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Diarmuid Ó Mathúna

Integrable Systems in Celestial Mechanics

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Integrable Systems in Celestial Mechanics

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I gcuimhne ar mo thuismitheoirí
(To the memory of my parents)

Contents

Preface

Direct involvement with the subject area of the present work dates from my years with NASA at its Electronics Research Center (ERC) in Cambridge, Massachusetts, in the 1960s. However, my approach to the problems of mathematical physics had been shaped earlier in my time as a graduate student in the Mathematics Department at MIT. The passage of time tends merely to further enhance my appreciation of that graduate study program, where I had the benefit of the intensive courses from Norman Levinson, C.-C. Lin, Jürgen Moser, and Eric Reissner. In the case of Reissner, my years as research assistant were a formative apprenticeship — one could say on the "shop-floor".

The stimulus to organize my convictions in book form came from my friends at Birkhäuser Boston, and I wish to thank Ann Kostant for providing me with the opportunity and support in producing it; a special thanks goes to Edwin Beschler, formerly of Birkhäuser, for his consistent encouragement over the years.

In the course of writing, I had the good-humored support and invaluable help from my one-time teacher and long-time friend, Vincent Hart. He read each chapter as it was written and his sharp eye picked up my many slips and errors. More significantly, his persistent questioning on the original form of Chapter 3 forced me to address all parameter ranges and provide solution forms to cover all possibilities for the case of negative energy.

He also raised the issue of providing illustrations of the various orbit types, a task I was reluctant to undertake. My only escape from the pressure was to invite him to do that part — to which he readily agreed. At that point a young colleague, Seán Murray, provided his precious expertise with Maple; the Appendix is the product of their fruitful cooperation.

In dealing with some difficulties with Euler's Latin, I had the ready assistance from the Medieval Latin team at the Royal Irish Academy, and I wish to record my thanks to Anthony Harvey, Angela Malthouse, and Jane Power.

The heroic task of typesetting was performed by Martin Stock, and the excellent layout of difficult material speaks for his committed perfectionism. Working from my manuscript was not easy, but his good humor prevailed over all difficulties. As one progresses into the script, the going does not get any

easier, but Martin's professionalism ensured that he finished with a flourish. For such an accomplishment, I am at a loss for words to adequately express my appreciation.

With such help and support, what deficiencies remain must be my sole responsibility.

Diarmuid Ó Mathúna
Institiúid Ard-Léinn Bhaile átha Cliath
Baile átha Cliath, 4

Samhain 2007

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General Introduction

> Mathematics is nothing if not
> a historical subject par excellence.
>
> — *Gian-Carlo Rota,* Indiscrete Thoughts*, p. 101*

In the AMS series *What's Happening in the Mathematical Sciences*, issue No. 5 (2002) includes a lively account of recent results in celestial mechanics under the title "A Celestial Pas des Trois". Therein the author, Barry Cipra, reiterates the observation " … the only part of Celestial Mechanics that is completely understood is the motion of two bodies" [C2, p. 72]. The same survey article recounts the "choreographies" recently discovered for the three-body problem — and also for the many-body problem — obtained through a combination of mathematical analysis and computer simulation. The number of new solution configurations of various patterns now runs into the millions [C2, p. 70].

In a larger frame, the story of the subject area over three centuries, with a particular weighting on developments since Poincaré, is engagingly told in *Celestial Encounters* by Diacu and Holmes (1996) [D3]. Their survey has a sharp focus on analytic developments over the latter half of the twentieth century, of which it gives an excellent account, both technically and historically. In spite of the intense progress on both fronts, analytical and computer-based simulation, the three-body problem — and in particular the restricted three-body problem — both remain far from being "completely understood." Such a claim would be possible only when there is a complete, explicit description of all solution forms.

Intermediate between the (integrable) two-body problem and the nonintegrable restricted three-body problem sits the problem of a body moving in the gravitational field of two fixed attracting centers. This problem — the "two-fixed-center problem" — was shown to be integrable, first by Euler in the years 1760–65 [E2: d,e] and also in the slightly later work (1766–69) of Lagrange [L2: a,b]. While Euler and Lagrange held each other in the highest mutual regard, we shall presently observe, in connection with their interest in this problem, that they express views that are in marked contrast — in fact, quite polarized. These contrasting attitudes have had their respective adherents in the generations since their time.

Over almost two and one-half centuries this problem has received attention that, although occasionally quite intense, has nevertheless been somewhat intermittent, with intervals during which no interest is evident. This may, at

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least partly, follow from some lingering discomfort with the artifice of a configuration of two fixed centers of gravitational attraction. However, it has been observed — and more than once — that in the case of the planar restricted three-body problem, if one considers the ratio of the mean motion of the primaries to the mean motion of the planetoid, then, in the limit as this ratio tends to zero, one has the planar problem of two fixed centers. The latter problem may, therefore, serve as a natural platform, providing a basis of approach to the restricted three-body problem. However, in the present work we stop short of that consideration; our aim is confined to the derivation of explicit representations for the solutions of the integrable problems.

Closely related to the two-fixed-center problem is a problem now identified with the dynamics of the near-earth satellite. Considering the separation parameter characterizing the distance between the two attracting centers, if one replaces the square of this parameter by its negative, one still has an integrable dynamical problem and a gravitational field that gives an excellent approximation to the gravitational field of the earth. Because of its specific significance, and also for the distinct features in the representation of the solution, it merits the separate treatment given to it here.

Following well-established tradition, we refer to the analysis of the dynamics of the two-body problem as the Kepler problem. The dynamical problem for the gravitational field of two fixed attracting centers we refer to as the Euler problem: Euler was the first on record to effect the separation of the problem, though Euler himself states that the problem had received attention from many of the "summi ingenii" of his time. The integrable problem associated with the earth-satellite was first recognized and given its present form by Vinti, and we refer to it as the Vinti problem.

The integrable problems of celestial mechanics are easily counted, namely,

I the Kepler problem
II the Euler problem
III the Vinti problem

with the latter two being transforms of each other.

We now give a brief historical survey of the evolution of these problems.

Background Survey

I The Kepler Problem

The conic section solutions to the Kepler problem were arrived at by Newton (*Principia*, 1687) through purely geometrical methods, making no appeal to either differential or integral calculus. His procedure is discussed in the paper by Hauser and Lang [H2], where it is emphasized that "Greek geometry is all that is required, without any use of vector analysis or calculus."

In a communication to Johann Bernoulli in 1710, Jacob Hermann [H3] proposed an analytic approach to the problem, which was quickly followed by the fuller treatment in Bernoulli's response [B2]. In both cases the geometrical configuration and energy considerations are used to establish the first constant of the motion: the subsequent analysis is essentially an exercise in the integral calculus. While Hermann's paper is extremely brief, the framework of the procedure is fully laid out; as he overlooked one constant of integration, his solution is accordingly deficient. In this paper of Hermann's, the Newtonian law of motion is set down in differential terms — apparently for the first time [S4]. The response of Johann Bernoulli gives a full and detailed analysis, and therein Bernoulli points out that, except for supplying the missing constant, his analysis follows the pattern laid out by Hermann (. . . "pour le reste, je le fais comme vous"). The analysis was further amplified in the paper by Varignon [V1].

For comparison with a later discussion of the Euler problem, we here observe that at a certain point in Bernoulli's reduction, the separated integral appears with an integrand that is the reciprocal of the square root of a quartic from which the *linear and constant terms are both absent*: this latter feature permits integration in terms of elementary (trigonometric) functions. Also implicit in Bernoulli's reduction is the representation of the solution in terms of an angle — a feature that was given its full explicit form a generation later in the comprehensive analysis by Euler [E2, a,b,c], where the solution is expressed in terms of trigonometric functions of the true anomaly — what has become the standard form. These developments are discussed in the account by Speiser [S4], previously cited. The transformation of the independent variable from time to the true anomaly is what would later be recognized as a regularizing transformation.

Subsequent analytic investigations of the Kepler problem would include both the generalization and extension of the spatial context. The fact that the closed periodic orbits survive in a space of constant curvature — be it positive or negative — was first explored by Paul Serret [S1] and later by Appel [A6] and Liebmann [L5]. These developments are reviewed in the survey by Kozlov and Harin [K3] and note should also be taken of the later paper of Kozlov [K2].

The Kepler problem in a general n-dimensional Euclidean space has been investigated by Moser: following the introduction of the regularizing transformation, he finds that "the energy surface is a smooth manifold on which, for negative energy, the closed orbits provide a fibration" [M1]. Other procedures exploring the regularization potentialities have received attention: we refer, in particular, to the book by Stiefel and Scheiffele [S6], where the focus is on spinor regularization, and also to the detailed exploration and results recorded by Souriau [S3].

An alternative perspective may be found in the overview by Albouy in the Recife Lectures 2002 [A3]: it includes further aspects of the history of this most unusual problem, which has every possible degeneracy. Lastly, the exhaustive treatment of the Kepler problem in all its aspects by Cordani [C4] includes a comprehensive bibliography.

II The Euler Problem

In the gravitational context, the next "larger" problem, having the Kepler problem as the degenerate case, is the problem of the motion of a body in the gravitational field of two fixed attracting centers. This problem collapses on to the Kepler problem when the separation parameter measuring the distance between the two attracting centers tends to zero. Cognizance of this requirement provides the motivation for the approach followed in the present work.

We give the outline of the history of this problem in several stages.

The Eighteenth Century

(i) Euler (1760–67): The work of Euler on this problem is recorded in the series of papers in the 1760s [E2, d,e], wherein there is a strong plea for attention to the significance of the problem — the solution of which would provide the basis for further potential developments in astronomy [E2:e(ii), p. 153]. It would appear that Euler saw possible applications in the development of lunar theory — and there is explicit mention of "Satelliti Terrae" [E2:e(i), p. 208]. He also mentions that the problem had attracted the attention of some of the greatest analysts of his time — "summi ingenii" — without success. In the printed record, however, Euler has unquestioned priority for this problem.

Having formulated the problem in a Cartesian coordinate framework, Euler, exhibiting that ingenuity of which he was master, shows by a series of quite involved transformations that the problem can be put in a separated form. Each of the separated integrals involves the reciprocal of the square root of a *general* quartic expression. The second integration he cannot effect by means of any of the (then) known functions, but he expresses the hope that some light may be thrown on the solution by means of approximate evaluations "per arcus sectionum conicarum" — he recognizes that he has an elliptic integral. Several of his contributions in the subsequent issues of *Novi Commentarii Petropolitanae* are directed at the evaluation of such integrals.

Of the three papers dealing with the dynamical problem of two fixed centers, the earlier ones [E2:d,e(i)] have their focus on the planar case: having achieved separation by that remarkable sequence of transformations, Euler then explores several special cases. In the later paper [E2:e(ii)], which also aims to include the three-dimensional case, Euler, after effecting the reduction, then proposes a "methodus succintior" for achieving separation. Using notation different from that of Euler, we denote the distance of the moving body from the respective centers by r_1 and r_2: Euler introduces the derived coordinates (called r and s respectively by Euler), which we denote by ρ_1 and ρ_2, by setting

$$\rho_1 = \tfrac{1}{2}(r_1 + r_2), \quad \rho_2 = \tfrac{1}{2}(r_1 - r_2) \tag{0.1a,b}$$

in terms of which the separation is promptly effected and the problem is reduced to

$$\frac{\mathrm{d}\rho_1}{\sqrt{\mathsf{P}_1}} = \frac{\mathrm{d}\rho_2}{\sqrt{\mathsf{P}_2}} \tag{0.2}$$

where P_1 and P_2 are general quartic expressions of ρ_1 and ρ_2, respectively.

The $\rho_1 - \rho_2$ coordinate system is of particular significance and plays a crucial role in the subsequent history of the problem. Clearly, the level surfaces of such a coordinate system are the confocal ellipses and hyperbolae anchored to the foci at the two fixed centers. It is a curious feature of Euler's analysis that these coordinates appear at a relatively late stage in his investigations.

(ii) Lagrange (1766–69, 1788): At about the same time, the problem had attracted the attention of the much younger Lagrange [L2:a,b]. In contrast to Euler's plea for the crucial and basic relevance of the problem to the future development of astronomy, Lagrange puts in an opening disclaimer regarding its applicability to any real system and rests the case for attention to the problem purely on its analytic interest [L2:b(v), p. 390]. However, there is no doubt that Lagrange's analysis and reduction are far more elegant and transparent than that of Euler.

Lagrange, the natural analyst and pioneer of the generalized coordinate system in the development of analytic mechanics, formulates the three-dimensional problem in terms of the spherical polar coordinate system based at one of the attracting centers. The system of differential equations resulting therefrom involves the two gravitational forces as well as the distance from the other center. From appropriate combinations of these equations and the utilization of the relation subsisting between the distances from the two centers (which we term r_1 and r_2 as before), the reduction leads to second-order differential equation for the radial coordinate r_1, involving the respective gravitational forces R_1 and R_2 and the two distance variables r_1 and r_2.

Symmetry considerations lead to a second similar differential equation for r_2 involving a corresponding expression wherein the gravitational forces R_1 and R_2 have been interchanged as also have been r_1 and r_2. Following a further reduction of this pair of conjugate equations, Lagrange then introduces the derived coordinates but without the $\frac{1}{2}$-factor included by Euler. Using notation different from that of Lagrange, and referring to relations (0.1) above, we set

$$\rho_1^* = 2\rho_1 = r_1 + r_2, \quad \rho_2^* = 2\rho_2 = r_1 - r_2 \tag{0.3a,b}$$

and, in terms of the $\rho_1^* - \rho_2^*$ coordinate system, Lagrange effects the separation of the first integrals in a form identical with that of Euler (except for the $\frac{1}{2}$-factor already noted).

Next, Lagrange shows that the integrability/separability is unaffected by the addition of a nongravitational "elastic-type" force (varying linearly with distance) directed toward the center point between the two attracting masses. There is then noted the existence of some particular elliptic orbits around each of the attracting masses — a phenomenon that Lagrange understandably records as “very remarkable” [L2, b(v), p. 397]. There follows a discussion of some “time of traverse” features of particular orbits in relation to some general results of Lambert [L1].

Lagrange’s derivation is remarkable for its clarity and the general elegance of its presentation. We record verbatim some of Lagrange’s concluding remarks in his treatment of the Euler problem, from the English translation by Boissonade and Vagliente [L2, b(v), p. 400] of the second edition [L2, b(ii)].

> The problem which we just solved was first solved by Euler for the case where there are only two fixed centers which attract inversely proportional to the square of the distances and where the body moves in plane containing the two centers (*Mémoires de Berlin* for 1760). His solution is specifically remarkable for the skill he has shown in using various substitutions to reduce the differential equations to the first order and to integration. These differential equations could not be solved by known methods because of their complexity.
>
> By giving a different form to these equations, I obtained directly the same results and I was even able to generalize them to the case where the curve is not in the same plane and where there is also a force proportional to the distance and directed toward a fixed center located in the middle of the two other centers. The reader should refer to the Fourth Volume of the old *Mémoires de Turin*, from which the preceding analysis is taken and in which the case is found where one of the centers is moving toward infinity and the force directed toward this center becomes uniform and acts along parallel lines. It is surprising that in this case the solution is not greatly simplified. Only the radicals, which enter in the denominators of the individual equations, contain only the third degree of these variables rather than the fourth degree, which also makes their integration dependent on the rectification of conic sections.

This latter modification of the physical problem might even find an application (the Stark effect), a century after Lagrange had written his last word.

(iii) The Coordinate System: Before we close our survey of the eighteenth century, it is appropriate to add a further comment on the coordinate system.

The coordinate system (r_1, r_2), on which Lagrange based his formulation and which Euler had eventually recognized as a basis for his “methodus succinctior” leading to separation, may by its nature be termed the bipolar coordinate system. Both of the derived systems (ρ_1, ρ_2) and (ρ_1^*, ρ_2^*) are elliptic coordinate systems: these two systems are identical except for the $\frac{1}{2}$-factor, and we shall confine our attention to the (ρ_1, ρ_2) system. We shall refer to this system as the Euler–Lagrange elliptic coordinate system or more frequently as the EL elliptic coordinate system.

This coordinate system plays a central role in the subsequent history of the problem. It is not surprising that this should be so, as this system — or some simple variant thereof — is necessary to facilitate separation. However, the accepted centrality of its role in effecting the transition to the first integrals may also have precluded the possibility of effecting the second integration to yield a general solution. This is an issue to which we shall return.

The Nineteenth Century

(i) Lagrange/Legendre: With the appearance of Lagrange's landmark work *Mechanique Analytique* in 1788, the problem and its analysis received a much wider exposure. In the subsequent years, Lagrange devoted a good deal of his time and attention to the revision and expansion of his magnum opus for a second edition. This was prepared in two volumes — the first issued in 1811 and the second, which contained the work on dynamics, appeared in print a few weeks after the author's death in 1815.

Meantime, the problem had attracted the attention of Legendre, who had independently observed and commented on the existence of elliptic orbits about each of the primaries — the "very remarkable" phenomenon noted by Lagrange. These and other observations were recorded by Legendre in his respective treatises on calculus and elliptic functions [L3, (a) and (b)]. A generation later, these curious orbits would be recognized as particular cases of the consequences of what became known as Bonnet's theorem [B3].

(ii) Jacobi: The next milestone in the history of the Euler problem is marked by the investigations of Jacobi, as recorded in his lecture course at the University of Königsberg. Following the appearance of Hamilton's "General Method" in 1834–35 [H1], Jacobi promptly developed the general procedure for the analysis of dynamical systems through a first-order partial differential equation, since known as the Hamilton–Jacobi equation and procedure. He demonstrated the method in his lecture courses. The lectures he delivered at Königsberg in 1842–43 were recorded by C.W. Borchardt and were published, under the editorship of A. Clebsch, over two decades later in 1866 [J1].

In his analysis of the Euler problem, Jacobi takes as his framework the bipolar coordinate system as laid out by Lagrange and formulates the Hamilton–Jacobi equation for the problem in these coordinates. Then transforming to the EL coordinate system, he is led to the separated form of the Hamilton–Jacobi differential equation (lecture 25).

Motivated by the larger context of dynamical systems, extending beyond the confines of celestial mechanics, Jacobi is led to investigate the separation issue. In the absence of any general method, he must pursue a partially inverse procedure — "on finding a remarkable substitution, look for the problems to which it can be applied." In the next three lectures (26–28), he introduces and develops a general system of coordinates, since known as Jacobi's ellipsoidal coordinates, and points out their applicability to a number of identified problems. Specifically, he frames the scheme to investigate the motion of a particle on a general ellipsoid in R^n.

Having noted the particular form taken by Jacobi's ellipsoidal coordinates in two- and three-dimensional Euclidean space, he then returns to the Euler problem (lecture 29). The planar form of Jacobi's elliptic coordinates, denoted by (λ_1, λ_1), is related to the EL elliptic coordinates (ρ_1, ρ_2) as follows:

$$\tfrac{1}{2}(r_1 + r_2) = \rho_1 = \sqrt{a + \lambda_1}, \quad \tfrac{1}{2}(r_1 - r_2) = \rho_2 = \sqrt{a + \lambda_2} \qquad (0.4\text{a,b})$$

where a is the length parameter of the Jacobi system. He then demonstrates the separability of the Hamilton–Jacobi partial differential equation for the Euler problem in the (slightly more general) Jacobi elliptic coordinate system, before finally reverting to the EL elliptic coordinate system for the expression of the first integrals.

(iii) Liouville: Influenced by the ideas of Jacobi, Liouville recognized the relevance of the essential concepts to the utilization of general orthogonal coordinates in the exploration of geodesics on the general triaxial ellipsoid [L6, a]. In line with that analysis, he investigated the parallel applicability to the Lagrangian formulation of dynamical systems. The crucial feature was separability, and it led to his formulation of sufficient conditions for separability of dynamical systems in general orthogonal coordinates [L6, b].

Liouville noted that most of the known integrable problems met his conditions for separability. In particular, he focused on the applicability to the Euler problem, including the problem as extended by Lagrange, and cited the use of Jacobi's ellipsoidal coordinates. Moreover, in the planar case, he showed that it admitted the additional modification by permitting the inclusion of an additional attractive force, normal to the axis defined by the two centers, and proportional to the inverse of the cube of the distance therefrom. He also raises the question of allowing the two primaries to rotate [L6, a(ii), pp. 440–1].

The attention of Jacobi and especially that of Liouville were crucial in maintaining interest in the problem, and many of the investigations in the latter half of the century reflect back to them. Liouville was probably the most influential mathematician of his time, and we shall see that in the ensuing years, many of the points of note are linked to his former students and associates — cf. Lützen [L8].

A curious feature of their work is that neither Jacobi nor Liouville made any move toward the second integration for a complete solution; we shall return to this point later.

(iv) J.A. Serret, J.L.F. Bertrand, J.G. Darboux: Also in the 1840s, Lagrange's work had attracted the attention of the young Joseph Alfred Serret[1] — a student of Liouville. In his thesis, presented at the Faculty of Sciences at Paris, he pointed out that part of Lagrange's analysis of the Euler problem needed some clarification and amplification in order to give a rigorous treatment and a satisfactory derivation of the results: cf. [L2, b(v), note 26, p. 590].

During the same decade, when decisions were being made for a third edition of Lagrange's *Mécanique Analytique*, the one chosen as editor was Joseph Bertrand — another protegé of Liouville. Bertrand prepared a significantly expanded edition with copious notes on the more recent developments — specifically, the Hamilton–Jacobi theory as well as the results of Poisson and Liouville. In the discussion of the Euler problem, Bertrand included the above-mentioned modifications and clarifications of J.A. Serret. The third edition, in two volumes, was published in Paris in the years 1852–55.

[1] Joseph Alfred Serret and Paul Serret, previously cited [S1], were brothers.

As already noted, the lectures of Jacobi appeared in print in Berlin in 1866. Later the complete works of Jacobi were prepared under the general editorship of E. Lettner. These were published in Berlin in 1884; they included a reissue of the Königsberg lectures with a foreword by K.T. Weierstrass.

Meanwhile, the publication of the complete works of Lagrange was undertaken by Le Ministère de l'Instruction Publique, with Gauthier-Villars (Paris) as Imprimeur-Libraire. The project (*Oeuvres de Lagrange*) comprised 14 volumes that appeared over the years 1867–92. The general editor was J.A. Serret who, by the time of his death in 1885, had prepared volumes I–X and seen them through to publication. Volumes XI and XII constitute the fourth edition of *Mécanique Analytique* under the editorship of Jean Gaston Darboux, and these made their appearance in the anniversary year 1888. The final two volumes, which deal with the correspondence of Lagrange, were edited by L. Lafaune and appeared in 1892.

It would seem that these publications may have led to a revival of interest in the works of Liouville. In particular, there appeared the investigations of Stäckel on separability in 1890 and 1891 [S5] and the subsequent work of Levi-Civita [L4] in 1904. In Stäckel's analysis, it is shown that for general orthogonal coordinates, Liouville's requirements for separability are both necessary and sufficient, while (independently) Levi-Civita has shown that if the potential depends on all coordinates, then for the system to be separable these coordinates must be orthogonal. The term "Liouvillian" has since been applied to many systems satisfying "Liouville's conditions" in a variety of forms more restrictive than that of Liouville's general formulation [L8, pp. 704–5].

In 1889 appeared the work of Velde [V3], which is a reconsideration of a form of the generalization of the Euler problem as formulated by Liouville. More significant is the analysis of Darboux [D1], which appeared in 1901. Therein it is shown that separability, in the EL elliptic coordinate system, persists for the Euler problem if there are added two further complex-valued masses that are conjugates of each other and situated respectively at the imaginary foci of the elliptic coordinate system. This extension has relevance to the Vinti problem to be discussed presently.

For celestial mechanics, the end of the century is marked by the massive investigations of Poincaré [P3]; however, there is no indication that the Euler problem held any interest for him.

The Twentieth Century — and Later

(i) Charlier, Hiltebeitel, Plummer: The opening decade of the twentieth century brought the two-volume treatise on celestial mechanics by Charlier [C1], the volumes appearing respectively in 1902 and 1907. The first volume is devoted to analysis while the second is concerned with specific problems in astronomy. In volume one, there is a thorough investigation of the Euler problem — the first such thorough treatment since that given in the lectures of Jacobi [J1] a half-century earlier.

It is worth paying particular attention to the layout of Charlier's first volume. Following the introduction of Lagrangian mechanics and the separability theorem of Liouville, Charlier immediately cites the Euler problem in EL elliptic coordinates as an example for the applicability of the latter theorem of Liouville — all in the first chapter. The second chapter introduces the Hamilton–Jacobi procedure and, following the determination of the separability criterion and the citation of Stäckel's theorem, moves to an investigation of periodic solutions. The third chapter is devoted entirely to the Euler problem for the various ranges (negative, zero, and positive) of the energy constant. Here the analysis is particularly focused on the possible ranges for the zeros of the relevant quartics appearing in the elliptic integrals. Examples are cited and samples of typical orbits for specific ranges are illustrated. Not until chapter four is the two-body (Kepler) problem discussed. This is followed in chapter five by an analysis of the three-body problem. The later chapters six and seven deal with perturbation theory.

Thus, in Charlier's presentation, the Euler problem has priority and takes center stage, with the Kepler problem taking a secondary role as the degenerate case. The significance of this unique feature of Charlier's approach does not seem to have drawn much comment.

A few years later in 1911, the paper by Hiltebeitel [H4] includes a detailed discussion of the Euler problem and its generalizations. Therein may be found the general formulation where the cases treated by Darboux, Liouville, Lagrange, and Euler can be seen in the context of the general problem, each one of those mentioned being a particular case of the preceding one. The analysis follows Charlier in its concentration on the ranges for the roots of the relevant quartics; again examples are cited and some typical orbits are illustrated.

In Plummer's treatise [P2] the Euler problem makes a brief but significant appearance. At the end of his discussion of the restricted three-body problem, he notes that if in the reduced equations, one sets the mean motion of the primaries to zero, one obtains the separated form of the Euler problem. It is convenient to quote his own words:

> The other case represents the problem of two centres of attraction fixed in space, so that $n = 0$. Then the equations become simply
>
> $$\begin{aligned}\frac{\mathrm{d}^2 u}{\mathrm{d}T^2} &= (\mu - \nu)c \sin u - c^2 h \sin 2u\\ \frac{\mathrm{d}^2 v}{\mathrm{d}T^2} &= (\mu + \nu)c \sinh v - c^2 h \sinh 2v.\end{aligned}$$
>
> Here the variables u, v are separated and the equations lead immediately to a solution in elliptic functions. The comparison of this problem with the simplest case of the problem of three bodies is instructive as to the difficulty of the latter.

The latter remark is worth bearing in mind for its (however slim) acknowledgment of the significance of the Euler problem.

The comment that the separated equations "lead immediately to a solution in elliptic functions" is typical of what is generally said — if anything at all is said — by the several authors at this point where separation of the first integrals has been achieved. In the various textbooks issued throughout the twentieth century, the reader is told that the solution of the Euler problem is expressible in terms of elliptic functions, without any specification of what type of elliptic function or any indication of how such a form is to be attained. As will be noted presently, there is one textbook that at least partially breaks this pattern in that it identifies the functions as those of Jacobi — but without any indication of how this is to be done; and that text appears in the twenty-first century [C3].

(ii) The Quantum Connection — and Some Corrections: Returning to the chronology, in the 1920s the Euler problem received attention from another quarter. Within a decade of the appearance of Bohr's quantum theory, the molecular analog of the Euler problem was being explored by two young students. The quantum problem is that of the hydrogen molecule ion H_2^+ and the students were K.F. Niessen [N1, 1923] at Utrecht and W. Pauli [P1, 1922] at Münich. In Pauli's work, the EL elliptic coordinates are normalized with the separation parameter. In both cases, the authors rely on the lectures of Jacobi and on Charlier's treatise; the several types of orbit are classified and energy calculations are made for particular orbits. Their work was noted by Sommerfeld in the fourth edition of his *Atombau und Spectralinien* (1924) and also in Born's *Vorlesungen über Atommechanik* (1925), where the discrepancies between energy calculations and the measured values are noted. Further work was done by Teller, Burrau, Richardson, and others. We shall not attempt to do justice to this story; it is ably told in the biography of Pauli by Charles Enz [E1, pp. 63–70]. Therein it is noted that the H_2^+ problem is both "the simplest molecular problem" and also the most complicated problem that the old quantum theory could handle.

Shortly thereafter, in the USSR it was observed that there were some inaccuracies in Charlier's analysis of the Euler problem. This appeared in the 1927 paper of Tallquist [T1] who besides making the appropriate corrections in the work of Charlier also refined the classification of the orbits. This correction and refinement was further extended in the following decade in the work of Badalyan [B1, 1934].

(iii) Prange, Whittaker, Wintner: The general issue of integration procedures in analytic mechanics was fully surveyed in the comprehensive article prepared by Prange for the *Encyclopedia of Mathematical Sciences* appearing in 1933 [P4]. Also there appeared the textbooks of Whittaker and Wintner, respectively. The first edition of Whittaker's treatise appeared in 1904, with later editions in 1917, 1927, and 1937; a German translation appeared in 1924. His treatment of the Euler problem is brief: having reduced the problem to the first integrals, he mentions that "the solution can be expressed in terms of elliptic functions" [W2, iv: p. 99]. In the book by Wintner (1947) [W3, pp. 145–7],

following a formulation in terms of bipolar coordinates, the separation is effected by transforming to EL elliptic coordinates; some qualitative features of the motion are mentioned.

(iv) 1950–2000: In the second half of the century, the major analytic advance was initiated by Kolmogorov [K1, a,b], particularly in his celebrated 1954 ICM address [K1, (b)], an English translation of which appears in Appendix D of reference [A1].

In that 1954 address, Kolmogorov set out the structure of his approach in the context of the classical integrable problems. In citing the Euler problem, he notes that

> the extremely instructive qualitative analysis of the problem on the attraction by two immovable centers that was made in Charlier's well-known treatise has proven to be incomplete and partially erroneous. It has been twice corrected [by Tallquist and Badalyan].
>
> [A1, Appendix D; p. 269, footnote]

He then goes on to emphasize that

> the real significance for Classical Mechanics of the analysis that I have made of Dynamical Systems on T^2 depends on whether there are sufficiently important examples of canonical systems with two degrees of freedom that cannot be integrated by classical methods and in which invariant (with respect to the transformation S^t) two-dimensional tori play a significant role.
>
> [A1, Appendix D; p. 270]

The strength of his approach will reveal itself in the nonintegrable case. This led to the major results of Arnold and Moser and what became known as the KAM theory, recorded in the book by Abraham and Marsden (1967) [A1]. The KAM theory is detailed in the textbook of Siegel • Moser (1971) [S2], and also in that of Arnold, Kozlov, and Neishtadt (1985; 1993; 1997) [A8]. As previously noted, the KAM theory is also discussed in the survey "Celestial Encounters" (1996) [D1].

This may be an appropriate point to note that the first English translation of the *Mécanique Analytique* of Lagrange appeared in 1996 [L2, b(v)].

Meantime, the previously noted idea of Paul Serret [S1] in generalizing the context of the Kepler problem to a space of constant curvature was taken up and applied to the Euler problem by Alekseev (1965) [A4], which led to further exploration in that area, mainly within the USSR. The recent developments in this area, together with an account of the work over the intervening years, may be found in the book devoted to the topic by Vozmischeva (2003) [V5].

The advent of the space age in the 1950s with the consequent new interest in space dynamics stimulated the quest for specific solutions and led to a renewal of interest in the Euler problem. There were investigations in two areas — an examination of specific cases of the Euler problem and an investigation of the potential use of solutions of the Euler problem as a basis of approximation for solutions of the restricted three-body problem. In the former, we mention especially the work of Deprit (1962) [D2]; and in the latter where the solution of the Euler problem is to be used as an "intermediate orbit," we note the work of Arenstorf and Davidson (1963) [A7]. Considerable further work on these lines is surveyed in the book by Szebehely (1967) [S8], which is particularly informative especially in the notes and reference lists at the close of Chapters 3 and 10.

The Euler problem gets mention in the book by Arnold (1974) [A8] and gets brief coverage in the book by Arnold, Kozlov, and Neishtadt (1985) [A9], where there is strong emphasis on according due recognition to the achievement of Jacobi [A9, p. 128].

(v) The Quantum Connection Revisited: Following the appearance of Wave Mechanics in the latter half of the 1920s, interest in the classical approach to the hydrogen ion, as a basis for "semiclassical" quantization, seems to have waned. After the lapse of one-half century, there appeared the curiously interesting — and somewhat baffling — work of Strand and Reinhardt (1979) [S7].[2] Therein the problem is normalized by the separation length (b) and formulated in terms of the EL coordinate system — also normalized by b.[3] We shall presently discuss the limitations inherent in such a formulation. Here we confine ourselves to the results presented in [S7].

The features of the work we wish to note are twofold:

1. the formulation incorporates the introduction of the regularizing transformation;
2. the solutions are presented in terms of Jacobian elliptic functions.

Following a discussion of the accessible/nonaccessible regions based on an analysis of the quartics, the solution forms are then presented in terms of Jacobian elliptic functions with the modulus expressed in terms of the roots of the quartics. The solution forms for ρ_2/b are given in relations (2.17) while those for ρ_1/b are given in relations (2.18).

Relation (2.17a) as it stands is baffling and defies any attempt to see how it could arise; however, if it is taken that in that equation the symbol dn is a misprint for what should read sn, then the two relations (2.17a,b) by appropriate rescaling can be brought into conformity with the solution forms for the case $\beta = 0$, presented in the present work, namely, relation (6.9) and (6.13) of Section 6, Chapter 3.

Relations (2.18) for ρ_1/b present other problems. The form (2.18b) given for the case where two complex roots arise appears simpler than the form (2.18a) given for the case of four real roots. Complex roots do not arise in the planar case; in the analysis presented here, it is shown how the general case can be reduced to the planar case (Chapter 4). If complex roots arise, it is not clear how they give rise to a solution, simpler in form to that of the planar case. Accordingly, it is not obvious what is the realm of relevance of the

---

[2] This work has been brought to my attention by Professor A.J. Bracken of the University of Queensland, which I gratefully acknowledge.

[3] The authors use the symbol c for the separation length; for consistency with the notation of the present work, we use the symbol b.

solution form (2.18b). We concentrate therefore on the solution form (2.18a) for the case where all four roots are real. Without giving any indication how such a solution form could arise, the solution is presented as a quotient of linear functions of the factor sn^2 as a function of the half-argument; if one notes that

$$\mathrm{sn}^2 \tfrac{1}{2}u = \frac{1 - \mathrm{cn}\, u}{1 + \mathrm{dn}\, u}$$

then clearly the solution form (2.18a) can be expressed as a quotient of linear functions of the elliptic functions cn and dn of the full argument. With the appropriate rescaling of all the quantities involved, this solution could be brought into conformity with the solution forms given in Chapter 3, e.g., relation (9.29) of Section 9 or (9A.33) of Section 9A, depending on the identity of ξ_2 and ξ_3. However, the form in which the solution (2.18a) in [S7] is expressed has the negative feature that in the Kepler limit as b tends to zero, two of the roots tend to infinity as does also the left side $\xi = \rho_1/b$. It is regrettable that this work has been left in such an unattractive form — at least partly from the choice of coordinate system and normalization procedure; further comment is not possible without a clearer indication of how the solution was arrived at.

The authors provide "representative" trajectories for the planar case in Figs. 7, 8, and 9 of [S7]. These correspond respectively to cases B1, B2, and A4 of the solution forms presented in Chapter 3 of the present work. The trajectories produced from the results of Chapter 3 for the appropriate values of the relevant parameters can be compared with those of [S7]. Considering that an initial point is not specified, the agreement is remarkable (see the figure below). This would lend credence to the likelihood that the appearance of the dn factor in relation (2.17a) of [S7] is a misprint.

This paper by Strand and Reinhardt is noted in the book by Gutzwiller (1990) [G1], where the representative trajectories are reproduced and — perhaps significantly — the solution forms are not exhibited. It is further remarked by Gutzwiller,

> The solution of this problem can be rated, with only slight exaggeration, as the most important in quantum mechanics, because if an energy level with a negative value E can be found, the chemical bond between two protons by a single electron has been explained.

It would appear that the interest from the Quantum Connection may match that from the perspective of Celestial Mechanics.

(vi) 2000–Present: Returning to the classical context, we note the recent (2004) work by Varvoglis, Vosikis, and Wodner [V2] wherein there is a classification of orbits in accordance with a specified scheme. In particular, collision orbits are identified and a transformation is proposed for the regularization of close approaches to facilitate (numerical) integration.

Also in this context, we mention the (2004) analytic work by Waalkens, Dullin, and Richter [W1], which identifies the foliation of phase space for the Euler problem with arbitrary relative strength of the two centers — corresponding to arbitrary values of the ratio β in the present work. This paper

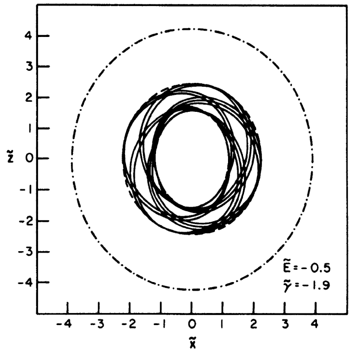
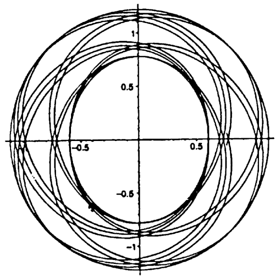


FIG. 7. A portion of a typical trajectory of class $P1$.



e=.22

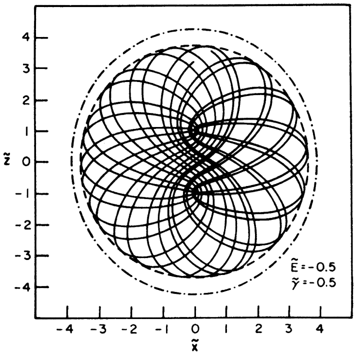
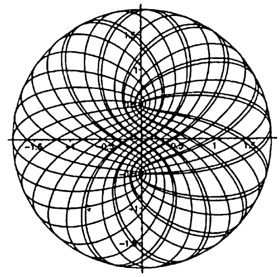


FIG. 8. A portion of a typical trajectory of class $P2$.



e=.866

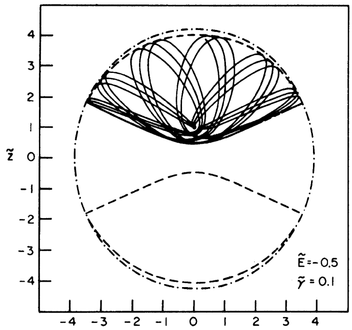
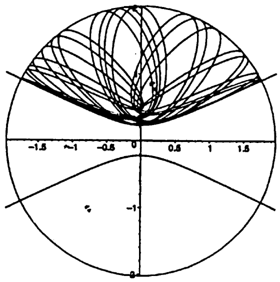


FIG. 9. A portion of a typical trajectory of class $P3$.

M. P. Strand and W. P. Reinhardt:
J. Chem. Phys., Vol 70, No. 8, 15 April 1979



e=1.05

Corresponding Trajectories from results of Chapter 3, Sections 6 and 9 with λ =.5: (λ=.5 corresponds to $\tilde{E}$=-.5,and here β=0; also $\tilde{\gamma}=(e^{\lambda}-1)/\lambda$.)

Representative trajectories from Strand and Reinhardt [S7], side by side with corresponding trajectories derived from Chapter 3 of the present work.

[W1] as well as the previously cited [S7] both include reference lists to related work.

Finally we come to the book by Cordani [C4] published in 2001. As would be expected from the title, the concentration is on the Kepler problem. But it is noted that the Kepler problem is also integrable in elliptic coordinates, and at a later stage there is a discussion — though quite brief — of the Euler problem as an integrable perturbation of the Kepler problem in these elliptic coordinates. Having arrived at the first integrals, he states,

> The point is to find a transformation $w \to z(w)$ such that the transformed integral acquires the Legendre canonical form of an elliptic integral of the first kind.... This last integral is solved invoking the inverse of Jacobi elliptical functions.

But there is no indication of how this is to be done. The statement implicitly acknowledges that the elliptic function solution form cannot be extracted from the integrals expressed in terms of the system of elliptic coordinates used, which is a form of the EL coordinate system with a normalized separation constant.

In the subsequent discussion, Cordani mentions that the sought-for transformation would depend on the roots of the quartic. There is then reference to the computer program EULER, which is used to generate the orbits that are exhibited. There are attractive illustrations of the orbits for the various ranges of the energy values. For details on the computer program, the reader is referred to the thesis of Codegoni at Milano (2000) [C3]. On the whole, this discussion of Cordani is the most interesting for some time.

The problem of transforming the quartic to Legendre canonical form is the crux of the matter. We shall return to this point later when we come to the aim and outline of the present work.

III The Vinti Problem

The Vinti problem took its form in the series of papers by J.P. Vinti [V4] in the years 1959–66. The motivation arose from the necessity to track Earth-orbiting satellites launched from the space program. Vinti showed that the problem of the earth satellite could be formulated as a dynamical problem wherein the potential approximates the geopotential and where the Hamilton–Jacobi equation is separable. Later it was recognized as a transform of the Euler problem. Furthermore, in its symmetric form (symmetry with respect to the equatorial plane), the Vinti problem can be viewed as a particular case of Darboux's generalization of the Euler problem [D1] already mentioned. However, there is no indication in Darboux's work that he was in any way aware of this potential application. The priority here belongs to Vinti, who presented approximate solutions to the problem in terms of infinite series. An alternative form of solution was given by Izsak (1963) [I1].

Some years after the appearance of Vinti's analysis, corresponding results appeared independently in publications in the USSR starting with the work of Aksenev, Grebenikov, and Demin (1964) [A2].

In the late 1960s, an alternate solution form for the Vinti problem was proposed by the present author [Ó M1]. With the analog for the Vinti problem of the true anomaly serving as independent variable, the solution is presented in terms of Jacobian elliptic functions of that generalized true anomaly. In the case of two of the coordinates, this representation is exact, while for the third coordinate, involving an elliptic integral of the third kind, an approximation procedure (in terms of the oblateness parameter) is necessary; the same approximate procedure is applied to the "time-angle" relation. At the time, it was assumed by the author that the idea would be taken up for application to the Euler problem. That this has not happened is one of the reasons for the present work.

The Vinti problem has continued to receive attention. We note in particular the work by Alfriend, Dasenbrock, Pickard, and Deprit (1977) [A5], the paper by Jezewski (1983) [J2], that of Livesey, Ó hÉighearta, and Vandyck (1994) [L7], and the doctoral dissertation of Floria [F1] presented at the Universidad de Valladolid in 1993.

Aim of the Present Work

The present work aims to show that the solutions to the three integrable problems specified (Kepler, Euler, Vinti) can be put in a form that admits the general representation of the orbits and where all three share a definite pattern. In the case of the Kepler problem, this form has been known for almost three centuries from the work of Hermann, Bernoulli, and Euler. In its standard form, that solution may be characterized as follows: with the true anomaly as the independent variable, the solution forms for the three coordinates of the spherical system are expressed in terms of trigonometric function of the true anomaly. The solution is completed by the time-angle relation expressing time in terms of elementary functions of the true anomaly.

Here we propose that with an appropriate generalization of the true anomaly in each of the two problems of Euler and Vinti, and with the proper choice of the coordinate system, each of the three coordinates can be expressed in terms of Jacobian elliptic functions of the generalized true anomaly; this solution form is also complemented by the appropriate generalization of the time-angle relation, also involving Jacobian elliptic functions.

In the case of two of the coordinates, the representation is exact: the third coordinate involving an elliptic integral of the third kind requires an approximation procedure, which is clearly suggested by the context; the approximation procedure also applies to the time-angle relation. In each case, the form of the solution permits the visual inspection of the manner in which the form collapses onto that of the Kepler problem in the degenerate case.

The question naturally arises as to why, in spite of the long history of the Euler problem, this had not already been done. The answer would appear to lie in two sources of “blockage” to the pursuit of such a solution form:

1. the normalization generally adopted
2. the coordinate system used.

With regard to the first point, those authors (e.g., Wintner) who normalize the geometric configuration do so by means of the separation length, thus normalizing the separation to unity. This practice has been established — even embedded — in the standard analysis and reduction of the restricted three-body problem. However, this normalization immediately locks the geometric configuration in a form that precludes the observation of what happens when the separation vanishes, that is, when the problem collapses to the Kepler problem. This is not a desirable situation.

In the traditional reduction of the Kepler problem, the normalization is effected by means of the geometric constants manifested by the semi–latus rectum and semimajor axis and the relation between them expressed through the eccentricity. These length scales are the geometric reflection of the physical constants of the motion, namely energy and angular momentum. In the case of the Euler and Vinti problems, no such geometric interpretation is evident. But there still are the two physical constants of motion — one again reflecting the energy and the other still having the dimension of angular momentum (but without the obvious physical interpretation). These constants permit the definition of the corresponding length scales, facilitating the corresponding normalization and a reduction that follows the pattern set in the Kepler problem. This is the route to be followed here.

The second suggested source of blockage — the coordinate system used — is even more crucial. While it is recognized that the reduction to the first integrals must be effected through some form of the EL elliptic coordinate system, it needs also to be recognized that in the case of the vanishing of the separation factor (thus degenerating to the Kepler case), we shall have

$$r_1 \to r, \quad r_2 \to r \quad \text{so that} \quad \rho_1 \to r, \quad \rho_2 \to 0 \tag{0.5}$$

and clearly the coordinate system itself becomes degenerate. This feature immediately precludes the possibility of any solution representation that would allow inspection of the form taken by the solution in the degenerate case.

Accordingly, we here formulate the Euler and Vinti problems in a coordinate system that we term the spheroidal coordinate system and which is clearly recognizable as a deformation of the standard spherical coordinate system. It is then necessary to transform to the EL coordinate system — or, in this case, a variant thereof — in order to effect the separation of the first integrals. But once the first integrals have been established, we revert to the spheroidal system, where the reduction of the second integration to a complete solution can be effected following clearly the pattern set in the Kepler case. Transforming the reduced (first integral) equations to a form amenable to solution in terms of Jacobian elliptic functions (Legendre canonical form) is then a strictly algebraic problem — though, in the general case, it is quite a challenging exercise.

We have seen that in the middle of the nineteenth century the Euler problem had the attention of the two great masters of both analytic mechanics and elliptic functions, namely Jacobi and Liouville. There is an irony in the fact that neither of the two pursued the analysis to the second integration to express the solution in terms of the functions named for the former.

Finally, in order to focus attention on the essential points of the procedure in the present work, we confine our analysis exclusively to the case of negative energy, leading to bound orbits. The necessary adjustment for the cases of zero and positive energy would follow the pattern long established in the Kepler problem. *The present derivation is confined to negative energy.*

Outline of the Present Work

The present work takes the following form:

- Following a brief outline of Lagrangian mechanics in Chapter 1, we give in Chapter 2 an analysis of the Kepler problem along the lines indicated with a view to having the same reduction procedure applied to the Euler and Vinti problems.

- Chapter 3, which may be considered the central chapter of the work, gives a full analysis of the planar Euler problem, yielding an exact solution form in terms of Jacobian elliptic functions complemented by the “time-angle” relation. The solution is a clear generalization of the form of the solution in the Kepler case.

- Chapter 4 deals with the Euler problem in the three-dimensional context. It is shown that, for two of the coordinates, the solution can by algebraic manipulation be reduced to the planar case. The formula for the third coordinate (longitude) involving an elliptic integral of the third kind is shown to be amenable to an approximation procedure.

- In Chapter 5, there is carried out a corresponding analysis and reduction of the Vinti problem.

- Chapter 6 deals with certain orbits of the Vinti problem that require or merit special attention.

In summary, once the essential features in the solution procedure for the Kepler problem have been taken aboard, then, in the spheroidal coordinate systems, the solutions for the Euler and Vinti problems, after the necessary digression to achieve separation of the first integrals, can be derived in an identical manner. These solution forms, expressed in terms of Jacobian elliptic functions, besides exhibiting the orbit form, have the satisfying feature of exhibiting the Kepler solution in terms of trigonometric functions as the degenerate case.

1

Lagrangian Mechanics

> Lagrange has perhaps done more than any other analyst to give extent and harmony to such deductive researches, by showing that the most varied consequences respecting the motions of systems of bodies may be derived from one radical formula; the beauty of the method so suiting the dignity of the results, as to make of his great work a kind of scientific poem.
>
> — *William Rowan Hamilton* [H1, (a) p. 247]

1 Lagrangian Systems

We take as the basis of our analysis the Lagrangian formulation of Dynamical Systems. When the energy of the system is resolved into its kinetic and potential components with the former denoted by T and the latter by V, then the Lagrangian function is defined as the difference

$$L = T - V \tag{1.1}$$

and the dynamical system is characterized by the minimization of the integral of the Lagrangian function

$$\delta \int L(q_i, \dot{q}_i, t)\, \mathrm{d}t = 0 \tag{1.2}$$

where the Lagrangian is a function of the generalized position coordinates q_i, $i = 1, \ldots, n$, and of the corresponding generalized velocity/momentum components $\dot{q}_i$, $i = 1, \ldots, n$, as well as of the time t.

In the class of problems with which we shall be concerned, the Lagrangian does not involve the time explicitly, so that the variational characterization has the form

$$\delta \int L(q_i, \dot{q}_i)\, \mathrm{d}t = 0 \tag{1.3}$$

which yields the system of variational equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \tag{1.4}$$

termed the Lagrange equations of the system.

It may be immediately noted that

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$$\frac{dL}{dt} = \sum_{i=1}^{n}\left[\frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right] \tag{1.5}$$

and if we use the Lagrange equations (1.4) to substitute for $\frac{\partial L}{\partial q_i}$, we find

$$\frac{dL}{dt} = \sum_{i=1}^{n}\left[\dot{q}_i\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right] = \sum_{i=1}^{n}\frac{d}{dt}\left[\dot{q}_i\frac{\partial L}{\partial \dot{q}_i}\right] \tag{1.6}$$

which, on integration, yields

$$\sum_{i=1}^{n}\dot{q}_i\frac{\partial L}{\partial \dot{q}_i} - L = \mathcal{E} \tag{1.7}$$

where $\mathcal{E}$ is the constant of integration. In the case where the potential energy V is independent of the velocity components, we have

$$T = T(q_i, \dot{q}_i), \quad V = V(q_i) \tag{1.8}$$

and there follows that

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} \tag{1.9}$$

and equation (1.7) may be written

$$\sum_{i=1}^{n}\dot{q}_i\frac{\partial T}{\partial \dot{q}_i} - (T - V) = \mathcal{E}\,. \tag{1.10}$$

In all cases under consideration in this work, the kinetic energy T is a homogeneous function of degree 2 in the velocity coordinates without cross-terms, so that

$$\sum_{i=1}^{n}\dot{q}_i\frac{\partial T}{\partial \dot{q}_i} = 2T \tag{1.11}$$

and hence relation (1.10) reads

$$T + V = \mathcal{E} \tag{1.12}$$

and the constant of integration $\mathcal{E}$ measures the total energy of the system. The above integral (1.12) is the energy integral.

2 Ignorable Coordinates

The case when a particular coordinate, which we take to be q_n, does not appear explicitly in the Lagrangian is worthy of special attention. In such a case, we let the Lagrangian be denoted by L^*, and the nth Lagrange equation reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L^*}{\partial \dot{q}_n}\right) = \frac{\partial L^*}{\partial q_n} = 0 \tag{2.1}$$

so that, with c_n denoting the constant of integration, there follows

$$\frac{\partial L^*}{\partial \dot{q}_n} = c_n\,. \tag{2.2}$$

This latter equation can be used to solve for $\dot{q}_n$ in terms of the remaining coordinates, and we may write

$$\dot{q}_n = \dot{q}_n(q_1, \ldots, q_{n-1}, \dot{q}_1, \ldots, \dot{q}_{n-1}, c_n)\,. \tag{2.3}$$

We now consider the variation of the Lagrangian, which takes the form

$$\delta L^* = \sum_{k=1}^{n-1} \frac{\partial L^*}{\partial q_k}\delta q_k + \sum_{k=1}^{n-1} \frac{\partial L^*}{\partial \dot{q}_k}\delta \dot{q}_k + \frac{\partial L^*}{\partial \dot{q}_n}\delta \dot{q}_n \tag{2.4}$$

which can be rearranged as

$$\begin{aligned}\sum_{k=1}^{n-1} \frac{\partial L^*}{\partial q_k}\delta q_k + \sum_{k=1}^{n-1} \frac{\partial L^*}{\partial \dot{q}_k}\delta \dot{q}_k &= \delta L^* - \frac{\partial L^*}{\partial \dot{q}_n}\delta \dot{q}_n \\ &= \delta\left[L^* - \dot{q}_n \frac{\partial L^*}{\partial \dot{q}_n}\right] + \dot{q}_n \delta\left(\frac{\partial L^*}{\partial \dot{q}_n}\right).\end{aligned} \tag{2.5}$$

We may now introduce the solution (2.3) for $\dot{q}_n$ into the factor in square brackets on the right of (2.5) to eliminate $\dot{q}_n$ from this expression. The resulting modified Lagrangian is a function of the quantities $q_1, \ldots, q_{n-1}, \dot{q}_1, \ldots,$ $\dot{q}_{n-1}$, c_n; denoting this modified Lagrangian by $L(q_1, \ldots, q_{n-1}, \dot{q}_1, \ldots, \dot{q}_{n-1}, c_n)$ we recall (2.2), and a rearrangement of the right-hand side leads to the replacement of (2.5) by

$$\begin{aligned}\sum_{k=1}^{n-1} \frac{\partial L^*}{\partial q_k}\delta q_k + \sum_{k=1}^{n-1} \frac{\partial L^*}{\partial \dot{q}_k}\delta \dot{q}_k &= \delta L + \dot{q}_n \delta c_n \\ &= \sum_{k=1}^{n-1}\left[\frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k\right] + \frac{\partial L}{\partial c_n}\delta c_n + \dot{q}_n \delta c_n\,.\end{aligned} \tag{2.6}$$

As this relation must hold for arbitrary variations, it follows that

$$\frac{\partial L}{\partial q_k} = \frac{\partial L^*}{\partial q_k}, \quad \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L^*}{\partial \dot{q}_k}, \quad k = 1, \ldots, n-1, \quad \frac{\partial L}{\partial c_n} = -\dot{q}_n\,. \tag{2.7}$$

Hence the problem is reduced to a Lagrangian of $(n-1)$ position and velocity variables, leading to the $(n-1)$ Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}, \quad k = 1, \ldots, (n-1) \tag{2.8a}$$

complemented by the equation for the nth velocity component

$$\dot{q}_n = -\frac{\partial L}{\partial c_n}\,. \tag{2.8b}$$

This procedure, outlined above for one ignorable coordinate, can be extended to any number of such coordinates.

3 Separable Systems

Our focus will be on integrable systems, and the most readily integrable class includes those characterized as separable. In such cases each of the energy functions (kinetic and potential) is the sum of distinct components, where every component of the potential energy involves but one position coordinate, and where every component of the kinetic energy involves but one position coordinate together with the square of the associated velocity coordinate.

Accordingly the separable system may be characterized by the following forms for the kinetic and potential energies:

$$T = \sum_{k=1}^{n} \tfrac{1}{2}\overline{u}_k(\overline{q}_k)\cdot\dot{\overline{q}}_k^2, \qquad V = \sum_{k=1}^{n} \overline{v}_k(\overline{q}_k) \tag{3.1a,b}$$

where we have used the "barred" notation on the variables later to be transformed. There follows

$$\frac{\partial L}{\partial \dot{\overline{q}}_k} = \overline{u}_k(\overline{q}_k)\cdot\dot{\overline{q}}_k, \qquad \frac{\partial L}{\partial \overline{q}_k} = \tfrac{1}{2}\overline{u}_k'(\overline{q}_k)\cdot\dot{\overline{q}}_k^2 - \overline{v}_k'(\overline{q}_k) \tag{3.2a,b}$$

and hence, from (3.2a),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\overline{q}}_k}\right) = \frac{\mathrm{d}}{\mathrm{d}t}[\overline{u}_k(\overline{q}_k)\cdot\dot{\overline{q}}_k] = \overline{u}_k(\overline{q}_k)\cdot\ddot{\overline{q}}_k + \overline{u}_k'(\overline{q}_k)\cdot\dot{\overline{q}}_k^2\,. \tag{3.3}$$

Combining (3.2b) with (3.3), we have for the Lagrange equations

$$\overline{u}_k(\overline{q}_k)\cdot\ddot{\overline{q}}_k + \overline{u}_k'(\overline{q}_k)\cdot\dot{\overline{q}}_k^2 = \tfrac{1}{2}\overline{u}_k'(\overline{q}_k)\cdot\dot{\overline{q}}_k^2 - \overline{v}_k'(\overline{q}_k) \tag{3.4}$$

or, after a rearrangement,

$$\overline{u}_k(\overline{q}_k)\cdot\ddot{\overline{q}}_k + \tfrac{1}{2}\overline{u}_k'(\overline{q}_k)\cdot\dot{\overline{q}}_k^2 + \overline{v}_k'(\overline{q}_k) = 0\,. \tag{3.5}$$

If we multiply across by $\dot{\overline{q}}_k$, we have

$$\overline{u}_k\cdot\dot{\overline{q}}_k\cdot\ddot{\overline{q}}_k + \tfrac{1}{2}\overline{u}_k'(\overline{q}_k)\cdot\dot{\overline{q}}_k^3 + \overline{v}_k'(\overline{q}_k)\cdot\dot{\overline{q}}_k = 0 \tag{3.6}$$

which may be rewritten

$$\frac{\mathrm{d}}{\mathrm{d}t}[\tfrac{1}{2}\overline{u}_k(\overline{q}_k)\cdot\dot{\overline{q}}_k^2 + \overline{v}_k(\overline{q}_k)] = 0 \tag{3.7}$$

which integrates to yield

$$\tfrac{1}{2}\overline{u}_k(\overline{q}_k)\cdot\dot{\overline{q}}_k^2+\overline{v}_k(\overline{q}_k)=c_k,\quad k=1,\ldots,n \tag{3.8}$$

where the c_k are the constants of integration. If we sum this system of equations over the indices $k=1,\ldots,n$, we obtain, recalling (3.1),

$$T+V=\sum_{k=1}^{n}c_k \tag{3.9}$$

so that the constants of integration c_k are subject to the constraint

$$\sum_{k=1}^{n}c_k=\mathcal{E} \tag{3.10}$$

implied by the energy integral (1.12).

Returning to equations (3.8), we may rearrange as follows:

$$\dot{\overline{q}}_k=\sqrt{\frac{2[c_k-\overline{v}_k(\overline{q}_k)]}{\overline{u}_k(\overline{q}_k)}}\,. \tag{3.11}$$

This leads to the procedure of integration when we rewrite it in the form

$$\mathrm{d}t=\sqrt{\frac{\overline{u}_k(\overline{q}_k)}{2[c_k-\overline{v}_k(\overline{q}_k)]}}\,\mathrm{d}\overline{q}_k\,. \tag{3.12}$$

This form of the equation suggests the change of variable

$$\mathrm{d}q_k=\sqrt{\overline{u}_k(\overline{q}_k)}\,\mathrm{d}\overline{q}_k\quad\text{or}\quad\dot{q}_k=\sqrt{\overline{u}_k(\overline{q}_k)}\,\dot{\overline{q}}_k \tag{3.13a,b}$$

and, if we further write

$$v_k(q_k)=\overline{v}_k(\overline{q}_k) \tag{3.14}$$

then (3.12) reads

$$\mathrm{d}t=\frac{\mathrm{d}q_k}{\sqrt{2[c_k-v_k(q_k)]}}\quad\text{or}\quad t=\int\frac{\mathrm{d}q_k}{\sqrt{2[c_k-v_k(q_k)]}}\,. \tag{3.15a,b}$$

If the variables $\overline{q}_k$ in relations (3.1) are replaced by the transformed variables q_k, it is evident that we may write

$$T=\sum_{k=1}^{n}\tfrac{1}{2}\dot{q}_k^2,\qquad V=\sum_{k=1}^{n}v_k(q_k)\,. \tag{3.16a,b}$$

Hence, for a separable system, there is no loss of generality in taking the energy in the forms (3.16) rather than in the forms (3.1).

4 Liouville Systems

It was observed by Liouville[1] that the class of separable systems of the form (3.1) can be extended by a further generalization.

Having noted that there is no loss of generality in replacing a system of the form (3.1) by one of the form (3.16), we now consider the system where

$$T = Q(q_1,\dots,q_n)\sum_{k=1}^{n}\tfrac{1}{2}\dot{q}_k^2, \qquad V = V(q_1,\dots,q_n) \tag{4.1a,b}$$

and the Lagrange equations have the form

$$\frac{\mathrm{d}}{\mathrm{d}t}[Q(q)\dot{q}_k] = \frac{\partial Q}{\partial q_k}\sum_{k=1}^{n}\tfrac{1}{2}\dot{q}_k^2 - \frac{\partial V}{\partial q_k}, \quad k = 1,\dots,n\,. \tag{4.2}$$

If we multiply across by $2Q(q)\dot{q}_k$, we have

$$2Q\dot{q}_k\frac{\mathrm{d}}{\mathrm{d}t}[Q\dot{q}_k] = Q\left(\sum_{k=1}^{n}\dot{q}_k^2\right)\frac{\partial Q}{\partial q_k}\dot{q}_k - 2Q\dot{q}_k\frac{\partial V}{\partial q_k} \tag{4.3a}$$

$$\text{and, using (4.1a),} \quad = 2T\frac{\partial Q}{\partial q_k}\dot{q}_k - 2Q\dot{q}_k\frac{\partial V}{\partial q_k} \tag{4.3b}$$

$$\text{and, applying (1.12),} \quad = 2(\mathcal{E} - V)\frac{\partial Q}{\partial q_k}\dot{q}_k - 2Q\dot{q}_k\frac{\partial V}{\partial q_k} \tag{4.3c}$$

so that, on rearrangement, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[Q\dot{q}_k]^2 = 2\mathcal{E}\frac{\partial Q}{\partial q_k}\dot{q}_k - 2\dot{q}_k\frac{\partial(QV)}{\partial q_k}\,. \tag{4.4}$$

It is evident that for the integrability of equation (4.4), it suffices that both Q and QV be expressible in separated form, namely,

$$Q(q_1,\dots,q_n) = \sum_{k=1}^{n} Q_k(q_k), \quad QV = \sum_{k=1}^{n} v_k(q_k)\,. \tag{4.5}$$

Then equation (4.4) can be replaced by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[Q\dot{q}_k]^2 &= 2\mathcal{E}\frac{\mathrm{d}Q_k}{\mathrm{d}q_k}\dot{q}_k - 2\frac{\mathrm{d}v_k}{\mathrm{d}q_k}\dot{q}_k \\ &= 2\frac{\mathrm{d}}{\mathrm{d}t}[\mathcal{E}Q_k - v_k]\end{aligned} \tag{4.6}$$

which, on integration, yields

$$\tfrac{1}{2}Q^2\dot{q}_k^2 - \mathcal{E}Q_k + v_k = c_k, \quad k = 1,\dots,n \tag{4.7}$$

[1] J. Liouville, *Journal de Mathématiques*, XIV (1849), p. 257. Actually, Liouville's result is far more general — see General Introduction and note reference [L8].

where the c_k are the constants of integration.

If we sum equations (4.7) over $k = 1, \ldots, n$, we have

$$\begin{aligned}\sum_{k=1}^{n} c_k &= Q^2 \sum_{k=1}^{n} \tfrac{1}{2}\dot{q}_k^2 - \mathcal{E}Q + QV \\ &= QT - \mathcal{E}Q + QV = Q(T + V - \mathcal{E}) = 0\end{aligned} \tag{4.8}$$

so that the energy integral (1.12) requires that the relation

$$\sum_{k=1}^{n} c_k = 0$$

must hold among the constants of the integrals (4.7).

Systems of the type (4.1) where Q and QV can be written in the form (4.5) we term Liouville systems. When one comes to apply the procedure in specific cases, the motivation is generally quite clear.

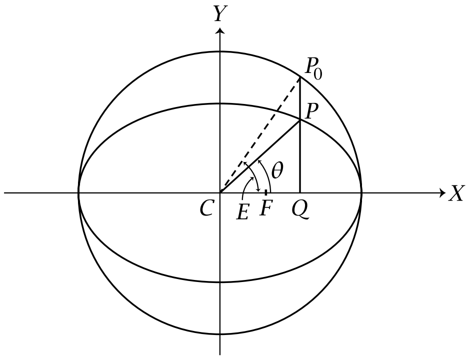
2

The Kepler Problem

For the Newtonian $1/r^2$ force law, a miracle occurs — all of the solutions are periodic instead of just quasi-periodic. To put it another way, the two-dimensional tori are further decomposed into invariant circles. This highly degenerate situation seems unbelievable from the point of view of general theory, yet it is the most interesting feature of the problem.

— *Richard Moeckel,* Bull. AMS., **41**:1 (2003), pp. 121–2. Review of *Classical and Celestial Mechanics, the Recife Lectures,* Cabral and Diacu (eds.), Princeton University Press, 2002.

A necessary preliminary to a full understanding of the Kepler problem is a full familiarity with the geometric and analytic features of the conics — particularly those of the ellipse.



1 Features of the Ellipse: Geometry and Analysis

Placing the origin at the center C, with X- and Y-coordinate axes coinciding respectively with the major and minor axes of the ellipse, then in terms of these Cartesian coordinates, the equation of the ellipse reads

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1 \tag{1.1}$$

where a and b measure the semimajor and semiminor axes, respectively. The equation can be characterized parametrically in the form

D. Ó Mathúna, *Integrable Systems in Celestial Mechanics*, doi: 10.1007/978-0-8176-4595-3_2,
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$$X = a\cos E, \quad Y = b\sin E. \tag{1.2}$$

The line PQ normal to the major axis through an arbitrary point $P(X,Y)$ meets the circumscribed circle at $P_0(X, Y_0)$. With E denoting the angle subtended at the center C between CP_0 and CQ, the interpretation of (1.2) is clear, and furthermore we see that

$$CQ = a\cos E, \quad P_0Q = a\sin E, \quad PQ = b\sin E. \tag{1.3}$$

For the radius vector $CP = R$ from the center to the arbitrary point $P(X,Y)$ of the ellipse, we have

$$R^2 = X^2 + Y^2 = a^2 - (a^2 - b^2)\sin^2 E. \tag{1.4}$$

The eccentricity e of the ellipse may be defined by

$$b^2 = a^2(1 - e^2) \tag{1.5}$$

so that for (1.4), we may write

$$R^2 = a^2[1 - e^2\sin^2 E] \tag{1.6a}$$

or

$$R = a[1 - e^2\sin^2 E]^{1/2} \tag{1.6b}$$

as the equation for the ellipse in terms of the "eccentric angle" E.

For the corresponding equation in terms of center-based polar coordinates (R, θ), we note

$$X = R\cos\theta, \quad Y = R\sin\theta \tag{1.7}$$

and equation (1.1) becomes

$$R^2\left[\frac{b^2}{a^2}\cos^2\theta + \sin^2\theta\right] = b^2 \tag{1.8}$$

which, on the introduction of (1.5) yields

$$R^2[1 - e^2\cos^2\theta] = a^2(1 - e^2) \tag{1.9a}$$

$$R = \frac{a\sqrt{1 - e^2}}{[1 - e^2\cos^2\theta]^{1/2}} \tag{1.9b}$$

as the required equation.

The point $F(ae, 0)$ is a focus of the ellipse. Moving the origin to the focus through the translation

$$x = X - ae, \quad y = Y \tag{1.10}$$

the Cartesian equation (1.1) becomes

$$\frac{(x + ae)^2}{a^2} + \frac{y^2}{b^2} = 1\,. \tag{1.11}$$

Substituting from (1.5) into (1.11) and rearranging yields

$$x^2 + y^2 = [a(1 - e^2) - ex]^2\,. \tag{1.12}$$

We now introduce polar coordinates (r, f) centered at the focus so that

$$x = r\cos f, \quad y = r\sin f \tag{1.13}$$

and relation (1.12) may be written

$$r = a(1 - e^2) - er\cos f = e\left[\frac{a(1 - e^2)}{e} - r\cos f\right]. \tag{1.14}$$

If we consider the line $x = a(1 - e^2)/e$ (parallel to the y-axis), which we call the directrix, then the factor in square brackets on the right of (1.14) measures the distance from an arbitrary point on the ellipse to the directrix. Hence equation (1.14) merely states that for an arbitrary point on the curve, the ratio of the distance from the focus to the distance from the directrix is given by the eccentricity e. This, in fact, can be taken as the general definition of a conic, which for $e < 1$ is an ellipse, whereas for $e > 1$ it is a hyperbola. Returning to (1.14), we note that it can be put in the neater — and possibly more recognizable — form

$$r = \frac{a(1 - e^2)}{1 + e\cos f} \tag{1.15}$$

which, with $e < 1$, we take as the standard equation for the ellipse.

For the corresponding relation in terms of the "eccentric angle" E,

$$\begin{aligned} r^2 = x^2 + y^2 &= (X - ae)^2 + Y^2 \\ &= X^2 + Y^2 - 2aeX + a^2e^2 \\ &= R^2 - 2a^2e\cos E + a^2e^2\,. \end{aligned} \tag{1.16}$$

Introducing R from (1.6) into (1.16) yields

$$r^2 = a^2[1 - e^2\sin^2 E - 2e\cos E + e^2] = a^2[1 - e\cos E]^2 \tag{1.17}$$

so that

$$r = a[1 - e\cos E] \tag{1.18}$$

as the sought-for relation.

For the ellipse, therefore, we note the following:

1. Equation (1.6) relates the center-based radius vector R at the point P to the angle-parameter E, being the angle subtended at the center between the major axis and the radius to the point where the normal to the major axis through P meets the circumscribed circle.

2. Equation (1.9) gives the equation of the ellipse in terms of the center-based polar coordinates (R, θ).
3. Equation (1.15) gives the equation of the ellipse in terms of the focus-based polar coordinates (r, f).
4. Equation (1.18) relates the radial coordinate r of the focus-based system (r, f) of item 3 above to the angle parameter E referred to in item 1 above. The attractive simplicity of (1.18) must be balanced against its mixed nature, involving coordinate systems of different origins.

A straightforward exercise yields the relation between the angles E and f. Since

$$x = r\cos f = r\left[2\cos^2\frac{f}{2} - 1\right] = r\left[1 - 2\sin^2\frac{f}{2}\right] \tag{1.19}$$

we have

(1) $$\begin{aligned} 2r\cos^2\frac{f}{2} &= r + x = r + X - ae = a(1 - e\cos E) + a\cos E - ae \\ &= a(1-e)[1+\cos E] = 2a(1-e)\cos^2\frac{E}{2} \end{aligned} \tag{1.20}$$

and hence

$$r\cos^2\frac{f}{2} = a(1-e)\cos^2\frac{E}{2}. \tag{1.21}$$

(2) $$\begin{aligned} 2r\sin^2\frac{f}{2} &= r - x = r - X + ae = a(1 - e\cos E) - a\cos E + ae \\ &= a(1+e)[1-\cos E] = 2a(1+e)\sin^2\frac{E}{2} \end{aligned}$$

and hence

$$r\sin^2\frac{f}{2} = a(1+e)\sin^2\frac{E}{2}. \tag{1.22}$$

Dividing (1.22) by (1.21) yields

$$\tan^2\frac{f}{2} = \frac{1+e}{1-e}\tan^2\frac{E}{2}, \qquad \tan^2\frac{E}{2} = \frac{1-e}{1+e}\tan^2\frac{f}{2}. \tag{1.23a,b}$$

This latter relation can now be used to derive the equation for R in terms of f, but its algebraic complexity limits its utility.

Returning to the standard equation (1.15), we see that (with prime denoting differentiation with respect to f)

$$r' = \frac{\mathrm{d}r}{\mathrm{d}f} = \frac{ae(1-e^2)\sin f}{(1+e\cos f)^2}. \tag{1.24}$$

Hence $r' = 0$ for $f = 0, \pm\pi, \ldots, \pm n\pi$. It can be easily checked that $f = 0$ is a minimum point for r (as also are $f = \pm 2n\pi$) while $f = \pi$ (as well as

$f = \pm(2n+1)\pi$) is the maximum point for r. The point $f = 0$, at which $r = a(1-e)$, we shall call the *pericenter*; the point $f = \pi$, at which $r = a(1+e)$, we shall call the *apocenter*.

At the extremity of the semiminor axis, we have

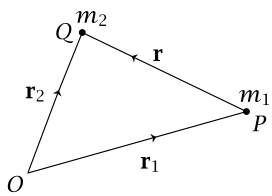
$$x = -ae, \quad y = b = a\sqrt{1-e^2} \tag{1.25}$$

from which it follows that, at that extremity,

$$r = a, \quad \cos f = -e \tag{1.26}$$

and hence we have that $[a, \arccos e]$ are the focus-based polar coordinates of the extremity of the positive semiminor axis.

2 The Two-Body Problem



We consider the motion of two bodies moving under the influence of their mutual attraction. Denoting the masses of the two bodies by m_1 and m_2, with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, referred to the origin at 0, we write

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \,. \tag{2.1}$$

In accordance with the inverse square law governing the gravitational attraction of m_1 and m_2, the equations of motion for m_1 and m_2 are given respectively by

$$m_1\ddot{\mathbf{r}}_1 = \frac{Gm_1m_2}{r^2}\mathbf{e}_r = \frac{Gm_1m_2}{r^3}\mathbf{r}, \qquad \text{and hence} \quad \ddot{\mathbf{r}}_1 = \frac{Gm_2}{r^3}\mathbf{r} \tag{2.2a}$$

$$m_2\ddot{\mathbf{r}}_2 = -\frac{Gm_1m_2}{r^2}\mathbf{e}_r = -\frac{Gm_1m_2}{r^3}\mathbf{r}, \qquad \text{and hence} \quad \ddot{\mathbf{r}}_2 = -\frac{Gm_1}{r^3}\mathbf{r} \tag{2.2b}$$

where we have used the “dot” to denote differentiation with respect to time t, and where the unit vector $\mathbf{e}_r$ is defined by $\mathbf{r} = |\mathbf{r}|\,\mathbf{e}_r = r\mathbf{e}_r$. Subtracting (2.2a) from (2.2b), we have

$$\ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1 = \ddot{\mathbf{r}} = -\frac{G(m_1+m_2)}{r^3}\mathbf{r} \tag{2.3}$$

and as the equation is unaltered by the replacement of $\mathbf{r}$ by $-\mathbf{r}$, or by the interchange of m_1 and m_2, equation (2.3) describes the motion of either body relative to the other. Moreover, equation (2.3) shows that the problem has been reduced to that of the motion of a particle of unit mass in the gravitational field of a body of mass m, situated at the origin, where

$$m = m_1 + m_2 \tag{2.4}$$

and if we set

$$\mu = G(m_1 + m_2) = Gm \tag{2.5}$$

then equation (2.3) reads

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} = -\frac{\mu}{r^2}\mathbf{e}_r \tag{2.6}$$

which is the standard form.

In the case of planetary motion, one may think of m_1 as the Sun and m_2 as the planet. In that case, we may write

$$m = m_1 + m_2 = m_1\left(1 + \frac{m_2}{m_1}\right) \tag{2.7}$$

and (2.6) describes the motion of the planet in the heliocentric coordinate system. We may also note that the dominance of the mass of the Sun would permit the approximation

$$m \approx m_1, \quad \mu \approx Gm_1 \tag{2.8}$$

when such an approximation is appropriate.

At this point, we introduce the gravitational potential. At an arbitrary point P in the gravitational fields of a mass m at Q, the function U defined by

$$U = \frac{Gm}{|PQ|} = \frac{Gm}{r} = \frac{\mu}{r} \tag{2.9}$$

is the potential per unit mass: it has the feature that the force defined by the gradient of this function U is in fact the Newtonian gravitational force acting on a particle of unit mass, namely

$$F = \nabla U = -\frac{Gm}{r^2}\mathbf{e}_r = -\frac{Gm}{r^3}\mathbf{r} = -\frac{\mu}{r^3}\mathbf{r} \tag{2.10}$$

so that, for the equation of motion of a particle P of unit mass, we have

$$\ddot{\mathbf{r}} = -\frac{Gm}{r^3}\mathbf{r} = -\frac{\mu}{r^3}\mathbf{r} \tag{2.11}$$

identical with (2.6).

In case of several masses m_i, $i = 1, \ldots, n$, situated respectively at Q_i, $i = 1, \ldots, n$, the potential function per unit mass at P is given by

$$U = \sum_{i=1}^{n} \frac{Gm_i}{|PQ_i|} \tag{2.12}$$

to which we shall have occasion to refer later.

In the next section when we encounter the conservation of energy, we shall see that the potential energy V per unit mass for a particle in the gravitational field of a mass m is given by

$$V = -\frac{Gm}{r} = -\frac{\mu}{r} = -U \tag{2.13}$$

so that the potential function is the negative of the potential energy.

The problem defined by the differential equations (2.6) with μ given by (2.5) is known as the *Kepler problem*.

3 The Kepler Problem: Vectorial Treatment

In the class of problems in Celestial Mechanics, the Kepler problem is distinguished by several features: it has every possible "degeneracy" — the "frequencies" associated with all three coordinates coincide so that all bound orbits are periodic (except for collision orbits); but more relevant at this point is the fact that the motion is always planar. This means that it admits a vectorial treatment to which other problems are not amenable.

In terms of a (heliocentric) spherical coordinate system (r, θ, φ) with unit base vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_\varphi$, it follows from

$$\mathbf{r} = r\mathbf{e}_r \tag{3.1}$$

that the velocity vector $\mathbf{v}$ is given by

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + r\sin\theta \cdot \dot{\varphi}\mathbf{e}_\varphi \tag{3.2}$$

where again the dot denotes differentiation with respect to time; there follows

$$\mathbf{r} \cdot \dot{\mathbf{r}} = r\dot{r} \tag{3.3a}$$

$$v^2 = \mathbf{v} \cdot \mathbf{v} = \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\,. \tag{3.3b}$$

We note that the fundamental equation (2.6) admits an immediate first integral — which we shall recognize as the energy integral. Taking the scalar product of (2.6) with the velocity vector $\dot{\mathbf{r}}$, we find

$$\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \cdot \dot{\mathbf{r}} = -\frac{\mu}{2r^3}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \cdot \mathbf{r}) = -\frac{\mu}{2r^3}\frac{\mathrm{d}}{\mathrm{d}t}(r^2) = -\frac{\mu}{r^2}\dot{r} \tag{3.4}$$

and so

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu}{r}\right) \tag{3.5}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\tfrac{1}{2}v^2 - \frac{\mu}{r}\right] = 0\,. \tag{3.6}$$

Letting $\mathcal{E}$ denote the constant of integration, we therefore have the energy integral in the form

$$\tfrac{1}{2}v^2 - \frac{\mu}{r} = \mathcal{E}\,. \tag{3.7}$$

For a particle of unit mass, the first term is clearly the kinetic energy and the second term is the potential energy; accordingly, if we use T to denote the kinetic and V the potential energy, then

$$T = \tfrac{1}{2}v^2, \qquad V = -\frac{\mu}{r}, \qquad T + V = \mathcal{E} \tag{3.8a,b,c}$$

and the definition of V is consistent with (2.13).

Rewriting (3.7) in the form

$$\tfrac{1}{2}v^2 = \mathcal{E} + \frac{\mu}{r} \tag{3.9}$$

and noting that the left-hand side is always positive, then if $\mathcal{E}$ is negative, relation (3.9) sets the lower limit on μ/r: if we exhibit the case of negative energy by writing

$$\mathcal{E} = -\alpha^2 \tag{3.10}$$

and define a length scale a by setting

$$a = \frac{\mu}{2\alpha^2} \tag{3.11}$$

then we have that

$$\frac{\mu}{r} - \alpha^2 \geq 0 \quad \text{implying} \quad \frac{\mu}{r} \geq \alpha^2 \tag{3.12}$$

and hence

$$r \leq \frac{\mu}{\alpha^2} = 2a \tag{3.13}$$

giving the corresponding upper limit on r: negative energy implies bound orbits, and these shall be the main focus of our attention.

Returning to relations (3.1) and (3.2) we form the angular momentum vector $\mathbf{C}$ by taking the cross product of $\mathbf{r}$ and $\mathbf{v}$, to find

$$\mathbf{C} = \mathbf{r} \times \mathbf{v} = \mathbf{r} \times \dot{\mathbf{r}} = -r^2 \sin\theta\dot{\varphi}\mathbf{e}_\theta + r^2\dot{\theta}\mathbf{e}_\varphi \tag{3.14}$$

and we further note that

$$\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times \dot{\mathbf{r}}) = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = 0 - \mathbf{r} \times \frac{\mu}{r^3}\mathbf{r} = 0\,. \tag{3.15}$$

Hence in the central gravitational field, the angular momentum vector $\mathbf{C}$ is constant. At this point, we observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{e}_r) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right) = \frac{r\dot{\mathbf{r}} - \dot{r}\mathbf{r}}{r^2} = \frac{r^2\dot{\mathbf{r}} - r\dot{r}\mathbf{r}}{r^3}$$
$$= \frac{(\mathbf{r}\cdot\mathbf{r})\dot{\mathbf{r}} - (\mathbf{r}\cdot\dot{\mathbf{r}})\mathbf{r}}{r^3} = \frac{\mathbf{r}\times(\dot{\mathbf{r}}\times\mathbf{r})}{r^3} = -\frac{\mathbf{r}\times\mathbf{C}}{r^3} = \frac{\mathbf{C}\times\mathbf{r}}{r^3}. \quad (3.16)$$

When $\mathbf{C} = 0$, the above relation implies that, in that case, the unit vector $\mathbf{e}_r$ is constant — hence the motion is rectilinear along the radius vector toward the origin, leading to collision. When $\mathbf{C} \neq 0$, it follows from (3.14) that

$$\mathbf{r}\cdot\mathbf{C} = \mathbf{r}\cdot(\mathbf{r}\times\dot{\mathbf{r}}) = 0 \quad (3.17)$$

so that $\mathbf{r}$ remains normal to the fixed vector $\mathbf{C}$; hence the motion takes place in the plane defined by the fixed (constant) vector $\mathbf{C}$.

It further follows from (3.14) that

$$C^2 = \mathbf{C}\cdot\mathbf{C} = (\mathbf{r}\times\dot{\mathbf{r}})\cdot(\mathbf{r}\times\dot{\mathbf{r}}) = r^4\sin^2\theta\dot{\varphi}^2 + r^4\dot{\theta}^2$$
$$= r^2[r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2] = r^2[v^2 - \dot{r}^2] \quad (3.18)$$

and we have a second integral, this one involving the magnitude of the angular momentum vector $\mathbf{C}$, namely

$$r^2[v^2 - \dot{r}^2] = C^2. \quad (3.19)$$

Moreover, rewriting the latter as an expression for v^2, and recalling the energy integral (3.7), we have

$$\tfrac{1}{2}v^2 = \tfrac{1}{2}\left[\frac{C^2}{r^2} + \dot{r}^2\right] = \mathcal{E} + \frac{\mu}{r} \quad (3.20)$$

giving the relation between the constants C and $\mathcal{E}$.

Returning to (3.16) and again applying the gravitational equation (2.6) and also noting that $\dot{\mathbf{C}} = 0$, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{e}_r) = \frac{\mathbf{C}\times\mathbf{r}}{r^3} = -\frac{\mathbf{C}\times\ddot{\mathbf{r}}}{\mu} = -\frac{1}{\mu}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{C}\times\dot{\mathbf{r}}) = \frac{1}{\mu}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}\times\mathbf{C}). \quad (3.21)$$

If we let $\mathbf{e}$ denote the arbitrary constant vector introduced by the integration of this latter vector differential equation, we have

$$\mu(\mathbf{e}_r + \mathbf{e}) = \mathbf{v}\times\mathbf{C} = \dot{\mathbf{r}}\times(\mathbf{r}\times\dot{\mathbf{r}}) = (\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})\mathbf{r} - (\mathbf{r}\cdot\dot{\mathbf{r}})\dot{\mathbf{r}} = v^2\mathbf{r} - r\dot{r}\dot{\mathbf{r}}. \quad (3.22)$$

Again, we note in passing that if $\mathbf{C} = 0$, then $\mathbf{e} = -\mathbf{e}_r$, so that $\mathbf{e}$ is the unit vector along the radius vector toward the origin. For $\mathbf{C} \neq 0$, we take the scalar product with $\mathbf{C}$ across (3.22), and noting that $\mathbf{C}$ is normal to both $\mathbf{r}$ and $\dot{\mathbf{r}}$, we find

$$\mathbf{e}\cdot\mathbf{C} = 0 \quad (3.23)$$

which implies that the vector $\mathbf{e}$ lies in the plane of the motion.

Taking the scalar product with $\mathbf{r}$ across (3.22) gives

$$\mu(r + \mathbf{e} \cdot \mathbf{r}) = v^2 r^2 - r^2 \dot{r}^2 = r^2(v^2 - \dot{r}^2) = C^2 \tag{3.24}$$

wherein we have introduced (3.18); we now rewrite (3.24) in the form

$$\mathbf{e} \cdot \mathbf{e}_r = \frac{C^2}{\mu r} - 1\,. \tag{3.25}$$

If, in the plane of the motion, we let the vector $\mathbf{e}$, whose magnitude we denote by e, define a base axis and if we let f denote the angle in this plane between this base vector and the radius vector $\mathbf{r}$, then (r, f) constitute a polar coordinate basis in the plane of the motion, and equation (3.25) can be written in the form

$$r[1 + e \cos f] = \frac{C^2}{\mu}\,. \tag{3.26}$$

For $e = 0$, the motion is circular. For $e \neq 0$, we rewrite (3.26) [in accord with (1.14)] as

$$r = e\left[\frac{C^2}{e\mu} - r\cos f\right] \tag{3.27}$$

which [referring to equation (1.14) and the subsequent paragraph] defines a conic with a directrix at a distance $C^2/\mu e$ from the origin and with eccentricity e. And for $e < 1$, this conic is an ellipse, and the vector $\mathbf{e}$ is the vector based at the focus (origin) directed at the pericenter and with magnitude e.

The vector $\mathbf{e}$ is known as the *Runge–Lenz vector* and also the *eccentric axis vector.*

There is one more exercise to be performed on relation (3.22). We recall that since $\mathbf{C}$ is normal to $\mathbf{v}$, there follows that

$$|\mathbf{v} \times \mathbf{C}| = vC, \quad (\mathbf{v} \times \mathbf{C})^2 = v^2 C^2\,. \tag{3.28}$$

Accordingly, if we square both sides of (3.22), then on reversing the order we find

$$\begin{aligned} v^2 C^2 &= \mu^2(\mathbf{e} + \mathbf{e}_r)^2 = \mu^2[1 + e^2 + 2\mathbf{e} \cdot \mathbf{e}_r] \\ &= \mu^2\left[1 + e^2 + 2\left(\frac{C^2}{\mu r} - 1\right)\right] = \mu^2(e^2 - 1) + 2\mu\frac{C^2}{r} \end{aligned} \tag{3.29}$$

in which we have introduced (3.25) and rearranged. Hence

$$\mu^2(1 - e^2) = -2C^2\left[\tfrac{1}{2}v^2 - \frac{\mu}{r}\right] = -2C^2\mathcal{E} \tag{3.30}$$

from which it is immediately evident that

$$e \lesseqgtr 1 \quad \text{corresponds to} \quad \mathcal{E} \lesseqgtr 0 \tag{3.31}$$

i.e., negative/positive energy corresponds to elliptic/hyperbolic orbits — as anticipated earlier.

Restricting our attention to bound orbits (negative energy), we introduce (3.10) and (3.11) into (3.30), to obtain

$$1 - e^2 = \frac{2C^2}{\mu^2}\alpha^2 = \frac{C^2}{\mu} \Big/ \frac{\mu}{2\alpha^2} = \frac{C^2}{\mu} \cdot \frac{1}{a} \tag{3.32}$$

and hence

$$\frac{C^2}{\mu} = a(1 - e^2) = p \tag{3.33}$$

where we introduce the symbol p to denote the semi–latus rectum — the value of r at $f = \pi/2$. In terms of these length parameters, equation (3.26) reads

$$r = \frac{p}{1 + e\cos f} = \frac{a(1 - e^2)}{1 + e\cos f} \tag{3.34}$$

as an alternate form for the equation of the orbit, and we write

$$b = a\sqrt{1 - e^2} \tag{3.35}$$

as the length parameter of the semiminor axis.

The polar coordinates (r, f) in the orbit plane together with the axis normal to the plane constitute a cylindrical polar coordinate system. With base unit vectors $\mathbf{e}_r$ and $\mathbf{e}_f$ in the orbit plane together with the axial unit vector $\mathbf{e}_k$, we may write

$$\mathbf{r} = r\mathbf{e}_r \tag{3.36a}$$

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{f}\mathbf{e}_f \tag{3.36b}$$

and, for the angular momentum, we have

$$\mathbf{C} = \mathbf{r} \times \mathbf{v} = r^2\dot{f}\mathbf{e}_k\,. \tag{3.37}$$

It follows that, for the magnitude of the angular momentum, we have

$$r^2\dot{f} = C = \sqrt{\mu p} = \sqrt{\mu a(1 - e^2)} = \sqrt{\mu a}\sqrt{1 - e^2} \tag{3.38}$$

wherein we have introduced (3.33). If we let τ denote the time for a complete orbit and if we also introduce the mean motion n, measuring the frequency, by the relation

$$n = \frac{2\pi}{\tau} \tag{3.39}$$

and note that the area traced out in one orbit is πab, we have that the mean areal velocity over an orbit is given by

$$\frac{\pi ab}{\tau} = \pi ab \cdot \frac{n}{2\pi} = \tfrac{1}{2}nab = \tfrac{1}{2}na^2\sqrt{1 - e^2}\,. \tag{3.40}$$

However, the areal velocity is, in fact, given by one-half the angular momentum of (3.38). Identifying the quantity in (3.38) with twice the quantity in (3.40) gives, after cancellation of the common $\sqrt{1-e^2}$ factor,

$$na^2 = \sqrt{\mu a} \tag{3.41}$$

and hence the important relation

$$n^2a^3 = \mu = Gm = G(m_1 + m_2) \tag{3.42}$$

whence we have substituted for μ from (2.5).

We are now in a position to make some observations:

1. The motion takes place in a plane defined by the angular momentum vector, and for negative energy the orbit is the ellipse (3.34); this is Kepler's First Law.
2. The constancy of the angular momentum (3.38) implies a constant mean areal velocity; this is Kepler's Second Law.
3. If the approximation (2.8) were to be introduced into (3.42), we would have $n^2a^3 = Gm$, a constant for all planets; this is Kepler's Third Law, more usually stated as the square of the orbit period is proportional to the cube of the semimajor axis.

Recalling equation (3.20) for the case of negative energy so that $\mathcal{E} = -\alpha^2$, we rearrange to obtain

$$\begin{aligned} r^2\dot{r}^2 &= -[2\alpha^2r^2 - 2\mu r + C^2] \\ &= -2\alpha^2\left[r^2 - \frac{\mu r}{\alpha^2} + \frac{C^2}{2\alpha^2}\right]. \end{aligned} \tag{3.43}$$

The singularity at $r = 0$ in this differential equation can be regularized by means of a regularizing transformation whereby a new independent variable E is introduced through the defining relation

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\sqrt{2\alpha^2}}{r} \quad \text{so that} \quad r\frac{\mathrm{d}}{\mathrm{d}t} = \sqrt{2\alpha^2}\frac{\mathrm{d}}{\mathrm{d}E} \tag{3.44}$$

and, on the introduction of (3.44) and some rearrangement, equation (3.43) becomes

$$\begin{aligned} \left(\frac{\mathrm{d}r}{\mathrm{d}E}\right)^2 &= -\left[r^2 - \frac{\mu}{\alpha^2}r + \frac{C^2}{\mu}\cdot\frac{\mu}{2\alpha^2}\right] \\ &= -[r^2 - 2ar + a^2(1-e^2)] \\ &= -[(a-r)^2 - a^2e^2] \end{aligned} \tag{3.45}$$

where we have introduced (3.11) and (3.33). By means of the substitution $a - r = aeZ$, this immediately integrates, and we find

$$r = a[1 - e\cos E] \tag{3.46}$$

satisfying the condition that $E = 0$ when $r = a(1-e)$. Recalling relation (1.18), it is evident that E can be identified with the eccentric angle introduced in (1.2).

It remains to determine the relation between the angle E and the time t. From the defining relation (3.44), we have

$$\sqrt{2\alpha^2}\frac{\mathrm{d}t}{\mathrm{d}E} = r = a[1 - e\cos E] \tag{3.47}$$

so that, on integration

$$\sqrt{2\alpha^2}(t - t_0) = a[E - e\sin E] \tag{3.48}$$

satisfying the requirement that $E = 0$ when $t = t_0$. From (3.41), we note that

$$n^2a^2 = \frac{\mu}{a} = 2\alpha^2 \tag{3.49}$$

and hence

$$M = n(t - t_0) = E - e\sin E, \tag{3.50}$$

known as *Kepler's equation*. The eccentric angle E defined by (3.44) is, in Celestial Mechanics, called the *eccentric anomaly*, and the quantity $M = n(t - t_0)$ is called the *mean anomaly*. The angle f, introduced in equation (3.26), is called the *true anomaly*. We postpone to the next section the full treatment of the true anomaly.

The vectorial treatment gives a full account of the Kepler orbit in its plane. The fuller picture of the motion in space, including the orientation of the orbit plane, is more clearly seen in the Lagrangian analysis, which is the subject of the next section.

4 The Kepler Problem: Lagrangian Analysis

In terms of spherical coordinates (r, θ, φ) (of the heliocentric system), the three Cartesian coordinates can be expressed as

$$x = r\sin\theta\cos\varphi \tag{4.1a}$$

$$y = r\sin\theta\sin\varphi \tag{4.1b}$$

$$z = r\cos\theta \tag{4.1c}$$

from which it can readily be deduced that the metric coefficients g_{ij} are given by

$$g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = r^2\sin^2\theta, \quad g_{ij} = 0, \quad i \neq j\,. \tag{4.2}$$

Then for the kinetic and potential energies per unit mass, we have, respectively,

$$T = \tfrac{1}{2}v^2 = \tfrac{1}{2}[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2], \quad V = -\frac{\mu}{r} \tag{4.3}$$

and the Hamiltonian, reflecting the total energy, is

$$H = T + V = \tfrac{1}{2}[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\cdot\dot{\varphi}^2] - \frac{\mu}{r} \tag{4.4}$$

while, for the Lagrangian, we have

$$L = T - V = \tfrac{1}{2}[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2] + \frac{\mu}{r}. \tag{4.5}$$

From the latter there follows the system of Lagrangian equations, which takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}[\dot{r}] = r\dot{\theta}^2 + r\sin^2\theta\cdot\dot{\varphi}^2 - \frac{\mu}{r^2} \tag{4.6a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[r^2\dot{\theta}] = r^2\sin\theta\cos\theta\cdot\dot{\varphi}^2 \tag{4.6b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[r^2\sin^2\theta\cdot\dot{\varphi}] = 0. \tag{4.6c}$$

As the coordinate φ does not appear explicitly in the Lagrangian (4.5), it is an ignorable coordinate, and the procedure outlined in Chapter 1 may be followed; or we may proceed directly.

From (4.6c) there follows an immediate integration yielding

$$r^2\sin^2\theta\cdot\dot{\varphi} = C_3, \quad \text{or} \quad \dot{\varphi} = \frac{C_3}{r^2\sin^2\theta} \tag{4.7a,b}$$

where C_3 is the constant of integration and represents the polar component of angular momentum. The introduction of (4.7) into (4.6a,b) yields, respectively

$$\frac{\mathrm{d}}{\mathrm{d}t}[\dot{r}] = r\dot{\theta}^2 - \frac{\mu}{r^2} + \frac{C_3^2}{r^3\sin^2\theta} \tag{4.8a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[r^2\dot{\theta}] = C_3^2\frac{\cos\theta}{r^2\sin^3\theta}. \tag{4.8b}$$

Considering (4.8b), we multiply across by $r^2\dot{\theta}$ to obtain

$$r^2\dot{\theta}\frac{\mathrm{d}}{\mathrm{d}t}[r^2\dot{\theta}] = C_3^2\frac{\cos\theta\cdot\dot{\theta}}{\sin^3\theta} \tag{4.9}$$

which may be rearranged as

$$\frac{\mathrm{d}}{\mathrm{d}t}[r^2\dot{\theta}]^2 = -C_3^2\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\sin^2\theta}\right] \tag{4.10}$$

or alternatively

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(r^2\dot{\theta})^2 + \frac{C_3^2}{\sin^2\theta}\right] = 0. \tag{4.11}$$

This implies that the expression in square brackets is constant; however, if we substitute for C_3 in terms of $\dot{\varphi}$ from (4.7a), the expression becomes

$$r^2[r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\varphi}^2] = r^2(v^2 - \dot{r}^2) \tag{4.12}$$

and if we recall (3.18), we see that this constant is the square of the angular momentum, namely C^2. Accordingly, the integral of (4.11) may be written

$$r^4\dot{\theta}^2 + \frac{C_3^2}{\sin^2\theta} = C^2 \tag{4.13}$$

or alternatively

$$r\dot{\theta}^2 = \frac{1}{r^3}\left[C^2 - \frac{C_3^2}{\sin^2\theta}\right] \tag{4.14}$$

as the form appropriate for the reduction of (4.8a), which we effect prior to the integration of (4.13).

If we substitute for $r\dot{\theta}^2$ from (4.14) and for $\dot{\varphi}$ from (4.7b) into equation (4.8a), we see that the terms with C_3^2 cancel and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[\dot{r}] = \frac{C^2}{r^3} - \frac{\mu}{r^2} = \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\mu}{r} - \frac{1}{2}\frac{C^2}{r^2}\right]. \tag{4.15}$$

If we multiply across by $\dot{r}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}[\tfrac{1}{2}\dot{r}^2] = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mu}{r} - \frac{1}{2}\frac{C^2}{r^2}\right] \tag{4.16}$$

or on rearrangement

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\dot{r}^2 - \frac{\mu}{r} + \frac{1}{2}\frac{C^2}{r^2}\right] = 0 \tag{4.17}$$

and so the expression in square brackets must be constant. Again, recalling (3.19) we see that the expression

$$\frac{1}{2}\dot{r}^2 - \frac{\mu}{r} + \frac{1}{2}\frac{C^2}{r^2} = \frac{1}{2}\dot{r}^2 - \frac{\mu}{r} + \frac{1}{2}(v^2 - \dot{r}^2) = \frac{1}{2}v^2 - \frac{\mu}{r} \tag{4.18}$$

is in fact the energy integral, whose constant has already been designated as $\mathcal{E}$ (3.7) and for negative energy has been identified by $-\alpha^2$ (3.10). Accordingly, the integrated relation reads

$$\frac{1}{2}\dot{r}^2 - \frac{\mu}{r} + \frac{1}{2}\frac{C^2}{r^2} = -\alpha^2 \tag{4.19}$$

or alternatively

$$r^2\dot{r}^2 = -2\alpha^2\left[r^2 - \frac{\mu}{\alpha^2}r + \frac{C^2}{2\alpha^2}\right] \tag{4.20}$$

identical with the previously derived (3.43). The subsequent analysis leading to the solution (3.46)

$$r = a[1 - e\cos E] \tag{4.21}$$

follows an identical pattern.

In item 4 below relation (1.18), we have noted a shortcoming of this simple form of the equation of the ellipse: for the dynamic problem a second shortcoming is now coming into view. If one were to apply the transformation (3.44) to equation (4.13) for $\dot{\theta}$, we would still have a coupled equation, and if one were to substitute for r from (4.21), one has a differential equation that is not readily integrable.

In fact, an inspection of equation (4.13) suggests the form of the alternative regularizing transformation that will effect the uncoupling of equations (4.13) and (4.17), whose uncoupled form admits a ready integration in the case of each equation.

The singularity in the differential equation (4.13) can be regularized by means of the [regularizing] transformation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{C}{r^2}, \qquad C\frac{\mathrm{d}}{\mathrm{d}f} = r^2\frac{\mathrm{d}}{\mathrm{d}t} \tag{4.22a,b}$$

and with f as the new independent variable, and with prime denoting differentiation with respect to f, equation (4.13) becomes

$$C^2\theta'^2 + \frac{C_3^2}{\sin^2\theta} = C^2\,. \tag{4.23}$$

If we now introduce a new parameter ν, representing the inclination of the orbit plane, and defined by

$$\nu = \frac{C_3}{C} \tag{4.24}$$

then equation (4.23) may be written

$$\sin^2\theta\cdot\theta'^2 = (1-\nu^2) - \cos^2\theta \tag{4.25}$$

which, as we shall see, admits a straightforward integration.

Returning to equation (4.20), we multiply by a further r^2-factor to obtain

$$r^4\dot{r}^2 = -r^2[2\alpha^2r^2 - 2\mu r + C^2]\,. \tag{4.26}$$

If we utilize the transformation (4.22) to introduce the new independent variable f, then after dividing across by C^2 we have

$$r'^2 = -r^2\left[1 - \frac{2\mu}{C^2}r + \frac{2\alpha^2}{C^2}\right] \tag{4.27a}$$

$$= -r^2\left[1 - \frac{2}{p}r + \frac{1}{ap}r^2\right] \tag{4.27b}$$

where we have introduced the length scales from (3.11) and (3.33) into (4.27a) to obtain (4.27b).

The integration of (4.27b) is facilitated by the introduction of an auxiliary dependent variable u, defined by

$$u = \frac{1}{r}, \quad r = \frac{1}{u}, \quad r' = -\frac{1}{u^2}u' \tag{4.28}$$

and, after a little manipulation, equation (4.27b) becomes

$$u'^2 = -\left[\left(u - \frac{1}{p}\right)^2 + \frac{1}{ap} - \frac{1}{p^2}\right] = \left[\frac{e^2}{p^2} - \left(u - \frac{1}{p}\right)^2\right]. \tag{4.29}$$

By setting

$$u - \frac{1}{p} = \frac{e}{p}w \tag{4.30}$$

the differential equation for w reads

$$w'^2 = 1 - w^2 \tag{4.31}$$

with solution

$$w = \cos(f + \omega_0) \tag{4.32}$$

where ω_0 is the constant of integration. It follows from (4.30) that

$$u = \frac{1}{p}[1 + e\cos(f + \omega_0)] \tag{4.33}$$

and hence, noting (4.28), we have

$$r = \frac{p}{1 + e\cos(f + \omega_0)}. \tag{4.34a}$$

Except for the factor ω_0, this is identical with (1.15) for the ellipse, so the variable f has the obvious angular interpretation; moreover, if the angle is measured from the pericenter so that

$$f = 0 \quad \text{corresponds to} \quad r = a(1 - e) \tag{4.34b}$$

then clearly $\omega_0 = 0$ and we have

$$r = \frac{p}{1 + e\cos f} \tag{4.35}$$

as the solution for r, identical with (1.15).

Returning to equation (4.25), we note that the integration can be facilitated by setting

$$\cos\theta = \sqrt{1 - \nu^2}S \tag{4.36}$$

so that equation (4.25) becomes

$$S'^2 = 1 - S^2 \tag{4.37}$$

with solution

$$S = \sin(f + \omega) \tag{4.38}$$

where ω is the constant of integration. The point where the orbit crosses the z-plane is called the *node* and the line joining it to the focus is called the *nodal line*. The crossing of the z-plane corresponds to $\theta = \pi/2$, and so noting (4.36) and (4.38), this must correspond to $f = -\omega$; hence ω measures the angle in the orbit plane subtended at the focus between the major axis and the nodal line. And we may write

$$\cos\theta = \sqrt{1 - \nu^2}\sin(f + \omega) \tag{4.39}$$

as the complete solution for the θ-coordinate.

It remains to integrate equation (4.7) for the third coordinate φ. Writing (4.7) in the form

$$r^2\dot{\varphi} = \frac{C_3}{\sin^2\theta} \tag{4.40}$$

we introduce the regularizing transformation (4.22) replacing t as the independent variable by f. We then have

$$C\varphi' = \frac{C_3}{\sin^2\theta} \tag{4.41}$$

and if we divide across by C and note the defining relation (4.24) for ν, we obtain

$$\varphi' = \frac{\nu}{\sin^2\theta} = \frac{\nu}{1 - \cos^2\theta}\,. \tag{4.42}$$

If we introduce $\cos\theta$ in terms of f from (4.39), we obtain

$$\begin{aligned}\varphi' &= \frac{\nu}{1 - (1 - \nu^2)\sin^2(f + \omega)} \\ &= \frac{\nu}{\cos^2(f + \omega) + \nu^2\sin^2(f + \omega)} = \frac{\nu\sec^2(f + \omega)}{1 + \nu^2\tan^2(f + \omega)}\,.\end{aligned} \tag{4.43}$$

The integration of equation (4.43) is facilitated by the substitution

$$\tan\Phi = \nu\tan(f + \omega) \tag{4.44}$$

from which we have

$$\sec^2\Phi \cdot \Phi' = \nu\sec^2(f + \omega), \quad \sec^2\Phi = 1 + \nu^2\tan^2(f + \omega) \tag{4.45a,b}$$

and from (4.43) there follows

$$\varphi' = \Phi' \quad \text{implying} \quad \Phi = \varphi + \varphi_0 \tag{4.45c}$$

where φ_0 is the constant of integration. Hence (4.44) implies that

$$\tan(\varphi + \varphi_0) = \nu \tan(f + \omega)\,. \tag{4.46}$$

We have already noted that at the nodal crossing, $f = -\omega$; if we now let Ω denote the longitude at this nodal line, then from (4.46) there follows

$$\tan(\Omega + \varphi_0) = 0, \quad \text{implying} \quad \varphi_0 = -\Omega \tag{4.47}$$

and hence, from (4.46), we have

$$\tan(\varphi - \Omega) = \nu \tan(f + \omega) \tag{4.48}$$

as the solution for the third coordinate φ.

The completion of the solution requires the determination of the time-angle relation connecting the time with the true anomaly f. For this we introduce the expression (4.35) into the inverted form of the defining relation (4.22a), and if we substitute for C from (3.33), we find

$$\frac{\mathrm{d}t}{\mathrm{d}f} = \frac{r^2}{C} = \frac{1}{\sqrt{\mu a}\sqrt{1-e^2}}\,\frac{a^2(1-e^2)^2}{(1+e\cos f)^2}\,. \tag{4.49}$$

If we recall from (3.41) that $\sqrt{\mu a} = na^2$, it follows that

$$n\frac{\mathrm{d}t}{\mathrm{d}f} = \frac{(1-e^2)^{3/2}}{(1+e\cos f)^2}\,. \tag{4.50}$$

For the integration of this expression we first note that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}f}\left[\frac{e\sin f}{1+e\cos f}\right] &= \frac{(1+e\cos f)e\cos f + e^2\sin^2 f}{(1+e\cos f)^2} = \frac{e^2 + e\cos f}{(1+e\cos f)^2}\\ &= \frac{(1+e\cos f) - (1-e^2)}{(1+e\cos f)^2}\\ &= \frac{1}{1+e\cos f} - \frac{(1-e^2)}{(1+e\cos f)^2}\end{aligned} \tag{4.51}$$

and hence, on multiplying by $\sqrt{1-e^2}$ and rearranging, we have

$$\frac{(1-e^2)^{3/2}}{(1+e\cos f)^2} = \frac{\sqrt{1-e^2}}{1+e\cos f} - \frac{\mathrm{d}}{\mathrm{d}f}\left[\frac{e\sqrt{1-e^2}\sin f}{1+e\cos f}\right]. \tag{4.52}$$

For the integration of the first term on the right we note that if we set

$$\tan\chi = \frac{\sqrt{1-e^2}\sin f}{e+\cos f} \tag{4.53}$$

there follows

$$\sec^2 \chi = \frac{(1 + e\cos f)^2}{(e + \cos f)^2}, \quad \cos\chi = \frac{e + \cos f}{1 + e\cos f}, \tag{4.54a,b}$$

$$\sin\chi = \frac{\sqrt{1 - e^2}\sin f}{1 + e\cos f}\,. \tag{4.54c}$$

Taking the derivative of (4.53), we find

$$\begin{aligned}
\sec^2\chi \cdot \chi' &= \frac{(1 + e\cos f)^2}{(e + \cos f)^2}\chi' \\
&= \frac{\sqrt{1 - e^2}(e + \cos f)\cos f + \sqrt{1 - e^2}\sin^2 f}{(e + \cos f)^2} \\
&= \sqrt{1 - e^2}\frac{(1 + e\cos f)}{(e + \cos f)^2}
\end{aligned} \tag{4.55}$$

which, with (4.54a), yields

$$\chi' = \frac{\sqrt{1 - e^2}}{1 + e\cos f} \tag{4.56}$$

and hence, noting (4.53), we have

$$\int \frac{\sqrt{1 - e^2}}{1 + e\cos f}\,\mathrm{d}f = \chi = \arctan\left[\frac{\sqrt{1 - e^2}\sin f}{e + \cos f}\right]. \tag{4.57}$$

Accordingly, the integration of (4.50) is accomplished by combining (4.52) and (4.57) to yield

$$M = n(t - t_0) = \arctan\left[\frac{\sqrt{1 - e^2}\sin f}{e + \cos f}\right] - \frac{e\sqrt{1 - e^2}\sin f}{1 + e\cos f} \tag{4.58}$$

where t_0, reflecting the constant introduced by the integration, is the time of the pericenter passage, i.e., $t = t_0$ corresponds to $f = 0$.

3

The Euler Problem I — Planar Case

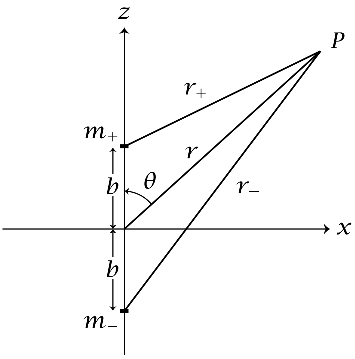
...exerceamus, certa spe freti, inde quicquam luminis ad tenebras illas dissipandas esse affulsorum.

— *L. Euler* [E2, e(i) p. 211]

[...hence, encouraged by a definite hope, let us deploy such light as may be cast to the dispelling of these obscurities.]

1 The Gravitational Field of Two Fixed Centers: Planar Case

Formulation: In the x-z plane, we consider the motion of a mass point P in the gravitational field induced by two fixed masses m_+ and m_- situated respectively at symmetrically placed points on the z-axis, $z = +b$ and $z = -b$.



The potential function per unit mass at the arbitrary point P is given by

$$U = G\Big(\frac{m_+}{r_+} + \frac{m_-}{r_-}\Big) \tag{1.1}$$

where G is the gravitational constant.

In terms of planar polar coordinates (r, θ), where the angle θ is measured with the positive z-axis as baseline, the application of the cosine law renders for the distances r_+ and r_-

$$r_+^2 = r^2 + b^2 - 2br\cos\theta \tag{1.2a}$$

$$r_-^2 = r^2 + b^2 + 2br\cos\theta. \tag{1.2b}$$

If we introduce planar prolate spheroidal coordinates (R, σ) based on the distance parameter b, then, in terms of the Cartesian coordinates (x, z) and also

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of the plane polars (r, θ), we have the defining relations[1]

$$r \sin\theta = x = \pm\sqrt{R^2 - b^2}\sin\sigma, \qquad r\cos\theta = z = R\cos\sigma \tag{1.3a,b}$$

with the implication that

$$r^2 = x^2 + z^2 = R^2 - b^2\sin^2\sigma\,. \tag{1.3c}$$

Then, in terms of the planar spheroidal coordinates (R, σ), relations (1.2) for r_+ and r_- take the form

$$\begin{aligned} r_+^2 &= r^2 + b^2 - 2br\cos\theta \\ &= R^2 + b^2\cos^2\sigma - 2bR\cos\sigma = (R - b\cos\sigma)^2 \end{aligned} \tag{1.4a}$$

$$\begin{aligned} r_-^2 &= r^2 + b^2 + 2br\cos\theta \\ &= R^2 + b^2\cos^2\sigma + 2bR\cos\sigma = (R + b\cos\sigma)^2 \end{aligned} \tag{1.4b}$$

so that, for the potential function per unit mass (1.1), we have

$$\begin{aligned} U &= G\left[\frac{m_+}{r_+} + \frac{m_-}{r_-}\right] = G\left[\frac{m_+}{R - b\cos\sigma} + \frac{m_-}{R + b\cos\sigma}\right] \\ &= G\left[\frac{R(m_+ + m_-) + b(m_+ - m_-)\cos\sigma}{R^2 - b^2\cos^2\sigma}\right] \\ &= G(m_+ + m_-)\left[\frac{R + b\left(\frac{m_+ - m_-}{m_+ + m_-}\right)\cos\sigma}{R^2 - b^2\cos^2\sigma}\right]. \end{aligned} \tag{1.5}$$

Accordingly, we introduce the dimensionless parameter β, measuring the asymmetry between the attracting masses, defined by the relation

$$\beta = \frac{m_+ - m_-}{m_+ + m_-} \tag{1.6a}$$

and if we write

$$\mu = G(m_1 + m_2) \tag{1.6b}$$

then formula (1.5) for the potential function per unit mass takes the compact form

$$U = \mu\frac{R + \beta b\cos\sigma}{R^2 - b^2\cos^2\sigma}\,. \tag{1.7}$$

Hence the potential energy per unit mass V for the mass point P in this gravitational field is given by

[1] In the planar case, a crucial distinction has to be noted between the spherical and spheroidal coordinate systems. In the former θ has the range $-\pi \le \theta \le \pi$, whereas in the latter σ is confined to the range $0 \le \sigma \le \pi$, necessitating the inclusion of the $\pm$ sign option on the right of (1.3a). This becomes of critical importance when utilizing the solution forms to obtain computer-generated graphical orbits.

$$V = -\mu \frac{R + \beta b \cos\sigma}{R^2 - b^2 \cos^2\sigma} \tag{1.8}$$

in accord with relation (2.13) of Chapter 2.

For the coordinate system defined by (1.3), we have

$$\frac{\partial x}{\partial R} = \pm \frac{R}{\sqrt{R^2 - b^2}} \sin\sigma, \qquad \frac{\partial z}{\partial R} = \cos\sigma \tag{1.9a,b}$$

$$\frac{\partial x}{\partial \sigma} = \pm\sqrt{R^2 - b^2} \cos\sigma, \qquad \frac{\partial z}{\partial \sigma} = -R\sin\sigma \tag{1.9c,d}$$

so that the metric coefficients are given by

$$g_{11} = \frac{R^2 - b^2\cos^2\sigma}{R^2 - b^2}, \qquad g_{12} = 0, \qquad g_{22} = R^2 - b^2\cos^2\sigma\,. \tag{1.10a,b,c}$$

It follows that the kinetic energy per unit mass, T, is given by

$$T = \tfrac{1}{2} \frac{R^2 - b^2\cos^2\sigma}{R^2 - b^2} \dot{R}^2 + \tfrac{1}{2}(R^2 - b^2\cos^2\sigma)\dot{\sigma}^2 \tag{1.11}$$

and the total energy per unit mass H for the mass point P is given by

$$\begin{aligned} H &= T + V \\ &= \tfrac{1}{2} \frac{R^2 - b^2\cos^2\sigma}{R^2 - b^2} \dot{R}^2 + \tfrac{1}{2}(R^2 - b^2\cos^2\sigma)\dot{\sigma}^2 - \mu \frac{R + \beta b\cos\sigma}{R^2 - b^2\cos^2\sigma}\,. \end{aligned} \tag{1.12}$$

The associated Lagrangian L is given by

$$\begin{aligned} L &= T - V \\ &= \tfrac{1}{2} \frac{R^2 - b^2\cos^2\sigma}{R^2 - b^2} \dot{R}^2 + \tfrac{1}{2}(R^2 - b^2\cos^2\sigma)\dot{\sigma}^2 + \mu \frac{R + \beta b\cos\sigma}{R^2 - b^2\cos^2\sigma} \end{aligned} \tag{1.13}$$

from which we derive the equations of motion. If we were to introduce an arbitrary mass m for the mass point, then this factor would clearly cancel in the Lagrange equations and so will be ignored.

In order to see that the Lagrange equations are integrable, it is necessary first to transform the Lagrangian into one of Liouville form, which is the topic of the next section.

2 The Lagrangian in Liouville Form: The Energy Integral

In order to transform the Lagrangian into one of Liouville form, we introduce the auxiliary variable ξ, defined by

$$R = b\cosh\xi \tag{2.1a}$$

so that

$$\dot{R} = b\sinh\xi \cdot \dot{\xi}, \qquad R^2 - b^2 = b^2\sinh^2\xi\,. \tag{2.1b,c}$$

Then the expressions (1.11) and (1.8) for the kinetic and potential energies become, respectively

$$T = b^2(\cosh^2 \xi - \cos^2 \sigma)[\tfrac{1}{2}\dot{\xi}^2 + \tfrac{1}{2}\dot{\sigma}^2], \qquad V = -\mu \frac{b \cosh \xi + \beta b \cos \sigma}{b^2 \cosh^2 \xi - b^2 \cos^2 \sigma} \tag{2.2a,b}$$

and the Lagrangian becomes

$$L = b^2(\cosh^2 \xi - \cos^2 \sigma)[\tfrac{1}{2}\dot{\xi}^2 + \tfrac{1}{2}\dot{\sigma}^2] + \mu \frac{b \cosh \xi + \beta b \cos \sigma}{b^2 \cosh^2 \xi - b^2 \cos^2 \sigma} \tag{2.3}$$

which is of the type recognized as integrable by Liouville. In the standard notation for such systems, already noted in Chapter 1, we write

$$Q_1(\xi) = b^2 \cosh^2 \xi, \qquad Q_2(\sigma) = -b^2 \cos^2 \sigma, \qquad Q = Q_1 + Q_2 \tag{2.4a,b,c}$$

$$V_1(\xi) = -\mu b \cosh \xi, \qquad V_2(\sigma) = -\mu \beta b \cos \sigma \tag{2.5a,b}$$

so that expressions (2.2), for the energies, take the form

$$T = Q[\tfrac{1}{2}\dot{\xi}^2 + \tfrac{1}{2}\dot{\sigma}^2], \qquad V = \frac{V_1 + V_2}{Q} \tag{2.6a,b}$$

and the Lagrangian (2.3) may be written

$$L = Q[\tfrac{1}{2}\dot{\xi}^2 + \tfrac{1}{2}\dot{\sigma}^2] - \frac{V_1 + V_2}{Q} \tag{2.7}$$

now in the standard Liouville form.

From the Lagrange equations for this system,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\xi}}\right) = \frac{\partial L}{\partial \xi}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\sigma}}\right) = \frac{\partial L}{\partial \sigma} \tag{2.8a,b}$$

we follow the standard procedure for the derivation of the energy integral. We multiply (2.8a) by $\dot{\xi}$ and (2.8b) by $\dot{\sigma}$, and add to obtain

$$\dot{\xi}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\xi}}\right) + \dot{\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\sigma}}\right) = \dot{\xi}\frac{\partial L}{\partial \xi} + \dot{\sigma}\frac{\partial L}{\partial \sigma} \tag{2.9}$$

from which there follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\dot{\xi}\frac{\partial L}{\partial \dot{\xi}} + \dot{\sigma}\frac{\partial L}{\partial \dot{\sigma}}\right] = \dot{\xi}\frac{\partial L}{\partial \xi} + \ddot{\xi}\frac{\partial L}{\partial \dot{\xi}} + \dot{\sigma}\frac{\partial L}{\partial \sigma} + \ddot{\sigma}\frac{\partial L}{\partial \dot{\sigma}} = \frac{\mathrm{d}L}{\mathrm{d}t} \tag{2.10}$$

which on integration yields

$$\dot{\xi}\frac{\partial L}{\partial \dot{\xi}} + \dot{\sigma}\frac{\partial L}{\partial \dot{\sigma}} - L = \mathcal{E} \tag{2.11}$$

— the energy integral in which $\mathcal{E}$ is the constant of integration.

When the Lagrangian has the form (2.7), we see that in this case

$$\dot{\xi}\frac{\partial L}{\partial \dot{\xi}} + \dot{\sigma}\frac{\partial L}{\partial \dot{\sigma}} = \dot{\xi}\frac{\partial T}{\partial \dot{\xi}} + \dot{\sigma}\frac{\partial T}{\partial \dot{\sigma}} = 2T \tag{2.12}$$

wherein we have noted the expression (2.6a) for T and the simplification resulting therefrom. When we introduce (2.12) into (2.11), we see that the energy integral may be written as

$$\mathcal{E} = 2T - L = 2T - (T - V) = T + V \tag{2.13}$$

showing that the constant $\mathcal{E}$ clearly measures the total energy (per unit mass) for the dynamical system.

We shall refer to the ξ-σ system as the Liouville coordinates.

3 The First Integrals in Liouville Coordinates

Returning to the Lagrange equations (2.8), we introduce the explicit form of the Lagrangian from (2.7), and the equations take the explicit form

$$\frac{\mathrm{d}}{\mathrm{d}t}(Q\dot{\xi}) = \frac{\mathrm{d}Q_1}{\mathrm{d}\xi}\left(\tfrac{1}{2}\dot{\xi}^2 + \tfrac{1}{2}\dot{\sigma}^2\right) - \frac{\partial V}{\partial \xi}, \quad \frac{\mathrm{d}}{\mathrm{d}t}(Q\dot{\sigma}) = \frac{\mathrm{d}Q_2}{\mathrm{d}\sigma}\left(\tfrac{1}{2}\dot{\xi}^2 + \tfrac{1}{2}\dot{\sigma}^2\right) - \frac{\partial V}{\partial \sigma}, \tag{3.1a,b}$$

We start by dealing with equation (3.1a). We multiply across by $Q\dot{\xi}$ and utilize (2.6a) to obtain

$$Q\dot{\xi}\frac{\mathrm{d}}{\mathrm{d}t}(Q\dot{\xi}) = \dot{\xi}\frac{\mathrm{d}Q_1}{\mathrm{d}\xi}T - \dot{\xi}Q\frac{\partial V}{\partial \xi} = \dot{\xi}\left[T\frac{\mathrm{d}Q_1}{\mathrm{d}\xi} - Q\frac{\partial V}{\partial \xi}\right]. \tag{3.2}$$

If we now use (2.13) to substitute for T in the square brackets on the right of (3.2), we have

$$\begin{aligned}\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[Q\dot{\xi}]^2 &= \dot{\xi}\left[(\mathcal{E} - V)\frac{\mathrm{d}Q_1}{\mathrm{d}\xi} - Q\frac{\partial V}{\partial \xi}\right] \\ &= \dot{\xi}\left[\mathcal{E}\frac{\mathrm{d}Q_1}{\mathrm{d}\xi} - \frac{\partial}{\partial \xi}(QV)\right] = \dot{\xi}\left[\mathcal{E}\frac{\mathrm{d}Q_1}{\mathrm{d}\xi} - \frac{\mathrm{d}V_1}{\mathrm{d}\xi}\right]\end{aligned} \tag{3.3}$$

where we have noted relations (2.4) and (2.6). A slight rearrangement puts (3.3) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\tfrac{1}{2}(Q\dot{\xi})^2 - \mathcal{E}Q_1 + V_1\right] = 0 \tag{3.4}$$

which integrates to yield

$$\tfrac{1}{2}(Q\dot{\xi})^2 - \mathcal{E}Q_1 + V_1 = C_1 \tag{3.5}$$

where C_1 is the constant of integration.

We next take the second equation (3.1b) and, after multiplying across by $Q\dot{\sigma}$, proceed in an identical manner to effect its integration; as the integrated equation for $\dot{\sigma}$, we obtain

$$\tfrac{1}{2}(Q\dot{\sigma})^2 - \mathcal{E}Q_2 + V_2 = C_2 \tag{3.6}$$

where C_2 is the constant of integration.

The addition of equations (3.5) and (3.6) shows that

$$\begin{aligned} C_1 + C_2 &= Q^2\left[\tfrac{1}{2}\dot{\xi}^2 + \tfrac{1}{2}\dot{\sigma}^2\right] - \mathcal{E}(Q_1 + Q_2) + V_1 + V_2 \\ &= QT - \mathcal{E}Q + QV = Q(T + V - \mathcal{E}) = 0 \end{aligned} \tag{3.7}$$

in which we have noted relations (2.4), (2.5), and (2.6). On noting relation (2.13), we see that the requirement

$$C_1 + C_2 = 0 \tag{3.8}$$

follows as a direct consequence of the energy integral.

Equations (3.5) and (3.6) together with the energy integral (2.13) are the first integrals of the system. From (3.8), it is clear that the constants are not independent; we shall see presently that two of them are independent. Having derived the integrals through the medium of the Liouville procedure, in terms of the Liouville coordinates ξ, σ, it is no longer convenient to retain them in this form. Accordingly, we next consider these integrals in terms of the original spheroidal coordinates R, σ.

4 The First Integrals in Spheroidal Coordinates

Recalling the defining relations (2.1) for the Liouville coordinate ξ, we see from relations (2.4) and (2.5) that, in terms of the spheroidal coordinates (R, σ), we have

$$Q_1 = R^2, \quad Q_2 = -b^2\cos^2\sigma, \quad Q = Q_1 + Q_2 = R^2 - b^2\cos^2\sigma \tag{4.1a,b,c}$$

$$V_1 = -\mu R, \quad V_2 = -\mu\beta b\cos\sigma, \quad QV = V_1 + V_2 = -\mu(R + \beta b\cos\sigma) \tag{4.2a,b,c}$$

and furthermore, from relations (2.1), there follows

$$\dot{\xi} = \frac{\dot{R}}{b\sinh\xi} = \frac{\dot{R}}{\sqrt{b^2\cosh^2\xi - b^2}} = \frac{\dot{R}}{\sqrt{R^2 - b^2}} \tag{4.3}$$

and hence we have

$$Q\dot{\xi} = \frac{\dot{R}}{\sqrt{R^2 - b^2}}(R^2 - b^2\cos^2\sigma) \tag{4.4a}$$

$$Q\dot{\sigma} = (R^2 - b^2\cos^2\sigma)\dot{\sigma}. \tag{4.4b}$$

Accordingly, in spheroidal coordinates, the equations for the first integrals (3.5) and (3.6) take the form

$$\tfrac{1}{2}\frac{(R^2 - b^2\cos^2\sigma)^2}{R^2 - b^2}\dot{R}^2 = \mathcal{E}R^2 + \mu R + C_1 \tag{4.5a}$$

$$\tfrac{1}{2}(R^2 - b^2\cos^2\sigma)^2\dot{\sigma}^2 = -\mathcal{E}b^2\cos^2\sigma + \mu\beta b\cos\sigma + C_2 \tag{4.5b}$$

with the restriction (3.8) that $C_1 + C_2 = 0$.

An inspection of (4.5b) in the limit as $b \to 0$ indicates that in that range the constant C_2 is positive, which immediately implies that C_1 is negative. Moreover, as we are primarily concerned with bound orbits, we focus on the case of negative energy and so we set

$$\mathcal{E} = -\alpha^2, \quad C_1 = -\tfrac{1}{2}C^2, \quad C_2 = \tfrac{1}{2}C^2 \tag{4.6a,b,c}$$

and again observing (4.5b), we see that C has the dimension of angular momentum. In terms of these constants, the equations for the first integrals (4.5) now read

$$\tfrac{1}{2}\frac{(R^2 - b^2\cos^2\sigma)^2}{R^2 - b^2}\dot{R}^2 = -\alpha^2R^2 + \mu R - \tfrac{1}{2}C^2 \tag{4.7a}$$

$$\tfrac{1}{2}(R^2 - b^2\cos^2\sigma)^2\dot{\sigma}^2 = \alpha^2b^2\cos^2\sigma + \mu\beta b\cos\sigma + \tfrac{1}{2}C^2. \tag{4.7b}$$

In effecting the reduction of these equations, we shall follow the signposts set down in the case of the Kepler problem in Chapter 2.

— ◇ —

In the subsequent analysis, we begin by focusing on the class of orbits that retain their validity in the Kepler limit. The following three sections are preparatory to the derivation of the solution forms. In Section 5 we define the length scales associated with the physical constants, and having introduced the regularizing transformation, we specify the dimensionless parameters as they naturally arise, leading to a normalized form of the differential equations. In Section 6, we discuss some special cases including the case $\beta = 0$, where the attracting primaries have identical mass — the case having particular relevance to the Quantum Connection. A complete analysis of the representative quartic is performed in Section 7 whereby we effect the reduction of the typical differential equation to Legendre canonical form.

With the establishment of the solution forms for orbits that retain their validity in the Kepler limit as developed in Sections 8 and 9, it becomes evident that their validity is confined to a definite range of the separation parameter. The solution forms valid in the supplementary range of the separation parameter are derived respectively in Sections 8A and 9A; as this latter range is defined by a lower limit to the separation parameter, the Kepler limit is not relevant — except in the singular case when $C = 0$, discussed later.

Following a limited analysis of the time-angle relation in Section 10, we return to a consideration of the set of orbits characterized by a choice of the angular momentum constant alternative to (4.6); the associated orbits that have no analog in the Kepler case are discussed in Sections 11 and 12.

5 Reduction of the Equations: The Regularizing Variable

First, in dealing with the R-equation, we rewrite equation (4.7a) in the form

$$(R^2 - b^2\cos^2\sigma)^2\dot{R}^2 = -C^2(R^2 - b^2)\Big[1 - 2\frac{\mu}{C^2}R + 2\frac{\alpha^2}{C^2}R^2\Big]. \tag{5.1}$$

If we introduce the length scales a and p, by setting

$$a = \frac{\mu}{2\alpha^2}, \quad p = \frac{C^2}{\mu}, \quad \text{and hence} \quad ap = \frac{C^2}{2\alpha^2} \tag{5.2a,b,c}$$

then the above equation (5.1) becomes

$$\frac{(R^2 - b^2\cos^2\sigma)^2}{C^2}\dot{R}^2 = -(R^2 - b^2)\Big[1 - \frac{2}{p}R + \frac{1}{ap}R^2\Big]. \tag{5.3}$$

Next, turning to the σ-equation, we rewrite (4.7b) in the form

$$(R^2 - b^2\cos^2\sigma)^2\dot{\sigma}^2 = C^2\Big[1 + 2\frac{\mu}{C^2}\beta b\cos\sigma + 2\frac{\alpha^2}{C^2}b^2\cos^2\sigma\Big] \tag{5.4}$$

so that, when written in terms of the length parameters (5.2),

$$\frac{(R^2 - b^2\cos^2\sigma)^2}{C^2}\dot{\sigma}^2 = 1 + 2\beta\frac{b}{p}\cos\sigma + \frac{b^2}{ap}\cos^2\sigma \tag{5.5}$$

we have the equation for σ, corresponding to equation (5.3) for R.

The form of the regularizing variable is suggested quite clearly in the above. We introduce the new independent variable f (corresponding to the true anomaly in the Kepler case) by the defining relation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\Lambda}{R^2 - b^2\cos^2\sigma} \tag{5.6}$$

where Λ is a parameter, having the dimension of angular momentum, and will be defined presently. It follows that

$$\frac{R^2 - b^2\cos^2\sigma}{\Lambda}\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}f} \tag{5.7}$$

so that with prime denoting differentiation with respect to f, the equations for the first integrals (5.3) and (5.5) become

$$\frac{\Lambda^2}{C^2}R'^2 = -(R^2 - b^2)\Big[1 - \frac{2}{p}R + \frac{1}{ap}R^2\Big] \tag{5.8a}$$

$$\frac{\Lambda^2}{C^2}\sigma'^2 = 1 + 2\beta\frac{b}{p}\cos\sigma + \frac{b^2}{ap}\cos^2\sigma. \tag{5.8b}$$

The next step is the introduction of a new dependent variable u into equation (5.8a) by setting

$$u = \frac{1}{R}, \quad \text{so that} \quad u' = -\frac{R'}{R^2}, \quad R' = -\frac{u'}{u^2} \tag{5.9a,b,c}$$

and in terms of u, equation (5.8a) becomes

$$\begin{aligned}\frac{\Lambda^2}{C^2}u'^2 &= -(1 - b^2u^2)\Big[u^2 - \frac{2}{p}u + \frac{1}{ap}\Big]\\ &= -(1 - b^2u^2)\Big[\Big(u - \frac{1}{p}\Big)^2 - \frac{1}{p^2}\Big(1 - \frac{p}{a}\Big)\Big].\end{aligned} \tag{5.10}$$

Still following the pattern set in the Kepler problem, we introduce a parameter e corresponding to the eccentricity in the Kepler case, by writing

$$p = a(1 - e^2), \quad \frac{p}{a} = 1 - e^2 \tag{5.11a,b}$$

and equation (5.10) then reads

$$\frac{\Lambda^2}{C^2}u'^2 = (1 - b^2u^2)\Big[\frac{e^2}{p^2} - \Big(u - \frac{1}{p}\Big)^2\Big]. \tag{5.12}$$

This suggests a substitution

$$u - \frac{1}{p} = \frac{e}{p}v, \quad u = \frac{1}{p}(1 + ev), \quad u' = \frac{e}{p}v' \tag{5.13a,b,c}$$

the introduction of which into (5.12) yields the differential equation for v, which, when we interchange the order of the factors on the right, takes the form

$$\frac{\Lambda^2}{C^2}v'^2 = (1 - v^2)\Big[1 - \frac{b^2}{p^2}(1 + ev)^2\Big]. \tag{5.14}$$

We now introduce the dimensionless parameter η, measuring the ratio of the separation factor b to the orbit-characteristic length p, namely

$$\eta^2 = \frac{b^2}{p^2} = \frac{b^2}{a^2(1 - e^2)^2} = \frac{b^2}{a^2}\cdot\frac{1}{\ell^2} \tag{5.15}$$

wherein we have written

$$\ell = 1 - e^2 \tag{5.16}$$

and equation (5.14) takes the form

$$\frac{\Lambda^2}{C^2}v'^2 = (1-v^2)[1-\eta^2(1+ev)^2] \tag{5.17a}$$

$$= (1-v^2)[(1-\eta^2) - 2\eta^2 ev - \eta^2 e^2 v^2]. \tag{5.17b}$$

Turning to the equation for the second integral (5.8b), it is immediately evident that if we multiply across by $\sin^2\sigma$ and set

$$S = \cos\sigma \tag{5.18}$$

then, in terms of S, equation (5.8b) becomes

$$\frac{\Lambda^2}{C^2}S'^2 = (1-S^2)[1+2\eta\beta S + \eta^2\ell S^2] \tag{5.19}$$

where we have introduced the parameters from (5.11), (5.15), and (5.16).

It remains to analyze and solve the reduced equations (5.17) and (5.19). At this point, we note that both equations fit the pattern of the generic equation

$$\frac{\Lambda^2}{C^2}y'^2 = (1-y^2)[(1-d^2) + 2sy + qy^2]. \tag{5.20}$$

Before proceeding to an analysis of (5.20), we first consider some particular cases and the insight gained therefrom.

6 Some Particular Cases

In order to get a sense of the direction in which one should go from here, it is worthwhile to pause and observe the shape things take in some particular cases.

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As a first example, we observe from (5.13) that in the particular case when $e = 0$, the solution implies a constant value for u and hence also for R. Explicitly,

$$e = 0 \text{ implies } p = a, \quad u = \frac{1}{p}, \quad u' = 0, \quad R = p. \tag{6.1}$$

If we note the implication of this by the introduction of $R = p$ into the defining equations of the coordinate system (1.3), we have

$$x = \sqrt{p^2 - b^2}\sin\sigma, \quad z = p\cos\sigma \tag{6.2}$$

and hence, noting (5.15),

$$\frac{x^2}{p^2-b^2} + \frac{z^2}{p^2} = 1, \quad \text{or} \quad \frac{x^2}{p^2(1-\eta^2)} + \frac{z^2}{p^2} = 1 \tag{6.3a,b}$$

gives the equation of the orbit in terms of the x-z Cartesian coordinate system. Clearly, the case $e = 0$ in this system corresponds to an orbit in the shape of a conic section of eccentricity η. When $\eta = 0$, we retrieve the circular orbit of the Kepler problem as the trivial case.

When $\eta^2 > 1$, equation (6.3b) should be written

$$\frac{z^2}{p^2} - \frac{x^2}{p^2(\eta^2 - 1)} = 1 \tag{6.4}$$

but as the reality of the coordinate system (6.2) is violated, there is no real orbit for this parameter range.

When $\eta^2 = 1$, the conic degenerates to $x = 0$, i.e., to the z-axis: here the motion is along the axis joining the two attracting masses, and we have a collision orbit. Clearly, the case $\eta^2 = 1$ requires special treatment and belongs with collision analysis.

When $\eta^2 < 1$, equation (6.3b) is the equation of a closed elliptic orbit of eccentricity η, with semimajor axis p.

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In the second example, we shall observe the simplification effected in equation (5.19) in the particular case when the two attracting centers have equal mass. Recalling the defining relation (1.6a), this case is characterized by

$$\beta = 0 \tag{6.5}$$

and equation (5.19) simplifies to

$$\frac{\Lambda^2}{C^2} S'^2 = (1 - S^2)[1 + \eta^2(1 - e^2)S^2]. \tag{6.6}$$

This equation admits solutions in terms of the Jacobian elliptic functions, but it is necessary to consider separately the cases $e^2 \gtreqless 1$. The case $e = 1$ again reduces to the equation for the Kepler case because of the coincidence that the multiplying factor $\eta^2(1 - e^2)$ vanishes for both $\eta = 0$ and $e = 1$.

Case A: $e^2 > 1$

For this case, we may set

$$\Lambda^2 = C^2, \quad k_{01}^2 = \eta^2(e^2 - 1) \tag{6.7}$$

and equation (6.6) takes the form

$$S'^2 = (1 - S^2)[1 - k_{01}^2 S^2] \tag{6.8}$$

with solution given by

$$S = \mathrm{sn}[f + \omega : k_{01}] \tag{6.9}$$

where sn is the Jacobian elliptic function of modulus k_{01} and ω is the constant of integration; this implies that $f = -\omega$ corresponds to the orbit's first crossing of the x-axis, namely, $z = 0$, at which $\cos\sigma = 0$, and hence $S = 0$.

Case B: $e^2 < 1$

In this case, $\eta^2(1-e^2)$ is positive, and equation (6.6) is not quite in amenable form. We rewrite (6.6) as follows:

$$\begin{aligned}\frac{\Lambda^2}{C^2}S'^2 &= (1-S^2)[1+\eta^2(1-e^2)-\eta^2(1-e^2)[1-S^2]] \\ &= [1+\eta^2(1-e^2)](1-S^2)\Big[1-\frac{\eta^2(1-e^2)}{1+\eta^2(1-e^2)}(1-S^2)\Big] \qquad (6.10)\end{aligned}$$

which is now amenable to solution in terms of the Jacobian elliptic functions. In this case, we proceed by setting

$$\Lambda^2 = C^2[1+\eta^2(1-e^2)], \quad k_{02}^2 = \frac{\eta^2(1-e^2)}{1+\eta^2(1-e^2)} \qquad (6.11\text{a,b})$$

and equation (6.10) becomes

$$S'^2 = (1-S^2)[1-k_{02}^2(1-S^2)]. \qquad (6.12)$$

The solution to equation (6.12) is the Jacobian elliptic function cn of modulus k_{02}, and for the general form we may write

$$S = \text{cn}[f+f_0 : k_{02}] \qquad (6.13)$$

where f_0 is the constant of integration. We shall see presently that we measure f from a "pericenter" appropriately defined when we come to a full analysis of the equation for R; here, following the precedent of the Kepler case, we let $-\omega$ denote the value of f at the point of the first crossing of the x-axis, that is,

$$z = 0, \quad \cos\sigma = 0, \quad S = 0, \quad \text{correspond to} \quad f = -\omega \qquad (6.14)$$

and hence

$$0 = \text{cn}[f_0-\omega : k_{02}] \qquad (6.15)$$

which is satisfied by taking

$$f_0-\omega = -K_{02}, \quad f_0 = \omega - K_{02} \qquad (6.16)$$

where K_{02} is the quarterperiod of the Jacobian elliptic function of modulus k_{02}; hence the solution (6.13) may be written

$$S = \text{cn}[f+\omega-K_{02} : k_{02}] \qquad (6.17)$$

which, on applying a standard relation for the function cn, may be rewritten

$$S = k'_{02}\frac{\text{sn}[f+\omega : k_{02}]}{\text{dn}[f+\omega : k_{02}]} \qquad (6.18)$$

where k'_{02} is the modulus complementary to k_{02}, determined by

$$k_{02}^2 + k'^2_{02} = 1. \qquad (6.19)$$

When $\eta = 0$, then $k_{02} = 0$, $k'_{02} = 1$, dn $= 1$, and we retrieve the solution form for the Kepler case.

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As a third example deserving special notice, we consider the instances where either

$$R' = 0 \quad \text{or} \quad \sigma' = 0. \tag{6.20a,b}$$

The situation (6.20a), apart from the singular case $R = b$, corresponds to the pair of bounding ellipses

$$R = a(1 - e), \quad R = a(1 + e) \tag{6.21a,b}$$

and we note that the inner ellipse (6.21a) disappears when

$$b > a(1 - e) = \frac{p}{1 + e}, \quad \text{or alternatively} \quad \eta > \frac{1}{1 + e}. \tag{6.22a,b}$$

The situation (6.20b), apart from the singular cases $S = \pm 1$, corresponds to the bounding hyperbolae

$$S = -\frac{\beta \pm \sqrt{e^2 + \beta^2 - 1}}{\eta(1 - e^2)}. \tag{6.23}$$

For the reality of the hyperbolae, the two conditions

$$e^2 + \beta^2 \geq 1, \quad \eta(1 - e^2) > \beta - \sqrt{e^2 + \beta^2 - 1} \tag{6.24a,b}$$

must be satisfied. Requirement (6.24b) has to be further subdivided:

$$\text{(i)} \qquad \beta - \sqrt{e^2 + \beta^2 - 1} < \eta(1 - e^2) < \beta + \sqrt{e^2 + \beta^2 - 1} \tag{6.25a}$$

implying the one-boundary hyperbola

$$S = -\frac{\beta - \sqrt{e^2 + \beta^2 - 1}}{\eta(1 - e^2)}. \tag{6.25b}$$

$$\text{(ii)} \qquad \eta(1 - e^2) > \beta + \sqrt{e^2 + \beta^2 - 1} > \beta - \sqrt{e^2 + \beta^2 - 1} \tag{6.26a}$$

implying the two-boundary hyperbolae

$$S = -\frac{\beta - \sqrt{e^2 + \beta^2 - 1}}{\eta(1 - e^2)}, \quad S = -\frac{\beta + \sqrt{e^2 + \beta^2 - 1}}{\eta(1 - e^2)}. \tag{6.26b}$$

The combination of the outer ellipse with either of the bounding hyperbolae constitute a limit set with critical points at the points of intersection of the ellipse with either hyperbola.

7 Analysis of the Generic Equation

Returning to the generic equation (5.20), we note that the quartic on the right of that equation is the product of two quadratic factors, from one of which the linear term is absent. The aim of this section is to produce a transformation that reduces the quartic to a product of two quadratic factors, from *both* of which the linear term will be absent. We proceed as follows.

Taking the first of the two factors on the right of (5.20), we write

$$1 - y^2 = J^2[(1-\delta y)^2 - (y-\delta)^2] = J^2(1-\delta^2)(1-y^2) \tag{7.1ab}$$

from which there immediately follows

$$J^2(1-\delta^2) = 1. \tag{7.2}$$

Turning to the second factor, we set

$$(1-d^2) + 2sy + qy^2 = J^2[A(1-\delta y)^2 + B(y-\delta)^2] \tag{7.3a}$$

$$= J^2[(A+B\delta^2) - 2\delta(A+B)y + (A\delta^2+B)y^2]. \tag{7.3b}$$

The latter equation leads to the three requirements

$$J^2(A+B\delta^2) = 1-d^2 \tag{a}$$

$$J^2\delta(A+B) = -s \tag{b}$$

$$J^2(A\delta^2+B) = q. \tag{c}$$

Adding (c) to (a) yields

$$J^2(1+\delta^2)(A+B) = (1-d^2) + q \tag{d}$$

while subtracting (c) from (a) yields

$$J^2(1-\delta^2)(A-B) = (1-d^2) - q$$

and noting (7.2), the latter becomes

$$A - B = (1-d^2) - q. \tag{e}$$

If we multiply (b) by 2, and add the result to (d), we find

$$J^2(1+\delta)^2(A+B) = (1-d^2) - 2s + q \tag{f}$$

while subtracting the same result from (d) yields

$$J^2(1-\delta)^2(A+B) = (1-d^2) + 2s + q. \tag{g}$$

Dividing (g) by (f) and denoting the quotient by ρ^2, we have

$$\rho^2 = \left(\frac{1-\delta}{1+\delta}\right)^2 = \frac{(1-d^2)+2s+q}{(1-d^2)-2s+q}. \tag{h}$$

Hence if we set

$$\rho = \sqrt{\frac{(1-d^2)+2s+q}{(1-d^2)-2s+q}} \tag{i}$$

we have

$$\frac{1-\delta}{1+\delta} = \pm\rho \tag{j}$$

with solution

$$\delta = \frac{1-\rho}{1+\rho} \tag{k}$$

as the second option arising from replacing ρ by $-\rho$ is excluded by the requirement that $\rho \to 1$ implies $\delta \to 0$. Accordingly we have

$$1-\delta^2 = \frac{4\rho}{(1+\rho)^2}, \qquad J^2 = \frac{(1+\rho)^2}{4\rho}. \tag{l}$$

It follows from (h) that

$$1-\rho^2 = -\frac{4s}{(1-d^2)-2s+q} \tag{m}$$

and from combining (h) with (i), the latter multiplied by 2, we have

$$\begin{aligned}(1+\rho)^2 = 1+\rho^2+2\rho &= \frac{2[(1-d^2)+q]}{[(1-d^2)-2s+q]} + 2\sqrt{\frac{(1-d^2)+2s+q}{(1-d^2)-2s+q}} \\ &= 2\left[\frac{[(1-d^2)+q]+\sqrt{[(1-d^2)-2s+q][(1-d^2)+2s+q]}}{[(1-d^2)-2s+q]}.\right]\end{aligned} \tag{n}$$

Next, we may write

$$\begin{aligned}[(1-d^2)-2s+q][(1-d^2)+2s+q] &= [(1-d^2)+q]^2 - 4s^2 \\ &= [(1-d^2)+q]^2\left[1-\frac{4s^2}{[(1-d^2)+q]^2}\right]\end{aligned} \tag{o}$$

in accordance with which we set

$$(1-2h)^2 = \left[1-\frac{4s^2}{[(1-d^2)+q]^2}\right] \tag{p}$$

which implies

$$h = \tfrac{1}{2}\left[1 - \sqrt{1 - \frac{4s^2}{[(1-d^2)+q]^2}}\,\right], \tag{7.4a}$$

$$1 - h = \tfrac{1}{2}\left[1 + \sqrt{1 - \frac{4s^2}{[(1-d^2)+q]^2}}\,\right] \tag{7.4b}$$

and further, that

$$[(1-d^2) - 2s + q][(1-d^2) + 2s + q] = [(1-d^2)+q]^2[(1-2h]^2$$

and hence

$$\sqrt{[(1-d^2) - 2s + q][(1-d^2) + 2s + q]} = [(1-d^2)+q][(1-2h]. \tag{q}$$

Recalling now relation (n), we see that, in terms of h, (n) can be rewritten as

$$\begin{aligned}(1+\rho)^2 &= 2\left[\frac{[(1-d^2)+q] + [(1-d^2)+q][1-2h]}{[(1-d^2) - 2s + q]}\right] \\ &= \frac{4(1-h)[(1-d^2)+q]}{[(1-d^2) - 2s + q]}.\end{aligned} \tag{r}$$

Returning to relation (k), we utilize both relations (m) and (r), and there follows that

$$\delta = \frac{1-\rho}{1+\rho} = \frac{1-\rho^2}{(1+\rho)^2} = -\frac{s}{[(1-d^2)+q][1-h]}. \tag{7.5}$$

From (f), if we introduce J^2 from (7.2), we find

$$[(1-d^2) - 2s + q] = \frac{(1+\delta)^2}{1-\delta^2}(A+B) = \frac{1+\delta}{1-\delta}(A+B) = \frac{A+B}{\rho}$$

so that, on the introduction of ρ from (i), we see that

$$A + B = \rho[(1-d^2) - 2s + q] = \sqrt{[(1-d^2) - 2s + q][(1-d^2) + 2s + q]}$$

into which we introduce relation (q) to obtain

$$A + B = [(1-d^2)+q][1-2h]. \tag{7.6}$$

From (e), we recall that

$$A - B = [(1-d^2) - q] \tag{7.7}$$

and so, for A and B we have, respectively,

$$A = (1-d^2)(1-h) - hq \tag{7.8a}$$

$$B = -h(1-d^2) + q(1-h). \tag{7.8b}$$

With h determined in terms of d, s, and q from (7.4a), then δ is determined in terms of these quantities from (7.5), and both A and B are determined from (7.8). With J^2 determined from (7.2), all quantities necessary for the decomposition (7.3a) are fully determined algebraically.

By the introduction of (7.1a) and (7.3a) into the model equation (5.20), we have

$$\frac{\Lambda^2}{C^2}y'^2 = J^4[(1-\delta y)^2 - (y-\delta)^2][A(1-\delta y)^2 + B(y-\delta)^2] \tag{7.9a}$$

$$= J^4(1-\delta y)^4\left[1-\left(\frac{y-\delta}{1-\delta y}\right)^2\right]\left[A + B\left(\frac{y-\delta}{1-\delta y}\right)^2\right] \tag{7.9b}$$

or, alternatively,

$$\frac{\Lambda^2}{C^2}\left[\frac{y'}{J^2(1-\delta y)^2}\right]^2 = \left[1-\left(\frac{y-\delta}{1-\delta y}\right)^2\right]\left[A + B\left(\frac{y-\delta}{1-\delta y}\right)^2\right]. \tag{7.10}$$

We may now write

$$Y = \frac{y-\delta}{1-\delta y}, \quad \text{so that} \quad y = \frac{Y+\delta}{1+\delta Y} \tag{7.11a,b}$$

and hence

$$Y'^2 = \frac{(1-\delta y)y' + \delta y'(y-\delta)}{(1-\delta y)^2} = \frac{(1-\delta^2)y'}{(1-\delta y)^2} = \frac{y'}{J^2(1-\delta y)^2} \tag{7.12}$$

and, in terms of Y, equation (7.10) becomes

$$\frac{\Lambda^2}{C^2}Y'^2 = (1-Y^2)[A + BY^2] \tag{7.13}$$

which is the desired form. The treatment of equation (7.13) will be determined by a closer scrutiny of A and B.

We next return to deal individually with the equations for the first integrals (5.17) and (5.19). It is convenient to consider first the latter equation (5.19) for $S = \cos\sigma$, for which we have already noted the form of the solution (6.18) for the particular case of $\beta = 0$.

8 The Equation for $S = \cos\sigma$: Specification of Λ

Equation (5.19) for S has the form (5.20) of the generic equation with the following identification of the parameters:

$$d^2 = 0, \quad s = \eta\beta, \quad q = \eta^2(1-e^2). \tag{8.1}$$

Using the subscript S to identify the algebraic quantities in this case, we have from (7.4)

$$h_S = \tfrac{1}{2}\left[1 - \sqrt{1 - \frac{4\eta^2\beta^2}{[1+\eta^2(1-e^2)]^2}}\right], \tag{8.2a}$$

$$1 - 2h_S = \sqrt{1 - \frac{4\eta^2\beta^2}{[1+\eta^2(1-e^2)]^2}}, \tag{8.2b}$$

$$1 - h_S = \tfrac{1}{2}\left[1 + \sqrt{1 - \frac{4\eta^2\beta^2}{[1+\eta^2(1-e^2)]^2}}\right]. \tag{8.2c}$$

From (7.5), there follows that for the δ-quantity we have

$$\begin{aligned}\delta_S &= -\frac{2\eta\beta}{[1+\eta^2(1-e^2)]\left[1 + \sqrt{1 - \frac{4\eta^2\beta^2}{[1+\eta^2(1-e^2)]^2}}\right]} \\ &= -\frac{2\eta\beta}{[1+\eta^2(1-e^2)] + \sqrt{[1+\eta^2(1-e^2)]^2 - 4\eta^2\beta^2}}.\end{aligned} \tag{8.3}$$

From (7.6) and (7.7) we have, respectively,

$$\begin{aligned}A_S + B_S &= [1+\eta^2(1-e^2)]\sqrt{1 - \frac{4\eta^2\beta^2}{[1+\eta^2(1-e^2)]^2}} \\ &= \sqrt{[1+\eta^2(1-e^2)]^2 - 4\eta^2\beta^2}\end{aligned} \tag{8.4a}$$

$$A_S - B_S = 1 - \eta^2(1-e^2) \tag{8.4b}$$

yielding

$$A_S = \tfrac{1}{2}[1 - \eta^2(1-e^2)] + \tfrac{1}{2}\sqrt{[1+\eta^2(1-e^2)]^2 - 4\eta^2\beta^2} \tag{8.5a}$$

$$B_S = -\tfrac{1}{2}[1 - \eta^2(1-e^2)] + \tfrac{1}{2}\sqrt{[1+\eta^2(1-e^2)]^2 - 4\eta^2\beta^2}. \tag{8.5b}$$

If we now set

$$\zeta = \frac{S - \delta_S}{1 - \delta_S S}, \quad S = \frac{\zeta + \delta_S}{1 + \delta_S \zeta} \tag{8.6a,b}$$

then, in accordance with the procedure leading to (7.13), we have the equation for ζ in the form

$$\frac{\Lambda^2}{C^2}\zeta'^2 = (1 - \zeta^2)[A_S + B_S\zeta^2] \tag{8.7}$$

which, for B_S negative, is in a form amenable to integration.

If we consider the quantity within the radical sign in (8.4a), we see that

$$[1+\eta^2(1-e^2)]^2 - 4\eta^2\beta^2 \lesseqgtr [1+\eta^2(1-e^2)]^2 - 4\eta^2(1-e^2) \quad \text{as} \quad \beta^2 \gtreqless 1 - e^2 \tag{8.8}$$

and hence

$$[1+\eta^2(1-e^2)]^2 - 4\eta^2\beta^2 \lesseqgtr [1-\eta^2(1-e^2)]^2 \quad \text{as} \quad \beta^2+e^2 \gtreqless 1. \tag{8.9}$$

It follows that

$$\sqrt{[1+\eta^2(1-e^2)]^2 - 4\eta^2\beta^2} \lesseqgtr [1-\eta^2(1-e^2)] \quad \text{as} \quad \beta^2+e^2 \gtreqless 1 \tag{8.10}$$

and hence, from (8.5b), we have that

$$B_S \lesseqgtr 0 \quad \text{as} \quad \beta^2+e^2 \gtreqless 1. \tag{8.11}$$

We shall consider the two cases separately.

Case A: $e^2+\beta^2 \geq 1$

In this case, where B_S is negative, we set

$$\Lambda^2 = C^2 A_S = \tfrac{1}{2}C^2\left[[1-\eta^2(1-e^2)] + \sqrt{[1+\eta^2(1-e^2)]^2 - 4\eta^2\beta^2}\right] = \Lambda_1^2 \tag{8.12a}$$

$$k_{S1}^2 = -\frac{B_S}{A_S} = \frac{[1-\eta^2(1-e^2)] - \sqrt{[1+\eta^2(1-e^2)]^2 - 4\eta^2\beta^2}}{[1-\eta^2(1-e^2)] + \sqrt{[1+\eta^2(1-e^2)]^2 - 4\eta^2\beta^2}} \tag{8.12b}$$

where (8.12a) is the defining relation for Λ_1^2. Then equation (8.7) takes the form

$$\zeta'^2 = (1-\zeta^2)[1-k_{S1}^2\zeta^2]. \tag{8.13}$$

In terms of the Jacobian elliptic function sn of modulus k_{S1}, the solution to equation (8.13) is given by

$$\zeta = \mathrm{sn}[f + f_{S0} : k_{S1}] \tag{8.14}$$

where f_{S0} is the constant of integration. From (8.6b) it follows that for S we have

$$S = \frac{\mathrm{sn}[f+f_{S0} : k_{S1}] + \delta_S}{1 + \delta_S\,\mathrm{sn}[f+f_{S0} : k_{S1}]}. \tag{8.15}$$

If we let $-\omega$ denote the value of f at which the orbit makes its first crossing of the x-axis, then

$$z = 0, \quad S = 0, \quad \text{for} \quad f = -\omega \tag{8.16}$$

and from (8.15) at $f = -\omega$, we have

$$\mathrm{sn}[f_{S0} - \omega : k_{S1}] + \delta_S = 0. \tag{8.17}$$

With f_{S0} determined from (8.17), we have the complete solution in (8.15).

We may note from (8.3) that when $\beta = 0$, $\delta_S = 0$, and the condition (8.16) is satisfied by taking $f_{S0} = \omega$, whereby we retrieve the solution (6.9) for that case.

Case B: $e^2 + \beta^2 \leq 1$

In this case where B_S is positive, we make (8.7) amenable to integration by writing it in the form

$$\begin{aligned} \frac{\Lambda^2}{C^2}\zeta'^2 &= (1-\zeta^2)[A_S + B_S - B_S(1-\zeta^2)] \\ &= (A_S + B_S)(1-\zeta^2)\left[1 - \frac{B_S}{A_S + B_S}(1-\zeta^2)\right]. \end{aligned} \tag{8.18}$$

Here, we define the parameters Λ and k_{S2} by setting

$$\Lambda^2 = C^2(A_S + B_S) = C^2\sqrt{[1 + \eta^2(1-e^2)]^2 - 4\eta^2\beta^2} = \Lambda_2^2 \tag{8.19a}$$

$$k_{S2}^2 = \frac{B_S}{A_S + B_S} = \tfrac{1}{2} - \tfrac{1}{2}\frac{1 - \eta^2(1-e^2)}{\sqrt{[1 + \eta^2(1-e^2)]^2 - 4\eta^2\beta^2}} \tag{8.19b}$$

where (8.19a) is the defining relation for Λ_2^2, and where we have introduced (8.4a) and (8.5b). In terms of these parameters, equation (8.18) reads

$$\zeta'^2 = (1-\zeta^2)[1 - k_{S2}^2(1-\zeta^2)]. \tag{8.20}$$

With an eye backward to the solution (6.17) for the particular case $\beta = 0$, we may write the solution to (8.20) in the form

$$\zeta = \text{cn}[f + f_{S0} - K_{S2} : k_{S2}] \tag{8.21}$$

where K_{S2} is the quarterperiod of the Jacobian elliptic function of modulus k_{S2} and f_{S0} is the arbitrary constant introduced by the integration.

Recalling relation (8.6b) for S expressed in terms of ζ, we see that the solution for S has the form

$$S = \frac{\text{cn}[f + f_{S0} - K_{S2} : k_{S2}] + \delta_S}{1 + \delta_S\,\text{cn}[f + f_{S0} - K_{S2} : k_{S2}]}. \tag{8.22}$$

For the Jacobian elliptic functions, we note the relation

$$\text{cn}[f + f_{S0} - K_{S2} : k_{S2}] = k'_{S2}\frac{\text{sn}[f + f_{S0} : k_{S2}]}{\text{dn}[f + f_{S0} : k_{S2}]} \tag{8.23}$$

where k'_{S2} is the complementary modulus, defined by

$$k'^2_{S2} = 1 - k^2_{S2}. \tag{8.24}$$

When (8.23) is introduced into (8.22), we have for S

$$S = \frac{k'_{S2}\,\text{sn}[f + f_{S0} : k_{S2}] + \delta_S\,\text{dn}[f + f_{S0} : k_{S2}]}{\text{dn}[f + f_{S0} : k_{S2}] + \delta_S k'_{S2}\,\text{sn}[f + f_{S0} : k_{S2}]}. \tag{8.25}$$

If we impose the condition that f takes the value $-\omega$ at the point where the orbit first crosses the x-axis, then

$$z = 0, \quad S = 0, \quad \text{when} \quad f = -\omega \tag{8.26}$$

and we see that the condition is satisfied if we set

$$k'_{S2}\,\mathrm{sn}[f_{S0} - \omega : k_{S2}] + \delta_S\,\mathrm{dn}[f_{S0} - \omega : k_{S2}] = 0. \tag{8.27}$$

Accordingly, in this case the solution is given by (8.25) with f_{S0} determined from (8.27).

From (8.3), we observe that when $\beta = 0$, $\delta_S = 0$, and in that case condition (8.27) is satisfied by taking $f_{S0} = \omega$, and we retrieve the solution (6.18) for that case.

Summary Remarks on the *S*-equation and the Range of β

1. When $\beta = 0$, the configuration is symmetric, and the two cases are respectively identical with the ranges $e^2 \gtreqless 1$ for the parameter e.
2. For $\beta \neq 0$, as β increases from 0 to 1, the range of Case A expands to $e^2 \geq 1 - \beta^2$, while the range of Case B shrinks to $e^2 \leq 1 - \beta^2$.
3. In the limit of $\beta \to 1$ (as the lighter mass-center shrinks to zero), the range of Case A becomes the entire range $e^2 \geq 0$, while the range of Case B disappears.

Finally to summarize the solution forms, we introduce the variable f_S by setting

$$f_S = f + f_{S0} \tag{8.28}$$

and the solutions are respectively

Case A: $e^2 + \beta^2 \geq 1$*:*

From (8.15) and (8.17)

$$S = \cos\sigma = \frac{\mathrm{sn}[f_S : k_{S1}] + \delta_S}{1 + \delta_S\,\mathrm{sn}[f_S : k_S]} \tag{8.29a}$$

with f_{S0} determined from

$$\mathrm{sn}[f_{S0} - \omega : k_{S1}] + \delta_S = 0. \tag{8.29b}$$

Case B: $e^2 + \beta^2 \leq 1$*:*

From (8.25) and (8.27)

$$S = \cos\sigma = \frac{k'_{S2}\,\mathrm{sn}[f_S : k_{S2}] + \delta_S\,\mathrm{dn}[f_S : k_{S2}]}{\mathrm{dn}[f_S : k_{S2}] + \delta_S k'_{S2}\,\mathrm{sn}[f_S : k_{S2}]} \tag{8.30a}$$

with f_{S0} determined from

$$k'_{S2}\,\mathrm{sn}[f_{S0}-\omega : k_{S2}] + \delta_S\,\mathrm{dn}[f_{S0}-\omega : k_{S2}] = 0. \tag{8.30b}$$

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At this point we note that reality for the quantities in (8.3), (8.5), and (8.12) is ensured if

$$[1+\eta^2(1-e^2)]^2 \geq 4\eta^2\beta^2 \tag{8.31a}$$

or alternatively

$$1+\eta^2(1-e^2) \geq 2\eta\beta. \tag{8.31b}$$

It is convenient to introduce the parameter λ, defined as

$$\lambda = \frac{b}{a} = \eta(1-e^2) \tag{8.32}$$

so that, in terms of λ, requirement (8.31) reads

$$\lambda^2 - 2\lambda\beta + (1-e^2) \geq 0. \tag{8.33}$$

The expression in λ, on the left of (8.33), has zeros

$$\lambda_1 = \beta - \sqrt{e^2+\beta^2-1}, \quad \lambda_2 = \beta + \sqrt{e^2+\beta^2-1} \tag{8.34a,b}$$

leading us again to consider the two cases separately:

Case A: $e^2+\beta^2 \geq 1$*:*

In this case, condition (8.33) is satisfied outside the interval between the zeros, that is, for

(i) $\lambda \leq \lambda_1$ and (ii) $\lambda \geq \lambda_2$.

When we come to analyze the system for $\lambda \geq \lambda_1$ in Subsection 8A, it will be evident that the entire range $\lambda \geq \lambda_1$ must be treated separately. Accordingly, the interval

$$0 \leq \lambda \leq \lambda_1 = \beta - \sqrt{e^2+\beta^2-1} \tag{8.35}$$

defines the *primary* λ*-range* in which the solution form (8.15) is valid for $e^2+\beta^2 \geq 1$.

Case B: $e^2+\beta^2 \leq 1$*:*

Here both zeros of (8.33) are complex, and the expression on the left of (8.33) is nonnegative for all real positive λ, so that the solution form (8.25) is valid over the entire λ-range, $\lambda \geq 0$ when $e^2+\beta^2 \leq 1$.

We now return to consider Case A in detail.

Case A: $e^2 + \beta^2 \geq 1$*:*

Rewriting the S-equation (5.19) in terms of λ, we have

$$(1-e^2)\frac{\Lambda^2}{C^2}S'^2 = (1-S^2)[\lambda^2 S^2 + 2\beta\lambda S + (1-e^2)] \tag{8.36a}$$

$$= (1-S)(1+S)[\lambda S + (\beta-\gamma)][\lambda S + (\beta+\gamma)] \tag{8.36b}$$

where, for convenience, we have introduced the notation

$$\gamma = \sqrt{e^2 + \beta^2 - 1}. \tag{8.37}$$

When we arrange the zeros of the quartic on the right of (8.36) in ascending order, we see that in the primary range (8.35), they form the sequence shown:

$$0 \leq \lambda \leq \beta - \gamma: \left\{-\frac{\beta+\gamma}{\lambda}, -\frac{\beta-\gamma}{\lambda}, -1, 1\right\}. \tag{8.38a:b}$$

We now recall that in effecting the reduction to the solution form (8.15), we began by dealing with the quartic as a product of two quadratic factors, respectively formed from the linear factors by combining the "upper" pair and the "lower" pair as indicated by the ordering of the zeros. Moreover, in this primary range (where $0 \leq \lambda \leq \beta - \gamma$), equation (8.36) balances for the full S-range, $-1 \leq S \leq 1$, so that the solution extends through the full range of $S(-1, 1)$.

When we move into the supplementary λ-range, $\lambda \geq \beta - \gamma$, we first consider the range

$$\beta - \gamma \leq \lambda \leq \beta + \gamma \tag{8.39}$$

and observe that the relative position of the second and third zeros in (8.38) is now reversed. This requires a restructuring of the quadratics in accordance with their formation from the linear factors by combining the "upper" and "lower" pairs as indicated by the sequence of zeros. The ordering now has the form

$$\beta - \gamma \leq \lambda \leq \beta + \gamma: \left\{-\frac{\beta+\gamma}{\lambda}, -1, -\frac{\beta-\gamma}{\lambda}, 1\right\}. \tag{8.40a:b}$$

In this λ-range, the solutions are restricted by the requirement

$$-1 \leq -\frac{\beta-\gamma}{\lambda} \leq S \leq 1 \ \left(\leq \frac{\beta+\gamma}{\lambda}\right). \tag{8.41}$$

Here there is a zone about the second (lighter) primary mass from which orbits are excluded. This zone is bounded by the hyperbola branch

$$S = -\frac{\beta-\gamma}{\lambda} \tag{8.42}$$

to the inside of which the orbits have no access. Outside the hyperbola, the orbits may perform some vigorous oscillations close to the bounding hyperbola as the particle is subjected to the twin forces of attraction and exclusion.

Moving further into the λ-range, when λ exceeds $\beta+\gamma$, the relative position of the first and second zeros in (8.40) is reversed; the ordering now has the form

$$\lambda \geq \beta+\gamma:\ \left\{-1, -\frac{\beta+\gamma}{\lambda}, -\frac{\beta-\gamma}{\lambda}, 1\right\} \tag{8.43a:b}$$

and clearly the formation of the quadratic factors from the linear factors (as indicated by the "upper" and "lower" pairs of zeros) is unaffected. Here there are two distinct orbit regions depending on the signs of the factors

$$S+\frac{\beta+\gamma}{\lambda}, \quad S+\frac{\beta-\gamma}{\lambda} \tag{8.44}$$

(i) when both factors are negative, we have the orbit region

$$-1 \leq S \leq -\frac{\beta+\gamma}{\lambda} \tag{8.45a}$$

(ii) when both factors are positive, we have the orbit region

$$-\frac{\beta-\gamma}{\lambda} \leq S \leq 1 \tag{8.45b}$$

(iii) when the factors have different signs, we have the

$$S\text{-Exclusion Zone:}\quad \left(-\frac{\beta-\gamma}{\lambda}, -\frac{\beta+\gamma}{\lambda}\right). \tag{8.45c}$$

Hence, when the λ-factor exceeds $\beta+\gamma$, a second bounding hyperbola appears defining a second boundary to the exclusion zone and introducing the second satellite system around the second (lighter) primary. We may refer to the two bounding hyperbola curves as the upper and lower boundaries of the exclusion zone, thereby defining a bipolar system of satellite orbits.

In the particular case when $e = 1$, the upper bounding hyperbola coincides with the x-axis, and the satellites of the first (heavier) primary are now confined to the upper half-plane. This case requires special attention and is analyzed in Section 12.

When the parameter e exceeds unity, the upper bounding hyperbola moves into the "upper" half-plane so that the bipolar system of satellite orbits becomes fully "polarized". This range also requires special attention and is fully analyzed in Section 11.

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The next step is to establish the solution form valid in the supplementary range

$$\lambda \geq \beta-\gamma \tag{8.46}$$

with the zeros ordered according to (8.40) or (8.43). This is the subject of Subsection 8A.

8A. Solution Form for the Supplementary Range: ($\lambda \geq \beta - \gamma$)

Rewriting the differential equation (8.36) in accord with the ordering (8.40) or (8.43), we have

$$\begin{aligned}\frac{(1-e^2)}{\lambda^2}\frac{\Lambda^2}{C^2}S'^2 &= -(S-1)\Big[S+\frac{\beta-\gamma}{\lambda}\Big](S+1)\Big[S+\frac{\beta+\gamma}{\lambda}\Big]\\ &= -\Big[S^2-\Big(1-\frac{\beta-\gamma}{\lambda}\Big)S-\frac{\beta-\gamma}{\lambda}\Big]\\ &\qquad\cdot\Big[S^2+\Big(1+\frac{\beta+\gamma}{\lambda}\Big)S+\frac{\beta+\gamma}{\lambda}\Big].\end{aligned} \tag{8A.1}$$

Considering the first quadratic factor on the right of (8A.1),

$$S^2-\Big(1-\frac{\beta-\gamma}{\lambda}\Big)S-\frac{\beta-\gamma}{\lambda} = \Big[S-\tfrac{1}{2}\Big(1-\frac{\beta-\gamma}{\lambda}\Big)\Big]^2-\tfrac{1}{4}\Big(1+\frac{\beta-\gamma}{\lambda}\Big)^2 \tag{8A.2}$$

which suggests that we set

$$S-\tfrac{1}{2}\Big(1-\frac{\beta-\gamma}{\lambda}\Big) = \tfrac{1}{2}\Big(1+\frac{\beta-\gamma}{\lambda}\Big)\zeta, \quad S = \tfrac{1}{2}\Big(1+\frac{\beta-\gamma}{\lambda}\Big)\zeta+\tfrac{1}{2}\Big(1-\frac{\beta-\gamma}{\lambda}\Big) \tag{8A.3a,b}$$

and the first quadratic factor (8A.2) expressed in terms of ζ becomes

$$\tfrac{1}{4}\Big(1+\frac{\beta-\gamma}{\lambda}\Big)^2(\zeta^2-1). \tag{8A.4}$$

With S given by (8A.3b), we have

$$S^2 = \tfrac{1}{4}\Big(1+\frac{\beta-\gamma}{\lambda}\Big)^2\zeta^2+\tfrac{1}{2}\Big[1-\Big(\frac{\beta-\gamma}{\lambda}\Big)^2\Big]\zeta+\tfrac{1}{4}\Big[1-\frac{\beta-\gamma}{\lambda}\Big]^2 \tag{8A.5a}$$

$$\Big(1+\frac{\beta+\gamma}{\lambda}\Big)S+\frac{\beta+\gamma}{\lambda} = \tfrac{1}{2}\Big[1+2\frac{\beta}{\lambda}+\frac{\beta^2-\gamma^2}{\lambda^2}\Big]\zeta+\tfrac{1}{2}\Big[1+2\frac{\beta}{\lambda}+4\frac{\gamma}{\lambda}-\frac{\beta^2-\gamma^2}{\lambda^2}\Big] \tag{8A.5b}$$

and, on combining the quantities in (8A.5), we find that the second factor on the right of (8A.1), when expressed in terms of ζ, reads

$$\tfrac{1}{4}\Big(1+\frac{\beta-\gamma}{\lambda}\Big)^2\zeta^2+\Big(1+\frac{\gamma}{\lambda}\Big)\Big(1+\frac{\beta-\gamma}{\lambda}\Big)\zeta+\tfrac{1}{4}\Big(3-\frac{\beta-\gamma}{\lambda}\Big)\Big(1+\frac{\beta+3\gamma}{\lambda}\Big). \tag{8A.6}$$

We introduce (8A.4) and (8A.6) into the right side of (8A.1), and if we also use (8A.3b) to substitute for S' on the left side, then, after multiplying across by $4\lambda^2$, we obtain the differential equation for ζ in the form

$$\begin{aligned}&4(1-e^2)\frac{\Lambda^2}{C^2}\zeta'^2 =\\ &\quad(1-\zeta^2)\Big[[3\lambda-(\beta-\gamma)][\lambda+\beta+3\gamma]+4(\lambda+\gamma)(\lambda+\beta-\gamma)\zeta\\ &\qquad\qquad+(\lambda+\beta-\gamma)^2\zeta^2\Big].\end{aligned} \tag{8A.7}$$

Comparing now the right side of (8A.7) with the right side of (5.20) shows that here we have

$$1 - d^2 = [3\lambda - (\beta - \gamma)][\lambda + \beta + 3\gamma],\ s = 2(\lambda + \gamma)[\lambda + \beta - \gamma],\ q = (\lambda + \beta - \gamma)^2 \tag{8A.8a,b,c}$$

from which there follows that

$$1 - d^2 + q = 4[(\lambda + \gamma)^2 + \beta(\lambda - \gamma)], \quad 1 - d^2 - q = 2[(\lambda + \gamma)^2 + 4\lambda\gamma - \beta^2]. \tag{8A.8d,e}$$

From relations (8A.8b,d) we have

$$\frac{2s}{1 - d^2 + q} = \frac{(\lambda + \gamma)(\lambda + \beta - \gamma)}{(\lambda + \gamma)^2 + \beta(\lambda - \gamma)} \tag{8A.9}$$

which (cf. relations above (7.4)) leads to

$$\begin{aligned}(1 - 2h)^2 &= 1 - \frac{4s^2}{(1 - d^2 + q)^2} = 1 - \frac{(\lambda + \gamma)^2(\lambda + \beta - \gamma)^2}{[(\lambda + \gamma)^2 + \beta(\lambda - \gamma)]^2} \\ &= \frac{[(\lambda + \gamma)^2 + \beta(\lambda - \gamma)]^2 - (\lambda + \gamma)^2(\lambda + \beta - \gamma)^2}{[(\lambda + \gamma)^2 + \beta(\lambda - \gamma)]^2}\end{aligned} \tag{8A.10}$$

The numerator is the difference between two squares and, when we expand and recombine, we find for the the numerator

$$4\lambda\gamma(\lambda + \beta + \gamma)[\lambda - (\beta - \gamma)] = 4\lambda\gamma[(\lambda + \gamma)^2 - \beta^2]. \tag{8A.11}$$

It follows from (8A.10) that

$$1 - 2h = \frac{2\sqrt{\lambda\gamma[(\lambda + \gamma)^2 - \beta^2]}}{(\lambda + \gamma)^2 + \beta(\lambda - \gamma)} \tag{8A.12}$$

and hence

$$h = \tfrac{1}{2} - \frac{\sqrt{\lambda\gamma[(\lambda + \gamma)^2 - \beta^2]}}{(\lambda + \gamma)^2 + \beta(\lambda - \gamma)} \tag{8A.13a}$$

$$1 - h = \tfrac{1}{2} + \frac{\sqrt{\lambda\gamma[(\lambda + \gamma)^2 - \beta^2]}}{(\lambda + \gamma)^2 + \beta(\lambda - \gamma)}. \tag{8A.13b}$$

Introducing relations (8A.9) and (8A.13b) into (7.5), we obtain for δ,

$$\delta_S^* = -\frac{(\lambda + \gamma)(\lambda + \beta - \gamma)}{(\lambda + \gamma)^2 + \beta(\lambda - \gamma) + 2\sqrt{\lambda\gamma[(\lambda + \gamma)^2 - \beta^2]}} \tag{8A.14}$$

and similarly from (7.6) and (7.7), we obtain

$$A_S^* + B_S^* = 8\sqrt{\lambda\gamma[(\lambda + \gamma)^2 - \beta^2]} \tag{8A.15a}$$

$$A_S^* - B_S^* = 2[(\lambda + \gamma)^2 + 4\lambda\gamma - \beta^2] \tag{8A.15b}$$

and hence

$$\begin{aligned} A_S^* &= 4\sqrt{\lambda\gamma[(\lambda+\gamma)^2-\beta^2]} + [(\lambda+\gamma)^2 + 4\lambda\gamma - \beta^2] \\ &= \left[\sqrt{(\lambda+\gamma)^2-\beta^2} + 2\sqrt{\lambda\gamma}\right]^2 \geq 0 \end{aligned} \tag{8A.16a}$$

$$\begin{aligned} B_S^* &= 4\sqrt{\lambda\gamma[(\lambda+\gamma)^2-\beta^2]} - [(\lambda+\gamma)^2 + 4\lambda\gamma - \beta^2] \\ &= -\left[\sqrt{(\lambda+\gamma)^2-\beta^2} - 2\sqrt{\lambda\gamma}\right]^2 \leq 0. \end{aligned} \tag{8A.16b}$$

For the reality of the quantities in (8A.14) to (8A.16), it suffices that

$$(\lambda+\gamma) \geq \beta \quad \text{or} \quad \lambda \geq \beta - \gamma \tag{8A.17}$$

which is the definition of the entire supplementary λ-range (8.46).

When we introduce the transformations

$$Y = \frac{\zeta - \delta_S^*}{1 - \delta_S^*\zeta}, \quad \zeta = \frac{Y + \delta_S^*}{1 + \delta_S^* Y} \tag{8A.18a,b}$$

then, in terms of Y, equation (8A.7) becomes

$$4(1-e^2)\frac{\Lambda^2}{C^2}Y'^2 = (1-Y^2)\left[A_S^* + B_S^* Y^2\right]. \tag{8A.19}$$

If we divide across by A_S^*, the equation reads

$$4(1-e^2)\frac{\Lambda^2}{C^2 A_S^*}Y'^2 = (1-Y^2)\left[1 - k_{S0}^2 Y^2\right] \tag{8A.20}$$

where we have written

$$k_{S0}^2 = -\frac{B_S^*}{A_S^*} = \left[\frac{2\sqrt{\lambda\gamma} - \sqrt{(\lambda+\gamma)^2-\beta^2}}{2\sqrt{\lambda\gamma} + \sqrt{(\lambda+\gamma)^2-\beta^2}}\right]^2 \leq 1 \tag{8A.21}$$

and we take k_{S0} to be the absolute value of the quantity exhibited within the square brackets on the right side of (8A.21). Hence if we make the identification

$$\Lambda^2 = \frac{C^2 A_S^*}{4(1-e^2)} = \frac{1}{4}\frac{C^2}{(1-e^2)}\left[2\sqrt{\lambda\gamma} + \sqrt{(\lambda+\gamma)^2-\beta^2}\right]^2 = \Lambda_0^2 \tag{8A.22}$$

then equation (8A.20) takes the standard form

$$Y'^2 = (1-Y^2)\left[1 - k_{S0}^2 Y^2\right] \tag{8A.23}$$

with solution in the form

$$Y = \mathrm{sn}[f + f_{S0} : k_{S0}] \tag{8A.24}$$

where again f_{S0} represents the constant of integration. From (8A.18b), we therefore have

$$\zeta = \frac{\mathrm{sn}[f + f_{S0}: k_{S0}] + \delta_S^*}{1 + \delta_S^* \,\mathrm{sn}[f + f_{S0}: k_{S0}]} \tag{8A.25}$$

and from (8A.3b), we have as solution for $S = \cos\sigma$,

$$S = \frac{1}{2\lambda} \times \frac{[(1+\delta_S^*)\lambda + (1-\delta_S^*)(\beta-\gamma)]\,\mathrm{sn}[f+f_{S0}: k_{S0}] + [(1+\delta_S^*)\lambda - (1-\delta_S^*)(\beta-\gamma)]}{1+\delta_S^*\,\mathrm{sn}[f+f_{S0}: k_{S0}]} \tag{8A.26}$$

with the constant of integration to be determined from the crossing of one or the other of the axes.

The solution form (8A.26) is valid in the range $\lambda > \beta - \gamma$; there is one point in the range that merits attention. The modulus k_{S0} as given by (8A.21) vanishes at

$$\lambda = (\beta + \gamma) \tag{8A.27}$$

and it can be easily checked that the factor within the square brackets on the right of (8A.21) is positive for $\lambda < \beta + \gamma$ and negative for $\lambda > \beta + \gamma$ and vanishes for $\lambda = \beta + \gamma$. Accordingly, when we write

$$k_{S0} = \pm \frac{2\sqrt{\lambda\gamma} - \sqrt{(\lambda+\gamma)^2 - \beta^2}}{2\sqrt{\lambda\gamma} + \sqrt{(\lambda+\gamma)^2 - \beta^2}} \leq 1 \tag{8A.28}$$

it is with the understanding that, in using the formula, we take

(i) the positive sign when $\beta - \gamma < \lambda < \beta + \gamma$ (8A.29a)

(ii) the negative sign when $\lambda > \beta + \gamma$ (8A.29b)

when inserting k_{S0} into the solution form (8A.26).

8B. Recapitulation of Solution Forms for the *S*-equation

For the S-equation, there are three distinct ranges.

Case A: $e^2 + \beta^2 \geq 1$:

*Range AS*1: $0 \leq \lambda \leq \beta - \gamma$, then with

(i) $\delta = \delta_S$ determined from (8.3)

(ii) $k = k_{S1}$ determined from (8.12b)

(iii) $\Lambda = \Lambda_1$ determined from (8.12a)

the solution form is given by (8.15).

*Range AS*2: $\lambda \geq \beta - \gamma$, then with

(i) $\delta = \delta_S^*$ determined from (8A.14)
(ii) $k = k_{S0}$ determined from (8A.21) or (8A.28/29)
(iii) $\Lambda = \Lambda_0$ determined from (8A.22)
the solution form is given by (8A.26).

Case B: $e^2 + \beta^2 \le 1$:

Range BS: $\lambda \ge 0$, then with

(i) $\delta = \delta_S$ determined from (8.3)
(ii) $k = k_{S2}$ determined from (8.19b)
(iii) $\Lambda = \Lambda_2$ determined from (8.19a)
the solution form is given by (8.25).

In Range $AS2$, there is a transition at $\lambda = \beta + \gamma$, where $k_{S0} = 0$, and the sign change indicated in (8A.28) occurs.

9 The Equation for *R*

Equation (5.17) for v has the form (5.20) of the generic equation with the following identification of the parameters:

$$d^2 = \eta^2, \quad s = -\eta^2 e, \quad q = -\eta^2 e^2. \tag{9.1}$$

We use the subscript v to identify the algebraic quantities in this case, and from the introduction of (9.1) into (7.4), we have

$$h_v = \tfrac{1}{2}\left[1 - \sqrt{1 - \frac{4\eta^4 e^2}{[1-\eta^2(1+e^2)]^2}}\right], \tag{9.2a}$$

$$1 - 2h_v = \sqrt{1 - \frac{4\eta^4 e^2}{[1-\eta^2(1+e^2)]^2}}, \tag{9.2b}$$

$$1 - h_v = \tfrac{1}{2}\left[1 + \sqrt{1 - \frac{4\eta^4 e^2}{[1-\eta^2(1+e^2)]^2}}\right]. \tag{9.2c}$$

From (7.5), there follows for the δ-quantity

$$\begin{aligned}\delta_v &= \frac{2\eta^2 e}{[1-\eta^2(1+e^2)]\left[1 + \sqrt{1 - \frac{4\eta^4 e^2}{[1-\eta^2(1+e^2)]^2}}\right]} \\ &= \frac{2\eta^2 e}{[1-\eta^2(1+e^2)] + \sqrt{[1-\eta^2(1+e^2)]^2 - 4\eta^4 e^2}}\end{aligned} \tag{9.3}$$

and from (7.6) and (7.7) we have, respectively,

$$\begin{aligned} A_v + B_v &= [1-\eta^2(1+e^2)]\sqrt{1-\frac{4\eta^4 e^2}{[1-\eta^2(1+e^2)]^2}} \\ &= \sqrt{[1-\eta^2(1+e^2)]^2-4\eta^4e^2} \\ &= \sqrt{[1-\eta^2(1-e^2)]^2-4\eta^2e^2} \qquad (9.4a) \end{aligned}$$

$$A_v - B_v = 1-\eta^2(1-e^2) \qquad (9.4b)$$

and so, for A_v and B_v individually, we have, respectively,

$$A_v = \tfrac{1}{2}[1-\eta^2(1-e^2)] + \tfrac{1}{2}\sqrt{[1-\eta^2(1-e^2)]^2-4\eta^2e^2} \qquad (9.5a)$$

$$B_v = -\tfrac{1}{2}[1-\eta^2(1-e^2)] + \tfrac{1}{2}\sqrt{[1-\eta^2(1-e^2)]^2-4\eta^2e^2} \qquad (9.5b)$$

and we note that A_v is always positive while B_v is always negative (or zero).

With δ_v as given by (9.3), we introduce the auxiliary variable w by setting

$$w = \frac{v-\delta_v}{1-\delta_v v}, \quad v = \frac{w+\delta_v}{1+\delta_v w} \qquad (9.6a,b)$$

and the differential equation for w, corresponding to (7.13), takes the form

$$\frac{\Lambda^2}{C^2}w'^2 = (1-w^2)[A_v + B_v w^2] \qquad (9.7a)$$

or, alternatively,

$$\frac{\Lambda^2}{C^2 A_v}w'^2 = (1-w^2)[1+\frac{B_v}{A_v}w^2]. \qquad (9.7b)$$

From Section 3.8, we now recall that for

Case A: $e^2+\beta^2 \geq 1$: we have defined Λ^2 by (8.12a), which we have denoted by Λ_1^2, while for

Case B: $e^2+\beta^2 \leq 1$: we have defined Λ^2 by (8.19a), which we have denoted by Λ_2^2.

From (8.12a) we see that

$$e^2+\beta^2 \geq 1: \frac{\Lambda_1^2}{C^2A_v} = \frac{[1-\eta^2(1-e^2)]+\sqrt{[1+\eta^2(1-e^2)]^2-4\eta^2\beta^2}}{[1-\eta^2(1-e^2)]+\sqrt{[1-\eta^2(1-e^2)]^2-4\eta^2e^2}} = \frac{1}{j_{v1}^2} \qquad (9.8a)$$

which is the defining relation for j_{v1}; similarly, from (8.19a) we have that

$$e^2+\beta^2 \leq 1: \frac{\Lambda_2^2}{C^2A_v} = \frac{2\sqrt{[1+\eta^2(1-e^2)]^2-4\eta^2\beta^2}}{[1-\eta^2(1-e^2)]+\sqrt{[1-\eta^2(1-e^2)]^2-4\eta^2e^2}} = \frac{1}{j_{v2}^2} \qquad (9.8b)$$

which is the defining relation for j_{v2}; we also set

$$k_v^2 = -\frac{B_v}{A_v} = \frac{[1-\eta^2(1-e^2)] - \sqrt{[1-\eta^2(1-e^2)]^2 - 4\eta^2 e^2}}{[1-\eta^2(1-e^2)] + \sqrt{[1-\eta^2(1-e^2)]^2 - 4\eta^2 e^2}} \tag{9.9}$$

and if we use the symbol j_v to cover both cases in (9.8), namely, we set

$$j_v = j_{v1} \qquad \text{when} \quad e^2 + \beta^2 \geq 1 \tag{9.10a}$$

$$j_v = j_{v2} \qquad \text{when} \quad e^2 + \beta^2 \leq 1, \tag{9.10b}$$

then equation (9.7b) may be written

$$\frac{1}{j_v^2} w'^2 = (1 - w^2)[1 - k_v^2 w^2]. \tag{9.11}$$

If we write

$$f_v = j_v f \tag{9.12}$$

then equation (9.11) may be written

$$\left(\frac{\mathrm{d}w}{\mathrm{d}f_v}\right)^2 = (1 - w^2)[1 - k_v^2 w^2]. \tag{9.13}$$

The solution to equation (9.13) has the form

$$w = \operatorname{sn}[f_v + f_{v0} : k_v] \tag{9.14}$$

where f_{v0} is the constant of integration. From (9.6), we therefore have

$$v = \frac{\operatorname{sn}[f_v + f_{v0} : k_v] + \delta_v}{1 + \delta_v \operatorname{sn}[f_v + f_{v0} : k_v]} \tag{9.15}$$

and it follows that

$$v' = j_v^2 (1 - \delta_v^2) \frac{\operatorname{cn}[f_v + f_{v0} : k_v] \cdot \operatorname{dn}[f_v + f_{v0} : k_v]}{\left[1 + \delta_v \operatorname{sn}[f_v + f_{v0} : k_v]\right]^2}. \tag{9.16}$$

Hence, if we impose the condition

$$f = 0 \quad (f_v = 0) \quad \text{implies} \quad v' = 0 \tag{9.17}$$

then the condition is satisfied, provided

$$\operatorname{cn}[f_{v0} : k_v] = 0. \tag{9.18}$$

We now recall that if K_v is the quarterperiod of the Jacobian elliptic function, then

$$\operatorname{cn}[f_v \pm K_v : k_v] = \mp k_v' \frac{\operatorname{sn}[f_v : k_v]}{\operatorname{dn}[f_v : k_v]} \tag{9.19}$$

where k'_v is the complementary modulus, given by

$$k_v'^2 = 1 - k_v^2. \tag{9.20}$$

Setting $f_v = 0$ in relation (9.19) yields

$$\text{cn}[\pm K_v : k_v] = \mp k'_v \, \text{sn}[0 : k_v] = 0 \tag{9.21}$$

so that condition (9.18) is satisfied if we take

$$f_{v0} = K_v \tag{9.22}$$

and the solution (9.14) for w assumes the form

$$w = \text{sn}[f_v + K_v : k_v] = \frac{\text{cn}[f_v : k_v]}{\text{dn}[f_v : k_v]} \tag{9.23}$$

and the solution form (9.15) for v becomes

$$v = \frac{\text{cn}[f_v : k_v] + \delta_v \, \text{dn}[f_v : k_v]}{\text{dn}[f_v : k_v] + \delta_v \, \text{cn}[f_v : k_v]} \tag{9.24}$$

which is the sought-for form of the solution.

With this form for v, and recalling (5.13), we have the following form for u, wherein we note the identification of f_v with $j_v f$ from (9.12)

$$u = \frac{1}{p}(1 + ev) = \frac{1}{p}\left[\frac{(1 + e\delta_v)\, \text{dn}[f_v : k_v] + (e + \delta_v)\, \text{cn}[f_v : k_v]}{\text{dn}[f_v : k_v] + \delta_v \, \text{cn}[f_v : k_v]}\right] \tag{9.25}$$

and recalling (5.9), we have the solution form for R:

$$R = p \cdot \frac{\text{dn}[f_v : k_v] + \delta_v \, \text{cn}[f_v : k_v]}{(1 + e\delta_v)\, \text{dn}[f_v : k_v] + (e + \delta_v)\, \text{cn}[f_v : k_v]}. \tag{9.26}$$

This may be put in an alternative form if we introduce the alternate set of parameters

$$p_* = \frac{p}{1 + e\delta_v}, \quad e_* = \frac{e + \delta_v}{1 + e\delta_v} \tag{9.27a,b}$$

so that

$$e = \frac{e_* - \delta_v}{1 - e_*\delta_v}, \quad p = \frac{1 - \delta_v^2}{1 - e_*\delta_v} p_* \tag{9.28a,b}$$

and the solution (9.26) may be written

$$R = p_* \frac{\text{dn}[f_v : k_v] + \delta_v \, \text{cn}[f_v : k_v]}{\text{dn}[f_v : k_v] + e_* \, \text{cn}[f_v : k_v]} \tag{9.29}$$

which takes the familiar Kepler form when $\eta \to 0$ so that $\delta_v \to 0$, $k_v \to 0$, and $j_v \to 1$.

Recalling formula (9.3) for δ_v, we set

$$d_0 = \frac{2}{[1 - \eta^2(1 + e^2)] + \sqrt{[1 - \eta^2(1 + e^2)]^2 - 4\eta^4 e^2}} \tag{9.30}$$

so that (9.3) can be written

$$\delta_v = \eta^2 d_0 e. \tag{9.31}$$

Introducing this form for δ_v into (9.27b), we have

$$e_* = e \cdot \frac{1 + \eta^2 d_0}{1 + \eta^2 d_0 e^2} \quad \text{or} \quad \frac{e}{e_*} = \frac{1 + \eta^2 d_0 e^2}{1 + \eta^2 d_0} \tag{9.32a,b}$$

and if we set

$$d = d_0 \frac{e}{e_*} = \frac{2}{[1 - \eta^2(1 + e^2)] + \sqrt{[1 - \eta^2(1 + e^2)]^2 - 4\eta^4 e^2}} \cdot \frac{1 + \eta^2 d_0 e^2}{1 + \eta^2 d_0} \tag{9.33}$$

we see that for δ_v, we can write

$$\delta_v = \eta^2 d e_* \tag{9.34}$$

which is the expression aligned with the solution form (9.29).

From (9.9), since the right-hand side vanishes for $\eta^2 = 0$ as well as for $e^2 = 0$, we may write

$$k_v^2 = \eta^2 e^2 g_0^2 \tag{9.35}$$

as the defining relation for g_0. By binomial expansion, it can be shown that for $\eta^2 < 1/(1 - e^2)$,

$$k_v^2 = \eta^2 e^2 [1 + 2\eta^2 + \eta^4(2 + e^2) + \cdots]$$

so that

$$g_0^2 = 1 + 2\eta^2 + \eta^4(2 + e^2) + \cdots .$$

If we set

$$g^2 = \left(\frac{e}{e_*}\right)^2 g_0^2 = \left(\frac{1 + \eta^2 d_0 e^2}{1 + \eta^2 d_0}\right)^2 g_0^2 \tag{9.36}$$

then we have

$$k_v^2 = \eta^2 g^2 e_*^2 \tag{9.37}$$

as the expression to match relation (9.34) for δ_v.

Although the form (9.29) for the solution has a neater appearance than does the form (9.26), we find it more convenient to use the latter form (9.26) when deriving the time-angle relation in Section 10.

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In the above treatment of the R-equation, we note that real values for the quantities (9.3) to (9.5) are ensured if

$$1 - \eta^2(1-e^2) \geq 2e\eta. \tag{9.38}$$

Expressed in terms of λ, as defined in (8.33), the requirement (9.38) becomes

$$\lambda^2 + 2e\lambda - (1-e^2) \leq 0 \tag{9.39}$$

so that the solution form (9.26) is valid in the λ-range

$$0 \leq \lambda \leq 1 - e. \tag{9.40}$$

Recalling equation (5.8a), we multiply across by $a^2(1-e^2)$ and introduce p from (5.11a); when we resolve the second quadratic factor on the right into its linear components, the R-equation takes the form

$$a^2(1-e^2)\frac{\Lambda^2}{C^2}R'^2 = -(R^2-b^2)[R - a(1-e)][R - a(1+e)]. \tag{9.41}$$

For this equation (9.41) to balance, R is constrained within the limits

$$a(1-e) \leq R \leq a(1+e) \tag{9.42}$$

showing that orbits are confined to the annular region defined by the two bounding ellipses $R = a(1-e)$ and $R = a(1+e)$.

Recalling that $R^2 \geq b^2$ throughout the system, we see that when the parameter λ moves into the supplementary range

$$1 - e \leq \lambda \leq 1 + e \tag{9.43}$$

the inner bounding ellipse $R = a(1-e)$ disappears and the only constraint on the orbits is the outer ellipse, namely

$$R = a(1+e). \tag{9.44}$$

For the λ-range (9.43), the orbits can fill the full elliptical range bounded by (9.44).

It remains to establish the solution form valid in the λ-range (9.43) supplementing the λ-range (9.40). The treatment for this supplementary range is the subject of Subsection 9A.

9A. Solution Form for the Supplementary Range, $\lambda \geq 1 - e$

If we resolve the first quadratic factor on the right of (9.41) into its linear components, we have

$$a^2(1-e^2)\frac{\Lambda^2}{C^2}R'^2 = -(R+b)(R-b)[R - a(1-e)][R - a(1+e)]. \tag{9A.1}$$

We observe that in the λ-range (9.40), the positive root represented in the third factor is greater than the positive root represented in the second factor. In the supplementary range

$$\lambda \geq 1 - e \tag{9A.2}$$

the relative position of these two roots is reversed. This "interleaving of zeros" requires a restructuring of the quadratic factors in the analysis.

In the range (9A.2), we effect this restructuring by combining the "upper" pair, namely, the second and fourth factors into one quadratic expression, and form the second quadratic expression from the "lower" pair, namely the first and third factors. Accordingly, we rewrite equation (9A.1) in the form

$$\begin{aligned} a^2(1-e^2)\frac{\Lambda^2}{C^2}R'^2 = \\ -\big[R^2 - [a(1+e)+b]R + ab(1+e)\big]\big[R^2 - [a(1-e)-b]R - ab(1-e)\big]. \end{aligned} \tag{9A.3}$$

If we scale the R-variable with the characteristic length-scale a, by setting

$$R = aP \tag{9A.4}$$

then, expressed in terms of P and λ, equation (9A.3) becomes

$$(1-e^2)\frac{\Lambda^2}{C^2}P'^2 = -\big[P^2 - [1+\lambda+e]P + \lambda(1+e)\big]\big[P^2 - [1-(\lambda+e)]P - \lambda(1-e)\big]. \tag{9A.5}$$

Considering the first quadratic factor on the right of (9A.5), we note that

$$P^2 - [1+\lambda+e]P + \lambda(1+e) = \big[P - \tfrac{1}{2}(1+\lambda+e)\big]^2 - \tfrac{1}{4}\big[1-(\lambda-e)\big]^2 \tag{9A.6}$$

which suggests that we set

$$P - \tfrac{1}{2}[1+\lambda+e] = \tfrac{1}{2}[1-(\lambda-e)]Q, \quad P = \tfrac{1}{2}[1-(\lambda-e)]Q + \tfrac{1}{2}[1+\lambda+e] \tag{9A.7a,b}$$

whereby the first quadratic factor (9A.6) may be written in terms of Q in the form

$$\tfrac{1}{4}\big[1-(\lambda-e)\big]^2(Q^2-1). \tag{9A.8}$$

In the second quadratic factor on the right of (9A.5), when we introduce P from (9A.7b), then, expressed in terms of Q, the expression becomes

$$\begin{aligned} &\tfrac{1}{4}\Big[[1-(\lambda-e)]Q + \tfrac{1}{2}[1+\lambda+e]\Big]^2 \\ &\qquad - \tfrac{1}{2}[1-(\lambda+e)]\Big[[1-(\lambda-e)]Q + [1+\lambda+e]\Big] - \lambda(1-e) \end{aligned} \tag{9A.9}$$

which, on expansion and rearrangement, reduces to the form

$$\tfrac{1}{4}[1-(\lambda-e)]^2Q^2+(\lambda+e)[1-(\lambda-e)]Q+\tfrac{1}{4}[3\lambda+e+1][\lambda+3e-1]. \tag{9A.10}$$

Accordingly, we rewrite the differential equation (9A.5) in terms of Q by introducing (9A.8) and (9A.10) into the right-hand side and using (9A.7b) to substitute for P' on the left; we find

$$4(1-e^2)\frac{\Lambda^2}{C^2}Q'^2= \tag{9A.11}$$
$$(1-Q^2)\Big[(3\lambda+e+1)(\lambda+3e-1)+4(\lambda+e)[1-(\lambda-e)]Q+[1-(\lambda-e)]^2Q^2\Big].$$

On comparing the right side of (9A.11) with that of the generic equation (5.20), we see that here we have

$$1-d^2=[3\lambda+e+1][\lambda+3e-1],\quad s=2(\lambda+e)[1-(\lambda-e)],\quad q=[1-(\lambda-e)]^2 \tag{9A.12a,b,c}$$

from which there follows that

$$1-d^2+q=4[(\lambda+e)^2-(\lambda-e)],\quad \frac{2s}{1-d^2+q}=\frac{(\lambda+e)[1-(\lambda-e)]}{[(\lambda+e)^2-(\lambda-e)]} \tag{9A.13a,b}$$

and hence

$$1-\frac{4s^2}{[1-d^2+q]^2}=\frac{4\lambda e[(\lambda+e)^2-1]}{[(\lambda+e)^2-(\lambda-e)]^2}. \tag{9A.14}$$

Recalling relations (7.4), we have

$$h=\tfrac{1}{2}\Big[1-\frac{2\sqrt{\lambda e}\sqrt{(\lambda+e)^2-1}}{[(\lambda+e)^2-(\lambda-e)]}\Big],\quad 1-h=\tfrac{1}{2}\Big[1+\frac{2\sqrt{\lambda e}\sqrt{(\lambda+e)^2-1}}{[(\lambda+e)^2-(\lambda-e)]}\Big]. \tag{9A.15a,b}$$

For these latter quantities to remain real, it suffices — as well as being necessary — that

$$\lambda\ge 1-e \tag{9A.16}$$

confirming validity in the range (9A.2).

From (7.5) we see that in this range, we have for δ

$$\delta_R=-\frac{(\lambda+e)[1-(\lambda-e)]}{(\lambda+e)^2-(\lambda-e)+2\sqrt{\lambda e}\sqrt{(\lambda+e)^2-1}}. \tag{9A.17}$$

Moreover, if we note from (9A.15) that

$$1-2h=\frac{2\sqrt{\lambda e}\sqrt{(\lambda+e)^2-1}}{(\lambda+e)^2-(\lambda-e)} \tag{9A.18}$$

then from (7.6) and (7.7), we have

$$A_R + B_R = 8\sqrt{\lambda e}\sqrt{(\lambda + e)^2 - 1} \tag{9A.19a}$$

$$A_R - B_R = 2[(\lambda + e)^2 + 4\lambda e - 1] \tag{9A.19b}$$

from which there follows that

$$A_R = \left[\sqrt{(\lambda + e)^2 - 1} + 2\sqrt{\lambda e}\right]^2 \tag{9A.20a}$$

$$B_R = -\left[\sqrt{(\lambda + e)^2 - 1} - 2\sqrt{\lambda e}\right]^2 \tag{9A.20b}$$

and we note that A_R is always positive while B_R is always negative (or zero).

Recalling the transformations (7.9) to (7.13), we see that if we set

$$Y = \frac{Q - \delta_R}{1 - \delta_R Q}, \quad Q = \frac{Y + \delta_R}{1 + \delta_R Y} \tag{9A.21a,b}$$

and apply these transformations to (9A.11), we obtain the differential equation for Y,

$$4(1 - e^2)\frac{\Lambda^2}{C^2}Y'^2 = (1 - Y^2)[A_R + B_R Y^2] \tag{9A.22}$$

or, on dividing across by A_R,

$$4(1 - e^2)\frac{\Lambda^2}{A_R C^2}Y'^2 = (1 - Y^2)[1 - k_R^2 Y^2] \tag{9A.23}$$

where we have written

$$k_R^2 = -\frac{B_R}{A_R} = \left[\frac{2\sqrt{\lambda e} - \sqrt{(\lambda + e)^2 - 1}}{2\sqrt{\lambda e} + \sqrt{(\lambda + e)^2 - 1}}\right]^2 \leq 1. \tag{9A.24}$$

When we set

$$\frac{1}{j_R^2} = 4(1 - e^2)\frac{\Lambda^2}{A_R C^2}, \quad f_R = j_R f \tag{9A.25a,b}$$

then equation (9A.23) reads

$$\left(\frac{\mathrm{d}Y}{\mathrm{d}f_R}\right)^2 = (1 - Y^2)[1 - k_R^2 Y^2] \tag{9A.26}$$

— in the standard form.

We now record the form taken by the factor j_R (9A.25a) over the parameter range (9.43) for λ:

AR2: Case A, $e^2 + \beta^2 \geq 1$, $1 - e \leq \lambda \leq \beta - \gamma$:

Here $\Lambda = \Lambda_1$ (8.12a) which we now express in terms of λ; with A_R given by (9A.20a), we find

$$\frac{1}{j_R^2} = \frac{1}{j_{R1}^2} = 2\frac{(1 - e^2 - \lambda^2) + \sqrt{(1 - e^2 + \lambda^2)^2 - 4\lambda^2\beta^2}}{[2\sqrt{\lambda e} + \sqrt{(\lambda + e)^2 - 1}]^2}. \tag{9A.27a}$$

AR3: Case A, $e^2 + \beta^2 \geq 1$, $\beta - \gamma \leq \lambda \leq 1 + e$:

Here $\Lambda = \Lambda_0$ (8A.22), and with A_R from (9A.20a), we obtain

$$\frac{1}{j_R^2} = \frac{1}{j_{R0}^2} = \left[\frac{2\sqrt{\lambda\gamma} + \sqrt{(\lambda+\gamma)^2 - \beta^2}}{2\sqrt{\lambda e} + \sqrt{(\lambda+e)^2 - 1}}\right]^2. \tag{9A.27b}$$

BR2: Case B, $e^2 + \beta^2 \leq 1$, $1 - e \leq \lambda \leq 1 + e$:

Here $\Lambda = \Lambda_2$ (8.19a), which we now express in terms of λ; with A_R from (9A.20a), we have

$$\frac{1}{j_R^2} = \frac{1}{j_{R2}^2} = 4\frac{\sqrt{[1 - e^2 + \lambda^2]^2 - 4\lambda^2\beta^2}}{[2\sqrt{\lambda e} + \sqrt{(\lambda+e)^2 - 1}]^2}. \tag{9A.27c}$$

The differential equation (9A.26) has the solution form

$$Y = \operatorname{sn}[f_R + f_{R0} : k_R] \tag{9A.28}$$

where f_{R0} is the constant of integration. If we choose f_R to be measured from "perihelion", then by reasoning identical with that leading from (9.14) to (9.24), we see that we may take

$$f_{R0} = K_R \tag{9A.29}$$

where K_R is the quarterperiod of the Jacobian elliptic functions of modulus k_R; we then have

$$Y = \operatorname{sn}[f_R + K_R : k_R] = \frac{\operatorname{cn}[f_R : k_R]}{\operatorname{dn}[f_R : k_R]} \tag{9A.30}$$

and hence, from (9A.21b)

$$Q = \frac{\operatorname{cn}[f_R : k_R] + \delta_R \operatorname{dn}[f_R : k_R]}{\operatorname{dn}[f_R : k_R] + \delta_R \operatorname{cn}[f_R : k_R]}. \tag{9A.31}$$

When we introduce this expression for Q into (9A.7b) and recall relation (9A.4), we obtain

$$\frac{R}{a} = \tfrac{1}{2}[1 - (\lambda - e)]\frac{\operatorname{cn}[f_R : k_R] + \delta_R \operatorname{dn}[f_R : k_R]}{\operatorname{dn}[f_R : k_R] + \delta_R \operatorname{cn}[f_R : k_R]} + \tfrac{1}{2}(1 + \lambda + e) \tag{9A.32}$$

or, alternatively,

$$\frac{R}{a} = \frac{1}{2}\frac{[(1+\lambda+e)+[1-(\lambda-e)]\delta_R]\operatorname{dn}[f_R : k_R]+[[1-(\lambda-e)]+[1+\lambda+e]\delta_R]\operatorname{cn}[f_R : k_R]}{\operatorname{dn}[f_R : k_R]+\delta_R \operatorname{cn}[f_R : k_R]} \tag{9A.33}$$

completing the solution form for the supplementary range.

9B. Recapitulation of the Solution Forms for the R-equation

It should be noted at the outset that in the parameter range where $e \le 1$, it is a simple exercise to show that

$$1 - e \le \beta - \gamma \le \beta + \gamma \le 1 + e.$$

For the R-equation there are five distinct ranges — three in Case A and two in Case B.

Case A: $e^2 + \beta^2 \ge 1$:

Range $AR1$: $0 \le \lambda \le 1 - e$, with

(i) δ_v determined from (9.3)

(ii) k_v determined from (9.9)

(iii) $j_v = j_{v1}$ determined from (9.8a), and with $f_v = j_{v1} f$, the solution form is given by (9.26).

Range $AR2$: $1 - e \le \lambda \le \beta - \gamma$, with

(i) δ_R determined from (9A.17)

(ii) k_R determined from (9A.24)

(iii) $j_R = j_{R1}$ determined from (9A.27a), and with $f_R = j_{R1} f$, the solution form is given by (9A.33).

Range $AR3$: $\beta - \gamma \le \lambda \le 1 + e$, with

(i) δ_R determined from (9A.17)

(ii) k_R determined from (9A.24)

(iii) $j_R = j_{R0}$ determined from (9A.27b), and with $f_R = j_{R0} f$, the solution form is given by (9A.33).

Case B: $e^2 + \beta^2 \le 1$:

Range $BR1$: $0 \le \lambda \le 1 - e$, with

(i) δ_v determined from (9.3)

(ii) k_v determined from (9.9)

(iii) $j_v = j_{v2}$ determined from (9.8b), and with $f_v = j_{v2} f$, the solution form is given by (9.26).

Range $BR2$: $1 - e \le \lambda \le 1 + e$, with

(i) δ_R determined from (9A.17)

(ii) k_R determined from (9A.24)

(iii) $j_R = j_{R2}$ determined from (9A.27c), and with $f_R = j_{R2} f$, the solution form is given by (9A.33).

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In Section 10 following, we outline the procedure for the derivation of the "time-angle" relation, namely the derivation of an explicit expression relating the "true anomaly" f to the original time variable t. Therein the procedure is illustrated for the solution forms valid in the primary range

$$\lambda \leq 1 - e$$

namely, the solution forms (8.29), (8.30), and (9.26); it is outlined solely for this parameter range. The derivation using the solution forms in the other orbit ranges follows a similar pattern.

Thereafter will follow the analysis and solution forms appropriate for the complementary range ($e \geq 1$, Section 11) and for the singular case ($e = 1$, Section 12). This will complete the analysis for the planar case.

10 The Time-Angle Relation

Recalling relations (8.12a) and (8.19a) respectively defining Λ_1^2 and Λ_2^2, we set

$$\text{Case A:} \quad j_{S1}^2 = \tfrac{1}{2}\left[[1 - \eta^2(1 - e^2)] + \sqrt{[1 + \eta^2(1 - e^2)]^2 - 4\eta^2\beta^2}\right] \tag{10.1a}$$

$$\text{Case B:} \quad j_{S2}^2 = \sqrt{[1 + \eta^2(1 - e^2)]^2 - 4\eta^2\beta^2} \tag{10.1b}$$

and we have

$$\text{Case A:} \quad e^2 + \beta^2 \geq 1: \qquad \Lambda^2 = j_{S1}^2 C^2 = \Lambda_1^2 \tag{10.2a}$$

$$\text{Case B:} \quad e^2 + \beta^2 \leq 1: \qquad \Lambda^2 = j_{S2}^2 C^2 = \Lambda_2^2 \tag{10.2b}$$

and we write

$$\Lambda^2 = j_S^2 C^2 \tag{10.3}$$

to cover both cases. Then, recalling relations (5.2b) and (5.11), we have

$$\Lambda^2 = j_S^2 C^2 = j_S^2 \mu \cdot \frac{C^2}{\mu} = j_S^2 \mu p = j_S^2 \mu a(1 - e^2). \tag{10.4}$$

Following the headline set in relation (3.41) of Chapter 2 for the Kepler problem, we define the factor n, corresponding to the mean motion in the Kepler case, by setting

$$n^2 a^3 = \mu \tag{10.5}$$

so that

$$\Lambda^2 = j_S^2 n^2 a^4 (1 - e^2), \quad \Lambda = j_S n a^2 \sqrt{1 - e^2} \tag{10.6a,b}$$

and hence

$$\frac{\Lambda}{p^2} = j_S \frac{na^2\sqrt{1-e^2}}{a^2(1-e^2)^2} = j_S n \cdot \frac{1}{(1-e^2)^{3/2}} \tag{10.7}$$

— a relation to be used below.

Recalling the defining relation (5.6) for the variable f, and noting (5.15), we have

$$\Lambda \frac{\mathrm{d}t}{\mathrm{d}f} = R^2 - b^2 \cos^2\sigma = p^2 \left[\left(\frac{R}{p}\right)^2 - \eta^2 \cos^2\sigma \right] \tag{10.8}$$

and dividing across by p^2, we introduce (10.7) above, and after a rearrangement, we have

$$j_S n \frac{\mathrm{d}t}{\mathrm{d}f} = (1-e^2)^{3/2} \left[\left(\frac{R}{p}\right)^2 - \eta^2 \cos^2\sigma \right] \tag{10.9}$$

which, on integration, yields

$$j_S n(t - t_0) = (1-e^2)^{3/2} \int_0^f \left[\left(\frac{R}{p}\right)^2 - \eta^2 \cos^2\sigma \right] \mathrm{d}f \tag{10.10}$$

where t_0 is the constant of integration, and we note that $t = t_0$ corresponds to $f = 0$, so that t_0 represents the time of "pericenter" passage.

Recalling from (9.26) that (R/p) is expressed in terms of $f_v = j_v f$ and from (8.29) and (8.30) that $\cos\sigma$ is expressed in terms of $f_S = f + f_{S0}$, it is convenient to multiply (10.10) across by j_v; if we also define the "mean anomaly" $\mathcal{M}$ by setting

$$\mathcal{M} = j_v j_S n(t - t_0) \tag{10.11}$$

then we may replace (10.10) by

$$\mathcal{M} = \mathcal{M}_0 - \eta^2 j_v (1-e^2)^{3/2} \mathcal{M}_1 \tag{10.12}$$

where

$$\mathcal{M}_0 = (1-e^2)^{3/2} \int_0 \left(\frac{R}{p}\right)^2 \mathrm{d}f_v, \quad \mathcal{M}_1 = \int \cos^2\sigma \, \mathrm{d}f_S\,. \tag{10.13a,b}$$

If we introduce (R/p) from (9.26), we have

$$\mathcal{M}_0 = (1-e^2)^{3/2} \int \left[\frac{\mathrm{dn}[f_v : k_v] + \delta_v \,\mathrm{cn}[f_v : k_v]}{(1 + e\delta_v)\,\mathrm{dn}[f_v : k_v] + (e + \delta_v)\,\mathrm{cn}[f_v : k_v]} \right]^2 \mathrm{d}f_v \tag{10.14}$$

and the introduction of the solution forms (8.29) and (8.30) into $\mathcal{M}_1$ yields respectively for the two cases

$$\mathcal{M}_1 = \int \left[\frac{\mathrm{sn}[f_S : k_S] + \delta_S}{1 + \delta_S \,\mathrm{sn}[f_S : k_S]} \right]^2 \mathrm{d}f_S, \qquad \text{for } e^2 + \beta^2 \geq 1, \tag{10.15a}$$

$$\mathcal{M}_1 = \int \left[\frac{k_S' \,\mathrm{sn}[f_S : k_S] + \delta_S \,\mathrm{dn}[f_S : k_S]}{\mathrm{dn}[f_S : k_S] + \delta_S k_S' \,\mathrm{sn}[f_S : k_S]} \right]^2 \mathrm{d}f_S, \qquad \text{for } e^2 + \beta^2 \leq 1. \tag{10.15b}$$

In view of the fact that the derivation of relation (4.58) of Chapter 2 in the Kepler case is a nontrivial exercise, the complexity of the above relations, while perhaps daunting, should not surprise us.

In terms of the parameter η, the factor $\mathcal{M}_1$ is a "correction" term to the "dominant" factor $\mathcal{M}_0$. We shall show how the dominant component in $\mathcal{M}_0$ can be separated out and indicate how a procedure for successive approximations may be pursued. This is done in Subsection 10A following.

The quantity $\mathcal{M}_1$ is the integral of the quotient of two second-order polynomials of elliptic functions: when the elliptic functions are expressed as Fourier series, then the integrand is expressible as the quotient of Fourier expansions.

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10A. The Analysis of $\mathcal{M}_0$

In dealing with the integral (10.14) for $\mathcal{M}_0$, we note that the elliptic functions appearing under the integral sign all have argument f_v as given by (9.12) with j_v given by (9.8) and have modulus k_v as given by (9.9); accordingly, the argument and modulus shall not be exhibited in this subsection. We start by considering the derivative

$$\frac{\mathrm{d}}{\mathrm{d}f_v}\left[\frac{e\,\mathrm{sn}}{(1+e\delta_v)\,\mathrm{dn}+(e+\delta_v)\,\mathrm{cn}}\right]$$
$$=\frac{[(1+e\delta_v)\,\mathrm{dn}+(e+\delta_v)\,\mathrm{cn}]e\,\mathrm{cn}\,\mathrm{dn}+e\,\mathrm{sn}[(1+e\delta_v)k_v^2\,\mathrm{sn}\,\mathrm{cn}+(e+\delta_v)\,\mathrm{sn}\,\mathrm{dn}]}{[(1+e\delta_v)\,\mathrm{dn}+(e+\delta_v)\,\mathrm{cn}]^2}. \tag{10A.1}$$

The numerator in the quotient on the right-hand side of (10A.1) may be rearranged as

$$\begin{aligned} e(e+\delta_v)\,\mathrm{dn}(\mathrm{sn}^2+\mathrm{cn}^2)&+e(1+e\delta_v)\,\mathrm{cn}(\mathrm{dn}^2+k_v^2\,\mathrm{sn}^2)\\ &=e(e+\delta_v)\,\mathrm{dn}+e(1+e\delta_v)\,\mathrm{cn}\\ &=(e^2+e\delta_v)\,\mathrm{dn}+(e+e^2\delta_v)\,\mathrm{cn}\\ &=(1+e\delta_v)\,\mathrm{dn}-(1-e^2)\,\mathrm{dn}+(e+\delta_v)\,\mathrm{cn}-\delta_v(1-e^2)\,\mathrm{cn}\\ &=(1+e\delta_v)\,\mathrm{dn}+(e+\delta_v)\,\mathrm{cn}-(1-e^2)[\mathrm{dn}+\delta_v\,\mathrm{cn}] \end{aligned} \tag{10A.2}$$

so that the quotient on the right-hand side of (10A.1) may be written

$$\frac{1}{(1+e\delta_v)\,\mathrm{dn}+(e+\delta_v)\,\mathrm{cn}}-\frac{(1-e^2)[\mathrm{dn}+\delta_v\,\mathrm{cn}]}{[(1+e\delta_v)\,\mathrm{dn}+(e+\delta_v)\,\mathrm{cn}]^2}. \tag{10A.3}$$

We insert (10A.3) into (10A.1) and multiply across by the factor $\sqrt{1-e^2}\cdot[\mathrm{dn}+\delta_v\,\mathrm{cn}]$; then, on performing a transposition, it follows from (10.14) that

$$\frac{\mathrm{d}\mathcal{M}_0}{\mathrm{d}f_v} = \frac{(1-e^2)^{3/2}[\mathrm{dn}+\delta_v\,\mathrm{cn}]^2}{[(1+e\delta_v)\,\mathrm{dn}+(e+\delta_v)\,\mathrm{cn}]^2}$$
$$= (\mathrm{dn}+\delta_v\,\mathrm{cn})\left[\frac{\sqrt{1-e^2}}{(1+e\delta_v)\,\mathrm{dn}+(e+\delta_v)\,\mathrm{cn}} - \frac{\mathrm{d}}{\mathrm{d}f_v}\left(\frac{e\sqrt{1-e^2}\,\mathrm{sn}}{(1+e\delta_v)\,\mathrm{dn}+(e+\delta_v)\,\mathrm{cn}}\right)\right]. \tag{10A.4}$$

In order to deal with the first term within the square brackets on the right of (10A.4), we first set, as the defining relation for $\bar{e}$,

$$1-\bar{e}^2 = (1-k_v^2)\big[(1+e\delta_v)^2-(e+\delta_v)^2\big] = k_v'^2\big[(1+e\delta_v)^2-(e+\delta_v)^2\big]$$
$$= k_v'^2(1-\delta_v^2)(1-e^2) = (1-k_v^2)(1-\delta_v^2)(1-e^2) \tag{10A.5}$$

in terms of which we introduce the function χ, defined by

$$\tan\chi = \frac{\sqrt{1-\bar{e}^2}\,\mathrm{sn}}{(e+\delta_v)\,\mathrm{dn}+(1+e\delta_v)\,\mathrm{cn}}. \tag{10A.6}$$

There follows that

$$\tan^2\chi = \frac{(1-\bar{e}^2)\,\mathrm{sn}^2}{[(e+\delta_v)\,\mathrm{dn}+(1+e\delta_v)\,\mathrm{cn}]^2} \tag{10A.7}$$

and hence, applying $\sec^2 = 1+\tan^2$, we have

$$\sec^2\chi = \frac{(e+\delta_v)^2\,\mathrm{dn}^2+(1+e\delta_v)\,\mathrm{cn}^2+(1-\bar{e}^2)\,\mathrm{sn}^2+2(e+\delta_v)(1+e\delta_v)\,\mathrm{dn}\,\mathrm{cn}}{[(e+\delta_v)\,\mathrm{dn}+(1+e\delta_v)^2\,\mathrm{cn}]^2}. \tag{10A.8}$$

If we introduce the defining relation (10A.5) for $1-\bar{e}^2$ into the numerator of the quotient on the right-hand side of (10A.8), then, for that numerator, we have

$$(e+\delta_v)^2[\mathrm{dn}^2-(1-k_v^2)\,\mathrm{sn}^2]$$
$$+ (1+e\delta_v)^2[\mathrm{cn}^2+(1-k_v^2)\,\mathrm{sn}^2] + 2(e+\delta_v)(1+e\delta_v)\,\mathrm{dn}\,\mathrm{cn}$$
$$= (e+\delta_v)^2\,\mathrm{cn}^2+(1+e\delta_v)^2\,\mathrm{dn}^2+2(e+\delta_v)(1+e\delta_v)\,\mathrm{dn}\,\mathrm{cn}$$
$$= [(1+e\delta_v)\,\mathrm{dn}+(e+\delta_v)\,\mathrm{cn}]^2 \tag{10A.9}$$

so that relation (10A.8) for $\sec^2\chi$ may be written

$$\sec^2\chi = \frac{[(1+e\delta_v)\,\mathrm{dn}+(e+\delta_v)\,\mathrm{cn}]^2}{[(e+\delta_v)\,\mathrm{dn}+(1+e\delta_v)\,\mathrm{cn}]^2}. \tag{10A.10}$$

Moreover, if we take the derivative across the defining relation (10A.6), we find

$$\sec^2\chi\frac{\mathrm{d}\chi}{\mathrm{d}f_v} = \sqrt{1-\bar{e}^2}\times$$
$$\left[\frac{\mathrm{cn}\,\mathrm{dn}[(e+\delta_v)\,\mathrm{dn}+(1+e\delta_v)\,\mathrm{cn}] + \mathrm{sn}[(e+\delta_v)k_v^2\,\mathrm{sn}\,\mathrm{cn}+(1+e\delta_v)\,\mathrm{sn}\,\mathrm{dn}]}{[(e+\delta_v)\,\mathrm{dn}+(1+e\delta_v)\,\mathrm{cn}]^2}\right]$$
$$= \sqrt{1-\bar{e}^2}\cdot\frac{[(e+\delta_v)\,\mathrm{cn}[\mathrm{dn}^2+k_v^2\,\mathrm{sn}^2] + (1+e\delta_v)\,\mathrm{dn}[\mathrm{sn}^2+\mathrm{cn}^2]}{[(e+\delta_v)\,\mathrm{dn}+(1+e\delta_v)\,\mathrm{cn}]^2}$$
$$= \sqrt{1-\bar{e}^2}\cdot\frac{(1+e\delta_v)\,\mathrm{dn}+(e+\delta_v)\,\mathrm{cn}}{[(e+\delta_v)\,\mathrm{dn}+(1+e\delta_v)\,\mathrm{cn}]^2}. \tag{10A.11}$$

Combining (10A.10) with (10A.11), we obtain

$$\frac{\mathrm{d}\chi}{\mathrm{d}f_v} = \frac{\sqrt{1-\bar{e}^2}}{[(1+e\delta_v)\,\mathrm{dn}+(e+\delta_v)\,\mathrm{cn}]} \tag{10A.12}$$

or, written in full, recalling (10A.6),

$$\frac{\mathrm{d}}{\mathrm{d}f_v}\left[\arctan\frac{\sqrt{1-\bar{e}^2}\,\mathrm{sn}}{(e+\delta_v)\,\mathrm{dn}+(1+e\delta_v)\,\mathrm{cn}}\right] = \frac{\sqrt{1-\bar{e}^2}}{[(1+e\delta_v)\,\mathrm{dn}+(e+\delta_v)\,\mathrm{cn}]}. \tag{10A.13}$$

We now introduce (10A.13) into (10A.4) to obtain

$$\frac{\mathrm{d}\mathcal{M}_0}{\mathrm{d}f_v} = (\mathrm{dn}+\delta_v\,\mathrm{cn})\frac{\mathrm{d}}{\mathrm{d}f_v}\left\{\sqrt{\frac{1-e^2}{1-\bar{e}^2}}\cdot\arctan\left[\frac{\sqrt{1-\bar{e}^2}\,\mathrm{sn}}{(e+\delta_v)\,\mathrm{dn}+(1+e\delta_v)\,\mathrm{cn}}\right]\right.$$
$$\left. - \frac{e\sqrt{1-e^2}\,\mathrm{sn}}{(1+e\delta_v)\,\mathrm{dn}+(e+\delta_v)\,\mathrm{cn}}\right\}. \tag{10A.14}$$

The difficulty in completing the integration arises from the multiplying factor $(\mathrm{dn}+\delta_v\,\mathrm{cn})$; it will prove convenient to define the quantity $\mathcal{M}_{00}$ by setting

$$\mathcal{M}_{00} = \tag{10A.15}$$
$$\sqrt{\frac{1-e^2}{1-\bar{e}^2}}\cdot\arctan\left[\frac{\sqrt{1-\bar{e}^2}\,\mathrm{sn}}{(e+\delta_v)\,\mathrm{dn}+(1+e\delta_v)\,\mathrm{cn}}\right] - \frac{e\sqrt{1-e^2}\,\mathrm{sn}}{(1+e\delta_v)\,\mathrm{dn}+(e+\delta_v)\,\mathrm{cn}}$$

so that (10A.14) may be rewritten as

$$\frac{\mathrm{d}\mathcal{M}_0}{\mathrm{d}f_v} = (\mathrm{dn}+\delta_v\,\mathrm{cn})\frac{\mathrm{d}\mathcal{M}_{00}}{\mathrm{d}f_v}$$
$$= \frac{\mathrm{d}}{\mathrm{d}f_v}[(\mathrm{dn}+\delta_v\,\mathrm{cn})\mathcal{M}_{00}] - \mathcal{M}_{00}\frac{\mathrm{d}}{\mathrm{d}f_v}(\mathrm{dn}+\delta_v\,\mathrm{cn}) \tag{10A.16a}$$
$$= \frac{\mathrm{d}}{\mathrm{d}f_v}[(\mathrm{dn}+\delta_v\,\mathrm{cn})\mathcal{M}_{00}] + \mathcal{M}_{00}[k_v^2\,\mathrm{sn}\,\mathrm{cn}+\delta_v\,\mathrm{sn}\,\mathrm{dn}] \tag{10A.16b}$$

and, recalling relations (9.31) and (9.35), we see that (10A.16) may be written

$$\frac{\mathrm{d}\mathcal{M}_0}{\mathrm{d}f_v} = \frac{\mathrm{d}}{\mathrm{d}f_v}[(\mathrm{dn} + \delta_v \,\mathrm{cn})\mathcal{M}_{00}] + \eta^2 e\mathcal{M}_{00}(d_0 \,\mathrm{dn} + eg_0^2 \,\mathrm{cn})\,\mathrm{sn}\,. \tag{10A.17}$$

We refer to the first term as the principal component, and in terms of the parameter η the second term is a correction term. The principal term is known explicitly in (10A.15); the correction term is the product of the known function $\mathcal{M}_{00}$ (10A.15) with a second-order polynomial of the elliptic functions. Setting

$$\mathcal{M}_{01} = \int \mathcal{M}_{00}(d_0 \,\mathrm{dn} + eg_0^2 \,\mathrm{cn})\,\mathrm{sn}\,\mathrm{d}f_v \tag{10A.18}$$

then we may write the integral of (10A.17) as

$$\mathcal{M}_0 = (\mathrm{dn} + \delta_v \,\mathrm{cn})\mathcal{M}_{00} + \eta^2 e\mathcal{M}_{01}\,. \tag{10A.19}$$

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Main Section 10 Continues

Following the analysis of $\mathcal{M}_0$ in Subsection 10A, we now return to the main section and introduce relation (10A.19) into formula (10.12) for the mean anomaly $\mathcal{M}$ to obtain

$$\mathcal{M} = (\mathrm{dn} + \delta_v \,\mathrm{cn})\mathcal{M}_{00} + \eta^2 e\mathcal{M}_{01} - \eta^2 j_v (1 - e^2)^{3/2}\mathcal{M}_1 \tag{10.16}$$

and with $\mathcal{M}_{00}$ given by (10A.15), $\mathcal{M}_{01}$ given by (10A.18), and $\mathcal{M}_1$ given by (10.15), we have the reduced form of the time-angle relation.

On inspection of the first term, and noting form (10A.15) for $\mathcal{M}_{00}$, we see that we have the appropriate generalization of relation (4.58) of Chapter 2 for the Kepler case.

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The procedure for successive approximations has merely been indicated here. In Chapter 5 we shall deal with the problem of the Earth satellite where a second-order approximation in terms of the relevant parameter is both appropriate and adequate; therein the approximation procedure is fully developed to that level of accuracy.

11 The Complementary Range

The foregoing analysis has been focused on the derivation of solution forms that follow from the choice of constants (4.6). We now proceed to the analysis for the range where the alternative choice of constants is admitted, namely where

$$\mathcal{E} = -\alpha^2, \quad C_1 = \tfrac{1}{2}C^2, \quad C_2 = -\tfrac{1}{2}C^2 \tag{11.1a,b,c}$$

yielding a complementary range of orbits that, however, have no analog in the Kepler case.

In terms of the above constants, equations (4.5) for the first integrals now assume the form

$$\tfrac{1}{2}\frac{(R^2-b^2\cos^2\sigma)^2}{R^2-b^2}\dot{R}^2 = -\alpha^2R^2+\mu R+\tfrac{1}{2}C^2 \tag{11.2a}$$

$$\tfrac{1}{2}(R^2-b^2\cos^2\sigma)^2\dot{\sigma}^2 = \alpha^2b^2\cos^2\sigma+\mu\beta b\cos\sigma-\tfrac{1}{2}C^2 \tag{11.2b}$$

and we may immediately note that the latter equation (11.2b) cannot balance for $b = 0$, and further that this equation indicates that there is a neighborhood of $\sigma = \pi/2$ that is excluded from the orbit range; thus, since the orbits cannot cross the x-axis, they are confined to the upper or lower half plane as satellite orbits about one or the other of the two “primaries”.

In effecting the reduction of equations (11.2), we follow the pattern already established in Sections 4–9, with appropriate modifications in the choices of parameters. When we rewrite equation (11.2a) in the form

$$(R^2-b^2\cos^2\sigma)^2\dot{R}^2 = C^2(R^2-b^2)\left[1+2\frac{\mu}{C^2}R-2\frac{\alpha^2}{C^2}R^2\right] \tag{11.3}$$

and, as in the previous analysis, we introduce the length scales a and p, by setting

$$a = \frac{\mu}{2\alpha^2},\quad p = \frac{C^2}{\mu},\quad \text{and hence}\quad ap = \frac{C^2}{2\alpha^2} \tag{11.4a,b,c}$$

then equation (11.3) may be written

$$\frac{(R^2-b^2\cos^2\sigma)^2}{C^2}\dot{R}^2 = (R^2-b^2)\left[1+\frac{2}{p}R-\frac{1}{ap}R^2\right] \tag{11.5a}$$

and the corresponding form of the σ-equation (11.2b) reads

$$\frac{(R^2-b^2\cos^2\sigma)^2}{C^2}\dot{\sigma}^2 = -\left[1-2\beta\frac{b}{p}\cos\sigma-\frac{b^2}{ap}\cos^2\sigma\right]. \tag{11.5b}$$

Introducing the regularizing variable f, defined by

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\overline{\Lambda}}{R^2-b^2\cos^2\sigma} \tag{11.6}$$

the pair of equations (11.5) take the form

$$\frac{\overline{\Lambda}^2}{C^2}R'^2 = (R^2-b^2)\left[1+\frac{2}{p}R-\frac{1}{ap}R^2\right] \tag{11.7a}$$

$$\frac{\overline{\Lambda}^2}{C^2}\sigma'^2 = -\left[1-2\beta\frac{b}{p}\cos\sigma-\frac{b^2}{ap}\cos^2\sigma\right] \tag{11.7b}$$

where again the prime symbol is used to denote differentiation with respect to f, and the parameter $\overline{\Lambda}$ is to be chosen presently.

In the R-equation (11.7a), we note that for the second factor on the right, we may write

$$1 + \frac{2}{p}R - \frac{1}{ap}R^2 = \left(1 + \frac{R}{p}\right)^2 - \frac{1}{p^2}\left(1 + \frac{p}{a}\right)R^2 \tag{11.8}$$

which suggests that, in this range, the appropriate parameterization takes the form

$$p = a(e^2 - 1) \text{ so that } 1 + \frac{p}{a} = e^2 \geq 1 \tag{11.9a,b}$$

and we may write

$$\begin{aligned} 1 + 2\frac{R}{p} - \frac{R^2}{ap} &= 1 + 2\frac{R}{p} - (e^2 - 1)\frac{R^2}{p^2} \\ &= -\frac{1}{a^2(e^2 - 1)}\left[R^2 - 2aR - a^2(e^2 - 1)\right] \\ &= -\frac{1}{a^2(e^2 - 1)}\left[R + a(e - 1)\right]\left[R - a(e + 1)\right]. \end{aligned} \tag{11.10}$$

Hence, on factoring the right side of (11.7a), we transpose and rearrange to obtain

$$a^2(e^2 - 1)\frac{\overline{\Lambda}}{C^2}R'^2 = -(R - b)\left[R - a(e + 1)\right](R + b)\left[R + a(e - 1)\right] \tag{11.11a}$$

$$\begin{aligned} &= -\Big[R^2 - [a(e + 1) + b]R + ab(e + 1)\Big] \\ &\quad \cdot \Big[R^2 + [a(e - 1) + b]R + ab(e - 1)\Big] \end{aligned} \tag{11.11b}$$

where we structured the quadratic factors in accord with the “upper” and “lower” pairs of zeros. We also rewrite the σ-equation (11.7b) in the form

$$(e^2 - 1)\frac{\overline{\Lambda}}{C^2}\sigma'^2 = \lambda^2 \cos^2\sigma + 2\beta\lambda\cos\sigma - (e^2 - 1) \tag{11.12a}$$

$$= \left[\lambda\cos\sigma - (\gamma - \beta)\right]\left[\lambda\cos\sigma + (\gamma + \beta)\right] \tag{11.12b}$$

where, having multiplied across by $(e^2 - 1)$, we apply relations (11.9a) and, consistent with previous notation, we have set

$$\lambda = \frac{b}{a}, \quad \gamma = \sqrt{\beta^2 + e^2 - 1} \geq \beta, \text{for } e \geq 1. \tag{11.13a,b}$$

Some observations immediately follow from equations (11.11) and (11.12):

(1) For equations (11.11) to balance, we must have

$$b \leq R \leq a(e + 1) \text{ so that } \lambda \leq e + 1 \tag{11.14a,b}$$

and the orbits are confined by the outer bounding ellipse, $R = a(e + 1)$ with no inner boundary.

(2) For (11.12) to balance, $\cos\sigma$ must be excluded from the range $(-\frac{\gamma+\beta}{\lambda}, \frac{\gamma-\beta}{\lambda})$. Hence $\cos\sigma$ is confined to the regions

$$\text{(a)}\quad \cos\sigma \text{ positive; } \frac{\gamma-\beta}{\lambda} \le \cos\sigma \le 1 \tag{11.15a}$$

$$\text{(b)}\quad \cos\sigma \text{ negative; } -1 \le \cos\sigma \le -\left(\frac{\gamma+\beta}{\lambda}\right) \tag{11.15b}$$

which, taken together with (11.14), means that the orbits are satellite orbits about one or the other of the primaries. We further note that, for $e > 1$,

$$e - 1 \le \gamma - \beta \le \gamma + \beta \le e + 1 \tag{11.16}$$

which can be easily checked.

If we now replace the variable σ by its cosine and scale the variable R with the length-scale a by setting

$$S = \cos\sigma, \quad R = aP; \tag{11.17a,b}$$

then, expressed in terms of S and λ, equation (11.12) becomes

$$\begin{aligned}(e^2-1)\frac{\overline{\Lambda}^2}{C^2}S'^2 &= (1-S^2)[\lambda^2S^2 + 2\beta\lambda S - (e^2-1)]\\ &= (1-S)[\lambda S - (\gamma-\beta)](1+S)[\lambda S + (\gamma+\beta)]\end{aligned} \tag{11.18}$$

and correspondingly equation (11.11) becomes

$$\begin{aligned}&(e^2-1)\frac{\overline{\Lambda}^2}{C^2}P'^2 =\\ &-\Big[P^2 - [1+(\lambda+e)]P + \lambda(e+1)\Big]\Big[P^2 - [1-(\lambda+e)]P + \lambda(e-1)\Big].\end{aligned} \tag{11.19}$$

A comparison of the right side of (11.18) with the right side of (8A.1) shows that these expressions are formally identical; similarly, the right side of (11.19) is formally identical with the right side of (9A.5). In fact, the only difference lies in the left sides where the coefficients have the factor $e^2 - 1$ rather than $(1 - e^2)$ appearing previously. Hence the reduction and solution of the above equations has been effected in Subsections 8A and 9A.

Accordingly for equation (11.18), we have $\overline{\delta}_S$ as given by (8A.14), namely,

$$\overline{\delta}_S = -\frac{(\lambda+\gamma)(\lambda+\beta-\gamma)}{(\lambda+\gamma)^2 + \beta(\lambda-\gamma) + 2\sqrt{\lambda\gamma[(\lambda+\gamma)^2-\beta^2]}} \tag{11.20}$$

and with $\overline{k}_S$, as given by (8A.21),

$$\overline{k}_S^2 = \left[\frac{2\sqrt{\lambda\gamma} - \sqrt{(\lambda+\gamma)^2-\beta^2}}{2\sqrt{\lambda\gamma} + \sqrt{(\lambda+\gamma)^2-\beta^2}}\right]^2 \tag{11.21}$$

equation (11.18) reduces to

$$4(e^2-1)\frac{\overline{\Lambda}^2}{C^2\overline{A}_S}Y'^2 = (1-Y^2)\left[1-\overline{k}_S^2Y^2\right] \tag{11.22}$$

where $\overline{A}_S$ is as in (8A.16a). Hence, when we make the identification

$$\overline{\Lambda}^2 = \tfrac{1}{4}\frac{C^2}{e^2-1}\left[2\sqrt{\lambda\gamma}+\sqrt{(\lambda+\gamma)^2-\beta^2}\right]^2 \tag{11.23}$$

equation (11.22) takes the standard form with the solution

$$Y = \mathrm{sn}[f+\overline{f}_S : \overline{k}_S] \tag{11.24}$$

where $\overline{f}_S$ represents the constant of integration. Then the solution for S takes the form

$$S = \frac{1}{2\lambda}\frac{[(1+\overline{\delta}_S)\lambda-(1-\overline{\delta}_S)(\gamma-\beta)]\,\mathrm{sn}[f+\overline{f}_S:\overline{k}_S]+[(1+\overline{\delta}_S)\lambda+(1-\overline{\delta}_S)(\gamma-\beta)]}{1+\overline{\delta}_S\,\mathrm{sn}[f+\overline{f}_S:\overline{k}_S]} \tag{11.25}$$

with the constant $\overline{f}_S$ to be determined. We note that the solution form (11.25) is formally identical with (8A.26) of Subsection 8A.

Turning to the P-equation (11.19) and referring to Subsection 9A, we have from (9A.17) that

$$\overline{\delta}_R = -\frac{(\lambda+e)(1-\lambda+e)}{(\lambda+e)^2-(\lambda-e)+2\sqrt{\lambda e}\sqrt{(\lambda+e)^2-1}} \tag{11.26}$$

and from (9A.24) the formula for $\overline{k}_R$, namely,

$$\overline{k}_R^2 = \left[\frac{2\sqrt{\lambda e}-\sqrt{(\lambda+e)^2-1}}{2\sqrt{\lambda e}+\sqrt{(\lambda+e)^2-1}}\right]^2 \tag{11.27}$$

whereby, in terms of the transformed variable Y, the differential equation is reduced to the present analog of (9A.23),

$$4(e^2-1)\frac{\overline{\Lambda}^2}{\overline{A}_RC^2}Y'^2 = (1-Y^2)[1-k_R^2Y^2] \tag{11.28}$$

with $\overline{A}_R$ given by (9A.20a). With $\overline{\Lambda}$ given by (11.23) and $\overline{A}_R$ from (9A.20a), we set

$$\frac{1}{\overline{j}_R^2} = 4(e^2-1)\frac{\overline{\Lambda}^2}{\overline{A}_RC^2} = \left[\frac{2\sqrt{\lambda\gamma}+\sqrt{(\lambda+\gamma)^2-\beta^2}}{2\sqrt{\lambda e}+\sqrt{(\lambda+e)^2-1}}\right]^2 \tag{11.29a}$$

$$f_R = \overline{j}_R f \tag{11.29b}$$

and the differential equation (11.28) reduces to

$$\left(\frac{\mathrm{d}Y}{\mathrm{d}f_R}\right)^2 = (1-Y^2)[1-\overline{k}_R^2Y^2] \tag{11.30}$$

— the standard form — with solution

$$Y = \operatorname{sn}[f_R + f_{R0} : \overline{k}_R]. \tag{11.31}$$

Hence, following the pattern leading from (9A.28) to (9A.33), we obtain

$$\frac{R}{a} = \frac{1}{2}\frac{[(1+\lambda+e)+[1-(\lambda-e)]\overline{\delta}_R]\,\mathrm{dn}[f_R : \overline{k}_R]+[[1-(\lambda-e)]+[1+\lambda+e]\overline{\delta}_R]\,\mathrm{cn}[f_R : \overline{k}_R]}{\mathrm{dn}[f_R : \overline{k}_R]+\overline{\delta}_R\,\mathrm{cn}[f_R : \overline{k}_R]} \tag{11.32}$$

completing the solution for the complementary range.

We note that the solution form (11.32) is formally identical with the form (9A.33) of Subsection 9A. This, together with the observations following (11.25) above, means that the solutions for the complementary range are already contained in Subsections 8A and 9A.

12 The Singular Case: $C = 0$

The class of orbits, wherein the angular momentum constant vanishes, requires a separate treatment, and specifically a modification of the constant appearing in the definition of the "true anomaly". In this case, we have

$$C_1 = C_2 = 0, \text{ and, as before } \mathcal{E} = -\alpha^2 \tag{12.1a,b,c}$$

and the pair of equations (4.5) take the form

$$\frac{(R^2 - b^2\cos^2\sigma)^2}{R^2 - b^2}\dot{R}^2 = -2\alpha^2R^2 + 2\mu R = -2\alpha^2(R^2 - 2aR) \tag{12.2a}$$

$$\begin{aligned}(R^2 - b^2\cos^2\sigma)^2\dot{\sigma}^2 &= 2\alpha^2b^2\cos^2\sigma + 2\mu\beta b\cos\sigma \\ &= 2\alpha^2[b^2\cos^2\sigma + 2\beta ab\cos\sigma]\end{aligned} \tag{12.2b}$$

where we have multiplied across by 2 and, as before, introduced the length-scale a, and further we introduce the parameter λ, by setting

$$a = \frac{\mu}{2\alpha^2}, \quad \lambda = \frac{b}{a}. \tag{12.3a,b}$$

We introduce the regularizing variable f by setting

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\Lambda_\alpha}{R^2 - b^2\cos^2\sigma} \tag{12.4}$$

in which the parameter Λ_α is to be specified presently. With f as the independent variable and again with the prime symbol denoting differentiation with respect to f, the pair of differential equations (12.2) take the form

$$\Lambda_\alpha^2 R'^2 = -2\alpha^2(R^2 - b^2)(R^2 - 2aR) \tag{12.5a}$$

$$\Lambda_\alpha^2 \sigma'^2 = 2\alpha^2 ab[\lambda\cos^2\sigma + 2\beta\cos\sigma]. \tag{12.5b}$$

When we scale the variable R with the characteristic length-scale a and replace the variable σ by its cosine, setting

$$R = aP, \quad S = \cos\sigma \tag{12.6a,b}$$

then the pair of equations (12.5) become

$$\frac{\Lambda_\alpha^2}{2\alpha^2 a^2} P'^2 = -(P^2 - \lambda^2)(P^2 - 2P) \tag{12.7a}$$

$$\frac{\Lambda_\alpha^2}{2\alpha^2 a^2} S'^2 = (1 - S^2)(\lambda^2 S^2 + 2\beta\lambda S) \tag{12.7b}$$

when expressed in terms of P and S.

Considering first the S-equation, we order the linear factors into "upper" and "lower" pairs and recombine to obtain

$$\begin{aligned}\frac{\Lambda_\alpha^2}{2\alpha^2 a^2} S'^2 &= -S(S-1)(S+1)[\lambda^2 S + 2\beta\lambda] \\ &= -(S^2 - S)[\lambda^2 S^2 + (\lambda^2 + 2\beta\lambda)S + 2\beta\lambda].\end{aligned} \tag{12.8}$$

For the first quadratic on the right of (12.8), we note that

$$S^2 - S = (S - \tfrac{1}{2})^2 - \tfrac{1}{4} \tag{12.9}$$

which suggests that we set

$$S - \tfrac{1}{2} = \tfrac{1}{2}\zeta, \quad S = \tfrac{1}{2}(\zeta + 1) \tag{12.10a,b}$$

and for the first quadratic factor, we have

$$S^2 - S = \tfrac{1}{4}(\zeta^2 - 1). \tag{12.11}$$

For the second factor on the right of (12.8), we have from (12.10b) that

$$S^2 = \tfrac{1}{4}(\zeta + 1)^2 = \tfrac{1}{4}\zeta^2 + \tfrac{1}{2}\zeta + \tfrac{1}{4} \tag{12.12a}$$

$$\begin{aligned}(\lambda^2 + 2\beta\lambda)S + 2\beta\lambda &= (\tfrac{1}{2}\lambda^2 + \beta\lambda)(\zeta + 1) + 2\beta\lambda \\ &= (\tfrac{1}{2}\lambda^2 + \beta\lambda)\zeta + (\tfrac{1}{2}\lambda^2 + 3\beta\lambda)\end{aligned} \tag{12.12b}$$

and hence

$$\lambda^2 S^2 + (\lambda^2 + 2\beta\lambda)S + 2\beta\lambda = \tfrac{1}{4}\lambda^2\zeta^2 + (\lambda^2 + \beta\lambda)\zeta + (\tfrac{3}{4}\lambda^2 + 3\beta\lambda). \tag{12.13}$$

Accordingly, if we substitute for S in terms of ζ in equation (12.8), we obtain as equation for ζ,

$$\frac{2\Lambda_\alpha^2}{\alpha^2 a^2}\zeta'^2 = (1 - \zeta^2)[3\lambda(\lambda + 4\beta) + 4\lambda(\lambda + \beta)\zeta + \lambda^2\zeta^2]. \tag{12.14}$$

Comparison of (12.14) with the generic equation (5.20) indicates that in this case, we have

$$1 - d^2 = 3\lambda(\lambda + 4\beta), \quad 2s = 4\lambda(\lambda + \beta), \quad q = \lambda^2 \tag{12.15a,b,c}$$

from which we have

$$1 - d^2 + q = 4\lambda(\lambda + 3\beta), \quad 1 - d^2 - q = 2\lambda(\lambda + 6\beta) \tag{12.16a,b}$$

so that

$$\frac{2s}{1 - d^2 + q} = \frac{\lambda + \beta}{\lambda + 3\beta} \leq 1 \tag{12.17}$$

and it follows that

$$1 - \frac{4s^2}{(1 - d^2 + q)^2} = \frac{4\beta(\lambda + 2\beta)}{(\lambda + 3\beta)^2}, \quad \sqrt{1 - \frac{4s^2}{(1 - d^2 + q)^2}} = \frac{2\sqrt{\beta}\sqrt{\lambda + 2\beta}}{\lambda + 3\beta}. \tag{12.18a,b}$$

From (7.4), in this case we have

$$h_{s\alpha} = \tfrac{1}{2}\left[1 - \frac{2\sqrt{\beta}\sqrt{\lambda + 2\beta}}{\lambda + 3\beta}\right] = \tfrac{1}{2}\frac{[\sqrt{\lambda + 2\beta} - \sqrt{\beta}]^2}{\lambda + 3\beta} \leq \tfrac{1}{2} \tag{12.19a}$$

$$1 - h_{s\alpha} = \tfrac{1}{2}\left[1 + \frac{2\sqrt{\beta}\sqrt{\lambda + 2\beta}}{\lambda + 3\beta}\right] = \tfrac{1}{2}\frac{[\sqrt{\lambda + 2\beta} + \sqrt{\beta}]^2}{\lambda + 3\beta} \geq \tfrac{1}{2} \tag{12.19b}$$

and from (7.5) we have

$$\delta_{s\alpha} = -\frac{\lambda + \beta}{[\sqrt{\lambda + 2\beta} + \sqrt{\beta}]^2} \tag{12.20}$$

and further, from (7.8)

$$A_{s\alpha} = \lambda\left[\sqrt{\lambda + 2\beta} + 2\sqrt{\beta}\right]^2 \tag{12.21a}$$

$$B_{s\alpha} = -\lambda\left[\sqrt{\lambda + 2\beta} - 2\sqrt{\beta}\right]^2. \tag{12.21b}$$

Accordingly if we set

$$Y = \frac{\zeta - \delta_{s\alpha}}{1 - \delta_{s\alpha}\zeta}, \quad \zeta = \frac{Y + \delta_{s\alpha}}{1 + \delta_{s\alpha}Y} \tag{12.22a,b}$$

then equation (12.14) for ζ is transformed to the equation for Y,

$$\frac{2\Lambda_\alpha^2}{\alpha^2 a^2} Y'^2 = (1 - Y^2)[A_{s\alpha} + B_{s\alpha}Y^2]. \tag{12.23}$$

When we divide across by $A_{s\alpha}$ and introduce the explicit forms from (12.21), we obtain

$$\frac{2\Lambda_\alpha^2}{\alpha^2 a^2 \lambda}\frac{1}{[\sqrt{\lambda + 2\beta} + 2\sqrt{\beta}]^2} Y'^2 = (1 - Y^2)\left[1 - \left(\frac{\sqrt{\lambda + 2\beta} - 2\sqrt{\beta}}{\sqrt{\lambda + 2\beta} + 2\sqrt{\beta}}\right)^2 Y^2\right]. \tag{12.24}$$

When we make the identification

$$\Lambda_\alpha^2 = \tfrac{1}{2}\alpha^2 a^2 \lambda \left[\sqrt{\lambda + 2\beta} + 2\sqrt{\beta}\right]^2 \tag{12.25}$$

and set

$$k_{s\alpha}^2 = \left[\frac{\sqrt{\lambda + 2\beta} - 2\sqrt{\beta}}{\sqrt{\lambda + 2\beta} + 2\sqrt{\beta}}\right]^2 \leq 1 \tag{12.26}$$

then equation (12.24) reduces to the standard form

$$Y'^2 = (1 - Y^2)(1 - k_{s\alpha}^2 Y^2) \tag{12.27}$$

with the solution

$$Y = \mathrm{sn}[f + f_{s0} \colon k_{s\alpha}] \tag{12.28}$$

where f_{s0} is the constant of integration. From (12.22b) therefore, we have for ζ

$$\zeta = \frac{\mathrm{sn}[f + f_{s0} \colon k_{s\alpha}] + \delta_{s\alpha}}{1 + \delta_{s\alpha}\,\mathrm{sn}[f + f_{s0} \colon k_{s\alpha}]} \tag{12.29}$$

and hence, from (12.10b) we have the solution for S in the form

$$S = \cos\sigma = \tfrac{1}{2}(1 + \delta_{s\alpha})\frac{1 + \mathrm{sn}[f + f_{s0} \colon k_{s\alpha}]}{1 + \delta_{s\alpha}\,\mathrm{sn}[f + f_{s0} \colon k_{s\alpha}]}. \tag{12.30}$$

If the angle of “perihelion” is represented by ω, so that

$$f = -\omega \ \text{ implies } \ \cos\sigma = 0, \tag{12.31}$$

then the condition

$$\mathrm{sn}[\omega + f_{s0} \colon k_{s\alpha}] + 1 = 0 \tag{12.32}$$

is to be applied for the determination of f_0.

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Subsection 12A: The Degenerate Case

We observe that when $\beta = 0$, then from (12.20)

$$\delta_{s\alpha} = -1 \tag{12A.1}$$

and the transformation (12.22) becomes degenerate. The above analysis is therefore valid for $\beta \neq 0$. The case

$$\beta = 0 \tag{12A.2}$$

requires the separate treatment of this subsection.

When $\beta = 0$, equation (12.7b) reads

$$\frac{\Lambda_\alpha^2}{2\alpha^2 a^2} S'^2 = \lambda^2 S^2 (1 - S^2) \tag{12A.3}$$

and consistent with (12.25), we make the identification

$$\Lambda_\alpha^2 = \tfrac{1}{2}\alpha^2 a^2 \lambda^2 \tag{12A.4}$$

and the normalized form of equation (12A.3) takes the form

$$\tfrac{1}{4} S'^2 = S^2 (1 - S^2) \tag{12A.5}$$

with the solution in the form

$$\cos\sigma = S = \operatorname{sech}[2(f + \omega)] \tag{12A.6}$$

where the constant ω is to be identified with the angle between the line of "perihelion" and the z-axis.

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Main Section 12 Continues

Turning now to equation (12.7a) for P, we resolve the right side into its linear components, which we regroup into "upper" and "lower" pairs of factors; based on this recombination into quadratic factors, we obtain

$$\frac{\Lambda_\alpha^2}{2\alpha^2 a^2} P'^2 = -(P - \lambda)(P - 2)P(P + \lambda) \tag{12.33a}$$

$$= -[P^2 - (\lambda + 2)P + 2\lambda][P^2 + \lambda P]. \tag{12.33b}$$

Considering the first quadratic factor on the right, we set

$$\begin{aligned} P^2 - (\lambda + 2)P + 2\lambda &= [P - (1 + \tfrac{1}{2}\lambda)]^2 - (1 + \tfrac{1}{2}\lambda)^2 + 2\lambda \\ &= [P - (1 + \tfrac{1}{2}\lambda)]^2 - (1 - \tfrac{1}{2}\lambda)^2 \end{aligned} \tag{12.34}$$

leading to the transformation

$$P - (1 + \tfrac{1}{2}\lambda) = (1 - \tfrac{1}{2}\lambda)Q, \quad P = (1 - \tfrac{1}{2}\lambda)Q + (1 + \tfrac{1}{2}\lambda) \tag{12.35a,b}$$

and the first quadratic factor (12.34) becomes

$$(1 - \tfrac{1}{2}\lambda)^2 (Q^2 - 1). \tag{12.36}$$

In the second quadratic factor, we note that

$$P^2 = (1 - \tfrac{1}{2}\lambda)^2 Q^2 + 2(1 - \tfrac{1}{4}\lambda^2)Q + (1 + \tfrac{1}{2}\lambda)^2 \tag{12.37}$$

and hence

$$P^2 + \lambda P = (1 - \tfrac{1}{2}\lambda)^2 Q^2 + (2 + \lambda - \lambda^2)Q + (1 + 2\lambda + \tfrac{3}{4}\lambda^2). \tag{12.38}$$

Introducing (12.36) and (12.38) into the right side of (12.33b) and using (12.35b) to substitute for P' on the left side, we obtain as an equation for Q,

$$\frac{\Lambda_\alpha^2}{2\alpha^2 a^2} Q'^2 = (1 - Q^2)\Big[(1 + 2\lambda + \tfrac{3}{4}\lambda^2) + (2 + \lambda - \lambda^2)Q + (1 - \tfrac{1}{2}\lambda)^2 Q^2\Big]. \quad (12.39)$$

Comparison with the generic equation (5.20) shows that here we have

$$1 - d^2 = 1 + 2\lambda + \tfrac{3}{4}\lambda^2, \quad 2s = 2 + \lambda - \lambda^2, \quad q = 1 - \lambda + \tfrac{1}{4}\lambda^2 \qquad (12.40\text{a,b,c})$$

and hence

$$1 - d^2 + q = 2 + \lambda + \lambda^2, \quad 1 - d^2 - q = 3\lambda + \tfrac{1}{2}\lambda^2 \qquad (12.41\text{a,b})$$

so that

$$\frac{2s}{1 - d^2 + q} = \frac{2 + \lambda - \lambda^2}{2 + \lambda + \lambda^2} \leq 1 \qquad (12.42)$$

from which there follows that

$$1 - \frac{4s^2}{(1 + d^2 + q)^2} = \frac{4\lambda^2(2 + \lambda)}{(2 + \lambda + \lambda^2)^2}, \quad \sqrt{1 - \frac{4s^2}{(1 - d^2 + q)^2}} = \frac{2\lambda\sqrt{2 + \lambda}}{2 + \lambda + \lambda^2}. \qquad (12.43\text{a,b})$$

From (7.4), we therefore have

$$h_{R\alpha} = \tfrac{1}{2}\left[1 - \frac{2\lambda\sqrt{2 + \lambda}}{2 + \lambda + \lambda^2}\right] = \tfrac{1}{2}\frac{(\sqrt{2 + \lambda} - \lambda)^2}{2 + \lambda + \lambda^2} \leq \tfrac{1}{2} \qquad (12.44\text{a})$$

$$1 - h_{R\alpha} = \tfrac{1}{2}\left[1 + \frac{2\lambda\sqrt{2 + \lambda}}{2 + \lambda + \lambda^2}\right] = \tfrac{1}{2}\frac{(\sqrt{2 + \lambda} + \lambda)^2}{2 + \lambda + \lambda^2} \geq \tfrac{1}{2} \qquad (12.44\text{b})$$

and, from (7.5), we find that

$$\delta_{R\alpha} = -\frac{2 + \lambda - \lambda^2}{(\sqrt{2 + \lambda} + \lambda)^2} = -\frac{(1 + \lambda)(2 - \lambda)}{(\sqrt{2 + \lambda} + \lambda)^2}. \qquad (12.45)$$

Relations (7.8) for A and B yield

$$\begin{aligned} A_{R\alpha} &= 1 - d^2 - h(1 - d^2 + q) = 1 + 2\lambda + \tfrac{3}{4}\lambda^2 - \tfrac{1}{2}(\sqrt{2 + \lambda} - \lambda)^2 \\ &= \tfrac{1}{4}[\lambda^2 + 6\lambda + 4\lambda\sqrt{2 + \lambda}] = \tfrac{1}{4}\lambda[\sqrt{2 + \lambda} + 2]^2 \qquad (12.46\text{a}) \\ B_{R\alpha} &= q - h(1 - d^2 + q) = 1 - \lambda + \tfrac{1}{4}\lambda^2 - \tfrac{1}{2}(\sqrt{2 + \lambda} - \lambda)^2 \\ &= -\tfrac{1}{4}[\lambda^2 + 6\lambda - 4\lambda\sqrt{2 + \lambda}] = -\tfrac{1}{4}\lambda[\sqrt{2 + \lambda} - 2]^2. \qquad (12.46\text{b}) \end{aligned}$$

Hence, when we set

$$Y = \frac{Q - \delta_{R\alpha}}{1 - \delta_{R\alpha} Q}, \quad Q = \frac{Y + \delta_{R\alpha}}{1 + \delta_{R\alpha} Y} \qquad (12.47\text{a,b})$$

equation (12.39) for Q is transformed into the following differential equation for Y:

$$\frac{\Lambda_\alpha^2}{2\alpha^2 a^2} Y'^2 = (1 - Y^2)[A_{R\alpha} + B_{R\alpha} Y^2]. \tag{12.48}$$

We next divide across by $A_{R\alpha}$ and introduce Λ_α^2 from (12.25); we also introduce $A_{R\alpha}$ and $B_{R\alpha}$ from (12.46) and thereby obtain

$$\left[\frac{\sqrt{\lambda + 2\beta} + 2\sqrt{\beta}}{\sqrt{\lambda + 2} + 2}\right]^2 Y'^2 = (1 - Y^2)\left[1 - \left[\frac{\sqrt{2 + \lambda} - 2}{\sqrt{2 + \lambda} + 2}\right]^2 Y^2\right]. \tag{12.49}$$

When we set

$$j_{R\alpha} = \frac{\sqrt{\lambda + 2} + 2}{\sqrt{\lambda + 2\beta} + 2\sqrt{\beta}}, \quad k_{R\alpha}^2 = \left[\frac{\sqrt{2 + \lambda} - 2}{\sqrt{\lambda + 2} + 2}\right]^2 \tag{12.50a,b}$$

and

$$f_\alpha = j_{R\alpha} f \tag{12.50c}$$

then equation (12.48) assumes the standard form

$$\left(\frac{\mathrm{d}Y}{\mathrm{d}f_\alpha}\right)^2 = (1 - Y^2)(1 - k_{R\alpha}^2 Y^2) \tag{12.51}$$

with the solution

$$Y = \mathrm{sn}[f_\alpha + f_{R0}\colon k_{R\alpha}] \tag{12.52}$$

where f_{R0} is the constant of integration.

From (12.47b) there follows that for Q we have

$$Q = \frac{\mathrm{sn}[f_\alpha + f_{R0}\colon k_{R\alpha}] + \delta_{R\alpha}}{1 + \delta_{R\alpha}\,\mathrm{sn}[f_\alpha + f_{R0}\colon k_{R\alpha}]} \tag{12.53}$$

and from (12.35b), and recalling (12.6a), we find

$$\frac{R}{a} = P = \frac{[(1 - \frac{1}{2}\lambda) + (1 + \frac{1}{2}\lambda)\delta_{R\alpha}]\,\mathrm{sn}[f_\alpha + f_{R0}\colon k_{R\alpha}] + [(1 + \frac{1}{2}\lambda) + (1 - \frac{1}{2}\lambda)\delta_{R\alpha}]}{1 + \delta_{R\alpha}\,\mathrm{sn}[f_R + f_{R0}\colon k_{R\alpha}]} \tag{12.54}$$

Again, following the procedure of Section 9, the analysis leading from (9.15) to (9.24) can be repeated to show that the condition that f_α be measured from "perihelion" can be satisfied by taking

$$f_{R0} = K_{R\alpha} \tag{12.55}$$

where $K_{R\alpha}$ is the quarterperiod of the Jacobian elliptic functions of modulus $k_{R\alpha}$. By repeated application of the relation

$$\mathrm{sn}[f_\alpha + K_{R\alpha}\colon k_{R\alpha}] = \frac{\mathrm{cn}[f_\alpha\colon k_{R\alpha}]}{\mathrm{dn}[f_\alpha\colon k_{R\alpha}]} \tag{12.56}$$

the solution form (12.54) can be recast as

$$\frac{R}{a} = \frac{[(1+\frac{1}{2}\lambda)+(1-\frac{1}{2}\lambda)\delta_{R\alpha}]\,\mathrm{dn}[f_\alpha\colon k_{R\alpha}] + [(1-\frac{1}{2}\lambda)+(1+\frac{1}{2}\lambda)\delta_{R\alpha}]\,\mathrm{cn}[f_\alpha\colon k_{R\alpha}]}{\mathrm{dn}[f_\alpha\colon k_{R\alpha}] + \delta_{R\alpha}\,\mathrm{cn}[f_\alpha\colon k_{R\alpha}]} \tag{12.57}$$

yielding the ultimate solution for R in the case $e = 1$.

For the case $\beta = 0$, it suffices to note that in this case, relation (12.50a) takes the form

$$j_{R\alpha} = \frac{1}{\sqrt{\lambda}}(\sqrt{\lambda + 2} + 2) \tag{12.58}$$

and the subsequent reduction remains the same.

Two further observations are worth making:

1. When we note relations (5.2) and (5.11) and consider the limit, for $e \to 1$, of the identification formula for $e \leq 1$ as given by (8A.22), we find it consistent with (12.25).
2. Similarly, when we note relations (11.4) and (11.9) and consider the limit for $e \to 1$ of the identification formula for $e > 1$ as given by (11.23), again we find it consistent with (12.25).

Accordingly, as long as due care is exercised, the results of Section 12 may be arrived at by taking limits of the results of Subsections 8A and 9A, or alternatively of taking the limits of the results of Section 11.

13 Summary of the Orbit Solutions

In Sections 11 and 12, respectively, we have noted that the solution forms for the complementary range ($e > 1$) and for the singular case ($e = 1$) are already included in the solution forms exhibited in Subsections 8A and 9A where the focus had been on a range of $e < 1$. Hence in the solution forms presented in Sections 8–8A and 9–9A, we have respectively the complete set of solution forms for the S- and R-equations covering the entire range of $e \geq 0$.

Accordingly, by appropriate combination of these S- and R-solution forms, we may now record the orbit solutions over the entire λ-range and note the main characteristic features of the orbits in each of the several range-segments of the separation parameter (λ). These range-segments are defined by the alignment and overlap of the λ-ranges specifying the ranges of validity for the several solution forms presented in Sections 8–8A and 9–9A. These solution forms together with their associated ranges of validity are listed in Subsections 8B and 9B, respectively.

The orbit solutions are formed from the combination of the S-solution form from Subsection 8B with the appropriate R-solution form from Subsection 9B for each of the appropriate λ-range segments. We shall list separately the orbits for Case A and Case B; there are four separate λ-segments to be considered in Case A, with two in Case B, making for six distinct orbit types in all.

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Case A: $e^2 + \beta^2 \geq 1$:

1. $A1$ *Orbit*: $AS1 \oplus AR1$ valid when $0 \leq \lambda \leq 1 - e$.

The orbits are restricted by the bounding inner and outer ellipses, with the inner ellipse defining an exclusion zone surrounding both primaries. The orbit loops, encircling both primaries, brushing off the inner and outer ellipses.

When $e \geq 1$, this orbit range disappears. In particular, when $\beta = 0$, which in this case implies that $e \geq 1$, this orbit range disappears.

2. $A2$ *Orbit*: $AS1 \oplus AR2$ valid when $1 - e \leq \lambda \leq \beta - \gamma$.

The bounding inner ellipse has disappeared. The orbit brushes off the outer bounding ellipse and loops around the primaries individually in a figure-of-eight pattern.

When $e \geq 1$, this orbit range disappears and as in $A1$, this happens in particular when $\beta = 0$.

3. $A3$ *Orbit*: $AS2 \oplus AR3$

(i) if $e \leq 1$, valid when $\beta - \gamma \leq \lambda \leq \beta + \gamma$;
(ii) if $e \geq 1$, valid when $\gamma - \beta \leq \lambda \leq \beta + \gamma$ ($A3^*$).

In this λ-range, a zone of exclusion appears about the minor primary; the zone is bounded by a hyperbola branch. The orbit, a satellite orbit of the major primary, brushes off the outer bounding ellipse and executes vigorous passes close to the bounding hyperbola.

When $e > 1$, the range of validity changes to the range $A3^*$ indicated above. In particular, when $\beta = 0$, so that necessarily $e \geq 1$, the range of validity shrinks to the single value $\lambda = \sqrt{e^2 - 1}$, which is covered by the range $A4$ following. Accordingly, when $\beta = 0$, this orbit range also disappears.

4. $A4$ *Orbit*: $AS2 \oplus AR3$ valid when $\beta + \gamma \leq \lambda \leq 1 + e$.

This combination is identical to $A3$ above. However, the orbit characteristics have now changed significantly. When the parameter λ moved through the transition point $\beta + \gamma$, a second bounding hyperbola appears around the minor primary, and the zone of exclusion now lies between the two hyperbolae. This second hyperbola in turn introduces a second satellite orbit system around the minor primary.

When $e = 1$, the upper hyperbola coincides with the x-axis, in accord with the separate analysis of Section 12.

When $e > 1$, the upper hyperbola moves into the upper half-plane, and two systems of satellite orbits are now fully polarized (Section 11).

When $\beta = 0$, then necessarily $e \geq 1$, and the range of validity becomes $\sqrt{e^2 - 1} \leq \lambda \leq 1 + e$.

Case B: $e^2 + \beta^2 \leq 1$:

5. $B1$ *Orbit*: $BS \oplus BR1$ valid when $0 \leq \lambda \leq 1 - e$.

The orbits are restricted by the bounding inner and outer ellipses, with the inner ellipse defining an exclusion zone surrounding both primaries. The orbit loops encircling both primaries brush off the inner and outer ellipses.

6. $B2$ *Orbit*: $BS \oplus BR2$ valid when $1 - e \leq \lambda \leq 1 + e$.

The inner bounding ellipse has disappeared, and the orbit brushing off the outer ellipse loops around each primary individually in a figure-of-eight pattern.

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We now record the orbit solution forms for each of the above six ranges. In this, all factors, including those from Sections 8 and 9, are expressed in terms of λ.

Case A: $e^2 + \beta^2 \geq 1$:

$A1$: valid for $0 \leq \lambda \leq 1 - e$.

$$\Lambda^2 = \Lambda_1^2 = \tfrac{1}{2}\frac{C^2}{1-e^2}\left[[1-e^2-\lambda^2] + \sqrt{[1-e^2+\lambda^2)]^2 - 4\lambda^2\beta^2}\right] \tag{8.12a}$$

$$\delta = \delta_S = -\frac{2\lambda\beta}{[1-e^2+\lambda^2] + \sqrt{[1-e^2+\lambda^2]^2 - 4\lambda^2\beta^2}} \tag{8.3}$$

$$k_S = k_{S1} = \sqrt{\frac{[1-e^2-\lambda^2] - \sqrt{[1-e^2+\lambda^2]^2 - 4\lambda^2\beta^2}}{[1-e^2-\lambda^2] + \sqrt{[1-e^2+\lambda^2]^2 - 4\lambda^2\beta^2}}} \tag{8.12b}$$

$$S = \frac{\mathrm{sn}[f + f_{S0} : k_{S1}] + \delta_S}{1 + \delta_S\,\mathrm{sn}[f + f_{S0} : k_{S1}]} \tag{8.15}$$

$$\delta = \delta_v = \frac{2\lambda^2 e}{[(1-\lambda^2) - e^2(2+\lambda^2) + e^4] + \sqrt{[(1-\lambda^2) - e^2(2+\lambda^2) + e^4]^2 - 4\lambda^4 e^2}} \tag{9.3}$$

$$k = k_v = \sqrt{\frac{[1-e^2-\lambda^2] - \sqrt{[1-e^2-\lambda^2]^2 - 4\lambda^2 e^2}}{[1-e^2-\lambda^2] + \sqrt{[1-e^2-\lambda^2]^2 - 4\lambda^2 e^2}}} \tag{9.9}$$

$$j_v = j_{v1} = \sqrt{\frac{(1-e^2-\lambda^2) + \sqrt{[(1-e^2-\lambda^2)^2 - 4\lambda^2 e^2]}}{(1-e^2-\lambda^2) + \sqrt{[(1-e^2+\lambda^2)^2 - 4\lambda^2\beta^2]}}} \tag{9.8a}$$

$$f_v = j_{v1} f$$

$$\frac{R}{a} = \frac{(1-e^2)[\mathrm{dn}[f_v : k_v] + \delta_v\,\mathrm{cn}[f_v : k_v]]}{(1+e\delta_v)\,\mathrm{dn}[f_v : k_v] + (e+\delta_v)\,\mathrm{cn}[f_v : k_v]} \tag{9.26}$$

Case A: $e^2 + \beta^2 \geq 1$:

$A2$: valid for $(1 - e) \leq \lambda \leq \beta - \gamma$.

$$\Lambda^2 = \Lambda_1^2 = \tfrac{1}{2}\frac{C^2}{1 - e^2}\left[[1 - e^2 - \lambda^2] + \sqrt{[1 - e^2 + \lambda^2)]^2 - 4\lambda^2\beta^2}\right] \tag{8.12a}$$

$$\delta = \delta_S = -\frac{2\lambda\beta}{[1 - e^2 + \lambda^2] + \sqrt{[1 - e^2 + \lambda^2]^2 - 4\lambda^2\beta^2}} \tag{8.3}$$

$$k_S = k_{S1} = \sqrt{\frac{[1 - e^2 - \lambda^2] - \sqrt{[1 - e^2 + \lambda^2]^2 - 4\lambda^2\beta^2}}{[1 - e^2 - \lambda^2] + \sqrt{[1 - e^2 + \lambda^2]^2 - 4\lambda^2\beta^2}}} \tag{8.12b}$$

$$S = \cos\sigma = \frac{\operatorname{sn}[f + f_{S0} : k_{S1}] + \delta_S}{1 + \delta_S \operatorname{sn}[f + f_{S0} : k_{S1}]} \tag{8.15}$$

$$\delta = \delta_R = -\frac{(\lambda + e)[1 - (\lambda - e)]}{(\lambda + e)^2 - (\lambda - e) + 2\sqrt{\lambda e}\sqrt{(\lambda + e)^2 - 1}} \tag{9A.17}$$

$$k = k_R = \frac{2\sqrt{\lambda e} - \sqrt{(\lambda + e)^2 - 1}}{2\sqrt{\lambda e} + \sqrt{(\lambda + e)^2 - 1}} \tag{9A.24}$$

$$j_R = j_{R1} = \frac{1}{\sqrt{2}}\frac{2\sqrt{\lambda e} + \sqrt{(\lambda + e)^2 - 1}}{\sqrt{[1 - e^2 - \lambda^2] + \sqrt{[1 - e^2 + \lambda^2]^2 - 4\lambda^2\beta^2}}} \tag{9A.27a}$$

$$f_R = j_{R1} f$$

$$\frac{R}{a} = \tfrac{1}{2}\frac{[[1+\lambda+e]+[1-(\lambda-e)]\delta_R]\operatorname{dn}[f_R : k_R] + [[1-(\lambda-e)]+[1+\lambda+e]\delta_R]\operatorname{cn}[f_R : k_R]}{\operatorname{dn}[f_R : k_R] + \delta_R \operatorname{cn}[f_R : k_R]} \tag{9A.33}$$

Case A: $e^2 + \beta^2 \geq 1$:

$A3$: $(\beta - \gamma) \leq \lambda \leq \beta + \gamma$;
$A4$: $(\beta + \gamma) \leq \lambda \leq 1 + e$;

$$\Lambda^2 = \Lambda_0^2 = \tfrac{1}{4}\frac{C^2}{1 - e^2}\left[2\sqrt{\lambda\gamma} + \sqrt{(\lambda + \gamma)^2 - \beta^2}\right] \tag{8A.22}$$

$$\delta = \delta_S^* = -\frac{(\lambda+\gamma)(\lambda+\beta-\gamma)}{(\lambda+\gamma)^2+\beta(\lambda-\gamma)+2\sqrt{\lambda\gamma[(\lambda+\gamma)^2-\beta^2]}} \tag{8A.14}$$

$$k_S = k_{S0} = \pm\frac{2\sqrt{\lambda\gamma}-\sqrt{(\lambda+\gamma)^2-\beta^2}}{2\sqrt{\lambda\gamma}+\sqrt{(\lambda+\gamma)^2-\beta^2}} \tag{8A.28}$$

$$\cos\sigma = S = \frac{1}{2\lambda}\frac{[(1+\delta_S^*)\lambda+(1-\delta_S^*)(\beta-\gamma)]\,\mathrm{sn}[f+f_{S0}:k_{S0}]+[(1+\delta_S^*)\lambda-(1-\delta_S^*)(\beta-\gamma)]}{1+\delta_S^*\,\mathrm{sn}[f+f_{S0}:k_{S0}]} \tag{8A.26}$$

$$\delta = \delta_R = -\frac{(\lambda+e)[1-(\lambda-e)]}{(\lambda+e)^2-(\lambda-e)+2\sqrt{\lambda e}\sqrt{(\lambda+e)^2-1}} \tag{9A.17}$$

$$k = k_R = (\pm)\frac{2\sqrt{\lambda e}-\sqrt{(\lambda+e)^2-1}}{2\sqrt{\lambda e}+\sqrt{(\lambda+e)^2-1}} \tag{9A.24}$$

$$j_R = j_{R0} = \frac{2\sqrt{\lambda e}+\sqrt{(\lambda+e)^2-1}}{2\sqrt{\lambda\gamma}+\sqrt{(\lambda+\gamma)^2-\beta^2}} \tag{9A.27b}$$

$$f_R = j_{R0}f$$

$$\frac{R}{a} = \frac{1}{2}\frac{[[1+\lambda+e]+[1-(\lambda-e)]\delta_R]\,\mathrm{dn}[f_R:k_R]+[[1-(\lambda-e)]+[1+\lambda+e]\delta_R]\,\mathrm{cn}[f_R:k_R]}{\mathrm{dn}[f_R:k_R]+\delta_R\,\mathrm{cn}[f_R:k_R]} \tag{9A.33}$$

Case B: $e^2+\beta^2 \le 1$:

$B1$: valid for $0 \le \lambda \le 1-e$.

$$\Lambda^2 = \Lambda_2^2 = \frac{C^2}{1-e^2}\sqrt{[1-e^2+\lambda^2]^2-4\lambda^2\beta^2} \tag{8.19a}$$

$$\delta = \delta_S = -\frac{2\lambda\beta}{[1-e^2+\lambda^2]+\sqrt{[1-e^2+\lambda^2]^2-4\lambda^2\beta^2}} \tag{8.3}$$

$$k = k_{S2} = \sqrt{\frac{1}{2}-\frac{1}{2}\frac{1-e^2-\lambda^2}{\sqrt{[1-e^2+\lambda^2]^2-4\lambda^2\beta^2}}} \tag{8.19b}$$

$$\cos\sigma = S = \frac{\sqrt{1-k_{S2}^2}\,\mathrm{sn}[f+f_{S0}:k_{S2}]+\delta_S\,\mathrm{dn}[f+f_{S0}:k_{S2}]}{\mathrm{dn}[f+f_{S0}:k_{S2}]+\delta_S\sqrt{1-k_{S2}^2}\,\mathrm{sn}[f+f_{S0}:k_{S2}]} \tag{8.25}$$

$$\delta = \delta_v = \frac{2\lambda^2 e}{[(1-\lambda^2) - e^2(2+\lambda^2) + e^4] + \sqrt{[(1-\lambda^2) - e^2(2+\lambda^2) + e^4]^2 - 4\lambda^4 e^2}} \quad (9.3)$$

$$k = k_v = \sqrt{\frac{[1-e^2-\lambda^2] - \sqrt{[1-e^2-\lambda^2]^2 - 4\lambda^2 e^2}}{[1-e^2-\lambda^2] + \sqrt{[1-e^2-\lambda^2]^2 - 4\lambda^2 e^2}}} \quad (9.9)$$

$$j = j_{v2} = \sqrt{\frac{[1-e^2-\lambda^2] + \sqrt{[1-e^2-\lambda^2]^2 - 4\lambda^2 e^2}}{2\sqrt{[1-e^2+\lambda^2]^2 - 4\lambda^2\beta^2}}} \quad (9.8b)$$

$$f_v = j_{v2} f$$

$$\frac{R}{a} = \frac{(1-e^2)[\mathrm{dn}[f_v : k_v] + \delta_v \,\mathrm{cn}[f_v : k_v]]}{(1+e\delta_v)\,\mathrm{dn}[f_v : k_v] + (e+\delta_v)\,\mathrm{cn}[f_v : k_v]} \quad (9.26)$$

Case B: $e^2 + \beta^2 \le 1$:

$B2$: valid for $1 - e \le \lambda \le 1 + e$.

$$\Lambda^2 = \Lambda_2^2 = \frac{C^2}{1-e^2}\sqrt{[1-e^2+\lambda^2]^2 - 4\lambda^2\beta^2} \quad (8.19a)$$

$$\delta = \delta_S = -\frac{2\lambda\beta}{[1-e^2+\lambda^2] + \sqrt{[1-e^2+\lambda^2]^2 - 4\lambda^2\beta^2}} \quad (8.3)$$

$$k = k_{S2} = \sqrt{\frac{1}{2} - \frac{1}{2}\frac{1-e^2-\lambda^2}{\sqrt{[1-e^2+\lambda^2]^2 - 4\lambda^2\beta^2}}} \quad (8.19b)$$

$$\cos\sigma = S = \frac{\sqrt{1-k_{S2}^2}\,\mathrm{sn}[f+f_{S0} : k_{S2}] + \delta_S\,\mathrm{dn}[f+f_{S0} : k_{S2}]}{\mathrm{dn}[f+f_{S0} : k_{S2}] + \delta_S\sqrt{1-k_{S2}^2}\,\mathrm{sn}[f+f_{S0} : k_{S2}]} \quad (8.25)$$

$$\delta = \delta_R = -\frac{(\lambda+e)[1-(\lambda-e)]}{(\lambda+e)^2 - (\lambda-e) + 2\sqrt{\lambda e}\sqrt{(\lambda+e)^2 - 1}} \quad (9A.17)$$

$$k = k_R = \frac{2\sqrt{\lambda e} - \sqrt{(\lambda+e)^2 - 1}}{2\sqrt{\lambda e} + \sqrt{(\lambda+e)^2 - 1}} \quad (9A.24)$$

$$j_R = j_{R2} = \frac{1}{2}\frac{2\sqrt{\lambda e} + \sqrt{(\lambda+e)^2 - 1}}{[[1-e^2+\lambda^2]^2 - 4\lambda^2\beta^2]^{1/4}} \quad (9A.27c)$$

$$f_R = j_{R2} f$$

$$\frac{R}{a} = \frac{1}{2} \frac{[[1+\lambda+e]+[1-(\lambda-e)]\delta_R]\,\mathrm{dn}[f_R : k_R] + [[1-(\lambda-e)]+[1+\lambda+e]\delta_R]\,\mathrm{cn}[f_R : k_R]}{\mathrm{dn}[f_R : k_R] + \delta_R\,\mathrm{cn}[f_R : k_R]} \tag{9A.33}$$

Finally, we make some observations regarding the ranges for the separation parameter λ in relation to a particular range of e and β.

We have already noted that in Case A ($e^2 + \beta^2 \geq 1$), if we have $\beta = 0$, so that $e \geq 1$, then the three orbit ranges A1, A2, and A3 disappear. However, when $\beta \neq 0$, no matter how small, then in Case A there is immediately admitted an associated e-range with $e \leq 1$, namely

$$\sqrt{1 - \beta^2} \leq e \leq 1$$

whereby the above three orbit ranges, namely A1, A2, and A3, reappear. With β small, the orbit ranges will be extremely narrow intervals for the separation parameter λ. Accordingly, for the e-parameter, there is an interval of extreme sensitivity

$$e_* \leq \sqrt{1 - \beta^2} \leq 1 + e_*.$$

Therein, as e traverses the parameter range and the separation parameter has its intervals specified accordingly, one can envision all six orbit ranges coming into play successively over a rather narrow overall range of the separation parameter λ.

It should be borne in mind that it would never be easy to guarantee that $\beta = 0$ exactly. Hence, these transitions in orbit types over a small but significant range of the parameter e could be of considerable interest.

4

The Euler Problem II — Three-dimensional Case

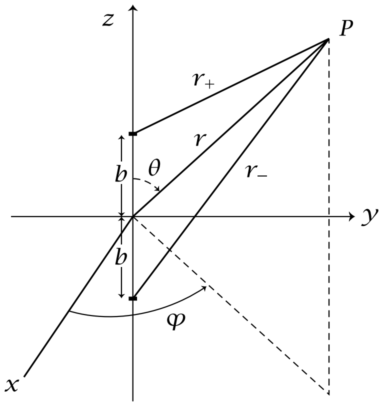
Cum autem semper cujusquam problematis quod a summis ingeniis frustra est tentatum, solutio maximi est momenti, tam vero haec questio ad eam Analyceos partem pertinet, ex qua sola nunc quidem omnia Astronomiae incrementa sunt expectanda.

— *L. Euler* [E2, e(ii)]

[Just as the solution of any problem, that has been tackled in vain by the highest intellects, is always of the greatest importance, so indeed this problem belongs to that branch of Analysis from which alone all developments in Astronomy are now, in fact, to be expected.]

1 The Gravitational Field of Two Fixed Centers: General Case

Referred to the Cartesian x-y-z triad, we consider the mass-point P in the gravitational field induced by the two fixed masses m_- and m_+ situated at the symmetrically placed points on the z-axis, namely at $z = -b$ and $z = +b$,



respectively. The potential function per unit mass at the arbitrary point P is again given by

$$U = G\left[\frac{m_-}{r_-} + \frac{m_+}{r_+}\right] \tag{1.1}$$

where G is the gravitational constant.

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In terms of the spherical polar coordinates (r, θ, φ) where the angle θ is measured from the z-axis as baseline, and φ is measured from the x-axis as baseline, the application of the cosine law renders for the distances r_- and r_+, respectively,

$$r_-^2 = r^2 + b^2 + 2br\cos\theta \tag{1.2a}$$

$$r_+^2 = r^2 + b^2 - 2br\cos\theta\,. \tag{1.2b}$$

If we now introduce spheroidal coordinates (R, σ, φ) based on the length parameter b, then in terms of the Cartesian coordinates x-y-z and of the spherical polar coordinates (r, θ, φ), we have

$$r\sin\theta\cos\varphi = x = \sqrt{R^2 - b^2}\sin\sigma\cos\varphi \tag{1.3a}$$

$$r\sin\theta\sin\varphi = y = \sqrt{R^2 - b^2}\sin\sigma\sin\varphi \tag{1.3b}$$

$$r\cos\theta = z = R\cos\sigma \tag{1.3c}$$

and

$$r^2 = x^2 + y^2 + z^2 = R^2 - b^2\sin^2\sigma\,. \tag{1.4}$$

In terms of the spheroidal coordinates (R, σ, φ), relations (1.2) for r_- and r_+ take the form

$$\begin{aligned} r_-^2 &= r^2 + b^2 + 2br\cos\theta \\ &= R^2 + b^2\cos^2\sigma + 2bR\cos\sigma = [R + b\cos\sigma]^2 \end{aligned} \tag{1.5a}$$

$$\begin{aligned} r_+^2 &= r^2 + b^2 - 2br\cos\theta \\ &= R^2 + b^2\cos^2\sigma - 2bR\cos\sigma = [R - b\cos\sigma]^2 \end{aligned} \tag{1.5b}$$

so that, as in the planar case (cf. Chapter 3), for the potential function we have

$$U = G\left[\frac{m_-}{R + b\cos\sigma} + \frac{m_+}{R - b\cos\sigma}\right] = G(m_+ + m_-)\left[\frac{R + \beta b\cos\sigma}{R^2 - b^2\cos^2\sigma}\right] \tag{1.6}$$

where again β is defined by

$$\beta = \frac{m_+ - m_-}{m_+ + m_-} \tag{1.7}$$

and measures the asymmetry between the attracting masses. As before we set

$$\mu = G(m_+ + m_-) \tag{1.8}$$

and we may write

$$U = \mu\frac{R + \beta b\cos\sigma}{R^2 - b^2\cos^2\sigma}\,. \tag{1.9}$$

Hence for a mass-point in this gravitational field, the potential energy per unit mass is given by

$$V^* = -\mu \frac{R + \beta b \cos\sigma}{R^2 - b^2 \cos^2\sigma}. \tag{1.10}$$

For the coordinate system defined in (1.3), there follows

$$\begin{aligned}
\frac{\partial x}{\partial R} &= \frac{R}{\sqrt{R^2-b^2}} \sin\sigma\cos\varphi, & \frac{\partial y}{\partial R} &= \frac{R}{\sqrt{R^2-b^2}} \sin\sigma\sin\varphi, & \frac{\partial z}{\partial R} &= \cos\sigma \\
\frac{\partial x}{\partial \sigma} &= \sqrt{R^2-b^2}\cos\sigma\cos\varphi, & \frac{\partial y}{\partial \sigma} &= \sqrt{R^2-b^2}\cos\sigma\sin\varphi, & \frac{\partial z}{\partial \sigma} &= -R\sin\sigma \\
\frac{\partial x}{\partial \varphi} &= -\sqrt{R^2-b^2}\sin\sigma\sin\varphi, & \frac{\partial y}{\partial \varphi} &= \sqrt{R^2-b^2}\sin\sigma\cos\varphi, & \frac{\partial z}{\partial \varphi} &= 0
\end{aligned}$$

so that the metric coefficients for the spheroidal coordinate system are given by

$$g_{11} = \frac{R^2 - b^2\cos^2\sigma}{R^2 - b^2}, \quad g_{12} = 0, \quad g_{13} = 0 \tag{1.11a}$$

$$g_{21} = 0, \quad g_{22} = R^2 - b^2\cos^2\sigma, \quad g_{23} = 0 \tag{1.11b}$$

$$g_{31} = 0, \quad g_{32} = 0, \quad g_{33} = (R^2 - b^2)\sin^2\sigma. \tag{1.11c}$$

Hence, the kinetic energy per unit mass is given by

$$T^* = \tfrac{1}{2}\frac{R^2 - b^2\cos^2\sigma}{R^2 - b^2}\dot{R}^2 + \tfrac{1}{2}(R^2 - b^2\cos^2\sigma)\dot{\sigma}^2 + \tfrac{1}{2}(R^2 - b^2)\sin^2\sigma\cdot\dot{\varphi}^2 \tag{1.12}$$

and for the total energy per unit mass, we have

$$\begin{aligned}
H &= T^* + V^* \\
&= \tfrac{1}{2}\frac{R^2 - b^2\cos^2\sigma}{R^2 - b^2}\dot{R}^2 + \tfrac{1}{2}(R^2 - b^2\cos^2\sigma)\dot{\sigma}^2 + \tfrac{1}{2}(R^2 - b^2)\sin^2\sigma\cdot\dot{\varphi}^2 \\
&\qquad - \mu\frac{R + \beta b\cos\sigma}{R^2 - b^2\cos^2\sigma}
\end{aligned} \tag{1.13}$$

and the associated Lagrangian is given by

$$\begin{aligned}
L^* &= T^* - V^* \\
&= \tfrac{1}{2}\frac{R^2 - b^2\cos^2\sigma}{R^2 - b^2}\dot{R}^2 + \tfrac{1}{2}(R^2 - b^2\cos^2\sigma)\dot{\sigma}^2 + \tfrac{1}{2}(R^2 - b^2)\sin^2\sigma\cdot\dot{\varphi}^2 \\
&\qquad + \mu\frac{R + \beta b\cos\sigma}{R^2 - b^2\cos^2\sigma},
\end{aligned} \tag{1.14}$$

from which the equations of motion are derived.

2 The Ignorable Coordinate: Liouville's Form and the Energy Integral

Since the coordinate variable φ does not appear explicitly in the Lagrangian, we can immediately utilize the fact that it is an ignorable coordinate and the consequences therefrom. The third Lagrangian equation reads

$$\frac{\mathrm{d}}{\mathrm{d}t}[(R^2 - b^2)\sin^2\sigma \cdot \dot{\varphi}] = 0, \tag{2.1}$$

which immediately yields the integral

$$(R^2 - b^2)\sin^2\sigma \cdot \dot{\varphi} = C_3 \tag{2.2}$$

where C_3 is the constant of integration. From (1.4) and (1.3c), we see that

$$\begin{aligned}(R^2 - b^2)\sin^2\sigma \cdot \dot{\varphi} &= [R^2 - b^2\sin^2\sigma - R^2\cos^2\sigma]\dot{\varphi} \\ &= [r^2 - r^2\cos^2\theta]\dot{\varphi} = r^2\sin^2\theta \cdot \dot{\varphi}\end{aligned} \tag{2.3}$$

from which it is evident that C_3 is the polar component of the angular momentum.

We now follow the standard procedure for dealing with the simplification following from the presence of the ignorable coordinate: we form the modified Lagrangian L, by setting

$$L = L^* - \dot{\varphi}\frac{\partial L^*}{\partial \dot{\varphi}} = L^* - (R^2 - b^2)\sin^2\sigma \cdot \dot{\varphi}^2 = L^* - \frac{C_3^2}{(R^2 - b^2)\sin^2\sigma} \tag{2.4}$$

in which we have substituted for $\dot{\varphi}$ from (2.2). By introducing L^* from (1.14) and again substituting for $\dot{\varphi}$ from (2.2), we write the modified Lagrangian (2.4) explicitly

$$\begin{aligned}L = \tfrac{1}{2}\frac{R^2 - b^2\cos^2\sigma}{R^2 - b^2}\dot{R}^2 + \tfrac{1}{2}(R^2 - b^2\cos^2\sigma)\dot{\sigma}^2 \\ + \mu\frac{R + \beta b\cos\sigma}{R^2 - b^2\cos^2\sigma} - \tfrac{1}{2}\frac{C_3^2}{(R^2 - b^2)\sin^2\sigma}\end{aligned} \tag{2.5}$$

and the modified form of the kinetic and potential energies are, respectively,

$$T = \tfrac{1}{2}\frac{R^2 - b^2\cos^2\sigma}{R^2 - b^2}\dot{R}^2 + \tfrac{1}{2}(R^2 - b^2\cos^2\sigma)\dot{\sigma}^2 \tag{2.6}$$

$$V = -\mu\frac{R + \beta b\cos\sigma}{R^2 - b^2\cos^2\sigma} + \tfrac{1}{2}\frac{C_3^2}{(R^2 - b^2)\sin^2\sigma} \tag{2.7a}$$

$$= -\frac{1}{R^2 - b^2\cos^2\sigma}\left[\mu(R + \beta b\cos\sigma) - \tfrac{1}{2}C_3^2\left(\frac{1}{\sin^2\sigma} + \frac{b^2}{R^2 - b^2}\right)\right]. \tag{2.7b}$$

If we introduce this latter modification into the modified Lagrangian (2.5), we have

$$\begin{aligned}L = \tfrac{1}{2}\frac{R^2 - b^2\cos^2\sigma}{R^2 - b^2}\dot{R}^2 + \tfrac{1}{2}(R^2 - b^2\cos^2\sigma)\dot{\sigma}^2 \\ + \frac{1}{R^2 - b^2\cos^2\sigma}\left[\mu(R + \beta b\cos\sigma) - \tfrac{1}{2}C_3^2\left(\frac{1}{\sin^2\sigma} + \frac{b^2}{R^2 - b^2}\right)\right],\end{aligned} \tag{2.8}$$

which is the required form.

The next step is the transformation of this modified Lagrangian into one of Liouville type through the introduction of the auxiliary variable ξ defined by

$$R = b\cosh\xi, \quad \dot{R} = b\sinh\xi \cdot \dot{\xi}, \quad R^2 - b^2 = b^2\sinh^s\xi, \tag{2.9a,b,c}$$

so that the form (2.8) is replaced by

$$\begin{aligned} L &= b^2(\cosh^2\xi - \cos^2\sigma)[\tfrac{1}{2}\dot{\xi}^2 + \tfrac{1}{2}\dot{\sigma}^2] \\ &+ \frac{1}{b^2(\cosh^2\xi - \cos^2\sigma)}\Big[\mu b(\cosh\xi + \beta\cos\sigma) - \tfrac{1}{2}C_3^2\Big(\frac{1}{\sin^2\sigma} + \frac{1}{\sinh^2\xi}\Big)\Big]. \end{aligned} \tag{2.10}$$

It will be convenient for subsequent manipulations to set

$$Q_1(\xi) = b^2\cosh^2\xi, \quad Q_2(\sigma) = -b^2\cos^2\sigma, \quad Q = Q_1 + Q_2 \tag{2.11a,b,c}$$

$$V_1(\xi) = -\Big[\mu b\cosh\xi - \tfrac{1}{2}C_3^2\frac{1}{\sinh^2\xi}\Big], \quad V_2 = -\Big[\mu\beta b\cos\sigma - \tfrac{1}{2}C_3^2\frac{1}{\sin^2\sigma}\Big] \tag{2.12a,b}$$

and so with

$$QV = V_1 + V_2, \quad T = Q[\tfrac{1}{2}\dot{\xi}^2 + \tfrac{1}{2}\dot{\sigma}^2] \tag{2.13a,b}$$

the Lagrangian (2.10) may be written

$$L = Q[\tfrac{1}{2}\dot{\xi}^2 + \tfrac{1}{2}\dot{\sigma}^2] - V = Q[\tfrac{1}{2}\dot{\xi}^2 + \tfrac{1}{2}\dot{\sigma}^2] - \frac{1}{Q}[V_1 + V_2] \tag{2.14}$$

which is now in the standard Liouville form.

The derivation of the energy integral from the Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial\dot{\xi}}\Big) = \frac{\partial L}{\partial\xi}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial\dot{\sigma}}\Big) = \frac{\partial L}{\partial\sigma} \tag{2.15a,b}$$

will follow the standard procedure already outlined in Chapter 3, Section 2, to which one can refer for the details omitted here. We multiply (2.15a) by $\dot{\xi}$ and (2.15b) by $\dot{\sigma}$ and add; after some rearrangement this yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\dot{\xi}\frac{\partial L}{\partial\dot{\xi}} + \dot{\sigma}\frac{\partial L}{\partial\dot{\sigma}}\Big] = \frac{\mathrm{d}L}{\mathrm{d}t} \tag{2.16}$$

which yields the integral

$$\dot{\xi}\frac{\partial L}{\partial\dot{\xi}} + \dot{\sigma}\frac{\partial L}{\partial\dot{\sigma}} = L + \mathcal{E} \tag{2.17}$$

where $\mathcal{E}$ is the constant of integration. When the Lagrangian has the form (2.14), then, noting (2.13b), the left-hand side of (2.17) can clearly be replaced by $2T$, and so we have

$$T + V = \mathcal{E} \tag{2.18}$$

showing that the constant $\mathcal{E}$ represents the total energy of the system.

3 The First Integrals in Liouville Coordinates

If we introduce the explicit form (2.14) into the Lagrange equations (2.15), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(Q\dot{\xi}) = \frac{\mathrm{d}Q_1}{\mathrm{d}\xi}\left(\tfrac{1}{2}\dot{\xi}^2 + \tfrac{1}{2}\dot{\sigma}^2\right) - \frac{\partial V}{\partial \xi}, \quad \frac{\mathrm{d}}{\mathrm{d}t}(Q\dot{\sigma}) = \frac{\mathrm{d}Q_2}{\mathrm{d}\sigma}\left(\tfrac{1}{2}\dot{\xi}^2 + \tfrac{1}{2}\dot{\sigma}^2\right) - \frac{\partial V}{\partial \sigma}, \tag{3.1a,b}$$

the integration of which will follow the pattern set in Chapter 3, Section 3.

Recapitulating, we multiply (3.1a) by $Q\dot{\xi}$, and noting relation (2.13a), we obtain

$$\begin{aligned} Q\dot{\xi}\frac{\mathrm{d}}{\mathrm{d}t}(Q\dot{\xi}) &= \dot{\xi}\left[T\frac{\mathrm{d}Q_1}{\mathrm{d}\xi} - Q\frac{\partial V}{\partial \xi}\right] = \dot{\xi}\left[\mathcal{E}\frac{\mathrm{d}Q}{\mathrm{d}\xi} - \frac{\partial (QV)}{\partial \xi}\right] \\ &= \dot{\xi}\left[\mathcal{E}\frac{\mathrm{d}Q_1}{\mathrm{d}\xi} - \frac{\mathrm{d}V_1}{\mathrm{d}\xi}\right] = \frac{\mathrm{d}}{\mathrm{d}t}[\mathcal{E}Q_1 - V_1], \end{aligned} \tag{3.2}$$

which yields the ξ-first integral

$$\tfrac{1}{2}(Q\dot{\xi})^2 - \mathcal{E}Q_1 + V_1 = C_1 \tag{3.3}$$

where C_1 is the constant of integration. Similarly, by multiplying (3.1b) by $Q\dot{\sigma}$ and proceeding in like manner, we obtain the σ-first integral

$$\tfrac{1}{2}(Q\dot{\sigma})^2 - \mathcal{E}Q_2 + V_2 = C_2. \tag{3.4}$$

The addition of (3.3) and (3.4) yields

$$Q^2\left[\tfrac{1}{2}\dot{\xi}^2 + \tfrac{1}{2}\dot{\sigma}^2\right] - \mathcal{E}(Q_1 + Q_2) + V_1 + V_2 = C_1 + C_2 \tag{3.5}$$

or alternatively, on recalling (2.13) and (2.11c), we have the compact form

$$Q[T + V - \mathcal{E}] = C_1 + C_2, \tag{3.6}$$

which from (2.18) clearly implies the relation

$$C_1 + C_2 = 0 \tag{3.7}$$

between the constants of the first integrals.

Equations (3.3), (3.4) together with the energy integral (2.18) are the first integrals of the modified Lagrangian (2.14). However, as the three integrals when combined imply relation (3.7), the three integrals are not independent. The third independent integral for the dynamical system is the polar component of angular momentum (2.2).

Having derived the first integrals by means of the Liouville procedure, facilitated by the introduction of the auxiliary “Liouville” variable ξ, the latter has now served its purpose and will be dispensed with. Substituting for ξ, we revert to the original R-σ spheroidal coordinate system.

4 The First Integrals in Spheroidal Coordinates

Recalling the defining relations (2.9), it follows from (2.11) and (2.12) that, in terms of the spheroidal coordinates R, σ, we have

$$Q_1(R) = R^2, \quad Q_2(\sigma) = -b^2\cos^2\sigma, \quad Q = R^2 - b^2\cos^2\sigma \tag{4.1a,b,c}$$

$$V_1(R) = -\Big[\mu R - \tfrac{1}{2}\frac{C_3^2 b^2}{R^2 - b^2}\Big], \quad V_2(\sigma) = -\Big[\mu\beta b\cos\sigma - \tfrac{1}{2}\frac{C_3^2}{\sin^2\sigma}\Big] \tag{4.2a,b}$$

and furthermore, from (2.9b), we have

$$\dot{\xi} = \frac{\dot{R}}{b\sinh\xi} = \frac{\dot{R}}{\sqrt{R^2 - b^2}} \tag{4.3}$$

and hence

$$Q\dot{\xi} = \frac{(R^2 - b^2\cos^2\sigma)}{\sqrt{R^2 - b^2}}\dot{R}, \quad Q\dot{\sigma} = (R^2 - b^2\cos^2\sigma)\dot{\sigma}. \tag{4.4a,b}$$

Accordingly, in terms of the spheroidal coordinates (R, σ), the equations for the first integrals (3.3) and (3.4) read

$$\tfrac{1}{2}\frac{(R^2 - b^2\cos^2\sigma)^2}{R^2 - b^2}\dot{R}^2 = \mathcal{E}R^2 + \Big[\mu R - \tfrac{1}{2}\frac{C_3^2 b^2}{R^2 - b^2}\Big] + C_1 \tag{4.5a}$$

$$\tfrac{1}{2}(R^2 - b^2\cos^2\sigma)^2\dot{\sigma}^2 = -\mathcal{E}b^2\cos^2\sigma + \Big[\mu\beta b\cos\sigma - \tfrac{1}{2}\frac{C_3^2}{\sin^2\sigma}\Big] + C_2 \tag{4.5b}$$

with the restriction (3.7) that $C_1 + C_2 = 0$.

An inspection of (4.5b) in the limit as $b \to 0$ indicates that in that range, C_2 must be positive, which immediately implies that C_1 is negative. Moreover, we shall keep the focus of our attention mainly on bound orbits, corresponding in this range with negative energy. Accordingly, we set

$$\mathcal{E} = -\alpha^2, \quad C_1 = -\tfrac{1}{2}C^2, \quad C_2 = \tfrac{1}{2}C^2 \tag{4.6}$$

and observing (4.5b) we note that C has the dimension of angular momentum. In terms of these constants, equations (4.5) take the form

$$\tfrac{1}{2}\frac{(R^2 - b^2\cos^2\sigma)^2}{R^2 - b^2}\dot{R}^2 = -\alpha^2R^2 + \Big[\mu R - \tfrac{1}{2}\frac{b^2C_3^2}{R^2 - b^2}\Big] - \tfrac{1}{2}C^2 \tag{4.7a}$$

$$\tfrac{1}{2}(R^2 - b^2\cos^2\sigma)^2\dot{\sigma}^2 = \alpha^2b^2\cos^2\sigma + \Big[\mu\beta b\cos\sigma - \tfrac{1}{2}\frac{C_3^2}{\sin^2\sigma}\Big] + \tfrac{1}{2}C^2. \tag{4.7b}$$

In pursuing the reduction and analysis of these equations, we shall follow the pattern set out in the planar case of Chapter 3, Section 5.

We note that if we set $\sigma = \frac{1}{2}\pi$ in (4.7b), we have

$$\tfrac{1}{2}C^2 - \tfrac{1}{2}C_3^2 = \tfrac{1}{2}R^2\dot{\sigma}^2 \geq 0 \tag{4.8}$$

so that

$$C_3^2 \leq C^2 \tag{4.9}$$

with equality holding only for the equatorial orbit when $\dot{\sigma} = 0$; relation (4.9) will be noted below.

— ◇ —

In this chapter, the analysis is performed only for the range corresponding to Sections 8 and 9 of Chapter 3; the extension of the analysis to the supplementary and complementary ranges corresponding to Sections 8A, 9A, 11, and 12 of Chapter 3, follows the established pattern.

5 Reduction of Equations: The Regularization

First, we consider the R-equation and rewrite equation (4.7a) in the form

$$(R^2 - b^2\cos^2\sigma)^2\dot{R}^2 = -C^2(R^2 - b^2)\Big[1 - \frac{2\mu}{C^2}R + \frac{2\alpha^2}{C^2}R^2 + \frac{b^2}{R^2 - b^2}\frac{C_3^2}{C^2}\Big]. \quad (5.1)$$

Again we introduce length scales a and p, together with a dimensionless parameter ν — the latter corresponding to the inclination in the Kepler case: we set

$$a = \frac{\mu}{2\alpha^2}, \quad p = \frac{C^2}{\mu} \quad \Big(\text{so that } ap = \frac{C^2}{2\alpha^2}\Big), \quad \nu = \frac{C_3}{C} \quad (5.2)$$

and from (4.9) we note that $\nu^2 \le 1$. In terms of these quantities, equation (5.1) reads

$$\frac{(R^2 - b^2\cos^2\sigma)^2}{C^2}\dot{R}^2 = -(R^2 - b^2)\Big[1 - \frac{2}{p}R + \frac{1}{ap}R^2\Big] - b^2\nu^2. \quad (5.3)$$

Next, for the σ-equation we rewrite (4.7b) in the form

$$\begin{aligned}\frac{(R^2 - b^2\cos^2\sigma)^2}{C^2}\dot{\sigma}^2 &= 1 + \frac{2\mu}{C^2}\beta b\cos\sigma + \frac{2\alpha^2}{C^2}b^2\cos^2\sigma - \frac{C_3^2}{C^2}\frac{1}{\sin^2\sigma} \\ &= 1 + 2\beta\frac{b}{p}\cos\sigma + \frac{b^2}{ap}\cos^2\sigma - \frac{\nu^2}{\sin^2\sigma} \quad (5.4)\end{aligned}$$

and, as in the planar case, the form of the regularizing transformation is indicated by the multiplying factor on the left-hand side of (5.3) and (5.4).

We introduce the new independent variable f by the defining equation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\Lambda}{R^2 - b^2\cos^2\sigma} \quad (5.5)$$

where Λ is a parameter with the dimension of angular momentum to be defined presently; it follows that

$$\frac{R^2 - b^2\cos^2\sigma}{\Lambda}\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}f} \quad (5.6)$$

so that, with prime denoting differentiation with respect to f, the equations for the first integrals (5.3) and (5.4) become, respectively,

$$\frac{\Lambda^2}{C^2}R'^2 = -(R^2 - b^2)\left[1 - \frac{2}{p}R + \frac{1}{ap}R^2\right] - b^2\nu^2 \tag{5.7}$$

$$\frac{\Lambda^2}{C^2}\sigma'^2 = 1 + 2\beta\frac{b}{p}\cos\sigma + \frac{b^2}{ap}\cos^2\sigma - \frac{\nu^2}{\sin^2\sigma}. \tag{5.8}$$

If, in the latter equation, we multiply across by $\sin^2\sigma$ and set

$$S = \cos\sigma, \text{ so that } S' = -\sin\sigma \cdot \sigma', \tag{5.9}$$

then equation (5.8) becomes

$$\frac{\Lambda^2}{C^2}S'^2 = (1 - S^2)\left[1 + 2\beta\frac{b}{p}S + \frac{b^2}{ap}S^2\right] - \nu^2, \tag{5.10}$$

which is the required form.

Equations (5.7) and (5.10) put into clear focus the significant complication arising in the three-dimensional case as distinct from the planar case, namely the factor with ν^2 appearing in each equation. Although it does not introduce any analytical difficulty, it presents a crucial algebraic complication: the decomposition of the quartic on the right-hand side into a product of two quadratic factors — obvious in the planar case when $\nu = 0$ — is no longer obvious in the general case and, in fact, is quite tedious to put into effect.

In order to see that the difficulties arising in both equations run parallel to each other, we first reduce both equations to dimensionless form.

6 Normalization of the Quartics

We introduce dimensionless parameters

$$\eta = \frac{b}{p}, \quad \frac{p}{a} = 1 - e^2 = \ell, \quad \lambda = \frac{b}{a} = \frac{b}{p}\frac{p}{a} = \eta\ell, \quad \frac{b^2}{ap} = \eta^2\ell \tag{6.1a,b,c,d}$$

in terms of which equation (5.10) becomes

$$\frac{\Lambda^2}{C^2}S'^2 = (1 - S^2)\left[1 + 2\beta\eta S + \eta^2\ell S^2\right] - \nu^2. \tag{6.2}$$

This form of the equation has the visual advantage that we observe immediately how in the case of $\eta = 0$, the problem collapses to the Kepler case $S'^2 = (1 - \nu^2) - S^2$ with solution $S = \sqrt{1 - \nu^2}\sin(f + \omega)$. However, in the general case, in order to effect the algebraic normalization, it becomes necessary to forego this visual feature, and it becomes convenient to introduce the auxiliary parameters β^* and $\overline{\lambda}$, by setting

$$\beta^* = \frac{\beta}{\eta\ell} = \frac{\beta}{\lambda}, \quad \overline{\lambda} = \frac{1}{\eta^2\ell} = \frac{\ell}{\lambda^2}. \tag{6.3a,b}$$

If we now divide across equation (6.2) by the factor $\eta^2\ell$, we obtain the following form of the equation:

$$\overline{\lambda}\frac{\Lambda^2}{C^2}S'^2 = (1 - S^2)\left[S^2 + 2\beta^* S + \overline{\lambda}\right] - \overline{\lambda}\nu^2, \tag{6.4}$$

which on expansion reads

$$\overline{\lambda}\frac{\Lambda^2}{C^2}S'^2 = -\left[S^4 + 2\beta^* S^3 - (1 - \overline{\lambda})S^2 - 2\beta^* S - \overline{\lambda}(1 - \nu^2)\right] \tag{6.5}$$

in which the quartic on the right exhibits the three independent parameters β^*, $\overline{\lambda}$, and ν, in terms of which the reduction of this quartic is to be effected.

Turning next to the R-equation (5.7), we first scale the R-factor by setting

$$R = aY \tag{6.6}$$

so that the new dependent variable Y is dimensionless. In terms of Y, equation (5.7) becomes

$$\frac{\Lambda^2}{C^2}Y'^2 = -(Y^2 - \eta^2\ell^2)\left[1 - \frac{2}{\ell}Y + \frac{1}{\ell}Y^2\right] - \eta^2\ell^2\nu^2 \tag{6.7a}$$

$$= -\frac{(Y^2 - \eta^2\ell^2)}{\ell}[Y^2 - 2Y + \ell] - \eta^2\ell^2\nu^2, \tag{6.7b}$$

or alternatively,

$$\ell\frac{\Lambda^2}{C^2}Y'^2 = -(Y^2 - \eta^2\ell^2)[Y^2 - 2Y + \ell] - \eta^2\ell^3\nu^2. \tag{6.8}$$

If we now note that

$$\lambda^2 = \eta^2\ell^2 = \frac{b^2}{p^2}\cdot\frac{p^2}{a^2} = \frac{b^2}{a^2}, \text{ and hence } \eta^2\ell^3 = \ell\lambda^2, \tag{6.9a,b}$$

then equation (6.8) takes the form

$$\ell\frac{\Lambda^2}{C^2}Y'^2 = -(Y^2 - \lambda^2)[Y^2 - 2Y + \ell] - \ell\lambda^2\nu^2 \tag{6.10a}$$

$$= -[Y^4 - 2Y^3 + (\ell - \lambda^2)Y^2 + 2\lambda^2 Y - \ell\lambda^2(1 - \nu^2)] \tag{6.10b}$$

in which the quartic on the right exhibits the dependence on the three independent parameters ℓ, λ^2, and ν.

The next step toward a solution would be the resolution of the quartics on the right of the above equations (6.5) and (6.10) into the product of two quadratic factors; from thence the succeeding step would be to introduce the transformation that would change such a product into a product of two quadratic factors, from one of which the linear term would be absent. This sequence presents the same difficulty in both cases, and the algebra becomes quite tedious.

In the case of the R-equation, there is no escape from the difficulty in the sense that the only possible simplification would remove the difficulty — and the generality — entirely. However, in the case of the S-equation, a considerable simplification is effected in the particular (symmetric) case where we take

$$\beta = 0 \tag{6.11}$$

and the equation becomes relatively tractable, while at the same time including all the analytic features. We shall explore this case in detail in the next section.

Before concluding this section, we set down a slightly generalized form of the quartic appearing in (6.10); namely, we consider

$$Y^4 - 2dY^3 + n(\ell - \lambda^2)Y^2 + 2gY - \ell\lambda^2(1 - \nu^2) \tag{6.12}$$

where we have introduced the the new parameters d, n, and g, respectively, into the second, third, and fourth terms. We may note:

Case I: If we take $d = 1$, $n = 1$, $g = \lambda^2$, then the quartic (6.12) becomes identical with that appearing in (6.10).

Case II: If we take $d = g = -\beta^*$, $n = -1$, set $\ell = 1$, and replace λ^2 by $\overline{\lambda}$, then the quartic (6.12) becomes identical with that appearing in (6.5).

Accordingly, the quartic (6.12) may be used as the generic model to cover both separated equations as they appear in (6.5) and (6.10).

Following our treatment of the σ-equation for the case $\beta = 0$ in the next section, we shall then return to a full consideration of the R-equation represented by (6.10). As the corresponding treatment for the σ-equation in its full generality would follow the same pattern as that for the R-equation, we shall not pursue that reduction here.

7 The σ-equation in the Case $\beta = 0$

In dealing with the σ-equation for the case $\beta = 0$, we may start either from (6.2) or (6.5); if we start from the latter, we have when $\beta = 0$ implying $\beta^* = 0$,

$$\begin{aligned} \frac{1}{\eta^2\ell}\frac{\Lambda^2}{C^2}S'^2 &= -\left[S^4 - (1 - \overline{\lambda})S^2 - \overline{\lambda}(1 - \nu^2)\right] \\ &= -\left[S^4 + \frac{(1 - \eta^2\ell)}{\eta^2\ell}S^2 - \frac{(1 - \nu^2)}{\eta^2\ell}\right]. \end{aligned} \tag{7.1}$$

Considering the form in square brackets as a quadratic in S^2, we may perform the factorization: the notation is facilitated if we set

$$2m_0 = 1 + \sqrt{1 + \frac{4\eta^2\ell(1 - \nu^2)}{(1 - \eta^2\ell)^2}} \tag{7.2}$$

and the equation may be written in the form

$$\frac{1}{\eta^2\ell}\frac{\Lambda^2}{C^2}S'^2 = -\left[S^2 - \frac{1 - \nu^2}{(1 - \eta^2\ell)m_0}\right]\left[S^2 + \frac{(1 - \eta^2\ell)}{\eta^2\ell}m_0\right]. \tag{7.3}$$

This suggests that we set

$$S = \sqrt{\frac{1-\nu^2}{(1-\eta^2\ell)m_0}}\,\zeta, \tag{7.4}$$

which if substituted into equation (7.3) renders that equation in the form

$$\frac{\Lambda^2}{C^2}\zeta'^2 = (1-\eta^2\ell)m_0(1-\zeta^2)\Big[1+\frac{\eta^2\ell(1-\nu^2)}{(1-\eta^2\ell)^2 m_0^2}\zeta^2\Big]. \tag{7.5}$$

The equation is now in a form that permits clarification with regard to the form of the solution.

Case A: When ℓ is negative, that is, when $e^2 > 1$, then we can write

$$\Lambda^2 = C^2(1-\eta^2\ell)m_0, \quad k_0^2 = -\frac{\eta^2\ell(1-\nu^2)}{(1-\eta^2\ell)^2 m_0^2} \tag{7.6}$$

and equation (7.5) becomes

$$\zeta'^2 = (1-\zeta^2)[1-k_0^2\zeta^2] \tag{7.7}$$

with the solution $\zeta = \mathrm{sn}[f+\omega : k_0]$, which gives S in the form

$$S = \sqrt{\frac{1-\nu^2}{(1-\eta^2\ell)m_0}}\,\mathrm{sn}[f+\omega : k_0] \tag{7.8}$$

where $-\omega$ is the value of f at the first equatorial crossing.

Case B: When ℓ is positive, i.e., when $e^2 < 1$, then we rearrange (7.5) to write

$$\begin{aligned}\frac{\Lambda^2}{C^2}\zeta'^2 &= (1-\eta^2\ell)m_0(1-\zeta^2)\Big[1+\frac{\eta^2\ell(1-\nu^2)}{(1-\eta^2\ell)^2 m_0^2}-\frac{\eta^2\ell(1-\nu^2)}{(1-\eta^2\ell)^2 m_0^2}(1-\zeta^2)\Big]\\ &= (1-\eta^2\ell)m_0\Big[1+\frac{\eta^2\ell(1-\nu^2)}{(1-\eta^2\ell)^2 m_0^2}\Big]\\ &\quad\times(1-\zeta^2)\Big[1-\frac{\eta^2\ell(1-\nu^2)}{(1-\eta^2\ell)^2 m_0^2+\eta^2\ell(1-\nu^2)}(1-\zeta^2)\Big].\end{aligned} \tag{7.9}$$

Hence we can choose the parameters as follows:

$$\Lambda^2 = C^2(1-\eta^2\ell)m_0\Big[1+\frac{\eta^2\ell(1-\nu^2)}{(1-\eta^2\ell)^2 m_0^2}\Big] \tag{7.10a}$$

$$k_0^2 = \frac{\eta^2\ell(1-\nu^2)}{(1-\eta^2\ell)^2 m_0^2+\eta^2\ell(1-\nu^2)} \tag{7.10b}$$

so that equation (7.9) is replaced by

$$\zeta'^2 = (1-\zeta^2)[1-k_0^2(1-\zeta^2)] \tag{7.11}$$

with solution

$$\zeta = \mathrm{cn}[f + f_0 : k_0] \tag{7.12}$$

where f_0 is the constant introduced by the integration. If we let $-\omega$ denote the value of f at the first crossing of the x-y plane

$$\zeta = 0, \quad S = 0, \quad z = 0, \quad \cos\sigma = 0, \quad \sigma = \pi/2, \quad \text{at} \quad f = -\omega \tag{7.13}$$

which is satisfied if we take

$$f_0 - \omega = -K_0 \tag{7.14}$$

where K_0 is the quarterperiod of the Jacobian elliptic function $\mathrm{cn}[f : k_0]$ of modulus k_0. Hence the general solution (7.12) may be written

$$\begin{aligned} \zeta &= \mathrm{cn}[f + \omega - K_0 : k_0] \\ &= k_0' \frac{\mathrm{sn}[f + \omega : k_0]}{\mathrm{dn}[f + \omega : k_0]} \end{aligned} \tag{7.15}$$

where k_0' is the complementary modulus determined by

$$k_0^2 + k_0'^2 = 1. \tag{7.16}$$

The solution for $S = \cos\sigma$ is obtained as

$$S = \sqrt{\frac{1 - \nu^2}{(1 - \eta^2 \ell) m_0}}\, k_0' \frac{\mathrm{sn}[f + \omega : k_0]}{\mathrm{dn}[f + \omega : k_0]} \tag{7.17}$$

where we have introduced (7.15) into (7.4).

8 The *R*-equation

Recalling the reduced form of the *R*-equation (6.10b), namely

$$\ell \frac{\Lambda^2}{C^2} Y'^2 = -[Y^4 - 2Y^3 + (\ell - \lambda^2)Y^2 + 2\lambda^2 Y - \ell\lambda^2(1 - \nu^2)] = -f(Y), \tag{8.1}$$

we now address the decomposition of the quartic within the square brackets.

8A. The Decomposition of the Quartic

We start with a transformation that eliminates the cubic term from the quartic. Setting

$$Y = X + \tfrac{1}{2} \tag{8A.1}$$

then, we find that

$$\begin{aligned} g(X) = f(Y) &= X^4 + (\ell - \lambda^2 - \tfrac{3}{2})X^2 \\ &\quad + (\ell + \lambda^2 - 1)X + \tfrac{1}{4}[\ell + 3\lambda^2 - \tfrac{3}{4} - 4\ell\lambda^2(1 - \nu^2)]. \end{aligned} \tag{8A.2}$$

Following the procedure of Descartes, we now effect the standard decomposition

$$\begin{aligned} g(X) &= [X^2 + UX + V][X^2 - UX + W] \\ &= X^4 + [V + W - U^2]X^2 + U(W - V)X + VW \end{aligned} \tag{8A.3}$$

which implies

$$4VW = \ell + 3\lambda^2 - \tfrac{3}{4} - 4\ell\lambda^2(1 - \nu^2) \tag{i}$$
$$V + W - U^2 = \ell - \lambda^2 - \tfrac{3}{2} \Rightarrow V + W = U^2 + (\ell - \lambda^2 - \tfrac{3}{2}) \tag{ii}$$
$$U(V - W) = 1 - \ell - \lambda^2. \tag{iii}$$

From (iii) there follows

$$(V - W)^2 = \frac{(1 - \ell - \lambda^2)^2}{U^2} \tag{8A.4}$$

which together with (i) implies for $(V + W)^2$

$$(V + W)^2 = \frac{(1 - \ell - \lambda^2)^2}{U^2} + [\ell + 3\lambda^2 - \tfrac{3}{4} - 4\ell\lambda^2(1 - \nu^2)]\,. \tag{8A.5}$$

Moreover from (ii), we have

$$(V + W)^2 = [U^2 + (\ell - \lambda^2 - \tfrac{3}{2})]^2\,. \tag{8A.6}$$

Identifying (8A.5) with (8A.6) and setting $Z = U^2$, we have as the equation for Z

$$Z[Z + (\ell - \lambda^2 - \tfrac{3}{2})]^2 - (1 - \ell - \lambda^2)^2 = [\ell + 3\lambda^2 - \tfrac{3}{4} - 4\ell\lambda^2(1 - \nu^2)]Z$$

which can be rearranged as

$$\begin{aligned} Z^3 + 2Z^2(\ell - \lambda^2 - \tfrac{3}{2}) - (1 - \ell - \lambda^2)^2 \\ &= [\ell + 3\lambda^2 - \tfrac{3}{4} - 4\ell\lambda^2(1 - \nu^2) - (\ell - \lambda^2 - \tfrac{3}{2})^2]Z \\ &= -3Z[1 - \tfrac{4}{3}\ell + \tfrac{1}{3}(\ell - \lambda^2)^2 + \tfrac{4}{3}\ell\lambda^2(1 - \nu^2)] \end{aligned} \tag{8A.7}$$

so that the equation for Z reads

$$\begin{aligned} &Z^3 - 3Z^2[1 - \tfrac{2}{3}(\ell - \lambda^2)] \\ &\quad + 3Z[1 - \tfrac{4}{3}\ell + \tfrac{1}{3}(\ell - \lambda^2) + \tfrac{4}{3}\ell\lambda^2(1 - \nu^2)] - [1 - \ell - \lambda^2]^2 = 0\,. \end{aligned} \tag{8A.8}$$

As an exercise-check we note that when $\nu = 0$, then $Z = 1$ is a root of this equation; in that case

$$U = 1, \quad V = \tfrac{1}{4} - \lambda^2, \quad W = \ell - \tfrac{3}{4} \tag{8A.8a}$$

and the factorization is

$$g(X) = [X^2 + X + (\tfrac{1}{4} - \lambda^2)][X^2 - X + (\ell - \tfrac{3}{4})] \tag{8A.8b}$$

and in terms of Y

$$f(Y) = (Y^2 - \lambda^2)(Y^2 - 2Y + \ell) \tag{8A.8c}$$

and we retrieve the result from the planar case.

The next step in dealing with equation (8A.8) is the elimination of the quadratic term. For this it is convenient to write

$$A = 1 - \tfrac{2}{3}(\ell - \lambda^2), \quad B = 1 - \tfrac{4}{3}\ell + \tfrac{1}{3}(\ell - \lambda^2)^2 + \tfrac{4}{3}\ell\lambda^2(1 - \nu^2),$$
$$C = 1 - \ell - \lambda^2, \tag{8A.9a,b,c}$$

so that we are dealing with the equation

$$Z^3 - 3AZ^2 + 3BZ - C^2 = 0\,. \tag{8A.10}$$

In this equation, we set

$$Z = Z_0 + A \tag{8A.11}$$

and, in terms of Z_0, the above equation becomes

$$Z_0^3 + 3(B - A^2)Z_0 + 3AB - 2A^3 - C^2 = 0\,. \tag{8A.12}$$

If the substitutions from (8A.9) are inserted, we obtain for the coefficient of Z_0

$$3(B - A^2) = -[\tfrac{1}{3}(\ell - \lambda^2)^2 + 4\lambda^2(1 - \ell) + 4\ell\lambda^2\nu^2] \tag{8A.13a}$$

and in computing the constant term, we note that

$$\begin{aligned} 3B - 2A^2 &= 1 - \tfrac{4}{3}\ell - \tfrac{8}{3}\lambda^2 + \tfrac{1}{9}(\ell - \lambda^2)^2 + 4\ell\lambda^2(1 - \nu^2) \\ A(3B - 2A^2) &= 1 - 2\ell - 2\lambda^2 + \ell^2 + \tfrac{2}{3}\lambda^2\ell - \tfrac{5}{3}\lambda^4 - \tfrac{2}{27}(\ell - \lambda^2)^3 \\ &\quad + 4\ell\lambda^2(1 - \nu^2)[1 - \tfrac{2}{3}(\ell - \lambda^2)] \end{aligned}$$

and hence

$$\begin{aligned} &3AB - 2A^3 - C^2 \\ &\quad = -\Big[\tfrac{2}{27}(\ell - \lambda^2)^3 - \tfrac{8}{3}\lambda^2(\ell - \lambda^2)(1 - \ell) + 4\ell\lambda^2\nu^2[1 - \tfrac{2}{3}(\ell - \lambda^2)]\Big]. \end{aligned} \tag{8A.13b}$$

It will prove convenient to arrange relations (8A.13) in a somewhat different format, namely,

$$3(B - A^2) = -\tfrac{1}{3}(\ell - \lambda^2)^2\Big[1 + \frac{12\lambda^2[1 - \ell(1 - \nu^2)]}{(\ell - \lambda^2)^2}\Big] \tag{8A.14a}$$

$$\begin{aligned} 3AB - 2A^3 - C^2 &= -\tfrac{2}{27}(\ell - \lambda^2)^3\Big[1 - \frac{36\lambda^2[1 - \ell(1 - \nu^2)]}{(\ell - \lambda^2)^2} + \frac{54\lambda^2\,\ell\nu^2}{(\ell - \lambda^2)^3}\Big] \\ &= -\tfrac{2}{27}(\ell - \lambda^2)^3\Big[1 - \frac{36\lambda^2[1 - \ell(1 - \nu^2)]}{(\ell - \lambda^2)^2}\Big[1 - \frac{3}{2}\frac{\ell\nu^2}{(\ell - \lambda^2)[1 - \ell(1 - \nu^2)]}\Big]\Big]. \end{aligned} \tag{8A.14b}$$

This suggests that we set

$$\lambda_* = \frac{12\lambda^2[1-\ell(1-\nu^2)]}{(\ell-\lambda^2)^2}, \quad s = 1 - \frac{3}{2}\frac{\ell\nu^2}{(\ell-\lambda^2)[1-\ell(1-\nu^2)]} \tag{8A.15a,b}$$

in terms of which we can write

$$3(B-A^2) = -\tfrac{1}{3}(\ell-\lambda^2)^2[1+\lambda_*] \tag{8A.16a}$$

$$3AB - 2A^3 - C^2 = -\tfrac{2}{27}(\ell-\lambda^2)^3[1-3\lambda_* s] \tag{8A.16b}$$

and equation (8A.12) for Z_0 may be written

$$Z_0^3 - \tfrac{1}{3}(\ell-\lambda^2)^2(1+\lambda_*)Z_0 - \tfrac{2}{27}(\ell-\lambda^2)^3(1-3\lambda_* s) = 0. \tag{8A.17}$$

We note from (8A.9a) and (8A.11) that in the particular case of $\nu = 0$ (so that $s = 1$), the root $Z = 1$ of (8A.8) would in that case correspond to the root $Z_0 = \frac{2}{3}(\ell - \lambda^2)$ of (8A.17) with $s = 1$. Accordingly, equation (8A.17) can be normalized by setting

$$Z_0 = \tfrac{1}{3}(\ell-\lambda^2)Z_* \tag{8A.18a}$$

and in terms of Z_*, equation (8A.17) becomes

$$Z_*^3 - 3(1+\lambda_*)Z_* - 2(1-3\lambda_* s) = 0 \tag{8A.18b}$$

and we note that when $\nu = 0$, then $s = 1$ and in that case the sought-for root is $Z_* = 2$. For compactness, we introduce the notation

$$A_* = 1+\lambda_*, \quad B_* = 1-3\lambda_* s \tag{8A.19a,b}$$

and the cubic to be solved (8A.18) takes the form

$$Z_*^3 - 3A_*Z_* - 2B_* = 0 \tag{8A.20}$$

for the solution of which we now follow the standard procedure.

We take Z_* in the form

$$Z_* = X_* + W_* \tag{8A.21}$$

and (8A.20) becomes

$$X_*^3 + W_*^3 + 3(X_*+W_*)(X_*W_* - A_*) - 2B_* = 0. \tag{8A.22}$$

By taking

$$X_*W_* = A_*, \quad W_* = \frac{A_*}{X_*} \tag{8A.23a,b}$$

equation (8A.22) reads

$$X_*^3 + \frac{A_*^3}{X_*^3} - 2B_* = 0 \tag{8A.24}$$

or, alternatively,

$$(X_*^3)^2 - 2B_*(X_*^3) + A_*^3 = 0\,. \tag{8A.25}$$

Solving this equation as a quadratic in X_*^3, we have the solution in the form

$$X_*^3 = B_* \pm \sqrt{B_*^2 - A_*^3} = B_* \pm i\sqrt{A_*^3 - B_*^2} \tag{8A.26a}$$

from which it immediately follows that

$$W_*^3 = \frac{A_*}{X_*} = \frac{A_*}{B_* \pm i\sqrt{A_*^3 - B_*^2}} = B_* \mp i\sqrt{A_*^3 - B_*^2}\,. \tag{8A.26b}$$

By substituting from (8A.19), a straightforward calculation shows that

$$A_*^3 - B_*^2 = \lambda_*[\lambda_*^2 + 3\lambda_*(1 - 3s^2) + 3(1 + 2s)]$$

and hence we may write

$$X_* = \left[(1 - 3\lambda_* s) + i\sqrt{\lambda_*}\left[\lambda_*^2 + 3\lambda_*(1 - 3s^2) + 3(1 + 2s)\right]^{1/2}\right]^{1/3} \tag{8A.27a}$$

$$W_* = \left[(1 - 3\lambda_* s) - i\sqrt{\lambda_*}\left[\lambda_*^2 + 3\lambda_*(1 - 3s^2) + 3(1 + 2s)\right]^{1/2}\right]^{1/3} \tag{8A.27b}$$

and hence, noting (8A.21), we have

$$\begin{aligned} Z_* = &\left[(1 - 3\lambda_* s) + i\sqrt{\lambda_*}\left[\lambda_*^2 + 3\lambda_*(1 - 3s^2) + 3(1 + 2s)\right]^{1/2}\right]^{1/3} \\ &+ \left[(1 - 3\lambda_* s) - i\sqrt{\lambda_*}\left[\lambda_*^2 + 3\lambda_*(1 - 3s^2) + 3(1 + 2s)\right]^{1/2}\right]^{1/3}, \end{aligned} \tag{8A.28}$$

a purely real quantity; moreover, as it is an even function of $\sqrt{\lambda_*}$, it is strictly a function of λ_*.

Having determined Z_*, we have thereby determined Z_0 (from (8A.18a)), and hence also we have determined Z from (8A.11); having determined Z, we know U, and thereby also V and W. Accordingly, we have effected the decomposition (8A.3) in the form

$$g(X) = [X^2 + UX + V][X^2 - UX + W]\,. \tag{8A.29}$$

We can now by a further transformation reduce the above product (8A.29) to a product of two quadratic factors, from one of which the linear term will be absent: this can be achieved by setting

$$X = Y_* - \tfrac{1}{2}U \tag{8A.30a}$$

so that, recalling from (8A.1) that $X = Y - 1/2$, we have

$$X + \tfrac{1}{2} = Y = Y_* + \tfrac{1}{2}(1 - U)\,. \tag{8A.30b}$$

We introduce (8A.30a) into (8A.29) and denote the resulting function of Y_* by $f_*(Y_*)$; accordingly, we have

$$f(Y) = g(X) = f_*(Y_*) = \left[Y_*^2 - (\tfrac{1}{4}U^2 - V)\right]\left[Y_*^2 - 2UY_* + (\tfrac{3}{4}U^2 + W)\right] \tag{8A.31}$$

which is the decomposition that we sought.

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We now return to the main target, namely the reduction of the R-equation. In terms of Y_*, the dimensionless form of the R-equation (8.1) may be written

$$\ell\frac{\Lambda^2}{C^2}Y_*'^2 = -[Y_*^2 - (\tfrac{1}{4}U^2 - V)][Y_*^2 - 2UY_* + (\tfrac{3}{4}U^2 + W)]. \tag{8.2}$$

We now recall from (6.6) that

$$R = aY = a[Y_* + \tfrac{1}{2}(1-U)] \tag{8.3}$$

where we have also noted (8A.30b). We now define R_* by setting

$$R_* = aY_* = R - \tfrac{1}{2}(1-U)a \tag{8.4}$$

so that if we multiply across equation (8.2) by a^2, we have, when written in terms of R_*,

$$\ell\frac{\Lambda^2}{C^2}R_*'^2 = -\Big[R_*^2 - (\tfrac{1}{4}U - V)a^2\Big]\Big[\frac{R_*^2}{a^2} - 2U\frac{R_*}{a} + (\tfrac{3}{4}U^2 + W)\Big] \tag{8.5}$$

or alternatively,

$$\frac{\ell}{\tfrac{3}{4}U^2 + W}\frac{\Lambda^2}{C^2}R_*'^2 = -\Big[R_*^2 - (\tfrac{1}{4}U - V)a^2\Big]\Big[1 - \frac{2U}{\tfrac{3}{4}U^2 + W}\frac{R_*}{a} + \frac{1}{\tfrac{3}{4}U^2 + W}\Big(\frac{R_*}{a}\Big)^2\Big]. \tag{8.6}$$

Here, as an exercise-check, we recall from (8A.8) that

$$\nu = 0 \text{ implies } U \to 1, \quad V \to \tfrac{1}{4} - \lambda^2, \quad W \to \ell - \tfrac{3}{4} \tag{8.7a}$$

$$\text{which implies } \tfrac{1}{4}U - V \to \lambda^2, \quad \tfrac{3}{4}U^2 + W \to \ell \tag{8.7b}$$

and we retrieve the relation (5.8a) of Chapter 3, a form already familiar from the planar case.

Consistent with this guideline, and recalling relations (6.9), we set

$$\Lambda_*^2 = \frac{\ell}{\tfrac{3}{4}U^2 + W}\Lambda^2, \quad p_* = \frac{\tfrac{3}{4}U^2 + W}{U\ell}p = \frac{\tfrac{3}{4}U^2 + W}{U}a \tag{8.8a,b}$$

$$a_* = Ua, \text{ so that } a_*p_* = \frac{\tfrac{3}{4}U^2 + W}{\ell}ap = (\tfrac{3}{4}U^2 + W)a^2 \tag{8.8c,d}$$

$$b_*^2 = \frac{\tfrac{1}{4}U - V}{\lambda^2}b^2 = (\tfrac{1}{4}U - V)a^2. \tag{8.8e}$$

wherein we have utilized relation (6.9a). In terms of the parameters and length scales defined in (8.8), equation (8.6) assumes the form

$$\frac{\Lambda_*^2}{C^2} R_*'^2 = -(R_*^2 - b_*^2)\Big[1 - \frac{2}{p_*} R_* + \frac{1}{a_* p_*} R_*^2\Big] \tag{8.9}$$

which is in the form recognizable from (5.8a) of Chapter 3, for the planar case.

We may now introduce dimensionless parameters analogous to those of (6.1), by setting

$$\eta_* = \frac{b_*}{p_*}, \quad \frac{p_*}{a_*} = \ell_* = 1 - e_*^2, \quad \frac{b_*}{a_*} = \frac{b_*}{p_*}\frac{p_*}{a_*} = \eta_* \ell_*, \quad \frac{b_*^2}{a_* p_*} = \eta_*^2 \ell_* \tag{8.10}$$

and, furthermore, analogous to (6.9) we may set

$$\lambda_*^{*2} = \eta_*^2 \ell_*^2 = \frac{b_*^2}{p_*^2}\frac{p_*^2}{a_*^2} = \frac{b_*^2}{a_*^2}, \text{ which implies } \eta_*^2 \ell_*^3 = \ell_* \lambda_*^{*2}. \tag{8.11}$$

The set of relations (8.10) and (8.11) provide the quantities in terms of which the subsequent analysis of equation (8.9) is to be framed.

Referring again to equation (5.8a) of Chapter 3, we see that equation (8.9) is identical in form with equation (5.8a) for the planar case; hence the procedure followed in arriving at the solution for the planar case is equally applicable here.

We have reduced the *R*-equation in the three-dimensional problem to a form identical with that arising in the planar case. All the modifications have gone into transforming the coefficients, parameters, and length scales, while the analytic problem remains unchanged, so that the subsequent analysis would follow an identical path. In fact, recalling the solution (9.25) of Chapter 3, we see that, for the relevant parameter range, we may write

$$\frac{R_*}{p_*} = \frac{\mathrm{dn}[j_v^* f : k_v^*] + \delta_v^* \,\mathrm{cn}[j_v^* f : k_v^*]}{(1 + e_* \delta_v^*)\,\mathrm{dn}[j_v^* f : k_v^*] + (e_* + \delta_v^*)\,\mathrm{cn}[j_v^* f : k_v^*]} \tag{8.12}$$

as the form of the solution, where the starred factors are to be determined in a manner identical with that outlined for the corresponding quantities in the planar case. Further, we note that if we define the factor u_* by setting

$$u_* = \frac{1}{R_*} \tag{8.13}$$

then we have the alternate form

$$p_* u_* = \frac{(1 + e_* \delta_v^*)\,\mathrm{dn}[j_v^* f : k_v^*] + (e_* + \delta_v^*)\,\mathrm{cn}[j_* f : k_v^*]}{\mathrm{dn}[j_v^* f : k_v^*] + \delta_v^* \,\mathrm{cn}[j_v^* f : k_v^*]}, \tag{8.14}$$

which may prove convenient in certain situations.

Finally, we come to the appropriate expression for *R*: from (8.4) there follows that

$$\frac{R}{p} = \frac{R_*}{p} + \frac{1}{2}\frac{1-U}{\ell} = \frac{p_*}{p}\frac{R_*}{p_*} + \frac{1}{2}\frac{1-U}{\ell} \tag{8.15a}$$

and if we note (8.8b), this implies

$$\frac{R}{p} = \frac{\frac{3}{4}U^2 + W}{U\ell}\frac{R_*}{p_*} + \frac{1}{2}\frac{1-U}{\ell} = \frac{\frac{3}{4}U^2 + W}{U\ell}\left[\frac{R_*}{p_*} + \frac{1}{2}\frac{U(1-U)}{\frac{3}{4}U^2 + W}\right]. \tag{8.15b}$$

Accordingly we set

$$j_R^* = \frac{\frac{3}{4}U^2 + W}{U\ell}, \quad q = \frac{1}{2}\frac{U(1-U)}{\frac{3}{4}U^2 + W} \tag{8.16a,b}$$

and recalling (8A.8a), we note that

$$\nu \to 0 \text{ implies } j_R^* \to 1, \quad q \to 0. \tag{8.16c}$$

Relations (8.15) may thereby be written

$$\frac{R}{p} = j_R^*\left[\frac{R_*}{p_*} + q\right] \tag{8.17}$$

and if we substitute from (8.12), then, omitting for the moment the explicit display of the argument $j_v^* f$ and modulus k_v^*, we have

$$\begin{aligned} \frac{R}{p} &= j_R^*\left[\frac{\mathrm{dn} + \delta_v^* \,\mathrm{cn}}{(1 + e_*\delta_v^*)\,\mathrm{dn} + (e_* + \delta_v^*)\,\mathrm{cn}} + q\right] \\ &= j_R^* \frac{[1 + q(1 + e_*\delta_v^*)]\,\mathrm{dn} + [\delta_v^* + q(e_* + \delta_v^*)]\,\mathrm{cn}}{(1 + e_*\delta_v^*)\,\mathrm{dn} + (e_* + \delta_v^*)\,\mathrm{cn}} \\ &= j_R^* \frac{[1 + q(1 + e_*\delta_v^*)]}{1 + e_*\delta_v^*} \frac{\mathrm{dn} + \frac{\delta_v^* + q(e_* + \delta_v^*)}{1 + q(1 + e_*\delta_v^*)}\,\mathrm{cn}}{\mathrm{dn} + \frac{e_* + \delta_v^*}{1 + e_*\delta_v^*}\,\mathrm{cn}}. \end{aligned} \tag{8.18}$$

This suggests that we set

$$j_R = j_R^* \frac{1 + q(1 + e_*\delta_v^*)}{1 + e_*\delta_v^*} = \left(q + \frac{1}{1 + e_*\delta_v^*}\right) j_R^* \tag{8.19a}$$

$$\overline{\delta} = \frac{\delta_v^* + q(e_* + \delta_v^*)}{1 + q(1 + e_*\delta_v^*)}, \quad \overline{e} = \frac{e_* + \delta_v^*}{1 + e_*\delta_v^*} \tag{8.19b,c}$$

so that (8.18) may now be written

$$\frac{R}{p} = j_R \cdot \frac{\mathrm{dn}[j_v^* f : k_v^*] + \overline{\delta}\,\mathrm{cn}[j_v^* f : k_v^*]}{\mathrm{dn}[j_v^* f : k_v^*] + \overline{e}\,\mathrm{cn}[j_v^* f : k_v^*]}, \tag{8.20}$$

a solution for R that is identical in form with the solution (8.12) for R_*.

A similar procedure is applicable to the σ-equation, but as already indicated, that will not be pursued here in its generality, apart from that already analyzed in Section 7 for the case $\beta = 0$, when the two attracting masses are identical.

It would appear that this class of problems provides ideal candidates for the ready application of symbolic programming; not the first time in history that Celestial Mechanics has provided the perfect challenge for contemporary computational techniques.

— ◇ —

We are now at a point where it is appropriate to make some overall observations:

1. The reduction of the R-equation to the form (8.9) has been followed by the derivation of the solution form (8.20). This solution form is valid in the parameter range corresponding to Section 9 of Chapter 3. The extension to the supplementary range corresponding to Section 9A of Chapter 3 is straightforward.

2. The analysis leading from the special σ-equation (7.1) to the solution form (7.17) can be extended to the general σ-equation (6.5) for both parameter ranges corresponding to Sections 8 and 8A of Chapter 3.

3. The further extension to the complementary parameter ranges corresponding to Sections 11and 12 of Chapter 3 is straightforward.

None of the above extensions will be detailed here.

The integration of the φ-equation, as outlined in Section 9 following, will be executed solely for the primary parameter range, using the solution forms as given earlier, namely for R as given (8.20) and for σ as given by (7.17). The integration for the other parameter ranges follows an identical procedure using the solution forms for R and σ appropriately derived.

9 The Integration of the Third (Longitude) Coordinate

The integration of the longitude coordinate requires the use of the solution forms for both the R- and the σ-coordinates. We confine our attention to the case $\beta = 0$, but again this is not a necessary restriction. The integration gives rise to certain terms, evidently “dominant” as explained presently as well as to correction terms involving elliptic integrals of both the second and third kinds. We shall see how the dominant terms may be identified and also see how to establish an approximation scheme for the correction terms that may be carried to any specified degree of accuracy in terms of a physically identifiable parameter.

From relation (2.2) we have

$$\dot{\varphi} = \frac{C_3}{(R^2 - b^2)\sin^2\sigma} \tag{9.1}$$

and if we introduce the regularizing variable f as the independent variable by means of (5.5), then with prime denoting differentiation with respect to f (as before), equation (9.1) becomes

$$\varphi' = \frac{C_3}{\Lambda}\frac{R^2 - b^2\cos^2\sigma}{(R^2 - b^2)\sin^2\sigma} = \frac{C_3}{\Lambda}\left[\frac{1}{\sin^2\sigma} + \frac{b^2}{R^2 - b^2}\right]. \tag{9.2}$$

For the determination of the parameters for this equation, we start by recalling the formulae for the modulus k_0, and for its complement $k_0' = \sqrt{1 - k_0^2}$, that arise in the solution form (7.17) for the σ-coordinate. From (7.10b), we recall that

$$k_0^2 = \frac{\eta^2\ell(1-\nu^2)}{(1-\eta^2\ell)^2 m_0^2 + \eta^2\ell(1-\nu^2)}, \quad k_0'^2 = \frac{(1-\eta^2\ell)^2 m_0^2}{(1-\eta^2\ell)^2 m_0^2 + \eta^2\ell(1-\nu^2)} \tag{9.3a,b}$$

while (7.10a) implies that

$$\Lambda^2 = C^2(1-\eta^2\ell)m_0\Big[\frac{(1-\eta^2\ell)^2 m_0^2 + \eta^2\ell(1-\nu^2)}{(1-\eta^2\ell)^2 m_0^2}\Big] = C^2\frac{(1-\eta^2\ell)m_0}{k_0'^2}\,. \tag{9.3c}$$

Noting the definition of ν in (5.2), we introduce the related quantity N_k by setting

$$\begin{aligned} N_k^2 = \frac{C_3^2}{\Lambda^2} &= \frac{C_3^2}{C^2}\frac{k_0'^2}{(1-\eta^2\ell)m_0} \\ &= \frac{k_0'^2\nu^2}{(1-\eta^2\ell)m_0} = \frac{(1-\eta^2\ell)m_0}{(1-\eta^2\ell)^2 m_0^2 + \eta^2\ell(1-\nu^2)}\cdot\nu^2 \end{aligned} \tag{9.4}$$

whereby equation (9.2) reads

$$\varphi' = N_k\Big[\frac{1}{\sin^2\sigma} + \frac{b^2}{R^2 - b^2}\Big] = N_k\Big[\frac{1}{\sin^2\sigma} + \frac{\eta^2}{(\frac{R}{p})^2 - \eta^2}\Big] \tag{9.5}$$

where we have rearranged the second term through the introduction of the parameter η^2 in accordance with (6.1a).

For the solution form (7.17) for $\cos\sigma$, we introduce a second parameter N_k', also associated with the inclination parameter ν, by setting

$$1 - N_k'^2 = \frac{(1-\nu^2)^2 k_0'^2}{(1-\eta^2\ell)m_0} = \frac{(1-\eta^2\ell)m_0}{(1-\eta^2\ell)^2 m_0^2 + \eta^2\ell(1-\nu^2)}(1-\nu^2) \tag{9.6}$$

and we note that

$$\begin{aligned} 1 + (N_k^2 - N_k'^2) &= N_k^2 + (1 - N_k'^2) \\ &= \frac{k_0'^2}{(1-\eta^2\ell)m_0} = \frac{(1-\eta^2\ell)m_0}{(1-\eta^2\ell)^2 m_0^2 + \eta^2\ell(1-\nu^2)}\,. \end{aligned} \tag{9.7}$$

Utilizing (9.6), we may write (7.17) in the form

$$S = \cos\sigma = \sqrt{1 - N_k'^2}\,\frac{\mathrm{sn}[f+\omega : k_0]}{\mathrm{dn}[f+\omega : k_0]} \tag{9.8}$$

from which there follows

$$\begin{aligned} \sin^2\sigma = 1 - \cos^2\sigma &= 1 - (1 - N_k'^2)\frac{\mathrm{sn}^2[f+\omega : k_0]}{\mathrm{dn}^2[f+\omega : k_0]} \\ &= \frac{\mathrm{cn}^2[f+\omega : k_0] + (N_k'^2 - k_0^2)\,\mathrm{sn}^2[f+\omega : k_0]}{\mathrm{dn}^2[f+\omega : k_0]} \end{aligned} \tag{9.9}$$

which suggests yet another inclination parameter N, defined by

$$N^2 = N_k'^2 - k_0^2 = k_0'^2 - (1 - N_k'^2) = k_0'^2\Big[1 - \frac{1-\nu^2}{(1-\eta^2\ell)m_0}\Big] \tag{9.10a}$$

$$= k_0'^2 \frac{(1-\eta^2\ell)m_0 - (1-\nu^2)}{(1-\eta^2\ell)m_0} = \frac{(1-\eta^2\ell)^2 m_0^2 - (1-\eta^2\ell)m_0(1-\nu^2)}{(1-\eta^2\ell)^2 m_0^2 + \eta^2\ell(1-\nu^2)} \tag{9.10b}$$

so that

$$\begin{aligned} 1 - N^2 &= \frac{(1-\eta^2\ell)m_0 + \eta^2\ell}{(1-\eta^2\ell)^2 m_0^2 + \eta^2\ell(1-\nu^2)}(1-\nu^2) \\ &= \frac{m_0 + \eta^2\ell(1-m_0)}{(1-\eta^2\ell)^2 m_0^2 + \eta^2\ell(1-\nu^2)}(1-\nu^2)\,. \end{aligned} \tag{9.11}$$

At this point, we observe that

$$\frac{k_0^2}{1-N^2} = \frac{\eta^2\ell}{m_0 + \eta^2\ell(1-m_0)} \tag{9.12}$$

wherein we note that the common factor $(1-\nu^2)$ has cancelled out: the above factor (9.12) appears frequently in the subsequent analysis, and here we recognize that the $(1-\nu^2)$-cancellation means that there does not arise any "small divisor" issue near $\nu = 1$ from this factor. In fact, we may set

$$\ell_0 = \frac{\ell}{m_0 + \eta^2\ell(1-m_0)}, \text{ so that } \frac{k_0^2}{1-N^2} = \eta^2\ell_0 = \eta_0^2 \tag{9.13a,b}$$

where the latter (9.13b) is to be taken as the defining relation for η_0^2, a parameter to be used in the sequel. Finally, taking the reciprocal of (9.9) and noting (9.10a), we have

$$\begin{aligned} \frac{1}{\sin^2\sigma} &= \frac{\mathrm{dn}^2[f+\omega : k_0]}{\mathrm{cn}^2[f+\omega : k_0] + N^2\,\mathrm{sn}^2[f+\omega : k_0]} \\ &= \frac{\mathrm{nc}^2[f+\omega : k_0]\,\mathrm{dn}^2[f+\omega : k_0]}{1 + N^2\,\mathrm{sc}^2[f+\omega : k_0]} \end{aligned} \tag{9.14}$$

which is the form to be substituted in the first term of (9.5).

For the second term in (9.5), we introduce the solution form (8.20) for (R/p) and find

$$\frac{\eta^2}{(R/p)^2 - \eta^2} = \tag{9.15}$$

$$\frac{\eta^2\big(\mathrm{dn}[j_v^* f : k_v^*] + \overline{e}\,\mathrm{cn}[j_v^* f : k_v^*]\big)^2}{j_R^2\big(\mathrm{dn}[j_v^* f : k_v^*] + \overline{\delta}\,\mathrm{cn}[j_v^* f : k_v^*]\big)^2 - \eta^2\big(\mathrm{dn}[j_v^* f : k_v^*] + \overline{e}\,\mathrm{cn}[j_v^* f : k_v^*]\big)^2}\,.$$

At this point, we should recall that η^2 is the parameter in terms of which the moduli k_0 and k_v are scaled. All the Jacobian elliptic functions may ultimately be expressed as Fourier series with coefficients and scale-factors expressed as (rapidly convergent) power series in the relevant modulus — ultimately as power series in η^2. With this in view, we set

$$\varphi_2 = \tag{9.16}$$
$$\int_{-\omega}^{f} \frac{\left(\mathrm{dn}[j_v^* f : k_v^*] + \overline{e}\,\mathrm{cn}[j_v^* f : k_v^*]\right)^2 \mathrm{d}(j_v^* f)}{\left(\mathrm{dn}[j_v^* f : k_v^*] + \overline{\delta}\,\mathrm{cn}[j_v^* f : k_v^*]\right)^2 - \frac{\eta^2}{j_R^2}\left(\mathrm{dn}[j_v^* f : k_v^*] + \overline{e}\,\mathrm{cn}[j_v^* f : k_v^*]\right)^2}$$

so that

$$\frac{\eta^2}{(R/p)^2 - \eta^2} = \eta^2 \frac{1}{j_R^2 j_v^*} \varphi_2' . \tag{9.17}$$

The contribution from the second term in (9.5) as exhibited in (9.15)–(9.17) has the multiplying factor $\eta^2(1/j_R^2 j_v^*)$ and so is a correction term in the sense that it vanishes in the limit $\eta^2 \to 0$. By means of the Fourier series representation of the Jacobian elliptic functions, the approximation can be executed to any specified degree of accuracy in η^2. Thus, although φ_2 as given by (9.16) is an elliptic integral of the third kind — which generally poses a computational challenge — in this case, an approximation scheme to any desired degree of accuracy in η^2 is a straightforward matter. For the first term in (9.5), we shall see that the procedure is not as straightforward — though still manageable.

Now by introducing (9.14) and (9.17), with φ_2 given by (9.16), into (9.5) we have

$$\varphi' = N_k \left[\frac{\mathrm{nc}^2[f+\omega : k_0]\,\mathrm{dn}^2[f+\omega : k_0]}{1 + N^2\,\mathrm{sc}^2[f+\omega : k_0]} + \eta^2 \frac{1}{j_R^2 j_v^*} \varphi_2' \right] \tag{9.18a}$$

$$= j_{N_k} \left[\frac{N\,\mathrm{nc}^2[f+\omega : k_0]\,\mathrm{dn}^2[f+\omega : k_0]}{1 + N^2\,\mathrm{sc}^2[f+\omega : k_0]} + \eta^2 N \frac{1}{j_R^2 j_v^*} \varphi_2' \right] \tag{9.18b}$$

$$= j_{N_k} \left[\varphi_1' + \eta^2 N \frac{1}{j_R j_v^*} \varphi_2' \right] \tag{9.18c}$$

where for φ_1 we have

$$\varphi_1' = \frac{N\,\mathrm{nc}^2[f+\omega : k_0]\,\mathrm{dn}^2[f+\omega : k_0]}{1 + N^2\,\mathrm{sc}^2[f+\omega : k_0]} \tag{9.19}$$

and the new parameter j_{N_k} is defined by

$$j_{N_k}^2 = \frac{N_k^2}{N^2} = \frac{k_0'^2 \nu^2}{(1-\eta^2\ell)m_0} \cdot \frac{(1-\eta^2\ell)m_0}{k_0'^2[(1-\eta^2\ell)m_0 - (1-\nu^2)]}$$
$$= \frac{\nu^2}{(1-\eta^2\ell)m_0 - (1-\nu^2)} \tag{9.20}$$

where we have utilized (9.4) for N_k^2 and (9.10) for N^2; from equation (7.2) we note that

$$\text{(i)} \quad \nu \neq 0, \eta \to 0, m_0 \to 1 \qquad \text{implies } j_{N_k}^2 \to \frac{\nu^2}{\nu^2} = 1, \tag{9.21a}$$

$$\text{(ii)} \quad \eta \neq 0, \nu \to 0 \qquad \text{implies } j_{N_k}^2 \to 0, \tag{9.21b}$$

the former corresponding to the Kepler case (Chapter 2) and the latter corresponding to the planar case of two fixed centers (Chapter 3) wherein the φ-coordinate does not appear.

— ◇ —

9A. The Integration of φ_1'

For the integration of φ_1', we first note that all elliptic functions appearing on the right of (9.19) have argument $f + \omega$ and modulus k_0; accordingly it is not necessary that these entities be exhibited explicitly in this subsection. If we set

$$\varphi_{10}' = \frac{N \operatorname{nc}^2[f + \omega : k_0]}{1 + N^2 \operatorname{sc}^2[f + \omega : k_0]} \tag{9A.1}$$

then relation (9.19) implies

$$\varphi_1' = \frac{N \operatorname{nc}^2 \operatorname{dn}^2}{1 + N^2 \operatorname{sc}^2} = \frac{N \operatorname{nc}^2}{1 + N^2 \operatorname{sc}^2}(1 - k_0^2 \operatorname{sn}^2) = (1 - k_0^2 \operatorname{sn}^2)\varphi_{10}' . \tag{9A.2}$$

We further observe that

$$\operatorname{sn}^2 \varphi_{10}' = \frac{N \operatorname{sc}^2}{1 + N^2 \operatorname{sc}^2} = \frac{N}{1 - N^2}\left(\frac{\operatorname{nc}^2}{1 + N^2 \operatorname{sc}^2} - 1\right) = \frac{1}{1 - N^2}(\varphi_{10}' - N) \tag{9A.3}$$

whereby the product $\operatorname{sn}^2 \varphi_{10}'$ is expressed as a linear function of φ_{10}' — a relation to be used both here and repeatedly in the subsequent analysis. Using (9A.3) and recalling (9.13b), we have from (9A.2) that

$$\varphi_1' = \varphi_{10}' - \frac{k_0^2}{1 - N^2}(\varphi_{10}' - N) = \varphi_{10}' - \eta_0^2(\varphi_{10}' - N) \tag{9A.4a}$$

$$= (1 - \eta_0^2)\varphi_{10}' + \eta_0^2 N . \tag{9A.4b}$$

For the integration of φ_{10}', we set

$$\varphi_0' = \frac{N \operatorname{nc}^2 \operatorname{dn}}{1 + N^2 \operatorname{sc}^2} \tag{9A.5}$$

which we recognize as an integrable entity and to which we return presently. From (9A.1) we see that

$$\varphi_{10}' = \varphi_0' + (1 - \operatorname{dn})\varphi_{10}' = \varphi_0' + \frac{1 - \operatorname{dn}^2}{1 + \operatorname{dn}}\varphi_{10}' = \varphi_0' + \frac{k_0^2 \operatorname{sn}^2}{1 + \operatorname{dn}}\varphi_{10}' \tag{9A.6a}$$

$$= \varphi_0' + \frac{k_0^2}{1 - N^2}\,\frac{(\varphi_{10}' - N)}{1 + \operatorname{dn}} = \varphi_0' + \eta_0^2 \frac{(\varphi_{10}' - N)}{1 + \operatorname{dn}} \tag{9A.6b}$$

where in the latter form (9A.6b) we have applied (9A.3). When we note that

$$\frac{1}{1+\mathrm{dn}} = \tfrac{1}{2}\left[1 + \frac{1-\mathrm{dn}}{1+\mathrm{dn}}\right] = \tfrac{1}{2}\left[1 + \frac{1-\mathrm{dn}^2}{(1+\mathrm{dn})^2}\right] = \tfrac{1}{2}\left[1 + \frac{k_0^2\,\mathrm{sn}^2}{(1+\mathrm{dn})^2}\right] \tag{9A.7}$$

we see that we can push the reciprocal factor $1/(1+\mathrm{dn})$ into the next-higher-order term in k_0^2 at the expense of increasing its power-index. The introduction of (9A.7) into (9A.6b) gives

$$\begin{aligned}\varphi_{10}' &= \varphi_0' + \tfrac{1}{2}\eta_0^2(\varphi_{10}' - N)\left[1 + \frac{k_0^2\,\mathrm{sn}^2}{(1+\mathrm{dn})^2}\right]\\ &= \varphi_0' + \tfrac{1}{2}\eta_0^2(\varphi_{10}' - N) + \tfrac{1}{2}\eta_0^2\frac{k_0^2\,\mathrm{sn}^2\,\varphi_{10}'}{(1+\mathrm{dn})^2} - \tfrac{1}{2}\eta_0^2\frac{k_0^2 N\,\mathrm{sn}^2}{(1+\mathrm{dn})^2}\end{aligned} \tag{9A.8}$$

to which we apply (9A.3) and obtain

$$\begin{aligned}\varphi_{10}' &= \varphi_0' + \tfrac{1}{2}\eta_0^2(\varphi_{10}' - N) + \tfrac{1}{2}\eta_0^4\frac{(\varphi_{10}' - N)}{(1+\mathrm{dn})^2} - \tfrac{1}{2}\eta_0^2 k_0^2\frac{N\,\mathrm{sn}^2}{(1+\mathrm{dn})^2}\\ &= \varphi_0' + (\varphi_{10}' - N)\left[\tfrac{1}{2}\eta_0^2 + \tfrac{1}{2}\frac{\eta_0^4}{(1+\mathrm{dn})^2}\right] - \tfrac{1}{2}\eta_0^2 k_0^2\frac{N\,\mathrm{sn}^2}{(1+\mathrm{dn})^2}\,.\end{aligned} \tag{9A.9}$$

Here it is appropriate to pause and note the emerging pattern: in (9A.6), we have the expression for φ_{10}' as a first-order expression in η_0^2 or k_0^2, with the η_0^2-term having denominator $(1+\mathrm{dn})$; in (9A.9), we have the expression for φ_{10}' as a second-order expression in η_0^2 with the η_0^4-terms having denominator $(1+\mathrm{dn})^2$. If we iterate once more, we should first note that

$$\frac{1}{(1+\mathrm{dn})^2} = \tfrac{1}{4}\left[1 + 2k_0^2\frac{\mathrm{sn}^2}{(1+\mathrm{dn})^2} + k_0^4\frac{\mathrm{sn}^4}{(1+\mathrm{dn})^4}\right], \tag{9A.10}$$

the introduction of which renders for (9A.9)

$$\begin{aligned}\varphi_{10}' &= \varphi_0' + \tfrac{1}{2}\eta_0^2(\varphi_{10}' - N) + \tfrac{1}{8}\eta_0^4(\varphi_{10}' - N)\left[1 + \frac{2k_0^2\,\mathrm{sn}^2}{(1+\mathrm{dn})^2} + \frac{k_0^4\,\mathrm{sn}^4}{(1+\mathrm{dn})^4}\right]\\ &\quad - \tfrac{1}{8}\eta_0^2 k_0^2 N\,\mathrm{sn}^2\left[1 + \frac{2k_0^2\,\mathrm{sn}^2}{(1+\mathrm{dn})^2} + \frac{k_0^4\,\mathrm{sn}^4}{(1+\mathrm{dn})^4}\right]\\ &= \varphi_0' + \tfrac{1}{2}\eta_0^2(\varphi_{10}' - N) + \tfrac{1}{8}\eta_0^4(\varphi_{10}' - N) + \tfrac{1}{4}\eta_0^4 k_0^2\frac{\mathrm{sn}^2\,\varphi_{10}'}{(1+\mathrm{dn})^2}\\ &\quad + \tfrac{1}{8}\eta_0^4 k_0^4\frac{\mathrm{sn}^4\,\varphi_{10}'}{(1+\mathrm{dn})^4} - \tfrac{1}{8}\eta_0^4 N\left[\frac{2k_0^2\,\mathrm{sn}^2}{(1+\mathrm{dn})^2} + k_0^4\frac{\mathrm{sn}^4}{(1+\mathrm{dn})^4}\right]\\ &\quad - \tfrac{1}{8}\eta_0^2 k_0^2 N\,\mathrm{sn}^2\left[1 + \frac{2k_0^2\,\mathrm{sn}^2}{(1+\mathrm{dn})^2} + \frac{k_0^4\,\mathrm{sn}^4}{(1+\mathrm{dn})^4}\right].\end{aligned} \tag{9A.11}$$

Again applying (9A.3) repeatedly to the terms with $\mathrm{sn}^2\,\varphi_{10}'$ we have, after systematic regrouping,

$$\begin{aligned}
\varphi'_{10} &= \varphi'_0 + \tfrac{1}{2}\eta_0^2(\varphi'_{10} - N) + \tfrac{1}{8}\eta_0^4(\varphi'_{10} - N) \\
&\quad + \tfrac{1}{4}\eta_0^6 \frac{(\varphi'_{10} - N)}{(1 + \mathrm{dn})^2} + \tfrac{1}{8}\eta_0^6 k_0^2 \frac{\mathrm{sn}^2(\varphi'_{10} - N)}{(1 + \mathrm{dn})^4} \\
&\quad - \tfrac{1}{8}\eta_0^2 k_0^2 N\,\mathrm{sn}^2\left[1 + \frac{2\eta_0^2}{(1 + \mathrm{dn})^2} + \frac{2k_0^2\,\mathrm{sn}^2}{(1 + \mathrm{dn})^2} + \frac{\eta_0^2 k_0^2\,\mathrm{sn}^2}{(1 + \mathrm{dn})^4} + \frac{k_0^4\,\mathrm{sn}^4}{(1 + \mathrm{dn})^4}\right] \\
&= \varphi'_0 + \tfrac{1}{2}\eta_0^2(\varphi'_{10} - N) + \tfrac{1}{8}\eta_0^4(\varphi'_{10} - N) + \tfrac{1}{4}\eta_0^6\frac{(\varphi'_{10} - N)}{(1 + \mathrm{dn})^2} + \tfrac{1}{8}\eta_0^8\frac{(\varphi'_{10} - N)}{(1 + \mathrm{dn})^4} \\
&\quad - \tfrac{1}{8}\eta_0^2 k_0^2 N\,\mathrm{sn}^2\left[\left(1 + \frac{2\eta_0^2}{(1 + \mathrm{dn})^2} + \frac{\eta_0^4}{(1 + \mathrm{dn})^4}\right)\right. \\
&\qquad\qquad \left. + \frac{2k_0^2\,\mathrm{sn}^2}{(1 + \mathrm{dn})^2}\left[1 + \tfrac{1}{2}\frac{\eta_0^2}{(1 + \mathrm{dn})^2}\right] + \frac{k_0^4\,\mathrm{sn}^4}{(1 + \mathrm{dn})^4}\right]. \qquad (9A.12)
\end{aligned}$$

The procedure can be continued indefinitely to any specified degree of accuracy in terms of powers of the parameters η_0^2 or k_0^2, and it becomes evident that the coefficients decrease more rapidly than the powers of 1/2. At this point, we perform a slight regrouping of (9A.12) as follows:

$$\begin{aligned}
\varphi'_{10} &= \varphi'_0 + (\varphi'_{10} - N)\left[\tfrac{1}{2}\eta_0^2 + \tfrac{1}{8}\eta_0^4 + \tfrac{1}{4}\frac{\eta_0^6}{(1 + \mathrm{dn})^2} + \tfrac{1}{8}\frac{\eta_0^8}{(1 + \mathrm{dn})^4}\right] \\
&\quad - \tfrac{1}{8}\eta_0^2 k_0^2 N\,\mathrm{sn}^2\left[\left(1 + \frac{2\eta_0^2}{(1 + \mathrm{dn})^2} + \frac{\eta_0^4}{(1 + \mathrm{dn})^4}\right)\right. \\
&\qquad\qquad \left. + \frac{2k_0^2\,\mathrm{sn}^2}{(1 + \mathrm{dn})^2}\left[1 + \tfrac{1}{2}\frac{\eta_0^2}{(1 + \mathrm{dn})^2}\right] + \frac{k_0^4\,\mathrm{sn}^4}{(1 + \mathrm{dn})^4}\right], \qquad (9A.13)
\end{aligned}$$

which will be the basis of the subsequent treatment.

In the context of the approximation scheme, we now illustrate the procedure in the case where accuracy to order η^6 (or η_0^6) is appropriate, so that we neglect terms of order η_0^8 and higher. Then in the first bracket in (9A.13), we omit the last term (with η_0^8), and in the term with η_0^6 it is consistent with this level of accuracy to replace $1/(1 + \mathrm{dn})^2$ by 1/4. In the second bracket, in view of the multiplying factor of order η^4, it is consistent with this level of accuracy to ignore terms of order η^4, namely, those terms with η_0^4, k_0^4, and $\eta_0^2 k_0^2$, while in terms with η_0^2 and k_0^2, we may again replace $1/(1+\mathrm{dn})^2$ by 1/4. Accordingly, for an approximation to order η^6, we may replace (9A.13) by

$$\varphi'_{10} = \varphi'_0 + (\varphi'_{10} - N)\left[\tfrac{1}{2}\eta_0^2 + \tfrac{1}{8}\eta_0^4 + \tfrac{1}{16}\eta_0^6\right] - \tfrac{1}{8}\eta_0^2 k_0^2 N\,\mathrm{sn}^2\left[(1 + \tfrac{1}{2}\eta_0^2) + \tfrac{1}{2}k_0^2\,\mathrm{sn}^2\right]. \qquad (9A.14)$$

If we set

$$j_{10} = 1 + \tfrac{1}{4}\eta_0^2 + \tfrac{1}{8}\eta_0^4 \qquad (9A.15a)$$

so that

$$\tfrac{1}{2}\eta_0^2 + \tfrac{1}{8}\eta_0^4 + \tfrac{1}{16}\eta_0^6 = \tfrac{1}{2}\eta_0^2 j_{10} \qquad (9A.15b)$$

then (9A.14) may be written

$$\varphi'_{10} = \varphi'_0 + \tfrac{1}{2}\eta_0^2 j_{10}(\varphi'_{10} - N) - \tfrac{1}{8}\eta_0^2 k_0^2 N \,\mathrm{sn}^2\left[(1 + \tfrac{1}{2}\eta_0^2) + \tfrac{1}{2}k_0^2\,\mathrm{sn}^2\right]. \quad (9A.16)$$

If we regroup to bring all terms with φ'_{10} to the left-hand side, we have

$$\begin{aligned}(1 - \tfrac{1}{2}\eta_0^2 j_{10})\varphi'_{10} = \varphi'_0 - \tfrac{1}{2}\eta_0^2 j_{10} N& \\ - \tfrac{1}{8}\eta_0^2 k_0^2 N\,\mathrm{sn}^2\left[(1 + \tfrac{1}{2}\eta_0^2) + \tfrac{1}{2}k_0^2\,\mathrm{sn}^2\right].& \qquad (9A.17)\end{aligned}$$

Recalling (9A.4), we utilize it to express φ'_{10} in terms of φ'_1, from which it follows that

$$\frac{1 - \frac{1}{2}\eta_0^2 j_{10}}{1 - \eta_0^2}\varphi'_1 = (1 - \tfrac{1}{2}\eta_0^2 j_{10})\varphi'_{10} + \eta_0^2\frac{(1 - \frac{1}{2}\eta_0^2 j_{10})}{1 - \eta_0^2}N, \qquad (9A.18a)$$

which on the introduction of (9A.17) is rendered

$$\begin{aligned}&= \varphi'_0 - \eta_0^2 N\left(\tfrac{1}{2}j_{10} - \frac{1 - \frac{1}{2}\eta_0^2 j_{10}}{1 - \eta_0^2}\right) - \tfrac{1}{8}\eta_0^2 k_0^2 N\,\mathrm{sn}^2\left[(1 + \tfrac{1}{2}\eta_0^2) + \tfrac{1}{2}k_0^2\,\mathrm{sn}^2\right] \\ &= \varphi'_0 + \frac{1 - \frac{1}{2}j_{10}}{1 - \eta_0^2}\eta_0^2 N - \tfrac{1}{8}\eta_0^2 k_0^2 N\,\mathrm{sn}^2\left[(1 + \tfrac{1}{2}\eta_0^2) + \tfrac{1}{2}k_0^2\,\mathrm{sn}^2\right]. \qquad (9A.18b)\end{aligned}$$

Hence dividing (9A.18) across by the multiplier of φ'_1, we have

$$\begin{aligned}\varphi'_1 = \frac{1 - \eta_0^2}{1 - \frac{1}{2}\eta_0^2 j_{10}}\varphi'_0 + \frac{1 - \frac{1}{2}j_{10}}{1 - \frac{1}{2}\eta_0^2 j_{10}}\eta_0^2 N - \frac{1}{8}\frac{(1 + \frac{1}{2}\eta_0^2)(1 - \eta_0^2)}{1 - \frac{1}{2}\eta_0^2 j_{10}}\eta_0^2 k_0^2 N\,\mathrm{sn}^2& \\ - \frac{1}{16}\frac{1 - \eta_0^2}{1 - \frac{1}{2}\eta_0^2 j_{10}}\eta_0^2 k_0^4 N\,\mathrm{sn}^4.& \quad (9A.19)\end{aligned}$$

It is therefore convenient to set

$$j_{10}^* = \frac{1 - \eta_0^2}{1 - \frac{1}{2}\eta_0^2 j_{10}}, \quad j_{N_1} = \frac{1 - \frac{1}{2}j_{10}}{1 - \frac{1}{2}\eta_0^2 j_{10}}, \quad j_{S_2}^* = \frac{(1 + \frac{1}{2}\eta_0^2)(1 - \eta_0^2)}{1 - \frac{1}{2}\eta_0^2 j_{10}}, \qquad (9A.20a,b,c)$$

so that (9A.19) may be written

$$\varphi'_1 = j_{10}^*\varphi'_0 + j_{N_1}\eta_0^2 N - \tfrac{1}{8}j_{S_2}^*\eta_0^2 k_0^2 N\,\mathrm{sn}^2 - \tfrac{1}{16}j_{10}^*\eta_0^2 k_0^4 N\,\mathrm{sn}^4, \qquad (9A.21)$$

which is the form to be introduced into relation (9.18c) for φ'.

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Returning now to the main section, we divide across (9.18c) by the factor j_{N_k} to obtain (bearing in mind the defining relation (9.16) for φ_2)

$$\frac{1}{j_{N_k}}\varphi' = \varphi'_1 + \frac{1}{j_R^2 j_v^*}\eta^2 N\varphi'_2 \qquad (9.22)$$

into which we introduce relation (9A.21) for φ_1' and find, recalling (9.13b),

$$\frac{1}{j_{N_k}}\varphi' = j_{10}^*\varphi_0' + j_{N_1}\eta_0^2 N\left(1 + \frac{1}{j_{N_1} j_R^2 j_v^* \ell_0}\varphi_2'\right) \\ - \tfrac{1}{8} j_{S_2}^* \eta_0^2 k_0^2 N \,\mathrm{sn}^2 - \tfrac{1}{16} j_{10}^* \eta_0^2 k_0^4 N \,\mathrm{sn}^4 \,. \tag{9.23}$$

After dividing across by j_{10}^*, then a transposition of terms and a rearrangement yields

$$\varphi_0' = \frac{1}{j_{N_k} j_{10}^*}\varphi' - \eta_0^2 N \frac{j_{N_1}}{j_{10}^*}\left(1 + \frac{1}{j_{N_1} j_R^2 j_v^* \ell_0}\varphi_2'\right) \\ + \tfrac{1}{8}\eta_0^2 k_0^2 N \frac{j_{S_2}^*}{j_{10}^*}\,\mathrm{sn}^2 + \tfrac{1}{16}\eta_0^2 k_0^4 N \,\mathrm{sn}^4 \tag{9.24a}$$

$$= \frac{1}{j_{N_k} j_{10}^*}\left\{\varphi' - \eta_0^2 N j_{N_1} j_{N_k}\left(1 + \frac{1}{j_{N_1} j_R^2 j_v^* \ell_0}\varphi_2'\right)\right. \\ \left. + \tfrac{1}{8}\eta_0^2 k_0^2 N j_{N_k} j_{S_2}^* \,\mathrm{sn}^2 + \tfrac{1}{16}\eta_0^2 k_0^4 N j_{N_k} j_{10}^* \,\mathrm{sn}^4 \right\}. \tag{9.24b}$$

It is now convenient to set

$$j_\varphi = \frac{1}{j_{N_k} j_{10}^*}, \quad j_N = j_{N_1} j_{N_k}, \quad j_{\varphi_2} = \frac{1}{j_{N_1} j_R^2 j_v^* \ell_0} \tag{9.25a,b,c}$$

$$j_{S_2} = j_{N_k} j_{S_2}^*, \quad j_{S_4} = j_{N_k} j_{10}^* \ (= \frac{1}{j_\varphi}) \tag{9.25c,d}$$

so that (9.24b) reads

$$\varphi_0' = j_\varphi\left\{\varphi' - \eta_0^2 N j_N (1 + j_{\varphi_2}\varphi_2') + \tfrac{1}{8}\eta_0^2 k_0^2 N j_{S_2} \,\mathrm{sn}^2 + \tfrac{1}{16}\eta_0^2 k_0^4 N j_{S_4} \,\mathrm{sn}^4 \right\}. \tag{9.26}$$

If we write

$$\Omega_2 = \int_{-\omega}^{f} \mathrm{sn}^2[f + \omega : k_0]\,\mathrm{d}f, \quad \Omega_4 = \int_{-\omega}^{f} \mathrm{sn}^4[f + \omega : k_0]\,\mathrm{d}f \tag{9.27}$$

then relation (9.26) may be written

$$\varphi_0' = j_\varphi \frac{\mathrm{d}}{\mathrm{d}f}\left\{(\varphi - \Omega_0) - \eta_0^2 N j_N[(f + \omega) + j_{\varphi_2}\varphi_2] \right. \\ \left. + \tfrac{1}{8}\eta_0^2 k_0^2 N j_{S_2}\Omega_2 + \tfrac{1}{16}\eta_0^2 k_0^4 N j_{S_4}\Omega_4\right\} \tag{9.28}$$

where Ω_0 is the (anticipated) constant of integration and represents the angle of the first nodal crossing, i.e., at $f = -\omega$. We now define the function Ω by setting

$$\Omega = \Omega_0 + \eta_0^2 N j_N[(f + \omega) + j_{\varphi_2}\varphi_2] \\ - \tfrac{1}{8}\eta_0^2 k_0^2 N j_{S_2}\Omega_2 - \tfrac{1}{16}\eta_0^2 k_0^2 N j_{S_4}\Omega_4 \tag{9.29}$$

and relation (9.28) may be written

$$\varphi_0' = j_\varphi \frac{\mathrm{d}}{\mathrm{d}f}(\varphi - \Omega)\,. \tag{9.30}$$

Referring to the defining relation (9A.5) for φ_0, it is a straightforward exercise to see that the integration of (9A.5) yields

$$\varphi_0 = \arctan\left[N\,\mathrm{sc}[f + \omega : k_0]\right] \tag{9.31}$$

so that the integration of (9.30) yields

$$\arctan\left[N\,\mathrm{sc}[f + \omega : k_0]\right] = j_\varphi(\varphi - \Omega) \tag{9.32}$$

or alternatively,

$$\tan j_\varphi(\varphi - \Omega) = N\,\mathrm{sc}[f + \omega : k_0],$$

which is the result in the form we seek, where it is immediately evident that it is the sought-for generalization of the corresponding result (4.48) of Chapter 2 for the Kepler problem.

10 The Time-Angle Relation

The time-angle relation follows from the integration of relation (5.5) in the form

$$\Lambda \frac{\mathrm{d}t}{\mathrm{d}f} = R^2 - b^2\cos^2\sigma \tag{10.1}$$

wherein the solution forms for R and σ in terms of f are to be introduced. However, we have already seen that these solution forms are formally identical to those derived in Chapter 3 for the planar case. Accordingly, with the appropriate (algebraic) modification in the relevant constants, the integration of (10.1) will follow that already outlined in Section 10 of Chapter 3 for the planar case and so will not be repeated here.

5

The Earth Satellite — General Analysis

If you do it right, then it should be easy;
if you do it wrong, then it can be hard.

— *Mathematics teacher*

1 The Geopotential and the Density Distribution

The standard representation of the gravitational potential function of a planet is given, in terms of a planet-centered spherical coordinate system, in the form

$$U_G = \frac{\mu}{r}\Bigg[1 - \sum_{n=1}^{\infty} J_n \left(\frac{r_0}{r}\right)^n P_n(\cos\theta) \\ - \sum_{n=1}^{\infty}\sum_{m=1}^{n} J_{nm}\left(\frac{r_0}{r}\right)^n P_n^m(\cos\theta)\cos(\varphi - \alpha_{mn})\Bigg], \quad (1.1)$$

where r_0 denotes the mean radius of the planet, the quantities J_n, J_{nm} are the planetary potential coefficients, and the quantities α_{mn} are further constants to be determined. The term with unity in the above expansion is clearly the Kepler term, and the higher coefficients may be inferred from satellite tracking. Not surprisingly, the potential coefficients for the earth are more thoroughly "mapped" than are those for the other planets. If the origin is set at the center of mass, then the term with J_1 vanishes.

In the case of the earth, which in this context is referred to as the *geoid*, the coefficients in the expansion (1.1) are called the geopotential coefficients and those of low order have been determined to an accuracy that carries a fair level of confidence. It was early established that for the geoid, the term with J_2 is clearly the most significant, dominating in its effects all other coefficients by at least an order of magnitude. If in this case we restrict our attention to that part of the geopotential (1.1) that is both rotationally symmetric (independent of φ) and also symmetric with respect to the equatorial plane (accounted by the J_n terms with even index $n = 2k$), then that expression includes both the Kepler term and the residually dominant J_2-term. Denoting this symmetric part of the geopotential by U_{GS}, then the residual $U_G - U_{GS}$ is a perturbation requiring separate treatment.

Explicitly for the symmetric geoid, we have

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$$U_{GS} = \frac{\mu}{r}\left[1 - \sum_{k=1}^{\infty} J_{2k}\left(\frac{r_0}{r}\right)^{2k} P_{2k}(\cos\theta)\right] \tag{1.2}$$

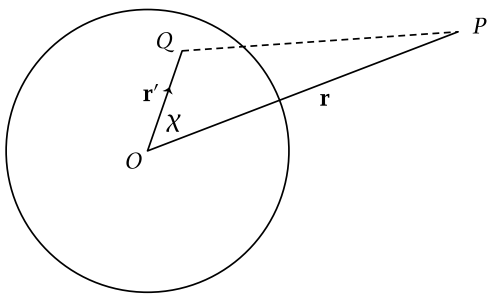
and for the lower-order coefficients, it is known that

$$J_2 = O(10^{-3}), \quad J_4 = -(1.4)J_2^2, \quad J_{2k} = O(10^{-6}), \quad k \geq 3\,. \tag{1.3a,b,c}$$

The form of the representation raises the question of a correspondence between the respective zonal harmonics and the density distribution within the geoid whereby the potential is induced. It is therefore worth exploring what is the mass distribution within a hypothetical geoid that gives rise to a geopotential function of the form (1.2).

One could pose the more general question as to what mass-distribution would induce a geopotential function of the form U_G of (1.1). However, while the restriction is not necessary to the procedure, it is more simply illustrated for the restricted potential U_{GS}.

Referred to a spherical coordinate system with origin at the center O of the sphere, we let $P(r, \theta, \varphi)$ denote an arbitrary exterior point and $Q(r', \theta', \varphi')$ an arbitrary interior point at which is posited the mass-element $\mathrm{d}m$. Denoting the position vectors of P and Q by $\mathbf{r}$ and $\mathbf{r}'$, respectively, we let χ denote the angle between $\mathbf{r}$ and $\mathbf{r}'$. If we let τ denote the mass-density at an arbitrary



interior point, then for the mass-element $\mathrm{d}m$, we have

$$\mathrm{d}m = \tau r'^2 \sin\theta' \,\mathrm{d}r' \,\mathrm{d}\theta' \,\mathrm{d}\varphi'\,. \tag{1.4}$$

Moreover, with a gravitational constant γ, the potential at P, due to the mass-element $\mathrm{d}m$ at Q, is given by

$$\mathrm{d}U_Q = \gamma \frac{\mathrm{d}m}{|r - r'|} = \gamma \frac{\mathrm{d}m}{[r^2 + r'^2 - 2rr'\cos\chi]^{1/2}} \tag{1.5a,b}$$

where, in deriving (1.5b), we have applied the cosine law to the triangle OPQ. Taking the factor r^2 outside the bracket, we have

$$\mathrm{d}U_Q = \gamma \frac{\mathrm{d}m}{r} \frac{1}{\left[1 + (\frac{r'}{r})^2 - 2(\frac{r'}{r})\cos\chi\right]^{1/2}} \tag{1.6}$$

and in the reciprocal of the square bracket, we recognize the generating function of the Legendre polynomials; hence

$$\mathrm{d}U_Q = \gamma \frac{\mathrm{d}m}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\chi) = \gamma \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} P_n(\cos\chi)\,\mathrm{d}m\,. \tag{1.7}$$

If we take the dot product of r with r', both directly and in component form, then from the identification of the two forms there follows

$$\cos\chi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi'), \tag{1.8}$$

which enables us to apply the addition theorem for Legendre polynomials, namely

$$\begin{aligned} P_n(\cos\chi) &= P_n(\cos\theta)P_n(\cos\theta') \\ &\quad + 2\sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta)P_n^m(\cos\theta')\cos m(\varphi - \varphi'). \end{aligned} \tag{1.9}$$

Introducing (1.9) into (1.7) and substituting for $\mathrm{d}m$ from (1.4), we find

$$\begin{aligned} \mathrm{d}U_Q = \gamma \Bigg\{ &\sum_{n=0}^{\infty} \frac{1}{r^{n+1}} [r'^n P_n(\cos\theta)P_n(\cos\theta')\tau r'^2 \sin\theta'\,\mathrm{d}r'\,\mathrm{d}\theta'\,\mathrm{d}\varphi'] \\ &+ 2\sum_{n=1}^{\infty} \frac{1}{r^{n+1}} \Bigg[r'^n \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta)P_n^m(\cos\theta') \\ &\qquad \times \cos m(\varphi - \varphi')\tau r'^2 \sin\theta'\,\mathrm{d}r'\,\mathrm{d}\theta'\,\mathrm{d}\varphi' \Bigg] \Bigg\}. \end{aligned} \tag{1.10}$$

To determine the potential at P due to the inhomogeneous sphere, it is necessary to perform the integration

$$U = \int \mathrm{d}U_Q \tag{1.11}$$

where the integration is taken through the solid sphere, namely r' ranges from 0 to r_0, θ' from 0 to π, and φ' from 0 to 2π. If τ is taken to be rotationally symmetric (i.e., independent of φ'), then the φ'-dimension in the first summation integrates to 2π while in the second summation the φ'-dimension integrates to 0. On completing the φ'-integration, we therefore have

$$U = 2\pi\gamma \sum_{n=0}^{\infty} \frac{P_n(\cos\theta)}{r^{n+1}} \int_0^{r_0}\int_0^{\pi} r'^{n+2} P_n(\cos\theta')\sin\theta'\tau\,\mathrm{d}r'\,\mathrm{d}\theta' \tag{1.12}$$

where we have interchanged the order of the integration and summation operations.

If we now further take it that τ does not depend on the radial coordinate r', then the r'-integration in (1.12) is immediate and we have

$$U = 2\pi\gamma r_0^2 \sum_{n=0}^{\infty} \left(\frac{r_0}{r}\right)^{n+1} \cdot \frac{P_n(\cos\theta)}{(n+3)} \cdot \int_0^{\pi} \tau P_n(\cos\theta') \sin\theta' \, \mathrm{d}\theta' \,. \tag{1.13}$$

While this assumption is consistent with the dominant terms in the geopotential, it is not presumed that this assumption is the only possibility or that it actually reflects physical reality.

The dependence of τ on θ' can be represented in the form

$$\tau = \tau_0\Big[1 + \sum_{\ell=1}^{\infty} \tau_\ell P_\ell(\cos\theta')\Big] \tag{1.14}$$

wherein τ_0 reflects the (constant) mean density and the τ_ℓ, $\ell \geq 1$, are the dimensionless measures of the higher moments. From the orthogonality of Legendre polynomials, there follows that

$$\begin{aligned}\int_0^{\pi} \tau P_n(\cos\theta') \sin\theta' \, \mathrm{d}\theta' &= \tau_0 \int_0^{\pi} P_n(\cos\theta')\Big[1 + \sum_{\ell=1}^{\infty} \tau_\ell P_\ell(\cos\theta')\Big] \sin\theta' \, \mathrm{d}\theta' \\ &\left.\begin{aligned} &= 2\tau_0 && \text{for } n = 0 \\ &= 2\tau_0\Big(\frac{\tau_n}{2n+1}\Big) && \text{for } n \geq 1. \end{aligned}\right\}\end{aligned} \tag{1.15}$$

By inserting (1.15) into (1.13) we have

$$\begin{aligned} U &= 2\pi\gamma r_0^2\Big[\frac{r_0}{r}\tfrac{2}{3}\tau_0 + 2\tau_0 \sum_{n=1}^{\infty} \Big(\frac{r_0}{r}\Big)^{n+1} \frac{\tau_n}{(2n+1)(n+3)} P_n(\cos\theta)\Big] \\ &= \tfrac{4}{3}\pi r_0^3 \tau_0 \frac{\gamma}{r}\Big[1 + \sum_{n=1}^{\infty} \Big(\frac{r_0}{r}\Big)^{n} \frac{3\tau_n}{(2n+1)(n+3)} P_n(\cos\theta)\Big]. \end{aligned} \tag{1.16}$$

We may now refer to the associated sphere of radius r_0 and constant density τ_0 as the mean sphere, and if we denote its mass by μ_0, we may introduce a normalized gravitational constant μ; explicitly we set

$$\mu_0 = \tfrac{4}{3}\pi r_0^3 \tau_0, \quad \mu = \gamma\mu_0, \tag{1.17a,b}$$

whereby (1.16) may be written

$$U = \frac{\mu}{r}\Big[1 + \sum_{n=1}^{\infty} \Big(\frac{r_0}{r}\Big)^{n} \frac{3\tau_n}{(2n+1)(n+3)} P_n(\cos\theta)\Big]. \tag{1.18}$$

From (1.18), it is clear that the τ_n can be chosen to fit an arbitrary axisymmetric potential; accordingly, if we impose the further requirement that the potential be symmetric with respect to the equatorial plane, then all coefficients of odd index must vanish, and we have

$$\tau_{2k+1} = 0, \quad k \geq 1 \tag{1.19}$$

and equation (1.18) takes the symmetric form

$$U_S = \frac{\mu}{r}\left[1 + \sum_{n=1}^{\infty} \frac{3\tau_{2n}}{(4n+1)(2n+3)}\left(\frac{r_0}{r}\right)^{2n} P_{2n}(\cos\theta)\right]. \tag{1.20}$$

Comparing (1.20) with (1.2), we see that U_S can be identified with U_{GS} if we take

$$\tau_{2n} = -(4n+1)(2n+3)\frac{J_{2n}}{3} \tag{1.21}$$

and all odd coefficients as zero as given in (1.19).

In (1.14), with the coefficients given by (1.19) and (1.21), we have produced a density distribution for the hypothetical geoid that gives rise to a potential function that matches the symmetric part U_{GS} of the geopotential function. It suggests a procedure that by successive refinement may lead to an approximation for the density distribution in the geoid (Earth) that, except for layers of sharp discontinuity, should give some insight into the actual distribution. The above procedure assumes continuity — in fact, presumes analyticity in the coordinate variables — for the density distribution.

2 The Vinti Potential

By an elegant device involving an excursion into complex parameters and the use of the generating functions for Legendre polynomials, Vinti arrived at a compact form for a potential function that incorporates the dominant elements in the geopotential function and for which the associated dynamical problem is integrable.

The framework for Vinti's approach is the oblate spheroidal coordinate system defined by

$$r\sin\theta\cos\varphi = x = \sqrt{R^2+b^2}\,\sin\sigma\cos\varphi \tag{2.1a}$$

$$r\sin\theta\sin\varphi = y = \sqrt{R^2+b^2}\,\sin\sigma\sin\varphi \tag{2.1b}$$

$$r\cos\theta = z = R\cos\sigma \tag{2.1c}$$

so that

$$r^2 = x^2 + y^2 + z^2 = R^2 + b^2\sin^2\sigma\,. \tag{2.2}$$

There follows that

$$\begin{aligned} r^2 - 2ibr\cos\theta - b^2 &= R^2 - b^2\cos^2\sigma - 2ibR\cos\sigma \\ &= [R - ib\cos\sigma]^2 \end{aligned} \tag{2.3a}$$

$$\begin{aligned} r^2 + 2ibr\cos\theta - b^2 &= R^2 - b^2\cos^2\sigma + 2ibR\cos\sigma \\ &= [R + ib\cos\sigma]^2, \end{aligned} \tag{2.3b}$$

the latter relations to be utilized below.

Considering next the generating function for Legendre polynomials

$$[1 - 2zh + h^2]^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(z) \tag{2.4}$$

we set

$$h^2 = -\kappa^2 \tag{2.5}$$

so that

$$h = i\kappa, \quad h = -i\kappa \tag{2.6a,b}$$

and we now consider separately the two cases. From the case of positive sign

$$\begin{aligned}
[1 - 2i\kappa z - \kappa^2]^{-1/2} &= \sum_{n=0}^{\infty} (i\kappa)^n P_n(z) \\
&= \sum_{k=0}^{\infty} (i\kappa)^{2k} P_{2k}(z) + i\kappa \sum_{k=0}^{\infty} (i\kappa)^{2k} P_{2k+1}(z) \\
&= \sum_{k=0}^{\infty} (-\kappa^2)^k P_{2k}(z) + i\kappa \sum_{k=0}^{\infty} (-\kappa^2)^k P_{2k+1}(z)
\end{aligned} \tag{2.7a}$$

and from the case of negative sign,

$$\begin{aligned}
[1 + 2i\kappa z - \kappa^2]^{-1/2} &= \sum_{n=0}^{\infty} (-i\kappa)^n P_n(z) \\
&= \sum_{k=0}^{\infty} (i\kappa)^{2k} P_{2k}(z) - i\kappa \sum_{k=0}^{\infty} (i\kappa)^{2k} P_{2k+1}(z) \\
&= \sum_{k=0}^{\infty} (-\kappa^2)^k P_{2k}(z) - i\kappa \sum_{k=0}^{\infty} (-\kappa^2)^k P_{2k+1}(z)\,.
\end{aligned} \tag{2.7b}$$

Introducing the arbitrary complex constant A_0, we form the real combination

$$\begin{aligned}
&A_0[1 - 2i\kappa z - \kappa^2]^{-1/2} + \overline{A}_0[1 + 2i\kappa z - \kappa^2]^{-1/2} \\
&\quad = (A_0 + \overline{A}_0) \sum_{k=0}^{\infty} (-\kappa^2)^k P_{2k}(z) + i\kappa(A_0 - \overline{A}_0) \sum_{k=0}^{\infty} (-\kappa^2)^k P_{2k+1}(z) \\
&\quad = 2A\Big[\sum_{k=0}^{\infty} (-\kappa^2)^k P_{2k}(z) - \kappa\beta \sum_{k=0}^{\infty} (-\kappa^2)^k P_{2k+1}(z) \Big]
\end{aligned} \tag{2.8}$$

where we have written

$$2A = A_0 + \overline{A}_0 (\text{real}), \quad 2i\beta A = A_0 - \overline{A}_0 (\text{pure imaginary}). \tag{2.9a,b}$$

If we now set

$$\kappa = \frac{b}{r}, \quad z = \cos\theta \tag{2.10}$$

and reverse the order in equation (2.8), we have

$$\begin{aligned}
2A\Big[\sum_{k=0}^{\infty}\Big(-\frac{b^2}{r^2}\Big)^k P_{2k}(\cos\theta) - \beta\frac{b}{r}\sum_{k=0}^{\infty}\Big(-\frac{b^2}{r^2}\Big)^k P_{2k+1}(\cos\theta)\Big]& \\
= A_0\Big[1 - 2i\frac{b}{r}\cos\theta - \frac{b^2}{r^2}\Big]^{-1/2} + \overline{A}_0\Big[1 + 2i\frac{b}{r}\cos\theta - \frac{b^2}{r^2}\Big]^{-1/2}& \\
= r\Big\{A_0\Big[r^2 - 2ibr\cos\theta - b^2\Big]^{-1/2} + \overline{A}_0\Big[r^2 + 2ibr\cos\theta - b^2\Big]^{-1/2}\Big\}.&
\end{aligned} \tag{2.11}$$

We now introduce relations (2.3) that express the factors within square brackets as perfect squares, and find

$$\begin{aligned}
2A\Big[\sum_{k=0}^{\infty}\Big(-\frac{b^2}{r^2}\Big)^k P_{2k}(\cos\theta) - \beta\frac{b}{r}\sum_{k=0}^{\infty}\Big(-\frac{b^2}{r^2}\Big)^k P_{2k+1}(\cos\theta)\Big]& \\
= r\Big[\frac{A_0}{R - ib\cos\sigma} + \frac{\overline{A}_0}{R + ib\cos\sigma}\Big] = 2Ar\frac{R - \beta b\cos\sigma}{R^2 + b^2\cos^2\sigma}.&
\end{aligned} \tag{2.12}$$

Dividing by the factor $2Ar$ and multiplying by the gravitation factor μ yields

$$\begin{aligned}
\frac{\mu}{r}\Big[\sum_{k=0}^{\infty}\Big(-\frac{b^2}{r^2}\Big)^k P_{2k}(\cos\theta) - \beta\frac{b}{r}\sum_{k=0}^{\infty}\Big(-\frac{b^2}{r^2}\Big)^k P_{2k+1}(\cos\theta)\Big]& \\
= \mu\frac{R - \beta b\cos\sigma}{R^2 + b^2\cos^2\sigma}.&
\end{aligned} \tag{2.13}$$

The entity in (2.13) is the Vinti potential function in its general form, containing two free constants b and β. Recognizing that the associated dynamical problem is integrable, Vinti observed that the significance of the potential lay in the fact that the parameter b can be chosen so that the potential matches the geopotential both in the zeroth (Kepler) and in the second (residually dominant) zonal harmonic with coefficient J_2 and that β could be chosen to fit the first zonal harmonic with coefficient J_1 in the case where the origin does not coincide with the center of mass. Furthermore, he showed that when the origin coincides with the center of mass, an additional adjustment allows β to be chosen so that the potential matches the geopotential up to the third zonal harmonic.

It was soon recognized that the Vinti potential with the associated dynamical problem together constitute the natural conjugate of the corresponding situation in the case of the problem of two fixed centers treated in Chapter 4. Referring to that, we see that the transformation relations (2.1) could be obtained by replacing b^2 by $-b^2$ in relations (1.3) of Chapter 4; furthermore, if in formula (1.9) of Chapter 4 we make the following replacements,

$$b^2 \to -b^2, \quad b \to ib, \quad \beta \to i\beta \tag{2.14}$$

we obtain the Vinti potential (2.13) above.

However, the Vinti problem merits attention in its own right, and the distinct form assumed by the solution in terms of Jacobian elliptic functions as well as the distinct procedure for arriving at that solution justifies an independent analysis.

We shall confine our attention to the case where $\beta = 0$, although as before this restriction is not necessary for the effectiveness of the procedure. Setting $\beta = 0$ in (2.13), we have for the equatorially (latitudinally) symmetric Vinti potential function

$$\frac{\mu R}{R^2 + b^2\cos^2\sigma} = \frac{\mu}{r}\sum_{k=0}^{\infty}(-)^k\left(\frac{b^2}{r^2}\right)^k P_{2k}(\cos\theta)$$
$$= \frac{\mu}{r}\left[1 - \sum_{k=1}^{\infty}(-)^{k+1}\left(\frac{b^2}{r_0^2}\right)^k\left(\frac{r_0^2}{r^2}\right)^k P_{2k}(\cos\theta)\right]. \tag{2.15}$$

Comparing with (1.2), we see that (2.15) matches (1.2) up to the second zonal harmonic if we choose b so that

$$\frac{b^2}{r_0^2} = J_2 \tag{2.16}$$

whereby the coefficient of the fourth zonal harmonic becomes (i.e., for $k = 2$)

$$(-)^3\left(\frac{b^2}{r_0^2}\right)^2 = -\left(\frac{b^2}{r_0^2}\right)^2 = -J_2^2 \tag{2.17}$$

which, at least from an order of magnitude consideration, is in harmony with the known value (1.3b) though it undervalues it by a ratio of 5 : 7; all higher coefficients in the Vinti potential are of order $(J_2)^k$, and so for $k \geq 3$ these are much smaller than the known values. From the viewpoint of application to Earth-satellite orbits, therefore, it is redundant to seek accuracy finer than $(J_2)^2$ or $(b^2/r_0^2)^2$ in any approximation scheme applied to the entities to be calculated.

3 The Vinti Dynamical Problem: The Ignorable Coordinate and the Energy Integral

Associated with the coordinate system (2.1), we have the metric coefficients

$$g_{11} = \frac{R^2 + b^2\cos^2\sigma}{R^2 + b^2}, \quad g_{22} = R^2 + b^2\cos^2\sigma, \tag{3.1a,b}$$
$$g_{33} = (R^2 + b^2)\sin^2\sigma, \qquad g_{ij} = 0,\ i \neq j \tag{3.1c,d}$$

so that, for the kinetic energy, we have

$$T^* = \tfrac{1}{2}\frac{R^2 + b^2\cos^2\sigma}{R^2 + b^2}\dot{R}^2 + \tfrac{1}{2}(R^2 + b^2\cos^2\sigma)\dot{\sigma}^2 + \tfrac{1}{2}(R^2 + b^2)\sin^2\sigma \cdot \dot{\varphi}^2. \tag{3.2}$$

Moreover, with the potential function (2.15), we have for the potential energy (per unit mass)

$$V^* = -\frac{\mu R}{R^2 + b^2 \cos^2 \sigma} \tag{3.3}$$

from which, for the Lagrangian $L^* = T^* - V^*$, there follows

$$L^* = \tfrac{1}{2}\frac{R^2 + b^2 \cos^2 \sigma}{R^2 + b^2}\dot{R}^2 + \tfrac{1}{2}(R^2 + b^2 \cos^2 \sigma)\dot{\sigma}^2 + \tfrac{1}{2}(R^2 + b^2)\sin^2 \sigma \cdot \dot{\varphi}^2 + \frac{\mu R}{R^2 + b^2 \cos^2 \sigma} \tag{3.4}$$

wherein again φ is clearly an ignorable coordinate. The third Lagrange equation reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(R^2 + b^2)\sin^2 \sigma \cdot \dot{\varphi}\right] = 0 \tag{3.5}$$

which immediately yields the φ-first integral

$$(R^2 + b^2)\sin^2 \sigma \cdot \dot{\varphi} = C_3 \tag{3.6}$$

where C_3 is the constant of integration. From (2.2) and (2.1c), we have that

$$\begin{aligned}(R^2 + b^2)\sin^2 \sigma &= R^2 - R^2\cos^2 \sigma + b^2 \sin^2 \sigma \\ &= r^2 - r^2 \cos^2 \theta = r^2 \sin^2 \theta\end{aligned} \tag{3.7}$$

so that

$$C_3 = (R^2 + b^2)\sin^2 \sigma \cdot \dot{\varphi} = r^2 \sin^2 \theta \cdot \dot{\varphi} \tag{3.8}$$

and again in this case, we see that C_3 measures the polar component of angular momentum.

Following the standard procedure for dealing with the ignorable coordinate, we form the modified Lagrangian by setting

$$\begin{aligned}L &= L^* - \dot{\varphi}\frac{\partial L^*}{\partial \dot{\varphi}} = L^* - (R^2 + b^2)\sin^2 \sigma \cdot \dot{\varphi}^2 \\ &= \tfrac{1}{2}\frac{R^2 + b^2\cos^2 \sigma}{R^2 + b^2}\dot{R}^2 + \tfrac{1}{2}(R^2 + b^2\cos^2 \sigma)\dot{\sigma}^2 \\ &\quad - \tfrac{1}{2}(R^2 + b^2)\sin^2 \sigma \cdot \dot{\varphi}^2 + \frac{\mu R}{R^2 + b^2 \cos^2 \sigma}\end{aligned} \tag{3.9}$$

wherein we substitute for $\dot{\varphi}$ from (3.6) to obtain

$$\begin{aligned}L = {}& \tfrac{1}{2}\frac{R^2 + b^2\cos^2 \sigma}{R^2 + b^2}\dot{R}^2 + \tfrac{1}{2}(R^2 + b^2\cos^2 \sigma)\dot{\sigma}^2 \\ &+ \frac{\mu R}{R^2 + b^2\cos^2 \sigma} - \tfrac{1}{2}C_3^2\frac{1}{(R^2 + b^2)\sin^2 \sigma}\end{aligned} \tag{3.10}$$

so that in the modified Lagrangian involving only the R- and σ-coordinates, the modified kinetic and potential energies are, respectively,

$$T = \tfrac{1}{2}\frac{R^2 + b^2\cos^2\sigma}{R^2 + b^2}\dot{R}^2 + \tfrac{1}{2}(R^2 + b^2\cos^2\sigma)\dot{\sigma}^2 \tag{3.11a}$$

$$V = -\frac{\mu R}{R^2 + b^2\cos^2\sigma} + \tfrac{1}{2}C_3^2\frac{1}{(R^2 + b^2)\sin^2\sigma}. \tag{3.11b}$$

To put the modified Lagrangian into a form of Liouville type, we first write (3.10) in the form

$$\begin{aligned} L &= (R^2 + b^2\cos^2\sigma)\Big[\tfrac{1}{2}\frac{\dot{R}^2}{R^2 + b^2} + \tfrac{1}{2}\dot{\sigma}^2\Big] \\ &\quad + \frac{1}{R^2 + b^2\cos^2\sigma}\Big[\mu R - \tfrac{1}{2}C_3^2\Big(\frac{1}{\sin^2\sigma} - \frac{b^2}{R^2 + b^2}\Big)\Big]. \end{aligned} \tag{3.12}$$

Next we introduce the variable ξ by the transformation

$$R = b\sinh\xi; \quad \text{so that} \quad \dot{R} = b\cosh\xi\cdot\dot{\xi}, \ R^2 + b^2 = b^2\cosh^2\xi \tag{3.13a;b,c}$$

and, in terms of the Liouville coordinates ξ and σ, the modified Lagrangian becomes

$$\begin{aligned} L &= b^2(\sinh^2\xi + \cos^2\sigma)\Big[\tfrac{1}{2}\dot{\xi}^2 + \tfrac{1}{2}\dot{\sigma}^2\Big] \\ &\quad + \frac{1}{b^2(\sinh^2\xi + \cos^2\sigma)}\Big[\mu b\sinh\xi - \tfrac{1}{2}C_3^2\Big(\frac{1}{\sin^2\sigma} - \frac{1}{\cosh^2\xi}\Big)\Big] \\ &= b^2(\sinh^2\xi + \cos^2\sigma)\Big[\tfrac{1}{2}\dot{\xi}^2 + \tfrac{1}{2}\dot{\sigma}^2\Big] \\ &\quad + \frac{1}{b^2(\sinh^2\xi + \cos^2\sigma)}\Big[\mu b\sinh\xi + \tfrac{1}{2}C_3^2\frac{1}{\cosh^2\xi} - \tfrac{1}{2}C_3^2\frac{1}{\sin^2\sigma}\Big]. \end{aligned} \tag{3.14}$$

It is now clear that it is convenient to set

$$Q_1(\xi) = b^2\sinh^2\xi, \quad Q_2(\sigma) = b^2\cos^2\sigma, \quad Q = Q_1 + Q_2 \tag{3.15a,b,c}$$

$$V_1(\xi) = -\Big(\mu b\sinh\xi + \tfrac{1}{2}C_3^2\frac{1}{\cosh^2\xi}\Big), \quad V_2(\sigma) = \tfrac{1}{2}C_3^2\frac{1}{\sin^2\sigma} \tag{3.16a,b}$$

and the expressions for the modified kinetic and potential energies become

$$T = Q\big(\tfrac{1}{2}\dot{\xi}^2 + \tfrac{1}{2}\dot{\sigma}^2\big), \quad V = \frac{1}{Q}(V_1 + V_2) \tag{3.17a,b}$$

and the modified Lagrangian reads

$$L = Q\big(\tfrac{1}{2}\dot{\xi}^2 + \tfrac{1}{2}\dot{\sigma}^2\big) - V \tag{3.18a}$$

$$\quad = Q\big(\tfrac{1}{2}\dot{\xi}^2 + \tfrac{1}{2}\dot{\sigma}^2\big) - \frac{1}{Q}(V_1 + V_2) \tag{3.18b}$$

now clearly recognizable in its Liouville form.

The resulting Lagrangian equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\xi}}\right) = \frac{\partial L}{\partial \xi}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\sigma}}\right) = \frac{\partial L}{\partial \sigma} \tag{3.19a,b}$$

from which we derive the energy integral in the manner detailed in Section 2 of Chapter 3. Multiplying the first equation in (3.19) by $\dot{\xi}$, the second by $\dot{\sigma}$, then on addition, we find

$$\dot{\xi}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\xi}}\right) + \dot{\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\sigma}}\right) = \dot{\xi}\frac{\partial L}{\partial \xi} + \dot{\sigma}\frac{\partial L}{\partial \sigma} \tag{3.20a}$$

which, on rearrangement, becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\dot{\xi}\frac{\partial L}{\partial \dot{\xi}} + \dot{\sigma}\frac{\partial L}{\partial \dot{\sigma}}\right] = \dot{\xi}\frac{\partial L}{\partial \xi} + \dot{\sigma}\frac{\partial L}{\partial \sigma} + \ddot{\xi}\frac{\partial L}{\partial \dot{\xi}} + \ddot{\sigma}\frac{\partial L}{\partial \dot{\sigma}} = \frac{\mathrm{d}L}{\mathrm{d}t} \tag{3.20b}$$

immediately yielding the integral

$$\dot{\xi}\frac{\partial L}{\partial \dot{\xi}} + \dot{\sigma}\frac{\partial L}{\partial \dot{\sigma}} - L = \mathcal{E} \text{ (constant)} \tag{3.21}$$

— the energy integral in its general form with $\mathcal{E}$ as the constant of integration. When we note that the Lagrangian has the explicit form (3.18), and with the modified kinetic and potential energies given by (3.17), there follows

$$\dot{\xi}\frac{\partial L}{\partial \dot{\xi}} + \dot{\sigma}\frac{\partial L}{\partial \dot{\sigma}} = \dot{\xi}\frac{\partial T}{\partial \dot{\xi}} + \dot{\sigma}\frac{\partial T}{\partial \dot{\sigma}} = 2T \tag{3.22}$$

which, when introduced in (3.21), yields

$$T + V = \mathcal{E} \tag{3.23}$$

— the more familiar form of the energy integral. When the explicit forms of (3.11) for T and V are introduced into (3.23), we have

$$\begin{aligned} &\tfrac{1}{2}\frac{R^2 + b^2\cos^2\sigma}{R^2 + b^2}\dot{R}^2 + \tfrac{1}{2}(R^2 + b^2\cos^2\sigma)\dot{\sigma}^2 \\ &\qquad + \tfrac{1}{2}C_3^2\frac{1}{(R^2 + b^2)\sin^2\sigma} - \frac{\mu R}{R^2 + b^2\cos^2\sigma} = \mathcal{E} \end{aligned} \tag{3.24}$$

for the explicit form of the energy integral in terms of the R-σ spheroidal coordinates.

In summarizing this section, we note that in (3.6) and (3.24), we have two first integrals for the Vinti problem with constants C_3 and $\mathcal{E}$ representing respectively the polar component of angular momentum and the energy.

4 The Integration of the Lagrangian Equations

It remains to effect the integration of the two remaining Lagrangian equations (3.19). If we introduce the form (3.18a) for L into (3.19), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[Q\dot{\xi}] = \frac{\mathrm{d}Q_1}{\mathrm{d}\xi}[\tfrac{1}{2}\dot{\xi}^2 + \tfrac{1}{2}\dot{\sigma}^2] - \frac{\partial V}{\partial \xi} \tag{4.1a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[Q\dot{\sigma}] = \frac{\mathrm{d}Q_2}{\mathrm{d}\sigma}[\tfrac{1}{2}\dot{\xi}^2 + \tfrac{1}{2}\dot{\sigma}^2] - \frac{\partial V}{\partial \sigma} \tag{4.1b}$$

the integration of which follows the pattern detailed in Section 3 of Chapter 3. Following that procedure, we multiply (4.1a) by $Q\dot{\xi}$, and on utilizing (3.17) and (3.23), we obtain

$$\begin{aligned} Q\dot{\xi}\frac{\mathrm{d}}{\mathrm{d}t}[Q\dot{\xi}] &= \dot{\xi}\Big[T\frac{\mathrm{d}Q_1}{\mathrm{d}\xi} - Q\frac{\partial V}{\partial \xi}\Big] = \dot{\xi}\Big[\mathcal{E}\frac{\mathrm{d}Q_1}{\mathrm{d}\xi} - \frac{\partial}{\partial \xi}(QV)\Big] \\ &= \dot{\xi}\Big[\mathcal{E}\frac{\mathrm{d}Q_1}{\mathrm{d}\xi} - \frac{\mathrm{d}V_1}{\mathrm{d}\xi}\Big] = \frac{\mathrm{d}}{\mathrm{d}t}[\mathcal{E}Q_1 - V_1] \end{aligned} \tag{4.2}$$

yielding the first integral associated with the ξ-coordinate in the form

$$\tfrac{1}{2}[Q\dot{\xi}]^2 - \mathcal{E}Q_1 + V_1 = C_1 \tag{4.3}$$

where C_1 is the constant of integration. An application of the identical procedure to (4.1b) with a multiplying factor $Q\dot{\sigma}$ yields the first integral associated with the σ-coordinate in the form

$$\tfrac{1}{2}[Q\dot{\sigma}]^2 - \mathcal{E}Q_2 + V_2 = C_2 \tag{4.4}$$

where C_2 is the constant of integration. As we already have C_3 and $\mathcal{E}$ as two independent constants, the two new constants C_1 and C_2 cannot be independent; the addition of (4.3) and (4.4) yields

$$Q^2[\tfrac{1}{2}\dot{\xi}^2 + \tfrac{1}{2}\dot{\sigma}^2] - \mathcal{E}(Q_1 + Q_2) + (V_1 + V_2) = C_1 + C_2 \tag{4.5}$$

or alternatively, on noting (3.17)

$$Q[T + V - \mathcal{E}] = C_1 + C_2 \tag{4.6}$$

which with the energy equation (3.23) implies

$$C_1 + C_2 = 0 \tag{4.7}$$

indicating the interdependence of the constants.

We next turn to express the integrated relations (4.3) and (4.4) in terms of the original spheroidal R-σ coordinates. Recalling the defining relations (3.13a), it follows from (3.15) and (3.16) that in terms of the spheroidal coordinates, we have

$$Q_1(R) = R^2, \quad Q_2(\sigma) = b^2\cos^2\sigma, \quad Q = R^2 + b^2\cos^2\sigma \tag{4.8a,b,c}$$

$$V_1(R) = -\Big[\mu R + \tfrac{1}{2}C_3^2\frac{b^2}{R^2 + b^2}\Big], \quad V_2(\sigma) = \tfrac{1}{2}C_3^2\frac{1}{\sin^2\sigma}. \tag{4.9a,b}$$

From (3.13b), we further note that

$$\dot{\xi} = \frac{\dot{R}}{b\cosh\xi} = \frac{\dot{R}}{\sqrt{R^2 + b^2}} \tag{4.10}$$

so that

$$Q\dot{\xi} = \frac{R^2 + b^2\cos^2\sigma}{\sqrt{R^2 + b^2}}\dot{R}, \quad Q\dot{\sigma} = (R^2 + b^2\cos^2\sigma)\dot{\sigma}. \tag{4.11a,b}$$

Accordingly, in terms of spheroidal R-σ coordinates, equations (4.3) and (4.4) become, respectively,

$$\tfrac{1}{2}\frac{(R^2 + b^2\cos^2\sigma)^2}{R^2 + b^2}\dot{R}^2 = \mathcal{E}R^2 + \left[\mu R + \tfrac{1}{2}C_3^2\frac{b^2}{R^2 + b^2}\right] + C_1 \tag{4.12a}$$

$$\tfrac{1}{2}(R^2 + b^2\cos^2\sigma)^2\dot{\sigma}^2 = \mathcal{E}b^2\cos^2\sigma - \tfrac{1}{2}C_3^2\frac{1}{\sin^2\sigma} + C_2 \tag{4.12b}$$

with the restriction (4.7) that $C_1 + C_2 = 0$.

An inspection of equation (4.12b) in the Kepler limit when $b \to 0$ shows that in that range, C_2 must be positive, which immediately implies that C_1 is negative. Moreover, if we focus our attention on bound orbits corresponding to negative energy, we may set

$$\mathcal{E} = -\alpha^2, \quad C_1 = -\tfrac{1}{2}C^2, \quad C_2 = \tfrac{1}{2}C^2 \tag{4.13a,b,c}$$

and again from an inspection of (4.12b), it is clear that C has the dimension of angular momentum.

In terms of the constants introduced in (4.13), equations (4.12) take the form

$$\tfrac{1}{2}\frac{(R^2 + b^2\cos^2\sigma)^2}{R^2 + b^2}\dot{R}^2 = -\alpha^2R^2 + \left[\mu R + \tfrac{1}{2}C_3^2\frac{b^2}{R^2 + b^2}\right] - \tfrac{1}{2}C^2 \tag{4.14a}$$

$$\tfrac{1}{2}(R^2 + b^2\cos^2\sigma)^2\dot{\sigma}^2 = -\alpha^2b^2\cos^2\sigma - \tfrac{1}{2}C_3^2\frac{1}{\sin^2\sigma} + \tfrac{1}{2}C^2. \tag{4.14b}$$

In the reduction of these equations, we shall follow the pattern set out in Chapter 3 for the planar case of two fixed centers.

5 Reduction of the Equations; Regularization; Normalization

In considering the R-equation, we start by rewriting (4.14a) in the form

$$(R^2 + b^2\cos^2\sigma)^2\dot{R}^2 = -C^2(R^2 + b^2)\left[1 - 2\frac{\mu}{C^2}R + \frac{2\alpha^2}{C^2}R^2 - \frac{C_3^2}{C^2}\frac{b^2}{R^2 + b^2}\right]. \tag{5.1}$$

We now introduce the standard length scales a and p_0, and the dimensionless parameter ν corresponding to the inclination in the Kepler case, by setting

$$a = \frac{\mu}{2\alpha^2}, \quad p_0 = \frac{C^2}{\mu}; \text{ so that } ap_0 = \frac{C^2}{2\alpha^2} : \nu = \frac{C_3}{C} \tag{5.2a,b;c:d}$$

in terms of which (5.1) reads

$$\frac{(R^2+b^2\cos^2\sigma)^2}{C^2}\dot{R}^2 = -(R^2+b^2)\Big[1-\frac{2}{p_0}R+\frac{1}{ap_0}R^2\Big]+b^2\nu^2. \tag{5.3}$$

Likewise, in terms of the parameters (5.2), we may write the σ-equation (4.14b) in the form

$$\begin{aligned}\frac{(R^2+b^2\cos^2\sigma)^2}{C^2}\dot{\sigma}^2 &= 1-2\frac{\alpha^2}{C^2}b^2\cos^2\sigma-\frac{C_3^2}{C^2}\frac{1}{\sin^2\sigma}\\ &= 1-\frac{b^2}{ap_0}\cos^2\sigma-\frac{\nu^2}{\sin^2\sigma}.\end{aligned} \tag{5.4}$$

Again the form of the regularizing transformation is indicated by the multiplying factor on the left side of (5.3) and (5.4).

We introduce the new independent variable f defined by the relation

$$\frac{\mathrm{d}f}{\mathrm{d}t}=\frac{\Lambda}{R^2+b^2\cos^2\sigma} \tag{5.5}$$

where the parameter Λ, having the dimension of angular momentum, is to be defined presently as a multiple of C. From (5.5), it follows that

$$\frac{R^2+b^2\cos^2\sigma}{\Lambda}\frac{\mathrm{d}}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}f} \tag{5.6}$$

and with prime denoting differentiation with respect to f, the equations for the first integrals (5.3) and (5.4) may be written

$$\frac{\Lambda^2}{C^2}R'^2 = -(R^2+b^2)\Big[1-\frac{2}{p_0}R+\frac{1}{ap_0}R^2\Big]+b^2\nu^2 \tag{5.7}$$

$$\begin{aligned}\frac{\Lambda^2}{C^2}\sigma'^2 &= 1-\frac{b^2}{ap_0}\cos^2\sigma-\frac{\nu^2}{\sin^2\sigma}\\ &= \frac{1}{\sin^2\sigma}\Big[(1-\cos^2\sigma)\Big[1-\frac{b^2}{ap_0}\cos^2\sigma\Big]-\nu^2\Big].\end{aligned} \tag{5.8}$$

At this point, it is appropriate to take note that in the case where $\nu = 0$, corresponding to $C_3 = 0$, we have the case of polar orbits, and the solution in terms of Jacobian elliptic functions in that case is immediate as the resolution of the quartics into quadratic factors is self-evident.

We now introduce dimensionless parameters defined by the relations

$$\eta=\frac{b}{p_0},\quad \ell = 1-e_0^2=\frac{p_0}{a} \tag{5.9a,b}$$

so that

$$\eta\ell=\frac{b}{p_0}\frac{p_0}{a}=\frac{b}{a},\quad \eta^2\ell=\frac{b^2}{ap_0}. \tag{5.10a,b}$$

In terms of these, the σ-equation (5.8) takes the form

$$\frac{\Lambda^2}{C^2}\sin^2\sigma \cdot \sigma'^2 = (1-\cos^2\sigma)[1-\eta^2\ell\cos^2\sigma] - \nu^2. \tag{5.11}$$

For the R-equation, we further introduce the dimensionless variable Y, defined by

$$R = aY \tag{5.12}$$

in terms of which the R-equation (5.7) takes the dimensionless form

$$\ell\frac{\Lambda^2}{C^2}Y'^2 = -(Y^2+\eta^2\ell^2)[Y^2-2Y+\ell] + \eta^2\ell^3\nu^2. \tag{5.13}$$

In this latter equation, it is convenient to introduce a further parameter λ^2, defined by

$$\lambda^2 = \eta^2\ell^2 \tag{5.14}$$

and equation (5.13) may be written

$$\begin{aligned}\ell\frac{\Lambda^2}{C^2}Y'^2 &= -(Y^2+\lambda^2)[Y^2-2Y+\ell] + \lambda^2\ell\nu^2\\ &= [Y^4-2Y^3+(\ell+\lambda^2)Y^2-2\lambda^2Y+\lambda^2\ell(1-\nu^2).]\end{aligned} \tag{5.15}$$

Equations (5.11) and (5.15) must now be dealt with individually — starting with the σ-equation (5.11).

6 The σ-Equation: Definition of Λ

If, in equation (5.11) for the σ-coordinate, we set

$$S = \cos\sigma \quad \text{so that} \quad S' = -\sin\sigma\cdot\sigma', \tag{6.1}$$

then, in terms of S, equation (5.11) takes the form

$$\begin{aligned}\frac{\Lambda^2}{C^2}S'^2 &= (1-S^2)(1-\eta^2\ell S^2) - \nu^2\\ &= (1-\nu^2) - (1+\eta^2\ell)S^2 + \eta^2\ell S^4\end{aligned} \tag{6.2}$$

or, alternatively,

$$\frac{1}{\eta^2\ell}\frac{\Lambda^2}{C^2}S'^2 = \frac{1-\nu^2}{\eta^2\ell} - \frac{1+\eta^2\ell}{\eta^2\ell}S^2 + S^4. \tag{6.3}$$

If we consider the expression

$$y^2 - \frac{1+\eta^2\ell}{\eta^2\ell}y + \frac{1-\nu^2}{\eta^2\ell} \tag{6.4}$$

the pair of roots is given by

$$y_+ = \tfrac{1}{2}\frac{1+\eta^2\ell}{\eta^2\ell}\Big[1+\sqrt{1-\frac{4\eta^2\ell}{(1+\eta^2\ell)^2}(1-\nu^2)}\Big]$$
$$y_- = \tfrac{1}{2}\frac{1+\eta^2\ell}{\eta^2\ell}\Big[1-\sqrt{1-\frac{4\eta^2\ell}{(1+\eta^2\ell)^2}(1-\nu^2)}\Big]. \tag{6.5}$$

It is therefore convenient to set

$$2m = 1+\sqrt{1-\frac{4\eta^2\ell}{(1+\eta^2\ell)^2}(1-\nu^2)} = 2+O(\eta^2) \tag{6.6}$$

and a straightforward calculation shows that

$$\frac{1}{2m} = \frac{(1+\eta^2\ell)^2}{4\eta^2\ell(1-\nu^2)}\Big[1-\sqrt{1-\frac{4\eta^2\ell}{(1+\eta^2\ell)^2}(1-\nu^2)}\Big] \tag{6.7}$$

and hence

$$1-\sqrt{1-\frac{4\eta^2\ell}{(1+\eta^2\ell)^2}(1-\nu^2)} = \frac{4\eta^2\ell}{(1+\eta^2\ell)^2}(1-\nu^2)\frac{1}{2m}. \tag{6.8}$$

Accordingly, the roots of the expression (6.4) are given in terms of m by

$$y_+ = \frac{1+\eta^2\ell}{\eta^2\ell}m, \quad y_- = \frac{1}{1+\eta^2\ell}\frac{1-\nu^2}{m}. \tag{6.9}$$

In light of this resolution, it follows that the S-equation (6.3) may be written

$$\frac{1}{\eta^2\ell}\frac{\Lambda^2}{C^2}S'^2 = \Big(\frac{1+\eta^2\ell}{\eta^2\ell}m - S^2\Big)\Big[\frac{1-\nu^2}{(1+\eta^2\ell)m} - S^2\Big]. \tag{6.10}$$

If we now introduce the substitution

$$S = \sqrt{\frac{1-\nu^2}{(1+\eta^2\ell)m}}\,\zeta \tag{6.11}$$

then, when written as an equation for ζ, and reversing the order of the factors, equation (6.10) becomes

$$\begin{aligned}\frac{1}{\eta^2\ell}\frac{\Lambda^2}{C^2}\zeta'^2 &= (1-\zeta^2)\Big[\frac{1+\eta^2\ell}{\eta^2\ell}m - \frac{1-\nu^2}{(1+\eta^2\ell)m}\zeta^2\Big] \\ &= \frac{1+\eta^2\ell}{\eta^2\ell}m(1-\zeta^2)\Big[1-\frac{\eta^2\ell}{(1+\eta^2\ell)^2}\frac{1-\nu^2}{m^2}\zeta^2\Big]\end{aligned} \tag{6.12}$$

and hence

$$\frac{\Lambda^2}{C^2}\zeta'^2 = (1+\eta^2\ell)m(1-\zeta^2)\Big[1-\frac{\eta^2\ell}{(1+\eta^2\ell)^2}\frac{1-\nu^2}{m^2}\zeta^2\Big]. \tag{6.13}$$

This equation clearly suggests the form for Λ. We define Λ by the relation

$$\Lambda^2 = (1 + \eta^2 \ell) m C^2 \tag{6.14}$$

and we further set

$$k_2^2 = \frac{\eta^2 \ell}{(1 + \eta^2 \ell)^2} \cdot \frac{1 - \nu^2}{m^2} \tag{6.15}$$

and equation (6.13) then reads

$$\zeta'^2 = (1 - \zeta^2)(1 - k_2^2 \zeta^2) \tag{6.16}$$

for which we have the solution

$$\zeta = \mathrm{sn}[f + \omega : k_2] \tag{6.17}$$

where ω is the constant of integration. Clearly

$$f + \omega = 0 \quad \text{implies} \quad \zeta = 0 \tag{6.18}$$

so that, recognizing that the equatorial crossing is indicated by

$$\cos\sigma = 0, \quad S = 0, \quad \zeta = 0 \tag{6.19}$$

we see that $-\omega$ is to be interpreted as the value of f at the first equatorial crossing. If we further set

$$1 - N^2 = \frac{1 - \nu^2}{(1 + \eta^2 \ell) m} \tag{6.20}$$

then combining (6.1), (6.11), and (6.17), we have

$$\cos\sigma = \sqrt{1 - N^2}\, \mathrm{sn}[f + \omega : k_2] \tag{6.21}$$

as the solution for the σ-coordinate. We note that when $\eta = 0$, $k_2 = 0$, the above solution collapses as the familiar form of the solution for the Kepler problem.

7 The *R*-Equation

Returning to the dimensionless form of the *R*-equation (5.15), namely

$$\ell \frac{\Lambda^2}{C^2} Y'^2 = -[Y^4 - 2Y^3 + (\ell + \lambda^2)Y^2 - 2\lambda^2 Y + \ell\lambda^2(1 - \nu^2)] = -f(y) \tag{7.1}$$

we may compare this with equation (8.1) of Chapter 4, and as already indicated, the above equation could be obtained by replacing λ^2 by $-\lambda^2$ (or equivalently, as $\lambda^2 = \eta^2 \ell^2$, by replacing η^2 by $-\eta^2$) in the corresponding equation of Chapter 4. Accordingly, we may take the results of Subsection 8A of Chapter 4 and, replacing λ^2 by $-\lambda^2$, apply the relations, mutatis mutandis, therefrom: we include these relations here in Subsection 7A.

7A. The Decomposition of the Quartic (7.1)

The transformation that eliminates the cubic term from the quartic on the right of (7.1), namely

$$Y = X + \tfrac{1}{2} \tag{7A.1}$$

results in

$$f(Y) = g(x) = \\ X^4 + (\ell + \lambda^2 - \tfrac{3}{2})X^2 + (\ell - \lambda^2 - 1)X + \tfrac{1}{4}[\ell - 3\lambda^2 - \tfrac{3}{4} + 4\ell\lambda^2(1 - \nu^2)]. \tag{7A.2}$$

Following the procedure of Descartes for the decomposition, we set

$$g(X) = [X^2 + UX + V][X^2 - UX + W] \tag{7A.3}$$

which implies

$$V - W = \frac{1 - \ell + \lambda^2}{U} \tag{7A.4a}$$

$$V + W = U^2 + (\ell + \lambda^2 - \tfrac{3}{2}) \tag{7A.4b}$$

and with

$$Z = U^2 \tag{7A.5}$$

the resulting equation for Z reads

$$Z^3 - 3[1 - \tfrac{2}{3}(\ell + \lambda^2)]Z^2 \\ + 3[1 - \tfrac{4}{3}\ell + \tfrac{1}{3}(\ell + \lambda^2)^2 - \tfrac{4}{3}\ell\lambda^2(1 - \nu^2)]Z - [1 - \ell + \lambda^2]^2 = 0. \tag{7A.6}$$

It is convenient to set

$$A = 1 - \tfrac{2}{3}(\ell + \lambda^2), \quad B = 1 - \tfrac{4}{3}\ell + \tfrac{1}{3}(1 + \lambda^2)^2 - \tfrac{4}{3}\ell\lambda^2(1 - \nu^2), \\ C = 1 - \ell + \lambda^2, \tag{7A.7a,b,c}$$

so that (7A.6) may be written

$$Z^3 - 3AZ^2 + 3BZ - C^2 = 0. \tag{7A.8}$$

The elimination of the quadratic term is effected by setting

$$Z = Z_0 + A \tag{7A.9}$$

and in terms of Z_0, equation (7A.8) becomes

$$Z_0^3 + 3(B - A^2)Z_0 + (3AB - 2A^3 - C^2) = 0 \tag{7A.10}$$

where the coefficients are given as follows:

$$3(B-A^2) = -\tfrac{1}{3}(\ell+\lambda^2)^2\left[1-12\lambda^2\frac{[1-\ell(1-\nu^2)]}{(\ell+\lambda^2)^2}\right] \tag{7A.11a}$$

$$3AB-2A^3-C^2 = -\tfrac{2}{27}(\ell+\lambda^2)^3\left[1+36\lambda^2\frac{[1-\ell(1-\nu^2)]}{(\ell+\lambda^2)^2}\right.$$
$$\left.\times\left[1-\tfrac{3}{2}\frac{\ell\nu^2}{(\ell+\lambda^2)[1-\ell(1-\nu^2)]}\right]\right]. \tag{7A.11b}$$

If for equations (7A.11) we introduce the notation

$$\lambda_* = 12\lambda^2\frac{[1-\ell(1-\nu^2)]}{(\ell+\lambda^2)^2} = 12\eta^2\frac{[1-\ell(1-\nu^2)]}{(1+\eta^2\ell)^2} \tag{7A.12a}$$

$$s = 1-\tfrac{3}{2}\frac{\ell\nu^2}{(\ell+\lambda^2)[1-\ell(1-\nu^2)]} = 1-\tfrac{3}{2}\frac{\nu^2}{(1+\eta^2\ell)[1-\ell(1-\nu^2)]} \tag{7A.12b}$$

wherein we have noted relation (5.14) expressing $\lambda^2 = \eta^2\ell^2$, then equation (7A.10) may be written

$$Z_0^3 - \tfrac{1}{3}(\ell+\lambda^2)^2(1-\lambda_*)Z_0 - \tfrac{2}{27}(\ell+\lambda^2)^3[1+3\lambda_* s] = 0. \tag{7A.13}$$

To normalize the expression in (7A.13), we set

$$Z_0 = \tfrac{1}{3}(\ell+\lambda^2)Z_* \tag{7A.14}$$

and, in terms of Z_*, the above equation (7A.13) becomes

$$Z_*^3 - 3(1-\lambda_*)Z_* - 2[1+3\lambda_* s] = 0 \tag{7A.15}$$

which could have been obtained from equation (8A.18) of Chapter 4 by replacing λ_* with $-\lambda_*$ therein.

Accordingly, from an inspection of (8A.28) of Chapter 4, we may write the solution of equation (7A.15) in the form

$$Z_* = \left[(1+3\lambda_* s) - \sqrt{\lambda_*}[\lambda_*^2 - 3\lambda_*(1-3s^2)+3(1+2s)]^{\frac{1}{2}}\right]^{\frac{1}{3}}$$
$$+\left[(1+3\lambda_* s) + \sqrt{\lambda_*}[\lambda_*^2 - 3\lambda_*(1-3s^2)+3(1+2s)]^{\frac{1}{2}}\right]^{\frac{1}{3}} \tag{7A.16}$$

and we note that the root is given as an even function of $\sqrt{\lambda_*}$ and hence is strictly a function of λ_* and hence of η^2.

7B. The Approximate Formulae

For application to the case of the Earth satellite, an approximation valid to second order in η^2 is adequate — and, in fact, sufficient. The determination of such an approximation at this stage is a straightforward exercise in binomial expansion. We introduce the factor D by setting

$$D(1+3\lambda_* s) = \sqrt{\lambda_*}[3(1+2s) - 3\lambda_*(1-3s^2) + \lambda_*^2]^{\frac{1}{2}} = O(\eta) \tag{7B.1}$$

so that, in terms of D, the solution form (7A.16) may be written

$$Z_* = (1+3\lambda_* s)^{\frac{1}{3}}[(1-D)^{\frac{1}{3}} + (1+D)^{\frac{1}{3}}] \tag{7B.2}$$

and we further note from (7B.1) that

$$D^2(1+3\lambda_* s)^2 = \lambda_*[3(1+2s) - 3\lambda_*(1-3s^2) + \lambda_*^2] = O(\eta^2). \tag{7B.3}$$

By direct expansion, we see that

$$(1+3\lambda_* s)^{-2} = 1 - 6\lambda_* s + O(\eta^4) \tag{7B.4}$$

so that

$$D^2 = 3\lambda_*[(1+2s) - \lambda_*(1+3s)^2] + O(\eta^6) \tag{7B.5a}$$

and

$$D^4 = 9\lambda_*^2(1+2s)^2 + O(\eta^6). \tag{7B.5b}$$

Direct expansions also reveal that

$$(1+D)^{\frac{1}{3}} = 1 + \tfrac{1}{3}D - \tfrac{1}{9}D^2 + \tfrac{5}{81}D^3 - \tfrac{10}{243}D^4 + \tfrac{22}{729}D^5 + O(\eta^6) \tag{7B.6a}$$

$$(1-D)^{\frac{1}{3}} = 1 - \tfrac{1}{3}D - \tfrac{1}{9}D^2 - \tfrac{5}{81}D^3 - \tfrac{10}{243}D^4 - \tfrac{22}{729}D^5 + O(\eta^6) \tag{7B.6b}$$

from which there follows

$$(1-D)^{\frac{1}{3}} + (1+D)^{\frac{1}{3}} = 2\left[1 - \tfrac{1}{9}D^2 - \tfrac{10}{243}D^4 + O(\eta^6)\right] \tag{7B.7}$$

into which we introduce D^2 and D^4 from (7B.5) to obtain (except for terms of $O(\eta^6)$)

$$(1-D)^{\frac{1}{3}} + (1+D)^{\frac{1}{3}} = 2\left[1 - \tfrac{1}{3}\lambda_*(1+2s) - \tfrac{1}{27}\lambda_*^2(1-14s-41s^2)\right]. \tag{7B.8}$$

A further direct expansion shows that

$$(1+3\lambda_* s)^{\frac{1}{3}} = 1 + \lambda_* s - \lambda_*^2 s^2 + O(\eta^6) \tag{7B.9}$$

so that inserting (7B.8) and (7B.9) into (7B.2), we obtain for Z_*

$$Z_* = 2\left[1 - \tfrac{1}{3}\lambda_*(1-s) - \tfrac{1}{27}\lambda_*^2(1-5s+4s^2)\right] + O(\eta^6) \tag{7B.10}$$

or, alternatively, except for terms of $O(\eta^6)$,

$$\frac{Z_*}{2} = 1 - \tfrac{1}{3}\lambda_*(1-s)\left[1 + \tfrac{1}{9}\lambda_*(1-4s)\right] \tag{7B.11}$$

and again we note that when $\nu = 0$, $s = 1$ and $Z_* = 2$.

Into relation (7A.9) we now introduce A from (7A.7a) and Z_0 from (7A.14) and rearrange to obtain

$$Z = 1 - \tfrac{2}{3}(\ell + \lambda^2)\Big[1 - \tfrac{1}{2}Z_*\Big] \tag{7B.12}$$

into which we now introduce Z_* from (7B.11) to obtain

$$Z = 1 - \tfrac{2}{9}(\ell + \lambda^2)\lambda_*(1 - s)\Big[1 + \tfrac{1}{9}\lambda_*(1 - 4s)\Big]. \tag{7B.13}$$

This expression can be put in more compact form if we first note from (7A.12) that

$$\tfrac{1}{9}\lambda_*(1 - s) = 2\frac{\lambda^2\ell\nu^2}{(\ell + \lambda^2)^3} = 2\frac{\eta^2\nu^2}{(1 + \eta^2\ell)^3} \tag{7B.14a,b}$$

where we have also noted (5.14). Accordingly, expression (7B.13) may be written

$$\begin{aligned} Z &= 1 - \frac{4\lambda^2\ell\nu^2}{(\ell + \lambda^2)^2}\Big[1 + \tfrac{1}{9}\lambda_*(1 - 4s)\Big] \\ &= 1 - \frac{4\eta^2\ell\nu^2}{(1 + \eta^2\ell)^2}\Big[1 + \tfrac{1}{9}\lambda_*(1 - 4s)\Big]. \end{aligned} \tag{7B.15}$$

Next we introduce the factor h_* by the defining algebraic relation

$$1 + \eta^2 h_* = \frac{1}{(1 + \eta^2\ell)^2}\Big[1 + \tfrac{1}{9}\lambda_*(1 - 4s)\Big] \tag{7B.16}$$

and a straighforward calculation following the introduction of (7A.12) shows that, to second order in η^2, we may write

$$1 + \eta^2 h_* = 1 - 2\eta^2(2 - \ell)(1 - 2\nu^2) + \eta^4\ell[(16 - 13\ell) - 8\nu^2(5 - 2\ell)] \tag{7B.17a}$$

or, to first order in η^2

$$h_* = -2(2 - \ell)(1 - 2\nu^2) + \eta^2\ell[(16 - 13\ell) - 8\nu^2(5 - 2\ell)]. \tag{7B.17b}$$

With h_* thus specified, we may replace the expression (7B.15) for Z by

$$Z = 1 - 4\eta^2\ell\nu^2(1 + \eta^2 h_*) \tag{7B.18}$$

Recalling from (7A.5) that $Z = U^2$, we derive the following expressions (to second order in η^2):

$$\begin{aligned} U &= 1 - 2\eta^2\ell\nu^2(1 + \eta^2 h_*) - 2\eta^4\ell^2\nu^4 \\ &= 1 - 2\eta^2\ell\nu^2 - 2\eta^4\ell\nu^2(h_* + \ell\nu^2) \end{aligned} \tag{7B.19a}$$

$$\begin{aligned} U^{-1} &= 1 + 2\eta^2\ell\nu^2[1 + \eta^2(h_* + \ell\nu^2)] + 4\eta^4\ell^2\nu^4 \\ &= 1 + 2\eta^2\ell\nu^2 + 2\eta^4\ell\nu^2(h_* + 3\ell\nu^2) \end{aligned} \tag{7B.19b}$$

$$\begin{aligned} (1 - \ell + \lambda^2)U^{-1} &= (1 - \ell) + \eta^2\ell[\ell + 2\nu^2(1 - \ell)] \\ &\quad + 2\eta^4\ell\nu^2[\ell^2 + (1 - \ell)(h_* + 3\ell\nu^2)]. \end{aligned} \tag{7B.19c}$$

Again recalling that $U^2 = Z$, it follows from combining (7A.4b) with (7B.18) that

$$V + W = -\tfrac{1}{2} + \ell + \eta^2\ell^2 - 4\eta^2\ell\nu^2(1 + \eta^2 h_*) \tag{7B.20a}$$

while from (7A.4a) and (7B.19c) we have

$$\begin{aligned} V - W &= (1 - \ell) + \eta^2\ell[\ell + 2\nu^2(1 - \ell)] \\ &\quad + 2\eta^4\ell\nu^2[\ell^2 + (1 - \ell)(h_* + 3\ell\nu^2)]. \end{aligned} \tag{7B.20b}$$

Relations (7B.20) render for V and W, respectively,

$$\begin{aligned} V &= \tfrac{1}{4} + \eta^2\ell[\ell - \nu^2(1 + \ell)] \\ &\quad + \eta^4\ell\nu^2[\ell^2 - (1 + \ell)h_* + 3\ell\nu^2(1 - \ell)] \end{aligned} \tag{7B.21a}$$

$$\begin{aligned} W &= -\tfrac{3}{4} + \ell - \eta^2\ell\nu^2(3 - \ell) \\ &\quad - \eta^4\ell\nu^2[\ell^2 + (3 - \ell)h_* + 3\ell\nu^2(1 - \ell)]. \end{aligned} \tag{7B.21b}$$

When, in the factored quartic (7A.3), we revert to the Y-variable by setting $X = Y - \frac{1}{2}$, then for $f(Y)$ we have

$$\begin{aligned} &f(Y) = \\ &\Big[Y^2 - (1 - U)Y + (\tfrac{1}{4} - \tfrac{1}{2}U + V)\Big]\Big[Y^2 - (1 + U)Y + (\tfrac{1}{4} + \tfrac{1}{2}U + W)\Big]. \end{aligned} \tag{7B.22}$$

For the elements in this factorization, we note from (7B.19a) that

$$\begin{aligned} 1 - U &= 2\eta^2\ell\nu^2 + 2\eta^4\ell\nu^2(h_* + \ell\nu^2) \\ &= 2\eta^2\ell\nu^2[1 + \eta^2(h_* + \ell\nu^2)] \end{aligned} \tag{7B.23a}$$

$$\begin{aligned} 1 + U &= 2[1 - \eta^2\ell\nu^2 - \eta^4\ell\nu^2(h_* + \ell\nu^2)] \\ &= 2[1 - \eta^2\ell\nu^2[1 + \eta^2(h_* + \ell\nu^2)]] \end{aligned} \tag{7B.23b}$$

and further from combining (7B.19a) with (7B.21), we find

$$\tfrac{1}{4} - \tfrac{1}{2}U + V = \eta^2\ell^2\Big[1 - \nu^2[(1 + \eta^2 h_*) - \eta^2[\ell + \nu^2(4 - 3\ell)]]\Big] \tag{7B.24a}$$

$$\begin{aligned} \tfrac{1}{4} + \tfrac{1}{2}U + W = \ell\Big[1 - \eta^2\nu^2[(4 - \ell)(1 + \eta^2 h_*) \\ + \eta^2\ell[\ell + \nu^2(4 - 3\ell)]]\Big]. \end{aligned} \tag{7B.24b}$$

Relations (7B.23) and (7B.24) suggest that we introduce the factors

$$\begin{aligned} h_0 &= 1 + \eta^2(h_* + \ell\nu^2) \\ &= 1 - \eta^2[2(2 - \ell) - \nu^2(8 - 3\ell)] + \eta^4\ell[(16 - 13\ell) - 8\nu^2(5 - 2\ell)] \end{aligned} \tag{7B.25a}$$

$$\begin{aligned} h_1 &= 1 + \eta^2 h_* - \eta^2[\ell + \nu^2(4 - 3\ell)] \\ &= 1 - \eta^2[(4 - \ell)(1 - \nu^2)] + \eta^4\ell[(16 - 13\ell) - 8\nu^2(5 - 2\ell)] \end{aligned} \tag{7B.25b}$$

$$\begin{aligned} h_2 &= (4 - \ell)(1 + \eta^2 h_*) + \eta^2\ell[\ell + \nu^2(4 - 3\ell)] = 4(1 + \eta^2 h_*) - \ell h_1 \\ &= (4 - \ell)[1 + \eta^2\ell(1 - \nu^2) + \eta^4\ell[(16 - 13\ell) - 8\nu^2(5 - 2\ell)]] \\ &\qquad - 8\eta^2(2 - \ell)(1 - 2\nu^2) \end{aligned} \tag{7B.25c}$$

and we may note here the three auxiliary factors

$$\begin{aligned} h_0 + h_1 &= 2(1+\eta^2 h_*) - \eta^2[\ell + 4\nu^2(1-\ell)] + O(\eta^4) && \text{(7B.26a)} \\ 1 - \nu^2 h_1 &= (1-\nu^2)[1 + \eta^2\nu^2(4-\ell)] + O(\eta^4) && \text{(7B.26b)} \\ h_2 - \ell h_0 &= 4(1+\eta^2 h_*) - \ell(h_0 + h_1) \\ &= (4 - 2\ell) + O(\eta^2) = 2(2-\ell) + O(\eta^2) && \text{(7B.26c)} \end{aligned}$$

which arise later.

We may write the approximate formulae for the coefficients appearing in the quadratic factors on the right of (7B.22) by means of the quantities h_0, h_1, and h_2 as follows:

$$1 - U = 2\eta^2\nu^2\ell h_0, \quad 1 + U = 2(1 - \eta^2\nu^2\ell h_0) \tag{7B.27a,b}$$

$$\tfrac{1}{4} - \tfrac{1}{2}U + V = \eta^2\ell(1-\nu^2 h_1), \quad \tfrac{1}{4} + \tfrac{1}{2}U + W = \ell(1 - \eta^2\nu^2 h_2) \tag{7B.27c,d}$$

which completes the derivation of the approximate formulae.

— ◇ —

If we now return to the dimensionless form of the R-equation, namely (7.1), and introduce the factorization (7B.22) with coefficients given by (7B.27), we have

$$\begin{aligned} \ell\frac{\Lambda^2}{C^2}Y'^2 = -[&Y^2 - 2\eta^2\nu^2\ell h_0 Y + \eta^2\ell^2(1-\nu^2 h_1)] \\ &\cdot [Y^2 - 2(1-\eta^2\nu^2\ell h_0)Y + \ell(1-\eta^2\nu^2 h_2)] \end{aligned} \tag{7.2}$$

and we note that in the case of a polar orbit when $\nu = 0$, we retrieve a factorization readily recognizable from (5.13) when we set $\nu = 0$ therein.

If we now revert to the dependent variable $R = aY$, and interchanging the order of the factors in (7.2), we recall from (5.9b) that $\ell = p_0/a$, then a rearrangement yields the equation (7.2) in terms of R as follows:

$$\begin{aligned} \ell\frac{\Lambda^2}{C^2}R'^2 = -\ell(1-\eta^2\nu^2 h_2)&\Big[1 - 2\frac{1-\eta^2\nu^2\ell h_0}{1-\eta^2\nu^2 h_2}\frac{R}{p_0} + \frac{1}{1-\eta^2\nu^2 h_2}\frac{R^2}{ap_0}\Big] \\ &\cdot \Big[R^2 - 2\eta^2\nu^2 h_0 p_0 R + \eta^2(1-\nu^2 h_1)p_0^2\Big] \end{aligned} \tag{7.3}$$

or, alternatively,

$$\begin{aligned} \frac{\Lambda^2}{C^2}\frac{1}{1-\eta^2\nu^2 h_2}R'^2 = -&\Big[1 - 2\frac{1-\eta^2\nu^2\ell h_0}{1-\eta^2\nu^2 h_2}\frac{R}{p_0} + \frac{1}{1-\eta^2\nu^2 h_2}\frac{R^2}{ap_0}\Big] \\ &\cdot \Big[R^2 - 2\eta^2\nu^2 h_0 p_0 R + \eta^2(1-\nu^2 h_1)p_0^2\Big]. \end{aligned} \tag{7.4}$$

We next introduce the new dependent variable u, by setting

$$u = \frac{1}{R}, \quad R = \frac{1}{u}, \quad R' = -\frac{u'}{u^2} \tag{7.5a,b,c}$$

so that in terms of u, the above equation reads

$$\frac{\Lambda^2}{C^2}\frac{1}{1-\eta^2\nu^2 h_2}u'^2 = -\Big[u^2 - 2\frac{1-\eta^2\nu^2\ell h_0}{1-\eta^2\nu^2 h_2}\frac{u}{p_0} + \frac{1}{1-\eta^2\nu^2 h_2}\frac{1}{ap_0}\Big]$$
$$\cdot\Big[1 - 2\eta^2\nu^2 h_0 p_0 u + \eta^2(1-\nu^2 h_1)p_0^2 u^2\Big]$$
$$= -\Big\{\Big[u - \frac{1-\eta^2\nu^2\ell h_0}{1-\eta^2\nu^2 h_2}\frac{1}{p_0}\Big]^2 - \frac{1}{p_0^2}\Big[\frac{1-\eta^2\nu^2\ell h_0}{1-\eta^2\nu^2 h_2}\Big]^2\Big[1 - \frac{\ell(1-\eta^2\nu^2 h_2)}{(1-\eta^2\nu^2\ell h_0)^2}\Big]\Big\}$$
$$\cdot\Big[1 - 2\eta^2\nu^2 h_0 p_0 u + \eta^2(1-\nu^2 h_1)p_0^2 u^2\Big]. \quad (7.6)$$

We now introduce the modified parameters (recalling (5.9b))

$$p_* = p_0\frac{1-\eta^2\nu^2 h_2}{1-\eta^2\nu^2\ell h_0}, \quad 1-e_*^2 = \frac{\ell(1-\eta^2\nu^2 h_2)}{(1-\eta^2\nu^2\ell h_0)^2} = \frac{1-\eta^2\nu^2 h_2}{(1-\eta^2\nu^2\ell h_0)^2}(1-e_0^2) \quad (7.7a,b)$$

so that the above equation takes the form

$$\frac{\Lambda^2}{C^2}\frac{1}{1-\eta^2\nu^2 h_2}u'^2$$
$$= -\Big[\Big(u - \frac{1}{p_*}\Big)^2 - \frac{e_*^2}{p_*^2}\Big]\Big[1 - 2\eta^2\nu^2 h_0 p_0 u + \eta^2(1-\nu^2 h_1)p_0^2 u^2\Big]. \quad (7.8)$$

Accordingly, we set

$$u - \frac{1}{p_*} = \frac{e_* w}{p_*}, \quad u = \frac{1}{p_*}(1+e_* w), \quad u' = \frac{e_*}{p_*}w' \quad (7.9a,b,c)$$

and, in terms of w, equation (7.8) becomes

$$\frac{\Lambda^2}{C^2}\frac{1}{1-\eta^2\nu^2 h_2}w'^2 =$$
$$(1-w^2)\Big[1 - 2\eta^2\nu^2 h_0\frac{p_0}{p_*}(1+e_* w) + \eta^2(1-\nu^2 h_1)\frac{p_0^2}{p_*^2}(1+e_* w)^2\Big]. \quad (7.10)$$

In the square bracket on the right of (7.10), we now introduce p_0/p_* from (7.7a). For this purpose, it suffices to take the formula merely to first order in η^2 so that we take

$$\frac{p_0}{p_*} = \frac{1-\eta^2\nu^2\ell h_0}{1-\eta^2\nu^2 h_2} = 1 + \eta^2\nu^2(h_2 - \ell h_0),$$
$$\Big(\frac{p_0}{p_*}\Big)^2 = 1 + 2\eta^2\nu^2(h_2 - \ell h_0). \quad (7.11)$$

If we then rearrange the expression within the square brackets of (7.10) as a polynomial in $e_* w$, we have for that expression

$$1 + \eta^2[(1-\nu^2 h_1)[1 + 2\eta^2\nu^2(h_2-\ell h_0)] - 2\nu^2 h_0[1 + \eta^2\nu^2(h_2-\ell h_0)]]$$
$$+ 2\eta^2[(1-\nu^2 h_1)[1 + 2\eta^2\nu^2(h_2-\ell h_0)] - \nu^2 h_0[1 + \eta^2\nu^2(h_2-\ell h_0)]]e_* w$$
$$+ \eta^2(1-\nu^2 h_1)[1 + 2\eta^2\nu^2(h_2-\ell h_0)]]e_*^2 w^2 \quad (7.12a)$$

and at this point it is appropriate to recall relations (7B.26) where we have recorded $(1-\nu^2 h_1)$ to first order in η^2 and $(h_2-\ell h_0)$ to zeroth order in η^2. If we introduce the formulae (7B.26), then the above expression takes the form

$$\begin{aligned} 1+\eta^2\Big[(1-\nu^2)[1+\eta^2\nu^2(12-5\ell)]-2\nu^2[1-\eta^2[2(2-\ell)-\nu^2(12-5\ell)]]\Big] \\ +2\eta^2\Big[(1-\nu^2)[1+\eta^2\nu^2(12-5\ell)]-\nu^2[1-\eta^2[2(2-\ell)-\nu^2(12-5\ell)]]\Big]e_* w \\ +\eta^2(1-\nu^2)[1+\eta^2\nu^2(12-5\ell)]e_*^2 w^2 \end{aligned} \tag{7.12b}$$

and we observe that the constant term and the coefficient of the quadratic term are both positive in the range of interest, while the linear term has a coefficient that assumes both positive and negative values in the range $0 \le \nu \le 1$. Accordingly, we introduce the notation

$$\begin{aligned} j_w^2 &= 1+\eta^2\Big[(1-\nu^2)[1+\eta^2\nu^2(12-5\ell)] \\ &\qquad -2\nu^2[1-\eta^2[2(2-\ell)-\nu^2(12-5\ell)]]\Big] \end{aligned} \tag{7.13a}$$

$$\begin{aligned} h &= \frac{1}{j_w^2}\Big[(1-\nu^2)[1+\eta^2\nu^2(12-5\ell)] \\ &\qquad -\nu^2[1-\eta^2[2(2-\ell)-\nu^2(12-5\ell)]]\Big] \end{aligned} \tag{7.13b}$$

$$q^2 = \frac{1}{j_w^2}[(1-\nu^2)[1+\eta^2\nu^2(12-5\ell)]] \tag{7.13c}$$

and, in particular, we note that when $\nu = 0$, h is positive, while at $\nu = 1$, h is negative for small values of η.

With this notation equation (7.10) may now be written

$$\frac{\Lambda^2}{C^2}\frac{1}{1-\eta^2\nu^2 h_2}\frac{1}{j_w^2}w'^2 = (1-w^2)\Big[1+2\eta^2 h e_* w+\eta^2 q^2 e_*^2 w^2\Big]. \tag{7.14}$$

In order to complete the setting of the differential equation for a straightforward solution, we now make the final adjustments to the quadratic factors on the right of equation (7.14). We first write

$$1-w^2 = J^2[(1-\delta w)^2-(w-\delta)^2] = J^2(1-\delta^2)(1-w^2) \tag{7.15}$$

so that we immediately have

$$J^2(1-\delta^2) = 1. \tag{7.16}$$

A similar decomposition of the second quadratic requires that

$$1+2\eta^2 h e_* w+\eta^2 q^2 e_*^2 w^2 = J^2[A(1-\delta w)^2+B(w-\delta)^2] \tag{7.17}$$

which implies the three relations

$$J^2(A+B\delta^2)=1, \quad J^2\delta(A+B) = -\eta^2 h e_*, \quad J^2(A\delta^2+B) = \eta^2 e_*^2 q^2. \tag{7.18a,b,c}$$

These may be combined, and in conjunction with (7.16) they yield

$$A - B = 1 - \eta^2 e_*^2 q^2, \quad \frac{1+\delta^2}{1-\delta^2}(A+B) = 1 + \eta^2 e_*^2 q^2, \quad \frac{\delta}{1-\delta^2}(A+B) = -\eta^2 e_* h. \tag{7.19a,b,c}$$

Dividing equation (7.19c) by equation (7.19b) yields the equation for δ, namely

$$\frac{\delta}{1+\delta^2} = -\frac{\eta^2 e_* h}{1+\eta^2 e_*^2 q^2}, \tag{7.20}$$

which may be solved as a quadratic in δ. However, for the approximation appropriate for the present problem, we may proceed as follows. Noting that $\delta = O(\eta^2)$, we see that, to second order in η^2, we have

$$\begin{aligned} \delta &= -\frac{\eta^2 e_* h}{1+\eta^2 e_*^2 q^2}(1 + \eta^4 e_*^2 h^2) \\ &= -\eta^2 e_* h[1 - \eta^2 e_*^2 q^2 + \eta^4 e_*^2 (e_*^2 q^4 + h^2)] \end{aligned} \tag{7.21}$$

and hence

$$\delta^2 = \eta^4 e_*^2 h^2 [1 - 2\eta^2 e_*^2 q^2 + \eta^4 e_*^2 (3 e_*^2 q^4 + 2h^2)] \tag{7.22}$$

so that, to third order in η^2, we have

$$\frac{1-\delta^2}{1+\delta^2} = \frac{1 - \eta^4 e_*^2 h^2 (1 - 2\eta^2 e_*^2 q^2)}{1 + \eta^4 e_*^2 h^2 (1 - 2\eta^2 e_*^2 q^2)} = 1 - 2\eta^4 e_*^2 h^2 (1 - 2\eta^2 e_*^2 q^2). \tag{7.23}$$

From (7.19b) there follows that, to third order in η^2,

$$\begin{aligned} A + B &= (1 + \eta^2 e_*^2 q^2)[1 - 2\eta^4 e_*^2 h^2 (1 - 2\eta^2 e_*^2 q^2)] \\ &= 1 + \eta^2 e_*^2 q^2 - 2\eta^4 e_*^2 h^2 + 2\eta^6 e_*^4 q^2 h^2 \end{aligned} \tag{7.24}$$

— the reason for going to third order in η^2 will be evident in the calculation of the B-factor: the combination of (7.19a) with (7.24) yields for the respective factors

$$A = 1 - \eta^4 e_*^2 h^2, \quad B = \eta^2 e_*^2 [q^2 - \eta^2 h^2 + \eta^4 e_*^2 q^2 h^2], \tag{7.25a,b}$$

both valid to second order in η^2.

The introduction of (7.15) and (7.17) into equation (7.14) yields

$$\begin{aligned} &\frac{\Lambda^2}{C^2(1 - \eta^2 \nu^2 h_2)} \frac{1}{j_w^2} w'^2 = \\ &\qquad J^4 [(1 - \delta w)^2 - (w - \delta)^2][A(1 - \delta w)^2 + B(w - \delta)^2] \end{aligned} \tag{7.26}$$

which, when we divide across by $J^4(1 - \delta w)^4$, becomes

$$\frac{\Lambda^2}{C^2(1-\eta^2\nu^2 h_2)}\frac{1}{j_w^2}\Big[\frac{w'}{J^2(1-\delta w)^2}\Big]^2 = \Big[1-\Big(\frac{w-\delta}{1-\delta w}\Big)^2\Big]\Big[A+B\Big(\frac{w-\delta}{1-\delta w}\Big)^2\Big]. \tag{7.27}$$

This latter form leads to the final transformation

$$v = \frac{w-\delta}{1-\delta w}, \quad w = \frac{v+\delta}{1+\delta v} \tag{7.28a,b}$$

and differentiation of (7.28a) yields

$$v' = \frac{(1-\delta^2)w'}{(1-\delta w)^2} = \frac{w'}{J^2(1-\delta w)^2} \tag{7.28c}$$

so that equation (7.27) becomes in terms of v

$$\frac{\Lambda^2}{C^2(1-\eta^2\nu^2 h_2)}\frac{1}{j_w^2}v'^2 = (1-v^2)(A+Bv^2) \tag{7.29a}$$

$$= (1-v^2)[(A+B)-B(1-v^2)]$$

$$= (A+B)(1-v^2)\Big[1-\frac{B}{A+B}(1-v^2)\Big]. \tag{7.29b}$$

(**) The procedure for dealing with equation (7.29) will depend on the sign of B. Referring to formula (7.25b) for B, and recalling relations (7.13) expressing the parameters q, h, and j_w in terms of ν, it is clear that, except for a narrow band near the equator ($\nu = 1$) where $1-\nu^2 = O(\eta^2)$, B is clearly positive. Dealing with orbits in the narrow equatorial band we defer to Chapter 6, where we focus on some special orbits. For the remainder of this chapter, we confine our attention to the range of orbits where

$$1-\nu^2 > O(\eta^2) \text{ so that } q^2-\eta^2h^2+\eta^4e_*^2q^2h^2 \geq 0 \tag{***}$$

ensuring that B is positive.

Consistent with this exclusion of the orbits in the equatorial band, we refer to equation (7.29b) and divide across by $A+B$, to obtain

$$\frac{\Lambda^2}{C^2(1-\eta^2\nu^2 h_2)}\frac{1}{j_w^2}\frac{1}{(A+B)}v'^2 = (1-v^2)[1-k_1^2(1-v^2)] \tag{7.30}$$

wherein we have written, to second order in η^2

$$k_1^2 = \frac{B}{A+B} = \eta^2 e_*^2\frac{q^2-\eta^2h^2+\eta^4e_*^2q^2h^2}{1+\eta^2e_*^2q^2-2\eta^4e_*^2h^2}. \tag{7.31}$$

If we recall the definition of Λ^2 from (6.14), we see that when we substitute for $(A+B)$ from (7.24), the coefficient of v'^2 in (7.30) may be written

$$\frac{1}{j_1^2} = \frac{(1+\eta^2\ell)m}{(1-\eta^2\nu^2h_2)[1+\eta^2e_*^2(q^2-2\eta^2h^2)]}\frac{1}{j_w^2} \tag{7.32}$$

in which m is given by (6.6) and j_w^2 by (7.13a). Equation (7.30) for v now takes the form

$$\left[\frac{\mathrm{d}v}{\mathrm{d}(j_1 f)}\right]^2 = \frac{1}{j_1^2}v'^2 = (1-v^2)[1-k_1^2(1-v^2)] \tag{7.33}$$

with solution

$$v = \mathrm{cn}[j_1(f+\omega_1): k_1] \tag{7.34}$$

where ω_1 is the constant of integration. If we define "perigee" as the point at which $v' = 0$ and choose the origin of the angle variable f at "perigee", then clearly

$$\omega_1 = 0 \tag{7.35}$$

and the solution (7.34) reads

$$v = \mathrm{cn}[j_1 f: k_1] \tag{7.36}$$

to be used below.

By combining (7.5b), (7.9b), and (7.28b), we obtain for the R-coordinate

$$R = \frac{1}{u} = \frac{p_*}{1+e_*w} = \frac{p_*(1+\delta v)}{(1+\delta e_*)+(e_*+\delta)v} \tag{7.37}$$

which, on the introduction of the solution form (7.3b) for v, renders the solution for R in the form

$$R = p_*\frac{1+\delta\,\mathrm{cn}[j_1 f: k_1]}{(1+\delta e_*)+(e_*+\delta)\,\mathrm{cn}[j_1 f: k_1]}. \tag{7.38}$$

We may now make a further adjustment to the parameters to put the solution in a somewhat neater form: we set

$$p = \frac{p_*}{1+\delta e_*}, \quad e = \frac{e_*+\delta}{1+\delta e_*} \tag{7.39a,b}$$

so that

$$e_* = \frac{e-\delta}{1-\delta e} \tag{7.40}$$

and the solution (7.38) takes the final form

$$R = p\frac{1+\delta\,\mathrm{cn}[j_1 f: k_1]}{1+e\,\mathrm{cn}[j_1 f: k_1]} \tag{7.41}$$

which takes the familiar Kepler form in the limit as $\eta^2 \to 0$, $\delta \to 0$, $k_1 \to 0$, $j_1 \to 1$. We also note that for the reciprocal auxiliary variable u, we have

$$pu = \frac{1 + e\,\mathrm{cn}[j_1 f : k_1]}{1 + \delta\,\mathrm{cn}[j_1 f : k_1]} \tag{7.42}$$

which will have repeated use in later sections.

We now revisit relations (7.21) and (7.31) for the parameters δ and k_1. In regard to (7.21), if we set

$$d_0 = h[1 - \eta^2 e_*^2 q^2 + \eta^4 e_*^2 (e_*^2 q^4 + h^2)] \tag{7.43}$$

then (7.21) takes the form

$$\delta = -\eta^2 e_* d_0. \tag{7.44}$$

Introducing this form for δ into (7.39b), we have

$$e = \frac{e_* - \eta^2 d_0 e_*}{1 - \eta^2 d_0 e_*^2} = e_* \frac{1 - \eta^2 d_0}{1 - \eta^2 d_0 e_*^2} \tag{7.45a}$$

and hence, substituting for d_0 from (7.43), we have, to second order in η^2

$$\begin{aligned} \frac{e_*}{e} = \frac{1 - \eta^2 d_0 e_*^2}{1 - \eta^2 d_0} &= 1 + \eta^2 d_0 (1 + \eta^2 d_0)(1 - e_*^2) \\ &= 1 + \eta^2 h[1 + \eta^2 (h - q^2 e_*^2)](1 - e_*^2). \end{aligned} \tag{7.45b}$$

If we write

$$\begin{aligned} &d = \frac{e_*}{e} d_0 = \\ &h[1 + \eta^2 h[1 + \eta^2 (h - q^2 e_*^2)](1 - e_*^2)][1 - \eta^2 e_*^2 q^2 + \eta^4 e_*^2 (q^2 e_*^4 + h^2)] \end{aligned} \tag{7.46}$$

then we may recast (7.44) in the form

$$\delta = -\eta^2 e d \tag{7.47}$$

which will be the form used in later sections.

Further, on squaring (7.45b), then, to second order in η^2 we have

$$\left(\frac{e_*}{e}\right)^2 = 1 + 2\eta^2 h(1 - e_*^2) + \eta^4 h(1 - e_*^2)[h(3 - e_*^2) - 2q^2 e_*^2]. \tag{7.48}$$

Referring now to formula (7.31) for k_1^2, with $(\frac{e_*}{e})$ as given by (7.48) above, we set

$$g^2 = \left(\frac{e_*}{e}\right)^2 \cdot \frac{q^2 - \eta^2 h^2 + \eta^4 e_*^2 q^2 h^2}{1 + \eta^2 e_*^2 q^2 - 2\eta^4 e_*^2 h^2} \tag{7.49}$$

so that formula (7.31) for k_1^2 may be written

$$k_1^2 = \eta^2 e^2 g^2 \tag{7.50}$$

which will be the form used in later sections.

8 The Integration of the φ-Coordinate

From relation (3.6), we have the differential equation for φ in the form

$$\dot{\varphi} = \frac{C_3}{(R^2 + b^2)\sin^2\sigma} \tag{8.1}$$

and if we introduce the regularizing variable f, defined in (5.5) and (5.6) as the independent variable, we have

$$\varphi' = \frac{C_3}{\Lambda}\frac{R^2 + b^2\cos^2\sigma}{(R^2 + b^2)\sin^2\sigma} = \frac{C_3}{\Lambda}\left[\frac{1}{\sin^2\sigma} - \frac{b^2}{R^2 + b^2}\right]. \tag{8.2}$$

When expressed in terms of the variable $u = 1/R$, equation (8.2) becomes

$$\varphi' = \frac{C_3}{\Lambda}\left[\frac{1}{\sin^2\sigma} - \frac{b^2u^2}{1 + b^2u^2}\right] = \frac{C_3}{\Lambda}\left[\frac{1}{\sin^2\sigma} - \frac{b^2}{p^2}\frac{(pu)^2}{1 + (\frac{b}{p})^2(pu)^2}\right]. \tag{8.3}$$

From (7.11a), taken together with (7B.25), we obtain, up to the terms of order η^2

$$\begin{aligned}\frac{p_0}{p_*} &= 1 + \eta^2\nu^2(h_2 - \ell h_0) = 1 + \eta^2\nu^2(4 - 2\ell)\\ &= 1 + 2\eta^2\nu^2(1 + e_0^2) = 1 + 2\eta^2\nu^2(1 + e^2)\end{aligned} \tag{8.4a}$$

wherein we have noted from (7.7b) and (7.39b) that

$$e_0 = e_* + O(\eta^2) = e + O(\eta^2). \tag{8.4b}$$

Furthermore, from (7.39a), we have, up to terms of η^2,

$$\frac{p_*}{p} = 1 + \delta e_* = 1 + \delta e = 1 - \eta^2 e^2(1 - 2\nu^2) \tag{8.5a}$$

wherein we have noted from (7.21) and (7.13) that, except for terms of order η^4,

$$\delta = -\eta^2 e_* h = -\eta^2 e(1 - 2\nu^2). \tag{8.5b}$$

Combining (8.4a) with (8.5a), we have (as defining relation for j_p), and again up to terms of order η^2,

$$\begin{aligned}j_p = \frac{p_0}{p} &= [1 + 2\eta^2\nu^2(1 + e^2)][1 - \eta^2 e^2(1 - 2\nu^2)]\\ &= 1 + \eta^2[2\nu^2 + e^2(4\nu^2 - 1)].\end{aligned} \tag{8.6}$$

On squaring, we have, up to terms of order η^2,

$$j_p^2 = \left(\frac{p_0}{p}\right)^2 = 1 + 2\eta^2[2\nu^2 + e^2(4\nu^2 - 1)] \tag{8.7a}$$

and it follows that

$$\frac{b^2}{p^2} = \left(\frac{b}{p_0}\right)^2\left(\frac{p_0}{p}\right)^2 = \eta^2 j_p^2 \tag{8.7b}$$

and we may write (8.3) in the form

$$\varphi' = \frac{C_3}{\Lambda}\left[\frac{1}{\sin^2\sigma} - \eta^2 j_p^2 \frac{(pu)^2}{1+\eta^2 j_p^2 (pu)^2}\right] \tag{8.8}$$

and hence, up to terms of order η^4,

$$\varphi' = \frac{C_3}{\Lambda}\left[\frac{1}{\sin^2\sigma} - \eta^2 j_p^2 (pu)^2[1-\eta^2(pu)^2]\right]. \tag{8.9}$$

It will prove convenient for the integration to have the factor N, as defined by (6.20), multiplying the factor $1/\sin^2\sigma$ on the right side. Accordingly, we rewrite (8.9) in the alternate form

$$\frac{\Lambda}{C_3}N\varphi' = \frac{N}{\sin^2\sigma} - \eta^2 j_p^2 N(pu)^2[1-\eta^2(pu)^2] \tag{8.10}$$

and introducing Λ from (6.14), and N from (6.20), we see that

$$\begin{aligned}\frac{\Lambda^2}{C_3^2}N^2 &= (1+\eta^2\ell)m\frac{C^2}{C_3^2}\left[1-\frac{1-\nu^2}{(1+\eta^2\ell)m}\right]\\ &= \frac{(1+\eta^2\ell)m-(1-\nu^2)}{\nu^2} = \frac{\nu^2+(1+\eta^2\ell)m-1}{\nu^2} = 1+O(\eta^2).\end{aligned} \tag{8.11}$$

Hence, if we set

$$j_N = \frac{[\nu^2+(1+\eta^2\ell)m-1]^{1/2}}{\nu} \tag{8.12a}$$

then using (6.6) we can show that

$$j_N = 1+\tfrac{1}{2}\eta^2\ell[1+\tfrac{1}{4}\eta^2\ell(3-4\nu^2)]; \tag{8.12b}$$

and equation (8.10) may be written

$$j_N\varphi' = \frac{N}{\sin^2\sigma} - \eta^2 j_p^2 N(pu)^2[1-\eta^2(pu)^2]. \tag{8.13}$$

Setting

$$\varphi_1' = \frac{N}{\sin^2\sigma}, \quad \varphi_2' = (pu)^2 - \eta^2(pu)^4 \tag{8.14a,b}$$

we have, in terms of φ_1 and φ_2

$$j_N\varphi' = \varphi_1' - \eta^2 j_p^2 N\varphi_2' \tag{8.15}$$

the components of which will be integrated separately: φ_1 is to be calculated to order η^4 whereas for φ_2 accuracy to order η^2 will suffice. We deal with φ_2 in Subsection 8A and with φ_1 in Subsection 8B.

8A. Determination of φ_2

From the solution form (7.42), there follows that up to terms of order η^2, we may write

$$
\begin{aligned}
pu &= [1 + e\,\mathrm{cn}[j_1 f : k_1]][[1 - \delta\,\mathrm{cn}[j_1 f : k_1]] \\
&= 1 + (e - \delta)\,\mathrm{cn}[j_1 f : k_1] - e\delta\,\mathrm{cn}^2[j_1 f : k_1]
\end{aligned} \tag{8A.1}
$$

and as these are the only elliptic functions that appear, in the remainder of this subsection we shall no longer exhibit the argument $j_1 f$ or the modulus k_1 except where necessary. If we now recall the approximate formula (8.5b) for δ, we see that (8A.1) may be rewritten

$$pu = 1 + e[1 + \eta^2(1 - 2\nu^2)]\,\mathrm{cn} + \eta^2 e^2 (1 - 2\nu^2)\,\mathrm{cn}^2 . \tag{8A.2}$$

There follows, from squaring relation (8A.2) and retaining terms up to order η^2, that we have

$$
\begin{aligned}
(pu)^2 = 1 &+ 2e[1 + \eta^2(1 - 2\nu^2)]\,\mathrm{cn} \\
&+ e^2[1 + 4\eta^2(1 - 2\nu^2)]\,\mathrm{cn}^2 + 2\eta^2 e^3(1 - 2\nu^2)\,\mathrm{cn}^3
\end{aligned} \tag{8A.3}
$$

and by squaring relation (8A.3), we obtain the formula for $(pu)^4$, which it suffices to calculate to zeroth order in η^2. Accordingly, we may write

$$
\begin{aligned}
(pu)^4 &= (1 + 2e\,\mathrm{cn} + e^2\,\mathrm{cn}^2)^2 \\
&= 1 + 4e\,\mathrm{cn} + 6e^2\,\mathrm{cn}^2 + 4e^3\,\mathrm{cn}^3 + e^4\,\mathrm{cn}^4 .
\end{aligned} \tag{8A.4}
$$

We now combine (8A.3) and (8A.4) in accordance with (8.14b) to obtain

$$
\begin{aligned}
\varphi_2' &= (pu)^2 - \eta^2 (pu)^4 \\
&= (1 - \eta^2) + 2[1 - \eta^2(1 + 2\nu^2)]e\,\mathrm{cn} + [1 - 2\eta^2(1 + 4\nu^2)]e^2\,\mathrm{cn}^2 \\
&\quad - 2\eta^2(1 + 2\nu^2)e^3\,\mathrm{cn}^3 - \eta^2 e^4\,\mathrm{cn}^4 .
\end{aligned} \tag{8A.5}
$$

At this point, it is covenient to record a number of approximate relations for the Jacobian elliptic functions. With modulus k and with the argument indicated by the (arbitrary) variable F, then

$$\mathrm{dn} = 1 - \tfrac{1}{2}k^2\,\mathrm{sn}^2 + O(k^4), \qquad 1 - \mathrm{dn} = \tfrac{1}{2}k^2\,\mathrm{sn}^2 + O(k^4), \tag{8A.6a,b}$$

$$1 + \mathrm{dn} = 2[1 - \tfrac{1}{4}k^2\,\mathrm{sn}^2] + O(k^4), \tag{8A.6c}$$

$$\frac{1}{1 + \mathrm{dn}} = \tfrac{1}{2}[1 + \tfrac{1}{4}k^2\,\mathrm{sn}^2] + O(k^4) = \tfrac{1}{2}[\mathrm{dn} + \tfrac{3}{4}k^2\,\mathrm{sn}^2] + O(k^4). \tag{8A.6d}$$

For the Fourier series approximations, we first note the series for the quarter-period K, given in terms of the modulus k, by the relation

$$\frac{\pi}{2K} = 1 - \tfrac{1}{4}k^2 - \tfrac{5}{64}k^4 + O(k^6) \tag{8A.7}$$

in terms of which is defined the modified variable G related to the fundamental variable F; to second order in k^2, this relation is expressed as

$$G = \frac{\pi}{2K}F = [1 - \tfrac{1}{4}k^2 - \tfrac{5}{64}k^4]F. \tag{8A.8}$$

In terms of this variable G, the Fourier series approximations take the form

$$\operatorname{sn} F = \sin G + O(k^2), \quad \operatorname{cn} F = \cos G + O(k^2), \quad \operatorname{dn} F = 1 + O(k^2) \tag{8A.9a,b,c}$$

and we also note the double-angle formula for cn^2

$$\begin{aligned}\operatorname{cn}^2 F &= \frac{\operatorname{dn} 2F + \operatorname{cn} 2F}{1 + \operatorname{dn} 2F}\\ &= \tfrac{1}{2}[\operatorname{dn} 2F + \tfrac{3}{4}k^2 \operatorname{sn}^2 2F][\operatorname{dn} 2F + \operatorname{cn} 2F] + O(k^4)\end{aligned} \tag{8A.10}$$

where, in the latter we have utilized relation (8A.6d), valid to the first order in k^2; and we recall from (7.31) that $k_1^2 = O(\eta^2)$.

We are now in a position to deal with the terms on the right of relation (8A.5) for φ_2'. We first write

$$F_1 = j_1 f, \quad G_1 = \frac{\pi}{2K_1}F_1 = [1 - \tfrac{1}{4}k_1^2 - \tfrac{5}{64}k_1^4 + O(\eta^6)]F_1. \tag{8A.11a,b}$$

If we note relations (7.31), (7.13), (7.49), and (7B.25), we see that, except for terms of order η^4,

$$k_1^2 = \eta^2 e^2 g^2 = \eta^2 e^2 (1 - \nu^2) \tag{8A.12}$$

and hence, if we apply (8A.6b), it follows that, up to terms of the first order in η^2, we have

$$\begin{aligned}\operatorname{cn} F_1 &= \operatorname{cn} F_1 \operatorname{dn} F_1 + (1 - \operatorname{dn} F_1)\operatorname{cn} F_1 = \operatorname{cn} F_1 \operatorname{dn} F_1 + \tfrac{1}{2}k_1^2 \operatorname{sn}^2 F_1 \operatorname{cn} F_1\\ &= \operatorname{cn} F_1 \operatorname{dn} F_1 + \tfrac{1}{2}\eta^2 e^2 (1 - \nu^2)\sin^2 G_1 \cos G_1\end{aligned} \tag{8A.13}$$

for the term with cn in (8A.5). For the term with cn^2, we note from (8A.10) that (omitting terms of $O(\eta^4)$),

$$\begin{aligned}\operatorname{cn}^2 F_1 &= \frac{\operatorname{dn} 2F_1 + \operatorname{cn} 2F_1}{1 + \operatorname{dn} 2F_1} = \tfrac{1}{2}[\operatorname{dn} 2F_1 + \tfrac{3}{4}k_1^2 \operatorname{sn}^2 2F_1][\operatorname{dn} 2F_1 + \operatorname{cn} 2F_1]\\ &= \tfrac{1}{2}[\operatorname{dn}^2 2F_1 + \operatorname{cn} 2F_1 \operatorname{dn} 2F_1 + \tfrac{3}{4}k_1^2 \operatorname{sn}^2 2F_1(\operatorname{dn} 2F_1 + \operatorname{cn} 2F_1)]\\ &= \tfrac{1}{2}[1 + \operatorname{cn} 2F_1 \operatorname{dn} 2F_1 - k_1^2 \operatorname{sn}^2 2F_1(1 - \tfrac{3}{4}[\operatorname{dn} 2F_1 + \operatorname{cn} 2F_1])]\\ &= \tfrac{1}{2}[1 + \operatorname{cn} 2F_1 \operatorname{dn} 2F_1 - k_1^2 \operatorname{sn}^2 2F_1(1 - \tfrac{3}{4}[1 + \cos 2G_1])]\\ &= \tfrac{1}{2}[1 + \operatorname{cn} 2F_1 \operatorname{dn} 2F_1 - \tfrac{1}{8}k_1^2(1 - \cos 4G_1) + \tfrac{3}{4}k_1^2 \sin^2 2G_1 \cos 2G_1]\\ &= \tfrac{1}{2}[(1 - \tfrac{1}{8}k_1^2) + \operatorname{cn} 2F_1 \operatorname{dn} 2F_1 + \tfrac{1}{8}k_1^2 \cos 4G_1 + \tfrac{3}{4}k_1^2 \sin^2 2G_1 \cos 2G_1]\\ &= \tfrac{1}{2}\Big[(1 - \tfrac{1}{8}\eta^2 e^2(1 - \nu^2)) + \operatorname{cn} 2F_1 \operatorname{dn} 2F_1\\ &\qquad + \tfrac{1}{8}\eta^2 e^2 (1 - \nu^2)[\cos 4G_1 + 6\sin^2 2G_1 \cos 2G_1]\Big].\end{aligned} \tag{8A.14}$$

For the terms with cn^3 and cn^4 in (8A.5), we note that except for terms of order η^2

$$\mathrm{cn}^3 F_1 = \cos^3 G_1 = \tfrac{1}{4}[3\cos G_1 + \cos 3G_1] \tag{8A.15a}$$

$$\mathrm{cn}^4 F_1 = \cos^4 G_1 = \tfrac{1}{8}[3 + 4\cos 2G_1 + \cos 4G_1]. \tag{8A.15b}$$

It remains to insert relations (8A.13)–(8A.15) into (8A.5).

We may list the terms consecutively, omitting terms of order η^4, and apart from the leading term, we have as follows:

$$\begin{aligned} 2[1-\eta^2(1+2\nu^2)]e\,\mathrm{cn}\,F_1 &= 2e[1-\eta^2(1+2\nu^2)]\,\mathrm{cn}\,F_1\,\mathrm{dn}\,F_1 \\ &\quad + \eta^2 e^3(1-\nu^2)\sin^2 G_1\cos G_1 \end{aligned} \tag{8A.16a}$$

$$\begin{aligned} [1-2\eta^2(1+4\nu^2)]e^2\,\mathrm{cn}^2 F_1 &= \tfrac{1}{2}e^2[1-2\eta^2(1+4\nu^2) - \tfrac{1}{8}\eta^2 e^2(1-\nu^2)] \\ &\quad + \tfrac{1}{2}e^2[1-2\eta^2(1+4\nu^2)]\,\mathrm{cn}\,2F_1\,\mathrm{dn}\,2F_1 \\ &\quad + \tfrac{1}{16}\eta^2 e^4(1-\nu^2)[\cos 4G_1 + 6\sin^2 2G_1\cos 2G_1] \end{aligned} \tag{8A.16b}$$

$$\begin{aligned} 2\eta^2(1+2\nu^2)&e^3\,\mathrm{cn}^3 F_1 + \eta^2 e^4\,\mathrm{cn}^4 F_1 \\ &= \eta^2 e^3\Big[\tfrac{1}{2}(1+2\nu^2)[3\cos G_1 + \cos 3G_1] \\ &\quad + \tfrac{1}{8}e[3 + 4\cos 2G_1 + \cos 4G_1]\Big]. \end{aligned} \tag{8A.16c}$$

By the appropriate combination of the factors (8A.16) in (8A.5) we obtain, after some rearrangement,

$$\begin{aligned} \varphi_2' = &\Big[(1+\tfrac{1}{2}e^2) - \eta^2[1 + e^2(1+4\nu^2) + \tfrac{1}{16}e^4(7-e^2)]\Big] \\ &+ 2e[1-\eta^2(1+2\nu^2)]\,\mathrm{cn}\,F_1\,\mathrm{dn}\,F_1 + \eta^2 e^3(1-\nu^2)\sin^2 G_1\cos G_1 \\ &+ \tfrac{1}{2}e^2[1-2\eta^2(1+4\nu^2)]\,\mathrm{cn}\,2F_1\,\mathrm{dn}\,2F_1 + \tfrac{3}{8}\eta^2 e^4\sin^2 2G_1\cos 2G_1 \\ &- \eta^2 e^3\Big[\tfrac{1}{2}(1+2\nu^2)[3\cos G_1 + \cos 3G_1] \\ &+ \tfrac{1}{2}e[\cos 2G_1 + \tfrac{1}{8}(1+\nu^2)\cos 4G_1]\Big]. \end{aligned} \tag{8A.17}$$

If we recall (8A.11), we have that $\mathrm{d}F_1/\mathrm{d}f = j_1$ and $\mathrm{d}G_1/\mathrm{d}F_1 = 1 + O(\eta^2)$, and hence, up to terms of order η^2, we have

$$\begin{aligned} \varphi_2' = j_1\frac{\mathrm{d}\varphi_2}{\mathrm{d}F_1} = \frac{\mathrm{d}}{\mathrm{d}F_1}\Big\{&\Big[(1+\tfrac{1}{2}e^2) - \eta^2[1 + e^2(1+4\nu^2) + \tfrac{1}{16}e^4(7-\nu^2)]\Big]F_1 \\ &+ 2e[1-\eta^2(1+2\nu^2)]\,\mathrm{sn}\,F_1 + \tfrac{1}{3}\eta^2 e^3(1-\nu^2)\sin^3 G_1 \\ &+ \tfrac{1}{4}e^2[1-2\eta^2(1+4\nu^2)]\,\mathrm{sn}\,2F_1 + \tfrac{1}{16}\eta^2 e^4(1-\nu^2)\sin^3 2G_1 \\ &- \eta^2 e^3\Big[\tfrac{1}{2}(1+2\nu^2)[3\sin G_1 + \tfrac{1}{3}\sin 3G_1] \\ &+ \tfrac{1}{4}e[\sin 2G_1 + \tfrac{1}{16}(1+\nu^2)\sin 4G_1]\Big]\Big\}. \end{aligned} \tag{8A.18}$$

If we utilize the relation

$$\sin^3 G_1 = \tfrac{1}{4}(3\sin G_1 - \sin 3G_1) \tag{8A.19}$$

and rearrange, then the execution of the integration of (8A.18) yields (apart from an additive constant)

$$\begin{aligned}
j_1\varphi_2 = \Big[&(1+\tfrac{1}{2}e^2) - \eta^2[1+e^2(1+4\nu^2)+\tfrac{1}{16}e^4(7-\nu^2)]\Big]F_1 \\
&+ 2e[1-\eta^2(1+2\nu^2)]\,\mathrm{sn}\,F_1 + \tfrac{1}{4}e^2[1-2\eta^2(1+4\nu^2)]\,\mathrm{sn}\,2F_1 \\
&- \tfrac{1}{4}\eta^2e^3[(5+13\nu^2)\sin G_1 + (1+\nu^2)\sin 3G_1] \\
&- \tfrac{1}{64}\eta^2e^4\Big[(13+3\nu^2)\sin 2G_1 \\
&+ (1+\nu^2)\sin 4G_1 + (1-\nu^2)\sin 6G_1\Big] \qquad (8A.20)
\end{aligned}$$

giving the formula for φ_2 if we divide across by j_1. In order to determine j_1 to the necessary accuracy, we first note from (6.6) that, except for terms of order η^4, we may write

$$m = 1-\eta^2\ell(1-\nu^2), \quad (1+\eta^2\ell)m = 1+\eta^2\ell\nu^2,$$
$$\frac{1}{(1+\eta^2\ell)m} = 1-\eta^2\ell\nu^2. \qquad (8A.21a,b,c)$$

From (7B.25), neglecting terms of order η^2, we have

$$h_0 = 1, \quad h_1 = 1, \quad h_2 = 4-\ell = 3+e^2 \qquad (8A.22a,b,c)$$

where in the latter, we have noted from (7.7b) and (7.39b) that to zeroth order in η^2 we may take

$$e_* = e, \quad \ell = 1-e_*^2 = 1-e^2. \qquad (8A.22d,e)$$

Introducing (8A.22) into relations (7.13) we have, to first order in η^2, that

$$j_w^2 = 1+\eta^2(1-3\nu^2) \qquad (8A.23a)$$

and from (7.49) and (7.13) we have, to zeroth order in η^2, that

$$g^2 = 1-\nu^2, \quad h = 1-2\nu^2. \qquad (8A.23b,c)$$

If we now utilize the approximate formulae (8A.22) and (8A.23) in relation (7.32), then on expansion and rearrangement, we obtain for j_1^2 the formula valid to first order in η^2,

$$j_1^2 = 1+\eta^2[(1+e^2)-\nu^2(7+e^2)] \qquad (8A.24a)$$

and hence, also to first order in η^2,

$$j_1 = 1+\tfrac{1}{2}\eta^2[(1+e^2)-\nu^2(7+e^2)] \qquad (8A.24b)$$
$$\frac{1}{j_1} = 1-\tfrac{1}{2}\eta^2[(1+e^2)-\nu^2(7+e^2)]. \qquad (8A.24c)$$

Then if we divide across by j_1 in relation (8.20), and utilizing (8A.24c) we rearrange, then up to first order in η^2 we have (again apart from the additive constant)

$$\begin{aligned}\varphi_2 = \Big[(1+\tfrac{1}{2}e^2) - \eta^2[1+e^2(1+4\nu^2)+\tfrac{1}{16}e^4(7-\nu^2)]\Big](f+\omega)\\ + 2e[1-\tfrac{1}{2}\eta^2(3+e^2)(1-\nu^2)]\,\mathrm{sn}\,F_1\\ + \tfrac{1}{4}e^2[1-\tfrac{1}{2}\eta^2[(5+e^2)+\nu^2(9-e^2)]]\,\mathrm{sn}\,2F_1\\ - \tfrac{1}{4}\eta^2 e^3[(5+13\nu^2)\sin G_1 + (1+\nu^2)\sin 3G_1]\\ - \tfrac{1}{64}\eta^2 e^4\Big[(13+3\nu^2)\sin 2G_1\\ + (1+\nu^2)\sin 4G_1 + (1-\nu^2)\sin 6G_1\Big].\end{aligned} \tag{8A.25}$$

It will prove convenient to have the notation

$$a_0 = (1+\tfrac{1}{2}e^2) - \eta^2[1+e^2(1+4\nu^2)+\tfrac{1}{16}e^4(7-\nu^2)] \tag{8A.26a}$$

$$a_1 = 2e[1-\tfrac{1}{2}\eta^2(3+e^2)(1-\nu^2)] \tag{8A.26b}$$

$$a_2 = \tfrac{1}{4}e^2[1-\tfrac{1}{2}\eta^2[(5+e^2)+\nu^2(9-e^2)]] \tag{8A.26c}$$

and

$$b_1 = \tfrac{1}{4}e^3(5+13\nu^2), \quad b_2 = \tfrac{1}{64}\eta^2 e^4(13+3\nu^2), \tag{8A.27a,b}$$

$$b_3 = \tfrac{1}{4}e^3(1+\nu^2), \quad b_4 = \tfrac{1}{64}e^4(1+\nu^2), \tag{8A.27c,d}$$

$$b_5 = 0, \qquad b_6 = \tfrac{1}{64}e^4(1-\nu^2) \tag{8A.27e,f}$$

so that relation (8A.25) can be written in the form

$$\varphi_2 = a_0(f+\omega) + \sum_{n=1}^{2} a_n\,\mathrm{sn}\,nF_1 - \eta^2\sum_{n=1}^{6} b_n \sin nG_1 \tag{8A.28}$$

which will be referred to later as the form for φ_2.

8B. Determination of φ_1

Turning to equation (8.14a) for φ_1, we introduce the solution form (6.21) as follows:

$$\begin{aligned}\varphi_1' &= \frac{N}{\sin^2\sigma} = \frac{N}{1-\cos^2\sigma} = \frac{N}{1-(1-N^2)\,\mathrm{sn}^2[f+\omega : k_2]}\\ &= \frac{N}{\mathrm{cn}^2[f+\omega : k_2] + N^2\,\mathrm{sn}^2[f+\omega : k_2]} = \frac{N\,\mathrm{nc}^2[f+\omega : k_2]}{1+N^2\,\mathrm{sc}^2[f+\omega : k_2]}\end{aligned} \tag{8B.1}$$

and as the elliptic functions in this equation for φ_1', all have argument $f+\omega$ and modulus k_2, these shall not be exhibited in this subsection except at the end of the analysis. We now set

$$\varphi_0' = \frac{N\,\mathrm{nc}^2\,\mathrm{dn}}{1+N^2\,\mathrm{sc}^2} \tag{8B.2}$$

which is immediately integrable as

$$\varphi_0 = \arctan\left[N\,\mathrm{sc}[f+\omega : k_2]\right] \tag{8B.3}$$

and we may write (8B.1) as

$$\varphi_1' = \varphi_0' + (1-\mathrm{dn})\varphi_1' = \varphi_0' + \frac{1-\mathrm{dn}^2}{1+\mathrm{dn}}\varphi_1' = \varphi_0' + \frac{k_2^2\,\mathrm{sn}^2}{1+\mathrm{dn}}\varphi_1' \tag{8B.4}$$

and if we apply formula (8A.6d) to the latter, we have

$$\varphi_1' = \varphi_0' + \tfrac{1}{2}k_2^2\,\mathrm{sn}^2\left[1+\tfrac{1}{4}k_2^2\,\mathrm{sn}^2\right]\varphi_1' \tag{8B.5}$$

valid to order k_2^4.

By combining relations (6.15) and (6.20), we see that

$$\frac{k_2^2}{1-N^2} = \frac{\eta^2\ell}{(1+\eta^2\ell)^2m^2}(1-\nu^2)\frac{(1+\eta^2\ell)m}{1-\nu^2} = \frac{\eta^2\ell}{(1+\eta^2\ell)m} = \eta^2\ell_0 \tag{8B.6a}$$

where we define the quantity ℓ_0 by the relation

$$\ell_0 = \frac{\ell}{(1+\eta^2\ell)m} \tag{8B.6b}$$

and we note in particular the cancellation of the $(1-\nu^2)$ factor. From (8B.1) we now observe that

$$\varphi_1'\,\mathrm{sn}^2 = \frac{N\,\mathrm{sc}^2}{1+N^2\,\mathrm{sc}^2} = \frac{N}{1-N^2}\left[\frac{\mathrm{nc}^2}{1+N^2\,\mathrm{sc}^2}-1\right] = \frac{1}{1-N^2}(\varphi_1'-N) \tag{8B.7}$$

and furthermore

$$\begin{aligned}\varphi_1'\,\mathrm{sn}^4 &= \mathrm{sn}^2\frac{1}{1-N^2}(\varphi_1'-N) = \frac{1}{1-N^2}[\varphi_1'\,\mathrm{sn}^2 - N\,\mathrm{sn}^2]\\ &= \frac{1}{1-N^2}\left[\frac{1}{1-N^2}(\varphi_1'-N) - N\,\mathrm{sn}^2\right]\\ &= \frac{1}{(1-N^2)^2}(\varphi_1'-N) - \frac{N}{1-N^2}\,\mathrm{sn}^2\,.\end{aligned} \tag{8B.8}$$

Noting relation (8B.5), we combine (8B.7) with (8B.8) in the form

$$\begin{aligned}&\left[\tfrac{1}{2}k_2^2\,\mathrm{sn}^2 + \tfrac{1}{8}k_2^4\,\mathrm{sn}^4\right]\varphi_1' =\\ &\qquad \tfrac{1}{2}\eta^2\ell_0(\varphi_1'-N) + \tfrac{1}{8}(\eta^2\ell_0)^2(\varphi_1'-N) - \tfrac{1}{8}(\eta^2\ell_0)^2N(1-N^2)\,\mathrm{sn}^2\end{aligned} \tag{8B.9}$$

where we have repeatedly applied relation (8B.6a). We now rearrange relation (8B.5) and introduce (8B.9) to obtain

$$\begin{aligned}\varphi_0' &= \left[1-\tfrac{1}{2}\eta^2\ell_0(1+\tfrac{1}{4}\eta^2\ell_0)\right]\varphi_1'\\ &\qquad + \tfrac{1}{2}\eta^2\ell_0(1+\tfrac{1}{4}\eta^2\ell_0)N + \tfrac{1}{8}(\eta^2\ell_0)^2N(1-N^2)\,\mathrm{sn}^2\,.\end{aligned} \tag{8B.10}$$

It is consistent with an approximation to order η^4 to take, in the last term of (8B.10),

$$\mathrm{sn}^2[f+\omega\colon k_2] = \sin^2 G_2 = \tfrac{1}{2}[1-\cos 2G_2] \tag{8B.11a}$$

where

$$G_2 = \frac{\pi}{2K_2}(f+\omega) = [1 - \tfrac{1}{4}k_2^2 - \tfrac{5}{64}k_2^4](f+\omega) \tag{8B.11b}$$

so that we can replace (8B.10) by

$$\begin{aligned}\varphi_0' &= [1 - \tfrac{1}{2}\eta^2\ell_0(1+\tfrac{1}{4}\eta^2\ell_0)]\varphi_1' \\ &\quad + \tfrac{1}{2}\eta^2\ell_0 N[1+\tfrac{1}{8}\eta^2\ell_0(3-N^2)] - \tfrac{1}{16}(\eta^2\ell_0)^2 N(1-N^2)\cos 2G_2.\end{aligned} \tag{8B.12}$$

We may write the above relation (8B.12) in terms of the more basic parameters ℓ and e by proceeding as follows. If we introduce (8A.21c) into both (8B.6) and (6.20), we find, up to first order in η^2,

$$\ell_0 = \ell(1-\eta^2\ell\nu^2), \quad 1-N^2 = (1-\nu^2)(1-\eta^2\ell\nu^2) \tag{8B.13a,b}$$

and hence

$$N^2 = \nu^2[1+\eta^2\ell(1-\nu^2)]. \tag{8B.13c}$$

There follows that, up to first order in η^2,

$$\ell_0(1+\tfrac{1}{4}\eta^2\ell_0) = \ell[1+\tfrac{1}{4}\eta^2\ell(1-4\nu^2)] \tag{8B.14a}$$

$$\ell_0[1+\tfrac{1}{8}\eta^2\ell_0(3-N^2)] = \ell[1+\tfrac{3}{8}\eta^2\ell(1-3\nu^2)]. \tag{8B.14b}$$

It is therefore consistent with an approximation to second order in η^2 to replace (8B.12) by

$$\begin{aligned}\varphi_0' &= [1 - \tfrac{1}{2}\eta^2\ell[1+\tfrac{1}{4}\eta^2\ell(1-4\nu^2)]]\varphi_1' \\ &\quad + \tfrac{1}{2}\eta^2\ell N[1+\tfrac{3}{8}\eta^2\ell(1-3\nu^2)] - \tfrac{1}{16}\eta^4 N\ell^2(1-\nu^2)\cos 2G_2.\end{aligned} \tag{8B.15}$$

If we set

$$j_0 = 1 - \tfrac{1}{2}\eta^2\ell[1+\tfrac{1}{4}\eta^2\ell(1-4\nu^2)] \tag{8B.16a}$$

$$\beta_0 = \tfrac{1}{2}\ell[1+\tfrac{3}{8}\eta^2\ell(1-3\nu^2)], \quad \beta_2 = \tfrac{1}{32}\ell^2(1-\nu^2) \tag{8B.16b,c}$$

then we may rewrite (8B.15) as

$$\varphi_0' = j_0\varphi_1' + \eta^2 N\beta_0 - 2\eta^4 N\beta_2\cos 2G_2. \tag{8B.17}$$

On performing the integration of (8B.17), we transpose terms to obtain, apart from an additive constant,

$$j_0\varphi_1 = \varphi_0 - \eta^2 N\beta_0(f+\omega) + \eta^4 N\beta_2 \sin 2G_2 \tag{8B.18}$$

completing the integration for φ_1.

— ◇ —

Returning to equation (8.15) then, apart from an additive constant, we have the integrated form

$$j_N\varphi = \varphi_1 - \eta^2 N j_p^2 \varphi_2. \tag{8.16}$$

Noting relation (8B.18), we see that it is convenient to multiply across (8.16) by the factor j_0 to obtain

$$j_0 j_N \varphi = j_0\varphi_1 - \eta^2 N j_0 j_p^2 \varphi_2. \tag{8.17}$$

Taking j_p^2 from (8.7a) and j_0 from (8B.16a), it follows immediately that to first order in η^2, we have for the relation defining j_u (noting $\ell = 1 - e_0^2 = 1 - e_*^2 + O(\eta^2) = 1 - e^2 + O(\eta^2)$),

$$j_u = j_0 j_p^2 = 1 - \tfrac{1}{2}\eta^2[1 + 3e^2 - 8\nu^2(1 + 2e^2)]. \tag{8.18}$$

Moreover, from combining (8.12b) with (8B.16a), then for the defining relation for j_3 (to second order in η^2), we have

$$\begin{aligned} j_3 = j_0 j_N &= [1 - \tfrac{1}{2}\eta^2\ell[1 + \tfrac{1}{4}\eta^2\ell(1 - 4\nu^2)]][1 + \tfrac{1}{2}\eta^2\ell[1 + \tfrac{1}{4}\eta^2\ell(3 - 4\nu^2)]] \\ &= 1 + O(\eta^6). \end{aligned} \tag{8.19}$$

With j_u as given by (8.18) and j_3 given by (8.19), we can replace (8.17) by

$$\varphi(= j_3\varphi) = j_0\varphi_1 - \eta^2 N j_u \varphi_2 \tag{8.20}$$

where we still keep the additive constant in reserve.

We now introduce $j_0\varphi_1$ from (8B.18) and φ_2 from (8A.28) into (8.20) to obtain

$$\begin{aligned} \varphi &= \varphi_0 - \eta^2 N\beta_0(f + \omega) + \eta^4 N\beta_2 \sin 2G_2 \\ &\qquad - \eta^2 N j_u\Big[a_0(f + \omega) + \sum_{n=1}^{2} a_n \operatorname{sn} nF_1 - \eta^2 \sum_{n=1}^{6} b_n \sin nG_1\Big] \\ &= \varphi_0 - \eta^2 N(\beta_0 + j_u a_0)(f + \omega) - \eta^2 N j_u \sum_{n=1}^{2} a_n \operatorname{sn} nF_1 \\ &\qquad + \eta^4 N\Big[j_u \sum_{n=1}^{6} b_n \sin nG_1 + \beta_2 \sin 2G_2\Big]. \end{aligned} \tag{8.21}$$

We now introduce the additive constant in a "distributed" manner and, recalling relations (8A.11) and (8B.11b), we replace (8.21) by

$$\begin{aligned} (j_3)(\varphi - \Omega_0) &= \varphi_0 - \eta^2 N(\beta_0 + j_u a_0)(f + \omega) \\ &\qquad - \eta^2 N j_u \sum_{n=1}^{2} a_n(\operatorname{sn}[n j_1 f : k_1] + \operatorname{sn}[n j_1 \omega : k_1]) \\ &\quad + \eta^4 N\Big[j_u \sum_{n=1}^{6} b_n\Big(\sin\frac{n\pi}{2K_1} j_1 f + \sin\frac{n\pi}{2K_1} j_1\omega\Big) + \beta_2 \sin\frac{\pi}{K_2}(f + \omega)\Big] \end{aligned} \tag{8.22}$$

and recalling the solution form (8B.3) for φ_0, we see that $f = -\omega$ corresponds to $\varphi = \Omega_0$ — the line of the first modal crossing. We may now define the angle Ω by setting

$$\Omega = \Omega_0 - \eta^2 N(\beta_0 + j_u a_0)(f + \omega)$$
$$- \eta^2 N j_u \sum_{n=1}^{2} a_n(\mathrm{sn}[nj_1 f\colon k_1] + \mathrm{sn}[nj_1\omega\colon k_1])$$
$$+ \eta^4 N\Big[j_u \sum_{n=1}^{6} b_n\Big(\sin\frac{n\pi}{2K_1}j_1 f + \sin\frac{n\pi}{2K_1}j_1\omega\Big) + \beta_2 \sin\frac{\pi}{K_2}(f+\omega)\Big]. \quad (8.23)$$

Then when we introduce φ_0 from (8B.3), we see that (8.23) can be replaced by

$$\tan(\varphi - \Omega) = N\,\mathrm{sc}[f + \omega\colon k_2] \quad (8.24)$$

which is clearly the appropriate generalization of the formula for longitude in the Kepler case.

9 The Time-Angle Relation

From the defining relation (5.5), we have

$$\Lambda\frac{\mathrm{d}t}{\mathrm{d}f} = R^2 + b^2\cos^2\sigma = p^2\Big[\Big(\frac{R}{p}\Big)^2 + \frac{b^2}{p^2}\cos^2\sigma\Big] \quad (9.1a)$$

so that

$$\frac{\Lambda}{p^2}\frac{\mathrm{d}t}{\mathrm{d}f} = \Big(\frac{R}{p}\Big)^2 + \frac{b^2}{p^2}\cos^2\sigma. \quad (9.1b)$$

We recall from (6.14) that

$$\Lambda^2 = (1 + \eta^2\ell)mC^2 = (1 + \eta^2\ell)m\mu\frac{C^2}{\mu} = (1 + \eta^2\ell)m\mu p_0 \quad (9.2)$$

where in the latter we have introduced p_0 from (5.2b). Introducing the factor n, corresponding to the mean motion in the Kepler case, through the defining relation

$$\mu = n^2 a^3 \quad (9.3)$$

which, together with relation (5.9b) for p_0, when introduced into (9.2) yields

$$\Lambda^2 = (1 + \eta^2\ell)mn^2a^4(1 - e_0^2) \quad (9.4)$$

so that

$$\Lambda = \sqrt{(1+\eta^2\ell)m}\, na^2\sqrt{1-e_0^2} \tag{9.5}$$

and hence, again noting (5.9b),

$$\begin{aligned}\frac{\Lambda}{p^2} &= \sqrt{(1+\eta^2\ell)m}\,\frac{p_0^2}{p^2}\cdot n\cdot\frac{1}{(1-e_0^2)^{3/2}}\\ &= \sqrt{(1+\eta^2\ell)m}\Big(\frac{p_0}{p_*}\Big)^2\Big(\frac{p_*}{p}\Big)^2\Big[\frac{1-e^2}{1-e_*^2}\cdot\frac{1-e_*^2}{1-e_0^2}\Big]^{3/2}\frac{n}{(1-e^2)^{3/2}}.\end{aligned} \tag{9.6}$$

By combining relations (7.7a and b), we see that

$$\begin{aligned}\Big(\frac{p_0}{p_*}\Big)^2\Big[\frac{1-e_*^2}{1-e_0^2}\Big]^{3/2} &= \frac{(1-\eta^2\nu^2\ell h_0)^2}{(1-\eta^2\nu^2h_2)^2}\cdot\frac{(1-\eta^2\nu^2h_2)^{3/2}}{(1-\eta^2\nu^2\ell h_0)^3}\\ &= \frac{1}{(1-\eta^2\nu^2h_2)^{1/2}(1-\eta^2\nu^2\ell h_0)}\end{aligned} \tag{9.7a}$$

and a corresponding combination, of relations (7.39a and b) with relation (7.40), yields

$$\Big(\frac{p_*}{p}\Big)^2\Big(\frac{1-e^2}{1-e_*^2}\Big)^{3/2} = \frac{(1-\delta^2)^2}{(1-\delta e)^2}\cdot\frac{(1-\delta e)^3}{(1-\delta^2)^{3/2}} = (1-\delta^2)^{1/2}(1-\delta e) \tag{9.7b}$$

and hence

$$\Big(\frac{p_0}{p_*}\Big)^2\Big(\frac{p_*}{p}\Big)^2\Big[\frac{1-e^2}{1-e_*^2}\cdot\frac{1-e_*^2}{1-e_0^2}\Big]^{3/2} = \frac{(1-\delta^2)^{1/2}(1-\delta e)}{(1-\eta^2\nu^2h_2)^{1/2}(1-\eta^2\nu^2\ell h_0)}. \tag{9.8}$$

Accordingly, if we set

$$j_T = \sqrt{(1+\eta^2\ell)m}\,\frac{(1-\delta^2)^{1/2}}{(1-\eta^2\nu^2h_2)^{1/2}}\cdot\frac{1-\delta e}{1-\eta^2\nu^2\ell h_0} \tag{9.9}$$

then relation (9.6) can be written as

$$\frac{\Lambda}{p^2} = j_T\frac{n}{(1-e^2)^{3/2}}. \tag{9.10}$$

If we further set (again noting (7.7a) and (7.39))

$$\frac{p_0}{p_*}\frac{p_*}{p} = \frac{1-\eta^2\nu^2\ell h_0}{1-\eta^2\nu^2h_2}\cdot\frac{1-\delta^2}{1-\delta e} = j_\nu \tag{9.11}$$

then the introduction of (9.10) and (9.11) into the differential equation (9.1b) yields (on recalling (5.9a))

$$j_T n\frac{\mathrm{d}t}{\mathrm{d}f} = (1-e^2)^{3/2}\Big[\Big(\frac{R}{p}\Big)^2 + \eta^2 j_\nu^2\cos^2\sigma\Big] \tag{9.12}$$

which, on integration, takes the form

$$j_T n(t-t_0) = (1-e^2)^{3/2}\int_0^f \Big[\Big(\frac{R}{p}\Big)^2 + \eta^2 j_\nu^2\cos^2\sigma\Big]\,\mathrm{d}f \tag{9.13}$$

where t_0 is the constant of integration. Since $t = t_0$ corresponds to $f = 0$, t_0 is the time of the first “pericenter” passage.

From (7.41), recalling that (R/p) is expressed in terms of $f_1 = j_1 f$ and from (6.21) that $\cos\sigma$ is expressed in terms of $(f + \omega)$, it will prove convenient to multiply across (9.13) by j_1. If we then define the mean anomaly M and the auxiliary factor j_N by setting

$$M = j_1 j_T n(t - t_0), \quad j_N = j_1 j_\nu^2 \tag{9.14a,b}$$

then the integrated relation (9.13) takes the form

$$M = (1 - e^2)^{3/2} \int \left(\frac{R}{p}\right)^2 \mathrm{d}f_1 + \eta^2 j_N (1 - e^2)^{3/2} \int \cos^2\sigma \,\mathrm{d}f\,. \tag{9.15}$$

It will be necessary to deal with the two terms separately. Accordingly, we set

$$M_0 = (1 - e^2)^{3/2} \int \left(\frac{R}{p}\right)^2 \mathrm{d}f_1, \quad M_{11} = \int \cos^2\sigma \,\mathrm{d}f \tag{9.16a,b}$$

so that (9.15) reads

$$M = M_0 + \eta^2 j_N (1 - e^2)^{3/2} M_{11}. \tag{9.17}$$

We now evaluate the two components individually: M_0 is to be evaluated to second order in η^2, while M_{11} is to be evaluated to first order.

Evaluation of M_0

If we introduce the solution form (7.41) for (R/p) into the defining relation (9.16a) for M_0, we have

$$M_0 = (1 - e^2)^{3/2} \int \left[\frac{1 + \delta\,\mathrm{cn}[f_1 : k_1]}{1 + e\,\mathrm{cn}[f_1 : k_1]}\right]^2 \mathrm{d}f_1 = (1 - e^2)^{3/2} I_0 \tag{9.18}$$

and we note that all elliptic functions appearing in (9.18) have argument f_1 and modulus k_1; accordingly, these will not be exhibited in this subsection except where it becomes necessary. Noting relation (7.47) for δ, it can readily be checked that

$$\left[\frac{1 + \delta\,\mathrm{cn}}{1 + e\,\mathrm{cn}}\right]^2 = \frac{(1 + \eta^2 d)^2}{(1 + e\,\mathrm{cn})^2} - \frac{2\eta^2 d(1 + \eta^2 d)}{1 + e\,\mathrm{cn}} + \eta^4 d^2. \tag{9.19}$$

Moreover, a straightforward differentiation shows that

$$\frac{\mathrm{d}}{\mathrm{d}f_1}\left[\frac{e\,\mathrm{sn}}{1 + e\,\mathrm{cn}}\right] = \left[\frac{1}{1 + e\,\mathrm{cn}} - \frac{1 - e^2}{(1 + e\,\mathrm{cn})^2}\right] \mathrm{dn} \tag{9.20}$$

from which there follows

$$\frac{1 - e^2}{(1 + e\,\mathrm{cn})^2} = \frac{1}{1 + e\,\mathrm{cn}} - \frac{1}{\mathrm{d}n}\frac{\mathrm{d}}{\mathrm{d}f_1}\left[\frac{e\,\mathrm{sn}}{1 + e\,\mathrm{cn}}\right]. \tag{9.21}$$

If we substitute for $1/(1+e\,\mathrm{cn})^2$ from (9.21) into (9.19), we can rearrange: on setting

$$j_4 = 1 + 2\eta^2 de^2 - \eta^4 d^2(1-2e^2) \tag{9.22a}$$

$$j_5 = (1+\eta^2 d)^2 = 1 + 2\eta^2 d + \eta^4 d^2 \tag{9.22b}$$

we find that

$$\left[\frac{1+\delta\,\mathrm{cn}}{1+e\,\mathrm{cn}}\right]^2 = \frac{1}{1-e^2}\left[j_4\frac{1}{1+e\,\mathrm{cn}} - j_5\frac{1}{\mathrm{d}n}\frac{\mathrm{d}}{\mathrm{d}f_1}\left(\frac{e\,\mathrm{sn}}{1+e\,\mathrm{cn}}\right)\right] + \eta^4\,\mathrm{d}^2 \tag{9.23}$$

so that the evaluation of the integral in (9.18) is reduced to the evaluation of the two integrals

$$I_1 = \int\frac{\mathrm{d}f_1}{1+e\,\mathrm{cn}},\quad I_2 = \int\frac{1}{\mathrm{dn}}\frac{\mathrm{d}}{\mathrm{d}f_1}\left[\frac{e\,\mathrm{sn}}{1+e\,\mathrm{cn}}\right]\mathrm{d}f_1\,. \tag{9.24a,b}$$

For the evaluation of I_1, we note that

$$\begin{aligned} I_1 &= \int\frac{\mathrm{dn}}{1+e\,\mathrm{cn}}\,\mathrm{d}f_1 + \int\frac{(1-\mathrm{dn})\,\mathrm{d}f_1}{1+e\,\mathrm{cn}} \\ &= \int\frac{\mathrm{dn}}{1+e\,\mathrm{cn}}\,\mathrm{d}f_1 + \int\frac{k_1^2\,\mathrm{sn}^2}{1+e\,\mathrm{cn}}\,\frac{1}{1+\mathrm{dn}}\,\mathrm{d}f_1 = I_{11} + k_1^2 I_{12} \end{aligned} \tag{9.25}$$

where the definitions of I_{11} and I_{12} are self-evident. It can readily be verified that

$$I_{11} = \int\frac{\mathrm{dn}}{1+e\,\mathrm{cn}}\,\mathrm{d}f_1 = \frac{1}{\sqrt{1-e^2}}\arctan\frac{\sqrt{1-e^2}\,\mathrm{sn}}{e+\mathrm{cn}} \tag{9.26a}$$

and we record for I_{12}

$$I_{12} = \int\frac{\mathrm{sn}^2}{1+e\,\mathrm{cn}}\,\frac{1}{1+\mathrm{dn}}\,\mathrm{d}f_1 \tag{9.26b}$$

which remains to be evaluated. Turning to relation (9.24b) for I_2, we integrate by parts to obtain

$$I_2 = \frac{1}{\mathrm{dn}}\frac{e\,\mathrm{sn}}{1+e\,\mathrm{cn}} - \int\frac{e\,\mathrm{sn}}{1+e\,\mathrm{cn}}\,\frac{k_1^2\,\mathrm{sn}\,\mathrm{cn}}{\mathrm{dn}^2}\,\mathrm{d}f_1 = I_{22} - k_1^2 I_{21} \tag{9.27}$$

with

$$I_{22} = \frac{1}{\mathrm{dn}}\,\frac{e\,\mathrm{sn}}{1+e\,\mathrm{cn}},\quad I_{21} = \int\frac{e\,\mathrm{sn}^2\,\mathrm{cn}}{1+e\,\mathrm{cn}}\,\frac{1}{\mathrm{dn}^2}\,\mathrm{d}f_1. \tag{9.28a,b}$$

Recapitulating, for the integral I_0 indicated in (9.18) we have, on noting (9.23), (9.25), and (9.27),

$$\begin{aligned} I_0 &= \frac{1}{1-e^2}[j_4 I_1 - j_5 I_2] + \eta^4 d^2 f_1 \\ &= \frac{1}{1-e^2}\Big[j_4 I_{11} - j_5 I_{22} + k_1^2[(1+2\eta^2 de^2)I_{12} + (1+2\eta^2 d)I_{21}]\Big] + \eta^4 d^2 f_1 \\ &= \frac{1}{1-e^2}\Big[j_4 I_{11} - j_5 I_{22} + k_1^2[(1+2\eta^2 d)(I_{12}+I_{21})] \\ &\qquad\qquad - 2\eta^2 d(1-e^2)I_{12}\Big] + \eta^4 d^2 f_1 \end{aligned} \tag{9.29}$$

where, in the factor multiplying k_1^2, we have utilized the fact that it is consistent with an approximation to second order in η^2 to neglect terms with η^4 from j_4 and j_5 as these appear in that factor. The approximation requires that $(I_{12} + I_{21})$ be evaluated to first order in η^2, whereas, for the last term with I_{12}, an evaluation to zeroth order in η^2 suffices.

Combining (9.26b) and (9.28b), we see that

$$\begin{aligned}
I_{12} + I_{21} &= \int \Big[\frac{\mathrm{sn}^2}{1 + e\,\mathrm{cn}} \frac{1}{1 + \mathrm{dn}} + \frac{e\,\mathrm{sn}^2\,\mathrm{cn}}{1 + e\,\mathrm{cn}} \frac{1}{\mathrm{dn}^2} \Big] \mathrm{d}f_1 \\
&= \int \frac{\mathrm{sn}^2}{1 + e\,\mathrm{cn}} \Big[\frac{1}{1 + \mathrm{dn}} + \frac{e\,\mathrm{cn}}{\mathrm{dn}^2} \Big] \mathrm{d}f_1 \\
&= \int \frac{\mathrm{sn}^2}{1 + e\,\mathrm{cn}} \Big[\frac{1}{1 + \mathrm{dn}} - \frac{1}{\mathrm{dn}^2} + \frac{1 + e\,\mathrm{cn}}{\mathrm{dn}^2} \Big] \mathrm{d}f_1 \\
&= \int \frac{\mathrm{sn}^2}{1 + e\,\mathrm{cn}} \Big[\frac{1}{1 + \mathrm{dn}} - \frac{1}{\mathrm{dn}^2} \Big] \mathrm{d}f_1 + \int \frac{\mathrm{sn}^2}{\mathrm{dn}^2} \mathrm{d}f_1 \\
&= \int \frac{\mathrm{sn}^2}{\mathrm{dn}^2} \mathrm{d}f_1 + \int \frac{1}{e^2} \Big[1 - e\,\mathrm{cn} - \frac{1 - e^2}{1 + e\,\mathrm{cn}} \Big] \Big[\frac{1}{1 + \mathrm{dn}} - \frac{1}{\mathrm{dn}^2} \Big] \mathrm{d}f_1 .
\end{aligned} \tag{9.30}$$

Next, we note the following relations:

$$\frac{1}{\mathrm{dn}^2} = \mathrm{dn} + k_1^2 \frac{\mathrm{sn}^2}{\mathrm{dn}^2} \Big[1 + \frac{\mathrm{dn}^2}{1 + \mathrm{dn}} \Big] = \mathrm{dn} + \tfrac{3}{2} k_1^2\,\mathrm{sn}^2 + O(k_1^4) \tag{9.31a}$$

and, alternatively

$$\frac{1}{\mathrm{dn}^2} = 1 + k_1^2 \frac{\mathrm{sn}^2}{\mathrm{dn}^2} = 1 + k_1^2\,\mathrm{sn}^2 + O(k_1^4); \tag{9.31b}$$

also

$$\frac{1}{1 + \mathrm{dn}} = \tfrac{1}{2} \Big[\mathrm{dn} + k_1^2\,\mathrm{sn}^2 \frac{2 + \mathrm{dn}}{(1 + \mathrm{dn})^2} \Big] = \tfrac{1}{2} [\mathrm{dn} + \tfrac{3}{4} k_1^2\,\mathrm{sn}^2] + O(k_1^4) \tag{9.32a}$$

and, alternatively,

$$\frac{1}{1 + \mathrm{dn}} = \tfrac{1}{2} \Big[1 + k_1^2 \frac{\mathrm{sn}^2}{(1 + \mathrm{dn})^2} \Big] = \tfrac{1}{2} [1 + \tfrac{1}{4} k_1^2\,\mathrm{sn}^2] + O(k_1^4). \tag{9.32b}$$

By combining the forms (9.31a) with (9.32a) and also the form (9.31b) with (9.32b), we obtain the two forms for the difference within the second bracket in (9.30). Reversing the order in the difference and neglecting terms of order k_1^4, we have the alternate forms

$$\frac{1}{\mathrm{dn}^2} - \frac{1}{1 + \mathrm{dn}} = \tfrac{1}{2} [\mathrm{dn} + \tfrac{9}{4} k_1^2\,\mathrm{sn}^2] \tag{9.33a}$$

$$= \tfrac{1}{2} [1 + \tfrac{7}{4} k_1^2\,\mathrm{sn}^2]. \tag{9.33b}$$

These relations (9.31) to (9.33) are now to be introduced judiciously into the integrals in (9.30). In the first integral in (9.30), we use the form (9.31b) for

$1/\,\mathrm{dn}^2$; in the second integral, with the term unity in the first bracket, we use the form (9.33b) for the multiplying difference; and with the second and third terms in that first bracket, we use the form (9.33a) for the multiplying difference. Then the integral (9.30) may be written

$$\begin{aligned} I_{12} + I_{21} = \int (\mathrm{sn}^2 + k_1^2\,\mathrm{sn}^4)\,\mathrm{d}f_1 - \frac{1}{2e^2}\int [1 + \tfrac{7}{4}k_1^2\,\mathrm{sn}^2]\,\mathrm{d}f_1 \\ + \frac{1}{2e^2}\int\Big[\frac{1-e^2}{1+e\,\mathrm{cn}} + e\,\mathrm{cn}\Big]\Big[\mathrm{dn} + \tfrac{9}{4}k_1^2\,\mathrm{sn}^2\Big]\,\mathrm{d}f_1. \end{aligned} \tag{9.34}$$

If we introduce k_1^2 from (7.50) into the above, and recall the defining relation (9.26a) for I_{11}, then on rearrangement, we see that we may rewrite (9.34) as

$$\begin{aligned} I_{12} + I_{21} &= (1 - \tfrac{7}{8}\eta^2 g^2)\int \mathrm{sn}^2\,\mathrm{d}f_1 + \eta^2 g^2 e^2 \int \mathrm{sn}^4\,\mathrm{d}f_1 - \frac{1}{2e^2}\int \mathrm{d}f_1 + \frac{1-e^2}{2e^2} I_{11} \\ &\quad + \frac{1}{2e}\int \mathrm{cn}\,\mathrm{dn}\,\mathrm{d}f_1 + \tfrac{9}{8}\eta^2 g^2 e \int \mathrm{sn}^2\,\mathrm{cn}\,\mathrm{d}f_1 + \tfrac{9}{8}\eta^2 g^2 (1-e^2)\int \frac{\mathrm{sn}^2}{1+e\,\mathrm{cn}}\,\mathrm{d}f_1 \\ &= \frac{1-e^2}{2e^2} I_{11} + \frac{1}{2e}\,\mathrm{sn} - \frac{1}{2e^2} f_1 + (1 - \tfrac{7}{8}\eta^2 g^2)\int \mathrm{sn}^2\,\mathrm{d}f_1 + \eta^2 g^2 e^2 \int \mathrm{sn}^4\,\mathrm{d}f_1 \\ &\quad + \tfrac{9}{8}\eta^2 g^2 e \int \mathrm{sn}^2\,\mathrm{cn}\,\mathrm{d}f_1 + \tfrac{9}{8}\eta^2 g^2 (1-e^2)\int \frac{\mathrm{sn}^2}{1+e\,\mathrm{cn}}\,\mathrm{d}f_1. \end{aligned} \tag{9.35}$$

We further observe from (9.26b), when taken together with (9.32b), we obtain the zeroth order approximation for I_{12} in the form

$$I_{12} = \tfrac{1}{2}\int \frac{\mathrm{sn}^2}{1+e\,\mathrm{cn}}\,\mathrm{d}f_1 \tag{9.36}$$

from which we have omitted all terms of order η^2.

We are now in a position to form the combination constituting the factor multiplying k_1^2 within the square bracket in (9.29). Neglecting terms of order η^4, on utilizing (9.35) and (9.36) we have, on regrouping,

$$\begin{aligned} &(1 + 2\eta^2 d)(I_{12} + I_{21}) - 2\eta^2 d(1-e^2) I_{12} = \\ &\qquad (1 + 2\eta^2 d)\Big[\frac{1-e^2}{2e^2} I_{11} + \frac{1}{2e}\,\mathrm{sn} - \frac{1}{2e^2} f_1\Big] \\ &\qquad + [1 + \eta^2(2d - \tfrac{7}{8}g^2)]\int \mathrm{sn}^2\,\mathrm{d}f_1 + \eta^2 g^2 e^2 \int \mathrm{sn}^4\,\mathrm{d}f_1 \\ &\qquad + \tfrac{9}{8}\eta^2 g^2 e \int \mathrm{sn}^2\,\mathrm{cn}\,\mathrm{d}f_1 - \eta^2(1-e^2)(d - \tfrac{9}{8}g^2)\int \frac{\mathrm{sn}^2}{1+e\,\mathrm{cn}}\,\mathrm{d}f_1. \end{aligned} \tag{9.37}$$

It remains to evaluate $\int \mathrm{sn}^2\,\mathrm{d}f_1$ to first order in η^2 and the remaining integrals to zeroth order in η^2.

For the evaluation of $\int \mathrm{sn}^2\,\mathrm{d}f_1$ to first order in η^2, we start from the double argument relation

$$\mathrm{sn}^2[f_1 : k_1] = \frac{1 - \mathrm{cn}[2f_1 : k_1]}{1 + \mathrm{dn}[2f_1 : k_1]} \tag{9.38}$$

and hence

$$\int \mathrm{sn}^2[f_1 : k_1]\,\mathrm{d}f_1 = \int \frac{\mathrm{d}f_1}{1+\mathrm{dn}[2f_1 : k_1]} - \int \frac{\mathrm{cn}[2f_1 : k_1]\,\mathrm{d}f_1}{1+\mathrm{dn}[2f_1 : k_1]}. \tag{9.39}$$

In reducing the integrals on the right side, we use the analog for the double angle of relations (9.32). In the first integral we apply the analog of (9.32b), while in the second integral we apply the analog of (9.32a); we find

$$\begin{aligned}
\int \mathrm{sn}^2[f_1 : k_1]\,\mathrm{d}f_1 &= \tfrac{1}{2}\int \left(1 + \tfrac{1}{4}k_1^2\,\mathrm{sn}^2[2f_1 : k_1]\right)\mathrm{d}f_1 \\
&\quad - \tfrac{1}{2}\int \mathrm{cn}[2f_1 : k_1]\left(\mathrm{dn}[2f_1 : k_1] + \tfrac{3}{4}k_1^2\,\mathrm{sn}^2[2f_1 : k_1]\right)\mathrm{d}f_1 \\
&= \tfrac{1}{2}f_1 + \tfrac{1}{8}k_1^2\int \mathrm{sn}^2[2f_1 : k_1] - \tfrac{1}{4}\,\mathrm{sn}[2f_1 : k_1] - \tfrac{3}{8}k_1^2\int \mathrm{cn}[2f_1 : k_1]\,\mathrm{sn}^2[2f_1 : k_1]\,\mathrm{d}f_1 \\
&= \tfrac{1}{2}f_1 - \tfrac{1}{4}\,\mathrm{sn}[2f_1 : k_1] + \tfrac{1}{8}k_1^2\int [\sin^2 2G_1 - 3\sin^2 2G_1\cos 2G_1]\,\mathrm{d}G_1
\end{aligned} \tag{9.40a}$$

where, in accordance with the level of approximation, G_1 is given by (8A.11),

$$G_1 = \frac{\pi}{2K_1}f_1 = [1 - \tfrac{1}{4}k_1^2 - \tfrac{5}{64}k_1^4]f_1 \tag{9.40b}$$

and it is consistent with the order of approximation to replace $\mathrm{d}f_1$ by $\mathrm{d}G_1$ in the latter integral. Noting that

$$\sin^2 2G_1 = \frac{1-\cos 4G_1}{2} \tag{9.41}$$

we can execute the integration and regroup to obtain

$$\begin{aligned}
&\int \mathrm{sn}^2[f_1 : k_1]\,\mathrm{d}f_1 = \\
&\qquad \tfrac{1}{2}(1 + \tfrac{1}{8}k_1^2)f_1 - \tfrac{1}{4}\,\mathrm{sn}[2f_1 : k_1] - \tfrac{1}{16}k_1^2\Big[\frac{\sin 4G_1}{4} + \sin^3 2G_1\Big]
\end{aligned} \tag{9.42}$$

and noting that

$$\sin^3 2G_1 = \tfrac{1}{4}[3\sin 2G_1 - \sin 6G_1] \tag{9.43}$$

we finally have

$$\begin{aligned}
&\int \mathrm{sn}^2[f_1 : k_1]\,\mathrm{d}f_1 = \\
&\tfrac{1}{2}(1 + \tfrac{1}{8}k_1^2)f_1 - \tfrac{1}{4}\,\mathrm{sn}[2f_1 : k_1] - \tfrac{1}{64}k_1^2[3\sin 2G_1 + \sin 4G_1 - \sin 6G_1]
\end{aligned} \tag{9.44}$$

valid to the first order in η^2.

For the three remaining integrals in (9.37), to be calculated to zeroth order in η^2, we have

$$\int \mathrm{sn}^4[f_1 : k_1]\,\mathrm{d}f_1 = \int \sin^4 G_1\,\mathrm{d}f_1 = \int [\tfrac{3}{8} - \tfrac{1}{2}\cos 2G_1 + \tfrac{1}{8}\cos 4G_1]\,\mathrm{d}f_1$$
$$= \tfrac{3}{8}f_1 - \tfrac{1}{2}\int \cos 2G_1\,\mathrm{d}G_1 + \tfrac{1}{8}\int \cos 4G_1\,\mathrm{d}G_1$$
$$= \tfrac{3}{8}f_1 - \tfrac{1}{4}\sin 2G_1 + \tfrac{1}{32}\sin 4G_1 \tag{9.45a}$$

$$\int \mathrm{sn}^2[f_1 : k_1]\,\mathrm{cn}[f_1 : k_1]\,\mathrm{d}f_1 = \int \sin^2 G_1 \cos G_1\,\mathrm{d}G_1$$
$$= \tfrac{1}{3}\sin^3 G_1 = \tfrac{1}{4}\sin G_1 - \tfrac{1}{12}\sin 3G_1 \tag{9.45b}$$

$$\int \frac{\mathrm{sn}^2[f_1 : k_1]\,\mathrm{d}f_1}{1 + e\,\mathrm{cn}[f_1 : k_1]} = \frac{1}{e^2}\int \Big[1 - e\,\mathrm{cn}[f_1 : k_1] - \frac{1 - e^2}{1 + e\,\mathrm{cn}[f_1 : k_1]}\Big]\,\mathrm{d}f_1$$
$$= \frac{1}{e^2}f_1 - \frac{1}{e}\int \cos G_1\,\mathrm{d}G_1 - \frac{1 - e^2}{e^2}I_{11}$$
$$= \frac{1}{e^2}f_1 - \frac{1}{e}\sin G_1 - \frac{1 - e^2}{e^2}I_{11}. \tag{9.45c}$$

If we introduce relations (9.44)–(9.45) into (9.37) and regroup the terms, then, after considerable rearrangement, we obtain

$$\begin{aligned}
&(1 + 2\eta^2 d)(I_{12} + I_{21}) - 2\eta^2 d(1 - e^2)I_{12} \\
&\quad = \frac{1 - e^2}{2e^2}I_{11}[(1 + 2\eta^2 d) + 2\eta^2(1 - e^2)(d - \tfrac{9}{8}g^2)] \\
&\quad + \frac{1 + 2\eta^2 d}{2e}\,\mathrm{sn}[f_1 : k_1] - \tfrac{1}{4}[1 + \eta^2(2d - \tfrac{7}{8}g^2)]\,\mathrm{sn}[2f_1 : k_1] \\
&- \frac{1 - e^2}{2e^2}f_1[1 + \eta^2(4d - \tfrac{9}{4}g^2 + \tfrac{7}{8}g^2e^2)] - \tfrac{1}{64}\eta^2 g^2 e^2[19\sin 2G_1 - \sin 4G_1 - \sin 6G_1] \\
&\quad + \eta^2\frac{1}{e}\Big[[d(1 - e^2) - \tfrac{9}{32}g^2(4 - 5e^2)]\sin G_1 - \tfrac{3}{32}g^2e^2\sin 3G_1\Big].
\end{aligned} \tag{9.46}$$

If we multiply across by $k_1^2 = \eta^2 g^2 e^2$ (7.50), then for the correction term within the square bracket in (9.29), we have, on a slight rearrangement,

$$\begin{aligned}
&k_1^2\Big[(1 + 2\eta^2 d)(I_{12} + I_{21}) - 2\eta^2 d(1 - e^2)I_{12}\Big] \\
&\quad = \tfrac{1}{2}\eta^2 g^2(1 - e^2)[(1 + 2\eta^2 d) + 2\eta^2(1 - e^2)(d - \tfrac{9}{8}g^2)]I_{11} \\
&+ \tfrac{1}{2}\eta^2 g^2 e(1 + 2\eta^2 d)\,\mathrm{sn}[f_1 : k_1] - \tfrac{1}{4}\eta^2 g^2 e^2[1 + \eta^2(2d - \tfrac{7}{8}g^2)]\,\mathrm{sn}[2f_1 : k_1] \\
&\quad - \tfrac{1}{2}\eta^2 g^2(1 - e^2)[1 + \eta^2(4d - \tfrac{9}{4}g^2 + \tfrac{7}{8}g^2e^2)]f_1 \\
&\quad - \tfrac{1}{64}\eta^4 g^4 e^4[19\sin 2G_1 - \sin 4G_1 - \sin 6G_1] \\
&\quad + \eta^4 g^2 e[d(1 - e^2) - \tfrac{9}{32}g^2(4 - 5e^2)]\sin G_1 - \tfrac{3}{32}\eta^4 g^4 e^3\sin 3G_1.
\end{aligned} \tag{9.47}$$

The above can be put in more compact form if we set

$$q_1 = \tfrac{1}{2}g^2 e(1+2\eta^2 d), \quad q_2 = -\tfrac{1}{4}g^2e^2[1+\eta^2(2d-\tfrac{7}{8}g^2) \tag{9.48a,b}$$

$$m_1 = -g^2e[d(1-e^2) - \tfrac{9}{32}g^2(4-5e^2)], \quad m_2 = \tfrac{19}{64}g^4e^4 \tag{9.49a,b}$$

$$m_3 = \tfrac{3}{32}g^4e^3, \quad m_4 = -\tfrac{1}{64}g^4e^4, \quad m_5 = 0, \quad m_6 = -\tfrac{1}{64}g^4e^4 \tag{9.49c,d,e,f}$$

$$\begin{aligned} j_6 &= 1 + 2\eta^2 d + 2\eta^2(1-e^2)(d-\tfrac{9}{8}g^2) \\ &= 1 + 2\eta^2[d + (1-e^2)(d-\tfrac{9}{8}g^2)] \end{aligned} \tag{9.50}$$

in terms of which (9.47) may be written

$$\begin{aligned} &k_1^2\Big[(1+2\eta^2 d)(I_{12}+I_{21}) - 2\eta^2 d(1-e^2)I_{12}\Big] \\ &\quad = \tfrac{1}{2}\eta^2 g^2(1-e^2)j_6 I_{11} - \tfrac{1}{2}\eta^2 g^2(1-e^2)[1+\eta^2(4d-\tfrac{9}{4}g^2+\tfrac{7}{8}g^2e^2)]f_1 \\ &\qquad + \eta^2\sum_{n=1}^{2} q_n \,\mathrm{sn}[nf_1 : k_1] - \eta^4 \sum_{n=1}^{6} m_n \sin nG_1. \end{aligned} \tag{9.51}$$

Observing how the above factor appears in (9.29), we divide across by $(1-e^2)$ and add the factor $\eta^4 d^2 f_1$ to the result, to obtain

$$\begin{aligned} &\frac{k_1^2}{1-e^2}\Big[(1+2\eta^2 d)(I_{12}+I_{21}) - 2\eta^2 d(1-e^2)I_{12}\Big] + \eta^4 d^2 f_1 \\ &\quad = \tfrac{1}{2}\eta^2 g^2 j_6 I_{11} - \tfrac{1}{2}\eta^2\Big[g^2 - \eta^2[2d^2 - g^2(4d-\tfrac{9}{4}g^2+\tfrac{7}{8}g^2e^2)]\Big]f_1 \\ &\qquad + \frac{1}{1-e^2}\Big[\eta^2\sum_{n=1}^{2} q_n \,\mathrm{sn}[nf_1 : k_1] - \eta^4 \sum_{n=1}^{6} m_n \sin nG_1\Big]. \end{aligned} \tag{9.52}$$

For the coefficient in the secular term, we introduce the notation

$$g_m^2 = g^2 - \eta^2[2d^2 - g^2(4d - \tfrac{9}{4}g^2 + \tfrac{7}{8}g^2e^2)] \tag{9.53}$$

whereby (9.52) may be written

$$\begin{aligned} &\frac{k_1^2}{1-e^2}\Big[(1+2\eta^2 d)(I_{11}+I_{22}) - 2\eta^2 d(1-e^2)I_{12}\Big] + \eta^4 d^2 f_1 \\ &\quad = \tfrac{1}{2}\eta^2 j_6 g^2 I_{11} - \tfrac{1}{2}\eta^2 g_m^2 f_1 \\ &\qquad + \frac{1}{1-e^2}\eta^2\Big[\sum_{n=1}^{2} q_n \,\mathrm{sn}[nf_1 : k_1] - \eta^2 \sum_{n=1}^{6} m_n \sin nG_1\Big]. \end{aligned} \tag{9.54}$$

It follows from (9.29) that

$$
\begin{aligned}
I_0 &= \frac{1}{1-e^2}[j_4 I_{11} - j_5 I_{22}] + \tfrac{1}{2}\eta^2 j_6 g^2 I_{11} - \tfrac{1}{2}\eta^2 g_m^2 f_1 \\
&\quad + \frac{1}{1-e^2}\eta^2\Big[\sum_{n=1}^{2} q_n \operatorname{sn}[nf_1 : k_1] - \eta^2 \sum_{n=1}^{6} m_n \sin nG_1\Big] \\
&= \frac{1}{1-e^2}[[j_4 + \tfrac{1}{2}\eta^2 j_6 g^2 (1-e^2)]I_{11} - j_5 I_{22}] - \tfrac{1}{2}\eta^2 g_m^2 f_1 \\
&\quad + \frac{1}{1-e^2}\eta^2\Big[\sum_{n=1}^{2} q_n \operatorname{sn}[nf_1 : k_1] - \eta^2 \sum_{n=1}^{6} m_n \sin nG_1\Big]
\end{aligned} \tag{9.55}
$$

and hence, from (9.18), we have

$$
\begin{aligned}
M_0 &= \sqrt{1-e^2}\Big[[j_4 + \tfrac{1}{2}\eta^2 j_6 g^2(1-e^2)]I_{11} - j_5 I_{22}\Big] - \tfrac{1}{2}\eta^2 g_m^2 (1-e^2)^{3/2} f_1 \\
&\quad + \eta^2\sqrt{1-e^2}\Big[\sum_{n=1}^{2} q_n \operatorname{sn}[nf_1 : k_1] - \eta^2 \sum_{n=1}^{6} m_n \sin nG_1\Big].
\end{aligned} \tag{9.56}
$$

If we introduce I_{11} from (9.26a) and I_{22} from (9.28a), we obtain

$$
\begin{aligned}
M_0 &= [j_4 + \tfrac{1}{2}\eta^2 j_6 g^2 (1-e^2)] \arctan \frac{\sqrt{1-e^2}\operatorname{sn}[f_1 : k_1]}{e + \operatorname{cn}[f_1 : k_1]} \\
&\quad - j_5 e\sqrt{1-e^2}\frac{1}{\operatorname{dn}[f_1 : k_1]}\frac{\operatorname{sn}[f_1 : k_1]}{1 + e\operatorname{cn}[f_1 : k_1]} - \tfrac{1}{2}\eta^2 g_m^2 (1-e^2)^{3/2} f_1 \\
&\quad - \eta^2\sqrt{1-e^2}\Big[\sum_{n=1}^{2} q_n \operatorname{sn}[nf_1 : k_1] - \eta^2 \sum_{n=1}^{6} m_n \sin nG_1\Big]
\end{aligned} \tag{9.57}
$$

as the final form for M_0.

Evaluation of M_{11}

For the evaluation of M_{11}, which it suffices to carry out to first order in η^2, we introduce the solution form (6.21) for $\cos\sigma$ into the defining relation (9.16b) for M_{11}. Setting $F_2 = f + \omega$, we have

$$
M_{11} = \int (1-N^2)\operatorname{sn}^2[F_2 : k_2]\,\mathrm{d}F_2 = (1-N^2)\int \operatorname{sn}^2[F_2 : k_2]\,\mathrm{d}F_2. \tag{9.58}
$$

In the above integral, the elliptic function has modulus k_2 — as will all elliptic functions appearing in this subsection. Accordingly, the modulus k_2 will be exhibited only as it becomes necessary. In effecting the integration, we follow the procedure that led from (9.39) to (9.44). From the double-angle formula, we have

$$
\operatorname{sn}^2 F_2 = \frac{1 - \operatorname{cn} 2F_2}{1 + \operatorname{dn} 2F_2} = \frac{1}{1 + \operatorname{dn} 2F_2} - \frac{\operatorname{cn} 2F_2}{1 + \operatorname{dn} 2F_2}. \tag{9.59}
$$

In the first factor we apply relation (9.32b), and in the second factor we apply relation (9.32a). To first order in η^2, we find

$$\begin{aligned}\mathrm{sn}^2 F_2 &= \tfrac{1}{2}[1 + \tfrac{1}{4}k_2^2\,\mathrm{sn}^2\,2F_2] - \tfrac{1}{2}\,\mathrm{cn}\,2F_2[\mathrm{dn}\,2F_2 + \tfrac{3}{4}k_2^2\,\mathrm{sn}^2\,2F_2] \\ &= \tfrac{1}{2}[1 - \mathrm{cn}\,2F_2\,\mathrm{dn}\,2F_2] + \tfrac{1}{8}k_2^2[\mathrm{sn}^2\,2F_2 - 3\,\mathrm{sn}^2\,2F_2\,\mathrm{cn}\,2F_2] \\ &= \tfrac{1}{2}[1 - \mathrm{cn}\,2F_2\,\mathrm{dn}\,2F_2] + \tfrac{1}{8}k_2^2[\sin^2 2G_2 - 3\sin^2 2G_2\cos 2G_2]\end{aligned} \tag{9.60}$$

where the relation of G_2 to F_2 is given by (8A.11), namely

$$G_2 = \frac{\pi}{2K_2}F_2 = [1 - \tfrac{1}{4}k_2^2 - \tfrac{5}{64}k_2^4]F_2. \tag{9.61}$$

Noting that $\sin^2 2G_2 = \frac{1}{2}(1 - \cos 4G_2)$, we can rewrite (9.60) in the form

$$\begin{aligned}\mathrm{sn}^2 F_2 &= \tfrac{1}{2}(1 + \tfrac{1}{8}k_2^2) - \tfrac{1}{2}\,\mathrm{cn}\,2F_2\,\mathrm{dn}\,2F_2 - \tfrac{1}{16}k_2^2[\cos 4G_2 + \frac{\mathrm{d}}{\mathrm{d}F_2}(\sin^3 2G_2)] \\ &= \tfrac{1}{2}(1 + \tfrac{1}{8}k_2^2) - \tfrac{1}{4}\frac{\mathrm{d}}{\mathrm{d}F_2}(\mathrm{sn}\,2F_2) - \tfrac{1}{64}k_2^2\frac{\mathrm{d}}{\mathrm{d}F_2}[\sin 4G_2 + 4\sin^3 2G_2] \\ &= \tfrac{1}{2}(1 + \tfrac{1}{8}k_2^2) \\ &\qquad - \frac{\mathrm{d}}{\mathrm{d}F_2}\Big[\tfrac{1}{4}\,\mathrm{sn}\,2F_2 + \tfrac{1}{64}k_2^2[3\sin 2G_2 + \sin 4G_2 - \sin 6G_2]\Big]\end{aligned} \tag{9.62}$$

and hence, on integration, apart from an additive constant, we have

$$\begin{aligned}&\int \mathrm{sn}^2 F_2\,\mathrm{d}F_2 = \\ &\tfrac{1}{2}(1 + \tfrac{1}{8}k_2^2)F_2 - \tfrac{1}{4}\,\mathrm{sn}[2F_2\colon k_2] - \tfrac{1}{64}k_2^2[3\sin 2G_2 + \sin 4G_2 - \sin 6G_2].\end{aligned} \tag{9.63}$$

The additive constant is to be chosen so that the integral vanishes for $f = 0$.

If we let

$$\gamma_2 = [1 - \tfrac{1}{4}k_2^2 - \tfrac{5}{64}k_2^4]\omega \tag{9.64}$$

and set

$$\begin{aligned}M_1 &= \int_0^f \mathrm{sn}^2[f + \omega\colon k_2]\,\mathrm{d}f \qquad (9.65) \\ &= \tfrac{1}{2}(1 + \tfrac{1}{8}k_2^2)f - \tfrac{1}{4}[\mathrm{sn}[2(f + \omega)\colon k_2] - \mathrm{sn}[2\omega\colon k_2]] \\ &\quad - \tfrac{1}{64}k_2^2\Big[3(\sin 2G_2 - \sin 2\gamma_2) + (\sin 4G_2 - \sin 4\gamma_2) - (\sin 6G_2 - \sin 6\gamma_2)\Big]\end{aligned}$$

then, from (9.58)

$$M_{11} = (1 - N^2)M_1. \tag{9.66}$$

— ◇ —

With M_0 given by (9.57) and M_1 given by (9.65), then for the mean anomaly M defined by relation (9.17), we have

$$M = M_0 + \eta^2 j_N(1 - e^2)^{3/2}(1 - N^2)M_1, \tag{9.67}$$

completing the formula for the time-angle relation to order η^4.

6

The Earth Satellite — Some Special Orbits

What makes the "critical inclination" so critical?

— *Mathematics student*

In this chapter, we discuss some particular orbits for the Earth satellite. Specifically, we investigate equatorial orbits, polar orbits, and orbits near the so-called "critical" inclination.

1 Orbits in the Near Equatorial Band

Prior to investigating the equatorial orbit, it is necessary to return to equation (7.29) of Chapter 5 and to develop the procedure for the reduction of that equation when the parameter B is negative, i.e., when

$$B = \eta^2 e_*^2 [q^2 - \eta^2 h^2 + \eta^4 e_*^2 q^2 h^2] \leq 0 \tag{1.1a}$$

where we have recalled (7.25b) of Chapter 5. Referring to the defining relations for h and q in (7.13) of Chapter 5, it is clear that the range (1.1a) includes the equatorial orbit, and we note that

$$B \to -\eta^4 e_*^2 h^2 \tag{1.1b}$$

when $q \to 0$ as $\nu \to 1$.

Taking (7.29) of Chapter 5 as the starting point, if we divide across by A, then the equation reads

$$\frac{\Lambda^2}{C^2(1 - \eta^2 \nu^2 h_2)} \frac{1}{j_w^2} \frac{1}{A} v'^2 = (1 - v^2)[1 - k_1^2 v^2] \tag{1.2}$$

where in this case we have set

$$k_1^2 = -\frac{B}{A} = -\eta^2 e_*^2 \frac{q^2 - \eta^2 h^2 + \eta^4 e_*^2 q^2 h^2}{1 - \eta^4 e_*^2 h^2} \tag{1.3}$$

which is clearly positive in the range defined by (1.1a) and is the analog, for the case of B negative, of relation (7.31) of Chapter 5, which applies to the case of B positive. We recall the definition of Λ^2 from (6.14) of Chapter 5, and when

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we substitute for A from (7.25a) of Chapter 5, we see that the coefficient of v'^2 in relation (1.2) may be written

$$\frac{1}{j_1^2} = \frac{(1+\eta^2\ell)m}{(1-\eta^2\nu^2 h_2)(1-\eta^4 e_*^2 h^2)}\frac{1}{j_w^2} \tag{1.4}$$

in which m is given by (6.6) and j_w^2 is given by (7.13a), both from Chapter 5.

Having defined k_1 in (1.3) and j_1 in (1.4), then, with the auxiliary variable f_1 defined by

$$f_1 = j_1 f, \tag{1.5}$$

equation (1.2) takes the form

$$\left(\frac{\mathrm{d}v}{\mathrm{d}f_1}\right)^2 = \frac{1}{j_1^2}v'^2 = (1-v^2)(1-k_1^2 v^2) \tag{1.6}$$

whose solution is given by

$$v = \mathrm{sn}[f_1 + f_0 \colon k_1] = \mathrm{sn}[j_1 f + f_0 \colon k_1] \tag{1.7}$$

in which f_0 is the arbitrary constant introduced by the integration. There follows that

$$v' = j_1\,\mathrm{cn}[f_1 + f_0 \colon k_1]\,\mathrm{dn}[f_1 + f_0 \colon k_1] \tag{1.8a}$$

$$v'' = -j_1^2\,\mathrm{sn}[f_1 + f_0 \colon k_1][\mathrm{dn}^2[f_1 + f_0 \colon k_1] + k_1^2\,\mathrm{cn}^2[f_1 + f_0 \colon k_1]]. \tag{1.8b}$$

With K_1 denoting the quarterperiod of the Jacobian elliptic function of modulus k_1, we see that if we take

$$f_0 = K_1 \tag{1.9}$$

and denoting the complementary modulus by $k_1' = \sqrt{1-k_1^2}$, we have

$$\begin{aligned} v' &= j_1\,\mathrm{cn}[f_1 + K_1 \colon k_1]\,\mathrm{dn}[f_1 + K_1 \colon k_1] \\ &= -j_1 k_1' \frac{\mathrm{sn}[f_1 \colon k_1]}{\mathrm{dn}[f_1 \colon k_1]}\frac{k_1'}{\mathrm{dn}[f_1 \colon k_1]}. \end{aligned} \tag{1.10}$$

It is clear that at $f_1 = 0$, then $v' = 0$ and $v'' < 0$, so that, with f_0 chosen in accordance with (1.9), the point $f = 0$ is a maximum point for v. From (7.28c) of Chapter 5, this implies that $f = 0$ is a maximum point for w; from (7.9c) of Chapter 5, it follows that $f = 0$ is a maximum point for u; and finally, from (7.5c) of Chapter 5, it follows that $f = 0$ is a minimum point for R.

Accordingly, the condition that the "angle" f be measured from "perigee", defined as the first point at which R reaches a minimum point, is satisfied by taking f_0 in accordance with (1.9). Returning to the solution form (1.7) with f_0 given by (1.9), we have

$$v = \operatorname{sn}[f_1 + K_1 : k_1] = \frac{\operatorname{cn}[f_1 : k_1]}{\operatorname{dn}[f_1 : k_1]} = \frac{\operatorname{cn}[j_1 f : k_1]}{\operatorname{dn}[j_1 f : k_1]}. \tag{1.11a,b,c}$$

From (7.28b) of Chapter 5, there follows that

$$w = \frac{\operatorname{cn}[j_1 f : k_1] + \delta \operatorname{dn}[j_1 f : k_1]}{\operatorname{dn}[j_1 f : k_1] + \delta \operatorname{cn}[j_1 f : k_1]} \tag{1.12}$$

and, hence, recalling (7.9b) of Chapter 5, we have

$$u = \frac{1}{p_*} \frac{(1 + \delta e_*) \operatorname{dn}[j_1 f : k_1] + (e_* + \delta) \operatorname{cn}[j_1 f : k_1]}{\operatorname{dn}[j_1 f : k_1] + \delta \operatorname{cn}[j_1 f : k_1]}. \tag{1.13}$$

Recalling (7.5b) of Chapter 5, we see that on taking the reciprocal of (1.13), there follows

$$R = p_* \frac{\operatorname{dn}[j_1 f : k_1] + \delta \operatorname{cn}[j_1 f : k_1]}{(1 + \delta e_*) \operatorname{dn}[j_1 f : k_1] + (e_* + \delta) \operatorname{cn}[j_1 f : k_1]}. \tag{1.14}$$

This suggests the form for the final parameters. Recalling that p_* and e_* are given respectively by (7.7a,b) of Chapter 5, we set

$$p = \frac{p_*}{1 + \delta e_*}, \quad e = \frac{e_* + \delta}{1 + \delta e_*} \tag{1.15a,b}$$

whereby the solution (1.14) reads

$$R = p \frac{\operatorname{dn}[j_1 f : k_1] + \delta \operatorname{cn}[j_1 f : k_1]}{\operatorname{dn}[j_1 f : k_1] + e \operatorname{cn}[j_1 f : k_1]} \tag{1.16}$$

while (1.13) may be replaced by

$$pu = \frac{\operatorname{dn}[j_1 f : k_1] + e \operatorname{cn}[j_1 f : k_1]}{\operatorname{dn}[j_1 f : k_1] + \delta \operatorname{cn}[j_1 f : k_1]}. \tag{1.17}$$

For the present case where B is negative, the solution forms (1.16) and (1.17) above respectively replace the solution forms (7.41) and (7.42) of Chapter 5, there derived for the case where B is positive.

The subsequent discussion of the parameters follows the same pattern as that following (7.42) of Chapter 5, except that now k_1 is given by (1.3) above, whereas in the former case it was given by (7.31) of Chapter 5. (Also j_1 is given by (1.4) instead of (7.32) of Chapter 5; however, j_1 does not appear in the subsequent discussion.) Accordingly, as in (7.46) of Chapter 5, we have

$$d = h[1 + \eta^2 h[1 + \eta^2 (h - q^2 e_*^2)](1 - e_*^2)][1 - \eta^2 q^2 e_*^2 + \eta^4 e_*^2 (q^4 e_*^2 + h^2)] \tag{1.18a}$$

and again

$$\delta = -\eta^2 e d. \tag{1.18b}$$

However, in place of (7.49) of Chapter 5, we now have

$$g^2 = -\left(\frac{e_*}{e}\right)^2 \frac{q^2 - \eta^2 h^2 + \eta^4 e_*^2 q^2 h^2}{1 - \eta^4 e_*^2 h^2} \tag{1.19a}$$

with

$$k_1^2 = \eta^2 e^2 g^2 \tag{1.19b}$$

as in the former case. Relations (1.18) and (1.19) are the forms to be used with any utilization of the solution forms (1.16) and (1.17).

In noting the distinct forms (1.16) and (1.17) above, for the case of B negative and in distinguishing them from the forms (7.41–2) of Chapter 5 for B positive, we recognize that the use of these solution forms in the integration of the φ-coordinate and in the derivation of the time-angle relation will require appropriate modification of the analysis performed in Sections 8 and 9 of Chapter 5. However, a retrospect shows that the solution (1.16) is formally identical with the solution (9.29) of Chapter 3. Hence the problem of two fixed centers provides the more aligned model for this component of the subsequent analysis. In the derivation of the time-angle relation, the substitution for R will follow that given in the analysis of M_0 in Section 10 of Chapter 3. In the integration of the φ-coordinate, the substitution for R will follow that laid out in the integration of φ_2 in Section 9 of Chapter 4.

The equatorial orbit when $\nu = 1$ and hence from (7.13c) of Chapter 5 when $q = 0$ is clearly in the range where B from (1.1a) is negative. Hence for that case, the above analysis is applicable in dealing with the R-equation. The equatorial orbit is the focus of the next section.

2 The Equatorial Orbit

In dealing with the equatorial orbit when

$$\nu = 1 \tag{2.1}$$

we start by considering the equation for $S = \cos\sigma$ by referring to Section 6 of Chapter 5. Setting $\nu = 1$ in equation (6.2) therein, we have

$$\begin{aligned} \frac{\Lambda^2}{C^2} S'^2 &= -(1 + \eta^2 \ell) S^2 + \eta^2 \ell S^4 \\ &= -S^2 [1 + \eta^2 \ell (1 - S^2)] \end{aligned} \tag{2.2}$$

which admits as the only real solution

$$S = 0, \quad S' = 0 \tag{2.3}$$

or in terms of σ, the expected result

$$\cos\sigma = 0, \quad \sigma = \pi/2. \tag{2.4a,b}$$

From (6.6) and (6.15) of Chapter 5, we see that when $\nu = 1$, we have

$$m = 1, \quad k_2 = 0 \tag{2.5a,b}$$

and we also note that

$$\Lambda^2 = C^2(1 + \eta^2 \ell), \quad N = 1 \tag{2.6a,b}$$

which respectively follow from (6.14) and (6.20) of Chapter 5.

Turning next to the R-equation, we first refer to Section 7 of Chapter 5. Setting $\nu = 1$ in relation (7B.17b) therein, we have

$$h_* = 2(2 - \ell) - 3\eta^2 \ell(8 - \ell) \tag{2.7}$$

while from (7B.25) when $\nu = 1$, we have

$$h_0 = 1 + \eta^2(4 - \ell) - 3\eta^4 \ell(8 - \ell) \tag{2.8a}$$

$$h_1 = 1 - 3\eta^4 \ell(8 - \ell) \tag{2.8b}$$

$$h_2 = (4 - \ell) + 8\eta^2(2 - \ell) - 3\eta^4 \ell(8 - \ell)(4 - \ell). \tag{2.8c}$$

With these values for h_0, h_1, and h_2 and with $\nu = 1$, relations (7.7) of Chapter 5 for p_* and e_* take the form

$$p_* = p_0 \frac{1 - \eta^2 h_2}{1 - \eta^2 \ell h_0}, \quad 1 - e_*^2 = \frac{1 - \eta^2 h_2}{(1 - \eta^2 \ell h_0)^2}(1 - e_0^2). \tag{2.9a,b}$$

Further, setting $\nu = 1$ in relations (7.13) of Chapter 5, we find for the equatorial orbit

$$j_w^2 = 1 - 2\eta^2[1 + \eta^2(8 - 3\ell)] \tag{2.10a}$$

$$h = -\frac{1}{j_w^2}[1 + \eta^2(8 - 3\ell)] \tag{2.10b}$$

$$q = 0 \tag{2.10c}$$

and from relation (7.21) of Chapter 5, we have to second order in η^2,

$$\delta = -\eta^2 e_* h[1 + \eta^4 e_*^2]. \tag{2.11}$$

Noting relation (1.1) of the present chapter, we see that when $\nu = 1$ and hence $q = 0$, relation (1.3) for k_1^2, to second order in η^2, takes the form

$$k_1^2 = \frac{\eta^4 e_*^2 h^2}{1 - \eta^4 e_*^2}. \tag{2.12}$$

Similarly, when we consider relation (1.4) of the present chapter, with $\nu = 1$, and recall from (2.5a) above that for the equatorial orbit $m = 1$, then, to second order in η^2, we have

$$\frac{1}{j_1^2} = \frac{1 + \eta^2 \ell}{(1 - \eta^2 h_2)(1 - \eta^4 e_*^2)} \frac{1}{j_w^2}. \tag{2.13}$$

In relations (2.11)–(2.13), we note that h_2 as well as h_0 and h_1 are given by (2.8), e_* is related to e_0 through (2.9b), and j_w and h are given by (2.10).

When we introduce $\nu = 1$ together with its corollary $q = 0$ into relations (1.18) of the present chapter, we obtain the forms taken by d and δ for the equatorial orbit, which to second order in η^2 may be written

$$d = h[1 + \eta^2 h(1 + \eta^2)(1 - e_*^2)][1 + \eta^4 e_*^2] \tag{2.14a}$$

$$\delta = -\eta^2 ed \tag{2.14b}$$

with e given in terms of e_* by

$$e = \frac{e_* + \delta}{1 + \delta e_*}. \tag{2.14c}$$

Similarly, with relations (1.19) when we introduce $\nu = 1$, $q = 0$ we see that, to second order in η^2

$$g^2 = \eta^2 \left(\frac{e_*}{e}\right)^2 \frac{h^2}{1 - \eta^4 e_*^2} \tag{2.15a}$$

with

$$k_1^2 = \eta^2 e^2 g^2 \tag{2.15b}$$

wherein h is given by (2.10b) above.

For the equatorial orbit, the solution for the R-equation is given by the form (1.16) wherein p and e are obtained from combining (1.15) with (2.9), δ is given by (2.14), j_1 is given by (2.13), and k_1 is given by (2.15), each of which assumes a form simpler than in the general case.

It is further to be noted that for the equatorial orbit, there is a significant simplification in the integration both of the equation for the φ-coordinate and of the time-angle relation. Referring to Section 8 of Chapter 5, we see that as $N = 1$ and $\sigma = \pi/2$, it follows from (8.14a) of Chapter 5 that, in this case,

$$\varphi_1' = 1 \tag{2.16a}$$

and hence, apart from an additive constant,

$$\varphi_1 = f. \tag{2.16b}$$

And noting that now (pu) is given by (1.17) above, the integration of (8.14b) of Chapter 5 will follow the pattern outlined in Section 9 of Chapter 4.

For the time-angle relation, it is clear from (9.16b) of Chapter 5 that

$$M_{11} = 0 \tag{2.17}$$

and the problem is reduced to the determination of M_0 from (9.16a) of Chapter 5, which will follow the pattern outlined in Section 10 of Chapter 3, noting that here pu is given by (1.17) above.

Finally, we observe that as in this case $\cos\sigma = 0$, there is only one independent frequency associated with the equatorial orbit, namely that of the Jacobian elliptic function of modulus k_1 as given by (2.15) having the quantity K_1 as its associated quarterperiod.

3 The Polar Orbit

In the case of polar orbits, the polar component of the angular momentum vanishes so that

$$C_3 = 0, \quad \nu = 0. \tag{3.1a,b}$$

If, in equations (5.7) and (5.8) of Chapter 5, we set $\nu = 0$, then the equations for R and σ take the form

$$\frac{\Lambda^2}{C^2} R'^2 = -(R^2 + b^2)\Big[1 - 2\frac{R}{p_0} + \frac{R^2}{ap_0}\Big] \tag{3.2a}$$

$$\frac{\Lambda^2}{C^2} \sin^2\sigma\,\sigma'^2 = (1 - \cos^2\sigma)\Big[1 - \frac{b^2}{ap_0}\cos^2\sigma\Big]. \tag{3.2b}$$

When we introduce the notation of (5.9) and (5.10) of Chapter 5 and set $S = \cos\sigma$, we have the alternate forms

$$\frac{\Lambda^2}{C^2}\frac{R'^2}{p_0^2} = -\Big[\Big(\frac{R}{p_0}\Big)^2 + \eta^2\Big]\Big[1 - 2\frac{R}{p_0} + (1 - e_0^2)\Big(\frac{R}{p_0^2}\Big)^2\Big] \tag{3.3a}$$

$$\frac{\Lambda^2}{C^2} S'^2 = (1 - S^2)[1 - \eta^2 \ell S^2]. \tag{3.3b}$$

On setting

$$\Lambda = C, \quad k_2^2 = \eta^2\ell = \eta^2(1 - e_0^2), \quad k_2 = \eta\sqrt{1 - e_0^2} \tag{3.4a,b,c}$$

there follows that

$$S = \cos\sigma = \mathrm{sn}[f + \omega : k_2] \tag{3.5}$$

as the solution for the σ-equation (3.3b).

Turning to the R-equation (3.3a) and setting $\Lambda = C$ in accord with (3.4a), then on factoring the second quadratic on the right, we have

$$\Big(\frac{R'}{p_0}\Big)^2 = -\Big[\Big(\frac{R}{p_0}\Big)^2 + \eta^2\Big]\Big[1 - (1 + e_0)\frac{R}{p_0}\Big]\Big[1 - (1 - e_0)\frac{R}{p_0}\Big]. \tag{3.6}$$

If we denote the zeros of the second quadratic by R_1 and R_2 respectively, so that

$$R_1 = \frac{p_0}{1 + e_0}, \quad R_2 = \frac{p_0}{1 - e_0}, \quad R_1 \le R_2 \tag{3.7a,b,c}$$

we see that the range of the orbits is restricted to the elliptic annulus bounded on the "inside" by the ellipse $R = R_1$ and on the "outside" by the ellipse $R = R_2$. In the limit when $\eta \to 0$, these boundaries coincide with the bounding circles $p_0/(1 + e_0)$ and $p_0/(1 - e_0)$ of the Kepler problem.

We next note the values assumed by the relevant parameters when $\nu = 0$. Recalling the defining relations (5.14) of Chapter 5

$$\lambda^2 = \eta^2 \ell^2 = \eta^2 (1 - e_0^2)^2 \tag{3.8}$$

we see from (7A.12) of Chapter 5 that, when $\nu = 0$

$$\lambda_* = 12\lambda^2 \frac{1-\ell}{(\ell + \lambda^2)^2} = 12\eta^2 \frac{1-\ell}{(1+\eta^2\ell)^2}, \quad s = 1 \tag{3.9a,b}$$

and equation (7A.15) of Chapter 5 becomes

$$Z_*^3 - 3(1-\lambda_*)Z_* - 2(1+3\lambda_*) = 0 \tag{3.10a}$$

or, in its factored form,

$$(Z_* - 2)[Z_* + (1 - i\sqrt{3\lambda_*})][Z_* + (1 + i\sqrt{3\lambda_*})] \tag{3.10b}$$

and clearly the root

$$Z_* = 2 \tag{3.11}$$

corresponds to the root determined by (7A.16) of Chapter 5.

For the approximate formulae derived in Section 7B of Chapter 5, we note that corresponding to relation (7B.1) we have, when $\nu = 0$, $s = 1$,

$$D(1 + 3\lambda_*) = \sqrt{\lambda_*}(9 + 6\lambda_* + \lambda_*^2)^{\frac{1}{2}} = \sqrt{\lambda_*}(\lambda_* + 3) = O(\eta) \tag{3.12}$$

and relation (7B.10) of Chapter 5 is now replaced by (3.11) above. It follows that relations (7B.12) and (7B.19) of Chapter 5 are respectively replaced in the present case ($\nu = 0$) by

$$Z = 1 \tag{3.13}$$

and

$$U = 1, \quad U^{-1} = 1, \quad (1 - \ell + \lambda^2)U^{-1} = 1 - \ell + \lambda^2. \tag{3.14a,b,c}$$

It further follows that relations (7B.21) of Chapter 5 now have the form

$$V = \tfrac{1}{4} + \lambda^2 = \tfrac{1}{4} + \eta^2\ell^2, \quad W = -\tfrac{3}{4} + \ell \tag{3.15a,b}$$

and the factorization (7B.22) of Chapter 5 for $f(Y)$ here takes the form

$$f(Y) = [Y^2 + \eta^2\ell^2][Y^2 - 2Y + \ell] \tag{3.16}$$

consistent with (3.3a) above.

Setting $s = 1$ in the defining relation (7B.16) of Chapter 5, we find for the present case ($\nu = 0$),

$$1 + \eta^2 h_* = \frac{1}{(1+\eta^2\ell)^2}(1 - \tfrac{1}{3}\lambda_*) \tag{3.17}$$

from which there follows

$$h_* = -2(2-\ell) + \eta^2\ell(16-13\ell) \tag{3.18a}$$
$$= -2(1+e_0^2) + \eta^2\ell(3+13e_0^2) \tag{3.18b}$$

which could be obtained directly by setting $\nu = 0$ in relation (7B.17b).

For the present case $\nu = 0$, we further have

$$h_0 = 1 + \eta^2 h_* = 1 - 2\eta^2(2-\ell) + \eta^4\ell(16-13\ell) \tag{3.19a}$$
$$h_1 = 1 + \eta^2(h_* - \ell) = 1 - \eta^2(4-\ell) + \eta^4\ell(16-13\ell) \tag{3.19b}$$
$$\begin{aligned} h_2 &= 4(1+\eta^2 h_*) - \ell h_1 \\ &= (4-\ell)[1+\eta^2\ell + \eta^4\ell(16-13\ell)] - 8\eta^2(2-\ell) \\ &= (4-\ell) - \eta^2[\ell^2 - 12\ell + 16] + \eta^4\ell(4-\ell)(16-13\ell) \end{aligned} \tag{3.19c}$$

corresponding to relations (7B.25) of Chapter 5.

Returning now to equation (3.3a) and setting $\Lambda = C$ in accord with (3.4a), the R-equation reads

$$R'^2 = -[R^2 + \eta^2 p_0^2]\Big[1 - 2\frac{R}{p_0} + (1-e_0^2)\Big(\frac{R}{p_0}\Big)^2\Big]. \tag{3.20}$$

When we introduce the variable u by setting

$$u = \frac{1}{R}, \quad u' = -\frac{1}{R^2}R'; \qquad R = \frac{1}{u}, \quad R' = -\frac{1}{u^2}u' \tag{3.21a,b;c,d}$$

then in terms of u, equation (3.20) becomes

$$\begin{aligned} u'^2 &= -\Big(u^2 - \frac{2}{p_0}u + \frac{1}{ap_0}\Big)(1+\eta^2 p_0^2 u^2) \\ &= -\Big[\Big(u - \frac{1}{p_0}\Big)^2 - \frac{1}{p_0^2}(1-\ell)\Big][1+\eta^2 p_0^2 u^2] \\ &= -\Big[\Big(u - \frac{1}{p_0}\Big)^2 - \frac{e_0^2}{p_0^2}\Big][1+\eta^2 p_0^2 u^2]. \end{aligned} \tag{3.22}$$

Relations (7.7) of Chapter 5 show that in this case when $\nu = 0$

$$p_* = p_0, \quad e_* = e_0 \tag{3.23a,b}$$

and we may set

$$u - \frac{1}{p_0} = \frac{e_0}{p_0}w, \quad u = \frac{1}{p_0}(1 + e_0 w) \tag{3.24a,b}$$

so that, when expressed in terms of w, the above differential equation becomes

$$\begin{aligned} w'^2 &= (1-w^2)[1+\eta^2(1+e_0w)^2] \\ &= (1+\eta^2)(1-w^2)\Big[1 + \frac{2\eta^2}{1+\eta^2}e_0 w + \frac{\eta^2}{1+\eta^2}e_0^2 w^2\Big]. \end{aligned} \tag{3.25}$$

In the decomposition of the quadratic factors, we have

$$1 - w^2 = J^2[(1-\delta w)^2 - (w-\delta)^2] = J^2(1-\delta^2)(1-w^2) \tag{3.26a}$$

and

$$\begin{aligned} 1 + \frac{2\eta^2}{1+\eta^2}e_0 w + \frac{\eta^2}{1+\eta^2}e_0^2 w^2 &= J^2[A(1-\delta w)^2 + B(w-\delta)^2] \\ &= J^2[(A+B\delta^2) - 2(A+B)\delta w + (A\delta^2+B)w^2] \end{aligned} \tag{3.26b}$$

from which there follows respectively

$$J^2(1-\delta^2) = 1 \tag{3.27}$$

$$J^2(A+B\delta^2) = 1, \quad J^2(A+B)\delta = -\frac{\eta^2}{1+\eta^2}e_0, \quad J^2(A\delta^2+B) = \frac{\eta^2}{1+\eta^2}e_0^2. \tag{3.28a,b,c}$$

Forming the difference $(3.28a) - (3.28c)$, we find

$$A - B = J^2(1-\delta^2)(A-B) = 1 - \frac{\eta^2}{1+\eta^2}e_0^2 \tag{3.29a}$$

while from the sum $(3.28a) + (3.28c)$, we have

$$\frac{1+\delta^2}{1-\delta^2}(A+B) = J^2(1+\delta^2)(A+B) = 1 + \frac{\eta^2}{1+\eta^2}e_0^2 \tag{3.29b}$$

and from (3.28b) we have

$$\frac{\delta}{1-\delta^2}(A+B) = -\frac{\eta^2 e_0}{1+\eta^2}. \tag{3.29c}$$

Dividing (3.29c) by (3.29b) yields

$$\frac{\delta}{1+\delta^2} = -\frac{\eta^2 e_0}{1+\eta^2(1+e_0^2)} = -\eta^2 e_1 \tag{3.30a}$$

where we have written

$$e_1 = \frac{e_0}{1+\eta^2(1+e_0^2)}. \tag{3.30b}$$

From (3.30a), we have as equation for δ

$$1 + \frac{1}{\eta^2 e_1}\delta + \delta^2 = 0 \tag{3.31a}$$

with solutions

$$\delta = -\frac{1}{2\eta^2 e_1}\left[1 \mp \sqrt{1-(2\eta^2 e_1)^2}\right]. \tag{3.31b}$$

When we discard the + sign within the square bracket as irrelevant since it is meaningless in the limit as $\eta \to 0$, we have

$$\delta = -\frac{1}{2\eta^2 e_1}\left[1 - \sqrt{1 - (2\eta^2 e_1)^2}\right] = -\eta^2 e_1[1 + \eta^4 e_1^2 + O(\eta^8)] \tag{3.32}$$

and hence to second order in η^2 we have, on recalling (3.30b),

$$\delta = -\eta^2 e_0[1 - \eta^2(1 + e_0^2) + \eta^4(1 + 3e_0^2 + e_0^4)] = -\eta^2 e_0 d_0 \tag{3.33a}$$

wherein we have written

$$d_0 = 1 - \eta^2(1 + e_0^2) + \eta^4(1 + 3e_0^2 + e_0^4). \tag{3.33b}$$

From (3.31a), we immediately have

$$1 + \delta^2 = -\frac{1}{\eta^2 e_1}\delta = [1 + \eta^2(1 + e_0^2)]d_0. \tag{3.34}$$

We now recall from (3.29) above that

$$A + B = \frac{1 - \delta^2}{1 + \delta^2}\left[1 + \frac{\eta^2}{1 + \eta^2}e_0^2\right] \tag{3.35a}$$

$$A - B = 1 - \frac{\eta^2}{1 + \eta^2}e_0^2 \tag{3.35b}$$

from which, by addition and subtraction, we obtain

$$A = \frac{1}{1 + \delta^2}\left[1 - \frac{\eta^2}{1 + \eta^2}e_0^2\delta^2\right] \tag{3.36a}$$

$$B = \frac{1}{1 + \delta^2}\left[\frac{\eta^2 e_0^2}{1 + \eta^2} - \delta^2\right] = \frac{\eta^2 e_0^2}{1 + \delta^2}\left[\frac{1}{1 + \eta^2} - \eta^2 d_0^2\right] \tag{3.36b}$$

so that, to second order in η^2, we may write

$$A = \frac{1}{1 + \delta^2} = \frac{1}{[1 + \eta^2(1 + e_0^2)]d_0} \tag{3.37a}$$

$$B = \frac{\eta^2 e_0^2}{1 + \delta^2} \cdot \frac{1 - \eta^2(1 + \eta^2)d_0^2}{1 + \eta^2} = \eta^2 e_0^2 d_1^2 \tag{3.37b}$$

where we have written

$$d_1^2 = \frac{1 - \eta^2(1 + \eta^2)d_0^2}{(1 + \eta^2)(1 + \delta^2)}. \tag{3.37c}$$

Finally, we note from (3.35a) that, to second order in η^2, we may write

$$A + B = \frac{(1 - \delta^2)[1 + \eta^2(1 + e_0^2)]}{(1 + \delta^2)(1 + \eta^2)} = (1 - 2\eta^4 e_0^2 d_0^2)\frac{1 + \eta^2(1 + e_0^2)}{1 + \eta^2} \tag{3.38}$$

and hence, noting (3.37b),

$$\frac{B}{A+B} = \eta^2 e_0^2 \frac{d_1^2(1+\eta^2)}{[1-2\eta^4 e_0^2 d_0^2][1+\eta^2(1+e_0^2)]} = \eta^2 e_0^2 d_2^2 \tag{3.39a}$$

where we have written

$$d_2^2 = \frac{(1+\eta^2)d_1^2}{[1-2\eta^4 e_0^2 d_0^2][1+\eta^2(1+e_0^2)]}. \tag{3.39b}$$

With the above values for the parameters, we introduce the decompositions (3.26) into the differential equation (3.25) so that we have

$$\frac{w'^2}{1+\eta^2} = J^4[(1-\delta w)^2 - (w-\delta)^2][A(1-\delta w)^2 + B(w-\delta)^2] \tag{3.40}$$

which when we divide across by $J^4(1-\delta w)^4$ becomes

$$\frac{1}{1+\eta^2}\left[\frac{w'}{J^2(1-\delta w)}\right]^2 = \left[1-\left(\frac{w-\delta}{1-\delta w}\right)^2\right]\left[A + B\left(\frac{w-\delta}{1-\delta w}\right)^2\right] \tag{3.41}$$

and when we introduce the transformation

$$v = \frac{w-\delta}{1-\delta w}, \quad v' = \frac{w'}{J^2(1-\delta w)^2} \tag{3.42a,b}$$

takes the form

$$\begin{aligned} \frac{1}{1+\eta^2} v'^2 &= (1-v^2)(A+Bv^2) \\ &= (1-v^2)[(A+B) - B(1-v^2)] \end{aligned} \tag{3.43}$$

and hence

$$\frac{1}{(1+\eta^2)(A+B)} v'^2 = (1-v^2)\left[1 - \frac{B}{A+B}(1-v^2)\right]. \tag{3.44}$$

Accordingly, we write

$$\begin{aligned} j_1^2 = (1+\eta^2)(A+B) &= \frac{(1-\delta^2)}{1+\delta^2}[1+\eta^2(1+e_0^2)] \\ &= (1-2\eta^4 e_0^2 d_0^2)[1+\eta^2(1+e_0^2)] \end{aligned} \tag{3.45a}$$

and

$$k_1^2 = \frac{B}{A+B} = \eta^2 e_0^2 d_2^2 \tag{3.45b}$$

and, noting (3.39a), the differential equation (3.44) reads

$$\left(\frac{\mathrm{d}v}{\mathrm{d}f_1}\right)^2 = (1-v^2)[1-k_1^2(1-v^2)] \tag{3.46}$$

with solution

$$v = \mathrm{cn}[j_1 f : k_1]. \tag{3.47}$$

There follows that

$$w = \frac{\mathrm{cn}[j_1 f : k_1] + \delta}{1 + \delta\,\mathrm{cn}[j_1 f : k_1]} \tag{3.48a}$$

$$u = \frac{1}{p_0}\frac{(1 + e_0\delta) + (e_0 + \delta)\,\mathrm{cn}[j_1 f : k_1]}{1 + \delta\,\mathrm{cn}[j_1 f : k_1]}. \tag{3.48b}$$

When we set

$$p = \frac{p_0}{1 + e_0\delta}, \quad e = \frac{e_0 + \delta}{1 + e_0\delta} \tag{3.49a,b}$$

then (3.48b) takes the form

$$u = \frac{1}{p}\frac{1 + e\,\mathrm{cn}[j_1 f : k_1]}{1 + \delta\,\mathrm{cn}[j_1 f : k_1]} \tag{3.50}$$

and for R we have the solution

$$R = p\frac{1 + \delta\,\mathrm{cn}[j_1 f : k_1]}{1 + e\,\mathrm{cn}[j_1 f : k_1]}. \tag{3.51}$$

In the derivation of the time-angle relation, there is a corresponding simplification in the parametric relations, but the analysis remains as outlined in Section 9 of Chapter 5.

4 The "Critical" Inclination

When perturbation theories are applied to the differential equations of Celestial Mechanics, then in the integration of expressions in which there is interaction between the associated basic frequencies, there may arise the problem of the "small divisor". From the manner in which it arises, the feature is frequently compared to the dynamic phenomenon of resonance. The comparison may not always be fruitful as dynamic resonance generally reflects physical reality, whereas the "small divisor" may be a consequence of the particular expansion scheme adopted.

In Celestial Mechanics, a resonance may result from the interaction of the frequencies of the motions of the respective body-masses. However, the problem of the "small divisor" may arise from the interaction of the frequencies of the respective coordinates of the motion of an individual particle. We shall indicate in the present case how this may follow from the attempt to force the solution into a representation not natural to its integrated form.

In the case of the Earth satellite when the perturbation is applied, the problem of the "small divisor" arises in the vicinity of the so-called "critical" inclination determined — to a first approximation — as $\nu^2 = 1/5$. It is natural to ask what light may be shed on this issue by the present integrated solution to the problem in its separated form.

At this point, it should be noted that perturbation theories are (generally) formulated as differential equations for the variation of the (slowly varying) instantaneous Kepler elements. We shall explore how the present integrated solution is transformed as we track it through the attempt to fit it into conformity with the instantaneous Kepler elements in a spherical coordinate system. We shall focus on this question as it is manifested in the solution for the σ-coordinate of the *spheroidal* coordinate system.

We recall the solution form (6.21) of Chapter 5 for the σ-coordinate in the *spheroidal* system, namely,

$$\cos\sigma = \sqrt{1 - N^2}\,\mathrm{sn}[f + \omega : k_2] \tag{4.1a}$$

in which the regularizing independent variable f is given by relation (5.5) of Chapter 5, namely,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\Lambda}{R^2 + b^2\cos^2\sigma}. \tag{4.1b}$$

In terms of the slowly varying instantaneous Kepler elements ν_k, ω_k, the corresponding relations in the *spherical* coordinate system would have the form

$$\cos\theta = \sqrt{1 - \nu_k^2}\,\sin[f_k + \omega_k] \tag{4.2a}$$

in which the regularizing independent variable f_k is defined for the instantaneous Kepler orbit in the form

$$\frac{\mathrm{d}f_k}{\mathrm{d}t} = \frac{\lambda_k}{r^2}. \tag{4.2b}$$

The form of relations (4.2) are a reflection of setting $\eta = 0$ in (4.1). The perturbation is now to be explored through the variation of the Kepler elements ω_k and ν_k.

To facilitate comparison, it is appropriate to reset the solution form (4.1) in terms of the spherical coordinate system by recalling relations (2.1c) and (2.2) of Chapter 5; there follows that

$$\cos\theta = \frac{R}{r}\cos\sigma = \frac{R}{\sqrt{R^2 + b^2\sin^2\sigma}}\cos\sigma$$

and with p given by (7.39a) combined with (7.7a), both of Chapter 5, then, writing $\overline{\eta} = b/p$, we have

$$\cos\theta \doteq \frac{1}{\sqrt{1 + \frac{b^2}{p^2}(pu)^2(1 - \cos^2\sigma)}}\cos\sigma$$

which, on the introduction of $\cos\sigma$ from (6.21) and pu from (7.42), both of Chapter 5, becomes

$$\cos\theta = \frac{\sqrt{1-N^2}\,\mathrm{sn}[f+\omega : k_2]}{\sqrt{1+\overline{\eta}^2\left[\frac{1+e\,\mathrm{cn}[j_1 f : k_1]}{1-\eta^2 ed\,\mathrm{cn}[j_1 f : k_1]}\right]^2\left[\mathrm{cn}^2[f+\omega : k_2]+N^2\,\mathrm{sn}^2[f+\omega : k_2]\right]}}. \tag{4.3}$$

From the identification of representation (4.2a) with (4.3), there follow the respective identifications of the amplitudes and of the normalized oscillation. From the identification of the amplitudes, we have

$$\nu_k = \sqrt{1-\frac{1-N^2}{1+\overline{\eta}^2\left[\frac{1+e\,\mathrm{cn}[j_1 f : k_1]}{1-\eta^2 de\,\mathrm{cn}[j_1 f : k_1]}\right]^2\left[\mathrm{cn}^2[f+\omega : k_2]+N^2\,\mathrm{sn}^2[f+\omega : k_2]\right]}} \tag{4.4}$$

and from the identification of the normalized oscillation, we have

$$\sin(f_k+\omega_k) = \mathrm{sn}[f+\omega : k_2] = \sin\mathrm{am}[f+\omega : k_2] \tag{4.5}$$

which for the angle element ω_k implies

$$\omega_k = \mathrm{am}[f+\omega : k_2] - f_k. \tag{4.6}$$

The final step would be the determination of f in terms of f_k and the introduction of that representation into relations (4.4) and (4.6); in the sequel, we confine our attention to the latter — namely, to the relation (4.6) for ω_k.

It is first necessary to determine f_k in terms of f. This is done by combining relation (4.1b) with relation (4.2b) to obtain

$$\begin{aligned}\frac{\mathrm{d}f_k}{\mathrm{d}f} &= \frac{\lambda_k}{\Lambda}\cdot\frac{R^2+b^2\cos^2\sigma}{r^2} = \frac{\lambda_k}{\Lambda}\cdot\frac{R^2+b^2\cos^2\sigma}{R^2+b^2\sin^2\sigma}\\ &= \frac{\lambda_k}{\Lambda}\cdot\frac{1+\overline{\eta}^2(pu)^2\cos^2\sigma}{1+\overline{\eta}^2(pu)^2(1-\cos^2\sigma)}\end{aligned} \tag{4.7}$$

which to first order in η^2 (or $\overline{\eta}^2$) may be written

$$\frac{\mathrm{d}f_k}{\mathrm{d}f} = \frac{\lambda_k}{\Lambda}[1-\overline{\eta}^2(pu)^2(1-2\cos^2\sigma)+O(\eta^4)]. \tag{4.8}$$

In the above expression, it is evident that the two frequencies, associated with the solution forms for the respective coordinates, will interact. Hence, in the integration there arises the possibility of the “small divisor”, which can arise when the two frequencies coincide, i.e., near a periodic solution.

The question then becomes one of determining what values of the parameters lead to the coincidence of the frequencies of the two coordinates R and σ — or in the solution forms for (pu) and $\cos\sigma$. The quarterperiod for $\cos\sigma$ is clearly K_2 while that for (pu) is K_1/j_1. Hence, the question becomes one of determining when the expression

$$j_1 K_2 - K_1 \tag{4.9}$$

vanishes. We shall carry out the calculations to first order in η^2.

From relation (6.6) of Chapter 5, there follows that, to first order in η^2

$$(1+\eta^2\ell)m = 1 + \eta^2\ell\nu^2 \tag{4.10a}$$

$$\frac{1}{(1+\eta^2\ell)m} = 1 - \eta^2\ell\nu^2 \tag{4.10b}$$

$$\frac{1}{(1+\eta^2\ell)^2 m^2} = 1 - 2\eta^2\ell\nu^2 \tag{4.10c}$$

and hence, recalling relation (6.15) of Chapter 5, we have

$$k_2^2 = \eta^2\ell(1-\nu^2) \tag{4.11}$$

to first order in η^2.

Next we note that from relation (7.13a) of Chapter 5, we find, to first order in η^2, that

$$j_w^2 = 1 + \eta^2(1-3\nu^2), \quad \frac{1}{j_w^2} = 1 - \eta^2(1-3\nu^2) \tag{4.12a,b}$$

and from combining (7.13b) with (7.13c), both of Chapter 5, there follows that

$$q^2 - \eta^2 h^2 = \frac{1}{j_w^2}\Big[(1-\nu^2)[1+\eta^2\nu^2(12-5\ell)] - \frac{\eta^2(1-2\nu^2)^2}{j_w^2}\Big].$$

Consistent with neglecting terms of order η^4, we may, in the last term within the large square bracket, take $j_w^2 \approx 1$; on regrouping, we find

$$q^2 - \eta^2 h^2 = \frac{1}{j_w^2}\Big[(1-\nu^2) - \eta^2[1 - \nu^2(16-5\ell) + \nu^4(16-5\ell)]\Big].$$

We now introduce $1/j_w^2$ from (4.12b) and expand the product; on neglecting terms of order η^4 a rearrangement yields

$$q^2 - \eta^2 h^2 = (1-\nu^2) - \eta^2[2 - 5\nu^2(4-\ell) + \nu^4(19-5\ell)]. \tag{4.13}$$

We further note from (7.7b) of Chapter 5 that it is consistent with neglecting terms of order η^4 to take $\eta^2 e_*^2 = \eta^2(1-\ell)$ so that recalling relation (7.13c) for q^2, we may set

$$\frac{1}{1+\eta^2 e_*^2 q^2} = 1 - \eta^2(1-\ell)q^2 = 1 - \eta^2(1-\ell)(1-\nu^2) \tag{4.14}$$

and hence, on combining (4.13) with (4.14), we have

$$\frac{q^2-\eta^2h^2}{1+\eta^2 e_*^2 q^2} = (1-\nu^2) - \eta^2[(3-\ell) - \nu^2(22-7\ell) + 2\nu^4(10-3\ell)]. \tag{4.15}$$

If we now recall the defining relation (7.31) of Chapter 5 for k_1^2, then from (4.15) above we see that, to first order in η^2, we may write

$$k_1^2 = \eta^2(1-\ell)(1-\nu^2). \tag{4.16}$$

It remains to determine j_1 to the same order of accuracy. In the defining relation (7.32) of Chapter 5, we introduce h_2 from (7B.25c) and q^2 from (7.13c), both from Chapter 5: on neglecting terms of order η^4 [so that we may take $\eta^2 e_*^2 = \eta^2(1-\ell)$], we obtain

$$\begin{aligned}\frac{1}{j_1^2} &= \frac{(1+\eta^2\ell)m}{[1-\eta^2\nu^2(4-\ell)][1+\eta^2(1-\ell)(1-\nu^2)]}\cdot\frac{1}{j_w^2}\\ &= \frac{(1+\eta^2\ell)m}{j_w^2}\cdot[1-\eta^2[(1-\ell)-\nu^2(5-2\ell)]].\end{aligned}$$

When we introduce $(1+\eta^2\ell)m$ from (4.10a) and $1/j_w^2$ from (4.12b), we have

$$\begin{aligned}\frac{1}{j_1^2} &= (1+\eta^2\ell\nu^2)[1-\eta^2(1-3\nu^2)][1-\eta^2[(1-\ell)-\nu^2(5-2\ell)]]\\ &= 1-\eta^2[2(1-4\nu^2)-\ell(1-\nu^2)].\end{aligned} \tag{4.17}$$

There follows that, to first order in η^2,

$$j_1^2 = 1+\eta^2[2(1-4\nu^2)-\ell(1-\nu^2)] \tag{4.18}$$

$$j_1 = 1+\eta^2[(1-4\nu^2)-\tfrac{1}{2}\ell(1-\nu^2)]. \tag{4.19}$$

We now recall that, to first order in terms of the modulus k, the quarter-period K of the Jacobian elliptic function is given by

$$K = \frac{\pi}{2}[1+\tfrac{1}{4}k^2] \tag{4.20}$$

so that for K_2 we have

$$K_2 = \frac{\pi}{2}[1+\tfrac{1}{4}k_2^2] = \frac{\pi}{2}[1+\tfrac{1}{4}\eta^2\ell(1-\nu^2)] \tag{4.21}$$

wherein we have introduced k_2 from (4.11). Hence, combining (4.19) with (4.21), we find

$$j_1K_2 = \frac{\pi}{2}[1+\tfrac{1}{4}\eta^2\ell(1-\nu^2)][1+\eta^2[(1-4\nu^2)-\tfrac{1}{2}\ell(1-\nu^2)]]$$

which to first order in η^2 may be replaced by

$$j_1K_2 = \frac{\pi}{2}[1+\eta^2[(1-4\nu^2)-\tfrac{1}{4}\ell(1-\nu^2)]]. \tag{4.22}$$

For the quarterperiod K_1, we find, on the introduction of k_1 from (4.16) into (4.20),

$$\begin{aligned} K_1 &= \frac{\pi}{2}[1 + \tfrac{1}{4}\eta^2(1-\ell)(1-\nu^2)] \\ &= \frac{\pi}{2}\Big[1 + \eta^2[\tfrac{1}{4}(1-\nu^2) - \tfrac{1}{4}\ell(1-\nu^2)]\Big]. \end{aligned} \tag{4.23}$$

The subtraction of (4.23) from (4.22) yields

$$j_1 K_2 - K_1 = \frac{\pi}{2}\Big[\eta^2[\tfrac{3}{4} - \tfrac{15}{4}\nu^2]\Big] = \frac{3\pi}{8}\eta^2(1-5\nu^2) \tag{4.24}$$

indicating how the "small divisor" problem arises for perturbation series in the vicinity of the "critical" inclination defined by $\nu^2 = 1/5$.

We already have the solution in the spheroidal coordinate system, given in its integrated form. It would appear that the "small divisor" problem arises from an attempt to force this solution into conformity with the solution for the instantaneous elements in a spherical coordinate system that is the framework of perturbation theory.

Appendix: Calculation and Exhibition of Orbits; The Time-Angle Relation

by Vincent G. Hart[1] *and Seán Murray*[2]

Raffiniert ist der Herr Gött, aber boshaft ist er nicht.
[God may be subtle, but he is not malicious.]

— *Albert Einstein*

Preliminary Remarks

In this Appendix are presented graphs of some orbits based on the analytic results derived in this book — primarily in Chapter 3 but some also from Chapters 5 and 6. We also add some results of integrations for the time-angle relation defined in Section 10 of Chapter 3. Prior to a detailed description of the orbits, it is useful to consider the possibilities in approaching the solution of the S- and R-equations as the parameters are varied.

First, consider the S-equation of equation (5.19) in Chapter 3. Recalling that $\ell = 1 - e^2$, the decomposition of the right side is straightforward as the zeros are clearly shown by the pairs

$$\mp 1, \quad -\frac{\beta \mp \gamma}{\lambda} \tag{1.1a}$$

where

$$\lambda = \eta\ell = b/a, \quad \gamma = (\beta^2 + e^2 - 1)^{1/2}. \tag{1.1b}$$

From observing the second pair of zeros, we see that if

$$\beta^2 + e^2 \begin{cases} \geq 1 & \text{there are 4 real zeros} \\ = 1 & \text{there are 3 real zeros (one a double-zero)} \\ \leq 1 & \text{there are 2 real and 2 complex zeros.} \end{cases}$$

By drawing the nominal graphs of S'^2 versus S, it becomes evident that for the entire λ-range and with $\beta \leq 1$, the solution for S must be constrained by

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$$-1 \leq S_0 \leq S \leq 1 \tag{1.2}$$

implying that the real angle σ lies in the range $0 \leq \sigma \leq \overline{\sigma} \leq \pi$.

The situation is different in the case of the R-equation as represented by equation (5.8a) of Chapter 3. When divided across by p^2, the differential equation for R/p has a right side with zeros (with $\eta = b/p$, $p/a = \ell$)

$$-\eta, \quad \eta, \quad 1/(1+e) = R_1/p, \quad 1/(1-e) = R_2/p,$$

and all four zeros are real. Here the graph of $(R'/p)^2$ versus R/p shows two cases to be considered:

(i) $0 \leq \eta \leq 1/(1+e) = R_1/p$ or $0 \leq \lambda \leq 1-e = R_1/a$
(ii) $R_1/p = 1/(1+e) \leq \eta \leq 1/(1-e) = R_2/p$ or $R_1/a = 1-e \leq \lambda \leq 1+e = R_2/a$.

On scaling with a rather than p, it is clear from the nominal graph case (i), a region of real values of R/a occurs in the interval $(1-e, 1+e)$, whereas in case (ii), real values of R/a occur in the interval $(\lambda, 1+e)$. These intervals are of primary interest in this study. In the text, the ascending order of the zeros is noted, and the zeros are grouped into a lower pair and an upper pair. Thus with the a scaling

(i) the lower pair is $(-\lambda, \lambda)$ and the upper pair is $(1-e, 1+e)$
(ii) the lower pair is $(-\lambda, 1-e)$ and the upper pair is $(\lambda, 1+e)$.

To ensure real solutions, it is crucial that the quartic be decomposed appropriately into the two quadratic factors. For example, in Subsection 9A of Chapter 3, the range $\lambda \geq 1-e$ or $\eta \geq 1/(1+e)$ case (ii) above is applicable, where we also require $\lambda \leq 1+e$ or $\eta \leq 1/(1-e)$. Here the process beginning at equation (9A.7), where the reduction begins with the upper pair of zeros λ and $1+e$, yields the real solutions. Beginning with a pair of zeros not in the upper or lower classes, for example $-\lambda$ and λ in this case, would lead to a quartic and solutions with undesirable real or complex values. With this preference in mind, a similar process can be followed in the arrangement of the S-equation to yield a solution valid in the range of interest $|S| \leq 1$.

An alternative procedure for the reduction of a quartic as a product of quadratic factors each of which is a sum of squares is outlined in Whittaker and Watson's *Modern Analysis* [Ww, Section 22.71]. That procedure admits two types of solution for the typical equation such as the S- and R-equation. Which type is to be rejected can only be determined by inspection as leading to values of $R/b < 1$, or $S > 1$. As mentioned, the method followed in the present work enables one to select the correct procedure at the outset.

The orbits described are drawn mainly from the solutions found in Chapter 3, which provides orbits for the planar Euler problem. Calculations have also been made for the Earth satellite problems discussed in the later chapters. It should be emphasized that all examples presume negative energy, which generally leads to bound orbits. In the numerical work, the extensive Maple suite of Jacobian elliptic functions was of the greatest assistance.

1 Orbits in Chapter 3

In Chapter 3, the plane Cartesian coordinates for points on the orbit are derived from the spheroidal coordinates by the relations

$$x/a = \left[(R/a)^2 - \lambda^2\right]^{1/2} \sin\sigma, \quad z/a = (R/a)\cos\sigma \tag{1.3}$$

and we note that R = constant yields ellipses, whereas σ = constant yields hyperbolae. Note also that while σ need vary only between 0 and π to identify all points on a hyperbola, we need both positive and negative values of x/a. The latter are obtained by realizing that these coordinates in (1.3) are derived from the three-dimensional forms in (1.3), Chapter 4, and negative and positive values of x/a follow by allowing φ to vary in the range $(-\pi, \pi)$. Change of sign of x/a is allowed for in the Maple program codes used to generate the orbits.

In Chapter 3, six classes of orbits are identified — as summarized in Section 13. We note that $0 \le \beta \le 1$, and γ is given in (1.1b). If $e < 1$, and γ is real, $\gamma < \beta$.

When $e^2 + \beta^2 \ge 1$,

$$A1: \ 0 < \lambda < 1 - e, \qquad A2: \ 1 - e < \lambda < \beta - \gamma$$
$$A3: \ \beta - \gamma < \lambda < \beta + \gamma, \qquad A4: \ \beta + \gamma < \lambda < 1 + e.$$

When $e^2 + \beta^2 \le 1$,

$$B1: \ 0 < \lambda < 1 - e, \qquad B2: \ 1 - e < \lambda < 1 + e.$$

If $e > 1$, γ is real, $\gamma > \beta$, we have in place of $A3$:

$$A3^*: \ \gamma - \beta < \lambda < \gamma + \beta.$$

When $\beta = 0$, the classes $A1$ and $A2$ disappear, and $A3$ becomes $A3^*$.

We show representative orbits of the above types in Figures 1–12. The axes are understood to show x/a and z/a. **Note that in all graphs, the mass points lie at $z/a = \pm\lambda$. If $\beta = 0$, the masses are equal, and if $\beta > 0$, the larger mass is uppermost.** Some general remarks can be made as β increases from zero.

A1: For small β we observe that the orbits can be perceived as close to a set of rotating ellipses that tighten up as β increases. Figure 2 is near the upper limit $\beta = 1$, and the orbit is almost circular.

A2: Orbits are wide loops, looping in a figure-of-eight fashion between and around the two mass centers.

A3: Orbits are wide loops encircling the heavier mass and bounded by the limit set consisting of the bounding ellipse and the bounding hyperbola associated with the lighter mass.

A4: The orbits appear as satellites of either one of the masses and are confined by the bounding ellipse and both bounding hyperbolae. The distinct pattern between $e < 1$ and $e > 1$ is indicated in the illustrated Figures 5 and 6.

B1: For small β and small $e < 1$, overlapping near-circles appears typical; as e increases, a system of rotating near-ellipses appears.

B2: Orbits looping between and around each of the attracting centers in a figure-of-eight fashion are the norm.

As representative figures for Chapter 3 orbits, the following 12 sets of parameters in the Table of Orbital Parameters serve to illustrate the above remarks. Whereas 50 points sufficed for most of the orbits plotted, we used 110 points in Figure 2 to illustrate close overlapping of a near-circular orbit.

Table of Orbital Parameters

Class	β	e	λ	Figure	Comment
A1	0.75	0.7	0.2	1	
A1	0.9753	0.3	0.182	2	Earth–Moon mass ratio 80 : 1
A2	0.9753	0.5	0.51	3	"
A3	0.9753	0.3	0.819	4	"
A4	0.75	0.7	1.2	5	
A4	0.75	1.4	2.2	6	
B1	0	0.5	0.4	7	Equal masses, cf. Strand & Reinhardt, Fig. 7
B1	0.3	0.3	0.2	8	
B1	0.5	0.8	0.1	9	
B2	0	0.5	0.75	10	Equal masses
B2	0.5	0.8	1.0	11	
B2	0.8	0.5	1.0	12	

It is interesting to note that Strand and Reinhardt [S7] obtained orbits shown in their Figures 7, 8, and 9 that are typical of our basic classes A4, B1, and B2. Since they considered only equal mass points ($\beta = 0$), our orbit classes A1, A2, and A3 were not accessible to them. Their Figure 7 is reproduced in our Figure 7.

2 Orbits in Chapters 5 and 6

For a plane satellite orbit as described in Chapter 6, we take parameters [Chapter 5, (5.9a,b)]

$$\ell = 0.9, \quad \eta^2 = 4 \times 10^{-5}, \quad \nu = 0\,.$$

Here $\ell = 1 - e_0^2$. Since the moduli of the elliptic functions involved in the solution are sufficiently small, we may substitute circular functions, and equations (3.5), (3.51) of Chapter 6 become

$$S = \cos\sigma = \sin(f+\omega),$$
$$R/p_0 = 1.000004\left(\frac{1+\delta\cos(j_1 f)}{1+e\cos(j_1 f)}\right),$$

where

$$\delta = -1.2649\times10^{-5}, \quad e = 0.316216382, \quad j_1 = 1.000022.$$

Here the coordinates for the orbit are [Chapter 5, (2.1)]

$$x/p_0 = \left[(R/p_0)^2+\eta^2\right]^{1/2}\sin\sigma\cos\varphi,$$
$$y/p_0 = \left[(R/p_0)^2+\eta^2\right]^{1/2}\sin\sigma\sin\varphi,$$
$$z/p_0 = (R/p_0)\cos\sigma,$$

where $p_0 = p(1+e_0\delta)$.

Since the orbit is plane, we may set $y = 0$, and both positive and negative values of x follow by allowing φ to vary around a full circle. An almost circular plane orbit is found as shown in Figure 13.

Chapter 5 concerns general satellite orbits. We calculated one for parameters

$$\ell = 0.9, \quad \eta^2 = 4\times10^{-5}, \quad \nu = 0.5,$$

and found that the orbit is almost identical with that shown in Figure 13 but in a plane inclined at 30° to the Oz axis. The latter feature follows from the S-equation:,

$$\sin f = (1-N^2)^{-1/2}\cos\sigma,$$

with $(1-N^2)^{-1/2} = 1.154705735$ in this case. It follows that σ is restricted to the interval (30°, 150°).

3 The Time-Angle Relation

(a) Chapter 3, Section 10

We substitute results for S and R into the integrals in (10.13a,b), and the Curtis–Clenshaw numerical integration process in Maple gives the dimensionless time M as in equation (10.12). The limits to the integrals are 0 to f_v for $(R/p)^2$, and 0 to f_s for S^2, with $f_v = j_v f_s$.

For $\beta = 79/81$, $e = 0.3$, $\lambda = 0.182$, we find $j_v = 0.998941015$, and the results are shown in Figure 14. Here $\sigma = \pi/2 - f_s$. For one complete orbit, we find $M = 6.1508$.

(b) Chapter 6

For a plane polar satellite orbit, the relevant equations are (9.14a) and (9.15) of Chapter 5, and the results appear in Figure 15 for parameters

$$\ell = 0.9, \quad \eta^2 = 4\times10^{-5}, \quad \nu = 0.$$

Here $\sigma = \pi/2 - f$. For one complete orbit $M = 6.2833$.

4 Orbits Derived from Given Initial Conditions in Chapter 3

We recollect that in order to plot an orbit, we need numerical values of the parameters β, λ, and e and of the arbitrary constants f_{S0} in S (see for example (8.25)), and f_{v0} in R (9.22). We assume that the total energy $\mathcal{E}$ (1.12) is negative and that f_{v0} is always chosen as $K(k_R)$ so that the R curve is symmetric about the Oz axis.

In the following we show that, given initial values of coordinates and velocity components at a point x_0/a, z_0/a and one of $\dot{x}_0/\alpha$, $\dot{z}_0/\alpha$ (or the direction of the orbit initially, that is, $\dot{x}_0/\dot{z}_0$), together with the parameters β (< 1) and λ (> 0), a number of orbits can be calculated by use of the energy equation.

There are two special simpler cases,

(1) the orbit starts on the x-axis in a direction orthogonal to that axis, and
(2) the orbit starts on the z-axis in a direction orthogonal to that axis.
(3) The general case is that where the orbit starts at an arbitrary point (x_0, z_0) off the axes, in an arbitrary direction.

The simpler cases are readily described.

(1) We assume here that $z_0 = 0$, $\dot{x}_0 = 0$, and x_0/a is given. Then $\sigma_0 = \pi/2$, so that by (1.3a) we find that

$$x_0/a = \left[(R_0/a)^2 - \lambda^2\right]^{1/2} \tag{IC.1}$$

and R_0/a can be calculated.

Next the energy equation (1.12) is written in the form

$$\mathcal{E} = -\alpha^2 = \tfrac{1}{2}(\dot{x}_0^2 + \dot{z}_0^2) - \mu/R_0 \tag{IC.2}$$

at the initial position. But by (5.2a) $\mu = 2a\alpha^2$, so that division by α^2 yields

$$\tfrac{1}{2}\left((\dot{x}_0/\alpha)^2 + (\dot{z}_0/\alpha)^2\right) = 2a/R_0 - 1 \tag{IC.3}$$

and use of the initial conditions gives $\dot{z}_0/\alpha$. Then by (4.7b), noting that in this case (1.3b) implies $\dot{z}_0 = -R_0\dot{\sigma}_0$, we find

$$C^2 = R_0^4\dot{\sigma}_0^2 = R_0^2\dot{z}_0^2. \tag{IC.4}$$

Finally, by (5.2b), we calculate

$$p/a = C^2/(a\mu) = \tfrac{1}{2}(R_0/a)^2(\dot{z}_0/\alpha)^2. \tag{IC.5}$$

Now there are two possibilities: either $e < 1$ and $p/a = 1 - e^2$ by (5.11b), or $e > 1$ and $p/a = e^2 - 1$ by (11.9a). But in each case e is determined, and knowing both β and λ, a variety of orbits can be plotted corresponding to different choices of the constant f_{s0}. Also, since β is not required at the outset, a further arbitrariness is admitted.

As an *example* we assume parameters $\beta = 79/81$, $\lambda = 0.182$, and initial values

$$z_0 = 0, \quad x_0/a = -0.6791, \quad \dot{x}_0 = 0.$$

Then we find $R_0/a = 0.7030$, $(\dot{z}_0/\alpha)^2 = 3.6896$, and $p/a = 0.9118$. This gives either $e = 0.2970$ or $e = 1.383$.

If we use the first value of e together with the chosen values of β and λ, an orbit very similar to a previously found almost-circular orbit with $e = 0.3$ is obtained (with $f_{s0} = 0$; see Figure 2).

However, the larger value of e cannot be used with the chosen values of β and λ, since when $e > 1$, we must also take $\lambda > \gamma - \beta$, and this last number is 0.3998 in this case [see (11.13b), Chapter 3].

(2) We next assume that $x_0 = 0$, $\dot{z}_0 = 0$, and z_0/a is assigned. Then $\sigma_0 = 0$, and by (1.3b) $R_0/a = z_0/a$. Here the energy equation (1.12) is

$$\mathcal{E} = -\alpha^2 = \tfrac{1}{2}(\dot{x}_0^2 + \dot{z}_0^2) - \mu(R_0 + b\beta)/(R_0^2 - b^2), \tag{IC.6}$$

which yields

$$\dot{x}_0^2/\alpha^2 = 4\frac{R_0/a + \beta\lambda}{(R_0/a)^2 - \lambda^2} - 2. \tag{IC.7}$$

By differentiating (1.3a), we note that

$$\dot{x}_0 = (R_0^2 - b^2)^{1/2}\dot{\sigma}_0 \tag{IC.8}$$

and (4.7b) then gives

$$C^2 = (R_0^2 - b^2)\dot{x}_0^2 - 2b^2\alpha^2 - 2b\mu\beta\,. \tag{IC.9}$$

Using (IC.7), we then find that

$$p/a = C^2/(a\mu) = \tfrac{1}{2}[(R_0/a)^2 - \lambda^2)](\dot{x}_0/\alpha)^2 - \lambda^2 - 2\beta\lambda\,. \tag{IC.10}$$

Again the two possible values of e are found.

In an *example* we assume

$$\beta = 79/81, \quad \lambda = 0.182$$

and initial values

$$x_0 = 0, \quad \dot{z}_0 = 0, \quad z_0/a = 0.7\,.$$

Then $R_0/a = 0.7$, $(\dot{x}_0/\alpha)^2 = 5.6827$, and we find that $p/a = 0.9100$. This yields either $e = 0.3$ or $e = 1.382$. The first value of e with the above parameters and $f_{s0} = 0$ gives a previously plotted almost-circular orbit (Figure 2). The larger value of e is disallowed with the above λ — as in the previous example.

Note that by contrast with the previous case (1), we must specify β at the outset in case (2).

(3) The general case comprises given values of x_0/a and z_0/a, both nonzero, also given is that one of $\dot{x}_0/\alpha$, $\dot{z}_0/\alpha$ is nonzero, or the ratio $\dot{x}_0/\dot{z}_0$ is known.

In this case, R_0 and σ_0 must be found from knowledge of the coordinates.

Equations (1.3) can be used to eliminate σ_0. We find after division by a

$$\frac{(x_0/a)^2}{(R_0/a)^2 - \lambda^2} + \frac{(z_0/a)^2}{(R_0/a)^2} = 1\,. \tag{IC.11}$$

This yields a quadratic equation for $(R_0/a)^2$ with solution

$$(R_0/a)^2 = (\lambda^2/2)\Big[1 + Z \pm \big((1+Z)^2 - 4(z_0/a)^2 \,/\, \lambda^2\big)^{1/2}\Big], \tag{IC.12}$$

where

$$Z = \big((x_0/a)^2 + (z_0/a)^2\big) \,/\, \lambda^2\,. \tag{IC.13}$$

A similar result follows by eliminating R_0 from (1.3):

$$cos^2\sigma_0 = \tfrac{1}{2}\Big[1 + Z \pm \big((1+Z)^2 - 4(z_0/a)^2 \,/\, \lambda^2\big)^{1/2}\Big]. \tag{IC.14}$$

We take the positive sign before the radical for R_0 and the negative sign for σ_0.

Next we differentiate with respect to time in equations (1.3a,b):

$$\begin{aligned}\dot{x}_0 &= (R_0\dot{R}_0 \sin\sigma_0) \,/\, (R_0^2 - b^2)^{1/2} + (R_0^2 - b^2)^{1/2}\cos\sigma_0\dot{\sigma}_0\\ \dot{z}_0 &= \dot{R}_0\cos\sigma_0 - R_0\sin\sigma_0\dot{\sigma}_0\,.\end{aligned} \tag{IC.15}$$

These equations can be solved for $\dot{\sigma}_0$ and $\dot{R}_0$, and the latter solution yields

$$\begin{aligned}\dot{R}_0/\alpha = {}& (\dot{x}_0/\alpha)\Big[R_0(R_0^2 - b^2)^{1/2} \,/\, (R_0^2 - b^2\cos\sigma_0)\Big]\sin\sigma_0\\ &+ (\dot{z}_0/\alpha)\Big[(R_0^2 - b^2) \,/\, (R_0^2 - b^2\cos^2\sigma_0)\Big]\cos\sigma_0\end{aligned} \tag{IC.16}$$

The energy equation (1.12) now reads

$$(\dot{x}_0/\alpha)^2 + (\dot{z}_0/\alpha)^2 = 4[(R_0/a) + \beta\lambda\cos\sigma_0] \,/\, ((R_0/a)^2 - \lambda^2\cos^2\sigma_0) - 2 \tag{IC.17}$$

and knowing one of the velocity components, or their ratio, enables us to find the other component.

The last equation needed is (4.7a), which reads, after division by $a^2\alpha^2$,

$$\begin{aligned}p/a = C^2/(2a^2\alpha^2) = {}& -(R_0/a)^2 + 2(R_0/a)\\ &- \tfrac{1}{2}(\dot{R}_0/\alpha)^2[(R_0/a)^2 - \lambda^2\cos^2\sigma_0] \,/\, [(R_0/a)^2 - \lambda^2]\,.\end{aligned} \tag{IC.18}$$

Then, since we know $\dot{R}_0/\alpha$ from (IC.16), the parameter e can be found. Again β must be prescribed from the outset in case (3).

In the following *example* we take

$$\beta = 79/81, \quad \lambda = 0.728$$

and initial values

$$x_0/a = -0.2548, \quad z_0/a = 0.67522\,.$$

At this point, we also suppose that $\dot{x}_0 = \dot{z}_0$, or that the orbit is equally inclined to the axes at the outset.

Then the above work shows that $R_0/a = 0.8432$, $\cos\sigma_0 = 0.8008$. Since x_0/a is negative, we take $\sin\sigma_0 = -0.5989$. Then the energy equation gives $\dot{x}_0/\alpha = \dot{z}_0/\alpha = 2.5707$, and $(\dot{R}_0/\alpha)^2 = 0.2346$. We find $e = 0.337$, or $e = 1.373$. The first value with the values of β and λ above together with $f_{s0} = 0$ yields an orbit somewhat different to one previously obtained with $e = 0.3$. Using instead the second value of e, we obtain a Type A3 orbit that is tightly constrained in the upper left quadrant in a series of overlapping near-ellipses.

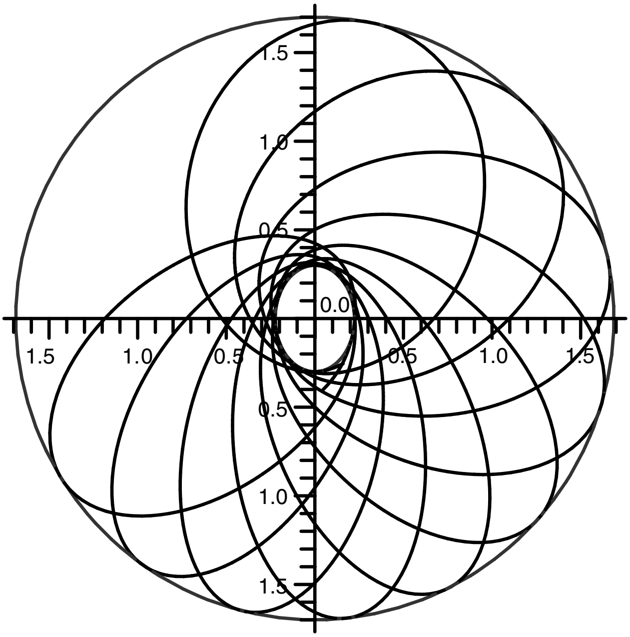

Fig. 1.

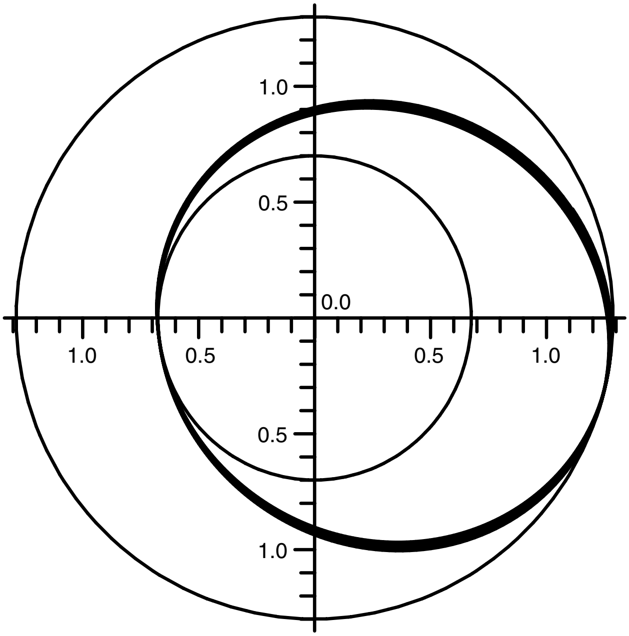

Fig. 2.

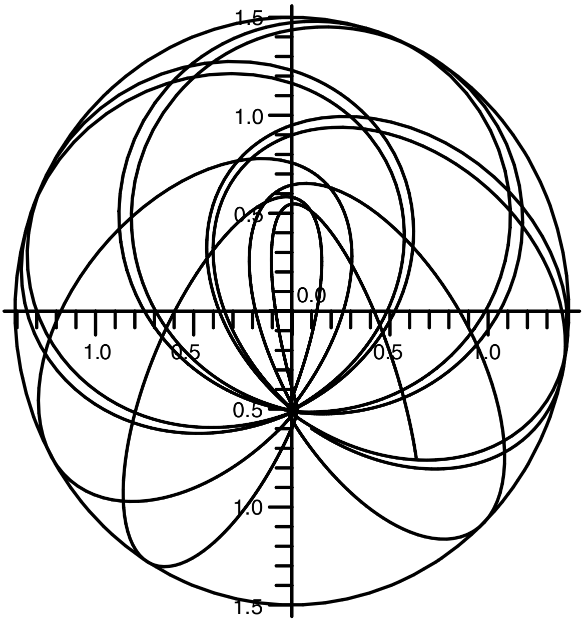

Fig. 3.

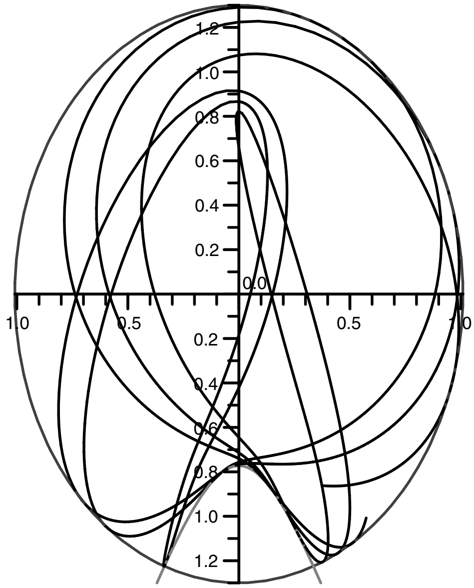
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Fig. 4.

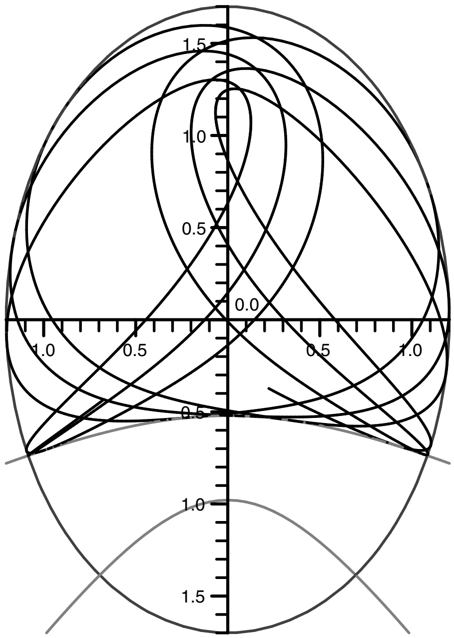
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Fig. 5.

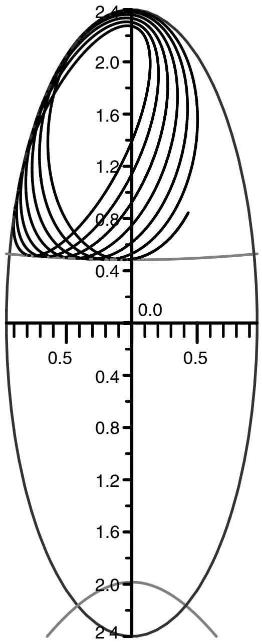
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Fig. 6.

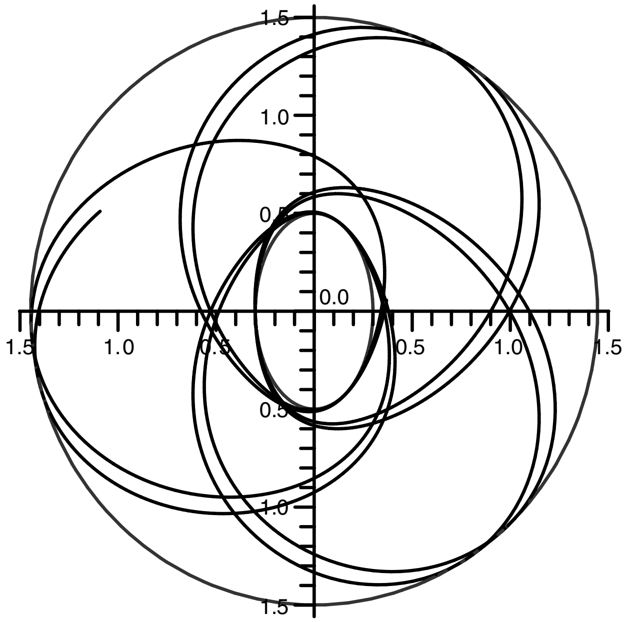
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Fig. 7.

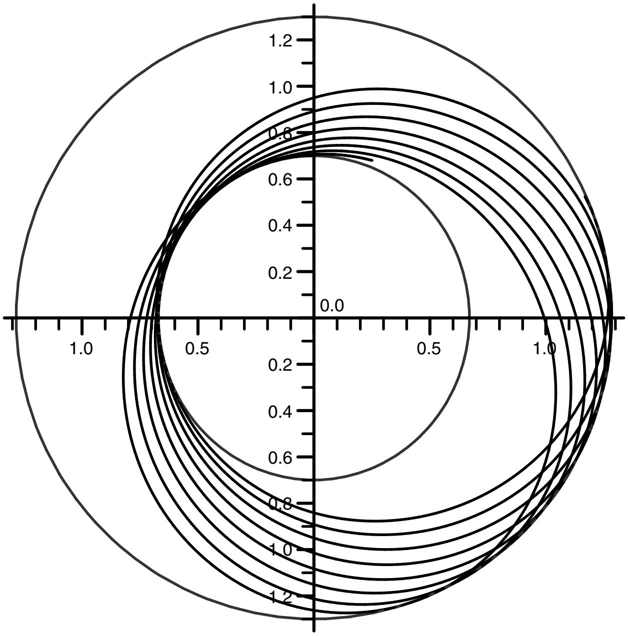
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Fig. 8.

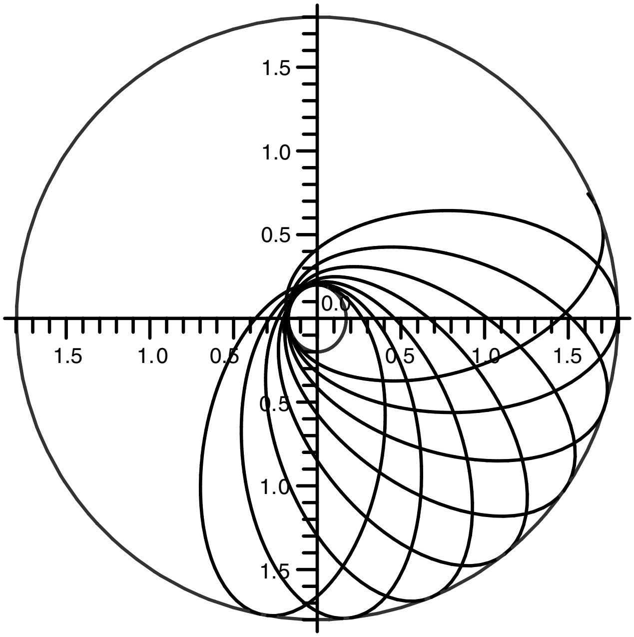
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Fig. 9.

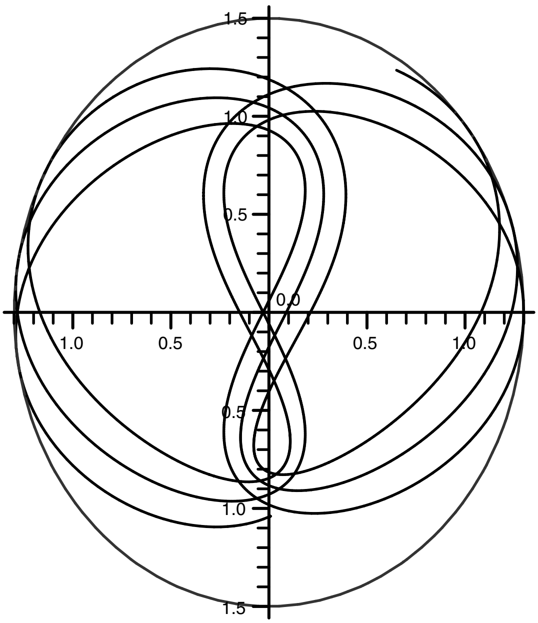

Fig. 10.

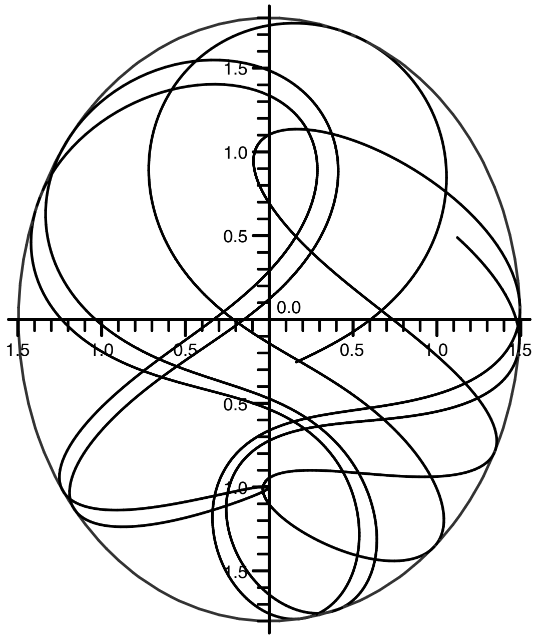

Fig. 11.

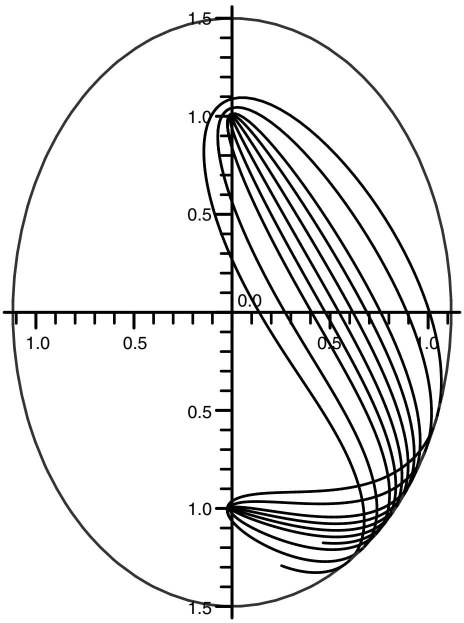

Fig. 12.

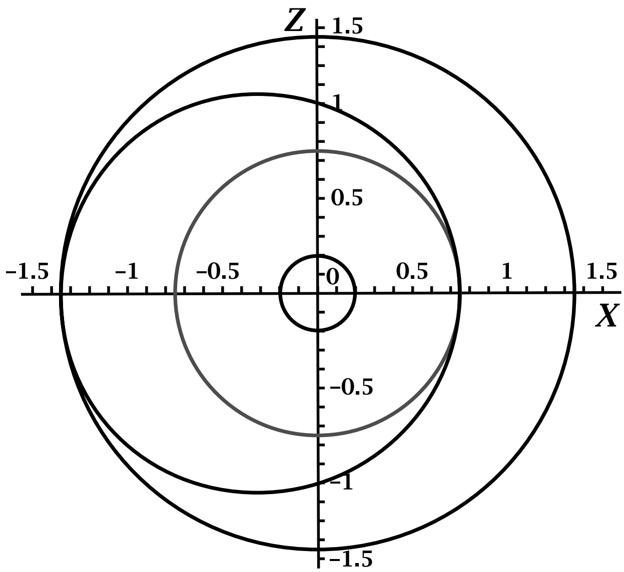

Fig. 13. Satellite polar orbit from Chapter 6. Parameters: $\ell = 0.9$, $\eta^2 = 0.00004$, $\nu = 0$. The smallest circle represents the earth, and the inner and outer ellipses are almost-circles centered on the origin. Here $X = x/p_0$ and $Z = z/p_0$.

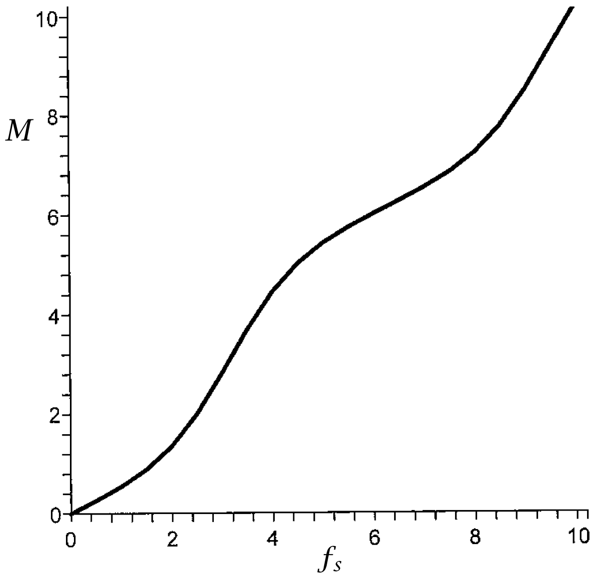

Fig. 14. Time-angle relation for a Chapter 3 orbit with $\beta = 79/81$, $e = 0.3$, $\lambda = 0.182$. The "time" M is defined by equation (10.12). Here $\sigma = \pi/2 - f_s$. For one complete orbit, $M = 6.1508$.

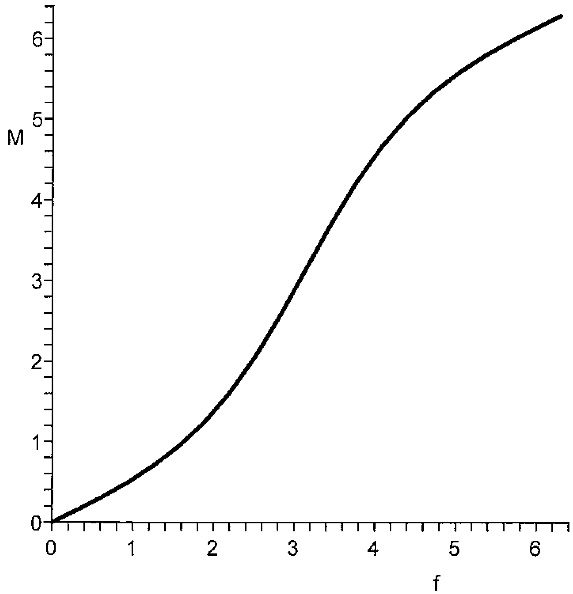

Fig. 15. Time-angle relation for a Chapter 6 plane polar satellite orbit with $\ell = 0.9$, $\eta^2 = 0.00004$, $\nu = 0$. The "time" M is defined by equations (9.14a) and (9.15) of Chapter 5. Here $\sigma = \pi/2 - f$. For one complete orbit, $M = 6.2833$.

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