the minimal sets of scattering data $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}$. To do this, we use the compact expressions for the scattering data variations, which follow from (8.146) and (8.147):

$$\frac{\delta \widetilde{\tau}^{\pm}(t,\lambda)}{\delta S^T(x)} \equiv \nabla_S \widetilde{\tau}^{\pm}(t,\lambda) = \frac{\mp i\lambda}{(\widetilde{a}^{\pm}(\lambda))^2} \widetilde{\boldsymbol{\Psi}}^{\pm}(x,t,\lambda) \; , \tag{8.243a}$$

$$\nabla_S \lambda_k^{\pm} = \mp i \lambda_k^{\pm} \widetilde{C}_k^{\pm} \widetilde{\boldsymbol{\Psi}}_k^{\pm}(x,t) \; , \tag{8.243b}$$

$$\nabla_S \widetilde{M}_k^{\pm} = \frac{\mp i}{(\dot{\widetilde{a}}_k^{\pm})^2} \left(\dot{\widetilde{\boldsymbol{\Psi}}}_k^{\pm}(x,t) - \frac{\ddot{\widetilde{a}}_k^{\pm}}{\dot{\widetilde{a}}_k^{\pm}} \widetilde{\boldsymbol{\Psi}}_k^{\pm}(x,t) \right) \; , \tag{8.243c}$$

and

$$\nabla_S \widetilde{\rho}^{\pm}(t,\lambda) = \frac{\mp i}{(\widetilde{a}^{\pm}(\lambda))^2} \widetilde{\boldsymbol{\Phi}}^{\pm}(x,t,\lambda) \; , \tag{8.244a}$$

$$\nabla_S \lambda_k^{\pm} = \pm i \lambda_k^{\pm} \widetilde{M}_k^{\pm} \widetilde{\boldsymbol{\Phi}}_k^{\pm}(x,t) \; , \tag{8.244b}$$

$$\nabla_S \widetilde{C}_k^{\pm} = \frac{\mp i}{(\dot{\widetilde{a}}_k^{\pm})^2} \left(\dot{\widetilde{\boldsymbol{\Phi}}}_k^{\pm}(x,t) - \frac{\ddot{\widetilde{a}}_k^{\pm}}{\dot{\widetilde{a}}_k^{\pm}} \widetilde{\boldsymbol{\Phi}}_k^{\pm}(x,t) \right) \; , \tag{8.244c}$$

In addition, for the variations of $\widetilde{\kappa}(\lambda)$ and $\widetilde{\eta}(\lambda)$ we find:

$$\nabla_S \widetilde{\eta}(\lambda) = i\widetilde{\boldsymbol{P}}(x,t,\lambda), \qquad \nabla_S \widetilde{\eta}_k^{\pm} = i\widetilde{\boldsymbol{P}}_k^{\pm}(x,t) \; , \tag{8.245a}$$

$$\nabla_S \widetilde{\kappa}(t,\lambda) = i\widetilde{\boldsymbol{Q}}(x,t,\lambda), \qquad \nabla_S \widetilde{\kappa}_k^{\pm} = i\widetilde{\boldsymbol{Q}}_k^{\pm}(x,t) \; , \tag{8.245b}$$

where $\widetilde{\boldsymbol{P}}(x,t,\lambda)$, $\widetilde{\boldsymbol{Q}}(x,t,\lambda)$ etc. are the elements of the symplectic basis (8.141).

From the above relations, there follows that the Poisson brackets between the scattering data are expressed through the skew-symmetric scalar products of the corresponding "squared" solutions. So, in order to evaluate them, we need to recall the results from Sect. 5.5.1 and Table 5.2 and apply to them the gauge transformations. This gives:

$$\{\widetilde{\rho}^{+}(t,\lambda), \widetilde{\tau}^{+}(t,\mu)\}_{\widetilde{(0)}} = -\frac{i [\![\widetilde{\boldsymbol{\Phi}}^{+}(x,t,\lambda), \widetilde{\boldsymbol{\Psi}}^{+}(x,t,\nu)]\!]_{\sim}}{(\widetilde{a}^{+}(\lambda))^2 (\widetilde{a}^{+}(\mu))^2} \; ,$$

$$= -i\pi\delta(\lambda-\mu)\,, \tag{8.246a}$$

$$\{\widetilde{\rho}^{-}(t,\lambda),\widetilde{\tau}^{-}(t,\mu)\}_{\widetilde{(0)}} = -\frac{i\left[\!\left[\widetilde{\boldsymbol{\Phi}}^{-}(x,t,\lambda),\widetilde{\boldsymbol{\Psi}}^{-}(x,t,\nu)\right]\!\right]_{\sim}}{(\widetilde{a}^{-}(\lambda))^2(\widetilde{a}^{-}(\mu))^2}.$$

$$= i\pi\delta(\lambda-\mu)\,. \tag{8.246b}$$

and

$$\{\widetilde{\eta}(\lambda),\widetilde{\kappa}(\mu)\}^{\mathbb{C}}_{\widetilde{(0)}} = i\delta(\lambda-\mu), \qquad \{\widetilde{\eta}(\lambda),\widetilde{\kappa}^{\pm}_k\}^{\mathbb{C}}_{\widetilde{(0)}} = 0\,,$$
$$\{\widetilde{\kappa}(\lambda),\widetilde{\eta}^{\pm}_k\}^{\mathbb{C}}_{\widetilde{(0)}} = 0, \qquad \{\widetilde{\eta}^{\pm}_k,\widetilde{\kappa}^{\pm}_m\}^{\mathbb{C}}_{\widetilde{(0)}} = i\delta_{km}, \tag{8.247a}$$

$$\{\widetilde{\eta}(\lambda),\widetilde{\eta}(\mu)\}^{\mathbb{C}}_{\widetilde{(0)}} = 0, \qquad \{\widetilde{\eta}^{\pm}_k,\widetilde{\eta}^{\pm}_m\}^{\mathbb{C}}_{\widetilde{(0)}} = 0, \qquad \{\widetilde{\eta}(\lambda),\widetilde{\eta}^{\pm}_k\}^{\mathbb{C}}_{\widetilde{(0)}} = 0\,, \tag{8.247b}$$

$$\{\widetilde{\kappa}(\lambda),\widetilde{\kappa}(\mu)\}^{\mathbb{C}}_{\widetilde{(0)}} = 0, \qquad \{\widetilde{\kappa}^{\pm}_k,\widetilde{\kappa}^{\pm}_m\}^{\mathbb{C}}_{\widetilde{(0)}} = 0, \qquad \{\widetilde{\kappa}(\lambda),\widetilde{\kappa}^{\pm}_k\}^{\mathbb{C}}_{\widetilde{(0)}} = 0\,. \tag{8.247c}$$

Thus we find that the set of variables $\{\widetilde{\eta}(\lambda),\widetilde{\kappa}(\lambda),\widetilde{\eta}^{\pm}_k,\widetilde{\kappa}^{\pm}_k\}$ satisfy canonical Poisson brackets. We also proved that $\{\widetilde{\eta}(\lambda),\widetilde{\eta}^{\pm}_k\}$ are in involution (see (8.247b) above). For evolution, defined by a generic NLEE, they are also time independent due to (8.63). The variables $\{\widetilde{\kappa}(\lambda),\widetilde{\kappa}^{\pm}_k\}$ are also in involution but depend on time linearly; see (8.63). Thus, these two sets of variables have all the necessary properties to be global "action-angle" variables for the NLEEs (8.201).

From the trace identities (8.209) and (8.210), we know that the integrals of motion $\widetilde{C}_k$ are expressed in terms of $\widetilde{\eta}(\lambda)$ and $\widetilde{\eta}^{\pm}_k$ only; therefore, they are in involution, i.e.

$$\{\widetilde{C}_n,\widetilde{C}_m\}^{\mathbb{C}}_{\widetilde{(0)}} = 0\,, \tag{8.248}$$

for $n,m=\pm1,\pm2,\dots$.

Thus we conclude that the NLEE (8.201) are infinite dimensional completely integrable complex Hamiltonian systems with respect to the canonical Poisson brackets defined by (8.229) on $\widetilde{\mathcal{M}}^{\mathbb{C}}$.

In treating the complete integrability of an infinite-dimensional system, the most difficult point is to ensure that the action-angle variable really spans the whole phase space $\mathcal{M}^{\mathbb{C}}$. We are now going to present a more rigorous proof of this fact, which is based on the completeness relation for the symplectic basis.

Again, the most straightforward way to derive the action-angle variables of the NLEE (8.201) is to insert the right- hand side of (8.199b) into the expression for $\widetilde{\Omega}^{\mathbb{C}}_{(0)}$. This gives:

$$\widetilde{\Omega}^{\mathbb{C}}_{(0)} = \frac{i}{2}\left[\!\!\left[[S,\delta S(x)] \mathop{\wedge}_{,} \left(i\int_{-\infty}^{\infty}\frac{d\lambda}{\lambda}\,\left(\delta\widetilde{\eta}(\lambda)\widetilde{\boldsymbol{Q}}(x,\lambda)-\delta\widetilde{\kappa}(\lambda)\widetilde{\boldsymbol{P}}(x,\lambda)\right)\right.\right.\right.$$

$$+ i \sum_{k=1}^{N} \Big(\frac{\delta \widetilde{\eta}_k^+}{\lambda_k^+} \widetilde{\boldsymbol{Q}}_k^+(x) - \frac{\delta \widetilde{\kappa}_k^+}{\lambda_k^+} \widetilde{\boldsymbol{P}}_k^+(x) + \frac{\delta \widetilde{\eta}_k^-}{\lambda_k^-} \widetilde{\boldsymbol{Q}}_k^-(x) - \frac{\delta \widetilde{\kappa}_k^-}{\lambda_k^-} \widetilde{\boldsymbol{P}}_k^-(x) \Big) \Big) \Big]\!\Big]_{\sim}$$

$$= -\frac{1}{2} \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda} \Big(\delta \widetilde{\eta}(\lambda) \wedge [\![[S, \delta S(x)], \widetilde{\boldsymbol{Q}}(x,\lambda)]\!]_{\sim}$$

$$- \delta \widetilde{\kappa}(\lambda) \wedge [\![[S, \delta S(x)], \widetilde{\boldsymbol{P}}(x,\lambda)]\!]_{\sim} \Big)$$

$$- \frac{1}{2} \sum_{k=1}^{N} \Big((\delta \widetilde{\eta}_k^+ \wedge [\![[S, \delta S(x)], \widetilde{\boldsymbol{Q}}_k^+(x)]\!]_{\sim} - \delta \widetilde{\kappa}_k^+ \wedge [\![[S, \delta S(x)], \widetilde{\boldsymbol{P}}_k^+(x)]\!]_{\sim}$$

$$- \frac{1}{2} \delta \widetilde{\eta}_k^- \wedge [\![[S, \delta S(x)], \widetilde{\boldsymbol{Q}}_k^-(x)]\!]_{\sim} - \delta \widetilde{\kappa}_k^- \wedge [\![[S, \delta S(x)], \widetilde{\boldsymbol{P}}_k^-(x))]\!]_{\sim} \Big)$$

$$= \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda^2} \delta \widetilde{\kappa}(\lambda) \wedge \delta \widetilde{\eta}(\lambda) + \sum_{k=1} \left(\frac{\delta \widetilde{\kappa}_k^+ \wedge \delta \widetilde{\eta}_k^+}{(\lambda_k^+)^2} + \frac{\delta \widetilde{\kappa}_k^- \wedge \delta \widetilde{\eta}_k^-}{(\lambda_k^-)^2} \right) . \tag{8.249}$$

In the above derivation, we made use of the inversion formulae (8.190) for the symplectic basis with $\widetilde{X}(x) = [S(x), \delta S(x)]$.

From (8.249), we see also that the 2-form $\widetilde{\Omega}_{(0)}^{\mathbb{C}}$, just like $\Omega_{(-2)}^{\mathbb{C}}$, has $\lambda^{-2}\widetilde{\eta}(\lambda)$, $\widetilde{\kappa}(\lambda)$, $(\lambda_k^{\pm})^{-2}\widetilde{\eta}_k^{\pm}$ and $\widetilde{\kappa}_k^{\pm}$ as canonical coordinates; the extra factors $(\lambda_k^{\pm})^{-2}$ can be taken care of by conveniently redefining $\widetilde{\eta}_k^{\pm}$ as follows; see (8.195):

$$\frac{\delta \widetilde{\eta}_k^{\pm}}{(\lambda_k^{\pm})^2} = \pm 2i \delta \left(\frac{1}{\lambda_k^{\pm}} \right) . \tag{8.250}$$

Let us recall now the trace identities (8.209), (8.210). From them, there follows that the Hamiltonian $H^{\mathbb{C}}$ of the NLEE depends only on the variables $\widetilde{\eta}(\lambda)$, $\widetilde{\eta}_k^{\pm}$:

$$H^{\mathbb{C}} = -i \int_{-\infty}^{\infty} d\mu \, f(\mu) \widetilde{\eta}(\mu) - 2 \sum_{k=1}^{N} \left(\widetilde{F}_k^+ - \widetilde{F}_k^- \right) \tag{8.251}$$

where

$$\widetilde{F}_k^{\pm} = \widetilde{F}(\lambda_k^{\pm}), \qquad \widetilde{F}(\lambda) = \int^{\lambda} d\lambda' \widetilde{f}(\lambda') . \tag{8.252}$$

Remark 8.9. The equations of motion written in terms of the variables $\widetilde{\eta}(\lambda)$, $\widetilde{\kappa}(\lambda)$, $\widetilde{\eta}_k^{\pm}$ and $\widetilde{\kappa}_k^{\pm}$ run as follows:

$$\frac{d\widetilde{\eta}}{dt} = 0, \qquad \frac{d\widetilde{\eta}_k^{\pm}}{dt} = 0, \qquad i\frac{d\widetilde{\kappa}}{dt} - 2f(\lambda) = 0, \qquad i\frac{d\widetilde{\kappa}_k^{\pm}}{dt} - 2f(\lambda_k^{\pm}) = 0 . \tag{8.253}$$

As already mentioned, we see that the "action" variables $\widetilde{\eta}$, $\widetilde{\eta}_k^{\pm}$ are time-independent while the "angle" variables $\widetilde{\kappa}$, $\widetilde{\kappa}_k^{\pm}$ are linear functions of t. However, here both types of variables are complex-valued. The "angle" variables $\widetilde{\kappa}(\lambda)$, $\widetilde{\kappa}_k^{\pm}$ (see (8.195b)) can be written as:

$$\widetilde{\kappa}(\lambda) = \frac{1}{2}\ln\left|\frac{\widetilde{b}^{+}(\lambda)}{\widetilde{b}^{-}(\lambda)}\right| + \frac{i}{2}\arg\frac{\widetilde{b}^{+}(\lambda)}{\widetilde{b}^{-}(\lambda)}, \qquad \widetilde{\kappa}_k^{\pm} = \pm\ln|\widetilde{b}_k^{\pm}| \pm i\arg\widetilde{b}_k^{\pm}\,, \quad (8.254)$$

from which we find that only their imaginary parts can be viewed as real angles taking values in the range $[0, 2\pi]$.

The completeness relation of the symplectic basis ensures: i) the uniqueness and the invertibility of the mapping from $\{S(x,t)\}$ to $\widetilde{\mathcal{T}}$; ii) the nondegeneracy of the 2-form $\widetilde{\Omega}_{(0)}^{\mathbb{C}}$ on $\widetilde{\mathcal{M}}^{\mathbb{C}}$.

Since we look at $S(x)$ as local coordinates on $\widetilde{\mathcal{M}}^{\mathbb{C}}$, any generic functional F or G on $\widetilde{\mathcal{M}}^{\mathbb{C}}$ can be expressed in terms of $S(x)$. Their variations δF and δG are the analogs of 1-forms over $\mathcal{M}^{\mathbb{C}}$. They can be expressed in terms of the "gradients" by:

$$\delta F = [\![\nabla_S F, \delta S]\!]_{\sim}, \qquad \delta G = [\![\nabla_S G, \delta S]\!]_{\sim}\,, \quad (8.255)$$

The "gradients" $\nabla_S F$ and $\nabla_S G$ are elements of the tangent space $T_S\widetilde{\mathcal{M}}^{\mathbb{C}}$.

Since the mapping $S \to \widetilde{\mathcal{T}}$ is one-to-one, it is possible to express F and G in terms of the scattering data. To this end, we consider the expansions of $\nabla_S F$ and $\nabla_S G$ over the symplectic basis:

$$\begin{aligned} \nabla_S F = i\int_{-\infty}^{\infty}\frac{d\lambda}{\lambda}\left(\widetilde{\eta}_F(\lambda)\widetilde{\boldsymbol{Q}}(x,\lambda) - \widetilde{\kappa}_F(\lambda)\widetilde{\boldsymbol{P}}(x,\lambda)\right) \\ + i\sum_{k=1}^{N}{}^{\pm}\left(\frac{\widetilde{\eta}_{F,k}^{\pm}}{\lambda_k^{\pm}}\widetilde{\boldsymbol{Q}}_k^{\pm}(x) - \frac{\widetilde{\kappa}_{F,k}^{\pm}}{\lambda_k^{\pm}}\widetilde{\boldsymbol{P}}_k^{\pm}(x)\right)\,, \end{aligned} \quad (8.256)$$

$$\widetilde{\eta}_F(\lambda) = i[\![\widetilde{\boldsymbol{P}}(x,\lambda), \nabla_S F]\!]_{\sim}, \qquad \widetilde{\kappa}_F(\lambda) = i[\![\widetilde{\boldsymbol{Q}}(x,\lambda), \nabla_S F]\!]_{\sim}\,,$$

$$\widetilde{\eta}_{F,k}^{\pm} = i[\![\widetilde{\boldsymbol{P}}_k^{\pm}(x), \nabla_S F]\!]_{\sim}, \qquad \widetilde{\kappa}_{F,k}^{\pm} = i[\![\widetilde{\boldsymbol{Q}}_k^{\pm}(x), \nabla_S F]\!]_{\sim}\,. \quad (8.257)$$

Similar expansion for $\nabla_S G$ is obtained from (8.256) by changing F to G. Such expansions will hold true provided F and G are restricted in such a way that the expansion coefficients $\widetilde{\eta}_F(\lambda)$ and $\widetilde{\kappa}_F(\lambda)$ are smooth and fall off fast enough for $\lambda \to \pm\infty$. In what follows, we shall assume that the functionals F and G satisfy.

Condition $\widetilde{\mathrm{C}}4$. The functionals F and G are restricted by the following implicit condition: The expansion coefficients $\widetilde{\eta}_F(\lambda)$ and $\widetilde{\kappa}_F(\lambda)$ and $\widetilde{\eta}_G(\lambda)$ and $\widetilde{\kappa}_G(\lambda)$ are Schwartz-type functions of λ for real λ.

Under the above assumptions the biquadratic relations satisfied by the elements of the symplectic basis (8.140), we can express the Poisson brackets between F and G in terms of their expansion coefficients as follows:

$$\begin{aligned}\{F,G\}^{\mathbb{C}}_{\widetilde{(0)}} &= -i[\![\nabla_S F, \nabla_S G]\!]_{\sim} && (8.258)\\ &= \int_{-\infty}^{\infty} d\lambda\, (\widetilde{\eta}_F\widetilde{\kappa}_G - \widetilde{\kappa}_F\widetilde{\eta}_G)(\lambda) + \sum_{k=1}^{N}{}^{\pm}\left(\widetilde{\eta}^{\pm}_{F,k}\widetilde{\kappa}^{\pm}_{G,k} - \widetilde{\kappa}^{\pm}_{F,k}\widetilde{\eta}^{\pm}_{G,k}\right).\end{aligned}$$

In particular, if we choose $F = H^{\mathbb{C}}$, then from (8.245) we find that $\widetilde{\eta}_{H^{\mathbb{C}}}(\lambda) = 0$, $\widetilde{\eta}^{\pm}_{H^{\mathbb{C}},k} = 0$ and

$$\widetilde{\kappa}_{H^{\mathbb{C}},k}(\lambda) = -2f(\lambda), \qquad \widetilde{\kappa}^{\pm}_{H^{\mathbb{C}},k} = -2f(\lambda^{\pm}_k),$$

which gives:

$$\{H,G\}^{\mathbb{C}}_{\widetilde{(0)}} = 2\int_{-\infty}^{\infty} d\lambda\, f(\lambda)\widetilde{\eta}_G(\lambda) + 2\sum_{k=1}^{N}(f(\lambda^+_k)\widetilde{\eta}^+_{G,k} + f(\lambda^-_k)\widetilde{\eta}^-_{G,k})\,. \quad (8.259)$$

Equation (8.259) allows us to describe the set of functionals G that are in involution with all integrals of motion of the generic NLEE. Indeed, the right-hand side of (8.259) will vanish identically for all choices of the dispersion law $f(\lambda)$ only provided the expansion coefficients of $\nabla_S G$ satisfy:

$$\widetilde{\eta}_g(\lambda) = 0, \qquad \lambda \in \mathbb{R}; \qquad \widetilde{\eta}^{\pm}_{G,k} = 0, \qquad \forall k = 1, \ldots, N\,. \quad (8.260)$$

One can also describe the Hamiltonian vector fields in terms of the symplectic basis:

$$\begin{aligned}\widetilde{X}_{H^{\mathbb{C}}}\cdot &\equiv -\{H^{\mathbb{C}}, \cdot\}^{\mathbb{C}}_{\widetilde{(0)}} && (8.261)\\ &= -2i\int_{-\infty}^{\infty} d\lambda\, f(\lambda)[\![\widetilde{\boldsymbol{P}}(x,\lambda), \nabla_S\,\cdot\,]\!]_{\sim} - 2i\sum_{k=1}^{N}{}^{\pm} f(\lambda^{\pm}_k)[\![\widetilde{\boldsymbol{P}}^{\pm}_k(x), \nabla_S\,\cdot\,]\!]_{\sim}\,,\end{aligned}$$

where $f(\lambda)$ is the dispersion law of the generic NLEE with Hamiltonian $H^{\mathbb{C}}$.

We end this section by underlying the special role of the subspace $\widetilde{\mathcal{L}}^{\mathbb{C}} \subset \widetilde{\mathcal{M}}^{\mathbb{C}}$ spanned by $\widetilde{\boldsymbol{P}}(x,\lambda)$ and $\widetilde{\boldsymbol{P}}^{\pm}_k(x)$, $k = 1, \ldots, N$, i.e. by "half" of the elements of the symplectic basis. All Hamiltonian vector fields with Hamiltonians of the form (8.241) induce dynamics which is tangent to $\widetilde{\mathcal{L}}^{\mathbb{C}}$. This is the subspace of maximal dimension in $\widetilde{\mathcal{M}}^{\mathbb{C}}$ on which the symplectic form $\widetilde{\Omega}^{\mathbb{C}}_{(0)}$ is degenerated. Therefore, $\widetilde{\mathcal{L}}^{\mathbb{C}}$ is the Lagrange submanifold of $\widetilde{\mathcal{M}}^{\mathbb{C}}$.

8.9 Hamiltonian Hierarchies and Gauge Transformations

The complete integrability of the generic NLEEs we described makes them rather special. They have an infinite number of integrals of motion C_n, which are in involution and these integrals satisfy the relation:

$$\nabla_S C_{n+m} = \widetilde{\Lambda}^m \nabla_S C_n \ , \tag{8.262}$$

which generalizes the Lenard relation (8.221). The important fact here is that the recursion operator Λ is a universal one and does not depend on either n or m. This important fact has far-reaching consequences, which we discuss below.

The first topic we would like to mention is the possibility introducing a hierarchy of Poisson brackets as follows:

$$\{F, G\}^{\mathbb{C}}_{\widetilde{(m)}} = \frac{1}{i} [\![\nabla_S F, \widetilde{\Lambda}^m \nabla_S G]\!]_{\sim} \ . \tag{8.263}$$

Naturally, we must prove that these brackets are indeed Poisson brackets.

The fact that the brackets are skew-symmetric follows from the property that Λ is "self-adjoint" with respect to the skew-symmetric scalar product. Indeed:

$$\begin{aligned} \{F, G\}^{\mathbb{C}}_{\widetilde{(m)}} &= \frac{1}{i} [\![\nabla_S F, \widetilde{\Lambda}^m \nabla_S G]\!]_{\sim} \\ &= -\frac{1}{i} [\![\widetilde{\Lambda}^m \nabla_S G, \nabla_S F]\!]_{\sim} \\ &= -\frac{1}{i} [\![\nabla_S G, \widetilde{\Lambda}^m \nabla_S F]\!]_{\sim} = -\{G, F\}^{\mathbb{C}}_{\widetilde{(m)}} \ . \end{aligned} \tag{8.264}$$

We also have the Leibnitz rule:

$$\begin{aligned} \{FG, H\}^{\mathbb{C}}_{\widetilde{(m)}} &= \frac{1}{i} [\![\nabla_S (FG), \widetilde{\Lambda}^m \nabla_S H]\!]_{\sim} \\ &= \frac{1}{i} F [\![\nabla_S G, \Lambda^m \nabla_S H]\!]_{\sim} + \frac{1}{i} [\![\nabla_S F, \widetilde{\Lambda}^m \nabla_S H]\!]_{\sim} G \\ &= F \{G, H\}^{\mathbb{C}}_{\widetilde{(m)}} + \{F, H\}^{\mathbb{C}}_{\widetilde{(m)}} G \ , \end{aligned} \tag{8.265}$$

the second line being a consequence from $\nabla_S (FG) = F \nabla_S G + (\nabla_S F) G$.

Finally, using the expansion (8.256) of $\nabla_S F$, an analogous one for $\nabla_S G$, and the fact that the elements $\widetilde{\boldsymbol{P}}(x, \lambda)$ and $\widetilde{\boldsymbol{Q}}(x, \lambda)$ are eigenfunctions of $\widetilde{\Lambda}$, (see (8.175)) we find:

$$\{F, G\}^{\mathbb{C}}_{\widetilde{(m)}} = -i [\![\nabla_S F, \widetilde{\Lambda}^{m-2} \nabla_S G]\!]_{\sim} \tag{8.266}$$

$$= \int_{-\infty}^{\infty} d\lambda \, \lambda^m \left(\widetilde{\eta}_F \widetilde{\kappa}_G - \widetilde{\kappa}_F \widetilde{\eta}_G \right)(\lambda) + \sum_{k=1}^{N}{}^{\pm} (\lambda_k^{\pm})^{m-2} \left(\widetilde{\eta}^{\pm}_{F,k} \widetilde{\kappa}^{\pm}_{G,k} - \widetilde{\kappa}^{\pm}_{F,k} \widetilde{\eta}^{\pm}_{G,k} \right) .$$

The Jacobi identity is not trivial to check in these notations. However, after we consider the corresponding symplectic form, this fact will become clear. For this reason, we postpone its discussion until later.

The hierarchy of Poisson brackets suggests that there must exist also hierarchy of vector fields, symplectic forms etc. Indeed, combining the symplectic form $\Omega^{\mathbb{C}}_{(m)}$ with $H^{\mathbb{C}}$ one gets the following Hamiltonian vector field:

$$\begin{aligned} &\widetilde{X}^{(m)}_{H^{\mathbb{C}}} \cdot \equiv -\{H^{\mathbb{C}}, \cdot\}^{\mathbb{C}}_{\widetilde{(m)}} \qquad (8.267)\\ &= -2i\int_{-\infty}^{\infty} d\lambda\, \lambda^{m-1} f(\lambda) [\![\widetilde{\boldsymbol{P}}(x,\lambda), \nabla_S \cdot]\!]_{\sim} -2i \sum_{k=1}^{N}{}^{\pm} (\lambda_k^{\pm})^{m-1} f(\lambda_k^{\pm}) [\![\widetilde{\boldsymbol{P}}^{\pm}_k(x), \nabla_S \cdot]\!]_{\sim} , \end{aligned}$$

The corresponding equation of motion is one of the higher generic NLEEs, namely:

$$i\widetilde{\Lambda}^m \frac{\partial S}{\partial t} + 2\widetilde{\Lambda}^m f(\widetilde{\Lambda}) \pi_S \sigma_3 = 0 . \qquad (8.268)$$

The same NLEE (8.268) can be obtained also using the canonical Poisson brackets $\{\cdot\,,\,\cdot\}^{\mathbb{C}}_{\widetilde{(0)}}$ with the Hamiltonian $H^{\mathbb{C}}_{(m)}$ given by:

$$H^{\mathbb{C}}_{(m)} = \sum_k 4i f_k C_{k+m+1} , \qquad (8.269)$$

Indeed, from the Lenard relation and from (8.242) there follows that

$$\nabla_S H^{\mathbb{C}}_{(m)} = 2\widetilde{\Lambda}^m f(\widetilde{\Lambda}) \pi_S \sigma_3 . \qquad (8.270)$$

It is also easy to check using (8.262) that we have the infinite chain of relations:

$$\begin{aligned} \cdots &= \widetilde{\Lambda}^{-1} \nabla_S H^{\mathbb{C}}_{(m+1)} = \nabla_S H^{\mathbb{C}}_{(m)} = \widetilde{\Lambda} \nabla_S H^{\mathbb{C}}_{(m-1)} \\ &= \cdots = \widetilde{\Lambda}^m \nabla_S H^{\mathbb{C}}_{(0)} = \widetilde{\Lambda}^{m+1} \nabla_S H^{\mathbb{C}}_{(-1)} = \cdots , \end{aligned} \qquad (8.271)$$

where we have put $H^{\mathbb{C}}_{(0)} = H^{\mathbb{C}}$. The elements of this infinite chain are well defined only if the potential $S(x)$ satisfies conditions $\widetilde{\mathrm{C}}1$ and $\widetilde{\mathrm{C}}4$.

Using the "self-adjoint" properties of $\widetilde{\Lambda}$ with respect to the skew-symmetric scalar product, we can write formally:

$$\begin{aligned} \widetilde{X}^{(m)}_{H^{\mathbb{C}}} \cdot &\equiv -i [\![\nabla_S H^{\mathbb{C}}_{(0)}, \widetilde{\Lambda}^m \nabla_S \cdot]\!]_{\sim} \\ &= -i [\![\widetilde{\Lambda}^p \nabla_S H^{\mathbb{C}}_{(0)}, \widetilde{\Lambda}^{m-p} \nabla_S \cdot]\!]_{\sim} = \{\nabla_S H^{\mathbb{C}}_{(p)}, \cdot\}^{\mathbb{C}}_{\widetilde{(m-p)}} , \end{aligned} \qquad (8.272)$$

for all $p = \pm 1, \pm 2, \ldots$. But each Hamiltonian vector field determines uniquely the corresponding equations of motion. Therefore, the chain of relations (8.271) shows that each generic NLEE allows a hierarchy of Hamiltonian formulations:

$$\frac{\partial S^+}{\partial t} = \{H^{\mathbb{C}}_{(p)}, S^+(x,t)\}^{\mathbb{C}}_{(-p)} , \qquad (8.273)$$

$$-\frac{\partial S^-}{\partial t} = -\{H^{\mathbb{C}}_{(p)}, S^-(x)\}^{\mathbb{C}}_{(-p)} , \tag{8.274}$$

The hierarchy of Poisson brackets entails that there must exist also a hierarchy of 2–forms:

$$\widetilde{\Omega}^{\mathbb{C}}_{(m)} = -\frac{i}{8} [\![[S, \delta S(x)] \underset{,}{\wedge} \widetilde{\Lambda}^m [S, \delta S(x)]]\!]_{\sim} . \tag{8.275}$$

These 2-forms are not canonical. The proof of the fact that $\delta\widetilde{\Omega}^{\mathbb{C}}_{(m)} = 0$, which is equivalent to the Jacobi identity for the corresponding Poisson brackets is performed by recalculating these forms in terms of the "action-angle" variables. For this, we follow the same idea as in the calculation of $\widetilde{\Omega}^{\mathbb{C}}_{(0)}$; see (8.249). We insert the expansion for $[S, \delta S(x)]$ over the symplectic basis and act on this expansion by $\widetilde{\Lambda}^m$. This is easy to do because of (8.175) and the result is:

$$\begin{aligned} \widetilde{\Lambda}^m [S, \delta S(x)] &= i \int_{-\infty}^{\infty} d\lambda\, \lambda^{m-1} \left(\delta\widetilde{\eta}(\lambda) \widetilde{\boldsymbol{Q}}(x,\lambda) - \delta\widetilde{\kappa}(\lambda) \widetilde{\boldsymbol{P}}(x,\lambda) \right) \\ &\quad + i \sum_{k=1}^{N} \left((\lambda_k^+)^{m-1} \left(\delta\widetilde{\eta}_k^+ \widetilde{\boldsymbol{Q}}_k^+(x) - \delta\widetilde{\kappa}_k^+ \widetilde{\boldsymbol{P}}_k^+(x) \right) \right. \\ &\quad \left. + (\lambda_k^-)^{m-1} \left(\delta\widetilde{\eta}_k^- \widetilde{\boldsymbol{Q}}_k^-(x) - \delta\widetilde{\kappa}_k^- \widetilde{\boldsymbol{P}}_k^-(x) \right) \right) . \end{aligned} \tag{8.276}$$

In calculating the skew-symmetric scalar products of $[S, \delta S(x)]$ with the right-hand side of (8.276), we again obtain the variations of the variables $\widetilde{\eta}$ and $\widetilde{\kappa}$ with the result:

$$\begin{aligned} \widetilde{\Omega}^{\mathbb{C}}_{(m)} &= i \int_{-\infty}^{\infty} d\lambda\, \lambda^{m-2} \delta\widetilde{\kappa}(\lambda) \wedge \delta\widetilde{\eta}(\lambda) \\ &\quad + i \sum_{k=1} ((\lambda_k^+)^{m-2} \delta\widetilde{\kappa}_k^+ \wedge \delta\widetilde{\eta}_k^+ + (\lambda_k^-)^{m-2} \delta\widetilde{\kappa}_k^- \wedge \delta\widetilde{\eta}_k^-) . \end{aligned} \tag{8.277}$$

Remark 8.10. The right-hand sides of (8.277) are well defined for all $m \geq 0$ for potentials $S(x)$ satisfying condition $\widetilde{\mathrm{C}}1$. This condition ensures that $\widetilde{\kappa}(\lambda)$ and $\widetilde{\eta}(\lambda)$ are Schwartz-type functions of λ.

Remark 8.11. For negative values of m, the existence of the integrals in (8.277) is ensured only provided we put additional restrictions on $S(x)$, which would ensure that $\lim_{\lambda\to 0} \lambda^m \delta\widetilde{\kappa}(\lambda) \wedge \delta\widetilde{\eta}(\lambda)$ exist for all $m < 0$.

Now it is easy to prove that.

Proposition 8.2 *Let potential $S(x)$ satisfy condition $\widetilde{\mathrm{C}}1$. Then the forms $\widetilde{\Omega}^{\mathbb{C}}_{(m)}$ for $m \geq 0$ are closed, i.e.*

$$\delta \widetilde{\Omega}^{\mathbb{C}}_{(m)} = 0, \qquad m = 0, 1, 2, \dots \ . \tag{8.278}$$

If in addition $S(x)$ satisfies the condition in remark 8.11, then each of the forms $\widetilde{\Omega}^{\mathbb{C}}_{(m)}$ is closed also for $m < 0$.

Proof. Indeed, the condition in proposition 8.2 is such that the integral in the right-hand side of (8.277) is well defined, so we can interchange the integration with the operation of taking the external differential δ. Therefore, we have:

$$\begin{aligned}
\delta \widetilde{\Omega}^{\mathbb{C}}_{(m)} &= i \int_{-\infty}^{\infty} d\lambda\, \lambda^{m-2} \delta \left(\delta \widetilde{\kappa}(\lambda) \wedge \delta \widetilde{\eta}(\lambda) \right) \\
&\quad + i \sum_{k=1} \delta \left((\lambda_k^+)^{m-2} \delta \widetilde{\kappa}_k^+ \wedge \delta \widetilde{\eta}_k^+ + (\lambda_k^-)^{m-2} \delta \widetilde{\kappa}_k^- \wedge \delta \widetilde{\eta}_k^- \right) \\
&= 0 \ ,
\end{aligned} \tag{8.279}$$

where we used the simple fact that $\delta(\delta g(\lambda)) \equiv 0$ for any $g(\lambda)$ and that $\delta \lambda_k^{\pm} \wedge \delta \widetilde{\eta}_k^{\pm} = 0$ due to (8.195).

Corollary 8.1 *The Poisson brackets $\{ \cdot, \cdot \}^{\mathbb{C}}_{\widetilde{(m)}}$ satisfy the Jacobi identity.*

Now, we are in position to establish the relationship between the gauge-equivalent hierarchies of Hamiltonian structures. The easiest way to do this is to compare the explicit expressions for $\widetilde{\Omega}^{\mathbb{C}}_{(m)}$ (8.279) and $\Omega^{\mathbb{C}}_{(m)}$ (7.74) and to use the relationship between the two sets of action-angle variables. From (8.195), (6.5) and (8.71), we find:

$$\begin{aligned}
\widetilde{\eta}(\lambda) &= \eta(\lambda), \qquad & \widetilde{\kappa}(\lambda) &= \kappa(\lambda) - \ln a_0^+ \ , \\
\widetilde{\eta}_k^{\pm} &= \eta_k^{\pm} , \qquad & \widetilde{\kappa}_k^{\pm} &= \kappa_k^{\pm} - \ln a_0^+ \ .
\end{aligned} \tag{8.280}$$

Inserting this result into (8.280) one easily gets:

$$\begin{aligned}
\widetilde{\Omega}^{\mathbb{C}}_{(m)} &= i \int_{-\infty}^{\infty} d\lambda\, \lambda^{m-2} (\delta \kappa(\lambda) \wedge \delta \eta(\lambda) - \delta \ln a_0^+ \wedge \delta \eta(\lambda)) \\
&\quad + i \sum_{k=1} ((\lambda_k^+)^{m-2} \delta \kappa_k^+ \wedge \delta \eta_k^+ + (\lambda_k^-)^{m-2} \delta \kappa_k^- \wedge \delta \eta_k^-) \qquad (8.281) \\
&\quad - i \delta \ln a_0^+ \wedge \sum_{k=1} ((\lambda_k^+)^{m-2} \delta \eta_k^+ - (\lambda_k^-)^{m-2} \delta \eta_k^-) \\
&= \Omega^{\mathbb{C}}_{(m-2)} - \delta \ln a_0^+ \wedge \delta C_{m-1}
\end{aligned} \tag{8.282}$$

Since $\ln a_0^+$ and C_{m-1} are integrals of motion for our NLEEs, the forms $\widetilde{\Omega}^{\mathbb{C}}_{(m)}$ and $\Omega^{\mathbb{C}}_{(m-2)}$ have the same Hamiltonian vector fields, written in different variables. See the discussion about it in Corollary 15.20, in Sect. 15.3.2, in the second part.

Thus, we have shown that the gauge transformation (8.34) preserves the Hamiltonian hierarchies as a whole but acts as a shift on the index m.

8.10 Involutions and Hierarchies

It is natural to expect that the involutions of the Zakharov–Shabat system $L(\lambda)$ will have their counterparts for $\widetilde{L}(\lambda)$. We start by recalling the main results of Sect. 6.3 and their consequences for the spectral data of $\widetilde{L}(\lambda)$.

The first involution considered there was (see (6.49)):

$$U^*(x,t,\lambda) = -\epsilon^{-1}U(x,t,\lambda)\epsilon, \qquad \epsilon = \begin{pmatrix} 0 & 1 \\ -\varepsilon_0 & 0 \end{pmatrix} , \tag{8.283}$$

where $U(x,t,\lambda) = q(x,t) - \lambda\sigma_3$ and $\varepsilon_0 = \pm 1$.

The fundamental solution of $L(\lambda)$ then must satisfy (6.50), which means that $g(x,t)$ must satisfy:

$$g^*(x,t) = \epsilon^{-1}g(x,t)\epsilon \; . \tag{8.284}$$

As a result the potential $\widetilde{U}(x,t,\lambda) = \lambda S(x,t)$ of $\widetilde{L}(\lambda)$ must satisfy:

$$\widetilde{U}^*(x,t,\lambda^*) = -\epsilon^{-1}\widetilde{U}(x,t,\lambda)\epsilon \tag{8.285}$$

or, equivalently:

$$S^*(x,t) = -\epsilon^{-1}S(x,t)\epsilon \; . \tag{8.286}$$

The corresponding constraints on the scattering data follow directly from (6.52), (6.53), and (8.71):

$$\widetilde{a}^-(\lambda) = (\widetilde{a}^+(\lambda^*))^*, \qquad \widetilde{b}^-(t,\lambda) = \varepsilon_0(\widetilde{b}^+(t,\lambda^*))^* \; , \tag{8.287a}$$

$$\widetilde{\rho}^-(t,\lambda) = \varepsilon_0(\widetilde{\rho}^+(t,\lambda^*))^*, \qquad \widetilde{\tau}^-(t,\lambda) = \varepsilon_0(\widetilde{\tau}^+(t,\lambda^*))^* \; , \tag{8.287b}$$

$$\widetilde{\eta}^-(\lambda) = \varepsilon_0(\widetilde{\eta}^+(\lambda^*))^*, \qquad \widetilde{\kappa}^-(t,\lambda) = -(\widetilde{\kappa}^+(t,\lambda^*))^* \; , \tag{8.287c}$$

The second involution (see (6.63)) was of the form:

$$U(x,t,\lambda) = \sigma U(x,t,-\lambda)\sigma^{-1}, \qquad \sigma = \begin{pmatrix} 0 & 1 \\ \varepsilon_1 & 0 \end{pmatrix} , \tag{8.288}$$

where $\varepsilon_1 = \pm 1$. Then the fundamental solution of $L(\lambda)$ satisfies, (6.64) which means that:

$$g(x,t) = \sigma g(x,t)\sigma^{-1} \; , \tag{8.289}$$

and consequently:

$$\widetilde{U}(x,t,\lambda) = \sigma\widetilde{U}(x,t,-\lambda)\sigma^{-1} \; , \tag{8.290}$$

or

$$S(x,t) = -\sigma S(x,t)\sigma^{-1} \; . \tag{8.291}$$

The restrictions on the scattering data which follow from (6.65) to (6.67) and (8.71) take the form:

$$\widetilde{a}^-(\lambda) = \widetilde{a}^+(-\lambda), \qquad \widetilde{b}^-(t,\lambda) = -\varepsilon_1 \widetilde{b}^+(t,-\lambda), \tag{8.292a}$$

$$\widetilde{\rho}^-(t,\lambda) = -\varepsilon_1 \widetilde{\rho}^+(t,-\lambda), \qquad \widetilde{\tau}^-(t,\lambda) = -\varepsilon_1 \widetilde{\tau}^+(t,-\lambda) \ , \tag{8.292b}$$

$$\widetilde{\eta}^-(\lambda) = \widetilde{\eta}^+(-\lambda), \qquad \widetilde{\kappa}^-(t,\lambda) = -\widetilde{\kappa}^+(t,-\lambda) \ . \tag{8.292c}$$

Taking into account (8.71), and the fact that $a_0^\pm \neq 0$, we conclude that the zeroes of the functions $\widetilde{a}^\pm(\lambda)$ and $a^\pm(\lambda)$, whenever present, coincide. We formulate this as.

Corollary 8.2 *The gauge-equivalent operators $L(\lambda)$ and $\widetilde{L}(\lambda)$ have the same set of discrete eigenvalues.*

8.10.1 The Involutions $S_3 = S_3^*$, $S_+ = \pm(S_-)^*$

Written for the components of $S(x,t)$ the involution (8.286) with $\varepsilon_0 = 1$ takes the form:

$$S_3(x,t) = S_3^*(x,t), \qquad S_-(x,t) = S_+^*(x,t) \ . \tag{8.293a}$$

This means that all components of the vector $\mathbf{S}(x,t)$ become real and the corresponding 2×2 matrix becomes hermitian:

$$S(x,t) = S^\dagger(x,t) \ . \tag{8.293b}$$

As usual, we assume that conditions $\widetilde{\mathrm{C}}1$–$\widetilde{\mathrm{C}}3$ hold so that corollary 8.2 holds and both $L(\lambda)$ and $\widetilde{L}(\lambda)$ have the same set of discrete eigenvalues

$$\lambda_k^\pm = \lambda_{0,k} \pm i\lambda_{1,k} \in \mathbb{C}_\pm \ , \tag{8.294}$$

where μ_k and ν_k are real and $\nu_k > 0$. Note that the symmetry $\lambda_k^+ = (\lambda_k^-)^*$ is a consequence of the imposed involution. Besides, we need to take into account (8.287) and to check how they affect the data on the discrete spectrum. Skipping the details, we find that its effect on the action-angle variables are as follows:

$$\begin{aligned} &\widetilde{\eta}(\lambda) = \widetilde{\eta}^*(\lambda), \qquad \widetilde{\kappa}(\lambda) = -\widetilde{\kappa}^*(\lambda), \\ &\widetilde{\eta_k}^+ = (\widetilde{\eta_k}^-)^*, \qquad \widetilde{\kappa_k}^+ = (\widetilde{\kappa_k}^-)^*, \end{aligned} \tag{8.295a}$$

where $k = 1, \dots, N$, and

$$\begin{aligned} \widetilde{\eta}(\lambda) = \tfrac{1}{\pi}\ln(1 + |\widetilde{\rho}^+(t,\lambda)|^2), \qquad &\widetilde{\kappa}(\lambda) = i \arg \widetilde{b}^+(t,\lambda), \qquad \lambda \in \mathbb{R}, \\ \widetilde{\eta}_k^\pm = 2\lambda_{1,k} \mp i\lambda_{0,k}, \qquad &\widetilde{\kappa}_k^\pm = \ln|\widetilde{b}_k^+| \pm i \arg \widetilde{b}_k^+ . \end{aligned} \tag{8.295b}$$

Thus, we conclude that $\widetilde{\Omega}_{(0)}$ becomes purely real and has the form:

$$\widetilde{\Omega}_{(0)} = -\int_{-\infty}^{\infty} d\lambda \, \delta\widetilde{\eta}(\lambda) \wedge \delta \arg \widetilde{b}^+(t,\lambda)$$

$$- 4\sum_{k=1}^{N} \left(\delta\nu_k \wedge \delta \arg \widetilde{b}_k^+(t) + \delta\mu_k \wedge \delta \ln |\widetilde{b}_k^+(t)| \right) . \tag{8.296}$$

As regards the integrals of motion, they also become real, (see (8.209)):

$$\widetilde{C}_p = -\frac{1}{2} \int_{-\infty}^{\infty} d\lambda\, \lambda^{p-1} \widetilde{\eta}(\lambda) - \frac{2}{p} \sum_{k=1}^{N} \mathrm{Im}\, (\lambda_k^+)^p , \tag{8.297}$$

and so the Hamiltonians $H_{(0)}$ become real:

$$\begin{aligned} H_{(0)} &= 4 \sum_p f_p \widetilde{C}_{p+1} \\ &= -2 \int_{-\infty}^{\infty} d\mu\, f(\mu) \widetilde{\eta}(\mu) - 8 \sum_{k=1}^{N} \mathrm{Im}\, (\widetilde{F}_k^+) , \end{aligned} \tag{8.298}$$

Here $f(\lambda)$ is the dispersion law and $\widetilde{F}(\lambda)$ and $\widetilde{F}_k^+$ are introduced in (8.252).

Analogously for the hierarchy $\widetilde{\Omega}_{(m)}$ we get:

$$\widetilde{\Omega}_{(m)} = - \int_{-\infty}^{\infty} d\lambda\, \lambda^{m-2} \delta \widetilde{\eta}(\lambda) \wedge \delta \arg \widetilde{b}^+(t, \lambda) \tag{8.299}$$

$$- \frac{4}{m-1} \sum_{k=1}^{N} \left(\delta \mathrm{Im}\, (\lambda_k^+)^{m-1} \wedge \delta \arg \widetilde{b}_k^+(t) - \delta \mathrm{Re}\, (\lambda_k^+)^{m-1} \wedge \delta \ln |\widetilde{b}_k^+(t)| \right) .$$

The hierarchy of Hamiltonians is provided by:

$$\begin{aligned} H_{(m)} &= 4 \sum_p f_p \widetilde{C}_{p+m+1} \\ &= -2 \int_{-\infty}^{\infty} d\mu\, \mu^m f(\mu) \widetilde{\eta}(\mu) - 8 \sum_{k=1}^{N} \mathrm{Im}\, (\widetilde{F}_k^{(m),+}) . \end{aligned} \tag{8.300}$$

where $\widetilde{F}_k^{(m),+} = \int^{\lambda} d\lambda'\, \lambda'^{,m-2} \widetilde{f}(\lambda')$.

Thus, we see that the overall effect of the reduction is to "decrease twice" the number of the dynamical variables both on the continuous and discrete spectrum. Now two of the three types of angle variables: $\arg b^+(t, \lambda)$ and $\arg \widetilde{b}_k^+$ are real and take values in the interval $[0, 2\pi]$; the variables of the third type $\ln |\widetilde{b}_k^+|$ are also real but may take arbitrary values.

The reduction imposes restrictions also on the dispersion law of the NLEE; see (8.63b). The reduction (8.283) admits only dispersion laws whose expansion coefficients are real, such as:

$$f(\lambda) = \sum_p f_p \lambda^p, \qquad f_p = f_p^* . \tag{8.301}$$

The most important examples of NLEE obtained by this reduction are the HF equation (8.1), its higher analog

$$i\frac{\partial S}{\partial t} - \frac{if_0}{8}\left\{\left[S, \frac{\partial^3 S}{\partial x^3}\right] - \frac{\partial S}{\partial x}\langle S_{xx}, S(x,t)\rangle + 2\left[S(x,t), \frac{\partial S}{\partial x}\right]\langle S_x, S_x\rangle\right\} = 0\,, \tag{8.302}$$

which is gauge equivalent to the complex mKdV equation (7.97), and their combination equation (8.58), which is gauge equivalent to the NLS-cmKdV equation (7.98).

The second choice $\varepsilon_0 = -1$ in (8.286) leads to:

$$S_3(x,t) = S_3^*(x,t), \qquad S_-(x,t) = -S_+^*(x,t)\,. \tag{8.303a}$$

For the components of the vector $\mathbf{S}(x,t)$, this means that only $S_3(x,t)$ remains real, while $S_{1,2}(x,t)$ becomes purely imaginary:

$$S_3(x,t) = S_3^*(x,t), \qquad S_1(x,t) = -S_1^*(x,t), \qquad S_2(x,t) = -S_2^*(x,t)\,, \tag{8.303b}$$

and the corresponding 2×2 matrix becomes “quasi-hermitian”:

$$S(x,t) = \sigma_3 S^\dagger(x,t)\sigma_3\,. \tag{8.303c}$$

The change of sign of ε_0 is crucial because, as was explained in Sect. 6.3 above, this involution makes $L(\lambda)$ equivalent to a self-adjoint eigenvalue problem (6.54), which can have no complex discrete eigenvalues. In addition, the functions $a^\pm(\lambda)$ can have no eigenvalues on the real axis, which means that $L(\lambda)$ has no discrete spectrum. Since the gauge transformations are isospectral the same facts must be true also for $\widetilde{L}(\lambda)$. As a consequence, the corresponding NLEE can have no soliton solutions. However, their complete integrability is preserved. Their action–angle variables are provided by:

$$\eta(\lambda) = \tfrac{1}{\pi}\ln(1 - |\rho^+(\lambda)|^2), \qquad \kappa(\lambda) = i \arg b^+(\lambda), \qquad \lambda \in \mathbb{R}, \tag{8.304}$$

From (6.55b), there follows that $|\rho^+(\lambda)| < 1$ which makes the action variables well defined for all real λ.

The integrals of motion remain purely real, (see (8.209)):

$$\widetilde{C}_p = -\frac{1}{2}\int_{-\infty}^{\infty} d\lambda\, \lambda^{p-1}\widetilde{\eta}(\lambda)\,, \tag{8.305}$$

and the Hamiltonians become:

$$H_{(m)} = 4\sum_p f_p \widetilde{C}_{p+m+1} = -2\int_{-\infty}^{\infty} d\mu\, \mu^m \widetilde{f}(\mu)\widetilde{\eta}(\mu)\,. \tag{8.306}$$

The hierarchy of Hamiltonian structures is given by the sequence of symplectic forms:

$$\widetilde{\Omega}_{(m)} = -\int_{-\infty}^{\infty} d\lambda\, \lambda^{m-2} \delta\widetilde{\eta}(\lambda) \wedge \delta \arg \widetilde{b}^{+}(\lambda) \,. \tag{8.307}$$

Obviously they are closed.

Again, the overall effect of the reduction is to decrease twice the number of the dynamical variables. Now, we have "true" action-angle variables: $\widetilde{\eta}(\mu)$ and $\arg \widetilde{b}^{+}(t,\lambda)$ take values in the interval $[0, 2\pi]$. Therefore, the phase space $\widetilde{\mathcal{M}}$ in some sense is isomorphic to an infinite-dimensional torus.

The HF-type equations constrained by the involution (8.303) are gauge equivalent to the NLS and cmKdV equations:

$$iu_t + u_{xx} - 2|u|^2 u(x,t) = 0 \,, \tag{8.308}$$

$$u_t + u_{xxx} - 6|u|^2 u\hat{x}(x,t) = 0 \,, \tag{8.309}$$

with "wrong" signs of the nonlinearity, compared to (7.96) and (7.97). Due to the lack of soliton solutions, they are not so attractive to physicists.

8.10.2 The Involutions $S^{+} = \pm S^{-}$

This involution is obtained from (8.288) or (8.291) with $\varepsilon_1 = \mp 1$. We consider two possibilities: $\varepsilon_1 = 1$ and $\varepsilon_1 = -1$, which means that there are no restrictions on $S_3(x,t)$ but

$$S^{+}(x,t) = -\varepsilon_1 S^{-}(x,t) \,. \tag{8.310}$$

Then the condition $S^2(x,t) = \mathbb{1}$ means that

$$S_3^2(x,t) - \varepsilon_1 (S^{+})^2(x,t) = 1 \,, \tag{8.311}$$

Therefore, $S_3(x,t)$ and $S^{+}(x,t)$ can be parametrized as trigonometric functions of one variable:

$$S_3(x,t) = \cos(\alpha(x,t)), \qquad S^{+}(x,t) = S^{-}(x,t) = \sin(\alpha(x,t)) \,, \tag{8.312a}$$

for $\varepsilon_1 = -1$ and

$$S_3(x,t) = \cosh(\beta(x,t)), \qquad S^{+}(x,t) = -S^{-}(x,t) = \sinh(\beta(x,t)) \,. \tag{8.312b}$$

for $\varepsilon_1 = 1$. Note that since $S_3(x,t)$ and $S^{\pm}(x,t)$ are complex-valued functions, then $\alpha(x,t)$ and $\beta(x,t)$ may also be complex-valued. Since $S_3(x,t)$ must tend to 1 for $x \to \pm\infty$ (see (8.7)), then we must have:

$$\lim_{x\to\pm\infty} \alpha(x,t) = 2k_{\pm}\pi, \qquad k_{\pm} = 0, \pm 1, \pm 2, \ldots, \tag{8.313a}$$

$$\lim_{x\to\pm\infty} \beta(x,t) = 2p_{\pm} i\pi, \qquad p_{\pm} = 0, \pm 1, \pm 2, \ldots \,, \tag{8.313b}$$

where $k_{\pm}$, $p_{\pm}$ are integers. The differences

$$k \equiv k_+ - k_- = \int_{-\infty}^{\infty} dx\, \frac{\partial \alpha}{\partial x} \tag{8.314a}$$

$$p \equiv p_+ - p_- = \int_{-\infty}^{\infty} dx\, \frac{\partial \beta}{\partial x} \tag{8.314b}$$

are integrals of motion for all NLEE of the corresponding class. Such kinds of invariants were first discovered and analyzed for the s-G equation and were called topological charges, because they reflect the complicated topological properties of the corresponding phase space. Indeed, $\widetilde{\mathcal{M}}^{\mathbb{C}}$ splits into union of invariant subspaces:

$$\widetilde{\mathcal{M}}^{\mathbb{C}} = \bigcup_{k=-\infty}^{\infty} \widetilde{\mathcal{M}}_k^{\mathbb{C}}, \qquad \widetilde{\mathcal{M}}_k^{\mathbb{C}} \cap \widetilde{\mathcal{M}}_s^{\mathbb{C}} = \emptyset \quad \text{for} \quad k \neq s\ , \tag{8.315}$$

each subspace $\widetilde{\mathcal{M}}_k^{\mathbb{C}}$ has fixed value of k. The dynamics of any of the NLEE leaves all this subspaces invariant. Any solution that has nonvanishing value for k (or p) is considered as topologically nontrivial. For this reason, the value of k (or p) is known as the topological charge. Indeed, they are conserved not because of the special dynamics prescribed by the NLEEs but due to the special choice of the initial conditions. We shall come back to this below.

For this involutions, one can calculate also the gauge function $g(x,t)$ related to $S(x,t)$ through (8.34c) in terms of α or β, respectively. Since $g(x,t)$ must satisfy (8.34c), then it has only two independent matrix elements constrained by the requirement $\det g(x,t) = 1$:

$$g(x,t) = \begin{pmatrix} \cos(\alpha/2) & \sin(\alpha/2) \\ -\sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix}, \qquad \text{for } \varepsilon_1 = -1\ , \tag{8.316a}$$

$$g(x,t) = \begin{pmatrix} \cosh(\beta/2) & \sinh(\beta/2) \\ \sinh(\beta/2) & \cosh(\beta/2) \end{pmatrix}, \qquad \text{for } \varepsilon_1 = 1\ . \tag{8.316b}$$

The scattering data of $\widetilde{L}(\lambda)$ must satisfy the restrictions analogous to (6.66) and (6.68), namely:

$$\begin{gathered} \widetilde{a}^-(\lambda) = \widetilde{a}^+(-\lambda), \qquad \widetilde{b}^-(t,\lambda) = -\epsilon_1 \widetilde{b}^+(t,-\lambda)\ , \\ \widetilde{\eta}(\lambda) = \widetilde{\eta}(-\lambda), \qquad \widetilde{\kappa}(t,\lambda) = -\widetilde{\kappa}(t,-\lambda), \end{gathered} \tag{8.317}$$

for the data on the continuous spectrum and

$$\begin{array}{ll} \lambda_k^+ = -\lambda_k^-, & \widetilde{b}_k^+(t) = -\epsilon_1 \widetilde{b}_k^-(t)\ , \\ \widetilde{\eta}_k^+ = \widetilde{\eta}_k^-, & \widetilde{\kappa}_k^+(t) = -\widetilde{\kappa}_k^-(t). \end{array}, \qquad k = 1, \ldots, N\ . \tag{8.318}$$

The consequences of this involutions on the Hamiltonian structures and on the conservation laws are as follows. From (8.296), there follows that the canonical symplectic form $\widetilde{\Omega}_{(0)}$ becomes identically zero: $\widetilde{\Omega}_0 \equiv 0$. In fact, from (8.277) and (8.292), we find that for all symplectic forms with even indices:

$$\widetilde{\Omega}_{(2p)} \equiv 0 \,, \tag{8.319}$$

As for the forms with odd indices, we get:

$$\begin{aligned} \widetilde{\Omega}_{(2p+1)} &= 2i \int_0^\infty d\lambda\, \lambda^{2p-1} \delta \ln \frac{\widetilde{b}^+(t,\lambda)}{\widetilde{b}^+(t,-\lambda)} \wedge \delta\widetilde{\eta}(\lambda) \\ &\quad + \frac{2}{p-1} \sum_{k=1}^{N} \delta\widetilde{\kappa}_k^+ \wedge \delta(\lambda_k^+)^{2p} \,. \end{aligned} \tag{8.320}$$

The reason why the degeneracy (8.319) occurs is that the integrand becomes an odd function of λ; besides, the terms under the summation sign due to (8.317) and (8.318) cancel pairwise. For odd values of $m = 2p+1$ these terms add up.

Similar situation occurs for the integrals of motion. From (8.317) and (8.318) it follows that

$$\widetilde{C}_{2p} = 0 \,, \tag{8.321}$$

$$\widetilde{C}_{2p+1} = -\int_0^\infty d\lambda\, \lambda^{2p-1} \widetilde{\eta}(\lambda) - \frac{2}{2p-1} \sum_{k=1}^{N} (\lambda_k^+)^{2p-1} \,. \tag{8.322}$$

The reduction now requires $\widetilde{f}(\lambda)$ to be an odd function of λ

$$\widetilde{f}(\lambda) = -\widetilde{f}(-\lambda), \qquad \text{or,} \qquad \widetilde{f}(\lambda) = \sum_p f_{2p-1} \lambda^{2p-1} \,; \tag{8.323}$$

but the coefficients f_{2p-1} may take complex values. It is easy to see that due to this, such reductions are not applicable, e.g. to the NLS and HF equations which contain second-order derivatives with respect to x.

As examples of interesting NLEE related to this reduction in Chap. 7, we mentioned the mKdV and sG. Here, we present the equation gauge equivalent to the mKdV equation:

$$\frac{\partial^2 S}{\partial x \partial t} - \left\langle \frac{\partial^2 S}{\partial x \partial t}, S(x,t) \right\rangle S + \frac{1}{2}[\sigma_3, S] S(x,t) = 0 \,, \tag{8.324}$$

with one of the following additional conditions:

$$\langle S, \sigma_1 \rangle = 0, \qquad \varepsilon_2 = -1 \,, \tag{8.325a}$$

$$\langle S, \sigma_1 \rangle = 0, \qquad \varepsilon_1 = 1 \,. \tag{8.325b}$$

8.10.3 Applying Both Involutions

Of course, one can ask whether it is possible to impose both involutions simultaneously. The answer is positive only if they commute. Indeed, we can calculate the interrelation between $\widetilde{U}(x,t,-\lambda^*)$ and $\widetilde{U}(x,t,\lambda)$ in two ways: (i) first applying involution (8.283) and then (8.284) and (ii) first applying (8.284) then (8.283). The two results will be identical if the matrices ϵ and σ commute, i.e. the involutions are compatible provided:

$$\varepsilon_0 = -\varepsilon_1 \; . \tag{8.326}$$

Again, we can consider two possibilities. The first one is:

$$\varepsilon_0 = 1, \qquad \varepsilon_1 = -1 \; , \tag{8.327}$$

which means that $S_3^*(x,t) = S_3(x,t)$ and

$$(S^+(x,t))^* = S^-(x,t) = S^+(x,t), \qquad \text{or} \qquad S^\pm(x,t) = \sin(\alpha(x,t)) \; , \tag{8.328}$$

where $\alpha(x,t)$ is a real-valued function.

In this case, the scattering data must satisfy both sets of restrictions (8.287) and (8.292) at the same time. This has to be done with special care for the discrete spectrum. Indeed, the discrete spectrum of L may contain two types of eigenvalues:

1. pairs of purely imaginary eigenvalues:

$$\lambda_k^\pm = \pm i s_k, \qquad k = 1,\dots,N_0 \; , \tag{8.329}$$

2. quadruplets of complex eigenvalues, lying on the vertices of a quadrangle in the complex λ–plane:

$$\lambda_k^\pm = \lambda_{0,k} \pm i\lambda_{1,k}, \qquad \lambda_{k+N_1}^\pm = -\lambda_{0,k} \pm i\lambda_{1,k}, \qquad k = N_0+1,\dots,N_0+N_1 \; . \tag{8.330}$$

The functions $\widetilde{\eta}(\lambda)$ and $\widetilde{\kappa}(t,\lambda)$ as before have the properties:

$$\widetilde{\kappa}(t,\lambda) = \widetilde{\kappa}(-\lambda), \qquad \widetilde{\eta}(\lambda) = -\widetilde{\eta}(-\lambda) \; , \tag{8.331}$$

(compare (6.67c) and (8.195)) so that only the 2-forms $\widetilde{\Omega}_{(2p+1)}$ are nondegenerate. A bit more care is needed to calculate the contribution of the discrete spectrum to $\widetilde{\Omega}_{(2p+1)}$. The result is:

$$\begin{aligned} \widetilde{\Omega}_{(2p+1)} &= -2\int_0^\infty d\lambda\, \lambda^{2p-1} \delta\widetilde{\kappa}(t,\lambda)\wedge\delta\widetilde{\eta}(\lambda) + \frac{2(-1)^p}{p}\sum_{k=1}^{N_0} \delta \ln \widetilde{b}_k^+ \wedge \delta s_k^{2p} \\ &+ \frac{4}{p}\sum_{k=N_0+1}^{N_0+N_1} \left(\delta(\ln|\widetilde{b}_k^+|)\wedge\delta(\mathrm{Re}\,(\lambda_k^+)^{2p} - \delta\arg\widetilde{b}_k^+ \wedge \delta(\mathrm{Im}\,(\lambda_k^+)^{2p}\right) \; . \end{aligned} \tag{8.332}$$

for $p = 0,1,2,\dots$.

Of course, these formulae are not valid for $p = 0$. In this case, we have:

$$\widetilde{\Omega}_{(1)} = -2\int_0^\infty \frac{d\lambda}{\lambda}\delta\arg\widetilde{b}^+(t,\lambda)\wedge\delta\widetilde{\eta}(\lambda) + 2\sum_{k=1}^{N_0}\delta\ln\widetilde{b}_k^+\wedge\delta\ln s_k$$
$$+\,4\sum_{k=N_0+1}^{N_0+N_1}\Big(\delta(\ln|\widetilde{b}_k^+|)\wedge\delta(\ln|\lambda_k^+| - \delta\arg\widetilde{b}_k^+\wedge\delta(\arg(\lambda_k^+)\Big)\ . \qquad (8.333)$$

The 2-form $\widetilde{\Omega}_{(1)}$ is well defined only for the class of potentials for which the reflection coefficients $\widetilde{\rho}^{\pm}(\lambda=0)=0$ and $\widetilde{\tau}^{\pm}(\lambda=0)=0$.

Analogously, for the integrals of motion we get:

$$C_{2p-1} = -\int_0^\infty d\lambda\,\lambda^{2p-2}\widetilde{\eta}(\lambda) - \frac{4}{2p-1}\sum_{k=1}^{N_0}(-1)^p s_k^{2p-1}$$
$$-\frac{8}{2p-1}\sum_{k=N_0+1}^{N_0+N_1}\mathrm{Im}\,(\lambda_k^+)^{2p-1}\ . \qquad (8.334)$$

The second choice for the involution parameters is

$$\varepsilon_0 = -1, \qquad \varepsilon_1 = 1\ , \qquad (8.335)$$

which preserves the restriction on $S_3^*(x,t) = S_3(x,t)$ but changes the ones on $S^{\pm}(x,t)$ to:

$$(S^+(x,t))^* = -S^-(x,t) = S^+(x,t) = \sinh(\beta(x,t))\ , \qquad (8.336)$$

where $\beta(x,t)$ is a real-valued function. In this case, the Lax operator does not have discrete eigenvalues. Skipping the details, we note that the corresponding action-angle variables are given by:

$$\widetilde{\eta}(\lambda) = \frac{1}{\pi}\ln(1-|\widetilde{\rho}^+(t,\lambda)|^2), \qquad \widetilde{\kappa}(\lambda) = i\arg\widetilde{b}^+(t,\lambda), \quad 0\le\lambda\ . \ (8.337)$$

The symplectic forms $\widetilde{\Omega}_{(2p)}$ vanish identically and $\widetilde{\Omega}_{(2p-1)}$ reduce to

$$\widetilde{\Omega}_{(2p+1)} = 2\int_0^\infty d\lambda\,\lambda^{2p-1}\delta\widetilde{\eta}(\lambda)\wedge\delta\arg\widetilde{b}^+(t,\lambda)\ . \qquad (8.338)$$

The integrals of motion $\widetilde{C}_{2p}$ become identically zero, while $\widetilde{C}_{2p+1}$ remains nontrivial and purely real (see (7.105)):

$$\widetilde{C}_{2p-1} = -\frac{1}{2}\int_{-\infty}^\infty d\lambda\,\lambda^{2p-2}\widetilde{\eta}(\lambda)\ . \qquad (8.339)$$

The corresponding Hamiltonians $H_{(2p-1)}$ then take the form:

$$H_{(2p-1)} = 4\sum_p f_{2p}\widetilde{C}_{2p-1} = -2\int_0^\infty d\mu\,\mu^{2p-2}\widetilde{f}(\mu)\widetilde{\eta}(\mu)\ , \qquad (8.340)$$

where the dispersion law $\widetilde{f}(\lambda)$ is an odd function of λ.

We end this Chapter with a remark concerning the Lax representations of the sine-Gordon and sinh-Gordon equations [2]. They possess the symmetry mentioned in the remark 8.8 above. Indeed, these equations are symmetric under the exchange $x \leftrightarrow t$. Using as $L(\lambda)$ the Zakharov-Shabat system with the involutions $q(x) = v(x,t)\sigma$ and $v = \pm v^*$, we get that the M-operator takes the form:

$$M = i\frac{\partial}{\partial t} + \frac{1}{\lambda}S(x,t) \; . \tag{8.341}$$

where $S(x,t)$ is given by the proper formula in (8.312) above. If we insert (8.312a) (resp. (8.312b)) into the matrix equation (8.324), after some calculations we derive the s-G equation for $\alpha(x,t)$ (resp. the sinh-G equation for $\beta(x,t)$).

The compatibility condition $[L(\lambda), M(\lambda)] = 0$ leads to an interrelation between $q(x,t)$ and $S(x,t)$, which may be solved explicitly. Indeed, inserting (8.316) into $q(x,t) = -ig_x\hat{g}(x,t)$ we get:

$$q(x,t) = -\frac{i}{2}\begin{pmatrix} 0 & \alpha_x \\ -\alpha_x & 0 \end{pmatrix}, \qquad \text{for s-G case } \varepsilon_1 = -1 \; , \tag{8.342a}$$

$$q(x,t) = -\frac{i}{2}\begin{pmatrix} 0 & \beta_x \\ \beta_x & 0 \end{pmatrix}, \qquad \text{for sinh-G case } \varepsilon_1 = 1, \tag{8.342b}$$

i.e. we reproduce the Lax representation for the s-G and sinh-G equations with x and t interchanged. In this particular case, the gauge transformation just interchanges the operators L and M in the Lax representation.

8.11 Comments and Bibliographical Review

1. The equivalence between the NLS and the HF equation was discovered by Lakshmanan [3]. The notion of gauge transformations for the Lax representation was introduced by Zakharov and Takhtadjan [1]. Using it, the equivalence between seemingly different hierarchies of NLEE became clear [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]; it was applied also for the discrete evolution equations [16, 17].
2. The direct and inverse scattering method for the gauge-equivalent operator $\widetilde{L}$ was developed in [1]. The corresponding Wronskian relations, the explicit form of the squared solutions, and their completeness relations were derived by the present authors [2, 10]. There also it was shown that the interpretation of the ISM as a generalized Fourier transform is valid also for the gauge-equivalent systems. This fact can be generalized for ZS and pole-type Lax operators related to any simple Lie algebra [6].
3. The gauge-equivalent soliton equations, e.g. the HF-type equations also admit Hamiltonian formulation [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67,

68, 69, 70, 71, 72]. However, the gauge transformations act nontrivially on the hierarchies of Hamiltonian structures of the NLEE. The interrelation between the hierarchies Hamiltonian structures of NLS and the HF equations was discovered by Kulish and Reyman [4].

4. A gauge-covariant formulation of the theory of recursion operators $\Lambda_{\pm}$ for the Zakharov–Shabat system is proposed in [2, 7, 10]. There, the operators, $\widetilde{\Lambda}_{\pm}$, corresponding to the gauge equivalent system in the pole gauge were explicitly calculated. Thus, the unified approach to the nonlinear Schrödinger-type equations based on Λ can be automatically reformulated with the help of $\tilde{\Lambda}$ for the HF-type equations. Consequently, it is established that the conserved densities for the HF-type equations are polynomial in $S(x)$ and its x-derivatives. Special attention is paid to the interrelation between the hierarchies of symplectic structures corresponding to the above-mentioned families of gauge-equivalent equations.
5. Along with the standard and the pole gauge, there exist also other possibilities to fix up the gauge. Some of these gauge transformations may look somewhat exotic, but the important property is whether they are one-to-one mappings; we will call such gauges admissible. If that is true, then all the ideas displayed in the first part of this monograph such as Wronskian relations, expansions over squared solutions, Hamiltonian hierarchies, etc. can be transferred to any admissible gauge. As examples for such hierarchies, we can point out the relation between the HF-type equations and the Wadati-Konno-Ichikawa-Shimizu (WKIS) equations [73]. Another example of such nontrivial gauge-like transformations was used in interrelating the properties of the KdV-type equations with the ones of the Camassa–Holm hierarchy [74, 75, 76, 77, 78, 79, 80, 81].

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9

The Classical r-Matrix Method

In this chapter, we shall outline the modern approach to the Hamiltonian properties of the NLEE based on the classical r-matrix. In the first section, we introduce the notion of the classical r-matrix for the ZS system on finite interval. The same method is applicable to its gauge-equivalent system. Next we outline how the r-matrix method can be extended to the ZS system on the whole axis with vanishing boundary conditions and calculate the Poisson brackets between the matrix elements of the scattering matrix. In Sect. 9.3, we derive the classical Yang–Baxter equation as the condition on the r-matrix that ensures the Jacobi identity on the relevant Poisson brackets. The classical r-matrices entertwining the ZS system and its gauge-equivalent one can be naturally generalized to higher rank Lie algebras $\mathfrak{g}$. New solutions of the classical Yang–Baxter equation can be obtained from already known ones by averaging over the action of Lie algebra automorphisms. Combining these facts with the properties of the fundamental representations of $\mathfrak{g}$, one can prove that the principal series of the integrals of motion for a given gZS system are in involution.

9.1 The Classical r-Matrix and the NLEE of NLS Type

In this chapter, we shall extensively use the notion of tensor product of matrices. The tensor product $A \otimes B$ of the $n \times n$ matrix A with the $m \times m$ matrix B can be defined in different ways; two of them are [1]:

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B & \dots & A_{1n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}B & A_{n2}B & \dots & A_{nn}B \end{pmatrix} \tag{9.1}$$

or

$$A \otimes B = \begin{pmatrix} B_{11}A & B_{12}A & \dots & B_{1m}A \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1}A & B_{m2}A & \dots & B_{nn}A \end{pmatrix} \tag{9.2}$$

Gerdjikov, V.S. et al.: *The Classical* r*-Matrix Method*. Lect. Notes Phys. **748**, 315–354 (2008)
DOI 10.1007/978-3-540-77054-1_9

With both definitions one ends up with an $nm \times nm$ matrices; the right-hand sides of (9.1) and (9.2) are related by a similarity transformation [1].

The results displayed below are independent of the definition for $A \otimes B$ we are using.

One of the definitions of the classical r-matrix is based on the Lax representation for the corresponding NLEE. We shall start from this definition, but first we shall introduce the following notation:

$$\left\{ U(x,\lambda) \underset{,}{\otimes} U(y,\mu) \right\} , \tag{9.3}$$

which is an abbreviated record for the Poisson bracket between all matrix elements of $U(x,\lambda)$ and $U(y,\mu)$

$$\left\{ U(x,\lambda) \underset{,}{\otimes} U(y,\mu) \right\}_{ik,lm} = \{U_{ik}(x,\lambda), U_{lm}(y,\mu)\} . \tag{9.4}$$

In particular, if $U(x,\lambda)$ is of the form:

$$U(x,\lambda) = q(x) - \lambda\sigma_3, \qquad q(x) = q^+\sigma_+ + q^-\sigma_- , \tag{9.5}$$

and the matrix elements of $q(x)$ satisfy (7.23), then:

$$\left\{ U(x,\lambda) \underset{,}{\otimes} U(y,\mu) \right\} = i\left(\sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_+\right) \delta(x-y) . \tag{9.6}$$

The classical r-matrix in our case is an element from $sl(2) \otimes sl(2)$ defined through the relation:

$$\left\{ U(x,\lambda) \underset{,}{\otimes} U(y,\mu) \right\} = i\left[r(\lambda-\mu), U(x,\lambda) \otimes \mathbb{1} + \mathbb{1} \otimes U(y,\mu)\right] \delta(x-y) . \tag{9.7}$$

Equation (9.7) can be understood as a system of 16 equations for the 16 matrix elements of $r(\lambda-\mu)$. However, these relations must hold identically with respect to λ, μ and $q^\pm(x)$, i.e. (9.7) is an overdetermined system of algebraic equations for the matrix elements of r. It is far from obvious whether such $r(\lambda-\mu)$ exists; still less obvious is that it depends only on the difference $\lambda-\mu$. In other words, far from any choice for $U(x,\lambda)$ and any choice for the Poisson brackets between its matrix elements a classical r-matrix can be found. In the next Section, we shall discuss in greater detail the question as to which types of $U(x,\lambda)$ allow r-matrices. In our case, system (9.7) allows a solution, with the r-matrix given by:

$$r(\lambda-\mu) = -\frac{P}{\lambda-\mu} , \tag{9.8}$$

where P is a constant 4×4 matrix:

$$P = \frac{1}{2}\left(\mathbb{1} + \sum_{\alpha=1}^{3} \sigma_\alpha \otimes \sigma_\alpha\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{9.9}$$

The matrix P possesses the following special properties:

$$P(X \otimes Y) = (Y \otimes X)P, \qquad P^2 \equiv \mathbb{1} , \tag{9.10}$$

i.e. it interchanges the positions of the elements in the tensor product. By using these properties of P we get:

$$[P, q(x) \otimes \mathbb{1} + \mathbb{1} \otimes q(x)] = 0 , \tag{9.11}$$

showing that the right-hand side of (9.7) does not contain $q(x)$. Besides:

$$\begin{aligned} &[P, \lambda\sigma_3 \otimes \mathbb{1} + \mu\mathbb{1} \otimes \sigma_3] = (\lambda - \mu)\,[P, \sigma_3 \otimes \mathbb{1}] \\ &= -2(\lambda - \mu)\,(\sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_+) , \end{aligned} \tag{9.12}$$

where we used the commutation relations between the Pauli matrices:

$$[\sigma_\alpha, \sigma_\beta] = 2i\epsilon_{\alpha\beta\gamma}\sigma_\gamma, \qquad \alpha, \beta, \gamma = 1, 2, 3 . \tag{9.13}$$

The comparison between (9.11), (9.12), and (9.7) shows that $r(\lambda - \mu)$ (9.8) indeed satisfies the definition (9.7).

Let us now show that the classical r-matrix, if it exists of course, is a very effective tool for calculating the Poisson brackets between the matrix elements of the scattering matrix $T(\lambda, t)$. It will be more convenient for us to start with the Zakharov–Shabat problem with periodic boundary conditions:

$$i\frac{\partial T_\ell}{\partial x} + U(x, \lambda)T_\ell(x, y, \lambda) = 0, \qquad U(x + 2\ell, \lambda) = U(x, \lambda) , \tag{9.14}$$

By $T_\ell(x, y, \lambda)$, we denote the fundamental solution of (9.14), satisfying the boundary condition:

$$T_\ell(x, x, \lambda) = \mathbb{1} . \tag{9.15}$$

Obviously $T_\ell(x, y, \lambda)$ can be written down as the ratio of two Jost solutions of (9.14):

$$T_\ell(x, y, \lambda) = \psi(x, \lambda)\hat{\psi}(y, \lambda) . \tag{9.16}$$

Periodic boundary conditions require special treatment, which we shall not display in detail. We just mention that the proper analog of the scattering matrix $T(\lambda, t)$ here is the transfer matrix $T_\ell(\lambda, t)$ for one period $-\ell \leq x \leq \ell$, i.e.

$$T_\ell(t, \lambda) = T_\ell(-\ell, \ell, t, \lambda) = \begin{pmatrix} a_\ell^+(\lambda) & -b_\ell^-(t, \lambda) \\ b_\ell^+(t, \lambda) & a_\ell^-(\lambda) \end{pmatrix} \tag{9.17}$$

Now we prove the following relation:

$$\left\{ T_\ell(x,y,\lambda) \underset{,}{\otimes} T_\ell(x,y,\mu) \right\} = [r(\lambda-\mu), T_\ell(x,y,\lambda) \otimes T_\ell(x,y,\mu)]\, \delta(x-y) \ , \tag{9.18}$$

for $-\ell \le x < y \le \ell$. In order to do this we shall make use of the general properties of the Poisson brackets, from which there follows:

$$\begin{aligned} \left\{ T_\ell(x,y,\lambda) \underset{,}{\otimes} T_\ell(x,y,\mu) \right\}_{ab,cd} &= \{T_{\ell,ab}(x,y,\lambda), T_{\ell,cd}(x,y,\mu)\} \\ = \int_{-\infty}^{\infty} dz\, dz' \, \frac{\delta T_{\ell,ab}(x,y,\lambda)}{\delta U_{jk}(z,\lambda)} &\{U_{jk}(z,\lambda), U_{lm}(z',\mu)\} \, \frac{\delta T_{\ell,cd}(x,y,\lambda)}{\delta U_{lm}(z',\lambda)} \ . \end{aligned} \tag{9.19}$$

In order to calculate the variational derivatives of $T_{\ell,ab}(x,y,\lambda)$, we shall use the relations:

$$\begin{aligned} & i\frac{d\delta T_\ell}{dx} = U(x,\lambda)\delta T_\ell(x,y,\lambda) + \delta U(x,\lambda) T_\ell(x,y,\lambda) \ , \\ & \delta T_\ell(x,x,\lambda) = 0 \ , \end{aligned} \tag{9.20}$$

which are obtained by taking the variation of (9.14). The formal solution of (9.20) is given by:

$$\delta T_\ell(x,y,\lambda) = -i \int_{-\infty}^{\infty} dz T_\ell(x,z,\lambda) \delta U(z,\lambda) T_\ell(z,y,\lambda) \ , \tag{9.21}$$

i.e. for the variational derivatives $\delta(T_{\ell,ab}(x,y,\lambda))/\delta U_{jk}(z,\lambda)$ we get:

$$\frac{\delta T_{\ell,ab}(x,y,\lambda)}{\delta U_{jk}(z,\lambda)} = -i T_{\ell,aj}(x,z,\lambda) T_{\ell,kb}(z,y,\lambda) \ . \tag{9.22}$$

Let us now insert (9.22) into (9.20) and write down the result in terms of tensor products:

$$\begin{aligned} &\left\{ T_\ell(x,y,\lambda) \underset{,}{\otimes} T_\ell(x,y,\mu) \right\} = - \int_y^x dz \int_y^x dz' \, (T_\ell(x,z,\lambda) \otimes T_\ell(z',y,\mu)) \\ &\qquad \left\{ U(z,\lambda) \underset{,}{\otimes} U(z',\mu) \right\} (T_\ell(z,y,\lambda) \otimes T_\ell(z',y,\mu)) \\ &= -i \int_y^x dz \, (T_\ell(x,z,\lambda) \otimes T_\ell(z,y,\mu)) \\ &\qquad [r(\lambda-\mu), U(z,\lambda) \otimes \mathbb{1} + \mathbb{1} \otimes U(z,\mu)] \, (T_\ell(z,y,\lambda) \otimes T_\ell(z,y,\mu)) \\ &= \int_y^x dz \left(\{T_\ell(x,z,\lambda) \otimes T_\ell(z,y,\mu)) \, r(\lambda-\mu) \frac{d}{dz} \, (T_\ell(z,y,\lambda) \otimes T_\ell(z,y,\mu)) \right. \end{aligned}$$

$$
\begin{aligned}
&+ i\,(T_\ell(x,z,\lambda)\otimes T_\ell(z,y,\mu))\,(U(z,\lambda)\otimes \mathbb{1} \\
&+ \mathbb{1}\otimes U(z,\mu))\,r(\lambda-\mu)\,(T_\ell(z,y,\lambda)\otimes T_\ell(z,y,\mu)))
\end{aligned} \tag{9.23}
$$

For the next step we use the fact that if $T_\ell(x,y,\lambda)$ is a solution of (9.14), (9.15), then $T_\ell(x,z,\lambda)$ will satisfy:

$$
\frac{dT_\ell(x,z,\lambda)}{dz} = -T_\ell(x,z,\lambda)U(z,\lambda), \qquad T_\ell(x,x,\lambda) = \mathbb{1}\,. \tag{9.24}
$$

This allows us to write down the integrand in the last equality in (9.23) as a total z-derivative:

$$
\begin{aligned}
&\left\{T_\ell(x,y,\lambda) \underset{,}{\otimes} T_\ell(x,y,\mu)\right\} \\
&= \int_y^x dz \frac{d}{dz}\left\{(T_\ell(x,z,\lambda)\otimes T_\ell(x,z,\mu)\right\} r(\lambda-\mu) \\
&\qquad (T_\ell(z,y,\lambda)\otimes T_\ell(z,y,\mu))\} \\
&= \left[r(\lambda-\mu), T_\ell(x,y,\lambda)\otimes T_\ell(x,y,\mu)\right]\ ,
\end{aligned} \tag{9.25}
$$

after which it coincides with (9.19). In order to calculate the Poisson brackets between the matrix elements of $T_\ell(\lambda)$, it is enough to put $x=\ell$, $y=-\ell$:

$$
\left\{T_\ell(\lambda) \underset{,}{\otimes} T_\ell(\mu)\right\} = \left[r(\lambda-\mu), T_\ell(\lambda)\otimes T_\ell(\mu)\right]\ . \tag{9.26}
$$

An elementary consequence of this result is the involutivity of the integrals of motion $I_{\ell,k}$. Indeed, let us multiply both sides of (9.26) by $C\otimes C$ with $C = (\mathbb{1}+\sigma_3)/2$ and take the tr on both sides. Then its right-hand side obviously vanishes, and the left-hand side gives:

$$
\left\{a_\ell^+(\lambda), a_\ell^+(\mu)\right\} = 0\ . \tag{9.27}
$$

Consider the asymptotic expansions of $\ln a_\ell^+(\lambda)$

$$
\ln a_\ell^+(\lambda) = \sum_{k=1}^{\infty} I_{\ell,k}\lambda^{-k}\ , \tag{9.28}
$$

(compare with (6.87)) and insert it into (9.27). Since (9.27) holds identically with respect to λ and μ we obtain:

$$
\{I_{\ell,k}, I_{\ell,m}\} = 0, \qquad k,m = 1,2,\dots \tag{9.29}
$$

One can choose also $C = (\mathbb{1}-\sigma_3)/2$, which immediately gives us the involutivity also of:

$$
\left\{a_\ell^+(\lambda), a_\ell^-(\mu)\right\} = \left\{a_\ell^-(\lambda), a_\ell^-(\mu)\right\} = 0\ . \tag{9.30}
$$

Let us also write down the Poisson brackets between the remaining matrix elements of $T_\ell(\lambda)$:

$$\begin{aligned}
&\left\{b_\ell^+(\lambda), b_\ell^+(\mu)\right\} = \left\{b_\ell^-(\lambda), b_\ell^-(\mu)\right\} = 0\ ,\\
&\left\{a_\ell^+(\lambda), a_\ell^+(\mu)\right\} = \left\{a_\ell^-(\lambda), a_\ell^-(\mu)\right\} = 0\ ,\\
&\left\{a_\ell^+(\lambda), a_\ell^-(\mu)\right\} = \frac{1}{\lambda-\mu}\left(b_\ell^+(\lambda)b_\ell^-(\mu) - b_\ell^-(\lambda)b_\ell^+(\mu)\right)\ ,\\
&\left\{b_\ell^+(\lambda), b_\ell^-(\mu)\right\} = \frac{1}{\lambda-\mu}\left(a_\ell^+(\lambda)a_\ell^-(\mu) - a_\ell^-(\lambda)a_\ell^+(\mu)\right)\ ,\\
&\left\{b_\ell^+(\lambda), a_\ell^\pm(\mu)\right\} = \frac{\pm 1}{\lambda-\mu}\left(b_\ell^+(\lambda)a_\ell^\pm(\mu) - a_\ell^\pm(\lambda)b_\ell^+(\mu)\right)\ ,\\
&\left\{b_\ell^-(\lambda), a_\ell^\pm(\mu)\right\} = \frac{\mp 1}{\lambda-\mu}\left(b_\ell^-(\lambda)a_\ell^\pm(\mu) - a_\ell^\pm(\lambda)b_\ell^-(\mu)\right)\ ,
\end{aligned} \tag{9.31}$$

Taking the limit $\ell \to \infty$, we are able to transfer these results also for the case of potentials with zero boundary conditions at infinity. Such procedure is not trivial. Here, we note that the scattering matrix $T(\lambda, t)$ is related to $T_\ell(\lambda, t)$ through:

$$T(\lambda) = \lim_{\substack{x\to-\infty\\ y\to\infty}} E^{-1}(x,\lambda)T_\ell(x,y,t,\lambda)E(y,\lambda), \qquad E(x,\lambda) = e^{-i\lambda x\sigma_3}\ . \tag{9.32}$$

Therefore, we multiply (9.19) by $E(y,\lambda)\otimes E(y,\mu)$ on the right and by $E^{-1}(x,\lambda)\otimes E^{-1}(x,\mu)$ on the left and calculate the limit for $x\to\infty$, $y\to-\infty$ taking into account (9.32) and the well-known formulae:

$$\lim_{x\to\pm\infty}\frac{e^{ix(\lambda-\mu)}}{\lambda-\mu} = \pm i\pi\delta(\lambda-\mu)\ . \tag{9.33}$$

The answer is given by:

$$\begin{aligned}
&\left\{T(\lambda) \underset{,}{\otimes} T(\mu)\right\}\\
&= r_+(\lambda-\mu)T(\lambda)\otimes T(\mu) - T(\lambda)\otimes T(\mu) r_-(\lambda-\mu)\ , \qquad (9.34)\\
&r_\pm(\lambda-\mu) = \lim_{x\to\pm\infty}\left(E^{-1}(x,\lambda)\otimes E^{-1}(x,\mu)\right) r(\lambda-\mu)\left(E(y,\lambda)\otimes E(y,\mu)\right)\\
&= -\frac{\sigma_0\otimes\sigma_0+\sigma_3\otimes\sigma_3}{2(\lambda-\mu)} \mp i\pi\delta(\lambda-\mu)\left(\sigma_+\otimes\sigma_- - \sigma_-\otimes\sigma_+\right)\ . \qquad (9.35)
\end{aligned}$$

Written in components the Poisson brackets (9.34) have the form:

$$\left\{a^+(\lambda), a^+(\mu)\right\} = \left\{a^-(\lambda), a^-(\mu)\right\} = \left\{a^+(\lambda), a^-(\mu)\right\} = 0\ ,$$

$$
\begin{aligned}
&\left\{b^+(\lambda), b^+(\mu)\right\} = \left\{b^-(\lambda), b^-(\mu)\right\} = 0 \,, \\
&\left\{b^+(\lambda), a^\pm(\mu)\right\} = \pm\frac{b^+(\lambda)a^\pm(\mu)}{2(\lambda-\mu)} + i\pi\delta(\lambda-\mu)a^\pm(\lambda)b^+(\lambda) \,, \\
&\left\{b^-(\lambda), a^\pm(\mu)\right\} = \mp\frac{b^-(\lambda)a^\pm(\mu)}{2(\lambda-\mu)} - i\pi\delta(\lambda-\mu)a^\pm(\lambda)b^-(\lambda) \,, \\
&\left\{b^+(\lambda), b^-(\mu)\right\} = -2i\pi\delta(\lambda-\mu)a^+(\lambda)a^-(\lambda) \,,
\end{aligned}
\tag{9.36}
$$

From the equalities in the first line of (9.36), there immediately follows that the integrals of motion $I_{\ell,k}$ retain their involutivity after the limit $\ell \to \infty$. The relations (9.36) allow us also to calculate the Poisson brackets between:

$$
\eta(\lambda) = \frac{1}{\pi}\ln a^+(\lambda)a^-(\lambda) = -\frac{1}{\pi}\ln\left(1+\rho^+(\lambda)\rho^-(\lambda)\right) \,, \tag{9.37}
$$

and

$$
\kappa(\lambda) = \frac{1}{2}\ln\left(b^+(\lambda)/b^-(\lambda)\right) \,, \tag{9.38}
$$

and to assert that

$$
\{\kappa(\lambda), \eta(\mu)\} = i\delta(\lambda-\mu), \qquad \{\kappa(\lambda), \kappa(\mu)\} = \{\eta(\lambda), \eta(\mu)\} = 0 \tag{9.39}
$$

for λ, $\mu \in \mathbb{R}$. Generically speaking, the Zakharov–Shabat system L (9.14b) possesses also discrete eigenvalues. The variables $\eta(\lambda)$, $\kappa(\lambda)$ (9.37) and (9.38) provide the scattering data on the continuous spectrum of L; the data on its discrete spectrum is characterized by:

$$
\eta_k^\pm = \mp 2i\lambda_k^\pm, \qquad \kappa_k^\pm = \pm\ln b_k^\pm, \qquad k = 1, \dots, M \,, \tag{9.40}
$$

where $\lambda_k^\pm \in \mathbb{C}_\pm$ are the discrete eigenvalues of L. We assume that these are finite number of simple eigenvalues and that the normalization of the Jost solutions $\psi_k^\pm(x) \equiv \psi(x, \lambda_k^\pm)$ are determined by the constants $b_k^\pm$. The fact that the eigenvalues $\lambda_k^\pm$ are simple means that $\lambda_k^\pm$ are simple zeroes of $a^\pm(\lambda)$, i.e. in the neighborhood of $\lambda_k^\pm$ $a^\pm(\lambda)$ has the expansion:

$$
a^\pm(\lambda) = \left(\lambda - \lambda_k^\pm\right)\dot{a}_k^\pm + \frac{1}{2}\left(\lambda - \lambda_k^\pm\right)^2 \ddot{a}_k^\pm + \dots \tag{9.41}
$$

The calculation of the Poisson brackets between $\eta_k^\pm$, $\kappa_k^\pm$, $\eta(\lambda)$ and $\kappa(\lambda)$ requires a deeper knowledge of the spectral theory of the system L which was outlined in Chaps. 3 and 4 of the present monograph. Here, we shall only write down the nontrivial (i.e. the non-vanishing) Poisson brackets:

$$
\left\{\eta_k^+, \kappa_l^+\right\} = \left\{\eta_k^-, \kappa_l^-\right\} = \delta_{kl} \,. \tag{9.42}
$$

The comparison between (2.54), (9.37), and (9.38) shows that if $q(x)$ is a solution of the NLEE (6.7), then $\eta_k^\pm$, $\kappa_k^\pm$, $\eta(\lambda)$ and $\kappa(\lambda)$ will satisfy:

$$
i\eta_t = 0, \qquad i\kappa_t = f(\lambda), \qquad i\eta_{k,t}^\pm = 0, \qquad i\kappa_{k,t}^\pm = f(\lambda_k^\pm), \tag{9.43}
$$

This makes evident the fact that the set $\{\eta(\lambda), \kappa(\lambda), \eta_k^{\pm}, \kappa_k^{\pm}\}$ may be considered as a generalization of the action-angle variables for systems with an infinite number of degrees of freedom as seen in Chap. 7.

9.2 The Classical r-Matrix and the NLEE of Heisenberg Ferromagnet Type

The method, which we used in the previous Section to introduce the classical r-matrix is not explicitly gauge covariant. We shall show that not only the Zakharov–Shabat system but also its gauge-equivalent one also allows an r-matrix formulation. We start by formulating the system $\tilde{L}$, gauge equivalent to L:

$$\tilde{L}\tilde{T}_\ell(x,y,\lambda) \equiv \left(i\frac{d}{dx} - \lambda S(x)\right)\tilde{T}_\ell(x,y,\lambda) = 0, \qquad \tilde{T}_\ell(x,x,\lambda) = 0\ , \quad (9.44)$$

where

$$S(x) = g^{-1}(x)\sigma_3 g(x) = \sum_{\alpha=1}^{3} S_\alpha(x)\sigma_\alpha\ , \qquad (9.45)$$

and $g(x) = \tilde{T}_\ell(x,y,\lambda=0)$. The symplectic structure in the phase space $\tilde{\boldsymbol{\Phi}}$ of the NLEE (8.203) is introduced by the following Poisson brackets:

$$\left\{S_\alpha(x), S_\beta(y)\right\}^{(0)} = -\epsilon_{\alpha\beta\gamma}S_\gamma(x)\delta(x-y)\ . \qquad (9.46)$$

(compare with (9.13)). The system (9.44) can be written down in the form $id\tilde{\psi}/dx + \tilde{U}(x,\lambda)$ with $\tilde{U}(x,\lambda) = -\lambda S(x)$. By inserting this expression for $\tilde{U}(x,\lambda)$ into (9.4), we obtain the following form of (9.6) related to the system $\tilde{L}$ and to the HFE:

$$\left\{\tilde{U}(x,\lambda) \underset{,}{\otimes} \tilde{U}(y,\mu)\right\}^{(0)} = -\lambda\mu\sum_{\alpha,\beta,\gamma}\epsilon_{\alpha\beta\gamma}\sigma_\alpha\otimes\sigma_\beta S_\gamma(x)\delta(x-y) \qquad (9.47)$$

We introduce the corresponding classical r-matrix through:

$$\left\{U(x,\lambda) \underset{,}{\otimes} U(y,\mu)\right\}^{(0)} = i\left[\tilde{r}(\lambda-\mu), \tilde{U}(x,\lambda)\otimes 1\!\!1 + 1\!\!1\otimes\tilde{U}(y,\mu)\right]\delta(x-y)$$

$$(9.48)$$

and consider (9.48), just like in the previous Section, as a system of 16 equations for the 16 matrix elements of $\tilde{r}(\lambda-\mu)$. They must hold identically with respect to λ, μ and $S(x)$. Just like before, it is far from obvious whether this system allows solution, but we can see that

$$\tilde{r}(\lambda-\mu) = -\frac{\lambda\mu P}{\lambda-\mu}, \qquad P = \frac{1}{2}\sum_{\alpha=0}^{3}\sigma_\alpha\otimes\sigma_\alpha\ , \qquad (9.49)$$

indeed satisfies (9.48). At first glance $\tilde{r}(\lambda-\mu)$ is not a function of the difference $\lambda-\mu$; but (9.49) can be written down as:

$$\tilde{r}(\lambda-\mu) = -\frac{P}{1/\lambda - 1/\mu} . \tag{9.50}$$

After changing the variables λ and μ by $1/\lambda$ and $1/\mu$ we see that $\tilde{r}(1/\lambda, 1/\mu)$ again depends only on the difference $1/\lambda - 1/\mu$ and, moreover, has the same matrix structure as the classical r-matrix for the NLSE.

Next, we derive the analogue to relation (9.26) for system (9.44). In order to calculate the Poisson brackets between the matrix elements of $\tilde{T}_\ell(x,y,\lambda)$, we need to know their variational derivatives with respect to $S_\gamma(x)$; these will be calculated by making use of (9.44) and the relation:

$$\begin{aligned} & i\frac{d\delta\tilde{T}_\ell}{dx} - \lambda S(x,y)\delta\tilde{T}_\ell(x,y,\lambda) = \lambda\delta S(x)\tilde{T}_\ell(x,y,\lambda) , \\ & \delta\tilde{T}_\ell(x,x,\lambda) = 0 , \end{aligned} \tag{9.51}$$

following from (9.44). The solution of (9.51) is given by the expression:

$$\delta\tilde{T}_\ell(x,y,\lambda) = -i\int_{-\infty}^{\infty} dz \tilde{T}_\ell(x,z,\lambda)\delta S(z)\tilde{T}_\ell(z,y,\lambda) , \tag{9.52}$$

i.e. the variational derivatives $\delta(T_{\ell,ab}(x,y,\lambda))/\delta S_0(z)$ are equal to

$$\frac{\delta\tilde{T}_{\ell,ab}(x,y,\lambda)}{\delta S_c(z,\lambda)} = -i\lambda\tilde{T}_\ell(x,z,\lambda)\sigma_c\tilde{T}_\ell(z,y,\lambda) . \tag{9.53}$$

The rest of the calculations are analogous to the ones in Sect. 9.1 and lead to:

$$\left\{\tilde{T}_\ell(x,y,\lambda) \underset{,}{\otimes} \tilde{T}_\ell(x,y,\mu)\right\}^{(0)} = \left[\tilde{r}(\lambda-\mu), \tilde{T}_\ell(x,y,\lambda)\otimes\tilde{T}_\ell(x,y,\mu)\right] , \tag{9.54}$$

for $-\ell < x < y < \ell$. In order to calculate the Poisson brackets between the matrix elements of $T_\ell(\lambda)$ it is enough to assume $x=\ell$, $y=-\ell$

$$\left\{\tilde{T}_\ell(\lambda) \underset{,}{\otimes} \tilde{T}_\ell(\mu)\right\}^{(0)} = \left[\tilde{r}(\lambda-\mu), \tilde{T}_\ell(\lambda)\otimes\tilde{T}_\ell(\mu)\right] . \tag{9.55}$$

From the properties of P (see (9.10), (9.11), (9.50), and (9.55), there immediately follows the involutivity of the integrals of motion $\tilde{I}_{\ell,k}$ with respect to Poisson brackets (9.46). Indeed, let us take the tr from both sides of (9.55); its right-hand side obviously vanishes, and denoting $\operatorname{tr}\tilde{T}_\ell(\lambda) = \tilde{F}_\ell(\lambda)$, the left-hand side gives:

$$\left\{\tilde{F}_\ell(\lambda), \tilde{F}_\ell(\mu)\right\}^{(0)} = 0 , \tag{9.56a}$$

From the gauge equivalence between L and $\tilde{L}$, we get $\tilde{a}_\ell^\pm(\lambda) = a_\ell^\pm(\lambda)/a_\ell^\pm(0)$. Therefore, the asymptotic expansions for $\ln \tilde{a}_\ell^\pm(\lambda)$ reads:

$$\ln \tilde{a}_\ell^\pm(\lambda) = \sum_{k=0}^{\infty} I_{\ell,k}\lambda^{-k}, \qquad I_{\ell,0} = \ln a_\ell^\pm(0) \; . \tag{9.56b}$$

Next, we derive the analog of (9.27):

$$\left\{ \tilde{a}_\ell^+(\lambda), \tilde{a}_\ell^+(\mu) \right\}^{(0)} = 0 \; . \tag{9.56c}$$

which must hold identically with respect to λ and μ. This gives, in complete analogy with (9.27),

$$\left\{ I_{\ell,k}, I_{\ell,m} \right\}^{(0)} = 0, \qquad k, m = 0, 1, 2, \ldots \tag{9.56d}$$

The formulae (9.56) are not an immediate consequence of (9.27), (9.28), (9.29) and (9.30), although they resemble them very much. The essential difference consists in that the gauge covariance does not transfer Poisson brackets (7.23) into Poisson brackets (9.46). Later we shall formulate the nontrivial interrelation between the Hamiltonian structures, defined by these two Poisson brackets. Now, we write down all Poisson brackets between the matrix elements of $T_\ell(\lambda)$, which follow from (9.55)

$$\begin{aligned}
\left\{ \tilde{a}_\ell^+(\lambda)\tilde{a}_\ell^+(\mu) \right\}^{(0)} &= \left\{ \tilde{a}_\ell^-(\lambda)\tilde{a}_\ell^-(\mu) \right\}^{(0)} = 0 \\
\left\{ \tilde{a}_\ell^+(\lambda)\tilde{a}_\ell^-(\mu) \right\}^{(0)} &= \frac{\lambda\mu}{\lambda-\mu}\left(\tilde{b}_\ell^+(\lambda)\tilde{b}_\ell^-(\mu) - \tilde{b}_\ell^-(\lambda)\tilde{b}_\ell^+(\mu) \right) , \\
\left\{ \tilde{b}_\ell^+(\lambda)\tilde{b}_\ell^-(\mu) \right\}^{(0)} &= \frac{\lambda\mu}{\lambda-\mu}\left(\tilde{a}_\ell^+(\lambda)\tilde{a}_\ell^-(\mu) - \tilde{a}_\ell^-(\lambda)\tilde{a}_\ell^+(\mu) \right) , \\
\left\{ \tilde{b}_\ell^+(\lambda)\tilde{a}_\ell^\pm(\mu) \right\}^{(0)} &= \pm\frac{\lambda\mu}{\lambda-\mu}\left(\tilde{b}_\ell^+(\lambda)\tilde{a}_\ell^\pm(\mu) - \tilde{a}_\ell^\pm(\lambda)\tilde{b}_\ell^+(\mu) \right) , \\
\left\{ \tilde{b}_\ell^-(\lambda)\tilde{a}_\ell^\pm(\mu) \right\}^{(0)} &= \mp\frac{\lambda\mu}{\lambda-\mu}\left(\tilde{b}_\ell^-(\lambda)\tilde{a}_\ell^\pm(\mu) - \tilde{a}_\ell^\pm(\lambda)\tilde{b}_\ell^-(\mu) \right) ,
\end{aligned} \tag{9.57}$$

The limit $\ell \to \infty$ can be taken like in (9.33) using the analogue of (9.32)

$$\tilde{T}(\lambda) = \lim_{x\to\infty} \lim_{y\to-\infty} E^{-1}(x,\lambda)\tilde{T}_\ell(x,y,\lambda)E(y,\lambda) \; . \tag{9.58}$$

This limit will also allow transfer of the results to the case of potentials with vanishing boundary conditions. As a result, we obtain:

$$\left\{ \tilde{T}_\ell(\lambda) \underset{,}{\otimes} T(\mu) \right\}^{(0)} = \tilde{r}_+(\lambda-\mu)\tilde{T}_\ell(\lambda)\otimes\tilde{T}_\ell(\mu) - \tilde{T}_\ell(\lambda)\otimes\tilde{T}_\ell(\mu)\tilde{r}_-(\lambda-\mu) \; , \tag{9.59}$$

where

$$\begin{aligned}
&\tilde{r}_{\pm}(\lambda-\mu) \\
&= \lim_{x\to\pm\infty} \left(E^{-1}(x,\lambda)\otimes E^{-1}(x,\mu)\right)\tilde{r}(\lambda-\mu)\left(E(x,\lambda)\otimes E(x,\mu)\right) \\
&= -\frac{\lambda\mu}{2(\lambda-\mu)}\left(\sigma_0\otimes\sigma_0+\sigma_3\otimes\sigma_3\right) \\
&\qquad \mp i\pi\lambda^2\delta(\lambda-\mu)\left(\sigma_+\otimes\sigma_- - \sigma_-\otimes\sigma_+\right)\ .
\end{aligned} \tag{9.60}$$

Written in components, Poisson brackets (9.59) have the form:

$$\left\{\tilde{a}^+(\lambda),\tilde{a}^+(\mu)\right\}^{(0)} = \left\{\tilde{a}^+(\lambda),\tilde{a}^-(\mu)\right\}^{(0)} = \left\{\tilde{a}^-(\lambda),\tilde{a}^-(\mu)\right\}^{(0)} = 0$$

$$\left\{\tilde{b}^+(\lambda),\tilde{b}^+(\mu)\right\}^{(0)} = \left\{\tilde{b}^-(\lambda),\tilde{b}^-(\mu)\right\}^{(0)} = 0\ , \tag{9.61}$$

$$\left\{\tilde{b}^+(\lambda),\tilde{a}^{\pm}(\mu)\right\}^{(0)} = \pm\frac{\lambda\mu}{2(\lambda-\mu)}\tilde{b}^+(\lambda)\tilde{a}^{\pm}(\mu) + i\pi\delta(\lambda-\mu)\lambda^2\tilde{a}^{\pm}(\lambda)\tilde{b}^+(\lambda)\ ,$$

$$\left\{\tilde{b}^-(\lambda),\tilde{a}^{\pm}(\mu)\right\}^{(0)} = \mp\frac{\lambda\mu}{2(\lambda-\mu)}\tilde{b}^-(\lambda)\tilde{a}^{\pm}(\mu) - i\pi\delta(\lambda-\mu)\lambda^2\tilde{a}^{\pm}(\lambda)\tilde{b}^-(\lambda)\ ,$$

$$\left\{\tilde{b}^+(\lambda),\tilde{b}^-(\mu)\right\}^{(0)} = -2i\pi\delta(\lambda-\mu)\lambda^2\tilde{a}^+(\lambda)\tilde{a}^-(\lambda)\ ,$$

In analogy with what we had before, there follows that the integrals of motion $I_{\ell,k}$ retain their involutivity also after the limit $\ell\to\infty$ is taken. From them there follows that the Poisson brackets between

$$\tilde{\eta}(\lambda) = \frac{1}{\pi\lambda^2}\ln a^+(\lambda)a^-(\lambda) = -\frac{1}{\pi\lambda^2}\left(1+\rho^+(\lambda)\rho^-(\lambda)\right)\ , \tag{9.62}$$

and

$$\tilde{\kappa}(\lambda) = \frac{1}{2}\ln\left(\tilde{b}^+(\lambda)/\tilde{b}^-(\lambda)\right)\ , \tag{9.63}$$

for λ, $\mu\in\mathbb{R}$ are equal to:

$$\left\{\tilde{\kappa}(\lambda),\tilde{\eta}(\mu)\right\}^{(0)} = i\delta(\lambda-\mu), \qquad \left\{\tilde{\eta}(\lambda),\tilde{\eta}(\mu)\right\}^{(0)} = \left\{\tilde{\kappa}(\lambda),\tilde{\kappa}(\mu)\right\}^{(0)} = 0\ . \tag{9.64}$$

A well-known fact from the scattering theory is that the gauge transformations preserve the spectrum of the operators, i.e. L and $\tilde{L}$ have the same continuous spectrum (filling up the real λ-axis) and the same sets of eigenvalues $\lambda_k^{\pm}$. The scattering data, characterizing the discrete spectrum of $\tilde{L}$, are introduced in analogy to $\eta_k^{\pm}$, $\kappa_k^{\pm}$ through:

$$\tilde{\eta}_k^{\pm} = \eta_k^{\pm}\left(\lambda_k^{\pm}\right)^{-2} = \mp 2i\left(\lambda_k^{\pm}\right)^{-1}, \qquad \tilde{\kappa}_k^{\pm} = \pm\ln\tilde{b}_k^{\pm}, \qquad k=1,\dots,M\ , \tag{9.65}$$

The nontrivial Poisson brackets between $\tilde{\eta}(\lambda)$, $\tilde{\kappa}(\lambda)$, $\tilde{\eta}_k^{\pm}$ and $\tilde{\kappa}_k^{\pm}$ are given by the expressions:

$$\left\{\tilde{\eta}_k^+, \tilde{\kappa}_l^+\right\}^{(0)} = \left\{\tilde{\eta}_k^-, \tilde{\kappa}_k^-\right\}^{(0)} = \delta_{kl} \; . \tag{9.66}$$

The relations (9.64), (9.66) can be formally obtained as a consequence of (9.39), (9.42) by taking into account the interrelations between the scattering data of both problems and by using the equivalence between the Hamiltonian structures related to L and $\tilde{L}$. This equivalence is a consequence of the following identities between the two gauge-equivalent systems for periodic boundary conditions (compare with (8.68), (8.69)):

$$\tilde{T}(x,y,t,\lambda) = g^{-1}(x,t)T(x,y,t,\lambda)g(y,t) \; , \tag{9.67a}$$

$$\tilde{T}_\ell(\lambda) = \tilde{T}_\ell(\ell,-\ell,t,\lambda) = T_\ell(t,\lambda)g(-\ell,t) = T_\ell(t,\lambda)T_\ell^{-1}(t,0)$$

in the periodic case, and by

$$\tilde{\psi}(x,t,\lambda) = g^{-1}(x,t)\psi(x,t,\lambda), \qquad \tilde{T}(t,\lambda) = T(t,\lambda)T^{-1}(t,0) \; ,$$

$$\tilde{\phi}(x,t,\lambda) = g^{-1}(x,t)\phi(x,t,\lambda)T^{-1}(t,0) \; , \tag{9.67b}$$

in the case of vanishing boundary conditions.

After some calculations we obtain:

$$\tilde{\eta}(\lambda) = \lambda^{-2}\eta(\lambda), \qquad \tilde{\kappa}(\lambda) = \kappa(\lambda) - \frac{1}{2}\ln\left(a^+(0)/a^-(0)\right) \tag{9.68}$$

and

$$\tilde{\eta}_k^{\pm} = \left(\lambda_k^{\pm}\right)^{-2}\eta_k^{\pm}, \qquad \tilde{\kappa}_k^{\pm} = \kappa_k^{\pm} - \frac{1}{2}\ln\left(a^+(0)/a^-(0)\right) \; . \tag{9.69}$$

9.3 Jacobi Identity and the Classical Yang–Baxter Equations

The canonical Poisson brackets (7.23) also allow us to calculate the Poisson brackets between any two admissible functionals A and B of $q^+(x)$ and $q^-(x)$. By admissible functionals, we mean functionals, which depend only on the values of $q^+(x)$ and $q^-(x)$ *inside the interval* $-\ell < x < \ell$; this means that the boundary conditions do not contribute to the Poisson brackets.[1] For example, if A and B are 2×2 matrix-valued functions, we obtain:

$$\left\{A \underset{,}{\otimes} B\right\} = i\int_{-\ell}^{\ell} dx \left(\frac{\delta A}{\delta v(x)} \otimes \frac{\delta B}{\delta v^*(x)} - \frac{\delta A}{\delta v^*(x)} \otimes \frac{\delta B}{\delta v(x)}\right) \; . \tag{9.70}$$

[1] The notion of admissible functionals and its importance to the investigation of the complete integrability of the NLEE is studied in detail in [2], Chap. 3, Part I.

The Poisson brackets possess the following properties:

(a) skew-symmetry:

$$\left\{A \underset{,}{\otimes} B\right\} = -P\left\{B \underset{,}{\otimes} A\right\}P\,, \tag{9.71}$$

(b) differentiability:

$$\left\{A \underset{,}{\otimes} BC\right\} = \left\{A \underset{,}{\otimes} B\right\}(\mathbb{1} \otimes C) + (\mathbb{1} \otimes B)\left\{A \underset{,}{\otimes} C\right\} \quad ; \tag{9.72}$$

(c) Jacobi identity:

$$\begin{aligned}&\left\{A \underset{,}{\otimes} \left\{B \underset{,}{\otimes} C\right\}\right\} + P_{13}P_{23}\left\{C \underset{,}{\otimes} \left\{A \underset{,}{\otimes} B\right\}\right\}P_{23}P_{13} + \\ &P_{13}P_{12}\left\{C \underset{,}{\otimes} \left\{A \underset{,}{\otimes} B\right\}\right\}P_{12}P_{13} = 0\,.\end{aligned} \tag{9.73}$$

Let us make clear the notations introduced in (9.73). Each summand in the right-hand side of (9.73) acts in the tensor cube $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, and by P_{kl}, we have denoted the linear operator (the matrix) P (9.9), which acts nontrivially only in the spaces labeled by k and l. In other words:

$$\begin{aligned}&P_{12} = \frac{1}{2}\sum_{k=0}^{3}\sigma_\alpha \otimes \sigma_\alpha \otimes \mathbb{1}\,, \qquad P_{13} = \frac{1}{2}\sum_{k=0}^{3}\sigma_\alpha \otimes \mathbb{1} \otimes \sigma_\alpha\,, \\ &P_{23} = \frac{1}{2}\sum_{k=0}^{3}\mathbb{1} \otimes \sigma_\alpha \otimes \sigma_\alpha\,,\end{aligned} \tag{9.74}$$

The proofs of (9.71), (9.72) and (9.73) follow from the properties of the Poisson brackets for the c-number functionals and also from the fact that each 2×2 matrix can be written down as:

$$A = \sum_{k,l} A_{kl}E_{kl}, \qquad B = \sum_{m,n} B_{mn}E_{mn}\,, \tag{9.75}$$

where the matrices E_{kl} are given by $(E_{kl})_{st} = \delta_{ks}\delta_{lt}$. Obviously:

$$\begin{aligned}\left\{A \underset{,}{\otimes} B\right\} &= \sum_{k,l,m,n}\left\{A_{kl}E_{kl} \underset{,}{\otimes} B_{mn}E_{mn}\right\} \\ &= \sum_{k,l,m,n}\{A_{kl}, B_{mn}\}\,E_{kl} \otimes E_{mn} = -\sum_{k,l,m,n}\{B_{mn}, A_{kl}\}\,E_{kl} \otimes E_{mn} \\ &= -\sum_{k,l,m,n}\{B_{mn}, A_{kl}\}\,P\,(E_{mn} \otimes E_{kl})\,P = -P\left\{B \underset{,}{\otimes} A\right\}P\,.\end{aligned} \tag{9.76}$$

Here we used the property (9.10) of P to interchange the positions of the factors in the tensor product and the antisymmetry of the Poisson brackets. Quite analogous, but more lengthy is the proof of (9.73). Here, we note that $P_{kl}(\dots)P_{kl}$ interchanges the positions of the factors, taking the k-th and the l-th positions in the tensor cube; for example:

$$P_{13}\,(X_1 \otimes X_2 \otimes X_3)\,P_{13} = X_3 \otimes X_2 \otimes X_1 \ . \tag{9.77}$$

Let us consider the Poisson brackets, given by:

$$\left\{U(x,\lambda) \underset{,}{\otimes} U(y,\mu)\right\} = i\,[r(\lambda-\mu), U(x,\lambda)\otimes \mathbb{1} + \mathbb{1}\otimes U(x,\mu)]\,\delta(x-y) \ , \tag{9.78}$$

which allow an r-matrix of the form:

$$r(\lambda,\mu) = \sum_{\alpha,\beta=0}^{3} r_{\alpha\beta}(\lambda,\mu)\sigma_\alpha \otimes \sigma_\beta \ , \tag{9.79}$$

and let us try to understand what restrictions on $r(\lambda,\mu)$ are imposed by properties (9.71), (9.72) and (9.73).

The fact that the right-hand side of (9.78) is proportional to $\delta(x-y)$ is known as the ultralocality of the initial Poisson brackets. In this Chapter, we shall deal only with ultralocal cases, so it will be enough to analyze only the coefficients in front of the δ-functions – they must be the same on both sides of the equations.

The property (9.71), where $A = U(x,\lambda)$ and $B = U(x,\mu)$, leads immediately to:

$$r(\lambda,\mu) = -Pr(\mu,\lambda)P \ . \tag{9.80}$$

We also make use of the fact that from (9.10) there follows:

$$P(U(x,\mu)\otimes \mathbb{1} + \mathbb{1}\otimes U(x,\lambda))P = U(x,\lambda)\otimes \mathbb{1} + \mathbb{1}\otimes U(x,\mu) \ . \tag{9.81}$$

The next property (9.72) does not impose restrictions on $r(\lambda,\mu)$, the reason being the fact that the Lie algebraic bracket $[X,Y]$ satisfies:

$$[X,YZ] = [X,Y]Z + Y[X,Z] \ . \tag{9.82}$$

For greater convenience in analyzing the consequences of the Jacobi identity for $r(\lambda,\mu)$, we shall denote by $J(A,B,C)$ the left-hand side of (9.73) and will introduce also the notations:

$$\begin{aligned} &A = U_1(x,\lambda) \ , \qquad B = U_2(x,\mu) \ , \qquad C = U_3(x,\nu) \ , \\ &U_1 = U(x,\lambda)\otimes \mathbb{1}\otimes \mathbb{1} = A\otimes \mathbb{1}\otimes \mathbb{1} \ , \\ &U_2 = \mathbb{1}\otimes U(x,\mu)\otimes \mathbb{1} = \mathbb{1}\otimes B\otimes \mathbb{1} \ , \\ &U_3 = \mathbb{1}\otimes \mathbb{1}\otimes U(x,\nu) = \mathbb{1}\otimes \mathbb{1}\otimes C, \ . \end{aligned} \tag{9.83}$$

In view of (9.78), each of the three terms in the left-hand side of (9.73) is proportional to $\delta(x-y)\delta(y-z)$. We shall omit this factor remembering that because of it we may assume $x = y = z$. The coefficient in front of the first term in the left-hand side of (9.73) with $A = U_1(x,\lambda)$, $B = U_2(y,\mu)$, $C = U_3(z,\nu)$ will be denoted by $\{U_1, \{U_2, U_3\}\}$ and is equal to:

$$\begin{aligned}
&\left\{U_1 \underset{,}{\otimes} \left\{U_2 \underset{,}{\otimes} U_3\right\}\right\} = i\left\{U_1 \underset{,}{\otimes} [r_{23}(\mu,\nu), U_2 + U_3]\right\} \\
&i\left[r_{23}(\mu,\nu), \left\{U_1 \underset{,}{\otimes} U_2\right\} + \left\{U_1 \underset{,}{\otimes} U_3\right\}\right] \\
&= -\left[r_{23}(\mu,\nu), [r_{12}(\lambda,\mu), U_1 + U_2] + [r_{13}(\lambda,\nu), U_1 + U_3]\right] \qquad (9.84) \\
&= [r_{23}(\mu,\nu), [r_{12}(\lambda,\mu) + r_{13}(\lambda,\nu), U_1] + [r_{12}(\lambda,\mu), U_2] + [r_{13}(\lambda,\nu), U_3]] \ .
\end{aligned}$$

From (9.74) and (9.80) we find that for $k \neq l$ and $l \neq m$

$$P_{kl} r_{lm}(\lambda,\mu) P_{kl} = r_{ml}(\mu,\lambda) \ , \qquad P_{kl} r_{lm}(\lambda,\mu) P_{kl} = r_{km}(\mu,\lambda) \ , \qquad (9.85)$$

where $r_{lm}(\lambda,\mu)$ is defined in analogy with P_{lm} (9.74). With this definition for $l \neq m$ we shall have $r_{lm}(\lambda,\mu) = -r_{ml}(\mu,\lambda)$. Thus for $l = 2$ and $m = 1$, we have:

$$\begin{aligned}
r_{12}(\lambda,\mu) &= \sum_{\alpha,\beta=0}^{3} r_{\alpha\beta}(\lambda,\mu)\sigma_\alpha \otimes \sigma_\beta \otimes \mathbb{1} \ , \\
r_{21}(\lambda,\mu) &= \sum_{\alpha,\beta=0}^{3} r_{\alpha\beta}(\mu,\lambda)\sigma_\beta \otimes \sigma_\alpha \otimes \mathbb{1} \ . \qquad (9.86)
\end{aligned}$$

and $r_{12}(\lambda,\mu) = -r_{21}(\mu,\lambda)$ is a consequence of (9.80). Relations (9.85) allow us to calculate easily the second and the third summands in (9.73). The second summand equals to:

$$[r_{12}, [r_{13}, U_1 + U_3] + [r_{23}, U_2 + U_3]] \ , \qquad (9.87)$$

and the third – to:

$$[r_{13}, [r_{23}, U_2 + U_3] - [r_{12}\ , U_1 + U_2]] \ . \qquad (9.88)$$

where we have omitted the arguments λ, μ, ν. They can be easily restored, if we keep in mind that λ is associated with the index 1, μ – with the index 2, ν – with the index 3. By summing (9.84), (9.87), and (9.88), we obtain:

$$\begin{aligned}
J(A,B,C) = [r_{23}, &[r_{12} + r_{13}, U_1] + [r_{12}, U_2] + [r_{13}, U_3]] \\
&+ [r_{12}, [r_{13}, U_1 + U_3] + [r_{23}, U_2 + U_3]] \\
&+ [r_{13}, [r_{23}, U_2 + U_3] - [r_{12}, U_1 + U_2]] \qquad (9.89)
\end{aligned}$$

Let us first consider those terms in the right-hand side of (9.89), which contain U_1 and let us make use of the Jacobi identity and the commutativity between r_{23} and U_1. After analogous transformation applied to the other terms in the right-hand side of (9.89) we find that (9.73) is equivalent to:

$$[U_1 + U_2 + U_3, [r_{12}, r_{13} + r_{23}] + [r_{13}, r_{23}]] = 0 \tag{9.90}$$

Obviously, if $r(\lambda, \mu)$ satisfies the equation:

$$[r_{12}, r_{13} + r_{23}] + [r_{13}, r_{23}] = 0 \tag{9.91}$$

then Poisson brackets (9.78) satisfy the Jacobi identity. The equation (9.91) is known in the literature as the classical Yang–Baxter equation.

By using the explicit form of P (9.9) and P_{kl} (9.74), it is not difficult to check that $r(\lambda, \mu)$ defined by (9.8) satisfies all the requirements for an r-matrix. The same is true also for $\tilde{r}(\lambda, \mu)$ (9.49).

Of course, these examples do not exhaust all known classical r-matrices. An effective method for constructing new solutions of the classical Yang–Baxter equation is based on a sort of 'averageing' procedure which we shall describe below.

Let us introduce the lattice $(n\omega, n \in \mathbb{Z})$, where ω is some constant (see [2], Part II, Chap. 4) and the cyclic group $\mathbb{Z}_2$, acting on the basis of $sl(2)$ by:

$$\mathcal{A}(\sigma_k) = \sigma_3 \sigma_k \sigma_3, \qquad \mathcal{A}^2(\sigma_k) = \sigma_k \ , \tag{9.92}$$

Let us now define $r^a(\lambda, \mu)$ by:

$$r^a(\lambda, \mu) = -\sum_{n=-\infty}^{\infty} (\mathcal{A}^n \otimes \mathbb{1})\, r(\lambda - \mu - n\omega) \ . \tag{9.93}$$

Inserting formally (9.93) into (9.91) shows that $r^a(\lambda, \mu)$ is also a solution of the classical Yang–Baxter equation. However, we must keep in mind that in proving this we have to change the order of summations, which is rigorously justified only if the series in the right-hand side of (9.93) is absolutely convergent.

Let us try to check whether this is so for $r^a(\lambda, \mu)$ (9.8):

$$\begin{aligned} r^a(\lambda - \mu) &= -\sum_{n=-\infty}^{\infty} \frac{(\mathcal{A}^n \otimes \mathbb{1})P}{\lambda - \mu - n\omega} \\ &= -\frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{\mathbb{1} \otimes \mathbb{1} + \sigma_3 \otimes \sigma_3 + (-1)^n (\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2)}{\lambda - \mu - n\omega} \ . \end{aligned} \tag{9.94}$$

The series in the right-hand side of (9.94) are not absolutely convergent, but have well-defined principle value:

$$r^a(\lambda-\mu) = \frac{\cos(\lambda-\mu)\,(\sigma_3\otimes\sigma_3+\sigma_0\otimes\sigma_0)+\sigma_1\otimes\sigma_1+\sigma_2\otimes\sigma_2}{2\sin(\lambda-\mu)} \ . \qquad (9.95)$$

A rigorous treatment of the problem of convergence for the series shows that (9.94) will satisfy the classical Yang– Baxter equation, only if the automorphism $\mathcal{A}$ satisfies an additional constraint, namely, the elements of $sl(2)$, which are stable with respect to $\mathcal{A}$ have to form an abelian subalgebra. In our case, this constraint is satisfied and consequently $r^a(\lambda,\mu)$ (9.95) satisfies (9.91). We can check this also by a direct calculation.

The solution (9.95) of (9.91) is an example of a *trigonometric* classical r-matrix.

The above procedure can be performed to a lattice of form ($n\omega_1+m\omega_2$, $n,m\in\mathbb{Z}$; $Im(\omega_2/\omega_1)>0$). We need also two automorphisms $\mathcal{A}_1$, $\mathcal{A}_2$ acting by:

$$\mathcal{A}_1(\sigma_k)=\sigma_3\sigma_k\sigma_3, \qquad \mathcal{A}_2(\sigma_k)=\sigma_1\sigma_k\sigma_1 \ , \qquad (9.96)$$

Then the averaging is done as follows:

$$r^a(\lambda-\mu) = -\sum_{n,m=-\infty}^{\infty} (\mathcal{A}_1^n\mathcal{A}_2^m\otimes \mathbb{1})\,r(\lambda-\mu-n\omega_1-m\omega_2) \ . \qquad (9.97)$$

If we insert here $r(\lambda-\mu)$ (9.8), we again encounter the problem about the absolute convergence of the series in the right-hand side of (9.97). This imposes restrictions on the automorphisms $\mathcal{A}_1$ and $\mathcal{A}_2$, namely, $\mathcal{A}_1$ and $\mathcal{A}_2$ must not have common fixed points. In our case, this is fulfilled and the averaging (9.97) leads to a new solution of the classical Yang–Baxter equation, known as the *elliptic* r-matrix

$$r^a(\eta) = \frac{(\mathrm{cn}\,(\eta,k)\,(\sigma_3\otimes\sigma_3+\mathbb{1}\otimes\mathbb{1})+\sigma_1\otimes\sigma_1+\mathrm{dn}\,(\eta,k)\sigma_2\otimes\sigma_2)}{2\mathrm{sn}\,(\eta,k)} \ , \qquad (9.98)$$

where $\mathrm{sn}\,(\eta,k)$, $\mathrm{dn}\,(\eta,k)$, $\mathrm{cn}\,(\eta,k)$ are elliptic functions of $\eta=\lambda-\mu$ with module k, which is related in the standard way to their periods ω_1 and ω_2.

9.4 The Classical r-Matrix and the Lax Representation

In this section, we shall analyze the interrelation between the classical r-matrix approach and the Lax approach. Here by a classical r-matrix, we shall mean a solution of the classical Yang–Baxter equation:

$$[r_{12}(\lambda-\mu), r_{13}(\lambda-\nu)+r_{23}(\mu-\nu)] + [r_{13}(\lambda-\nu), r_{23}(\mu-\nu)] = 0 \qquad (9.99)$$

Our main aim will be to show that with each solution of (9.99) we can relate at least one class of NLEE, which allow Lax representation and whose Hamiltonian structure is defined by $r(\lambda-\mu)$. In the beginning, we shall not use the specifics of the algebra $sl(2)$ but will consider the general case, related to an arbitrary semisimple Lie algebra $\mathfrak{g}$. We shall introduce the basis and the structure constants of $\mathfrak{g}$ in the standard way by:

$$[X_a, X_b] = C_{ab}^c X_c \, , \tag{9.100}$$

Let us assume that

$$r(\lambda - \mu) = \sum_{a,b} r_{ab}(\lambda - \mu) X_a \otimes X_b \, , \tag{9.101}$$

is a solution of (9.99) and let us introduce the Lax operator:

$$L\psi \equiv i\frac{d\psi}{dx} + U(x, \lambda)\psi(x, \lambda) = 0 \, , \tag{9.102}$$

where

$$U(x, \lambda) = \sum_{a,b} r_{ab}(\lambda) X_a u_b(x) \, . \tag{9.103}$$

To L we can relate a class of exactly solvable NLEE for the variables $u_b(x)$. The class of allowed coefficients $u_b(x)$ will define the phase space $\mathbf{\Phi}$ of these NLEEs, and the Hamiltonian structure on $\mathcal{M}$ we shall introduce in a natural algebraic way (compare with (9.100))

$$\{u_a(x), u_b(y)\} = -iC_{abc} u_c(x)\delta(x - y) \, . \tag{9.104}$$

We shall show, that the Lax operator (9.102) satisfies the intertwining condition (9.7) with the classical r-matrix (9.101). We shall see that this condition is an immediate consequence of (9.99). Indeed, each of the summands in (9.99) takes values in the tensor cube $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$. Besides, the structure of (9.99) is such that in order to calculate the commutators in the left-hand side we need to know only the commutation relations (9.100). In other words, the classical Yang–Baxter equation does not depend on the choice of the representation for the algebra g.

Let us introduce the mapping:

$$\mathfrak{g} \to \mathcal{M} \, , \tag{9.105a}$$

under which

$$X_a \to -u_a(x), \qquad [,] \to \frac{1}{i}\{\, , \,\} \, . \tag{9.105b}$$

Then, we consider (9.99) and apply the mapping (9.105a) to the third factors in the tensor product:

$$\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g} \otimes \mathcal{M} \, . \tag{9.106}$$

We shall also omit the second sign for tensor product in writing down the elements of $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathcal{M}$. This mapping does not influence $r_{12}(\lambda - \mu)$ but applied to $r_{13}(\lambda - \nu)$ and $r_{23}(\mu - \nu)$ gives:

$$r_{13}(\lambda-\nu) = \sum_{a,b} r_{ab}(\lambda-\nu) X_a \otimes \mathbb{1} \otimes X_b$$

$$\to \sum_{a,b} r_{ab}(\lambda-\nu) u_b(x) X_a \otimes \mathbb{1} = U(x,\lambda-\nu) \otimes \mathbb{1} \tag{9.107}$$

$$r_{23}(\mu-\nu) = \sum_{a,b} r_{ab}(\mu-\nu) \mathbb{1} \otimes X_a \otimes X_b$$

$$\to \sum_{a,b} r_{ab}(\mu-\nu) u_b(x) \mathbb{1} \otimes X_a = \mathbb{1} \otimes U(x,\mu-\nu) \tag{9.108}$$

Multiplying the commutator $[r_{13}(\lambda-\nu), r_{23}(\mu-\mu)]$ by $\delta(x-y)$, changing $\lambda-\nu$ and $\mu-\nu$ to λ and μ, respectively, and making use of (9.104) we see, that:

$$\delta(x-y)\,[r_{13}(\lambda), r_{23}(\mu)] = \sum_{a,b,c,d} r_{ab}(\lambda) r_{cd}(\mu) X_a \otimes X_c \otimes [X_b, X_a]$$

$$\to \sum_{a,b} \sum_{c,d} r_{ab}(\lambda) r_{cd}(\mu) X_a \otimes X_c \{u_b(x), u_d(y)\} \ , \tag{9.109}$$

i.e.

$$\delta(x-y)\,[r_{13}(\lambda), r_{23}(\mu)] \to i \left\{ U(x,\lambda) \underset{,}{\otimes} U(y,\mu) \right\} \ . \tag{9.110}$$

Thus applying mapping (9.105a) to (9.99) we obtain:

$$[r_{12}(\lambda-\mu), U(x,\lambda) \otimes \mathbb{1} + \mathbb{1} \otimes U(x,\mu)]\,\delta(x-y) + i \left\{ U(x,\lambda) \underset{,}{\otimes} U(y,\mu) \right\} = 0 \ . \tag{9.111}$$

This completes our first task – we showed that to each solution $r(\lambda-\mu)$ of (9.99) one can associate at least one Lax operator L. As examples for this construction, we may consider the r-matrices (9.8), (9.95), and (9.98). Applied to these r-matrices the construction leads to the following auxiliary linear problems of the type of (9.102)[2]

$$U(x,\lambda) = -\frac{1}{2\lambda} \sum_{\alpha=1}^{3} \sigma_\alpha S_\alpha(x) \ , \tag{9.112}$$

$$U(x,\lambda) = \frac{1}{2\sin\lambda} \left(\sigma_3 S_3(x) \cos\lambda + \sigma_1 S_1(x) + \sigma_2 S_2(x) \right) \ , \tag{9.113}$$

$$U(x,\lambda) = \frac{1}{2\mathrm{sn}\,(\lambda,k)} \left(\sigma_1 S_1(x) + \sigma_2 S_2(x) \mathrm{dn}\,(\lambda,k) \right.$$
$$\left. + \sigma_3 S_3(x) \mathrm{cn}\,(\lambda,k) \right) \ . \tag{9.114}$$

[2] Here we omit insignificant terms proportional to $\mathbb{1} \otimes \mathbb{1}$.

The system (9.102), (9.112), as we have seen, allows us to apply the inverse scattering method to the Heisenberg ferromagnet equation (8.1) of Chap. 2, while (9.102), (9.113) allow us to solve the anisotropic Heisenberg ferromagnet equation.

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx} + \mathbf{S} \times J\mathbf{S}(x) \; ; \qquad \mathbf{S}(x) = (S_1, S_2, S_3) \ , \tag{9.115}$$

where $J = \operatorname{diag}\,(J_1, J_2, J_3)$. The last system, (9.102), (9.113), allows us to solve the Landau–Lifshitz equation, which differs from (9.115) only in that the diagonal matrix J is chosen to be generic with three different eigenvalues: $J = \operatorname{diag}\,(J_1, J_1, J_3)$, $J_1 < J_2 < J_3$. In all these three cases, the components of the vector $\mathbf{S}(x)$ satisfy the canonical Poisson brackets (9.46).

The fact that each of the above-mentioned three equations possess Lax representation is well known (see e.g. [2], Part II). Indeed, even a stronger proposition holds, which makes evident the interrelation between the $\mathcal{M}$-operators in the Lax representation and the classical r-matrix. More accurately, using the classical r-matrix we shall construct the generating functional for the $\mathcal{M}$-operator-operators of the NLEE. We shall show that this is true for the case of the Zakharov– Shabat system with periodic boundary conditions. The generalization to the systems of the type of (9.102), (9.103) requires technically more complicated construction. Simultaneously, we shall show that the Lax representation of a given NLEE can be written down in an explicitly Hamiltonian form. Namely, we shall show that the Hamiltonian equations of motion:

$$\frac{dq^+(x)}{dt} = -\left\{p_\ell(\mu), q^+(x,t)\right\} , \qquad \frac{dq^-(x)}{dt} = -\left\{p_\ell(\mu), q^-(x,t)\right\} \ , \tag{9.116}$$

where

$$p_\ell(\mu) = \arccos\left(\frac{1}{2} F_\ell(\mu)\right) \ , \tag{9.117}$$

are equivalent to the compatibility condition (8.11). To do this, we shall calculate the Poisson brackets:

$$\left\{T_\ell(x,y,\mu) \underset{,}{\otimes} U(z,\lambda)\right\} , \qquad -\ell < y < z < x < \ell \, , \tag{9.118}$$

by using (9.22) and the obvious property:

$$\frac{\delta U_{ab}(z,\lambda)}{\delta U_{kj}(z',\lambda)} = \delta_{ak}\delta_{bj}\delta(z-z') \ . \tag{9.119}$$

This leads to the following result for (9.118):

$$\begin{aligned}
&\frac{1}{i}\left\{T_\ell(x,y,\mu) \underset{,}{\otimes} U(z,\lambda)\right\} \\
&= (T_\ell(x,y,\mu)\otimes \mathbb{1})[r(\lambda-\mu), U(z,\lambda)\otimes \mathbb{1} + \mathbb{1}\otimes U(z,\lambda)](T_\ell(x,y,\mu)\otimes \mathbb{1})
\end{aligned}$$

$$= [\mathcal{M}(z,x,y,\lambda,\mu), \mathbb{1}\otimes U(z,\lambda)] \\ + (T_\ell(x,y,\mu)\otimes\mathbb{1})[r(\lambda-\mu), U(z,\mu)\otimes\mathbb{1}](T_\ell(z,y,\mu)\otimes\mathbb{1}) \quad (9.120)$$

where

$$M(z,x,y,\lambda,\mu) = (T_\ell(x,z,\mu)\otimes\mathbb{1})r(\lambda-\mu)(\mathbb{1}\otimes T_\ell(z,y,\mu)) \ . \quad (9.121)$$

In order to calculate the second term in the last line of (9.120), we use again (9.24) and (9.26)

$$\begin{aligned} & i(T_\ell(x,z,\mu)\otimes\mathbb{1})[r(\lambda-\mu), U(z,\mu)\otimes\mathbb{1}](T_\ell(z,y,\mu)\otimes\mathbb{1}) \\ &= i(T_\ell(x,z,\mu)\otimes\mathbb{1})r(\lambda-\mu)(U(z,\mu)T_\ell(z,y,\mu)\otimes\mathbb{1}) \\ &\quad - i(T_\ell(x,z,\mu)U(z,\mu)\otimes\mathbb{1})r(\lambda-\mu)(T_\ell(z,y,\mu)\otimes\mathbb{1}) \\ &= -\frac{d}{dz}\left\{(T_\ell(x,z,\mu)\otimes\mathbb{1})r(\lambda-\mu)(T_\ell(z,y,\mu)\otimes\mathbb{1})\right\} \\ &= -\frac{d}{dz}M(z,x,y,\lambda,\mu) \ . \end{aligned} \quad (9.122)$$

Now we can rewrite (9.118) in the form:

$$\left\{T_\ell(x,y,\mu) \underset{,}{\otimes} U(z,\lambda)\right\} = i[M(z,x,y,\lambda,\mu), \mathbb{1}\otimes U(z,\lambda)] + \frac{dM}{dz} \ . \quad (9.123)$$

Let us now assume that $x = \ell$, $y = -\ell$, and let us take tr_1 on both sides of (9.123). We immediately get:

$$\{F_\ell(\mu), U(z,\lambda)\} = i[\tilde{V}(z,\lambda,\mu), U(z,\lambda)] + \frac{d\tilde{V}(z,\lambda,\mu)}{dz} \ , \quad (9.124)$$

where

$$\tilde{V}(z,\lambda,\mu) = \mathrm{tr}_1 M(z,\ell,-\ell,\lambda,\mu) = \mathrm{tr}_1(T(x,z,\mu)\otimes\mathbb{1})r(\lambda-\mu)(T(z,y,\mu)\otimes\mathbb{1}) \ . \quad (9.125)$$

It is obvious from (9.124) that the Hamiltonian equation of motion:

$$\frac{\partial U}{\partial t} = -\{F_\ell(\mu), U(x,\lambda)\} \ , \qquad F_\ell(\mu) = \frac{1}{2}\mathrm{tr}\, T_\ell(\mu) \ , \quad (9.126)$$

which is nothing else but system (9.116) written down in a matrix form, acquiring the form:

$$i\frac{\partial U}{\partial t} - i\frac{d\tilde{V}(x,\lambda,\mu)}{\partial x} + [U(x,\lambda), \tilde{V}(x,\lambda,\mu)] = 0 \ , \quad (9.127)$$

This equation resembles compatibility condition (8.11) very much. Indeed, if we expand $\tilde{V}(x, \lambda, \mu)$ over the negative powers of μ:

$$\tilde{V}(x, \lambda, \mu) = \sum_{k=1}^{\infty} \mu^{-k} \tilde{V}_k(x, \lambda) , \tag{9.128}$$

we shall get a family of Lax equations:

$$i \frac{\partial U}{\partial t} - i \frac{d\tilde{V}_k(x, \lambda, \mu)}{\partial x} - [U(x, \lambda), \tilde{V}_k(x, \lambda, \mu)] = 0 . \tag{9.129}$$

These equations are Hamiltonian with respect to the Poisson brackets (7.23) and have, as Hamiltonians, the expansions coefficients $F_{\ell,k}$ of

$$F_\ell(\mu) = \sum_{k=0}^{\infty} F_{\ell,k} \mu^{-k} . \tag{9.130}$$

Let us now rewrite $\tilde{V}(x, \lambda, \mu)$ in a more compact form. To this end, we shall use the explicit form of $r(\lambda - \mu)$ (9.8). We get:

$$\begin{aligned} \tilde{V}(x, \lambda, \mu) &= -\frac{1}{\lambda - \mu} \mathrm{tr}_1 (T(\ell, x, \mu) \otimes \mathbb{1}) P (T(x, -\ell, \mu) \otimes \mathbb{1}) \\ &= -\frac{1}{\lambda - \mu} \mathrm{tr}_1 P (T(x, -\ell, \mu) P T(\ell, x, \mu) \otimes \mathbb{1}) \\ &= -\frac{1}{\lambda - \mu} \mathrm{tr}_1 (\mathbb{1} \otimes T(x, -\ell, \mu) T(\ell, x, \mu)) P \\ &= -\frac{1}{\lambda - \mu} T(x, -\ell, \mu) T(\ell, x, \mu)) \mathrm{tr}_1 P \end{aligned} \tag{9.131}$$

But from (9.9) we easily get that:

$$\mathrm{tr}_1 P = \frac{1}{2} \mathrm{tr}_1 \left(\mathbb{1} \otimes \mathbb{1} + \sum_{\alpha=1}^{3} \sigma_\alpha \otimes \sigma_\alpha \right) = \mathbb{1} . \tag{9.132}$$

i.e.

$$\tilde{V}(x, \lambda, \mu) = -\frac{1}{\lambda - \mu} T(x, -\ell, \mu) T(\ell, x, \mu) \tag{9.133}$$

As one should expect, $\tilde{V}(x, \lambda, \mu)$ is a periodic function of x.

The construction proposed above has the following drawback: It defines the Lax representations for a class of nonlocal NLEEs, since the coefficients $F_{\ell,k}$ in (9.130) are nonlocal functionals of $q(x)$. In order to obtain the Lax representations for the local NLEEs, we shall again use the representation for $T_\ell(x, y, \mu)$ (6.94), (6.95) and (6.96). Inserting it in the right-hand side of (9.133) we obtain:

$$\begin{aligned}
&\tilde{V}(x,\lambda,\mu)\\
&= -\frac{1}{\lambda-\mu}(\mathbb{1}\otimes W(x,\mu))\exp(iZ(x,-\ell,\mu)+iZ(\ell,x,\mu))\,(\mathbb{1}\otimes W(x,\mu))^{-1}\\
&= -\frac{1}{\lambda-\mu}(\mathbb{1}\otimes W(x,\mu))\exp(iZ(\ell,-\ell,\mu))\,(\mathbb{1}\otimes W(x,\mu))^{-1}\ ,
\end{aligned}\tag{9.134}$$

where we again made use of the fact that $W(x,\lambda)$ is periodic and also the expression (6.96) for $Z(x,y,\mu)$. But, as we have already shown, tr $Z(\ell,-\ell,\mu)$ vanishes and consequently:

$$\exp(iZ(\ell,-\ell,\mu)) = \exp(i\sigma_3 p_\ell(\mu)) = \cos(p_\ell(\mu))\mathbb{1} + i\sin(p_\ell(\mu))\sigma_3\ . \tag{9.135}$$

As a result, we see that $\tilde{V}(x,\lambda,\mu)$ can be written down in the form:

$$\begin{aligned}
\tilde{V}(x,\lambda,\mu) = -\frac{1}{\lambda-\mu}&\{\cos(p_\ell(\mu))\mathbb{1}\\
&+ i\sin(p_\ell(\mu))(\mathbb{1}+W(x,\mu))\sigma_3(\mathbb{1}+W(x,\mu))^{-1}\}\ .
\end{aligned}\tag{9.136}$$

The first term in (9.136) is x-independent and is proportional to the unit matrix, therefore it does not contribute to the compatibility condition (9.127). Now we can introduce:

$$V(x,\lambda,\mu) = -\frac{1}{2(\lambda-\mu)}(\mathbb{1}+W(x,\mu))\sigma_3(\mathbb{1}+W(x,\mu))^{-1} \simeq \frac{-i\tilde{V}(x,\lambda,\mu)}{2\sin p_\ell(\mu)}\ , \tag{9.137}$$

where $\simeq$ reminds us that we have neglected some terms, which are not important for the Lax representation. The analog of (9.126) is now:

$$-\frac{i}{2\sin p_\ell(\mu)}\{F_\ell(\mu),U(x,\lambda)\} = [V(x,\lambda,\mu),U(x,\lambda)] + i\frac{\partial V}{\partial x}\ , \tag{9.138}$$

where:

$$p_\ell(\mu) = \arccos\frac{1}{2}F_\ell(\mu) \qquad \text{or} \qquad F_\ell(\mu) = 2\cos p_\ell(\mu)\ . \tag{9.139}$$

It remains to replace this expression in the left-hand side of (9.138) and to make use of the properties of the Poisson brackets; this leads to:

$$\{p_\ell(\mu),U(x,\lambda)\} = [V(x,\lambda,\mu),U(x,\lambda)] + i\frac{\partial V(x,\lambda,\mu)}{\partial x}\ . \tag{9.140}$$

Since $p_\ell(\mu)$ is the generating functional for the local integrals of motion, then $V(x,\lambda,\mu)$ will be the generating functional for the Lax representations of the

NLEEs of NLS type (6.7). Let us consider the expansion over the negative powers of μ of the Hamiltonian equation:

$$-\, i\, \{p_\ell(\mu), U(x,\lambda)\} \equiv iU_t(x,\lambda) = [U(x,\lambda), V(x,\lambda,\mu)] + i\frac{\partial V}{\partial x}\,, \tag{9.141}$$

and let us consider the coefficient of μ^{-1}. If

$$V(x,\lambda,\mu) = \sum_{k=1}^{\infty} \mu^{-k} V_k(x,\lambda)\,, \tag{9.142}$$

then

$$i\,\{I_k, U(x,\lambda)\} \equiv iU_t(x,\lambda) = [U(x,\lambda), V_k(x,\lambda)] + i\frac{\partial V_k}{\partial x}\,, \tag{9.143}$$

which is the Lax representation for the NLEE with the Hamiltonian $H = I_k$.

9.5 The Classical r-Matrix and the Involutivity of the Integrals of Motion in the Case of Arbitrary Semisimple Lie Algebra

Here, we shall shortly discuss the generalizations of the r-matrix approach to systems of Zakharov–Shabat type, related to the semisimple Lie algebras. This section requires some knowledge of the structural theory of the Lie algebras and their representations. The reader who does not have that knowledge can safely omit it.

Let us consider the system:

$$L\psi \equiv \left(i\frac{d}{dx} + q(x) - \lambda J \right) \psi(x,\lambda) = 0\,, \tag{9.144}$$

and its gauge equivalent:

$$\tilde{L}\psi \equiv \left(i\frac{d}{dx} - \lambda S(x) \right) \tilde{\psi}(x,\lambda) = 0\,, \tag{9.145}$$

where the functions $q(x)$, $S(x)$ take values in the semisimple Lie algebra $\mathfrak{g}$ with rank r; J is a constant real element of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, $\dim_{\mathbb{C}} \mathfrak{h} = r$. The relation between (9.144) and (9.145) is given by:

$$\begin{aligned} &\tilde{\psi}(x,\lambda) = g^{-1}(x)\psi(x,\lambda), \qquad S(x) = g^{-1}(x) J g(x), \\ &ig_x + q(x)g(x) = 0\,. \end{aligned} \tag{9.146}$$

Below, we shall use the well-known notations for the Cartan–Weyl basis (see [3, 4]) E_α, H_j, $\alpha \in \Delta$, $j = 1, \dots, r$, where Δ is the set of roots of $\mathfrak{g}$.

We introduce an ordering in Δ with the help of a convenient regular element $J_{\rm reg} \in \mathfrak{h}$, namely, we shall say that the root α is positive (negative) if $\alpha(J_{\rm reg}) > 0$ ($\alpha(J_{\rm reg}) < 0$). Of course, for this, we must require that all $\alpha(J_{\rm reg})$ are real and nonvanishing. The simple roots will be denoted by α_1, …, α_r, and the fundamental weights – by ω_1, …, ω_r. By definition:

$$\frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \omega_j)} = \delta_{ij}, \qquad 1 \le i, \quad j \le r \,, \tag{9.147}$$

and E_α, $H_j \in \mathfrak{h}$ satisfy the following commutation relations:

$$[H_j, E_\alpha] = \alpha(H_j) E_\alpha, \qquad [E_\alpha, E_{-\alpha}] = H_\alpha, \qquad [E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta} \,, \tag{9.148}$$

and $N_{\alpha,\beta} = 0$ if $\alpha + \beta$ is not a root. We also assume that element J is close in some sense to $J_{\rm reg}$, i.e. if $\alpha > 0$ ($\alpha < 0$), then also $\alpha(J) > 0$ ($\alpha(J) < 0$). In g, we can introduce metrics with the help of the Killing form $\langle\, ,\, \rangle$; moreover $\alpha(J) = (a, \alpha)$ and:

$$\langle H_i, H_j \rangle = (\alpha_i, \alpha_j), \qquad \langle E_\alpha, E_{-\beta} \rangle = \delta_{\alpha\beta}, \qquad N_{\alpha\beta} = -N_{-\alpha,-\beta} \,. \tag{9.149}$$

To each of the systems, (9.144), (9.145), one can relate NLEE, generalizing the NLSE and the HFE correspondingly. The natural way to fix gauge (9.144) requires that:

$$q(x) = \sum_{\alpha \in \Delta, (a,\alpha) \neq 0} q_\alpha(x) E_\alpha, \qquad q_\alpha(x) = \langle q(x), E_{-\alpha} \rangle \,, \tag{9.150}$$

where a is the vector in the root space $\mathbb{E}^r$, corresponding to the element $J \in \mathfrak{h}$. A very representative in this respect is the case when J has only two different eigenvalues: 1 and -1, with multiplicities n_1 and n_2.[3] Now we can introduce an additional symmetry using an involutive automorphism of g. In other words, the construction of L can be associated with any of the symmetric spaces, related to g. They provide the most representative generalizations of the NLSE [5]:

$$i[J, q_t] + q_{xx} - \frac{1}{2} q^3(x) = 0, \qquad J = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad q = \begin{pmatrix} 0 & q^+ \\ q^- & 0 \end{pmatrix} \,, \tag{9.151}$$

where q^+ and q^- are $n_1 \times n_2$ and $n_2 \times n_1$ matrices and also of the generalizations of the HFE:

$$S_t + [S, S_{xx}] = 0 \,, \qquad S^2(x) = \mathbb{1} \,. \tag{9.152}$$

[3] If $n_1 \neq n_2$, then $\operatorname{tr} J \neq 0$; we can easily transfer to the case with $\operatorname{tr} J = 0$ by a gauge transformation. For our purposes, this form of J is more convenient, since $J^2 = \mathbb{1}$.

In order to solve these NLEEs, we also can apply the inverse scattering method. The classical approach to the solution of the inverse scattering problem, based on the Gelfand–Levitan–Marchenko equation, is not fully adequate. Its natural generalization, proposed by Zakharov, Manakov, and Shabat [6, 7] also provides an effective method for the construction of the soliton solutions in the general case (arbitrary real J). It is not possible to present this approach here. We only note that NLEE (9.151) and (9.152) are equivalent to the following linear equations for the scattering matrices related to (9.144) and (9.145), respectively:

$$i\frac{dT}{dt} + 2\lambda^2[J, T(t,\lambda)] = 0 \, ,$$

$$i\frac{d\tilde{T}}{dt} + 2\lambda^2[J, \tilde{T}(\lambda,t)] = 0 \, , \qquad \tilde{T}(\lambda,t) = T(\lambda,t)\widehat{T}(0,t) \, . \tag{9.153}$$

The interrelation between both scattering matrices, given in (9.153), generalizes (8.68) in Chap. 8. The conditions $\lim_{x\to\pm\infty} S(x,t) = J$, just like before, requires that $[T(0,t), J] = 0$. But, assuming that $\lambda = 0$ in (9.153) we see, that $T(0,t)$ is an integral of motion for the NLSE.

From the explicit form of L (9.144) and (9.151), there follows that $T(\lambda,t)$ has a block-matrix structure, i.e. we may assume that $T(\lambda,t)$ is given by the right-hand side of (9.17), where $a^{\pm}(\lambda)$, $b^{\pm}(\lambda)$ are matrices of the corresponding size. From (9.153), we find that each of the matrix elements of $a^{+}(\lambda)$ and $a^{-}(\lambda)$ can be considered as a generating functional of the integrals of motion.

The generalizations of the NLSE and the HFE (9.151) and (9.152) possess hierarchies of Hamiltonian structures; the canonical ways to introduce symplectic structures in the corresponding phase spaces $\mathcal{M}_{\mathfrak{g}}$ and $\tilde{\mathcal{M}}_{\mathfrak{g}}$ are based on the following Poisson brackets:

$$\{q_\alpha(x), q_\beta(x)\} = -i\delta_{\alpha,-\beta}\delta(x-y) \, , \tag{9.154}$$

$$\{S_\alpha(x), S_\beta(x)\} = -iN_{\alpha\beta}S_{\alpha+\beta}(x)\delta(x-y) \, , \tag{9.155}$$

$$\{S_\alpha(x), S_{-\alpha}(x)\} = -i\sum_{j=1}^{r}(\alpha,\alpha_j)S_j(x)\delta(x-y) \, , \tag{9.156}$$

(compare with (9.104)), where

$$S_\alpha(x) = \langle S(x), E_{-\alpha}\rangle \, , \qquad S_j(x) = \langle S(x), H_j^\vee\rangle \, , \tag{9.157}$$

and $H_j^\vee$, $j = 1,\dots,r$ is a basis in $\mathfrak{h}$, biorthogonal to H_k, i.e. $\langle H_k, H_j^\vee\rangle = \delta_{jk}$.

These symplectic structures allow the r-matrix approach, i.e. both $U(x,\lambda) = q(x) - \lambda J$ and $\tilde{U}(x,\lambda) = -\lambda S(x)$ satisfy (9.7) and (9.48) with $r(\lambda-\mu)$ and $\tilde{r}(\lambda-\mu)$, respectively, where

$$r(\lambda-\mu) = -\frac{\Pi}{\lambda-\mu}, \qquad \tilde{r}(\lambda-\mu) = \frac{-\lambda\mu\Pi}{\lambda-\mu} \, , \tag{9.158}$$

$$\Pi = \sum_{j=1}^{r} H_j \otimes H_j^{\vee} + \sum_{\alpha\in\Delta} E_\alpha \otimes E_{-\alpha} \, , \tag{9.159}$$

Using (9.148), we see that both $r(\lambda,\mu)$ and $\tilde{r}(\lambda,\mu)$ satisfy the classical Yang–Baxter equation (9.99). As a consequence, we obtain without difficulty that the corresponding monodromy matrices $T_\ell(\lambda)$, $\tilde{T}_\ell(\lambda)$ and scattering matrices $T(\lambda)$, $\tilde{T}(\lambda)$ in the case of zero boundary conditions for $q(x)$ and $S(x)-J$ satisfy:

$$\left\{ T_\ell(\lambda) \underset{,}{\otimes} T_\ell(\mu) \right\} = \left[r(\lambda-\mu), T_\ell(\lambda) \otimes T_l(\mu) \right] , \tag{9.160}$$

$$\left\{ \tilde{T}_\ell(\lambda) \underset{,}{\otimes} \tilde{T}_\ell(\mu) \right\} = \left[\tilde{r}(\lambda-\mu), \tilde{T}_\ell(\lambda) \otimes \tilde{T}_\ell(\mu) \right] , \tag{9.161}$$

and

$$\begin{aligned} &\left\{ T(\lambda) \underset{,}{\otimes} T(\mu) \right\} \\ &= r_+(\lambda-\mu) T(\lambda) \otimes T(\mu) - T(\lambda) \otimes T(\mu) r_-(\lambda-\mu) \, , \end{aligned} \tag{9.162}$$

$$\begin{aligned} &\left\{ \tilde{T}(\lambda) \underset{,}{\otimes} \tilde{T}(\mu) \right\} \\ &= \tilde{r}_+(\lambda-\mu) \tilde{T}(\lambda) \otimes \tilde{T}(\mu) - \tilde{T}(\lambda) \otimes \tilde{T}(\mu) \tilde{r}_-(\lambda-\mu) \, . \end{aligned} \tag{9.163}$$

The limits $\ell \to \infty$ in calculating $r_\pm(\lambda-\mu)$ (9.35) are given by:

$$\begin{aligned} &r_\pm(\lambda-\mu) \\ &= \lim_{\substack{x\to\infty \\ y\to-\infty}} \left(E^{-1}(x,\lambda) \otimes E^{-1}(x,\mu) \right) r(\lambda-\mu) \left(E(y,\lambda) \otimes E(y,\mu) \right) \qquad (9.164) \\ &= -\frac{1}{\lambda-\mu} \sum_{j=1}^{r} H_j \otimes H_j^{\vee} \mp i\pi\delta(\lambda-\mu) \sum_{\alpha\in\Delta} \operatorname{sign}\alpha E_\alpha \otimes E_{-\alpha} \, , \qquad (9.165) \end{aligned}$$

with $E(x,\lambda) = \exp(-i\lambda J x)$ and analogically for $\tilde{r}(\lambda-\mu)$:

$$\tilde{r}_\pm(\lambda-\mu) = -\frac{\lambda\mu}{\lambda-\mu} \sum_{j=1}^{r} H_j \otimes H_j^{\vee} \mp i\pi\lambda^2\delta(\lambda-\mu) \sum_{\alpha\in\Delta} \operatorname{sign}\alpha E_\alpha \otimes E_{-\alpha} \, . \tag{9.166}$$

To derive them we made use of commutation relations (9.148) and relations (9.33).

As already noted, the integrals of motion of the NLSE (9.151) and the HFE (9.152) are generated by the matrix elements of the diagonal blocks $a^\pm(\lambda)$ and $\tilde{a}^\pm(\lambda)$ of $T^\pm(\lambda)$ and $\tilde{T}^\pm(\lambda)$. The difference with respect to the $g \cong sl(2)$ case is that not all these generating functionals are in involution.

We shall formulate below r series of involutive functionals; they are related to $T(\lambda)$ and are defined as follows:

$$\mathcal{A}_j(\lambda) = \langle \omega_j | \, T(\lambda) \, | \omega_j \rangle \, , \qquad a_j(\lambda) = \ln \mathcal{A}_j(\lambda) \; , \tag{9.167}$$

where $T(\lambda)$ is considered in the representation $V^{(j)}$ with highest weight $\omega = \omega_j$, and by $|\omega_j\rangle$ we have denoted the highest weight vector in $V^{(j)}$. We remind that by definition the highest weight vector $|\omega_j\rangle$ has the properties:

$$\begin{aligned} E_\alpha |\omega_j\rangle = 0 \; , \qquad & (H_k - (\alpha_k, \omega_j))|\omega_j\rangle = 0 \; , \\ \langle \omega_j | E_{-\alpha} = 0 \; , \qquad & \langle \omega_j |(H_k - (\alpha_k, \omega_j)) = 0 \; . \end{aligned} \tag{9.168}$$

for $\alpha > 0$.

As an immediate consequence of (9.162) and (9.163), we shall prove that $\mathcal{A}_j(\lambda)$ and $\mathcal{A}_k(\mu)$ are in involution for any choice of λ, μ, j, and k. To this purpose, we shall use the fact that (9.162) and (9.163) are valid, irrespective of the choice of the representation of $\mathfrak{g}$. Let us consider $T(\lambda)$ in the representation $V^{(j)}$, and $T(\mu)$ in the representation $V^{(k)}$. The corresponding classical r-matrix will again be given by (9.159); only the generators in the first factors in the tensor product are taken in the representation $V^{(j)}$, and the ones in the second factors, in the representation $V^{(k)}$. Let us consider the average of (9.162) between the vectors $\langle \omega_j | \otimes \langle \omega_k |$ and $|\omega_j\rangle \otimes |\omega_k\rangle$, where $r_\pm(\lambda - \mu)$ are given by (9.164). It is not difficult to check that in view of (9.168), we have:

$$\begin{aligned} r_+(\lambda-\mu)|\omega_j\rangle \otimes |\omega_k\rangle = r_-(\lambda-\mu)|\omega_j\rangle \otimes |\omega_k\rangle = f_{jk}|\omega_j\rangle \otimes |\omega_k\rangle \; , \\ \langle \omega_j | \otimes \langle \omega_k | r_+(\lambda-\mu) = \langle \omega_j | \otimes \langle \omega_k | r_-(\lambda-\mu) = f_{jk}\langle \omega_j | \otimes \langle \omega_k | \; , \end{aligned} \tag{9.169}$$

where

$$f_{jk} = -\frac{1}{\lambda-\mu} \sum_{p=1}^{r} (\omega_j, \alpha_p)(\omega_p, \omega_k) = \frac{(\omega_j, \alpha_p)(\omega_j, \omega_k)}{\lambda - \mu} \; . \tag{9.170}$$

The comparison between (9.167) and (9.169) immediately gives, [27]:

$$\{\mathcal{A}_j(\lambda), \mathcal{A}_k(\mu)\} = 0 \qquad \text{or} \qquad \{a_j(\lambda), a_k(\mu)\} = 0 \; . \tag{9.171}$$

We also note that precisely $a_j(\lambda)$ are generating functionals of the local integrals of motion. The Hamiltonian of the NLSE (9.151) is a linear combination of their expansion coefficients.

Analogous procedure can be applied also to the gauge-equivalent equations (9.152). In this case, the integrals of motion are given by:

$$\begin{aligned} \tilde{\mathcal{A}}_j(\lambda) &= \langle \omega | \tilde{T}(\lambda) | \omega_j \rangle = \langle \omega_j | T(\lambda) T^{-1}(0) | \omega_j \rangle \; , \\ \tilde{a}_j(\lambda) &= \ln \tilde{\mathcal{A}}_j(\lambda) = a_j(\lambda) - a_j(0) \; . \end{aligned} \tag{9.172}$$

In order that the last equation in (9.172) holds, we required additional condition – $T(0)$ must be diagonal, and not just block-diagonal. The proof of the involutivity:

$$\left\{\tilde{\mathcal{A}}_j(\lambda), \tilde{\mathcal{A}}_k(\mu)\right\} = 0 \qquad \text{or} \qquad \{\tilde{a}_j(\lambda), \tilde{a}_k(\mu)\} = 0 \;, \tag{9.173}$$

goes along the same lines, using the relations (9.163) and (9.166). The only difference is that in (9.169) and (9.170) f_{jk} must be replaced by $\tilde{f}_{jk} = \lambda\mu f_{jk}$.

9.6 Possibilities for Generalizations of the r-Matrix Approach. A Short Review

We shall finish this chapter with a short review on the perspectives of the classical r-matrix approach.

Up to now, we considered two gauge-equivalent linear problems (9.144) and (9.145) and their particular cases for $g \cong sl(2)$. In both cases $U(x,\lambda)$ and $\tilde{U}(x,\lambda)$ depend linearly on the spectral parameter λ, and the canonical Poisson brackets defined on the coefficients of $U(x,\lambda)$ and $\tilde{U}(x,\lambda)$ are directly related to the algebra g, which is the ground for $\mathcal{L}$. This may give the wrong impression that to each auxiliary linear problem L (or $\tilde{L}$) there corresponds specific r-matrix (of course, if it exists at all). Actually it comes out that a whole class of auxiliary linear problems lead to the same r-matrix.

1. Let us consider linear problem of the type of (9.144), where $U(x,\lambda)$ is a rational function of λ

$$U(x,\lambda) = \sum_{k=-M}^{N} u_k(x)\lambda^k \qquad \text{for} \quad M \geq 0, N > 0 \;, \tag{9.174}$$

   and the coefficients $u_k(x)$ take values in the semisimple Lie algebra g. We first impose on $u_k(x)$ a condition, which fixes the gauge of L. This can be done in many natural ways; here, we only list two of them: (a) $u_N = J \in h$, $J = \text{const}$ and (b) $u_{-M} = J$, $M > 0$. The gauge degrees of freedom, considered from the point of view of the Hamiltonian approach to the NLEE, form an invariant subspace $\boldsymbol{\Phi}_\mathbf{0}$ in the phase space $\boldsymbol{\Phi}_g \equiv \{u_k\}$. Moreover, as a rule their evolution is trivial. That is why it is natural to fix up the gauge and to focus our attention on the other invariant subspace $\boldsymbol{\Phi}_g \backslash \boldsymbol{\Phi}_0$, where the dynamics of the corresponding NLEE takes place. The formal fixing up of the gauge imposes constraints on $\boldsymbol{\Phi}_g$, and this procedure must be performed according to Dirac's prescription. In $\boldsymbol{\Phi}_g \backslash \boldsymbol{\Phi}_0$, we can introduce the following Poisson brackets between the coefficients $u_k^a(x) =< u_k(x), X_a >$ of $U(x,\lambda)$:

$$\left\{u_k^a(x), u_l^b(y)\right\} = -iC_{ab}^c u_{k+l}^c(x)\delta(x-y) \;, \tag{9.175}$$

where C^c_{ab} are the structure constants of the algebra g. By X_a we have denoted the basis in g, which is orthonormal with respect to the Killing form

$$[X_a, X_b] = C^c_{ab} X_c, \qquad < X_a, X_b >= \delta_{ab} \ , \tag{9.176}$$

In this basis, the element Π (9.159) has the form

$$\Pi = \sum_a X_a \otimes X_a \ . \tag{9.177}$$

The Poisson brackets (9.175) are a natural generalization of (9.156). We note here that if $k+l$ is out of the interval $(-M, N)$, then the corresponding right-hand side of (9.175) must vanish.

After a somewhat lengthy calculation we find that if $M > 0$, then (9.174) and (9.175) do not allow a classical r-matrix. An r-matrix exists for the deformed Poisson brackets:

$$\left\{u^a_k(x), u^b_l(y)\right\}^{(0)} = \begin{cases} -iC^c_{ab} u^c_{k+l}(x)\delta(x-y) & \text{for } k \geq 0, l \geq 0 \ , \\ iC^c_{ab} u^c_{k+l}(x)\delta(x-y) & \text{for } k < 0, l < 0 \ , \\ 0 & \text{otherwise} \ , \end{cases} \tag{9.178}$$

It is well known that there exist duality between the Poisson brackets and the Lie-algebraic operation in g. That is why the existence of r can be interpreted as a definition of a second *Lie-algebraic* structure on g. Let us explain in more detail this fact, following the ideas in [9, 10]. We start by noting that the Poisson brackets (9.175) are dual to the commutation relations between the generators of the algebra $g_c = g \otimes \mathbb{C}[\lambda, \lambda^{-1}]$. The elements of g_c are the Laurent series $X(\lambda)$, $Y(\lambda)$ over the powers of λ

$$X(\lambda) = \sum_{k=-\infty}^{N} u^a_k(x)\lambda^k X_a \ , \qquad Y(\lambda) = \sum_{k=-M}^{\infty} v^a_k(x)\lambda^k X_a \ , \tag{9.179}$$

truncated at least on one side, whose coefficients take values in g [11, 12, 13, 14]. As basis in g_c, we can consider $X_{a,k} = \lambda^k X_a$. Then, the commutation relations and the metric in g_c are introduced by:

$$\begin{aligned} &[X_{a,k}, X_{b,l}] = C^c_{ab} X_{c,k+l} \ , \\ &\langle\langle X(\lambda), Y(\lambda)\rangle\rangle = \operatorname*{Res}_{\lambda=0} \frac{< X(\lambda), Y(\lambda) >}{\lambda} = \sum_{k=-\infty}^{\infty} u^a_k v^a_{-k} \ . \end{aligned} \tag{9.180}$$

The Poisson brackets (9.175), corresponding to these commutation relations, do not allow one to introduce an r-matrix.

Conversely, to the Poisson brackets (9.178), which allow an r-matrix there correspond the second Lie-algebraic structure, which can be introduced in g_c. It is obtained from the first one after a deformation with the linear operator R as follows:

$$[X, Y]_R = [RX, Y] + [X, RY] \,, \tag{9.181}$$

where $RX_{a,k} = X_{a,k}$ for $k \geq 0$ and $RX_{a,k} = -X_{a,k}$ for $k < 0$. If we denote by X_+ (X_-) the part of the series (9.179), which contains only the nonnegative (the negative) powers of λ, we can rewrite (9.181) in the form:

$$[X, Y]_R = [X_+, Y_+] - [X_-, Y_-] \,, \tag{9.182}$$

The Poisson brackets (9.178) are dual precisely to this Lie-algebraic operation.

In order that (9.182) defines a Lie bracket, it is necessary that it satisfies the Jacobi identity:

$$[X, B(Y, Z)] + [Y, B(Z, X)] + [Z, B(X, Y)] = 0 \,, \tag{9.183}$$

where

$$B(X, Y) = [RX, RY] - R([X, Y]_R) \,. \tag{9.184}$$

The classical Yang–Baxter equation (9.91) is obtained, if we equate to zero *each of the three* summands in (9.183). But (9.91) is only a *sufficient,* but not necessary, condition for (9.183) to hold. Actually (9.183) allows also another nontrivial solution:

$$B(X, Y)[RX, RY] - R([X, Y]_R) = -[X, Y] \,. \tag{9.185}$$

If we insert (9.185) in the left-hand side of (9.183), we obtain the Jacobi identity for X, Y, Z with respect to the initial Lie-algebraic structure (9.180). Equation (9.185) is known as the modified Yang–Baxter equation [10], and the algebra with the two Lie-algebraic structures (9.180) and (9.181) we shall call a Lie-bialgebra.

The concrete realization of the operator R, given above, corresponds to the simplest classical r-matrix (9.158), where the function $(\lambda - \mu)^{-1}$ must be understood as a principal value. This function, considered as the kernel of an integral operator, gives the well-known Hilbert transform.

Another possible realization of R as an element of the ring of linear operators is the following: $RX = X$, if X is a differential operator, and $RX = -X$, if X is an integral Volterra-type operator. The corresponding r-matrix allows one to describe the Hamiltonian structures of the NLEE in the approach of Gelfand and Dorfman [15].

2. Just like the inverse scattering method, the classical r-matrix approach can be formulated and applied to the solution of difference evolution equations (DEE) [2, 10, 16, 17, 18, 19]. We shall mention only the simplest case – the difference analogue of the Zakharov–Shabat system. In this case, we require the compatibility between:

$$L_d \equiv \psi_{n+1}(z) = U_n(z)\psi_n(z) \,, \qquad U_n(z) = \begin{pmatrix} z & q_n^+ \\ q_n^- & 1/z \end{pmatrix} \tag{9.186}$$

and

$$\mathcal{M}_d \equiv \frac{d\psi_n}{dt} = V_n(z)\psi_n(z) \ . \tag{9.187}$$

The compatibility condition is of the form:

$$\frac{dU_n}{dt} + U_n(z,t)V_n(z,t) - V_{n+1}(z,t)U_n(z,t) = 0 \ . \tag{9.188}$$

Here, it is z that plays the role of the spectral parameter, and $U_n(z,t)$ takes values not in the algebra but in the group $SL(2)$. As an example of a DEE, which can be solved with the help of (9.188) we write down the difference analogue of the NLSE:

$$\begin{aligned} &iq^+_{n,t} + \left(1 - q^+_n q^-_n\right)\left(q^+_{n-1} + q^+_{n+1}\right) - 2q^+_n = 0 \ , \\ &- iq^-_{n,t} + \left(1 - q^+_n q^-_n\right)\left(q^-_{n-1} + q^-_{n+1}\right) - 2q^-_n = 0 \ , \end{aligned} \tag{9.189}$$

Here we need an alternative natural definition of the classical r–matrix [10, 18, 19]:

$$\left\{U_k(z) \underset{,}{\otimes} U_l(\zeta)\right\} = [r(z,\zeta), U_k(z) \otimes U_l(\zeta)]\delta_{kl} \ , \tag{9.190}$$

since the ultralocal Poisson brackets between q^+_n and q^-_n are introduced by:

$$\{q^+_k, q^-_l\} = -i\left(1 - q^+_k q^-_l\right)\delta_{kl} \ , \tag{9.191}$$

One may check that Poisson brackets (9.191) allow classical r-matrix defined by (9.190), where $r(z,\zeta)$ is obtained from the trigonometric r-matrix $r^a(\lambda-\mu)$ (9.96) by assuming $z = e^{i\lambda}$, $\zeta = e^{i\mu}$ [19].

3. A substantial constraint on the r-matrix approach is the ultralocality, imposed on the initial Poisson brackets. Lately, there have been successful attempts to overcome this difficulty and to introduce r-matrix also for nonultralocal Poisson brackets [20, 21]. Now in the definition of the r-matrix there enter two (not just one) independent functions:

$$\begin{aligned} \left\{U(x,\lambda) \underset{,}{\otimes} U(y,\mu)\right\} &= [u(y,\lambda,\mu) - v(x,\lambda,\mu), \mathbb{1} \otimes U(z)]\delta'(x-y) \\ &+ \ \{[u(x,\lambda,\mu), U_k(x,\lambda) \otimes \mathbb{1}] + [v(x,\lambda,\mu), \mathbb{1} \otimes U_k(z)]\}\,\delta(x-y) \end{aligned} \tag{9.192}$$

In order that these brackets are antisymmetric, u and v must satisfy:

$$Pv(x,\lambda,\mu)P = -u(x,\mu,\lambda), \qquad Pu(x,\lambda,\mu)P = -v(x,\mu,\lambda) \ , \tag{9.193}$$

and the Jacobi identity leads to the following generalization of the Yang–Baxter equation:

$$
\begin{aligned}
&[v_{23}(x,\mu,\nu), v_{12}(x,\lambda,\mu)] + [v_{23}(x,\mu,\nu), v_{13}(x,\lambda,\mu)] \\
&+ [v_{13}(x,\lambda,\nu), u_{12}(x,\lambda,\mu)] \\
&+ H^{(v)}_{1,23}(x,\lambda,\mu,\nu) - H^{(v)}_{2,13}(x,\mu,\lambda,\nu) = 0 \,,
\end{aligned} \tag{9.194}
$$

where

$$
H^{(v)}_{1,23}(x,\lambda,\mu,\nu)\delta(x-y) = \{U_1(x,\lambda), v_{23}(y,\mu,\nu)\} \ . \tag{9.195}
$$

The Poisson brackets between the matrix elements of the monodromy matrices also acquire a more general form.

4. Another possibility is to introduce an r-matrix with x-dependence [6], defined through:

$$
\begin{aligned}
&\left\{ U(x,\lambda) \underset{,}{\otimes} U(y,\mu) \right\} \\
&= i\left([r(x,\lambda,\mu), U(x,\lambda)\otimes \mathbb{1} + \mathbb{1}\otimes U(y,\mu)] - \frac{dr}{dx} \right) \delta(x-y)
\end{aligned} \tag{9.196}
$$

Then the corresponding monodromy matrices satisfy:

$$
\begin{aligned}
\left\{ T(x,y,\lambda) \underset{,}{\otimes} T(x,y,\mu) \right\} &= r(x,\lambda,\mu) T(x,y,\lambda)\otimes T(x,y,\mu) \\
&- T(x,y,\lambda)\otimes T(x,y,\mu) r(y,\lambda,\mu)
\end{aligned} \tag{9.197}
$$

5. Immediately after the introduction of the notion of the classical r-matrix, and the realization of the fact that the classical Yang–Baxter equation is of pure algebraic nature, there appeared Belavin and Drinfeld's paper [22], which classifies all nondegenerate solutions of this equation. The main result of this paper consists in that to each semisimple Lie algebra g and to each Coxeter automorphism of g one can relate a classical r-matrix. In other words, to each Kac–Moody algebra, there corresponds a classical r-matrix of the form (9.158), (9.159). To each of these r-matrices, one can apply the averaging procedure, which leads to the so-called trigonometric r-matrices. Finally, for the algebras $sl(n)$, it is possible to repeat the averaging procedure and to construct the corresponding elliptic r-matrices.
6. We considered two classes of boundary conditions for the auxiliary linear problem L – periodic and decaying for $x \to \pm\infty$. In papers [23, 24], Sklyanin proposes a new version of the r-matrix approach, which is independent of the choice of the boundary conditions for $\mathcal{L}$.
7. A wide scope of results, concerning the classical r-matrix approach have been obtained as consequences from the quantum R-matrix approach and from the quantum version of the Yang–Baxter equation [25]

$$
R_{12}(\lambda,\mu)R_{13}(\lambda,\nu)R_{23}(\mu,\nu) = R_{23}(\mu,\nu)R_{13}(\lambda,\nu)R_{12}(\lambda,\mu) \ . \tag{9.198}
$$

This equation lies in the foundation of the recently introduced quantum groups [9, 26]. As an introduction to this domain, one can use [18, 19, 25, 27, 28, 29], and many other references, cited therein.

9.7 Comments and Bibliographical Review

1. In mathematics soliton equations stimulated the development of the theory of the Kac–Moody and other infinite-dimensional graded algebras [11, 13, 14, 30, 31, 32, 33, 34, 35]. Eventually the discovery of the quantum version of the ISM introduced the so-called quantum groups on one side, which were directly related to a special class of solvable models in statistical physics. Closely related to this approach is the method of the τ-function; see e.g. [36, 37, 38] and the reference therein.
2. It is only natural that the Lax representation, which is purely algebraic in nature, stimulated the development of the infinite-dimensional algebras known also as Kac–Moody algebras. The well-known method coming from the theory of simple Lie groups and algebras [1, 3, 4, 39, 40, 41, 42, 43] were found very useful in a number of aspects in soliton theory [16, 33, 34, 35, 38, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104].
3. At the end of the 1970s, a new branch in the soliton theory was started. Today it is known as the quantum inverse scattering method [22, 25, 27, 61, 71, 80, 105, 106, 107, 108, 109, 111, 112, 113, 114, 115, 116] which soon created another mathematical field known as quantum groups; see [9, 22, 38, 117].
4. The classical r–matrix [2, 22, 27, 75, 103, 106, 108, 110, 118] came up as the quasi-classical approximation of the quantum R–matrix of Baxter [25]. It quickly drew the attention of physicists and mathematicians, because it provided an effective tool for the study of the Hamiltonian properties of the soliton equations. In particular, for Lax operators whose potentials allow ultralocal Poisson brackets, it allowed a very efficient way to calculate the Poisson brackets between the scattering matrix elements [2, 27].
5. The fact that the principal minors of the scattering matrix $T(\lambda)$ of the generalized ZS system are analytic functions of λ, and their logarithms can be viewed as generating functionals for the integrals of motion has been known to Zakharov, Shabat, and Manakov (see [118]) for $\mathfrak{g} \simeq sl(n)$. These results are naturally generalized for any semisimple Lie algebra $\mathfrak{g}$, see [34, 35, 119, 120]. Using the well-known facts about the fundamental representations of $\mathfrak{g}$ it was possible to prove that these integrals are in involution [8].

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10

Introduction

The geometric ideas have always been an intrinsic part of the theory of the Differential Equations. Just in order to be more specific, suppose that we consider the following system of ordinary differential equations

$$\frac{dy}{dt} = X(y) , \tag{10.1}$$

where $y = (y^1, y^2, \dots y^n)$ belongs to some open subset Ω of $\mathbb{R}^n$, $t \in \mathbb{R}$ is the time and $X(y)$ is differentiable function $X : \Omega \to \mathbb{R}^n$. It is well known how fruitful may be the interpretation of the right-hand side of this equation as vector field X over Ω and the interpretation of the solutions of the above equation as integral curves of X. In particular, the geometric viewpoint proved to be very useful in such topics as the search for symmetries and conservation laws for (10.1).

The geometric ideas have even stronger influence on the theory of systems of evolution equations if these equations are Hamiltonian – then the geometric techniques and ideas are indispensable, and such notions as Poisson brackets are one of the necessary tools in the theory. The natural generalization of these classical topics from open sets in $\mathbb{R}^n$ to spaces that are only locally as $\mathbb{R}^n$ leads to the theory of dynamical systems on manifolds, which is now a developed branch of Mathematics with many applications; see for example the series [1].

Now consider the system of the evolutionary partial differential equations

$$\frac{\partial f}{\partial t}(x,t) = X(f(x,t), f_x(x,t), \dots) . \tag{10.2}$$

In the above, the right-hand side

$$X = (X_1(f, f_x, \dots), X_2(f, f_x, \dots), \dots, X_n(f, f_x, \dots))$$

is a smooth vector function in all of its arguments, and $f = (f_1, f_2, \dots, f_n)$ is a set of smooth scalar functions f_i, depending on the time and one additional spatial variable x. Usually x belongs to some interval I on the line.

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DOI 10.1007/978-3-540-77054-1_10

The attempts to apply the techniques developed for system (10.1) to system (10.2) are very old and were in the scope of such mathematicians as Lie and Volterra. The idea suggests to consider the right-hand side of the above system of evolution equations (10.2) for the functions f_i; $i = 1, 2, \dots n$ as vector field on the infinite-dimensional manifold on which the functions $f(x)$ are "points". This manifold will be referred below as the manifold of potentials. Of course, to pose the problem properly, one must give additional information about the manifold of potentials, which include the boundary conditions for f, the class of functions to which f belong, etc.

However, maybe because of the difficulties to treat infinite-dimensional case which leads to some lack of rigor in the considerations, the development of these ideas has been relatively slow, and only in the last decades they have again started to attract the attention they deserve. In our opinion, the discovery of the Inverse Scattering Method was one of the reasons that caused the renaissance. It turned out that the equations solvable through Inverse Scattering Method can be investigated more closely than the "usual" nonlinear evolution equations. Besides, they have number of interesting properties, for example, they are Hamiltonian and for them have been discovered hierarchies of conservation laws. It seems that because of these circumstances the glances turned again to the geometric ideas. Nowadays, the questions of the conservation laws and Hamiltonian structures, as well as the questions about complete integrability, which when the Inverse Scattering Method (ISM) was discovered were treated within its techniques, are investigated through several different approaches. Among them are the geometric theory of the generating (recursion) operators [2]; the so-called Adler scheme, which exploits the existence of two Lie algebraic structures over a given Lie algebra and hence the existence of two Poisson-Lie brackets on the corresponding coalgebra; the approach based on the properties of the classical R-matrix, etc. All these approaches have many interrelations and frequently overlap, even consider the same objects giving them different names, but at least at the present moment each of them has its own theory and applications. Among the approaches listed above, we shall consider in detail only the theory of the generating operators. For convenience, we shall refer to the approach based on the existence of two Poisson-Lie brackets as algebraic approach, though all the algebraic procedures in them have geometric meaning.[1] The geometric theory of the generating operators originates from the beautiful result of F. Magri [3], who found the second Hamiltonian structure for the KdV equation. He noticed that the important properties of the generating operators, namely, that they are producing integrable evolution equations, densities of conservation laws and hierarchies of symplectic structures, are due to the fact that they can be considered special mixed tensor fields over the manifolds of potentials, fields with vanishing

[1] It is interesting to note that one of the principal identities in the R-matrix formalism, the so-called modified Yang-Baxter identity; see [4], for an endomorphism R of a Lie algebra coincides with the definition of the Nijenhuis tensor if $R^2 = -\mathbf{1}$.

Nijenhuis bracket. Such fields are called Nijenhuis tensor fields (Nijenhuis tensors). It turned out that the generating operators are in fact dual objects to the Nijenhuis tensors. The development of the original ideas have brought into consideration new geometric structures – the so-called Poisson-Nijenhuis structures, which we shall introduce in Sect. 14.4. We shall be mainly interested in the infinite-dimensional dynamical systems, the finite dimensional ones considered very thoroughly in number of monograph books; see for example [5] or [6]. However, frequently we shall consider the finite dimensional systems as an illustration or when they present some interesting features.

In order to avoid possible misunderstanding, we must stress again that when we use the word "geometric," it refers to the idea of treating the solution of an evolution partial differential equation as a curve on an infinite-dimensional manifold. There are a number of other approaches and techniques, which are also geometric and which must not be mixed with the geometric approach we use in this book. For example, in order to study partial differential equations, one can use the so-called jet spaces and then these equations are considered as submanifolds in them. These ideas also showed a renaissance in the past years; see for example [7] or [8] for the general theory. To the so-called soliton equations (which are one of the main objects in this book) these techniques have been also successfully applied; see [9]. As geometric must of course be classified the techniques that allow to find large classes of solutions to the soliton equations – the so-called finite-gap solutions; see [10]. In them the basic objects are some Riemann manifolds associated with the analytical solutions of the corresponding auxiliary linear problems. There are also other related techniques for integration, algebraic and geometric; see [11]. However, all these various approaches focus on different topics, and we shall not discuss them. There are of course other monographs that (at least partially) adopt the similar viewpoint as we do, for example [12]. However, as already mentioned, our book is about the recursion operators and their interpretation as Nijenhuis tensors. The Nijenhuis tensors and the Poisson-Nijenhuis manifolds are, therefore, the central objects in our geometric picture and the ones that unite the two parts of this book together.

Considerable space in this part is dedicated to the geometric theory of the generating operators associated with the so-called generalized Zakharov-Shabat linear system (for shortness GZS)

$$L\psi = (i\frac{\partial}{\partial x} + q(x) - \lambda J)\psi = 0, \quad x \in \mathbb{R} \, . \tag{10.3}$$

Here, the potential function

$$q(x) = \sum_{\alpha\in\Delta} q_\alpha(x) E_\alpha \tag{10.4}$$

takes values in some semisimple Lie algebra $\mathfrak{g}$, with Cartan subalgebra $\mathfrak{h}$, which we assume fixed, as well as the corresponding root system Δ and the corresponding root vectors E_α related to $\mathfrak{h}$. We assume that J is *constant, real, regular* element from the Cartan subalgebra $\mathfrak{h}$, and for simplicity we also assume that the functions $q_\alpha(x)$ are Schwartz-type functions on the line. The fact that the element J is regular means that

$$\mathfrak{h} = \ker \mathrm{ad}_J = \{X \in \mathfrak{g} : \mathrm{ad}_J(X) = [J, X] = 0\} . \tag{10.5}$$

The Zakharov-Shabat system, considered earlier in this book, is a special case of (10.3) when $\mathfrak{g} = \mathrm{sl}(2, \mathbb{C}); J = \sigma_3$. In order to describe the linear space of potential functions, we introduce the orthogonal splitting

$$\mathfrak{g} = \overline{\mathfrak{g}} \oplus \mathfrak{h} \tag{10.6}$$

with respect to the Killing form $\langle\ ,\ \rangle = \mathrm{tr}\,(\mathrm{ad}_X \circ \mathrm{ad}_Y)$, and then of course we obtain that the linear space of all potentials $q(x)$ is the space of all Schwartz-type functions on the line taking values in $\overline{\mathfrak{g}}$.

Remark 10.1. We shall denote the Killing form of a Lie algebra either by $B(X, Y)$ or by $\langle X, Y\rangle$. The first notation will be used, when there is some other pairing, already denoted by $\langle ., .\rangle$.

The treatment of the general case of arbitrary semisimple Lie algebra requires some knowledge about the structural theory of the semisimple Lie algebras, and maybe this circumstance makes some of the arguments difficult to follow. In order to make things easier, we advise the inexperienced reader each time he sees the words semisimple Lie algebra to have in mind the example of the algebra $\mathrm{sl}\,(n, \mathbb{C})$ – the algebra of traceless $n \times n$ matrices with complex entries. Then one can assume that the element J is of the form

$$J = \mathrm{diag}\,(j_1, j_2, \ldots, j_n), \quad \sum_{i=1}^{n} j_i = 0 . \tag{10.7}$$

The regularity of J means that $j_i - j_k \neq 0$ if $i \neq k$. Then

$$\ker(\mathrm{ad}_J) = \{\xi : \xi \in \mathrm{sl}\,(n, \mathbb{C}), [J, \xi] = 0\} \tag{10.8}$$

is the subalgebra of the diagonal matrices. This subalgebra is usually taken as the Cartan subalgebra $\mathfrak{h}$ for $\mathfrak{g} = \mathrm{sl}\,(n, \mathbb{C})$. The Killing form for $\mathfrak{g}$

$$\langle \xi, \eta\rangle = \mathrm{tr}\,(\mathrm{ad}_\xi \circ ad_\eta); \quad \xi, \eta \in \mathfrak{g} \tag{10.9}$$

in the case $\mathfrak{g} = \mathrm{sl}\,(n, \mathbb{C})$ is equal to

$$\langle \xi, \eta\rangle = \frac{1}{2n}\mathrm{tr}\,(\xi\eta); \quad \xi, \eta \in \mathrm{sl}\,(n, \mathbb{C}) . \tag{10.10}$$

Further, the orthogonal completion of $\mathfrak{h}$ with respect to the Killing form (10.9), denoted above by $\overline{\mathfrak{g}}$, in the case $\mathfrak{g} = \mathrm{sl}\,(n, \mathbb{C})$, is simply the subspace of off-diagonal $n \times n$ matrices. The fact that this subspace is invariant with respect to the action of the diagonal matrices $\mathfrak{h}$ means that if

$$H = \operatorname{diag}(h_1, h_2, \ldots, h_n), \quad \sum_{i=1}^{n} h_i = 0 \,. \tag{10.11}$$

then

$$[H, \bar{\mathfrak{g}}] \subset \bar{\mathfrak{g}} \,, \tag{10.12}$$

which can be easily verified. Let us take now the usual basis $\{E_{ij};\ 1 \le i, j \le n\}$ in the space of the $n \times n$ matrices, that is, the entries $(E_{ij})_{ks}$ of E_{ij} are:

$$(E_{ij})_{ks} = \delta_{ik}\delta_{js} \,, \tag{10.13}$$

(δ_{ik} is the Kronecker symbol). The matrices $E_{ij};\ i \ne j$ form a basis in the subspace of the off-diagonal matrices. It is not difficult to check that

$$[H, E_{ij}] = \alpha_{ij}(H)E_{ij}; \quad i \ne j \tag{10.14}$$

where $\alpha_{ij};\ i \ne j$ are the following linear functions on the space $\mathfrak{h}$:

$$\alpha_{ij}(H) = h_i - h_j \,. \tag{10.15}$$

In the case $\mathfrak{g} = \mathrm{sl}\,(n, \mathbb{C})$, the functions α_{ij} are the roots, the root system Δ is the set of all α_{ij}'s, and the corresponding root vectors are E_{ij}. Thus, for $\mathfrak{g} = \mathrm{sl}\,(n, \mathbb{C})$ (10.4) means that

$$q(x) = \sum_{i \ne j} q_{ij}(x)E_{ij} \,. \tag{10.16}$$

Remember also we have assumed that $q_{ij}(x)$ are Schwartz-type functions on the line.

If the reader has in mind what was said in the above lines, he can safely "translate" all statements for the general case into the corresponding ones for the case $\mathrm{sl}\,(n, \mathbb{C})$ and follow most of the argument without the help of the theory of the semisimple Lie algebras.

We must recall also some facts about the soliton equations that can be solved with the help of auxiliary linear problem (10.3). Suppose we are looking for all soliton equations having Lax form

$$[L, M] = 0 \,, \tag{10.17}$$

where L is the operator defined in (10.3), and the operator M is of the form

$$\begin{aligned} M &= i\frac{\partial}{\partial t} + \sum_{k=0}^{N} \lambda^k M_k \\ M_N &\in \mathfrak{h}, \quad M_N = \text{const} \,. \end{aligned} \tag{10.18}$$

As M_N is a fixed constant element from the Cartan subalgebra, from (10.17) we get a system of equations for the coefficients M_k, which can be resolved recursively. At each step, finding the projection of the coefficient functions M_{k+1} over $\overline{\mathfrak{g}}$, we see that it is obtained via one and the same integro-differential operator $\Lambda_\pm$, that is, one finds that

$$\pi_0 M_{k+1} = \Lambda_\pm(\pi_0 M_k) \ , \tag{10.19}$$

where π_0 is the orthogonal projector onto the space $\overline{\mathfrak{g}}$. As to the operators $\Lambda_\pm$, they have the form

$$\begin{aligned} &\Lambda_\pm(X(x)) = \\ &\mathrm{ad}_J^{-1}\left(i\frac{\partial X}{\partial x} + \pi_0[q,X] + i\,\mathrm{ad}_q \int_{\pm\infty}^{x} (\mathbf{1}-\pi_0)[q(y),X(y)]dy. \right) , \end{aligned} \tag{10.20}$$

where $\mathrm{ad}_q(X) = [q,X]$. The relation (10.19) is one of the reasons the above operators are called *generating operators* – they generate the coefficients M_k starting from the first one. Finally, if we find all the coefficients, we obtain that the evolution equations (10.17) can be written into one of the following equivalent forms:

$$\begin{aligned} &\text{a)} \quad i\,\mathrm{ad}_J^{-1}\frac{\partial q}{\partial t} + \Lambda_+^N\left(\mathrm{ad}_J^{-1}[M_N,q]\right) = 0 \\ &\text{b)} \quad i\,\mathrm{ad}_J^{-1}\frac{\partial q}{\partial t} + \Lambda_-^N\left(\mathrm{ad}_J^{-1}[M_N,q]\right) = 0 \ . \end{aligned} \tag{10.21}$$

The above suggests that the operators $\Lambda_\pm$ play an important role in the theory of equations (10.21). For the first time, they were introduced in the famous paper [13] for the case of the algebra $\mathrm{sl}(2,\mathbb{C})$ (Zakharov-Shabat system) and then were generalized for the case of arbitrary semisimple algebra; see for example [14, 15, 16, 17]. For the operators $\Lambda_\pm$, there exists developed spectral theory, due to fact that their eigenfunctions (the so-called *squares* or *adjoint solutions*) can be obtained from the fundamental analytical solutions of the GZS linear problem (10.3). We have seen in the first part of this book that the expansions over the adjoint solutions play a very important role. From one side they are closely related to the spectral decomposition of $\Lambda_\pm$. From the other, the coefficients of the expansions over the adjoint solutions of the potential function $q(x)$ can be used as scattering data for (10.3), and thus the map from the potential function $q(x)$ to the scattering data can be regarded as generalized Fourier transform. In terms of the coefficients of the expansions over the adjoint solutions (10.21) are linear. Then, the Inverse Scattering Method for the equations associated with (10.3) can be regarded as an analog of the Fourier transform method for solving linear partial differential equations. In this approach, the soliton equations can be treated via spectral theory of the operators. How it can be done is explained in detail in the first part of this book.

However, except for beautiful spectral properties, the operators $\Lambda_\pm$ have interesting geometric meaning. The geometric methods that reveal its properties practically do not depend on the spectral properties of $\Lambda_\pm$ and can be developed independently. This is how it happened historically. One of the aims of this book is to bring together the spectral and the geometric approaches to the operators $\Lambda_\pm$. In this part, we shall concentrate upon the geometric theory and from the list of the properties of the generating operators, we underline those important for the geometric approach:

- The equations (10.21) are Hamiltonian with respect to the following hierarchy of symplectic forms:

$$\Omega^{(m)}(X_1, X_2) = \int_{-\infty}^{+\infty} \langle X_1, \Lambda^m \mathrm{ad}_J^{-1} X_2 \rangle dx \; , \tag{10.22}$$

$$\Lambda = \frac{1}{2}(\Lambda_+ + \Lambda_-) \; . \tag{10.23}$$

- The equations (10.21) possess infinite series of conservation laws.
- The Hamiltonian functions for (10.21) are in involution, that is, their Poisson brackets vanish. Consequently, the flows of (10.21) commute.

We shall show that the above properties are a consequence of special geometric structures defined on the manifold of potentials of the GZS system.

We shall construct also the geometric theory of the generating operators for the following system, gauge-equivalent to the GZS system (10.3):

$$\begin{aligned} &\tilde{L}\tilde{\psi} = (i\frac{\partial}{\partial x} - \lambda S)\tilde{\psi} = 0, \quad x \in \mathbb{R} \\ &\tilde{\psi}(x,\lambda) = \psi_0^{-1}(x)\psi(x,\lambda), \quad S(x) = \psi_0^{-1} J \psi_0 \; , \end{aligned} \tag{10.24}$$

where $\psi_0(x)$ is the Jost solution for (10.3) evaluated at $\lambda = 0$. The Jost solution for (10.3) is a fundamental solution obeying :

$$\psi(x,\lambda)e^{ix\lambda} \to \mathbf{1}, \quad \text{as} \quad x \to +\infty \; . \tag{10.25}$$

In the case $\mathfrak{g} = \mathrm{su}\,(2)$ and the Heisenberg ferromagnet equation the Lax pair is written in terms of $\mathfrak{g} = \mathrm{su}\,(2)$. Then one of most frequently used boundary conditions, arising naturally in the physical models in which Heisenberg ferromagnet is obtained, are $\lim_{x\to\pm\infty} S(x) = \sigma_3$. So, it seems reasonable to consider the following boundary conditions for $S(x)$ in the general case:

$$\lim_{x\to\pm\infty} S(x) = J \; . \tag{10.26}$$

In order to ensure the above relations, we must impose the following requirement on the solution $\psi_0(x)$:

$$\lim_{x\to+\infty} \psi_0 = \exp H, \quad H \in \mathfrak{h} \; , \tag{10.27}$$

which, of course, is an implicit restriction on the potential function q. Actually the generalizations we are considering are such that when $\mathfrak{g} = \mathrm{su}\,(2)$, one obtains from (10.24) the auxiliary linear problem for the Heisenberg ferromagnet equation, gauge-equivalent to the Zakharov-Shabat linear problem; see [18]. It is known that each of the equations associated with L and having Lax representation $[L, M] = 0$ has an analog – the equation that can be written into the form

$$\begin{aligned} [\tilde{L}, \tilde{M}] &= 0 \\ \tilde{L} = \psi_0^{-1} L \psi_0, &\quad \tilde{M} = \psi_0^{-1} M \psi_0 \,. \end{aligned} \tag{10.28}$$

Such pairs of equations are called gauge-equivalent. The equations associated with $\tilde{L}$ possess the same Hamiltonian properties as equations (10.17) or if one prefers, as equations (10.21). In the simplest case $\mathfrak{g} = \mathrm{su}\,(2)$, the first two nonlinear equations in the hierarchies for L and $\tilde{L}$ are the Nonlinear Schrödinger equation (NLS) and the Heisenberg ferromagnet Equation respectively. As mentioned already, their gauge equivalence, established in [18], is one of the celebrated facts in the theory of the soliton equations.

Remark 10.2. One must have in mind that in both the operators $\tilde{L}$ and $\tilde{M}$, all the coefficients must be expressed only through the new potential function $S(x)$ and its x-derivatives; otherwise one cannot effectively use them. However, to perform it is not so simple, though there is a clear procedure for it, [19] (see also [20] for the case of the algebra $\mathrm{sl}\,(3, \mathbb{C})$).

It can be proved then that the hierarchy of equations associated with the system $\tilde{L}$, (10.24) have the form

$$-i\mathrm{ad}_S^{-1} \frac{\partial S}{\partial t} + (\tilde{\Lambda}_\pm)\tilde{\Lambda}_\pm \tilde{\pi}_0 M_N = 0 \,, \tag{10.29}$$

where

$$\begin{aligned} (\tilde{\Lambda}_\pm) &= \mathrm{Ad}\,(\psi_0^{-1})\Lambda_\pm \mathrm{Ad}\,(\psi_0) \\ \tilde{\pi}_0 &= \mathrm{Ad}\,(\psi_0^{-1})\pi_0 \mathrm{Ad}\,(\psi_0) \end{aligned} \tag{10.30}$$

and by $\mathrm{Ad}\,(\psi_0^{-1})X$ we mean $\psi_0^{-1} X \psi_0$.

The evolution equations (10.29) are in fact the equations gauge-equivalent to (10.21) and have been considered in the first part of the book. We have:

- Equations (10.29) are Hamiltonian with respect to the following hierarchy of symplectic forms:

$$\tilde{\Omega}^{(m)}(\tilde{X}_1, \tilde{X}_2) = \int_{-\infty}^{+\infty} \langle \tilde{X}_1, \tilde{\Lambda}^m \mathrm{ad}_S^{-1}, \tilde{X}_2 \rangle dx \,, \tag{10.31}$$

$$\tilde{\Lambda} = \frac{1}{2}(\tilde{\Lambda}_+ + \tilde{\Lambda}_-) \,. \tag{10.32}$$

- Equations (10.29) possess infinite series of conservation laws.
- The Hamiltonian functions for (10.29) are in involution, that is, their Poisson brackets are zero, and as a consequence, the flows of equations (10.29) commute.

The theory of the generating operators $\tilde{\Lambda}_{\pm}$ associated with the gauge-equivalent system (10.24) can be developed in complete analogy with the theory of the generating operators for system (10.3) and any result for one of these systems have its counterpart for the other system. We shall mention only the relation between the hierarchies of symplectic structures; the details can be found in [19]. In order to introduce this relation, we need, however, some information about the spectral properties of the system L.

First of all, the Jost solutions $\psi(x,\lambda), \phi(x,\lambda)$ for system (10.3) are fundamental solutions, uniquely defined for real λ by their asymptotic behavior:

$$\begin{aligned} &\psi(x,\lambda)e^{-i\lambda Jx} \to \mathbf{1}, \qquad &&\phi(x,\lambda)e^{-i\lambda Jx} \to \mathbf{1} \\ &\quad x \to +\infty, &&x \to -\infty\,. \end{aligned} \tag{10.33}$$

Then, in the standard way is defined the transition matrix $T(\lambda)$:

$$T(\lambda) = \psi^{-1}\phi\,. \tag{10.34}$$

As known, starting from the Jost solutions, which do not possess any analytical properties with respect to the spectral parameter λ, one can construct using the Gauss decomposition for $T(\lambda)$, see for example [21, 22], another couple of fundamental solutions – $\chi^{\pm}(x,\lambda)$. The Gauss decomposition we use here is a generalization of that in the general linear group $\mathrm{GL}\,(n)$; for the case of semisimple Lie groups, see [23, 24]. The solutions $\chi^{\pm}(x,\lambda)$ are defined for λ on the real line by

$$\begin{aligned} \chi^{+}(x,\lambda) &= \psi(x,\lambda)T^{-}(\lambda)D^{+}(\lambda) = \phi(x,\lambda)S^{+}(\lambda) \\ \chi^{-}(x,\lambda) &= \psi(x,\lambda)T^{+}(\lambda)D^{-}(\lambda) = \phi(x,\lambda)S^{-}(\lambda)\,. \end{aligned} \tag{10.35}$$

If one considers λ as complex, it can be proved that $\chi^{+}(x,\lambda)$ allows analytic continuation in the upper half-plane, and $\chi^{-}(x,\lambda)$ allows analytic continuation in the lower half-plane. As a matter of fact, these are the solutions used in the spectral theory for L and $\Lambda_{\pm}$ and the Riemann-Hilbert problem constructions for finding the solutions of the nonlinear evolution equations solvable through the auxiliary linear problem L; see [21, 22] for the case of the algebra $\mathrm{sl}\,(n,\mathbb{C})$ and real regular J, [25, 26, 27, 28, 29] for $\mathrm{sl}\,(n,\mathbb{C})$, and complex regular J and [30] for the situation of an arbitrary complex semisimple Lie algebra and arbitrary regular J. In the formulae (10.35), $T^{\pm}, S^{\pm}, D^{\pm}$ are the factors of the following Gauss decomposition for $T(\lambda)$:

$$T(\lambda) = T^{-}(\lambda)D^{+}(\lambda)(S^{+}(\lambda))^{-1} = T^{+}(\lambda)D^{-}(\lambda)(S^{-}(\lambda))^{-1}\,. \tag{10.36}$$

where $T^{\pm}, S^{\pm}, D^{\pm}$ have the form:

$$S^{\pm}(\lambda) = \exp \sum_{\alpha\in\Delta_+} s^{\pm}_{\pm}(\lambda) E_{\pm\alpha}$$

$$T^{\pm}(\lambda) = \exp \sum_{\alpha\in\Delta_+} t^{\pm}_{\pm}(\lambda) E_{\pm\alpha}$$

$$D^{\pm} = I \exp \sum_{i=1}^{r} \Delta^{\pm}_j(\lambda) H_j \ . \tag{10.37}$$

Here, $\{H_i, E_\alpha;\ i = 1, 2, \ldots, r;\ \alpha \in \Delta\}$ is a basis of the algebra $\mathfrak{g}$, (the so-called Cartan-Weyl basis) such that $\{H_i;\ i = 1, 2, \ldots, r\}$ form a basis of $\mathfrak{h}$, and the rest of the vectors $\{E_\alpha, \alpha \in \Delta\}$ are basis in $\bar{\mathfrak{g}}$. As to the splitting of the root system

$$\Delta = \Delta_+ \cup \Delta_- \tag{10.38}$$

into the subsystems of positive Δ_+ and negative Δ_+ roots, it is defined as follows:

$$\begin{aligned} \Delta_+ &= \{\alpha : \alpha(J) > 0\} \\ \Delta_- &= \{\alpha : \alpha(J) < 0\} \ . \end{aligned} \tag{10.39}$$

The element I in the Gauss decompositions belongs to the universal center of the Lie group G corresponding to the algebra $\mathfrak{g}$.

Both $D^{\pm}(\lambda)$ do not depend on t, and they are generating functions for the conservation laws of the evolution equations (10.21). The construction is the following. In the first place, it can be shown that the functions $\Delta^{\pm}_j(\lambda)$ have identical power series (in λ^{-1}) asymptotic expansions for $\lambda \to \infty$. Then, since the coefficients Δ^k_j in these expansions do not depend on time, they are conserved quantities for evolution equations (10.21). These coefficients can be explicitly calculated using the recursion operators. It turns out that their linear combinations are in fact the Hamiltonian functions for (10.21).

After these preliminaries let

$$F : [S(x)] \to [q(x)] \tag{10.40}$$

be the inverse of the map $[q(x)] \to [S(x)]$. (It maps the potential function $q(x)$ for (10.3) into the potential function $S(x)$ for (10.24)). Let $D(F)$ be the Gateau derivative of F. It can be proved that

$$D(F) = -\mathrm{Ad}\,(\psi_0) \circ \mathrm{ad}_{\,S} \circ \tilde{\Lambda}_- \circ \mathrm{ad}^{-1}_{\,S} \ . \tag{10.41}$$

As a consequence, the following relation between the hierarchy of symplectic structures $\Omega^{(m)}$; $m \in \mathbb{Z}$ associated with L and the hierarchy of symplectic structures $\tilde{\Omega}^{(n)}$; $n \in \mathbb{Z}$ associated with $\tilde{L}$ can be established:

$$F^* \Omega^{(m)} = \tilde{\Omega}^{(m+2)} - \frac{i}{2} \sum_{k,l=1}^{r} \langle H_k, H_l \rangle d\Delta_l^{(m+1)} \wedge d\Delta_{(k)}(0) \ . \tag{10.42}$$

Here, as usual $F^*\Omega^{(m)} = \Omega^{(m)}(D(F)\cdot, D(F)\cdot)$ and the wedge $\wedge$ denotes the exterior product. As to the quantities $\Delta_k(0)$, it can be shown that assumption (10.27) is equivalent to $T(0) = D^+(0) = D^-(0)$, and then we put $\Delta_k(0) = \Delta_k^+(0) = \Delta_k^-(0)$.

Let us remind again that we shall refer to system (10.3) as generalized Zakharov-Shabat system (GZS) in canonical gauge and to system (10.24) as generalized Zakharov-Shabat system (GZS) in pole gauge.

With this, we finish our quick survey of the properties of the generating operators and the nonlinear evolution equations associated with the auxiliary linear problems L and $\tilde{L}$.

Let us say also some words about how this part of the book is organized.

- In Chap. 11, we remind some facts from the Differential Geometry and introduce the notation we use. We have planned the material in chapter 11 in such a way that the reader who is familiar with the basic facts and ideas of Differential Geometry will recall the main properties of the geometric objects and should be able to follow the argument in the next chapters, without constantly referring to other books. Some of the following chapters, however, are devoted to more specific geometric objects, and for that reason we discuss them in greater detail at the corresponding place. Such writing can be regarded as a part of our effort to make this book accessible to a larger audience and in general to make it more "readable." However, the mathematical methods used are of great variety and this complicates things. Together with the geometric methods, we use notions and results from the theory of the Lie algebras, especially the semisimple. Lie algebras. We were unable to give definitions of all these notions, as it would have increased considerably the volume of the book but refer the reader to some book on Lie algebras. Though the task of making the book self-consistent was not very hard to achieve in the first chapters, as the material develops, more and more things are needed and the final chapters are more difficult to follow. For that reason we tried to involve the knowledge of things not defined in the book as to say "at the last stage" so that at least the ideas could be easily understandable. But if something is wrong in the language or grammar you can change it preserving the meaning.
- In Chap. 12, we discuss the possibilities of defining Poisson brackets, the usual one, based on the existence of symplectic structure and the second one, based on the existence of Poisson structures. Here also introduce the properties of the fundamental fields of symplectic structures and Poisson structures and discuss briefly the problems of restriction of the symplectic structure and Poisson structure on submanifolds, a process that is sometimes called Hamiltonian reduction. Interesting and nontrivial examples of such restrictions are given in Sect. 12.2, which is dedicated to the discussion of the complexifications of real dynamical systems and their real forms. At the end of this chapter, in Sect. 12.4, we make a short introduction to the integrability of Hamiltonian systems and give the motivations for introducing a new geometric object – the generating

operator (the Nijenhuis tensor), which plays a central role in this part of the book.

- In Chap. 13, we develop the mathematical tools for the study of mixed tensor fields – the theory of the vector-valued differential forms. Then we proceed with the vector-valued differential forms of degree 1, introduce the important notion of Nijenhuis tensor (Nijenhuis structure), and give some applications of the general theory, such as the Haantjes theorem and the Nijenhuis theorem. These theorems treat the case when the Nijenhuis tensor (operator) is semisimple. In Sect. 13.3, we introduce the principal properties of the Nijenhuis tensors which will be used throughout this part of the book.
- In Chap. 14, we continue the discussion about the relation between the integrability and the Nijenhuis structures. We consider the case when the Nijenhuis operator (tensor) is not semisimple and give some applications. In 14.1 and in 14.3, we dedicate some attention to the finite dimensional case, when there exist "enough" integrals of motion, but they are not in involution. In Sect. 14.4, we consider the interrelation between Poisson structure and Nijenhuis structure. We introduce here the notion of compatible Poisson pairs (structures) and define what Poisson-Nijenhuis manifolds (P-N manifolds) are. Having in mind the applications in the next section, we also discuss here the construction of Nijenhuis tensor, provided we have two compatible Poisson tensors. We also discuss the principal properties of the P-N manifolds, their fundamental fields, the mechanism, the hierarchies of Poisson structures on a P-N manifold arise, and some other related topics.
- Chapter 15 is entirely devoted to the geometric theory of the equations associated with the generalized Zakharov-Shabat system. First, in Sect. 15.1, we apply the geometric theory developed in the previous chapter to the investigation of the Hamiltonian structures for the hierarchy of equations, associated with the generalized Zakharov-Shabat system (GZS) in canonical gauge. More specifically, we show that the manifold of potentials for GZS is P-N manifold. In the next section, Sect. 15.2, we discuss the properties of the so-called momentum map, which is a classical object in the Hamiltonian Dynamics. Here, it is applied to reveal the interrelation between P-N structures on Lie groups and P-N structures on the dual spaces of the corresponding Lie algebras. More specifically, we show that the two compatible Poisson tensors P^0 and Q^0 that generate the P-N structure of the equations associated with the GZS system arise on a certain coalgebra using a suitably chosen momentum map. We make some comparisons with other approaches and give a simple proof of the compatibility of P^0 and Q^0 in 15.2.3. In Sect. 15.3, we apply the theory of P-N structures defined on Lie groups and algebras and establish the interrelations between three remarkable manifolds: the manifold of potentials for the generalized Zakharov-Shabat system in canonical gauge, the manifold of potentials for the generalized Zakharov-Shabat system in pole gauge, and the manifold

of the Jost solutions for $\lambda = 0$ for GZS system in canonical gauge. At the end of this chapter, in Sect. 15.4, we find explicitly sets of fundamental fields for these structures. According to the general theory these fields give rise to the hierarchies of integrable systems on each of these manifolds, closely related between themselves. On the manifold of potentials for GZS system and on the manifold of potentials for GZS system in pole gauge these vector fields generate exactly (10.21) and (10.29).

- In Chap. 16, we consider the so-called linear bundles of Lie algebras, an algebraic concept that has attracted attention recently. Roughly speaking, this means that there exists of a set of Lie algebra structures on the same underlying vector space in such a manner that these structures form a vector space. The reason we consider the linear bundles of Lie algebras is that in a straightforward way they lead to compatible Poisson tensors, Sect. 16.2. We give some examples and applications here in Sect. 16.5 to the so-called Clebsh and Neumann systems. The reader who is interested only in the infinite-dimensional case can pass directly to Sect. 16.5.2, where the case of the algebra $\mathrm{o}\,(4)$ is considered. This case is very special and needed in Sect. 16.6, where the hierarchies of the so-called $O(3)$ Chiral System (CF) and the Landau-Lifshitz equation (LL) are considered. The recursion operators (Nijenhuis tensors) for CF hierarchy is obtained as an application of the same technique which was applied already to the hierarchies associated with GZS system.
- Finally, we give a short list of the abbreviations we are using.
- Appendix A is devoted to generalizations of the approaches we have presented. It is written in a somewhat different style from the rest of the book; it is rather concise. We have tried to illustrate the generalizations by examples rather than give all the appropriate definitions and theorems. We believe that both the specialists and nonspecialists in the field will find this chapter useful. In Sect. A.1, it is shown that it is possible to give purely algebraic formulation to many aspects of the theory of the classical dynamical systems by exploiting the existing duality between $\mathcal{F}(\mathcal{M})$ and $\mathcal{M}$, where $\mathcal{F}(\mathcal{M})$ is the ring of smooth functions over the manifold $\mathcal{M}$. This allows to develop an "abstract" approach to integrability of dynamical systems, choosing $\mathcal{F}$ to be simply an associative algebra. Naturally, the new constructions reduce to the familiar ones when $\mathcal{F}$ is realized as a ring of functions on a specific "carrier" manifold $\mathcal{M}$. An advantage of the algebraic approach is the ease with which we can add fermionic degrees of freedom, though of course one should be careful in assuming that results true in the bosonic situation hold true in the fermionic case too [31]. In Sect. A.2, the role of $(1,1)$ graded tensor field N that could play the role of a Nijenhuis tensor in the analysis of complete integrability of dynamical systems with fermionic variables is discussed. It is shown that such a tensor can be a recursion operator if and only if N as a graded map is even. We clarify this fact by constructing an odd tensor for two examples, the supersymmetric Toda chain and the supersymmetric harmonic oscillator.

We explicitly show that an odd tensor cannot be a recursion operator, since it does not allow to build new integrals of motion in contrast to what usually happens in the ordinary, i.e. nongraded systems.

Through this part we have tried to give (at the proper places) very short historical remarks about the origin and development of the geometric notions and ideas we are speaking about. However, the literature referring to these topics is so vast, and the approaches used so diverse that we have little hope that we have been able to do justice to these interesting topics. We have even less hope that we have been able to mention everybody who contributed significantly to the development of these ideas in the last decades, a circumstance that we regret and for which we apologize.

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11

Smooth Manifolds

In this chapter, we give some important facts about Differential Geometry and introduce the notation we use. In the literature, there are often differences either in the signs or in the coefficients for the expressions defining some basic operations and that is why some notes about them are indispensable. We could limit ourselves to listing the differences and refer the reader to some excellent books [1, 2, 3, 4] on Differential Geometry and Hamiltonian systems, introducing only the less common objects, as the Nijenhuis tensors, for example. However, since we try to bring together the spectral and the geometric approaches for the recursion operators, we have made an effort to give the reader all the information that will help them to read this part of the book independently, introducing gradually, first, the basic notions and facts from Differential Geometry, Symplectic Geometry and the Hamiltonian Systems Theory and then the more specific topics.

There is also another point that we would like to make clear from the beginning. The theory of the infinite-dimensional manifolds is far more intricate than the corresponding finite dimensional one and not so "classic" and not so "ideologically" clear. Besides, when one tries to perform some calculations, one immediately gets a little disappointed in the general theory, because just to carry out the simplest constructions one must restrict oneself to some class of spaces, usually Banach spaces – see for example [2] – and to check fulfilment of conditions that take one far from the original task and get one in the area of the Functional Analysis. Of course, one needs Banach spaces in order to have the Implicit Function Theorem, which is so important for Differential Geometry, but in the applications the spaces are usually not Banach spaces, and it can be even difficult to decide what spaces to take as tangent and cotangent spaces to some set that we want to treat as "manifold", in order that future constructions do not lead to discrepancies. Indeed, even in the case of the Zakharov–Shabat system, which except for the Sturm-Liouville eigenvalue problem is the best studied auxiliary linear problem, one is not able to give a clear geometric description of the so-called generic potentials to which, as assumed, belong the solutions of the corresponding evolution equations.

Gerdjikov, V.S. et al.: *Smooth Manifolds*. Lect. Notes Phys. **748**, 373–406 (2008)
DOI 10.1007/978-3-540-77054-1_11

In other words, we give the spectral properties of the fundamental solutions for these potentials, but before solving the linear problem we are not able to say if a given potential is "generic" or not. However, these potentials are the "points" in our manifold $\mathcal{M}$, from which one must start the geometric analysis. Some of the important facts, as for example, that the "generic" potentials are dense in some Banach space or Fréchet space are helpful indeed, but the rigorous theory will be difficult to construct. All this is added to the usual predicament that in the infinite-dimensional case, the vector spaces E we have are usually not reflexive ($(E^*)^* \neq E$), and the usual tensor calculus identifications, strictly speaking, cannot be used. In a rigorous approach all this must be taken into account. Such an approach (if possible) cannot be reconciliated with our view of how this book must be written, that is, the excessive rigor will only complicate the original ideas without adding to them new features and will restrict greatly the scope of questions we would like to discuss. So, we shall adopt quite a utilitarian viewpoint – we shall try to keep the analogy with the finite dimensional case as far as is possible and shall show how the things work mostly on the calculative level.

11.1 Basic Objects: Manifolds, Vector and Tensor Fields

The structure of infinite of dimensional manifold (over one of the classical fields $\mathbb{R}$ or $\mathbb{C}$) on a Hausdorff topological space $\mathcal{M}$ can be defined in analogy with the corresponding structure in the finite dimensional case with the help of a set of homeomorphisms $\varphi_i : U_i \to \varphi_i(U_i) \subset E$, where U_i are open subsets of $\mathcal{M}$ covering the space $\mathcal{M}$:

$$\bigcup_{i \in I} U_i = \mathcal{M} \tag{11.1}$$

and E is some "model" infinite-dimensional vector space. (Usually Banach space). The pair (U_i, φ_i) is called as usual a chart of $\mathcal{M}$ (or parametrization of $\mathcal{M}$) and U_i is called a parameterized neighborhoods of $\mathcal{M}$. The collection of the charts $\{(U_i, \varphi_i)\}_{i \in I}$ covering $\mathcal{M}$ is called an *atlas* of $\mathcal{M}$. Of course, in the case of a finite dimensional differentiable manifold of dimension n (without boundary) $E = \mathbb{R}^n$ and then

$$\varphi_i(m) = (x^1(m), x^2(m), \ldots, x^n(m)) \; . \tag{11.2}$$

The functions $m \mapsto x^i(m)$ are called local coordinates in the chart (local coordinate system, or frame) (U_i, φ_i).

Usually one imposes also the property of $\mathcal{M}$ to be paracompact, in order to be able to construct partition of the unity, subordinate to any cover – an important tool for "global" constructions; see for example [1, 3]. But we shall not enter here in such a detail.

In order to define the structure of differential manifold one must assume that the transition maps

$$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_j \cap U_i) \to \varphi_i(U_i \cap U_j) \tag{11.3}$$

are infinitely differentiable in the Fréchet sense (of course we consider those i, j for which $U_i \cap U_j \neq \emptyset$).

Recall that the Fréchet derivative $D(f)|_m$ of the map $f : U \to E_2$, where U is open in E_1 and E_1, E_2 are Banach or Fréchet spaces, is a continuous linear map $E_1 \mapsto E_2$, such that $f(m+h) - f(m)$ is approximated by $D(f)|_m h$ up to the terms of order higher than h, that is, in the Banach case for example

$$||f(m+h)f(m) - D(f)|_m h||_2 = o(||h||_1) \ , \tag{11.4}$$

where $||.||_i$ $i = 1, 2$ are the norms in E_1, E_2. When $E_1 = \mathbb{R}^n$, $E_2 = \mathbb{R}^m$, $D(f)|_m$ is the Jacobi matrix of f.

Since we do not want to enter into the subject of complex manifolds, we shall in general limit ourselves to the real case, that is, with real spaces E_1, E_2 (the single exception is Sect. 12.2). Even when the corresponding space consists of complex functions (vectors), we understand them as a couple of real ones – the real and imaginary part of the corresponding functions or the real and imaginary part of the corresponding vectors in some fixed basis.

Also, as already explained, we shall pay practically no attention to the topological aspects of the theory and concentrate on the rich calculus existing in Differential Geometry, so this is the right moment to discuss the derivatives that shall be used. Often in the literature instead of the Fréchet derivative $D(f)$ is used the so-called variational derivative – especially in the case when E_2 is $\mathbb{R}$ or $\mathbb{C}$. The interrelation of these two objects is the following: Suppose E is some functional space of functions $u(y)$ and let $F : E \to \mathbb{C}$,

$$F(u) = \int_{-\infty}^{+\infty} K(y, u(y), u'(y), \ldots u^{(n)}(y))dy \tag{11.5}$$

where K is differentiable in all its arguments. Then F is functional on E. In order to perform the necessary calculations, we assume that E consists of functions defined on the line, having the derivatives up to a certain order and decaying at infinity as fast as we need. To be even more explicit, let E consist of scalar functions. Then the usual variational derivative of F at the point $u \in E$ is equal to

$$\frac{\delta F}{\delta u} = \sum_{k=0}^{n} (-1)^k \frac{d^k}{dx^k} \frac{\partial K}{\partial u^{(k)}} \ , \tag{11.6}$$

that is, *it is a function.* On the other hand, the derivative $D(F)$ is the following *linear map* on E (provided it is continuous):

$$h \to D(F)|_u(h) = \int\limits_{-\infty}^{+\infty} \frac{\delta K}{\delta u}(y)h(y)dy\ . \tag{11.7}$$

However, often the continuous linear functionals on E, that is, the elements of E^*, can be embedded into some other functional space $\bar{E}$. It is the case when there is some natural pairing (bilinear continuous map)

$$B : E \times \bar{E} \to \mathbb{C}\ . \tag{11.8}$$

(In the above example, it is the integral of the product of two functions). Then if $\alpha \in E^*$ and there exists unique $\varphi \in \bar{E}$ such that

$$\alpha(h) = B(h, \varphi) \tag{11.9}$$

for all $h \in E$, we identify α and φ. We also can speak about the identification of α and φ if we are interested not in $\alpha(h)$ for all possible $h \in E$ but only in h that belong to some subspace of E, etc. The possibilities here may be various but the idea is quite clear, and in case we identify functional from E^* with vector from $\bar{E}$, the variational derivative coincides with the Fréchet derivative in the sense we explained. For this reason, we shall use the variational derivative symbol in the case when the pairing B is natural, and when we believe that this will simplify the notation. One can also adopt the equivalent viewpoint that the derivative $\frac{\delta F}{\delta u}$ is generalized function (distribution) and then it coincides with $D(F)$.

Returning to the general definitions, we recall that the differentiable maps between smooth manifolds are introduced in complete analogy to the finite dimensional case, that is, the corresponding map is differentiable if its expressions (its representatives) in the charts are differentiable. In more detail, let $h : \mathcal{M} \to \mathcal{N}$ is continuous at $m \in \mathcal{M}$. Let (U, φ) is a chart at the neighborhood U of $m \in \mathcal{M}$, and (V, ψ) is a chart of $\mathcal{N}$ at the neighborhood V of the point $h(m)$, $\psi : V \to F$. Then the map h is called differentiable at m if its representative in these charts, namely the map

$$\psi \circ h \circ (\varphi)^{-1} : E \to F \tag{11.10}$$

(in its natural domain) is differentiable at the point $\varphi(m)$. Map h is called differentiable (smooth) if it is differentiable at each point. Below, we shall assume that all the maps are C^∞ differentiable.

Special case of manifolds are the Lie groups. Lie group G is a manifold, such that at the same time it is a group, and the group operations are smooth. In other words, if $(g, h) \mapsto gh$ is the group operation in G the map $M : G{\times}G \mapsto G$

$$M(g, h) = gh^{-1}; \quad g, h \in G \tag{11.11}$$

is smooth. As easily seen, for each fixed $g \in G$ the maps $L_g : G \mapsto G$ and $R_g : G \mapsto G$, as well as the map $I_G : G \mapsto G$:

$$L_g(h) = gh, \quad R_g h = hg, \quad I_G(h) = h^{-1} \tag{11.12}$$

are smooth maps. L_g are called left translations of the group, R_g right translations of the group and I_G is called the inversion of the group G.

As in the finite dimensional case a tangent vector at the point $m \in \mathcal{M}$ is defined through its representatives in the corresponding system of charts – $X_{i,m} \in E$, $m \in U_i$. Two representatives $X_{i,m}, X_{j,p}$ correspond to the same vector X_m if $p = m$ and

$$X_{i,m} = D(\varphi_{ij})|_m(X_{j,m}); \quad i, j \in I . \tag{11.13}$$

The tangent space $T_m(\mathcal{M})$ is then the set of all tangent vectors at the point $m \in \mathcal{M}$ and naturally is a vector space. For the sake of brevity it sometimes is denoted simply by T_m.

In a similar way, one can define the cotangent space $T^*_m(\mathcal{M})$ (sometimes simply denoted by T^*_m), that is, the set of covectors at the point $m \in \mathcal{M}$. Indeed, let us consider the family of covectors

$$\alpha_{i,m} \in E^*, \quad i \in I, \quad m \in U_i,$$

where E^* is the dual space of E, or in other words the set of continuous linear maps $E \to \mathbb{R}$ (if the vector space E is over $\mathbb{R}$). Let $D^*(\varphi_{ji})|_m$ be the dual (adjoint) map of $D(\varphi_{ij})|_m$, then

$$D^*(\varphi_{ij})|_m : E^* \to E^*.$$

If

$$\alpha_{i,m} = D^*(\varphi_{ji})|_m(\alpha_{j,m}); \quad i, j \in I \tag{11.14}$$

we say that $\alpha_{i,m}$ is the representative of the covector α_m in the chart (U_i, φ_i). It is readily seen that $\langle \alpha_{i,m}, X_{i,m} \rangle$ does not depend on the chart, and therefore one can put

$$\langle \alpha_{i,m}, X_{i,m} \rangle = \langle \alpha_m, X_m \rangle . \tag{11.15}$$

(In this formula $\langle\ ,\ \rangle$ denotes the canonical pairing between E and E^*). In a similar way one can define the space of (r, s) tensors (r-times contravariant and s-times contravariant) at the point $m \in \mathcal{M}$ as elements from the tensor product

$$T^{(r,s)}_m(\mathcal{M}) = \underbrace{T_m \otimes T_m \otimes \ldots \otimes T_m}_{r \text{ times}} \underbrace{\otimes T^*_m \otimes T^*_m \otimes \ldots \otimes T^*_m}_{s \text{ times}} . \tag{11.16}$$

If R_m is (r, s)-type tensor at the point $m \in \mathcal{M}$ then its representatives $R_{i,m}$ are tensors from the tensor product

$$E^{(r,s)} = \underbrace{E \otimes E \otimes \ldots \otimes E}_{r \text{ times}} \underbrace{\otimes E^* \otimes E^* \otimes \ldots \otimes E^*}_{s \text{ times}} \tag{11.17}$$

and in the different charts are related as follows:

$$R_{i,m} = [A_{m,ij} \otimes B_{m,ij}](R_{j,m}); \quad i, j \in I \tag{11.18}$$
$$A_{m,ij} = \underbrace{D(\varphi_{ij})|_m \otimes D(\varphi_{ij})|_m \otimes \ldots \otimes D(\varphi_{ij})|_m}_{r \text{ times}}$$
$$B_{m,ij} = \underbrace{D^*(\varphi_{ji})|_m \otimes D^*(\varphi_{ji})|_m \otimes \ldots \otimes D^*(\varphi_{ji})|_m}_{s \text{ times}} .$$

The corresponding tangent and the cotangent bundles, as well as the corresponding tensor bundles over an infinite-dimensional manifold $\mathcal{M}$, can also be introduced as in the finite dimensional case. For example, consider the space

$$T(\mathcal{M}) = \{(m, \xi_m) : m \in \mathcal{M},\ \xi_m \in T_m(\mathcal{M})\} . \tag{11.19}$$

If p_M is the projection $p_M(m, \xi_m) = m$ then the charts of $T(\mathcal{M})$ are defined in the following way: Suppose that (U_i, φ_i) is a chart of $\mathcal{M}$ about the point m. Let us put $W_i = p_M^{-1}(U_i)$ and let Ψ_i be the map

$$\begin{aligned} &\Psi_i : W_i \to U_i \times E \\ &\Psi_i(m, \xi_m) = (m, \xi_{i,m}) . \end{aligned} \tag{11.20}$$

The maps Ψ_i are called local trivializations of $T(\mathcal{M})$ at W_i, as locally $T(\mathcal{M})$ is identified with Cartesian product of two spaces. In unique way, the space $T(\mathcal{M})$ can be endowed with a topology in which all the maps Ψ_i are homeomorphisms over their images. Clearly,

$$\bigcup_{i \in I} W_i = T(\mathcal{M}) . \tag{11.21}$$

Besides, from the definition of representative $\xi_{i,m}$, it follows that if $W_i \cap W_j \neq \emptyset$ the maps $\Psi_i \circ \Psi_j^{-1}$ and $\Psi_j \circ \Psi_i^{-1}$ are smooth. Now recall that there is the following classical result; see for example [3]:

Theorem 11.1. *Let $\mathcal{N}$ be topological manifold and there exists a set of local homeomorphisms $\Psi_i : W_i \to N_i$ into the manifolds N_i, where $W_i \subset \mathcal{N}$ are open and*

$$\bigcup_{i \in J} W_i = \mathcal{N} . \tag{11.22}$$

Suppose that for every pair of indices (i, j) the maps $\Psi_i \circ \Psi_j^{-1}$ are differentiable. Then on $\mathcal{N}$ there exists unique structure of smooth manifold such that all the maps $\Psi_i : W_i \to N_i$ are diffeomorphisms.

Via this theorem, $T(\mathcal{M})$ is endowed with unique structure of differential manifold and the trivializations of $T(\mathcal{M})$ are local diffeomorphisms. The charts of $T(\mathcal{M})$ are constructed from the local trivializations Ψ_i and the charts of $\mathcal{M}$. We put

$$\Phi_i = (\varphi_i \oplus \mathbf{1}_E) \circ \Psi_i , \tag{11.23}$$

or in other words

$$\Phi_i(m, \xi_m) = (\varphi_i(m), \xi_{i,m}) . \tag{11.24}$$

The manifold $T(\mathcal{M})$ is called total space of the tangent bundle $\tau(\mathcal{M})$, where $\tau(\mathcal{M})$ is the triple

$$\tau(\mathcal{M}) = (T(\mathcal{M}), p_M, \mathcal{M}) \tag{11.25}$$

(tangent bundle with total space $T(\mathcal{M})$, base $\mathcal{M}$ and projection p_M).

The same construction is applied almost without any changes to obtain the cotangent bundle:

$$\tau^*(\mathcal{M}) = (T^*(\mathcal{M}), q_M, \mathcal{M}) \tag{11.26}$$

(cotangent bundle with total space $T^*(\mathcal{M})$, base $\mathcal{M}$ and projection q_M). Here

$$T^*(\mathcal{M}) = \{(m, \alpha_m) : m \in \mathcal{M},\ \alpha_m \in T^*_m(\mathcal{M})\} , \tag{11.27}$$

q_M is the projection $q_M(m, \alpha_m) = m$ and the trivializations $\bar{\Psi}_i$ are defined in the following way:

$$\begin{aligned} &\bar{\Psi}_i : q_M^{-1}(U_i) = \bar{W}_i \to U_i \times E^* \\ &\bar{\Psi}_i(m, \alpha_m) = (m, \alpha_{i,m}) . \end{aligned} \tag{11.28}$$

As to the charts of $T^*(\mathcal{M})$, they are introduced as follows:

$$\bar{\Phi}_i = (\varphi_i \oplus \mathbf{1}_{E^*}) \circ \bar{\Psi}_i , \tag{11.29}$$

or equivalently

$$\bar{\Phi}_i(m, \alpha_m) = (\varphi_i(m), \alpha_{i,m}) . \tag{11.30}$$

The tensor bundless of tensors of arbitrary type (p, q) are defined in a similar way.

Smooth vector fields, differential forms, and in general tensor fields are considered as smooth sections of the corresponding vector bundles with base $\mathcal{M}$, see below. For example, smooth vector field X over the smooth manifold $\mathcal{M}$ is a smooth map

$$X : \mathcal{M} \to T(\mathcal{M}), \quad p_M \circ X = id_{\mathcal{M}} . \tag{11.31}$$

(We say that the map X is a smooth section of $\tau(\mathcal{M})$ over $\mathcal{M}$).

Remark 11.2. Since all the objects we consider are smooth (differentiable) the words smooth (differentiable) are sometimes omitted in the future.

We also denote the value of the vector field X at $m \in \mathcal{M}$ by $X(m)$ or by $X|_m$. As one can see from the definition, a vector field X is represented in the local charts by the differentiable functions

$$m \to X_{i,m} \in E , \tag{11.32}$$

such that (11.13) are true.

In the same manner, a differential 1-form α (also simply 1-form, linear form or field of covectors) is defined as a smooth section of the cotangent bundle, that is, as a smooth map

$$\alpha : \mathcal{M} \to T^*(\mathcal{M}), \quad q_M \circ \alpha = id_{\mathcal{M}} \,. \tag{11.33}$$

We shall denote the value of the 1-form α at $m \in \mathcal{M}$ by $\alpha(m)$ or by $\alpha|_m$. Also, one can see from the definition that a 1-form α is represented in some local chart by the differentiable functions

$$m \to \alpha_{i,m} \in E^* \,, \tag{11.34}$$

such that (11.14) are satisfied.

The definitions of smooth tensor fields are similar – they can be represented as set of differentiable functions

$$m \to R_{i,m} \in E^{(r,s)} \,, \tag{11.35}$$

such that (11.18) are satisfied.

11.2 Basic Operations with Tensor Fields

In future, the set of all smooth vector fields on $\mathcal{M}$ shall be denoted by $\mathcal{T}(\mathcal{M})$ and the set of all smooth differential 1-forms (or simply smooth 1-forms) on $\mathcal{M}$ by $\Lambda^1(\mathcal{M})$. Further, we denote by $\mathcal{D}(\mathcal{M})$ the ring of all smooth functions on $\mathcal{M}$. Then $\mathcal{T}(\mathcal{M})$ and $\Lambda^1(\mathcal{M})$ can be regarded as modules over $\mathcal{D}(\mathcal{M})$. Another important module is the module of the p-times covariant skew-symmetric tensors, also called the module of (scalar) differential p-forms (forms of degree p) over $\mathcal{M}$. This module shall be denoted by $\Lambda^p(\mathcal{M})$. The operations in the above algebraical structures are defined in natural way – pointwise. If for example X and Y are vector fields: $(X, Y \in \mathcal{T}(\mathcal{M}))$, f is smooth function $(f \in \mathcal{D}(\mathcal{M}))$ and α, β are p-forms $(\alpha, \beta \in \Lambda^p(\mathcal{M}))$ then

$$\begin{aligned} &(X+Y)(m) = X(m) + Y(m); \quad m \in \mathcal{M} \\ &fX(m) = f(m)X(m); \quad m \in \mathcal{M} \\ &(\alpha + \beta)(m) = \alpha(m) + \beta(m); \quad m \in \mathcal{M} \,. \end{aligned} \tag{11.36}$$

In case we are considering p-forms, there is one additional and very important algebraic operation – the exterior product. It is induced by the corresponding operation defined on $\mathbf{A}^p(E) \subset E^{(0,p)}$; $p = 1, 2, \ldots$, where $\mathbf{A}^p(E)$ are the spaces of p-linear skew-symmetric forms on the "model" vector space E. If $\mathbb{K}$ is the field of numbers, then $\beta \in \mathbf{A}^p(E)$ is a function

$$\beta : (x_1, x_2, \ldots, x_p) \to \beta(x_1, x_2, \ldots, x_p) \in \mathbb{K} \tag{11.37}$$

such that

1. It is linear in each of its arguments.
2. For each σ belonging to the group of permutations G_p of the elements $(1, 2, \ldots, p)$, we have

$$\beta(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(p)}) = \epsilon(\sigma)\beta(x_1, x_2, \ldots, x_p) , \tag{11.38}$$

where $\epsilon(\sigma)$ is the parity of the permutation σ.

By definition $\mathbf{A}^0(E) = \mathbb{K}$.

The exterior product (wedge product "$\wedge$") is then the bilinear operation

$$\wedge : \mathbf{A}^p \times \mathbf{A}^q \to \mathbf{A}^{(p+q)} . \tag{11.39}$$

defined in the following way: For $\alpha \in \mathbf{A}^p, \beta \in \mathbf{A}^q$; $p, q \geq 1$, the wedge product $\alpha \wedge \beta$ of α and β is a $p+q$ form such that if x_i; $i = 1, 2, \ldots, p+q$ are elements of E we have:

$$\begin{aligned} &\alpha \wedge \beta(x_1, x_2, \ldots, x_{p+q}) = \\ &\frac{1}{p!q!} \sum_{\sigma \in G_{p+q}} \epsilon(\sigma)\alpha(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(p)})\beta(x_{\sigma(p+1)}, x_{\sigma(p+2)}, \ldots, x_{\sigma(p+q)}) , \end{aligned} \tag{11.40}$$

where G_{p+q} is the group of the permutations of the elements $1, 2, \ldots, p+q$. For $p = 0$ we set

$$\alpha \wedge \beta = \alpha\beta . \tag{11.41}$$

Then with respect to the wedge product, the module

$$\mathbf{A}(E) = \overset{\infty}{\underset{k=0}{\oplus}} \mathbf{A}^k(E) , \tag{11.42}$$

is an algebra – the algebra of the exterior forms or simply the exterior algebra.[1] This algebra is graded anticommutative associative algebra. Graded, because of (11.39), anticommutative due to the identity:

$$\alpha \wedge \beta = (-1)^{pq}\beta \wedge \alpha; \quad \alpha \in \mathbf{A}^p(E), \beta \in \mathbf{A}^q(E) , \tag{11.43}$$

and associative, because for any forms α, β, γ

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma) . \tag{11.44}$$

We introduce now another property of the wedge product which is used frequently. Suppose that i_x denotes the contraction with element $x \in E$ (the interior product) $i_x : \mathbf{A}^p(E) \to \mathbf{A}^{p-1}(E)$ defined as:

[1] When E is finite dimensional vector space with dimension r then for $k > r$ we have $\mathbf{A}^k(E) = 0$, and the direct sum is finite. If E is infinite the sum is also infinite.

$$
\begin{aligned}
&(i_x\beta)(x_1, x_2, \ldots, x_{p-1}) = \beta(x, x_1, x_2, \ldots, x_{p-1}); \quad \beta \in \mathbf{A}^p(E), p > 1 \\
&i_x\beta = 0, \quad \beta \in \mathbf{A}^0(E) \; .
\end{aligned} \tag{11.45}
$$

Then for $\alpha \in \mathbf{A}^p(E)$, $\beta \in \mathbf{A}^q(E)$, $x \in E$

$$
i_x(\alpha \wedge \beta) = (i_x\alpha) \wedge \beta + (-1)^p \alpha \wedge i_x\beta \; . \tag{11.46}
$$

Remark 11.3. The notation $x\rfloor\alpha$ instead of $i_x\alpha$ is also frequently used. More generally, by $R\rfloor S$ is denoted the contraction between an $(r, 0)$ tensor R and $(s, r+k)$ tensor S involving the first r covariant indices of S. Of course, $R\rfloor S$ is then a (s, k) tensor.

The pointwise wedge product operation defines the wedge product in the module

$$
\Lambda(\mathcal{M}) = \mathop{\oplus}_{k=0}^{\infty} \Lambda^k(\mathcal{M}) \; , \tag{11.47}
$$

thus endowing it with an algebra structure. It is called the algebra of differential forms over $\mathcal{M}$ or also simply the exterior algebra over $\mathcal{M}$. By definition $\Lambda^0(\mathcal{M}) = \mathcal{D}(\mathcal{M})$. The formula (11.43) is then true again for α and β being now a p-form and a q-form, respectively. The interior product (11.45) has also an analog – the interior product i_X of the algebra $\Lambda(\mathcal{M})$ with the same properties as (11.45). In more detail, if γ is p-form, X – vector field, then $i_X\gamma$ is $(p-1)$-form defined by

$$
i_X\gamma(X_1, X_2, \ldots, X_{p-1}) = \gamma(X, X_1, X_2, \ldots, X_{p-1}) \; . \tag{11.48}
$$

For 0-form (function) we have by definition $i_X f = 0$. The equality (11.46) is still true but now α and β must be p-form and q-form, respectively, and one must put instead of element $x \in E$ the vector field X. The formula (12.199) is also true for any three differential forms. Of course, in all this formulae α, β, γ, X are fields on the same manifold $\mathcal{M}$.

From the above, we conclude that the algebra of the differential forms $\Lambda(\mathcal{M})$ is graded anticommutative associative algebra.

In a similar way, one can also define the graded algebra of the contravariant skew-symmetric tensors (the product in it is also denoted by "$\wedge$") but we shall use this operation only in some formulae, in order to write them more concisely.

Some additional operations with vector fields and differential forms arise if one uses the tangent map. In order to introduce it, let $h : \mathcal{M} \to \mathcal{N}$ be a differentiable map between the manifolds $\mathcal{M}$ and $\mathcal{N}$. Then dh_m is a linear map between $T_m(\mathcal{M})$ and $T_{h(m)}(\mathcal{N})$. It is most easily defined, if one considers how it acts on the representatives of the vectors in the charts of $\mathcal{M}$ and $\mathcal{N}$. If (U_i, φ_i) and (V_k, ψ_k) are charts of $\mathcal{M}$ and $\mathcal{N}$, respectively, mapping the open sets U_i and V_k onto the model spaces E and F then for $m \in U_i$, $h(m) \in V_k$, we set

$$
(dh_m(X_m))_{k,h(m)} = D(\psi_k \circ h \circ \varphi_i^{-1})|_{\varphi_i(m)}(X_{i,m}) \; . \tag{11.49}
$$

It follows that

$$\begin{aligned} dh_m : \quad & T_m(\mathcal{M}) \to T_{h(m)}(\mathcal{N}) \\ dh_m^* : \quad & T_{h(m)}^*(\mathcal{N}) \to T_m^*(\mathcal{N}) \ . \end{aligned} \tag{11.50}$$

The linear maps dh_m are defined for each $m \in \mathcal{M}$ and induce the map h^T (called a tangent map) between the tangent bundles $\tau(\mathcal{M})$ and $\tau(\mathcal{N})$, such that the following diagram is commutative:

$$\begin{array}{ccc} & h^T & \\ T(\mathcal{M}) & \to & T(\mathcal{N}) \\ p_M \quad \downarrow & & \downarrow \quad p_N \\ & h & \\ \mathcal{M} & \to & \mathcal{N} \end{array} \tag{11.51}$$

In other words,

$$T(\mathcal{M}) \ni (m, \xi_m) \to h^T(m, \xi_m) = (h(m), dh_m(\xi_m)) \in T(\mathcal{N}) \ . \tag{11.52}$$

Definition 11.4. *If h is a diffeomorphism and X is a vector field, $h^T \circ X \circ h^{-1}$ is also a vector field. X is called invariant under h if $h^T \circ X \circ h^{-1} = X$.*

The tangent map also induces a map $h^* : \Lambda^p(\mathcal{N}) \to \Lambda^p(\mathcal{M})$, defined as follows. Suppose that $X_i \in \mathcal{T}(\mathcal{M})$, $i = 1, 2, \ldots p$, and $\alpha \in \Lambda^p(\mathcal{N})$. Then for arbitrary $m \in \mathcal{M}$,

$$\begin{aligned} & [h^* \alpha(X_1, X_2, \ldots, X_p)] (m) = \\ & \alpha|_m(dh_m(X_1|_m), dh_m(X_2|_m), \ldots, dh_m(X_p|_m)) \ . \end{aligned} \tag{11.53}$$

The map h^* is a homomorphism $\Lambda(\mathcal{N}) \to \Lambda(\mathcal{M})$, that is, for $\alpha, \beta \in \Lambda(\mathcal{N})$

$$h^*(\alpha \wedge \beta) = (h^* \alpha) \wedge (h^* \beta) \ , \tag{11.54}$$

and is called the pull-back map.

Definition 11.5. *If $h : \mathcal{M} \mapsto \mathcal{M}$ is a diffeomorphism, a form $\omega \in \Lambda^p(\mathcal{M})$ is called invariant under h if $h^* \omega = \omega$.*

The tangent maps in the case when $\mathcal{N}$ is the field of the scalars $\mathbb{R}$ are of particular importance. If $f : \mathcal{M} \to \mathbb{R}$, then df_m maps $T_m(\mathcal{M})$ into $\mathbb{R}$. With its help one can introduce the directional derivative $Xf = X(f)$ of f along the vector field X (also denoted by $L_X f$ and called the Lie derivative of the function f):

$$[X(f)](m) = [Xf](m) = [L_X f](m) = \langle df_m, X_m \rangle \ . \tag{11.55}$$

Then the notion of the Lie brackets of two vector fields X and Y can be introduced in the standard way:

$$[X,Y]_{i,m} = D\left(Y_{i,\varphi_i^{-1}(x)}\right)\big|_{x=\varphi_i(m)}(X_{i,m}) - D\left(X_{i,\varphi_i^{-1}(x)}\right)\big|_{x=\varphi_i(m)}(Y_{i,m})\ , \tag{11.56}$$

and it is easy to see that for each function f on the manifold $\mathcal{M}$, and for each two vector fields X and Y, we have

$$[X,Y]f = X(Yf) - Y(Xf)\ . \tag{11.57}$$

The Lie bracket endows the vector space $\mathcal{T}(\mathcal{M})$ with Lie algebra structure, as can be seen from the following properties:

$$\begin{aligned} &1.\ \ [aX,Y] = a[X,Y] \quad X \in \mathcal{T}(\mathcal{M}), \quad a \in \mathbb{R} \\ &2.\ \ [X,Y] = -[Y,X], \quad X,Y \in \mathcal{T}(\mathcal{M}) \\ &3.\ \ [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0\ , \end{aligned} \tag{11.58}$$

for any $X,Y,Z \in \mathcal{T}(\mathcal{M})$. Actually the first of these equations is a particular case of more general relation

$$[X,fY] = f[X,Y] + (Xf)Y; \quad f \in \mathcal{D}(\mathcal{M})\ , \tag{11.59}$$

which can be easily proved.

Since it is important for some generalizations, we shall discuss it at the end of this book, but we also remind that a vector field X over a real manifold $\mathcal{M}$ can be considered as a derivation of the ring of the differentiable functions $\mathcal{D}(\mathcal{M})$. Indeed, derivation D of $\mathcal{D}(\mathcal{M})$ is defined as $\mathbb{R}$-linear map $\mathcal{D}(\mathcal{M}) \mapsto \mathcal{D}(\mathcal{M})$, for which

$$D(f.g) = D(f).g + f.D(g); \quad f,g \in \mathcal{D}(\mathcal{M}) \tag{11.60}$$

$$D(f)(m) = 0, \quad \text{if } f = \text{const} \text{ in some neighborhood of } m\ . \tag{11.61}$$

The derivations of $\mathcal{D}(\mathcal{M})$ do not form an associative algebra but form a Lie algebra with respect to the commutator.

If X is a smooth vector field $f \mapsto L_X(f) = Xf$ as easily checked is a derivation. In the finite dimensional case, the converse is also true, that is, each derivation D of $\mathcal{D}(\mathcal{M})$ is of the form L_Y where Y is a smooth vector field. Thus (11.57) shows that the correspondence between the derivations and vector fields is a Lie algebra isomorphism between the derivations (as Lie algebra with respect to the commutator) and the vector fields (with respect to the Lie bracket).

Further, as we have already defined the Lie bracket of vector fields the exterior derivative (Cartan derivative) $d\omega$ of p-form ω is defined by the formula:

$$\begin{aligned} &d\omega(X_1, X_2, \ldots, X_{p+1}) = \\ &\sum_{i=1}^{p+1} (-1)^{i-1} X_i \left(\omega(X_1, X_2, \ldots, \hat{X}_i, \ldots, X_{p+1})\right) + \\ &\sum_{i<j} (-1)^{(i+j)} \omega([X_i, X_j], X_1, X_2, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1})\ . \end{aligned} \tag{11.62}$$

Here, X_n are vector fields, and the hat over the symbol of the vector field means that in the corresponding expression this field must be omitted. Thus, $d\omega$ is $(p+1)$-form, and it is easy to see that d has the well-known coboundary property $d^2 = 0$. We have then the usual definitions of closed and exact forms:

Definition 11.6. *The p-form α is called closed if $d\alpha = 0$.*

Definition 11.7. *The p-form α is called exact if there exists (p-1)-form β, such that $d\beta = \alpha$.*

From $d^2 = 0$ follows that each exact form is closed, and one can introduce cohomologies. The closed p forms are the p-cocycles $Z^p(\mathcal{M})$, the exact forms are the coboundaries $B^p(\mathcal{M})$, and the de Rham cohomologies are defined as the quotient spaces

$$H^p(\mathcal{M}) = Z^p(\mathcal{M})/B^p(\mathcal{M}) \ . \tag{11.63}$$

If $H^p(\mathcal{M}) = 0$ each closed form is exact, but generally speaking this is not true. Locally, however, it is always true. In the finite dimensional case, the famous Poincaré lemma states that for each point there exists an open neighborhood of it, such that on it each closed form is exact. It has also analog in the infinite-dimensional case [2].

A very important property of the exterior product is that d commutes with the pull-back map h^*. In other words if $h : \mathcal{M} \to \mathcal{N}$ is differentiable map between two manifolds then

$$d \circ h^* = h^* \circ d \ . \tag{11.64}$$

(Of course, one of these ds acts on the forms over $\mathcal{M}$ and the other on the forms over $\mathcal{N}$). The exterior derivative already defined, we can define the Lie derivative of p-form ω by the formula

$$L_X\omega = i_X d\omega + d i_X \omega \ . \tag{11.65}$$

It is readily seen that from the definition it follows

$$d \circ L_X - L_X \circ d = 0 \ . \tag{11.66}$$

After we have defined the Lie derivative for differential forms, we define it on vector fields as

$$L_X(Y) = [X, Y] \tag{11.67}$$

and then extend this operation on tensor fields using the requirement that L_X behaves like differentiation with respect to the tensor product and the contraction between tensor fields. All this procedure is quite classic; see for example [1, 5], where it is applied in the finite dimensional case.

Example 11.8. Let $m \to S_m$ be $(1,1)$ tensor field, such tensor fields we shall call also mixed tensor fields. Then for each $X \in \mathcal{T}(\mathcal{M})$ the contraction $X\rfloor S$ is a vector field, more precisely:

$$m \to (X \rfloor S)(m) \tag{11.68}$$

is vector field. (This is the reason the $(1,1)$ tensor fields are often referred as fields of operators and $X \rfloor S$ is written also as $S(X)$ or $S.X$). Then for all $X, Y \in \mathcal{T}(\mathcal{M})$

$$L_Y(S(X)) = L_Y(S(X)) = L_Y(S)(X) + S(L_Y(X)) \,. \tag{11.69}$$

Hence

$$L_Y(S)(X) = [Y, S(X)] - S([Y,X]), \quad X, Y \in \mathcal{T}(\mathcal{M}) \,, \tag{11.70}$$

and usually this equation is used if one wants to find $L_Y S$.

The Lie derivative has also the following useful properties: For any vector fields X and Y

$$\begin{aligned} [L_X, i_Y] &= L_X \circ i_Y - i_Y \circ L_X = i_{[X,Y]} \\ [L_X, L_Y] &= L_X \circ L_Y - L_Y \circ L_X = L_{[X,Y]} \,. \end{aligned} \tag{11.71}$$

(Of course, in the first of these formulae is implied that L_X and i_Y act on the module of differential forms $\Lambda(\mathcal{M})$).

As mentioned already, we purposely avoid topological aspects and concentrate only on "calculus." However, there are some constructions that are difficult to generalize, and here we must say a few words about it. As mentioned, the infinite-dimensional vector spaces often are not reflexive. This means that if E is a topological vector space one can no more identify E^{**} and E. Therefore, a continuous linear map on E^* is no more necessarily element from E. Due to this circumstance one must be cautious, as definitions which are equivalent in the finite dimensional case may be essentially different in the infinite-dimensional case. For example, if R is continuous linear operator then in the finite dimensional case it is identified with tensor $\hat{R} \in E^{(1,1)}$ and vice-versa. The relation between the two objects is the following

$$\langle \alpha, R(x) \rangle = \hat{R}(x, \alpha); \quad x \in E, \ \alpha \in E^* \,. \tag{11.72}$$

In the above formula the tensor $\hat{R}$ is understood as bilinear map

$$\hat{R} : E \times E^* \to \mathbb{K} \,, \tag{11.73}$$

where $\mathbb{K}$ is one of the classical fields ($\mathbb{R}$ or $\mathbb{C}$) and $\langle \ , \ \rangle$ is the canonical pairing between E^* and E. In the infinite-dimensional case, provided that $\hat{R}$ is given, from (11.72) one can define only the linear map $x \to R(x) \in E^{**}$ and $E^{**} \neq E$. There are also some other difficulties; see [2, 4].

11.3 Local Flows

If X is a vector field over $\mathcal{M}$ the smooth curve $\gamma : I \to \mathcal{M}$ (here I is an interval on the real axis) is called integral curve for X if

$$\frac{d\gamma}{dt} = d\gamma_t(1) = X(\gamma(t)) \ . \tag{11.74}$$

The system of the differential equations (11.74) is determined by X and vice versa. For this reason, we shall call the field X dynamical system, and the above equations – the equations corresponding to the dynamical system X. In other words, in the future, we identify the vector fields and the systems of differential equations that correspond to them. The points m for which $X(m) = 0$ are called critical points of the dynamical system X.

From the local existence and uniqueness theorem about the solution of differential equation on E (again we deliberately do not speak about the restrictions one must impose in order to have such theorem for X; see [2] for such a discussion) it follows that at least locally (in some neighborhood of each point $m \in \mathcal{M}$) there exists unique maximal solution of the differential equation (11.74) such that $\gamma(0) = m$. Let us denote this solution by $\varphi_t(m)$. It is known then that from the smooth dependence of the solution of the differential equation (11.74) on the initial conditions we obtain some additional properties of $\varphi_t(m)$. We collect all of them in the following theorem [2, 3, 4]:

Theorem 11.9. *The map $(t, m) \to \varphi_t(m)$ is defined on the open subset V of $\mathbb{R} \times \mathcal{M}$, such that $(\mathbb{R} \times \{m\}) \cap V$ is of the type $I_m \times \{m\}$ where $I_m \subset \mathbb{R}$ is interval containing the zero. The function φ_t satisfies:*

$$\frac{d\varphi_t}{dt}(m) = X(\varphi_t(m)), \quad t \in I_m \ . \tag{11.75}$$

and has the properties

$$\begin{aligned} &\varphi_0 = id_{\mathcal{M}} \\ &\varphi_{t_1+t_2} = \varphi_{t_1} \circ \varphi_{t_2}, \quad t_1, t_2 \in \mathbb{R} \\ &\varphi_t^{-1} = \varphi_{-t}, \quad t \in \mathbb{R} \ . \end{aligned} \tag{11.76}$$

(All the equations are true for those m and t for which the corresponding maps are defined).

Definition 11.10. *The map $\varphi_t(m)$ is called local 1-parametric group (of diffeomorphisms) generated by the vector field X or local flow of X.*

Remark 11.11. Often the local flow is denoted by $\exp tX(m)$.

Definition 11.12. *A vector field X over $\mathcal{M}$ is called complete, if $\varphi_t(m)$ is defined for each $m \in \mathcal{M}$ and $t \in \mathbb{R}$.*

If the manifold $\mathcal{M}$ is finite dimensional and compact, each field X over it is complete.

Suppose that $\alpha \in \Lambda(\mathcal{M})$, then one has the following important relations:

$$\lim_{t \to 0} \frac{1}{t}(\varphi_t^* \alpha - \alpha)|_m = L_X \alpha|_m \ . \tag{11.77}$$

Also, if Y is vector field then

$$\lim_{t \to 0} \frac{1}{t}(\varphi_{-t}^T \circ Y \circ \varphi_t - Y)|_m = [X, Y]|_m \ . \tag{11.78}$$

These relations make natural the definitions that will follow.

Definition 11.13. *The function f is called a first integral, constant of motion or conservation law for the vector field X (the dynamical system X) if $L_X f = Xf = 0$. The closed 1-form α is called first integral (sometimes generalized first integral) of the vector field X (dynamical system) if $i_X \alpha = 0$.*

If the closed form α is first integral, then locally, in some neighborhood $U \subset \mathcal{M}$ of the point m there exists $F \in \mathcal{D}(U)$, such that $\alpha|_U = dF$. From (11.77) it follows that $F(\varphi_t(m))$ does not depend on t, that is F is first integral for the differential equation (11.74) in the "normal" sense.

If the function G is first integral $G(\varphi_t(m)) = G(m) = \text{const}$, that is, G is constant on the integral curves of X, which explains the term "constant of motion."

We shall say that the vector field X (the dynamical system defined by X) on the real, n-dimensional manifold $\mathcal{M}$ can be integrated in quadratures (or is completely integrable) if it has $n-1$ independent first integrals. In the case of forms, this means that there are $n-1$ closed forms β_s, $1 \le s \le n-1$, such that for each $m \in \mathcal{M}$ the covectors $\beta_s(m)$ are linearly independent. This condition can be written also as:

$$(\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_{n-1})(m) \neq 0; \quad m \in \mathcal{M} \ . \tag{11.79}$$

If we are speaking of integrals of motion that are functions, then $\beta_s = df_s$ and the condition that β_s are linearly independent implies that f_s are functionally independent.

Definition 11.14. *The field Y is called a symmetry of the vector field X if $L_X Y = [X, Y] = 0$.*

It can be shown that if Y is a symmetry of X, then if we denote the flow generated by Y with ψ_τ, the (11.78) entails

$$\begin{aligned} \psi_{-\tau}^T \circ X \circ \psi_\tau &= X \\ \psi_\tau \circ \varphi_t &= \varphi_t \circ \psi_\tau \end{aligned} \tag{11.80}$$

which explains the term "symmetry."

We can also define the following 1-parametric group of maps acting on the tensor fields Q of type (r,s):

$$(\psi_t^{(r,s)}Q)|_m = \underbrace{d\varphi_{-t} \otimes d\varphi_{-t} \otimes \ldots \otimes d\varphi_{-t}}_{r \text{ times}} \underbrace{\otimes\, d\varphi_t{}^* \otimes d\varphi_t{}^* \otimes \ldots \otimes d\varphi_t{}^*}_{s \text{ times}}(Q|_{\varphi_t(m)})\ . \tag{11.81}$$

Then we have a more general definition:

Definition 11.15. *If Q field of type (r,s) on the manifold $\mathcal{M}$ and $h : \mathcal{M} \mapsto \mathcal{M}$ is a diffeomorphism, then the field*

$$m \mapsto Q^h|_m = \underbrace{dh_m \otimes dh_m \otimes \ldots \otimes dh_m}_{r \text{ times}} \underbrace{\otimes\, dh^{-1}{}^*_m \otimes dh^{-1}{}^*_m \otimes \ldots \otimes dh^{-1}{}_m{}^*}_{s \text{ times}}(Q|_{h^{-1}(m)})\ . \tag{11.82}$$

is also a tensor of type (r,s). We say that Q is invariant under h if $Q = Q^h$. This definition generalizes the definitions of invariant vector field and invariant differential form we had earlier.

Remark 11.16. One can see that if Q is invariant under the diffeomorphism h, then Q is invariant under h^{-1} as well.

Returning to the flow corresponding to the field X, it is not difficult to show that one has the formula below, which generalizes the results we have introduced earlier for the case of forms and vector fields:

$$L_X Q|_m = \lim_{t\to 0} \frac{1}{t}(\psi_t^{(r,s)}(Q) - Q)|_m\ . \tag{11.83}$$

Definition 11.17. *We say that the vector field $X \in \mathcal{T}(\mathcal{M})$ is fundamental for the (r,s)-type tensor field Q (or that X is an infinitesimal symmetry of Q) if*

$$L_X Q = 0\ . \tag{11.84}$$

As can be seen from (11.83), the vector field X is fundamental for the tensor field Q if and only if Q is invariant under the one-parametric group of transformations $\psi_t^{(r,s)}$.

The above formulae are in agreement with the already mentioned property that if P and Q are two tensor fields, $P \otimes Q$ is their tensor product and by dot "." we denote the contraction between certain number of covariant and contravariant components in $P \otimes Q$, then one has

$$L_X(P \otimes Q) = (L_X P) \otimes Q + P \otimes (L_X Q) \tag{11.85}$$
$$L_X(P.Q) = (L_X P).Q + P.(L_X Q)\ . \tag{11.86}$$

Let us give some examples of the above constructions which shall be useful for our purposes. Suppose we have $(1,1)$ type tensor field N over a manifold $\mathcal{M}$. First of all we remark that this field can be regarded in two ways:

1. As a field $m \to \bar{N}_m$ of linear maps (operators)

$$\begin{aligned}
&\bar{N}_m : T_m(\mathcal{M}) \to T_m(\mathcal{M}) \\
&\langle \alpha_m, \bar{N}_m(X_m) \rangle = N_m(X_m, \alpha_m) = N_m.(X_m \otimes \alpha_m) \\
&X_m \in T_m(\mathcal{M}), \quad \alpha_m \in T_m^*(\mathcal{M})\ .
\end{aligned} \tag{11.87}$$

2. As a field of linear maps (adjoint operators) $m \to \bar{N}_m^*$:

$$\begin{aligned}
&\bar{N}_m^* : T_m^*(\mathcal{M}) \to T_m^*(\mathcal{M}) \\
&\langle \bar{N}_m^*(\alpha_m), X_m \rangle = N_m(X_m, \alpha_m) = N_m.(X_m \otimes \alpha_m) \\
&X_m \in T_m(\mathcal{M}), \quad \alpha_m \in T_m^*(\mathcal{M})\ .
\end{aligned} \tag{11.88}$$

One can define Lie derivative of the fields $m \to \bar{N}_m$ and $m \to \bar{N}_m^*$ as:

$$\begin{aligned}
L_X \bar{N}|_m &= \lim_{t \to 0} \frac{1}{t} \left(d\varphi_{-t} \circ \bar{N} \circ d\varphi_t - \bar{N} \right)|_m \\
L_X \bar{N}|_m &= \lim_{t \to 0} \frac{1}{t} \left(d\varphi_t^* \circ \bar{N}^* \circ d\varphi_{-t}^* - \bar{N}^* \right)|_m\ ,
\end{aligned} \tag{11.89}$$

but it is not difficult to see that

$$\begin{aligned}
\overline{\psi_t^{(1,1)} N} &= d\varphi_{-t} \circ \bar{N} \circ d\varphi_t \\
(\overline{\psi_t^{(1,1)} N})^* &= d\varphi_t^* \circ \bar{N}^* \circ d\varphi_{-t}^*\ ,
\end{aligned} \tag{11.90}$$

and we see that the action of the flow on the $(1,1)$ tensor(1-1) field N coincides with the adjoint action of flow on $\bar{N}$ and the coadjoint action of the flow on $\bar{N}^*$. Taking the above into account, we obtain the following proposition which often is useful in the calculations:

Proposition 11.18. *Let X be vector field, N be $(1,1)$ tensor field as above. Then*

$$(L_X \bar{N})^* = L_X \bar{N}^*\ . \tag{11.91}$$

Corollary 11.19. *If the vector field X is fundamental for the field $\bar{N}$ then X is fundamental for the field $\bar{N}^*$.*

Remark 11.20. In what follows, if there is no possibility of misunderstanding we shall denote the $(1,1)$ tensor field N and the field of operators $\bar{N}$ by the same letter.

11.4 Distributions (Fields of Subspaces)

The situation when we have a curve $\gamma(t)$ which is tangent to the vector field can be generalized in the sense that we can consider a set of vector fields $X_1, X_2, \ldots X_p$ which are tangent to a submanifold $\mathcal{N} \subset \mathcal{M}$. We give the following definition:

Definition 11.21. *The subset $\mathcal{N} \subset \mathcal{M}$ of the manifold $\mathcal{M}$ is called submanifold if it itself is a manifold and the inclusion map $j : \mathcal{N} \to \mathcal{M}$ is an immersion (that is dj_m is injective at each $m \in \mathcal{N}$).*

(Recall that we assume all the maps to be smooth.)

Remark 11.22. Our definition for submanifold is what is usually called immersed submanifold. If in addition we require that the topology on $\mathcal{N} \subset \mathcal{M}$ is the induced topology, the submanifold is called a submanifold embedded in $\mathcal{M}$. Since we are not going to discuss topological aspects the difference between immersed and embedded submanifolds will not play a role in our considerations.

Example 11.23. If U is some open set in $\mathcal{M}$, it is obviously a submanifold.

Suppose that $\mathcal{N} \subset \mathcal{M}$ is a submanifold and let $j : \mathcal{N} \mapsto \mathcal{M}$ be the canonical inclusion map. It is readily seen that j is differentiable. We have

Definition 11.24. *For $m \in \mathcal{N} \subset \mathcal{M}$ a vector $X_m \in T_m(\mathcal{M})$ is called tangent to $\mathcal{N}$ if $X_m \in dj_m(T_m(\mathcal{N}))$. Since dj_m is injective, sometimes $T_m(\mathcal{N})$ and $dj_m(\mathcal{N})$ are identified and one just writes:*

$$T_m(\mathcal{N}) \subset T_m(\mathcal{M}), \quad m \in \mathcal{N} \,. \tag{11.92}$$

The vector field X over $\mathcal{M}$ is tangent to the submanifold $\mathcal{N}$ if for each $m \in \mathcal{N}$, $X(m)$ is tangent to $\mathcal{N}$. In this case, X induces on $\mathcal{N}$ unique vector field Y, such that $j^T \circ Y = X \circ j$, called sometimes the restriction of X on $\mathcal{N}$.

Definition 11.25. *If ω is a p-form on $\mathcal{M}$, then $j^*\omega$ is a p-form on $\mathcal{N}$, called the restriction of ω on $\mathcal{N}$. In particular, for 0-form f (function) on $\mathcal{M}$, we have $j^*f = f \circ j$.*

Why we call $j^*\omega$ a restriction becomes clear, if we recall the identification leading to (11.92).

Now, we shall give definition for distribution or as it is often called – field of subspaces. The definitions and results that follow below are simplified and, strictly speaking, are such that ought to be only for the finite dimensional case. As already explained, there are some subtleties in the infinite-dimensional case, see [2]. This subtleties are present both in the formulation and in the proof of the Frobenius theorem, which is the main result about distributions, but we deliberately avoid to speak about them here. However, when in the future we use these results, we shall usually find the integral submanifolds explicitly, so the lack of rigor will not affect negatively our future constructions.

Suppose that $\mathcal{M}$ is a differentiable manifold, which is locally diffeomorphic to some "model" space E (Banach space for example). Then for each m the space $T_m(\mathcal{M})$ is isomorphic to E. We shall denote the "model" spaces we are working with by $\mathcal{E}$ and the set of their dual spaces by $\mathcal{E}^*$.

Definition 11.26. *A differentiable distribution (field of subspaces) on a manifold $\mathcal{M}$ is a field $\mathcal{S} : m \mapsto \mathcal{S}_m$ where $\mathcal{S}_m$ is some subspace of T_m belonging to $\mathcal{E}$ and satisfying the conditions:*

1. *For each $m \in \mathcal{M}$ there exists E'_m such that $E = E'_m \oplus \mathcal{S}_m \sim E'_m \times \mathcal{S}_m$, where $E'_m \in \mathcal{E}$.*
2. *For each $m \in \mathcal{M}$ and $X_m \in \mathcal{S}_m$ there exists a neighborhood U of $\mathcal{M}$ and a smooth vector field X on U such that $X(m) = X_m$ and for each $q \in U$, $X(q) \in \mathcal{S}_q$.*

If the spaces E'_m, $\mathcal{S}'_m$ are isomorphic to some constant spaces E' and S belonging to $\mathcal{E}$, we shall call the distribution a regular distribution. In the finite dimensional case regular means that $\mathcal{S}_m$ has dimension that does not depend on the point, so in this case we speak about distributions of constant dimension $p = dim(\mathcal{S}_m)$ and call p dimension of the distribution $\mathcal{S}$.

Definition 11.27. *A differentiable Pfaffian system on a manifold $\mathcal{M}$ is a field of subspaces $\mathcal{J} : m \mapsto \mathcal{J}_m$ where $\mathcal{J}_m$ is some subspace of T^*_m of belonging to $\mathcal{E}^*$ and having the properties:*

1. *For each $m \in \mathcal{M}$, there exists S'_m such that $E^* = S'_m \oplus \mathcal{J}_m \sim S'_m \times \mathcal{J}_m$, where S'_m is a space belonging to $\mathcal{E}^*$.*
2. *For each $m \in \mathcal{M}$ and $\alpha_m \in \mathcal{J}_m$, there exists a neighborhood U of $\mathcal{M}$ and a smooth covector field α on U such that $\alpha(m) = \alpha_m$ and for each $q \in U$, $\alpha(q) \in \mathcal{J}_q$.*

If the spaces S'_m, $\mathcal{J}_m$ are isomorphic to some constant spaces S' belonging to $\mathcal{E}^*$ we shall call the Pfaffian system a regular Pfaffian system. In the finite dimensional situation regularity implies that the dimension p of $\mathcal{J}_m$ does not depend on the point. In this case we speak about Pfaffian systems of constant rank p and p is called the rank of $\mathcal{J}$.

If $\mathcal{S}$ is a distribution, then for $m \in \mathcal{M}$ we can define

$$\mathcal{S}^{\perp}_m = \{\alpha_m \in T^*_m : \langle \alpha_m, X_m \rangle = 0 : X_m \in \mathcal{S}_m\} . \tag{11.93}$$

Then, as it is easily checked $m \mapsto \mathcal{S}^{\perp}_m$ is a differentiable Pfaffian system. If $\mathcal{J}$ is a Pfaffian system, then for $m \in \mathcal{M}$ we can define

$$\ker \mathcal{J}_m = \{X_m \in T_m : \langle \alpha_m, X_m \rangle = 0 : \alpha_m \in \mathcal{J}_m\} . \tag{11.94}$$

Then $m \mapsto \ker \mathcal{J}_m$ is a differentiable distribution.

Definition 11.28. *The submanifold $\mathcal{N}$ is called integral (sub)manifold for the distribution $m \to \mathcal{S}_m$ if for any $m \in \mathcal{N}$ we have $\mathcal{S}_m = dj_m(T_m(\mathcal{N}))$, where $j : \mathcal{N} \mapsto \mathcal{M}$ is the inclusion map.*

Definition 11.29. *The distribution $m \to \mathcal{S}_m$ is called completely integrable (integrable in Frobenius sense or simply integrable) if at each point there exists integral submanifold. For the sake of brevity, we shall usually say that the distribution is integrable. We shall say that a Pfaffian system $\mathcal{J}$ is integrable if the corresponding distribution $m \mapsto ker\,\mathcal{J}_m$ is integrable.*

The following theorem, called the Frobenius theorem, plays a central role in the theory of distributions:

Theorem 11.30. *Let the set of the model spaces $\mathcal{E}$ consist of complete spaces (Banach for example) and let $m \to \mathcal{S}_m$ be a regular distribution. It is integrable if and only if from $X, Y \in \mathcal{S}$ (that is for each $m \in \mathcal{M}$ we have $X(m), Y(m) \in \mathcal{S}_m$) follows that $[X, Y] \in \mathcal{S}$.*

The above theorem has numerous applications in Geometry and Mechanics.

Suppose we have integrable field $m \to S_m \subset T_m(\mathcal{M})$ of subspaces of constant dimension (distribution) on the finite dimensional manifold $\mathcal{M}$. The maximal integral submanifolds of this distribution are usually called (integral) leafs of the distribution and the resulting geometric structure, that is the representation of $\mathcal{M}$ as disjoint union of leafs is called *foliation of $\mathcal{M}$*.

In case there exists the manifold $\mathcal{N}$ and smooth map $p : \mathcal{M} \to \mathcal{N}$ in such a way that $p^{-1}(m)$; $m \in \mathcal{N}$ is either empty or coincides with exactly one leaf of the foliation $\mathcal{F}$, the foliation $\mathcal{F}$ is called *projectable through p*. A special case of the above situation is when there exists submanifold $\mathcal{N} \subset \mathcal{M}$, such that $\mathcal{N}$ intersects with the leafs only once (it is of course possible that $\mathcal{N}$ does not intersect all the leafs) and is transversal to the foliation, that is, the tangent space of $T_m(\mathcal{N})$ and the vector space S_m define splitting of $T_m(\mathcal{M})$

$$T_m(\mathcal{M}) = T_m(\mathcal{N}) \oplus S_m; \quad m \in \mathcal{N} . \tag{11.95}$$

(Here $T_m(\mathcal{N})$ is identified by $dj_m(T_m(\mathcal{M}))$ and $j : \mathcal{N} \mapsto \mathcal{M}$ is the inclusion map). The above is evidently related to the general concept of transversality of manifolds:

Definition 11.31. *Two submanifolds $\mathcal{N}, \mathcal{P} \subset \mathcal{M}$ are called transversal if at the points $m \in \mathcal{N} \cap \mathcal{P}$ we have*

$$T_m(\mathcal{N}) \oplus T_m(\mathcal{P}) = T_m(\mathcal{M}) . \tag{11.96}$$

When $\mathcal{N}$ is transversal to the foliation $\mathcal{F}$, they say sometimes that it is a *section* of the foliation. In case $\mathcal{N}$ is a section of the foliation $\mathcal{F}$, the projection map p is defined in the following way. For $m \in \mathcal{M}$, we take the leaf $\mathcal{L}_m$ that passes through m. Then the domain of p is the set of all points m for which $\mathcal{L}_p$ intersects $\mathcal{N}$ and $p(m) \in \mathcal{N}$ is the unique intersection point of $\mathcal{N}$ and $\mathcal{L}_m$.

The Frobenius theorem has also a dual formulation, which we shall not introduce in all generality but only in the following situation. Assume that we have a manifold $\mathcal{M}$ and suppose that on $\mathcal{M}$ is defined a finite family of independent 1-forms $\{\theta_i\}_{i=1}^k$, such that the spaces

$$\mathcal{S}_m = \bigcap_{i=1}^{k} \ker \theta_i(m), \quad m \in \mathcal{M} , \tag{11.97}$$

are isomorphic to some constant space. Naturally, $\{\theta_i\}_{i=1}^k$ generate a Pfaffian system $\mathcal{J}$. If $\mathcal{N}$ is arbitrary integral manifold and $j : \mathcal{N} \mapsto \mathcal{M}$ is the inclusion map we have $j^*\theta_i = 0$.

In the finite dimensional situation, when $\dim(\mathcal{M}) = d$ the field $m \mapsto \mathcal{S}_m$ defines a $(d-k)$-dimensional distribution $\mathcal{S}$. One can check that the Frobenius theorem implies that $\mathcal{J}$ is integrable if and only if for each i we have

$$d\theta_i(X, Y) = 0 \tag{11.98}$$

for any vector fields X and Y belonging to $\mathcal{S}$. In the finite dimensional situation, this condition is equivalent to the condition that the form $d\theta_i$ belongs to the ideal, generated by the family $\{\theta_i\}_{i=1}^k$ in the algebra of the exterior forms. In particular, if all θ_i are closed, for each point m there exists local coordinates x^s; $s = 1, 2, \dots, \dim(\mathcal{M})$ about m, such that locally $\theta_i = dx^i$ for $i = 1, 2, \dots, k$. If in addition k is equal to d – the dimension of the manifold, the family $\{\theta_i\}_{i=1}^d$ is a basis in the space of 1-forms (it is often called a frame). If θ_i are closed, the frame $\{\theta_i\}_{i=1}^d$ is called *holonomic* and is called *nonholonomic* if this is not true.

11.5 Related Tensor Fields

Let $\mathcal{M}$ and $\mathcal{N}$ be two manifolds and $h : \mathcal{M} \to \mathcal{N}$ be some differentiable map.

Definition 11.32. *Two vector fields $X \in \mathcal{T}(\mathcal{M})$, $Y \in \mathcal{T}(\mathcal{N})$ are called related through h (or h-related) if*

$$h^T \circ X = Y \circ h . \tag{11.99}$$

In other words, for arbitrary $m \in \mathcal{M}$, $h^T(X(m)) = Y(h(m))$.

Of course, if X is vector field over $\mathcal{M}$ a vector field Y, h-related to X may fail to exist. With the new notion the definition of the tangent field to a submanifold can be cast in the following form:

Definition 11.33. *Let $\mathcal{N} \subset \mathcal{M}$ be a submanifold and let j be the inclusion map $j : \mathcal{N} \mapsto \mathcal{M}$. We say that the vector field $X \in \mathcal{T}(\mathcal{M})$ is tangent to the submanifold $\mathcal{N}$ if X is j-related to some vector field Y on $\mathcal{N}$.*

Example 11.34.
Let h be diffeomorphism $h : \mathcal{M} \to \mathcal{N}$. Let X be vector field over $\mathcal{M}$. Then the fields X and $h_*(X) = h^T \circ X \circ h^{-1}$ are h-related.

One can prove the following without much difficulty.

Proposition 11.35. *If the fields $X_1, X_2 \in \mathcal{T}(\mathcal{M})$ are h-related to the fields $Y_1, Y_2 \in \mathcal{T}(\mathcal{N})$, then $[X_1, X_2]$ is h-related to $[Y_1, Y_2]$.*

Definition 11.36. *If $\alpha \in \Lambda^p(\mathcal{M})$ and $\beta \in \Lambda^p(\mathcal{N})$ are two p-forms over $\mathcal{M}$ and $\mathcal{N}$ respectively we say that they are h-related if $\alpha = h^*\beta$.*

Remark 11.37. We again emphasize that for given α the form β may fail to exist. However, it can be proved that if $\mathcal{N} \subset \mathcal{M}$ is a submanifold of $\mathcal{M}$ and $j : \mathcal{N} \mapsto \mathcal{M}$ is the corresponding inclusion map, then locally for each p-form γ on $\mathcal{N}$ there exist a p-form α on $\mathcal{M}$ such that $j^*\alpha = \gamma$. More exactly, for $m \in \mathcal{N}$ there exists a neighborhood V_m in $\mathcal{N}$, a neighborhood W_m of m in $\mathcal{M}$ such that $V_m \subset W_m$ and a form α defined on W_m such that $\gamma|_{V_m} = j^*\alpha|_{V_m}$.

It is not hard to prove.

Proposition 11.38. *If the vector fields $X \in \mathcal{T}(\mathcal{M})$ and $Y \in \mathcal{T}(\mathcal{N})$ are h-related, and if $\alpha \in \Lambda^p(\mathcal{M})$, $\beta \in \Lambda^p(\mathcal{N})$ are two h-related p-forms, then*

$$i_X\alpha = i_X h^*\beta = h^* i_Y \beta \ . \tag{11.100}$$

Corollary 11.39. *Let $X_1, X_2, \ldots, X_p \in \mathcal{T}(\mathcal{M})$; $Y_1, Y_2, \ldots, Y_p \in \mathcal{T}(\mathcal{N})$ be pairwise h-related. Then if the forms $\alpha \in \Lambda^p(\mathcal{M})$, $\beta \in \Lambda^p(\mathcal{N})$ are two h-related p-forms we have*

$$\alpha(X_1, X_2, \ldots, X_p)(m) = \beta(Y_1, Y_2, \ldots, Y_p)(h(m)) \ . \tag{11.101}$$

The notion of h-related tensor fields can be defined along the same lines.

Definition 11.40. *Let $m \to R_m$ and let $n \to S_n$ be tensor fields of the same type (q, p), over $\mathcal{M}$ and over $\mathcal{N}$ respectively, that is $R_m \in T_m^{(q,p)}(\mathcal{M})$, $S_n \in T_n^{(p,q)}(\mathcal{N})$. We shall say that they are h-related if for any h-related sets of fields $X_1, X_2, \ldots, X_p \in \mathcal{T}(\mathcal{M})$ and $Y_1, Y_2, \ldots, Y_p \in \mathcal{T}(\mathcal{N})$ and for each set of* 1*-forms $\beta_1, \beta_2, \ldots, \beta_q$ over $\mathcal{N}$ at each point $m \in \mathcal{M}$ we have*

$$\begin{aligned} &\langle S, X_1 \otimes X_2 \otimes \ldots \otimes X_p \otimes h^*\beta_1 \otimes h^*\beta_2 \otimes \ldots \otimes h^*\beta_q \rangle(m) = \\ &\langle R, Y_1 \otimes Y_2 \otimes \ldots \otimes Y_p \otimes \beta_1 \otimes \beta_2 \otimes \ldots \otimes \beta_q \rangle(h(m)) \ . \end{aligned} \tag{11.102}$$

Taking into account the definitions of h-related fields and the properties of the Lie derivative and the exterior derivative one can check that.

Proposition 11.41. *Let R and S be h-related tensor fields over $\mathcal{M}$ and over $\mathcal{N}$, respectively, and let X and Y be two h-related vector fields. Then the tensor fields $L_X R$ and $L_Y S$ are also h-related. If α and β are h-related forms then the forms $d\alpha$ and $d\beta$ are also h-related.*

We see that if two tensor fields are h-related they have similar properties, and if the sets of h-related 1-forms and vector fields are sufficiently large then any relation over $\mathcal{M}$ written in terms of the operations d and L_X for the first set of these tensor fields has its analog over the manifold $\mathcal{N}$ for the second set of tensor fields.

Definition 11.42. *Let $\mathcal{N} \subset \mathcal{M}$ be submanifold and let j be the inclusion map. Let Q be a tensor field over $\mathcal{M}$. We shall say that the tensor field $\bar{Q}$ over $\mathcal{N}$ is restriction of Q on $\mathcal{N}$ if Q and $\bar{Q}$ are j-related.*

11.6 Local Form of the Geometric Objects

Often it is more convenient to work in a fixed system of charts, and so the local form of the geometric objects is needed. The "representatives" of the vector fields and tensor fields we have introduced earlier (see (11.32), (11.34), (11.35)) are in fact the local forms, but they must be expressed through the coordinates. Let us consider first the finite dimensional case. As the space E is of dimension n it is isomorphic to the Euclidean space $\mathbb{R}^n$ (here we consider the real case). Then we can safely assume that $E = \mathbb{R}^n$ and let e_i; $i = 1, 2, \ldots, n$ be the standard orthonormal basis in $\mathbb{R}^n$. Let us consider the chart (U_i, φ_i) of the manifold $\mathcal{M}$. Let x^s be the following functions on $U_i \subset \mathcal{M}$:

$$x^s = p^s \circ \varphi_i \,, \tag{11.103}$$

where p^s is the projector onto the s-th component in the Cartesian product $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$. Then, it is readily seen that $(\varphi_i(m))^s = x^s(m)$; $s = 1, 2, \ldots, n$. The functions $x^s(m)$ are called coordinates of the point $m \in \mathcal{M}$. Let $f(m)$ be a function on $\mathcal{M}$. We shall write $f(x)$ for the function $f(\varphi_i^{-1}(x))$ defined on $\varphi_i(U_i)$. The use of the same symbol for two different functions is slightly confusing, but it is convenient and universally accepted. (The function $f(x)$ is the same as $f(m)$, but in different variables). $f(x)$ is called the expression of f in the coordinates x^s or local form of f in the chart (U_i, φ_i). (Earlier we called it also representative of f in the chart (U_i, φ_i)).

Let now ξ_m be a vector in $T_m(\mathcal{M})$ and his representative in the above chart be $\xi \in \mathbb{R}^n$. The definition of the tangent map shows that

$$dx^s(\xi_m) = p^s(\xi) = \xi^s \,. \tag{11.104}$$

Then clearly the 1-forms dx^s form a basis of 1-forms on $T^*_m(\mathcal{M})$ and each covector α_m can be written as :

$$\alpha_m = \sum_{s=1}^{n} a_s dx^s, \quad a^s \in \mathbb{R} \,. \tag{11.105}$$

This is called local form of the covectors in the chart (U_i, φ_i).

It is readily seen that dx^s are in fact smooth fields of 1-forms in U_i and, therefore, for $m \in U_i$ each field of covectors (1-form) α can be written into the form

$$\alpha(x) = \sum_{s=1}^{n} a_s(x) dx^s \ , \tag{11.106}$$

where $a^s(x)$ are smooth functions on $\varphi_i(U_i) \subset \mathbb{R}^n$.

Here, we have adopted the same convention as above and written $\alpha(x)$ for the function $\alpha(\varphi_i^{-1}(x))$. The expression (11.106) is called local form of the 1-form α in the chart (U_i, φ_i). In particular, if f is a function one obtains

$$df(x) = \sum_{s=1}^{n} \frac{\partial f}{\partial x^s} dx^s \ . \tag{11.107}$$

For arbitrary p-form β one has similar expression:

$$\beta(x) = \sum_{1 \le i_1 < i_2 < \ldots < i_p \le n} a_{i_1 i_2 \ldots i_p}(x) dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p} \ . \tag{11.108}$$

If ξ is the representative of $\xi_m \in T_m(\mathcal{M})$ from the above considerations, we easily get the following expression:

$$df_m(\xi_m)|_{m=\varphi^{-1}(x)} = \sum_{s=1}^{n} \frac{\partial f}{\partial x^s} \xi^s \ . \tag{11.109}$$

Now let $X(m)$ be vector field with representative $X_{i,m}$. As in the above let us denote by $X(x)$ the following vector field on $\varphi_i(U_i) \subset \mathbb{R}^n$:

$$x \mapsto X(x) = X_{i,m=(\varphi_i^{-1}(x))} \ . \tag{11.110}$$

This field is called local form of the vector field X in the chart (U_i, φ_i). Usually it is written using different notation which we shall introduce now. They are quite natural if we calculate the local form of the function Xf, that is, $Xf(x) = df(X)(x)$ in the chart (U_i, φ_i). We find that

$$Xf(x) = \sum_{s=1}^{n} \frac{\partial f}{\partial x^s} X^s(x) \ , \tag{11.111}$$

and in particular

$$Xx^m = dx^m(X) = \sum_{s=1}^{n} \frac{\partial x^m}{\partial x^s} X^s(x) = X^m(x) \ . \tag{11.112}$$

Suppose now Z_s are vector fields on U_i, such that $Z_s(x) = e_s$, where e_s are the vectors of the canonical basis in $\mathbb{R}^n$. These fields restricted to each tangent space form a basis in it and a basis for the module of the vector fields on U_i. We have

$$Z_s f = \frac{\partial f}{\partial x^s} \tag{11.113}$$

and in particular

$$Z_s x^m(x) = dx^m \left(\frac{\partial x^m}{\partial x^s} \right) = \delta_s^m \ . \tag{11.114}$$

Thus Z_s form a basis in $\mathcal{T}(U_i)$, dual to the basis of the 1-forms dx^s in $\Lambda^1(U_i)$. Because of (11.113) the vector fields Z_s are denoted in the following way:

$$Z_s = \frac{\partial}{\partial x^s} \ . \tag{11.115}$$

Using this notation, we see that

$$dx^s \left(\frac{\partial}{\partial x^m} \right) = \delta_m^s \tag{11.116}$$

and if we apply this to (11.112) we get that

$$X(x) = \sum_{s=1}^{n} X^s(x) \frac{\partial}{\partial x^s} \ . \tag{11.117}$$

This expression is then called a local form of the vector field X in the chart (U_i, φ_i).

In the same manner, one can obtain the local form of tensor fields of arbitrary type. For example, here are the local forms of:

1. $(1,1)$ tensor field N (field of operators $N_m : T_m(\mathcal{M}) \to T_m(\mathcal{M})$) :

$$N = \sum_{k,m=1}^{n} N_k^m(x) \frac{\partial}{\partial x^m} \otimes dx^k \ , \tag{11.118}$$

where $N_k^m(x)$ are smooth functions on $\varphi_i(U_i)$,

2. $(1,1)$ tensor field S (field of operators $S_m : T_m^*(\mathcal{M}) \to T_m^*(\mathcal{M})$) :

$$S = \sum_{k,m=1}^{n} S_k^m(x) dx^k \otimes \frac{\partial}{\partial x^m} \ , \tag{11.119}$$

where $S_k^m(x)$ are smooth functions on $\varphi_i(U_i)$,

3. 2-form ω:

$$\omega = \sum_{1 \le k < m \le n} a_{km}(x)\, dx^k \wedge dx^m \ , \tag{11.120}$$

where $a_{km}(x)$ are smooth functions on $\varphi_i(U_i)$.

The fields dx^s and $\frac{\partial}{\partial x^i}$ we have just introduced are also used to produce the coordinates on the tangent and cotangent bundles. Indeed, suppose in the chart (U_i, φ_i) we have the coordinates x^s; $s = 1, 2, \ldots, n$, then taking into account (11.28) the following coordinates on $q_M^{-1}(U_i) \subset T^*(\mathcal{M})$ are used: If $a = (m, \alpha_m) \in q_M^{-1}(U_i)$ we introduce

$$\begin{aligned} q^s(a) &= p^s \circ \pi^1 \circ \bar{\Phi}_i(a) = p^s \circ \varphi(m) = x^s(m) \\ p_s(a) &= \left\langle \frac{\partial}{\partial x^s}, \pi^2 \circ \bar{\Phi}_i(a) \right\rangle = \left\langle \frac{\partial}{\partial x^s}, \alpha_{i,m} \right\rangle , \end{aligned} \tag{11.121}$$

where q_M is the projection of $T^*(\mathcal{M})$ over $\mathcal{M}$: $q_M(m, \alpha_m) = m$; π^1, π^2 are the projections onto the first and the second component of the Cartesian product $U_i \times E^*$, respectively, and p^s is the projection on the s-th component of the Cartesian product $\mathbb{R}^n$. Then (q, p) are coordinates on $q_M^{-1}(U_i)$.

Quite in the same way, on $p_M^{-1}(U_i) \subset T(\mathcal{M})$, taking into account (11.24), we have the following coordinates: For $b = (m, \xi_m) \in p_M^{-1}(U_i)$ we set

$$\begin{aligned} q^s(b) &= p^s \circ \pi^1 \circ \Phi_i(b) = p^s \circ \varphi(m) \\ r_i(b) &= dx^s(\pi^2 \circ \Phi_i(b)) = dx^s(\xi_m) , \end{aligned} \tag{11.122}$$

where p_M is the projection of $T(\mathcal{M})$ over $\mathcal{M}$: $q_M(m, \xi_m) = m$; π^1, π^2 are the projections onto the first and second component of the Cartesian product $U_i \times E$, respectively, and p^s is the projection on the s-th component of the Cartesian product $\mathbb{R}^n$. Then (q, r) are coordinates on $p_M^{-1}(U_i)$.

The coordinates introduced in the above on $T(\mathcal{M})$ and $T^*(\mathcal{M})$ are associated with the choice of the coordinates about the point $m \in \mathcal{M}$ and are usually called canonical. As a matter of fact they are so convenient that seldom some different coordinates are used.

Example 11.43 (The Liouville form). Let $T^*(\mathcal{M})$ be the total space of the cotangent bundle of $\mathcal{M}$. If $a = (m, \alpha_m)$ is a point in $T^*(\mathcal{M})$, and τ_a is a vector at a then $q_M^T(\tau_a) \in T_m(\mathcal{M})$; $m = q_M(a)$, where q_M is the projection onto the base $\mathcal{M}$ of $T^*(\mathcal{M})$. The Liouville form λ is defined as:

$$\lambda(\tau_a) = \langle q_M^T(\tau_a), \alpha_{q_M(a)} \rangle = \alpha_m(q_M^T(\tau_a)) . \tag{11.123}$$

In the canonical coordinates (q, p), associated with some chart (U_i, φ_i), we have

$$\lambda = \sum_{s=1}^{n} p_s dq^s . \tag{11.124}$$

For the infinite-dimensional case one can chose notation (though it is not obligatory) making the formulae for the local form of the geometric objects as similar as possible to the finite dimensional ones. In order to fix the ideas suppose the "model" space E is the vector space of Schwartz-type r-dimensional vector-functions on the line. In other words $u \in E$, if $u = (u^1, u^2, \ldots, u^r)$, where $u^i = u^i(x)$ are Schwartz-type functions on the line. If (U, φ) is a chart of some manifold, for which E is "model" space, then "coordinates" of m are now the functions

$$\varphi(m) = u = (u^1(x), u^2(x), \dots u^r(x)).$$

As we have seen in the above, if F is differentiable functional, then the variational derivative can be regarded as kernel of the integral operator dF (sometimes in the sense of distributions). Similar notation can be adopted for another geometric objects. For example, we have the "basis" of 1-forms $\delta u^i(x)$; $x \in \mathbb{R}$, $i = 1, 2 \dots, r$. Here by $\delta u^i(x)$ is denoted the functional

$$v = (v^1(y), v^2(y), \dots, v^r(y)) \to \delta u^i(x)(v) = v^i(x), \quad x - \text{fixed} . \tag{11.125}$$

Then, for a field of covectors α (1 form α) we have the local expression

$$\alpha = \int_{-\infty}^{+\infty} dy \sum_{i=1}^{r} a_i[u](y) \delta u^i(y) \tag{11.126}$$

The symbol $[u]$ in $a_i[u](y)$ simply reminds us that the functions $a_i[u](y)$ depend on the point $u = \varphi(m)$. This formal notation is interpreted as follows: If V_m is tangent vector at the point m with representative

$$v = (v^1(y), v^2(y), \dots, v^r(y))$$

then

$$\alpha(V_m) = \int_{-\infty}^{+\infty} dy \sum_{i=1}^{r} a_i[u](y) \delta u^i(y)(v) = \int_{-\infty}^{+\infty} dy \sum_{i=1}^{r} a_i[u](y) v_i(y) . \tag{11.127}$$

As mentioned before, we put aside the question about what conditions $a_i(y)$ must satisfy to ensure that the form α is differentiable. It is clear that in the general case $a_i(y)$ are generalized functions (distributions). If $F : \mathcal{M} \mapsto \mathbb{R}$ is a function (of course it is a functional here) for dF we obtain:

$$dF = \int_{-\infty}^{+\infty} dy \sum_{i=1}^{r} \frac{\delta F}{\delta u_i}[u](y) \delta u_i(y) \tag{11.128}$$

in accordance with what we had earlier.

In order to keep the analogy with the finite dimensional case, we also write on E vector fields "dual" to the 1-forms $\delta u^i(x)$. So we introduce the fields

$$\frac{\delta}{\delta u^j(z)} , \tag{11.129}$$

that satisfy the relations

$$\delta u^i(x) \left(\frac{\delta}{\delta u^j(z)} \right) = \delta^i_j \delta(x - z) \tag{11.130}$$

where $\delta(x-z)$ is the Dirac "function." Of course, there are no such elements in E, and thus we just introduce here some formal expressions and formal calculus with them. These expressions have no other justification than the fact that when one works properly with them they lead to results which themselves have real meaning. For example, the vector field V with representative

$$V(m) = (v^1(m), v^2(m), \ldots, v^r(m)) = v[u], \quad u = \varphi_i(m)$$

has local form

$$V = \int_{-\infty}^{+\infty} dz \sum_{j=1}^{r} v^j[u](z) \frac{\delta}{\delta u^j(z)} \tag{11.131}$$

and acting according to the rule (11.130) for the value of $\alpha(V)$ we get (11.127). With these notations we write for the local forms of:

1. $(1,1)$ tensor field N (field of operators $N_m : T_m(\mathcal{M}) \to T_m(\mathcal{M})$) :

$$N = \sum_{k,s=1}^{n} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dxdy N_k^s[u](x,y) \frac{\delta}{\delta u^s(x)} \otimes \delta u_k(y) \,, \tag{11.132}$$

2. $(1,1)$ tensor field P (field of operators $P_m : T_m^*(\mathcal{M}) \to T_m^*(\mathcal{M})$) :

$$P = \sum_{k,s=1}^{n} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dxdy P_k^s[u](x,y) \delta u^k(y) \otimes \frac{\delta}{\delta u^s(x)} \,, \tag{11.133}$$

3. 2-form ω

$$\omega = \sum_{1\le k<s\le n} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dxdy\ a_{ks}[u](x,y) \delta u^k(x) \wedge \delta u^s(y) \,. \tag{11.134}$$

Let $\mathcal{M} = E = \mathcal{S}$ be the space of the Schwartz-type functions on the line. Then of course it is covered by one chart $(\mathcal{S}, \mathrm{id}_{\mathcal{S}})$, and in the above formulae $m = u$; $r = 1$. Thus:

- Local expressions for vector fields:

$$\xi = \int_{-\infty}^{+\infty} dy \xi[u](y) \frac{\delta}{\delta u(y)} \,. \tag{11.135}$$

- Local expressions for differential 1-forms:

$$\alpha = \int_{-\infty}^{+\infty} dy \alpha[u](y) \delta u(y) \,. \tag{11.136}$$

- As a more explicit example, on the same manifold, consider the smooth function (functional)

$$F(u) = \int_{-\infty}^{+\infty} (u_x)^2 dx \; . \tag{11.137}$$

Then

$$dF_u(\xi) = 2 \int_{-\infty}^{+\infty} \xi_x u_x dx; \quad \xi \in T_u(\mathcal{S}) = \mathcal{S} \; . \tag{11.138}$$

and the local expression of dF is

$$dF = -2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dy dx \delta'(x-y) u_y \delta u(x) \; . \tag{11.139}$$

- As a particular example of 2-form let us define

$$\omega(\xi, \eta) = \int_{-\infty}^{+\infty} \xi(x) \eta_x(x) dx; \quad \xi, \eta \in T_u(\mathcal{S}) = \mathcal{S} \; . \tag{11.140}$$

Its local expression is

$$\omega = \frac{1}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx dy \delta'(x-y) \delta u(x) \wedge \delta u(y) \; . \tag{11.141}$$

- Consider also the field of linear maps $u \to P_u$; $P_u : T_u^*(\mathcal{S}) \to T_u(\mathcal{S})$ defined as:

$$P_u(\alpha) = \alpha_x; \quad \alpha(x) \in T_u^*(\mathcal{S}) \; . \tag{11.142}$$

The local form of P is

$$P = - \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx dy \delta'(x-y) \frac{\delta}{\delta u(y)} \otimes \frac{\delta}{\delta u(x)} \; . \tag{11.143}$$

The tensor P is in fact Poisson tensor (see Chap. 3) and defines the Hamiltonian structure of the Korteweg de Vries and Burgers equations hierarchies [6, 7].

With the above notation, the calculations become very similar to those in the finite dimensional case and that is in fact the main reason they have been introduced.

We cannot avoid the question of the Stokes formula, because we use it in some comments and illustrations of some ideas, and we shall give a brief sketch of topic. The missing details can be found in practically any book of Differential Geometry; we refer to one of the following monographs: [1, 2, 3, 4].

We must introduce some necessary notions and definitions, first of which will be the notion of a n-dimensional real manifold with a boundary. The definition of a finite dimensional manifold $\mathcal{M}$ with a boundary is essentially different from the one given above for the manifold without a boundary. As before, we have a topological space $\mathcal{M}$ and a family of parameterizations (atlas) $\{(U_i, \varphi_i)\}_{i\in I}$ of it. However, this time φ_i are homeomorphisms of the open sets $U_i \subset \mathcal{M}$ into $\varphi_i(U_i)$ which are open sets in the half-space

$$\mathbb{H}^n = \left\{x^1, x^2, \ldots, x^n : x^n \geq 0\right\} \tag{11.144}$$

and not as before open sets in $\mathbb{R}^n$. The transition maps

$$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_j \cap U_i) \to \varphi_i(U_i \cap U_j) \tag{11.145}$$

are again assumed to be differentiable, but since $\varphi_j(U_j \cap U_i)$ are open in $\mathbb{H}^n$ and not in $\mathbb{R}^n$, we must say what a differentiable map is in this case. We assume that $\varphi : U \mapsto \mathbb{R}^m$ where $U \subset \mathbb{H}^n$ is differentiable if there exists W – open in $\mathbb{R}^n$, such that $V \subset W \cap \mathbb{H}^n$ and a differentiable map $f : W \mapsto \mathbb{R}^m$ such that $f|_U = \varphi$.

The boundary $\partial\mathbb{H}^n$ of $\mathbb{H}^n$ is then the plane $x^n = 0$ and is identified with $\mathbb{R}^{n-1}$. By definition, the boundary of $\mathcal{M}$ is the collection of points that are pre-images of $\mathbb{R}^{n-1}$ under the parameterizations that is

$$\partial\mathcal{M} = \underset{i\in I}{\cup} \varphi_i^{-1}(\partial\mathbb{H}^n \cap \varphi_i(U_i)) \,. \tag{11.146}$$

An important fact is that the boundary $\partial\mathcal{M}$ is a manifold of dimension $n-1$ (without boundary). Indeed, an atlas for it can be obtained restricting on the boundary the charts of any atlas of $\mathcal{M}$. In more detail, if $\mathcal{A} = (U_i, \varphi_i)_{i\in I}$ is an atlas of $\mathcal{M}$, the family of charts

$$\{(\partial\mathbb{H}^n \cap \varphi_i(U_i), \varphi|_{\partial\mathbb{H}^n \cap \varphi_i(U_i)})\}_{i\in I_1} \,, \tag{11.147}$$

where I_1 labels those charts, for which $\partial\mathbb{H}^n \cap \varphi_i(U_i) \neq \emptyset$ is an atlas of $\partial\mathcal{M}$. With these definitions almost everything that was introduced for manifolds without a boundary, together with the basic operations with the tensor fields, can be introduced for manifolds with a boundary. An obvious exception is the theorem of the existence of the local flow, which must be modified, because we must consider only vector fields that at the points of the boundary are tangent to it. We mention also that using the definition of the differential structure on the manifolds $\mathcal{M}$ and $\partial\mathcal{M}$ it is easily proved that the inclusion map $j : \partial\mathcal{M} \mapsto \mathcal{M}$ is differentiable.

Definition 11.44. *An n-dimensional real manifold $\mathcal{N}$ is called orientable if on it there exists n-form α, different from zero at each point. Such a form is called a volume form.*

For example, in $\mathbb{R}^n$ with coordinates $(x^1, x^2, \ldots, x^n)$, we have the following volume form, which is called the *canonical volume form of* $\mathbb{R}^n$:

$$\alpha = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n \ . \tag{11.148}$$

It can be shown that orientability is equivalent to the existence of an atlas $\{(U_i, \varphi_i)\}_{i\in I}$ of $\mathcal{N}$, such that the Jacobi matrices of all the transition maps $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}$ (see (11.3)), have positive determinants:

$$\det D(\varphi_{ij})|_{\varphi_j(m)} > 0; \quad m \in U_i \cap U_j \tag{11.149}$$

(of course, for all i, j for which φ_{ij} makes sense). Such an atlas is called oriented atlas and a choice of an oriented atlas is called an *orientation* of $\mathcal{N}$. It can also be proved that an orientation of an orientable manifold can also be fixed, if we fix a volume form β. Then one can pick an atlas $\mathcal{A} = (U_s, \varphi_s)_{s\in J}$ such that in the local expression

$$\beta|_{U_i} = f_i(x^1, x^2, \ldots, x^n) dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n \tag{11.150}$$

of β in each $(U_i, \varphi_i) \in \mathcal{A}$ all the coefficient functions $f_i(x^1, x^2, \ldots, x^n) > 0$. Conversely, if we have an oriented atlas $\mathcal{A}$ there exist a volume form β such that in the charts of the atlas the coefficient functions f_i defined as in (11.150) are always positive.

For example, $\mathbb{R}^n$ is always oriented using the volume form (11.148), and in the atlas consisting of a single chart $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$ the coefficient function is equal to 1.

Definition 11.45. *We say that two orientations, defined by two oriented atlases $\mathcal{A} = (U_s, \varphi_s)_{s\in I}$ and $\mathcal{B} = (V_k, \psi_k)_{k\in J}$ are equivalent (or are the same) if the determinants of the Jacobi matrices of each of the transition maps $\varphi_s \circ \psi_k^{-1}$ are positive. (Of course, for those $s \in I$, $k \in J$ for which $\varphi_s \circ \psi_k^{-1}$ makes sense)*

Definition 11.46. *Two volume forms α_1 and α_2 are called equivalent, if there exists a positive number c such that $\alpha_1 = c\alpha_2$.*

The above relations are indeed relations of equivalence, and classes of equivalent orientations and classes of equivalent volume forms become one to-one correspondence. In case the atlas $\mathcal{A} = (U_s, \varphi_s)_{s\in I}$ and the volume form β define the same orientation, the coefficient functions f_i from the local expressions (11.150) are positive. We say that the volume form is compatible with the orientation.

One additional remark here is that if α is a volume form, then $-\alpha$ is a volume form too. The orientations defined by α and $-\alpha$ are called *opposite orientations*. It is not hard to prove that when $\mathcal{N}$ is connected on $\mathcal{N}$ there are exactly two different orientations.

In what follows $\mathcal{N}$ will be connected, orientable, with a fixed orientation, that is with a fixed orientated atlas $\mathcal{A} = (U_i, \varphi_i)_{i\in I}$. If $\mathcal{N}$ is compact,

n-dimensional manifold, or the n-form β has compact support, then for one can define an integral of β over $\mathcal{N}$. Suppose first that the support of β is contained in the open set U_i, where (U_i, φ_i) is one of the charts belonging to some oriented atlas, defining the orientation of $\mathcal{N}$. Then for $p \in U_i$ we have on coordinates $x^s(p) = [\varphi_i(p)]^s$, and the form β has local form:

$$\beta|_{U_i} = h_i(x^1, x^2, \ldots, x^n) dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n \ . \tag{11.151}$$

We set:

$$\int_{\mathcal{N}} \beta = \int_{\varphi_i(U_i)} h_i(x^1, x^2, \ldots, x^n) dx^1 dx^2 \ldots dx^n \ , \tag{11.152}$$

where the integral is understood in the Riemann sense. The fact that we have an oriented atlas can be used to show that this value is the same, if the support of β is in some other parameterized neighborhood, and we use some different parametrization for calculating the above integral. In the general case, when the support of β is not in some U_i one can divide the support of β into pieces, each of which has the above property, which is a little old technique or use the so-called partition of the unity to write β as a locally finite sum of forms, each of which have support lying in some U_i to define the integral. As can be seen, the map:

$$\alpha \mapsto \int_{\mathcal{N}} \alpha \tag{11.153}$$

is a linear functional from the linear space of the n-forms having compact support (in case $\mathcal{N}$ is compact this is simply $\Lambda^n(\mathcal{N})$) to $\mathbb{R}$.

The fact that if α is a volume form implies that there exists oriented atlas such that in the local expressions (11.150) defined by that atlas all the functions f_i are positive. From here one can get the following important fact:

Proposition 11.47. *Let $\mathcal{N}$ be oriented compact manifold and α is a volume form compatible with the orientation of $\mathcal{N}$. Then*

$$\int_{\mathcal{N}} \alpha > 0 \ . \tag{11.154}$$

When $\mathcal{N}$ is a manifold with a boundary $\partial\mathcal{N}$, the orientation of $\mathcal{N}$ defines an orientation of $\partial\mathcal{N}$ in the sense that if we have an oriented atlas of $\mathcal{N}$ its restriction to $\partial\mathcal{N}$ is oriented atlas of $\partial\mathcal{N}$. The resulting orientation is called the *induced orientation.*

If the orientation of $\mathcal{N}$ is given the so-called *canonical orientation* of $\partial\mathcal{N}$ it is defined as follows: If $n = \dim(\mathcal{N})$ is odd, the canonical orientation is the induced orientation and when n is even, it is the opposite to the induced orientation. With all these definitions and conventions, we have the following result, known as Stokes formula (theorem):

Theorem 11.48. *Let $\mathcal{N}$ be oriented, connected and compact n-dimensional manifold with boundary $\partial\mathcal{N}$, canonically oriented. Let $j : \partial\mathcal{N} \mapsto \mathcal{N}$ be the inclusion map. Then if β is a $n-1$ form on $\mathcal{N}$ we have*

$$\int_{\mathcal{N}} d\beta = \int_{\partial\mathcal{N}} j^*\beta \, . \tag{11.155}$$

In the case the manifold $\mathcal{N}$ has no boundary, the above result reads:

$$\int_{\mathcal{N}} d\beta = 0 \, . \tag{11.156}$$

Concluding this section we stress again that apart from some difficulties of topological and analytical character, almost all the principal geometric objects can be generalized in a natural way to the infinite-dimensional case.

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12

Hamiltonian Dynamics

It is well known that the principal objects needed to define Hamiltonian dynamics are the Poisson brackets, [1, 2]. In this chapter, we discuss the different ways one can define them – through symplectic structures and more generally through Poisson structures. Naturally, here arise the questions of restriction of these structures on submanifolds, and we give some attention to this topic, since we shall use the restriction techniques heavily in the future. Finally, we discuss the questions of integrability of Hamiltonian systems and introduce in relation with the integrability questions the principal geometric object we study in this part of the book – the Nijenhuis tensor.

12.1 Symplectic Structures

The classical way to define Poisson brackets, known from any course of Mechanics, is the following. Let $\mathbb{R}^{2n} = \mathbb{R}^n_p \times \mathbb{R}^n_q$ be the $2n$ dimensional Euclidean space. The notation $\mathbb{R}^{2n} = \mathbb{R}^n_p \times \mathbb{R}^n_q$ means that the points x of $\mathbb{R}^{2n}$ are written into the form:

$$x = (p, q) = (p_1, p_2, \ldots, p_n, q^1, q^2, \ldots, q^n) \ . \tag{12.1}$$

Suppose $f(x), g(x)$ are smooth functions over $\mathbb{R}^{2n}$. Then the classical Poisson bracket $\{f, g\}$ is defined as

$$\{f, g\} = \sum_{i=1}^{n} (\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q^i}) \ . \tag{12.2}$$

The Poisson bracket of f and g is a bilinear operation and $\{f, g\}$ has the properties:

$$\{f, g\} = -\{f, g\} \quad \text{(skew-symmetry)}$$
$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0 \quad \text{(Jacobi identity)}$$

Gerdjikov, V.S. et al.: *Hamiltonian Dynamics*. Lect. Notes Phys. **748**, 407–458 (2008)
DOI 10.1007/978-3-540-77054-1_12

$$\{f, gh\} = \{f, g\}h + g\{f, h\} \quad \text{(Leibnitz rule)} \ . \tag{12.3}$$

In Classical Mechanics, the equations of motion of a mechanical system can be written into the so-called Hamiltonian or canonical form:

$$\dot{p}_i = \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}, \qquad \dot{q}^i = \frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}; \qquad i = 1, 2, \dots n \ , \tag{12.4}$$

where H is some function on (p, q) called Hamiltonian function (or simply Hamiltonian) of the system. The variables p_i are then called the generalized momenta and q^i the generalized coordinates of the corresponding mechanical system. Together, (p, q) are called canonical coordinates. The above form of the equations of motion is called the canonical form of these equations.

As immediately checked, the equations of motion can be cast also into the equivalent form

$$\frac{dp_i}{dt} = \{H, p_i\}, \qquad \frac{dq^i}{dt} = \{H, q^i\}; \qquad i = 1, 2, \dots n \ , \tag{12.5}$$

and moreover, if $(p(t), q(t)) = x(t)$ is solution of the canonical equations then for each function $f = f(x)$, we have

$$\frac{df(x(t))}{dt} = \{H, f\}(x(t)) \ . \tag{12.6}$$

Then on condition that the evolution equation

$$\frac{df}{dt} = F(f(x(t))) \tag{12.7}$$

can be written into the form (12.6), we say that it is in Hamiltonian form with Hamiltonian function H.

Thus, for writing the evolution equation for some function on $\mathbb{R}^{2n}$ (the phase space), one needs only to know how to calculate Poisson brackets. This simple observation permits to make immediately important generalizations. Indeed, suppose that we have some way of defining Poisson brackets on some manifold $\mathcal{M}$, that is, we can define on the space of the smooth functions on $\mathcal{M}$, a bracket operation satisfying (12.3). Then, one can write equations of motion and generalize the whole Hamiltonian Mechanics, that is, one is no more restricted to $\mathbb{R}^{2n}$, the splitting into p and q coordinates, and so on. In case we have Poisson brackets and the evolution equation of the type (12.7) can be written in Hamiltonian form with some H, we say that (12.7) has (or possesses) Hamiltonian structure.

Now we discuss how one can define Poisson brackets.

First of all, one must mention the direct generalization of the classical Poisson brackets. It turns out that the existence of Poisson brackets for functions over $\mathbb{R}^{2n}$ is due to the existence of the following 2-form

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq^i \ , \tag{12.8}$$

called canonical symplectic form of $\mathbb{R}^{2n}$. It actually permits to establish isomorphism between the differentials of functions *df* and the so-called Hamiltonian vector fields: $df \mapsto X_f$, where X_f is defined uniquely by the property $X_f(g) = \{f, g\}$. Taking a general point of view, we have

Definition 12.1. *A symplectic form ω over the real manifold $\mathcal{M}$ is a closed, nondegenerate 2-form. The manifold $\mathcal{M}$ equipped with symplectic form is called a symplectic manifold and denoted by $(\mathcal{M}, \omega)$. We say also that on $\mathcal{M}$ it is defined a symplectic structure ω.*

The nondegeneracy means that for each $\alpha \in \Lambda^1(\mathcal{M})$, there exists unique vector field X_α over $\mathcal{M}$ such that

$$i_{X_\alpha}\omega = \alpha \ . \tag{12.9}$$

As can be seen, the nondegeneracy of ω entails that the dimension of the manifold is even, so below we shall suppose that it is equal to $2n$. Also, as $\omega^n \neq 0$ (the product here is of course the exterior product), such manifolds are always orientable; see definition (11.44).

Definition 12.2. *If $(\mathcal{M}, \omega)$ is a symplectic manifold, it is called exact if the form ω is exact, that is, if there exists 1-form α, such that $d\alpha = \omega$.*

All orientable geometric surfaces (the connected manifolds of dimension 2 embedded in $\mathbb{R}^3$) are symplectic manifolds. In particular, the sphere $\mathbb{S}^2$ is a symplectic manifold. Indeed, as their dimension equals 2, for these manifolds being orientable is equivalent to being symplectic.

Applying the Stokes theorem (11.155) to a compact, orientable geometric surface S, we see that it cannot be an exact symplectic manifold. Indeed, if $\omega = d\alpha$, then

$$\int_S \omega = \int_{\partial S} \alpha = 0 \tag{12.10}$$

as the boundary $\partial S = \emptyset$. But this is a contradiction, since ω in that case is a volume form and that integral cannot be equal to zero.

There exist manifolds of even dimension, which are not symplectic, for example, the Möbius strip is a geometric surface which is not orientable, hence it is not symplectic. Another example is provided by the spheres $\mathbb{S}^{2k}$, $k > 1$, which are not symplectic manifolds, though they are orientable. Indeed, it is known that for $k > 1$ the de Rham cohomologies $H^2(\mathbb{S}^{2k}) = 0$, that is, all 2-forms on $\mathbb{S}^{2k}$ are exact. Suppose that $\omega = d\alpha$ is a symplectic form. In that case, it is easily seen that since $d\omega = 0$, we have $d(\alpha \wedge \omega^{k-1}) = \omega^k$. Then an argument, analogous to that we used before, applied to the volume form ω^k leads to contradiction, since the integral of a volume form over $\mathbb{S}^{2k}$ cannot be equal to zero.

Definition 12.3. *If H is some smooth function, then the unique vector field X_H satisfying $i_{X_H}\omega = -dH$ is called the Hamiltonian vector field corresponding to the Hamiltonian function H (or simply to the Hamiltonian H). If α is a closed* 1*-form, then the field X_α such that $i_{X_\alpha}\omega = \alpha$ is called the Hamiltonian vector field corresponding to α (sometimes generalized Hamiltonian vector field corresponding to α).*

Remark 12.4. Since locally each closed form is exact, it is clear that the two notions of Hamiltonian fields locally coincide.

The theory of symplectic manifolds is rich and well developed; see [3, 4, 5] and the series [6, 7]. Even a brief review of it is not an easy task and is not within the scope of the present work. The reader can find extensive bibliography, for example, in [5], where important generalizations as Poisson manifolds (we shall discuss them later) are also discussed, together with some topological results. There are also generalizations for the infinite-dimensional case, see for example, [8, 9], where the Hamiltonian structures of the soliton equations are studied. The entire second part of the present book is also related to that issue. The interest in Hamiltonian structures arose, after it was realized that the soliton equations are Hamiltonian systems. A little later came the understanding that for the infinite-dimensional system one can introduce the notion of Liouville integrability [10]. Further development came with the discovery of the so-called Adler scheme and its variations (see [11, 12, 13, 14, 15, 16, 17, 18]) which use Hamiltonian structures related to Lie algebras in a crucial way. Finally, what is most important for our approach, F. Magri discovered the bi-Hamiltonian formulation of the KdV equation [19]. This brought into light another important geometric object – that of the Nijenhuis tensor.

Let us introduce some important notions and elementary facts for the finite dimensional case.

It can be shown [3, 4, 5] that at least locally a finite dimensional symplectic manifold is isomorphic to $\mathbb{R}^{2n}$, equipped with the symplectic structure (12.8), that is, (at least locally) there exists coordinate frame (local chart) with local coordinates (p_i, q^j), $1 \le i, j \le n$ in which the expression of ω is exactly given by (12.8). This result is called *Darboux theorem*, the corresponding frame is called *Darboux frame*, and the corresponding coordinates (p_i, q^j) *canonical coordinates*. One can check that in a Darboux frame the differential equations corresponding to the dynamical system X_H are exactly (12.8).

From what has been said it follows that nondegenerate 2-form ω defines field of invertible linear maps[1] $m \to \bar{\omega}_m$:

$$\bar{\omega}_m : T_m(\mathcal{M}) \to T_m(\mathcal{M})^* \;, \tag{12.11}$$

such that

$$\omega(X, Y) = \langle \bar{\omega}(X), Y \rangle = -\langle X, \bar{\omega}(Y) \rangle; \quad X, Y \in \mathcal{T}(\mathcal{M}) \;. \tag{12.12}$$

[1] Sometimes if no confusion is possible, we shall denote it by the same letter ω

Example 12.5. Consider the manifold $\mathcal{M} = \mathbb{R}^{2n}$. Each symplectic form over this manifold must be of the form

$$\omega = \frac{1}{2} \sum_{i,j=1}^{2n} \omega_{ij}(x) dx^i \wedge dx^j \ , \tag{12.13}$$

where $(\omega_{ij}(x))_{1 \leq i,j \leq 2n}$ is some skew-symmetric ($\omega_{ij}(x) = -\omega_{ji}(x)$), nondegenerate matrix, possibly depending on x. The requirement that ω is closed is equivalent to the requirement

$$\frac{\partial}{\partial x^i}\omega_{jk}(x) + \frac{\partial}{\partial x^j}\omega_{ki}(x) + \frac{\partial}{\partial x^k}\omega_{ij}(x) = 0 \ , \tag{12.14}$$

for each three different indices i, j, k. Then the Hamiltonian vector field corresponding to the function $f \in \mathcal{D}(\mathcal{M})$ is

$$X_f = - \sum_{j=1,j}^{2n} \omega^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j} \ , \tag{12.15}$$

where the matrix $(\omega^{ij}(x))_{1 \leq i,j \leq 2n}$ is the inverse of the matrix $(\omega_{ij}(x))_{1 \leq i,j \leq 2n}$.

Example 12.6. Let $\mathcal{M} = E = \mathcal{S}$ be space of the Schwartz-type functions on the line. By analogy with the finite dimensional case, we can assume that a symplectic form can be written locally as

$$\omega = \frac{1}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx\, dy \omega[u](x,y) \delta u(x) \wedge \delta u(y) \ , \tag{12.16}$$

the function $\omega[u](x, y)$ being some skew-symmetric, nondegenerate kernel. The nondegeneracy means that if for some element $f \in \mathcal{S}$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \omega[u](x,y) f(x) g(y) dx dy = 0 \tag{12.17}$$

for all $g \in \mathcal{S}$ then $f = 0$. The requirement $d\omega = 0$ is equivalent to

$$\frac{\delta \omega[u](x,y)}{\delta u(z)} + \frac{\delta \omega[u](y,z)}{\delta u(x)} + \frac{\delta \omega[u](z,x)}{\delta u(y)} = 0 \ . \tag{12.18}$$

Then the vector field X_H will be Hamiltonian (corresponding to Hamiltonian function H) if:

$$i_{X_H} \omega + dH = 0 \ , \tag{12.19}$$

which means that

$$\int_{-\infty}^{+\infty} \omega(x,y) X_H[u](y) dy = -\frac{\delta H}{\delta u(x)} . \tag{12.20}$$

or

$$\bar{\omega}(X_H)([u]) = -dH[u] , \tag{12.21}$$

where the operator $\bar{\omega}[u] : T_u(\mathcal{M}) \to T^*_u(\mathcal{M})$ is given by

$$\xi(x) \mapsto \int_{-\infty}^{+\infty} \omega(x,y)\xi(y) dy . \tag{12.22}$$

Let us take an example from the theory of the KdV equation

$$u_t = -uu_x - u_{xxx} . \tag{12.23}$$

The vector field defining this equation is:

$$X = \int_{-\infty}^{\infty} X[u](x) \frac{\delta}{\delta u(x)} dx, \qquad X[u](x) = -uu_x - u_{xxx} . \tag{12.24}$$

As mentioned before, the operator $\bar{\omega}$ for the KdV equation is given by

$$\bar{\omega} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx dy \delta'(x-y) \delta u(x) \otimes \delta u(y) , \tag{12.25}$$

where $\delta(x-y)$ is the Dirac "function," and one can check that the Hamiltonian for X is

$$H[u] = -\int_{-\infty}^{+\infty} \left(-\frac{1}{6}u^2 + \frac{1}{2}u_x^2\right) dx . \tag{12.26}$$

Coming back to the general situation, the symplectic form on $\mathcal{M}$, or as it is usually called, a symplectic structure on $\mathcal{M}$, permits to define Poisson brackets on the ring $\mathcal{D}(\mathcal{M})$ of the smooth functions over $\mathcal{M}$. Let us remind this classical construction; for more details see [3, 4].

On the first place, the condition $d\omega = 0$, written in terms of the field $\bar{\omega}$ runs as follows:

$$\begin{aligned} &[d\bar{\omega}(X_1)](X_2, X_3) + [d\bar{\omega}(X_2)](X_3, X_1) + [d\bar{\omega}(X_3)](X_1, X_2) \\ &+ X_1\langle\bar{\omega}(X_2), X_3\rangle + X_2\langle\bar{\omega}(X_3), X_1\rangle + X_3\langle\bar{\omega}(X_1), X_2\rangle = 0 , \end{aligned} \tag{12.27}$$

for arbitrary vector fields X_1, X_2, X_3.

We need now one more identity. Suppose $\alpha = \bar{\omega}(X)$ is closed. Calculating, we get that

$$d[\bar{\omega}(X)](Y,Z) = Y\langle\bar{\omega}(X), Z\rangle - Z\langle\bar{\omega}(X), Y\rangle - \langle\bar{\omega}(X), [Y,Z]\rangle \, . \tag{12.28}$$

This means that the condition $d[\bar{\omega}(X)](Y,Z) = 0$ is equivalent to

$$Y\langle\bar{\omega}(X), Z\rangle - Z\langle\bar{\omega}(X), Y\rangle - \langle\bar{\omega}(X), [Y,Z]\rangle = 0 \tag{12.29}$$

for arbitrary vector fields Y and Z.

Now let us introduce

Definition 12.7. *Let α_1, α_2 be two closed 1-forms, and let the corresponding Hamiltonian fields be X_1, X_2. Then, we define the Poisson bracket of α_1, α_2 as*

$$\{\alpha_1, \alpha_2\} = -d\langle\bar{\omega}(X_1), X_2\rangle = -d\langle\alpha_1, X_2\rangle = d\langle\alpha_2, X_1\rangle \, . \tag{12.30}$$

We have

Theorem 12.8. *The set of closed forms over a symplectic manifold is a Lie algebra with respect to the Poisson bracket operation.*

Proof. Since the bracket is skew-symmetric, it is clear that it is enough to check that the brackets satisfy the Jacobi identity. For this, let X_i, $i = 1,2,3$ be vector fields, such that the forms $\alpha_i = \bar{\omega}(X_i)$ are closed. Then, from (12.27) we get

$$\begin{aligned} &X_1\langle\bar{\omega}(X_2), X_3\rangle + X_2\langle\bar{\omega}(X_3), X_1\rangle + X_3\langle\bar{\omega}(X_1), X_2\rangle = \\ &\langle X_1, \{\alpha_2, \alpha_3\}\rangle + \langle X_2, \{\alpha_3, \alpha_1\}\rangle + \langle X_3, \{\alpha_1, \alpha_2\}\rangle = 0 \, . \end{aligned} \tag{12.31}$$

Finally, applying the exterior derivative d, we get the Jacobi identity for the bracket (12.30). The theorem is proved.

We have the following property of the bracket operation.

Proposition 12.9. *If $\alpha_1 = \bar{\omega}(X_1)$, $\alpha_2 = \bar{\omega}(X_2)$ and $d\alpha_1 = d\alpha_2 = 0$ then $\{\alpha_1, \alpha_2\} = \bar{\omega}([X_1, X_2])$.*

Proof. We use again (12.27) with arbitrary X_3 and X_1, X_2 as in the above and get

$$d[\bar{\omega}(X_3)](X_1, X_2) + X_2\langle\bar{\omega}(X_3), X_1\rangle - X_1\langle\bar{\omega}(X_3), X_2\rangle + X_3\langle\bar{\omega}(X_1), X_2\rangle = 0 \, . \tag{12.32}$$

Taking into account (12.28), we deduce that

$$\langle\bar{\omega}(X_3), [X_1, X_2]\rangle = X_3\langle\bar{\omega}(X_1), X_2\rangle = \langle X_3, \{\alpha_1, \alpha_2\}\rangle \, . \tag{12.33}$$

In order to complete the proof, it remains to note that X_3 is arbitrary.

The above proposition shows that the existence of the Poisson bracket structure is due to the Lie algebra structure over the module $\mathcal{T}(\mathcal{M})$. It is transferred to $\Lambda^1(\mathcal{M})$ via the isomorphism $\bar{\omega}$. We can define now Poisson brackets of functions:

Definition 12.10. *Let f_1, f_2 be two functions, and let X_1, X_2 be vector fields, such that $\bar{\omega}(X_i) = -df_i$; $i = 1, 2$. Then let us set*

$$\{f_1, f_2\} = \langle \bar{\omega}(X_1), X_2 \rangle = \omega(X_1, X_2) = X_1 f_2 = -X_2 f_1 \ . \tag{12.34}$$

Theorem 12.11. *The formula (12.34) defines Poisson brackets on the set of smooth functions $\mathcal{D}(\mathcal{M})$ over $\mathcal{M}$.*

Proof. Again, it suffices to prove only the Jacobi identity. Let f_1, f_2, f_3 be three functions. Let us put $\alpha_i = -df_i; \quad i = 1, 2, 3$. Clearly α_i are closed, and for them relation (12.31) is satisfied. This relation is exactly the Jacobi identity. The theorem is proved.

Let us now give some definitions about the first integrals for the Hamiltonian systems (Hamiltonian vector fields).

Definition 12.12. *Let α_1, α_2 be closed 1 forms on a symplectic manifold. We say that they are in involution if $\{\alpha_1, \alpha_2\} = 0$. Let f_1, f_2 be two smooth functions on a symplectic manifold. We say that they are in involution if $\{f_1, f_2\} = 0$.*

If X_1, X_2 are the Hamiltonian vector fields that correspond to closed 1-forms α_1, α_2 the proposition (12.9) shows that the relation $\{\alpha_1, \alpha_2\} = 0$ implies $[X_1, X_2] = 0$, and, therefore, the flows of the fields X_1, X_2 commute. If $\alpha_1 = -df_1$, $\alpha_2 = -df_2$ and $\{f_1, f_2\} = 0$, we can say more; in this case, the properties of the Poisson brackets (12.34) show that $X_1 f_2 = 0$ and $X_2 f_1 = 0$, that is, f_2 is first integral for X_1 and f_1 is first integral for X_2. From the Jacobi identity follows a well-known property of the first integrals for a Hamiltonian system.

Proposition 12.13. *Let X_H be the dynamical system corresponding to the Hamiltonian function H and let f_1, f_2 be first integrals of X_H. Then their Poisson bracket $\{f_1, f_2\}$ is also a first integral of X_H. Therefore, the first integrals of X_H form a Lie algebra.*

From the general theory of systems of first-order differential equations, one would expect that on symplectic manifold of dimension $2n$, in order to integrate a Hamiltonian vector field X_H with Hamiltonian H in quadratures, one needs $2n-1$ functionally independent first integrals. As we shall see later, the situation with a Hamiltonian vector field X_H is different, for it, one needs only $n-1$ first integrals in involution, which together with H form a set of n functionally independent first integrals.

12.1.1 Fundamental Fields of Symplectic Form

Let $(\mathcal{M}_1, \omega_1)$ and $(\mathcal{M}_2, \omega_2)$ be symplectic manifolds, and let $h : \mathcal{M}_1 \mapsto \mathcal{M}_2$ be a differentiable map. The map is called a symplectic map, if $h^* \omega_2 = \omega_1$. It is called a symplectomorphism if h is symplectic and is a diffeomorphism.

Also, when $\mathcal{M}_1 = \mathcal{M}_2$ and $\omega_1 = \omega_2$ instead of symplectomorphism, some authors say that the map h is a canonical map (transformation). We shall prefer to say that the map is symplectomorphism, since there can be confusion with the maps (transformations) that preserve only the form of the canonical equations (but not necessarily the Hamiltonian functions). These transformations are also called canonical transformations, but they form a bigger group than the group of symplectomorphisms. As is well known, the canonical transformations play a crucial role in the Hamilton-Jacobi equation formalism for solving the canonical equations.

As is common for geometric structures, it is interesting to study the fields whose flows preserve this structure. Such fields are called fundamental fields of the corresponding structure. In our case, we consider the fields whose flows preserve the symplectic form of a given symplectic manifold $(\mathcal{M}, \omega)$. Definition (11.17) and equation (11.83) show that X is a fundamental field for ω if that $L_X\omega = 0$. This means that

$$L_X\omega = i_X d\omega + d i_X\omega = d i_X\omega = 0 , \tag{12.35}$$

which shows that the fundamental fields correspond, through the isomorphism is $\bar{\omega}$ to the closed 1-forms. If the local flow of X is φ_t, we get

$$\varphi_t^*\omega = \omega , \tag{12.36}$$

that is, the maps φ_t are symplectic maps.

Probably the most important example of a symplectic structure is the so-called canonical symplectic structure on the cotangent bundle $T^*(\mathcal{M})$ of a given finite dimensional manifold $\mathcal{M}$ of dimension n. If λ is the Liouville form over $T^*(\mathcal{M})$, (see 11.124) one can define a symplectic structure setting $\omega = d\lambda$. In the coordinates p_i, q^i, (see 11.121) $d\lambda$ has exactly the form (12.8), that is, equals

$$d\lambda|_{p_M^{-1}(U_i)} = d\left(\sum_{i=1}^{n} p_i dq^i\right) = \sum_{i=1}^{n} dp_i \wedge dq^i \tag{12.37}$$

and the nondegeneracy of $d\lambda$, as well as its closure are evident. The symplectic structure defined by $d\lambda$ plays a central role in the Classical Hamiltonian Mechanics; see for example [3, 4]. In case of $\mathbb{R}^{2n}$ endowed with the canonical form (12.8), the volume form ω^n is proportional to the canonical volume form of $\mathbb{R}^{2n}$:

$$\alpha = dp_1 \wedge dp_2 \wedge \ldots \wedge dp_n \wedge dq^1 \wedge dq^2 \wedge \ldots \wedge dq^n . \tag{12.38}$$

Though simple, the above facts have important applications. For example, a direct consequence from it and (12.35), (12.36) is the Liouville theorem, which is basic in the Classical Statistical Mechanics.

Theorem 12.14. *The local flow of a Hamiltonian vector field X over a symplectic manifold $\mathcal{M}$ of dimension $2n$ with symplectic structure ω preserves the volume form ω^n. In other words, if $G \subset \mathcal{M}$ is a bounded region in $\mathcal{M}$, then*

$$\int\limits_{G} \omega^n = \int\limits_{\varphi_t(G)} \omega^n \,. \tag{12.39}$$

12.1.2 Restriction of a Symplectic Structure on Submanifold

The question of restriction of symplectic structure on submanifolds is of great importance for the applications, because the restriction on submanifold reduces the problem of dynamics with constraints. It is known that a number of physical theories are formulated in terms of the so-called singular Lagrangian functions, and this does not allow immediately a Hamiltonian formulation over a symplectic manifold, [20]. However, the canonical coordinates are very crucial for the quantization and for this reason, the technique devised by Dirac (Dirac brackets, see below) and also other techniques of reduction, (see for example [21]) play an important role in the quantization. We shall not treat the Lagrangian formalism here, and so we cast the problem into purely "symplectic" terms.

Let $\mathcal{N} \subset \mathcal{M}$ be a smooth submanifold of the symplectic manifold $(\mathcal{M}, \omega)$. Let $j : \mathcal{N} \to \mathcal{M}$ be the inclusion map. Then, though the restricted form $j^*\omega$ is closed, nothing guarantees that it will be a symplectic form on $\mathcal{N}$ since it can be degenerate. At that point naturally two questions arise.

First, assume that $j^*\omega$ is nondegenerate. Then, since we want to formulate everything in terms of the quantities related of $\mathcal{N}$, we want to know how to calculate the Poisson brackets of the restrictions $f^* = f \circ j$, $g^* = g \circ j$ of functions f, g defined on $\mathcal{M}$ with respect to the new structure. This is the so-called Dirac brackets problem.

The second question arises if $j^*\omega$ is degenerate. Then, it cannot be used to define Poisson brackets on $\mathcal{N}$, so we are forced to consider submanifolds of $\mathcal{N}$ on which the restriction of ω is nondegenerate. We now concentrate only on this issue and consider the following problem. Suppose $\mathcal{N}$ is a manifold and ω_0 is closed, but degenerate 2-form over it. Is it possible to restrict ω_0 over some submanifold of $\mathcal{N}$, in order to obtain a symplectic form? For simplicity, we shall assume that the kernel of the map $(\bar{\omega}_0)_m : T_m(\mathcal{N}) \to T^*_m(\mathcal{N})$ defined by ω_0 has constant dimension.[2] They say also that the 2-form ω has constant rank.

Remark 12.15. The rank of $(\omega_0)_m$ is the rank of the linear map $(\bar{\omega}_0)_m$: $T_m(\mathcal{N}) \mapsto T^*_m(\mathcal{N})$ that corresponds to ω_0.

[2] In the future, when there is no possibility of confusion, we shall denote by the same letter ω a 2-form ω and the field of linear maps from $T_m(\mathcal{N})$ to $T^*_m(\mathcal{N})$ induced by it, but at present we keep the notation $\bar{\omega}$ for the field of the linear maps.

We see that if the rank is constant the field of subspaces

$$m \to \ker(\bar{\omega}_0)_m \tag{12.40}$$

is a regular distribution.

Definition 12.16. *A manifold $\mathcal{N}$ equipped with closed form ω_0 having constant rank is called a presymplectic manifolds and the form ω_0 a presymplectic form or presymplectic structure. The presymplectic manifold equipped with a form ω_0 is denoted by $(\mathcal{N}, \omega_0)$.*

It is instructive to consider first the situation for vector spaces. So let E be a vector space, equipped with 2-form ω_0. If ω_0 is nondegenerate we call E symplectic vector space. If F is a subspace in E then, as easily seen, the restriction of ω_0 on F will be nondegenerate if and only if

$$F^{\perp} \cap F = \{0\} \ , \tag{12.41}$$

where

$$F^{\perp} = \{X : \omega_0(X, Y) = 0, \quad \text{for all} \ \ Y \in F\} \ . \tag{12.42}$$

The space $F^{\perp}$ is sometimes called *skew-orthogonal to* F, or in case ω_0 is symplectic, *symplectically orthogonal to* F. Let us give some definitions related to the notions we introduced.

Definition 12.17. *Let E be a vector space endowed with a skew-symmetric form ω_0 (not necessarily symplectic). Then*

- *If $W \subseteq W^{\perp}$, W is called isotropic.*
- *If $W \supseteq W^{\perp}$, W is called coisotropic.*
- *If (E, ω_0) is a symplectic vector space of dimension $2n$ and W is isotropic and coisotropic, then W is called a Lagrangian subspace. As easily seen, in this case $\dim(W) = n$.*

If we have a symplectic structure on E, one can see that the maximal dimension of an isotropic vector subspace is the half of the dimension of E, because $\dim W^{\perp} = \dim E - \dim W$ and the definition of an isotropic space entails $\dim W \leq \dim W^{\perp}$. For manifolds, the analogs of the above definitions are as below.

Definition 12.18. *Let $\mathcal{N}$ be some submanifold of the presymplectic manifold $(\mathcal{M}, \omega)$.*

- *$\mathcal{N}$ is called isotropic (coisotropic) if for each point $p \in \mathcal{N}$ the space $dj_p(T_p(\mathcal{N})) \subset T_p(\mathcal{M})$, $p \in \mathcal{N}$ is isotropic (coisotropic) with respect to $\omega(p)$, where j is the inclusion map $j : \mathcal{N} \mapsto \mathcal{M}$.*
- *A submanifold $\mathcal{N}$ of the symplectic manifold $(\mathcal{M}, \omega)$ is called Lagrangian if for each $p \in \mathcal{N}$ the space $dj_p(T_p(\mathcal{N})) \subset T_p(\mathcal{M})$, $p \in \mathcal{N}$ is Lagrangian with respect to $\omega(p)$.*

Now let us return to the case of a vector space E with a 2-form ω_0 on it. In case we are interested in knowing whether the restriction of ω_0 is nondegenerate on some fixed F, we must simply check the relation (12.41). But if we are just looking for a space F on which ω_0 is nondegenerate, there are immediate candidates for such spaces. Indeed, suppose G is some space complementary to $\ker(\bar{\omega}_0)$, that is

$$G \oplus \ker(\bar{\omega}_0) = E\ . \tag{12.43}$$

Then ω_0, restricted to G, is nondegenerate. Indeed, suppose that it is not true. Then there exists $X \neq 0$, $X \in G$, such that $\omega_0(X, Y) = 0$ for all $Y \in G$. Since arbitrary element $Z \in E$ can be written into the form $Z = Z_1 + Z_2$ with $Z_1 \in G$, $Z_2 \in \ker(\bar{\omega}_0)$, we have $\omega_0(X, Z) = 0$ for all $Z \in E$. This means that $X \in \ker(\bar{\omega}_0)$, which is a contradiction.

Now, we want to present an analog of the above construction for manifolds. Suppose that we have presymplectic structure ω_0 on the manifold $\mathcal{N}$. Then we have a field of subspaces $m \to \ker(\bar{\omega}_0)_m$, which as we have seen is a regular distribution. From the fact that ω_0 is closed, there follows that this distribution is integrable in the Frobenius sense. Indeed, if $X, Y, Z \in \mathcal{T}(\mathcal{N})$ and $X_m, Y_m \in \ker(\bar{\omega}_0)_m$ then $d\omega_0(X, Y, Z) = 0$ gives

$$\begin{aligned} &X\omega_0(Y,Z) - Y\omega_0(X,Z) + Z\omega_0(X,Y) - \omega_0([X,Y],Z) \\ &+ \omega_0([X,Z],Y) - \omega_0([Y,Z],X) = 0\ . \end{aligned} \tag{12.44}$$

Evaluating at the point m we get that $\omega_0([X, Y], Z)|_m = 0$ for arbitrary Z. This of course means that $[X, Y]_m$ belongs to $\ker(\bar{\omega}_0)_m$.

Now suppose that $m \mapsto S_m$ is some field of subspaces, transversal to $\ker(\bar{\omega}_0)_m$ that is at each point $m \in \mathcal{N}$

$$\ker(\bar{\omega}_0)_m \oplus S_m = T_m(\mathcal{N})\ . \tag{12.45}$$

As the restriction of ω_0 onto S_m is nondegenerate, it is clear that if there exists a submanifold $\mathcal{S} \subset \mathcal{N}$ such that at each point $m \in \mathcal{S}$ its tangent space $T_m(\mathcal{S})$ is equal to S_m, restricting ω_0 on $\mathcal{S}$, we shall obtain nondegenerate 2-form. In this case, we say that $\mathcal{S}$ *is transversal to the distribution (12.40)*. The above discussion can be summarized as follows.

Theorem 12.19. *If $(\mathcal{M}, \omega_0)$ is presymplectic manifold than each submanifold $\mathcal{N} \subset \mathcal{M}$ transversal to the distribution (12.40) is a symplectic manifold.*

Let us see how the constructions we outlined in the above works.
Let us define the following presymplectic structure on $\mathbb{R}^3 \setminus \{0\}$. For $\mathbf{x} \in \mathbb{R}^3 \setminus \{0\}$ we set

$$\omega_{\mathbf{x}}(\mathbf{z}_1, \mathbf{z}_2) = \langle \mathbf{x}, \mathbf{z}_1 \times \mathbf{z}_2 \rangle; \quad \mathbf{z}_1, \mathbf{z}_2 \in T_{\mathbf{x}}(\mathbb{R}^3 \setminus \{0\}) = \mathbb{R}^3\ . \tag{12.46}$$

(Here $\langle\ ,\ \rangle$ stands for the inner product and $\times$ for the cross product in $\mathbb{R}^3$). The form ω is not symplectic, which is clear even from the fact that the manifold is odd-dimensional. The distribution $\mathbf{x} \to \ker(\bar{\omega}_{\mathbf{x}}) = \mathbb{R}\mathbf{x}$ is integrable, and the integral manifolds for it are the straight lines through the origin. Equally simple is to find transversal manifolds – these are the spheres of radii r embedded in $\mathbb{R}^3 \setminus \{0\}$ in the standard way:

$$\mathbb{S}^2_r = \left\{ \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : \quad x_1^2 + x_2^2 + x_3^2 = r^2 > 0 \right\} . \tag{12.47}$$

According to the above theorem, these manifolds are symplectic manifolds. Let us consider for example the unit sphere $\mathbb{S}^2 = \mathbb{S}^2_1$. The Poisson brackets of the coordinate functions x_i on $\mathbb{S}^2$ can be calculated without difficulties. They are:

$$\{x_i, x_j\} = \epsilon_{ijk} x_k , \tag{12.48}$$

where ϵ_{ijk} is the antisymmetric Levi-Civita symbol.[3] The structure we have just described is often used to give the manifold

$$S^{2N} = \underbrace{\mathbb{S}^2 \times \mathbb{S}^2 \times \ldots \times \mathbb{S}^2}_{N \quad times} \tag{12.49}$$

symplectic structure (taking it in each component to be that introduced in the above). This structure plays an important role in some physical models describing dynamics in magnetic chains.

The next theorem, named after Dirac, is about the brackets now called Dirac brackets, [21, 22, 23], which as we mentioned have numerous applications in Theoretical Physics and Mechanics.

Theorem 12.20. *Let $(\mathcal{M}, \omega)$ be $2n$-dimensional symplectic manifold. Let the submanifold $\mathcal{N} \subset \mathcal{M}$ be defined by the set of $2k$ smooth relations (constraints) $\varphi_s(m) = 0$, or in other words:*

$$\mathcal{N} = \{m : \varphi_s(m) = 0, \ s = 1, 2, \ldots, 2k\} \subset \mathcal{M} . \tag{12.50}$$

Assume that the matrix constructed from the Poisson brackets of the constraints is nondegenerate, that is, for each $m \in \mathcal{N}$:

$$\det\{\varphi_l, \varphi_s\}(m)_{1 \le l,s \le 2k} \neq 0 . \tag{12.51}$$

Then

- *The symplectic form ω has nondegenerate restriction $\bar{\omega}$ on the submanifold $\mathcal{N}$, thus equipping it with symplectic structure.*
- *If f, g are two smooth functions on $\mathcal{M}$ and f^*, g^* are their restrictions to $\mathcal{N}$, then their Poisson bracket with respect to the symplectic structure over $\mathcal{N}$ can be calculated according to the following formula (Dirac Brackets' Formula):*

$$\{f^*, g^*\}(m) = \{f, g\}(m) - \sum_{i,j=1}^{2k} \{f, \varphi_i\} Q^{ij} \{\varphi_j, g\}(m) , \tag{12.52}$$

[3] $\epsilon_{ijk} = 0$, if at least two of the indices coincide, and if (ijk) is a permutation of the indices $1, 2, 3$, then ϵ_{ijk} is equal to the parity of that permutation.

where the matrix $(Q^{ij}(m))_{1\le l,s\le 2k}$ *is the inverse of* $(\{\varphi_l,\varphi_s\}(m))_{1\le l,s\le 2k}$ *and* $m\in\mathcal{N}$*.*

Proof. First, we note that from the assumptions of the theorem, it follows that the 1-forms $d\varphi_i$, $i=1,2,\dots,2k$ are linearly independent, otherwise the matrix $(\{\varphi_l,\varphi_s\})_{1\le l,s\le 2k}$ will be degenerate. Therefore the matrix $(Q^{ij}(m))_{1\le l,s\le 2k}$ exists. Next, as the symplectic structure defines an isomorphism between the modules of vector fields and the module of 1-forms, the Hamiltonian vector fields $X_i=X_{\varphi_i}$, corresponding to the functions φ_i, will be linearly independent too. Also, by the definition of $\mathcal{N}$

$$T_m(\mathcal{N})=\{X_m:\ d\varphi_s|_m(X_m)=0;\ s=1,2,\dots 2k\}\ . \tag{12.53}$$

Clearly, the dimension of $\mathcal{N}$ is $2(n-k)$. It is not difficult to see that since $d\varphi_s(X_l)=\{\varphi_s,\varphi_l\}$ the vector fields X_s are not tangent to the submanifold $\mathcal{N}$ and as $d\varphi_s(X)=\omega(X,X_s)$, at the points $m\in\mathcal{N}$ they span the space $(T_m(\mathcal{N}))^{\perp}$. Therefore,

$$T_m(\mathcal{N})\oplus(T_m(\mathcal{N}))^{\perp}=T_m(\mathcal{M}).$$

Next, let us consider how the vectors decompose in the above splitting. If $X_m\in T_m(\mathcal{M})$, we can write

$$\begin{aligned} X_m&=(X_m-\sum_{i=1}^{2k}c_i(m)X_i(m))+\sum_{i=1}^{2k}c_i(m)X_i(m)\\ c_i(m)&=\sum_{s=1}^{2k}Q^{si}(m)d\varphi_s|_m(X_m)\ . \end{aligned} \tag{12.54}$$

A simple calculation shows that the first term in the brackets belongs to $T_m(\mathcal{N})$ and the second to $(T_m(\mathcal{N}))^{\perp}$. According to our previous discussion, (12.53) ensures that the restriction of ω on $\mathcal{N}$ endows it with symplectic structure. Let us construct the bracket, corresponding to this structure. Suppose that $f,g\in\mathcal{D}(\mathcal{M})$ are two smooth functions, and let X_f,X_g be their Hamiltonian fields. For $Y_m\in T_m(\mathcal{N})$ consider the relations:

$$\begin{aligned} &\omega_m(X_f(m)-\sum_{i,s=1}^{2k}Q^{is}(m)\{f,\varphi_s\}(m)X_i(m),\ Y_m)=\\ &\omega(X_f(m),Y_m)+\sum_{i,s=1}^{2k}Q^{is}(m)\{f,\varphi_s\}(m)d\varphi|_m(Y_m)=\\ &\omega_m(X_f(m),Y_m)=-df|_m(Y_m). \end{aligned}$$

Thus $X_f-\sum\limits_{i,s=1}^{2k}Q^{is}\{f,\varphi_s\}X_i$ is exactly the Hamiltonian vector field corresponding to the restriction f^* of f on $\mathcal{N}$. But then,

$$\{f^*, g^*\} = dg(X_f - \sum_{i,s=1}^{2k} Q^{is}\{f, \varphi_s\}X_i) =$$

$$\{f, g\} - \sum_{i,j=1}^{2k} \{f, \varphi_i\}Q^{ij}\{\varphi_j, g\}.$$

This completes the proof.

12.2 Real and Complex Hamiltonian Systems

The questions about complex Hamiltonian structures and complex Hamiltonian Dynamics and their relation with the real ones is an interesting topic but frequently not given much attention in the books about Hamiltonian Systems (there are exceptions of course; see for example [18]). The relations of a complex Hamiltonian system with the real Hamiltonian system are important, as can be seen, for example, from the first part, where there have been considered reductions of some general complex hierarchies of equations. The above topics also provide interesting examples of restrictions to submanifolds. We shall consider the finite dimensional case; the infinite-dimensional is treated along the same lines. However, in order to avoid the complications related to complex manifolds, we shall consider only the case of finite dimensional symplectic vector spaces.

12.2.1 Complexified Hamiltonian Systems

Let us start with the standard formulation of the Hamiltonian equations of motion for a real dynamical system with n degrees of freedom:

$$\frac{dq^k}{dt} = \{H, q^k\} = \frac{\partial H}{\partial p_k}, \qquad \frac{dp_k}{dt} = \{H, p_k\} = -\frac{\partial H}{\partial q^k}\ , \tag{12.55}$$

where we have the canonical Poisson brackets

$$\{q^k, q^m\} = \{p_k, p_m\} = 0, \qquad \{p_k, q^m\} = \delta_k^m\ , \tag{12.56}$$

between the variables q^k, p_s; $k, s = 1, 2 \ldots, n$, which we shall denote by (p, q). As we know, these brackets exist, because on the phase space $\mathcal{M}^{(n)} = \mathbb{R}^{2n}$, there exist canonical symplectic structure, defined by the symplectic form

$$\omega = \sum_{k=1}^{n} dp_k \wedge dq^k\ . \tag{12.57}$$

In what follows, we shall consider the case when $H = H(p, q)$ is a real-analytic function on $(p, q) \in \mathbb{R}^{2n} = \mathcal{M}^{(n)}$. In order to follow the argument better, the reader can assume at the beginning that the Hamiltonian is of the form:

$$H(p,q) = c + \sum_{m+k\geq 1}^{2n} \sum_{\substack{i_1,\ldots,i_k=1 \\ j_1,\ldots,j_m=1}}^{n} H^{(k,m)}_{i_1,\ldots,i_k;j_1,\ldots,j_m} q^{i_1}\ldots q^{i_k} p_{j_1}\ldots p_{j_m} \ , \tag{12.58}$$

where the coefficients $H^{(k,m)}_{i_1,\ldots,i_k;j_1,\ldots,j_m}$ are real. Such choice contains nontrivial examples and outlines perfectly the ideas.

The Hamiltonian function H defines the Hamiltonian vector field X_H, that is, the unique field satisfying the relation $-dH = i_{X_H}\omega$. The field X_H has the form

$$X_H = \sum_{k=1}^{n} \left(\frac{\partial H}{\partial p_k}\frac{\partial}{\partial q^k} - \frac{\partial H}{\partial q^k}\frac{\partial}{\partial p_k} \right) \ . \tag{12.59}$$

Our next step is to complexify the phase space and to consider complex-valued dynamical variables. To this end, we construct :

$$\mathbb{C}\mathcal{M}^{(n)} = \mathcal{M}^{(2n)}_{\mathbb{C}} = \mathcal{M}^{(n)} \oplus i\mathcal{M}^{(n)} \ . \tag{12.60}$$

The above relation means that the k-th component z_k of $\mathbf{z} \in \mathcal{M}^{(2n)}_{\mathbb{C}}$ is represented uniquely as $x_k + iy_k$ with real x_k and y_k, or in other words that $\mathbf{z} = \mathbf{x} + i\mathbf{y}$, where $\mathbf{x}, \mathbf{y} \in \mathcal{M}^{(n)}$. In terms of the coordinates (p,q), the same will be written as

$$p_k = p_k^0 + ip_k^1, \qquad q^k = q_0^k + iq_1^k; \qquad k = 1,2,\ldots,n \ , \tag{12.61}$$

or introducing n-dimensional column vectors $\mathbf{p}, \mathbf{q}$, the same can be written into the form

$$\mathbf{p} = \mathbf{p}^0 + i\mathbf{p}^1, \qquad \mathbf{q} = \mathbf{q}^0 + i\mathbf{q}^1 \ . \tag{12.62}$$

In other words, we denote by superscript (or subscript) "0" the real part and with superscript (or subscript) "1" the imaginary part of the corresponding variable.

We can treat $\mathbf{z}$, $\mathbf{p}$ and $\mathbf{q}$ also as real column vectors with twice more components:

$$\mathbf{p} = \mathbf{p}^0 + i\mathbf{p}^1 \longrightarrow \left(\mathbf{p}^0, \mathbf{p}^1\right)^t, \qquad \mathbf{q} = \mathbf{q}^0 + i\mathbf{q}^1 \longrightarrow \left(\mathbf{q}^0, \mathbf{q}^1\right)^t \ , \tag{12.63}$$

where the superscript "t" denotes the transposition. Then naturally $\mathbf{z} = \mathbf{x}^0 + i\mathbf{y}^1$ will be equivalent to a $4n$-component column vector:

$$\mathbf{z} = \mathbf{x} + i\mathbf{y} \longrightarrow \left(\mathbf{p}^0, \mathbf{p}^1, \mathbf{q}^0, \mathbf{q}^1\right)^t \ , \tag{12.64}$$

in case we regard the space $\mathcal{M}^{(2n)}_{\mathbb{C}}$ as $4n$ dimensional real vector space. The fact that we denote by the same letter the vectors in the real and complex space will not lead to ambiguity, as it will be clear what case we have in mind.

Each function F on $\mathcal{M}^{(n)}$, that is real-analytic in (p,q), can be extended uniquely to analytic functions on $\mathcal{M}^{(2n)}_{\mathbb{C}}$ simply setting the arguments $\mathbf{p}, \mathbf{q}$

in F to be complex. In order to distinguish the extended functions from the original ones, we shall write $F^{\mathbb{C}} = F^0 + iF^1$ for the extended function. By assumption, among these functions is the Hamiltonian H, so we write

$$H^{\mathbb{C}} = H^0\left(\mathbf{p}^0, \mathbf{p}^1, \mathbf{q}^0, \mathbf{q}^1\right) + iH^1(\mathbf{p}^0, \mathbf{p}^1, \mathbf{q}^0, \mathbf{q}^1)\,, \tag{12.65}$$

where H^0 and H^1 are real and depend on twice more real variables then before.

We can introduce also a complex symplectic form, simply assuming that p, q are as in (12.62):

$$\begin{aligned}\omega_{\mathbb{C}} = \omega_0 + i\omega_1 = \sum_{k=1}^{n} dp_k \wedge dq^k = \\ \sum_{k=1}^{n}\left(dp_k^0 \wedge dq_0^k - dp_k^1 \wedge dq_1^k\right) + i\sum_{k=1}^{n}\left(dp_k^0 \wedge dq_1^k + dp_k^0 \wedge dq_1^k\right),\end{aligned} \tag{12.66}$$

where

$$\omega_0 = \sum_{j=1}^{n}(dp_j^0 \wedge dq_0^j - dp_j^1 \wedge dq_1^j), \quad \omega_1 = \sum_{j=1}^{n}(dp_j^1 \wedge dq_0^j + dp_j^0 \wedge dq_1^j)\,. \tag{12.67}$$

This of course means that we have two symplectic forms, ω_0 and ω_1, defined on $\mathcal{M}_{\mathbb{C}}^{(2n)}$ considered as a real vector space.

Now, in (12.58), we can assume that all the values p, q are complex and H is changed to $H^{\mathbb{C}}$ but of course, the time continues to be real. Thus, we obtain a new dynamical system, which we call the complexified Hamiltonian system. The system reads

$$\begin{aligned}\frac{dp_s^0}{dt} + i\frac{dp_s^1}{dt} &= -\frac{\partial H^{\mathbb{C}}}{\partial q^s} = -\left(\frac{\partial H^0}{\partial q_0^s} + i\frac{\partial H^1}{\partial q_0^s}\right)\,, \\ \frac{dq_0^s}{dt} + i\frac{dq_1^s}{dt} &= \frac{\partial H^{\mathbb{C}}}{\partial p_s} = \left(\frac{\partial H^0}{\partial p_s^0} + i\frac{\partial H^1}{\partial p_s^0}\right)\,.\end{aligned} \tag{12.68}$$

We shall see that the complexified system can be cast in the form of a real Hamiltonian system, but with twice more degrees of freedom. To this end, we use the analyticity properties of $H^{\mathbb{C}}$, which entail that $H^{\mathbb{C}}$ satisfies the Cauchy–Riemann equations with respect to each of the complex variables:

$$\begin{aligned}\frac{\partial H^0}{\partial q_0^k} &= \frac{\partial H^1}{\partial q_1^k}, & \frac{\partial H^1}{\partial q_0^k} &= -\frac{\partial H^0}{\partial q_1^k}\,, \\ \frac{\partial H^0}{\partial p_k^0} &= \frac{\partial H^1}{\partial p_k^1}, & \frac{\partial H^1}{\partial p_k^0} &= -\frac{\partial H^0}{\partial p_k^1}\,,\end{aligned} \tag{12.69}$$

for all $k = 1.\ldots, n$. These relations allow to write the right-hand sides of (12.68) only in terms of the function H^0, considered as a function on the real variables $p_k^0, p_k^1, q_0^k, q_1^k$. A simple calculation shows that (12.68) can be cast into the block matrix form:

$$\frac{d\mathbf{z}}{dt} = S_0 \frac{\partial}{\partial \mathbf{z}} H^{(2n)}, \qquad S_0 = \begin{pmatrix} 0 & 0 & -\mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 \\ 0 & -\mathbf{1} & 0 & 0 \end{pmatrix}, \tag{12.70}$$

where $\mathbf{z}$ is given by the right-hand side of (12.64), $\frac{\partial}{\partial \mathbf{z}} H^{(2n)}$ is the gradient of $H^{(2n)}$, and in order to distinguish the new situation from the old one, we use the notation:

$$H^{(2n)} = H^0(\mathbf{p}^0, \mathbf{q}^0, \mathbf{p}^1, \mathbf{q}^1) . \tag{12.71}$$

The equations (12.70) contain only the real variables $\mathbf{q}^0$, $\mathbf{p}^0$, $\mathbf{q}^1$, $\mathbf{p}^1$, and as can be readily checked, they have the form of equations of motion for a real Hamiltonian system with $2n$ degrees of freedom. This system has Hamiltonian given by (12.71), and the corresponding symplectic form is

$$\omega^{(2n)} = \omega_0 = \sum_{k=1}^{n} dp_k^0 \wedge dq_0^k - \sum_{k=1}^{n} dp_k^1 \wedge dq_1^k . \tag{12.72}$$

This form is defined on the space $\mathcal{M}_{\mathbb{C}}^{(2n)}$ considered a $4n$-dimensional real vector space. The corresponding Poisson brackets between the coordinate functions read:

$$\begin{aligned} &\{p_k^0, q_0^m\}_{\omega_0} = \delta_k^m, \quad \{p_k^1, q_1^m\}_{\omega_0} = -\delta_k^m , \\ &\{p_k^0, q_1^m\}_{\omega_0} = 0, \quad \{p_k^1, q_0^m\}_{\omega_0} = 0 , \end{aligned} \tag{12.73}$$

and the Hamiltonian vector field, corresponding to $H^{(2n)}$, has the form:

$$X_H^{\mathbb{C}} = \sum_{j=1}^{n} \left(\frac{\partial H^0}{\partial p_j^0} \frac{\partial}{\partial q_0^j} - \frac{\partial H^0}{\partial p_j^1} \frac{\partial}{\partial q_1^j} - \frac{\partial H^0}{\partial q_0^j} \frac{\partial}{\partial p_j^0} + \frac{\partial H^0}{\partial q_1^j} \frac{\partial}{\partial p_j^1} \right) . \tag{12.74}$$

The Cauchy–Riemann equations can be used also to express the right-hand sides of (12.68) in terms of H^1, considered a function of the real variables p_k^0, p_k^1, q_0^k, q_1^k. Everything is done in analogous way, as in the case we expressed the right-hand sides through H^0, and we obtain a Hamiltonian system with Hamiltonian H^1. This system is defined on the $\mathcal{M}_{\mathbb{C}}^{(2n)}$ considered a $4n$-dimensional real symplectic vector space but this time with the symplectic form is ω_1. Of course, we have another Poisson bracket here, and we denote it by $\{F, G\}_{\omega_1}$. We leave it to the reader to show that the Hamiltonian vector field, X_{H^1}, corresponding to H^1, after taking into account the Cauchy–Riemann equations, is equal to $X_H^{\mathbb{C}}$, and therefore it defines the same dynamics. Also, one easily checks that the Cauchy–Riemann equations entail

$$\{H^0, H^1\}_{\omega_0} = \{H^0, H^1\}_{\omega_1} = 0 . \tag{12.75}$$

Summarizing, we have obtained that the complexification of a generic Hamiltonian system with a real-analytic Hamiltonian, having n degrees of freedom (on $2n$-dimensional phase space), can be rewritten as a real Hamiltonian system, having $2n$ degrees of freedom (on $4n$-dimensional phase space) in two different ways.

Liouville Integrability[4]

If the initial Hamiltonian system is Liouville integrable, i.e. possesses n independent integrals of motion I_j in involution , then this property can be inherited by $X_H^{\mathbb{C}}$, provided I_j are also real-analytic functions on $\mathcal{M}^{(n)}$. Indeed, let us denote by $I_j^{\mathbb{C}} = I_j^0 + iI_j^1$ their analytic continuation to $\mathcal{M}_{\mathbb{C}}^{(2n)}$. Using the Cauchy–Riemann equations, one verifies that H^0 and H^1, as well as all I_j^0 and I_j^1, are in involution with respect to both the brackets we have introduced:

$$\{H^r, I_j^s\}_{\omega_0} = \{H^r, I_j^s\}_{\omega_1} = 0, \quad \{I_j^r, I_k^s\}_{\omega_0} = \{I_j^r, I_k^s\}_{\omega_1} = 0 \,, \tag{12.76}$$

for $r, s = 0, 1$ and $1 \le j \le n$. Consequently, if the I_0-s and the I_1-s are functionally independent, $X_H^{\mathbb{C}}$ becomes a real Hamiltonian system with $2n$ degrees of freedom, which is Liouville integrable.

Even without complete integrability, given a solution

$$p_s(t; p_k(0), q^k(0)), \quad q^s(t; p_k(0), q^k(0)); \qquad s, k = 1, 2, \ldots, n \tag{12.77}$$

of the initial system (12.55), which depends analytically on the initial conditions $(q(0), p(0))$, this solution can be extended to a solution of the complexified system, and also to a solution of (12.70). (It is enough to consider $q^k(0)$, $p_k(0)$ complex). This simple remark will become useful when we discuss the real Hamiltonian forms of (12.68).

We shall establish now the relation between the dynamics defined by $X_H^{\mathbb{C}}$ (the complexified dynamics) and by X_H (the original one). To this end, we use again that the function H is real analytic. From this property easily follows, that

$$\begin{aligned}
&\overline{H^{\mathbb{C}}}(\mathbf{p}^0 + i\mathbf{p}^1, \mathbf{q}_0 + i\mathbf{q}_1) = \\
&H^{\mathbb{C}}(\overline{\mathbf{p}^0 + i\mathbf{p}^1},\ \overline{\mathbf{q}_0 + i\mathbf{q}_1}) = H^{\mathbb{C}}(\mathbf{p}^0 - i\mathbf{p}^1, \mathbf{q}_0 - i\mathbf{q}_1) \,,
\end{aligned} \tag{12.78}$$

where the "bar" stands for the complex conjugation. As a result,

$$\begin{aligned}
H^0(\mathbf{p}^0, \mathbf{p}^1, \mathbf{q}_0, \mathbf{q}_1) &= H^0(\mathbf{p}^0, -\mathbf{p}^1, \mathbf{q}_0, -\mathbf{q}_1) \\
H^1(\mathbf{p}^0, \mathbf{p}^1, \mathbf{q}_0, \mathbf{q}_1) &= -H^1(\mathbf{p}^0, -\mathbf{p}^1, \mathbf{q}_0, -\mathbf{q}_1) \,,
\end{aligned} \tag{12.79}$$

[4] We shall discuss the Liouville integrability later, in Sect. 12.4, so the reader who is not familiar with this topic can go to 12.4 for the necessary definitions.

and we see that on $\mathcal{M}^{(n)}$, where by definition $\mathbf{p}^1 = 0$ and $\mathbf{q}_1 = 0$, we have $H^1 = 0$. Moreover, the above equations, together with the Cauchy–Riemann equations, entail that on $\mathcal{M}^{(n)}$ we have

$$\frac{\partial H^0}{\partial \mathbf{p}^1} = \frac{\partial H^0}{\partial \mathbf{q}_1} = 0 \, . \tag{12.80}$$

Looking back at (12.74), we see that the Hamiltonian vector field $X_H^{\mathbb{C}}$ is tangent to the subspace $\mathcal{M}^{(n)} \subset \mathcal{M}_{\mathbb{C}}^{(2n)}$ and on it $X_H^{\mathbb{C}}$ equals X_H. Thus the complexified dynamics, projected on the real space $\mathcal{M}^{(n)}$, that is putting

$$\mathbf{p}^1 = 0, \quad \mathbf{q}_1 = 0, \quad \frac{\partial H^0}{\partial \mathbf{p}^1} = 0, \quad \frac{\partial H^0}{\partial \mathbf{q}_1} = 0 \, , \tag{12.81}$$

gives the old dynamics. We see that the two dynamics are consistent, but apparently we did not obtain something new. However, one can regard these results in a completely different way. The point is that one and the same complex space can be obtained complexifying different real spaces (there are different real forms of the same complex space) and the same complex dynamics is obtained through different real ones. Thus complexifying one real dynamics and then "projecting" onto different real form can give interesting results.

In order to make the considerations of complexifying and projecting to the real form more transparent, we shall cast all that was done until now in a coordinate free form. For this, we shall use the vector field $X_H^{\mathbb{C}}$, see (12.74), defining the complexified dynamics on the space $\mathcal{M}_{\mathbb{C}}^{(2n)}$ (considered a real vector space). In the first place, we have

$$i_{X_H^{\mathbb{C}}} \, \omega_0 = -dH^0 \, , \tag{12.82}$$

where ω_0 is given in (12.67) because $X_H^{\mathbb{C}}$ is a Hamiltonian vector field. Next, one can check that the Cauchy–Riemann equations are equivalent to the following relation

$$i_{X_H^{\mathbb{C}}} \, \omega_1 = -dH^1 \, , \tag{12.83}$$

where ω_0 is as in (12.67). We can write these two equations together:

$$i_{X_H^{\mathbb{C}}} \, \omega_{\mathbb{C}} = -dH^{\mathbb{C}} \, , \tag{12.84}$$

where, $H^{\mathbb{C}}$ was introduced in (12.65) and $\omega_{\mathbb{C}}$ was introduced in (12.66); needless to say, we consider all the coordinates $p_k^0, p_k^1, q_0^k, q_1^k$ as real. Of course, similar relations can be written for any function F which is real analytic on $\mathcal{M}^{(n)}$ and has extension $F^{\mathbb{C}}$ on $\mathcal{M}_{\mathbb{C}}^{(2n)}$. The equations (12.82), (12.83) give

$$\{H^0, H^1\}_{\omega_0} = \{H^0, H^1\}_{\omega_1} = 0,$$

and also, if I_j, I_k are real-analytic on $\mathcal{M}^{(n)}$, one immediately recovers (12.76).

Now, let us introduce on $\mathcal{M}_{\mathbb{C}}^{(2n)}$ the conjugation J defined by the complexification (12.60), that is,

$$J(\mathbf{p}^0 + i\mathbf{p}^1) = \mathbf{p}^0 - i\mathbf{p}^1, \quad J(\mathbf{q}_0 + i\mathbf{q}_1) = \mathbf{q}_0 - i\mathbf{p}_1 \ . \tag{12.85}$$

In other words J, considered on $\mathcal{M}_{\mathbb{C}}^{(2n)}$ as real space, acts as

$$J(\mathbf{p}^0) = \mathbf{p}^0, \quad J(\mathbf{p}^1) = -\mathbf{p}^1, \quad J(\mathbf{q}_0) = \mathbf{q}_0, \quad J(\mathbf{q}_1) = -\mathbf{q}_1 \ . \tag{12.86}$$

Naturally, J is involutive linear map ($J^2 = \mathrm{id}$). It is immediately checked that

$$J^*\omega_0 = \omega_0, \quad J^*\omega_1 = -\omega_1 \ , \tag{12.87}$$

where J^* is the pull-back map. If we introduce

$$\bar{\omega}_{\mathbb{C}} = \omega_0 - i\omega_1 \ , \tag{12.88}$$

we can write both equations in (12.86) as

$$J^*\omega_{\mathbb{C}} = \bar{\omega}_{\mathbb{C}} \ . \tag{12.89}$$

Now, the condition of real analyticity of H can be written with the help of J in the following concise way

$$J^*H^{\mathbb{C}} = \overline{H^{\mathbb{C}}} \ , \tag{12.90}$$

or, in terms of H^0 and H^1, as

$$J^*H^0 = H^0, \quad J^*H^1 = -H^1 \ . \tag{12.91}$$

Applying the well-known property of tensor fields related through a diffeomorphism (this time it is J), and the fact that the pull-back map commutes with the exterior derivative d, from (12.84) we get the relation

$$J^*\left(i_{X_H^{\mathbb{C}}}\ \omega_{\mathbb{C}}\right) = i_{(J^{-1})_*(X_H^{\mathbb{C}})}(J^*\omega_{\mathbb{C}}) = -dJ^*H^{\mathbb{C}} \ , \tag{12.92}$$

where as usual for arbitrary vector field X, we write

$$((J^{-1})_*X)(\mathbf{z}) = dJ^{-1}(X(J(\mathbf{z}))).$$

But, as J is involutive, this expression is equal to $J(X(J(\mathbf{z})))$ and the properties (12.89), (12.90) of J permit to cast (12.92) into the form:

$$i_{X_{JH}^{\mathbb{C}}}\bar{\omega}_{\mathbb{C}} = -d\overline{H^{\mathbb{C}}} \ , \tag{12.93}$$

where

$$X_{JH}^{\mathbb{C}}(\mathbf{z}) = JX_H^{\mathbb{C}}(J\mathbf{z}) \ . \tag{12.94}$$

The relation (12.94) can be written also into the form

$$i_{X^{\mathbb{C}}_{JH}}\omega_{\mathbb{C}} = -dH^{\mathbb{C}} \ , \tag{12.95}$$

which shows that $X^{\mathbb{C}}_{JH} = X^{\mathbb{C}}_{H}$. Thus, the vector field $X^{\mathbb{C}}_{H}$ satisfies:

$$X^{\mathbb{C}}_{H}(\mathbf{z}) = JX^{\mathbb{C}}_{H}(J\mathbf{z}) \ . \tag{12.96}$$

Now, consider this relation on the subspace $\mathcal{M}^{(n)} \subset \mathcal{M}^{(2n)}_{\mathbb{C}}$. On it $J(\mathbf{z}) = \mathbf{z}$, and, therefore,

$$X^{\mathbb{C}}_{H}(\mathbf{z}) = JX^{\mathbb{C}}_{H}(\mathbf{z}) \ . \tag{12.97}$$

This means that on $\mathcal{M}^{(n)}$ all the components, corresponding to the vectors

$$\frac{\partial}{\partial q^j_1}, \quad \frac{\partial}{\partial p^1_j}; \qquad j = 1, 2, \ldots, n,$$

are zero, or, in other words, that on $\mathcal{M}^{(n)}$ we have the equations (12.80). This shows of course that the complex dynamics projects into the initial real one. Also, (12.91), (12.89) immediately give that restricting on $\mathcal{M}^{(n)} \subset \mathcal{M}^{(2n)}_{\mathbb{C}}$ we have $H^1 = 0$, $H^0 = H$ and $\omega_1 = 0$, $\omega_0 = \omega$ (we do not write in these formulae the canonical inclusion map $j : \mathcal{M}^{(n)} \mapsto \mathcal{M}^{(2n)}_{\mathbb{C}}$, because we do not want to complicate them unnecessarily). So, we have obtained the coordinate-free formulation which we shall use considering other real forms. Let us first remind some facts about them.

12.2.2 Real Hamiltonian Forms

The construction of real forms of complex symplectic spaces follows the usual scheme for real forms of vector spaces and real forms of Lie algebras. This is not surprising, as the Poisson brackets on $\mathcal{M}^{(n)}$ define actually a Lie algebra structure. As is well known, if V is a complex vector space, a map $J : V \mapsto V$ is called a conjugation map (or simply conjugation), if it satisfies:

$$\begin{aligned} &J(\lambda\mathbf{x}) = \bar{\lambda}\mathbf{x}; \quad \mathbf{x} \in V, \lambda \in \mathbb{C} \\ &J(\mathbf{x} + \mathbf{y}) = \mathbf{x} + \mathbf{y}; \quad \mathbf{x}, \mathbf{y} \in V \\ &J^2 = \mathrm{id}_V \ . \end{aligned} \tag{12.98}$$

If V is considered a real space with a double dimension, the conjugation becomes a linear map. A real form of V is a real vector space, $W \subset V$, such that after a complexification it gives V, in other words, any $\mathbf{z} \in V$ has unique representation $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ with $\mathbf{x}, \mathbf{y} \in W$. Such real form naturally defines a conjugation

$$J(\mathbf{x} + i\mathbf{y}) = \mathbf{x} - i\mathbf{y} \ . \tag{12.99}$$

Conversely, each conjugation J defines a real form W_J, one simply must put

$$W_J = \{\mathbf{z} : J(\mathbf{z}) = \mathbf{z}\} \subset V \ . \tag{12.100}$$

Different conjugations J_1, J_2 give different real forms. However, as readily checked, the map $S = J_1 \circ J_2^{-1}$ is an involution, that is, $S : V \mapsto V$ is linear, and $S^2 = \mathrm{id}_V$. This circumstance shows that the study of the real forms can be reduced to the study of one fixed real form and the involutions of V. The things, however, are not so simple and are additionally complicated when we have a Lie algebra structure, in this case S must be also an isomorphism of the Lie algebra. We cannot enter in all detail in this theory; we shall only mention that in the semisimple case, roughly speaking, the above can be done, and the things can be reduced to the study of one real form W_{J_1} of V and an involutive automorphism[5] S, leaving this form invariant (this happens when J_1 and J_2 commute). As a result, the restriction of S on W_{J_1} will also be an involutive automorphism of W_{J_1}, commuting with J_2. Precisely, this situation will be interesting for us.

For reasons that we believe now have become clear, we shall assume that on the space $\mathcal{M}^{(n)} = \mathbb{R}^{2n}$ there exists an involutive linear map $\mathcal{C}$, acting canonically on $\mathcal{M}^{(n)}$:

$$\mathcal{C}^*\omega = \omega; \quad \mathcal{C}^2 = \mathrm{id}\ . \tag{12.101}$$

We shall require also that H is invariant under $\mathcal{C}$, that is :

$$\mathcal{C}^*(H) = H\ . \tag{12.102}$$

Remark 12.21. Note that the requirement (12.101) on $\mathcal{C}$ implies that for any smooth functions F, G, we have $\mathcal{C}^*\{F, G\} = \{\mathcal{C}^*F, \mathcal{C}^*G\}$, that is, property (12.101) is an analog of the requirement to have involutive automorphism in the case of Lie algebras.

Now, the map $\mathcal{C}$ is naturally extended to $\mathcal{M}_{\mathbb{C}}^{(2n)}$. We denote it by the same letter and, as easily seen, it satisfies

$$\mathcal{C}^*\omega_{\mathbb{C}} = \omega_{\mathbb{C}}; \quad \mathcal{C}^2 = \mathrm{id}\ . \tag{12.103}$$

Also, for the analytically extended $H^{\mathbb{C}}$ we have:

$$\mathcal{C}^*(H^{\mathbb{C}}) = H^{\mathbb{C}}\ . \tag{12.104}$$

But, on $\mathcal{M}_{\mathbb{C}}^{(2n)}$ there exists the canonical conjugation J (defined by the real form $\mathcal{M}^{(n)}$), and we can combine it with $\mathcal{C}$ in order to obtain another conjugation. In other words, we assume that $\mathcal{C}$ and J commute, and we define the map $\tilde{J}$:

$$\tilde{J} = \mathcal{C} \circ J = J \circ \mathcal{C}\ . \tag{12.105}$$

As readily seen, it is again a conjugation, that is, $\tilde{J}$ is involutive, and $\tilde{J}(\lambda\mathbf{z}) = \bar{\lambda}\tilde{J}(\mathbf{z})$. The properties of J, listed at the end of the previous subsection permit to obtain that $\tilde{J}$ satisfies:

[5] We speak about automorphisms, because there is an additional algebraic structure on V.

$$\tilde{J}^* \omega_{\mathbb{C}} = \bar{\omega}_{\mathbb{C}} \, , \tag{12.106}$$

$$\tilde{J}^* H^{\mathbb{C}} = \overline{H^{\mathbb{C}}} \, . \tag{12.107}$$

Absolutely in the same way as before, applying the pull-back map $\tilde{J}^*$ to the identity

$$i_{X_H^{\mathbb{C}}} \, \omega_{\mathbb{C}} = -dH^{\mathbb{C}} \, , \tag{12.108}$$

we get that the vector field $X_H^{\mathbb{C}}$ satisfies

$$X_H^{\mathbb{C}}(\mathbf{z}) = (\tilde{J}^{-1})_* X_H^{\mathbb{C}}(\mathbf{z}) = \tilde{J} X_H^{\mathbb{C}}(\tilde{J}(\mathbf{z}) \, . \tag{12.109}$$

This time, instead of looking for a space on which $J\mathbf{z} = \mathbf{z}$, we look for a space such that $\tilde{J}\mathbf{z} = \mathbf{z}$, that is, we define the real form of $\mathcal{M}_{\mathbb{C}}^{(2n)}$, as:

$$\mathcal{M}_{\mathbb{R}}^{(n)} = \{\mathbf{z} \in \mathcal{M}_{\mathbb{C}}^{(2n)} : \tilde{J}(\mathbf{z}) = \mathbf{z}\} \, , \tag{12.110}$$

and we have the corresponding splitting:

$$\mathcal{M}_{\mathbb{C}}^{(2n)} = \mathcal{M}_{\mathbb{R}}^{(n)} \oplus i\mathcal{M}_{\mathbb{R}}^{(n)} \tag{12.111}$$

(of course, as real vector spaces). On the real vector space $\mathcal{M}_{\mathbb{R}}^{(n)}$ we have

$$X_H^{\mathbb{C}}(\mathbf{z}) = \tilde{J} X_H^{\mathbb{C}}(\mathbf{z}); \quad \mathbf{z} \in \mathcal{M}_{\mathbb{R}}^{(n)} \, , \tag{12.112}$$

which means that $X_H^{\mathbb{C}} \in \mathcal{M}_{\mathbb{R}}^{(n)}$, that is, the projection of $X_H^{\mathbb{C}}$ on $i\mathcal{M}_{\mathbb{R}}^{(n)}$ are zero. This ensures that the dynamics defined by $X_H^{\mathbb{C}}$ is projectable on $\mathcal{M}_{\mathbb{R}}^{(n)}$. In order to see that the projected dynamics is Hamiltonian, we apply the pull-back map $\tilde{J}^*$ to the Equation (12.84). Taking into account (12.109) yields

$$i_{X^{\mathbb{C}}_{CH}} \, \tilde{J}^* \omega_{\mathbb{C}} = -d \, \tilde{J}^* H^{\mathbb{C}} \, , \tag{12.113}$$

where in order to write simpler expressions, we have put

$$X^{\mathbb{C}}_{CH}(\mathbf{z}) = \tilde{J} X_H^{\mathbb{C}}(\tilde{J}(\mathbf{z})) \, . \tag{12.114}$$

Restricting on $\mathcal{M}_{\mathbb{R}}^{(n)}$, we get that the dynamics is Hamiltonian, with Hamiltonian function given by

$$H_{\mathbb{R}}(\mathbf{z}) = \tilde{J}^* H^0(\mathbf{z}) = (j^* H^0) = H^0(j(\mathbf{z})) \, . \tag{12.115}$$

where j is the canonical inclusion map $j : \mathcal{M}^{(n)} \mapsto \mathcal{M}_{\mathbb{C}}^{(2n)}$. The relation (12.115) simply means that

$$H_{\mathbb{R}}(\mathbf{z}) = H^0(\mathbf{z}); \quad \mathbf{z} \in \mathcal{M}_{\mathbb{R}}^{(n)} \, . \tag{12.116}$$

The Hamiltonian vector field is of course $X_H^{\mathbb{C}}(\mathbf{z})$, and the corresponding symplectic form is:

$$\omega_{\mathbb{R}} = \tilde{J}^* \omega_0 = j^* \omega_0 \ . \tag{12.117}$$

As in the case of the initial real form, relations (12.107), (12.106) give that restricting on $\mathcal{M}_{\mathbb{R}}^{(n)} \subset \mathcal{M}_{\mathbb{C}}^{(2n)}$ and we have analogs of all the relations we had when we restricted on the real form $\mathcal{M}^{(n)}$, namely $j^* H^1 = 0$, $j^* H^0 = H_{\mathbb{R}}$ and $j^* \omega_1 = 0$, $j^* \omega_0 = \omega_{\mathbb{R}}$, where as before j is the canonical inclusion map $j : \mathcal{M}_{\mathbb{R}}^{(n)} \mapsto \mathcal{M}_{\mathbb{C}}^{(2n)}$.

In order to calculate the new symplectic form and the new Hamiltonian, we need to calculate the map j. To this end, let us note that since $\mathcal{C}$ is involution on $\mathcal{M}^{(n)}$, its eigenvalues are ± 1, and $\mathcal{M}^{(n)}$ splits into two eigenspaces of $\mathcal{C}$: $\mathcal{M}^{(n)} = \mathcal{M}_+^{(n_+)} \oplus \mathcal{M}_-^{(n_-)}$, where $2n_\pm = \dim \mathcal{M}_\pm^{(n_\pm)}$. The above subspaces are indeed of even dimension, for the identity $\mathcal{C}^* \omega = \omega$ is equivalent to $\omega(\mathbf{x}, \mathbf{y}) = \omega(\mathcal{C}\mathbf{x}, \mathcal{C}\mathbf{y})$, where $\mathbf{x}, \mathbf{y}$ are arbitrary vectors. Taking $\mathbf{x} \in \mathcal{M}_+^{(n_+)}$, $\mathbf{y} \in \mathcal{M}_-^{(n_-)}$, we see that $\omega(\mathbf{x}, \mathbf{y}) = 0$, and since ω is nondegenerate, its restriction on the spaces $\mathcal{M}_\pm^{(n_\pm)}$ must be nondegenerate too. The last circumstance entails that their dimension is even. Next, we have also the following splitting of $\mathcal{M}_{\mathbb{R}}^{(n)}$

$$\mathcal{M}_{\mathbb{R}}^{(n)} = \mathcal{M}_+^{(n_+)} \oplus i\mathcal{M}_-^{(n_-)} \ , \tag{12.118}$$

for any element of $\mathcal{M}_{\mathbb{R}}^{(n)}$ can be represented as:

$$\mathbf{z} = \mathbf{x} + i\mathbf{y} \ , \tag{12.119}$$

where $\mathbf{x}, \mathbf{y} \in \mathcal{M}^{(n)}$, and the condition $\tilde{J}(\mathbf{z}) = \mathbf{z}$ readily gives that in this case we must have $\mathbf{x} \in \mathcal{M}_+^{(n_+)}$ and $\mathbf{y} \in \mathcal{M}_-^{(n_-)}$. We have already seen that the involution $\mathcal{C}$ guarantees that each of the subspaces $\mathcal{M}_+^{(n_+)}$, $\mathcal{M}_-^{(n_-)}$ is a symplectic subspace of $\mathcal{M}^{(n)}$. Let us assume that on the spaces $\mathcal{M}_+^{(n_+)}$, $\mathcal{M}_-^{(n_-)}$ we have chosen canonical coordinates

$$\left(p_k^+, q_+^k\right)_{k=1}^{n_+} \quad \text{on} \quad \mathcal{M}_+^{(n_+)}, \qquad \left(p_k^-, q_-^k\right)_{k=1}^{n_-} \quad \text{on} \quad \mathcal{M}_-^{(n_-)} \ . \tag{12.120}$$

This means that the symplectic forms $\omega_\pm$ (the restrictions of ω) on the spaced $\mathcal{M}_+^{(n_+)}$ and $\mathcal{M}_-^{(n_-)}$ can be written into the form:

$$\omega_+ = \sum_{k=1}^{n_+} dp_k^+ \wedge dq_+^k, \qquad \omega_- = \sum_{k=1}^{n_-} dp_k^- \wedge dq_-^k \ . \tag{12.121}$$

and that $\omega = \omega_+ + \omega_-$. For the sake of brevity we shall denote the coordinates on $\mathcal{M}_+^{(n_+)}$, $\mathcal{M}_-^{(n_-)}$, respectively, by $z_{k,+}$ and $z_{k,-}$. The fact that the above spaces are eigenspaces of the involution $\mathcal{C}$ entails that $p_k^\pm$, $q_\pm^k$ satisfy:

$$\mathcal{C}^*(p_k^\pm) = \pm p_k^\pm, \qquad \mathcal{C}^*(q_\pm^k) = \pm q_\pm^k \ , \tag{12.122}$$

for all $k = 1, 2, \ldots, n_\pm$. The same, written in terms of $z_{k,\pm}$, reads $\mathcal{C}^*(z_{k,\pm}) = \pm z_{k,\pm}$. Now it is easy to perform the restriction. Indeed, we must do the following. First, we must complexify the space, that is, assume that all $z_{k;\pm}$ are complex. We write $z_{k;\pm} = z^0_{k;\pm} + iz^1_{k;\pm}$ and then express the form ω_0 and the function $H^{\mathbb{C}}$ through the variables $z^{\alpha}_{k;\pm}$, $\alpha = 0, 1$. Next, we set in all the formulae $z_{k,+}$ to be real and instead of $z_{k,-}$ we set $iz_{k,-}$, in other words, if $\mathbf{z} = (z_{k,+}, z_{s,-})$ (real) are chosen to be coordinates on $\mathcal{M}^{(n)}_{\mathbb{R}}$, the canonical inclusion map $j : \mathcal{M}^{(n)}_{\mathbb{R}} \mapsto \mathcal{M}^{(2n)}_{\mathbb{C}}$ is given by

$$j(\mathbf{z})^0_{k,+} = z_{k,+}, \quad j(\mathbf{z})^1_{k,+} = 0, \quad j(\mathbf{z})^0_{k,-} = 0, \quad j(\mathbf{z})^1_{k,-} = z_{k,-} \,. \tag{12.123}$$

Now let us find the symplectic form. We have $\omega_{\mathbb{C}} = \omega_0 + i\omega_1$, where

$$\begin{aligned} \omega_0 &= \sum_{k=1}^{n_+} \left(dp^0_{k,+} \wedge dq^{k,+}_0 - dp^1_{k,+} \wedge dq^{k,+}_1 \right) \\ &\quad + \sum_{k=1}^{n_-} \left(dp^0_{k,-} \wedge dq^{k,-}_0 - dp^1_{k,-} \wedge dq^{k,-}_1 \right) \,, \\ \omega_1 &= \sum_{k=1}^{n_+} \left(dp^0_{k,+} \wedge dq^{k,+}_1 + dp^1_{k,+} \wedge dq^{k,+}_0 \right) \\ &\quad + \sum_{k=1}^{n_-} \left(dp^1_{k,-} \wedge dq^{k,-}_0 + dp^0_{k,-} \wedge dq^{k,-}_1 \right) \,. \end{aligned} \tag{12.124}$$

Then (12.123) readily yields:

$$\omega_{\mathbb{R}} = \sum_{k=1}^{n_+} dp^+_k \wedge dq^k_+ - \sum_{k=1}^{n_-} dp^-_k \wedge dq^k_- \,. \tag{12.125}$$

Finally, if

$$H^0(\mathbf{z}^0_{k,+} + i\mathbf{z}^1_{k,+}\,, \mathbf{z}^0_{k,-} + i\mathbf{z}^1_{k,-}) = H^0(\mathbf{z}^0_{k,+}\,, \mathbf{z}^1_{k,+}, \mathbf{z}^0_{k,-}, \mathbf{z}^1_{k,-}) \,, \tag{12.126}$$

then

$$H_{\mathbb{R}}(\mathbf{z}_{k,+}, \mathbf{z}_{k,-}) = H^0(\mathbf{z}_{k,+}, 0, 0, \mathbf{z}_{k,-}) \,, \tag{12.127}$$

or simply

$$H_{\mathbb{R}} = H(\mathbf{z}_{k,+}, i\mathbf{z}_{k,-}) = H(p^+_k, q^k_+, ip^-_k, iq^k_-) \,. \tag{12.128}$$

Let us recapitulate. We started from real Hamiltonian system $\{(\mathcal{M}^{(n)}, \omega), H\}$ and an involutive symplectic linear map $\mathcal{C}$, and we were able to construct another real Hamiltonian system $\{(\mathcal{M}^{(n)}_{\mathbb{R}}, \omega_{\mathbb{R}}), H_{\mathbb{R}}\}$:

$$\begin{aligned} \mathcal{M}^{(n)}_{\mathbb{R}} &= \mathcal{M}^{(n_+)}_+ \oplus i\mathcal{M}^{(n_-)}_- \,, \\ \omega_{\mathbb{R}} &= \sum_{k=1}^{n_+} dp^+_k \wedge dq^k_+ - \sum_{k=n_++1}^{n} dp^-_k \wedge dq^k_- \end{aligned}$$

$$H_{\mathbb{R}} = \ H(p_k^+, q_+^k, ip_k^-, iq_-^k)\ , \tag{12.129}$$

which we called a real Hamiltonian form of the complex dynamical system defined by the initial system.

Both systems have the same number of degrees of freedom. Moreover, the solutions of $\{(\mathcal{M}^{(n)}, \omega), H\}$, depending analytically on the initial values $\mathbf{q}(0) = \mathbf{q}(t)|_{t=0}$, $\mathbf{p}(0) = \mathbf{p}(t)|_{t=0}$, can be mapped onto solutions of $\{(\mathcal{M}_{\mathbb{R}}^{(n)}, \omega_{\mathbb{R}}), H_{\mathbb{R}}\}$, because due to the analyticity properties they will pass through both the operations:

- complexification and extension to $\mathcal{M}_{\mathbb{C}}^{(2n)}$;
- projection on the real form $\mathcal{M}_{\mathbb{R}}^{(n)}$ of the complex space $\mathcal{M}_{\mathbb{C}}^{(2n)}$. The construction amounts to putting instead of $(\mathbf{p}_+^0(0), \mathbf{q}_+^0(0), \mathbf{p}_-^1(0), \mathbf{q}_-^1(0))$ the values $(\mathbf{p}_+^0(0), \mathbf{q}_+^0(0), i\mathbf{p}_-^1(0), i\mathbf{q}_-^1(0))$.

Let us illustrate these ideas on a particular Hamiltonian system; see [24]. It is interesting, because it is completely integrable. This will demonstrate also that our construction preserves the integrability The example is provided by the Toda chain (TC), related to the algebra $\mathrm{sl}\,(n)$. Its Hamiltonian and symplectic form are given by

$$H_{\mathrm{TC}} = \sum_{k=1}^{n} \frac{p_k^2}{2} + \sum_{k=1}^{n-1} \exp\langle \mathbf{q}, \alpha_k\rangle + c_0 \exp\langle \mathbf{q}, \alpha_0\rangle, \tag{12.130}$$

$$\omega_{\mathrm{TC}} = \sum_{k=1}^{n} dp_k \wedge dq^k = (d\mathbf{p} \wedge d\mathbf{q})\ , \tag{12.131}$$

where[6] $\alpha_k = e_k - e_{k+1}$ for $k = 1, \ldots n-1$, $\alpha_0 = e_1 - e_n$. The vectors e_k form the canonical orthonormal basis in $\mathbb{R}^n$ and by $\langle ., .\rangle$ is denoted the standard inner product. Depending on the choice of the real constant c_0 we have two different versions of the Toda chain:

- $c_0 = 0$ corresponds to Toda chain with free ends;
- $c_0 = 1$ gives the so-called affine Toda chain.

Our considerations will be valid for both these cases.

Let us first complexify the TC model. As we explained above, the complexified system can be considered as real Hamiltonian system with $2n$ degrees of freedom. In our case this system is characterized by the following Hamiltonian and symplectic form:

$$H_{\mathrm{CTC}} = \frac{\langle \mathbf{p}^0, \mathbf{p}^0\rangle}{2} - \frac{\langle \mathbf{p}^1, \mathbf{p}^1\rangle}{2} + \sum_{k=1}^{n-1} \exp\langle \mathbf{q}^0, \alpha_k\rangle \cos\langle \mathbf{q}^1, \alpha_k\rangle$$

[6] Those familiar with the theory of the semisimple Lie algebras will recognize in the set α_k, $k = 1, \ldots, n-1$ the set of the simple roots of $\mathrm{sl}\,(n)$ and in α_0 the minimal root of $\mathrm{sl}\,(n)$.

$$+ c_0 \exp\langle \mathbf{q}, \alpha_0 \rangle \cos\langle \mathbf{q}^1, \alpha_0 \rangle \,, \tag{12.132}$$

$$\omega_{\mathrm{CTC}} = (d\mathbf{p}^0 \wedge d\mathbf{q})^0 - (d\mathbf{p}^1 \wedge d\mathbf{q}^1) \,, \tag{12.133}$$

where, as above, $\mathbf{p}^0$, $\mathbf{p}^1$ and $\mathbf{q}^0$, $\mathbf{q}^1$ are the real and the imaginary parts of the complexified vectors $\mathbf{p}$, $\mathbf{q}$. Now, we introduce an involution for which $\mathcal{C}^*(H) = H$ holds. We take:

$$\mathcal{C}^*(p_k) = -p_{\bar{k}}, \qquad \mathcal{C}^*(q^k) = -q^{\bar{k}} \,. \tag{12.134}$$

where $\bar{k} = n + 1 - k$. Here, it is convenient to consider separately the cases of even and odd values of n.

(1) $n = 2r$. On the subspaces $\mathcal{M}_{\pm}^{(n_\pm)}$, we can choose the coordinates :

$$p_k^{\pm} = \frac{1}{\sqrt{2}} \left(p_k \mp p_{\bar{k}} \right), \qquad q_{\pm}^k = \frac{1}{\sqrt{2}} \left(q^k \mp q^{\bar{k}} \right) \,, \tag{12.135}$$

for $k = 1, \ldots, r$. We have $\dim \mathcal{M}_+^{(n_+)} = \dim \mathcal{M}_-^{(n_-)} = 2r$ and in terms of $p_k^{\pm}$, $q_{\pm}^k$ the initial Hamiltonian and 2-form can be expressed as:

$$\begin{aligned} H_{\mathrm{TC1}} = {} & \frac{1}{2} \sum_{k=1}^{r} ((p_k^+)^2 + (p_k^-)^2) + \exp(-\sqrt{2} q_+^r) + c_0 \exp(\sqrt{2} q_+^1) \\ & + 2 \sum_{k=1}^{r-1} \exp((q_+^{k+1} - q_+^k)/\sqrt{2}) \cosh \frac{q_-^{k+1} - q_-^k}{\sqrt{2}} \,, \end{aligned} \tag{12.136}$$

$$\omega_{\mathrm{TC1}} = \sum_{k=1}^{r} dp_k^+ \wedge dq_+^k + \sum_{k=1}^{r} dp_k^- \wedge dq_-^k . \tag{12.137}$$

The function H_{TC1} (12.136) is even function on the variables q_-^k and p_k^-, and, therefore, the condition (12.102) is satisfied; see (12.122). The corresponding real form RTC1 of the complexified TC model can be obtained from (12.136), (12.137), replacing q_-^k, p_k^- with iq_-^k and ip_k^-, respectively, which yields:

$$\begin{aligned} H_{\mathrm{RTC1}} = {} & \frac{1}{2} \sum_{k=1}^{r} ((p_k^+)^2 - (p_k^-)^2) + \exp(-\sqrt{2} q_+^r) \\ & + 2 \sum_{k=1}^{r-1} \exp((q_+^{k+1} - q_+^k)/\sqrt{2}) \cos \frac{q_-^{k+1} - q_-^k}{\sqrt{2}} \,, \end{aligned} \tag{12.138}$$

$$\omega_{\mathrm{RTC1}} = \sum_{k=1}^{r} dp_k^+ \wedge dq_+^k - \sum_{k=1}^{r} dp_k^- \wedge dq_-^k \,. \tag{12.139}$$

(2) $n = 2r + 1$. Then in the subspaces $\mathcal{M}_{\pm}^{(n_\pm)}$, one can choose the coordinates:

- On $\mathcal{M}_+^{(n_+)}$:

$$p_k^+ = \frac{1}{\sqrt{2}}(p_k - p_{\bar{k}}), \; q_+^k = \frac{1}{\sqrt{2}}(q^k - q^{\bar{k}}) \, , \tag{12.140}$$

- On $\mathcal{M}_-^{(n_-)}$:

$$\begin{aligned} p_k^- &= \frac{1}{\sqrt{2}}(p_k + p_{\bar{k}}), \; q_-^k = \frac{1}{\sqrt{2}}(q^k + q^{\bar{k}}); \\ p_{r+1}^- &= p_{r+1}, \; q_-^{r+1} = q^{r+1} \, , \end{aligned} \tag{12.141}$$

for $1 \leq k \leq r$. Note that $\dim \mathcal{M}_+^{(n_+)} = 2r$, $\dim \mathcal{M}_-^{(n_-)} = 2r+2$. In terms of $p_k^\pm$, $q_\pm^k$, the initial Hamiltonian and 2-form can be expressed as:

$$\begin{aligned} H_{\mathrm{TC2}} = \; & \frac{1}{2}\sum_{k=1}^{r}((p_k^+)^2 + (p_k^-)^2) + 2\exp(-q_+^r/\sqrt{2})\cosh\left(q_-^{r+1} - \frac{q_-^r}{\sqrt{2}}\right) \\ & + 2\sum_{k=1}^{r-1}\exp((q_+^{k+1} - q_+^k)/\sqrt{2})\cosh\frac{q_-^{k+1} - q_-^k}{\sqrt{2}} + c_0\exp(\sqrt{2}q_+^1) \, , \end{aligned} \tag{12.142}$$

$$\omega_{\mathrm{TC2}} = \sum_{k=1}^{r} dp_k^+ \wedge dq_+^k + \sum_{k=1}^{r+1} dp_k^- \wedge dq_-^k . \tag{12.143}$$

H_{TC2} (12.142) is also even function of q_-^k and p_k^- and the condition (12.122) is satisfied. The corresponding real form RTC2 of the complexified TC model can be obtained from (12.142), (12.143) by replacing q_-^k, p_k^- with iq_-^k and ip_k^- respectively, yielding :

$$\begin{aligned} H_{\mathrm{RTC2}} = & \frac{1}{2}\sum_{k=1}^{r}((p_k^+)^2 - (p_k^-)^2) + 2\exp(-q_+^r/\sqrt{2})\cos\left(q_-^{r+1} - \frac{q_-^r}{\sqrt{2}}\right) \\ & + 2\sum_{k=1}^{r-1}\exp((q_+^{k+1} - q_+^k)/\sqrt{2})\cos\frac{q_-^{k+1} - q_-^k}{\sqrt{2}} + c_0\exp(\sqrt{2}q_+^1), \end{aligned} \tag{12.144}$$

$$\omega_{\mathrm{RTC2}} = \sum_{k=1}^{r} dp_k^+ \wedge dq_+^k - \sum_{k=1}^{r+1} dp_k^- \wedge dq_-^k \; . \tag{12.145}$$

These two models are generalizations of the well-known Toda chains, related to the simple Lie algebras[7]. The reader, familiar with the theory of simple Lie algebras, will identify the action of the automorphism $\mathcal{C}$ (12.134) with the dual action of the unique outer automorphism of $\mathrm{sl}\,(n)$. The roots that are fixed under this action define the simple roots system of the algebras $\mathrm{sp}\,(2r)$ and $\mathrm{o}\,(2r+1)$. Therefore, it is not accidental that after putting $q_-^k = 0$ and $p_k^- = 0$ in (12.138) and (12.144), we get the Toda chains associated to the Lie algebras $\mathrm{sp}\,(2r)$ and $\mathrm{o}\,(2r+1)$, respectively.

[7] The simple Lie algebras are semisimple Lie algebras that do not have nontrivial ideals

12.3 Poisson Structures

As already discussed, the requirement that a closed form ω on $\mathcal{M}$ is non-degenerate sometimes is too restrictive. From the other side, if ω is degenerate additional constructions are needed to define Poisson brackets, and these brackets are defined only on some submanifolds. Certainly, if we want to consider Poisson brackets of functions on $\mathcal{M}$, this is a flaw. Fortunately, there exists another way to define the Poisson brackets on manifolds based on the so-called Poisson tensors. The most simple way to approach this topic is to see that actually we do not need the tensor field $m \to \bar{\omega}_m$ for the construction of Poisson brackets, but we need its inverse $m \to P_m$, that is, the field that satisfies: $P_m \circ \bar{\omega}_m = id_{T_m(\mathcal{M})}$; $\bar{\omega}_m \circ P_m = id_{T^*_m(\mathcal{M})}$. Indeed, suppose that α_i; $i = 1, 2, 3$ are 1-forms and X_i; $i = 1, 2, 3$ are fields, such that $\bar{\omega}(X_i) = \alpha_i$. The condition (12.27) then gives:

$$\begin{aligned}
&d\alpha_1(X_2, X_3) + X_3\langle \alpha_1, X_2\rangle + d\alpha_2(X_3, X_1)\\
&+X_1\langle \alpha_2, X_3\rangle + d\alpha_3(X_1, X_2) + X_2\langle \alpha_3, X_1\rangle =\\
&\langle L_{X_2}\alpha_1, X_3\rangle + \langle L_{X_3}\alpha_2, X_1\rangle + \langle L_{X_1}\alpha_3, X_2\rangle = 0\ .
\end{aligned} \tag{12.146}$$

A brief calculation shows that the above expression can be put into the form

$$\langle \alpha_1, PL_{P\alpha_3}\alpha_2\rangle + \langle \alpha_2, PL_{P\alpha_1}\alpha_3\rangle + \langle \alpha_3, PL_{P\alpha_2}\alpha_1\rangle = 0\ , \tag{12.147}$$

for arbitrary choice of the 1-forms α_i, $i = 1, 2, 3$. But the last formula is written only in terms of P. One can check that if P satisfies (12.147), then one can define the Poisson brackets of closed forms and functions exactly as it was done before, that is, setting

$$\begin{aligned}
&\{\alpha_1, \alpha_1\}_P = d\langle P(\alpha_1), \alpha_2\rangle\\
&\{f_1, f_2\}_P = -\langle P(df_1), df_2\rangle\ ,
\end{aligned} \tag{12.148}$$

for two closed 1-forms α_1, α_2 and two functions f_1, f_2. Thus, our attention is driven to another geometric object – the tensor field P, called Poisson tensor; see [25] for modern introduction into this topic. More precisely,

Definition 12.22. *The Poisson tensor is a field of tensors of type* $(2, 0)$*, that is a field of linear maps :*

$$m \to P_m : T^*_m(\mathcal{M}) \to T_m(\mathcal{M})\ , \tag{12.149}$$

satisfying the following conditions:

$$\begin{aligned}
&a) \quad P^* = -P\\
&b) \quad [P, P]_S = 0\ .
\end{aligned} \tag{12.150}$$

A manifold $\mathcal{M}$, equipped with Poisson tensor, is called Poisson manifold or P-manifold. It is said also that on $\mathcal{M}$ is defined Poisson structure.

In (12.150), we have denoted by $[\ ,\ ]_S$ the so-called Schouten–Nijenhuis bracket of two tensor fields, (see [26, 27]), which in the case of $(2,0)$ tensors P and Q can be defined through the relation

$$[P,Q]_S(\gamma_1,\gamma_2,\gamma_3) = \langle QL_{P\gamma_1}(\gamma_2),\gamma_3\rangle + \langle PL_{Q\gamma_1}(\gamma_2),\gamma_3\rangle + \mathrm{cycl}\,(1,2,3)\ , \tag{12.151}$$

which must be satisfied for arbitrary choice of the 1-forms $\gamma_1,\gamma_2,\gamma_3$.

One can check that $[P,Q]_S$ is 3 times contravariant tensor, in other words, tensor field of type (3,0). The symbol $\mathrm{cycl}\,(1,2,3)$ means that to the expression before cycl one must add all the analogous expressions that can be obtained through cyclic permutation of the indices 1,2,3. As one can see $[P,P]_S = 0$ is equivalent to (12.147). Also, one immediately checks that if P is a Poisson tensor, then for arbitrary constant $c \neq 0$ the tensor cP is also a Poisson tensor.

It is rather difficult to trace where the notion of the Poisson tensor has appeared for the first time. According to [28, 29] the S. Lie's book, [30] contains the essentials of the theory of the Poisson manifolds (called in [30] "function groups") and not only, as generally believed, the so-called "Poisson-Lie" structures on duals of Lie algebras (we introduce them later in (12.172), see also the discussion before that formula). Also, in a book by Carathéodory, [31] there has been a rather complete exposition of the theory based on even earlier work of S. Lie.

As for modern sources, the properties of the Poisson tensors in general, as mentioned before, are studied in [25], the properties of the algebras of vector fields associated with Poisson tensor structure are also studied in [26, 29, 32], together with some cohomological aspects arising from the above bracket, see also [33] for this topic. Comprehensive bibliography and the most important results about the finite dimensional Poisson manifolds can be found in [5]. There exist also some interesting generalization of the notion of the Poisson structure, see [34, 35] and [36].

Below and in the next subsection we introduce some of the principal properties of the Poisson manifolds.

Let us start with the observation, made in [37, 38], that the condition $[P,P]_S = 0$ allows equivalent formulation:

$$P[L_{P(\alpha)}(\beta) - L_{P(\beta)}(\alpha) + d\,(\langle\alpha, P(\beta)\rangle)] + L_{P(\beta)}(P(\alpha)) = 0\ , \tag{12.152}$$

for arbitrary choice of the 1-forms α,β. Indeed, the equation $[P,P]_S = 0$ means that for arbitrary γ_i

$$\langle PL_{P\gamma_1}(\gamma_2),\gamma_3\rangle + \langle PL_{P\gamma_2}(\gamma_3),\gamma_1\rangle + \langle PL_{P\gamma_3}(\gamma_1),\gamma_2\rangle = 0\ . \tag{12.153}$$

We have

$$\begin{aligned} &\langle PL_{P\gamma_2}(\gamma_3),\gamma_1\rangle = -\langle L_{P\gamma_2}(\gamma_3), P\gamma_1\rangle = \\ &L_{P\gamma_2}\langle\gamma_1, P\gamma_3\rangle + \langle L_{P\gamma_2}(P\gamma_1),\gamma_3\rangle\ , \\ &\langle PL_{P\gamma_3}(\gamma_1),\gamma_2\rangle = -\langle L_{P\gamma_3}(\gamma_1), P\gamma_2\rangle = \end{aligned} \tag{12.154}$$

$$\begin{aligned}&\langle L_{P\gamma_3}(P\gamma_2),\gamma_1\rangle - L_{P\gamma_3}\langle\gamma_1,P\gamma_2\rangle =\\&\langle\gamma_3,Pd\langle\gamma_1,P\gamma_2\rangle\rangle - \langle L_{P\gamma_2}(P\gamma_3),\gamma_1\rangle =\\&\langle\gamma_3,Pd\langle\gamma_1,P\gamma_2\rangle\rangle - L_{P\gamma_2}\langle\gamma_1,P\gamma_3\rangle - \langle PL_{P\gamma_2}(\gamma_1),\gamma_3\rangle\ .\end{aligned} \tag{12.155}$$

Inserting (12.154, 12.155) in (12.153) we get

$$\langle P[L_{P(\gamma_1)}(\gamma_2) - L_{P(\gamma_2)}(\gamma_1) + d\,(\langle\gamma_1,P(\gamma_2)\rangle)] + L_{P(\gamma_2)}(P(\gamma_1)),\gamma_3\rangle = 0\ . \tag{12.156}$$

Finally, taking into account that $\gamma_i;\ i=1,2,3$ are arbitrary, we arrive at the relation (12.152).

The relation (12.152) is convenient, in order to see why the Poisson tensor defines Poisson brackets. First of all, note that if the 1-form α is closed, then for arbitrary vector field X we have

$$L_X\alpha - d\langle\alpha,X\rangle = 0 \tag{12.157}$$

and vice versa. Indeed,

$$L_X\alpha = di_X\alpha + i_Xd\alpha = d\langle\alpha,X\rangle\ . \tag{12.158}$$

Suppose now that $X_\alpha = P\alpha,\ X_\beta = P\beta$ where α and β are closed 1-forms. Then

$$L_{P(\alpha)}(\beta) - L_{P(\beta)}(\alpha) + d\langle\alpha,P(\beta)\rangle = d\langle\beta,P(\alpha)\rangle \tag{12.159}$$

and from (12.152), we get that

$$-\{\alpha,\beta\}_P = Pd\langle\beta,P(\alpha)\rangle = [P\alpha,P\beta]\ . \tag{12.160}$$

Thus again, as it was in the case of a symplectic structure, the tensor field P "transfers" the Lie algebra structure from the module of vector fields to the module of 1-forms. It is then natural to give the following definitions.

Definition 12.23. *The field $X_\alpha = P(\alpha)$ is called the Hamiltonian vector field corresponding to the* 1*-form α, and the field $X_f = -P(df)$ is called the Hamiltonian vector field corresponding to the function f.*

Of course, if P is invertible (P_m is invertible at any point m) the tensor field P^{-1} defines symplectic structure. Thus Poisson and symplectic structures are dual, and some authors call the Poisson structure (the Poisson tensor) *implectic* structure.

Let us mention also that if $\{f,g\};\ f,g\in\mathcal{D}(\mathcal{M}$ is Poisson bracket (Poisson bracket structure) on the manifold $\mathcal{M}$, then $\{f,g\}_c = c\{f,g\}$, where $c\neq 0$ is some constant is also a Poisson bracket. We shall say that these Poisson brackets are not essentially different. If the brackets $\{f,g\}$ are defined through a Poisson tensor P, then $\{f,g\}_c$ are defined through the Poisson tensor cP. For this reason, we shall call P and cP not essentially different. Also, by the same logic, we shall call two symplectic forms essentially different if they are not proportional. As we shall see later, it is possible to have two essentially different Poisson brackets on the same manifold, and this situation is very interesting and important.

As mentioned, it seems that the notion of Poisson tensor goes back to S. Lie, [30], though at that time it was not called by that name. Actually, S. Lie has introduced on a manifold $\mathcal{M}$ skew-symmetric tensor field of type $(2,0)$ (which in the finite dimensional case is the same as introducing field of linear maps $P_m : T^*_m(\mathcal{M}) \to T_m(\mathcal{M})$). In local coordinates x^i; $i = 1, 2, \ldots n$, and in modern notation this field can be written as:

$$P = \sum_{i,j=1}^{n} P^{ij}(x) \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}, \quad P^{ij} = -P^{ji} . \tag{12.161}$$

S. Lie then proved that the expression

$$\{f_1, f_2\} = \sum_{i,j=1}^{n} P^{ij}(x) \frac{\partial f_1}{\partial x^i} \frac{\partial f_2}{\partial x^j} \tag{12.162}$$

defines Poisson brackets if and only if

$$\sum_{s=1}^{n} P^{si} \frac{\partial P^{jk}}{\partial x^s} + \mathrm{cycl}\,(i,j,k) = 0, \tag{12.163}$$

which is the coordinate expression of the condition $[P, P]_S = 0$.

S. Lie has also discovered remarkable Poisson structure looking for Poisson tensor P^{ij} over $\mathbb{R}^n$ with linear dependance on the coordinates:

$$P^{ij} = \sum_{k=1}^{n} C^k_{ij} x^k . \tag{12.164}$$

The constants C^k_{ij}, which of course must be skew-symmetric with respect to the lower indices, satisfy the condition (12.162) if

$$\sum_{s=1}^{n} \left(C^m_{si} C^s_{jk} + C^m_{sj} C^s_{ki} + C^m_{sk} C^s_{ij} \right) = 0 . \tag{12.165}$$

But this means that C^k_{ij} are structure constants of some Lie algebra. In other words, if $\mathfrak{g}$ is n-dimensional vector space (for example over $\mathbb{R}$) with basis $\{I_j\}_1^n$, one can define the bracket of two vectors

$$X = \sum_{s=1}^{n} x^s I_s, \qquad Y = \sum_{s=1}^{n} y^s I_s \tag{12.166}$$

as

$$[X, Y] = \sum_{i,j,s=1}^{n} x^i y^j C^s_{ij} I_s . \tag{12.167}$$

and then $\mathfrak{g}$ will be Lie algebra. Indeed, due to the skew-symmetry of the coefficients, C^k_{ij}, one has $[X, Y] = -[Y, X]$ and (12.165) ensures the Jacobi identity

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0; \quad X,Y,Z \in \mathfrak{g} . \tag{12.168}$$

However, Lie's construction apparently has been forgotten, and later the same result has been rediscovered by Kirillov, Souriau, and Kostant, [39, 40, 41]. Now different names are used for it – Poisson-Lie structure, Kirillov structure, Berezin structure, etc. We shall prefer the name Poisson-Lie structure and for the corresponding tensor field P the name Kirillov's tensor or Poisson-Lie tensor. The modern way to introduce the above notions is the following. Let $\mathfrak{g}$ be a Lie algebra (over $\mathbb{R}$ or $\mathbb{C}$) and let $\mathfrak{g}^*$ be its dual space (called also the coalgebra). Consider the adjoint representation of $\mathfrak{g}$. The adjoint representation is linear map $\mathfrak{g} \to \mathrm{Hom}\,(\mathfrak{g}, \mathfrak{g}) : X \to \mathrm{ad}_X$, where

$$\mathrm{ad}_X(Y) = [X,Y]; \quad X,Y \in \mathfrak{g} . \tag{12.169}$$

The coadjoint representation of $\mathfrak{g}$ is then the linear map $\mathfrak{g} \to \mathrm{Hom}\,(\mathfrak{g}^*, \mathfrak{g}^*)$: $X \to -\mathrm{ad}^*_X$, where ad^*_X is the adjoint of ad_X. The adjoint map ad^*_X is defined in the usual way

$$\langle \mathrm{ad}^*_X(\mu), Y\rangle = \langle \mu, \mathrm{ad}_X(Y)\rangle = \langle \mu, [X,Y]\rangle \ , \tag{12.170}$$

for $\mu \in \mathfrak{g}^*$; $X, Y \in \mathfrak{g}$ where by $\langle\ ,\ \rangle$ is denoted the canonical pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. Let us identify $\mathfrak{g}^{**}$ with $\mathfrak{g}$. Then the Kirillov tensor K is a field $\mu \to K_\mu$ of linear maps

$$K_\mu : \mathfrak{g}^{**} \sim \mathfrak{g} \to \mathfrak{g}^*, \qquad K_\mu(X) = \mathrm{ad}^*_X \mu \ . \tag{12.171}$$

The Poisson tensor already defined, the Poisson bracket of two functions $f_1(\mu)$,$f_2(\mu)$ over the dual space $\mathfrak{g}^*$ can be constructed according to the general scheme. In more detail, the procedure is the following: First, we take the derivatives $df_1(\mu), df_2(\mu)$. They are elements of $\mathfrak{g}^{**}$ and, therefore, can be regarded as vectors from $\mathfrak{g}$. Finally, we put

$$\{f, g\}(\mu) = \langle \mathrm{ad}^*_{df_1}(\mu), df_2\rangle = \langle \mu, [df_1(\mu), df_2(\mu)]\rangle \ . \tag{12.172}$$

As already mentioned, this bracket is called Poisson-Lie bracket or Kirillov bracket. The above Poisson structure is quite important, both from the theoretical viewpoint and for the applications; see [1, 4, 41].

The case when $\mathfrak{g}$ is a semisimple Lie algebra is of particular interest, since in that case there is canonical way to identify $\mathfrak{g}$ and $\mathfrak{g}^*$. As is well known, (see [42, 43]) the semisimple Lie algebras over $\mathbb{R}$ and $\mathbb{C}$ are characterized by the fact that the following form, called the Killing form of the algebra $\mathfrak{g}$, is nondegenerate:

$$B(X,Y) = \mathrm{tr}\,(\mathrm{ad}_X \circ \mathrm{ad}_Y); \quad X,Y \in \mathfrak{g} \ . \tag{12.173}$$

From the Jacobi identity it follows that each ad_X is skew-symmetric with respect to the Killing form, that is

$$B(\mathrm{ad}_X(Y), Z) = -B(Y, \mathrm{ad}_X(Z)); \quad X, Y, Z \in \mathfrak{g} \ . \tag{12.174}$$

Since B is nondegenerate, $\mathfrak{g}^*$ and $\mathfrak{g}$ can be identified through it. In detail, for $\mu \in \mathfrak{g}^*$ there exists a unique element $X \in \mathfrak{g}$, such that

$$\langle \mu, Y \rangle = B(X, Y) \ , \tag{12.175}$$

for all $Y \in \mathfrak{g}$ and we identify μ with X. We shall denote $B(X,Y)$ by $\langle X, Y \rangle$, using the same notation as for the canonical pairing between $\mathfrak{g}$, $\mathfrak{g}^*$, thus showing explicitly that we made the above identification. The property (12.174) allows then to identify also the adjoint and the coadjoint representation. Finally, since $\mathfrak{g}^* \sim \mathfrak{g}$, we can define Poisson brackets for functions on $\mathfrak{g}$. The construction is the following. If $f_1(X), f_2(X)$ are two functions on $\mathfrak{g}$, then $df_1(X), df_2(X) \in \mathfrak{g}^* \sim \mathfrak{g}$, and we set

$$\{f_1, f_2\}(X) = \langle [df_1(X), df_2(X)], X \rangle \ . \tag{12.176}$$

12.3.1 Fundamental Fields of a Poisson Tensor

Now let us consider the question about the fundamental fields of a Poisson tensor. Having in mind the properties of the Hamiltonian fields in the case of a symplectic structure the following result is not surprising.

Proposition 12.24. *The Hamiltonian vector fields are fundamental fields for the tensor field P.*

Indeed, it is easy to see that (12.152) can be cast into the following equivalent forms

$$P[L_{P(\alpha)}(\beta) + d\left(\langle \alpha, P(\beta) \rangle\right)] + L_{P(\beta)}(P)(\alpha) = 0 \tag{12.177}$$

(true for arbitrary $\alpha, \beta \in \Lambda^1(\mathcal{M})$) or

$$P[i_{P(\alpha)} d\beta] + L_{P\beta}(P)(\alpha) = 0 \ , \tag{12.178}$$

for arbitrary $\alpha, \beta \in \Lambda^1(\mathcal{M})$. Let β be some fixed closed 1-form. Then from (12.178) follows $L_{P\beta}(P) = 0$, and, therefore, $P\beta$ is fundamental field for the Poisson tensor P. The proposition is proved.

12.3.2 Restriction of Poisson Tensor on Submanifold

It often happens that the bracket $\{\ ,\ \}_P$ defined through the Poisson tensor P is degenerate, that is, there exists nonconstant functions g, such that

$$\{f, g\}_P = 0$$

for arbitrary function f. (Such functions are called Casimir functions.) Naturally, if Casimir functions exist, the Poisson tensor has a nontrivial kernel. But, as we explained before, sometimes we need P to be nondegenerate, so we can to try to restrict the Poisson tensor onto submanifold, where it will be nondegenerate. Thus we naturally arrive at another type of restriction problem. The main result in this direction is given by a theorem, proved in general form, in [44]; see also [45, 46]. We shall use a simplified version of it, proved in [37, 38]. In the same Chap. 2 numerous applications of the restriction techniques to the soliton equation theory are considered, see also [38, 47] for such applications. We present below the version of [37, 38] and in the future refer to it as *The Restriction Theorem*:

Theorem 12.25. *Let $\mathcal{M}$ be Poisson manifold and $\mathcal{N} \subset \mathcal{M}$ be a submanifold. Let us denote by j the inclusion map of $\mathcal{N}$ into $\mathcal{M}$, by $\mathcal{X}^*_P(\mathcal{N})_m$ the subspace of covectors $\alpha \in T^*_m(\mathcal{M})$ such that*

$$P_m(\alpha) \in dj_m(T_m(\mathcal{N})) = \mathcal{I}\mathrm{m}\,(dj_m); \quad m \in \mathcal{N}\,, \tag{12.179}$$

(where $\mathcal{I}\mathrm{m}$ denotes the image) and by $T^{\perp}(\mathcal{N})_m$ – the set of all covectors at $m \in \mathcal{M}$ vanishing on the subspace $\mathcal{I}\mathrm{m}\,(dj_m)$, $m \in \mathcal{N}$ (also called the annihilator of $\mathcal{I}\mathrm{m}\,(dj_m)$ in $T^*_m(\mathcal{M})$). Let the following relations hold:

$$\mathcal{X}^*_P(\mathcal{N})_m + T^{\perp}(\mathcal{N})_m = T^*_m(\mathcal{M}); \quad m \in \mathcal{N} \tag{12.180}$$

$$\mathcal{X}^*_P(\mathcal{N})_m \cap T^{\perp}(\mathcal{N})_m \subset ker\,(P_m); \quad m \in \mathcal{N}\,. \tag{12.181}$$

Then there exists unique Poisson tensor $\overline{P}$ on $\mathcal{N}$, j-related to P, that is

$$P_m = dj_m \circ \overline{P}_m \circ (dj_m)^*; \quad m \in \mathcal{N}\,. \tag{12.182}$$

Proof. Let γ be a covector from $T^*_m(\mathcal{N})$. As the map $dj_m : T_m(\mathcal{N}) \to T_m(\mathcal{M})$ is injective, the map $(dj_m)^* : T^*_m(\mathcal{M}) \to T^*_m(\mathcal{N})$ is subjective and $\ker\,(dj_m)^* = T^{\perp}(\mathcal{N})_m$. Therefore, there exists $\epsilon \in T^*_m(\mathcal{M})$ such that $\gamma = (dj_m)^*\epsilon$. According to our assumptions

$$\epsilon = \epsilon_1 + \epsilon_2; \quad \epsilon_1 \in \mathcal{X}^*_P(\mathcal{N})_m,\ \epsilon_2 \in T^{\perp}(\mathcal{N})_m.$$

We define

$$\overline{P}_m(\gamma) = P_m(\epsilon_1)\,. \tag{12.183}$$

If γ has another representation of the same type:

$$\gamma = (dj_m)^*\mu, \quad \mu = \mu_1 + \mu_2; \quad \mu_1 \in \mathcal{X}^*_P(\mathcal{N})_m,\ \mu_2 \in T^{\perp}(\mathcal{N})_m,$$

then $(\epsilon - \mu) \in T^{\perp}(\mathcal{N})_m$, which due to (12.181) entails $(\epsilon_1 - \mu_1) \in \ker\,(P_m)$, and, thus, the definition of $\overline{P}_m$ contains no ambiguity. It is not difficult to check that $\overline{P}_m$ is linear and that the field $m \to \overline{P}_m$ is smooth. It can also be checked that $\overline{P}$ obeys the first condition in (12.150). Thus in order to prove that it is a Poisson tensor, it suffices to prove that $[\overline{P},\overline{P}]_S = 0$. Let γ_i; $i = 1,2,3$ be 1-forms on $\mathcal{N}$. We know (see remark (11.37) in Sect. (11.5)) that, at least locally, there exist 1-forms β_i on $\mathcal{M}$ such that γ_i and β_i are j-related. Then, according to the definition of $\overline{P}$ the fields $\overline{P}\gamma_i$ and $P\beta_i$ are j-related. Therefore, $L_{\overline{P}\gamma_i}\gamma_s$ is j-related to $L_{P(\beta_i)}\beta_s$ (see proposition (11.41)) and as a result

$$\langle L_{\overline{P}\gamma_i}\gamma_s, \gamma_k\rangle(m) = \langle L_{P\beta_i)}\beta_s, \beta_k\rangle(j(m)) \; .$$

Finally, from the expression of the Schouten–Nijenhuis bracket (12.151), one gets that

$$[\overline{P},\overline{P}]_S(\gamma_1,\gamma_2,\gamma_3)(m) = [P,P]_S(\beta_1,\beta_2,\beta_3)(j(m)) = 0 \; .$$

The theorem is proved.

Corollary 12.26. *If the distribution $m \to \mathcal{I}\mathrm{m}\,(P_m)$ is regular (has constant dimension) it is integrable in the Frobenius sense. The Poisson bracket can be restricted on any of the integral leaves of this distribution, and the corresponding restricted Poisson tensor is nondegenerate.*

Proof. The fact that the distribution $m \to \mathcal{I}\mathrm{m}\,(P_m)$ is integrable is a simple consequence from the definition of the Poisson tensor; see (12.152). Suppose $\mathcal{N}$ is some integral leaf and suppose $j : \mathcal{N} \mapsto \mathcal{M}$ is the inclusion map. For each $m \in \mathcal{N}$ we have

$$\mathcal{X}^*_P(\mathcal{N})_m = T^*_m(\mathcal{M}) \; . \tag{12.184}$$

On the other hand, using the condition $P^* = -P$, we easily get that $T^{\perp}(\mathcal{N})_m = \mathcal{I}\mathrm{m}\,(P_m)^{\perp} = \ker(P_m)$. The theorem we just proved applies immediately, and we see that the Poisson tensor can be restricted onto $\mathcal{N}$. The resulting tensor $\overline{P}_m$ is kernel free. Indeed, from the construction of $\overline{P}_m$, it follows that $\overline{P}_m(\gamma) = P_m(\epsilon)$, $\gamma = (dj_m)^*\epsilon$. Then $\overline{P}_m(\gamma) = 0$ entails $P_m(\epsilon) = 0$. This means that $\epsilon \in \mathcal{I}\mathrm{m}\,(P_m)^{\perp}$ and then $\gamma = (dj_m)^*\epsilon = 0$. This completes the proof.

Remark 12.27. Closely related to the above result is the so-called Marsden–Weistein theorem, [5, 46, 48] showing how to reduce a Hamiltonian system with symmetry group to a symplectic space having lower dimension than the original one.

Let us say a few words about the infinite-dimensional case. We have mentioned already that in this case the property

$$T^{**}_m(\mathcal{M}) = T_m(\mathcal{M})$$

holds no more. Thus strictly speaking, we cannot write $P^* = -P$ and must substitute this condition with the weaker one

$$\langle \alpha, P(\beta) \rangle = -\langle P(\alpha), \beta \rangle \tag{12.185}$$

for arbitrary 1-forms α, β, assuming that P is linear and continuous with respect to some suitable topology. However, usually, we write as before $P^* = -P$, having in mind the above restrictions. An other thing that must be taken into account is that in the infinite-dimensional case, even if the kernel of a continuous liner map is zero, the inverse continuous map may fail to exist. Thus the duality between Poisson structure and symplectic structure is not so simple as in the finite dimensional case, and the corresponding maps can be inverted only on some dense subset of "good" elements.

12.4 Mixed Tensor Fields and Integrability

We pass now to the principal geometric object of this book – the Nijenhuis tensor. However, we think that it is not instructive to introduce it just by a formal definition, so we shall try to show how one can come to it in a natural way, considering the question of integrability of dynamical systems.

A classical integrability criterion for Hamiltonian systems is given by the Liouville theorem, which we shall remind below in the form given by Arnold, (see [5]), and because of this it is called the Liouville–Arnold theorem.[8]

Theorem 12.28. *Let $(\mathcal{M}, \omega)$ be $2n$-dimensional symplectic manifold, let r_i, $i = 1, 2 \ldots, n$ be smooth functions defined on $\mathcal{M}$ such that $\{r_i, r_j\} = 0$, $i, j = 1, 2 \ldots, n$. Let $dr_1, \ldots, dr_n$ be linearly independent for each point belonging to the level manifold $\mathcal{M}_h$*

$$\mathcal{M}_h = \{m : r_i(m) = h_i = const; \quad i = 1, 2, \ldots, n\} \subset \mathcal{M} \tag{12.186}$$

(or equivalently, the functions r_i are functionally independent). Let $H_0 = r_1$, and let us consider the Hamiltonian field X_{H_0} (Hamiltonian system) corresponding to H_0 and the flow corresponding to it[9]. We shall call it simply the flow of H_0.

Then

- *$\mathcal{M}_h$ is a smooth manifold, invariant under the flow of H_0.*
- *If the submanifold $\mathcal{M}_h$ is compact and connected, it is diffeomorphic to the n-dimensional torus*

$$\mathbb{T}^n = \{(\phi^1 \pmod{2\pi}, \phi^2 \pmod{2\pi}, \ldots, \phi^n \pmod{2\pi})\} . \tag{12.187}$$

(The functions ϕ are called angle variables).

[8] We shall introduce later another variant of the same theorem, called Liouville–Cartan theorem.

[9] Of course, we can choose here H_0 to be equal to any of the functions r_i, and everything remains the same.

- *The Hamiltonian flow, generated by H_0, defines on $\mathcal{M}_h$ almost periodic motion, that is, the evolution of the angle coordinates is given by*

$$\frac{d\phi_i}{dt} = \nu^i(h); \quad i = 1, 2, \ldots, n \,, \tag{12.188}$$

where $h = (h_1, h_2, \ldots, h_n)$.
- *The Hamiltonian equations for H_0 are integrable in quadratures.*

Definition 12.29. *In the above situation, the Hamiltonian system X_{H_0}, corresponding to the Hamiltonian functions H_0 is called completely integrable in the Liouville sense.*

The functions $(r_i, \phi^j)_{1\le i,j\le n}$ can be taken locally as coordinates, but in the above situation, there are some other very useful coordinates, the so-called action variables $(H_1, H_2, \ldots, H_n)$, which together with the coordinates ϕ^j form the so-called set of action-angle coordinates (variables) $(H_i, \phi^j)_{1\le i,j\le n}$. In the realms of the Liouville–Arnold theorem, when the submanifold $\mathcal{M}_h$ is compact, they are defined in its neighborhood and have the properties:

1. The symplectic form ω in terms of the action-angle variables has canonical form:
$$\omega = \sum_{i=1}^{n} dH_i \wedge d\phi^i \,, \tag{12.189}$$
that is, the action-angle coordinates are Darboux coordinates for ω.
2. The functions ν^i are functions only on $H = (H_1, H_2, \ldots H_n)$. Thus, in the action-angle coordinates the Hamiltonian vector field X, corresponding to H_0, takes the form
$$X = \sum_{i=1}^{n} \nu^i(H)\frac{\partial}{\partial\phi^i} = \sum_{i=1}^{n} \nu^i(H) X_i \,, \tag{12.190}$$
where $X_i = \frac{\partial}{\partial\phi^i}$ are the Hamiltonian vector fields, corresponding to the Hamiltonian functions H_i.

Perusing the proof of this theorem (see for example [5]) one can see that such a beautiful splitting of the evolution is due to the following circumstances:

(a) There are n linearly independent fields Y_i (the Hamiltonian fields corresponding to the functions r_i)
(b) The fields Y_i define an integrable distribution (each $\mathcal{M}_h$ is their integral leaf).
(c) The fact that $\mathcal{M}_h$ is compact entails that the fields Y_i are complete, that is, the corresponding flows exist for all values of the time and commute ($\{r_i, r_j\} = 0$ leads to $[Y_i, Y_j] = 0$). Finally, since for each h, the manifold $\mathcal{M}_h$ is connected, the flows of Y_i define on it transitive action of the Abelian group $\mathbb{R}^n$. After factoring over a discrete stationary subgroup of some point, we obtain the torus that appears in the Liouville–Arnold theorem.

Some additional effort is needed to obtain the action-angle variables, but the circumstances we outlined are the most important.

We are able to present now conditions that create the same situation as above, without involving a symplectic structure (at least from the beginning). We describe this construction in the following two sections and also show how within this frame arise a multi-Hamiltonian description of some type of dynamical systems and an interesting new object.

Let $\mathcal{M}$ be a smooth $2n$-dimensional manifold, and suppose that on it there exist n linearly independent vector fields $X_1, X_2 \ldots, X_n$ (at each point) and n functionally independent functions $F_1, F_2 \ldots, F_n$. We can represent these conditions in the analytic form

$$\begin{aligned} X_1 \wedge X_2 \wedge \ldots \wedge X_n(m) &\neq 0 \\ dF_1 \wedge dF_2 \wedge \ldots \wedge dF_n(m) &\neq 0 , \end{aligned} \tag{12.191}$$

for each $m \in \mathcal{M}$, where in the first equation, the symbol $\wedge$ is used for the skew-symmetric tensor product (defined similarly as the wedge product of forms). Next, we require that $F_i, X_j;\ i, j = 1, 2 \ldots n$ satisfy

$$[X_i, X_j] = 0, \quad L_{X_i} F_j = 0 . \tag{12.192}$$

(If it turns out that (12.191, 12.192) are satisfied only on some open submanifold $\mathcal{N}$, then we can restrict the considerations to $\mathcal{N}$). Now, assume that the level sets of the submersion map

$$F : \mathcal{M} \to \mathbb{R}^n, \quad F = (F_1, F_2, \ldots, F_n) \tag{12.193}$$

are compact and connected and suppose that a dynamical system X on $\mathcal{M}$ has the form

$$X = \sum_{i=1}^{n} \nu^i X_i, \quad \nu^i = \nu^i(F_1, F_2, \ldots F_n) . \tag{12.194}$$

We are able to show that X is integrable on $\mathcal{M}$ in the same sense as Liouville–integrable systems are. Indeed, the vector fields X_i are tangent to each level surface $F^{-1}(a)$, $a \in \mathbb{R}^n$, and since these surfaces are compact and connected the fields X_i are complete on them and define a transitive action of the Abelian group $\mathbb{R}^n$ just as before. Even if the surfaces $F^{-1}(a)$ are not compact, on each of them we can find 1-forms $\alpha^1, \ldots, \alpha^n$, such that

$$\alpha^i(X_j) = \delta^i_j; \quad i, j = 1, 2, \ldots, n . \tag{12.195}$$

The forms α^i are closed. Indeed,

$$d\alpha^i(X_s, X_k) = X_s(\alpha^i(X_k)) - X_k(\alpha^i(X_s)) - \alpha^i([X_s, X_k]) = 0 \tag{12.196}$$

and the fields $\{X_i\}_1^n$ form a basis in the module of the vector fields. Next, at least locally, there exist functions ϕ^i such that $d\phi^j = \alpha^j$, and also locally one can choose as coordinates the functions $(F_i, \phi^j)_{1\leq i,j\leq n}$. As a result, the vector field X in (12.194) can be explicitly integrated in some neighborhood of each point on the surface $F^{-1}(a)$, because in the coordinates $(F_i, \phi^j)_{1\leq i,j\leq n}$ its integral curves satisfy

$$\dot{\phi}^i = \nu^i(F_1, F_2, \ldots, F_n), \qquad \dot{F}_i = 0 \ , \tag{12.197}$$

where, as is usually done in Mechanics, the time derivative is denoted by a dot. The solutions of this system are

$$\phi^i(t) = t\nu^i(F(m_0)) + \phi^i(m_0), \qquad F_i(t) = F_i(m_0) \ , \tag{12.198}$$

where $m_0 \in \mathcal{M}$ is the initial point. When the variables ϕ^j are of angle type and the integral curves are closed, the functions ν^i play the role of frequencies, so in what follows we shall call them frequencies.

The above construction, though very simple, is important for understanding integrability. It stresses the fact that the dynamics of the system in some coordinate frame separates – some of the coordinates are constants and the evolution of the others is simple (usually linear).

There are also some additional points related to the Liouville theorem that attracted attention only recently. In order to introduce them, let us remark that considering some dynamical system, the symplectic structure is supposed to be already given. In Classical Analytical Mechanics, we obtain it in some standard and canonical way. The system is considered on the phase space $T^*(\mathcal{P})$, which is the cotangent space of the configuration space $\mathcal{P}$, and $T^*(\mathcal{P})$ has canonical symplectic structure on it defined by the Liouville form. However, if we consider again the situation of the Liouville–Arnold theorem, we can see that the systems X_i and X introduced there are completely integrable not only with respect to the structure (12.189) but also to the set of structures defined by

$$\omega_f = \sum_{i=1}^{n} df_i(H) \wedge d\phi^i \ , \tag{12.199}$$

where $f_i(H) = f_i(H_1, H_2, \ldots, H_n)$ are smooth functions, satisfying only the requirement that ω_f must be nondegenerate. The above discussion suggests that the integrability in Liouville–Arnold sense is related to the existence of several symplectic (or Poisson) structures, for which the same dynamical system can be written in Hamiltonian form.

12.4.1 Multi-Hamiltonian Formulations of the Integrable Systems

We shall show now how we can obtain the so-called recursion operators (Nijenhuis tensors in geometric terminology) in case we have an integrable system. To this end, we shall assume that we have the situation described in the previous subsection when we tried to generalize the constructions of the Liouville–Arnold theorem. In other words, let on the manifold $\mathcal{M}$ there exist sets of functions $F_1, F_2 \ldots, F_n$ and fields $X_1, X_2 \ldots, X_n$ obeying the conditions (12.191, 12.192). Let us also have the dynamical system (the vector field) X as in (12.194). Maintaining the analogy with the Liouville–Arnold case, we can define then a class of closed 2-forms by the formula

$$\omega_f = \sum_{i=1}^{n} df_i(F) \wedge \alpha^i \ , \tag{12.200}$$

where f_i are some smooth functions we shall specify later, while $F = (F_1, F_2 \ldots, F_n)$ and α^i are the same as before. It is not hard to see that ω_f is nondegenerate if

$$df_1 \wedge \ldots \wedge df_n \neq 0 \ . \tag{12.201}$$

For all 2-forms of the above type, the action of the Abelian group $\mathbb{R}^n$ defined by the flows of the fields X_i, is canonical. Indeed, by the construction of ω_f we have

$$i_{X_j}\omega_f = -df_j; \quad j = 1, 2, \ldots, n \ , \tag{12.202}$$

which shows that X_j are Hamiltonian vector fields. For the vector field X in (12.194), we have

$$i_X\omega_f = -\sum_{i=1}^{n} \nu^i df_i \ . \tag{12.203}$$

Hence X can be cast in Hamiltonian form if $i_X\omega_f$ is exact. A necessary condition for this is $di_X\omega_f = 0$, which leads to the equation

$$\sum_{i=1}^{n} d\nu^i \wedge df_i = 0 \ . \tag{12.204}$$

If this equation holds, then at least locally there exists a Hamiltonian for X. The solutions $f_1, f_2, \ldots, f_n$ of (12.204) satisfying also (12.201) will yield Hamiltonian formulations for the dynamical system X. If it is already Hamiltonian with respect to some given structure, we shall have alternative formulations. If not, then we can cast it in a Hamiltonian form.

Moreover, on each integral leaf $F_i = \text{const}\,;\ i = 1, 2 \ldots n$ (the level surface $F_i = \text{const}\,;\ i = 1, 2 \ldots n$ is clearly an integral leaf for the distribution defined by the fields X_i), the field X will be completely integrable in the Liouville–Arnold sense. Indeed, as easily checked, the functions $f_1, f_2, \ldots, f_n$ due to (12.192) are integrals of motion in involution:

$$\{f_i, f_j\}_f = \omega_f(X_i, X_j) = L_{X_i} f_j = 0 \ , \tag{12.205}$$

and the sets of functions f_j and F_i can be expressed through each other because of (12.201).

Definition 12.30. *A symplectic form for which X is Hamiltonian vector field shall be called admissible form.*

There exist two limit cases when it is easy to obtain solutions of (12.204).

1. *The constant case.* All the frequencies ν^i are constants, then $d\nu^i = 0$ and (12.204) is satisfied.

In this case any 2-form of the type (12.200) is admissible symplectic form and the corresponding Hamiltonian function for X is given by

$$H = \sum_{i=1}^{n} \nu^i f_i \; . \tag{12.206}$$

Example 12.31. A system for which the above situation takes place is the n-dimensional harmonic oscillator. We can write the field X on the manifold $\mathcal{M} = \mathbb{R}^n_p \times \mathbb{R}^n_q$ in the standard way

$$\begin{aligned} X &= \sum_{i=1}^{n} \eta^i Z_i \\ Z_i &= \frac{1}{\sqrt{m_i k_i}} p_i \frac{\partial}{\partial q_i} - \sqrt{m_i k_i} q_i \frac{\partial}{\partial p_i}; \quad \eta^i = \frac{k_i}{m_i} \; . \end{aligned} \tag{12.207}$$

For each i, the parameters m_i and k_i are the mass and the elastic constant of the i-th oscillator and the functions F_i are the "partial" Hamiltonians

$$F_i = \frac{1}{2}(\frac{p_i^2}{m_i} + k_i q_i^2); \quad i = 1, 2, \ldots, n \; . \tag{12.208}$$

The frequencies are the constants η^i.

2. *The nonresonant case.* None of the frequencies ν^i is constant and

$$d\nu^1 \wedge \cdots \wedge d\nu^n \neq 0 \; . \tag{12.209}$$

The above, of course, means that $\det(\frac{\partial \nu^i}{\partial F_j}) \neq 0$, so we can choose ν^i as part of the coordinates. We can assume F_j to be functions on ν^i, $i = 1, 2, \ldots, n$, so we can write everything in terms of ν^i instead of F_i. In the nonresonant case, there exists a class of simple solutions of (12.204) consisting of linear functions

$$\begin{aligned} f_i &= \sum_{j=1}^{n} A_{ij} \nu^j; \quad i = 1, 2 \ldots n \\ A_{ij} &= A_{ji}, \quad \det(A_{ij}) \neq 0. \end{aligned} \tag{12.210}$$

The Hamiltonian form of X is given then by the quadratic Hamiltonian function

$$H_A = \frac{1}{2} \sum_{i,j=1}^{n} A_{ij} \nu^i \nu^j \; , \tag{12.211}$$

and the symplectic form can be chosen to be

$$\omega_A = \sum_{i,j=1}^{n} A_{ij} d\nu^i \wedge \alpha^j \ . \tag{12.212}$$

Still in the nonresonant case, we can also consider any other symplectic structure of the form

$$\omega_f = \sum_{i=1}^{n} df_i(\nu^i) \wedge \alpha^i \ , \tag{12.213}$$

where for each i the function f_i depends only on the frequency ν^i. Then ω_f will be admissible as long as it is nondegenerate, i.e. as long as $df_1 \wedge \ldots \wedge df_n \neq 0$. The corresponding Hamiltonian functions depend on the explicit form of the functions f_i. For example, if $f_i = \frac{\partial G_i}{\partial \nu^i}(\nu^i)$, the Hamiltonian can be written as

$$H_G = \sum_{i=1}^{n} (G_i - \nu^i \frac{\partial G_i}{\partial \nu^i}) \ . \tag{12.214}$$

Example 12.32. The n-dimensional harmonic oscillator can again be used as an illustration. We can write it as

$$X = \sum_{i=1}^{n} F_i Y_i \ , \tag{12.215}$$

where F_i is given by (12.208) and $Y_i = \eta_i(F_i)^{-1} Z_i$; Z_i as in (12.207). We can see that now the partial Hamiltonians F_i play the role of frequencies.

Remark 12.33. The intermediate cases are more involved, see [49].

Remark 12.34. It is worth while mentioning that there can exist admissible Hamiltonian structures for X, which cannot be obtained through the construction we have outlined.

12.4.2 Recursion Operators for Integrable Systems

As we have seen, given the dynamical system (12.194), we can construct infinitely many Hamiltonian structures, for example, as in (12.200) or (12.213). If a system can be cast into Hamiltonian form in two essentially different ways (that is, with respect to two essentially different Poisson brackets), we shall say that it allows (has) a bi-Hamiltonian formulation, has a bi-Hamiltonian structure or simply that it is bi-Hamiltonian. We have seen that integrability of Liouville–Arnold type implies that we have bi-Hamiltonian properties (and even multi-Hamiltonian properties). As we shall also see, the existence of two essentially different structures for which a given dynamical system is Hamiltonian allows to construct also mixed tensor fields, interlacing two such structures and these tensor fields (recursion operators, Nijenhuis tensors) play an important role in the integrability. Let us consider again the two limit cases.

1. *The constant case*: $d\nu^i = 0$; $i = 1, 2 \ldots, n$.

Two admissible symplectic structures can be obtained from (12.200):

$$\omega_1 = \sum_{i,j=1}^{n} \delta_{ij} dF_i \wedge \alpha^j \ = \sum_k \omega_{(k)} \ , \tag{12.216}$$

where $\omega_k = dF_k \wedge \alpha^k$, and

$$\omega_f = \sum_{i,j=1}^{n} \delta_{ij} f_i(F_i) dF_i \wedge \alpha^j \ = \sum_k f_k(F_k)\omega_{(k)} \ , \tag{12.217}$$

with the condition $df_1 \wedge \ldots \wedge df_n \neq 0$. Using the above structures, we can construct a $(1,1)$ (mixed) tensor field N^* on $\mathcal{M}$, which relates the two symplectic structures:

$$N^* = \bar{\omega}_f \circ \bar{\omega}_1^{-1} = \sum_{k=1}^{n} f_k(F_k)\mathbf{I}_k \ , \tag{12.218}$$

where by $\mathbf{I}_k$ is denoted the identity operator on the k-th two dimensional "plane" of $T^*(\mathcal{M})$ with "coordinates" (dF_k, α^k).

2. *The nonresonant case*: $d\nu^1 \wedge \cdots \wedge d\nu^n \neq 0$.

In this case again, two admissible symplectic structures are obtained from (12.213):

$$\omega_0 = \sum_{i,j=1}^{n} \delta_{ij} d\nu^i \wedge \alpha^j = \sum_k \omega_{(k)} \ , \tag{12.219}$$

where as before $\omega_{(k)} = dF_k \wedge \alpha^k$, and

$$\omega_f = \sum_{i,j=1}^{n} \delta_{ij} f^i(\nu^i) d\nu^i \wedge \alpha^j = \sum_k f_k(\nu^k)\omega_{(k)} \ , \tag{12.220}$$

with the requirement $df_1 \wedge \ldots \wedge df_n \neq 0$. From these structures, we can construct a $(1,1)$ tensor field N^* on $\mathcal{M}$ through the formula

$$N^* = \bar{\omega}_f \circ \bar{\omega}_0^{-1} = \sum_{k=1}^{n} f_k(\nu^k)\ \mathbf{I}_k \ , \tag{12.221}$$

where $\mathbf{I}_k$ is the identity operator on the k-th two-dimensional "plane" of $T^*(\mathcal{M})$ with "coordinates" $(d\nu^k, \alpha^k)$, and according to our conventions $\bar{\omega}$, $\bar{\omega}_0$ stand for the fields of linear maps

$$m \mapsto \bar{\omega}_m, (\bar{\omega}_0)_m : T_m(\mathcal{M}) \mapsto T_m^*(\mathcal{M})$$

corresponding to the forms ω, ω_0. The fields $m \mapsto N_m^*$ are fields of endomorphisms of the vector space $T_m^*(\mathcal{M})$.

Liouville–Arnold Integrable Systems

Let us consider again the situation we had in the Liouville–Arnold theorem. Let $(\mathcal{M}, \omega_0)$ be a symplectic manifold, let the vector fields $X_1, \ldots, X_n$ we used in the construction that is described after (12.192) be complete vector fields corresponding to the independent functions $H_1, \ldots, H_n$, which are in involution. (This ensures the conditions (12.192) with $F_i = H_i$). In this situation, one can find "*angle*" 1-forms $\alpha^1, \ldots, \alpha^n$, such that $\alpha^i(X_j) = \delta^i_j$ and $d\alpha^i = 0$ (if we use the action-angle variables, we shall have of course $\alpha^i = d\phi^i$).

Suppose the dynamical vector field X on $\mathcal{M}$ has the form (12.194), where we must put now $F_i = H_i$. Suppose that X is Hamiltonian system corresponding to the Hamiltonian H.

If F is arbitrary function of the H_j, we have $dF \wedge dH_1 \wedge \ldots \wedge dH_n = 0$. Suppose now that F is such that $\det(\frac{\partial^2 F}{\partial H_i \partial H_j}) \neq 0$ (we shall call such F nondegenerate). Then one can see that the exact 2-form

$$\omega_F = \sum_{i=1}^{n} d(\frac{\partial F}{\partial H_i}\alpha^i) \tag{12.222}$$

is an admissible symplectic form for all the dynamical systems X_i. In particular, if

$$F = \frac{1}{2}\sum_{i=1}^{n} H_i^2 \tag{12.223}$$

we recover the original structure. If we use the existing *action-angle* variables (H_k, ϕ^k) we shall have:

$$X = \sum_{k=1}^{n} \nu^k \frac{\partial}{\partial \phi^k}$$

$$\omega_0 = \sum_{k=1}^{n} dH_k \wedge d\phi^k$$

$$i_X\omega_0 = \sum_{k=1}^{n} \nu^k dH_k = -\sum_{k=1}^{n} \frac{\partial H}{\partial H_k} dH_k = -dH\ , \tag{12.224}$$

and we obtain

$$\nu^k = \frac{\partial H}{\partial H_k}; \quad k = 1, 2 \ldots, n.$$

The condition $d\nu^1 \wedge \ldots \wedge d\nu^n \neq 0$ is equivalent to $\det(\frac{\partial \nu^k}{\partial H_l}) \neq 0$ and, as easily seen, is also equivalent to the nondegeneracy of the Hamiltonian function, that is, to $\det\left(\frac{\partial^2 H}{\partial H_i \partial H_j}\right) \neq 0$. Thus the *nonresonant case* corresponds to nondegenerate H. We also see that we can use the functions ν^k as part of the coordinates and write immediately one admissible symplectic structure:

$$\omega_\nu = \sum_{k=1}^{n} d\nu^k \wedge d\phi^k \ . \tag{12.225}$$

Then the Hamiltonian for X will be the quadratic function

$$H_\nu = \frac{1}{2}\sum_{k=1}^{n}(\nu^k)^2 \ . \tag{12.226}$$

In the same way, we can obtain that in the *nonresonant case* the completely integrable system X has infinitely many admissible symplectic structures, some of them having the form

$$\omega_f = \sum_{i=1}^{n} df_i(\nu^i) \wedge d\phi^i \ , \tag{12.227}$$

where, of course, the functions f_i satisfy $df^1 \wedge \ldots \wedge df^n \neq 0$. Note, however, that in general ω_0 cannot be obtained in this way.

Thus the systems we are speaking about admit recursion operators given by expression (12.221).

Example 12.35. Consider now the following completely integrable system, introduced in [50].

Let $\mathcal{M} = \mathbb{R}^2 \times \mathbb{T}^2 = \{(x, y, \theta, \eta)\}$. ($\mathbb{T}^2$ is the two-dimensional torus). The manifold $\mathcal{M}$ is endowed a structure of symplectic manifold by the 2-form

$$\omega_0 = dx \wedge d\theta + dy \wedge d\eta \ . \tag{12.228}$$

Let the dynamical system be defined by the Hamiltonian $H = x^3 + y^3 + xy$. The vector field corresponding to H is easily calculated:

$$\begin{aligned} X &= \nu_\theta \frac{\partial}{\partial\theta} + \nu_\eta \frac{\partial}{\partial\eta} \\ \nu_\theta &= 3x^2 + y, \quad \nu_\eta = 3y^2 + x \ . \end{aligned} \tag{12.229}$$

The previous discussion shows that this system admits infinitely many alternative Hamiltonian formulations on the dense open submanifold $\mathcal{M}_0$ of $\mathcal{M}$, characterized by the condition $d\nu_\theta \wedge d\nu_\eta \neq 0$, that is, by $36xy - 1 \neq 0$. $\mathcal{M}_0$ coincides with the submanifold on which H is nondegenerate. Two symplectic structures of the type we consider are given by

$$\omega_1 = d\nu_\theta \wedge d\theta + d\nu_\eta \wedge d\eta \tag{12.230}$$

$$\omega_2 = f(\nu_\theta) d\nu_\theta \wedge d\theta + g(\nu_\eta) d\nu_\eta \wedge d\eta \ , \tag{12.231}$$

where f and g are arbitrary functions such that $df \wedge dg \neq 0$. The corresponding recursion operators are given by

$$N^* = \bar{\omega}_2 \circ \bar{\omega}_1^{-1} = f(\nu_\theta) \left(d\nu_\theta \otimes \frac{\partial}{\partial \nu_\theta} + d\theta \otimes \frac{\partial}{\partial \theta} \right) + g(\nu_\eta) \left(d\nu_\eta \otimes \frac{\partial}{\partial \nu_\eta} + d\eta \otimes \frac{\partial}{\partial \eta} \right) . \qquad (12.232)$$

We point out that ω_0 does not belong to the family of symplectic structures given in (12.231) and that our recursion operators (12.232) cannot be "factored" through ω_0, that is, written into the form $\bar{\omega} \circ \bar{\omega}_0^{-1}$.

After the above discussion and examples, our attention is now driven to the mixed tensor fields N^*, we have obtained in (12.232),(12.217), and (12.221). We have seen that they interrelate different symplectic structures. For reasons that shall become clear later, we prefer to state not the properties of the fields N^* but the properties of the fields of their adjoint operators:

$$m \mapsto N_m : T_m(\mathcal{M}) \mapsto T_m(\mathcal{M}).$$

They are fields of endomorphisms of the tangent spaces $T_m(\mathcal{M})$. In all the examples we have given, the fields $m \mapsto N_m$ have the following properties:

- N is *invariant* under X ($L_X N = 0$), where X is the dynamical system we consider, the same that possesses bi-Hamiltonian formulation.
- N is semisimple and has double degenerate eigenvalues without critical points. This is a short way of saying that at each point m the operator N_m is semisimple, its eigenvalues $\lambda_i(m)$ are smooth functions, the corresponding eigenspaces are two-dimensional and $d\lambda_i|_m \neq 0$. The expression "without critical points" comes from the fact that when $d\lambda_i|_m \neq 0$ the Hamiltonian vector fields that correspond to the functions λ_i do not have critical points.
- N has vanishing Nijenhuis torsion R_N (or Nijenhuis bracket), that is

  $$[NX, NY] - N[NX, Y] - N[X, NY] + N^2[X, Y] = 0$$

  for any choice of the vector fields X, Y.

From the above properties, only the third one needs a proof, but that can be done by simple, though tedious calculations. We shall not do that here, because later we shall be able to give some general results which ensure that the Nijenhuis torsion vanishes, so the above will be a simple consequence from the general theory.

Tensor fields as obtained above are known to arise in the theory of the soliton equations and are called recursion operators, generating operators, Λ-operators, hereditary operators, or Nijenhuis operators (tensors). Usually what is called Λ-operators are the adjoint operators N^*; recursion operators is used both for N and N^*, and in the geometric approaches for N is used the name Nijenhuis tensor. Their role in the geometric theory of the integrable systems has been realized after the pioneer work of F. Magri [19]; see also

[51], and the works of another authors as I. Ya. Dorfman [52]; A. S. Fokas and B. Fuchssteiner, [53, 54] (under the name of hereditary operators); V.E. Zakharov and B. G. Konopelchenko [55] (under the name recursion operators). We are trying to cite the earliest sources here, but for the contemporary developments of the theory see the monographs [56, 57, 58]; in the monographs [58], the Nijenhuis tensor is a central geometric object and the finite and infinite case are studied on the same grounds. We refer only to sources (and maybe in an incomplete way), treating the geometric and Hamiltonian properties of the field N (N^*). As already mentioned, the same object appeared under the name Λ-operator in the spectral approaches. Probably the understanding of the geometric importance of N in the theory of integrable systems could have been discovered earlier, if the rarely cited works of A.P. Stone [59, 60] had been widely known, because they contain a lot of the necessary information for that understanding. However, historically only after the work of F. Magri [19], the central role of the operator (tensor) N in the geometric approaches as an operator, relating two symplectic (Poisson) structures became clear.

From the other side, tensor fields with vanishing Nijenhuis torsion also turned out to be known earlier in the Differential Geometry, but their study has been motivated by their applications to some other topics (almost complex structures for example). We prefer to elaborate on the abstract properties of the Nijenhuis tensors in a separate section, namely, Sect. 13.3 of the next chapter. Here we just intend to convince the reader that these interesting fields already arise naturally in the theory of finite dimensional completely integrable Hamiltonian systems. We hope to justify further this opinion in Chap. 14. Up to now, however, the most interesting applications of the Nijenhuis tensors (operators) have been in the study of the infinite-dimensional dynamical systems.

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13

Vector-Valued Differential Forms

The goal of this chapter is to introduce the mixed tensor fields with vanishing Nijenhuis bracket, which we do at the end. However, at the beginning, we adopt a more general viewpoint, and together with the Nijenhuis bracket of two fields of operators, we describe the more general notion of the bracket of two vector-valued differential forms. We believe that the applications of rich calculus related to these objects is not limited only to the Nijenhuis tensors, and that is one of the reasons we present them here. Another reason is that this topic is more specific, and usually it is treated only in journal papers. Finally, this more general viewpoint will be useful in the generalizations we are going to present in the Appendix.

We present the theory of vector-valued differential forms following [1].

13.1 Derivations of the Graded Ring of the Exterior Forms

Let $\mathcal{M}$ be a manifold and $\Lambda(\mathcal{M})$ be the exterior algebra over $\mathcal{M}$. We have considered already the operations i_X, d, L_X defined on $\Lambda(\mathcal{M})$ and their properties, but it turns out that these operations are particular cases of more general operations – the so-called derivations of $\Lambda(\mathcal{M})$. In order to avoid confusion, we must clarify the notation we use. First, the set of the constant functions on $\mathcal{M}$ is naturally isomorphic to $\mathbb{R}$ and for that reason we shall denote it by $\mathbb{R}$. Then $\mathbb{R} \subset \Lambda(\mathcal{M})$ is a subring of $\Lambda(\mathcal{M})$. We have the following:

Definition 13.1. *A derivation D of degree r of the ring $\Lambda(\mathcal{M})$ related to the subring $\mathbb{R}$ (or simply derivation of degree r) is an endomorphism, such that*

$$\begin{aligned} &(1)\ D(\mathbb{R}) = 0 \\ &(2)\ D(\Lambda^s(\mathcal{M})) \subset \Lambda^{s+r}(\mathcal{M}); \quad s = 0, 1, 2 \ldots \\ &(3)\ D(\omega \wedge \pi) = D(\omega) \wedge \pi + (-1)^{qr} \omega \wedge D(\pi)\ , \end{aligned} \tag{13.1}$$

where $\omega \in \Lambda^q(\mathcal{M}), \pi \in \Lambda^p(\mathcal{M})$.

Gerdjikov, V.S. et al.: *Vector-Valued Differential Forms*. Lect. Notes Phys. **748**, 459–471 (2008)
DOI 10.1007/978-3-540-77054-1_13

All the operations we have introduced – the interior product, the exterior derivative (Cartan derivative), and the Lie derivative are derivations of $\Lambda(\mathcal{M})$ of degrees -1, $+1$ and 0, respectively. If $f \in \mathcal{D}(\mathcal{M})$ is a smooth function and D is a derivation of $\Lambda(\mathcal{M})$ of degree r, then fD is derivation of degree r too, so the set $\mathrm{Der}^{\,r}(\Lambda(\mathcal{M}))$ of the derivations of degree r is a module over $\mathcal{D}(\mathcal{M})$, as well as the set of all derivations $\mathrm{Der}\,(\Lambda(\mathcal{M}))$. However, $\mathrm{Der}\,(\Lambda(\mathcal{M}))$ possesses additional algebraic structure, as stated by the next proposition.

Proposition 13.2. *Let D_1 and D_2 are two derivations of degree r_1 and r_2, respectively. Then the endomorphism*

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{r_1 r_2} D_2 \circ D_1 \tag{13.2}$$

is a derivation of degree $r_1 + r_2$ called the graded commutator of D_1 and D_2.

(The proof of the proposition amounts to simple calculation).

The definition of $[D_1, D_2]$ entails

$$[D_1, D_2] = -(-1)^{r_1 r_2}[D_2, D_1]\ , \tag{13.3}$$

and, therefore, with respect to the commutator, the module of the derivations of $\Lambda(\mathcal{M})$ is graded anticommutative ring.

Also by a computation one can prove the following:

Proposition 13.3. *For each choice of the derivations D_1, D_2, D_3 of degrees r_1,r_2,r_3, respectively, the following identity, called the Jacobi identity, holds*

$$(-1)^{r_3 r_1}[D_1, [D_2, D_3]] + \mathrm{cycl}(1,2,3) = 0\ . \tag{13.4}$$

Here, as usual, by $\mathrm{cycl}\,(1, 2, 3)$, we denote the set of two additional terms that can be obtained from the first one by cyclic permutation of the indices $(1, 2, 3)$. Thus, with respect to the commutator $\mathrm{Der}\,(\Lambda(\mathcal{M}))$ is also a graded Lie algebra over $\mathbb{R}$.

It is important that one can construct derivations of degree $n-1$ from the so-called vector-valued differential forms of degree n. Let us introduce these new objects.

Definition 13.4. *We call vector-valued differential form of degree $n \geq 1$ a smooth field N of linear skew-symmetric maps:*

$$\mathcal{M} \ni m \to N_m : T_m(\mathcal{M})^n \to T_m(\mathcal{M})\ . \tag{13.5}$$

One can also say that a vector-valued form N of degree n is $(1, n)$-type tensor field over the manifold $\mathcal{M}$, skew-symmetric with respect to the covariant indices. In a natural way, one can define the multiplication of N by function, and then clearly the set of all vector-valued differential forms $V^{(n)}(\mathcal{M})$ of degree n becomes a module over $\mathcal{D}(\mathcal{M})$. We also set by definition $V^{(0)}(\mathcal{M})$ to be the module $\mathcal{T}(\mathcal{M})$ of the vector fields over $\mathcal{M}$ and then $V^{(n)}(\mathcal{M})$ is defined for $n \geq 0$.

Definition 13.5. *The module*

$$V(\mathcal{M}) = \overset{\infty}{\underset{n=0}{\oplus}} V^{(n)}(\mathcal{M}) \tag{13.6}$$

is called the module of vector-valued differential forms.

There are also some natural operations between vector-valued forms and between vector-valued forms and differential forms. The first operation simply generalizes the exterior product. If $\omega \in \Lambda^p(\mathcal{M})$, $p \geq 1$; $N \in V^{(n)}(\mathcal{M})$, $n \geq 1$ and if $X_1, X_2, \ldots, X_{p+n} \in T_m(\mathcal{M})$, we define $m \to (\omega \wedge N)_m$ to be the field of skew-symmetric maps:

$$\begin{aligned}&(\omega \wedge N)_m(X_1, X_2, \ldots, X_{p+n}) = \\ &\frac{1}{p!m!} \sum_{\sigma \in G_{p+n}} \epsilon(\sigma)\omega_m(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(p)}) \times \\ &N_m(X_{\sigma(p+1)}, X_{\sigma(p+2)}, \ldots, X_{\sigma(p+n)}) \ .\end{aligned} \tag{13.7}$$

Here, as in the definition of the wedge product, G_{p+n} stands for the permutation group of $\{1, 2, \ldots, p+q\}$ and $\epsilon(\sigma)$ is the parity of $\sigma \in G_{p+n}$. Further, for $X \in V^{(0)}(\mathcal{M})$, $\omega \in \Lambda^p(\mathcal{M})$, $p \geq 0$ we put

$$\omega \wedge X = \omega \otimes X \ , \tag{13.8}$$

where the tensor product is taken over the ring $\mathcal{D}(\mathcal{M})$ and for $f \in \Lambda^0(\mathcal{M})$, $N \in V^{(n)}(\mathcal{M})$, $n \geq 0$ we set

$$f \wedge N = fN \ . \tag{13.9}$$

We also define

$$N \wedge \omega = (-1)^{pn} \omega \wedge N \tag{13.10}$$

for $\omega \in \Lambda^p(\mathcal{M})$, $N \in V^{(n)}(\mathcal{M})$. Of course, if $\omega \in \Lambda^p(\mathcal{M})$, $N \in V^{(n)}(\mathcal{M})$ then $\omega \wedge N \in V^{(n+p)}(\mathcal{M})$.

Thus, the operation introduced in the above has properties similar to the wedge product of differential forms.

There is also another operation with vector-valued forms. Suppose we have $P \in V^{(p)}(\mathcal{M})$, $p \geq 1$; $N \in V^{(n)}(\mathcal{M})$, $n \geq 1$. Then for

$$X_1, X_2, \ldots, X_{p+n-1} \in T_m(\mathcal{M})$$

we define $m \to (P \odot N)_m$ to be the field of skew-symmetric maps:

$$\begin{aligned}&(P \odot N)_m(X_1, X_2, \ldots, X_{p+n-1}) = \\ &\frac{1}{(n-1)!p!} \sum_{\sigma \in G_{p+n-1}} \epsilon(\sigma) N_m \left(P_m(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(p)})\right. \\ &\left.X_{\sigma(p+1)}, X_{\sigma(p+2)}, \ldots, X_{\sigma(p+n-1)}\right) \ .\end{aligned} \tag{13.11}$$

It is clear that $P \odot N$ is vector-valued differential form of degree $p + n - 1$.

Next, for $M \in V^{(0)}(\mathcal{M})$, we put $i_N M = 0$, and for $X \in V^{(0)}(\mathcal{M})$ (vector field), $M \in V^{(p)}(\mathcal{M})$, $p \geq 1$, we define

$$i_X M(X_1, X_2, \ldots, X_{p-1}) = M(X, X_1, X_2, \ldots, X_{p-1}) \,. \tag{13.12}$$

Quite in the same way, if $P \in V^{(p)}(\mathcal{M})$, $p \geq 1$; $\alpha \in \Lambda^n(\mathcal{M})$, $n \geq 1$ and $X_1, X_2, \ldots, X_{p+n-1} \in T_m(\mathcal{M})$, we define $m \to (i_P\alpha)_m$ to be the following differential form:

$$\begin{aligned}&(i_P\alpha)_m(X_1, X_2, \ldots, X_{p+n-1}) = \\ &\frac{1}{(n-1)!p!} \sum_{\sigma \in G_{p+n-1}} \epsilon(\sigma)\alpha_m \big(P_m(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(p)}), \\ &X_{\sigma(p+1)}, X_{\sigma(p+2)}, \ldots, X_{\sigma(p+n-1)}\big) \,.\end{aligned} \tag{13.13}$$

It is easily seen that $i_P\alpha$ is differential form of degree $p+n-1$. For $\alpha \in \Lambda^0(\mathcal{M})$, we put $i_N\alpha = 0$, and for $X \in V^{(0)}(\mathcal{M})$, we define i_X to be the interior product.

Example 13.6. Let I be the field of identity operators and let $N \in V^{(n)}(\mathcal{M})$, then

$$I \odot N = nN, \quad N \odot I = N \,. \tag{13.14}$$

The example shows that generally speaking $N \odot M \neq M \odot N$.

Example 13.7. Let I be again the field of identity operators, let $N \in V^{(1)}(\mathcal{M})$, suppose that $X_1, X_2, \ldots, X_{p+n-1} \in T_m(\mathcal{M})$ and $\alpha \in \Lambda^p(\mathcal{M})$, then

$$\begin{aligned}&i_I\alpha = p\alpha \\ &[i_N\alpha]_m(X_1, X_2, \ldots, X_p) = \\ &\sum_{j=1}^{p} \alpha_m(X_1, X_2, \ldots, X_{j-1}, N(X_j), X_{j+1}, X_{j+2}, \ldots, X_p) \,.\end{aligned} \tag{13.15}$$

It is not hard to establish some useful properties of the operations we have introduced:

1. For $N \in V^{(n)}(\mathcal{M})$, $\alpha \in \Lambda^p(\mathcal{M})$, $\beta \in \Lambda^q(\mathcal{M})$, we have:

$$\begin{aligned}\alpha \wedge (\beta \wedge N) &= (\alpha \wedge \beta) \wedge N \\ i_{\alpha \wedge N}\beta &= \alpha \wedge i_N\beta \\ i_N[\alpha \wedge \beta] &= (i_N\alpha) \wedge \beta + (-1)^{(n-1)p}\alpha \wedge i_M\beta \,.\end{aligned} \tag{13.16}$$

2. For $N \in V^{(n)}(\mathcal{M})$, $M \in V^{(m)}(\mathcal{M})$, hold the relations

$$i_N i_M - (-1)^{(n-1)(m-1)} i_N i_M = i_{M \odot N} - (-1)^{(n-1)(m-1)} i_{N \odot M} \,. \tag{13.17}$$

The last relation in (13.16) shows that i_N is a derivation of degree $n-1$. Using the graded commutator, (13.17) can be written into the form

$$[i_N, i_M] = i_{M \odot N} - (-1)^{(n-1)(m-1)} i_{N \odot M} \, . \tag{13.18}$$

We shall present now the most essential properties of the derivations of the graded ring of the differential forms. One of the main facts is presented by the following theorem [1].

Theorem 13.8. *Any derivation D of the graded ring $\Lambda(\mathcal{M})$ is completely determined by its action on $\Lambda^0(\mathcal{M}) \oplus \Lambda^1(\mathcal{M})$. Any map satisfying (13.1) for $\omega \in \Lambda^p(\mathcal{M})$, $\pi \in \Lambda^q(\mathcal{M})$, $p+q \leq 1$ can be extended uniquely to a derivation of degree r.*

From these results it easily follows that nontrivial derivations of degree less than -1 do not exist and that all the derivations of degree -1 act trivially on $\Lambda^0(\mathcal{M})$. Another essential result of the theory is the following.

Theorem 13.9. *Every derivation of degree r acting trivially on Λ^0 is of the type i_N, where N is vector-valued differential form of degree $r+1$.*

The particular case of this result for $r = -1$ is classical, it means that only the interior products i_X, where X is some vector field, are derivations of degree -1.

Remark 13.10. Sometimes different terminology is used, and the derivations having even degree are called "derivations", while the derivations of odd degree are called "antiderivations" [2].

Definition 13.11. *The derivations of the type i_M, where M is vector-valued differential form of degree m, are called derivations of type "i."*

Definition 13.12. *The derivations of the type*

$$d_M = [i_M, d] = i_M d - (-1)^{(m-1)} d i_M \, , \tag{13.19}$$

where M is vector-valued differential form of degree m and d is the exterior derivative, are called derivations of type "d."

One can prove the following results; see [1]:

Theorem 13.13. *The derivations of type "d", and only these derivations, commute with the Cartan derivative d. Derivations of this type are completely determined by their action on $\Lambda^0(\mathcal{M})$.*

Theorem 13.14. *Each derivation can be written in a unique way as a sum of two derivations, the first one being of the type "i" and the second one of the type "d."*

In other words, we have the decomposition

$$\begin{aligned}\operatorname{Der}(\Lambda(\mathcal{M})) &= \operatorname{Der}_i(\Lambda(\mathcal{M})) \oplus \operatorname{Der}_d(\Lambda(\mathcal{M}))\\ \operatorname{Der}_i(\Lambda(\mathcal{M})) &= \bigoplus_{s=-1}^{\infty} \operatorname{Der}_i^s(\Lambda(\mathcal{M}))\\ \operatorname{Der}_d(\Lambda(\mathcal{M})) &= \bigoplus_{s=0}^{\infty} \operatorname{Der}_d^s(\Lambda(\mathcal{M}))\ ,\end{aligned} \tag{13.20}$$

where $\operatorname{Der}_i^s(\Lambda(\mathcal{M}))$ and $\operatorname{Der}_d^s(\Lambda(\mathcal{M}))$ are the sets of derivations of degree s of the types "i" and "d", respectively. The relation (13.18) and the Jacobi identity show that

$$\begin{aligned}[\operatorname{Der}_i(\Lambda(\mathcal{M})), \operatorname{Der}_i(\Lambda(\mathcal{M}))] &\subset \operatorname{Der}_i(\Lambda(\mathcal{M}))\\ [\operatorname{Der}_d(\Lambda(\mathcal{M})), \operatorname{Der}_d(\Lambda(\mathcal{M}))] &\subset \operatorname{Der}_d(\Lambda(\mathcal{M}))\ ,\end{aligned} \tag{13.21}$$

or, in other words, that $\operatorname{Der}_i(\Lambda(\mathcal{M}))$ and $\operatorname{Der}_d(\Lambda(\mathcal{M}))$ are subalgebras of $\operatorname{Der}(\Lambda(\mathcal{M}))$. Therefore, the Lie algebra operation, defined over $\operatorname{Der}(\Lambda(\mathcal{M}))$, gives rise to two Lie bracket operations over the graded module of the vector-valued differential forms $V(\mathcal{M})$, which we denote below by bracket $[\ ,\ ]$ and by asterisk $*$

$$\begin{aligned}[i_M, i_N] &= i_{M*N}\\ [d_M, d_N] &= d_{[M,N]}\ .\end{aligned} \tag{13.22}$$

The equation (13.17) shows that

$$M * N = M \odot N - (-1)^{(n-1)(m-1)} N \odot M\ . \tag{13.23}$$

As to the operation $(M,N) \mapsto [M,N]$, it is a new one. Both the above operations we introduced are bilinear and obey the following identities which are a consequence of the "skew-symmetry" of the commutator and the Jacobi identity:

$$\begin{aligned}M * N &= -(-1)^{(n-1)(m-1)} N * M\\ [M,N] &= -(-1)^{nm}[N,M]\ ,\end{aligned} \tag{13.24}$$

for any vector-valued forms M and N of degrees m and n, respectively, and

$$\begin{aligned}&(-1)^{(n_3-1)(n_1-1)} N_1 * (N_2 * N_3) + (-1)^{(n_1-1)(n_2-1)} N_2 * (N_3 * N_1) +\\ &(-1)^{(n_2-1)(n_3-1)} N_3 * (N_1 * N_2) = 0\end{aligned} \tag{13.25}$$

$$\begin{aligned}&(-1)^{n_3 n_1}[N_1,[N_2,N_3]] + (-1)^{n_1 n_2}[N_2,[N_3,N_1]] +\\ &(-1)^{n_2 n_3}[N_3,[N_1,N_2]] = 0\end{aligned} \tag{13.26}$$

for any three vector-valued forms N_1, N_2, N_3, of degrees n_1, n_2, n_3, respectively. The operation $*$ is of pure algebraic character and does not seem to have much application in Differential Geometry. In what follows, we shall concentrate on the operation $[\ ,\ ]$, which endows the module of vector-valued differential forms $V(\mathcal{M})$ with a structure of a graded Lie algebra over $\mathbb{R}$.

Definition 13.15. *If M and N are vector-valued forms of degrees m and n, respectively, then $[M, N]$ is called differential concomitant or Nijenhuis bracket of M and N.*

It can be verified that in the case $m = n = 0$, the Nijenhuis bracket coincides with the Lie bracket of two vector fields.

Example 13.16. If I is the field of identity maps, then $d_I = d$, and as for arbitrary M, we have $[d, d_M] = 0$ it follows that $[I, M] = 0$.

There are some more relations between the objects we have already introduced. The first part of them concerns the multiplication of vector-valued forms with scalar forms. We have the following

Proposition 13.17. *The multiplication $(\alpha, D) \mapsto \alpha \wedge D$, defined by*

$$(\alpha \wedge D)\beta = \alpha \wedge (D(\beta)) \tag{13.27}$$

endows the module of the derivations $Der(\Lambda(\mathcal{M}))$ with a structure of a graded module over the graded ring $\Lambda(\mathcal{M})$, that is

$$\Lambda^n(\mathcal{M}) \wedge Der^p(\Lambda(\mathcal{M})) \subset Der^{n+p}(\Lambda(\mathcal{M}))\ , \tag{13.28}$$

where $Der^p(\Lambda(\mathcal{M}))$ is the set of derivation of degree p. The submodule of the forms of type "i" is invariant under the above multiplication:

$$\Lambda(\mathcal{M}) \wedge Der_i(\Lambda(\mathcal{M})) \subset Der_i(\Lambda(\mathcal{M})),$$

in fact,

$$\alpha \wedge i_N = i_{\alpha \wedge N}\ , \tag{13.29}$$

and the submodule of the derivations $Der_d(\Lambda(\mathcal{M}))$ is invariant provided α is closed, since for $\alpha \in \Lambda^p(\mathcal{M})$, $N \in V^{(n)}(\mathcal{M})$, we have

$$\alpha \wedge d_N = d_{\alpha \wedge N} + (-1)^{(p+n-1)} i_{d\alpha \wedge N}\ . \tag{13.30}$$

It is not hard to establish the following formulae:

$$\begin{aligned} [\alpha \wedge i_M, i_N] &= \alpha \wedge [i_M, i_N] - (-1)^{(m+p-1)(n-1)} (i_N \alpha) \wedge i_M \\ [\alpha \wedge d_M, d_N] &= \alpha \wedge [d_M, d_N] - (-1)^{(m+p)n} (d_N \alpha) \wedge d_M\ , \end{aligned} \tag{13.31}$$

where M and N are two vector-valued forms of degree m and n, respectively, and α is p-form.

The second type of relations originates from the decomposition of the commutator $[i_M, d_N]$ into a sum of derivation of type "i" and derivation of type "d." We have

Theorem 13.18. *If M and N are vector-valued forms of degree m and n, respectively, then*

$$[i_M, d_N] = d_{M \odot N} + (-1)^n i_{[M,N]}\ . \tag{13.32}$$

Applying this theorem to the second relation in (13.31), one can get the following identity

$$[\alpha \wedge M, N] = \alpha \wedge [M, N] + (-1)^{(p+m)} d\alpha \wedge M \odot N \\ -(-1)^{(p+m)n}(d_N\alpha) \wedge M \ , \tag{13.33}$$

for two vector-valued forms M and N of degree m and n, respectively, and for a scalar form α of degree p.

Also, with the help of the above theorem, and making use of Jacobi identity for the derivations d_L, d_M, i_N, where $L \in V^{(l)}(\mathcal{M})$, $M \in V^{(m)}(\mathcal{M})$, and $N \in V^{(n)}(\mathcal{M})$, we get

$$[N \odot L, M] + (-1)^{l(n-1)}[L, N \odot M] - N \odot [L, M] = \\ (-1)^{(l-1)m}[N, M] \odot L + (-1)^{(l-1)}[N, L] \odot M \ . \tag{13.34}$$

13.2 Vector-Valued Forms of Degree One

We apply now the general results for the case of vector-valued differential forms of degree 1 and we obtain some useful identities. First of all, let us note that in case N and M are vector-valued forms of degree 1 (fields of operators) some of the operations introduced earlier have natural meaning. For example, it is readily seen that $N \odot M = M \circ N = MN$, where $M \circ N = MN$ is the field $m \mapsto M_m \circ N_m$. If S is vector-valued differential form of degree 2, we have

$$N \odot S(X_1, X_2) = S(NX_1, X_2) + S(X_1, NX_2); \quad X_1, X_2 \in \mathcal{T}(\mathcal{M}).$$

Taking into account the above, the relation (13.34) between three fields of operators reads

$$[N, L \circ M](X_1, X_2) + [N \circ M, L](X_1, X_2) \\ -[N, L](MX_1, X_2) - [N, L](X_1, MX_2) = \\ N \circ [L, M](X_1, X_2) + L \circ [N, M](X_1, X_2) \ , \tag{13.35}$$

for any two vector fields X_1, X_2.

Remark 13.19. The notation $[M, N]$, which is universally accepted, can unfortunately be confused with the commutator. In order to avoid misunderstanding, note that if M and N are vector-valued forms of degree 1, then $[M, N]$ is vector-valued form of degree 2, that is, we have $[M, N](X_1, X_2) = -[M, N](X_2, X_1)$, for arbitrary vector fields X_1, X_2. However, see (13.36), we have $[M, N] = [N, M]$.

Applying directly the identity (13.32), we get that if X_1, X_2 are two arbitrary vector fields then

$$
\begin{aligned}
&[N, M](X_1, X_2) = \\
&[NX_1, MX_2] + [MX_1, NX_2] - M[NX_1, X_2] - N[MX_1, X_2] \\
&-M[X_1, NX_2] - N[X_1, MX_2] + N \circ M[X_1, X_2] + M \circ N[X_1, X_2].
\end{aligned}
\tag{13.36}
$$

Here, as usual, the symbol $[X_1, X_2]$ stands for the Lie bracket of two vector fields). The equation (13.36) is often taken as an independent definition of the Nijenhuis bracket in the case when both N and M are of degree 1. However, if (13.36) is given as definition, one must prove that the right-hand side of the above equation depends linearly on the components of the vector fields and does not depend on their derivatives – that is to show that $[N, M]$ is indeed vector-valued differential form of degree 2. This of course can be done by routine calculations.

The case $N = M$ is of particular interest for us. If $N = M$ then

$$
\begin{aligned}
&\frac{1}{2}[N, N](X_1, X_2) = R_N(X_1, X_2) = \\
&[NX_1, NX_2] + N^2[X_1, X_2] - N[NX_1, X_2] - N[X_1, NX_2] \, .
\end{aligned}
\tag{13.37}
$$

Therefore, R_N is a $(1, 2)$ tensor field, called (just as $[N, N]$) *Nijenhuis bracket of N or Nijenhuis torsion of N* and usually denoted again by $[N, N]$. Fortunately, this notation is used mainly when $[N, N] = 0$, and thus the difference does not create problems.

Definition 13.20. *Let N be $(1, 1)$ tensor field (field of operators). The mixed $(1, 1)$ tensor filed N is called Nijenhuis tensor if its Nijenhuis torsion vanishes, that is if $[N, N] = 0$.*

We see that if N is Nijenhuis tensor, then for any two vector fields X and Y we have

$$
[NX, NY] + N^2[X, Y] - N[NX, Y] - N[X, NY] = 0 \, . \tag{13.38}
$$

We give also the expression of the above condition in local coordinates. If x^i, $i = 1, 2, \ldots, n$ are local coordinates on the n-dimensional manifold $\mathcal{M}$, and the local expression of N in these coordinates is

$$
N = \sum_{i,j=1}^{n} N_i^j(x) \frac{\partial}{\partial x^j} \otimes dx^i \, , \tag{13.39}
$$

then

$$
\sum_{h=1}^{n} (N_i^h \frac{\partial N_j^k}{\partial x^h} - N_j^h \frac{\partial N_i^k}{\partial x^h} + N_h^k \frac{\partial N_i^h}{\partial x^j} - N_h^k \frac{\partial N_j^h}{\partial x^i}) = 0 \tag{13.40}
$$

is the expression of (13.38) in local coordinates.

Now, let us mention one simple result which is immediate consequence of (13.38). One can consider the tensor field as a linear map from the module $\mathcal{T}(\mathcal{M})$ into itself. Then (13.38) yields

Proposition 13.21. *The image of $\mathcal{T}(\mathcal{M})$ under N is a Lie subalgebra in $\mathcal{T}(\mathcal{M})$ (with respect to the Lie bracket structure).*

13.3 Nijenhuis Tensors

Definition 13.22. *If a manifold is equipped with Nijenhuis tensor, we say that it is Nijenhuis manifold or that on the manifold is defined a Nijenhuis structure and sometimes refer to the tensor N as Nijenhuis operator. We shall describe now the principal properties of the Nijenhuis tensors.*

We shall describe now the principal properties of the Nijenhuis tensors and start with the fundamental fields of a Nijenhuis tensor N. By definition, the fundamental fields of N are the fields for which $L_X N = 0$. Of course, this means that the local flow φ_t of X preserves the field N. The fundamental fields of a Nijenhuis tensor, possess remarkable "hereditary" property, stated in the following proposition:

Proposition 13.23. *If the vector field X is fundamental for the Nijenhuis tensor N then the vector field NX is also fundamental field for N.*

The proof easily follows from the fact that the relation (13.38) is equivalent to the relation:

$$[L_{NX}(N) - NL_X(N)]\,(Y) = 0 \tag{13.41}$$

for two arbitrary vector fields X, Y or that

$$L_{NX}(N) - NL_X(N) = 0 \tag{13.42}$$

for arbitrary vector field X. Then if $L_X N = 0$, it follows that $L_{NX}(N) = 0$.

From (13.42) and the properties of the Lie derivative, it follows that the condition $[N, N] = 0$ can be written into another form:

$$L_{NX}(N^*) - L_X(N^*)N^* = 0\,. \tag{13.43}$$

This permits to establish another interesting property of the Nijenhuis tensor field.

Proposition 13.24. *If α is 1-form and $d\alpha = 0$, $dN^*\alpha = 0$, then $d(N^*)^2\alpha = 0$ too. As a consequence, all the forms $(N^*)^k\alpha$; $k = 1, 2, \ldots$ are closed.*

Proof. The condition $d\alpha = 0$ is equivalent to the condition $L_X\alpha = d[\alpha(X)] = d\langle\alpha, X\rangle$ for arbitrary vector field X. Having this in mind, let X be arbitrary vector field. We have then

$$\begin{aligned}
&L_X[(N^*)^2\alpha] - d\langle(N^*)^2\alpha, X\rangle = L_X[N^*N^*\alpha] - d\langle N^*\alpha, NX\rangle = \\
&L_X(N^*)(N^*\alpha) + N^*L_X(N^*\alpha) - d\langle N^*\alpha, NX\rangle = \\
&[L_X(N^*)N^*]\alpha + N^*d\langle N^*\alpha, X\rangle - d\langle N^*\alpha, NX\rangle\,.
\end{aligned} \tag{13.44}$$

As $L_X(N^*)N^* = L_{NX}N^*$, the above expression can be written into the form

$$
\begin{aligned}
&[L_{NX}N^*](\alpha) + N^* d\langle N^*\alpha, X\rangle - d\langle N^*\alpha, NX\rangle = \\
&L_{NX}(N^*\alpha) - N^*(L_X\alpha) + N^* d\langle N^*\alpha, X\rangle - d\langle N^*\alpha, NX\rangle = 0\ . \qquad (13.45)
\end{aligned}
$$

This shows that $(N^*)^2\alpha$ is closed which proves the first part of the proposition. Repeating the argument we get the second part of the proposition.

We would like to mention some other properties of the Nijenhuis tensor:

Proposition 13.25. *If N is Nijenhuis tensor then N^s; $s = 1, 2, \ldots$ are also Nijenhuis tensors.*

Proof. Let us consider $[N^{n+1}, N^{n+1}]$, where n is a natural number. For arbitrary vector fields X, Y we have

$$
[N^{n+1}, N^{n+1}](X, Y) = -N^{n+1}L_X(N^{n+1})Y + L_{N^{n+1}X}(N^{n+1})Y\ . \qquad (13.46)
$$

Next, from one side

$$
-N^{n+1}L_X(N^{n+1})Y = -N^{n+1}\{\sum_{m=0}^{n} N^m L_X(N)N^{n-m}\}Y\ . \qquad (13.47)
$$

From the other side

$$
\begin{aligned}
&L_{N^{n+1}X}(N^{n+1})Y = \{\sum_{p=0}^{n} N^p L_{N^{n+1}X}(N)N^{n-p}Y\} = \\
&\{\sum_{p=0}^{n} N^p \left(NL_{N^nX}(N)N^{n-p}Y + [N, N](N^nX, N^{n-p}Y)\right)\} = \\
&\{\sum_{p=0}^{n} N^{n+1}N^p L_X(N)N^{n-p}Y\} \\
&+\{\sum_{p=0}^{n} N^p \sum_{s=0}^{n} N^s[N, N](N^{n-s}X, N^{n-p}Y)\}\ . \qquad (13.48)
\end{aligned}
$$

Inserting (13.47) and (13.48) into (13.46) we get

$$
[N^{n+1}, N^{n+1}](X, Y) = \{\sum_{p=0}^{n} N^p \sum_{s=0}^{n} N^s[N, N](N^{n-s}X, N^{n-p}Y)\}\ . \qquad (13.49)
$$

Since $[N, N] = 0$, the tensor $[N^{n+1}, N^{n+1}]$ is also equal to zero.

Other results that can be obtained immediately, are the following:

Proposition 13.26. *If N is Nijenhuis tensor, then N^{-1} (if it exists) is also a Nijenhuis tensor.*

Corollary 13.27. *If N is invertible Nijenhuis tensor, then*

$$N^s; \quad s = \pm 1, \pm 2, \dots$$

are also Nijenhuis tensors.

One of the applications of Nijenhuis bracket is described in the following theorem, formulated in its present form by Haantjes [3]. The proof we present here is due to Frölicher and Nijenhuis [1] and concerns the finite dimensional case. We believe that the infinite-dimensional case can be treated along the same lines, but there are some elements in the proof that are not easy to generalize.

Theorem 13.28 (Haantjes). *Let N be a field of linear maps on the manifold $\mathcal{M}$:*

$$m \to N_m : T_m(\mathcal{M}) \to T_m(\mathcal{M}),$$

(field of operators). Let for each $m \in \mathcal{M}$ the operator N_m be semisimple with real eigenvalues $\lambda(m), \mu(m) \dots$ and let the multiplicity of each eigenvalue be constant. Let S_λ be the field of eigenspaces corresponding to the eigenvalue λ

$$m \to S_\lambda(m) \subset T_m(\mathcal{M}); \quad m \in \mathcal{M}$$

and let

$$S_\lambda \oplus S_\mu \oplus \dots \oplus S_\nu$$

be the field of the direct sums

$$m \mapsto S_\lambda(m) \oplus S_\mu(m) \oplus \dots \oplus S_\nu(m); \quad m \in \mathcal{M}.$$

Let us assume that the functions $\lambda, \mu \dots$ are smooth and that the corresponding fields of subspaces are also smooth distributions. Then each S_λ and each $S_\lambda \oplus S_\mu \oplus \dots \oplus S_\nu$ is integrable (in Frobenius sense) if and only if for any vector fields X, Y

$$\begin{aligned}&\mathcal{R}_N(X,Y) = \\ &N^2 R_N(X,Y) + R_N(NX,NY) - NR_N(NX,Y) - NR_N(X,NY) = 0\end{aligned} \tag{13.50}$$

where R_N is the torsion of the tensor field N.

Proof. Let X, Y be two vector fields which are fields of eigenvectors for N. Let the eigenvalue of X be λ, and the eigenvalue of Y be μ (of course, since N is a field of linear maps the eigenvalues are scalar fields, that is functions). Then one can calculate that

$$R_N(X,Y) = (N-\lambda)(N-\mu)[X,Y] + (\lambda-\mu)((Y\lambda)X + (X\mu)Y) \ . \tag{13.51}$$

Making use of $(N-\lambda)X = (N-\mu)Y = 0$ we find from here that

$$\mathcal{R}_N(X,Y) = (N-\lambda)^2(N-\mu)^2[X,Y] = 0 \ . \tag{13.52}$$

Since N is semisimple, the above means that

$$[X,Y] \in S_\lambda \oplus S_\mu,$$

which is the necessary and sufficient condition for the integrability of the distribution $m \mapsto S_\lambda(m) \oplus S_\mu(m)$. If $\lambda = \mu$, then

$$\mathcal{R}_N(X,Y) = (N-\lambda)^4[X,Y] = 0,$$

and arguments similar to those we used in the above show that $[X,Y] \in S_\lambda$ which is the necessary and sufficient condition for $m \mapsto S_\lambda(m)$ to be integrable.

The following theorem (often referred to as Nijenhuis theorem) is a direct consequence from the Haantjes theorem.

Theorem 13.29 (Nijenhuis). *Let N be Nijenhuis tensor on the manifold $\mathcal{M}$. Suppose the eigenvalues and eigenspaces of N have the same properties as in theorem (13.28). Then using the same notation as in the above we have:*

- *Each S_λ and each $S_\lambda \oplus S_\mu \oplus \ldots \oplus S_\nu$ is integrable in the Frobenius sense.*
- *If $\lambda \neq \mu$, $Y \in S_\mu$ then $Y(\lambda) = L_Y\lambda = 0$. In other words, the eigenvalue λ depends only on the coordinates of the integral submanifold of the distribution $m \to S_\lambda(m)$.*

Proof. It is evident that one needs to prove only the second statement of the theorem. To this end, let, as in the above, the eigenvalue of X be λ, and the eigenvalue of Y be μ. (Of course, they are smooth functions over $\mathcal{M}$). Then

$$R_N(X,Y) = (N-\lambda)(N-\mu)[X,Y] + (\lambda-\mu)((Y\lambda)X + (X\mu)Y) = 0\ . \quad (13.53)$$

From the Haantjes' theorem, we know already that $[X,Y] \in S_\lambda \oplus S_\mu$, so the first term in the above equation vanishes. Then $(\lambda-\mu)((Y\lambda)X+(X\mu)Y) = 0$, and if $\lambda \neq \mu$ it follows that $Y\lambda = X\mu = 0$. This completes the proof.

References

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14

Integrability and Nijenhuis Tensors

In this chapter, we link the theory of the Nijenhuis tensors and the questions of integrability, trying to give some general view on integrability based on the theory of Nijenhuis tensors and their fundamental fields. In the example of several finite and infinite dimensional dynamical systems, we show how these ideas work. Finally, we show the natural way the Nijenhuis tensors arise when we have compatible Poisson tensors. The development of these ideas leads to the notion of Poisson-Nijenhuis manifold (P-N manifold), which we introduce here. This notion is crucial for our treatment of the integrable Hamiltonian systems.

14.1 Integrability Criteria and Nijenhuis Tensors

Nowadays the inverse scattering transform method (IST) is universally recognized as one of the important techniques for integration of partial differential evolution equations [1], provided the corresponding equation can be cast into Lax form. Its applications are numerous, and without exaggeration it is one of the significant mathematical discoveries of the 20-th century. However, in spite of its success, a compact a priori criterion of complete integrability is, to date, not available. On the other hand, if we consider the soliton equations as dynamical systems on infinite-dimensional manifolds, the IST can be regarded as transformation so to say from “generic coordinates” (potentials) to action-angle variables [2], and this gives hopes that such criterion could be found in terms of the original Lax pairs. These hopes are strengthened by the natural way, in which, using the Lax representation, one calculates an important operator, (the Λ operator, squared eigenfunctions operator, recursion operator), as has been shown in the first part of the present book. The point is that, as we shall see, the adjoint of this operator fits in the geometric picture as a mixed tensor field N on the phase manifold $\mathcal{M}$ and plays on it the same role as it does in the finite dimensional case. Such a tensor field N has usually the following properties:

Gerdjikov, V.S. et al.: *Integrability and Nijenhuis Tensors*. Lect. Notes Phys. **748**, 473–513 (2008)
DOI 10.1007/978-3-540-77054-1_14

1. It is invariant under the dynamics (this is also known as N being a strong symmetry, or in other words, if Γ is the dynamical system (vector field), we have $L_\Gamma(N) = 0$.
2. Its Nijenhuis torsion vanishes (also known as hereditary property).
3. It has a doubly degenerate continuous spectrum (in a certain well-defined manner), and when dealing with generic potentials it has a finite number of discrete eigenvalues. The set of the discrete eigenvalues defines the so-called soliton sector. The eigenspaces corresponding to the discrete eigenvalues are invariant with respect to the evolution and generically are two-dimensional. We have two possibilities here, which have been investigated up to now:
   (a). There exist exactly two eigenvectors in each such space. We shall call this case a *diagonalizable case.*
   (b). Each eigenspace in the soliton sector is spanned by a "true" eigenvector and a generalized one; this corresponds to a matrix with a single 2×2 Jordan block. We call this case *nondiagonalizable.*

   In both cases, the set of the eigenvectors (true and generalized) in the soliton sector, together with eigenvectors for the continuous spectrum, forms a complete set.

Using the first two properties there has been constructed a geometric integrability scheme, see [3, 4], and in the recent years much attention has been given to the diagonalizable case, because when we have it, we are able to construct sequences of conservation laws, Abelian algebras of symmetries and hierarchies of integrable nonlinear equations [5, 6, 7, 8].

However, in spite of the fact that the diagonalizable case occurs more frequently, the nondiagonalizable case can also occur. That is why we would like to propound an a priori separability criterion, which can include this new spectral option. In what follows, we deal with the nondiagonalizable case; from the very construction, it will be clear how one can deal with the diagonalizable one. We shall see also that, as far as soliton dynamics is concerned, integrability can be proved without further hypotheses. For background-radiation dynamics (the part described by the continuous spectrum), it is still unclear how to formulate a priori integrability criterion. The considerations we give below probably can be formulated directly in the terms of the corresponding Lax representations (considered in terms of the bundles for which the phase manifold is the base) [9], provided these representations could be put into the frames of the above geometric picture, but in our opinion up to now conclusive results in that direction have not been obtained.

Suppose that on the manifold $\mathcal{M}$ one has a vector field Γ, and Nijenhuis tensor field N (that is a mixed tensor field with vanishing Nijenhuis bracket) which is invariant for the flow of Γ, that is $L_\Gamma N = 0$. Suppose that N has the properties listed in the above. We propose the following.

Integrability criterion. *The dynamics defined by the vector field Γ completely separates into 1-degree of freedom dynamics. The components associated* *to the degrees of freedom corresponding to eigenvalues λ which are not stationary (depend on the time) are integrable and Hamiltonian* [10].

Below, we shall give a sketch how the above criterion can be obtained, assuming that the calculations we perform can be justified.

Denote by λ^i the generic discrete eigenvalue of N, and just in order to fix the ideas, assume that the continuous spectrum of N fills the real semiaxis $\mathbb{R}_+$ and does not overlap with the discrete spectrum. Then the vanishing of the Nijenhuis torsion R_N associated with N means that for all $\alpha \in \Lambda^1(\mathcal{M})$ and $X, Y \in \mathcal{T}(\mathcal{M})$

$$R_N(\alpha, Y, X) = \langle \alpha, [(L_{NX}N) - N(L_X N)]Y \rangle = 0 \,. \tag{14.1}$$

(Here $\langle .,. \rangle$ is the natural pairing between the 1-forms and vector fields). According to our assumptions there exists a basis

$$\{e_i, \epsilon_i, f_{1,(k)}, f_{2,(k)}; \quad i = 1, 2, \ldots, n, \quad k \in \mathbb{R}^+\}$$

of $\mathcal{T}(\mathcal{M})$, such that

$$\begin{aligned} &Ne_i = \lambda^i e_i \\ &N\epsilon_i = \lambda^i \epsilon_i + e_i; \quad i = l, 2, \ldots, n \\ &Nf_{l,(k)} = kf_{l,(k)}; \quad l = 1, 2, \quad \text{and} \quad k \in \mathbb{R}_+ \,. \end{aligned} \tag{14.2}$$

Let us now introduce the corresponding dual basis,

$$\{t^i, \theta^i, \gamma^{1,(k)}, \gamma^{2,(k)}; \quad i = 1, 2, \ldots, n, \quad k \in \mathbb{R}_+\} \tag{14.3}$$

of $\Lambda^1(\mathcal{M})$, that is a basis, for which

$$\begin{aligned} &\langle t^i, e_j \rangle = \langle \theta^i, \epsilon_j \rangle = \delta^i_j \\ &\langle \gamma^{l,(k)}, f_{p,(h)} \rangle = \delta^{l,(k)}{}_{p,(h)} \\ &\langle t^i, \epsilon_j \rangle = \langle \theta^i, e_j \rangle = \langle t^i, f_{l,(k)} \rangle = 0 \\ &\langle \theta^i, f_{l,(k)} \rangle = \langle \gamma^{l,(k)}, e_i \rangle = \langle \gamma^{l,(k)}, \epsilon_i \rangle = 0 \,, \end{aligned} \tag{14.4}$$

where $i, j = 1, 2, \ldots, n$, and $\delta^{l,(k)}{}_{p,(h)} = \delta^l_p \delta(k-h)$, where $\delta(k-h)$ is the Dirac "function." The relations (14.4) written in terms of the above 1-forms read

$$\begin{aligned} &N^* t^i = \lambda^i t^i + \theta^i \\ &N^* \theta_i = \lambda^i \theta_i; \quad i = l, 2, \ldots, n \\ &N^* \gamma^{l,(k)} = k\gamma^{l,(k)}; \quad l = 1, 2, \quad \text{and} \quad k \in \mathbb{R}_+ \,, \end{aligned} \tag{14.5}$$

where N^* denotes the formal adjoint of N. As we shall see, no more ingredients are needed to prove the separability into one degree of freedom dynamics, and (except for the assumption that the λ^i's are not stationary at any point) the integrability of the discrete part of it can also be proved. Our analysis starts with the observation that the condition (14.1) can be cast into the following form:

$$
\begin{aligned}
&L_{e_i}\lambda^j = 0, \quad L_{f_{l,(k)}}\lambda^i = 0, \quad (\lambda^i - \lambda^j)L_{\epsilon_i}\lambda^j = 0 \\
&(N-\lambda^i)(N-\lambda^i)[e_i, e_j] = 0, \quad (N-\lambda^i)(N-\lambda^j)^2[e_i, \epsilon_j] = 0 \\
&(N-\lambda^i)^2(N-\lambda^j)^2[\epsilon_i, \epsilon_j] = 0, \quad (N-k)(N-h)[f_{l,(k)}, f_{p,(h)}] = 0 \\
&(N-\lambda^i)(N-k)[e_i, f_{l,(k)}] = 0, \quad (N-\lambda^i)(N-k)[\epsilon_i, f_{l,(k)}] = 0\,. \quad (14.6)
\end{aligned}
$$

Then it is easily seen that the (14.6) are equivalent to

$$\theta^i \wedge d\theta^i = \theta^i \wedge t^i \wedge dt^i = \gamma^{1,(k)} \wedge \gamma^{2,(k)} \wedge \delta\gamma^{l,(k)} = 0\,, \qquad (14.7)$$

this implying, by the Frobenius theorem, that without loss of generality, the θ's, t's and γ's can be considered to be closed forms, or as it is also said , the basis

$$\{e_i, \epsilon_i, f_{1,(k)}, f_{2,(k)}; \quad i = 1, 2, \ldots, n, \quad k \in \mathbb{R}_+\},$$

can be chosen to be a holonomic frame.

On the other hand, from the first line in (14.6), we get that $d\lambda^i = (L_{\epsilon_i}\lambda^i)\theta^i$, this implying

$$N^* d\lambda^i = \lambda^i d\lambda^i\,. \qquad (14.8)$$

In particular, this means that the θ's can be chosen to be equal to the $d\lambda^i$'s if, as we assumed, we have $d\lambda^i \neq 0$. Furthermore, the fact that the frame is holonomic implies that the set of functions $\lambda^1, \lambda^2, \ldots, \lambda^n$ can be completed to form a local coordinate system

$$(\lambda^1, \lambda^2, \ldots, \lambda^n, \phi^1, \phi^2, \ldots, \phi^n, \psi^{1,k}, \psi^{2,k}; \quad k \in \mathbb{R}_+)$$

(some of the variables may be periodic) in such a way, that

$$e_i = \frac{\partial}{\partial \phi^i}, \quad \epsilon_i = \frac{\partial}{\partial \lambda^i}, \quad f_{l,(k)} = \frac{\delta}{\delta \psi^{l,(k)}}\,. \qquad (14.9)$$

Then the tensor N can be written into the following canonical form

$$
\begin{aligned}
N = &\sum_i \lambda_i \left(\frac{\partial}{\partial \lambda^i} \otimes d\lambda^i + \frac{\partial}{\partial \phi^i} \otimes d\phi^i + \frac{\partial}{\partial \phi^i} \otimes d\lambda^i \right) + \\
&\sum_{l=1}^{2} \int_0^\infty dk\, k \frac{\delta}{\delta \psi_k^\ell} \otimes \delta\psi^\ell(k)\,. \qquad (14.10)
\end{aligned}
$$

Now, it can be checked that the condition $L_\Gamma N = 0$ we have on N and Γ is equivalent to the following system of equations:

$$\langle d\lambda^i, \Gamma \rangle = 0, \quad \frac{\partial}{\partial \phi^j}\langle d\phi^i, \Gamma \rangle = 0, \quad \frac{\delta}{\delta \psi^{l,(k)}}\langle d\phi^i, \Gamma \rangle = 0$$

$$
\begin{aligned}
&(\lambda^i - \lambda^j)\frac{\partial}{\partial \lambda^j}\langle d\phi^i, \Gamma\rangle = 0, \quad \frac{\partial}{\partial \phi^i}\langle \delta\psi^{l,(k)}, \Gamma\rangle = 0 \\
&\frac{\delta}{\delta \lambda^i}\langle \delta\psi^{l,(k)}, \Gamma\rangle = 0, \quad (k-h)\frac{\delta}{\delta \psi^{l,(h)}}\langle \delta\psi^{p,(k)}, \Gamma\rangle = 0\,. \qquad (14.11)
\end{aligned}
$$

From the above equations separability and integrability follow. More specifically, the first equation in the first line of (14.11) means the vanishing of the "λ-components" of Γ, the second and the third equation in the first line mean the independence of the ϕ-components on the ϕs, and on the continuous coordinates. In the second line, the first equation means that each ϕ^i-component depends only on the corresponding λ^i. The last set of equations shows that the continuous components cannot depend on the discrete variables and that each continuous component can only be a function of the continuous variables with the same continuous index. The most general form of Γ is then

$$
\Gamma = \sum_{i=1}^{n} \Gamma^i(\lambda^i)\frac{\partial}{\partial \phi^i} + \sum_{\ell=1}^{2}\int_0^\infty dk \Gamma^\ell(k)\Big(\psi^1(k), \psi^2(k)\Big)\frac{\delta}{\delta\psi^\ell(k)}\,. \qquad (14.12)
$$

The dynamic equations decouple in the following second-order systems for the continuous degrees of freedom (background radiation dynamics):

$$
\begin{aligned}
\dot{\psi}^{1,(k)} &= \Gamma^{1,k}(\psi^{1,(k)}, \psi^{2,(k)}) \\
\dot{\psi}^{2,(k)} &= \Gamma^{2,k}(\psi^{1,(k)}, \psi^{2,(k)})\,, \qquad (14.13)
\end{aligned}
$$

and the following (trivially integrable) equations:

$$
\dot{\phi}^i = \Gamma^i(\lambda^i), \qquad \dot{\lambda}^i = 0\,, \qquad (14.14)
$$

for the discrete part (the soliton dynamics). Consequently, the discrete part of the dynamics (the soliton dynamics) is Hamiltonian with respect to the family of symplectic forms

$$
\omega_0 = \sum_i f_i(\lambda^i) d\lambda^i \wedge d\phi^i\,, \qquad (14.15)
$$

where f is a function that does not vanish on the points of the discrete spectrum.

Remark 14.1. Γ is not supposed to be a Hamiltonian system. Its admissible Hamiltonian structures are "generated" by the hypothesis that the eigenspaces of N are bidimensional and the requirements $d\lambda^i \neq 0$.

14.2 Recursion Operators in Dissipative Dynamics

We have seen that a nonlinear evolution equation $u_t = \Gamma[u]$, defined by the vector field $\Gamma[u]$, is integrable if there exists a mixed tensor field N on $\mathcal{M}$, satisfying the conditions described at the beginning of the preceding section and that this even imply the existence of symplectic forms (at least on the soliton sector) with respect to which the dynamics is Hamiltonian. In that case, the integrability is of the Liouville-Arnold type. On the other hand, we mentioned also that there are many interesting cases, in which the dynamics is not an integrable Hamiltonian one and in which suitable generalization of the above geometric scheme could still be useful. We are going to present one such example, in which an invariant mixed tensor field is used in order to analyze dissipative dynamics. It seems that one needs only to remove that part of the assumptions on N, which lead to the existence of constants of motion. Thus we shall assume that N has zero Nijenhuis bracket and is invariant under Γ but is not real-diagonalizable, that is, the eigenvalues are complex. A good example illustrating this situation is provided by the Burgers equation

$$u_t = 2uu_x + u_{xx} \; . \tag{14.16}$$

As is well known, this equation is the simplest one in which nonlinearity and diffusion effects compensate in such a way that we have no undesirable effects on the evolution.

14.2.1 The Burgers Equation Hierarchy

It is a classical result, (see [11, 12] or [13]), that the Burgers equation linearizes through the so-called Cole-Hopf transformation (map)

$$u = h[v] = \frac{v_x}{v} \; , \tag{14.17}$$

that is, if u is as in the above, and if v satisfies the Heat Equation

$$v_t = v_{xx} \; , \tag{14.18}$$

then, u satisfies the Burgers equation.

It can be shown, (see [14]), that the Burgers Equation is a member of a hierarchy of nonlinear evolution equations which linearize, through the same transformation (14.17) and which are reduced to the equations of the following type

$$v_t = D^n v; \quad n = 1, 2, \ldots \; . \tag{14.19}$$

where D stands for the x-derivative operator. The elements of this hierarchy have different properties. The even elements of (14.19) are equations with dissipative dynamics. The odd elements are equations with integrable Hamiltonian systems with respect to the following symplectic form:

$$\Omega[\delta_1 v, \delta_2 v] = \int_{-\infty}^{+\infty} \delta_1 v(x)(D^{-1}\delta_2 v)(x) dx \; , \tag{14.20}$$

where

$$(D^{-1}f)(x) = \int_{-\infty}^{x} f(y)dy \ , \tag{14.21}$$

and the corresponding Hamiltonian functions (functionals) are given by

$$H_p[v] = \frac{1}{2}\int_{-\infty}^{+\infty} (D^p v)^2 dx \ . \tag{14.22}$$

In order that (14.20), (14.22) make sense, some assumptions on the functional space $\mathcal{M}$ must be made, for example, to assume that $\mathcal{M}$ consists of Schwartz-type functions on the line. Then one easily checks that

$$\hat{N}[v] = D \tag{14.23}$$

is a Nijenhuis Γ-invariant tensor for the heat equation hierarchy. In the geometric approach, (14.17) plays the role of a map from one manifold into another, and thus a Nijenhuis Γ-invariant tensor operator for the Burgers hierarchy is readily obtained from $\hat{N}[v]$ (see [14, 15]). In other words, requiring N and $\hat{N}$ to be $v[u]$-related

$$N[u] = (\frac{\delta v}{\delta u})^{-1}\hat{N}[v](\frac{\delta v}{\delta u}) \ , \tag{14.24}$$

easily yields

$$N[u] = D + DuD^{-1} \ . \tag{14.25}$$

The Burgers hierarchy is obtained by repeated applications of the operator (14.25) to the field $\Gamma_0 = u_x$, that is

$$\Gamma_k = N^k \Gamma_0 \ . \tag{14.26}$$

The first fields of the hierarchy are

$$\begin{aligned} \Gamma_0 &= u_x \\ \Gamma_1 &= 2uu_x + u_{xx} \\ \Gamma_2 &= (3u^3 + 3uu_x + u_{xx})_x \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{14.27}$$

From a geometric point of view this hierarchy is $v[u]$-related to the linear one, and roughly speaking (the manifolds for the Burgers hierarchy and the linear hierarchy are not diffeomorphic), one can "translate" what can be said for (14.19) to the Burgers hierarchy. In this way, one obtains that (14.26) splits into the following two subhierarchies, which we call the Dissipative and the Hamiltonian hierarchy:

- Dissipative hierarchy

$$N\Gamma_0, N^3\Gamma_0, \ldots, N^{2n+1}\Gamma_0, \ldots \tag{14.28}$$

- Hamiltonian hierarchy

$$\Gamma_0, N^2\Gamma_0, \ldots, N^{2n}\Gamma_0, \ldots \; . \tag{14.29}$$

The equations from these hierarchies are sequences of dissipative and Hamiltonian vector fields, respectively. The above situation can be better understood examining the spectral properties of N. Its "block diagonal form" is

$$N = \int_0^\infty k\left(e_{(k)} \otimes \theta'^{,(k)} - e'_{(k)} \otimes \theta^k\right) dk \; , \tag{14.30}$$

where

$$e_{(k)}[u] = \int_{-\infty}^{\infty} dx(-u\cos kx - k\sin kx) \times \exp\Big[-\int_{-\infty}^{x} udy\Big] \frac{\delta}{\delta u(x)} \tag{14.31}$$

$$e'_{(k)}[u] = \int_{-\infty}^{\infty} dx(-u\sin kx + k\cos kx) \times \exp\Big[-\int_{-\infty}^{x} udy\Big] \frac{\delta}{\delta u(x)} \tag{14.32}$$

form a basis in the generic invariant subspace of N, and $\{\theta^{(k)}\theta'^{(k)}\}$ is the corresponding dual basis:

$$\begin{aligned} &Ne_{(k)} = -ke'_{(k)}, \quad Ne'_{(k)} = ke_{(k)} \\ &\langle \theta'^{(k)}, e'_{(h)} \rangle = \langle \theta^{(k)}, e_{(h)} \rangle = \delta(h-k) \\ &\langle \theta'^{(k)}, e_{(h)} \rangle = \langle \theta^{(k)}, e'_{(h)} \rangle = 0 \; . \end{aligned} \tag{14.33}$$

The conditions :

$$[e_{(k)}, e_{(h)}] = [e'_{(k)}, e'_{(h)}] = [e'_{(k)}, e_{(h)}] = 0 \; , \tag{14.34}$$

imply that the frame is holonomic, that is, ensure (at least locally) the existence of coordinates $(q^{(k)}, p^{(k)})$, such that:

$$e_{(k)} = \frac{\delta}{\delta q^{(k)}}, \quad e'_{(k)} = \frac{\delta}{\delta p^{(k)}} \; . \tag{14.35}$$

The operator N can be restricted to the two-dimensional integral manifold spanned by $\{e_{(k)}, e'_{(k)}\}$, then it reads:

$$\frac{\delta}{\delta J^{(k)}} \otimes \delta\varphi^{(k)} - \frac{\delta}{\delta\varphi^{(k)}} \otimes \delta J^{(k)} \tag{14.36}$$

where

$$J^{(k)} = \frac{1}{2}\left(q^{(k)^2} + p^{(k)^2}\right), \qquad \varphi^{(k)} = \tan^{-1}\left(\frac{q^{(k)}}{p^{(k)}}\right) \tag{14.37}$$

are action-angle type variables. Then, one can see that operator N moves a dissipative integrable field of the type

$$X_D^{(k)} = J^{(k)} \frac{\delta}{\delta J^{(k)}} , \tag{14.38}$$

into a Hamiltonian one

$$X_H^{(k)} = J^{(k)} \frac{\delta}{\delta \varphi^{(k)}} \tag{14.39}$$

and vice versa.
The above "alternating" character of N is responsible for the splitting of (14.26) into two subhierarchies. Furthermore, we observe that

- The operator N has two-dimensional invariant spaces, but is not real diagonalizable.
- The operator N^2, generating the Hamiltonian subhierarchy, is diagonalizable, with doubly degenerate constant eigenvalues.

For none of these subhierarchies the integrability criterion we have propounded holds. However, we note that the projections of dissipative fields define finite degree of freedom dynamics on the two-dimensional invariant spaces, while for the Hamiltonian fields, the existence of a functional $J^{(k)}[u]$ (integrals of motion) ensures the integrability of the corresponding projection. It is worthy of remark also that the functionals $J^{(k)}[u]$ play the role of a Liapunov functional for the projection of the dissipative dynamics on the two-dimensional invariant sub-manifolds, ensuring the asymptotic stability of the solutions for which $J^{(k)}[u] = 0$.

Let us consider more closely the Hamiltonian subhierarchy (14.29). First of all, some care is needed for the appropriate choice of the functional space $\mathcal{M}$ on which the dynamics is defined. It is natural to take $\mathcal{M}$ to be the functional space, whose elements u tend to constant as $x \to \pm\infty$, as it is the space on which lies the typical solitary wave of Burgers hierarchy. However, with such a choice, it would not be possible to introduce a Hamiltonian structure on $\mathcal{M}$. Indeed, if we go back to the linear hierarchy of the Heat Equation, we shall see that the functions v (see the transformation (14.17)) behave like $\exp(kx)$ as $x \to \pm\infty$. But then the Hamiltonian which we had written in (14.22) is not defined. Thus, it is natural to restrict $\mathcal{M}$ in such a way that both the symplectic structures and Hamiltonian function (14.22) make sense. This can be accomplished by considering the phase space to be the space of functions $v(x)$ that tend to some constants fast enough as $x \to +\infty$ and $x \to -\infty$ (and then the functions $u(x)$ vanish as $x \to \pm\infty$). Also, we can circumvent the difficulties related to the ambiguities inverting the differentiation operator passing to the corresponding space of equivalence classes, requiring that in each class the integral on the whole axis of $v(x)$ has some fixed value.

The above construction ensures the existence of a symplectic form such that the subhierarchy, which we called Hamiltonian, is indeed a hierarchy of Hamiltonian vector fields. Note also the interesting fact that despite the fact that the eigenvalues of the operator N^2 are constant, N^2 generates a sequence of integrals of motion. The example we have given and its analysis shows the importance of the hypothesis on the spectrum of the invariant mixed tensor field N for the properties of the corresponding dynamical systems. We have seen that if we drop the hypothesis that N is diagonalizable, we can consider also dissipative dynamics within the geometric scheme. However, the spectral properties depend strongly on the particular choice of N, and their investigation is not an easy matter. For example, the first part of this book is almost entirely dedicated to this topic for the ZS. So, we shall limit ourselves to the study of the geometric constructions permitting to obtain Nijenhuis tensors N. From the next chapter onwards the book is dedicated to this.

14.3 Noncommutative Integrability Criteria

Let us consider again the finite dimensional case, that is, we assume that we have a symplectic manifold $(\mathcal{M}, \omega)$ of dimension $2n$ and a Hamiltonian vector field Γ on it. We shall discuss now how the usual scheme of integrability can be generalized. Such generalization is necessary, because if the number of independent first integrals of Γ is greater than n, they cannot be anymore in involution. Indeed, for such a set of functions their Hamiltonian vector fields must be independent and are orthogonal with respect to ω, that is, at each point they span an isotropic space. We know, however, that the dimension of such space cannot exceed n. From the other side, the Poisson bracket of two first integrals ia a first integral too, so it is natural to assume that Γ possesses a set of first integrals $\{f_a\}_{1\le a\le m} \subset \mathcal{D}(\mathcal{M})$ which close a (non-commutative) Lie algebra $\mathfrak{A}$ over $\mathbb{R}$ with respect the Poisson bracket. We shall say that two elements $f, g \in \mathfrak{A}$ Poisson-commute if their Poisson bracket is zero.

Let us recall now some notions we shall use. If $\mathfrak{g}$ is a finite dimensional Lie algebra and $\mu \in \mathfrak{g}^*$ is a covector, then the annihilator $\mathrm{Ann}\,(\mu)$ of μ is the space

$$\{X \in \mathfrak{g} : \mathrm{ad}\,^*_X \mu = 0\} \; . \tag{14.40}$$

As readily seen, $\mathrm{Ann}\,(\mu)$ is a subalgebra. Next, we define the index $\mathrm{ind}\,(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$.

$$\mathrm{ind}\,(\mathfrak{g}) = \min_{\mu\in\mathfrak{g}^*} \dim \mathrm{Ann}(\mu) \; . \tag{14.41}$$

It can be shown that when the algebra is semisimple, the index coincides with the rank of the algebra, that is, with the dimension of the Cartan subalgebra.

Let us denote by r the index $\mathrm{ind}\,(\mathfrak{A})$ of our algebra of first integrals. Now we are ready to state the noncommutative generalization of the Liouville-Arnold theorem; see [16, 17, 18]:

Theorem 14.2. *Suppose that Hamiltonian vector field Γ on a symplectic manifold $\left(\mathcal{M}^{2n}, \omega\right)$ possesses $f_1, f_2, \ldots, f_k$ functionally independent first integrals, which span a finite dimensional real Lie algebra $\mathfrak{A}$ of dimension k. In addition, let*

$$\mathit{dim}\,(\mathfrak{A}) + \mathit{ind}\,(\mathfrak{A}) = k + r = \mathit{dim}\,\mathcal{M} = 2n \; . \tag{14.42}$$

Then the submanifolds defined by the conditions $f_i = const$, $i = 1, 2, \dots, k$ *are invariant manifolds for* Γ. *In in some neighborhood of any such manifold one can define canonical coordinates* (λ, χ, p, q) *such that*

1. *The* χ*'s are coordinates on the invariant manifolds.*
2. *The Hamiltonian equations corresponding to* Γ *take the form*

$$\begin{aligned} &\dot{\lambda}_i = 0, \quad \dot{\chi}^i = \nu_i(\lambda), \quad \dot{p}_\alpha = 0, \quad \dot{q}^\alpha = 0 \\ &i = 1, 2, \dots, r; \quad \alpha = r+1, r+2, \dots, n \,. \end{aligned} \tag{14.43}$$

As in the commutative case, if the invariant manifolds are compact and connected, one can prove that they are tori, and the χ*'s can be chosen to be angle variables. The canonical coordinates are called, in this case, "generalized action – angle variables."*

The Liouville-Arnold theorem can be recovered from the above formulation. Indeed, when $\mathfrak{A}$ is Abelian, its index coincides with its dimension and then the condition (14.42) shows that $\dim(\mathfrak{A}) = n$. However, it is interesting that if the assumption (14.42) holds, and if the algebra satisfies the so-called FI condition, (for more details see [16, 18]), on the symplectic manifold $\mathcal{M}$ there exist $n = \frac{1}{2}\dim \mathcal{M}$ independent first integrals of Γ, which are in involution, that is, actually we have the situation as in the Liouville-Arnold theorem.

The proofs of the above results can be found in [18, 19, 20, 21]; here we shall limit ourselves with a brief sketch of the ideas of the proof of the Theorem 14.2.

Let us discuss first what happens in the Liouville-Arnold case, when $\mathfrak{A}$ is commutative of dimension n. Then the level surfaces of the first integrals f_i define an invariant Lagrangian foliation $\mathcal{F}_1$ of $\mathcal{M}$, that is a disjoint union of integral leaves, each of which is a Lagrangian manifold; see Definitions (12.17), (12.18). The Hamiltonian vector fields X_i associated with the functions f_i are commuting vector fields, tangent to the leaves, and at each point of a given leaf form a basis of its tangent space. They can be used to define local coordinates χ^i on the leaves. The fields X_i commute with the Hamiltonian vector field Γ, which is also tangent to the leaves. Consequently, in a neighborhood of a point $p \in \mathcal{M}$, the field Γ can be written into the form $\Gamma = \sum_i \nu^i(f) X_i$ where the set (χ, f) are canonical coordinates, which means that the Hamiltonian equations for Γ take the following form:

$$\dot{\chi}^i = \nu^i, \quad \dot{f}_i = 0 \,. \tag{14.44}$$

In the noncommutative case, we have $2n - r$ first integrals f_a. The equations $f_a = \text{const}$ still define a foliation $\mathcal{F}_1$, invariant with respect to the flow of Γ, but the leaves now have dimension $r \leq n$, and the Hamiltonian vector fields X_a, associated with the first integrals f_a are not necessarily tangent to the leaves of this foliation. (The fields X_a of course close an algebra, isomorphic to $\mathfrak{A}$).

However, the condition $\dim(\mathfrak{A}) + \operatorname{ind}(\mathfrak{A}) = \dim\mathcal{M}$ ensures that for each leaf $\mathcal{L}$, there exists a subalgebra of $\mathfrak{A}_{\mathcal{L}} \subset \mathfrak{A}$, whose associated vector fields commute with all X_a on $\mathcal{L}$. The Hamiltonian vector fields $\bar{X}_i$ corresponding to some basis of $\mathfrak{A}_{\mathcal{L}}$, yield a basis of tangent vector fields for $\mathcal{L}$. We have that $\omega\left(\bar{X}_i, X_a\right)\big|_{\mathcal{L}} = 0$ and therefore $\omega\left(\bar{X}_i, \bar{X}_j\right)\big|_{\mathcal{L}} = 0$ if $\bar{X}_i, \bar{X}_j \in \mathfrak{A}_{\mathcal{L}}$ so that each leaf will be an isotropic manifold. The foliation $\mathcal{F}_1$ is then called isotropic. To obtain a set of canonical coordinates in a neighborhood of $p \in \mathcal{L}$ and eventually of the whole of $\mathcal{L}$, one needs to exploit further the properties of $\mathcal{F}_1$. At each $p \in \mathcal{L}$ consider the subspace $T_p(\mathcal{L}) \subseteq T_p(\mathcal{M})$ and the distribution of symplectically orthogonal subspaces $p \mapsto (T_p(\mathcal{L}))^{\perp}$. Since $\omega\left(\bar{X}_i, X_a\right)\big|_{\mathcal{L}} = 0$, this distribution is generated, for all leaves, by the vector fields X_a. Furthermore, since X_a satisfies the hypotheses of the Frobenius theorem, we obtain a distribution which defines a second foliation – a coisotropic foliation $\mathcal{F}_2$,[1] whose leaves are themselves foliated by those of the first foliation $\mathcal{F}_1$. One can now prove (at least locally) the existence of canonical coordinates $\left(\lambda_i, \chi^i, p_\alpha, q^\alpha\right)$, such that the symplectic structure and the dynamical vector field take the form

$$\omega = \sum_i d\lambda_i \wedge d\chi^i + \sum_\alpha dp_\alpha \wedge dq^\alpha, \;\; \Gamma = \sum_i \nu^i(\lambda)\, X_i \;, \tag{14.45}$$

so that the equations of motion become

$$\dot{\lambda}_i = 0, \quad \dot{\chi}^i = \nu^i, \quad \dot{p}_\alpha = 0, \quad \dot{q}^\alpha = 0 \,. \tag{14.46}$$

The functions λ_i describe locally $\mathcal{F}_2$, and their associated Hamiltonian vector fields X_i define coordinates χ^i on $\mathcal{F}_1$. The fields X_i are independent and commute between themselves and with Γ. To understand better this canonical coordinates, let us note that the momentum map[2] $J : \mathcal{M} \to \mathfrak{A}^*$, defined by $J : x \to \xi_x \in \mathfrak{A}^*$, where $\xi_x(f) = f(x)$, $f \in \mathfrak{A}$, defines foliation of some neighborhood $\mathcal{U}$ of any leaf of $\mathcal{F}_2$, where the leaves are $\mathcal{L}_x = J^{-1}(\xi_x)$, that is, they are the leaves of $\mathcal{F}_1$. Then the neighborhood $\mathcal{U}$ can be chosen to be of the type $\mathcal{L}_x \times \mathcal{P} \times \mathcal{O}$, where $\mathcal{O}$ is the coadjoint orbit through ξ_x defined by the connected Lie group A, corresponding to $\mathfrak{A}$, and $\mathcal{P}$ is some manifold transversal to $\mathcal{O}$. The symplectic structure ω, restricted to $\mathcal{O}$, coincides with the restriction of the Poisson-Lie structure (the Kirillov structure); the coordinates we had in the above (p_α, q^α) are canonical coordinates on $\mathcal{O}$ and λ_i are coordinates on $\mathcal{P}$. It can be proved [20] that all needed for the existence of such local coordinates is actually the existence of the double foliation, namely, that $\mathcal{M}$ possesses an isotropic foliation, such that the distribution of the subspaces, symplectically orthogonal to the tangent spaces to its leaves, is integrable.

[1] This means that its leaves are coisotropic submanifolds.

[2] The reader who is not familiar with this subject can look into subsection 15.2.1 for some relevant definitions.

14.3.1 The Invariant Tensor Fields in the Noncommutative Case

We shall try to characterize now the noncommutative integrability in another way.

Theorem 14.3. *Let Γ be a dynamical system on a $2n$-dimensional manifold $\mathcal{M}$ which admits a $(1,1)$ mixed tensor field N, having the properties:*

- *N is Γ-invariant, that is*
$$L_\Gamma N = 0 \ . \tag{14.47}$$
- *At each point $p \in \mathcal{M}$, N_p is diagonalizable with only simple or doubly degenerate eigenvalues, whose differentials are linearly independent.*
- *If by $S(p)$, $p \in \mathcal{M}$, is denoted the sum of all the eigenspaces, associated with the doubly degenerate eigenvalues of $N(p)$, then for arbitrary vector field, such that $X(p) \in S(p)$, and for arbitrary vector field $Y \in \mathcal{T}(\mathcal{M})$ and 1-form γ, we have*

$$\mathcal{N}_N(\gamma, X, Y) = \langle \gamma, R_N(X,Y)\rangle = 0 \ , \tag{14.48}$$

(or equivalently, $R_N(X,Y) = 0$), where R_N is the Nijenhuis torsion of N, see (13.37). Then the vector field Γ is Hamiltonian and defines integrable "separable" dynamics.

Proof. Let us denote by $\lambda_1, \lambda_2, .., \lambda_r$ the doubly degenerate eigenvalues and by $\mu_{2r+1}, ..., \mu_{2n}$ let us denote the simple eigenvalues. According to the assumptions, the tensor field N can be written into the form

$$N = \sum_{i=1}^{r} \lambda_i \left(e_i \otimes \vartheta^i + e_{i+r} \otimes \vartheta^{i+r}\right) + \sum_{\alpha=2r+1}^{2n} \mu_\alpha e_\alpha \otimes \vartheta^\alpha \ , \tag{14.49}$$

where the e's form a basis of eigenvectors of N and the $\vartheta' s$ are the elements of the dual basis. Thus,

$$\begin{aligned} &N e_i = \lambda_i e_i, \quad N e_{i+r} = \lambda_i e_{i+r}, \quad N e_\alpha = \mu_\alpha e_\alpha, \quad i \le r, \quad \alpha \ge 2r+1 \\ &N^* \vartheta^i = \lambda_i \vartheta^i, \quad N^* \vartheta^{i+r} = \lambda_i \vartheta^{i+r}, \quad N^* \vartheta^\alpha = \mu_\alpha \vartheta^\alpha, \quad i \le r, \quad \alpha \ge 2r+1. \end{aligned} \tag{14.50}$$

We know, see (13.37), that

$$R_N(X,Y) = [NX, NY] + N^2[X,Y] - N[NX,Y] - N[X,NY] \, . \tag{14.51}$$

Calculating the above expression on the basis vector fields $\{e_1, \dots, e_{2n}\}$ yields,

$$\begin{aligned} &R_N(e_i, e_j) = \\ &(N - \lambda_i)(N - \lambda_j)[e_i, e_j] + (\lambda_i - \lambda_j)\left[(L_{e_i}\lambda_j) e_j + \left(L_{e_j}\lambda_i\right) e_i\right] \end{aligned}$$

$$R_N(e_i, e_\alpha) = \\ (N - \lambda_i)(N - \mu_\alpha)[e_i, e_\alpha] + (\lambda_i - \mu_\alpha)[(L_{e_i}\mu_\alpha) e_\alpha + (L_{e_\alpha}\lambda_i) e_i], \tag{14.52}$$

where $i, j \leq 2r$ and $\alpha \geq 2r + 1$, and the conditions about the spectrum of N entail[3] the following relations:

$$\begin{aligned} &(N - \lambda_i)(N - \lambda_j)[e_i, e_j] = 0, \quad (\lambda_i - \lambda_j) e_i(\lambda_j) = 0 \\ &(N - \lambda_i)(N - \mu_\alpha)[e_i, e_\alpha] = 0, \quad e_i(\mu_\alpha) = e_\alpha(\lambda_i) = 0 . \end{aligned} \tag{14.53}$$

It follows that for any three vector fields e_i, e_j, e_α we have

$$[e_i, e_j] = ae_i + be_j + ce_{i+r} + de_{j+r}, \quad [e_i, e_\alpha] = fe_i + ge_{i+r} + he_\alpha . \tag{14.54}$$

Thus, any two vector fields e_i and e_{i+r}, belonging to the same eigenvalue λ_i, satisfy the relation:

$$[e_i, e_{i+r}] = c_i e_i + c_{i+r} e_{i+r}. \tag{14.55}$$

In this way, if $i \in \{1, \ldots, r\}$ the vector fields e_i, e_{i+r} form a local basis of a 2-dimensional integrable distribution and through each point of $\mathcal{M}$ passes 2-dimensional integral manifold of this distribution. On each such integral manifold one can find coordinates (ξ^i, η^i), such that

$$e_i = \frac{\partial}{\partial \xi^i}, \quad e_{i+r} = \frac{\partial}{\partial \eta^i} . \tag{14.56}$$

Then the relations (14.53) ensure that the basis on which the tensor field N diagonalize is "partially" holonomic. On the other hand, using equations (14.53) we get

$$d\lambda_i = \sum_j \vartheta^j e_j(\lambda_i) + \sum_\alpha \vartheta^\alpha e_\alpha(\lambda_i) = \sum_j \vartheta^j e_j(\lambda_i) = \vartheta^i e_i(\lambda_i) , \tag{14.57}$$

and, therefore,

$$N^* d\lambda_i = N^* \vartheta^i e_i(\lambda_i) = \vartheta^i \lambda_i e_i(\lambda_i) = \lambda_i d\lambda_i . \tag{14.58}$$

Next,

$$d\mu^\rho = d\mu_\rho = \sum_{k=1}^{2r} \vartheta^i e_i(\mu_\rho) + \sum_{\alpha=1}^{2n} \vartheta^\alpha e_\alpha(\mu_\rho) = \sum_{\alpha=1}^{2n} \vartheta^\alpha e_\alpha(\mu_\rho) , \tag{14.59}$$

Using the above relations, it is now possible to choose a holonomic basis in such a way, that N has the following expression

$$N = \sum_{j=1}^{r} \lambda_j \left(e_j \otimes \vartheta^j + e_{r+j} \otimes d\lambda^j\right) + \sum_{\rho,\sigma=2r+1}^{n} C_\rho^\sigma e_\sigma \otimes d\mu^\rho , \tag{14.60}$$

[3] See the proof of the Haantjes theorem.

with

$$C_\rho^\sigma = \sum_{\alpha=2r+1}^{2n} \mu_\alpha e_\alpha \left(\mu^\sigma\right) \left[e_\alpha \left(\mu^\rho\right)\right]^{-1} \qquad \text{and} \qquad d\vartheta^i = 0 \,. \tag{14.61}$$

In addition, in a neighborhood of each 2-dimensional submanifold, we can choose coordinates (λ, χ, μ), such that the tensor N can also be written into the form:[4]

$$N = \sum_{j=1}^{r} \lambda_j \left(\frac{\partial}{\partial \lambda_i} \otimes d\lambda_i + \frac{\partial}{\partial \chi^i} \otimes d\chi^i\right) + \sum_{\rho,\sigma=2r+1}^{n} C_\rho^\sigma \frac{\partial}{\partial \mu_\rho} \otimes d\mu_\sigma \,. \tag{14.62}$$

On the basis we have constructed, the vector field Γ can be written as

$$\Gamma = \sum_{i=1}^{r} \left(\Lambda_i \frac{\partial}{\partial \lambda_i} + \Phi^i \frac{\partial}{\partial \chi^i}\right) + \sum_{\alpha=2r+1}^{n} E^\alpha e_\alpha \,, \tag{14.63}$$

so that the condition $L_\Gamma N = 0$ implies $\Lambda_i = 0$, $E^\alpha = 0$. It follows that

$$\Gamma = \sum_{i=1}^{r} \Phi^i \left(\lambda_i, \chi^i\right) \frac{\partial}{\partial \chi^i} \,. \tag{14.64}$$

Finally, we note that one can find symplectic structures with respect to which the vector field Γ is Hamiltonian. Indeed, the closed 2-form

$$\omega = \sum_{k=1}^{r} G_k \left(\lambda_k, \chi^k\right) d\lambda_k \wedge d\chi^k + \sum_{\alpha,\beta=2r+1}^{2n} f_{\alpha\beta} \left(\mu_\alpha, \mu_\beta\right) d\mu_\alpha \wedge d\mu_\beta \,, \tag{14.65}$$

will be invariant if

$$\frac{\partial}{\partial \chi^i} \left(G_i \Phi^i\right) = 0 \,. \tag{14.66}$$

The nondegeneracy condition for ω can be cast into the form

$$\det \left(f_{\alpha\beta}\right) \prod_{k=1}^{r} G_k \neq 0 \,. \tag{14.67}$$

This is equivalent to the requirement that if $\Phi^i \left(\lambda_i, \chi^i\right)$ vanishes at some point, then it also vanishes on the integral curve of $\frac{\partial}{\partial \chi^i}$ through that point.

If the vector field Γ has no singular points, one can immediately point out a class of symplectic structures with respect to which it is Hamiltonian:

[4] The notation has been chosen to correspond to the geometric structure we previously described.

$$\omega=\sum_{k=1}^{r}\frac{g_k\left(\lambda_k\right)}{\Phi^k\left(\lambda_k,\chi^k\right)}d\lambda_k\wedge d\chi_k+\sum_{\alpha,\beta=2r+1}^{2n}f_{\alpha\beta}\left(\mu_\alpha,\mu_\beta\right)d\mu_\alpha\wedge d\mu_\beta\ , \tag{14.68}$$

where g_k and $f_{\alpha\beta}$ are arbitrary functions such that

$$\det\left(f_{\alpha\beta}\right)\prod_{k=1}^{r}\frac{g_k}{\Phi^k}\neq 0\ . \tag{14.69}$$

Remark 14.4. If Φ_k is identically zero for some index k, we can define

$$\begin{aligned}\omega=&\sum_i g_i\left(\lambda_i\right)d\lambda_i\wedge d\chi_i+\sum_j\frac{g_j\left(\lambda_j\right)}{\Phi^j\left(\lambda_j,\chi_j\right)}d\lambda_j\wedge d\chi_j\\&+\sum_{\alpha,\beta=2r+1}^{2n}f_{\alpha\beta}\left(\mu_\alpha,\mu_\beta\right)d\mu_\alpha\wedge d\mu_\beta\ ,\end{aligned} \tag{14.70}$$

where the index i in the sum runs over those eigenspaces, for which $\Phi^j=0$. When Γ has zeroes but does not vanish identically, we have to exclude from the original manifold the subset of the zeroes of Γ. The resulting manifold is invariant with respect to the flow of Γ, and we can proceed as before.

If the submanifold $\mu=\text{const}$ is compact and connected, we can introduce as usual action-angle coordinates (J,φ) so that the vector field Γ and the symplectic structure ω, in the coordinates (J,φ,μ) take the following form:

$$\Gamma=\sum_{i=1}^{r}\Gamma^i\left(J_i\right)\frac{\partial}{\partial\varphi^i} \tag{14.71}$$

$$\omega=\sum_{k=1}^{r}f_k\left(J_k\right)dJ_k\wedge d\varphi^k+\sum_{\alpha,\beta=2r+1}^{2n}f_{\alpha\beta}\left(\mu_\alpha,\mu_\beta\right)d\mu_\alpha\wedge d\mu_\beta\ . \tag{14.72}$$

In this case, the family of symplectic structures with respect to which Γ is Hamiltonian is described in [20, 22, 23]. The tensor field N can be used to generate compatible invariant symplectic structures according to

$$\omega_N\left(X,Y\right)=\omega_1\left(NX,Y\right)+\omega_1\left(X,NY\right)+\omega_2\left(X,Y\right) \tag{14.73}$$

with

$$\omega_1=\sum_{k=1}^{r}f_k\left(J_k\right)dJ_k\wedge d\varphi^k,\quad \omega_2=\frac{1}{2}\sum_{\alpha,\beta=2r+1}^{2n}f_{\alpha\beta}\left(\mu_\alpha,\mu_\beta\right)d\mu_\alpha\wedge d\mu_\beta\ . \tag{14.74}$$

Let us outline now the construction of mixed tensor fields in the case of noncommutative integrable systems. Suppose we have a noncommutative integrable system with properties as in Theorem 14.2. By the analysis we had in the above, we can assume that they have the symplectic structure

$$\omega = \sum_i d\lambda_i \wedge d\chi^i + \sum_\alpha dp_\alpha \wedge dq^\alpha$$

and the dynamics is defined by the equations of motion

$$\begin{aligned} &\dot{\lambda}_i = 0, \quad \dot{\chi}^i = \nu_i, \quad \dot{p}_\alpha = 0, \quad \dot{q}^\alpha = 0 \\ &i = 1, 2, \ldots, r; \quad \alpha = r+1, r+2, \ldots, n \,. \end{aligned} \tag{14.75}$$

Denoting by μ the collection of the p's and q's, we have

$$\dot{\lambda}_i = 0, \quad \dot{\chi}_i = \nu_i, \quad \dot{\mu}_\alpha = 0 \,. \tag{14.76}$$

Consider now the following tensor field

$$N = \sum_{j=1}^{r} \lambda_j \left(\frac{\partial}{\partial \lambda_i} \otimes d\lambda_i + \frac{\partial}{\partial \chi^i} \otimes d\chi^i\right) + \sum_{\rho,\sigma}^{2n} C_\rho^\sigma(\mu) \frac{\partial}{\partial \mu_\rho} \otimes d\mu_\sigma \,, \tag{14.77}$$

where $C_\rho^\sigma(\mu) = \delta_\rho^\sigma \mu_\sigma$ is diagonal matrix. It can be verified that N has a vanishing torsion and is invariant, provided the Hamiltonian function is "separable", that is, if H can be written into the form:

$$H = K_1(\lambda) + K_2(\mu) \,, \tag{14.78}$$

with

$$K_1(\lambda) = \sum_{i=1}^{r} H_i(\lambda_i) \,. \tag{14.79}$$

If K_1 is not separable but

$$\det\left(\frac{\partial^2 K_1}{\partial \lambda_j \partial \lambda_i}\right) \neq 0 \,, \tag{14.80}$$

the construction of the invariant tensor field can be done as in [4]. This shows that in the noncommutative case, an invariant torsion free tensor field can also be found. Of course, such a tensor field always generates (by repeated application) Abelian algebras of symmetries. Thus, regardless of the vanishing of the torsion on the whole space the noncommutative features are related to the nondegenerate eigenvalues and are still described by the term $\sum_{\rho,\sigma} C_\rho^\sigma(\mu) \frac{\partial}{\partial \mu_\rho} \otimes d\mu_\sigma$.

14.3.2 The Kepler Dynamics

The Kepler problem is one of the celebrated problems of Classical Mechanics, and all theoretical constructions in Mechanics and Symplectic Geometry are tested on it; see for example the recent monograph [24]. It is interesting to obtain for it a recursion operator. We believe that its construction deserves a special subsection, since it is not elementary and has been done quite recently.

As is known, the phase space for the Kepler problem is the symplectic manifold $(\mathcal{M}_0, \omega)$, where $\mathcal{M}_0 = T^*(\mathbb{R}^3 \setminus \{0\}) = (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^3$ and $\omega = \sum_{i=1}^3 dp_i \wedge dx_i$. The dynamical system (the Kepler dynamics) is defined by the Hamiltonian function

$$H = \frac{1}{2m}\mathbf{p}^2 - \frac{k}{r} \ . \tag{14.81}$$

Here $\mathbf{p} = (p_1, p_2, p_3)$ is the linear momentum; $\mathbf{x} = (x_1, x_2, x_3)$ is the position vector which length is denoted by r; k is a positive constant and m is the mass of the particle. Through this section $||\ ||$ will denote the usual $\mathbb{R}^3$ norm defined by the standard inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^3 x_i y_i$, so r=$||\mathbf{x}||$. It is well known that the angular momentum $\mathbf{L} = \mathbf{x} \times \mathbf{p}$ is a vector constant of motion:

$$\{H, \mathbf{L}\} = 0 \ , \tag{14.82}$$

which is a short way to write the relations $\{H, L_i\} = 0$; $i = 1, 2, 3$. The above implies that the trajectory lies in the plane orthogonal to $\mathbf{L} = \text{const}$, passing through the origin. The Kepler problem, however, possesses another vector first integral, the so-called Laplace-Runge-Lenz vector, given by:

$$\mathbf{B} = \frac{1}{m}\mathbf{p} \times \mathbf{L} - k\frac{\mathbf{x}}{r}. \tag{14.83}$$

In other words, $\{H, \mathbf{B}\} = 0$. Of course, since we cannot have 3 additional independent integrals of motion, there must be a relation between the integrals. Indeed, one can observe that

$$\langle \mathbf{B}, \mathbf{L} \rangle = 0 \ , \tag{14.84}$$

which implies that the Laplace-Runge-Lenz vector lies in the plane of the motion. We also have

$$\{L_i, B_j\} = \sum_{s=1}^{3} \epsilon_{ijs} B_s \ , \tag{14.85}$$

where ϵ_{ijs} is the Levi-Civita symbol. One verifies also that

$$\{B_i, B_j\} = -\frac{2H}{m} \sum_{s=1}^{3} \epsilon_{ijs} L_s \ . \tag{14.86}$$

It is well known that for a negative energies the motion is bounded and the orbits are ellipses. We shall be interested in these kind of motions, so we restrict our symplectic manifold to the set $\mathcal{M}$, where $H < 0$. Since $\mathcal{M}$ is an open subset of $\mathcal{M}_0$ it inherits all its structures. On $\mathcal{M}$, we have $H < 0$, and, therefore, we can introduce the vector

$$\mathbf{A} = \left(-\frac{2H}{m}\right)^{-\frac{1}{2}} \mathbf{B} \,, \tag{14.87}$$

and the Poisson brackets we had before become:

$$\{L_i, A_j\} = \sum_{s=1}^{3} \epsilon_{ijs} A_s, \quad \{A_i, A_j\} = \sum_{s=1}^{3} \epsilon_{ijs} L_s \,. \tag{14.88}$$

As a consequence, for the components of the vector-functions

$$\mathbf{I} = \frac{1}{2}(\mathbf{L} + \mathbf{A}), \quad \mathbf{J} = \frac{1}{2}(\mathbf{L} - \mathbf{A}) \tag{14.89}$$

we have the following Poisson brackets

$$\begin{aligned} \{I_h, I_s\} &= \epsilon_{hsl} I_l \\ \{J_h, J_s\} &= \epsilon_{hsl} J_l \\ \{I_h, J_s\} &= 0 \,. \end{aligned} \tag{14.90}$$

The above shows that the Lie algebra of the symmetries for the Kepler dynamics is the algebra $\mathrm{o}\,(3) \oplus \mathrm{o}\,(3) \sim \mathrm{o}\,(4)$ or, which is the same, $\mathrm{su}\,(2) \oplus \mathrm{su}\,(2)$. In particular, the Hamiltonian H can be written into the form

$$H = -\frac{mk^2}{2\,(L^2 + A^2)} = -\frac{mk^2}{4\,(I^2 + J^2)} \,. \tag{14.91}$$

In terms of the generators of $\mathrm{o}\,(4)$: $L_{hk} = -L_{kh}$, $(1 \le k, h \le 4)$, $k \ne h$ defined by:

$$\begin{aligned} L_{hs} &= \epsilon_{hsi} L_i; \qquad h, s = 1, 2, 3 \\ L_{h4} &= -L_{4h} = A_h; \qquad h = 1, 2, 3 \end{aligned} \tag{14.92}$$

the Hamiltonian H becomes

$$H = -\frac{mk^2}{C_1} \,, \tag{14.93}$$

where $C_1 = \sum_{i,j=1}^{4} L_{ij} L_{ij}$ is the so-called first Casimir element of $\mathrm{o}\,(4)$.

Our next step is to construct the action-angle variables (J_s, φ^s); $s = 1, 2, 3$. The idea to find them is the following. Let us assume that the Liouville form $\lambda = \sum_{i=1}^{3} p_i dx_i$ in the action-angle coordinates equals

$$\lambda = \sum_{s=1}^{3} J_s d\varphi^s \tag{14.94}$$

where $0 \le \varphi^s \le 2\pi$.

Remark 14.5. Since $d(\sum_{s=1}^{3} J_s d\varphi^s) = \omega = d\lambda$, the form written in the right-hand side of (14.94) and the form λ differ by a closed form γ. Assuming that γ is exact, we have that $\sum_{s=1}^{3} J_s d\varphi^s = \lambda + dF$, where F is some function. Then dF will give no contribution in the integration by which we find the action variables (see (14.95) below), and we can simply assume that (14.94) holds.

If β_s is the curve obtained when we vary φ_s from 0 to 2π, while the other variables remain constant, then

$$J_s = \frac{1}{2\pi} \int_{\beta_s} \lambda \, . \tag{14.95}$$

We now note that we shall obtain some linear combinations of the J_i's if instead of the curves β_i we use the curves $\alpha_i = n_i^1 \beta_1 + n_i^2 \beta_2 + n_i^3 \beta_3$ with some integers n_i^j so it remains to guess some closed curve on the torus where the motion occurs. This is relatively simple, since the motion defined by our Hamiltonian is always periodic. After the variables J_i are obtained, we must check that the Hamiltonian H depends only on J_i's as should be. The next step will be to find φ^i. This would be easy if we are using the Hamilton-Jacobi equation formalism. How this idea works will be seen below; see also [25].

Let us we consider the Hamilton-Jacobi equation in spherical coordinates (r, ϑ, φ) whose origin coincides with $(0, 0, 0)$. The spherical coordinates are chosen of course because the Hamilton-Jacobi equation for the Kepler problem in spherical coordinates allows separation of variables, and we can find the solution explicitly. In spherical coordinates, the symplectic form, the Hamiltonian, and the Hamiltonian vector field corresponding to H take the form:

$$\omega = dp_r \wedge dr + dp_\vartheta \wedge d\vartheta + dp_\varphi \wedge d\varphi \tag{14.96}$$

$$H = \frac{1}{2m}(p_r^2 + \frac{p_\vartheta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2 \vartheta}) - \frac{k}{r} \tag{14.97}$$

$$\begin{aligned} \Gamma = \frac{1}{m} \Bigg[& p_r \frac{\partial}{\partial r} + \frac{p_\vartheta}{r^2} \frac{\partial}{\partial \vartheta} + \frac{p_\varphi}{r^2 \sin^2 \vartheta} \frac{\partial}{\partial \varphi} \\ & - \frac{1}{r^2} \left(p_\vartheta^2 + \frac{p_\varphi^2}{\sin^2 \vartheta} \right) \frac{\partial}{\partial p_r} - \frac{p_\varphi^2 \cos \vartheta}{r^2 \sin^3 \vartheta} \frac{\partial}{\partial p_\vartheta} - \frac{k}{r^2} \frac{\partial}{\partial p_\varphi} \Bigg] \, . \end{aligned} \tag{14.98}$$

Since the Hamiltonian does not depend explicitly on time, the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \left(\left(\frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial S}{\partial \vartheta} \right)^2 + \frac{1}{r^2 \sin^2 \vartheta} \left(\frac{\partial S}{\partial \varphi} \right)^2 \right) - \frac{k}{r} = 0 \tag{14.99}$$

can be reduced, setting $S = W - Et$, to the form

$$\frac{1}{2m} \left(\left(\frac{\partial W}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial W}{\partial \vartheta} \right)^2 + \frac{1}{r^2 \sin^2 \vartheta} \left(\frac{\partial W}{\partial \varphi} \right)^2 \right) - \frac{k}{r} = E \, . \tag{14.100}$$

We look now for a solution $W(r,\vartheta,\varphi)$ of the type

$$W(r,\vartheta,\varphi) = W_r(r) + W_\vartheta(\vartheta) + W_\varphi(\varphi) \ , \tag{14.101}$$

where W_r, W_ϑ and W_φ, depend only on the corresponding variable r, ϑ and φ. In this way, the Hamilton-Jacobi equation for W becomes

$$\frac{1}{2m}\left(\left(\frac{dW_r}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dW_\vartheta}{d\vartheta}\right)^2 + \frac{1}{r^2\sin^2\vartheta}\left(\frac{dW_\varphi}{d\varphi}\right)^2\right) - \frac{k}{r} = E \ . \tag{14.102}$$

Standard reasoning leads to the conclusion that the above equation is equivalent to the system:

$$\begin{aligned}
\frac{dW_\varphi}{d\varphi} &= \pi_\varphi \\
\left(\frac{dW_\vartheta}{d\vartheta}\right)^2 &= \pi_\vartheta^2 - \frac{\pi_\varphi^2}{\sin^2\vartheta} \\
\left(\frac{dW_r}{dr}\right)^2 &= 2m\left[E + \frac{k}{r}\right] - \frac{\pi_\vartheta^2}{r^2}.
\end{aligned} \tag{14.103}$$

where $\pi_\varphi, \pi_\vartheta$ are constants of integration. The constant π_φ has clear physical meaning – it is simply the projection of the angular momentum on the z-axis. The constant π_ϑ is equal to the length of the angular momentum, because

$$||\mathbf{L}||^2 = ||\mathbf{x}\times\mathbf{v}||^2 = p_\vartheta^2 + \frac{p_\varphi^2}{\sin^2\vartheta} \ . \tag{14.104}$$

If α denotes the angle between the plane of the orbit and the (x,y) plane, we have

$$\begin{aligned}
p_\varphi &= ||\mathbf{L}||\cos\alpha \\
\pi_\varphi &= \pi_\vartheta \cos\alpha \ .
\end{aligned} \tag{14.105}$$

Therefore, the solution of the Hamilton-Jacobi equation is $S = -Et + W$:

$$W = \pi_\varphi\varphi + \int d\vartheta\sqrt{\pi_\vartheta^2 - \frac{\pi_\varphi^2}{\sin^2\vartheta}} + \int dr\sqrt{2mE + \frac{2mk}{r} - \frac{\pi_\vartheta^2}{r^2}} \ . \tag{14.106}$$

As a consequence,

$$\begin{aligned}
p_\varphi &= \pi_\varphi \\
p_\vartheta &= \sqrt{\pi_\vartheta^2 - \frac{\pi_\varphi^2}{\sin^2\vartheta}} \\
p_r &= \sqrt{2mE + \frac{2mk}{r} - \frac{\pi_\vartheta^2}{r^2}} \ .
\end{aligned} \tag{14.107}$$

Now we implement the scheme for finding the action-angle variables we outlined in the above. First we find the "action variables"

$$J_s = \frac{1}{2\pi}\int_{\gamma_s}\lambda; \quad i = 1,2,3 \tag{14.108}$$

where

$$\lambda = p_\varphi d\varphi + p_\vartheta d\vartheta + p_r dr \tag{14.109}$$

and the closed curves $\gamma_1, \gamma_2, \gamma_3$ define 3 independent cycles in the phase space – the space of the variables $(r, \vartheta, \varphi, \pi_\varphi, \pi_\vartheta, \pi_r)$, for which the Hamiltonian has negative values. These expressions will become most simple if γ_1 is given parametrically by $p_\varphi = p_\varphi(\varphi)$, while the other variables except p_φ, φ remain constant; γ_2 is given parametrically by $p_\vartheta = p_\vartheta(\vartheta)$, while the other variables except p_ϑ, ϑ remain constant and the same for γ_3. In this case, we have the action variables

$$\begin{aligned} J_\varphi &= \frac{1}{2\pi}\oint_{\gamma_1}\pi_\varphi d\varphi \\ J_\vartheta &= \frac{1}{2\pi}\oint_{\gamma_2}\sqrt{\pi_\vartheta^2 - \frac{\pi_\varphi^2}{\sin^2\vartheta}}\, d\vartheta \\ J_r &= \frac{1}{2\pi}\oint_{\gamma_3}\sqrt{2mE + \frac{2mk}{r} - \frac{\pi_\vartheta^2}{r^2}}\, dr. \end{aligned} \tag{14.110}$$

To guess γ_1 is easy; we can choose it to be parameterized by φ, since when φ varies from 0 to 2π, while the other variables remain constant, we have a closed curve. Thus we obtain

$$J_\varphi = \pi_\varphi\ . \tag{14.111}$$

As to the curves γ_2, γ_3, since the movement is periodic, they can be chosen to join the turning points, that is, they are fixed by the requirement that the corresponding velocities vanish or, better, the corresponding momenta p_ϑ and p_r (expressed, of course, in terms of variables π_ϑ and π_φ) vanish. We have:

$$\begin{aligned} p_\vartheta^2 &= \pi_\vartheta^2 - \frac{\pi_\varphi^2}{\sin^2\vartheta} = 0 \\ p_r^2 &= 2mE + \frac{2mk}{r} - \frac{\pi_\vartheta^2}{r^2} = 0 \end{aligned} \tag{14.112}$$

and the first equation defines two angles θ_1, θ_2, the second two values of r - r_1, r_2. Then the closed curve γ_2 parameterized by ϑ is defined by $\vartheta \mapsto p_\vartheta(\vartheta)$ when ϑ goes from ϑ_1 to ϑ_2 and $\vartheta \mapsto -p_\vartheta(\vartheta)$ when ϑ goes back from ϑ_2 to ϑ_1, while the rest of the variables remain constant. In the same way we define the curve γ_3. In order to find J_ϑ, we must integrate from ϑ_1 to ϑ_2 and multiply the result by 2. The same is true for the integral over r, but we integrate here from r_1 and r_2. Using (14.105) we reduce the equation for ϑ_1, ϑ_2 to

$$\sin^2\vartheta = \frac{\pi_\varphi^2}{\pi_\vartheta^2} = \cos^2\alpha\ . \tag{14.113}$$

Since ϑ always lies between 0 and π, we have $\sin\vartheta > 0$ and $\sin\vartheta_1 = \sin\vartheta_2 = \cos\alpha$. We find that $\vartheta_1 = \pi/2-\alpha$, $\vartheta_2 = \pi/2+\alpha$. But the integral from $\pi/2-\alpha$ to $\pi/2$ equals the integral from $\pi/2$ to $\pi/2-\alpha$, and we obtain

$$J_\vartheta = \frac{4\pi_\vartheta}{2\pi}\int_{\pi/2-\alpha}^{\pi/2}\frac{1}{\sin\vartheta}\sqrt{\sin^2\alpha-\cos^2\vartheta}\ d\vartheta \tag{14.114}$$

$$= \frac{2\pi_\vartheta}{\pi}\sin^2\alpha\int_0^{\pi/2}\frac{\cos^2\tau}{1-\sin^2\alpha\sin^2\tau}\ d\tau\ , \tag{14.115}$$

where the new variable τ is introduced, related to the old one by

$$\cos\vartheta = \sin\alpha\sin\tau\ . \tag{14.116}$$

Putting $x=\tan\tau$, we transform the last integral into the form

$$J_\vartheta = \frac{2\pi_\vartheta}{\pi}\int_0^{+\infty}\left[\frac{1}{1+x^2}-\frac{\cos^2\alpha}{1+x^2\cos^2\alpha}\right]dx \tag{14.117}$$

$$= \frac{2\pi_\vartheta}{\pi}\left(\frac{\pi}{2}-\frac{\pi}{2}\cos\alpha\right) = \pi_\vartheta\left(1-\cos\alpha\right)\ . \tag{14.118}$$

and finally, using again (14.105), we get

$$J_\vartheta = \pi_\vartheta - \pi_\varphi\ . \tag{14.119}$$

The integral giving J_r can be calculated using the Residue Theorem. We first remark that the roots r_1 and r_2 of the equation

$$2mE + \frac{2mk}{r} - \frac{\pi_\vartheta^2}{r^2} = 0\ , \tag{14.120}$$

which gives the integration limits, are positive if $E<0$ (as we have seen they correspond to the radii of the turning points of the motion). Next, the function

$$f(z) = \sqrt{2mE + \frac{2mk}{z} - \frac{\pi_\vartheta^2}{z^2}}\ , \tag{14.121}$$

allows analytic continuation from the real axis to the complex plane. It has two branch points at

$$z_\pm = -\frac{k}{2E}\left[1\pm\sqrt{1+\frac{2\pi_\vartheta^2 E}{mk^2}}\right] \tag{14.122}$$

and a simple pole at $z=0$. We cut the complex plane on the line segment $[r_1, r_2]$, and we choose the branch of $f_+(z)$ of $f(z)$, which on the upper side of the cut $(z = x+i0^+)$ equals

$$-f(x) = -\sqrt{2mE + \frac{2mk}{x} - \frac{\pi_\vartheta^2}{x^2}} \ , \tag{14.123}$$

and on the lower side ($z = x - i0^+$) equals

$$f(x) = \sqrt{2mE + \frac{2mk}{x} - \frac{\pi_\vartheta^2}{x^2}} \ . \tag{14.124}$$

The function $f_+(z)$ is analytic in $\mathbb{C} \setminus ([r_1, r_2] \cup \{0\})$ and has a simple pole at $z = 0$. The integral over r, first integrating from r_1 to r_2 and then integrating again from r_2 to r_1, changing the sign of the integrand, can be considered as complex integral of the function $(1/2\pi) f_+(z)$, first going on the lower side of the cut from r_1 to r_2 and then back on the upper side from r_2 to r_1. The Cauchy theorem ensures that the same value will be obtained, if we take the complex integral $(1/2\pi) \int_\gamma f_+(z) dz$ over a closed contour γ, oriented anticlockwise, which encompasses the segment $[r_1, r_2]$ and does not encompass the pole $z = 0$. Such integral can be calculated using the Residue theorem, and this gives:

$$J_r = i \left(\mathrm{Res(f;0)} + \mathrm{Res(f;\infty)}\right) \ . \tag{14.125}$$

Now, since $\mathrm{Res(f_+;0)} = \sqrt{-\pi_\vartheta^2}$ and $\mathrm{Res(f_+;\infty)} = \mathrm{mk}/\sqrt{\mathrm{2mE}}$, we obtain

$$J_r = -\pi_\vartheta + \frac{mk}{\sqrt{-2mE}} \ . \tag{14.126}$$

Then from $J_\varphi + J_\vartheta = \pi_\vartheta$, we get

$$J_\varphi + J_\vartheta + J_r = \frac{mk}{\sqrt{-2mE}} \ . \tag{14.127}$$

The above relation shows that the Hamiltonian function can be written in terms of the action-angle variables in the following way:

$$H = -\frac{mk^2}{2\left(J_\varphi + J_\vartheta + J_r\right)^2} \ . \tag{14.128}$$

Finally, replacing in (14.106) the variables $E, \pi_\vartheta, \pi_\varphi$ with their expressions in terms of action coordinates

$$\begin{aligned} E &= -\frac{mk^2}{2\left(J_\varphi + J_\vartheta + J_r\right)^2} \\ \pi_\vartheta &= J_\vartheta - J_\varphi \\ \pi_\varphi &= J_\varphi \ , \end{aligned} \tag{14.129}$$

we get the function W as a function of the $J's$. This allows us to define the corresponding angle variables:

$$\varphi^h = \frac{\partial W}{\partial J_h}; \quad h = 1, 2, 3 \ , \tag{14.130}$$

where in the above and below we write $J_1 = J_r$, $J_2 = J_\vartheta$, $J_3 = J_\varphi$ in order to cast the expressions in compact form. After performing the integrations, we obtain a symplectic map from the original coordinates $(p_r, p_\vartheta, p_\varphi, r, \vartheta, \varphi)$ to the action-angle coordinates $\left(J_h, \varphi^h\right)$; $h = 1, 2, 3$. This map is implicitly given by:

$$J_1 = -\sqrt{p_\vartheta^2 + \frac{p_\varphi^2}{\sin^2\vartheta}} + mk\left(\frac{2mk}{r} - \frac{p_\vartheta^2}{r^2} - \frac{p_\varphi^2}{r^2\sin^2\vartheta} - p_r^2\right)^{-1} \tag{14.131}$$

$$J_2 = \sqrt{p_\vartheta^2 + \frac{p_\varphi^2}{\sin^2\vartheta}} - p_\varphi \tag{14.132}$$

$$J_3 = p_\varphi \tag{14.133}$$

$$\varphi^1 = \varphi - J^{-2}\left[-m^2k^2r^2 + 2mkJ^2r - J^2(J_\vartheta + J_\varphi)^2\right]^{\frac{1}{2}} \tag{14.134}$$

$$+ \arcsin\frac{mkr - J^2}{J\sqrt{J^2 - (J_\vartheta + J_\varphi)^2}} \tag{14.135}$$

$$\varphi^2 = \varphi^1 - \arcsin\frac{[mkr - (J_\vartheta + J_\varphi)^2]J}{\sqrt{J^2 - (J_\vartheta + J_\varphi)^2}} \tag{14.136}$$

$$- \arcsin\frac{(J_\vartheta + J_\varphi)\cos\vartheta}{\sqrt{(J_\vartheta + J_\varphi)^2 - J_\varphi}} \tag{14.137}$$

$$\varphi^3 = \varphi^2 + \arcsin\frac{J_\vartheta \cot\vartheta}{\sqrt{(J_\vartheta + J_\varphi)^2 - J_\varphi}} + \varphi^1 \ , \tag{14.138}$$

where

$$J = J_1 + J_2 + J_3 = J_r + J_\vartheta + J_\varphi \ . \tag{14.139}$$

In order to define the map in an explicit way, in the last three relations $J_r, J_\vartheta, J_\varphi$ must be expressed in terms of $(p_r, p_\vartheta, p_\varphi, r, \vartheta, \varphi)$, using the first three of the above equations.

The Hamiltonian H, the symplectic form ω, and the Hamiltonian vector field $\mathit{\Gamma}$ can be written through the action-angle variables as follows:

$$\begin{aligned} H &= -mk^2 J^{-2} \\ \omega &= \sum_{h=1}^{3} dJ_h \wedge d\varphi^h \\ \mathit{\Gamma} &= 2mk^2 J^{-3}\left(\frac{\partial}{\partial\varphi^1} + \frac{\partial}{\partial\varphi^2} + \frac{\partial}{\partial\varphi^3}\right) \ . \end{aligned} \tag{14.140}$$

As already seen, the analysis of a given Hamiltonian system and the search for alternative symplectic forms and Nijenhuis tensors is much easier in action-angle variables which was the reason we calculated them here for the Kepler problem. As one can check, (see [25]), the vector field $\mathit{\Gamma}$ is Hamiltonian also with respect to the symplectic form ω_1:

$$\omega_1 = \sum_{h,k} R_k^h \, dJ_h \wedge d\varphi^k \, , \tag{14.141}$$

where the matrix $R = (R_k^h)_{1 \leq k,h \leq 3}$ is defined by:

$$R = \frac{1}{2} \begin{pmatrix} J_r & J_\vartheta & J_\varphi \\ J_\vartheta - J_\varphi & J_r + J_\varphi & J_\varphi \\ J_\varphi - J_\vartheta & J_\vartheta & J_r + J_\vartheta \end{pmatrix} . \tag{14.142}$$

(We prefer to write $J_r, J_\varphi, J_\vartheta$ instead of J_1, J_2, J_3 in the final expressions). The form ω_1 generates new Poisson brackets

$$\{f, g\}_1 = \sum_{k,h} (R^{-1})_h^k \left(\frac{\partial f}{\partial J_h} \frac{\partial g}{\partial \varphi^k} - \frac{\partial f}{\partial \varphi^k} \frac{\partial g}{\partial J_h} \right) , \tag{14.143}$$

and Γ corresponds to the new Hamiltonian function H_1:

$$H_1 = -2mk^2 J^{-1} . \tag{14.144}$$

In the original coordinates (p, q) the symplectic form ω_1 is simply written as:

$$\omega_1 = \sum_i dK_i \wedge d\varphi^i \tag{14.145}$$

where the functions $K_i(p, q)$ and $\alpha^i(p, q)$ are given by:

$$\begin{aligned} K_1 &= \frac{1}{4}[J_r + (J_\vartheta - J_\varphi)^2] \\ K_1 &= \frac{1}{4}[J_r + (J_\vartheta - J_\varphi)^2] \\ K_3 &= \frac{1}{2} J_\varphi [J_r + J_\vartheta] . \end{aligned} \tag{14.146}$$

(Recall that $J_i = J_i(p, q)$, $\varphi^i = \varphi^i(p, q)$). Since ω is nondegenerate, one can construct a mixed invariant tensor field N_0, such that $\omega(N_0(X), Y) = \omega_1(X, Y)$. The tensor N_0 has the form

$$N_0 = \sum_{h,k} \left[R_k^h \, dJ_h \otimes \frac{\partial}{\partial J_k} + R_h^k \, d\varphi^h \otimes \frac{\partial}{\partial \varphi^k} \right] , \tag{14.147}$$

As easily verified, N_0 has double degenerate eigenvalues and vanishing Nijenhuis torsion, the last property being equivalent to the compatibility of the symplectic structures ω and ω_1. However, one can check, see [25], that the action of N_0^* does not give new integrals of motion as one would expect, for example

$$N_0^* dH = d\left(k\sqrt{-\frac{m}{H}} \right) . \tag{14.148}$$

This is not what we would like to obtain, and so we must look for another Nijenhuis tensor. It can be found changing the action-angle variables to some others, closely related to them. As already seen, the Kepler dynamics possesses five independent first integrals, for example, those chosen among Hamiltonian and the components of the angular momentum and the components of the Laplace-Runge-Lenz vector. In action-angle coordinates (J, φ) we can choose the following independent five first integrals

$$J_r,\ J_\vartheta,\ J_\varphi,\ \varphi^1 - \varphi^2,\ \varphi^2 - \varphi^3\ . \tag{14.149}$$

Now, using the so-called Delauney action-angle variables, (I_i, α_i); $i = 1, 2, 3$, (see [24]), which are given by

$$\begin{aligned}
I_1 &= J_r + J_\vartheta + J_\varphi = \lambda_1 \\
I_2 &= J_\vartheta + J_\varphi = \mu_3 \\
I_3 &= J_\varphi = \mu_4 \\
\alpha_1 &= \varphi^1 = \chi_1 \\
\alpha_2 &= \varphi^2 - \varphi^1 = \mu_5 \\
\alpha_3 &= \varphi^3 - \varphi^2 = \mu_6\ ,
\end{aligned} \tag{14.150}$$

and introducing the new variables $\lambda_1, \chi_1,\ \mu_\alpha$; $\alpha = 3, 4, 5, 6$, we can construct the following invariant tensor field

$$N = \lambda_1 \left(\frac{\partial}{\partial \lambda_1} \otimes d\lambda_1 + \frac{\partial}{\partial \chi_1} \otimes d\chi_1 \right) + \sum_{\alpha=3}^{6} \mu_\alpha \frac{\partial}{\partial \mu_\alpha} \otimes d\mu_\alpha . \tag{14.151}$$

As easily checked, the Nijenhuis torsion of this tensor field is zero. This tensor field can be used as Nijenhuis tensor for the Kepler Dynamics – Its eigenvalues give all the first integrals of the field $\varGamma$.

The above example illustrates the general concept that the integrability is closely related to the existence of a mixed invariant tensor field with vanishing Nijenhuis torsion, though sometimes it is hidden, and it is not an easy task to find it.

14.4 Compatible Poisson Structures and Poisson-Nijenhuis Structures

Following [26] we shall say that on the manifold $\mathcal{M}$ is defined P-N structure (Poisson-Nijenhuis structure) if on $\mathcal{M}$ are defined simultaneously Poisson tensor P and Nijenhuis tensor N, satisfying the following coupling conditions:

$$\begin{aligned}
&(a)\ NP = PN^* \\
&(b)\ PL_{N(X)}(\alpha) - PL_X(N^*\alpha) + L_{P(\alpha)}(N)(X) = 0\ ,
\end{aligned} \tag{14.152}$$

for arbitrary choice of the vector field X and the 1-form α. We also introduce the following $(2,1)$ type tensor field $[P,N]$ through the property

$$[P,N](X,\alpha) = PL_{N(X)}(\alpha) - PL_X(N^*\alpha) + L_{P(\alpha)}(N)(X) \,. \tag{14.153}$$

A manifold endowed with a P-N structure will be called Poisson-Nijenhuis manifold or for shortness P-N manifold.

We present here the condition (b) exactly as it has been initially introduced by Magri in the first work that mentioned P-N structures [26] (for more recent developments see [27, 28]). In order to understand its meaning, let us assume that P is invertible, then $B = (P)^{-1} : \Lambda^1(\mathcal{M}) \mapsto \mathcal{T}(\mathcal{M})$ defines a symplectic form through $\omega_B(X,Y) = \langle B(X), Y\rangle$. The coupling condition (b) then is a consequence from $d\omega_B = 0$, the first coupling condition, and the relation $d_N(\omega_B) = 0$. For symplectic form that satisfies $\omega(NX,Y) = \omega(X,NY)$ the second coupling condition is equivalent to the requirement $d_N(\omega_B) = 0$. Recall that the operation d_N has been introduced earlier in (13.19), and according to its definition for a symplectic form, the condition $d_N\omega_B = 0$ is equivalent to $di_N(\omega_B) = 0$ which means that the form $i_N\omega$ is closed. Since $i_N\omega(X,Y) = \omega(NX,Y) + \omega(X,NY)$, all this simply shows that the second coupling condition ensures that $\omega_N(X,Y) = \omega(NX,Y)$ is a closed 2-form.

The second coupling condition has also natural interpretation in terms of the so-called Lie bi-algebroid structures [29], which we are not going to introduce here.

The structure we have introduced seems very specific. However, it turns out that for the so-called soliton equations it arises in a natural way. The point is that almost in each approach to the theory of completely integrable systems one can notice the crucial role played by the so-called compatible Poisson tensors (see [26, 30, 31, 32, 33]), or as also called Hamiltonian pairs (see [34, 35]).

Definition 14.6. *Two Poisson tensors P and Q are compatible if the tensor $P+Q$ is Poisson tensor too.*

It is evident that for this it is necessary and sufficient that

$$[P,Q]_S = 0 \,. \tag{14.154}$$

If P,Q are compatible it easily follows that for arbitrary constants a,b the field $aP+bQ$ is also a Poisson tensor.

After these preliminaries we prove now theorem [26], showing how P-N structures arise when we have a compatible Poisson pair.

Theorem 14.7. *Let P and Q be Poisson tensors on $\mathcal{M}$. Let Q^{-1} exist, that is, we assume that there exists a smooth field of linear maps $m \to Q_m^{-1}$. Then the tensor fields $N = P\circ Q^{-1}$ and Q endow the manifold with P-N structure.*

Proof. First of all, let us note that if $N = P\circ Q^{-1}$, then the first coupling condition is already satisfied, and it remains to prove the second coupling condition and the fact that N is Nijenhuis tensor. Let us begin with the second coupling condition.

Let $P = NQ$ and Q be compatible Poisson tensors. This means that simultaneously $[Q,Q]_S = 0$, $[NQ,NQ]_S = 0$ and $[Q,NQ]_S = 0$. In more detail

$$\langle QL_{Q\gamma_1}\gamma_2,\gamma_3\rangle + \langle QL_{Q\gamma_2}\gamma_3,\gamma_1\rangle + \langle QL_{Q\gamma_3}\gamma_1,\gamma_2\rangle = 0 \tag{14.155}$$

for any three 1-forms $\gamma_1,\gamma_2,\gamma_3$;

$$\langle NQL_{NQ\gamma_1}\gamma_2,\gamma_3\rangle + \langle NQL_{NQ\gamma_2}\gamma_3,\gamma_1\rangle + \langle NQL_{NQ\gamma_3}\gamma_1,\gamma_2\rangle = 0 \tag{14.156}$$

for any three 1-forms $\gamma_1,\gamma_2,\gamma_3$;

$$\begin{aligned}&\langle QL_{NQ\gamma_1}\gamma_2,\gamma_3\rangle + \langle QL_{NQ\gamma_2}\gamma_3,\gamma_1\rangle + \langle QL_{NQ\gamma_3}\gamma_1,\gamma_2\rangle +\\ &\langle NQL_{Q\gamma_1}\gamma_2,\gamma_3\rangle + \langle NQL_{Q\gamma_2}\gamma_3,\gamma_1\rangle + \langle NQL_{Q\gamma_3}\gamma_1,\gamma_2\rangle = 0\end{aligned} \tag{14.157}$$

for any three 1-forms $\gamma_1,\gamma_2,\gamma_3$. Making use of (14.155) we have:

$$\begin{aligned}&\langle NQL_{Q\gamma_1}\gamma_2,\gamma_3\rangle = \langle QL_{Q\gamma_1}\gamma_2,N^*\gamma_3\rangle =\\ &-\langle QL_{Q\gamma_2}(N^*\gamma_3),\gamma_1\rangle - \langle QL_{QN^*\gamma_3}\gamma_1,\gamma_2\rangle =\\ &-\langle QL_{Q\gamma_2}(N^*\gamma_3),\gamma_1\rangle - \langle QL_{NQ\gamma_3}\gamma_1,\gamma_2\rangle\ .\end{aligned}$$

Analogously,

$$\begin{aligned}\langle NQL_{Q\gamma_2}\gamma_3,\gamma_1\rangle &= -\langle QL_{Q\gamma_3}(N^*\gamma_1),\gamma_2\rangle - \langle QL_{NQ\gamma_1}\gamma_2,\gamma_3\rangle\\ \langle NQL_{Q\gamma_3}\gamma_1,\gamma_2\rangle &= -\langle QL_{Q\gamma_1}(N^*\gamma_2),\gamma_3\rangle - \langle QL_{NQ\gamma_2}\gamma_3,\gamma_1\rangle.\end{aligned}$$

Inserting the above expressions into (14.157) we get the following equation

$$\langle QL_{Q\gamma_1}(N^*\gamma_2),\gamma_3\rangle + \langle QL_{Q\gamma_2}(N^*\gamma_3),\gamma_1\rangle + \langle QL_{Q\gamma_3}(N^*\gamma_1),\gamma_2\rangle = 0\ . \tag{14.158}$$

Thus we see that provided $NQ = QN^*$ and $Q^* = -Q$ are true the relations $[Q,Q]_S = 0$ $[Q,NQ]_S = 0$ lead to the equation (14.158) and conversely, this equation, together with $[Q,Q]_S = 0$ leads to $[Q,NQ]_S = 0$.

Remark 14.8. It is easy to see that we can introduce the $(3,0)$ tensor field $[Q,N]_S$ by requirement:

$$\begin{aligned}&[Q,N]_S(\gamma_1,\gamma_2,\gamma_3) =\\ &\langle QL_{Q\gamma_1}(N^*\gamma_2),\gamma_3\rangle + \langle QL_{Q\gamma_2}(N^*\gamma_3),\gamma_1\rangle + \langle QL_{Q\gamma_3}(N^*\gamma_1),Q\gamma_2\rangle\ ,\end{aligned} \tag{14.159}$$

then the calculation we have done show that for arbitrary 1-forms $\gamma_1,\gamma_2,\gamma_3$ we have:

$$\begin{aligned}&[NQ,Q]_S(\gamma_1,\gamma_2,\gamma_3) + [Q,N]_S(\gamma_1,\gamma_2,\gamma_3) =\\ &+([Q,Q]_S(N^*\gamma_1,\gamma_2,\gamma_3) + \mathrm{cycl}\,(1,2,3))\ .\end{aligned} \tag{14.160}$$

We shall show now that if Q is invertible, the relation (14.158) is equivalent to the coupling condition (14.152 b). Now, in order to obtain (14.152 b) we transform the three terms in the left-hand side of (14.158). For the second one, using some simple identities, we get

$$\begin{aligned}
&\langle QL_{Q\gamma_2}(N^*\gamma_3),\gamma_1\rangle = -\langle L_{Q\gamma_2}(N^*\gamma_3),Q\gamma_1\rangle = \\
&-L_{Q\gamma_2}\langle N^*\gamma_3,Q\gamma_1\rangle + \langle N^*\gamma_3,L_{Q\gamma_2}(Q\gamma_1)\rangle = \\
&-L_{Q\gamma_2}\langle \gamma_3,NQ\gamma_1\rangle + \langle \gamma_3,N[Q\gamma_1,Q\gamma_2]\rangle = \\
&-\langle L_{Q\gamma_2}\gamma_3,NQ\gamma_1\rangle - \langle L_{Q\gamma_2}(N)(Q\gamma_1),\gamma_3\rangle = \\
&\langle QL_{Q\gamma_2}\gamma_3,N^*\gamma_1\rangle - \langle L_{Q\gamma_2}(N)(Q\gamma_1),\gamma_3\rangle \ .
\end{aligned} \tag{14.161}$$

If we put in (14.155) γ_3 instead of γ_1, $N^*\gamma_1$ instead of γ_2 and γ_2 instead of γ_3, the third term can be cast into the form

$$\langle QL_{Q\gamma_3}(N^*\gamma_1),\gamma_2\rangle = -\langle QL_{Q\gamma_2}\gamma_3,N^*\gamma_1\rangle - \langle QL_{QN^*\gamma_1}\gamma_2,\gamma_3\rangle \ . \tag{14.162}$$

Finally, inserting (14.161), (14.162) into (14.158) we arrive at:

$$\langle QL_{Q\gamma_1}(N^*\gamma_2) - L_{Q\gamma_2}(N)(Q\gamma_1) - QL_{NQ\gamma_1}\gamma_2,\gamma_3\rangle = 0 \ . \tag{14.163}$$

It remains to take into account that $\gamma_1,\gamma_2,\gamma_3$ are arbitrary and that Q is invertible to obtain that for each 1-form α and each vector field X we have

$$QL_X(N^*\alpha) - L_{Q\alpha}(N)(X) - QL_{NX}(\alpha) = 0 \ , \tag{14.164}$$

which is none but the coupling condition.

Remark 14.9. With the help of the tensor field $[N,Q]_S$ we have introduced in the remark (14.8), the calculations that we have preformed can be written as

$$[Q,N]_S(\gamma_1,\gamma_2,\gamma_3) = [Q,Q]_S(N^*\gamma_1,\gamma_2,\gamma_3) + \langle [Q,N](Q\gamma_1,\gamma_2),\gamma_3\rangle \ . \tag{14.165}$$

Now we must show that N is Nijenhuis tensor. For this, we shall consider (14.156) but first we shall make some preparations. First

$$\begin{aligned}
&\langle NQL_{NQ\gamma_1}(\gamma_2),\gamma_3\rangle = -\langle L_{QN^*\gamma_1}(\gamma_2),QN^*\gamma_3\rangle = \\
&-L_{QN^*\gamma_1}\langle \gamma_2,QN^*\gamma_3\rangle - \langle \gamma_2,[NQ\gamma_3,NQ\gamma_1]\rangle = \\
&-L_{QN^*\gamma_1}\langle N^*\gamma_2,Q\gamma_3\rangle - \langle \gamma_2,[NQ\gamma_3,NQ\gamma_1]\rangle = \\
&\langle QL_{QN^*\gamma_1}(N^*\gamma_2),\gamma_3\rangle - \langle \gamma_2,N[NQ\gamma_1,Q\gamma_3]\rangle - \langle \gamma_2,[NQ\gamma_3,NQ\gamma_1]\rangle.
\end{aligned}$$

Next, using (14.155), (14.158) we obtain:

$$\begin{aligned}
&\langle NQL_{NQ\gamma_2}(\gamma_3),\gamma_1\rangle = \langle QL_{QN^*\gamma_2}(\gamma_3),N^*\gamma_1\rangle = \\
&-\langle QL_{Q\gamma_3}(N^*\gamma_1),N^*\gamma_2\rangle - \langle QL_{QN^*\gamma_1}(N^*\gamma_2),\gamma_3\rangle = \\
&\langle QL_{Q\gamma_1}((N^*)^2\gamma_2),\gamma_3\rangle + \langle QL_{QN^*\gamma_2}(N^*\gamma_3),\gamma_1\rangle \\
&-\langle QL_{QN^*\gamma_1}(N^*\gamma_2),\gamma_3\rangle = -\langle L_{Q\gamma_1}((N^*)^2\gamma_2),Q\gamma_3\rangle
\end{aligned}$$

$$
\begin{aligned}
&+\langle QL_{NQ\gamma_2}(N^*\gamma_3),\gamma_1\rangle - \langle QL_{QN^*\gamma_1}(N^*\gamma_2),\gamma_3\rangle = \\
&-L_{Q\gamma_1}\langle (N^*)^2\gamma_2, Q\gamma_3\rangle + \langle (N^*)^2\gamma_2, L_{Q\gamma_1}Q\gamma_3\rangle \\
&-\langle L_{NQ\gamma_2}(N^*\gamma_3),Q\gamma_1\rangle - \langle QL_{QN^*\gamma_1}(N^*\gamma_2),\gamma_3\rangle = \\
&-L_{Q\gamma_1}\langle \gamma_2, N^2Q\gamma_3\rangle - \langle \gamma_2, N^2[Q\gamma_3, Q\gamma_1]\rangle \\
&-\langle L_{NQ\gamma_2}(N^*\gamma_3),Q\gamma_1\rangle - \langle QL_{QN^*\gamma_1}(N^*\gamma_2),\gamma_3\rangle.
\end{aligned}
$$

We transform also the last term in (14.156) with the help of (14.155) as follows:

$$
\begin{aligned}
&\langle NQL_{NQ\gamma_3}(\gamma_1),\gamma_2\rangle = \langle QL_{QN^*\gamma_3}(\gamma_1), N^*\gamma_2\rangle = \\
&-\langle QL_{Q\gamma_1}(N^*\gamma_2), N^*\gamma_3\rangle - \langle QL_{QN^*\gamma_2}(N^*\gamma_3),\gamma_1\rangle = \\
&\langle L_{Q\gamma_1}(N^*\gamma_2), QN^*\gamma_3\rangle + \langle L_{QN^*\gamma_2}(N^*\gamma_3), Q\gamma_1\rangle = \\
&L_{Q\gamma_1}\langle (N^*\gamma_2), QN^*\gamma_3\rangle - \langle N^*\gamma_2, L_{Q\gamma_1}(QN^*\gamma_3)\rangle \\
&+\langle L_{QN^*\gamma_2}(N^*\gamma_3), Q\gamma_1\rangle = L_{Q\gamma_1}\langle \gamma_2, N^2Q\gamma_3\rangle \\
&+\langle \gamma_2, N[NQ\gamma_3, Q\gamma_1]\rangle + \langle L_{QN^*\gamma_2}(N^*\gamma_3), Q\gamma_1\rangle.
\end{aligned}
$$

Finally, we insert these expressions in (14.156) and get:

$$
\langle \gamma_2, [NX_3, NX_1] + N^2[X_3, X_1] - N[NX_3, X_1] - N[X_3, NX_1]\rangle = 0,
$$

where we have put $X_1 = Q\gamma_1, X_3 = Q\gamma_3$. As Q is invertible and $\gamma_1, \gamma_2, \gamma_3$ are arbitrary, we obtain exactly the zero Nijenhuis bracket condition

$$
[NX_3, NX_1] + N^2[X_3, X_1] - N[NX_3, X_1] - N[X_3, NX_1] = 0.
$$

The theorem is proved.

A construction similar to the one we have used in the above theorem can be applied also in the following situation. Suppose on the manifold $\mathcal{M}$ we have simultaneously a Poisson tensor P and a closed 2-form Ω (not necessarily nondegenerate), for example a presymplectic form. Let $\bar{\Omega}$ be the corresponding field of linear maps

$$
m \to \bar{\Omega}_m : T_m(\mathcal{M}) \to T^*_m(\mathcal{M}) \,. \tag{14.166}
$$

Then the following theorem (see [26]) holds:

Theorem 14.10. *If the form that corresponds to $\bar{\Omega} \circ P \circ \bar{\Omega}$ is closed, then the tensor fields P and $N = P \circ \bar{\Omega}$ define P-N structure on the manifold $\mathcal{M}$.*

The theorem can be proved using simple calculations, similar to what we have used in the above.

An interesting situation arises on symplectic manifold $\mathcal{M}$ with symplectic form ω, if in addition there is a nondegenerate Nijenhuis tensor N for which we have:

$$
\bar{\omega} \circ N = N^* \circ \bar{\omega} \,. \tag{14.167}
$$

This condition is an analog of the coupling condition (a) for the P-N structure. Of course, as usual, here $\bar{\omega}$ is the field of maps $m \mapsto \bar{\omega}_m : T_m(\mathcal{M}) \mapsto T^*_m(\mathcal{M})$ that corresponds to ω.

In this case, if the eigenvalues of N are smooth functions on $\mathcal{M}$ they generate a system of integrable vector fields, without the additional requirements usually imposed on ω and N, see for example [26]. In order to prove this other version of the Liouville theorem is used, called the Liouville-Cartan theorem,[5] [36].

Theorem 14.11. *Let $(\mathcal{M}, \omega)$ be a symplectic manifold of dimension $2n$, let α be a closed 1-form on $\mathcal{M}$ and let X be the Hamiltonian field that corresponds to it ($i_X\omega = \alpha$). Suppose that $\beta_1, \beta_2, \ldots, \beta_{n-1}$ are closed 1-forms that are first integrals of X ($\beta_s(X) = 0$, $1 \le s \le n-1$). Let us assume that the forms β_s have the properties:*

1. *The forms β_s are in involution (that is, $\{\beta_i, \beta_j\} = 0$).*
2. *The forms $\alpha, \beta_1, \beta_2, \ldots, \beta_{n-1}$ are linearly independent for each point m of an open set $U \subset \mathcal{M}$.*

Then

1. *On the set U there exist n linear forms (1-forms) $\gamma, \gamma_1, \gamma_2, \ldots, \gamma_{n-1}$, such that*

$$\omega_U = \alpha \wedge \gamma + \sum_{s=1}^{n-1} \beta_s \wedge \gamma_s \ . \tag{14.168}$$

2. *The 2-forms $d\gamma, d\gamma_1, d\gamma_2, \ldots, d\gamma_{n-1}$ belong to the ideal generated by the forms $\alpha, \beta_1, \beta_2, \ldots, \beta_{n-1}$ in the exterior algebra $\Lambda(\mathcal{M})$.*

The 1-forms $\alpha, \beta_1, \beta_2, \ldots, \beta_{n-1}$ generate on $U \subset \mathcal{M}$ integrable Pfaffian system. Let $\mathcal{N}$ be integral manifold for it and let $j : \mathcal{N} \mapsto \mathcal{M}$ be the inclusion map. Then

1. *The field X is tangent to $\mathcal{N}$ and induces on it a vector field Z (X and Z are j-related).*
2. *The forms $j^*\gamma, j^*\gamma_1, j^*\gamma_2, \ldots, j^*\gamma_{n-1}$ (restrictions of $\gamma, \gamma_1, \gamma_2, \ldots, \gamma_{n-1}$ on $\mathcal{N}$) are linearly independent at each point of $\mathcal{N}$.*
3. *The forms $j^*\gamma_1, j^*\gamma_2, \ldots, j^*\gamma_{n-1}$ are first integrals of Z ($j^*\gamma_s(Z) = 0$ and $dj^*\gamma_s = 0$).*
4. *$dj^*\gamma = 0$ and $j^*\gamma(Z) = 1$.*

The theorem states that one can find a submanifold $\mathcal{N}$ of $\mathcal{M}$ such that the field X is tangent to it and on $\mathcal{N}$ we have $n-1$ independent integrals (the γ's) for X. Thus X is integrable in quadratures (completely integrable).

Let us return now to the situation when we have a Nijenhuis tensor N on a symplectic manifold $(\mathcal{M}, \omega)$ coupled with the symplectic structure by

[5] Actually in the formulation below, we united two theorems, the so-called theorem of Gallissot and the Liouville-Cartan theorem.

(14.167). In order to simplify the notations, and to avoid repeating all the time that we have some property pointwise, we shall use the following conventions. Let N be a $(1,1)$ tensor on the manifold $\mathcal{M}$, that is, N is a field of linear operators $N_m : T_m(\mathcal{M}) \mapsto T_m(\mathcal{M})$. We shall call it semisimple if N_m is semisimple for each m. Next, let us suppose that the eigenvalues $\lambda_i(m)$ of N_m are smooth functions and the eigenspaces $S_i(m)$ corresponding to $\lambda_i(m)$ have constant dimensions. Then we shall refer to the distributions $m \mapsto S_i(m)$ as the eigenspaces and shall denote them by S_i. We shall call the dimension of $S_i(m)$ the dimension of S_i and we shall call the functions λ_i eigenvalues corresponding to S_i. Following the same logic, we shall denote the distribution $m \mapsto S_i(m) \oplus S_j(m)$ by $S_i \oplus S_j$. If ω is symplectic form on $\mathcal{M}$ we shall say that S_i and S_j are orthogonal with respect to ω if $S_i(m)$ and $S_j(m)$ are orthogonal with respect to ω_m. If X is a field we shall say that $X \in S_i$ if for each m $X(m) \in S_i(m)$ and so on, each property that holds pointwise for some fields of objects will be stated as property of the corresponding fields.

Now we have the following proposition (see [37] and [26]).

Proposition 14.12. *Let $(\mathcal{M}, \omega)$ be $2n$-dimensional symplectic manifold on which there exists Nijenhuis tensor N, such that $N^* \circ \bar{\omega} = \bar{\omega} \circ N$. Let N be semisimple and let its eigenvalues λ_i be smooth functions on $\mathcal{M}$. Let the dimension of the eigenspaces S_i corresponding to λ_i be constant on $\mathcal{M}$. Then:*

1. *The eigenspaces S_i, corresponding to the eigenvalues λ_i are orthogonal with respect to ω and have even dimension.*
2. *If none of the functions λ_i is locally equal to a constant, that is, there is no open subset $V \subset \mathcal{M}$ such that $\lambda_i|_V = const$, then the forms $d\lambda_i$ are independent and are in involution. The Hamiltonian vector fields X_i corresponding to λ_i ($i_{X_i}\omega = -d\lambda_i$) belong to the subspaces S_i.*
3. *If for each i $dim(S_i) = 2$, that is, if each eigenvalue is exactly double degenerate, and if these eigenvalues are not locally equal to constants then:*
   (a) *$\{\lambda_i\}_{i=1}^n$ is complete set of functions in involution and each vector field X_i is completely integrable Hamiltonian system.*
   (b) *The 2-form ω can be locally expressed in the following way:*
   $$\omega = \sum_{i=1}^{n} \omega_i, \quad \omega_i = \omega|_{S_i}, \quad \omega_i = d\lambda_i \wedge \gamma_i,$$
   where γ_i are 1-forms on $\mathcal{M}$. If Y_i are the vector fields corresponding to γ_i ($-\gamma_i = i_{Y_i}\omega$) then X_i, Y_i span the subspaces S_i.
   If the vector fields X_i, Y_i can be chosen in such a way that, $[X_i, Y_i] = 0$, then $L_{X_i}N = 0$.
   (c) *If the eigenvalues λ_i have no zeroes on $\mathcal{M}$, then the 2-forms $\omega^{(k)}$ corresponding to the tensor field*
   $$\overline{\omega^{(k)}} = \bar{\omega} \circ N^k; \quad k = 0, 1, 2, \ldots,$$
   are again symplectic forms on $\mathcal{M}$.

Proof. At the beginning we recall that a Nijenhuis tensor N, whose eigenspaces have constant dimension holds the Nijenhuis theorem. Now we start with the proof.

1. First of all, for $m \in \mathcal{M}$ let us consider, $Z_i \in S_i(m),\ Z_j \in S_j(m),\ i \neq j$. We have

$$\omega_m(N_m Z_i, Z_j) = \lambda_i(m)\omega_m(Z_i, Z_j) = \omega_m(Z_i, N_m Z_j) = \lambda_j(m)\omega_m(Z_i, Z_j).$$

From here it follows that $\omega_m(Z_i, Z_j) = 0$, and therefore S_i and S_j are orthogonal w.r.t. the symplectic form ω. Naturally, we have also $S_i \cap S_j = \{0\}$.

2. Let $i_{X_i}\omega = -d\lambda_i$. Then if $i \neq j$ from the Nijenhuis theorem, it follows that $\omega(X_i, S_j) = 0$. This entails that $X_i \in S_i$ and therefore $\{d\lambda_i\}$ are in involution. As $d\lambda_i \neq 0$ then X_i; $i = 1, 2, \ldots, n$ belong to different subspaces S_i and are linearly independent. It follows that $d\lambda_i$; $i = 1, 2, \ldots, n$ are linearly independent too.

3. Let now

$$\dim S_i = 2, \quad \omega = \sum_{i=1}^{n} \omega_i, \quad \omega_i = \omega|_{S_i}.$$

Then $\{d\lambda_i\}_{i=1}^{n}$ is a complete involutive set of closed 1-forms. Moreover, for each distribution

$$S = S_{i_1} \oplus S_{i_2} \oplus \ldots \oplus S_{i_k}$$

the 2-form $\omega_S = \omega|S$ is symplectic form on the corresponding integral submanifolds of S and the forms

$$d\lambda_{i_1}, d\lambda_{i_2}, \ldots, d\lambda_{i_k}$$

form a complete set in involution on that submanifold. According to the Liouville-Cartan theorem, (14.11), each field X_i is then completely integrable Hamiltonian system and

$$\omega_S = d\lambda_{i_1} \wedge \gamma_{i_1} + d\lambda_{i_2} \wedge \gamma_{i_2} \ldots + d\lambda_{i_k} \wedge \gamma_{i_k}.$$

Here γ_{i_l} are 1-forms, such that $d\gamma_{i_l}$ belong to the ideal generated by $d\lambda_{i_l}$ in the algebra of differential forms on the corresponding submanifold. It is not difficult to check that (at least locally) one can extend these forms to forms on $\mathcal{M}$ and choose them in such a way that $d\gamma_i = d\lambda_i \wedge \delta_i$, where δ_i is some 1-form and $d\lambda_i \wedge \gamma_i = \omega_i$. Let Y_i be the vector field corresponding to the 1-form γ_i $(-\gamma_i = i_{Y_i}\omega)$. Then

$$\omega(X_i, Y_i) = 1, \quad \omega(X_i, Y_j) = 0; \quad i \neq j.$$

For this reason Y_i belongs to S_i and X_i, Y_i span S_i.

Let us prove now that $L_{X_i} N = 0$. Since we have

$$(L_{X_i} N)(Y) = [X_i, NY] - N[X_i, Y],$$

then taking into account that X_i, Y_i span S_i it is enough to prove that the right-hand side of the above equation is zero for $Y = X_j, Y_j$. By assumption $[X_i, Y_i] = 0$ and then

$$
\begin{aligned}
&(L_{X_i}N)(X_j) = [d\lambda_j(X_i)]X_j = 0; \quad i \neq j \\
&(L_{X_i}N)(X_i) = [d\lambda_i(X_i)]X_i = -\omega(X_i, X_i)X_i = 0 \\
&(L_{X_i}N)(Y_j) = [d\lambda_j(X_i)]Y_i = 0 \,.
\end{aligned}
\tag{14.169}
$$

Finally, it is clear that

$$\omega^{(k)} = \sum_{i=1}^{n} \lambda_i^k \; d\lambda_i \wedge \gamma_i$$

and therefore $d\omega^{(k)} = 0$. The proposition is proved.

The proposition shows that spectral properties are sometimes so restrictive, that they ensure some of the requirements that we usually impose a priori.

14.5 Principal Properties of Poisson-Nijenhuis Manifolds

Let us return again to the P-N structures. The application of these structures are motivated by the interesting features of their fundamental fields.

Definition 14.13. *The field X is called fundamental for the P-N structure if*

$$L_X N = 0, \quad L_X P = 0 \,. \tag{14.170}$$

In other words, X is fundamental field for the P-N structure if it is fundamental for both the tensors P and N.

In the following theorem are collected the most essential properties of the fundamental fields, see for example [26]:

Theorem 14.14. *Let $\mathcal{M}$ be P-N manifold. Let χ_N^* be the set of* 1- *forms α satisfying the conditions:*

$$d\alpha = 0, \quad dN^*\alpha = 0 \,. \tag{14.171}$$

$\chi_N^*(\mathcal{M})$ will be called the set of fundamental forms. Then the set of vector fields $\chi_{PN}(\mathcal{M})$

$$X_\alpha = \{P(\alpha) : \alpha \in \Lambda^1(\mathcal{M}), \quad d\alpha = 0, \quad dN^*\alpha = 0\} \tag{14.172}$$

are fundamental for the P-N structure. The vector spaces $\chi_{PN}(\mathcal{M})$ and $\chi_N^*(\mathcal{M})$ are Lie algebras (with respect to the Lie bracket and Poisson bracket respectively) and P is homomorphism between these algebras:

$$[X_\alpha, X_\beta] = P\{\alpha, \beta\}_P \,. \tag{14.173}$$

Moreover, the above algebras are invariant under the action of N and N^*, respectively, and N (N^*) commute with the Lie algebra operation:

$$\begin{aligned} &N^*\{\alpha,\beta\}_P = \{N^*\alpha,\beta\}_P = \{\alpha, N^*\beta\}_P; \qquad \alpha,\beta \in \chi^*_N(\mathcal{M}) \\ &N[X_\alpha, X_\beta] = [NX_\alpha, X_\beta] = [X_\alpha, NX_\beta]; X_\alpha, X_\beta \in \chi_{PN}(\mathcal{M}) \ . \end{aligned} \tag{14.174}$$

Proof. Let α be 1-form such that $d\alpha = dN^*\alpha = 0$. As already seen, the condition $d\alpha = 0$ is equivalent to the condition

$$L_X\alpha = d[\alpha(X)] = d\langle\alpha, X\rangle \ , \tag{14.175}$$

for arbitrary vector field X. Then

$$L_{N(X)}\alpha - L_X(N^*\alpha) = d\langle\alpha, NX\rangle - d\langle N^*\alpha, X\rangle = 0.$$

But from the coupling condition

$$P(L_{N(X)}\alpha - L_X(N^*\alpha)) + L_{P(\alpha)}(N)(X) = 0$$

it follows that $L_{P(\alpha)}N = 0$. Thus, the vector field $P(\alpha)$ is fundamental for the tensor N. It is also fundamental for the tensor P (see Proposition 12.24), and therefore $P(\alpha)$ is fundamental field for the P-N structure. Next, if $\alpha,\beta \in \chi^*_N(\mathcal{M})$ we have the following chain of relations:

$$\begin{aligned} &N^*\{\alpha,\beta\} - \{N^*\alpha,\beta\} = N^*d\langle\alpha, P\beta\rangle - d\langle N^*\alpha, P\beta\rangle = \\ &N^*L_{P\beta}(\alpha) - L_{P\beta}(N^*\alpha) = -[L_{P\beta}(N^*)]\alpha = 0 \ . \end{aligned} \tag{14.176}$$

Taking into account that $\{\alpha,\beta\} = -\{\beta,\alpha\}$, we easily get the first line of (14.174). Thus we have also proved that $N^*\{\alpha,\beta\}$ is closed. What remains to prove is that $\chi^*_N(\mathcal{M})$ is invariant under the action of N^*. For this it is enough to prove that if $\alpha \in \chi^*_N(\mathcal{M})$, that is, if $d\alpha = dN^*\alpha = 0$, then $d(N^*)^2\alpha = 0$ too. But this is one of the properties of a Nijenhuis tensor, see proposition (13.24). Consider now the fundamental fields $P\alpha$. For the Poisson bracket of two closed forms α,β, we have

$$[X_\alpha, X_\beta] = -P\{\alpha,\beta\} \ . \tag{14.177}$$

Therefore, if $\alpha \in \chi^*_N(\mathcal{M})$

$$N[X_\alpha, X_\beta] = -NP\{\alpha,\beta\} = -PN^*\{\alpha,\beta\} = -P\{N^*\alpha,\beta\} \ . \tag{14.178}$$

From the other hand

$$[NX_\alpha, X_\beta] = [NP\alpha, X_\beta] = [PN^*\alpha, P\beta] = -P\{N^*\alpha,\beta\} \tag{14.179}$$

and finally we get

$$N[X_\alpha, X_\beta] = [NX_\alpha, X_\beta] = [X_\alpha, NX_\beta] \ . \tag{14.180}$$

The theorem is proved.

14.6 Hierarchies of Poisson Structures

It is remarkable that P-N structure generates also a hierarchy of Poisson structures. More specifically, we have the following.

Theorem 14.15. *Let the tensors P and N endow the manifold $\mathcal{M}$ with P-N structure. Then on $\mathcal{M}$ there exist infinite number of Poisson structures $P_k = N^k P = P(N^*)^k$, $k = 1, 2, \ldots$ and there are infinitely many P-N structures, defined by the pairs (P_k, N^s), $k, s = 1, 2, \ldots$.*

The proof of the above theorem readily follows from some remarkable identities, introduced in [26], which we shall present below. Let P and N be two fields of the type $P_m : T^*_m(\mathcal{M}) \to T_m(\mathcal{M})$; $N_m : T_m(\mathcal{M}) \to T_m(\mathcal{M})$ such that $NP = PN^*$. Recall that we defined the following tensors (tensor fields):
The $(3, 0)$ type tensor field $[P, P]_S$ (Schouten bracket):

$$\begin{aligned} &[P, P]_S : \Lambda^1(\mathcal{M}) \times \Lambda^1(\mathcal{M}) \times \Lambda^1(\mathcal{M}) \mapsto \mathcal{D}(\mathcal{M}) \\ &[P, P]_S(\alpha_1, \alpha_2, \alpha_3) = \langle \alpha_1, PL_{P\alpha_2}\alpha_3 \rangle + \mathrm{cycl}\,(1, 2, 3)\ , \end{aligned} \tag{14.181}$$

for $\alpha_1, \alpha_2, \alpha_3 \in \Lambda^1(\mathcal{M})$.
The $(2, 1)$ type tensor field $[P, N]$:

$$\begin{aligned} &[P, N] : \mathcal{T}(\mathcal{M}) \times \Lambda^1(\mathcal{M}) \mapsto \mathcal{T}(\mathcal{M}) \\ &\langle [P, N](X, \alpha), \beta \rangle = \\ &\langle P[L_{NX}(\alpha) - L_X(N^*\alpha)] + L_{P(\alpha)}(N)X, \beta \rangle\ , \end{aligned} \tag{14.182}$$

for $\alpha, \beta \in \Lambda^1(\mathcal{M}); X \in \mathcal{T}(\mathcal{M})$.
The $(1, 2)$ tensor field $[N, N]$ (Nijenhuis bracket):

$$\begin{aligned} &[N, N] :\ \mathcal{T}(\mathcal{M}) \times \mathcal{T}(\mathcal{M}) \mapsto \mathcal{T}(\mathcal{M}) \\ &[N, N](X_1, X_2) = L_{NX_1}(N)X_2 - NL_{X_1}(N)X_2 \end{aligned} \tag{14.183}$$

for $X_1, X_2 \in \mathcal{T}(\mathcal{M})$, or

$$\begin{aligned} &\langle [N, N](X_1, X_2), \alpha \rangle = \\ &\langle N^2([X_1, X_2]) + [N(X_1), N(X_2)], \alpha \rangle \\ &-\langle N[X_1, N(X_2)] + N[N(X_1), X_2], \alpha \rangle \end{aligned} \tag{14.184}$$

for $\alpha \in \Lambda^1(\mathcal{M}); X_1, X_2 \in \mathcal{T}(\mathcal{M})$.
Then we have

Proposition 14.16. *For the tensor fields P (of type $(2, 0)$) and N (of type $(1, 1)$) such that $NP = PN^*$ and for $\alpha_1, \alpha_2, \alpha_3, \alpha, \beta \in \Lambda^1(\mathcal{M})$ and $X \in \mathcal{T}(\mathcal{M})$, hold the following identities:*

$$
\begin{aligned}
&[NP,NP]_S(\alpha_1,\alpha_2,\alpha_3) = \\
&[P,P]_S(N^*\alpha_1,N^*\alpha_2,\alpha_3) + \langle\alpha_2,[N,N](P\alpha_1,P\alpha_3)\rangle \\
&+ \langle[P,N](P\alpha_3,\alpha_1),N^*\alpha_2\rangle, \qquad (14.185)\\
&\langle[NP,N](X,\alpha),\beta\rangle = \langle[P,N](X,\alpha),N^*\beta\rangle + \langle[N,N](P\alpha,X),\beta\rangle. \qquad (14.186)
\end{aligned}
$$

Proof. Let us consider $[NP,NP]_S$:

$$
\begin{aligned}
&[NP,NP]_S(\alpha_1,\alpha_2,\alpha_3) = \\
&\langle N^*\alpha_1, PL_{PN^*\alpha_2}\alpha_3\rangle + \langle N^*\alpha_2, PL_{PN^*\alpha_3}\alpha_1\rangle + \langle N^*\alpha_3, PL_{PN^*\alpha_1}\alpha_2\rangle = \\
&[P,P]_S(N^*\alpha_1,N^*\alpha_2,\alpha_3) - \langle\alpha_3, PL_{PN^*\alpha_1}(N^*\alpha_2)\rangle \\
&+\langle N^*\alpha_2, PL_{P\alpha_3}(N^*\alpha_1)\rangle - \langle N^*\alpha_2, PL_{PN^*\alpha_3}(\alpha_1)\rangle \\
&+\langle N^*\alpha_3, PL_{PN^*\alpha_1}(\alpha_2)\rangle = \\
&[P,P]_S(N^*\alpha_1,N^*\alpha_2,\alpha_3) + \langle\alpha_3, NPL_{PN^*\alpha_1}(\alpha_2) - PL_{PN^*\alpha_1}(N^*\alpha_2)\rangle \\
&+\langle\alpha_2, NP[L_{NP\alpha_3}(\alpha_1) - L_{P\alpha_3}(N^*\alpha_1)\rangle = \\
&[P,P]_S(N^*\alpha_1,N^*\alpha_2,\alpha_3) + \langle[P,N](P\alpha_3,\alpha_1),N^*\alpha_2\rangle \\
&-\langle\alpha_2, NL_{P\alpha_1}(N)(P\alpha_3)\rangle - \langle\alpha_3, PL_{PN^*\alpha_1}(N^*)\alpha_2\rangle = \\
&[P,P]_S(N^*\alpha_1,N^*\alpha_2,\alpha_3) + \langle[P,N](P\alpha_3,\alpha_1),N^*\alpha_2\rangle \\
&+\langle[N,N](P\alpha_1,P\alpha_3),\alpha_2\rangle - \langle\alpha_2, L_{NP\alpha_1}(N)P\alpha_3\rangle \\
&-\langle\alpha_3, PL_{NP\alpha_1}(N^*)\alpha_2\rangle\ . \qquad (14.187)
\end{aligned}
$$

But,

$$
\begin{aligned}
&-\langle\alpha_3, PL_{NP\alpha_1}(N^*)P\alpha_2\rangle = \langle P\alpha_3, L_{NP\alpha_1}(N^*)P\alpha_2\rangle = \\
&\langle[L_{NP\alpha_1}(N^*)]^*P\alpha_3,\alpha_2\rangle = \langle\alpha_2, L_{NP\alpha_1}(N)P\alpha_3\rangle\ , \qquad (14.188)
\end{aligned}
$$

and the last two terms in (14.187) cancel. This proves (14.185).

In order to prove (14.186) we consider the following chain of equalities:

$$
\begin{aligned}
&\langle[NP,N](X,\alpha),\beta\rangle = \\
&\langle NP[L_{NX}(\alpha) - L_X(N^*\alpha)] + L_{NP\alpha}(N)X,\beta\rangle = \\
&\langle NP[L_{NX}(\alpha) - L_X(N^*\alpha)] + [N,N](P\alpha,X) + NL_{P\alpha}(N)X,\beta\rangle = \\
&\langle N\{P[L_{NX} - L_X(N^*\alpha)] + L_{P\alpha}(N)X\},\beta\rangle + \langle[N,N](P\alpha,X),\beta\rangle = \\
&[P,N](X,\alpha),N^*\beta\rangle + \langle[N,N](P\alpha,X),\beta\rangle\ . \qquad (14.189)
\end{aligned}
$$

This completes the proof of the proposition. Finally, taking into account Theorem 13.25 the proof of the Theorem 14.15 is easily obtained by induction.

From the above considerations, we get two important corollaries, which show how the P-N structure generates hierarchies of commuting Hamiltonian vector fields and hierarchies of Poisson structures for them:

Corollary 14.17. *If $\alpha, \beta \in \chi_N^*(\mathcal{M})$ are in involution (X_α and X_β commute), then for arbitrary natural numbers k and n the forms $(N^*)^k\alpha$, $(N^*)^n\beta$ are also in involution (the fields $(N)^k X_\alpha$ and $(N)^n X_\beta$ commute).*

Corollary 14.18. *The fields of the type $(N)^k\alpha$, where $\alpha \in \chi_N^*(\mathcal{M})$, are Hamiltonian with respect to a hierarchy of Poisson structures $P, NP, \dots N^k P$. If N^{-1} exists the hierarchy is infinite and consists of the Poisson tensors of the type $N^r P$ where r is an integer.*

Remark 14.19. Naturally, when the manifold $\mathcal{M}$ is finite dimensional as a result of the Caley-Hamilton theorem, we can obtain principally new structures only up to some number p. After this number (for $k > p$) the Poisson tensors $N^k P$ are linear combinations of the Poisson tensors $N^s P$; $s = 0, 1, 2, \dots p$.

As we shall see, the corollaries (14.17), (14.18) have special application in the theory of soliton equations. They explain the fact that the soliton equations occur in hierarchies, have commuting flows, and are Hamiltonian with respect to a hierarchy of Poisson structures.

We finish the review of the properties of the P-N manifolds with the remark that the identities (14.160), (14.165) we established in the proof of theorem (14.7) show that we can invert it in the following way.

Theorem 14.20. *If $\mathcal{M}$ is a P-N manifold, endowed with Poisson tensor P and a Nijenhuis tensor N, then P and NP are compatible Poisson tensors.*

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15

Poisson–Nijenhuis Structures Related to the Generalized Zakharov–Shabat System

In this chapter, we show that the geometric constructions outlined in the previous section permit to identify the generating operators $\Lambda_{\pm}$ for the generalized Zakharov–Shabat system (GZS system) in the pole gauge with the operator N^*, which is the adjoint of a Nijenhuis operator N of some P–N structure on the manifold of potentials. In [1], this was done for the case when the algebra is $\mathrm{sl}\,(n)$. We first prove the same result in the case of arbitrary semisimple Lie algebra. This is achieved considering special compatible Poisson tensors, and later in this chapter, we show what is the algebraic mechanism that leads to the existence of the aforementioned compatible Poisson tensors. Next, the general theory about the momentum maps, which we briefly present here, permits to understand deeper the gauge transformation leading from canonical to the pole gauge and to give the same interpretation we gave to the operators $\Lambda_{\pm}$ to the generating operators $\tilde{\Lambda}_{\pm}$, that appear in the theory of the soliton equations associated with GZS system in pole gage – they are also adjoint to some Nijenhuis tensors. As a result, we are able to present a clear geometric picture showing the relation between three manifolds naturally endowed with P–N structures – (i) the manifold of potentials in canonical gauge, (ii) the manifold of potentials in pole gauge, and (iii) the manifold of the Jost solutions for $\lambda = 0$. We also point out an important subalgebra of the algebra of fundamental fields which generates on each of these manifolds hierarchies of integrable systems (soliton equations).

15.1 Poisson–Nijenhuis Structures for GZS System in Canonical Gauge

Let us denote by $\mathfrak{g}[x]$ the set of Schwartz-type functions:

$$f : \mathbb{R} \to \mathfrak{g} \ ,$$

where $\mathfrak{g}$ is a fixed semisimple Lie algebra. Clearly, $\mathfrak{g}[x]$ is Lie algebra too if we define the Lie bracket of two functions $f, g \in \mathfrak{g}[x]$ pointwise, that is, we put

Gerdjikov, V.S. et al.: *Poisson–Nijenhuis Structures Related to the Generalized Zakharov–Shabat System.* Lect. Notes Phys. **748**, 515–545 (2008)
DOI 10.1007/978-3-540-77054-1_15

$$[f,g](x) = [f(x), g(x)]; \quad x \in \mathbb{R}\ .$$

Admitting some lack of rigor, we shall identify $\mathfrak{g}[x]$ with $\mathfrak{g}[x]^*$ using the bilinear form

$$\langle\langle X, Y \rangle\rangle = \int_{-\infty}^{+\infty} \langle X(x), Y(x)\rangle dx; \quad X, Y \in \mathfrak{g}[x]\ , \tag{15.1}$$

where $\langle \xi, \eta \rangle = \mathrm{tr}\,(\mathrm{ad}_{\,\xi} \circ \mathrm{ad}_{\,\eta})$; $\xi, \eta \in \mathfrak{g}$ is the Killing form of the algebra $\mathfrak{g}$.

In addition to the above identification, below we shall denote by the same symbol the 2*-forms* ω *and the corresponding fields* $\bar{\omega}$ *of linear maps. Thus the two-forms and the Poisson tensor fields will appear as fields of operators.*

Now, taking into account the above conventions and identifications, we remark that, as is well known for the equations that can be solved with the help of auxiliary linear problem (10.3), we have the following compatible Poisson tensors,[1] for example [2]:

$$\begin{aligned} Q^0_q(\xi) &= -\mathrm{ad}_{\,\xi} J; \quad q(x), \xi(x) \in \mathfrak{g}[x] \\ P^0_q(\xi) &= -\mathrm{ad}_{\,\xi} q + i\partial_x \xi; \quad q(x), \xi(x) \in \mathfrak{g}[x]\ , \end{aligned} \tag{15.2}$$

where i is the imaginary unit.

To see that these tensors, which are defined on the manifold $\mathcal{M} = \mathfrak{g}[x]^*$, are tensors of the type we want, we note that since $\mathcal{M}$ is linear space the tangent space at each point coincides with $\mathfrak{g}[x]^*$ and the cotangent space with $\mathfrak{g}[x]^{**}$. Due to the convention we have made to identify vectors and covectors through (15.1), we can assume that both these spaces coincide with $\mathfrak{g}[x]$. Then in (15.2), $q \in \mathfrak{g}[x]^*$, $\xi \in T_q(\mathcal{M}) \sim \mathfrak{g}[x]$. It is easy to notice that the tensor Q^0 is not kernel free, and therefore we cannot find $(Q^0)^{-1}$ and carry out the construction of P–N manifold as in theorem (14.7). Fortunately, as it was already mentioned, it is possible to restrict the tensors (15.2) onto some integral leaf of the distribution $\mathcal{I}\mathrm{m}\,(Q^0)$. To perform it, we shall use the corollary (12.26) applying the construction of the Restriction Theorem (12.25) to the tensor Q^0. According to proposition (12.26), we must consider the distribution $\mathcal{J}$:

$$q \to \mathcal{I}\mathrm{m}\,(Q^0)_q = \mathcal{I}\mathrm{m}\,(\mathrm{ad}_{\,J})\ . \tag{15.3}$$

On the integral leaves of $\mathcal{J}$ the Poisson tensor Q^0 allows restriction Q which is nondegenerate. The elements of $\mathcal{I}\mathrm{m}\,(\mathrm{ad}_{\,J})$ are the functions belonging to $\mathfrak{g}[x]$ taking values in the orthogonal complement $\overline{\mathfrak{g}}$ of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ with respect to the Killing form (recall that J is regular). We shall denote the subspace of these elements by $\overline{\mathfrak{g}}[x]$. Then the integral leaves of the distribution (15.3) are the following submanifolds:

---

[1] Later, we shall obtain this fact as a consequence from some general algebraic construction.

$$\overline{\mathcal{M}}_c = \{q : q = c + \xi;\ c - \text{fixed},\ c \in \mathfrak{g}[x],\ \xi \in \overline{\mathfrak{g}}[x]\} . \qquad (15.4)$$

Let us choose the following leaf

$$\overline{\mathcal{M}}_c|_{c=0} = \overline{\mathcal{M}}_0 = \overline{\mathfrak{g}}[x] . \qquad (15.5)$$

and let $j : \overline{\mathcal{M}}_0 \mapsto \mathfrak{g}[x]$ be the inclusion map. Clearly $T_q(\overline{\mathcal{M}}_0) = \overline{\mathfrak{g}}[x]$ and having in mind the pairing (15.1), we can also assume that $T_q^*(\overline{\mathcal{M}}_0) = \overline{\mathfrak{g}}[x]$. If $\alpha \in T_q^*(\mathcal{M})$ then $\pi_0\alpha = \alpha$, where π_0 is the orthogonal projector (with respect to the Killing form) onto the space $\overline{\mathfrak{g}}$. Due to the identifications we have made $dj = \pi_0$ and $[dj]^* = \pi_0$. All this means that we can write $[dj]^*\alpha = \alpha$, and then the construction of proposition (12.26) means that

$$Q(\alpha) = Q^0([dj]^*\alpha) = Q^0(\alpha) . \qquad (15.6)$$

Thus $Q = \mathrm{ad}_J$. Note that now the tensor Q is nondegenerate, since on the space $\overline{\mathfrak{g}}$ there exists the inverse of ad_J – the operator ad_J^{-1}.

To restrict P^0, we again invoke the Restriction Theorem (12.25). We have seen that in order to perform the restriction on the submanifold $\overline{\mathcal{M}}_0$ the following conditions must hold:

$$\chi^*_{P^0}(\overline{\mathcal{M}}_0)_q + T^{\perp}(\overline{\mathcal{M}}_0)_q = T_q^*(\mathcal{M}); \quad q \in \overline{\mathcal{M}}_0 \qquad (15.7)$$

$$\chi^*_{P^0}(\overline{\mathcal{M}}_0)_q \cap T^{\perp}(\overline{\mathcal{M}}_0)_q \subset \ker(P_q), \quad q \in \overline{\mathcal{M}}_0 . \qquad (15.8)$$

A simple calculation shows that

$$T^{\perp}(\overline{\mathcal{M}}_0)_q = \{\alpha :\ \alpha \in T_q^*(\mathcal{M}),\ \langle\langle \alpha, \xi \rangle\rangle = 0,\ \xi \in \overline{\mathfrak{g}}[x]\} . \qquad (15.9)$$

In other words, $T^{\perp}(\overline{\mathcal{M}}_0)_q$ consists of functions taking values in $\mathfrak{h}$, and it is natural to denote the space of these functions by $\mathfrak{h}[x]$.

From the other hand

$$\chi^*_{P^0}(\overline{\mathcal{M}}_0)_q = \{\alpha :\ \alpha \in T_q^*(\mathcal{M}),\ i\partial_x\alpha + [q, \alpha] \in \overline{\mathfrak{g}}[x]\} . \qquad (15.10)$$

Therefore, $\alpha \in \chi^*_{P^0}(\overline{\mathcal{M}}_0)_q$ exactly when

$$(\mathbf{1} - \pi_0)(i\partial_x\alpha + [q, \alpha]) = 0 . \qquad (15.11)$$

If $\alpha \in \chi^*_{P^0}(\overline{\mathcal{M}}_0)_q \cap T^{\perp}(\overline{\mathcal{M}}_0)_q$, then $[\mathfrak{h}, \overline{\mathfrak{g}}] \subset \overline{\mathfrak{g}}$ entails that $\partial_x\alpha = 0$. Since $\lim_{x\to\pm\infty} \alpha(x) = 0$ we get $\alpha = 0$. Thus, we have proved that

$$T^{\perp}(\overline{\mathcal{M}}_0)_q \cap \chi^*_{P^0}(\overline{\mathcal{M}}_0)_q = \{0\} \qquad (15.12)$$

and the requirement (15.8) of the Restriction Theorem is fulfilled.

In order to prove that (15.7) is also true, let us remark that the condition (15.11) can be cast into the form:

$$(\mathbf{1} - \pi_0)\alpha = i(\mathbf{1} - \pi_0) \int_{-\infty}^{x} [q(y), \alpha(y)]dy + A(\alpha, q) \ , \tag{15.13}$$

where $A(\alpha, q)$ is some constant in x which in general can depend on α and the potential q. Since $\lim_{x\to\pm\infty} \alpha(x) = 0$, we must have

$$A(q, \alpha) = i(\mathbf{1} - \pi_0) \int_{-\infty}^{+\infty} [q(y), \alpha(y)]dy = 0 \ . \tag{15.14}$$

or equivalently:

$$\langle\langle [H, q], \alpha \rangle\rangle = 0; \quad H \in \mathfrak{h} \ . \tag{15.15}$$

The above relation imposes implicit restrictions on the cotangent vectors on $\overline{\mathcal{M}}_0$. Actually, if we choose a basis $\{H_i\}_1^r$ in $\mathfrak{h}$, then (15.15) is equivalent to r equations $\langle\langle [H_i, q], \alpha \rangle\rangle = 0$; $i = 1, 2 \ldots, r$. If we want to continue with the geometric constructions, we are forced to assume that the (15.15) is fulfilled.[2]

In order to stress that for our potentials we need to have the relations (15.15) in what follows we shall denote the manifold of potentials by $\mathcal{M}_0$.

One can prove that $\mathcal{M}_0$ is dense in $\overline{\mathcal{M}}_0$, but we shall not go into this matter. We simply want to prove that (15.7), (15.8) are true for the manifold $\mathcal{M}_0$ instead of $\overline{\mathcal{M}}_0$.

Let us consider now the requirement (15.7). For arbitrary $\alpha \in T_q^*(\mathcal{M}_0)$ we put

$$\gamma(\alpha) = -i(\mathbf{1} - \pi_0) \int_{-\infty}^{x} [q(y), \pi_0(\alpha)(y)]dy \ . \tag{15.16}$$

It is not difficult to notice that

$$\pi_0(\alpha) - \gamma(\alpha) \in \chi^*_{P^0}(\mathcal{M}_0)_q.$$

For that reason the identity

$$\alpha = (\pi_0(\alpha) - \gamma(\alpha)) + ((\mathbf{1} - \pi_0)\alpha + \gamma(\alpha)) \tag{15.17}$$

shows that

$$\chi^*_{P^0}(\mathcal{M}_0)_q \oplus T^{\perp}(\mathcal{M}_0)_q = T_q^*(\mathcal{M}) \ . \tag{15.18}$$

[2] Restrictions of this kind arise frequently in the theory of soliton equations and are important in order to define correctly the symplectic (or Poisson) structures for these equations; see for example [3, 4].

Thus, the conditions of the Restriction Theorem hold and P^0 allows restriction on $\mathcal{M}_0$. Let us denote this restriction by P. Now we are going to calculate it. Let $\alpha \in T_q^*(\mathcal{M}_0)$. As before $[dj]^*\alpha = \alpha$, and according to the restriction procedure we must put

$$P(\alpha) = P^0(\alpha - \gamma(\alpha))$$

It is easy to calculate that

$$P(\alpha) = i\frac{\partial}{\partial x}\alpha + \pi_0([q,\alpha]) + [q, i(\mathbf{1} - \pi_0) \int\limits_{-\infty}^{x} [q,\alpha](y)dy\ ]$$

$$\pi_0(\alpha) = \alpha \in T_q^*(\mathcal{M}_0)\ . \tag{15.19}$$

Now it is possible to obtain the Nijenhuis tensor $N = P \circ Q^{-1} = PQ^{-1}$:

$$N = \left[i\partial_x + \pi_0 \mathrm{ad}_{\,q} + i\mathrm{ad}_{\,q}(\mathbf{1} - \pi_0) \int\limits_{-\infty}^{x} \mathrm{ad}_{\,q}\ .\ dy \right] (\mathrm{ad}_{\,J})^{-1}\ . \tag{15.20}$$

We would like to note that as a consequence from the condition (15.15) in the lower bound of the above integral we could write $+\infty$ instead of $-\infty$ without changing the value of the expressions for P and N.

Now the tensor fields P and N endow $\mathcal{M}_0$ with a P–N structure. The adjoint of N is easily found:

$$N^* = (PQ^{-1})^* = (Q^*)^{-1}P^* = Q^{-1}P = Q^{-1}(PQ^{-1})Q = Q^{-1}NQ.$$

Now, after all this geometric theory, the comparison shows that N^* is exactly equal to the generating operators $\Lambda_\pm$ we had in the Introduction (compare with (10.20)) for the GZS system. (According to the additional assumptions on the covectors these operators act identically.) We can also write

$$N^* = \frac{1}{2}(\Lambda_+ + \Lambda_-) = \Lambda\ . \tag{15.21}$$

The geometric theory, however, is incomplete without the possibility to calculate the fundamental fields of the P–N structure. Later, see proposition (15.25) we shall show that the vector fields:

$$X_H : q \to X_H(q) = [H,q]; \quad H \in \mathfrak{h} \tag{15.22}$$

are fundamental fields. The corresponding fundamental forms are

$$\alpha_H : q \to \alpha_H(q) = \mathrm{ad}_{\,J}^{-1}[H,q]\ . \tag{15.23}$$

(Recall that from the results of part I it follows that the forms $\mathrm{ad}_J^{-1}[H,q]$ and $\Lambda_\pm \mathrm{ad}_J^{-1}[H,q]$ are closed.) In addition, from the relation

$$[X_{H_1}, X_{H_2}](q) = X_{[H_1,H_2]}(q) \ , \tag{15.24}$$

it follows that if $H_1, H_2 \in \mathfrak{h}$, then the Lie bracket of the fields X_{H_1} and X_{H_1} is zero, or equivalently, that the forms α_{H_1} and α_{H_2} are in involution. Then from corollaries 14.17, 14.18, we obtain the following:

Proposition 15.1. *Let all the quantities be as defined in the above. Then*

- *The vector fields $N^n X_H$; $n = 0, 1, \ldots$; $H \in \mathfrak{h}$ commute.*
- *The equations*

$$i\, ad_J^{-1} q_t + \Lambda^n ad_J^{-1}[H,q] = 0$$

  are Hamiltonian with respect to the hierarchy of symplectic structures: $\Omega^n = \Lambda^n (Q^0)^{-1} = \Lambda^n ad_J^{-1}$.

This proposition gives geometric interpretation of the results obtained using expansions over adjoint solutions and shows how the spectral, and geometric methods can be used to help and to clarify each other.

15.2 Poisson–Nijenhuis Structures on Lie Groups and Algebras

In the definitions of both Q^0 and P^0, a crucial role was played by some Lie algebra structure (the potentials are functions with values in some Lie algebra $\mathfrak{g}$). This circumstance is not accidental, and we shall see there exists a canonical construction allowing to endow the dual of a given Lie algebra with a P–N structure. It turns out to be related to a P–N structure on the corresponding Lie group. The interrelation is established with the help of the so-called momentum map, a classical object from the theory of the Lie groups acting by symplectomorphisms (or as also said canonically) on a symplectic manifold. Based on our knowledge, these ideas are suggested for the first time in [1], though as we shall see, most of them appear with some modifications in algebraical approaches to the soliton equations. At the outset, we need some additional facts and definitions.

15.2.1 The Momentum Map

Let $\mathcal{M}$ be manifold and let G be connected Lie group, acting from the left on $\mathcal{M}$. This means that there exists smooth map $G \times \mathcal{M} \to \mathcal{M}$,

$$(g, m) \to l_g(m) = l_g m \tag{15.25}$$

with the following properties:

$$\begin{aligned} l_{gh} &= l_g \circ l_h = l_g l_h \; ; \quad g, h \in G \\ l_e &= id_{\mathcal{M}} \; , \end{aligned} \tag{15.26}$$

where e is the unit element of G.

Remark 15.2. Right action $(g, m) \to r_g(m) = r_g m$ of a group on a manifold can be defined in a similar way, but instead of the relations (15.26), we require

$$\begin{aligned} r_{gh} &= r_h \circ r_g = r_h r_g; \quad g, h \in G \\ r_e &= id_{\mathcal{M}} \; . \end{aligned} \tag{15.27}$$

Usually only left or right action is considered, because if $(g, m) \mapsto r_g m$ is a right action then one can define a left action putting $l_g(m) = r_{g^{-1}} m = (r_g)^{-1} m$.

A tensor fields on $\mathcal{M}$ is called invariant with respect to the action of G, or simply G-invariant, if it is invariant under all the diffeomorphisms l_g; $g \in G$. For example, a differential form β is G-invariant, if $l_g^* \beta = \beta$, $g \in G$.

We say that G acts transitively if for any $m_1, m_2 \in \mathcal{M}$ there exists $g \in G$, such that $m_2 = l_g(m_1)$. We say that G acts simply transitively on $\mathcal{M}$, if the element g in the above relation is unique, provided m_1 and m_2 are given. If m_0 is a fixed point of $\mathcal{M}$, the group

$$G_0 = \{g : l_g(m_0) = m_0\} \subset G \tag{15.28}$$

is a closed subgroup of G (called the stability subgroup of m_0), and there is a bijection between $\mathcal{M}$ and G/G_0. As is known from the theory of the Lie groups (see for example [5]), G_0 is then a Lie group and G/G_0 is a smooth manifold. Then though there are cases when as manifolds $\mathcal{M}$ and G/G_0 are not diffeomorphic (even not homeomorphic) in general one can identify $\mathcal{M}$ with G/G_0.

Example 15.3. If G is a Lie group, then the left translations L_g define a left action of G on itself and the right translations R_g a right action of G on itself. Both actions are clearly simply transitive.

When G acts on two manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ we call a map $h : \mathcal{M}_1 \mapsto \mathcal{M}_2$ *equivariant* with respect to these actions if

$$\mathcal{L}_g \circ h = h \circ l_g; \quad g \in G \; . \tag{15.29}$$

(Here the action of G on $\mathcal{M}_1$ is denoted by l_g and on $\mathcal{M}_2$ by $\mathcal{L}_g$). As usual, when one has an action of a group G on a manifold $\mathcal{M}$ we call orbits of G the sets:

$$\mathcal{O}_{m_0} = \{m : \; m = l_g(m_0); \quad g \in G\} \subset \mathcal{M} \; . \tag{15.30}$$

where m_0 is a fixed point in $\mathcal{M}$ and $\mathcal{O}_{m_0}$ is called the orbit through m_0. When G is a Lie group, $\mathcal{M}$ is smooth and the action is also smooth, the orbits are manifolds.

The fundamental fields of the left action of the group G on $\mathcal{M}$ are then defined as follows:

$$X_\xi|_m = \frac{d}{dt} l_{\exp(-t\xi)} m|_{t=0} \ . \tag{15.31}$$

Here $\xi \in \mathfrak{g} = T_e(G)$-the Lie algebra of G and as usual for fixed ξ by $\exp(t\xi)$ is denoted the unique solution of the equation

$$\frac{d}{dt} g(t) = dL_{g(t)}(\xi); \quad g(0) = e \ , \tag{15.32}$$

where L_g is the left multiplication map (left translation): $L_g(h) = gh$. The fundamental fields are clearly tangent to the orbit of G through m.

Remark 15.4. The maps $t \mapsto \exp(t\xi)$; $\xi \in \mathfrak{g}$ are also called one-parametric subgroups of G.

It is easy to see that

$$\begin{aligned} X_{\xi+\eta} &= X_\xi + X_\eta; \quad \xi, \eta \in \mathfrak{g} \\ X_{c\xi} &= cX_\xi; \quad \xi \in \mathfrak{g},\ c \in \mathbb{R} \ . \end{aligned} \tag{15.33}$$

The fundamental fields have the following important properties:

- The vectors $X_\xi|_m$ span the subspace $\mathcal{I}\mathrm{m}\,(dl_g|_m)$.
- The set of fundamental fields is invariant with respect to the diffeomorphisms l_g. More precisely,

$$dl_g(X_\xi)|_{l_g^{-1}m} = X_{\mathrm{Ad}\,(g)\xi}|_m \ , \tag{15.34}$$

  where by $\mathrm{Ad}\,(g)\xi$ is denoted the adjoint action of the group G on its algebra:

$$\mathrm{Ad}\,(g)\xi = \frac{d}{dt}(g \exp(t\xi) g^{-1})|_{t=0}; \quad \xi \in \mathfrak{g} \ . \tag{15.35}$$

Now, let $(\mathcal{M}, \omega)$ be symplectic manifold with symplectic form ω, and let the Lie group G act from the left in such a way that all l_g are symplectic maps, that is

$$l_g^* \omega|_m = \omega|_{l_g m}(dl_g., dl_g.) = \omega|_m; \quad g \in G \ . \tag{15.36}$$

and since they are diffeomorphisms they are symplectomorphisms.

Remark 15.5. It can be readily seen that the construction of the momentum map (see below) can be extended also to the case when ω is not a symplectic form but simply a closed 2-form.

From the definition of the fundamental fields, one can see that we must have

$$L_{X_\xi} \omega = 0; \quad \xi \in \mathfrak{g} \ . \tag{15.37}$$

Then from (15.33) it follows that the equation

$$\langle \theta(.), \xi \rangle = -i_{X_\xi}\omega = \omega(X_\xi, .)\ ; \quad \xi \in \mathfrak{g} \tag{15.38}$$

has unique solution for θ. In (15.38), θ is a $\mathfrak{g}^*$-valued 1-form[3] on $\mathcal{M}$, or in other words, a field of linear maps

$$m \to \theta_m : T_m(\mathcal{M}) \to \mathfrak{g}^*$$

and $\langle\ ,\ \rangle$ is the canonical pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. It can be shown, see [6, 7], that the form θ has the following properties:

$$\begin{aligned} &l_g^*\theta|_m = \theta|_{l_g m}(dl_g.) = \mathrm{Ad}^*(g^{-1})\theta(.)|_m\ ; \quad m \in \mathcal{M},\ g \in G \\ &d\theta = 0\ , \end{aligned} \tag{15.39}$$

where $g \to \mathrm{Ad}^*(g^{-1})$ is the coadjoint action of the group G on the vector space $\mathfrak{g}^*$. By virtue of the Poincaré lemma, at least locally, there exists function a $\Phi_\omega : \mathcal{M} \to \mathfrak{g}$, such that

$$d\Phi_\omega = \theta\ . \tag{15.40}$$

In the special case when G acts transitively on $\mathcal{M}$, that is, when for any two points $m_1, m_2 \in \mathcal{M}$ there exists $g \in G$ such that $m_2 = l_g m_1$, the locally defined functions Φ_ω can be extended to a function on $\mathcal{M}$ [6, 7]. Then Φ_ω is called the momentum map of G (or more accurately, the momentum map, corresponding to the action of G). It can be proved that Φ_ω has the property

$$l_g^*\Phi_\omega(m) = \Phi_\omega(l_g m) = \mathrm{Ad}^*(g^{-1})\Phi_\omega + C(g); \quad g \in G\ , \tag{15.41}$$

where the function $C : G \to \mathfrak{g}$ does not depend on $m \in \mathcal{M}$. It is not hard to establish that for the functions $C(g)$ holds the following relation (called cocycle relation)

$$C(gh) = \mathrm{Ad}^*(g^{-1})C(h) + C(g); \quad g, h \in G\ . \tag{15.42}$$

Functions of that kind are called $\mathrm{Ad}^*(g^{-1})$-1-cocycles of G, see [6]. The above property of $C(g)$ allows to define another left action of G on $\mathfrak{g}^*$:

$$\mathcal{L}_g\mu = \mathrm{Ad}^*(g^{-1})\mu + C(g); \quad \mu \in \mathfrak{g}\ . \tag{15.43}$$

This action makes the momentum map equivariant, or in other words, for each $g \in G$ the following diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{l_g} & \mathcal{M} \\ \Phi_\omega \downarrow & & \downarrow \Phi_\omega \\ \mathfrak{g}^* & \xrightarrow[\mathcal{L}_g]{} & \mathfrak{g}^* \end{array} \tag{15.44}$$

is commutative.

[3] Instead of $\mathfrak{g}^*$-valued 1-form fixing a basis in $\mathfrak{g}^*$, one can speak about $n = \dim(\mathfrak{g}^*)$ "usual" 1-forms.

The momentum map plays an important role in the Classical Mechanics, it has permitted a new viewpoint of several classical topics [8] and is also a part of various theoretical constructions [9, 10, 11].

The main property of the momentum map is that it establishes one-to-one correspondence between manifolds on which given Lie group G acts transitively via symplectomorphisms from one side and the orbits of the coadjoint action of G from the other side. The corresponding theory is sometimes referred to as Kirillov–Kostant–Souriau theory. We remind that the coadjoint orbits (orbits in the coadjoint representation of G) are the manifolds:

$$\mathcal{O}_\mu = \{\eta = \mathrm{Ad}\,(g)^* \mu;\ g \in G\}\ . \tag{15.45}$$

On the coadjoint orbits there is canonical symplectic structure – the structure defined by the restriction of the Kirillov tensor and the momentum map is a symplectic map; for more details see [6, 10, 11].

Provided the manifold $\mathcal{M}$ is connected, the momentum map Φ_ω is defined up to an additive constant μ_0. If we add the constant μ_0 to Φ_ω, then in the expression for $C(g)$ appears an additional term, equal to

$$\mathrm{Ad}^*(g^{-1})\mu_0 - \mu_0 = B_{\mu_0}(g)\ . \tag{15.46}$$

For arbitrary choice of μ_0, the function $B_{\mu_0}(g)$ satisfies the cocycle relations (15.42), and thus we obtain 1-cocycles, called trivial 1-cocycles or coboundaries of the coadjoint action.

15.2.2 Momentum Maps on Lie Groups

Let us consider the case when $\mathcal{M}$ coincides with G and G acts on itself by left translations:

$$L_g h = gh; \quad g, h \in G\ . \tag{15.47}$$

Remark 15.6. The objects (functions, forms) that are invariant under all the left translations are called left-invariant. Usually only the left translations are considered. The reason is that left translations action is transformed into right translations action (and vice versa) using the inversion map $I_G : g \to g^{-1}$. Indeed, one has

$$I_G \circ L_g = R_g \circ I_G; \quad g \in G,$$

where $R_g(h) = hg$ is the right translation. In addition, if for example the function $f : G \to \mathbb{R}$ is left-invariant ($L_g^* f = f,\ g \in G$) then $I_G^* f$ is right-invariant. The same is true for tensors of arbitrary type.

In the special case $\mathcal{M} = G$ and left translations action, the 2-form ω is invariant if it is left-invariant:

$$L_g^* \omega|_h = \omega|_{gh}(dL_g.,dL_g.) = \omega|_h(.,.)\ . \tag{15.48}$$

It follows that

$$\omega_h = \omega|_h(.,.) = \omega|_e(dL_{h^{-1}}.,dL_{h^{-1}}.) \; . \tag{15.49}$$

Thus, like the left-invariant vector field, see below how a left-invariant form can be reconstructed using only its value at the point e-the unit element of the group. If we adopt a different viewpoint and consider ω as a field of linear maps

$$g \to \omega_g : T_g(G) \to T_g^*(G)$$

then the above formula can be written into equivalent form

$$\omega_h = (dL_h^{-1})^* \circ \omega_e \circ dL_h^{-1} \; . \tag{15.50}$$

The fundamental fields of the left-translations action are the right-invariant vector fields:

$$\frac{d}{dt}(\exp(-t\xi)h)|_{t=0} = -dR_h\xi = -\xi^r|_h \; , \tag{15.51}$$

where $\xi \in \mathfrak{g}$ and, as above, R_h means right multiplication by h. The vector field ξ^r is called right-invariant vector field corresponding to $\xi \in \mathfrak{g}$. Quite in the same way one can define left-invariant vector fields:

$$\frac{d}{dt}(h\exp(-t\xi))|_{t=0} = -dL_h\xi = -\xi^l|_h; \quad \xi \in \mathfrak{g} \; . \tag{15.52}$$

It is well known that for the left (right) invariant vector fields on Lie group associated with $\xi,\eta \in \mathfrak{g}$ hold the relations:

$$[\xi^l,\eta^l] = ([\xi,\eta])^l; \quad \xi,\eta \in \mathfrak{g} \tag{15.53}$$

$$[\xi^r,\eta^r] = -([\xi,\eta])^r; \quad \xi,\eta \in \mathfrak{g} \; . \tag{15.54}$$

Both left- and right-invariant vector fields at the point $g \in G$ span the tangent space $T_g(G)$ at g. For $\alpha \in \mathfrak{g}^*$, one also defines right-invariant and left-invariant 1-forms on G:

$$\alpha_g^r = \alpha^r|_g = dR_{g^{-1}}^*\alpha, \quad \alpha_g^l = \alpha^l|_g = dL_{g^{-1}}^*\alpha \; . \tag{15.55}$$

Between invariant fields and forms exist the following classical relations

$$\begin{aligned} L_{\xi^l}(\alpha^l) &= -(\mathrm{ad}_\xi^*\alpha)^l = -i_{\xi^l}(d\alpha^l) \\ L_{\xi^r}(\alpha^r) &= (\mathrm{ad}_\xi^*\alpha)^r = -i_{\xi^r}(d\alpha^r) \; . \end{aligned} \tag{15.56}$$

They are often called the *Maurer-Cartan identities*. It is also clear that for $\alpha \in \mathfrak{g}^*, \ \xi \in \mathfrak{g}$, we have

$$L_{\xi^r}(\alpha^l) = L_{\xi^l}(\alpha^r) = 0 \; . \tag{15.57}$$

As in the case with the left-invariant (right-invariant) vector fields the left-invariant (right-invariant) 1-forms at the point g span the cotangent space $T_g^*(G)$. In general, if ω is left (right) invariant form and ξ is right (left) invariant vector field we have $L_\xi\omega = 0$.

Let us return again to the momentum map, defined on a group G. In this case the map Φ_ω, introduced in (15.40) is itself a cocycle of the coadjoint action if we normalize it by the condition $\Phi_\omega(e) = 0$. This means that

$$\Phi_\omega(gh) = \mathrm{Ad}^*(g^{-1})\Phi_\omega(h) + \Phi_\omega(g) \; . \tag{15.58}$$

As could be expected, the group cocycles possesses algebraic analogs – algebraic cocycles. Let us remind some definitions; see for example [12]. Let $\mathfrak{g}$ be Lie algebra and (V, f) be some finite dimensional representation of $\mathfrak{g}$, that is, V is finite dimensional vector space and f is a linear map:

$$f \;:\; \mathfrak{g} \to \mathrm{Hom}\,(V, V) \; , \tag{15.59}$$

such that for arbitrary $X, Y \in \mathfrak{g}$

$$[f(X), f(Y)] = f([X, Y]) \; . \tag{15.60}$$

We sometimes say that f is a (linear) action of the algebra $\mathfrak{g}$ on V, since one can "integrate" the representation of the algebra in order to obtain representation of the corresponding group G, and then we shall have left action of G acting on V through linear maps.

Example 15.7. Adjoint representation (adjoint action): If $\mathfrak{g}$ is Lie algebra, then for $X \in \mathfrak{g}$ we define the linear map $\mathrm{ad}_X \in \mathrm{Hom}\,(\mathfrak{g}, \mathfrak{g})$:

$$\mathrm{ad}_X(Y) = [X, Y] \; . \tag{15.61}$$

One can check that $(\mathfrak{g}, \mathrm{ad})$ is a representation of $\mathfrak{g}$ called the adjoint representation of $\mathfrak{g}$.

Example 15.8. Coadjoint representation(coadjoint action): If $\mathfrak{g}$ is a Lie algebra, $\mathfrak{g}^*$ its dual space, then for $X \in \mathfrak{g}$ we define the linear map

$$X \to -\mathrm{ad}^*_X \in \mathrm{Hom}\,(\mathfrak{g}^*, \mathfrak{g}^*)$$

Example 15.9. Trivial representation: $f = 0$, V – some vector space.

Let $A^p(\mathfrak{g})$ be the set of all skew-symmetric p-linear maps:

$$\alpha : \underbrace{\mathfrak{g} \times \mathfrak{g} \times \ldots \times \mathfrak{g}}_{p \text{ times}} \to V.$$

By definition we put $A^0(\mathfrak{g}) = V$. The sets $A^p(\mathfrak{g})$; $p = 0, 1, \ldots \dim(\mathfrak{g})$ are clearly vector spaces, their elements are called cochains. One can define now a linear map

$$d \;:\; A^p(\mathfrak{g}) \to A^{p+1}(\mathfrak{g}) \tag{15.62}$$

through the formula

$$d\alpha(X_1, X_2, \dots, X_{p+1}) = \\ \sum_{i=1}^{n} (-1)^{i-1} f(X_i)(\alpha(X_1, X_2, \dots, \hat{X}_i, X_{i+1} \dots, X_{p+1})) + \\ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_1, X_2, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{p+1}) \ , \quad (15.63)$$

where $X_1, X_2, \dots, X_{p+1} \in \mathfrak{g}$ and $\alpha \in A^p(\mathfrak{g})$. The hat over a symbol means that in the sequence $X_1, X_2 \dots X_{p+1}$ this symbol is omitted. For $\alpha \in A^0(\mathfrak{g}) = V$ we put $d\alpha(X) = f(X)(\alpha) = f(X)\alpha$.

Then one can prove that $d^2 = 0$, and this allows to give the usual definitions of cocycles, coboundaries, and cohomologies of an algebra associated with the representation (V, f); see [12]:

- $\alpha \in A^p(\mathfrak{g})$ is called p-cocycle if $d\alpha = 0$. The set of all p-cocycles associated with the representation (f, V) is denoted by $Z^p(\mathfrak{g})_f$.
- $\alpha \in A^p(\mathfrak{g})$ is called coboundary if $\alpha = d\beta$, where $\beta \in A^{p-1}(\mathfrak{g})$. The set of all p-coboundaries associated with the representation (f, V) is denoted by $B^p(\mathfrak{g})_f$.
- The quotient space $H^p(\mathfrak{g})_f = Z^p(\mathfrak{g})_f / B^p(\mathfrak{g})_f$ is called the p-th cohomology space associated with the representation (f, V) of $\mathfrak{g}$.

It can be proved that all the cohomology spaces of the semisimple Lie algebras are trivial if the corresponding representation (V, f) is not trivial: $f \neq 0$; see [12]. Also, for arbitrary representation of a semisimple Lie algebra $\mathfrak{g}$, the first two cohomology spaces $H^1(\mathfrak{g})_f = 0$, $H^2(\mathfrak{g})_f = 0$ (these results are known as the Whitehead lemmas).

In order to demonstrate the relations between objects associated with the Lie groups and those for the corresponding Lie algebras, let us remark that for left-invariant 2-form ω^l the condition

$$d\omega^l(\xi_1^l, \xi_3^l, \xi_3^l) = 0; \quad \xi_1, \xi_2, \xi_3 \in \mathfrak{g},$$

is equivalent to

$$\omega_e([\xi_1, \xi_2], \xi_3) + \text{cycl}\,(1, 2, 3) = 0 \ . \tag{15.64}$$

But then $(\xi, \eta) \to \omega_e(\xi, \eta)$ is 2-cocycle of the trivial action of the algebra $\mathfrak{g}$ on $\mathbb{R}$. Further, one can see that to the exact 2-forms correspond coboundaries, that is left-invariant 2-forms ω^α which at e have the form:

$$\omega_e^\alpha(\xi, \eta) = \langle \alpha, [\xi, \eta] \rangle; \quad \alpha \in \mathfrak{g}^* \ . \tag{15.65}$$

It is not difficult to prove that

$$\omega^\alpha = d\alpha^l \ , \tag{15.66}$$

where α^l is the left-invariant 1-form corresponding to the covector $\alpha \in \mathfrak{g}^*$. Finally, from the definitions of the maps Φ_ω and $\mathcal{L}_g$ one can obtain that

$$\frac{d}{dt}\mathcal{L}_{(\exp t\xi)}q|_{t=0} = -(\mathrm{ad}^*_\xi q + \omega_e(\xi)); \quad \xi \in \mathfrak{g}; \quad q \in \mathfrak{g}^* \,. \tag{15.67}$$

According to the general constructions in the theory of momentum maps the right-hand side of this equation defines a Poisson tensor over $\mathfrak{g}^*$. (Since $\mathfrak{g}^{**} = \mathfrak{g}$, the elements $\xi \in \mathfrak{g}$ are treated as covectors over $\mathfrak{g}^*$.) Thus on the coalgebra $\mathfrak{g}^*$, we obtain natural Poisson structure generated by the symplectic structure on G. The above fact can be verified immediately, but we shall give another and more elegant proof in the next section.

The momentum map plays an important role in the theory of P–N manifolds on groups. The point is that there is a natural way to introduce P–N structure on Lie group and the momentum map establishes the relation of this structure with a similar structure defined on the dual space of the corresponding algebra; see [13]. We present below some of these results.

Theorem 15.10. *Let G be Lie group, let ω be left-invariant symplectic form and let Ω be a closed, right-invariant 2-form on G. Then*

1. *The group G can be endowed by a structure of P–N manifold given by the tensor fields*

$$\mathbf{P} = \omega^{-1}, \quad \mathbf{N} = \mathbf{P} \circ \Omega \,. \tag{15.68}$$

2. *Let us consider on the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G the following tensor fields*

$$\begin{aligned} q &\to P_q = -(ad^*_\xi q + \omega_e(\xi)) \\ q &\to Q_q = -\Omega_e \,. \end{aligned} \tag{15.69}$$

Then P, Q are compatible Poisson tensors on $\mathfrak{g}^*$, and the following relations hold:

$$\begin{aligned} [d\Phi_\omega] \circ \mathbf{P} \circ [d\Phi_\omega]^* &= P \\ [d\Phi_\omega] \circ \mathbf{N} \circ \mathbf{P} \circ [d\Phi_\omega]^* &= Q \end{aligned} \tag{15.70}$$

(In other words, $\mathbf{P}$ and $\mathbf{N} \circ \mathbf{P}$ are Φ_ω-related to P and Q).[4]

We shall not directly use this result, but the reader will see that the construction we shall use in the section dedicated to the manifold of the Jost solutions for the generalized Zakharov–Shabat system is inspired by the above theorem.

[4] Recall that we use the same letter for a 2-form β and the field of linear maps $\bar{\beta}$ that corresponds to it.

15.2.3 Algebraic Approaches. Gel'fand–Fuchs Cocycle

We show now how the Poisson structure, introduced earlier (see (15.67)), is obtained in some other the approaches to the soliton equations. For the sake of brevity and since crucial role in them play algebraic constructions we call them algebraic approaches; see [2, 14, 15, 16, 17] (the list of the references can be easily extended.) In the algebraic approaches, the Poisson brackets that are introduced are the Lie–Poisson brackets on the dual of some Lie algebra, for example some loop algebra[5] (algebra of formal power series on λ with coefficients in some finite dimensional Lie algebra). For the convenience of the reader, we remind that the Lie–Poisson bracket is defined on the dual $\mathfrak{g}^*$ of some Lie algebra $\mathfrak{g}$ according to the formula:

$$\{f,g\}(q) = -\langle q, [df|_q, dg|_q]\rangle; \quad q \in \mathfrak{g}^* \ , \tag{15.71}$$

where f, g are smooth functions on $\mathfrak{g}^*$. Their differentials at the point q – $df|_q, dg|_q$ are linear functions on $\mathfrak{g}^*$ and are considered elements of $\mathfrak{g}$. Geometrically, this means that the Lie-algebra structure allows to obtain the Poisson tensor on $\mathfrak{g}^*$ (the Kirillov tensor):

$$\begin{aligned} &q \to K_q : \quad K_q(\xi) = -\mathrm{ad}\,^*_\xi \\ &q \in \mathfrak{g}^*; \quad \xi \in T^*_q(\mathfrak{g}^*) = (\mathfrak{g}^*)^* \sim \mathfrak{g}\ . \end{aligned} \tag{15.72}$$

In the algebraic approaches, usually the main construction is some variant of the so-called Adler scheme [4, 14] (see also [18] Chap. 4), where there is a large bibliography. We shall not present this scheme here, but we must note that one of the most essential steps in the construction is the so-called extension with Abelian kernel; see [12]. This extension actually introduces the dependance on the additional (spacial) variable x, and without it the scheme can be applied only to the ordinary differential equations. Let us briefly describe what is meant by extension with Abelian kernel.

Let (V, f) be representation of the algebra $\mathfrak{g}$, that is, V is vector space and f is homomorphism $f : \mathfrak{g} \to \mathrm{End}\,(V)$. Extension of $\mathfrak{g}$ with Abelian kernel V is a pair $(\mathcal{E}, F)$, where $\mathcal{E}$ is Lie algebra and F is subjective homomorphism

$$F : \mathcal{E} \to \mathfrak{g},$$

such that $F^{-1}(\{0\}) = V$ and the representation (V, f) is reconstructed using the sections of F. (Sections of F are called linear maps $r : \mathfrak{g} \to \mathcal{E}$, such that $F \circ r = id_{\mathfrak{g}}$). In other words, for every section $\xi \to r(\xi)$

$$[r(\xi), v] = f(\xi)v; \quad \xi \in \mathfrak{g}; \quad v \in V\ . \tag{15.73}$$

[5] Loop algebras are called also affine algebras.

Basic result in the theory of extensions with Abelian kernel is the following theorem [12]:

Theorem 15.11. *All the extensions with Abelian kernels can be obtained in the following way: Let us take the direct sum $\mathcal{E} = \mathfrak{g} \oplus V$ (as vector spaces) and let us define on it the structure of Lie algebra according to the formula:*

$$[(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], f(\xi_1)v_2 - f(\xi_2)v_1 + \omega(\xi_1, \xi_2)), \tag{15.74}$$

where $\xi_1, \xi_1 \in \mathfrak{g}$; $v_1, v_2 \in V$ and ω is f-cocycle of $\mathfrak{g}$, that is ω is linear skew-symmetric map $\mathfrak{g} \times \mathfrak{g} \to V$ satisfying the condition

$$f(\xi_1)\omega(\xi_2, \xi_3) + cycl(1,2,3) = \omega([\xi_1, \xi_2], \xi_3) + cycl(1,2,3) \tag{15.75}$$

for any $\xi_1, \xi_2, \xi_3 \in \mathfrak{g}$. Two extensions are equivalent, if the difference of the corresponding cocycles ω_1, ω_2 is coboundary, that is, if there exist linear map $\alpha : \mathfrak{g} \to V$, such that

$$\begin{aligned}(\omega_1 - \omega_2)(\xi_1, \xi_2) = d\alpha(\xi_1, \xi_2) = \\ f(\xi_1)\alpha(\xi_2) - f(\xi_2)\alpha(\xi_2) - \alpha([\xi_1, \xi_2]) \ .\end{aligned} \tag{15.76}$$

(For more details see [12]). To our knowledge, in the algebraic approaches until now, only extensions with $f = 0, V = \mathbb{R}, \mathbb{C}$ have been considered. Then the extension is in fact a central extension as every element from $\mathbb{C}$ (if the field of scalars is $\mathbb{C}$) commutes with all the other elements; see (15.74). In this case, it is easy to see that the (15.75) coincides with (15.64). Equation (15.76) shows that the difference $\omega_1 - \omega_2$ is an exact 2-form. We shall denote the central extension defined through ω by $\mathfrak{g}^\omega$. We have

$$\mathfrak{g}^\omega = \mathfrak{g} \oplus \mathbb{C} \tag{15.77}$$

(as vector spaces). As for the bracket, it is given by

$$[(\xi, a), (\eta, b)] = ([\xi, \eta], \omega(\xi, \eta)) \ . \tag{15.78}$$

If we introduce the pairing

$$\langle (q, a), (\eta, b) \rangle_{\mathfrak{g}^\omega} = \langle q, \eta \rangle_{\mathfrak{g}} + ab \tag{15.79}$$

then we can assume that $(\mathfrak{g}^\omega)^* = \mathfrak{g}^* \oplus \mathbb{C}$. The coadjoint action is calculated without difficulties:

$$-\operatorname{ad}^*_{(\xi,a)}(q, b) = -\operatorname{ad}^*_\xi(q) - b\omega(\xi) \ . \tag{15.80}$$

(Here ω must be understood as linear map $\omega : \mathfrak{g} \to \mathfrak{g}^*$ and $\xi \in \mathfrak{g}, q \in \mathfrak{g}^*$).

The Kirillov tensor on $\mathfrak{g}^\omega$ is degenerate, because $\mathrm{ad}^*_{(0,a)}(q,b) = 0$ and for that reason it is usually restricted on the submanifolds of the type

$$\mathcal{M}^c = \mathfrak{g}^* + c = (\mathfrak{g}^*, c) \subset \mathfrak{g}^* \oplus \mathbb{C} . \tag{15.81}$$

For example on $\mathcal{M}^1 = \mathcal{M}^{c=1}$, we have the Poisson tensor (or if one prefers the coadjoint action of the central extension of $\mathfrak{g}$):

$$-\,\mathrm{ad}^*_{(\xi,a)}(q,1) = -\mathrm{ad}^*_\xi(q) - \omega(\xi) . \tag{15.82}$$

The comparison with (15.67) leads to the following result

Proposition 15.12. *The coadjoint action of algebra $\mathfrak{g}^\omega$ on the hyperplane $\mathcal{M}^1 \subset (\mathfrak{g}^\omega)^*$ coincides with the infinitesimal action of $\mathcal{L}_g$.*

We have formulated this proposition separately, in order to stress again that the geometric and the algebraic approaches often operate with the same objects and constructions but use different names for them. However, different points of view may lead to interesting results, even if in one of the approaches something seems trivial. For example, as a result of the above discussion we get the following corollaries.

Corollary 15.13. *If ω is a cocycle as discussed in the above and $\alpha = const$, $\alpha \in \mathfrak{g}^*$ then together with the tensor*

$$K^1_q(\xi) = -(ad^*_\xi q + \omega(\xi)) \tag{15.83}$$

$$\xi \in \mathfrak{g},\ \ q \in \mathfrak{g}^* , \tag{15.84}$$

we have also two other Poisson tensors

$$K^2_q(\xi) = -(ad^*_\xi q + \omega(\xi) + ad^*_\xi \alpha)$$

$$K^3_q(\xi) = -ad^*_\xi \alpha . \tag{15.85}$$

Indeed, these tensors are obtained using the trivial cocycle corresponding to α.

Remark 15.14. The last tensor can be obtained also from the momentum map Φ_ω in the case $\omega = 0$, normalized by the condition $\Phi_{\omega=0}(e) = \alpha$

Corollary 15.15. *The Poisson tensors K^1, K^2, K^3 are compatible.*

It happens that the compatible pair of Poisson tensors on the manifold of potentials for the generalized Zakharov–Shabat system (see (15.2)) arises in the way we just described. As a matter of fact, in (15.2), the role of $\mathfrak{g}$ is played by $\mathfrak{g}[x]$, where $\mathfrak{g}[x]$ is the Lie algebra of Schwartz-type functions with values in a finite dimensional semisimple Lie algebra $\mathfrak{g}$, and $\alpha = J$. As to the cocycle ω, it is the famous Gel'fand–Fuchs cocycle, defined as

$$\omega(\xi, \eta) = \langle\langle i\partial_x \xi, \eta \rangle\rangle; \quad \xi, \eta \in \mathfrak{g}[x] , \tag{15.86}$$

where by ∂_x is denoted the differentiation with respect to x. Thus, the compatibility of K^1, K^2, K^3 follows from algebraic considerations, and we can skip over the cumbersome calculations of the Schouten brackets $[K^i, K^j]_S$; $i, j = 1, 2, 3$. In the case of Zakharov–Shabat system we identify the algebra and the coalgebra and we can write the above tensors into the form (up to some changes in the constants):

$$\begin{aligned} K^1_q(\xi) &= -\operatorname{ad}_\xi q + i\partial_x \xi \\ K^2_q(\xi) &= -\operatorname{ad}_\xi q + i\partial_x \xi + \operatorname{ad}_\xi J \\ K^3_q(\xi) &= \operatorname{ad}_\xi J , \end{aligned} \tag{15.87}$$

where $\xi \in \mathfrak{g}$, $J \in \mathfrak{g}$; $J = \text{const}$. As far as we know, this elegant proof of the compatibility of the tensors K^1, K^2, K^3 was mentioned for the first time in [16], as a by-product of the algebraic scheme. One can recognize in K^1 and K^3 the tensors P^0 and Q^0 from (15.2), which we used to construct the Nijenhuis tensor for the GZS system in canonical gauge.

15.3 Poisson–Nijenhuis Structures on Coadjoint Orbits

15.3.1 The Manifold of Jost Solutions

Now we shall apply the general scheme of Sect. 15.2 and shall give geometric interpretation of the gauge transformation associated with ψ_0^{-1}; see Introduction. We shall see that the gauge transformation can be interpreted as a map from one P–N manifold into another and moreover that there is third P–N manifold – the manifold of the Jost solutions, which is intrinsically related to these two manifolds. Part of these results has been obtained in a sequence of works of the authors (see [19, 20, 21, 22]) and has been exposed briefly in [23] but have never been presented in full. All the notation we use here are same as in Sect. 15.1: G will be fixed connected semisimple Lie group with Lie algebra $\mathfrak{g}$, J – fixed regular element from the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, $\mathfrak{g}[x]$, $\mathfrak{h}[x]$ – the algebras of Schwartz-type functions on the line taking values in $\mathfrak{g}$ and $\mathfrak{h}$, respectively.

Let G be the connected group corresponding to the algebra $\mathfrak{g}$, and let us define the following groups:

- $G[x]$ – the group of smooth functions $g : \mathbb{R} \to G$, such that

$$\lim_{x \to \pm\infty} g(x) = e \tag{15.88}$$

- $G^c[x]$ – the group of smooth functions $g : \mathbb{R} \to G$, such that

$$\lim_{x\to+\infty} g(x) = e, \qquad \lim_{x\to-\infty} g(x) = \exp H_g; \quad H_g \in \mathfrak{h} \,. \tag{15.89}$$

(Of course, the operations in those groups are understood pointwise.) Some clumsiness in the definition of $G^c[x]$ results from our desire to have properly defined momentum map on the set of Jost solutions; see below. We assume that in both cases $g(x)$ converges to its limit values fast enough, and therefore we can assume that the Lie algebra corresponding to $G[x]$ is $\mathfrak{g}[x]$ and the Lie algebra corresponding to $G^c[x]$ is the algebra $\mathfrak{g}^c[x]$ of the smooth functions with values in $\mathfrak{g}$, whose elements $\xi(x)$ satisfy the relations

$$\lim_{x\to+\infty} \xi(x) = 0, \qquad \lim_{x\to-\infty} \xi(x) = H_\xi; \quad H_\xi \in \mathfrak{h} \,, \tag{15.90}$$

where $\xi(x)$ converges fast enough to its limit values. We shall assume also that $(\mathfrak{g}[x])^*$ is identified with $\mathfrak{g}[x]$ and $(\mathfrak{g}^c[x])^*$ is identified with $\mathfrak{g}^c[x]$, which, for example in the last case means that we consider only linear functionals f having the form

$$f(\xi) = \langle\langle \eta_f, \xi \rangle\rangle = \int_{-\infty}^{+\infty} \langle \eta_f(x), \xi(x) \rangle dx; \quad \eta_f \in \mathfrak{g}[x] \,. \tag{15.91}$$

Let us define now on $T_e(G[x]) = \mathfrak{g}[x]$ the following 2-form:

$$\omega(\xi, \eta)_e = i\langle\langle \partial_x \xi, \eta \rangle\rangle; \quad \xi, \eta \in \mathfrak{g}[x] \,. \tag{15.92}$$

and let us extend it by left-invariance on the group $G[x]$. As we have seen in Sect. 15.2 the resulting form will be a symplectic form. In the same manner, let us take the closed 2-form

$$\Omega_e(\xi, \eta) = \langle\langle [J, \xi], \eta \rangle\rangle; \quad \xi, \eta \in \mathfrak{g}[x] \,, \tag{15.93}$$

and let us extend it by right invariance to the whole group $G[x]$. It is not difficult to see that Ω can be extended not simply to $G[x]$ but even to $G^c[x]$. Unfortunately, the form ω cannot be extended to $G^c[x]$ because ω do not satisfy the cocycle condition on $\mathfrak{g}^c[x]$. Thus, it is not possible to apply directly the results of Sect. 15.2 and to obtain P–N structure from a right-invariant 2-form and left-invariant symplectic form. However, we shall try to apply the ideas outlined there, but with some modifications, inverting not the tensor ω, but Ω, of course after restricting it on some submanifold, where it is not degenerate.

The momentum map for ω, as far as we know, was pointed out for the first time in [1]. As it is not hard to prove, it is given by the formula:

$$\Phi_\omega(g) = -i(\partial_x g)g^{-1} \,. \tag{15.94}$$

The most easy way to check it is to fix some faithful matrix representations of the group G and work in it. Since the group G is semisimple, the adjoint and the coadjoint action can be identified and from the definition of Φ_ω, we derive that $\theta = d\Phi_\omega$ must satisfy

$$\theta(\xi^l_g) = -i\mathrm{Ad}^*(g^{-1})\xi_x = -i\mathrm{Ad}\,(g)\xi_x \,. \tag{15.95}$$

where ξ^l is an arbitrary left-invariant vector field. This means that for any vector η at g we have

$$\begin{aligned} &d\Phi_\omega|_g(\eta) = d\Phi_\omega|_g(g(g^{-1}\eta)) = \theta(g(g^{-1}\eta)) = \\ &-ig(g^{-1}\eta)_x g^{-1} = -id(g_x g^{-1})|_g(\eta) \,. \end{aligned} \tag{15.96}$$

Finally, taking into account that $\Phi_\omega(e) = 0$, we get that the momentum map is indeed (15.94). Now, though ω cannot be extended to $G^c[x]$, the map Φ_ω can be extended, and this observation will be our starting point. Let us consider Φ_ω more closely. If $q = \Phi(g)$, then $\Phi^{-1}(q) = g$ is found as solution of the differential equation

$$\begin{aligned} &ig_x + q(x)g(x) = 0 \\ &\lim_{x\to+\infty} g(x) = \mathbf{1}, \qquad \lim_{x\to-\infty} g(x) = \exp H_g;\ H_g \in \mathfrak{h}. \end{aligned} \tag{15.97}$$

This means that $g(x)$ is exactly the Jost solution ψ_0 for the GZS system; see Introduction. It follows that on the manifold of potentials $\mathcal{M}_0$ for the generalized Zakharov–Shabat linear system there exists the smooth function $\Phi^{-1}_\omega : \mathcal{M}_0 \to G^c[x]$, and we can unambiguously define the submanifold of Jost solutions

$$\mathcal{M}^G_0 = \Phi^{-1}_\omega(\mathcal{M}_0) \subset G^c[x] \,, \tag{15.98}$$

diffeomorphic to $\mathcal{M}_0$. Let us find the vectors that are tangent to the manifold $\mathcal{M}^G_0$, that is, let us find $T_g(\mathcal{M}^G_0)$. From the definition of $\mathcal{M}^G_0$ we get that

$$d\Phi_\omega(\xi^r_g) = -\mathrm{Ad}^*(g^{-1})\omega_e\mathrm{Ad}\,(g^{-1})\xi \,. \tag{15.99}$$

If we perform the calculation and take into account that $\mathrm{Ad}^*(g^{-1}) = \mathrm{Ad}\,(g)$ we obtain

$$d\Phi_\omega(\xi^r_g) = -(\xi_x + [q(x), \xi(x)]); \quad q = \Phi_\omega(g) \,. \tag{15.100}$$

Since for $q \in \mathcal{M}_0$ we have $(\mathbf{1} - \pi_0)q = 0$, we have that $d\Phi_\omega(\xi^r_g) \in T_q(\mathcal{M}_0)$ exactly when

$$(\mathbf{1} - \pi_0)(\xi_x + [q(x), \xi(x)]) = 0 \,. \tag{15.101}$$

Thus, $T_g(\mathcal{M}^G_0)$ is spanned by those ξ^r_g, for which the above relation is fulfilled. Note, that due to our assumption (15.15) we can write (15.101) into the form

$$(\mathbf{1}-\pi_0)\xi(x) = \int_{-\infty}^{x} (\mathbf{1}-\pi_0)[q(y),\xi(y)]dy = \int_{+\infty}^{x} (\mathbf{1}-\pi_0)[q(y),\xi(y)]dy \ . \quad (15.102)$$

From the above relation, taking into account that $\ker \Omega_g$ is spanned by the right-invariant vector fields ξ^r such that $\pi_0\xi = 0$, we see that

$$\ker \Omega_g \cap T_g(\mathcal{M}_0^G) = \{0\} \ . \quad (15.103)$$

Actually, one has

$$T_g(G^c) = \ker \Omega_g \oplus T_g(\mathcal{M}_0^G) \ , \quad (15.104)$$

this relation being none, but already discussed (15.18). However, here we have a "global" variant of the above splitting, that is, the above relation is a consequence of the fact that the manifold $\mathcal{M}_0^G$ is transversal to the foliation defined by $g \to \ker \Omega|_g$. In order to see it, let us first remark that $\ker \Omega_e$ in our case is equal to the algebra $\mathfrak{h}^c[x]$:

$$\mathfrak{h}^c[x] = \{\xi \in \mathfrak{g}^c[x], \ \pi_0\xi = 0\}. \quad (15.105)$$

It is reasonable to assume that the Lie group $H^c[x]$, corresponding to this Lie algebra, consists of the functions $\exp \beta$; $\beta \in \mathfrak{h}^c[x]$. Let us consider the left coset spaces of $H^c[x]$, that is, the spaces $H^c[x]g$. It is not difficult to understand that they are integral leaves of the distribution $\ker \Omega$: $g \to \ker \Omega_g$. Now we see that indeed $\mathcal{M}_0^G$ is transversal to the coset spaces, but we have a little more, and we are able to show that $\mathcal{M}_0^G$ intersects with every coset space only once. In order to prove it, let us take kg; $k = \exp \beta$, $\beta \in \mathfrak{h}^c[x]$. Then we have

$$(\mathbf{1}-\pi_0)\Phi_\omega(kg) = -i(\mathbf{1}-\pi_0)\left[k_x k^{-1} + \mathrm{Ad}\,(k)(g_x g^{-1})\right] =$$
$$-i\beta_x + (\mathbf{1}-\pi_0)\mathrm{Ad}\,(k)q = -i\beta_x = 0 \ , \quad (15.106)$$

because $\mathrm{Ad}\,(k(x))$, $k \in H^c[x]$ preserves the splitting $\mathfrak{g} = \mathfrak{h} \oplus \bar{\mathfrak{g}}$. Taking into account that $\beta(+\infty) = 0$, we get $\beta = 0$, and this completes the proof.

We summarize all that was said until now for the manifold $\mathcal{M}_0^G$, into the following

Proposition 15.16. *The submanifold $\mathcal{M}_0^G$ is transversal to the integral leafs of the distribution $\ker \Omega$: $g \to \ker \Omega_g$ and intersects with every leaf only once. The projector $\mathcal{P}_g$ onto the subspace $T_g(\mathcal{M}_0^G)$ corresponding to the splitting (15.104) is equal to :*

$$\mathcal{P}_g = dR_g \circ \left[\mathbf{1}. + i\int_{-\infty}^{x} (\mathbf{1}-\pi_0)[\Phi_\omega(g), .]dy\right] \circ \pi_0 \circ dR_g^{-1} \ . \quad (15.107)$$

The form of this projector suggests to define the following tensor fields on $\mathcal{M}_0^G$:

$$g \to Q_g^G = \mathcal{P}_g \circ dR_g \circ \mathrm{ad}\,_J^{-1} \circ (dR_g)^{-1}$$
$$g \to N_g^G = Q_g \circ \omega_g \; . \tag{15.108}$$

Then we are able to prove the following proposition:

Proposition 15.17. *The tensor fields Q^G, N^G defined on the manifold $\mathcal{M}_0^G$ are Φ_ω – related to the tensor fields N^2Q, Q on the manifold $\mathcal{M}_0$, that is :*

$$d\Phi_\omega \circ N^G \circ d\Phi_\omega^{-1} = N$$
$$d\Phi_\omega \circ Q^G \circ d\Phi_\omega^* = N^2 \circ Q = N^2 Q \; . \tag{15.109}$$

Proof. Making use of (15.99) we obtain the following relations

$$d\Phi_\omega = -\mathrm{Ad}\,^*(g^{-1}) \circ \omega_e \circ dL_g^{-1}$$
$$d\Phi_\omega^* = (dL_g^{-1})^* \circ \omega_e \circ \mathrm{Ad}\,(g^{-1}) \; . \tag{15.110}$$

For the sake of brevity let us introduce the notation:

$$\alpha = \pi_0 dR_g^* \omega(\xi^r) = \pi_0 \mathrm{Ad}\,^*(g^{-1})\omega_e \mathrm{Ad}\,(g^{-1})\xi = \pi_0 \mathrm{Ad}\,(g)\omega_e \mathrm{Ad}\,(g^{-1})\xi,$$

where $\xi \in \mathfrak{g}[x]$. Then taking into account that $\mathrm{Ad}\,^*(g^{-1}) = \mathrm{Ad}\,(g)$, for $d\Phi_\omega N^G(\xi_g^r)$ we get

$$-\mathrm{Ad}\,(g)\omega_e \mathrm{Ad}\,(g^{-1}) \left[\mathrm{ad}\,_J^{-1}\alpha_x + \int_{-\infty}^{x} (\mathbf{1} - \pi_0)[\Phi_\omega(g), \mathrm{ad}\,_J^{-1}\alpha] dy \right] =$$

$$-i\mathrm{ad}\,_J^{-1}\alpha_x - \pi_0[\Phi_\omega(g), \mathrm{ad}\,_J^{-1}\alpha] - i[\Phi_\omega(g), \int_{-\infty}^{x} (\mathbf{1} - \pi_0)[\Phi_\omega(g), \mathrm{ad}\,_J^{-1}\alpha] dy]$$

$$= - \left(N|_{\Phi_\omega(g)} \right)(\alpha).$$

If $\xi \in T_{\Phi_\omega}(\mathcal{M}_0)$, from (15.101) follows that

$$\alpha = \pi_0 \mathrm{Ad}\,^*(g^{-1})\omega_e \mathrm{Ad}\,(g^{-1})\xi = \mathrm{Ad}\,^*(g^{-1})\omega_e \mathrm{Ad}\,(g^{-1})\xi = -(d\Phi_\omega|g)(\xi^r).$$

If we use now the explicit form of the Nijenhuis tensor N on $\mathcal{M}_0$, see (15.20), we can write the above expression into the equivalent form

$$d\Phi_\omega \circ N^G = N \circ d\Phi_\omega \; , \tag{15.111}$$

which coincides with the first relation in (15.109). The second one is verified in a similar way. The proposition is proved.

The above proposition shows that the geometric properties of the tensor fields N^G, Q^G on the manifold of Jost solutions $\mathcal{M}_0^G$ for the system (10.3) are the same as the properties of the tensor fields N, N^2Q on the manifold of potentials $\mathcal{M}_0$ for the same system. Therefore N^G, Q^G endow $\mathcal{M}_0^G$ with P–N structure.

15.3.2 The Manifold of Potentials in Pole Gauge

We shall use now propositions (15.16,15.17) and shall map the manifold $\mathcal{M}_0^G$ onto an orbit of the group $G^c[x]$. Thus the results referred to in the Introduction concerning the gauge equivalent hierarchies of evolution equations related to $\tilde{L}$ (recall that $\tilde{L}$ is L in the pole gauge) will appear naturally in the geometric picture. First, let us introduce the manifold $\mathcal{O}_J$ – the orbit of the element J with respect to the adjoint representation of the group G:

$$\mathcal{O}_J = \mathrm{Ad}\,(G)J = \{v : v = \mathrm{Ad}^*(g^{-1})J = \mathrm{Ad}\,(g)J; g \in G\} . \qquad (15.112)$$

Our considerations have led us to the manifold $\mathcal{O}_J[x]$, consisting of all smooth functions $f(x)$ on the line, such that

$$f : \mathbb{R} \to \mathcal{O}_J, \qquad \lim_{x\to\pm\infty} f(x) = J . \qquad (15.113)$$

Let us define the map $\varPhi_\Omega : G^c[x] \to \mathcal{O}_J[x]$ by the formula below.[6]

$$g \to \mathrm{Ad}^*(g)J = \mathrm{Ad}\,(g^{-1})J . \qquad (15.114)$$

One can see that the foliation $g \to gH^c[x]$ is projectable through $\varPhi_\Omega$, that is, every leaf $gH^c[x]$ is mapped into a single point of the manifold $\mathcal{O}_J[x]$. The map $\varPhi_\Omega$ becomes injective when restricted to $\mathcal{M}_0^G$. Then we can endow the manifold $\mathcal{O}_J[x]$ with a P–N structure transferring the P–N structure from $\mathcal{M}_0^G$. In other words, we define on $\mathcal{O}_J[x]$ the tensor fields $\tilde{Q}, \tilde{N}$ by the formulae:

$$\begin{aligned}
&\tilde{N}|_S = d\varPhi_\Omega|_g \circ N^G|_g \circ (d\varPhi_\Omega|_g)^{-1} \\
&\tilde{Q}|_S = d\varPhi_\Omega|_g \circ Q^G|_g \circ d\varPhi_\Omega^*|_g \\
&S = \mathrm{Ad}^*(g)J = \mathrm{Ad}\,(g^{-1})J ,
\end{aligned} \qquad (15.115)$$

where $g \in \mathcal{M}_0^G$. Then we have

Proposition 15.18. *The following relations hold:*

$$\begin{aligned}
&\tilde{N}|_S = Ad\,(g)^* \circ N|_q \circ Ad^*(g^{-1}) \\
&\tilde{Q}|_S = ad_S \\
&S = Ad^*(g)J = Ad\,(g^{-1})J ,
\end{aligned} \qquad (15.116)$$

where $q = \varPhi_\omega(g)$, $g \in \mathcal{M}_0^G$.

Proof. Indeed, let ξ^r be right-invariant vector field. Then

$$d\varPhi_\Omega(\xi^r) = \mathrm{Ad}^*(g)\mathrm{ad}_J\xi . \qquad (15.117)$$

[6] The notation for this map is no accident. It is not difficult to see that $\varPhi_\Omega$ is the momentum map for the 2-form Ω, normalized by: $\varPhi_\Omega(e) = J$.

Therefore

$$d\Phi_\Omega|_g = \mathrm{Ad}^*(g) \circ \mathrm{ad}_J \circ dR_{g^{-1}}$$
$$d\Phi^*_\Omega|_g = -dR^*_{g^{-1}} \circ \mathrm{ad}_J \circ \mathrm{Ad}(g) \,. \tag{15.118}$$

It is not difficult to define also $d\Phi_\Omega^{-1}|_g$ on the tangent space $T_S(\mathcal{O}_J[x])$, where $S = \mathrm{Ad}^*(g)J$:

$$d\Phi_\Omega^{-1}|_g = dR_g \circ \left[\mathbf{1}. + i \int_{-\infty}^{x} (\mathbf{1} - \pi_0)[\phi_\omega(g), .]dy \right] \circ \mathrm{ad}_J^{-1} \circ \mathrm{Ad}^*(g^{-1}) \,. \tag{15.119}$$

We have

$$\tilde{N}|_S = V \circ W$$
$$V = \mathrm{Ad}^*(g) \circ \pi_0 \circ \left\{ \mathrm{Ad}^*(g^{-1}) \circ \omega_e \circ \mathrm{Ad}(g^{-1}) \right\}$$
$$W = \left\{ \mathbf{1}. + i \int_{-\infty}^{x} (\mathbf{1} - \pi_0)[\phi_\omega(g), .]dy \right\} \circ \mathrm{ad}_J^{-1} \circ \mathrm{Ad}^*(g^{-1}) \,. \tag{15.120}$$

Then after some calculation we get the first equation in (15.116). The second equation in (15.116) can be proved in the same manner, taking into account that

$$\mathrm{Ad}^*(g) \circ \mathrm{ad}_J \circ \mathrm{Ad}(g) = \mathrm{ad}_S \,. \tag{15.121}$$

The proposition is proved.

It is clear now that $\tilde{Q}, \tilde{N}$ endow $\mathcal{O}_J[x]$ with a P–N structure. Further, from the relation $N^* = \mathrm{ad}_J^{-1} \circ N \circ \mathrm{ad}_J$ it follows also that

$$\tilde{N}^* = \mathrm{Ad}(g^{-1}) \circ N^* \circ \mathrm{Ad}(g) = \mathrm{ad}_S^{-1} \circ \tilde{N} \circ \mathrm{ad}_S \,. \tag{15.122}$$

If we denote now $\tilde{N}^*$ by $\tilde{\Lambda}_\pm$, $\tilde{N}$ by $\tilde{\Lambda_\pm}^*$ and g by ψ_0 the above relations can be written into the form

$$\tilde{\Lambda}_\pm = \mathrm{Ad}(\psi_0^{-1}) \circ \Lambda_\pm \circ \mathrm{Ad}(\psi_0) = \mathrm{ad}_S^{-1} \circ \tilde{\Lambda_\pm}^* \circ \mathrm{ad}_S \,. \tag{15.123}$$

It is evident (see Introduction) that the operators we have obtained are the generating operators for the generalized Zakharov-Shabat system in pole gauge.

Now, combining (15.17,15.18) we easily get

Corollary 15.19.

$$\tilde{N} = d(\Phi_\Omega \circ \Phi_\omega^{-1}) \circ N \circ d(\Phi_\omega \circ \Phi_\Omega^{-1})$$
$$\tilde{Q} = ad_S = d(\Phi_\Omega \circ \Phi_\omega^{-1}) \circ N^2 \circ Q \circ d(\Phi_\Omega \circ \Phi_\omega^{-1})^* \,. \tag{15.124}$$

A simple comparison shows that the function $\Phi_\omega^{-1} \circ \Phi_\Omega$ we have in the above is what in the Introduction (see (10.40)) we denoted by F. Using this notation again, the second relation from (15.124) can be written as

$$\tilde{Q} = \mathrm{ad}_{S} = \tilde{N}^2 \circ [dF]^{-1} \circ Q \circ [dF^{-1}]^* \,. \tag{15.125}$$

In other words, for the symplectic forms $\Omega^{(m)} = \mathrm{ad}_J^{-1} \circ N^n$; $n = 0, 1, 2\ldots$ and $\tilde{\Omega}^{(m)} = \mathrm{ad}_S^{-1} \circ \tilde{N}^n$; $n = 0, 1, 2\ldots$ we have

Corollary 15.20.

$$F^* \Omega^{(m)} = \tilde{\Omega}^{(m+2)}; \quad m = 0, 1, 2, \ldots \,. \tag{15.126}$$

However, the last formula differs from the formula (10.42) in the Introduction, since in (10.42) there is additional term in the right-hand side. We shall show that this term gives no contribution, if we consider fields that obey the restriction (15.15). Indeed, let ξ be a tangent vector to $\mathcal{M}_0$ at the point q. Let $g = \Phi_\omega^{-1}(q)$. In order to simplify the calculations let us put

$$\xi = \delta q, \quad d(g)(\xi) = \delta g \,, \tag{15.127}$$

where $q \to g(q)$ is the inverse of Φ_ω, and let us work in some faithful matrix representation. According to (15.102), if δg is tangent to the manifold $\mathcal{M}_0^G$ then

$$(\mathbf{1} - \pi_0)(\delta g)g^{-1}(x) = \int_{-\infty}^{x} (\mathbf{1} - \pi_0)[q, (\delta g)g^{-1}]dx \,. \tag{15.128}$$

As $\lim_{x\to\infty} g = e$, it follows that

$$\int_{-\infty}^{+\infty} (\mathbf{1} - \pi_0)[q, (\delta g)g^{-1}]dx = 0 \,. \tag{15.129}$$

From the other hand, differentiating $(\delta g)g^{-1}$ with respect to x we have

$$\partial_x((\delta g)g^{-1}) = i\delta q + i[q, (\delta g)g^{-1}] \,, \tag{15.130}$$

and therefore,

$$-\lim_{x\to\infty} (\mathbf{1} - \pi_0)[(\delta g)g^{-1}] = i \int_{-\infty}^{+\infty} (\mathbf{1} - \pi_0)[q, (\delta g)g^{-1}]dx = 0.$$

Now recall that $g = \psi_0$, where ψ_0 is the Jost solution of (10.3). Then

$$\lim_{x\to-\infty} \psi_0 = T(0) = D^+(0) = D^-(0)$$

$$T(0) = \exp \sum_{j=1}^{r} \Delta_j(0) H_j \,, \tag{15.131}$$

and from the above it follows that on the manifold $\mathcal{M}_0$ we have

$$\delta\Delta_j(0) = D(\Delta_j(0))(\xi) = 0\,. \tag{15.132}$$

In other words, in order to perform our geometric constructions, we have made assumptions which automatically lead to the fact that the additional term we have spoken about is zero on $\mathcal{M}_0$. Finally, let us remark that all the quantities $\Delta_j(0)$ are integrals of motion for the hierarchy related to the GZS system. It follows that when we calculate the Hamiltonian vector fields, the additional term can be omitted. Indeed, suppose that ω_1 and ω_2 are symplectic forms on the manifold $\mathcal{N}$ and that

$$\omega_2 = \omega_1 + \sum_{i=1}^{N} df_i \wedge dg_i \tag{15.133}$$

where f_i, g_i are smooth functions on $\mathcal{N}$. Let X_f be the Hamiltonian vector field corresponding to the Hamiltonian f with respect to the symplectic form ω_1 and let f_i, g_i be integrals of motion for X_f. Then

$$i_{X_f}\omega_2 = i_{X_f}\omega_1 + \sum_{i=1}^{N}(X_f(f_i)dg_i - X_f(g_i)df_i) = -df,$$

or in other words, X_f is Hamiltonian vector field for the Hamiltonian function f also with respect to ω_2. From this, it follows that the Poisson brackets $\{f,g\}_1$, $\{f,g\}_2$ of f with some other function g, constructed with the help of the forms ω_1, ω_2 are equal.

Finally, we write the relations between the Poisson–Nijenhuis manifolds arising from the GZS system, namely,

- $\mathcal{M}_0$ – the manifold of potentials for the generalized Zakharov–Shabat system in canonical gauge
- $\mathcal{O}_J[x]$ – the manifold of potentials for the generalized Zakharov–Shabat system in pole gauge
- $\mathcal{M}_0^G$ – the manifold of the Jost solutions at $\lambda = 0$ for the generalized Zakharov-Shabat system in canonical gauge

into the following diagram

$$\begin{array}{ccccc} & \mathcal{M}_0^G & \xrightarrow{\;id_{\mathcal{M}_0^G}\;} & \mathcal{M}_0^G & \\ & (Q^G, N^G) & & (Q^G, N^G) & \\ \Phi_\omega & \downarrow & & \downarrow & \Phi_\Omega \\ & \mathcal{M}_0 & \xrightarrow[F = \Phi_\omega^{-1}\circ\Phi_\Omega]{} & \mathcal{O}_J[x] & \\ & (N^2Q, N) & & (\tilde{Q}, \tilde{N}) & \end{array} \tag{15.134}$$

15.4 Fundamental Fields for GZS System

We have mentioned already that without the knowledge of the fundamental fields for a given P–N structure, the theory would be incomplete, as one cannot construct explicitly the dynamical systems with commuting flows. In this section we shall show that all the P–N structures, related to the generalized Zakharov–Shabat system, are invariant under the action of r-parametric group of diffeomorphisms. This fact will lead us to the construction of r-parametric families of fundamental fields. The fact that the P–N structures on the manifolds $\mathcal{M}_0$, $\mathcal{M}_0^G$ and $\mathcal{O}_J[x]$ are interrelated allows one to operate simultaneously with all of them.

To begin with, let us define on the Lie group $G^c[x]$ the following diffeomorphism:

$$g \to F_H^G(g) = L_{\exp H} \circ R_{\exp H}^{-1}(g) = (\exp H)g(\exp H)^{-1} , \tag{15.135}$$

where $H \in \mathfrak{h}$. Note that although $\exp H \notin G^c[x]$, we have that $F_H^G(g) \in G^c[x]$. Let us define similar diffeomorphism on the algebra $\mathfrak{g}^c[x]$:

$$q \to F_H^A(q) = \mathrm{Ad}\,(\exp H)q; \quad H \in \mathfrak{h} . \tag{15.136}$$

It is not difficult to see that as a matter of fact we have defined action of the group

$$\mathbf{H} = \{\exp H,\ H \in \mathfrak{h}\} \tag{15.137}$$

on $G^c[x]$ and on $\mathfrak{g}^c[x]$ and $G^c[x]$.

Proposition 15.21. *The manifold $\mathcal{M}_0$ is invariant with respect to the action of the group $\mathbf{H}$.*

Proof. Indeed, if $q \in \mathcal{M}_0$ then $\bar{q} = F_H^A(q) = \mathrm{Ad}\,(\exp H)q$ is function of Schwartz type. We have also $(\mathbf{1}-\pi_0)\bar{q} = 0$. If the Jost solution for the potential function q is ψ_0, the Jost solution for $\bar{q}$ will be $F_H^G(\psi_0)$. Then one can verify that $\bar{q}$ belongs to $\mathcal{M}_0$ (the condition (15.15) are also fulfilled). This completes the proof.

Next we prove

Proposition 15.22. *The maps $\Phi_\omega : G^c[x] \to \mathfrak{g}[x]$ and $\Phi_\Omega : G^c[x] \to \mathcal{O}_J[x]$ are equivariant with respect to the action of the group $\mathbf{H}$.*

Proof. Let us consider for example $\Phi_\Omega(F_H^G(g))$. We have:

$$\begin{aligned}
&\Phi_\Omega(F_H^G(g)) = \\
&\mathrm{Ad}^*(\exp Hg\exp -H)J = \mathrm{Ad}^*(\exp -H)\mathrm{Ad}^*(g)\mathrm{Ad}^*(\exp H)J = \\
&\mathrm{Ad}\,(\exp H)\mathrm{Ad}\,(g^{-1})J = F_H^A(\Phi_\Omega^A(g)) ,
\end{aligned} \tag{15.138}$$

that is $\Phi_\Omega \circ F_H^G = F_H^G \circ \Phi_\Omega$, which means that Φ_Ω is equivariant. In a similar manner, one is able to show that Φ_ω is equivariant. The proposition is proved.

The theorem below is the central result in the present section:

Theorem 15.23. *The manifolds $\mathcal{M}_0$, $\mathcal{M}_0^G$, $\mathcal{O}_J[x]$ and their P–N structures are invariant with respect to the action of* $\mathbf{H}$.

Proof. The fact that the manifolds are invariant has been already proved. In order to prove the invariance of the P–N structures, let us note that

$$dF_H^G|_g(\xi^l|_g) = \Big([\mathrm{Ad}\,(\exp H)\xi]^l|_{F_H^G}(g)\Big)^l \tag{15.139}$$

$$dF_H^G|_g(\xi^r|_g) = \Big([\mathrm{Ad}\,(\exp H)\xi]^l|_{F_H^G}(g)\Big)^r \ . \tag{15.140}$$

For that reason

$$\begin{aligned} &dF_H^G|_g = dL_{F_H^G(g)} \circ \mathrm{Ad}\,(\exp H) \circ dL_{g^{-1}} = \\ &dR_{F_H^G(g)} \circ \mathrm{Ad}\,(\exp H) \circ dR_{g^{-1}} \ , \end{aligned} \tag{15.141}$$

$$\begin{aligned} &(dF_H^G)^*|_g = dL^*_{g^{-1}} \circ \mathrm{Ad}\,(\exp H) \circ dL^*_{F_H^G(g)} = \\ &dR^*_{g^{-1}} \circ \mathrm{Ad}\,(\exp H) \circ dR^*_{F_H^G(g)} \ . \end{aligned} \tag{15.142}$$

Now let us prove for example that the tensor Q^G is invariant.

$$\begin{aligned} &dF_H^G|_g \circ Q^G|_g \circ (dF_H^G|_g)^* = \\ &dR_{F_H^G(g)} \circ \mathrm{Ad}\,(\exp H) \circ [\mathrm{ad}_J^{-1}. + i\int_{-\infty}^{x} (\mathbf{1} - \pi_0)[\varPhi_\omega(g), \mathrm{ad}_J^{-1}.]dy] \circ \\ &\circ \pi_0 \circ \mathrm{Ad}\,(\exp -H) \circ dR^*_{F_H^G(g)} = \\ &dR_{F_H^G(g)} \circ [\mathrm{ad}_J^{-1}. + i\int_{-\infty}^{x} (\mathbf{1} - \pi_0)[\mathrm{Ad}\,(\exp H)(\varPhi_\omega(g)), \mathrm{ad}_J^{-1}.]dy] \circ \\ &\circ \pi_0 \circ dR^*_{F_H^G(g)} \ . \end{aligned} \tag{15.143}$$

Since the map $\varPhi_\omega$ is equivariant, $\mathrm{Ad}\,(\exp H)(\varPhi_\omega(g)) = \varPhi_\omega(F_H^G(g))$ and we obtain

$$dF_H^G|_g \circ Q^G|_g \circ (dF_H^G|_g)^* = Q^G_{F_H^G} \ . \tag{15.144}$$

In the same way we arrive at the relation

$$dF_H^G|_g \circ N^G|_g \circ (dF_H^G|_g)^{-1} = N^G_{F_H^G} \ , \tag{15.145}$$

which shows that the tensor field $g \to N^G$ is also invariant with respect to the action of $\mathbf{H}$. Now, as the maps Φ_ω, Φ_Ω are equivariant with respect to the action of $\mathbf{H}$ it follows that the P–N structures on the manifolds $\mathcal{M}_0$ and $\mathcal{O}_J[x]$ are also invariant with respect to the action of $\mathbf{H}$. The theorem is proved.

We have an immediate corollary

Corollary 15.24. *The following sets of vector fields are fundamental:*

- *For the P–N structure on $\mathcal{M}_0$ the fields:*

$$q \to [H, q]; \quad H \in \mathfrak{h} \, . \tag{15.146}$$

- *For the P–N structure on $\mathcal{M}_0^G$ the fields:*

$$g \to H^r(g) - H^l(g); \quad H \in \mathfrak{h} \, . \tag{15.147}$$

- *For the P–N structure on $\mathcal{O}_J[x]$ the fields:*

$$S \to [H, S]; \quad H \in \mathfrak{h} \, . \tag{15.148}$$

Proof. Let us fix $H \in \mathfrak{h}$ and let us consider the 1-parametric group in $\mathbf{H}$:

$$t \mapsto \exp{(tH)}.$$

Then the tensor fields (Q^G, N^G), (Q, N), and $(\tilde{Q}, \tilde{N})$ are invariant with respect to the action of this group. Therefore the fundamental fields of the action of the 1-parametric group are fundamental for the tensor fields listed above. These fundamental fields are:

- On the algebra $\mathfrak{g}[x]$:

$$q \mapsto \frac{d}{dt} \mathrm{Ad}\,(\exp{(tH)})(q)|_{t=0} = [H, q] \, . \tag{15.149}$$

- On the group $G^c[x]$:

$$g \mapsto \frac{d}{dt} \exp{(tH)} g \exp{(-tH)}|_{t=0} = H^r(g) - H^l(g) \, . \tag{15.150}$$

- On the orbit $\mathcal{O}_J[x]$:

$$S \mapsto \frac{d}{dt} \mathrm{Ad}\,(\exp{(tH)})(S)|_{t=0} = [H, S] \, . \tag{15.151}$$

We would like to note that the fields $g \to H^r(g)$ and $g \to H^l(g)$ do not satisfy the boundary conditions, resulting from the definition of $G^c[x]$, and, therefore, separately they do not represent vector field on $G^c[x]$. Their difference however, satisfies these conditions and $g \to H^r(g) - H^l(g)$ is a vector field on $G^c[x]$.

Corollary 15.25. • *For each $H \in \mathfrak{h}$ and $n = 0, 1, 2, \ldots$ the vector fields:*

$$q \to N^n([H, q]) = ad_J \circ (N^*)^n (ad_J^{-1}[H, q]) \tag{15.152}$$

on $\mathcal{M}_0$ commute and are Hamiltonian with respect to infinite hierarchy of Poisson structures $Q \circ N^n = (N^*)^n \circ Q$ (or symplectic structures $\Omega^n = ad_J^{-1} \circ N^n$).

• *For each $H \in \mathfrak{h}$ and $n = 0, 1, 2, \ldots$ the vector fields:*

$$g \to (N^G)^n (H^r(g) - H^l(g)) \tag{15.153}$$

on $\mathcal{M}_0^G$ commute and are Hamiltonian with respect to infinite hierarchy of Poisson structures $Q^G \circ (N^G)^n$.

• *For each $H \in \mathfrak{h}$ and $n = 0, 1, 2, \ldots$ the vector fields:*

$$S \to \tilde{N}^n([H, S]) = (\tilde{N}^*)^n (ad_S^{-1}[H, S]) = -(\tilde{N}^*)^n (\tilde{\pi}_0 H) \tag{15.154}$$

on $\mathcal{O}_{[}x]$ commute and are Hamiltonian with respect to infinite hierarchy of Poisson structures $\tilde{Q} \circ \tilde{N}^n = (\tilde{N}^*)^n \circ \tilde{Q}$ (or symplectic structures $\tilde{\Omega}^n = ad_S^{-1} \circ \tilde{N}^n$).

These results are in complete accordance with the properties of the hierarchies of nonlinear evolution equations related to the GZS system (10.3) and its gauge-equivalent system (10.24) and give to these results a beautiful geometric interpretation.

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16

Linear Bundles of Lie Algebras and Compatible Poisson Structures

The main goal of this chapter is to introduce two more examples of Nijenhuis operators, arising as before from compatible Poisson tensors. These are the Nijenhuis operators (tensors) that are related to the hierarchies of the so-called $O(3)$ chiral system (CF) and with the hierarchy of the so-called Landau–Lifshitz equation (LL). However, in order to introduce the pairs of compatible Poisson tensors that will give us the corresponding P-N manifolds we must first consider the algebraic structures that are responsible for their existence. For this reason, we first introduce the notion of linear bundles of Lie algebras and how this algebraic structure leads to compatible Poisson tensors in the finite in the infinite dimensional case. The aforementioned algebraic structure is interesting by itself, so we dedicate some space to it also.

16.1 Preliminaries

Let $\mathrm{Mat}\,(n, \mathbb{K}) = \mathrm{End}\,(\mathbb{K}^n)$ be the linear space of all $n \times n$ matrices over the field $\mathbb{K}$. In what follows $\mathbb{K}$ will be one of the classical fields – $\mathbb{R}$ or $\mathbb{C}$. We shall consider it as fixed and shall write it explicitly only if it is necessary. So we write $\mathrm{Mat}\,(n)$ instead of $\mathrm{Mat}\,(n, \mathbb{K})$ and so on. The space $\mathrm{Mat}\,(n)$ possesses a natural structure of associative algebra and as a consequence a structure of a Lie algebra defined by the commutator $[X, Y] = XY - YX$. Considered as Lie algebra the space $\mathrm{Mat}\,(n)$ is then usually denoted by $\mathrm{gl}\,(n)$. However, the structure of the associative algebra over $\mathrm{Mat}\,(n)$ is not unique, for example if we fix an element $J \in \mathrm{Mat}\,(n)$ then we can define the product $(X \circ Y)_J = XJY$ and with respect to the new product the vector space $\mathrm{Mat}\,(n)$ is again associative algebra. The new associative algebra structure naturally induces new Lie algebra structure, defined by the bracket

$$[X, Y]_J = XJY - YJX \ . \tag{16.1}$$

Thus we obtain a family of Lie brackets, labeled by the element J. It is readily seen that we have actually a linear space of Lie bracket structures or simply

Gerdjikov, V.S. et al.: *Linear Bundles of Lie Algebras and Compatible Poisson Structures.*
Lect. Notes Phys. **748**, 547–611 (2008)
DOI 10.1007/978-3-540-77054-1_16

linear space of Lie brackets, because evidently for any numbers a_1, a_2 and $J_1, J_2 \in \mathrm{Mat}\,(n)$ we have

$$a_1[X,Y]_{J_1} + a_2[X,Y]_{J_2} = [X,Y]_{a_1 J_1 + a_2 J_2} \,. \tag{16.2}$$

This construction can be applied even if X, Y, J are not square matrices. Indeed, if $X, Y \in \mathrm{Mat}\,(n,m)$ – the linear space of $n \times m$ matrices and $J \in \mathrm{Mat}\,(m,n)$ – the linear space of $n \times m$ matrices, then the expression (16.1) defines again linear space of Lie brackets or as they also say (16.1) defines linear bundle of Lie algebras.

It is difficult to trace where the above construction was used for the first time as it is too simple not to be discovered at the time when the intensive investigations of the Lie algebras had been started. Maybe the question was considered not so important, because as is readily seen, if J is not degenerate and there exists element $D \in \mathrm{gl}\,(n)$ such that $D^2 = J^{-1}$ then the map: $g : X \mapsto DXD$ is isomorphism between $\mathrm{gl}\,(n)$ (with respect to the bracket $[X,Y]_J$) and $\mathrm{gl}\,(n)$ with respect to the usual bracket:

$$g([X,Y]) = [g(X), g(Y)]_J \,. \tag{16.3}$$

For the bundles $(\mathrm{o}\,(n), \mathrm{sym}\,(n))$, $(\mathrm{sym}\,(n), \mathrm{o}\,(n))$ (the exact definitions are given below) similar considerations (see [1]) show that there are a finite number of nonisomorphic algebras in the bundle. But the possibility of having all these algebras simultaneously is interesting, and so this algebraic structure is by no means void of interest.

We believe also that apart from the applications related to compatible Poisson tensors, which we shall consider below, one can use the above algebra structures to study deformations of Lie algebras giving this a natural background.

Most of the information about the bundles of Lie brackets and their applications can be found in the monograph [1] and in [2]. In [1], there are also references to earlier works, all of which, however, are relatively recent. The linear bundles of Lie algebras in [1] are used mainly because of their applications to the integrable systems – bundles of Lie algebras naturally define compatible Poisson tensors and the cases $(\mathrm{o}\,(n), \mathrm{sym}\,(n))$ and $(\mathrm{sym}\,(n), \mathrm{o}\,(n))$ (see below) are considered thoroughly, there have been found the nonisomorphic algebras in these bundles and for them the index, center and the invariants of the coadjoint action.

We are interested in the bundles of the Lie (brackets) algebras for the same reason mentioned above – the bundles of Lie brackets generate compatible Poisson tensors, and we know that compatible Poisson tensors lead to Nijenhuis tensors. Even without finding some Nijenhuis fields the compatible Poisson tensors have important applications to the finite dimensional integrable systems since they can be used to establish Liouville-type integrability. In [1] there is a large collection of such applications. For example, recently the bracket $[X,Y]_J$ was used efficiently in [3] to reveal the bi-Hamiltonian structure of the Euler equations on $\mathrm{o}(n)$, where it was called "modified" Lie bracket, and the authors have been able to establish in a simple way the relation between the Mischenko and Manakov series of integrals for the n-dimensional rigid body problem. Together with the works that will be cited below (see the next paragraph), an application to the infinite-dimensional systems has been considered recently in [4].

The expression (16.1) naturally appeared in the description of the Hamiltonian structures for the chiral fields system hierarchy and Landau–Lifshitz equation hierarchy of integrable equations, cf. [5, 6], obtained via polynomial pencil of Lax pairs on the algebra $\mathrm{o}(4)$, though at the beginning the underlying algebraic structure has been overlooked. Recently [7, 8, 9], the "new" brackets have been used to describe the bi-Hamiltonian structures of the hierarchies we mentioned, and the corresponding Nijenhuis operator has been calculated. In the present chapter, we shall introduce these results, since they fit in the geometric picture we are discussing, but first we continue with some general results about the new brackets.

16.2 General Properties

We start with some definitions and theorems–those that are the same as in [1] and marked by TF. The rest of results presented here were obtained [2].

Definition 16.1 (TF). *Let $\mathfrak{g}$ and V be vector spaces and let for arbitrary $v \in V$ be defined the Lie bracket on $\mathfrak{g}$:*

$$(X, Y) \to L_v(X, Y) = [X, Y]_v = ad^v_X(Y) \,, \tag{16.4}$$

such that for any $v_1, v_2 \in V$, $a_1, a_2 \in \mathbb{K}$,

$$a_1[X, Y]_{v_1} + a_2[X, Y]_{v_2} = [X, Y]_{a_1 v_1 + a_2 v_2} \,. \tag{16.5}$$

We shall say that $(\mathfrak{g}, V)$ is a linear bundle of Lie algebras and if V is finite dimensional the dimension of V will be called the dimension of the linear bundle $(\mathfrak{g}, V)$.

The space $\mathfrak{g}$ endowed with the bracket L_v shall be denoted by $\mathfrak{g}_v$. When $\mathfrak{g}_v$ coincides with some classical matrix algebra with respect to the commutator we shall denote it by the notation that is used for this algebra and the bracket in it by bracket without subscript. In this case, we shall denote by the same letter the algebra $\mathfrak{g}$ and the underlying vector space. Also, we shall denote by d_v the coboundary operator associated with the adjoint representation of $\mathfrak{g}_v$ or with the trivial representation (the representation is usually clear from the context). Of course, d_v acts on the graded module of skew-symmetric maps from $\mathfrak{g}$ into $\mathfrak{g}$ (or $\mathbb{K}$) which play the role of cochains. As usual (see [10]), we denote by $\mathcal{L}^v_X$ the operator

$$\mathcal{L}^v_X = d_v \circ i_X + i_X \circ d_v; \quad X \in \mathfrak{g} \tag{16.6}$$

where i_X, acting on the k-cochain M gives the $k-1$ cochain:

$$i_X(M)(X_1, X_2, \ldots, X_{k-1}) = M(X, X_1, X_2, \ldots, X_{k-1}) \,. \tag{16.7}$$

It is easy to check that the fact that for $u, v \in V$ the expression $L_u + L_v$ is Lie bracket entails that for $X_1, X_2, X_3 \in \mathfrak{g}$ one has the identity:

$$[X_1, [X_2, X_3]_u]_v + [X_1, [X_2, X_3]_v]_u + \mathrm{cycl}\,(1,2,3) = 0 \,. \tag{16.8}$$

The notation cycl (1, 2, 3) means that one must add to the first two terms the expressions obtained from them by cyclic permutation of the indices 1,2,3.

The applications of the linear bundles of Lie algebras in the theory of the integrable systems are based on the following simple proposition, which, applied to the corresponding Poisson-Lie tensors, allows to construct sets of functions in involution.

Proposition 16.2 (TF). *Denote by $Z(\mathfrak{g}_u)$ the center of $\mathfrak{g}_u$; $u \in V$. We have:*

- *$Z(\mathfrak{g}_u)$ is subalgebra in all the algebras $\mathfrak{g}_v$; $v \in V$.*
- *If $X_1 \in Z(\mathfrak{g}_u)$ and $X_2 \in Z(\mathfrak{g}_v)$ then $[X_1, X_2]_u, [X_1, X_2]_v \in Z(\mathfrak{g}_w)$, where $w = \lambda u + \mu v$ and λ, μ are fixed numbers.*

Taking into account (16.8), it is not difficult to see that if V is a vector space and L_v; $v \in V$ is a family of Lie brackets on $\mathfrak{g}$ having the property that $L_{\mu v} = \mu L_v$, then $(\mathfrak{g}, V)$ will be linear bundle of Lie brackets if and only if

$$d_v L_w = d_w L_v = 0; \quad v, w \in V \,. \tag{16.9}$$

In other words, we have:

Proposition 16.3 (TF). *For $v, w \in V$ the bracket L_w is 2-cocycle for the coboundary operator d_v.*

Let us denote now by $\mathrm{ad}^{\,v}_X$ the adjoint action of the algebra defined by the bracket $[X, Y]_v$. Clearly, the map $\mathrm{ad}^{\,v}_X$ can be considered as 1-cocycle for the adjoint representation of $\mathfrak{g}_w$ for $w \in V$. A brief calculation shows that (16.9) implies that for $v, w \in V$ and $X \in \mathfrak{g}$ we have:

$$d_v \mathrm{ad}^{\,w}_X + d_w \mathrm{ad}^{\,v}_X = 0 \,. \tag{16.10}$$

Since $i_X L_w = \mathrm{ad}^{\,w}_X$ this also means that

$$\mathcal{L}^v_X L_w + \mathcal{L}^w_X L_v = 0 \,. \tag{16.11}$$

Let A be a linear operation on $\mathfrak{g}$ and B be a bilinear operation. The action of the linear operation A on the bilinear operation B can be defined as follows:

$$A(B)(X, Y) = A(B(X, Y)) - B(A(X), Y) - B(X, A(Y)); \quad X, Y \in \mathfrak{g} \,. \tag{16.12}$$

We have

Definition 16.4 (TF). *Let $(\mathfrak{g}, V)$ be a linear bundle of Lie brackets. Let $A_v(X)$ be the linear map $A_v(X)(Y) = [X,Y]_v$. We say that $(\mathfrak{g}, V)$ is closed, if the family of bilinear operations L_w; $w \in V$ is closed under the action of the linear operations $A_v(X)$; $v \in V,\ X \in \mathfrak{g}$.*

We cast this definition in a form more convenient for our purposes:

Definition 16.5. *A linear bundle of Lie algebras is called closed if there exists a map*

$$f : \mathfrak{g} \times V \times \mathfrak{g} \mapsto V \ , \tag{16.13}$$

such that

$$\mathcal{L}^w_X L_v = L_{f(w,Z,v)} \ , \tag{16.14}$$

for $v, w \in V$; $X \in \mathfrak{g}$.

In order to see that the two definitions are equivalent, it is enough to remark that

$$\mathcal{L}^w_X(L_v) = d_w i_X(L_v) + i_X d_w(L_v) = d_w(\mathrm{ad}\,^v_X) \ . \tag{16.15}$$

Let us turn our attention to the function $f(v, Z, w)$. Clearly, $f(v, Z, w)$ is linear in each argument. For closed bundles the (16.11) implies that

$$L_{(f(w,Z,v)-f(v,Z,w))} = 0; \quad v, w \in V,\ Z \in \mathfrak{g} \ . \tag{16.16}$$

Definition 16.6. *We call the bundle $(V, \mathfrak{g})$ of Lie algebras regular, if the only Abelian algebra in it is $\mathfrak{g}_{v=0}$.*

For a closed regular bundle, the relation (16.16) immediately gives that the function $f(v, Z, w)$ is skew-symmetric with respect to v and w.

Definition 16.7 (TF). *A linear bundle of Lie algebras is called irreducible if the only common ideals for all the algebras $\mathfrak{g}_v$; $v \in V$ are the trivial ones, that is $\{0\}$ and $\mathfrak{g}$.*

Usually, when one has some irreducible algebraic structures, one can look for some classification. In the literature, one can find citations about the following classification theorem for closed irreducible linear bundles of Lie algebras.[1]

Theorem 16.8. *All irreducible closed linear bundles of Lie algebras over $\mathbb{C}$ are comprised in the following list:*

- *$\mathfrak{g} = o(n)$ the linear space of all skew-symmetric matrices in $Mat(n)$, $V = sym(n)$ – the linear space of all symmetric matrices in $Mat(n)$, with the bracket*

$$[X, Y]_v = XvY - YvX \ . \tag{16.17}$$

[1] We cite this result according to [1], where it is given without a proof with reference to [11], where, however, there is again only a short abstract. This situation is a little embarrassing, as we actually have never seen the proof of this theorem.

- $\mathfrak{g} = sym(n)$ – *the linear space of all symmetric matrices in* $Mat(n)$, $V = o(n)$ – *the linear space of all skew-symmetric matrices in* $Mat(n)$, *with the bracket*
$$[X, Y]_v = XvY - YvX\ . \tag{16.18}$$
- $\mathfrak{g}$ – *the linear space of all* $n \times m$ *matrices,* V – *the linear space of all* $m \times n$ *matrices, with the bracket*
$$[X, Y]_v = XvY - YvX\ . \tag{16.19}$$
- $\mathfrak{g} = V^\omega$ – *symplectic linear space with symplectic form* ω, $V = V^\omega$, *with the bracket*
$$[X, Y]_v = \omega(v, X)Y - \omega(v, Y)X - \omega(X, Y)v\ . \tag{16.20}$$
- *One – dimensional bundle generated by a simple Lie algebra, that is* $\mathfrak{g}$ – *simple Lie algebra,* $V = \mathbb{C}$, *with the bracket*
$$[X, Y]_v = v[X, Y]\ . \tag{16.21}$$

However, in such classification-type theorems, we must know exactly the definition of equivalent objects, and we could not find what is meant by equivalent bundles of Lie algebras in [1]. Thus, the above result seems to be unclear, because if for example we adopt the point of view that in the bundles $(\mathfrak{g}_1, V_1)$ and $(\mathfrak{g}_2, V_2)$ are equivalent if in the families $(\mathfrak{g}_1)_v,\ v \in V_1$ and $(\mathfrak{g}_2)_w,\ w \in V_2$ are contained the same abstract Lie algebras, and if we adopt the viewpoint that the equivalence between $(\mathfrak{g}_1, V_1)$ and $(\mathfrak{g}_2, V_2)$ means that there exists a pair of linear maps:

$$\begin{aligned} \mathcal{F} &: \mathfrak{g}_1 \mapsto \mathfrak{g}_2 \\ F &: V_1 \mapsto V_2 \end{aligned} \tag{16.22}$$

such that F, $\mathcal{F}$ is isomorphism and

$$\mathcal{F}([X, Y]_v) = [\mathcal{F}(X), \mathcal{F}(Y)]_{F(v)}\ , \tag{16.23}$$

it is not evident that we shall have the same classes of equivalence. The choice of the appropriate definition for equivalent bundles is related to the notions of homomorphism between two bundles of Lie algebras and subbundle of a bundle of Lie algebras, and, here, we can also have different types of definitions. We believe that it is natural to accept the following.

Definition 16.9. *The pair* $(\mathcal{F}, F)$ *of linear maps as in (16.22) is called homomorphism of the linear bundles* $(\mathfrak{g}_1, V_1)$ *and* $(\mathfrak{g}_2, V_2)$ *if they satisfy (16.23). If both* F *and* $\mathcal{F}$ *are isomorphisms, we shall call the bundles isomorphic. The bundle* $(\mathfrak{g}_1, V_1)$ *will be called a subbundle of* $(\mathfrak{g}_2, V_2)$ *if* $\mathfrak{g}_1 \subset \mathfrak{g}_2$, $V_1 \subset V_2$ *and if (16.23) holds for the inclusion maps* $F : V_1 \mapsto V_2$ *and* $\mathcal{F} : \mathfrak{g}_1 \mapsto \mathfrak{g}_2$.

The the non-isomorphic algebras in the families $\mathrm{o}\,(n)_J$ and $\mathrm{sym}\,(n)_I$, contained in $(\mathrm{o}\,(n), \mathrm{sym}\,(n))$ and $(\mathrm{sym}\,(n), \mathrm{o}\,(n))$ over $\mathbb{C}$ are classified in [1] using the following simple considerations. Suppose C is nondegenerate $n \times n$ matrix, denote by the upper index "t" the transposition, and consider the map

$$A \mapsto h_C(A) = C^t A C \ . \tag{16.24}$$

Then the linear spaces of symmetric and skew-symmetric $n \times n$ matrices are invariant under h_C. Moreover, one can see that h_C is isomorphism between the algebra $\mathrm{o}\,(n)_{J'}$; $J' = CJC^t$ (J-symmetric) and the algebra $\mathrm{o}\,(n)_J$, and h_C is isomorphism between the algebra $\mathrm{sym}\,(n)_{J'}$, $J' = CJC^t$ (J-skew-symmetric) and the algebra $\mathrm{sym}\,(n)_J$. As any symmetric or skew-symmetric bilinear form over $\mathbb{C}$ can be put into canonical form we obtain:

- For every $J \in \mathrm{sym}\,(n)$ exists C such that $CJC^t = \mathrm{diag}\,(\mathbf{1}_p, \mathbf{0}) = H_p$ (the blocks are $p \times p$ and $q \times q$ dimensional, $p + q = n$).
- For every $J \in \mathrm{o}\,(n)$ there exists C such that $CJC^t = I_{k,q} = \mathrm{diag}\,(S_{2k}, \mathbf{0}_q)$, $2k + q = n$, where the block S_{2k} is $2k \times 2k$ dimensional and has the form

$$S_{2k} = \begin{pmatrix} 0 & \mathbf{1}_k \\ -\mathbf{1}_k & 0 \end{pmatrix} . \tag{16.25}$$

Thus one needs to study only the algebras $\mathrm{o}\,(n)_{H_p}$ and $\mathrm{sym}\,(n)_{I_{k,q}}$. For the algebras $\mathrm{o}\,(n)_{H_p}$ we have the following splitting of the vector space $\mathrm{o}\,(n)$:

$$\mathrm{o}\,(n) = \mathrm{o}\,(p) \oplus (V_1 \oplus V_2) \ , \tag{16.26}$$

where $\mathrm{o}\,(p)$ consists of the matrices of the type (in block form)

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad A \in \mathrm{Mat}\,(p), \quad A^t = -A \ , \tag{16.27}$$

V_1 consists of the matrices of the type

$$\begin{pmatrix} 0 & B \\ -B^t & 0 \end{pmatrix}, \quad B \in \mathrm{Mat}\,(p, n-p) \ , \tag{16.28}$$

and V_2 consists of the matrices of the type

$$\begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}, \quad D \in \mathrm{Mat}\,(n-p), \quad D^t = -D \ . \tag{16.29}$$

If we consider (16.26) we can see that actually $\mathrm{o}\,(n)_{H_p}$ is semidirect sum of $o(p)$ and the radical of $\mathrm{o}\,(n)_{H_p}$, which is equal to $V_1 \oplus V_2$. On V_1 the action of $\mathrm{o}\,(p)$ is simply $n - k$ times the canonical action of $\mathrm{o}\,(p)$ on $\mathbb{C}^p$, on V_2 the action of $o(p)$ is trivial, $[V_1, V_2]_{H_p} \subset V_2$ and V_2 is the center of the algebra $\mathrm{o}\,(n)_{H_p}$. In particular, when $p = n - 1$, the center is zero and $\mathrm{o}\,(n)_{H_{n-1}}$ is a direct sum of $\mathrm{o}\,(n-1)$ and the radical V_1, which is $(n-1)$-dimensional

Abelian subalgebra. The algebra $\mathrm{o}\,(n)_{H_{n-1}}$ is isomorphic to $\mathrm{e}\,(n-1)$, that is, to the algebra corresponding to the group of rigid body movements of the $(n-1)$-dimensional Euclidean space.

Considerations of the same type show that the algebra $\mathrm{sym}\,(n)_{I_{k,q}}$ is semidirect sum of the symplectic algebra $\mathrm{sp}\,(2k)$ and the radical, which is of the type $V = V_1 \oplus V_2$. On V_1 the action of $\mathrm{sp}\,(2k)$ is $n-2k$ times the canonical action of $\mathrm{sp}\,(2k)$ on $\mathbb{C}^{2k}$, on V_2 the action of $\mathrm{sp}\,(2k)$ is trivial, $[V_1, V_2]_{I_{k,q}} \subset V_2$ and V_2 is the center of the algebra $\mathrm{sym}\,(n)_{I_{k,q}}$.

The case of the semisimple algebras being clear, in the list from theorem (16.8) only the bundle $(\mathrm{Mat}\,(p,q), \mathrm{Mat}\,(q,p))$ and the symplectic space case $(\mathfrak{g} = V^\omega, V = V^\omega)$ remains. The symplectic case is not treated in [1], but it can be treated along the same lines. Indeed, it is well known, for any symplectic form there exists a symplectic basis, that is, basis of V^ω:

$$\{X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n\}; \quad 2n = \dim\,(V^\omega),$$

such that

$$\omega(X_i, Y_j) = \delta_{ij}, \quad \omega(X_i, X_j) = \omega(Y_i, Y_j) = 0\,. \tag{16.30}$$

Moreover, the vector X_1 in this basis can be chosen to be any vector fixed beforehand, provided it is not equal to zero. Since the case $v = 0$ is trivial let us consider the algebra structure defined by $v = X_1 \neq 0$, and let us denote the corresponding algebra by $\mathfrak{g}_v$. Then the brackets (16.20) read:

$$\begin{aligned}
&[X_i, X_j] = 0\\
&[X_i, Y_j] = -\delta_{1j} X_i - \delta_{1i} X_1\\
&[Y_i, Y_j] = \delta_{1i} Y_j - \delta_{1j} Y_i\,,
\end{aligned} \tag{16.31}$$

and we see that all the algebras $\mathfrak{g}_v$ $(v \neq 0)$ are isomorphic.

Denote by $\mathrm{span}\,\{Z_1, Z_2, \ldots, Z_p\}$ the vector space spanned by the vectors Z_i. Then the spaces:

$$\begin{aligned}
\mathfrak{g}_1 &= \mathrm{span}\,\{X_1, X_2, \ldots, X_n\}\\
\mathfrak{g}_2 &= \mathrm{span}\,\{Y_1, Y_2, \ldots, Y_n\}
\end{aligned} \tag{16.32}$$

are subalgebras, $\mathfrak{g}_v = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $\mathfrak{g}_1$ is Abelian ideal in $\mathfrak{g}_v$. Next, as

$$\mathfrak{g}_2^{(1)} = [\mathfrak{g}_2, \mathfrak{g}_2] \subset \mathrm{span}\,\{Y_2, Y_3, \ldots, Y_n\}, \quad \mathfrak{g}_2^{(2)} = 0\,, \tag{16.33}$$

we see that $\mathfrak{g}_v^{(3)} = 0$ and $\mathfrak{g}_v$ is solvable.

In the sequel, we present a simple construction giving closed linear bundles of Lie algebras, but before doing it let us consider once more the abstract case. Suppose we have the closed linear bundle $(\mathfrak{g}, V)$, suppose also $X, Y \in \mathfrak{g}_v$. It is well known that $[\mathcal{L}^v_X, \mathcal{L}^v_Y] = \mathcal{L}^v_{[X,Y]_v}$. Applying this relation to the bracket L_w we get:

Proposition 16.10. *If the closed bundle* $(\mathfrak{g}, V)$ *is regular, every algebra* $\mathfrak{g}_v$ *possesses a natural representation*

$$\mathfrak{g} \ni X \mapsto F^v(X) \in End(V) , \tag{16.34}$$

given by the formula:

$$F^v(X)w = f(v, X, w) . \tag{16.35}$$

Consider now $u, v \in V$ and fixed $X \in \mathfrak{g}$. The operations $\mathcal{L}^u_X$ and $\mathcal{L}^v_X$ are of zero order for the graded module of the skew-symmetric maps, and, therefore, the commutator $[\mathcal{L}^u_X, \mathcal{L}^v_X]$ is also of zero order. We easily get:

Proposition 16.11. *Let* $(\mathfrak{g}, V)$ *be closed, regular linear bundle of Lie brackets. If for arbitrary* $u, v \in V$ *and* $X \in \mathfrak{g}$ *there exist the* 3 *– linear map:*

$$\Psi : V \times \mathfrak{g} \times V \mapsto V , \tag{16.36}$$

such that for arbitrary $w \in V$ *we have:*

$$[\mathcal{L}^u_X, \mathcal{L}^v_X]L_w = \mathcal{L}^{\Psi(u,X,v)}_X L_w , \tag{16.37}$$

then for fixed X *the map* $(u, v) \mapsto \Psi(u, X, v)$ *defines a Lie algebra structure on* V *and*

$$u \mapsto \Phi^X(u) : \ \Phi^X(u)w = f(u, X, w) \tag{16.38}$$

is a representation of this algebra.

It is not very surprising that for all the algebras in the list from the theorem (16.8), and also for the algebras from the construction we shall present in the next section, we find that the representations Φ^X are actually adjoint representations, in fact for all these cases $(u, v) \mapsto f(u, X, v)$ is a Lie bracket and $\Psi(u, X, v) = f(u, X, v)$. More precisely:

- For the bundle $(\mathrm{o}\,(n), \mathrm{sym}\,(n))$, $f(v, X, w) = [v, w]_X$ – the bracket for the bundle $(\mathrm{sym}\,(n), \mathrm{o}\,(n))$.
- For the bundle $(\mathrm{sym}\,(n), \mathrm{o}\,(n))$, $f(v, X, w) = [v, w]_X$ – the bracket for $(\mathrm{sym}\,(n), \mathrm{o}\,(n))$.
- For the bundle $(\mathrm{Mat}\,(n, m), \mathrm{Mat}\,(m, n))$, $f(v, X, w) = [v, w]_X$ – the bracket for the bundle $(\mathrm{Mat}\,(m, n), \mathrm{Mat}\,(n, m))$.
- For the bundle (V^ω, V^ω) the function $f(v, X, w) = [v, w]_X$ is the bracket in the same bundle.
- For the trivial case with the bracket $v[X, Y]$ the function $f(v, X, w)$ is identically zero.

The representations F^v for the bundles $(\mathrm{o}\,(n), \mathrm{sym}\,(n))$ and $(\mathrm{sym}\,(n), \mathrm{o}\,(n))$ can be cast in more convenient form. For example, let us consider the case $\mathrm{o}\,(n)_v$ where v is some symmetric matrix of rank n. Then the map (16.24) with C such that $v = C^{-1}(C^t)^{-1}$ establishes isomorphism between $\mathrm{o}\,(n)$ and $\mathrm{o}\,(n)_v$ and the map

$$X \mapsto \Psi_C(X) = F^v \circ h_C(X) \in \mathrm{End}\,(\mathrm{sym}\,(n)) \tag{16.39}$$

is a representation of $\mathrm{o}\,(n)$. The calculation shows that

$$\Psi_C = h_{(C^t)^{-1}} \circ \Psi \circ h_{(C^t)}\ , \tag{16.40}$$

where Ψ is the classical representation of $\mathrm{o}\,(n)$ into the space of the symmetric matrices:

$$\begin{aligned} &X \mapsto \Psi(X) \in \mathrm{End}\,(\mathrm{sym}\,(n)) \\ &\Psi(X)w = Xw - wX \end{aligned} \tag{16.41}$$

and h was defined in (16.24).

More generally,

- The representations F^v of $\mathrm{o}\,(n)_v$ for v having rank p is equivalent to the representation

  $$\begin{aligned} &X \mapsto \Psi_{H_p}(X) \in \mathrm{End}\,(\mathrm{sym}\,(n)) \\ &\Psi_{H_p}(X)w = H_pXw - wXH_p \end{aligned} \tag{16.42}$$

  of $\mathrm{o}\,(n)_{H_p}$.
- The representations F^v of $\mathrm{sym}\,(n)_v$ for v-skew-symmetric and having rank $2k$ is equivalent to the representation

  $$\begin{aligned} &X \mapsto \Psi_{I_{k,q}}(X) \in \mathrm{End}\,(\mathrm{o}\,(n)) \\ &\Psi_{I_{k,q}}(X)w = I_{k,q}Xw - wXI_{k,q} \end{aligned} \tag{16.43}$$

  of the algebra $\mathrm{sym}\,(n)_{I_{k,q}}$, $(2k + q = n)$.

16.3 Construction of Closed Linear Bundles of Lie Algebras

Consider the space $\mathrm{Mat}\,(n)$. On it there are two natural algebraic structures, induced from the associative algebra structure: the Lie algebra structure, defined by the commutator, and the structure of commutative algebra, defined by the anti-commutator $X_1 * X_2 = X_1X_2 + X_2X_1$. To distinguish the two structures we denote $\mathrm{Mat}\,(n)$ by $\mathrm{gl}\,(n)$ in the first case and $\mathrm{mat}\,(n)$ in the second. There exist the following natural maps:

- The representation of $\mathrm{gl}\,(n)$ into $\mathrm{End}\,(\mathrm{gl}\,(n))$:

  $$X \to F(X) : F(X)Y = -X^tY - YX\ . \tag{16.44}$$

- The map of Mat (n) into End (Mat (n)):

$$X \to G(X) : G(X)Y = X^t Y - YX \ . \tag{16.45}$$

Since F is representation, for fixed S the subspace

$$\mathfrak{g}_S = \{X : F(X)S = 0\} \subset \mathrm{gl}\,(n)\ , \tag{16.46}$$

is a Lie subalgebra in gl (n). The map G possesses the following interesting property:

$$G(X_1 * X_2) = -F(X_1)G(X_2) - F(X_2)G(X_1) \tag{16.47}$$

and because of it the space

$$V_S = \{J : G(J)S = 0\} \subset \mathrm{mat}\,(n) \tag{16.48}$$

is subalgebra of the algebra mat (n) containing the unity $\mathbf{1}_n$. Let us denote $(X_1 * X_2)_Y = X_1 Y X_2 + X_2 Y X_1$ and $[X_1, X_2]_Y = X_1 Y X_2 - X_2 Y X_1$. For $Y = \mathbf{1}_n$, these operations are the usual commutator and anti-commutator. By straightforward calculations one establishes :

Proposition 16.12. *The operations $(X_1 * X_2)_Y$ and $[X_1, X_2]_Y$ have the properties:*

$$\begin{aligned}
&[\mathfrak{g}_S, V_S]_J \subset V_S, \quad (\mathfrak{g}_S * V_S)_J \subset \mathfrak{g}_S \\
&[\mathfrak{g}_S, \mathfrak{g}_S]_J \subset \mathfrak{g}_S, \quad (\mathfrak{g}_S * \mathfrak{g}_S)_J \subset V_S \\
&[V_S, V_S]_J \subset \mathfrak{g}_S, \quad (V_S * V_S)_J \subset V_S
\end{aligned} \tag{16.49}$$

for $J \in V_S$ and

$$\begin{aligned}
&[\mathfrak{g}_S, V_S]_X \subset \mathfrak{g}_S, \quad (\mathfrak{g}_S * V_S)_X \subset V_S \\
&[\mathfrak{g}_S, \mathfrak{g}_S]_X \subset V_S, \quad (\mathfrak{g}_S * \mathfrak{g}_S)_X \subset \mathfrak{g}_S \\
&[V_S, V_S]_X \subset V_S, \quad (V_S * V_S)_X \subset \mathfrak{g}_S
\end{aligned} \tag{16.50}$$

for $X \in \mathfrak{g}_S$.

Corollary 16.13. *If the matrix S is nondegenerate, then $\mathfrak{g}_S \cap V_S = \{0\}$ and the linear space $G = \mathfrak{g}_S \oplus V_S$ is Lie algebra and commutative algebra with respect to all the brackets $[X, Y]_M$ and products $(X * Y)_M$ for $M \in G$. In particular, (G, G) is closed linear bundle of Lie algebras.*

Corollary 16.14. *$(\mathfrak{g}_S, V_S)$ is closed linear bundle of Lie algebras with bracket $[X_1, X_2]_J$. The map f, introduced in the definition of closed linear bundle of Lie algebras, has the form $f(J_1, Z, J_2) = [J_1, J_2]_Z$ and for every fixed Z, it endows the space V_S with a structure of Lie algebra, that is $(V_S, \mathfrak{g}_S)$ is closed bundle of Lie algebras with respect to the bracket $[J_1, J_2]_Z$, $Z \in \mathfrak{g}_S$. If J is nondegenerate and if in V_S there exists an element D such that $D^2 = J^{-1}$ then the map: $h : X \mapsto DXD$ is isomorphism between $\mathfrak{g}_S$ as Lie algebra with respect to the bracket $[X, Y]_J$ and $\mathfrak{g}_S$ as Lie algebra with respect to the standard bracket.*

Let us note, that even when we restrict ourself to the class of bundles of the form $(\mathfrak{g}_S, V_S)$ this class is large enough. For example, one readily sees that if we choose $S = \mathbf{1}_n$ then $\mathfrak{g}_S = \mathrm{o}\,(n)$ – the orthogonal algebras, if we choose

$$S = S_{2n} = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix} \in \mathrm{gl}\,(2n) \ , \tag{16.51}$$

we get the symplectic algebras $\mathrm{sp}\,(2n)$.[2] Thus, on all the simple algebras from the classical series $B_n = \mathrm{o}\,(2n+1)$; $n \geq 2$, $C_n = \mathrm{sp}\,(2n)$; $n \geq 3$ and $D_n = \mathrm{o}\,(2n)$; $n \geq 4$, we can define closed linear bundles of Lie algebras. Of course, closed linear bundle of Lie algebras can be defined also on the algebras $\mathrm{o}\,(3)$, $\mathrm{o}\,(4)$, $\mathrm{o}\,(6)$, $\mathrm{sp}\,(2)$, which are not included in the series B_n, C_n, and D_n or because they are semisimple but not simple (the case of $\mathrm{o}\,(4)$), or because they are isomorphic to some algebras of the complete list of the classical series of simple Lie algebras. The list of the classical Lie algebras includes in addition to the algebras cited above the algebras $A_n = \mathrm{sl}\,(n+1)$; $n > 1$ for which it seems that the above construction cannot be applied. (The algebra $\mathrm{sl}\,(4)$ is isomorphic too (6), and since on it there exists a closed linear bundle structure, the algebra $\mathrm{sl}\,(4)$ is an exception).

For the algebra $o(n)$; $n > 2$ the definition of the space V_S, $S = \mathbf{1}_n$ entails that J must be symmetric, that is $V_{\mathbf{1}} = \mathrm{sym}\,(n)$ and for $\mathrm{sp}\,(2n)$ we have

$$V_{S_{2n}} = \left\{ J : \ J = \begin{pmatrix} A & B \\ C & A^t \end{pmatrix} ; \ A, B, C \in \mathrm{gl}\,(n), \ B^t = -B, \ C^t = -C \right\} . \tag{16.52}$$

The space $V_{S_{2n}}$ contains the unity, so the bundle $(\mathrm{sp}\,(2n), V_{S_{2n}})$ contains the usual bracket. Therefore for $n \geq 3$ this bundle is closed and irreducible. It is not difficult to see, however, that this bundle is isomorphic to $(\mathrm{sym}\,(2n), \mathrm{o}\,(2n))$, the isomorphism (H, h) being:

$$\begin{aligned} H &: \ X \in \mathrm{sym}\,(2n) \mapsto D_{2n} X D_{2n} \in \mathrm{sp}\,(2n) \\ h &: \ J \in \mathrm{o}\,(2n) \mapsto D_{2n}^{-1} J D_{2n}^{-1} \in V_{S_{2n}} \end{aligned} \tag{16.53}$$

where D_{2n} is the matrix:

$$D_{2n} = \frac{i\sqrt{2}}{2} \begin{pmatrix} \mathbf{1}_n & -\mathbf{1}_n \\ \mathbf{1}_n & \mathbf{1}_n \end{pmatrix} . \tag{16.54}$$

It is easy to check that $D_{2n}^{-1} = -D_{2n}^t$ and $D_{2n}^2 = S_{2n}$. This entails not only that from $X \in \mathrm{sym}\,(2n)$ follows $D_{2n} X D_{2n} \in \mathrm{sp}\,(2n)$ and that from $J \in \mathrm{o}\,(2n)$ follows $D_{2n}^{-1} J D_{2n}^{-1} \in V_{S_{2n}}$ but also that

$$H([X, Y]_J) = [H(X), H(Y)]_{h(J)} . \tag{16.55}$$

[2] The symplectic algebra $\mathrm{sp}\,(2n)$ over $\mathbf{R}$ or $\mathbf{C}$ consist of the $2n \times 2n$ matrices X having the property $\omega_0(Xv, w) + \omega_0(v, Xw) = 0$ for $v, w \in \mathbf{R}^{2n}$ ($\mathbf{C}^{2n}$), where ω_0 is the canonical symplectic form on $\mathbf{R}^{2n}$ ($\mathbf{C}^{2n}$).

In particular, $H([X,Y]_{S_{2n}}) = [H(X), H(Y)]$.

The case of the bundles $(Mat(p,q), Mat(q,p))$ from the theorem (16.8) can be obtained via the above construction with a slight modification. Suppose that v is such that $v^2 = \mathbf{1}_n$. Then

$$\phi : X \mapsto vXv^{-1} \tag{16.56}$$

is an involution of $\mathrm{Mat}\,(n)$. Therefore $\mathrm{Mat}\,(n)$ splits into the spaces $\mathrm{Mat}^{\,+}(n)$ and $\mathrm{Mat}^{\,-}(n)$ corresponding to the eigenvalues $+1$ and -1 of ϕ. If now $(\mathfrak{g}_S, V_S)$ is one of the bundles constructed in the above, then the following proposition holds:

Proposition 16.15. *The pairs of spaces:* $(\mathfrak{g}_S \cap Mat^-(n), V_S \cap Mat^-(n))$ *and* $(\mathfrak{g}_S \cap Mat^+(n), V_S \cap Mat^+(n))$ *are linear bundles of Lie algebras with respect to the brackets, induced from* $(\mathfrak{g}_S, V_S)$*.*

Now if we choose $n = p+q$, $S = \mathbf{1}_n$, $v = \mathrm{diag}\,(\mathbf{1}_p, -\mathbf{1}_q)$, we can see that the space $\mathfrak{g}_S \cap \mathrm{Mat}^{\,-}(n)$ consists of the matrices having the block form:

$$\begin{pmatrix} 0 & b \\ -b^t & 0 \end{pmatrix}$$

(the blocks on the diagonal are of type $p \times p$ and $q \times q$, respectively). The space $V_S \cap \mathrm{Mat}^{\,-}(n))$ is the space of matrices of the form:

$$\begin{pmatrix} 0 & s^t \\ s & 0 \end{pmatrix}.$$

The first space is naturally isomorphic to $\mathrm{Mat}\,(p,q)$ and the second to $\mathrm{Mat}\,(q,p)$. It remains to see that the bracket structure on $\mathfrak{g}_S \cap \mathrm{Mat}^{\,-}(n)$ is actually equivalent to $[b_1, b_2]_s = b_1 s b_2 - b_2 s b_2$ where $b_1, b_2 \in \mathrm{Mat}\,(p,q)$ and $s \in \mathrm{Mat}\,(q,p)$. This completes the proof.

Let us prove some general properties of the bundles we have constructed. We have seen (see theorem (16.3)) that if $(\mathfrak{g}, V)$ is arbitrary linear bundle of Lie algebras, the Lie bracket L_v is 2-cocycle for the adjoint representation defined by the bracket $[X,Y]_w$ (that is with respect to d_w). For the bundles $(\mathfrak{g}_S, V_S)$ we can say a little more.

Proposition 16.16. *For the bundle* $(\mathfrak{g}_S, V_S)$ *consider the following maps:*

$$\begin{aligned} \alpha_J \; &: \mathfrak{g}_S \; \to \mathfrak{g}_S \\ \alpha_J(X) &= \frac{1}{2}(JX + XJ) \end{aligned} \tag{16.57}$$

where $J \in V_S$*.(One can easily check that* $\alpha_J(X) \in \mathfrak{g}_S$ *so the map* α_J *is properly defined). Let* $J_1, J_2 \in V_S$*. Then*

$$[d_{J_2}\alpha_{J_1}](X,Y) = [d_{J_1}\alpha_{J_2}](X,Y) = \frac{1}{2}[X,Y]_{J_1 * J_2} \tag{16.58}$$

where the coboundary operator d_{J_1} *corresponds to the bracket* $[X,Y]_{J_1}$ *with respect to the adjoint representation.*

Corollary 16.17. *Consider the bundle* $(\mathfrak{g}_S, V_S)$. *Then:*

- *The map* $(X, Y) \to [X, Y]_J$ *is* 2*-coboundary for the adjoint representation of* $\mathfrak{g}_S$ *with respect to the usual bracket.*
- *If* H_1, H_2 *are two commuting matrices belonging to* V_S *and if* H_1^{-1} *exists and belongs to* V_S, *then the map* $(X, Y) \to [X, Y]_{H_2}$ *is* 2*-coboundary for the adjoint representation of* $\mathfrak{g}$ *with respect to the bracket* $[X, Y]_{H_1}$. *In particular, if* H, H^{-1} *belong to* V_S *the usual bracket* $[X, Y]$ *is* 2*-coboundary for the adjoint representation of* $\mathfrak{g}$ *with respect to the bracket* $[X, Y]_H$.

Proof. Indeed, it is enough to put in (16.58) $J_1 = \mathbf{1}$ and $J_2 = J$ to obtain the first statement and to put $J_1 = H_1$, $J_2 = H_1^{-1} H_2$, in order to obtain the second.

The corollary shows that actually the new brackets arise as some coboundaries of the adjoint representation for the usual structure on $\mathfrak{g}$ defined by the commutator.

We believe that the beautiful properties of the above algebraic structures can find a large area of applications, but here we limit ourselves only to the applications of the bundles $(\mathfrak{g}_S, V_S)$ in the construction of compatible Poisson tensors. Just to outline the general idea, we know that the equations having a Lax representation can be considered in the formalism referred to as Adler scheme. In it, there are considered loop algebras [12, 13, 14, 15] (called sometimes also affine algebras), that is, algebras of formal Laurent series in λ with finite singular part and coefficients in a fixed algebra $\mathfrak{g}$. Such an algebra is denoted by $\mathfrak{g} \otimes [\lambda, \lambda^{-1}]$ and its elements are series of the type:

$$P_n = \sum_{i=-n}^{\infty} \lambda^i X_i; \quad X_i \in \mathfrak{g} \,. \tag{16.59}$$

One can see that using the properties of the bundles $(\mathfrak{g}_S, V_S)$, we can define algebras, having similar properties as the loop algebras. For example, one of the possibilities is to consider the algebras of the type $\mathfrak{g}_S^{p_r} = \left(\mathfrak{g}_S \otimes [\lambda, \lambda^{-1}]\right) p_r$ where p_r is a fixed element from the vector space $V_S \otimes [\lambda, \lambda^{-1}]$ - the vector space of Laurent series with finite singular part and coefficients in V_S. The elements of these algebras are elements of the type $\bar{P}_n = P_n p_r$, where P_n is as in (16.59). A brief calculation shows that

$$[P_n p_r, Q_m p_r] = ([P_n, Q_m]_{p_r}) p_r \,. \tag{16.60}$$

Therefore, we can consider the algebra $\mathfrak{g}_S^{p_r}(\lambda)$ as an algebra having the same underlying space as $\mathfrak{g}_S \otimes [\lambda, \lambda^{-1}]$ but endowed with different bracket:

$$(P_n, Q_m) \mapsto [P_n, Q_m]_{p_r} \,. \tag{16.61}$$

These algebras have been used already, as we shall see the pencils of polynomial Lax pairs for the $O(3)$ chiral fields system and Landau–Lifshitz equation belongs to the algebra $\mathrm{o}(4)^{(\lambda + J)}$, [2, 6]. The applications of the bundle

$(\mathrm{o}\,(4), \mathrm{sym}\,(4))$ are due to the remarkable property that it possesses a bilinear form invariant to all algebras in the bundle. This circumstance allows to use the Gel'fand-Fuchs cocycle in the same way we used it before and to obtain compatible Poisson on the linear space $\mathrm{o}\,(4)^{(\lambda+J)}[x]$ – the space of all Schwartz-type functions from $\mathbb{R}$ into the algebra $\mathrm{o}\,(4)^{(\lambda+J)}$. Without involving the Gel'fand-Fuchs cocycle one cannot introduce the spatial variable x. We shall consider this interesting case below.

There exist even more general objects – the algebras of the type

$$\mathcal{A}_S = \left(\mathfrak{g}_S \otimes [\lambda, \lambda^{-1}]\right)\left(V_S \otimes [\lambda, \lambda^{-1}]\right) \tag{16.62}$$

that is, algebras generated by the elements of the type $P_n p_s$; $P_n \in \mathfrak{g}_S \otimes [\lambda, \lambda^{-1}]$ and $p_s \in V_S \otimes [\lambda, \lambda^{-1}]$. It is easy to see that

$$[P_n p_s, Q_m q_r] = ([P_n, Q_m]_{p_s})q_r + Q_m([p_s, q_r]_{P_n}) \tag{16.63}$$

and then $\mathcal{A}_S$ becomes a Lie algebra in which the algebras $\mathfrak{g}_S^{p_r}(\lambda)$ are subalgebras. We do not know whether such more general algebras have been considered.

Since the natural application of the bundles of Lie algebras are in the theory of Poisson-Lie tensors, see below, it is important to calculate the coadjoint action of the new algebra structures. In what follows, we shall often assume that the restriction of the symmetric bilinear form $\mathrm{tr}\,(XY)$ over the algebra $\mathfrak{g}_S$ is nondegenerate. It is well known to be true for all the algebras $\mathrm{o}\,(n)$; $n > 2$, $\mathrm{sl}\,(n)$; $n > 1$ and $\mathrm{sp}\,(2n)$, considered in their natural representation, so our considerations are justified. Moreover, it is known that the restriction of the form $\mathrm{tr}\,(XY)$ onto one of these algebras is invariant under the adjoint action of the usual Lie algebra structure:

$$\mathrm{tr}\,(\mathrm{ad}_X(Y)Z) + \mathrm{tr}\,(Y \mathrm{ad}_X(Z)) = 0; \quad X, Y, Z \in \mathfrak{g}_S\,. \tag{16.64}$$

Remark 16.18. Actually, as is well known, if the algebra $\mathfrak{g}$ is simple and it is irreducibly represented, the trace form is proportional to the Killing form

$$B(X, Y) = \mathrm{tr}\,(\mathrm{ad}_X \mathrm{ad}_Y)\,. \tag{16.65}$$

We readily obtain

Proposition 16.19. *Suppose that the trace form $tr(XY)$ is nondegenerate on $\mathfrak{g}_S$. Then if we identify the spaces $\mathfrak{g}_S$ and $\mathfrak{g}_S^*$ through the bilinear form $tr(XY)$ the coadjoint action for the Lie algebra structure $[X, Y]_J$ is given by*

$$-(ad_X^J)^*(Y) = JXY - YXJ \tag{16.66}$$
$$X \in \mathfrak{g}_S, \quad Y \in \mathfrak{g}_S^* \sim \mathfrak{g}_S.$$

16.4 Poisson-Lie Tensors Over the Algebras $\mathfrak{g}_S$

Recall that there is a canonical way to equip the dual space $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ with Poisson structure (the Poisson-Lie structure, given by the Kirillov tensor). If $\mu \in \mathfrak{g}^*$ is a generic point, then if we identify

$$T_\mu(\mathfrak{g}^*) = \mathfrak{g}^*, \quad T^*_\mu(\mathfrak{g}^*) = \mathfrak{g}^{**} = \mathfrak{g}\ , \tag{16.67}$$

the Kirillov tensor over $\mathfrak{g}^*$ will be defined by the following field of linear maps:

$$\begin{aligned} &\mu \to L_\mu \in \operatorname{Hom}(\mathfrak{g}, \mathfrak{g}^*) \\ &L_\mu(X) = -\mathrm{ad}^*_X \mu; \quad X \in \mathfrak{g}. \end{aligned} \tag{16.68}$$

This structure was introduced earlier in this book (see end of Sect. 12.3), formulae (12.172) and (12.176). If f, g are two smooth functions on $\mathfrak{g}^*$, then

$$\{f, g\}(q) = -\langle q, [df_q, dg_q]\rangle\ , \tag{16.69}$$

where $\langle , \rangle$ is the canonical pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$.

Suppose that there exists symmetric nondegenerate bilinear form $B(X, Y)$ on $\mathfrak{g}$, invariant with respect to the adjoint action of $\mathfrak{g}$ – for example in the case of semisimple algebras, $B(X, Y)$ can be taken to be the Killing form. Then one can identify in canonical way $\mathfrak{g}^*$ and $\mathfrak{g}$. In more detail, to every $\mu \in \mathfrak{g}^*$ corresponds unique $X_\mu \in \mathfrak{g}$ such that

$$\langle \mu, Y\rangle = \mu(Y) = B(X_\mu, Y); \quad Y \in \mathfrak{g}\ . \tag{16.70}$$

In this case, the adjoint and coadjoint action coincide, and as we have seen earlier the Poisson-Lie bracket (16.69) can be written into the form

$$\{f, g\}(q) = -B(q, [df_q, dg_q])\ . \tag{16.71}$$

We remark, however, that generally speaking, one cannot expect that some bilinear form, for example the trace form is invariant for all the algebras in the bundle.

Now taking into account proposition (16.19) we get:

Proposition 16.20. *Suppose $(\mathfrak{g}_S, V_S)$ is the closed linear bundle of Lie algebras, of the type defined in the above. Then*

1. *For arbitrary $J \in V_S$, we can define the following Poisson-Lie tensor on the dual space $\mathfrak{g}^*_S$:*

$$q \to A_q : P^J_q(X) = -\left(ad^J_X\right)^*(q) \tag{16.72}$$

(here $X \in \mathfrak{g}_S$, $q \in \mathfrak{g}^*_S$) and for different J all these tensors are compatible.

2. Suppose the matrix Lie algebra $\mathfrak{g}_S$ is identified with its dual space $\mathfrak{g}_S^*$ by the nondegenerate bilinear form $tr(XY)$. Then on $\mathfrak{g}_S$ exists the following family of compatible Kirillov tensors:

$$\begin{aligned} & q \to P_q^J : P_q^J(X) = (JXq - qXJ) \\ & X \in \mathfrak{g}_S, \quad q \in \mathfrak{g}_S^* \sim \mathfrak{g}_S. \end{aligned} \tag{16.73}$$

For example, the pair $P_q^{\mathbf{1}}, P_q^{J^2}$, where J is diagonal matrix, J^2 is its square and $\mathbf{1}$ is the matrix unity, was used in [3] to describe the bi-Hamiltonian structure of the Euler equations on the algebras $\mathrm{o}(n)$.

Let us give another simple proof of the fact that for the algebras for which $\mathrm{tr}(XY)$ is nondegenerate, the tensor $\mu \mapsto K_\mu^A(X) = AX\mu - \mu XA$ is Poisson tensor under the, assumption that $A \in V_S$ is such that $D = A^{-\frac{1}{2}}$ exists and belongs to V_S. (on the example $\mathfrak{g}_S = \mathrm{o}(n)$, V_S – the symmetric $n \times n$ matrices we see that the assumption is true for the elements in general position and then by continuity of the Schouten bracket with respect to A the result follows for arbitrary element of V_S).

Proposition 16.21. *Suppose that for the algebra $\mathfrak{g}_S$, the 2-form $tr(XY)$ is nondegenerate and $A \in V_S$ is such that $D = A^{-\frac{1}{2}}$ exists and also belongs to V_S. Then the map $h : R \mapsto DRD$ is a linear isomorphism of the vector space $\mathfrak{g}_S$. Suppose we have identified $\mathfrak{g}_S$ and $\mathfrak{g}_S^*$ using the trace form $tr(XY)$. Let us consider the Kirillov tensor*

$$\begin{aligned} & \mu \mapsto K_\mu \in Hom(\mathfrak{g}_S, \mathfrak{g}_S^*) \\ & K_\mu(X) = -ad_X^*\mu \sim [X, \mu] \end{aligned} \tag{16.74}$$

for the usual bracket on $\mathfrak{g}_S$ and let us denote by K^h the image of K under the transformation h (that is K and K^h are h-related). Then K^h is equal to the Kirillov tensor

$$\begin{aligned} & \mu \mapsto K_\mu^A \in Hom(\mathfrak{g}_S, \mathfrak{g}_S^*) \\ & K_\mu^A(X) = -[ad_X^A]^*\mu = AX\mu - \mu XA\ . \end{aligned} \tag{16.75}$$

for the bracket $[X, Y]_A$.

Proof. If $\mathfrak{g}_S$ and $\mathfrak{g}_S^*$ are identified as stated in the theorem, we easily calculate that the maps $d[h]_\mu^{-1}$ and $\left(d[h]_\mu^{-1}\right)^*$ coincide and are equal to h^{-1}. Inserting into the relation

$$[K^h]_\mu = d[h]_\mu^{-1} \circ K_{h(\mu)} \circ \left(d[h]_\mu^{-1}\right)^*$$

we obtain that $K^h = K^A$, and this completes the proof.

Though simple, the above result shows that one can find the integral leaves of the foliation defined by the distribution

$$\mu \mapsto T_\mu = \{AX\mu - \mu XA;\ X \in \mathfrak{g}_S\}$$

using the orbits of the usual adjoint action. The construction is the following: one takes $\mu_0 \in \mathfrak{g}_S^*$, then finds the orbit $O_{h(\mu_0)}$ and the needed leaf is

$$O^A_{\mu_0} = h^{-1}\left(O_{h(\mu_0)}\right) . \tag{16.76}$$

There is another algebraic mechanism we used earlier in this part of the book (see Sect. (15.2.3)), in order to obtain compatible Poisson tensors. The construction involves a 2-cocycle γ of the trivial action of a given Lie algebra $\mathfrak{g}$ in order to make an extension with Abelian kernel and use the Poisson-Lie bracket defined by this extension, appropriately restricted. One can use even trivial cocycles. Applying this construction to the case we consider now we get the field of Poisson tensors

$$\mu \to P_\mu : P_\mu(X) = -\mathrm{ad}^*_X\mu - \gamma(X, .);\quad X \in \mathfrak{g}, \mu \in \mathfrak{g}^* , \tag{16.77}$$

and in the case of trivial cocycle defined by $\mu_0 \in \mathfrak{g}^*$ the above reduces to

$$\mu \to P_\mu : P_\mu(X) = -\mathrm{ad}^*_X\mu - \mathrm{ad}^*_X\mu_0;\quad X \in \mathfrak{g} . \tag{16.78}$$

However, there are some specific points here. First, one cannot perform the construction simultaneously for all the algebras in the bundle. Indeed, if γ is a 2-cocycle for one of the algebra structures, say $\mathfrak{g}_v$ in $(\mathfrak{g}_S, V_S)$, it usually is not a cocycle for the other structures. Even a trivial cocycle is not generally a "simultaneous" cocycle. Indeed, let $J_1 \in V_S$. The skew-symmetric bilinear form:

$$[\mu_0, J_1](X, Y) = \langle \mu_0, [X, Y]_{J_1}\rangle , \tag{16.79}$$

is cocycle for the trivial representation of the algebra, if for if $X_1, X_2, X_3 \in \mathfrak{g}_S$ we have

$$d_{J_2} b[\mu_0, J_1](X_1, X_2, X_3) = \langle \mu_0, [X_1, X_2]_{[J_1,J_2]X_3}\rangle + \mathrm{cycl}\,(1, 2, 3) . \tag{16.80}$$

This expression generally speaking is not equal to zero. Also, as we have seen, one cannot usually find a form that is invariant under all the Lie algebras in the bundle, so apparently one cannot define the Gel'fand-Fuchs cocycle which was so important in constructing the compatible Poisson structures for the evolution equations with spatial variable we considered earlier. It is remarkable that for some bundles it still can be done.

16.5 Finite Dimensional Applications

16.5.1 The Clebsh and the Neumann System

Let us consider some applications of the above constructions. The first of them will be finite dimensional and related the so-called Clebsh and Neumann systems. The Clebsh system is the system of the following equations:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{x} \times \mathbf{p} \\ \dot{\mathbf{p}} &= \mathbf{x} \times \hat{A}\mathbf{x} ,\end{aligned} \tag{16.81}$$

where $\mathbf{x}, \mathbf{p} \in \mathbb{C}^3$; $\hat{A} = \mathrm{diag}\,(a_1, a_2, a_3)$, $a_i \in \mathbb{C}$,

$$\begin{aligned}\mathbf{x} &= (x_1, x_2, x_3)^t \\ \hat{A}\mathbf{x} &= (a_1x_1, a_2x_2, a_3x_3)^t .\end{aligned} \tag{16.82}$$

Remark 16.22. Actually, the Clebsh system is called the above system when the vectors $\mathbf{x}$, $\mathbf{p}$ are real and the numbers a_i are positive. We prefer to work with the complex numbers, in order to avoid consideration of the real forms of the corresponding algebras, but of course one can easily pass to real variables.

For $\mathbf{x}, \mathbf{y} \in \mathbb{C}^3$ denote

$$[\mathbf{x}]^+ = \begin{pmatrix} 0 & x_1 & x_2 & 0 \\ -x_1 & 0 & x_3 & 0 \\ -x_2 & -x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{16.83}$$

and

$$[\mathbf{y}]^- = \begin{pmatrix} 0 & 0 & 0 & y_3 \\ 0 & 0 & 0 & -y_2 \\ 0 & 0 & 0 & y_1 \\ -y_3 & y_2 & -y_1 & 0 \end{pmatrix} . \tag{16.84}$$

Consider now the splitting of the algebra $\mathrm{o}\,(4)$ into two subspaces:

$$\begin{aligned}\mathrm{o}\,(4) &= \mathfrak{h}^+ \oplus \mathfrak{h}^- \\ \mathfrak{h}^+ &= \{[\mathbf{x}]^+ : \ \mathbf{x} \in \mathbb{C}^3\} \\ \mathfrak{h}^- &= \{[\mathbf{y}]^- : \ \mathbf{y} \in \mathbb{C}^3\} .\end{aligned} \tag{16.85}$$

As this subspaces are the eigenspaces of the involutive automorphism $r : X \mapsto vXv^{-1}$ where $v = \mathrm{diag}\,(1, 1, 1, -1)$ we have

$$[\mathfrak{h}^+, \mathfrak{h}^+] \subset \mathfrak{h}^+, \quad [\mathfrak{h}^+, \mathfrak{h}^-] \subset \mathfrak{h}^-, \quad [\mathfrak{h}^-, \mathfrak{h}^-] \subset \mathfrak{h}^+ , \tag{16.86}$$

and the spaces $\mathfrak{h}^+, \mathfrak{h}^-$ are orthogonal with respect to any invariant inner product (with respect to the usual bracket) on $\mathrm{o}\,(4)$. If A is diagonal matrix we have $vAv^{-1} = A$ and we deduce that

$$[\mathfrak{h}^+, \mathfrak{h}^+]_A \subset \mathfrak{h}^+, \quad [\mathfrak{h}^+, \mathfrak{h}^-]_A \subset \mathfrak{h}^-, \quad [\mathfrak{h}^-, \mathfrak{h}^-]_A \subset \mathfrak{h}^+ . \tag{16.87}$$

Therefore, if we identify $\mathrm{o}\,(4)$ and $\mathrm{o}\,(4)^*$ using the Killing form, we have

$$[\mathrm{ad}\,^A_{\mathfrak{h}^+}]^*(\mathfrak{h}^+) \subset \mathfrak{h}^+, \quad [\mathrm{ad}\,^A_{\mathfrak{h}^+}]^*(\mathfrak{h}^-) \subset \mathfrak{h}^-, \quad [\mathrm{ad}\,^A_{\mathfrak{h}^-}]^*(\mathfrak{h}^-) \subset \mathfrak{h}^+ . \tag{16.88}$$

After some simple calculations we get:

Proposition 16.23. *The Clebsh system is equivalent to the system of matrix equations:*

$$\begin{aligned}&\frac{d}{dt}[\mathbf{p}]^+ = A([\mathbf{x}]^-)^2 - ([\mathbf{x}]^-)^2 A\\&\frac{d}{dt}[\mathbf{x}]^- = -A[\mathbf{x}]^-[\mathbf{p}]^+ + [\mathbf{p}]^+[\mathbf{x}]^- A = -[\mathbf{x}]^-[\mathbf{p}]^+ + [\mathbf{p}]^+[\mathbf{x}]^-\\&A = \mathrm{diag}(\mathrm{a}_3, \mathrm{a}_2, \mathrm{a}_1, 1)\ .\end{aligned} \tag{16.89}$$

Remark 16.24. It is easy to see that $A[\mathbf{x}]^-[\mathbf{p}]^+ = [\mathbf{x}]^-[\mathbf{p}]^+$ and $[\mathbf{p}]^+[\mathbf{x}]^- A = [\mathbf{p}]^+[\mathbf{x}]^-$.

If we denote $\mu = [\mathbf{p}]^+ + i[\mathbf{x}]^-$ where $i = \sqrt{-1}$ we arrive at the following

Corollary 16.25. *The Clebsh system can be written as a system on the coadjoint orbit:*

$$\frac{d}{dt}\mu = -\left[ad_\xi^A\right]^* \mu; \quad \xi = -\pi^-(\mu)\ , \tag{16.90}$$

for the Lie algebra structure defined by the bracket $[X, Y]_A = XAY - YAX = ad_X^A(Y)$ *if we identify* $o(4)$ *and* $o(4)^*$ *through the Killing form* $B(X, Y) = 2tr(XY)$. *Here* π^- *is the projection onto* $\mathfrak{h}^-$ *with respect to the splitting (16.85).*

The above form of the Clebsh system can be obtained from the Lax pair for the Clebsh system, used in [16] to find the general solution of the Clebsh system in Riemann θ-functions.

It is easy to see the integrals in involution for (16.89) needed for complete integrability. First of them is the Hamiltonian:

$$H_1 = \frac{i}{2}\mathrm{tr}\,(\pi^-(\mu)\pi^-(\mu)) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)\ . \tag{16.91}$$

Next, the ring $\mathcal{R}$ of the central functions for the Poisson-Lie bracket defined by $[X, Y]_A$ (A-diagonal and nondegenerate) on $\mathrm{o}\,(n)$ can easily be obtained for the corresponding ring of central functions for the usual Poisson-Lie bracket on $\mathrm{o}\,(n)$. The last is generated by

$$\mathrm{tr}\ \left[\mu A^{-1}\right]^{2l}; \quad 2l < n \tag{16.92}$$

when n is odd, and if n is even, we must add to the above family the function $\mathrm{Pf}(\mu A^{-1}) = \sqrt{\det(\mu A^{-1})}$. So in our case we have the integrals of motion

$$H_2 = -\frac{1}{2}(\det A)\mathrm{tr}\,(\mu A^{-1}\mu A^{-1}) = \left(\sum_{i=1}^{3} a_i p_i^2\right) - a_1a_2x_3^2 - a_1a_3x_2^2 - a_2a_3x_1^2 \tag{16.93}$$

and $\mathrm{Pf}(\mu A^{-1}) = \sqrt{\det(\mu A^{-1})}$, which up to a constant multiplier is equal to

$$H_3 = \frac{i}{8} B_T(\mu, \mu) = \sum_{j=1}^{3} p_j x_j \;, \tag{16.94}$$

(the form B_T is another invariant form on $\mathrm{o}(4)$ that shall be introduced later (see (16.132)). The integrals H_1, H_2, H_3 ensure that the Clebsh system is completely integrable. Of course, the complete integrability of the Clebsh system is well known, but here we obtained it using some "unusual" bracket, and this illustrates the possibilities one has working with different Poisson structures.

As another application, we shall consider one natural generalization of the so-called Neumann system. The Neumann system is the following system of ordinary differential equations

$$\begin{aligned} \dot{x}_i &= y_i \\ \dot{y}_i &= -a_i x_i + (< \hat{A}\mathbf{x}, \mathbf{x} > -||\mathbf{y}||^2) x_i \\ < \mathbf{x}, \mathbf{y} > &= 0, \quad ||\mathbf{x}||^2 = 1 \\ i &= 1, 2, \ldots, n \;. \end{aligned} \tag{16.95}$$

where $\hat{A} = \mathrm{diag}\,(a_1, a_2, \ldots, a_n)$, by $\mathbf{z}$ is denoted a column vector with components z_i and by $< .,. >$ – the standard inner product in $\mathbb{R}^n$, defining the norm $||.||$. In vector notation, the system will have the form:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{y} \\ \dot{\mathbf{y}} &= -\hat{A}\mathbf{x} + (< \hat{A}\mathbf{x}, \mathbf{x} > -||\mathbf{y}||^2)\mathbf{x} \\ < \mathbf{x}, \mathbf{y} > &= 0, \quad ||\mathbf{x}||^2 = 1 \;. \end{aligned} \tag{16.96}$$

Let us assume again that $\mathbf{x}, \mathbf{y}$ are complex and let us write $< \mathbf{z}, \mathbf{w} >$ for $\sum\limits_{i=1}^{n} z_i w_i$ and $||\mathbf{z}||^2$ for $\sum\limits_{i=1}^{n} z_i^2$. Of course, now $||\mathbf{z}||^2$ can be a complex number. As mentioned before, in similar situations the reduction to the real case can be performed without difficulties.

It is known that the Neumann system can be cast in the form suggested by K. Uhlenbeck:

$$\begin{aligned} &\dot{X} = [P, X], \qquad \dot{P} = [X, \hat{A}] \\ &< \mathbf{x}, \mathbf{y} >= 0, \quad ||\mathbf{x}||^2 = 1 \;, \end{aligned} \tag{16.97}$$

where

$$X = \mathbf{x}\mathbf{x}^t, \qquad P = \mathbf{y}\mathbf{x}^t - \mathbf{x}\mathbf{y}^t \;. \tag{16.98}$$

The above form of the Neumann system permitted T. Ratiu to give an interpretation of it as a system on coadjoint orbit for the semidirect product of Lie algebras: $\mathrm{o}(n) \times s(n)$ (the first one is considered the algebra of the skew-symmetric $n \times n$ matrices with respect of the commutator and $s(n)$ is an

Abelian algebra, which underlying vector space, is the space of the symmetric $n \times n$ matrices, see [17]). T. Ratiu proved that the orbit is $(2n-2)$-dimensional and found complete series of integrals in involution using the following Lax representation of the Neumann system:

$$\begin{aligned} &\dot{L} = [L, M] \\ &L = \lambda^2 \hat{A} + P\lambda - X, \qquad M = -\lambda\hat{A} - P \,. \end{aligned} \tag{16.99}$$

We shall cast now the Neumann system into another form. To this end, consider the Lie algebra structure defined on the algebra $\mathrm{o}\,(n+1)$ by the bracket $[R, S]_A = RAS - SAR$, where

$$A = \mathrm{diag}\,(a_1, a_2, \ldots, a_n, 1) \,. \tag{16.100}$$

In what follows, we shall write the elements of $\mathrm{o}\,(n+1)$ in block form:

$$S = \begin{pmatrix} Q & \mathbf{z} \\ -\mathbf{z}^t & 0 \end{pmatrix} , \tag{16.101}$$

where $Q \in \mathrm{o}\,(n)$ and $\mathbf{z} \in \mathbb{C}^n$. The matrix A is then written as

$$A = \begin{pmatrix} \hat{A} & 0 \\ 0 & 1 \end{pmatrix} . \tag{16.102}$$

Similar, to the case of the algebra $\mathrm{o}\,(4)$ we have the orthogonal splitting $\mathrm{o}\,(n+1) = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ (with respect to the Killing form), corresponding to the eigenspaces of the involutive automorphism $S \mapsto VSV$, where V is the matrix:

$$V = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & -1 \end{pmatrix} . \tag{16.103}$$

It can be checked that

$$\begin{aligned} \mathfrak{g}^+ &= \left\{ X : \; X = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} ; Q \in \mathrm{o}\,(n) \right\} \\ \mathfrak{g}^- &= \left\{ X : \; X = \begin{pmatrix} 0 & \mathbf{z} \\ -\mathbf{z}^t & 0 \end{pmatrix} ; \mathbf{z} \in \mathbb{C}^n \right\} . \end{aligned} \tag{16.104}$$

Exactly as in the case of the algebra $\mathrm{o}\,(4)$, we have the relations (16.86), (16.87) and (16.88), with $\mathfrak{g}^\pm$ instead $\mathfrak{h}^\pm$.

Let us denote the projections onto the spaces $\mathfrak{g}^\pm$ by the upper indices "+" and "−". Next, let us put

$$\begin{aligned} &S^+ = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} , \quad S^- = \begin{pmatrix} 0 & \mathbf{x} \\ -\mathbf{x}^t & 0 \end{pmatrix} \\ &P = \mathbf{y}\mathbf{x}^t - \mathbf{x}\mathbf{y}^t \,. \end{aligned} \tag{16.105}$$

Now a simple calculation shows that

Proposition 16.26. *The Neumann system is equivalent to the system of matrix equations:*

$$\frac{d}{dt}S^+ = A[S^-]^2 - [S^-]^2 A$$
$$\frac{d}{dt}S^- = -AS^-S^+ + S^+S^-A$$
$$< \mathbf{x}, \mathbf{y} > = 0, \quad ||\mathbf{x}||^2 = 1 \,. \tag{16.106}$$

Suppose now we identify $\mathrm{o}\,(n+1)$ and $\mathrm{o}^*(n+1)$ through the Killing form. Then remembering the coadjoint action for the bracket $[X, Y]_A$ and putting $\mu = iS^- + S^+$ we get:

Proposition 16.27. *The Neumann system is equivalent to the system on a coadjoint orbit of $o\,(n+1)^*$:*

$$\frac{d\mu}{dt} = -[ad_\xi^A]^*\mu, \quad \xi = \pi^-(\mu) \,, \tag{16.107}$$

restricted by the conditions

$$tr\,(\pi^-(\mu)\pi^-(\mu)) = tr\,(\xi^2) = 1$$
$$\pi^+(\mu) = \pi^+(\mu)[\pi^-(\mu)]^2 + [\pi^-(\mu)]^2\pi^+(\mu) \,, \tag{16.108}$$

where by $\pi^\pm$ are denoted the projections onto $\mathfrak{g}^\pm$.

It is clear that the system (16.107) is a generalization of the Neumann system and the later is obtained if we take the general system on the level surface S_0 corresponding to $H = \frac{i}{2}$, where the Hamiltonian function is H is equal to $\frac{i}{2}\mathrm{tr}\,(\pi^-(\mu)\pi^-(\mu))$. Further, there is one more restriction on the system, defined by the second equation in (16.108). It is easy to see that (16.108) is invariant under the Hamiltonian flow on S_0 so this condition is consistent with the evolution.

The Neumann system is completely integrable. We believe that it would be interesting to investigate whether the nonrestricted system is also completely integrable and if so, to obtain the Neumann system from the general case as Hamiltonian reduction.

16.5.2 The Algebra o (4)

Let us concentrate now on the algebra $\mathrm{o}\,(4)$, which is the Lie algebra of 4×4 skew-symmetric matrices. We have considered it into above with relation to the Clebsh system, but now we shall introduce on it a different splitting. The algebra $\mathrm{o}\,(4)$ and the dynamic systems that can be defined on it attracted considerable attention, and there exist a large number of interesting and deep results about the dynamic systems and their integrals of motion on $\mathrm{o}\,(4)$ (see for example [18]). A complete up to date bibliography about the integrable systems on $\mathrm{o}(4)$ can be found in [1]. However, as far as we know, the possibility of introducing on $\mathrm{o}(4)$ other symmetric nondegenerate bilinear form, not proportional to $\mathrm{tr}(XY)$ and having the remarkable property that it is invariant simultaneously with respect to the algebras of the 10-dimensional bundle of Lie algebras is unique and was not been revealed until recently, [2]. Due to the existence of such form the coadjoint action of these algebras coincides with their adjoint action and using Gel'fand-Fuchs cocycle, we are able to obtain new family of Poisson tensors. In [8], in order to calculate the Nijenhuis operator for the hierarchy of $O(3)$ chiral fields system, have been used the invariant form we are speaking about, but the understanding that there exist relations with the closed linear bundles of Lie algebras came later, [2]. So now we are able to present the whole algebraic picture.

In order to perform the calculations in an efficient way, let us make some preparations. For $\mathbf{u} \in \mathbb{K}^3$ we write

$$\mathbf{u} \to \{\mathbf{u}\}_I = \begin{pmatrix} 0 & u_1 & u_2 & u_3 \\ -u_1 & 0 & u_3 & -u_2 \\ -u_2 & -u_3 & 0 & u_1 \\ -u_3 & u_2 & -u_1 & 0 \end{pmatrix} , \tag{16.109}$$

$$\mathbf{v} \to \{\mathbf{v}\}_{II} = \begin{pmatrix} 0 & v_1 & v_2 & -v_3 \\ -v_1 & 0 & v_3 & v_2 \\ -v_2 & -v_3 & 0 & -v_1 \\ v_3 & -v_2 & v_1 & 0 \end{pmatrix} . \tag{16.110}$$

It is easy to show that every element $A \in \mathrm{o}(4)$ can be written into the form

$$A = \{\mathbf{u}\}_I + \{\mathbf{v}\}_{II} , \tag{16.111}$$

and this representation corresponds to the well known splitting of $\mathrm{o}(4)$ into direct sum of two $\mathrm{o}(3)$ algebras. In other words, the subalgebras

$$\begin{aligned} \mathfrak{g}_I &= \left\{ \{\mathbf{u}\}_I : \mathbf{u} \in \mathbb{K}^3 \right\} \subset \mathrm{o}(4) \\ \mathfrak{g}_{II} &= \left\{ \{\mathbf{u}\}_{II} : \mathbf{u} \in \mathbb{K}^3 \right\} \subset \mathrm{o}(4) \end{aligned} \tag{16.112}$$

are both isomorphic to the algebra $\mathrm{o}(3)$ and are ideals in $\mathrm{o}(4)$. As a consequence $[\mathfrak{g}_I, \mathfrak{g}_{II}] = 0$ and the subspaces $\mathfrak{g}_I, \mathfrak{g}_{II}$ are orthogonal with respect to the Killing form on $\mathrm{o}(4)$.

Remark 16.28. The above splitting of $\mathrm{o}(4)$ into a sum of two ideals is inspired from the left and right action of the body of quaternions $\mathbb{H}$ (considered an algebra over $\mathbb{C}$) on itself. If one takes the usual basis $\{1, i, j, k\}$ in $\mathbb{H}$ then the matrices of the left multiplications by the elements i, j, k generate $\mathfrak{g}_I$ and the matrices of the right multiplications by i, j, k generate $\mathfrak{g}_{II}$.

We now note that there exists a liner involution T, which defines the splitting (16.112). Indeed, let us define the linear map $T : \mathrm{o}(4) \to \mathrm{o}(4)$

$$(T(X))_{ij} = \frac{1}{2}\epsilon_{ijks}X_{ks}; \quad i,j,k,s = 1,2,3,4\ , \tag{16.113}$$

where ϵ_{ijks} is the skew-symmetric Levi-Civita symbol and as the rule about the summation over repeated indices is assumed.

Remark 16.29. It is interesting to note that T coincides with the Hodge-star operator, if one consider the matrices of $\mathrm{o}(4)$ as skew-symmetric tensors, but we cannot say if this is a mere coincidence or there is something deeper here.

The algebraic properties of T are listed in the following proposition.

Proposition 16.30. *Let T be the linear map $T : o(4) \to o(4)$ defined in (16.113). Then*

1. *T is an involution:*
$$T^2 = id_{o(4)} = E\ . \tag{16.114}$$
2. *T is symmetric with respect to the Killing form:*
$$B(T(X),Y) = B(X,T(Y)); \quad X,Y \in o(4)\ . \tag{16.115}$$
3. *$[T(X),T(Y)] = [X,Y]; \quad X,Y \in o(4)$.*
4. *The two $o(3)$ subalgebras of $o(4)$ are invariant under the action of T:*
$$T(\{\mathbf{u}\}_I) = \{\mathbf{u}\}_I, \quad T(\{\mathbf{u}\}_{II}) = -\{\mathbf{u}\}_{II}\ , \tag{16.116}$$
and, therefore, the orthogonal projectors P_I, $P_{II} = E - P_I$ on the subalgebras $\mathfrak{g}_I$ and $\mathfrak{g}_{II}$ (with respect to the Killing form) are given by
$$P_I = \frac{1}{2}(E+T), \quad P_{II} = \frac{1}{2}(E-T)\ , \tag{16.117}$$
where, as in the above, E is the identity map.
5. *The form $B([X,Y],T(Z))$ is a 3-cocycle for the trivial representation of the algebra $o(4)$.*

We can consider now the general structure of a linear bundle over $\mathrm{o}(4)$ defined by the bracket

$$\begin{aligned} [X,Y]_J &= XJY - YJX = d\alpha_J(X,Y) \\ \alpha_J(X) &= \frac{1}{2}(XJ + JX)\ , \end{aligned} \tag{16.118}$$

where J is symmetric matrix. Let us write the new bracket into the form

$$[X,Y]_J = X\hat{J}Y - Y\hat{J}X + \frac{1}{4}\mathrm{tr}\,(J)[X,Y]$$
$$\hat{J} = J - \frac{1}{4}\mathrm{tr}\,(J)\mathbf{1}\ , \tag{16.119}$$

the matrix $\hat{J}$ being now traceless. From this formula, it is clear that without loss of generality one can consider only the structures defined by traceless matrices. Let

$$\alpha_{\hat{J}}(X) = \frac{1}{2}(X\hat{J} + \hat{J}X)\ , \tag{16.120}$$

be the cocycle defined by $\hat{J}$. We now have:

Proposition 16.31. *If J is any symmetric, traceless 4×4 matrix, then*

$$T \circ \alpha_J = -\alpha_J \circ T\ . \tag{16.121}$$

Proof. It is enough to show that

$$\begin{aligned} T(\alpha_J(X)) &= -\alpha_J(X); \quad X \in \mathfrak{g}_I \\ T(\alpha_J(X)) &= \alpha_J(X); \quad X \in \mathfrak{g}_{II}\ . \end{aligned} \tag{16.122}$$

Let us prove for example the first identity. For this we write

$$[T(\alpha_J(X))]_{ij} = \frac{1}{4}\epsilon_{ijks}(J_{kl}X_{ls} + X_{kl}J_{ls}),$$

(here and below summation over repeated indices is assumed), and insert into it

$$X_{ls} = \frac{1}{2}\epsilon_{lspq}X_{pq}, \quad X_{kl} = \frac{1}{2}\epsilon_{klpq}X_{pq}.$$

Then after using the identity,

$$\epsilon_{ijkm}\epsilon_{pqrm} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix}$$

and taking into account that X is skew-symmetric, J – symmetric and traceless, we get exactly $-[\alpha_J(X)]_{ij}$ which completes the proof.

The proposition shows that α_J interchanges the algebras $\mathfrak{g}_I$ and $\mathfrak{g}_{II}$, that is

$$\alpha_J(\mathfrak{g}_I) \subset \mathfrak{g}_{II}, \quad \alpha_J(\mathfrak{g}_{II}) \subset \mathfrak{g}_I\ . \tag{16.123}$$

As an example, consider

$$J = \mathrm{diag}\,(-j_1 - j_2 + j_3, -j_1 + j_2 - j_3, j_1 - j_2 - j_3, j_1 + j_2 + j_3)\ . \tag{16.124}$$

The calculation shows that

$$\alpha_J(\{\mathbf{u}\}_I) = -\{K\mathbf{u}\}_{II}, \quad \alpha_J(\{\mathbf{u}\}_{II}) = -\{K\mathbf{u}\}_I \ , \tag{16.125}$$

where $K = \operatorname{diag}(j_1, j_2, j_3)$ and $(K\mathbf{u})_s = j_s u_s$; $s = 1, 2, 3$ (summation is not assumed). As $[X, Y]_J = d\alpha_J(X, Y)$ the expression for $\alpha_J(X)$ permits to calculate explicitly the brackets of the Lie-algebra structures in the bundle. According to what we know about the bundles $(\mathrm{o}\,(n), \mathrm{sym}\,(n))$ the algebra we shall obtain depends on the rank of the matrix J. Here one can check this explicitly. Indeed, let J be as above, and let us introduce a basis of $\mathrm{o}\,(4)$:

$$\{\mathbf{e}_i\}_I, \{\mathbf{e}_i\}_{II}; \quad i = 1, 2, 3,$$

where $\{\mathbf{e}_i\}_1^3$ is the standard basis in $\mathbb{C}^3$. Then the determinant Δ_J of the Gram matrix for the Killing form associated with the bracket $[X, Y]_J$ in the above basis is $\Delta_J = -64(\det J)^3$. This of course is in agreement with the fact that the algebra $\mathrm{o}\,(4)_J$ is isomorphic to $\mathrm{o}\,(4)$ (with usual bracket), if and only if J is not degenerate. For the case when the rank of J is equal to 3, the general theory shows that the algebra we obtain is isomorphic to $\mathrm{e}\,(3)$. In order to see it explicitly, we introduce in $\mathrm{o}\,(4)$ the basis

$$\mathbf{e}_i^+ = \{\mathbf{e}_i\}_I + \{\mathbf{e}_i\}_{II}, \quad \mathbf{e}_i^- = \{\mathbf{e}_i\}_I - \{\mathbf{e}_i\}_{II}; \quad i = 1, 2, 3 \ . \tag{16.126}$$

If

$$\mathbf{x}^+ = x_i \mathbf{e}_i^+, \quad \mathbf{y}^- = y_i \mathbf{e}_i^-$$

(summation over repeated indices is assumed) it can be calculated that

$$\begin{aligned}
[\mathbf{x}^+, \mathbf{y}^+]_J &= 2(j_3 + j_2 - j_1)(\mathbf{x} \times \mathbf{y})_1 \mathbf{e}_1^+ \\
&\quad + 2(j_3 + j_1 - j_2)(\mathbf{x} \times \mathbf{y})_2 \mathbf{e}_2^+ + 2(j_1 + j_2 - j_3)(\mathbf{x} \times \mathbf{y})_3 \mathbf{e}_3^+ \\
[\mathbf{x}^-, \mathbf{y}^-]_J &= -2(j_1 + j_2 + j_3)(\mathbf{x} \times \mathbf{y})^- \\
[\mathbf{x}^+, \mathbf{y}^-]_J &= -2[(j_3 - j_2)(x_2 y_3 + x_3 y_2) - j_1(\mathbf{x} \times \mathbf{y})_1] \mathbf{e}_1^- \\
&\quad - 2[(j_1 - j_3)(x_3 y_1 + x_1 y_3) - j_2(\mathbf{x} \times \mathbf{y})_2] \mathbf{e}_2^- \\
&\quad - 2[(j_2 - j_1)(x_1 y_2 + x_2 y_1) - j_3(\mathbf{x} \times \mathbf{y})_3] \mathbf{e}_3^- \ ,
\end{aligned} \tag{16.127}$$

where by $\mathbf{x} \times \mathbf{y}$ is denoted the cross product of the vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$.

We see that if $j_1 + j_2 + j_3 = 0$ the algebra spanned by $\{\mathbf{e}_i^-\}_{i=1}^3$ is Abelian. If the field of numbers is $\mathbb{C}$ and if $j_i \neq 0$, one can check that the generators

$$\begin{aligned}
&\mathbf{f}_1 = -\frac{\mathbf{e}_1^+}{4\sqrt{j_2 j_3}}, \quad \mathbf{f}_2 = -\frac{\mathbf{e}_2^+}{4\sqrt{j_1 j_3}}, \quad \mathbf{f}_3 = -\frac{\mathbf{e}_1^+}{4\sqrt{j_2 j_3}} \\
&\mathbf{p}_i = \frac{\mathbf{e}_i^-}{\sqrt{j_i}}; \quad i = 1, 2, 3
\end{aligned} \tag{16.128}$$

span over $\mathbb{R}$ the algebra e (3) of the group of the rigid body movements of the 3-dimensional Euclidean space:

$$[\mathbf{f}_i, \mathbf{f}_j]_J = -\epsilon_{ijk}\mathbf{f}_k, \quad [\mathbf{f}_i, \mathbf{p}_k]_J = \epsilon_{ijk}\mathbf{p}_k, \quad [\mathbf{p}_i, \mathbf{p}_j]_J = 0 \;, \tag{16.129}$$

where ϵ_{ijk} is the 3-dimensional Levi-Civita symbol.

Let us return to the bundle $(\mathrm{o}\,(4), \mathrm{sym}\,(4))$ and let α_J be the cocycle defined by J (traceless) and let $B(X,Y)$ be the Killing form of $\mathrm{o}\,(4)$. It is known that $B(X,Y) = 2\mathrm{tr}\,(XY)$, and so in the calculations below we actually use $\mathrm{tr}\,(XY)$. The following proposition holds

Proposition 16.32. *Suppose that J is symmetric and traceless. Then*

1. *The form $B(XJY - YJX, T(Z)) = B(ad^J_X Y, T(Z))$ is a 3-cocycle for the trivial representation of $o\,(4)$ in $\mathbb{K}$.*
2. *The linear map α_J is symmetric with respect to the Killing form*

$$B(\alpha_J(X), Y) = B(X, \alpha_J(Y)); \quad X, Y \in o\,(4) \;. \tag{16.130}$$

Corollary 16.33. *The form $\Omega_J(X,Y) = B(\alpha(X), T(Y))$ is skew-symmetric and with respect to it both the ideals $\mathfrak{g}_I$ and $\mathfrak{g}_{II}$ and the "graphs"*

$$\begin{aligned} \Gamma_I &= \{A + \alpha(A);\ A \in \mathfrak{g}_I\} \\ \Gamma_{II} &= \{A + \alpha(A);\ A \in \mathfrak{g}_{II}\} \;, \end{aligned} \tag{16.131}$$

are isotropic subspaces. In particular, if J is as in (16.153) and if $j_1 j_2 j_3 \neq 0$ the form $\Omega_J(X,Y)$ is a symplectic form and $\mathfrak{g}_I$, $\mathfrak{g}_{II}$, Γ_I, Γ_{II} are Lagrangian subspaces.

As T is symmetric with respect to the Killing form, we can see from the above that there is another symmetric bilinear form, except B, that plays important role, namely, the form

$$B_T(X,Y) = B(X, T(Y)) \;. \tag{16.132}$$

For the subsequent calculations, we note also that for arbitrary $\mathbf{a}, \mathbf{b} \in \mathbb{C}^3$

$$\begin{aligned} &B(\{\mathbf{a}\}_I, \{\mathbf{b}\}_I) = B_T(\{\mathbf{a}\}_I, \{\mathbf{b}\}_I) = -8\langle \mathbf{a}, \mathbf{b}\rangle \\ &B(\{\mathbf{a}\}_{II}, \{\mathbf{b}\}_{II}) = -B_T(\{\mathbf{a}\}_{II}, \{\mathbf{b}\}_{II}) = -8\langle \mathbf{a}, \mathbf{b}\rangle \\ &B(\{\mathbf{a}\}_I, \{\mathbf{b}\}_{II}) = B_T(\{\mathbf{a}\}_I, \{\mathbf{b}\}_{II}) = 0 \;. \end{aligned} \tag{16.133}$$

where $\langle \mathbf{a}, \mathbf{b}\rangle = \sum_{i=1}^{3} a_i b_i$.

Of course, if $X = X_I + X_{II}$, $Y = Y_I + Y_{II}$ (the indices correspond to the projections onto the $\mathfrak{g}_I, \mathfrak{g}_{II}$) one calculates

$$B_T(X,Y) = B(X_I, Y_I) - B(X_{II}, Y_{II}) \;, \tag{16.134}$$

so for the usual Lie algebra structure there is little difference between $B(X,Y)$ and $B_T(X,Y)$. However it is not so for the structures defined by $[X,Y]_J$ and the following property is quite unexpected:

Proposition 16.34. *$B_T(X,Y)$ is nondegenerate bilinear form which is invariant for all the brackets $[X,Y]_J$, J-symmetric.*

Proof. If $J = c\mathbf{1}$, all is trivial, so one must consider only the case when J is traceless. We can calculate the bracket $[X,Y]_J$ using (16.31) and the fact that the elements from $\mathfrak{g}_I$ and $\mathfrak{g}_{II}$ commute. For this, we write $X = X_I + X_{II}$, $Y = Y_I + Y_{II}$ and remembering that $[X,Y]_J = [d\alpha_J](X,Y)$ we get:

$$([X,Y]_J)_I = [X_I, \alpha_J(Y_{II})] - [Y_I, \alpha_J(X_{II})] - \alpha_J([X_{II}, Y_{II}])$$
$$([X,Y]_J)_{II} = [X_{II}, \alpha_J(Y_I)] - [Y_{II}, \alpha_J(X_I)] - \alpha_J([X_I, Y_I])\ . \tag{16.135}$$

Then taking into account (16.134) the invariance can be proved by simple calculation.

Corollary 16.35. *For J traceless, the 3-cocycle*

$$\gamma_J(X,Y,Z) = B(ad^J_X(Y), T(Z)) = B_T(ad^J_X(Y), Z) \tag{16.136}$$

(with respect to the trivial representation of $o(4)$ in $\mathbb{K}$) is a coboundary.

Proof. Indeed, from the above it easily follows that α_J is skew-symmetric with respect to B_T and then β_J:

$$\beta_J(X,Y) = B_T(\alpha_J(X), Y)\ , \tag{16.137}$$

is a 2-cocycle. The calculation shows that $\gamma_J = d\beta_J$, which completes the proof.

This property is also somewhat surprising, because as it can be shown that the cocycle $B(\mathrm{ad}_X(Y), Z)$, the same is not true.

Taking into account the existence of the invariant bilinear form $B_T(X,Y)$, we easily arrive to the following application of our constructions:

Proposition 16.36. *The linear bundle $(o(4), sym(4))$ allows to construct the following 10-parametric family of compatible Kirillov (Poisson-Lie) tensors on the manifold $o(4)$:*

$$q \to P^J_q \in End(o(4,\mathbb{K}))\ :\ \ P_q(X) = -ad^J_X q \tag{16.138}$$
$$X \in o(4,\mathbb{K});\quad q \in o(4,\mathbb{K})^* \sim o(4,\mathbb{K}).$$

As we have seen earlier, we can identify $\mathrm{o}(4)$ and $\mathrm{o}^*(4)$ using also the Killing form (the trace form) and then we obtain the Poisson tensors (16.75). It is, therefore, natural to ask what is the relation between the tensors we obtained in different ways. We have the following simple result:

Proposition 16.37. *The coadjoint action $[ad^J_X]^*(Y) = JXY - YXJ$ obtained identifying $o(4)$ and $o^*(4)$ through the Killing form is related to the adjoint action $ad^J_X(Y) = XJY - YJX$ (also a coadjoint action if we identify $o(4)$ and $o^*(4)$ through the form B_T) in the following way:*

$$[ad^J_X]^* = T \circ ad^J_X \circ T\ . \tag{16.139}$$

Thus the families (16.138) and (16.75) are equivalent (though not compatible between themselves) as Poisson tensor fields.

Corollary 16.38. *The operator* $Y \mapsto [ad_X^J]^*(Y)$ *is skew-symmetric with respect to the bilinear form* B_T*.*

The fact that we have invariant form for the algebras of the bundle related to $\mathrm{o}\,(4)$ permits to define the Gel'fand-Fuchs cocycle just in the way we have defined it on the algebras of the type $\mathfrak{g}[x]$, where the algebra $\mathfrak{g}$ is semisimple. For convenience of the reader, we remind here this construction. Suppose $\mathfrak{g}_0[x]$ is the infinite-dimensional manifold of the smooth functions $f(x)$ defined on the real line, taking their values in the Lie algebra $\mathfrak{g}$ and tending fast enough to some constant $f_0 \in \mathfrak{g}$ when $|x| \to \infty$. For obvious reasons, we take the tangent space $T_f(\mathfrak{g}_0[x])$ at the point $f \in \mathfrak{g}_0[x]$ to be the vector space consisting of all Schwartz type functions $X(x)$ on the line taking their values in $\mathfrak{g}$. We denote this space by $\mathfrak{g}[x]$. If on $\mathfrak{g}$ there exists a linear bundle of Lie algebra structures, then clearly such bundles exists on $\mathfrak{g}[x]$ and on $\mathfrak{g}_0[x]$. When $\mathfrak{g}$ is semisimple, on the algebra $\mathfrak{g}[x]$ one introduces Poisson structure involving the operator of differentiation ∂_x. The construction uses the Gel'fand-Fuchs cocycle on $\mathfrak{g}[x]$, see [12, 13, 14, 15], which is given by the expression:

$$\gamma(X,Y) = c \int_{-\infty}^{+\infty} B(\partial_x X(x) Y(x)) dx$$

$$X, Y \in \mathfrak{g}[x]; \quad c = \text{const}\,, \quad \partial_x = \frac{\partial}{\partial x}\,, \tag{16.140}$$

where $B(X,Y)$ is symmetric, invariant, nondegenerate bilinear form on $\mathfrak{g}$. Allowing some lack of rigor we identify $\mathfrak{g}[x]$ and $\mathfrak{g}[x]^*$ using the symmetric bilinear form on $\mathfrak{g}[x]$:

$$\langle\langle X(x), Y(x)\rangle\rangle = \int_{-\infty}^{+\infty} B(X(x), Y(x)) dx; \quad X, Y \in \mathfrak{g}[x]\,. \tag{16.141}$$

Then, see the discussion in the Sect. 15.2.3, and in particular proposition 15.13, as well as the relations (15.87); we get the following family of compatible Poisson tensors:

$$\begin{aligned} &P_q^{(q_0;a,b,c)}(X) = a\mathrm{ad}_X q + b\mathrm{ad}_X q_0 + c\partial_x X \\ &q, X \in \mathfrak{g}[x]; \quad a, b, c \;-\; \text{numbers.} \end{aligned} \tag{16.142}$$

where a, b, c are numbers and q_0 is a fixed element from $\mathfrak{g}$ (defining a trivial cocycle). Since for the bundle $\mathrm{o}\,(4)$ we have additional parameters labeling the different algebras in it, and the form B_T is invariant with respect to all the algebras in the bundle, we can repeat the outlined construction for the case of $\mathfrak{g} = \mathrm{o}\,(4)$ putting B_T instead B. Invoking proposition 15.13 and the relations (15.87), we get the following result:

Theorem 16.39. *On the manifold* $o\,(4)_0[x]$ *there exists a* 11*-parameter family of compatible Poisson tensors*

$$\begin{aligned}&P_q^{(J;a,b)}(X) = a\,ad_X^J q + b\partial_x X \\ &X \in o\,(4)[x],\ \ q \in o(4)_0[x], \\ &J \ - \ \text{symmetric};\ \ a, b \ - \ \textit{numbers.}\end{aligned} \tag{16.143}$$

We underline again that the possibility to use the extension defined by Gel'fand-Fuchs cocycle simultaneously for all the Lie algebras of the above 10 – dimensional linear bundle of Lie algebras became possible because of the existence of the invariant bilinear form B_T (if B_T is not invariant then we do not have a cocycle). Note, however, that now we cannot add trivial cocycle term $\mathrm{ad}_X^J q_0$ because $B_T(q_0, [X,Y]_J)$ is of course 2-cocycle for d_J, but generally speaking is not cocycle for d_H, where H is another symmetric matrix. Also, in order to avoid misunderstanding, we underline again that for the family $P^{(J;a,b)}$ one must use not the usual inner product but that defined by B_T, that is:

$$\langle\langle X(x), Y(x)\rangle\rangle = \int_{-\infty}^{+\infty} B_T(X(x), Y(x))dx; \quad X, Y \in \mathfrak{g}[x]\ . \tag{16.144}$$

The Poisson brackets are constructed in the following way. First, we identify any linear form β of the type

$$\beta(Y) = \langle\langle X_\beta, Y\rangle\rangle \tag{16.145}$$

with the function $X_\beta(x)$, taking values in $\mathrm{o}\,(4)$. (Of course, $X_\beta(x)$ can be also a distribution). Then if $F[A]$ and $G[A]$ are two functionals, we identify their differentials dF and dG with two $\mathrm{o}\,(4)$-valued functions $X_F(x)$ and $X_G(x)$. Finally, the Poisson bracket of F and G is given by

$$\{F, G\}_{(J;a,b)} = \tag{16.146}$$

$$\int_{-\infty}^{+\infty} B_T(a[A(x), X_F(x)]_J + b\ \partial_x X_F(x),\ X_G(x))dx\ . \tag{16.147}$$

The tensor fields from this family, namely $P^{(\mathbf{1};-\frac{1}{2},0)}$ and $P^{(J;\frac{1}{2},1)}$, with J as in (16.124) and $\mathbf{1}$ – the matrix unity, have been used in [7, 8] to describe the bi-Hamiltonian structures of the $O(3)$ chiral fields system hierarchy and the bi-Hamiltonian structures of Landau–Lifshitz equation hierarchy obtained via polynomial pencil of Lax pairs. We shall reproduce some of these results in the next section.

16.6 The Chiral Fields Hierarchy and the Associated Recursion Operators

We shall now apply the compatible Poisson structures we introduced in theorem 16.39 in order to obtain the generating operator (Nijenhuis tensor) on the manifold of potentials for the so-called chiral fields hierarchy. Our experience with the Zakharov–Shabat system and its generalizations shows that the existence of a Lax representation is quite helpful when looking for recursion (Nijenhuis) operators. So it is natural to start with the set of Lax pairs for the chiral fields hierarchy. The first equation (which gives the name of the hierarchy) is the $O(3)$ chiral fields system (CF):

$$\begin{aligned} \mathbf{u}_t + \mathbf{u}_x + \mathbf{u} \times P\mathbf{v} &= 0 \\ \mathbf{v}_t - \mathbf{v}_x + \mathbf{v} \times P\mathbf{u} &= 0 \, . \end{aligned} \tag{16.148}$$

Here $\mathbf{v}, \mathbf{u}$ are two vector fields depending on x, t and taking values on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, $\times$ is the cross product symbol and if $\mathbf{a}$ is a 3-dimensional vector with components (a_i) which we write as a column, then $P\mathbf{a}$ is a vector with components $p_i a_i$, where p_i are fixed real numbers, that is, P is the matrix:

$$P = \operatorname{diag}(p_1, p_2, p_3) \, . \tag{16.149}$$

The $O(3)$ chiral fields system describes dynamics in anti-ferromagnets, in liquid crystals, and has connections with another integrable systems. In [5], there have been introduced new Lax pairs both for this system and for the famous Landau–Lifshitz equation (LL) [19]; see (16.173) below. The new Lax pairs are polynomial in the spectral parameter in contrast to the ones that have been discovered earlier and which depend on the spectral parameter through elliptic functions [20, 21]. The new pairs will be the starting point for our considerations. Using the elliptic pairs it was shown that the Landau–Lifshitz equation and the system of $O(3)$ chiral fields system are completely integrable Hamiltonian systems, see [21, 22], and the hierarchies of equations related to the LL and CF systems, as well as their Hamiltonian structures, were investigated, see [23, 24, 25, 26]. There arise a number of interesting questions about the use of different Lax pairs, but we shall speak of them later.

We start with the chiral fields system. Let us fix the notation as in (16.109), (16.110), that is $\{\mathbf{u}\}_I, \{\mathbf{u}\}_{II}$ will denote the 4×4 matrices from (16.109), (16.110). Each element $A \in \mathrm{o}(4)$ can be written into the form

$$A = \{\mathbf{u}\}_I + \{\mathbf{v}\}_{II} \, , \tag{16.150}$$

and this representation corresponds to the splitting of $\mathrm{o}(4)$ into direct sum of two $\mathrm{o}(3)$ algebras.

The pencil of Lax pairs for the CF consists of the following hierarchy:

$$L = \frac{\partial}{\partial x} - U, \quad M_n = \frac{\partial}{\partial t} - V_n$$

$$
\begin{aligned}
U(\lambda) &= \frac{1}{2}A(\lambda + J) \\
V_n(\lambda) &= \frac{1}{2}(\lambda^n B_0 + \lambda^{n-1}B_1 + \ldots + B_n)(\lambda + J) \\
n &= 0, 1, 2, \ldots ,
\end{aligned}
\tag{16.151}
$$

where

$$
\begin{aligned}
A &= \{\mathbf{u}\}_I + \{\mathbf{v}\}_{II} \\
B_n &= \{\mathbf{b}_n\}_I + \{\mathbf{c}_n\}_{II} .
\end{aligned}
\tag{16.152}
$$

As in (16.124), J will be the diagonal matrix

$$
J = \mathrm{diag}\,(-j_1 - j_2 + j_3, -j_1 + j_2 - j_3, j_1 - j_2 - j_3, j_1 + j_2 + j_3) \tag{16.153}
$$

and $\mathbf{u}(x,t), \mathbf{v}(x,t) \in \mathbb{R}^3$ are smooth vector fields taking values on the unit sphere:

$$
(\mathbf{u})^2 = 1, \quad (\mathbf{v})^2 = 1 . \tag{16.154}
$$

The vector fields $\mathbf{u}(x,t), \mathbf{v}(x,t)$ obey the boundary conditions

$$
\begin{aligned}
&\lim_{x\to\pm\infty} \mathbf{u} = \mathbf{u}_0 = \mathrm{const} \\
&\lim_{x\to\pm\infty} \mathbf{v} = \mathbf{v}_0 = \mathrm{const} \\
&\lim_{x\to\pm\infty} \left(\frac{\partial}{\partial x}\right)^n \mathbf{u} = 0 \\
&\lim_{x\to\pm\infty} \left(\frac{\partial}{\partial x}\right)^n \mathbf{v} = 0 \\
&n = 1, 2, \ldots .
\end{aligned}
\tag{16.155}
$$

We assume also that $\mathbf{u}(x), \mathbf{v}(x)$ converge fast enough to their limit values, in order to be able to say that the tangent vectors to the manifold of potentials consist of vectors that converge fast enough to zero, which ensures the existence of the integrals we shall write. More precisely, we shall assume that the components of $\mathbf{u}(x) - \mathbf{u}_0$ and $\mathbf{v}(x) - \mathbf{u}_0$ are Schwartz-type functions on the line. Let us denote by $\mathcal{M}$ the set of the matrices of the type (16.152) with $\mathbf{u}(x), \mathbf{v}(x)$ obeying the conditions (16.154, 16.155). $\mathcal{M}$ is which is called the set of potentials.

The nonlinear evolution equations, corresponding to the hierarchy of Lax pairs introduced in (16.151), have the following matrix form:

$$
A_t = (B_n)_x - \frac{1}{2}(AJB_n - B_nJA) = \frac{1}{2}[A, B_{n+1}] , \tag{16.156}
$$

and can be written in an equivalent "vector" form

$$\begin{aligned}\mathbf{u}_t &= -\mathbf{u} \times \mathbf{b}_{n+1} \ , \\ \mathbf{v}_t &= -\mathbf{v} \times \mathbf{c}_{n+1} \ , \\ n &= 0, 1, 2, \dots \ ,\end{aligned} \tag{16.157}$$

where the functions $\mathbf{b}_n, \mathbf{c}_n; \quad n = 0, 1, \dots$ are the solutions of the infinite system:

$$\begin{aligned}\mathbf{u} \times \mathbf{b}_0 &= 0, \quad \mathbf{v} \times \mathbf{c}_0 = 0 \\ \mathbf{u} \times \mathbf{b}_{n+1} &= -(\mathbf{b}_n)_x - K(\mathbf{v} \times \mathbf{c}_n) + \mathbf{u} \times K(\mathbf{c}_n) - \mathbf{b}_n \times K(\mathbf{v}) \\ \mathbf{v} \times \mathbf{c}_{n+1} &= -(\mathbf{c}_n)_x - K(\mathbf{u} \times \mathbf{b}_n) + K(\mathbf{u}) \times \mathbf{c}_n - K(\mathbf{b}_n) \times \mathbf{v} \\ n &= 0, 1, 2, \dots \ .\end{aligned} \tag{16.158}$$

We call this system the CF chain system. In the above expression, K is the diagonal matrix $K = \operatorname{diag}(j_1, j_2, j_3)$ and $(K(\mathbf{a}))_i = j_i a_i$. The next proposition gives an algorithm for obtaining successively the functions $\mathbf{b}_n$, $\mathbf{c}_n$:

Proposition 16.40. *The CF chain system has the following solution :*

$$\begin{aligned}\mathbf{b}_0 &= \epsilon\mathbf{u}, \quad \mathbf{c}_0 = \mu\mathbf{v} \\ \mathbf{b}^u_{n+1} &= \mathbf{u} \times (\mathbf{b}^u_n)_x + \langle \mathbf{u}, \mathbf{b}_n \rangle \mathbf{u} \times \mathbf{u}_x + [K(\mathbf{c}_n{}^v)]^u \\ &\quad - (\langle \mathbf{u}, \mathbf{b}_n \rangle - \langle \mathbf{v}, \mathbf{c}_n \rangle)[K(\mathbf{v})]^u + \mathbf{u} \times K(\mathbf{v} \times \mathbf{c}_n{}^v) + \langle \mathbf{u}, K(\mathbf{v}) \rangle \mathbf{b}^u_n \\ \mathbf{c}^v_{n+1} &= \mathbf{v} \times (\mathbf{c}^v_n)_x + \langle \mathbf{v}, \mathbf{c}_n \rangle \mathbf{v} \times \mathbf{v}_x + [K(\mathbf{b}^u_n)]^v \\ &\quad + (\langle \mathbf{u}, \mathbf{b}_n \rangle - \langle \mathbf{v}, \mathbf{c}_n \rangle)[K(\mathbf{u})]^v + \mathbf{v} \times K(\mathbf{u} \times \mathbf{b}^u_n) + \langle \mathbf{u}, K(\mathbf{v}) \rangle \mathbf{c}_n{}^v \\ n &= 0, 1, 2, \dots \ ,\end{aligned} \tag{16.159}$$

where ϵ, μ are arbitrary constants and

$$\begin{aligned}\langle \mathbf{u}, \mathbf{b}_n \rangle &= \int_{\pm\infty}^{x} \left(\langle \mathbf{b}^u_n, \mathbf{u}_x \rangle + \langle \mathbf{u} \times K(\mathbf{v}), \mathbf{b}^u_n \rangle + \langle \mathbf{v} \times K(\mathbf{u}), \mathbf{c}_n{}^v \rangle \right) dx \\ \langle \mathbf{v}, \mathbf{c}_n \rangle &= \int_{\pm\infty}^{x} \left(\langle \mathbf{c}_n{}^u, \mathbf{v}_x \rangle + \langle \mathbf{u} \times K(\mathbf{v}), \mathbf{b}^u_n \rangle + \langle \mathbf{v} \times K(\mathbf{u}), \mathbf{c}_n{}^v \rangle \right) dx.\end{aligned} \tag{16.160}$$

In the above formulae, we denote by $\langle , \rangle$ the usual $\mathbb{R}^3$ inner product:

$$\langle \mathbf{a}, \mathbf{b} \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{16.161}$$

and by the upper indices "u" and "v" are denoted the projections onto the plains orthogonal to the vector fields $\mathbf{u}$ and $\mathbf{v}$, respectively. (Of course, as $\mathbf{u}$

and $\mathbf{v}$ depend on x these plains depend on x too). The proof of the above proposition is done by induction. Though laborious, it is not difficult, so we shall omit it.

Let us remark that the above recursion formulae entail the existence of the integro-differential operators $\mathbf{A}_{\pm}$, such that

$$\begin{pmatrix} \mathbf{b}_{n+1}^{u} \\ \mathbf{c}_{n+1}^{v} \end{pmatrix} = \mathbf{A}_{\pm} \begin{pmatrix} \mathbf{b}_{n}^{u} \\ \mathbf{c}_{n}^{v} \end{pmatrix} . \tag{16.162}$$

These operators have the form:

$$\mathbf{A}_{\pm} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} =$$

$$\begin{aligned}
&\begin{pmatrix} \mathbf{u} \times (\mathbf{a})_x + \mathbf{u} \times \mathbf{u}_x \int\limits_{\pm\infty}^{x} \left(\langle \mathbf{a}, \mathbf{u}_x \rangle + \langle \mathbf{u} \times R(\mathbf{v}), \mathbf{a} \rangle + \langle \mathbf{v} \times K(\mathbf{u}), \mathbf{b} \rangle\right) dx \\ \mathbf{v} \times (\mathbf{b})_x + \mathbf{v} \times \mathbf{v}_x \int\limits_{\pm\infty}^{x} \left(\langle \mathbf{b}, \mathbf{v}_x \rangle + \langle \mathbf{u} \times K(\mathbf{v}), \mathbf{a} \rangle + \langle \mathbf{v} \times K(\mathbf{u}), \mathbf{b} \rangle\right) dx \end{pmatrix} \\
&+ \begin{pmatrix} [K(\mathbf{b})]^u - [K(\mathbf{v})]^u \int\limits_{\pm\infty}^{x} \left(\langle \mathbf{a}, \mathbf{u}_x \rangle - \langle \mathbf{b}, \mathbf{v}_x \rangle\right) dx \\ [K(\mathbf{a})]^v + [K(\mathbf{u})]^v \int\limits_{\pm\infty}^{x} \left(\langle \mathbf{a}, \mathbf{u}_x \rangle - \langle \mathbf{b}, \mathbf{v}_x \rangle\right) dx \end{pmatrix} \\
&+ \begin{pmatrix} \mathbf{u} \times K(\mathbf{v} \times \mathbf{b}) + \langle \mathbf{u}, K(\mathbf{v}) \rangle \mathbf{a} \\ \mathbf{v} \times K(\mathbf{u} \times \mathbf{a}) + \langle \mathbf{u}, K(\mathbf{v}) \rangle \mathbf{b} \end{pmatrix} ,
\end{aligned} \tag{16.163}$$

where $\mathbf{a}, \mathbf{b}$ are two vector fields, such that $\langle \mathbf{a}(x), \mathbf{u}(x) \rangle = \langle \mathbf{b}(x), \mathbf{v}(x) \rangle = 0$. We shall see that the operators (16.162) are actually the recursion operators for the CF hierarchy. Finally, let us write the first two members of the CF hierarchy (we also show by that the chiral fields system belongs to the hierarchy):

1. $n = 0$. First system in the CF hierarchy :

$$\begin{aligned} \mathbf{u}_t &= \epsilon \mathbf{u}_x + (\epsilon - \mu)(\mathbf{u} \times K(\mathbf{v})) \\ \mathbf{v}_t &= \mu \mathbf{v}_x - (\epsilon - \mu)(\mathbf{v} \times K(\mathbf{u})) . \end{aligned} \tag{16.164}$$

   After the following choice of the parameters :
   $\epsilon = -1, \quad \mu = 1, \quad P = 2K$ and changing $\mathbf{u}$ to $-\mathbf{u}$ we obtain the $O(3)$ chiral fields system.
2. $n = 1$. Second system in the CF hierarchy :

$$\begin{aligned} \mathbf{u}_t = {} & \epsilon \mathbf{u} \times \mathbf{u}_{xx} + 2\epsilon(\langle \mathbf{u}, R(\mathbf{v}) \rangle - \langle \mathbf{u}_0, R(\mathbf{v}_0) \rangle)\mathbf{u}_x \\ & - \epsilon R(\mathbf{v}_x) + \epsilon \langle \mathbf{u}, R(\mathbf{v}_x) \rangle \mathbf{u} - \mu \mathbf{u} \times R(\mathbf{v} \times \mathbf{v}_x) \\ & + (\epsilon - \mu)(\langle \mathbf{u}, R(\mathbf{v}) \rangle - \langle + \mathbf{u}_0, R(\mathbf{v}_0) \rangle)(\mathbf{u} \times R(\mathbf{v})) - (\epsilon - \mu)\mathbf{u} \times R^2(\mathbf{u}) \end{aligned}$$

$$\begin{aligned}\mathbf{v}_t = & \mu\mathbf{v} \times \mathbf{v}_{xx} + 2\mu(\langle \mathbf{v}, R(\mathbf{u})\rangle - \langle \mathbf{v}_0, R(\mathbf{u}_0)\rangle)\mathbf{v}_x \\ & - \mu R(\mathbf{u}_x) + \mu\langle \mathbf{v}, R(\mathbf{u}_x)\rangle \mathbf{v} - \epsilon\mathbf{v} \times R(\mathbf{u} \times \mathbf{u}_x) \\ & - (\epsilon - \mu)(\langle \mathbf{v}, R(\mathbf{u})\rangle - \langle \mathbf{v}_0, R(\mathbf{u}_0)\rangle)(\mathbf{v} \times R(\mathbf{u})) + (\epsilon - \mu)\mathbf{v} \times R^2(\mathbf{v}) \, .\end{aligned} \tag{16.165}$$

- An interesting special case of this system is obtained for $\mu = 0$. Then we have

$$\begin{aligned}\mathbf{u}_t = & \epsilon\mathbf{u} \times \mathbf{u}_{xx} + 2\epsilon(\langle \mathbf{u}, K(\mathbf{v})\rangle - \langle \mathbf{u}_0, K(\mathbf{v}_0)\rangle)\mathbf{u}_x \\ & - \epsilon K(\mathbf{v}_x) + \epsilon\langle \mathbf{u}, K(\mathbf{v}_x)\rangle \mathbf{u} \\ & + \epsilon(\langle \mathbf{u}, K(\mathbf{v})\rangle - \langle \mathbf{u}_0, K(\mathbf{v}_0)\rangle)(\mathbf{u} \times K(\mathbf{v})) - \epsilon\mathbf{u} \times K^2(\mathbf{u}) \\ \mathbf{v}_t = & - \epsilon\mathbf{v} \times K(\mathbf{u} \times \mathbf{u}_x) - \epsilon(\langle \mathbf{v}, K(\mathbf{u})\rangle - \langle \mathbf{v}_0, K(\mathbf{u}_0)\rangle)(\mathbf{v} \times K(\mathbf{u})) \\ & + \epsilon\mathbf{v} \times K^2(\mathbf{v}) \, .\end{aligned} \tag{16.166}$$

- Another reduction of the general system (16.165) is obtained, if we assume that $\epsilon = \mu$. Then we have

$$\begin{aligned}\mathbf{u}_t = & \epsilon\mathbf{u} \times \mathbf{u}_{xx} + 2\epsilon(\langle \mathbf{u}, K(\mathbf{v})\rangle - \langle \mathbf{u}_0, K(\mathbf{v}_0)\rangle)\mathbf{u}_x \\ & - \epsilon K(\mathbf{v}_x) + \epsilon\langle \mathbf{u}, K(\mathbf{v}_x)\rangle \mathbf{u} - \epsilon\mathbf{u} \times K(\mathbf{v} \times \mathbf{v}_x) \\ \mathbf{v}_t = & \epsilon\mathbf{v} \times \mathbf{v}_{xx} + 2\epsilon(\langle \mathbf{v}, K(\mathbf{u})\rangle - \langle \mathbf{v}_0, K(\mathbf{u}_0)\rangle)\mathbf{v}_x \\ & - \epsilon K(\mathbf{u}_x) + \epsilon\langle \mathbf{v}, K(\mathbf{u}_x)\rangle \mathbf{v} - \epsilon\mathbf{v} \times K(\mathbf{u} \times \mathbf{u}_x) \, .\end{aligned} \tag{16.167}$$

16.7 Polynomial Lax Representation of the Landau–Lifshitz Hierarchy

Let us go now to the study of the Landau–Lifshitz equation, (16.173).

The Landau–Lifshitz equation can be obtained within the general scheme, described when we introduced the CF hierarchy, if we impose instead of the constraint $\mathbf{v}^2 = 1$ the constraint $\mathbf{v} = 0$. Unfortunately, the condition $\mathbf{v}^2 = 1$ is essential the construction of the recursion formulae. Thus one cannot simply insert $\mathbf{v} = 0$ in the solution for the $O(3)$– CF chain system, in order to obtain the solution for the corresponding chain system for LL equation and must consider it by its own right.

Remark 16.41. One can check that if instead of the constraints $\mathbf{v} = 0, \mathbf{u}^2 = 1$ we choose the constraints $\mathbf{u} = 0, \mathbf{v}^2 = 1$ we shall obtain the same hierarchy of Lax pairs. Thus in all the constructions there exists a symmetry between the two $\mathrm{o}(3)$ subalgebras in $\mathrm{o}(4)$.

In order to obtain the LL equation in the same terms as it is usually written, we shall change the notation in the Lax pairs we introduced earlier in relation with the CF system. We put $\mathbf{u} = \mathbf{S}$ and $\mathbf{v} = 0$, and then we get the following hierarchy, which is the hierarchy of the Lax pairs for the Landau–Lifshitz equation hierarchy, (LL hierarchy):

$$L = \frac{\partial}{\partial x} - U, \quad M_n = \frac{\partial}{\partial t} - V_n \tag{16.168}$$

$$\begin{aligned} U(\lambda) &= \frac{1}{2} A(\lambda + J) \\ V_n(\lambda) &= \frac{1}{2}(\lambda^n B_0 + \lambda^{n-1} B_1 + \ldots + B_n)(\lambda + J) \\ n &= 0, 1, 2, \ldots \end{aligned} \tag{16.169}$$

where

$$A = \{\mathbf{S}\}_I, \quad B_n = \{\mathbf{b}_n\}_I + \{\mathbf{c}_n\}_{II} \ , \tag{16.170}$$

J is the diagonal matrix introduced earlier in (16.153), which for convenience of the reader we remind again:

$$J = \operatorname{diag}\left(-j_1 - j_2 + j_3, -j_1 + j_2 - j_3, j_1 - j_2 - j_3, j_1 + j_2 + j_3\right) . \tag{16.171}$$

As to $\mathbf{S}(x) \in \mathbb{R}^3$, it is a smooth vector field depending on one spatial variable, taking values on the unit sphere $\mathbb{S}^2 = \{\mathbf{S} : \mathbf{S}^2 = 1\}$ and tending fast enough to some limit value $\mathbf{S}_0$ when $x \to \pm\infty$. Of course, the derivatives of $\mathbf{S}(x)$ go to zero when $x \to \pm\infty$. Usually $\mathbf{S}_0$ is assumed to be $(0, 0, 1)$, which we shall also assume. However, we shall require a condition that ensures the above, but is stronger. We shall assume that $\mathbf{S}(x)$ is a function taking its values on the sphere $\mathbb{S}^2$, such that the components of $\mathbf{S}(x) - \mathbf{S}_0$ are Schwartz type functions on the line. On the space $\mathcal{M}_S$ of the functions $\mathbf{S}(x)$, satisfying the above conditions, exist two famous infinite-dimensional integrable systems which attracted considerable attention in the past decades:

1. The Heisenberg ferromagnet equation (HF):

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx} \ , \tag{16.172}$$

   which was introduced earlier in this book in connection with the Nonlinear Schrödinger equation.
2. The Landau–Lifshitz equation (LL):

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx} + \mathbf{S} \times P\mathbf{S} \ . \tag{16.173}$$

   Here P is a diagonal matrix with entries, $p_i > 0$, and $(P\mathbf{S})_i = p_i S_i$, $i = 1, 2, 3$.

The LL equation [19] describes perturbations propagating in a direction orthogonal to the anisotropy axis in a ferromagnet and the boundary conditions $\lim_{x\to\pm\infty}\mathbf{S}(x) = \mathbf{S}_0$ for it naturally arise from the physical background. This equation, as the CF system, is closely related to some integrable systems of Classical Mechanics [27].

Lax pairs both for HF equation and LL equation can be obtained in the form (16.168), so we look now into it. Let us denote the set of the matrices of the type $A = \{\mathbf{S}\}_I$, with potential $\mathbf{S}(x)$ belonging to $\mathcal{M}_S$ by $\mathcal{M}_S^I$. In other words, $\mathcal{M}_S^I \subset \mathrm{o}\,(4)$ is the manifold of potentials. It is easily checked that the set of the nonlinear evolution equations, corresponding to the above Lax pairs, have the form:

$$A_t = (B_n)_x - \frac{1}{2}(AJB_n - B_nJA)\,, \tag{16.174}$$

where the matrix functions B_n satisfy the following infinite system of equations, which we shall call the LLp chain system ("p" to denote that it is obtained through polynomial pencil):

$$\begin{aligned}[A, B_0] &= 0\\ [A, B_{n+1}] &= 2(B_n)_x - (AJB_n - B_nJA); \quad n = 0, 1, \dots\,.\end{aligned} \tag{16.175}$$

The equations (16.174) can be written also into the equivalent form

$$A_t = \frac{1}{2}[A, B_{n+1}]\,. \tag{16.176}$$

The hierarchy (16.174), or (16.176), is the Landau–Lifshitz hierarchy of evolution equations (or simply LL hierarchy). We shall need to distinguish the hierarchies obtained through different pencils, so we call this hierarchy the LLp hierarchy.

Of course, the above relations can be written also in terms of the vector fields $\mathbf{S}$, $\mathbf{b}_n$ and $\mathbf{c}_n$ and take the form:

- The LLp hierarchy of evolution equations:

$$\mathbf{S}_t = (\mathbf{b}_n)_x - \mathbf{S}\times K(\mathbf{c}_n); \quad n = 0, 1, 2, \dots\,, \tag{16.177}$$

  or

$$\mathbf{S}_t = -\mathbf{S}\times \mathbf{b}_{n+1}; \quad n = 0, 1, 2, \dots\,. \tag{16.178}$$

- The LLp chain system

$$\begin{aligned}&\mathbf{S}\times\mathbf{b}_0 = 0\\ &\mathbf{S}\times\mathbf{b}_{n+1} = -(\mathbf{b}_n)_x + \mathbf{S}\times K(\mathbf{c}_n)\\ &(\mathbf{c}_n)_x = -K(\mathbf{S}\times\mathbf{b}_n) + K(\mathbf{S})\times\mathbf{c}_n\\ &n = 0, 1, \dots\,.\end{aligned} \tag{16.179}$$

In the above expressions, K is the same diagonal matrix we had before, that is, $K = \mathrm{diag}\,(j_1, j_2, j_3)$, the numbers j_i are the same as in the definition of J and $(K(\mathbf{a}))_i = j_i a_i$. Below, when same relations are expressed through 4×4 matrices in $\mathrm{o}\,(4)$, we shall say that we have matrix representation of the hierarchies, equations etc., when expressed in terms of vectors, we shall say that we have vector representation.

It can be shown that the LLp chain system possesses a solution at least if we begin with $\mathbf{b}_0 = \mathbf{S}$, $\mathbf{c}_0 = 0$. We discuss this solution later; now we just note that with this choice the second nontrivial equation of the hierarchy is the Landau–Lifshitz equation

$$\mathbf{S}_t = (\mathbf{b}_1)_x - \mathbf{S} \times \mathbf{c}_1 = \mathbf{S} \times \mathbf{S}_{xx} - \mathbf{S} \times P(\mathbf{S}) \,, \tag{16.180}$$

where $P = -K^2$, that is,

$$P = \mathrm{diag}\,(p_1, p_2, p_3), \quad p_i = -j_i^2; \quad i = 1, 2, 3 \,. \tag{16.181}$$

This means that, in order to obtain LL equation, the entries of K must be purely imaginary.

Let us consider now the LLp chain system (16.175). Using the properties of the cocycle $\alpha_J(X) = 1/2(XJ + JX)$, associated with J, see (16.153), (16.135), we first cast the hierarchy in the following form:

$$\begin{aligned}
&[A, F_0] = 0\\
&\frac{1}{2}[A, F_{n+1}] = (F_n)_x - \frac{1}{2}[A, \alpha_J(G_n)]\\
&(G_n)_x - \frac{1}{2}[\alpha_J(A), G_n] + \frac{1}{2}\alpha_J([A, F_n]) = 0\\
&n = 0, 1, 2, \ldots
\end{aligned} \tag{16.182}$$

$$A = \{\mathbf{S}\}_I, \quad F_n = \{\mathbf{b}_n\}_I, \quad G_n = \{\mathbf{c}_n\}_{II} \,, \tag{16.183}$$

or, equivalently,

$$\begin{aligned}
&\mathbf{S} \times \mathbf{b}_0 = 0\\
&\mathbf{S} \times \mathbf{b}_{n+1} = -(\mathbf{b}_n)_x + \mathbf{S} \times K(\mathbf{c}_n)\\
&(\mathbf{c}_n)_x - K(\mathbf{S}) \times \mathbf{c}_n = -K(\mathbf{S} \times \mathbf{b}_n)\\
&n = 0, 1, 2, \ldots \,.
\end{aligned} \tag{16.184}$$

Here, as in the rest of the text, $K = \mathrm{diag}\,(j_1, j_2, j_3)$. It is known (see [6]) that the LLp chain system has the following solution (obtained recursively)

$$\begin{aligned}
&\mathbf{b}_0 = \mathbf{S}, \quad \mathbf{c}_0 = 0\\
&\mathbf{b}_1 = \mathbf{S} \times \mathbf{S}_x, \quad \mathbf{c}_1 = K(\mathbf{S})\\
&\mathbf{b}_{n+1} = \mathbf{b}^S_{n+1} + \mathbf{S} \int_{\pm\infty}^{x} \langle \mathbf{b}^S_{n+1}, \mathbf{S}_x \rangle dx
\end{aligned}$$

$$\mathbf{b}_{n+1}^S = \tilde{\Lambda}_\pm(\mathbf{b}_n^S) + (K(\mathbf{c}_n))^S \tag{16.185}$$

$$\mathbf{c}_n = \sum_{q=1}^{n} (-1)^{q-1} K^{(q)}(\mathbf{b}_{n-q}) \tag{16.186}$$

$$n = 1, 2, \ldots .$$

Let us explain the notation and the meaning of the quantities that appear in the solution. In the first place, the superscript S means that we take the projection to the space orthogonal to S, that is:

$$\mathbf{a}^S = \mathbf{a} - \langle \mathbf{a}, \mathbf{S} \rangle \mathbf{S} = -\mathbf{S} \times (\mathbf{S} \times \mathbf{a}) \,. \tag{16.187}$$

Next, the sequence of diagonal matrices $K^{(n)}$; $n = 1, 2, \ldots$ is constructed recursively, starting from $K^{(1)} = K$. More specifically, the matrices $K^{(n)}$ are obtained from the requirement that for arbitrary $\mathbf{a}$ and $\mathbf{b}$ they satisfy the following equations:

$$K^{(1)}(\mathbf{a}) \times K^{(1)}(\mathbf{b}) = K^{(2)}(\mathbf{a} \times \mathbf{b}) \tag{16.188}$$

$$K^{(1)}(\mathbf{a} \times K^{(1)} K^{(1)}(\mathbf{b})) + K^{(1)}(\mathbf{a}) \times K^{(2)}(\mathbf{b}) = K^{(3)}(\mathbf{a} \times \mathbf{b})$$

$$K^{(1)}(\mathbf{a} \times K^{(1)} K^{(2)}(\mathbf{b})) + K^{(2)}(\mathbf{a} \times K^{(1)} K^{(1)}(\mathbf{b}))$$

$$+ K^{(1)}(\mathbf{a}) \times K^{(3)}(\mathbf{b}) = K^{(4)}(\mathbf{a} \times \mathbf{b})$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum_{i=1}^{n-2} K^{(i)}(\mathbf{a} \times K^{(1)} K^{(n-i-1)}(\mathbf{b})) + K^{(1)}(\mathbf{a}) \times K^{(n-1)}(\mathbf{b}) = K^{(n)}(\mathbf{a} \times \mathbf{b}),$$

where $K^{(1)} = K$ and in the last formula $n \geq 3$. The family of diagonal matrices $K^{(n)}$ is well defined (see [6]), and the entries $K_i^{(n)}$; $i = 1, 2, 3$ of $K^{(n)}$ are homogeneous polynomials of degree n in the variables j_1, j_2, j_3. For example, the first members of the family $K^{(n)}$ are:

$$\begin{aligned}
&K_1^{(1)} = j_1,\ K_1^{(2)} = j_2 j_3,\ K_1^{(3)} = j_1(j_2^2 + j_3^2),\ K_1^{(4)} = j_2 j_3 (2j_1^2 + j_2^2 + j_3^2) \\
&K_2^{(1)} = j_2,\ K_2^{(2)} = j_1 j_3,\ K_2^{(3)} = j_2(j_1^2 + j_3^2),\ K_2^{(4)} = j_1 j_3 (j_1^2 + 2j_2^2 + j_3^2) \\
&K_3^{(1)} = j_3,\ K_3^{(2)} = j_1 j_2,\ K_3^{(3)} = j_3(j_1^2 + j_2^2),\ K_3^{(4)} = j_1 j_2 (j_1^2 + j_2^2 + 2j_3^2).
\end{aligned} \tag{16.189}$$

Note that $KK^{(2)} = j_1 j_2 j_3 \mathbf{1}$ and that if $K = \mathbf{1}$, all the matrices $K^{(n)}$ are proportional to $\mathbf{1}$. If we have already the matrices $K^{(n)}$, then $\mathbf{c}_n$ is expressed as shown in (16.186) through $\mathbf{b}_s$; $s = 0, 1, \ldots, n-1$. We shall assume that the components of $\mathbf{b}_n$ and $\mathbf{c}_n$ are polynomials in the components of $\mathbf{S}, \mathbf{S}_x, \mathbf{S}_{xx}, \ldots$. Similar facts are typical in the theory of the soliton equations. However, one proves them later, after more careful examination of the properties of the corresponding recursion operators (see for example [28]), for the case of the Zakharov–Shabat system and its gauge-equivalent, or they follow from some other considerations, usually related to the auxiliary spectral problems. If we take such dependence on $\mathbf{S}(x)$ and its derivatives as granted, then the matrices B_n have the same behavior at $+\infty$ and $-\infty$:

$$\lim_{x\to-\infty} B_n(x) = \lim_{x\to+\infty} B_n(x) \,. \tag{16.190}$$

Finally, the operators $\tilde{\Lambda}_{\pm}$ are given by the formula

$$\tilde{\Lambda}_{\pm}(\mathbf{x}(x)) = \mathbf{S} \times \frac{\partial}{\partial x}\mathbf{x}(x) + (\mathbf{S}\times\mathbf{S}_x)\int\limits_{\pm\infty}^{x} \langle \mathbf{x}(x), \mathbf{S}_x\rangle dx \,, \tag{16.191}$$

and are the recursion operators for Heisenberg Ferromagnet (HF) chain system. They are familiar from the study of the Heisenberg Ferromagnet equation, we had earlier in this book, but here they are written in vector representation. The HF chain system has the form, [29, 30] or [24]:

$$\begin{aligned} &\mathbf{S}\times\mathbf{b}_0 = 0 \\ &\mathbf{S}\times\mathbf{b}_{n+1} = -(\mathbf{b}_n)_x \\ &n = 0, 1, \ldots \end{aligned} \tag{16.192}$$

and can be obtained as limit case of the LLp chain system putting $K = 0$, $\mathbf{c}_n = 0$. In other words, the solution of (16.192) is:

$$\begin{aligned} \mathbf{b}_0 &= \mathbf{S}, \quad \mathbf{b}_1 = \mathbf{S}\times\mathbf{S}_x \\ \mathbf{b}^S_{n+1} &= \tilde{\Lambda}_{\pm}(\mathbf{b}^S_n) \\ \mathbf{b}_{n+1} &= \mathbf{b}^S_{n+1} + \mathbf{S}\int\limits_{\pm\infty}^{x} \langle \mathbf{b}^S_{n+1}, \mathbf{S}_x\rangle dx \\ n &= 1, 2, \ldots \,. \end{aligned} \tag{16.193}$$

Coming back to the LLp chain system, the first few members of the solution are:

$$\begin{aligned} \mathbf{c}_1 &= K(\mathbf{S}) \\ \mathbf{b}_2 &= \mathbf{S}_{xx} - K^2(\mathbf{S}) + h_{LL}\mathbf{S} \\ \mathbf{c}_2 &= K(\mathbf{S}\times\mathbf{S}_x) - K^{(2)}(\mathbf{S}) \\ \mathbf{b}_3 &= -\mathbf{S}\times\mathbf{S}_{xxx} + \mathbf{S}\times K^2(\mathbf{S}_x) \\ &\quad + K^2(\mathbf{S}\times\mathbf{S}_x) + h_{LL}\mathbf{S}\times\mathbf{S}_x - \langle K^2(\mathbf{S}\times\mathbf{S}_x), \mathbf{S}\rangle\mathbf{S} \\ \mathbf{c}_3 &= K(\mathbf{b}_2) - K^{(2)}(\mathbf{b}_1) + K^{(3)}(\mathbf{b}_0) \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{16.194}$$

where

$$h_{LL} = \frac{1}{2}\langle \mathbf{S}_x, \mathbf{S}_x \rangle - (\frac{1}{2}\langle K^2(\mathbf{S}), \mathbf{S} \rangle - \frac{1}{2}\langle K^2(\mathbf{S}_0), \mathbf{S}_0 \rangle) \; . \tag{16.195}$$

The notation h_{LL} is chosen because h_{LL} is the Hamiltonian density for the LL equation with respect to the so-called first Hamiltonian structure on $\mathcal{M}_S$ (see the first equation in (16.234)). This means that the Hamiltonian function is equal to:

$$H_{LL} = \int_{-\infty}^{+\infty} h_{LL}(\mathbf{S}(x), \mathbf{S}_x(x))dx \; . \tag{16.196}$$

It is interesting to note the HF equation can be embedded in the LLp hierarchy not only if we set $K = 0$, but if we set $K = \mathbf{1}$. Then the HF equation is the second nontrivial equation in the hierarchy. The corresponding chain system, to which we refer as HFI chain system, is

$$\begin{aligned} &\mathbf{S} \times \mathbf{b}_0 = 0 \\ &\mathbf{S} \times \mathbf{b}_{n+1} = -(\mathbf{b}_n)_x + \mathbf{S} \times \mathbf{c}_n \\ &(\mathbf{c}_n)_x - \mathbf{S} \times \mathbf{c}_n = -\mathbf{S} \times \mathbf{b}_n \\ &n = 0, 1, \ldots \; , \end{aligned} \tag{16.197}$$

and as far as we know, never has been considered in relation to the HF equation. One can solve the HFI chain system but, as can be expected, the solution does not yield new equations. Indeed, if the n-th equation of HFI is $\mathbf{S}_t = Y_n(\mathbf{S}, \mathbf{S}_x, \ldots)$ and the n-th equation of the HF hierarchy is $\mathbf{S}_t = X_n(\mathbf{S}, \mathbf{S}_x, \ldots)$, then Y_n is a linear combination with coefficients that do not depend on x (but depend on n) of $X_1, X_2, \ldots, X_n$. This of course can be seen directly from the pencil of the Lax pairs, however, for reasons that will become clear later, we do it in terms of the chain system. When $K = \mathbf{1}$, all the matrices $K^{(s)} = k_s\mathbf{1}$, where k_s are positive integers, obtained recursively as follows:

$$\begin{aligned} k_1 &= k_2 = 1 \\ k_{n+2} &= k_1 k_n + k_2 k_{n-1} + \ldots + k_{n-1}k_2 + k_n k_1 + k_{n+1} \\ n &= 1, 2, \ldots \; . \end{aligned} \tag{16.198}$$

For example, $k_3 = 2$, $k_4 = 4$, $k_5 = 9$, $k_6 = 21$. Then for $n = 1, 2, \ldots$ we get

$$\mathbf{S} \times \mathbf{b}_{n+1} = -(\mathbf{b}_n)_x + \mathbf{S} \times \sum_{s=1}^{n} (-1)^{s-1} k_s \mathbf{b}_{n-s} \; , \tag{16.199}$$

and therefore:

$$\mathbf{b}^S_{n+1} = \tilde{\Lambda}_{\pm}(\mathbf{b}^S_n) + \sum_{s=1}^{n} (-1)^{s-1} k_s \mathbf{b}^S_{n-s}$$

$$\mathbf{b}_{n+1} = \mathbf{b}_{n+1}^S + \mathbf{S} \int_{\pm\infty}^{x} \langle \mathbf{b}_{n+1}^S, \mathbf{S}_x \rangle dx \ . \tag{16.200}$$

Since as before $\mathbf{b}_1 = \mathbf{S} \times \mathbf{S}_x$, the elements of the new hierarchy are linear combinations with constant coefficients of the elements of the hierarchy HF. We can also try to find a recursion operator here. From the HFI chain system we get

$$\mathbf{b}_{n+1}^S = \tilde{\Lambda}_\pm(\mathbf{b}_n^S) + \mathbf{c}_n^S \tag{16.201}$$

Unfortunately, when we want to express $\mathbf{c}_n^S$, the formula is more complicated:

$$\mathbf{b}_n^S = \mathbf{c}_n^S + \tilde{\Lambda}_\pm(\mathbf{c}_n^S) + (-1)^{n-1} k_n \mathbf{S} \times \mathbf{S}_x \ . \tag{16.202}$$

We can write formally:

$$\mathbf{c}_n^S = (\mathrm{id} + \tilde{\Lambda}_\pm)^{-1}[\mathbf{b}_n^S + (-1)^n k_n \mathbf{S} \times \mathbf{S}_x] \ , \tag{16.203}$$

but the above relation cannot be used to calculate recursion operators, since it involves the numbers k_n, which are found using another recursion process.

Additional difficulties arise also, because, as we have seen in the first part, the continuous spectrum of $\tilde{\Lambda}_\pm$ fills up the real line, and then the operator $(\mathrm{id} + \tilde{\Lambda}_\pm)^{-1}$ is not defined.

Fortunately there is another way to approach the problem, that works for arbitrary K, which we are going to present in the next section.

16.8 Recursion Operators for LLp Hierarchy

Let us compare first the LLp chain system with the chain system obtained in [24] for the elliptic pencil of Lax pairs which is the most usually cited in relation to the LL equation. We shall call it the LLe chain system. We are not introducing the Lax pairs, the details of the computations can be find in [24]. The chain system is given by:

$$\begin{aligned}
&\mathbf{S} \times \mathbf{a}_0 = 0 \\
&\mathbf{S} \times \mathbf{b}_n = (\mathbf{a}_n)_x \\
&\mathbf{S} \times \mathbf{a}_{n+1} = \frac{1}{4}[(\mathbf{b}_n)_x + C(\mathbf{S}) \times \mathbf{a}_n] \\
&n = 0, 1, 2, \dots \ .
\end{aligned} \tag{16.204}$$

Here C is the diagonal matrix,

$$C = \mathrm{diag}\,(p_1 - p_3, p_2 - p_3, 0) = P - p_3 \mathbf{1} \ , \tag{16.205}$$

and the hierarchy is calculated starting from $\mathbf{a}_0 = -\mathbf{S}$. Then $\mathbf{b}_0 = \mathbf{S} \times \mathbf{S}_x$, and the LLe hierarchy of evolution equations is given by

$$\begin{aligned}&\mathbf{S}_t = (\mathbf{b}_n)_x + C(\mathbf{S}) \times \mathbf{a}_n \\ &n = 0, 1, 2, \dots \, .\end{aligned} \tag{16.206}$$

The first equation in the hierarchy is the LL equation. The recursion operator, obtained in [24], and for long considered in the literature as the only possible recursion operator for the LL equation, has the form:

$$\begin{aligned}&\Phi_{\pm}^{LL}(\mathbf{h}) = \frac{1}{4}(\tilde{\Lambda}_{\pm})^2\mathbf{h} \\ &-\frac{1}{4}\mathbf{S} \times \left\{C(\mathbf{S}) \times \mathbf{h} + (\mathbf{S}_x)\partial_x^{-1}(\langle \mathbf{S}, C(\mathbf{S}) \times \mathbf{h}\rangle) + C(\mathbf{S}) \times \mathbf{S}\partial_x^{-1}(\langle \mathbf{S}, (\mathbf{h})_x\rangle)\right\} ,\end{aligned} \tag{16.207}$$

where it is assumed that $\mathbf{h}^S = \mathbf{h}$. According to the explanations of the authors of [24], ∂_x^{-1} is the inverse of the x-derivative operator ∂_x and $\partial_x^{-1}(0)$ is understood as constant. (See the discussion after $\mathcal{A}_-$ has been introduced in (16.222)). Then

$$\mathbf{c}_{n+1} = \Phi_{\pm}^{LL}(\mathbf{c}_n); \quad n = 0, 1, 2, \dots \, , \tag{16.208}$$

where

$$\mathbf{c}_n = -\mathbf{S} \times [(\mathbf{b}_n)_x + C(\mathbf{S}) \times \mathbf{a}_n] \, . \tag{16.209}$$

In [24] it has been shown by direct calculations that the adjoint operator $(\Phi_{\pm}^{LL})^*$ is a Nijenhuis tensor field relating to two Poisson structures.

We want to obtain now a recursion operator for the LLp chain system and to give to it geometric interpretation. We shall see that it is essentially different from (16.207). Now, though the formulae (16.185) permit to calculate the hierarchy, they are not what we are looking for, because the recursion operator must relate $\mathbf{b}_{n+1}^S$ and $\mathbf{b}_n^S$ as does the operator $\tilde{\Lambda}_{\pm}$ for the HF chain system. We start with the observation that in the equations (16.184) appears the operator $\mathcal{A}$:

$$\mathcal{A}(\mathbf{h}) = \frac{d\mathbf{h}}{dx} - K(\mathbf{S}) \times \mathbf{h} \, , \tag{16.210}$$

which relates $\mathbf{c}_n$ and $\mathbf{b}_n$. If it is invertible, one can eliminate $\mathbf{c}_n$ and arrive at the desired result, just, as has been done for the HFI system. For this reason, let us consider $\mathcal{A}$ more closely. The natural domain D_n of $\mathcal{A}$ consists of the differentiable functions $\mathbf{h} : \mathbb{R} \mapsto \mathbb{C}^3$. However, since we expect to use $\mathcal{A}$ in the LLp chain system, we restrict D_n to the set D of the functions $\mathbf{h}(x)$, such that $\mathbf{h}(x) - (0, 0, h_3^0)$ is a Schwartz-type function on the line. (Here h_3^0 is some constant.) In order to perform our constructions, we need additional assumptions on $\mathbf{S}(x)$. We shall assume not only that $\mathbf{S}(x) - (0, 0, 1)$ is a Schwartz-type function on the line but that

$$|S_{1,2}(x)| \leq H \exp(-(|\mathrm{Im}\,(j_3)| + \epsilon)|x|) \,, \tag{16.211}$$

where H, ϵ are some positive constants. (Less restrictive conditions can be introduced, but since we want only to justify our constructions and not to develop the spectral theory of $\mathcal{A}$, we shall use these.)

Proposition 16.42. *If the potential function* $\mathbf{S}(x)$ *satisfies the above conditions, then for any fixed constant* h_3^0 *there exists unique* $\mathbf{h}(x) \in D_n$*, such that*

$$\mathcal{A}\mathbf{h} = 0, \qquad \lim_{x \to -\infty} \mathbf{h}(x) = (0, 0, h_3^0).$$

Proof. Let $\mathbf{h} \in D_n$ satisfy

$$\frac{d\mathbf{h}}{dx} - K(\mathbf{S}) \times \mathbf{h} = 0, \qquad \lim_{x \to -\infty} \mathbf{h}(x) = (0, 0, h_3^0). \tag{16.212}$$

We reformulate our problem, in order to put it in a more convenient form. From the algebra $\mathbb{C}^3$ (with respect to the cross-product), we pass to the isomorphic algebra sl (2), that is, instead of $\mathbf{h}(x)$, $\mathbf{S}(x)$, we introduce the matrix functions $h(x), S(x)$ etc. defined as

$$\begin{aligned} h(x) &= \sum_{s=1}^{3} h_s(x)\sigma_s \\ S(x) &= \sum_{s=1}^{3} S_s(x)\sigma_s, \qquad S_0 = \sigma_3 \\ K(S) &= \sum_{s=1}^{3} j_s S_s(x)\sigma_s \,, \end{aligned} \tag{16.213}$$

where σ_s; $s = 1, 2, 3$ are the Pauli matrices. The inner product $\langle \mathbf{g}, \mathbf{h} \rangle = \sum_{s=1}^{3} h_s g_s$ of two vectors $\langle \mathbf{g}, \mathbf{h} \rangle$ can be written as $2B(h, g)$ where $B(h, g)$ is the Killing form of sl (2). Indeed, as is known, for sl (2) holds $B(h, g) = \frac{1}{4}\mathrm{tr}\,(hg)$ and so $B(\sigma_s, \sigma_l) = \frac{1}{2}\delta_{sl}$. We denote by the same letter the operator that corresponds to $\mathcal{A}$ under the isomorphism $\mathbb{C}^3 \sim \mathrm{sl}\,(2)$, that is, when $\mathcal{A}$ acts on sl (2)-valued functions, it takes the form:

$$\mathcal{A}(h) = \frac{dh}{dx} + \frac{i}{2}[K(S), h] \,. \tag{16.214}$$

Then the equation (16.212) reads

$$\frac{dh}{dx} + \frac{i}{2}[K(S), h] = 0 \,. \tag{16.215}$$

Next, we consider the following auxiliary linear problem, closely related to (16.215):

$$\frac{d\psi}{dx} + \frac{i}{2}K(S)\psi = 0$$
$$\lim_{x\to -\infty} \exp(-ix\frac{j_3}{2}\sigma_3)\psi(x) = \mathbf{1}\ . \tag{16.216}$$

If we put $\tilde{\psi}(x) = \exp(-ix\frac{j_3}{2}\sigma_3)\psi(x)$, then as easily checked, (16.216) is equivalent to the following integral equation for $\tilde{\psi}(x)$:

$$\tilde{\psi}(x) = \mathbf{1} - \frac{i}{2}\int_{-\infty}^{x} e^{(-iy\frac{j_3}{2}\sigma_3)}(K(S) - j_3\sigma_3)e^{(iy\frac{j_3}{2}\sigma_3)}\tilde{\psi}(y)dy\ , \tag{16.217}$$

or, written in components:

$$\tilde{\psi}(x) =$$
$$\mathbf{1} - \frac{i}{2}\int_{-\infty}^{x} \begin{pmatrix} j_3(S_3(y)-1) & (j_1S_1(y) - ij_2S_2(y))e^{iyj_3} \\ (j_1S_1(y) - ij_2S_2(y))e^{-iyj_3} & -j_3(S_3(y)-1) \end{pmatrix} \tilde{\psi}(y)dy. \tag{16.218}$$

The assumption (16.211) ensures that the above integral equation is of Volterra type, since its kernel (with respect to the matrix norm) is bounded by expression of the type $M\exp(-\epsilon|x|)$, $M = \text{const}$. Therefore, (16.211) has unique solution and this solution is invertible matrix. (For the properties of such integral equations see [30]; the above linear problem is similar to the auxiliary problem for the Heisenberg Ferromagnet equation considered there). As a consequence, (16.216) has unique solution and which also is invertible matrix. Suppose ψ is a solution of (16.216). Then the following properties are easily verified:

- If C is a constant 2×2 matrix, then $\psi C\psi^{-1}(x)$ satisfies the equation (16.215).
- Let $B(X,Y)$ be the Killing form of the algebra $\mathrm{sl}\,(2)$. Suppose $h_1(x), h_2(x)$ are arbitrary functions with values in $\mathrm{sl}\,(2)$ satisfying (16.215). Then the derivative of $r(x) = B(h_1(x), h_2(x))$ vanishes, that is, $r(x) = \text{const}$.

From the above it easily follows that if $h(x)$ is a function with values in $\mathrm{sl}\,(2)$ satisfying (16.215), it is necessarily of the form:

$$h(x) = \sum_{s=1}^{3} a_s\psi\sigma_s\psi^{-1}(x); \quad a_s = \text{const}\ . \tag{16.219}$$

Since the asymptotic when $x \to -\infty$ of the right hand side is

$$(a_1\cos(j_3x) + a_2\sin(j_3x))\sigma_1 + (-a_1\sin(j_3x) + a_2\cos(j_3x))\sigma_2 + a_3\sigma_3, \tag{16.220}$$

the function $h(x)$ tends to $h_3^0\sigma_3$ only if $a_s = 0$; $s = 1,2$ and $a_3 = h_3^0$. The proposition is proved.

Corollary 16.43. *If the limit* $\lim_{x\to\pm\infty} f(x)$ *is known, the function* $f(x) \in D$ *can be recovered uniquely from its image* $g = \mathcal{A}(f)$*. In particular, restricted on the vector space of Schwartz-type functions, the operator* $\mathcal{A}$ *has trivial kernel and is invertible.*

As to the calculation of the inverse operator $\mathcal{A}^{-1}$, it can be done using the function $\psi(x)$. Indeed, consider (16.215) with a right side, that is:

$$\frac{dh}{dx} + \frac{i}{2}[K(S), h] = g(x) \ , \tag{16.221}$$

where $g(x)$ is Schwartz type function. Then, since $B(\sigma_s, \sigma_l) = -\frac{1}{4}\delta_{sl}$, one can check that we have the following solution of our problem (provided that the integral converges and we can differentiate it):

$$\begin{aligned} h(x) &= \mathcal{R}_{(-)} g(x) \\ &= a\psi\sigma_3\psi^{-1}(x) + \frac{1}{2}\sum_{s=1}^{3} \psi\sigma_s\psi^{-1}(x)\int_{-\infty}^{x} B(g(y), \psi\sigma_s\psi^{-1}(y))dy. \end{aligned} \tag{16.222}$$

As it should be, the solution depends on one arbitrary parameter a. If we know that $\lim_{x\to-\infty} h(x) = 0$, then $a = 0$. In this case the solution is unique; the solution is also unique, if we require for it the asymptotic: $h(x) \to a\sigma_3$, $a \neq 0$ when $x \to -\infty$ and a is some fixed constant. In both these cases, one can write $\mathcal{R}_{(-)} = \mathcal{A}^{-1}$. However, we still have some difficulties. Indeed, for the geometric interpretations of the soliton equations hierarchies, we must ensure that if $\mathcal{A}^{-1}g(x)$ satisfies $\lim_{x\to-\infty} \mathcal{A}^{-1}g(x) = a$ then $\lim_{x\to+\infty} \mathcal{A}^{-1}g(x) = a$ too. We can see that the above, generally speaking, is not true. But if this happens, it simply means that g is not in the image of $\mathcal{A}$. We also note that using solutions $\varphi(x)$ of the differential equation in (16.216), but this time defined by their behavior at $+\infty$ instead at $-\infty$, we get another expression for the inverse. In order to distinguish these expressions we write $\mathcal{A}_-^{-1}$ and $\mathcal{A}_+^{-1}$. As a matter of fact, we have

$$\begin{aligned} h(x) = \mathcal{A}_+^{-1} g(x) = \mathcal{R}_{(+)} g(x) = a\varphi\sigma_3\varphi^{-1}(x) \\ + \frac{1}{2}\sum_{s=1}^{3} \varphi\sigma_s\varphi^{-1}(x)\int_{+\infty}^{x} B(g(y), \varphi\sigma_s\varphi^{-1}(y))dy \ . \end{aligned} \tag{16.223}$$

Since φ and ψ are fundamental solutions of the same system of differential equations, there exists a nondegenerate constant matrix T such that $\psi = \varphi T$. The properties of the Killing form allow to obtain that if $h(x)$ has the same asymptotic at $\pm\infty$, then

$$\begin{aligned} &\mathcal{A}_+^{-1} g = \\ &\mathcal{A}_{(-)}^{-1} g(x) + \frac{1}{2}\sum_{s=1}^{3} \varphi\sigma_s\varphi^{-1}(x)\int_{-\infty}^{+\infty} B(g(y), \varphi\sigma_s\varphi^{-1}(y))dy. \end{aligned} \tag{16.224}$$

There is no contradiction here, because in the case when $g(x) = \mathcal{A}(h)$, where h is such that $h(x) - a\sigma_3$ is a Schwartz-type function on the line, after integrating by parts and using the properties of the form B, one can show that the integral in the right-hand side vanishes and $\mathcal{A}_-^{-1} g = \mathcal{A}_+^{-1} g$. The above again shows that inverting $\mathcal{A}$ one must be sure that one acts on functions that belong to the image of $\mathcal{A}$.

The difficulties we have encountered in inverting some operators are common in the geometric approaches to the soliton equations and in the theory of the recursion operators, though they are simpler in the case when the potentials vanish at infinity. Even in that case, one usually must invert ∂_x (as in the case of the recursion operators for the HF), and of course the inverse is unique up to an additive constant. Different authors choose for the inverse one of the operators

$$\int_{-\infty}^{x}, \quad \int_{+\infty}^{x}, \quad \frac{1}{2}\left(\int_{-\infty}^{x} + \int_{+\infty}^{x}\right) \tag{16.225}$$

(the last expression is usually chosen when we need to ensure some symmetry properties) or simply write ∂_x^{-1}. All these expressions of course amount to the same, since each time one uses recursion operators it is possible to prove that the integrands are total derivatives of polynomial functions on the potential and its derivatives (at least it is the case for the Nonlinear Schrödinger equation hierarchy and the HF hierarchy it is so) or in other words, the integrands belong to the image of ∂_x. What happens in our case is similar. For example, solving

$$\frac{d\mathbf{c}_1}{dx} - K(\mathbf{S}) \times \mathbf{c}_1 = -K(\mathbf{S} \times \mathbf{b}_1) = K(\mathbf{S}_x(x)) \tag{16.226}$$

for $\mathbf{c}_1$ gives $K(\mathbf{S}) + \mathbf{h}$, where $\mathbf{h}$ satisfies the corresponding homogeneous equation. If we assume that the solution tends to $K(\mathbf{S}_0)$ when $x \to \pm\infty$ then $\mathbf{c}_1 = K(\mathbf{S}(x))$. The formula for the $\mathbf{c}_n$'s (see (16.186)) shows that we can find the limit of $\mathbf{c}_n(x)$ as $x \to \pm\infty$ and then $\mathbf{c}_n$ can be uniquely determined from the equation:

$$(\mathbf{c}_n)_x - K(\mathbf{S}) \times \mathbf{c}_n = -K(\mathbf{S} \times \mathbf{b}_n) \; . \tag{16.227}$$

Therefore, $\mathcal{A}_\pm^{-1}(K(\mathbf{S} \times \mathbf{b}_n))$ is well defined.

Remark 16.44. The formula (16.186) avoids solving the differential equation for $\mathbf{c}_n$, but it gives $\mathbf{c}_n$ as function of $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_{n-1}$. What we want, however, is an expression that depends only on $\mathbf{b}_n$.

Having in mind the properties of the operator $\mathcal{A}$, discussed in the above, the LLp chain system (16.184) can be written into the form:

$$\begin{aligned} &\mathbf{S} \times \mathbf{b}_0 = 0 \\ &\mathbf{S} \times \mathbf{b}_{n+1} = -(\mathbf{b}_n)_x - \mathbf{S} \times K\mathcal{A}_\pm^{-1} K(\mathbf{S} \times \mathbf{b}_n) \\ &n = 0, 1, \ldots \; , \end{aligned} \tag{16.228}$$

where $\mathcal{A}_{\pm}^{-1}$ is used in the sense we discussed. The difficulties we mentioned can be overcome only after a thorough study of the spectral theory of the operator L in the Lax representation; here we limit ourselves to the geometric interpretation of the hierarchies.

For arbitrary vector $\mathbf{q}$, the fact that $\mathbf{S} \in \mathbb{S}^2$ entails $\mathbf{q}^S = -\mathbf{S} \times (\mathbf{S} \times \mathbf{q})$, and we obtain:

Proposition 16.45. *The solution of LLp chain system can be put into the form:*

$$
\begin{aligned}
\mathbf{b}_0 &= \mathbf{S} \\
\mathbf{b}_1 &= \mathbf{S} \times \mathbf{S}_x \\
\mathbf{b}_{n+1}^S &= \Lambda_{\pm}^L(\mathbf{b}_n^S) \\
\mathbf{b}_{n+1} &= \mathbf{b}_{n+1}^S + \mathbf{S} \int_{\pm\infty}^{x} \langle \mathbf{b}_{n+1}^S, \mathbf{S}_x \rangle dx \qquad (16.229) \\
n &= 1, 2, \dots\ , \qquad (16.230)
\end{aligned}
$$

where

$$
\Lambda_{\pm}^L \mathbf{h} = \tilde{\Lambda}_{\pm}(\mathbf{h}) + \mathbf{S} \times [\mathbf{S} \times K\mathcal{A}_{\pm}^{-1} K(\mathbf{S} \times \mathbf{h})] = \tilde{\Lambda}_{\pm}(\mathbf{h}) + [K\mathcal{A}_{\pm}^{-1} K(\mathbf{S} \times \mathbf{h})]^S \qquad (16.231)
$$

(of course, here we assume $\langle \mathbf{h}(x), \mathbf{S}(x) \rangle = 0$).

The operator $\Lambda_{\pm}^L$ is a recursion operator for the LLp chain system. When $K = 0$, it reduces to $\tilde{\Lambda}_{\pm}$ – the recursion operator for the HF chain system, an advantage over the recursion operator obtained in [24], which reduces to $\tilde{\Lambda}_{\pm}^2$. Its flaw seems to be the presence of $\mathcal{A}_{\pm}^{-1}$, which cannot be expressed in terms of $\mathbf{S}(x)$ and its derivatives (in an explicit form). But since we have a nice recursion formula which gives the values $K\mathcal{A}_{\pm}^{-1} K(\mathbf{S} \times \mathbf{b}_n)$, this flaw is only apparent; actually one calculates the members of the LLp hierarchy at least as easily as one calculates them for LLe hierarchy. There is, however, another thing that makes both operators quite different. The point is that generally speaking, the LLp chain system is complex, and we cannot simply consider K real, because the system of interest (the LL equation) is obtained exactly when the entries of K are purely imaginary. But since the LL equation is real, it is desirable to have a real hierarchy. The expression for the generating operator, $\Lambda_{\pm}^L$ shows that it is complex too and hence the quantities $\mathbf{b}_n$ are complex. The same is deduced easily from the formula for $\mathbf{c}_n$, cf. (16.186). The entries of the matrices $K^{(q)}$ are homogeneous polynomials of degree q in the entries of K, see [6], and hence for even q they are real and from odd q they are imaginary. This circumstance, together with

$$
\mathbf{S} \times \mathbf{b}_{n+1} = -(\mathbf{b}_n)_x + \mathbf{S} \times K(\mathbf{c}_n)\ , \qquad (16.232)
$$

show that, generally speaking, $\mathbf{b}_n$ have imaginary parts. The imaginary terms vanish up to $n = 3$ when we start by $\mathbf{b}_0 = \mathbf{S}$, $\mathbf{c}_0 = 0$, because $KK^{(2)} = -i|j_1 j_2 j_3|\mathbf{1}$, and there are no terms proportional to i in the equation for $\mathbf{b}_3$. However, the same relation shows that in the expression for $\mathbf{b}_4$ there is imaginary term, proportional to $\mathbf{S} \times \mathbf{S}_x$. What happens is that in the imaginary part of the hierarchy, we have linear combinations of the same vector fields as in the real part. In the present context, they appear together and it is not clear how we can separate them. The above also shows that one cannot expect that the recursion operator for the elliptic pencil will be the square of the recursion operator we have introduced. There must be some more subtle connection between them, a connection that will be revealed after the relation between the polynomial and elliptic pencils is better understood.

Now, following closely [9], we are going to show that for $\Lambda^L = \Lambda^L_{\pm}$ (we consider the two operators as equivalent) we have the usual geometric interpretation of the recursion operators, that is, Λ^L relates two Poisson structures and as a consequence its adjoint is a Nijenhuis tensor.

16.9 LLp Hierarchy – the Poisson-Nijenhuis Structure

Looking at the LLp hierarchy of evolution equations, cf. (16.174), (16.176), and using the definition of the tensor field $P_A^{(J;a,b)}$ introduced earlier in theorem 16.39, one sees that they can be cast into the following equivalent forms

$$A_t = P_A^{(\mathbf{1};-\frac{1}{2},0)} B_{n+1}, \quad A_t = P_A^{(J;\frac{1}{2},1)} B_n; \quad n = 0, 1, \dots\ , \tag{16.233}$$

where the two tensors are compatible. One immediately has the idea to interpret B_n as 1-forms and from (16.233), it is clear that the aforementioned tensors play an important role. Now, for the sake of brevity, we shall drop the complicated upper indices and denote the tensors simply by P and Q, that is:

$$P_A^{(\mathbf{1};-\frac{1}{2},0)} = Q_A = \frac{1}{2}\mathrm{ad}_A, \quad P_A^{(J;\frac{1}{2},1)} = P_A = \partial_x - \frac{1}{2}\mathrm{ad}^{\,J}_{A}\,. \tag{16.234}$$

The tensor Q is familiar, and in vector representation it defines the so-called first Hamiltonian structure on $\mathcal{M}_S$; the tensor P is more "exotic."

Now we want to restrict these tensors on the space of potentials

$$\mathcal{M}_S^I = \{A = \{\mathbf{S}\}_I : \quad \mathbf{S}(x) \in \mathcal{M}_S\} \sim \mathcal{M}_S\,. \tag{16.235}$$

The first tensor (Q) is immediately restricted, and its form remains the same, so we denote the restriction by the same letter. It remains to restrict the second tensor (P), and to this end we use the Restriction Theorem (12.25). In order to check the conditions of (12.25), we calculate:

$$
\begin{aligned}
T_A(\mathcal{M}_S^I) &= \{X : X = [A, Y], Y \in \mathfrak{g}_I[x]\} \\
T_A^{\perp}(\mathcal{M}_S^I) &= \{fA + Y^* : f \in \mathcal{S}(\mathbb{R}),\ Y^* \in \mathfrak{g}_{II}[x]\} ,
\end{aligned}
\tag{16.236}
$$

where $\mathcal{S}(\mathbb{R})$ denotes the space of the Schwartz functions on the line. As explained, we identify the tangent and cotangent vectors using the form $\langle\langle\ ,\ \rangle\rangle$, but we write star as superscript, when we are dealing with covectors, in order to make the geometric meaning clearer. Next

$$
\mathcal{X}_P^*(\mathcal{M}_S^I)_A = \{X^* + Y^* : X^* \in \mathfrak{g}_I[x],\ Y^* \in \mathfrak{g}_{II}[x]\} , \tag{16.237}
$$

where X^* and Y^* satisfy:

$$
\begin{aligned}
&(X^*)_x - \frac{1}{2}[A, \alpha_J(Y^*)] \in T_A(\mathcal{M}_S^I) \\
&(Y^*)_x - \frac{1}{2}[\alpha_J(A), Y^*] = -\frac{1}{2}\alpha_J([A, X^*]) .
\end{aligned}
\tag{16.238}
$$

The first of this relations implies that

$$
B_T(X_x^*(x), A(x)) = 0 . \tag{16.239}
$$

If we put $X^* = \{\mathbf{b}\}_I$, $Y^* = \{\mathbf{c}\}_{II}$, the second one is equivalent to the vector equation:

$$
\mathbf{c}_x - K(\mathbf{S}) \times \mathbf{c} = -K(\mathbf{S} \times \mathbf{b}) . \tag{16.240}
$$

As one can see the operator $\partial_x - \frac{1}{2}\mathrm{ad}_{\,\alpha_J(A)}$ is the matrix form of the operator $\mathcal{A}$, introduced in (16.210), so we shall denote it by the same letter. Then the equation from (16.238) reads

$$
\mathcal{A}(Y^*) = -\frac{1}{2}\alpha_J([A, X^*]) . \tag{16.241}
$$

The domain D of the operator $\mathcal{A}$ is the set of the smooth functions $Y^*(x)$ taking values in $\mathfrak{g}_{II}$ and tending fast enough to some constants $\{(0,0,a)\}_{II}$. In this case, as we have seen, the operators $\mathcal{A}_{\pm}^{-1}$ are well defined on the image of $\mathcal{A}$ and are equal on this image. From the fact that $\mathcal{A}$ has trivial kernel when restricted to $\mathfrak{g}_{II}[x]$ we get

$$
(\mathcal{X}_P^*(\mathcal{M}_S^I)_A) \cap T_A^{\perp}(\mathcal{M}_S^I) = \{0\} . \tag{16.242}
$$

Finally, if $X^* + Y^*$ is a covector from $T_A^*(\mathrm{o}\,(4)_0[x])$, making use of (16.133) we obtain

$$
\begin{aligned}
&X^* + Y^* = Z_1^* + Z_2^* && (16.243)\\
&Z_1^* = X^* + \frac{1}{8}A\int_{\pm\infty}^{x} B_T(A(y), X_y(y))dy - \frac{1}{2}\mathcal{A}_{\pm}^{-1}(\alpha_J([A, X^*]))
\end{aligned}
$$

$$Z_2^* = Y^* - \frac{1}{8} A \int_{\pm\infty}^{x} B_T(A(y), X_y(y))dy + \frac{1}{2}\mathcal{A}_{\pm}^{-1}(\alpha_J([A, X^*])).$$

Since $Z_1^* \in \mathcal{X}_P^*(\mathcal{M}_S^I)_A$, $Z_2^* \in T_A^{\perp}(\mathcal{M}_S^I)$ the above shows that

$$(\mathcal{X}_P^*(\mathcal{M}_S^I)_A) + T_A^{\perp}(\mathcal{M}_S^I) = T_A^*(\mathrm{o}\,(4)_0[x]) \,. \tag{16.244}$$

Therefore the requirement of the theorem 12.25 are fulfilled and P allows restriction. According to this theorem, for $X^* \in T_A^*(\mathcal{M}_S^I)$, one must take first $j^*(X^*)$, where j is the canonical inclusion map $j : \mathcal{M}_S^I \mapsto \mathcal{M}_S$, then represent it as a sum $Y_1^* + Y_2^*$, where $Y_1^* \in \mathcal{X}_P^*(\mathcal{M}_S^I)_A$, $Y_2^* \in T_A^{\perp}(\mathcal{M}_S^I)$, and the restricted tensor will satisfy $\bar{P}_A(X^*) = P_A(Y_1^*)$. In our case $j^*(X^*) = X^*$ so we get:

$$\begin{aligned}\bar{P}_A(X^*) &= \partial_x(X^*) + \frac{1}{8} A_x \int_{\pm\infty}^{x} B_T(X_y^*(y), A(y))dy \\ &\quad + \frac{1}{8} B_T(X_x^*, A)\ A - \frac{1}{4}[A, (\alpha_J \circ \mathcal{A}_{\pm}^{-1} \circ \alpha_J)([A, X^*])] \\ &= \pi(\partial_x(X^*)) + \frac{1}{8} A_x \int_{\pm\infty}^{x} B_T(X_y^*(y), A(y))dy - \frac{1}{4}[A, \mathcal{A}_{\pm}^{-1}(\alpha_J([A, X^*])],\end{aligned} \tag{16.245}$$

where π denotes the projection in $\mathrm{o}\,(4)$ onto the subspace, orthogonal to A. Also, since $B_T(X^*(x), A(x)) = 0$, one has

$$B_T(X_x(x), A(x)) = -B_T(X(x), A_x(x)) \,. \tag{16.246}$$

Finally we obtain:

Proposition 16.46. *On the manifold of potentials $\mathcal{M}_S^I$ there exists a restriction $\bar{P}$ of the Poisson tensor P, having the form:*

$$\begin{aligned}\bar{P}_A(X^*) &= \pi(\partial_x(X^*)) - \frac{1}{8} A_x \int_{\pm\infty}^{x} B_T(X^*(y), A_y(y))dy \\ &\quad - \frac{1}{4}[A, (\alpha_J \circ \mathcal{A}_{\pm}^{-1} \circ \alpha_J)([A, X^*])] \,,\end{aligned} \tag{16.247}$$

where $X^* \in T_A^*(\mathcal{M}_S^I)$.

Since on $T_A(\mathcal{M}_S^I)$ the operator Q_A is invertible, we are able to calculate $\Lambda_{\pm}^L = Q_A^{-1} \circ \bar{P}_A$:

$$\begin{aligned}\Lambda^L_\pm(X^*) &= -\frac{1}{2}[A,\partial_x(X^*)] + \frac{1}{16}[A,A_x]\int\limits_{\pm\infty}^{x} B_T(X^*(y),A_y)dy \\ &\quad -\frac{1}{2}(\pi\circ\alpha_J\circ\mathcal{A}_\pm^{-1}\circ\alpha_J\circ\mathrm{ad}_A)(X^*) \\ &= \tilde{\Lambda}_\pm(X^*) - \frac{1}{2}(\pi\circ\alpha_J\circ\mathcal{A}_\pm^{-1}\circ\alpha_J\circ\mathrm{ad}_A)(X^*)\ . \end{aligned} \tag{16.248}$$

(Of course, in the above, we assume that X^* satisfies $B_T(X^*(x),A(x))=0$). Some explications are needed for the last formula. If we calculate the above in vector representation we shall get that the term

$$\Psi_\pm(X^*) = -\frac{1}{2}[A,\partial_x(X^*)] + \frac{1}{16}[A,A_x]\int\limits_{\pm\infty}^{x} B_T(X^*(y),A_y)dy\ , \tag{16.249}$$

in which $X^* = \{\mathbf{b}\}_I$ is equal to $\{\tilde{\Lambda}_\pm\mathbf{b}\}_I$, and the entire expression in (16.248) equals

$$\Lambda^L_\pm(X^*) = \{\tilde{\Lambda}_\pm(\mathbf{b}) + [(K\mathcal{A}_\pm^{-1}K)(\mathbf{S}\times\mathbf{b})]^S\}_I\ , \tag{16.250}$$

which explains why we denoted the operator $Q_A^{-1}\circ\bar{P}_A$ by $\Lambda^L_\pm$, and the operator $\Psi_\pm$ in (16.249) by $\tilde{\Lambda}_\pm$ – they are just the recursion operators for LLp and HF in vector representation, cf. (16.231) and (16.191). As we have seen, the action of $\mathcal{A}_+^{-1}$ and $\mathcal{A}_+^{-1}$ gives the same result, so below we write Λ^L instead of $\Lambda^L_\pm$. The general theory of compatible Poisson tensors then yields :

Theorem 16.47. *The pair $(Q, N^L = (\Lambda^L)^*)$ endows the manifold of potentials $\mathcal{M}^I_S$ (or $\mathcal{M}^I_S$) with a P-N structure.*

Let us consider again the equations from the LLp hierarchy, (16.174), (16.176). We want to reveal their Hamiltonian properties. According to (16.233), we can write them as

$$A_t = Q_A(B_{n+1}) = P_A(B_n);\quad n=0,1,\dots\ , \tag{16.251}$$

where $B_n = F_n + G_n$ ($F_n\in\mathfrak{g}_I$, $G_n\in\mathfrak{g}_{II}$) satisfy the chain system (16.175), or equivalently, (16.182), which for our convenience we write again:

$$\begin{aligned} &[A,F_0]=0 \\ &\frac{1}{2}[A,F_{n+1}] = (F_n)_x - \frac{1}{2}[A,\alpha_J(G_n)] \\ &(G_n)_x - \frac{1}{2}[\alpha_J(A),G_n] + \frac{1}{2}\alpha_J([A,F_n]) = 0\ . \end{aligned} \tag{16.252}$$

We want to give geometric interpretation to these relations. The third equation is equivalent to

$$G_n = -\frac{1}{2}\mathcal{A}_\pm(\alpha_J([A,\pi(F_n)])).$$

As to the second equation, first, it shows that $(F_n)_x \in T_A(\mathcal{M}_S^I)$ and therefore $B_T(A(x),(F_n(x))_x) = 0$. Next, since $F_n = \pi(F_n) + B_T(A, F_n)A$ and

$$B_T(A, F_n)(x) = \int_{\pm\infty}^{x} \partial_y B_T(A(y), F_n(y))dy = \int_{\pm\infty}^{x} B_T(A_y(y), F_n(y))dy,$$

it is easily seen that it can be written as:

$$Q_A(\pi(F_{n+1})) = \bar{P}_A(\pi(F_n)) \, . \qquad (16.253)$$

But then the hierarchy (16.174), (16.176) can also be cast into the form:

$$A_t = Q_A(\pi(F_{n+1})), \quad A_t = \bar{P}_A(\pi(F_n)) \, . \qquad (16.254)$$

We, therefore, can interpret $\pi(F_n)$ as 1-forms on $\mathcal{M}_S^I$ and the above equations as Hamiltonian equations, provided $\pi(F_n) = dH_n$, where H_n are the Hamiltonians for these equations. The theory of the P-N manifolds shows that if on a P-N manifold the 1-form β is fundamental (corresponds to fundamental field of the structure) then β is closed and if β, $N^*\beta$ are closed, then $(N^*)^k\beta$, $k \geq 2$ are closed too. In our case, this means that all $\pi(F_n)$ are closed forms. Indeed, $\pi(F_1)$ is closed and fundamental (it can be shown exactly as it is done for the form B_1 in the case of $O(3)$ chiral fields system (see below (16.298)); The form $\pi(F_2)$ is even exact, since it corresponds to the LL equation and it is Hamiltonian. So the equations (16.254), starting from the second one (the LL equation), are Hamiltonian and even bi-Hamiltonian in a generalized sense. As for the proof that there exist well-defined functionals H_n, such that $\pi(F_n) = dH_n$ (in general one has it starting from some n_0), this fact usually follows from some other considerations. For example, one gets this result for the HF equation and NLS equation from the spectral theory for the corresponding auxiliary liner problem and the spectral theory of the recursion operators. It is well known that for NLS equation and HF equation the Hamiltonians can be obtained through the recursion operators [29], though their locality is not immediately seen and needs to be proved [28]. We shall assume that H_n exist for $n \geq 2$. If so, the equations (16.254), together with (16.253), show that the nonlinear evolution equation hierarchy, related to the polynomial pencil is bi-Hamiltonian and is related to the P-N structure described in theorem 16.47. From the general theory of the P-N manifolds we get

Corollary 16.48. *The nonlinear equations from the hierarchy (16.176) (from the hierarchy (16.177)), are Hamiltonian, and their Hamiltonians are in involution.*

Finally, we note that the above corollary can be obtained also directly. Indeed, if we denote by $\{H_n, H_m\}_Q$, $\{H_n, H_m\}_P$ the Poisson brackets defined by Q and $\bar{P}$ respectively, we can write the equalities:

$$\{H_n, H_m\}_Q = \frac{1}{2}\langle\langle [A, \pi(F_n)], \pi(F_m)\rangle\rangle$$

$$\langle\langle \bar{P}(\pi(F_{n-1}))], \pi(F_m)\rangle\rangle = \{H_{n-1}, H_m\}_P = \tag{16.255}$$

$$\frac{1}{2}\langle\langle [A, \pi(F_n)], \pi(F_m)\rangle\rangle = \frac{1}{2}\langle\langle [A, B_n], B_m\rangle\rangle$$

$$\frac{1}{2}\int\limits_{-\infty}^{+\infty} B_T([A(x), B_n(x)], B_m(x))dx = \tag{16.256}$$

$$\{H_{n-1}, H_m\}_P = \langle\langle \bar{P}(\pi(F_{n-1}))], \pi(F_m)\rangle\rangle$$

$$\int\limits_{-\infty}^{+\infty} B_T(\partial_x B_{n-1}(x) - \frac{1}{2}[A(x), B_{n-1}(x)]_J, B_m(x))dx\ . \tag{16.257}$$

Now, we implement a technique described in [31] and frequently used in similar proofs. Let us suppose for definitiveness that $n > m$. Using that $\mathrm{ad}\,{}^J_X$ is skew-symmetric with respect to the bilinear form B_T, integrating by parts and taking into account the properties of the matrices B_n (see (16.190)), we get that

$$\{H_n, H_m\}_Q = \{H_{n-1}, H_{m+1}\}_Q\ . \tag{16.258}$$

Applying this identity sufficiently many times, we arrive at the equation of the type

$$\{H_n, H_m\}_Q = \{H_{n-1}, H_{m+1}\}_Q = \ldots = \{H_k, H_k\}_Q \tag{16.259}$$

if $n - m$ is even, or at the equation of the type

$$\{H_n, H_m\}_Q = \{H_{n-1}, H_{m+1}\}_Q = \ldots = \{H_m, H_n\}_Q = -\{H_n, H_m\}_Q \tag{16.260}$$

if $n - m$ is odd. In both cases, we conclude that $\{H_n, H_m\}_Q = 0$. From the proof it follows also that $\{H_n, H_m\}_P = 0$. This is exactly what we wanted to show.

16.10 Chiral Fields Hierarchy : the Poisson-Nijenhuis Structure

We are going to define now the Hamiltonian structures of the equations from the CF hierarchy. First of all let us remark that the set of potentials has natural structure of infinite-dimensional manifold. Indeed, the manifold of potentials for the CF hierarchy is the set of the smooth functions

$$\begin{aligned} &A(x) = \{\mathbf{u}(x)\}_I + \{\mathbf{v}(x)\}_{II} \\ &(\mathbf{u}(x))^2 = (\mathbf{v}(x))^2 = 1 \\ &\mathbf{u}(x), \mathbf{v}(x) - \text{real}\ , \end{aligned} \tag{16.261}$$

defined on the real line $\mathbb{R}$ and tending fast enough to some limit value

$$A_0 = \{\mathbf{u}_0\}_I + \{\mathbf{v}_0\}_{II},$$

as $|x| \to \infty$.

One can easily see that the requirements (16.261) simply mean that $A(x)$ takes its values in the following orbit of the adjoint representation of the group $SO(4, \mathbb{R})$ (the group of the orthogonal 4×4 matrices with unit determinant):

$$\mathcal{O}_{B_0} = \{A = Ad(g)B_0;\ g \in SO(4, \mathbb{R})\} \subset \mathrm{o}\,(4)\ , \tag{16.262}$$

where

$$B_0 = \begin{pmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \{(1, 0, 0)\}_I + \{(1, 0, 0)\}_{II}\ . \tag{16.263}$$

Therefore, the manifold of potentials is the set $\mathcal{O}_{B_0}[x]$ consisting of functions taking values in $\mathcal{O}_{B_0}$ and tending fast enough to some limit value as $|x| \to \infty$. Clearly $\mathcal{O}_{B_0}[x]$ is infinite dimensional manifold and submanifold of $\mathrm{o}\,(4, \mathbb{R})_0[x]$. This submanifold can be understood better, if one remarks that the orbit $\mathcal{O}_{B_0}$ have also the following representation:

$$\mathcal{O}_{B_0} = \{A : B(A, A) = -16, \quad B_T(A, A) = 0\} \subset \mathrm{o}\,(4, \mathbb{R})\ , \tag{16.264}$$

where $B(X, Y)$ and $B_T(X, Y)$ are the symmetric forms on $\mathrm{o}\,(4, \mathbb{R})$ we have introduced earlier. In order to see this, it is enough to calculate $B(A, A)$ and $B_T(A, A)$ using the relations (16.133).

Then one can see that $X(x) \in T_A(\mathcal{O}_{B_0}[x])$ when

$$B(A, X) = 0, \quad B_T(A, X) = B(A, T(X)) = B(T(A), X) = 0\ . \tag{16.265}$$

Let us make some preparations. We shall denote the points q of the algebra $\mathrm{o}\,(4)$ by column of two vectors

$$q = (\mathbf{a}, \mathbf{b})^t = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}$$

having in mind of course that this means $q = \{\mathbf{a}\}_I + \{\mathbf{b}\}_{II}$. Therefore the points $A(x) = \{\mathbf{u}(x)\}_I + \{\mathbf{v}(x)\}_{II}$ of the manifold $\mathcal{O}_{B_0}[x]$ are represented by a column $A(x) = (\mathbf{u}(x), \mathbf{v}(x))^t$ or simply by $A = (\mathbf{u}, \mathbf{v})^t$. Also, in order to write down in more convenient way some complicated expressions, we shall denote by lower indices I and II the following projections

$$\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}_I = \mathbf{a},\ \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}_{II} = \mathbf{b}. \tag{16.266}$$

With the new notations a vector $X(x)$ at the point $A(x) \in \mathcal{O}_{B_0}[x]$ is represented by a couple of Schwartz-type functions $(\mathbf{a}(x), \mathbf{b}(x))^t$, for which:

$$\langle \mathbf{u}(x), \mathbf{a}(x) \rangle = \langle \mathbf{v}(x), \mathbf{b}(x) \rangle = 0 \; , \tag{16.267}$$

where $\langle \; , \; \rangle$ is the canonical inner product in $\mathbb{R}^3$. According to our convention we identify the vectors and covectors using the pairing defined in (16.141), putting of course B_T instead of B. We easily obtain

$$\langle\langle (\mathbf{a}(x), \mathbf{b}(x))^t, (\mathbf{c}(x), \mathbf{d}(x))^t \rangle\rangle = -8 \int\limits_{-\infty}^{+\infty} [\langle \mathbf{a}(x), \mathbf{c}(x) \rangle - \langle \mathbf{b}(x), \mathbf{d}(x) \rangle] dx \; . \tag{16.268}$$

Thus, the above form is nondegenerate when restricted to the tangent space $T_A(\mathcal{O}_{B_0}[x])$, and through it we can identify the tangent space $T_A(\mathcal{O}_{B_0}[x])$ and the cotangent space $T^*_A(\mathcal{O}_{B_0}[x])$.

We shall try now to restrict two of the tensors from the 3-parametric family of Poisson tensors defined in Theorem 16.39 onto the submanifold $\mathcal{O}_{B_0}[x]$. These tensors are the already familiar from (16.234) tensors

$$Q_A = \frac{1}{2} \mathrm{ad}_{\,A} \tag{16.269}$$

$$P_A = -\frac{1}{2} \mathrm{ad}^{\,J}_{\,A} + \partial_x \; , \tag{16.270}$$

or equivalently,

$$P \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} \mathbf{a}_x + R(\mathbf{v} \times \mathbf{b}) - \mathbf{u} \times R(\mathbf{b}) + \mathbf{a} \times R(\mathbf{v}) \\ \mathbf{b}_x + R(\mathbf{u} \times \mathbf{a}) - \mathbf{v} \times R(\mathbf{a}) + \mathbf{b} \times R(\mathbf{v}) \end{pmatrix} \tag{16.271}$$

$$Q \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} -\mathbf{u} \times \mathbf{a} \\ -\mathbf{v} \times \mathbf{b} \end{pmatrix} \; . \tag{16.272}$$

Note that $Q^{-1} = -Q$.

The fact that Q allows nondegenerate restriction over $\mathcal{O}_{B_0}[x]$, and its form after the restriction does not change, is in fact the theorem that the Poisson-Lie tensor restricted to an orbit of the coadjoint representation is nondegenerate, so there is no need to prove it.

As to the tensor P, it cannot be restricted directly. In order to perform the restriction, we shall use again the Restriction Theorem 12.25 applying it this time to $\mathcal{M} = \mathrm{o}\,(4, \mathbb{R})_0[x]$ and $\mathcal{N} = \mathcal{O}_{B_0}[x]$. Let us find $\mathcal{X}^*_P(\mathcal{N})_A$ and $T^{\perp}_A(\mathcal{N})$ for a point A belonging to $\mathcal{N} = \mathcal{O}_{B_0}[x]$ (for the definitions of these spaces see theorem 12.25). Naturally, for the annihilator $T^{\perp}_A(\mathcal{N})$ we get

$$T^{\perp}_A(\mathcal{N}) = \{(f\mathbf{u}, g\mathbf{v})^t : \quad f, g \in \mathcal{S}\} \; , \tag{16.273}$$

where $\mathcal{S}$ is the set of all Schwartz type functions on the line. We can also say that

$$T_A^{\perp}(\mathcal{N}) = \{(\bar{f}A + \bar{g}T(A): \quad \bar{f}, \bar{g} \in \mathcal{S}\} . \tag{16.274}$$

According to the definition $X \in \mathcal{X}_P^*(\mathcal{N})_A$ if $P_A(X) \in T_A(\mathcal{O}_{B_0}[x])$ or in other words if the following equations hold

$$B(A, \partial_x X) - \frac{1}{2} B(A, \mathrm{ad}\,_A^J(X)) = 0 \tag{16.275}$$

$$B_T(A, \partial_x X) = 0 . \tag{16.276}$$

After some simple transformations we obtain that these equations are equivalent to

$$B(A, X) = \partial_x^{-1}[B(A_x, X) + \frac{1}{2} B(A, \mathrm{ad}\,_A^J(X))] \tag{16.277}$$

$$B_T(A, X) = B(T(A), X) = \partial_x^{-1} B(A_x, T(X)), \tag{16.278}$$

where ∂_x^{-1} stands for the inverse of the operator ∂_x. The choice of ∂_x^{-1} of course is not unique, and it is easy to see that we can use as inverse any of the operators:

$$\partial_x^{-1} = \tau \int_{-\infty}^{x} + (1-\tau) \int_{+\infty}^{x}, \quad \tau \in \mathbb{R} , \tag{16.279}$$

but we shall postpone the discussion about the appropriate choice for ∂_x^{-1}, in order to proceed with our geometric construction.

Let us remark that for $A \in \mathcal{O}_{B_0}[x]$ we have $B(A, T(A)) = B_T(A, A) = 0$ or in other words A and $T(A)$ are orthogonal with respect to the Killing form. Then taking into account (16.265), we see that the following orthogonal decomposition holds:

$$\mathrm{o}\,(4, \mathbb{R}) = [\mathbb{R}A(x) \oplus \mathbb{R}T(A(x))] \oplus T_A(\mathcal{O}_{B_0}[x]) . \tag{16.280}$$

(This decomposition obviously depends on x).

For fixed X, let us denote by X^A the orthogonal projection of X onto the space $T_A(\mathcal{O}_{B_0}[x])$ and the orthogonal projection of X over the space spanned by A and $T(A)$ by X_A. One can see that

$$X^A = X + \frac{1}{16} B(A, X)A + \frac{1}{16} B(T(A), X)T(A)$$

$$X_A = -\frac{1}{16} B(A, X)A - \frac{1}{16} B(T(A), X)T(A) . \tag{16.281}$$

If we return now to (16.277) and (16.278), then due to the fact that

$$B(A, \mathrm{ad}\,_A^J(T(A))) = B_T(T(A), \mathrm{ad}\,_A^J(T(A))) = 0 , \tag{16.282}$$

we see that in the right-hand sides we can put instead of X the projection X^A, and these equations actually show that if $X \in \mathcal{X}_P^*(\mathcal{N})_A$ the component X_A is expressed by the component X^A. Taking this into account we write

$$X = Y + Z \,, \tag{16.283}$$

where

$$\begin{aligned} Y = X^A &- \frac{1}{16} A\ \partial_x^{-1}[B(A_x, X^A) + \frac{1}{2} B(A, \mathrm{ad}\,^J_A(X^A))] \\ &- \frac{1}{16} T(A)\ \partial_x^{-1} B(A_x, T(X^A)) \end{aligned} \tag{16.284}$$

$$\begin{aligned} Z = X_A &+ \frac{1}{16} A\ \partial_x^{-1}[B(A_x, X^A) + \frac{1}{2} B(A, \mathrm{ad}\,^J_A(X^A))] \\ &\frac{1}{16} T(A)\ \partial_x^{-1} B(A_x, T(X^A)) \,. \end{aligned} \tag{16.285}$$

Using (16.282), one can check that $Y \in \mathcal{X}_P^*(\mathcal{N})_A$. As to the vector Z, it is a linear combination of A and $T(A)$ and hence $Z \in T_A^{\perp}(\mathcal{N})$. Moreover, from (16.283) and (16.277, 16.278) we see that

$$\begin{aligned} &T_A^{\perp}(\mathcal{N}) \oplus \mathcal{X}_P^*(\mathcal{N})_A = T_A^*(\mathcal{M}) \\ &T_A^{\perp}(\mathcal{N}) \bigcap \mathcal{X}_P^*(\mathcal{N})_A = \{0\} \subset \ker{(P_A)} \,. \end{aligned} \tag{16.286}$$

(Of course, here $\mathcal{M} = \mathrm{o}\,(4, \mathbb{R})_0[x]$ and $\mathcal{N} = \mathcal{O}_{B_0}[x]$.) Then the requirements of the Restriction Theorem 12.25 are fulfilled, and there exists restriction $\bar{P}$ of P defined over $\mathcal{N}$. According to the prescriptions of the Restriction Theorem, for $\alpha \in T_A^*(\mathcal{N})$, we must take $\beta = j^*\alpha$, then represent β as sum $\beta_1 + \beta_2$ in such a way that $\beta_1 \in \mathcal{X}_P^*(\mathcal{N})_A$ and $\beta_2 \in T_A^{\perp}(\mathcal{N})$ and finally put $\bar{P}(\alpha) \equiv P(\beta_1)$. As usual, j is the inclusion map $j : \mathcal{N} \to \mathcal{M}$.

Since in our case $T_A(\mathcal{M})$ and $T_A^*(\mathcal{M})$ are identified, the pull-back of the inclusion map j is simply the orthogonal projection $X \to X^A$. As is readily seen, the role of the component β_1 here is played by the expression (16.284) where we must put X instead of X^A in the integrands. Taking into account all that, we arrive at the following expression for the restricted Poisson tensor:

$$\begin{aligned} &\bar{P}_A(X) = \partial_x X - \frac{1}{16} A_x \partial_x^{-1}[B(A_x, X) + \frac{1}{2} B(A, \mathrm{ad}\,^J_A(X))] \\ &- \frac{1}{16} T(A_x) \partial_x^{-1} B(A_x, T(X)) - \frac{1}{16} A[B(A_x, X) + \frac{1}{2} B(A, \mathrm{ad}\,^J_A(X))] \\ &- \frac{1}{16} T(A) B(A_x, T(X)) - \frac{1}{2} \mathrm{ad}\,^J_A(X) \,, \end{aligned} \tag{16.287}$$

where $X \in T_A^*(\mathcal{N}) \sim T_A(\mathcal{N})$.

Remark 16.49. The function A_x tends to zero as $|x| \to \infty$, and $X(x)$ is a function of the Schwartz type, so the integrals in (16.287) exist. The same is true for the integrals in the expressions for Y and Z, see (16.284), (16.285).

We must ensure also that $\bar{P}$ is skew-symmetric, at least in a weak sense, that is, we must have

$$\langle\langle \bar{P}_A(X), Y\rangle\rangle = -\langle\langle X, \bar{P}_A(Y)\rangle\rangle \tag{16.288}$$

for $X, Y \in T_A(\mathcal{O}_{B_0}[x])$. A simple integration by parts shows that for this we must take

$$\partial_x^{-1} = \frac{1}{2}\left(\int_{-\infty}^{x} + \int_{+\infty}^{x}\right) . \tag{16.289}$$

The construction of Nijenhuis tensor N is now an easy task. We must calculate $N = \bar{P}Q^{-1} = -\bar{P}Q$ or $N^* = Q^{-1}\bar{P} = -Q\bar{P}$. We obtain:

$$\begin{aligned} N_A^*(X) = &-\frac{1}{2}[A, \partial_x X] + \frac{1}{32}[A, A_x]\partial_x^{-1}[B(A_x, X) + \frac{1}{2}B(A, \mathrm{ad}\,{}_A^J(X))] \\ &+\frac{1}{32}[A, T(A_x)]\partial_x^{-1}B(A_x, T(X)) \\ &+\frac{1}{32}[A, T(A)]B(A_x, T(X)) + \frac{1}{4}[A, \mathrm{ad}\,{}_A^J(X)] , \end{aligned} \tag{16.290}$$

where $X \in T_A^*(\mathcal{N}) \sim T_A(\mathcal{N})$. Now let us formulate our main result:

Theorem 16.50. *The tensor fields Q and $N = \bar{P}Q^{-1}$ endow the manifold of potentials $\mathcal{N} = \mathcal{O}_{B_0}[x]$ for the CF hierarchy with Poisson-Nijenhuis structure.*

We are going to apply now this result to the CF hierarchy, but first we must write the operators which we have obtained in terms of $\mathbf{u}, \mathbf{v}$. If we put $X = (\mathbf{a}, \mathbf{b})^t$ and assume $\langle \mathbf{u}, \mathbf{a}\rangle = \langle \mathbf{v}, \mathbf{b}\rangle = 0$ we get:

$$\begin{aligned} \left[\bar{P}_A\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}\right]_I = &[\partial_x\mathbf{a} + R(\mathbf{v}\times\mathbf{b}) + \mathbf{u}\times R(\mathbf{b}) + \mathbf{a}\times R(\mathbf{v})]^u \\ &+ \mathbf{u}\times R(\mathbf{v})\ \partial_x^{-1}[\langle \mathbf{u}_x, \mathbf{a}\rangle - \langle \mathbf{v}_x, \mathbf{b}\rangle] \\ &+ \mathbf{u}_x\ \partial_x^{-1}[\langle \mathbf{u}_x, \mathbf{a}\rangle + \langle \mathbf{u}\times R(\mathbf{v}), \mathbf{a}\rangle - \langle R(\mathbf{u})\times\mathbf{v}, \mathbf{b}\rangle] \end{aligned} \tag{16.291}$$

$$\begin{aligned} \left[\bar{P}_A\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}\right]_{II} = &[\partial_x\mathbf{b} + R(\mathbf{u}\times\mathbf{a}) + \mathbf{v}\times R(\mathbf{a}) + \mathbf{b}\times R(\mathbf{u})]^v \\ &+ \mathbf{v}\times R(\mathbf{u})\ \partial_x^{-1}[\langle \mathbf{v}_x, \mathbf{b}\rangle - \langle \mathbf{u}_x, \mathbf{a}\rangle] \\ &+ \mathbf{v}_x\ \partial_x^{-1}[\langle \mathbf{v}_x, \mathbf{b}\rangle + \langle \mathbf{v}\times R(\mathbf{u}), \mathbf{b}\rangle - \langle R(\mathbf{v})\times\mathbf{u}, \mathbf{a}\rangle] , \end{aligned} \tag{16.292}$$

where as before by upper indices u, v we denote the projections on the planes orthogonal to the vectors $\mathbf{u}$ and $\mathbf{v}$, respectively and $R = \mathrm{diag}\,(j_1, j_2, j_3)$. For the tensor field N^* we obtain:

$$\left[N_A^*\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}\right]_I = \mathbf{u}\times\partial_x\mathbf{a}+\mathbf{u}\times R(\mathbf{v}\times\mathbf{b})-[R(\mathbf{b})]^u+\mathbf{a}\langle R(\mathbf{v}),\mathbf{u}\rangle$$
$$-[R(\mathbf{v})]^u\,\partial_x^{-1}[\langle\mathbf{u}_x,\mathbf{a}\rangle-\langle\mathbf{v}_x,\mathbf{b}\rangle]$$
$$+(\mathbf{u}\times\mathbf{u}_x)\partial_x^{-1}[\langle\mathbf{u}_x,\mathbf{a}\rangle+\langle\mathbf{u}\times R(\mathbf{v}),\mathbf{b}\rangle-\langle R(\mathbf{u})\times\mathbf{v},\mathbf{b}\rangle] \tag{16.293}$$

$$\left[N_A^*\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}\right]_{II} = \mathbf{v}\times\partial_x\mathbf{b}+\mathbf{v}\times R(\mathbf{u}\times\mathbf{a})-[R(\mathbf{a})]^v+\mathbf{b}\langle R(\mathbf{u}),\mathbf{v}\rangle$$
$$-\ [R(\mathbf{u})]^v\,\partial_x^{-1}[\langle\mathbf{v}_x,\mathbf{b}\rangle-\langle\mathbf{u}_x,\mathbf{a}\rangle]$$
$$+\ (\mathbf{v}\times\mathbf{v}_x)\partial_x^{-1}[\langle\mathbf{v}_x,\mathbf{b}\rangle+\langle\mathbf{v}\times R(\mathbf{u}),\mathbf{b}\rangle-\langle R(\mathbf{v})\times\mathbf{u},\mathbf{a}\rangle]\,. \tag{16.294}$$

The comparison shows that the recursion operators (16.163) that appear in (16.162) are related with N^* in the following way:

$$N^*=\frac{1}{2}(\mathbf{A}_+ + \mathbf{A}_-)\,. \tag{16.295}$$

For the equations from the CF hierarchy, one can equivalently use $\mathbf{A}_+$ and $\mathbf{A}_-$ (the integrands in this formulae are always total derivatives), and, therefore, it is evident that one can use also N^*. Remember now that the equations from the CF hierarchy have the form (compare with (16.156) and (16.157)):

$$A_t=(B_n)_x-\frac{1}{2}(AJB_n-B_nJA)=\frac{1}{2}[A,B_{n+1}]=Q_A(B_{n+1})\,. \tag{16.296}$$

and

$$\begin{aligned}B_{n+1}&=N^*(B_n)\\ B_n&=\{\mathbf{b}_n\}_I+\{\mathbf{c}_n\}_{II};\quad n\geq 1\,.\end{aligned} \tag{16.297}$$

If we consider B_n as 1-forms, all these equations are Hamiltonian. The only thing that must be proved, in order to apply the general results about the P-N manifolds to our case is to show that the forms B_1 and B_2 are closed. But, the evolution equation corresponding to B_2 is up to some changes of the parameters the $O(3)$-chiral fields system. It is well known that it has Hamiltonian function, see [21] and therefore B_2 is closed. As to the form B_1 it is proportional to

$$\epsilon\{\mathbf{u}\times\mathbf{u}_x\}_I+\mu\{\mathbf{v}\times\mathbf{v}_x\}_{II}\ , \tag{16.298}$$

ϵ,μ being constants. One readily sees that it is enough to show that on the manifold $\mathcal{N}$ of the smooth vector functions $\mathbf{v}(x)$ taking values on the unit sphere $\mathbb{S}^2$ and tending sufficiently fast to some value $\mathbf{v}_0$ as $|x|\to\infty$ the covector field $\mathbf{v}\to\gamma_v$:

$$\gamma_v(\mathbf{a})=\int_{-\infty}^{+\infty}\langle\mathbf{v}\times\mathbf{v}_x,\mathbf{a}[v]\rangle dx;\quad \mathbf{a}[v]\in T_v(\mathcal{N}) \tag{16.299}$$

is closed. If $\mathbf{a}$ and $\mathbf{b}$ are two vector fields on $\mathcal{N}$ then the calculation shows that

$$[d\gamma]_v(\mathbf{a},\mathbf{b}) = \int\limits_{-\infty}^{+\infty} \partial_x \langle \mathbf{v}, \mathbf{a}[v] \times \mathbf{b}[v] \rangle dx - 3 \int\limits_{-\infty}^{+\infty} \langle \mathbf{v}_x, \mathbf{a}[v] \times \mathbf{b}[v] \rangle dx.$$

For the vector fields, $\mathbf{a}$ and $\mathbf{b}$ $\lim_{|x|\to 0} \mathbf{a}[v](x) = \lim_{|x|\to 0} \mathbf{b}[v](x) = 0$ and the first term in the right-hand side is zero. As to the second term it is zero simply because $\mathbf{v}_x(x), \mathbf{a}[v](x), \mathbf{b}[v](x)$ are orthogonal to $\mathbf{v}(x)$ and hence $\langle \mathbf{v}_x, \mathbf{a}[v] \times \mathbf{b}[v] \rangle = 0$. As a consequence, from the above considerations and from the properties of the fundamental fields of the Poisson-Nijenhuis manifolds, we get the following.

Theorem 16.51. *The right-hand sides of the equations from the CF hierarchy are fundamental fields for the P-N structure generated by the field $Q^{-1}B_1$. These equations are Hamiltonian with respect to the infinite hierarchy of Poisson structures, and the flows corresponding to these fields commute.*

The results for CF system and LL described in the above deserve some discussion. As we have mentioned, an interesting question is whether using essentially different Lax pairs we obtain the same facts and objects for the corresponding nonlinear evolution equations. The answer for the case of the CF hierarchy of equations and more precisely for their Hamiltonian structures and conservation laws is affirmative. We have obtained the same Poisson tensors P and Q over the manifold of potentials as have been obtained using the hierarchies of Poisson structures over elliptic algebras, see [26]. Hence we have the same hierarchy of equations and the same conservation laws for it. The results we are citing are obtained using different algebraic objects so let us say a little more about them. First, let us introduce the so-called elliptic algebras. The elliptic algebra $\mathfrak{g}_e$, used for describing the CF hierarchy of equations and their Hamiltonian structures is spanned by the generators

$$\begin{aligned} &X_\alpha^{2l+1} = \omega^{2l}\omega_\alpha X_\alpha \\ &X_\alpha^{2l+2} = \omega^{2l}\omega_\alpha^{-1}\omega_1\omega_2\omega_3 X_\alpha \\ &l \in \mathbb{Z}; \quad \alpha = 1,2,3\ . \end{aligned} \tag{16.300}$$

In the above formulae X_α are the generators of $\mathrm{o}\,(3) \sim \mathrm{su}\,(2)$ with commutation relations

$$[X_\alpha, X_\beta] = \epsilon_{\alpha\beta\gamma} X_\gamma$$

and $\epsilon_{\alpha\beta\gamma}$ is the three dimensional Levi-Civita symbol. As a rule, in the literature the generators X_α are expressed through the Pauli matrices. The quantities ω, ω_α, $\alpha = 1,2,3$, satisfy the quadratic relations

$$\omega_\alpha^2 - \omega_\beta^2 = p_\beta - p_\alpha, \quad \omega^2 - \omega_\alpha^2 = d_\alpha = p_\alpha - \frac{1}{3}(p_1 + p_2 + p_3)\ , \tag{16.301}$$

where $p_\alpha = -j_\alpha^2$. Usually the following natural parametrization for ω, ω_α is used:

$$\omega_\alpha = \sqrt{\wp(\lambda) - d_\alpha}, \quad \omega = \sqrt{\wp(\lambda)}\ , \tag{16.302}$$

where $\wp(\lambda)$ is the Weierstrass function: $(\wp')^2 = 4(\wp - d_1)(\wp - d_2)(\wp - d_3)$. We prefer the expressions through the Weierstrass function rather than the usual expressions with the Jacobi elliptic functions, but of course it is all the same. One has the following commutation relations of the generators (16.300) defining the so-called elliptic algebra:

$$\begin{aligned}
[X_\alpha^{2l}, X_\beta^{2m}] &= \epsilon_{\alpha\beta\gamma}(X_\gamma^{2(l+m)} - d_\gamma X_\gamma^{2(l+m-1)}) \\
[X_\alpha^{2l}, X_\beta^{2m+1}] &= \epsilon_{\alpha\beta\gamma}(X_\gamma^{2(l+m)+1} - d_\gamma X_\gamma^{2(l+m)-1}) \\
[X_\alpha^{2l+1}, X_\beta^{2m+1}] &= \epsilon_{\alpha\beta\gamma} X_\gamma^{2(l+m+1)}\ .
\end{aligned} \tag{16.303}$$

These relations are a consequence both from the commutation relations of su (2) and the properties of the elliptic functions.

The tensors P and Q we have used in the above arise as restrictions over some submanifold [26], of the natural Kirillov (Poisson-Lie) tensors for the elliptic algebra $\mathfrak{g}_e$ and its central extension with the help of Gel'fand-Fuchs cocycle.

Comparing the two approaches we note that all the results we have obtained here can be formulated in terms of graded algebras too. For example, in our approach, we in fact use the graded algebra $\mathfrak{g}_p$ generated by the elements

$$\begin{aligned}
N_\alpha^n &= -\frac{1}{2}\lambda^n \{\mathbf{e}_\alpha\}_I(\lambda + J) \\
M_\alpha^n &= -\frac{1}{2}\lambda^n \{\mathbf{e}_\alpha\}_{II}(\lambda + J) \\
n &\in \mathbb{Z},\ \alpha = 1, 2, 3\ ,
\end{aligned} \tag{16.304}$$

where $\mathbf{e}_\alpha$; $\alpha = 1, 2, 3$ is the canonical basis in $\mathbb{R}^3$, that is $(\mathbf{e}_\alpha)_\beta = \delta_{\alpha\beta}$. The commutation relations between this generators are

$$\begin{aligned}
[N_\alpha^n, N_\beta^m] &= \epsilon_{\alpha\beta\gamma}(N_\gamma^{n+m+1} + j_\gamma M_\gamma^{n+m}) \\
[M_\alpha^n, M_\beta^m] &= \epsilon_{\alpha\beta\gamma}(M_\gamma^{n+m+1} + j_\gamma N_\gamma^{n+m}) \\
[N_\alpha^n, M_\beta^m] &= -\epsilon_{\alpha\beta\gamma}(j_\beta N_\gamma^{n+m} + j_\alpha M_\gamma^{n+m})\ .
\end{aligned} \tag{16.305}$$

Remark 16.52. We must underline that writing the above relations we used implicitly the new Lie algebra structure we have introduced over the algebra o (4), and the above relations are not simple consequences from the usual Lie-algebra structure over o (4).

Then again the tensors P and Q can be obtained restricting the natural Kirillov (Poisson-Lie) tensors for $\mathfrak{g}_p$ and its central extension over some submanifold. One easily finds that the resulting Poisson submanifolds used in both constructions are isomorphic, and then the two approaches – based on the elliptic algebra $\mathfrak{g}_e$ and on the algebra $\mathfrak{g}_p$ are equivalent. So, at the present moment we have the "experimental" result that the two algebras $\mathfrak{g}_e$ and $\mathfrak{g}_p$ generate same Poisson structures over some submanifolds, but it is still an open question why this occurs.

As already explained, the situation with LL equation and the recursion operators we have obtained is far more complicated than the CF case. The recursion operators here are different from those obtained by Barouch and al. in [24] via elliptic pencil. It seems that both have equal "right" to be called recursion operators for the LL equation, but what is their relation is still not known. The above-mentioned facts make even more interesting than before the question whether there is equivalence between the two pencils containing L-A pairs for the Landau–Lifshitz equation – the elliptic pencil and the polynomial pencil.

Finally, we must mention that the spectral of the recursion operators for LL and CF has not been done yet, and this is of course another area of research.

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A

Appendix: Generalizations

In this appendix, we briefly discuss possible generalizations of the ideas that have been developed through the second part of the book. The first one is the possibility of developing a purely algebraic approach to the dynamical systems and the second is the possibility of developing a supersymmetric dynamics and supersymmetric generalizations of the objects we have considered. We hope that the discussion of the aforementioned topics, which are linked to each other, will be interesting for the specialists.

A.1 Algebraic Approach to Integrability

In recent years, there have been attempts to create a purely algebraic approach to differential calculus, which proved to be useful in gauge theories [1, 2, 3, 4, 5] and allows to construct analogs of the Lagrangian and Hamiltonian formalism [6, 7, 8, 9]. In particular, the algebraic approach proved to be useful in the treatment of topologically nontrivial solutions of gauge theories and a symmetry breaking, permitting to pass from the 't Hooft-Polyakov monopole to the Dirac monopole, and in general it allows to extend the usual differential calculus to the case when the variables do not commute [10, 11, 12].

Within the algebraic approach, it is also possible to treat successfully the problem of integrability. We briefly introduce these ideas here, extending some preliminary results, presented in [13].

From the beginning, we make a review the of the so-called algebraic differential calculus. Then we present an algebraic definition of a dynamical system and an algebraic Hamiltonian system. Within this framework, we discuss a possible generalization of the notion of integrability and give an integrability criterion for Hamiltonian systems. Finally, we present two interesting examples.

Gerdjikov, V.S. et al.: *Appendix: Generalizations.* Lect. Notes Phys. **748**, 613–634 (2008)
DOI 10.1007/978-3-540-77054-1_17

A.1.1 Algebraic Differentiable Calculus

The main idea behind the algebraic approaches is that a manifold $\mathcal{M}$ and the algebra $\mathcal{F} = \mathcal{D}(\mathcal{M})$ of smooth functions on the manifold $\mathcal{M}$ in some sense contain the same information. So, we can replace $\mathcal{M}$ with $\mathcal{F}$ in many situations, which do not depend strongly on the topology, but on the "calculus" – for example in the study of integrals of motion of dynamical systems. (We intentionally use now another notation for the algebra of the smooth functions, because we generalize the traditional concepts and want to distinguish the new situation.) Indeed, the manifold $\mathcal{M}$ and the algebra $\mathcal{F}$ can be considered to be "dual" to each other since the so-called evaluation map

$$\begin{aligned} ev : \mathcal{M} &\to \mathrm{Hom}_{\mathbb{R}}(\mathcal{F}, \mathbb{R}) \\ m &\mapsto ev_m : ev_m(f) = f(m), \end{aligned} \tag{A.1}$$

is a bijection, and, therefore, one can identify $\mathcal{M}$ and Hom $(\mathcal{F}(\mathcal{M}), \mathbb{R})$ (see e.g. [14] for the details).

As mentioned, the advantage of the above "replacement" is that $\mathcal{F}$ is a $\mathbb{R}$-algebra, and those concepts and constructions which depend only on the algebraic features of $\mathcal{F}$ are not strongly influenced by the topological characteristics of $\mathcal{M}$ or whether it is finite or infinite-dimensional. Also, since everything turns to be algebraic in nature, one can obtain highly nontrivial generalizations taking $\mathcal{F}$ to be not a commutative algebra (as it is the case when $\mathcal{F}$ is the algebra of the smooth functions over some manifold $\mathcal{M}$) but simply some associative algebra. Actually this is the idea behind the so-called Noncommutative Geometry. In the noncommutative case, however, Der $(\mathcal{F})$ is not anymore a left $\mathcal{F}$-module (see below), it is a module only over the center of $\mathcal{F}$. In what follows immediately, we restrict ourselves to the commutative case, that is, to the case $\mathcal{F}$ when is a commutative and associative algebra. The noncommutative generalization will be considered later.

In the new approach, the role of the algebra of the vector fields on $\mathcal{M}$ is played by the Lie algebra $\mathcal{X} = \mathrm{Der}\,(\mathcal{F})$ of the derivations of $\mathcal{F}$ and its $\mathcal{F}$-dual, denoted by $\mathcal{X}^* = (\mathrm{Der}\,(\mathcal{F}))^*$, is the counterpart of the module of the one-forms. Of course, we assume that each derivation acts trivially on $\mathbb{R}$ considered as imbedded in $\mathcal{F}$.

Out of the previous ingredients, one can construct an algebraic tensor calculus, which is an analogue of the usual tensor calculus on some manifold $\mathcal{M}$, [15, 16]. Let us introduce the main objects. First, the collection of all skew-symmetric $\mathcal{F}$-linear maps

$$\underbrace{\mathcal{X} \times \ldots \times \mathcal{X}}_{p \text{ times}} \mapsto \mathcal{F} \tag{A.2}$$

is denoted by $\Lambda^p(\mathcal{X}, \mathcal{F})$ with $\Lambda^0(\mathcal{X}, \mathcal{F}) = \mathcal{F}$, $\Lambda^1(\mathcal{X}, \mathcal{F}) = \mathcal{X}^*$. They are interpreted as the modules of the p-of forms. On $\Lambda^*(\mathcal{X}, \mathcal{F}) = \oplus_p \Lambda^p(\mathcal{X}, \mathcal{F})$, the *exterior derivative* d, the Lie derivative L_X and the inner product i_X are defined in the standard way. They are given by the following expressions:

- The exterior derivative, $d : \Lambda^p(\mathcal{X}, \mathcal{F}) \to \Lambda^{p+1}(\mathcal{X}, \mathcal{F})$.

$$(d\alpha)(X_1, X_2, \ldots, X_{p+1}) = \tag{A.3}$$
$$\sum_i (-1)^{i+1} X_i \cdot \alpha(X_1, X_2, \ldots, \hat{X}_i, \ldots, X_{p+1}) +$$
$$\sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}) \; , \tag{A.4}$$

  where as usual the "hat" means that the corresponding field must be omitted and $\alpha \in \Lambda^p(\mathcal{X}, \mathcal{F})$.
- The inner product, $i_{(\cdot)} : \mathcal{X} \times \Lambda^p(\mathcal{D}, \mathcal{F}) \to \Lambda^{p-1}(\mathcal{X}, \mathcal{F})$.

$$(i_X \alpha)(X_1, \ldots, X_{p-1}) = \alpha(X, X_1, \ldots, X_{p-1}). \tag{A.5}$$

- The Lie derivative, $L_{(\cdot)} : \mathcal{X} \times \Lambda^p(\mathcal{X}, \mathcal{F}) \to \Lambda^p(\mathcal{X}, \mathcal{F})$.

$$(L_X \alpha)(X_1, \ldots, X_p) = X \cdot \alpha(X_1, \ldots, X_p)$$
$$+ \sum_k (-1)^{k+1} \alpha([X, X_k], X_1, \ldots, \hat{X}_k, \ldots, X_p) \; . \tag{A.6}$$

The coboundary property $d^2 = 0$ follows from the fact that $\mathcal{X}$ is a Lie algebra; i.e. from the Jacobi identity. One also has the Cartan identity,

$$L_X = i_X d + d i_X ,$$

which as in the classical case, satisfies $[L_X, d] = 0$. In the usual way, considering the complex $(\Lambda(\mathcal{X}, \mathcal{F}), d)$, one can define closed and exact forms, as well as the corresponding cohomology spaces $H^p(\mathcal{X}, \mathcal{F})$. For the sake of brevity, these spaces are sometimes denoted also by $H^p(\mathcal{F})$ or $H^p(\mathcal{F}, d)$, if one wants to show explicitly the coboundary operator defining them. In a similar way, considering the $\mathcal{F}$-linear maps

$$\underbrace{\mathcal{X}^* \times \ldots \times \mathcal{X}^*}_{p \text{ times}} \mapsto \mathcal{F} \tag{A.7}$$

one defines the contravariant tensor fields and continuing like that, one is able to define tensors of arbitrary type and the tensor calculus.

Any derivation Γ of $\mathcal{F}$, being an analogue of a vector field, defines an "algebraic dynamical system" in the following way:

$$\dot{f} = L_\Gamma f; \quad f \in \mathcal{F}, \tag{A.8}$$

where the dot denotes the time-derivative. As is known, the solution of the above equation with initial condition f gives $\Phi_t^\Gamma(f)$ – the 1-parametric group of automorphisms of $\mathcal{F}$, or the flow associated with the derivation Γ. We expect that the solution is written as a series (formal at the beginning):

$$\Phi_t^\Gamma(f) = f + \sum_{k=1}^{\infty} \frac{t^k}{k!}(L_\Gamma)^k f; \qquad t \in \mathbb{R}, \quad f \in \mathcal{F}. \tag{A.9}$$

However, to treat the question of the convergence of the above series, we need a topology on our algebra $\mathcal{F}$ (in fact this is already necessary, if we want to speak about time derivatives). In order to avoid speaking about topology, we shall consider some situations, where the above formulae make sense without any topology.

Let us now give some definitions.

Definition A.1. *We say that the set* $\mathbf{f} = \{f_i\}_{i\in I} \subset \mathcal{F}$ *is a generating set for* $\mathcal{F}$ *if any element* $X \in \mathcal{X} = Der(\mathcal{F})$ *is completely determined by its action on the set* $\{f_i\}_{i\in I}$ *or, equivalently, if* $\{df_i\}_{i\in I}$ *generate* $\mathcal{X}^* = Der^*(\mathcal{F})$ *with coefficients in* $\mathcal{F}$*.*

The above means that we are able to determine the action of $X \in \mathcal{X}$ on $\mathcal{F}$ having only the action of X on $\{f_i\}_{i\in I}$, even if we are not able to express any $f \in \mathcal{F}$ in terms of $f_i, i \in I$.

We say that a finite generating set $\mathbf{f} = \{f_i\}_1^n$ for $\mathcal{F}$ provides a *linearization* for Γ if

$$L_\Gamma f_j = \sum_i A_j^i f_i; \qquad A_j^i - \text{real constants.} \tag{A.10}$$

In this case, we take the exponent $\exp tA$ of A (A is a real valued matrix, so the exponent exists) and we set for the flow of Γ:

$$\Phi_t^\Gamma(\mathbf{f}) = (\exp tA)\mathbf{f}\ . \tag{A.11}$$

Of course, the topological questions are just "hidden" here, since eventually we reduced everything to a finite-dimensional vector spaces, where the topology is uniquely defined, whatever the "ambient" space might be.

Another situation in which the flow can be calculated explicitly, because it is reduced to the finite-dimensional flow, is provided by generating sets on which Γ is *nilpotent*, that is, if we have

$$(L_\Gamma)^k f_j = 0; \qquad j \in I\ , \tag{A.12}$$

for some integer k that does not depend on the index $j \in I$. In this case, it is not necessary to have a finite generating set. Our formal series (A.9) becomes a polynomial and is well defined in the algebra $\mathcal{F}$. In particular,

Definition A.2. *We say that the subset* $\mathcal{F}_\Gamma \subset \mathcal{F}$ *is an integrating set for the derivation* Γ *if it generates* $\mathcal{F}$ *and if* $L_\Gamma L_\Gamma f = 0$ *for all* $f \in \mathcal{F}_\Gamma$*. In this case, the* 1*-parametric group of automorphisms on* $\mathcal{F}_\Gamma$ *is given by*

$$\Phi_t^\Gamma(f) = f + tL_\Gamma f; \qquad t \in \mathbb{R}, \qquad f \in \mathcal{F}_\Gamma\ . \tag{A.13}$$

We have seen that in each of the above cases, we can explicitly calculate the flow

$$\Phi_t^{\Gamma} : \mathbb{R} \times \mathcal{F} \to \mathcal{F}. \tag{A.14}$$

However, it is clear that such situations are exceptional, that is, we should not expect to be able to find "adapted" generating set for an arbitrary derivation.

Also, the previous constructions cannot be implemented, if $\mathcal{X}^* = \mathrm{Der}^*(\mathcal{F})$ cannot be generated by a set $\{df_i\}_{i \in I}$, for example, if the first cohomology group $H^1(\mathcal{F}, d) \neq 0$.

A.1.2 Poisson Rings, Bi-Hamiltonian Systems and Nijenhuis Tensors

Now we introduce what in this context is a Hamiltonian system and how to understand complete integrability. We need of course the notion of Poisson brackets on the ring $\mathcal{F}$.

Definition A.3. *Let the ring $\mathcal{F}$ be endowed with a Lie algebra structure over $\mathbb{R}$. We say that $\mathcal{F}$ is a Poisson ring if the Lie algebra structure and the ring structure are compatible, which means that the Lie bracket*

$$\{\ ,\ \} : \mathcal{F} \times \mathcal{F} \mapsto \mathcal{F} \tag{A.15}$$

is a derivation when we fix one of the arguments, that is, if

$$\{f_1 f_2, g\} = f_1\{f_2, g\} + \{f_1, g\} f_2, \tag{A.16}$$

for arbitrary $f_1, f_2, g \in \mathcal{F}$.

Then, as easily seen, the map $f \mapsto X_f$, where

$$X_f(g) = \{f, g\}\ , \tag{A.17}$$

defines a Lie algebra homomorphism $\mathcal{F} \mapsto \mathrm{Der}\,(\mathcal{F}) = \mathcal{X}$. As we are working with Lie algebras, in the sequel, we use the Lie algebra notation

$$\mathrm{ad}\,_f = X_f \in \mathrm{Der}\,(\mathcal{F}) = \mathcal{X}. \tag{A.18}$$

Similar to the classical case, a Poisson structure on $\mathcal{F}$ can be defined by an element $B \in \Lambda^2(\mathcal{X}^*, \mathcal{F})$, (Poisson tensor), provided B satisfies some requirements. In this case we set

$$\{f, g\} = i_{B(df)} dg = -i_{B(dg)} df; \qquad f, g \in \mathcal{F}. \tag{A.19}$$

Here we are using the same letter for the element $B \in \Lambda^2(\mathcal{X}^*, \mathcal{F})$ and the associated $\mathcal{F}$-linear map

$$B : \mathcal{X}^* \mapsto \mathrm{Der}\,(\mathcal{F})\ . \tag{A.20}$$

Then, like the classical situation, see (12.151), the Jacobi identity for the bracket is equivalent to the vanishing of the Schouten bracket of B; see (12.151):

$$[B, B]_S = 0. \tag{A.21}$$

Below we give some necessary definitions, which are similar to the corresponding ones in the classical case.
The centralizer $\mathcal{C}_A$ of $A \subset \mathcal{F}$ is defined as

$$\mathcal{C}_A = \{g \in \mathcal{F} : \{g, f\} = 0, f \in A\} \subset \mathcal{F}\ . \tag{A.22}$$

It is a Poisson subring of $\mathcal{F}$. In particular, when A consists of one element, say f, we shall write $\mathcal{C}_f$. The interpretation of $\mathcal{C}_f$ is clear – it is the set of constants of motion for the derivation X_f. Another example of the above definition is the centralizer of $\mathcal{F}$, denoted by $\mathcal{C}(\mathcal{F})$. As can be seen, it is actually the center of $\mathcal{F}$ (considered as Lie algebra):

$$\mathcal{C}(\mathcal{F}) = \{g \in \mathcal{F},\ \{g, f\} = 0; \quad f \in \mathcal{F}\}\ . \tag{A.23}$$

In case we have a Poisson algebra, $\mathcal{C}(\mathcal{F})$ is also called the Casimir subalgebra of $\mathcal{F}$ associated with the Poisson structure. Its elements are called Casimir elements. Naturally, the Casimir elements are integrals of motion for each derivation X_f, $f \in \mathcal{F}$.

Given a Poisson subring $\mathcal{A}$, we denote its centralizer $\mathcal{C}(\mathcal{A})$ also by $\mathcal{A}'$, so

$$\mathcal{A}' = \{f \in \mathcal{F} : \{f, g\} = 0; \quad f \in \mathcal{A}\}\ . \tag{A.24}$$

$\mathcal{A}'$ is also called the *polar* or *reciprocal* set of $\mathcal{A}$. Until now everything is as in the case of an arbitrary Lie algebra. Now, we give more specific definitions, keeping in mind the analogy with the classical case [17].

Definition A.4. *Suppose $\mathcal{A}$ is a Poisson subring and $\mathcal{A}'$ is its centralizer. We say that $\mathcal{A}$ is regular if the following two subalgebras of derivations*

$$\mathcal{X}_{\mathcal{A}'} = \{X_f : f \in \mathcal{A}'\} \tag{A.25}$$
$$\mathcal{N}_{\mathcal{A}} = \{Y \in Der(\mathcal{F}) : Y(g) = 0;\ g \in \mathcal{A}\} \tag{A.26}$$

coincide as $\mathcal{F}$-modules (in general one has only that $\mathcal{X}_{\mathcal{A}'} \subseteq \mathcal{N}_{\mathcal{A}}$).

In the classical situation, when the Poisson structure is defined by a symplectic form, a regular subring is such that on it the Poisson structure is nondegenerate, i.e. the Casimir subalgebra coincides with $\mathbb{R}$.

Definition A.5. *We say that*

(a) $\mathcal{A}$ is isotropic if $\mathcal{A} \subset \mathcal{A}'$.
(b) $\mathcal{A}$ is coisotropic if $\mathcal{A}' \subset \mathcal{A}$.

(c) A Poisson subring which is isotropic and coisotropic is called a Lagrangian subring.

(Note that a Lagrangian subring is also a maximal Abelian subalgebra in $\mathcal{A}$.)

Now we are ready for a generalization of the complete integrability.

Definition A.6. *We say that an element $f \in \mathcal{F}$ defines a completely integrable system Γ_f if its set of constants of motion $\mathcal{C}_f$, contains isotropic and coisotropic regular Poisson subring $\mathcal{A}_f$.*

Remark A.7. For a generic completely integrable system Γ_f, the subring $\mathcal{C}_f$ need not be isotropic and coisotropic.

Definition A.8. *On a Poisson ring $\mathcal{F}$ we define a Darboux set as a generating set for $\mathcal{F}$, denoted by $\mathcal{F}_1 \oplus \mathcal{F}_2$, having the properties*

1. $\mathcal{F}_1$ is regular, isotropic and coisotropic
2. $\mathcal{F}_2$ is regular, isotropic and coisotropic
3. for any $f \in \mathcal{F}_1$ there exists $g_f \in \mathcal{F}_2$, such that $\{f, g_f\} \in \mathbb{R}$.

Remark A.9. Of course, there is no guarantee that a Poisson ring possesses a Darboux generating set.

Definition A.10. *A Darboux set for $\mathcal{F}$ defines action-angles variables for the derivation Γ_f if it is an integrating set for Γ_f (see, Definition A.2).*

It can happen that the ring $\mathcal{F}$ has two different structures of Poisson ring, defined for example by two Poisson tensors B_1 and B_2. We say that B_1, B_2 are compatible (or that the brackets defined by them are compatible) if

$$\lambda\{\ ,\ \}_{B_1} + \mu\{\ ,\ \}_{B_2} : \mathcal{F} \times \mathcal{F} \mapsto \mathcal{F} \tag{A.27}$$

defines a Poisson ring structure on $\mathcal{F}$ for any $\lambda, \mu \in \mathbb{R}$ (see [18]).

A derivation X is said to allow (or have) bi-Hamiltonian formulation if there exist two elements of $\mathcal{F}$, f_1 and f_2 such that

$$X(g) = \{f_1, g\}_{B_1} = \{f_2, g\}_{B_2}. \tag{A.28}$$

for all $g \in \mathcal{F}$.

If on $\mathcal{F}$ there exist compatible brackets, we can define

$$\begin{aligned} \mathcal{C}^1_X &= \{f \in \mathcal{F} : \{f, f_1\}_{B_1} = 0\} \\ \mathcal{C}^2_X &= \{f \in \mathcal{F} : \{f, f_2\}_{B_2} = 0\}\ . \end{aligned} \tag{A.29}$$

In general $\mathcal{C}^1_X$ ($\mathcal{C}^2_X$) is not a Poisson subring with respect to B_2 (B_1) so that we can "construct" new constants of motion for X using $\mathcal{C}^1_X$ and $\mathcal{C}^2_X$ and both the brackets.

Let us have a pair (B_1, B_2) of Poisson tensors and one of them, say B_1, is invertible. Then we can define an endomorphism $N = B_2 \circ B_1^{-1}$ that makes the diagram below commutative

$$\begin{array}{ccc} \mathcal{X}^* & = & \mathcal{X}^* \\ B_2 \downarrow & & \uparrow B_1^{-1} \\ \mathcal{X} & \xrightarrow{N} & \mathcal{X} \end{array} \tag{A.30}$$

We can extend N to an operator i_N on $\Lambda(\mathcal{X}, \mathcal{F})$ by

$$(i_N\omega)(X_1, \ldots, X_p) = \sum_j \omega(X_1, \ldots, NX_j, \ldots, X_p) , \tag{A.31}$$

for $\omega \in \Lambda^p(\mathcal{X}, \mathcal{F})$ and $X_i \in \mathcal{X}$. Using the operator d of the complex $(\Lambda(\mathcal{X}, \mathcal{F}), d)$, we can also define the operator

$$d_N : \Lambda^p(\mathcal{X}, \mathcal{F}) \mapsto \Lambda^{p+1}(\mathcal{X}, \mathcal{F})$$

by the identity

$$d_N = i_N d - d i_N, \tag{A.32}$$

(compare with the classical definitions (13.13),(13.19)).

From the compatibility of the two Poisson structures B_1 and B_2, it follows that the Nijenhuis torsion of R_N

$$R_N(X, Y) = N^2[X, Y] - N[NX, Y] - N[X, NY] + [NX, NY] , \tag{A.33}$$

is identically zero. (Of course, here $X, Y \in \mathcal{X}$.) As noted in [19], this is equivalent to the fact that $(d_N)^2 = 0$ – compare also with the second equation in (13.22). As a consequence, N can be used as a recursion operator. We put $N^* = B_1^{-1} \circ B_2$ (formal adjoint) and with this notation have

Proposition A.11. *Let $\alpha \in \mathcal{X}^*$ be such that $d\alpha = 0$ and $dN^*\alpha = 0$. Then $d((N^*)^k\alpha) = 0$, $k = 1, 2, \ldots$.*

The proof follows exactly the lines of the classical one; see Proposition 13.24.

Let us assume that $H^1(\mathcal{F}, d) = 0$. Then from Proposition A.11 it follows

Proposition A.12. *If $f_0, f_1 \in \mathcal{F}$ are such that*

$$N^* df_0 = df_1 , \tag{A.34}$$

then there exist $f_k \in \mathcal{F}$ such that

$$(N^*)^k df_0 = df_k; \qquad k = 1, 2, \ldots \tag{A.35}$$

Proposition A.13. *The set $\{f_0, \ldots, f_k, \ldots\}$ defines an isotropic Poisson ring with respect to any Poisson tensor the type $\lambda B_1 + \mu B_2$; $\lambda, \mu = const$, that is*

$$\{f_j, f_k\}_{\lambda B_1 + \mu B_2} = 0; \quad j, k = 0, 1, \ldots \tag{A.36}$$

Proof. The equations (A.36) follows from the relations

$$\{f_j, f_k\}_{B_1} = \{f_{j-1}, f_k\}_{B_2}$$
$$\{f_j, f_k\}_{B_2} = \{f_j, f_{k+1}\}_{B_1}; \quad j > k,$$

which can be checked immediately. Assuming as before that B_1^{-1} exists, we can define a 2-form ω_{B_1} on $\mathcal{X}$, by

$$\omega_{B_1}(X, Y) = i_X(B_1^{-1}(Y)), \tag{A.37}$$

for $X, Y \in \mathcal{X}$. In particular,

$$\omega_{B_1}(B_1 df_1, B_1 df_2) = \{f_1, f_2\}_{B_1} \ . \tag{A.38}$$

One can check that if $X_0 \in \mathcal{X}$ and $k > l$ we have

$$\begin{aligned} &d\omega_{B_1} = 0, \quad \omega_{B_1}((N)^k X_0, (N)^{k-1} X_0) = 0 \\ &\omega_{B_1}((N)^k X_0, (N)^l X_0) = \omega_{B_1}((N)^{k-1} X_0, (N)^{l+1} X_0) \ . \end{aligned} \tag{A.39}$$

Finally, one can prove the following proposition

Proposition A.14. *Let B be a Poisson tensor, N be a $(1,1)$ tensor field such that $(d_N)^2 = 0$. Suppose that we have the derivation $\Gamma_f = \{f, \cdot\}_B$. Then the following properties hold:*

1. *The condition $L_{\Gamma_f} N = 0$ entails that $\mathcal{A}_{\Gamma_f} = \left\{\Gamma_f, N\Gamma_f, ..., N^k \Gamma_f, ...\right\}$ is an Abelian Lie algebra.*
2. *If B, NB are compatible Poisson tensors then NB gives alternative Hamiltonian formulation of Γ_f.*

As a final comment, we must add that when a Poisson tensor B is invertible, $\mathcal{A}_{\Gamma_f}$ is associated with the B-Poisson subring. Assuming that $H^1(\mathcal{F}, d) = 0$, we deduce that it is isotropic. However, in order to prove that Γ_f is completely integrable one needs to prove that $\mathcal{A}_{\Gamma_f}$ is regular and coisotropic. Checking this condition can be a highly nontrivial task.

Summarizing our discussion, in the algebraic approach, we can recover all the principal properties of Nijenhuis tensor and its application in obtaining integrals of motion in involution.

Remark A.15. If one has a symplectic structure, one can define a symplectic ring that happens to be a special kind of Poisson ring. In this case, one can define isotropic Lie and Lagrangian subalgebras in $\mathcal{X} = \mathrm{Der}\,(\mathcal{F})$ just as in the classical situation. Then the subalgebra and the polar subalgebra (or reciprocal subalgebra) generate the whole algebra of derivations.

Naturally, when $\mathcal{F}$ is realized as the ring of functions on a Poisson manifold $\mathcal{M}$, all our definitions and propositions reduce to the classical ones.

As an illustration of how the above scheme is implemented, we discuss two examples.

A Periodic Lattice of Particles Interacting with Harmonic Forces

This is an example of a derivation which can be put in a nilpotent form.

The ring $\mathcal{F}$ is defined by the following properties:

1. $\mathcal{F}$ is generated by $2n$ elements $\{h_i, f_i\}_{i=1}^n$, but in order to make the formulae more compact we shall understand the indices modulo n. We also assume that
$$dh_1 \wedge \ldots \wedge dh_n \wedge df_1 \wedge \ldots \wedge df_n \neq 0 \,. \tag{A.40}$$
2. The set of the derivations $\mathrm{Der}\,(\mathcal{F}) = \mathcal{X}$ is generated over $\mathcal{F}$ by the elements $\{X_i, Y_i\}_{i=1}^n$, defined by
$$L_{X_i} f_j = \delta_{ij}, \quad L_{X_i} h_j = 0, \quad L_{Y_i} f_j = 0, \quad L_{Y_i} h_j = \delta_{ij}, \tag{A.41}$$
where $i, j = 1, 2, \ldots, n$.

The "dynamics" is described by the derivation

$$\Gamma = \sum_{i=1}^n h_i X_i - \sum_{i=1}^n (f_i - f_{i+1})(Y_{i+1} - Y_i) \tag{A.42}$$

and the corresponding "equations of motion"

$$\begin{aligned} \dot{h}_i &= f_{i-1} - 2f_i + f_{i-1} \\ \dot{f}_i &= h_i; \quad i = 1, 2, \ldots, n \,, \end{aligned} \tag{A.43}$$

are linear. The flow of Γ can be constructed explicitly, through the exponent of suitable constant matrix. Indeed, "introducing new variables"

$$\begin{aligned} P_i &= h_i; \quad i = 1, 2, \ldots, n \\ Q_i &= \frac{1}{\sqrt{i(i+1)}}(f_1 + \ldots + f_i - i f_{i+1}); \quad i = 1, 2, \ldots, n-1 \\ Q_n &= \frac{1}{\sqrt{n}}(f_1 + \ldots + f_n) \,, \end{aligned} \tag{A.44}$$

we see that the derivation (A.42) can be written as

$$\Gamma = P_n \partial_{Q_n} + \sum_{i=1}^{n-1} (P_i \partial_{Q_i} - Q_i \partial_{P_i}) \,. \tag{A.45}$$

Here ∂_{Q_i} and ∂_{P_i} are the derivations dual to the 1-forms $\{dQ_i, dP_i\}$, that is, they are defined by: $dQ_i(\partial_{Q_j}) = dP_i(\partial_{P_j}) = \delta_{ij}$, $dQ_i(\partial_{P_j}) = dP_i(\partial_{Q_j}) = 0$. Let us introduce now another set of variables: I_k, α_k; $k = 1, 2, \ldots, n$.

$$I_k = P_k^2 + Q_k^2, \quad d\alpha_k = I_k^{-1}(P_k dQ_k - Q_k dP_k) \tag{A.46}$$

for $k = 1, 2, \ldots, n-1$ and $I_n = P_n$, $\alpha_n = Q_n$. It is almost immediate to see that on this set the derivation Γ is nilpotent. Indeed, for $1 \leq k \leq n$, we have

$$L_\Gamma I_k = 0, \quad L_\Gamma \alpha_k = 1 \,. \tag{A.47}$$

The Periodic Toda Lattice (As Before Indices Are Understood Modulo n)

This is an example of bi-Hamiltonian dynamical derivation admitting a recursion operator. Here we give only the basic ingredients and refer to [13] for the details.

The ring $\mathcal{F}$ is defined by the following properties:

1. $\mathcal{F}$ is generated by $2n$ elements $\{h_i, f_i\}$; $i = 1, 2, \ldots, n$, where
$$dh_1 \wedge \ldots \wedge dh_n \wedge df_1 \wedge \ldots \wedge df_n \neq 0.$$
Each element f_i possesses an inverse denoted by f_i^{-1}.
2. The set of derivations $\mathrm{Der}\,(\mathcal{F}) = \mathcal{X}$ is generated over $\mathcal{F}$ by the $2n$ elements $\{X_i, Y_i\}_{i=1}^n$, defined by
$$L_{X_i} f_j = \delta_{ij} f_i, \quad L_{X_i} h_j = 0, \quad L_{Y_i} f_j = 0, \quad L_{Y_i} h_j = \delta_{ij}. \tag{A.48}$$

The "dynamics" of the Toda Lattice is defined by the derivation
$$\Gamma = \sum_i h_i X_i + \sum_i \lambda_i f_i f_{i+1}^{-1} (Y_{i+1} - Y_i)\,, \tag{A.49}$$
where the λ_i; $i = 1, 2, \ldots, n$ are constant parameters.

The derivation Γ is bi-Hamiltonian with respect to the two symplectic structures on $\mathcal{X}$:
$$\begin{aligned} \omega_1 &= \sum_i f_i^{-1} df_i \wedge dh_i \\ \omega_2 &= \sum_i (-\lambda_i df_i \wedge df_{i+1}^{-1} + h_i f_i df_i^{-1} \wedge dh_i) - \sum_{i<j} dh_i \wedge dh_j \quad , \end{aligned} \tag{A.50}$$
with Hamiltonian functions
$$\begin{aligned} H_1 &= \sum_i \frac{1}{2} h_i^2 + \sum_i \lambda_i f_i f_{i+1}^{-1} \\ H_2 &= \sum_i \frac{1}{3} (h_i)^3 + \sum_i \lambda_i (h_i + h_{i+1}) f_i f_{i+1}^{-1}\,. \end{aligned} \tag{A.51}$$

The two symplectic structures are compatible and the dual of the corresponding recursion operator $N^* : \mathcal{X}^* \to \mathcal{X}^*$ is given by
$$\begin{aligned} & N^*(df_i) = h_i df_i - f_i \sum_j \varepsilon_{ij} dh_j \\ & N^*(dh_i) = \lambda_i f_i df_{i+1}^{-1} - \lambda_{i-1} f_i^{-1} df_{i-1} + h_i dh_i \\ & i = 1, 2, \ldots, n. \end{aligned} \tag{A.52}$$

The symbol ε_{ij} is skew-symmetric for any pair of indices i, j and $\varepsilon_{ij} = +1$ if $i > j$. The Nijenhuis operator $N : \mathcal{X} \to \mathcal{X}$ is then given by:

$$
\begin{aligned}
N(X_i) &= \lambda_i f_i f_{i+1}^{-1} Y_{i+1} - \lambda_{i-1} f_{i-1} f_i^{-1} Y_{i-1} + h_i X_i \\
N(Y_i) &= h_i Y_i + \sum_{j=1}^{n} \varepsilon_{ij} X_j \\
i &= 1, 2, \ldots, n.
\end{aligned}
\tag{A.53}
$$

A.2 Integrability of Dynamics with Fermionic Variables

Most of mathematical objects have super-counterparts, where both commutative and anticommutative variables appear. We have super-Lie algebras, differential supermanifolds, super-Lie groups, etc., of which the "even part" are classical Lie algebras, differential manifolds and Lie groups, respectively. The "odd parts" physically describe fermions and for that reason the super-objects become very interesting, see [10, 20]. There have been several attempts to analyze integrability of fermionic dynamical systems (see for example [21, 22, 23, 24]) and to extend to such systems [25], at least at algorithmic level, the results and techniques used for bosonic dynamics, based on the role of recursion operators. In particular, one would like to define a graded Nijenhuis torsion.

In what follows, we shall address this issue. We show that a mixed $(1,1)$ graded tensor field N can act as a recursion operator if and only if N is an even map.

There are dynamical systems, like supersymmetric Witten's dynamics [26], which allow a bi-Hamiltonian formulation with an even and odd Hamiltonian functions and in terms of an even and an odd Poisson structure respectively (so that the dynamical vector field is always even) [27, 28, 29]. This allows to construct an odd tensor field, which seems a good candidate for a recursion operator. We explicitly show that in fact such a tensor field cannot be a recursion operator.

As everywhere in this book, we are working only with smooth, i.e. C^∞ objects, in this subsection the notation follows as close as possible those we have already used. In particular, if $\mathcal{M}$ is a (finite dimensional) "ordinary" manifold, we denote by $\mathcal{D}(\mathcal{M})$ the ring of real valued functions on $\mathcal{M}$, by $\mathcal{T}(\mathcal{M})$ the Lie algebra of vector fields, by $\mathcal{T}(\mathcal{M})^*$ its dual, the module of 1-forms, and by $\mathcal{T}_1^1(\mathcal{M})$ the mixed $(1,1)$ tensor fields.

The main property of the mixed tensor field N (the Nijenhuis tensor), which proved up to now to be so important for the complete integrability, is the vanishing of its Nijenhuis torsion $R_N(X,Y) = 0$. One expects that a suitable generalization of such a condition could play an important role in analyzing the integrability of dynamical systems with fermionic degrees of freedom too. Moreover, it seems natural to think that such a generalization could come from a graded generalization of some of the following relations which are true in the bosonic case:

- If $R_N = 0$ then the image of N is a Lie subalgebra of $\mathcal{T}(\mathcal{M})$, see Proposition 13.21.
- If $R_N = 0$, α is a closed 1-form, and $N^*\alpha$ is closed, then $(N^*)k\alpha$ for $k \geq 2$ are closed too.
- If $R_N = 0$ then $d_N \circ d_N = 0$, see (13.19) and (13.22).
- If P_1 and P_2 are two Poisson tensors, $N = P_1^{-1} \circ P_2$ and $R_N = 0$, then $P_1 + P_2$ is also a Poisson tensor; see Theorem 14.7.
- Let $A(X,Y) = [NX,Y] + [X,NY] - N[X,Y]$, then $R_N = 0$ is equivalent to $NA(X,Y) = [NX,NY]$, and for any value of the real parameter λ, the expression
$$[X,Y]_\lambda = [X,Y] + \lambda A(X,Y) \tag{A.54}$$
satisfies the Jacobi identity.

One could expect of course that some of the above relations will not be true in the graded situation.

A.2.1 Graded Differential Calculus

Now, before the analysis of the graded zero Nijenhuis bracket condition, we shall give a brief review of the graded differential calculus on supermanifolds, followed by the study of some simple examples. Some fundamentals of supermanifold theory can be found in [30, 31] and for a mathematically coherent definition we refer the reader to [32, 33]. In the following, by graded we always mean $\mathbb{Z}_2$-graded.

The basic algebraic object for the graded calculus is some real exterior algebra $B_L = (B_L)_0 \oplus (B_L)_1$ with L generators; here $(B_L)_0$ is the set of the even elements and $(B_L)_1$ the set of the odd ones. Then, by definition, a (m,n) dimensional supermanifold is a topological manifold $\mathcal{S}$ modeled over the "vector superspace"
$$B_L^{m,n} = (B_L)_0^m \times (B_L)_1^n \tag{A.55}$$
through atlases, whose transition maps satisfy suitable "supersmoothness" conditions. A supersmooth function $f : U \subset B_L^{m,n} \to B_L$ has the usual superfield expansion
$$f(x^1 \dots x^m, \theta^1 \dots \theta^n) = f_0(x) + \sum_{\alpha=1}^{n} f_\alpha(x)\,\theta^\alpha + \cdots + f_{1\dots n}(x)\,\theta^1 \dots \theta^n, \tag{A.56}$$
where the xs are the even coordinates, and the θs are the odd ones. The dependence of the coefficient functions $f_{\dots}(x)$ on the even variables is fixed by their values for real arguments.

We denote by $\mathcal{G}(\mathcal{S})$ and $\mathcal{G}(U)$ the graded ring of supersmooth B_L-valued functions on $\mathcal{S}$ and $U \subset \mathcal{S}$, respectively.

The class of the supermanifolds which up to now has found applications in theoretical physics is given by the so-called De Witt supermanifolds. The basis of topology on them is given by the preimages of open sets in $\mathbb{R}^m$ through the "body map" $\sigma^{m,n} : B_L^{m,n} \mapsto \mathbb{R}^m$, that is, it is the roughest topology in which $\sigma^{m,n}$ are continuous. An (m,n)-supermanifold is De Witt supermanifold, if it has an atlas in which the images of the coordinate maps are open in the De Witt topology. A De Witt (m,n)-supermanifold is a locally trivial vector bundle over a classical m-manifold $\mathcal{S}_0$ (called the body of $\mathcal{S}$) [31]. Then it is not surprising that, modulo some technicalities, a De Witt supermanifold can be identified with a Berezin–Kostant supermanifold [34, 35].

The graded tangent space $T(\mathcal{S})$ is constructed in the following manner. For each $x \in \mathcal{S}$, let $\mathcal{G}(x)$ be the set of the germs of the functions at x and denote by $T_x(\mathcal{S})$ the space of graded B_L-linear maps $X : \mathcal{G}(x) \to B_L$, which satisfy the Leibnitz rule. Then, $T_x(\mathcal{S})$ is a free-graded B_L-module of dimension (m,n), and the disjoint union $\coprod_{x\in(\mathcal{S})} T_x(\mathcal{S})$ can be given the structure of a rank (m,n) super vector bundle over $\mathcal{S}$, denoted by $T(\mathcal{S})$. The set of the sections $\mathcal{X}(\mathcal{S})$ of $T(\mathcal{S})$ is a graded $\mathcal{G}(\mathcal{S})$-module and is identified with the graded Lie algebra $\operatorname{Der}\mathcal{G}(\mathcal{S})$ of the derivations of $\mathcal{G}(\mathcal{S})$. Derivations (or vector fields) are said to be even (or odd), if they are even (or odd) as maps $\mathcal{G}(\mathcal{S}) \to \mathcal{G}(\mathcal{S})$, (satisfying in addition a graded Leibnitz rule). A local basis is given by

$$\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial \theta^1}, \ldots, \frac{\partial}{\partial \theta^n}. \tag{A.57}$$

Remark A.16. Unless explicitly stated, by using a partial derivative, we always mean a left derivative, namely, a derivative acting from the left. In general, if $z^i = (x^j, \theta^k)$, when acting on any homogeneous function $f \in \mathcal{G}(\mathcal{S})$, left and right derivative are related by

$$\frac{\overrightarrow{\partial}}{\partial z^i} f = (-1)^{p(z^i)[p(f)+1]} f \frac{\overleftarrow{\partial}}{\partial z^i}; \qquad i = 1, 2, \ldots, m+n. \tag{A.58}$$

In a similar way, one defines the cotangent space and the cotangent bundle. Let $T_x^*(\mathcal{S})$ be the space of graded B_L-linear maps $T_x(\mathcal{S}) \to B_L$ and $T^*(\mathcal{S}) = \coprod_{x\in\mathcal{S}} T_x^*(\mathcal{S})$. Since $T_x^*(\mathcal{S})$ is a free-graded B_L-module of dimension (m,n) then $T^*(\mathcal{S})$ is a type (m,n) super vector bundle over $\mathcal{S}$. The set of the sections $\mathcal{X}(\mathcal{S})^*$ of $T^*(\mathcal{S})$ have a structure of a graded $\mathcal{G}(\mathcal{S})$-module and is identified with the set of the graded $\mathcal{G}(\mathcal{S})$-linear maps $\operatorname{Der}\mathcal{G}(\mathcal{S}) \to \mathcal{G}(\mathcal{S})$. These sections are the 1-forms on $\mathcal{S}$. Forms are said to be even (odd) if they are even (odd) as maps $\mathcal{X}(M) \to \mathcal{G}(\mathcal{S})$.

In general, a type (p,q) (p covariant and q contravariant) graded tensor is any graded $\mathcal{G}(\mathcal{S})$-(multi-linear) map

$$\alpha : \mathcal{X}(\mathcal{S}) \times \ldots \times \mathcal{X}(\mathcal{S}) \times \mathcal{X}(\mathcal{S})^* \times \ldots \times \mathcal{X}(\mathcal{S})^* \mapsto \mathcal{G}(\mathcal{S})$$

(p factors $\mathcal{X}(\mathcal{S})$ and q factors $\mathcal{X}(\mathcal{S})^*$). The collection of all type (p,q) tensor fields is a graded $\mathcal{G}(\mathcal{S})$-module. A graded p-form is a skew-symmetric covariant graded tensors of type $(0,p)$. We denote by $\Omega^p(\mathcal{S})$ the collection of all p-forms.

Then the exterior derivative on $\mathcal{S}$ is defined by setting $df(X) = X\rfloor df = X(f)$, for $f \in \mathcal{G}(\mathcal{S})$, $X \in \mathcal{X}(\mathcal{S})$ and is extended in the usual way to a map $\Omega^p(\mathcal{S}) \to \Omega^{p+1}(\mathcal{S})$, $p \geq 0$ which satisfies $d^2 = 0$. If $X_i \in \mathcal{X}(\mathcal{S})$ are homogeneous elements, then we have

$$(X_1, \ldots, X_{p+1})\rfloor d\varphi = \sum_{i=1}^{p+1} (-1)^{a(i)}\, X_i((X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})\rfloor\varphi) + \sum_{1\leq i<j\leq p} (-1)^{b(i,j)}([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1})\rfloor\varphi, \tag{A.59}$$

where

$$\begin{aligned} a(i) &= 1 + i + p(X_i)\sum_{h=1}^{i-1} p(X_h)\ , \\ b(i,j) &= i + j + p(X_i)\sum_{h=1}^{i-1} p(X_h) + p(X_j)\sum_{h=1;\ h\neq i}^{j-1} p(X_h). \end{aligned} \tag{A.60}$$

Also, $(X_1, \ldots, X_q)\rfloor\psi$ stands for $\psi(X_1, \ldots, X_q)$ and the "hat" as usual means that the corresponding element is omitted.

From the definition, one obtains that $p(d) = 0$.

Next, the Lie derivative $L_{(\cdot)}$ over the space of the forms is defined by

$$\begin{aligned} &L_{(\cdot)} : \mathcal{X}(\mathcal{S}) \times \Omega^p(\mathcal{S}) \to \Omega^p(\mathcal{S}) \\ &L_X = X\rfloor \circ d + d \circ X\rfloor; \quad X \in \mathcal{X}(\mathcal{S})\ . \end{aligned} \tag{A.61}$$

Clearly, $p(L_X) = p(X)$.

Finally the Lie derivative of any tensor product of two fields can be defined in the usual manner by requiring the Leibnitz rule to hold and extend by linearity.

Suppose now that we have a tensor $N \in \mathcal{T}_1^1(M)$ (that is a $(1,1)$ type tensor) that is homogeneous of degree $p(N)$. Again, we can define two graded endomorphisms of $\mathcal{X}(\mathcal{S})$ and $\mathcal{X}(\mathcal{S})^*$ by the usual expressions (in the following two formulae X, Y are homogeneous elements in $\mathcal{X}(\mathcal{S})$ while α is any element in $\mathcal{X}(\mathcal{S})^*$):

$$\begin{aligned} &\widehat{N} : \mathcal{X}(\mathcal{S}) \mapsto \mathcal{X}(\mathcal{S}), \quad \check{N} : \mathcal{X}(\mathcal{S})^* \mapsto \mathcal{X}(\mathcal{S})^* \\ &N(X, \alpha) = \widehat{N}X\rfloor\alpha = (-1)^{p(X)p(N)} X\rfloor\check{N}\alpha\ . \end{aligned} \tag{A.62}$$

One is tempted to define a graded Nijenhuis torsion of N by a relation analogous to the one we have in the bosonic case

$$
\begin{aligned}
&{}^{G}R_N(X,Y;\alpha) = {}^{G}R_N(X,Y)\rfloor\alpha \\
&{}^{G}R_N(X,Y) = N^2[X,Y] + (-1)^{p(N)p(X)}[NX,NY] - \\
&N[NX,Y] - (-1)^{p(N)p(X)}N[X,NY].
\end{aligned} \tag{A.63}
$$

Proposition A.17. *The map* ${}^{G}R_N : \mathcal{X}(\mathcal{S}) \times \mathcal{X}(\mathcal{S}) \to \mathcal{X}(\mathcal{S})$ *defined in (A.63) is* $\mathcal{G}(\mathcal{S})$*-linear and graded skew-symmetric if and only if* $p(N) = 0$.

The proof is obtained by simple computation.

Remark A.18. When $p(N) = 1$, the expression in the right-hand side of (A.63) is not Skewsymmetric and even it is not linear when restricted to the even vector fields. Therefore, (A.63) defines a graded tensor (which will be in addition graded skew-symmetric) if and only if $p(N) = 0$.

A.2.2 Poisson Supermanifolds and Super Nijenhuis Tensors

We briefly describe how one can introduce super Poisson structures on a (m, n) – dimensional supermanifold $\mathcal{S}$ [34, 36]. For a more profound discussion, see [37]. As before, we shall denote by $z^i = (x^j, \theta^k)$; $i = 1, 2, \ldots, m+n$ the local coordinates on $\mathcal{S}$. The following proposition was obtained in [34] and can be proved by direct calculation.

Proposition A.19. *Let* (ω^{ij}) *be a* $(m+n) \times (m+n)$ *matrix (depending on the point* $z \in \mathcal{S}$*), such that* ω^{ij} *are homogeneous with parity* $p(\omega^{ij}) = p(z^i) + p(z^j) + p(\omega)$ *and let* $p(\omega)$ *not depend on the indices* i *and* j*. Suppose also that the following properties hold:*

$$
\omega^{ji} = -(-1)^{[p(z^i)+p(\omega)][p(z^j)+p(\omega)]}\omega^{ij} \tag{A.64}
$$

$$
\begin{aligned}
&(-1)^{[p(z^i)+p(\omega)][p(z^l)+p(\omega)]}\omega^{is}\frac{\vec{\partial}}{\partial z^s}\omega^{jl} + \\
&(-1)^{[p(z^l)+p(\omega)][p(z^j)+p(\omega)]}\omega^{ls}\frac{\vec{\partial}}{\partial z^s}\omega^{ij} + \\
&(-1)^{[p(z^j)+p(\omega)][p(z^i)+p(\omega)]}\omega^{js}\frac{\vec{\partial}}{\partial z^s}\omega^{li} = 0\,.
\end{aligned} \tag{A.65}
$$

Then the bracket defined by:

$$
\{F, G\} = F\frac{\overleftarrow{\partial}}{\partial z^i}\omega^{ij}\frac{\vec{\partial}}{\partial z^j}G \tag{A.66}
$$

endows $\mathcal{G}(\mathcal{S})$ *with a Lie superalgebra structure (Poisson superstructure).*

We have two different kind of structures according to whether $p(\omega) = 0$ (even Poisson structure) or $p(\omega) = 1$ (odd Poisson structure). Indeed, one can check that the bracket (A.66) has the properties

$$\{F, G\} = -(-1)^{[p(F)+p(\omega)][p(G)+p(\omega)]}\{G, F\} \tag{A.67}$$

and also

$$\begin{aligned}
&(-1)^{[p(F)+p(\omega)][p(H)+p(\omega)]}\{\{F, G\}, H\} + \\
&(-1)^{[p(G)+p(\omega)][p(F)+p(\omega)]}\{\{G, H\}, F\} + \\
&(-1)^{[p(H)+p(\omega)][p(G)+p(\omega)]}\{\{H, F\}, G\} = 0\,.
\end{aligned} \tag{A.68}$$

From (A.67) and (A.68) it follows that if we consider the elements $\mathcal{G}(\mathcal{S})$ as elements of Poisson superalgebra, the homogeneous elements of $\mathcal{G}(\mathcal{S})$ preserve their parity if $p(\omega) = 0$, while they change it if $p(\omega) = 1$.

If the matrix (ω^{ij}) is regular, then its inverse (ω_{ij}), $\omega_{ij}\omega^{jk} = \delta_i^k$, gives the components of a symplectic form $\omega = \frac{1}{2}dz^i \wedge dz^j \omega_{ji}$. In other words ω is closed, nondegenerate and such that

$$\begin{aligned}
&p(\omega_{ij}) = p(z^i) + p(z^j) + p(\omega) \\
&\omega_{ji} = -(-1)^{p(z^i)p(z^j)}\omega_{ij}.
\end{aligned}$$

The form ω is homogeneous with parity exactly equal to $p(\omega)$.

A result that is an analogue of the Darboux theorem can also be obtained, see [36].

Proposition A.20. *Let $(\mathcal{S}, \omega)$ be a (m, n)-dimensional symplectic manifold with ω homogeneous. Then*

1. If $p(\omega) = 0$, then dim$(\mathcal{S}) = (2r, n)$ and there exist local coordinates such that written in them ω has the form

$$\omega = dq^i \wedge dp^i + d\xi^j \wedge d\xi^j, \qquad (\omega_{ij}) = \begin{pmatrix} 0 & \mathbf{I}_r & 0 \\ -\mathbf{I}_r & 0 & 0 \\ 0 & 0 & \mathbf{I}_n \end{pmatrix}. \tag{A.69}$$

2. If $p(\omega) = 1$, then dim$(\mathcal{S}) = (m, m)$ and there exist local coordinates such that

$$\omega = du^i \wedge d\xi^i, \qquad (\omega_{ij}) = \begin{pmatrix} 0 & \mathbf{I}_m \\ -\mathbf{I}_m & 0 \end{pmatrix}. \tag{A.70}$$

With a Poisson structure, we can construct Hamiltonian equations. If the bracket is defined as in (A.66), the Hamiltonian equations corresponding to the Hamiltonian H have the form

$$\dot{z}^i = \omega^{ij} \frac{\overrightarrow{\partial}}{\partial z^j} H\,. \tag{A.71}$$

Suppose we want construct the flow of (A.71). Since the evolution is defined by even vector field, we deduce that the Poisson structure and the Hamiltonian function must have the same parity, in particular, if the Poisson structure is odd, we need an odd Hamiltonian function. We are ready now to introduce the graded Nijenhuis condition, but we would like to study first some examples.

Mixed Bosonic-Fermionic Harmonic Oscillator

The mixed bosonic-fermionic harmonic oscillator in $(2,2)$ dimensions in coordinates (q,p,η,ξ) has the following equations of motion

$$\dot{q} = p, \quad \dot{p} = -q, \quad \dot{\eta} = \xi, \quad \dot{\xi} = -\eta. \tag{A.72}$$

The equations (A.72) can be cast in Hamiltonian form in two ways with the following Hamiltonian functions: the usual even one

$$H = \frac{1}{2}(p^2 + q^2) + i\xi\eta, \tag{A.73}$$

and an odd one

$$K = p\xi + q\eta. \tag{A.74}$$

The two Poisson structures that make this possible are

$$\Lambda_H = \omega_H^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \qquad \omega_H = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \tag{A.75}$$

and

$$\Lambda_K = \omega_K^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad \omega_K = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{A.76}$$

respectively.
We can now construct a mixed invariant tensor field N setting

$$N = \omega_H \circ \Lambda_K = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}. \tag{A.77}$$

However, this odd tensor field ($p(N) = 1$) is not a recursion operator. Indeed, one can easily find that

$$\begin{aligned} &NdK = dH \\ &NdH = -i(dq)\xi + i(dp)\eta - i(d\eta)p + i(d\xi)q; \quad d(NdH) \neq 0. \end{aligned}$$

If we calculate now the Poisson brackets of the coordinate variables with the two symplectic structure (A.75) and (A.76), we find that

$$\{q,p\}_H = 1, \quad \{p,q\}_H = -1, \quad \{\eta,\eta\}_H = i, \quad \{\xi,\xi\}_H = i \tag{A.78}$$

$$\{q,\xi\}_K = 1, \quad \{\xi,q\}_K = -1, \quad \{p,\eta\}_K = -1, \quad \{\eta,p\}_K = 1. \tag{A.79}$$

(the brackets that are not written are identically equal to zero). We see that the sum $\{\cdot,\cdot\}_+$ of the two structures is itself a Poisson structure with the property

$$\{F,G\}_+ = -(-1)^{p(F)p(G)}\{G,F\}_+ \tag{A.80}$$

but it does not have definite parity. Moreover, $\{\cdot,\cdot\}_+$ is degenerate.

The Witten Dynamics

There are some other interesting examples that come from supersymmetric dynamics. In [27, 28] it has been shown that the dynamics of Witten's Hamiltonian systems [26] can be given a bi-Hamiltonian formulation with an even Poisson bracket and Grassmann even Hamiltonian or with an odd bracket and Grassmann odd Hamiltonian. As an example of this situation we study a supersymmetric Toda chain with coordinates (q,p,η,ξ). Then, the even Hamiltonian is given by

$$H = \frac{1}{2}(p^2 + e^q) + \frac{1}{2} i\xi\eta e^{\frac{q}{2}}. \tag{A.81}$$

With the even Poisson structure

$$\Lambda_H = \omega_H^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \qquad \omega_H = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \tag{A.82}$$

the equations of motion read

$$\begin{aligned} \dot q = p, \quad \dot p &= -\frac{1}{2}e^q - \frac{1}{4} i\xi\eta e^{\frac{q}{2}} \\ \dot\eta = \frac{1}{2}\xi e^{\frac{q}{2}}, \quad \dot\xi &= -\frac{1}{2}\eta e^{\frac{q}{2}} \ . \end{aligned} \tag{A.83}$$

As easily checked, the following functions are constants of motion

$$K = p\xi + e^{\frac{q}{2}}\eta, \quad L = p\eta - e^{\frac{q}{2}}\xi, \quad F = i\xi\eta \ , \tag{A.84}$$

so we can use K in (A.84) (or L) as Hamiltonian functions. The symplectic structure, corresponding to K, can be written as

$$\begin{aligned} &\omega_K = dq \wedge d\xi + dp \wedge dq(e^{-\frac{q}{2}}\eta) + dp \wedge d\eta(-2e^{-\frac{q}{2}}) + df \wedge dH = \\ &d\{dq(-\xi) + dp(2e^{-\frac{q}{2}}\eta) + f dH\} \ , \end{aligned} \tag{A.85}$$

where $f(q,p,\eta,\xi)$ is a function, given by

$$\begin{aligned} &f = A\xi + B\eta \\ &A(q,p) = \frac{1}{p^2+e^q}\left(\frac{2p}{\sqrt{p^2+e^q}} \ln\left(\frac{e^{\frac{q}{2}}}{p+\sqrt{p^2+e^q}}\right) + \frac{2e^{\frac{q}{2}}}{\sqrt{p^2+e^q}} - 2 \right) \end{aligned}$$

$$B(q,p) = \frac{1}{p^2+e^q}\left(\frac{2e^{\frac{q}{2}}}{\sqrt{p^2+e^q}}\ln\left(\frac{e^{\frac{q}{2}}}{p+\sqrt{p^2+e^q}}\right) - \frac{2p}{\sqrt{p^2+e^q}} - 2pe^{-\frac{q}{2}}\right). \tag{A.86}$$

If Γ is the dynamical vector field of the Toda system given by (A.83), then the function f satisfies $i_\Gamma df = e^{-\frac{q}{2}}\eta$, and this ensures that $i_\Gamma \omega_K = dK$. A somewhat tedious calculation shows that the $(1,1)$ tensor field

$$T = \omega_K \circ \Lambda_H \ , \tag{A.87}$$

has the property

$$TdH = dK, \quad d(T^2 dH) \neq 0 \ . \tag{A.88}$$

The above shows that the operator T, defined in (A.87), is not a recursion operator (conjugate of a Nijenhuis operator).

As we have seen, one of the most attractive properties of a (not graded) $(1,1)$ tensor field N with vanishing Nijenhuis torsion is the possibility of generating sequences of exact 1-forms. Let $T = N^*$, then from $d(TdF) = 0$ follows that $d(T^k dF) = 0$. Let us analyze now the graded situation. Suppose T is a graded $(1,1)$ tensor field which is homogeneous of parity $p(T)$. Then, if α is any 1-form, by using the definition (A.59), after some (graded) algebra, one gets

$$\begin{aligned}
&(X \wedge Y)\rfloor d(T^2\alpha) = \\
&\{(-1)^{p(N)p(Y)} X \wedge NY + (-1)^{p(N)[p(X)+p(Y)]} NX \wedge Y\}\rfloor d(N\alpha) \\
&-(-1)^{p(N)[p(X)+p(N)]}(NX \wedge NY)\rfloor d\alpha \\
&-(-1)^{p(N)}\ {}^G R_N(X,Y)\rfloor \alpha \\
&+(-1)^{p(N)p(X)}[1-(-1)^{p(N)}] L_{NX}(NY\rfloor\alpha) \ ,
\end{aligned}$$

where ${}^G R_N$ is defined in (A.63). Then it is clear that for an $(1,1)$ odd tensor, the $(2,1)$ tensor corresponding to its torsion (super-Nijenhuis torsion), can be defined only when $p(N) = 0$. The same result is obtained with the use of the general approach (using the identity $d_N \circ d_N = 0$).

Summarizing, we have shown that there are examples of dynamical systems, whose dynamical vector field Γ admits two Hamiltonian formulations, odd and even, respectively, and that the tensor field N, constructed out of the corresponding Poisson structures, is not a recursion operator, since it cannot generate new integrals of motion after the first two. We have also shown that the above fact is general and that for a generic graded $(1,1)$ tensor field N a graded Nijenhuis torsion cannot be defined unless N is even. From the very proof it seems quite probable that similar property should be true also in the infinite-dimensional case. We must remark, however, that the "no go" result we have proved only shows that the things cannot be generalized so directly. The possibilities for generalizations are by no means exhausted, but the things become very complicated, and even in the simplest cases, we see that a considerable effort must be done to generalize for the graded case the geometric structures, which are natural in the nongraded situation.

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B

Abbreviations

For the convenience of the reader, we give here a list of the most common abbreviations we have used.

AKNS – Ablowitz, Kaup, Newell, Segur (method/approach)
BT – Bäcklund transformation
CF – chiral fields system
cmKdV – complexified modified Korteweg-de Vries (equation)
CTC – complex Toda chain
DEE – difference evolution equations
FAS – Fundamental analytic solutions
GFT – generalized Fourier transform
GLM – Gel'fand-Levitan-Marchenko (equation)
GmKdV – generalized modified Korteweg-de Vries (equation)
GMB – generalized Maxwell-Bloch (equation)
GNLS – generalized Nonlinear Schrödinger (equation)
GZS – generalized Zakharov-Shabat (system)
HF – Heisenberg ferromagnet (equation)
HFI, HFII – First and second Heisenberg Ferromagnet equation chain systems
ISM – Inverse Scattering Method
KdV – Korteweg-de Vries (equation)
LL – Landau-Lifshitz (equation)
LLe – Landau-Lifshitz elliptic chain system
LLp – Landau-Lifshitz polynomial chain system
mKdV – modified Korteweg-de Vries (equation)
NLS – Nonlinear Schrödinger equation
NLEEs – nonlinear evolution equations
P-N manifold – Poisson-Nijenhuis manifold
TC – Toda chain
RHP – Riemann-Hilbert problem
RTC – real Toda chain

RTC1, RCT2 – Real forms of the complexified Toda chain system (for odd and even dimension of the configuration space)

s-G – sine-Gordon (equation)

ZS – Zakharov-Shabat (system)

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