Progress in Mathematics
287

Alberto S. Cattaneo
Anthony Giaquinto
Ping Xu
Editors

Higher Structures in Geometry and Physics

In Honor of Murray Gerstenhaber and Jim Stasheff

Birkhäuser

Birkhäuser

Progress in Mathematics
Volume 287

Alberto S. Cattaneo
Anthony Giaquinto
Ping Xu
Editors

Higher Structures in Geometry and Physics

In Honor of Murray Gerstenhaber and Jim Stasheff

Birkhäuser

Editors
Alberto S. Cattaneo
Institut für Mathematik
Universität Zürich-Irchel
Winterthurerstrasse 190
8057 Zürich
Switzerland
alberto.cattaneo@math.uzh.ch

Ping Xu
Department of Mathematics
Pennsylvania State University
University Park, PA 16802
USA
ping@math.psu.edu

Anthony Giaquinto
Mathematics and Statistics
Loyola University Chicago
Chicago, IL 60626
USA
tonyg@math.luc.edu

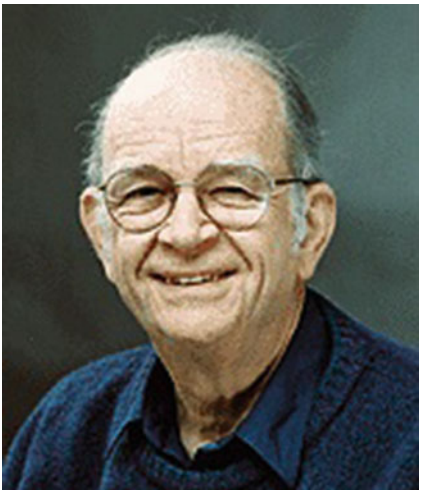
ISBN 978-0-8176-4734-6 e-ISBN 978-0-8176-4735-3
DOI 10.1007/978-0-8176-4735-3
Springer New York Dordrecht Heidelberg London

Mathematics Subject Classification (2010): 53D55, 16S80, 17B25, 53D17

Printed on acid-free paper

www.birkhauser-science.com


Murray Gerstenhaber


Jim Stasheff

Foreword

You have tendered us a great honor.

Beyond that, we have listened throughout this conference with fascination and pleasure to the advances those invited here have made in the bloodless battle against an ever-retreating foe[1] in which we are all engaged. In this battle, where we start from ground gained by our teachers and predecessors – some of whom held forth in these same halls – our only weapons are those of thought. We devote our lives to this struggle against ignorance driven mainly by the hope of occasional triumphs and the joy of experiencing them when they occur.

Rewards and honors are usually bestowed on individuals but should really be shared by our whole community of mathematicians. Without each other's support, little progress is possible. And so while we thank you for the recognition you have given us and acknowledge our indebtedness to the organizers Professors Alberto Cattaneo and Ping Xu, to the institutions which provided the funds that enabled us to gather here the European Science Foundation, the National Science Foundation, and the National Security Agency of the USA, and to the Institut Henri Poincaré – for providing the venue in which we gathered, we also recognize our indebtedness to the continuing community of scholars, which we have been privileged to join. We have freely used their ideas. We hope that those newly joined in the battle will share our enthusiasm, go farther tomorrow than we, and look on our yesterday with appreciation that the ground we won has been not just for ourselves but for them.

Again, you have tendered us a great honor, but the celebration is really one of our communal spirit. Our knowledge will never be perfect nor will our understanding of the mysteries of mathematics ever be complete, but our searching, too, will never cease. And may we and those who succeed us gather often to celebrate this restless spirit.

---

[1] André Weil.

Preface

This book arose from a meeting centered on higher algebraic structures that are now ubiquitous in various areas of mathematics (algebra, algebraic topology, differential geometry, algebraic geometry, mathematical physics) and theoretical physics (quantum field theory, string theory). These structures provide a common language essential for the study of deformation quantization, theory of algebroids and groupoids, symplectic field theory, and much more.

These higher algebraic structures first appeared in 1963, in Murray Gerstenhaber's[1] *The cohomology structure of an associative ring* and in Jim Stasheff's *Homotopy associativity of H-spaces. I, II*.[2] In these fundamental publications, one finds the introduction of the notions that were to be called a Gerstenhaber algebra (developed in part to understand algebraic deformation theory) and an A_∞ algebra (developed in part to understand higher homotopies). While the relation between these notions was not immediately recognized, the ideas of higher homotopies and algebraic deformation would merge decades later and they are now permanently intertwined. The ideas of Gerstenhaber and Stasheff are present in every contribution of this volume.

[1] Ann. Math. **78**, 267–288 (1963).
[2] Trans. Am. Math. Soc. **108**, 275–292 (1963).

Contents

List of Contributors

Lawrence Breen
Université Paris 13,
CNRS, UMR 7539 LAGA,
99, avenue J.-B. Clément,
93430 Villetaneuse, France
breen@math.univ-paris13.fr

Michel Cahen
Département de Mathématique,
Université Libre de Bruxelles,
Campus Plaine, C. P. 218,
Boulevard du Triomphe,
1050 Bruxelles, Belgium
mcahen@ulb.ac.be

Alberto S. Cattaneo
Institut für Mathematik,
Universität Zürich-Irchel,
Winterthurerstrasse 190,
8057 Zürich, Switzerland
alberto.cattaneo@math.uzh.ch

V.A. Dolgushev
Department of Mathematics,
University of California at Riverside,
900 Big Springs Drive,
Riverside, CA 92521, USA
and
Mathematics Department,
Northwestern University,
2033 Sheridan Rd.,
Evanston, IL 60208, USA
vald@math.ucr.edu

Giovanni Felder
Department of Mathematics,
ETH Zürich,
8092 Zürich, Switzerland
giovanni.felder@math.ethz.ch

Anthony Giaquinto
Mathematics and Statistics,
Loyola University Chicago,
Chicago, IL 60626, USA
tonyg@math.luc.edu

Simone Gutt
Département de Mathématique,
Université Libre de Bruxelles,
Campus Plaine, C. P. 218,
Boulevard du Triomphe,
1050 Bruxelles, Belgium
and
Département de Mathématique,
LMAM,
Université Paul Verlaine-Metz,
Ile du Saulcy,
57045 Metz Cedex 01, France
gutt@poncelet.univ-metz.fr

Johannes Huebschmann
UFR de Mathématiques,
Université des Sciences
et Technologies de Lille,
CNRS-UMR 8524,

59655 Villeneuve d'Ascq Cedex,
France
Johannes.Huebschmann@math.univ-lille1.fr

T. Kadeishvili
A. Razmadze Mathematical Institute,
1, M. Alexidze Str.,
Tbilisi, 0193 Georgia
kade@rmi.acnet.ge

Hiroshige Kajiura
Department of Mathematics and Informatics,
Chiba University, 263-8522 Japan
kajiura@math.s.chiba-u.ac.jp

Bernhard Keller
Institut de Mathématiques de Jussieu,
UMR 7586 du CNRS,
Université Paris 7-Denis Diderot,
Case 7012, 2, place Jussieu,
75251 Paris Cedex 05, France
keller@math.jussieu.fr

Yvette Kosmann-Schwarzbach
Centre de Mathématiques Laurent Schwartz, École Polytechnique,
91128 Palaiseau, France
yks@math.polytechnique.fr

Jean-Louis Loday
Institut de Recherche Mathématique Avancée,
CNRS et Université de Strasbourg,
7 rue R. Descartes,
67084 Strasbourg Cedex, France
loday@math.unistra.fr

S.A. Merkulov
Department of Mathematics,
Stockholm University,
10691 Stockholm, Sweden
sm@math.su.se

Daniel Sternheimer
Department of Mathematics,
Keio University,
3-14-1 Hiyoshi, Kohoku-ku,
Yokohama, 223-8522 Japan
Daniel.Sternheimer@u-bourgogne.fr

D.E. Tamarkin
Department of Mathematics,
University of California at Riverside,
900 Big Springs Drive,
Riverside, CA 92521, USA
and
Mathematics Department,
Northwestern University,
2033 Sheridan Rd.,
Evanston, IL 60208, USA
tamarkin@math.northwestern.edu

Charles Torossian
CNRS-Institut Mathématiques de Jussieu,
Université Paris 7,
Case 7012, 2 place Jussieu,
75251 Paris Cedex 05, France
torossian@math.jussieu.fr

B.L. Tsygan
Department of Mathematics,
University of California at Riverside,
900 Big Springs Drive,
Riverside, CA 92521, USA
and
Mathematics Department,
Northwestern University,
2033 Sheridan Rd., Evanston,
IL 60208, USA
tsygan@math.northwestern.edu

Ronald N. Umble
Department of Mathematics,

Millersville University
of Pennsylvania,
Millersville, PA 17551, USA
ron.umble@millersville.edu

Stefan Waldmann
Fakultät für Mathematik und Physik,
Albert-Ludwigs-Universität Freiburg,
Physikalisches Institut,
Hermann Herder Strasse 3,
79104 Freiburg, Germany
Stefan.Waldmann@physik.uni-freiburg.de

Topics in Algebraic Deformation Theory

Anthony Giaquinto

Mathematics and Statistics, Loyola University Chicago, Chicago, IL 60626, USA,
tonyg@math.luc.edu

Dedicated to Murray Gerstenhaber and Jim Stasheff

Abstract. We give a selective survey of topics in algebraic deformation theory ranging from its inception to current times. Throughout, the numerous contributions of Murray Gerstenhaber are emphasized, especially the common themes of cohomology, infinitesimal methods, and explicit global deformation formulas.

Key words: Deformation theory, Cohomology, Quantization

2010 Mathematics Subject Classification: 16S80

1 Introduction

The concept of deformation is pervasive in mathematics. Its aim is to study objects of some type by organizing them into continuous families and determine how objects in the same family are related. This is the classic problem of determining the "moduli" of an algebra or of a more general structure. The moduli are, roughly, the parameters on which the structure depends. The idea goes back at least to Riemann who, in his famous treatise [Ri57] on abelian functions, showed that the Riemann surfaces of genus g form a single continuous family of dimension $3g-3$. This family is the prototype of a moduli space, a concept central to deformation theory.

The modern era of deformations began with the pioneering work of Fröhlicher–Nijenhuis [FN57] and Kodaira–Spencer [KS58] on deformations of complex manifolds. In particular, we see in [FN57] the first formal use of infinitesimal (cohomological) methods in a deformation problem as the authors prove that if X is a complex manifold, T its sheaf of holomorphic tangent vectors, then there can be no perturbation of the complex structure whenever $H^1(X,T)=0$. In the monumental treatise [KS58], Kodaira and Spencer

A.S. Cattaneo et al. (eds.), *Higher Structures in Geometry and Physics*, Progress in Mathematics 287, DOI 10.1007/978-0-8176-4735-3_1,

then developed a systematic theory of deformations of complex manifolds, including the infinitesimal and obstruction theories. For the case of Riemann surfaces, there are no obstructions as $H^2(X, T) = 0$.

Algebraic deformation theory began with Gerstenhaber's seminal paper [Ge64]. Although the analytic theory served as a model, numerous new concepts lie within the realm of algebraic deformation theory. In fact, all formal aspects of analytic deformations of manifolds are special cases of those in the algebraic theory — this will be made precise in Section 8.

Infinitesimal methods for algebra deformations are governed by Hochschild cohomology. The study of infinitesimals led to the discovery of the Gerstenhaber algebra structure on $HH^*(A, A)$, see [Ge63]. The ingredients of a Gerstenhaber algebra — compatible graded Lie and commutative products — occupy a central position in "higher structures in mathematics and physics." Another key higher structure is that of the various infinity algebras: A_∞, L_∞ and their generalizations. These structures have roots in Stasheff's landmark treatise [Sta63], which coincidentally appeared in the same year as [Ge63]. While disjoint at the time, the ideas in Gerstenhaber's and Stasheff's 1963 papers would become closely intertwined in the years to come. Indeed, Gerstenhaber called the entire Hochschild cohomology $HH^*(A, A)$ the "infinitesimal ring" of A in [Ge63], even though only the components of $HH^i(A, A)$ with $i \leq 3$ had natural interpretations related to infinitesimals and obstructions. But more than 30 years later, it became well known that the entire Hochschild cohomology $HH^*(A, A)$ *is* the space of infinitesimals of deformations of A to an A_∞ algebra.

This survey only represents a sampling of ideas in algebraic deformation theory. None of the discussion is new, except for the results in the last section on algebra variations. Many ideas are only sketched and proofs are omitted. More topics are left out than included. In particular, the theory of deformation quantization is largely left out — the reader is referred to Sternheimer's contribution [Ste] in this volume and the references therein for the important physical perspective. Also left out is the deformation theory of infinity algebras, gerbes, stacks, chiral algebras, affine and dynamical quantum groups, and the like. More comprehensive surveys of algebraic deformation theory and quantization can be found in the excellent treatises [GS88], [CKTB05] and [DMZ07].

Algebraic Deformations

Let A be an associative algebra over a commutative ring k.

Definition 1. *A* formal deformation *of A is a $k[[t]]$-algebra A_t which is flat and t-adically complete as a $k[[t]]$-module, together with an isomorphism $A \simeq A_t/tA_t$.*

For every deformation, there is a $k[[t]]$-module isomorphism between A_t and $A[[t]]$. Once such an isomorphism is fixed, the multiplication in A_t is necessarily of the form

$$\mu_t : A[[t]] \otimes_{k[[t]]} A[[t]] \to A[[t]] \quad \text{with} \quad \mu_t(a,b) = ab + \mu_1(a,b)t + \mu_2(a,b)t^2 + \cdots$$

where ab represents the multiplication of A and each $\mu_i \in \mathrm{Hom}(A \otimes A, A)$ is extended to be $k[[t]]$-bilinear. Setting $\mu_0(a,b) = ab$, we have $\mu_t = \sum \mu_i t^i$ and $\mu_t(a,b)$ will be denoted $= a * b$.

It is clear that one can consider formal deformations of other algebraic structures (Lie algebras, bialgebras, algebra homomorphisms, etc.) by modifying the above definition to suit the appropriate category. The deformation of an algebraic structure is usually subjected to the same equational constraints as the original structure:

- Associative algebra A: $(a * b) * c = a * (b * c)$. If A is commutative, we can also require $a * b = b * a$.
- Lie algebra L: $[a,b]_t = -[b,a]_t$ and the Jacobi identity for $[a,b]_t = [a,b] + [a,b]_1 t + [a,b]_2 t^2 + \cdots$.
- Bialgebra B: associativity of $*$, coassociativity of $\Delta_t(a) = \Delta(a) + \Delta_1(a)t + \cdots$, and $\Delta_t(a * b) = \Delta_t(a) * \Delta_t(b)$.
- Algebra homomorphism $\phi : A \to A'$: $\phi_t(ab) = \phi_t(a)\phi_t(b)$ with $\phi_t(a) = \phi(a) + \phi_1(a)t + \phi_2(a)t^2 + \cdots$.

Even though in this note we are concerned with formal deformations, there are many important and explicit instances for which the deformed products converge or are even polynomial in t when $k = \mathbb{R}$ or $\mathbb{C}$.

2 The deformation philosophy of Gerstenhaber

The pioneering principle of Gerstenhaber is that the equational constraints above can be naturally interpreted in terms of the appropriate cohomology groups and higher structures on them. In particular, the infinitesimal (linear term of the deformation) is a cocycle in the cohomology group — Hochschild $HH^2(A,A)$ in the associative case, Harrison $Har^2(A,A)$ in the commutative case, Chevalley–Eilenberg $H^2_{CE}(L,L)$ in the Lie case, Gerstenhaber–Schack $H^2_{GS}(B,B)$ in the bialgebra case, and the diagram cohomology $H^2_d(\phi,\phi)$ in the algebra homomorphism case. Moreover, the obstructions to extending infinitesimal and n-th order deformations to global ones are controlled by the differential graded Lie algebra structure on the cohomology.

In the associative case, the graded Lie structure (and much more) was laid out in [Ge63]. There it was shown that the Hochschild cohomology $HH^*(A,A) = \bigoplus_{n \geq 0} H^n(A,A)$ has a remarkably rich structure consisting of two products,

- A graded commutative product where $\deg HH^p(A,A) = p$,
- A graded Lie product where $\deg HH^p(A,A) = p - 1$,
- $[\alpha, -]$ is a graded derivation of the commutative product.

A graded k-module satisfying the above conditions is a *Gerstenhaber algebra*. Other notable examples are $\bigwedge^* L$ (where L is a Lie algebra), $H^*(X, \bigwedge^* T)$ (where X is a manifold and T is its sheaf of tangent vectors), and the diagram cohomology $H_d^*(\mathbb{A}, \mathbb{A})$ of an arbitrary presheaf $\mathbb{A}$ of k-algebras (to be defined in Section 8). The Chevalley–Eilenberg, Harrison, and bialgebra cohomology cohomologies carry graded Lie brackets, but are not Gerstenhaber algebras in general.

In [Ge63], the commutative and Lie products on the Hochschild cohomology $HH^*(A, A)$ are defined at the cochain level and are proved to descend to the level of cohomology. An intrinsic interpretation of the graded Lie structure was given by Stasheff in [Sta93]. There he proved that the Gerstenhaber bracket coincides with the natural graded bracket on $Coder(BA, BA)$, where BA is the bar complex of A.

Returning to the equational constraints for a deformation of an algebra A, the associativity of μ_t can succinctly be expressed in terms of the Gerstenhaber bracket as $[\mu_t, \mu_t] = 0$. Writing $\mu_t = \mu_0 + \mu'$ it follows that $2[\mu_0, \mu'] + [\mu', \mu'] = 0$. Since the coboundary in the shifted Hochschild complex $C^*(A, A)[1]$ is $\delta = [\mu_0, -]$, the first summand is $2\delta\mu'$. Thus we arrive at the fundamental associativity equivalences

$$\mu_t \quad \text{associative} \quad \iff \quad [\mu_t, \mu_t] = 0 \quad \iff \quad \delta(\mu') + \frac{1}{2}[\mu', \mu'] = 0. \qquad (1)$$

Thus μ_t is associative if and only if μ' satisfies the Maurer–Cartan equation. Although not explicitly stated as such, the idea that deformations are governed by a differential graded Lie algebra and solutions to the Maurer–Cartan equation goes back to Gerstenhaber's original paper [Ge64].

3 Algebras with Deformations

The search for deformations of an algebraic structure A begins with the appropriate cohomology group (usually $H^2(A, A)$) which comprises the infinitesimals. Given an infinitesimal μ_1, the basic question is whether it can be integrated to a full deformation or not. In other words, is it possible to find $\mu_2, \mu_3, \ldots$ such that $\mu' = \sum_{i \geq 1} \mu_i t^i$ satisfies the Maurer–Cartan equation? Of course, the vanishing of the obstruction group (usually $H^3(A, A)$) guarantees that any infinitesimal is integrable, but this is rarely the case. A necessary condition in general is that the *primary obstruction*, $[\mu_1, \mu_1]$, must equal zero. There are then higher obstructions which must vanish in order for μ_1 to be integrable. Thus, a reasonable starting point for deformations is to first determine which infinitesimals have a vanishing primary obstruction. Remarkably, in several fundamentally important cases, all infinitesimals μ_1 with $[\mu_1, \mu_1] = 0$ can be integrated. In fact, some of the most celebrated theorems in deformation theory are expressions of this phenomenon.

Perhaps the most studied algebra deformations are those which lie in the realm of deformation quantization, a concept introduced in the seminal paper [BFFLS78]. Suppose X is a real manifold and $A = C^\infty(X)$. Then $HH^2(A, A)$ can be identified with the space of bivector fields $\alpha \in \Gamma(X, \wedge^2 T)$, and the primary obstruction to α is the Schouten bracket $[\alpha, \alpha]$. The condition $[\alpha, \alpha] = 0$ asserts that α determines Poisson structure on X. In [BFFLS78] it was asked whether any Poisson structure can be quantized. The affirmative answer to this question is one of the jewels of deformation theory.

Theorem 1 (Kontsevich [Ko97]). *Any Poisson manifold can be quantized. More generally, there is, up to equivalence, a canonical correspondence between associative deformations of the algebra A and formal Poisson structures $\alpha_t = \alpha_1 t + \alpha_2 t^2 + \cdots$ on A.*

Also in [Ko97] is a remarkable explicit quantization formula for $X = \mathbb{R}^n$. The formula involves certain weighted graphs which determine the $*$-product expansion. A physical interpretation of the deformation quantization formula in terms of path integrals of models in string theory was made precise by Cattaneo and Felder in [CF00]. In the case where X is a smooth algebraic variety, quantization of Poisson brackets is also possible, but significant modifications of the approach are necessary, see [Ko01], [VdB07], and [Ye05].

Another case where the primary obstruction to integrating an infinitesimal is the only one is in the realm of quantum groups. Consider a Lie bialgebra $\mathfrak{a}$. The cocommutator, $\delta : \mathfrak{a} \to \mathfrak{a} \otimes \mathfrak{a}$, can be extended to an infinitesimal deformation of the coalgebra structure of $U\mathfrak{a}$ which is compatible with the Lie bracket on $\mathfrak{a}$, and hence the multiplication in $U\mathfrak{a}$. The cocommutator may thus be viewed as an infinitesimal whose primary obstruction vanishes. Drinfel'd asked in [Dr92] whether any Lie bialgebra can be quantized. The affirmative answer to this question is another famous result in deformation theory.

Theorem 2 (Etingof–Kazhdan [EK96]). *Any Lie bialgebra can be quantized. That is, if $\mathfrak{a}$ is a Lie bialgebra, then there exists a Hopf algebra deformation of $U\mathfrak{a}$ whose infinitesimal is the cocommutator of $\mathfrak{a}$.*

The quantization of $U\mathfrak{a}$ depends on a choice of Drinfel'd associator. It is known that associators are not unique and are notoriously difficult to compute with.

Many of the results pertaining to quantization of solutions to the various types of classical Yang–Baxter equation can also be viewed as examples of the phenomenon that in certain situations, the primary obstruction to integrating an infinitesimal structure is the only one. Some of these instances will be discussed in Section 7.

In general, the condition $[\mu_1, \mu_1] = 0$ does not guarantee that an infinitesimal μ_1 is integrable. The earliest known example is geometric in nature and predates the algebraic theory. In [Do60], Douady exhibited an example of an infinitesimal deformation (in the Kodaira–Spencer sense) of the Heisenberg group whose primary obstruction vanishes, yet its secondary obstruction, a Lie–Massey bracket, fails to vanish. More recently and in the algebraic case, Mathieu has given examples of commutative Poisson algebras which cannot be quantized, see [Mat97].

4 Algebras without Deformations

A deformation A_t of an algebra A is *trivial* if there is a $k[[t]]$-algebra isomorphism $\Phi_t : A_t \to A[[t]]$ which reduces the identity modulo t. An algebra is *rigid* if it has no nontrivial deformations. The cohomology results of Section 2 provide the first elementary result in deformation theory.

Theorem 3. *If $H^2(A, A) = 0$, then A is rigid.*

Algebras which satisfy the hypothesis of Theorem 3 are called *absolutely rigid*. Here are some notable examples of absolutely rigid algebras in various categories.

- Any separable k-algebra A is rigid as these are characterized by $HH^n(A, -)$ for all $n \geq 1$.
- The enveloping algebra $U\mathfrak{g}$ of a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ is rigid as an algebra as $HH^n(U\mathfrak{g}, U\mathfrak{g}) = 0$ for $n \geq 1$. It does admit deformations as a Hopf algebra.
- The coordinate ring $\mathcal{O}(V)$ of a smooth affine variety V is rigid as a commutative algebra as $Har^n(\mathcal{O}(V), \mathcal{O}(V)) = 0$ for $n \geq 1$ – a more precise interpretation of the cohomology $\mathcal{O}(V)$ will be given in Section 6. It does admit noncommutative deformations, however.
- The m-th Weyl (Heisenberg) algebra A_m is rigid as $HH^n(A_m, A_m) = 0$ for $n \geq 1$.

The converse of Theorem 3 is known to be false in many instances. Richardson has provided in [Ri67] examples of rigid Lie algebras L with $H^2_{CE}(L, L) \neq 0$. In the associative case, Gerstenhaber and Schack have given examples of rigid associative algebras when $\mathrm{char}(k) = p$ in [GS86]. Remarkably, whether there exist rigid associative algebras A in characteristic zero with $HH^2(A, A) \neq 0$ is still an open question even in the case where A is a finite-dimensional $\mathbb{C}$-algebra.

The rigid algebras of Gerstenhaber and Schack in $\mathrm{char}(k) = p$ are not everyday examples. The smallest rigid algebra constructed with $HH^2(A, A) \neq 0$ is a 669-dimensional algebra over $\mathbb{F}_2$. The algebra is a poset algebra of a suspension of a triangulation of the projective plane.

The proof of the rigidity of these algebras despite nonzero $HH^2(A, A)$ is based on an elementary but fundamental theorem of [GS86] concerning *relative* Hochschild cohomology. If S is a subalgebra of A, then a cochain $F \in C^n(A, A)$ is *S-relative* if

$$F(sa_1,\dots,a_n) = sF(a_1,\dots,a_n), \quad F(a_1,\dots,a_n s) = F(a_1,\dots,a_n)s,$$
$$\text{and} \quad F(\dots,a_i s, a_{i+1},\dots) = F(\dots,a_i, sa_{i+1},\dots) \quad (2)$$

for all $s \in S$ and $a_i \in A$. Further, an S-relative cochain F is *reduced* if $F(a_1,\dots,a_n) = 0$ whenever any $a_i \in S$.

Theorem 4. *If S is a separable subalgebra of k, then the inclusion of the complex of reduced S-relative cochains into the full Hochschild complex induces an isomorphism of cohomology.*

The theorem significantly reduces cohomology computations whenever A has a large separable subalgebra. For example, a poset algebra is a subset of the algebra of $n \times n$ matrices and one may take S to be the diagonal matrices. Using Theorem 4 it is elementary to show that the Hochschild cohomology of the poset algebra coincides with the simplicial cohomology of the geometric realization or nerve (see Section 8) of the poset.

Another application of the theorem is in the computation of the cohomology of a crossed product algebra. Let V be a finite-dimensional k-vector space, SV its symmetric algebra, and G a finite group which acts on V, and hence on SV. For $x \in SV$ and $g \in G$ denote the action of g on x by x^g. The crossed product $SV \ltimes G$ has underlying space $SV \otimes kG$, with the usual multiplication in SV and kG, and relations $(1 \otimes g)(x \otimes 1) = x^g \otimes g$ for $x \in SV$ and $g \in G$. For simplicity, we omit the tensor product symbol and write an element $x \otimes g$ simply as xg.

When $|G|$ is invertible in k, then Maschke's Theorem asserts that kG is separable – let us assume this here. In this case, we may compute $HH^*(SV \ltimes G, SV \ltimes G)$ using reduced kG-relative cochains. For such a cochain F, we have

$$F(x_1g_1, x_2g_2, \dots, x_ng_n) = F(x_1, x_2^{g_1}, \dots, x_n^{g_1\cdots g_{n-1}})\, g_1 \cdots g_n.$$

The right side of the equation is an element of $C^n(SV, SV \ltimes kG)$ and it is easy to see that it is G-invariant in the sense that

$$gF(x_1,\dots,x_n)g^{-1} = F(gx_1g^{-1},\dots,gx_ng^{-1}).$$

Therefore we obtain

$$HH^*(SV \ltimes G, SV \ltimes G) \simeq (HH^*(SV, SV \ltimes G))^G.$$

A complete computation of the cohomology can be found in many sources, and one which explicitly uses kG-relative cohomology is [Pi06]. There is interest in crossed product cohomology and deformations as they have geometric implications for orbifolds (see [CGW04]) and in the theory of symplectic reflection algebras (see [EG02] and Section 9 of this survey).

It is interesting to note that some of the rigid algebras listed above naturally appear in parametrized families, a seeming contradiction to the general theory. For example, consider $\mathcal{O}_\lambda = \mathbb{C}[x,y]$ with $y^2 = x(x-1)(x-\lambda)$, the ring of regular functions on an affine elliptic curve. If λ is close to λ', then $\mathcal{O}_\lambda \ncong \mathcal{O}_{\lambda'}$. Nevertheless, $HH^2(\mathcal{O}_\lambda, \mathcal{O}_\lambda) = 0$. According to Kontsevich, the problem here is that the variety, being affine, is not compact and formal deformation theory for noncompact objects can give "nonsensical" results, see [Ko01]. In the associative case, a "compact" object is a finite-dimensional algebra and so we expect other nonsensical results for some infinite-dimensional algebras. Here is such an example: the first quantized Weyl algebra $A_q = \mathbb{C}\langle x,y\rangle/(qxy - yx - 1)$ is not isomorphic to A_1 for q near 1. However, as noted above, $HH^2(A_1, A_1) = 0$. To put this into a formal deformation theoretic perspective, let $q = 1+t$. Then there is indeed an analytic isomorphism $\phi : A_1[[t]] \to A_q[[t]]$, but it has zero radius of convergence. A similar phenomenon happens for the situation with O_λ and $O_{\lambda'}$. The problem is that passing to the formal power series versions of these algebras has trivialized the deformations.

The above examples suggest that the classic deformation theory of a single algebra does not always detect the dependency of an algebra on parameters. However, the more general *diagram* cohomology theory of Section 8 can detect such dependencies, but does not show how the algebras vary with the parameters. The construction of the algebras with varying moduli can sometimes be accomplished through the idea of a *variation* of algebras. This concept will be addressed in Section 10.

5 Universal Deformation Formulas

The process of constructing deformations using the infinite step-by-step procedure of extending deformations of order n to $n+1$ for each $n \geq 1$ is impractical. There are instances though in which a closed form for μ_t is known. One is the explicit quantization of Poisson brackets on $\mathbb{R}^n$ given in [Ko97]. Another comes from the use of "universal deformation formulas" which are, in essence, Drinfel'd twists which act on certain classes of algebras. The prototypical example of this type of formula was given by Gerstenhaber in [Ge68]. There it was observed that if ϕ and ψ are commuting derivations of any associative algebra (in characteristic zero), then $a * b = \sum \phi^n(a)\psi^n(b)\dfrac{t^n}{n!}$ is associative. The most famous use of this idea gives the Moyal product. For example, if $A = k[x,y]$ with $\phi = \partial_x$ and $\psi = \partial_y$, then we have $x * y - y * x = t$, and the deformation is isomorphic to the first Weyl algebra as long as $t \neq 0$. When $\phi = x\partial_x$ and $\psi = y\partial y$, then the deformation is graded and isomorphic to the skew-polynomial ring $k\langle x,y\rangle/(qxy - yx)$ with $q = e^t$. These examples can of course be extended to higher dimensions.

Definition 2. *Suppose $B = (B, \varDelta, 1, \epsilon)$ is a bialgebra with comultiplication $\varDelta$, unit 1, and counit ϵ. A universal deformation formula (UDF) based on B is an element $F \in (B \otimes B)[[t]]$ such that*

$$((\varDelta \otimes 1)(F))(F \otimes 1) = ((1 \otimes \varDelta)(F))(1 \otimes F) \quad \textit{and} \quad (\epsilon \otimes 1)F = (1 \otimes \epsilon)F = 1 \otimes 1.$$

The virtue of a UDF is that for any B-module algebra A, the product $a * b = \mu \circ F(a \otimes b)$ is associative and hence is a deformation of A, see [GZ98].

Example 1. Suppose B is commutative and let $r \in P \otimes P$ where P is the space of primitive elements. Then $F = \exp(tr)$ is a UDF. Primitive elements of B act as derivations of any B-module algebra and so this UDF gives a wide range of Moyal-type deformations.

Example 2. Let $B = U\mathfrak{s}$, where $\mathfrak{s}$ is the Lie algebra with basis $\{H, E\}$ and relation $[H, E] = E$. Set $H^{\langle n \rangle} = H(H + 1) \cdots (H + n - 1)$. Then $F = \sum \frac{t^n}{n!} H^{\langle n \rangle} \otimes E^n$ is UDF. For an example of its use, take $A = k[x, y]$ with the derivations $H = x\partial_x$ and $E = x\partial_y$. The deformed algebra has the relation $x * y - y * x = tx^2$ and is the Jordan quantum plane. Numerous generations of this UDF can be found in [KLO01], [LS02] and the references therein.

Example 3. Let $\mathfrak{g} \otimes \mathfrak{g}$ be a Lie algebra and let $r \in \mathfrak{g} \wedge \mathfrak{g}$ satisfy $[r, r] = 0$, where $[-, -]$ is the Schouten bracket. Drinfel'd has shown in [Dr83] that there exists a UDF $F = 1 \otimes 1 + tr + O(t^2)$. Examples 1 and 2 are of this form. It should be noted that $[r, r] = 0$ means that r is a skew-symmetric solution of the classical Yang–Baxter equation.

Example 4. Let B be the bialgebra generated by $\{D_1, D_2, \sigma\}$ with relations $D_1 D_2 = D_2 D_1, \quad D_i \sigma = q \sigma D_i$ $(i = 1, 2)$, and comultiplication

$$\varDelta(D_1) = D_1 \otimes \sigma + 1 \otimes D_1, \quad \varDelta(D_2) = D_2 \otimes 1 + \sigma \otimes D_1, \quad \varDelta(\sigma) = \sigma \otimes \sigma.$$

Then $F = \exp_q(tD_1 \otimes D_2)$ is a UDF, where the q-exponential is the usual exponential series with $n!$ replaced by $n_q!$.

Note that for any B-module algebra, σ acts as an automorphism and. the elements D_1, D_2 act as commuting skew derivations with respect to σ. Thus, this UDF provides q-Moyal type deformations. For example, it can be used to deform the quantum plane $k\langle x, y \rangle / (qxy - yx)$ to the first quantized Weyl algebra $A_q = k\langle x, y \rangle / (qxy - yx - 1)$. Formulas of this type were also used in [CGW04] to deform certain crossed products $SV \ltimes G$.

Recently, universal deformation formulas have arisen naturally in the work of Connes and Moscovici on Rankin–Cohen brackets and the Hopf algebra H_1 of transverse geometry, see [CM04]. Rankin–Cohen brackets are families of bi-differential operators on modular forms. These brackets can be assembled to give universal deformation formulas. Some applications appear in [CM04].

The formulas based on H_1 are also connected to certain topics in deformation quantization as it relates to the Poisson geometry of groupoids and foliations. see [BTY07].

6 Commutative Algebras and the Hodge Decomposition

Let A be a commutative algebra over a field of characteristic zero. In [Ba68], Barr proved that the Harrison cohomology $Har^n(A, A)$ is a direct summand of the Hochschild cohomology $HH^n(A, A)$. The key to this splitting was Barr's discovery of an idempotent e_n in $\mathbb{Q}S_n$, the rational group algebra of the symmetric group. The symmetric group acts on $C^n(A, M)$ (the Hochschild n-cochains of A with coefficients in a symmetric A-bimodule M) via $\sigma F(a_1, \ldots, a_n) = F(a_{\sigma 1}, \ldots, a_{\sigma n})$. Barr proved that $\delta(e_n F) = e_{n+1}(\delta F)$, where δ is the Hochschild coboundary operator. Thus $HH^n(A, M)$ splits as $e^n_{nHH}(A, M) \oplus (1 - e_n)HH^n(A, M)$, and the latter piece is $Har^n(A, M)$. Barr's work received little attention until 1987 when Gerstenhaber and Schack extended the splitting, see [GS87]. In $\mathbb{Q}S_n$ there are n mutually orthogonal idempotents $e_n(1), \ldots, e_n(n)$ with the property that $\delta(e_n(r)F) = e_{n+1}(r)(\delta F)$ for all $F \in C^n(A, M)$. The relation between the idempotents and coboundary give the following fundamental theorem.

Theorem 5 (Hodge Decomposition). *Suppose* $\mathrm{char}(k) = 0$ *and let* A *be a commutative algebra and* M *a symmetric* A*-bimodule. Then there is a splitting*

$$HH^n(A, M) = HH^{1,n-1}(A, M) \oplus HH^{2,n-2}(A, M) \oplus \cdots \oplus HH^{n,0}(A, M)$$

where $HH^{r,n-r}(A, M)$ *is the cohomology of the complex* $e_*(r)C^*(A, M)$.

Around the same time as the Gerstenhaber–Schack paper [GS87], Loday, using different techniques, exhibited a splitting of the Hochschild and cyclic cohomologies of a commutative algebra, see [Lo88], [Lo89].

The idempotents $e_n(r)$, which have independent interest apart from cohomology, are most easily described using the following elegant generating function discovered by Garsia in [Ga90]:

$$\sum e_n^{(r)} x^r = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)(x - d_\sigma)(x - d_\sigma + 1) \cdots (x - d_\sigma + n - 1)\, \sigma$$

where d_σ is the number of descents in σ, i.e., the number of i with $\sigma(i) > \sigma(i + 1)$.

The following diagram, in which $HH^{i,n-i}(A, M)$ is abbreviated as $H^{i,n-i}$, is instructive in understanding the Hodge decomposition.

$$
\begin{array}{ccccccccccc}
 & & & & & & & & & & \iddots \\
 & & & & & & & & \boxed{H^{5,0}} & \longrightarrow & \cdots \\
 & & & & & & \boxed{H^{4,0}} & \longrightarrow & \boxed{H^{4,1}} & \longrightarrow & \cdots \\
 & & & & \boxed{H^{3,0}} & \longrightarrow & \boxed{H^{3,1}} & \longrightarrow & \boxed{H^{3,2}} & \longrightarrow & \cdots \\
 & & \boxed{H^{2,0}} & \longrightarrow & \boxed{H^{2,1}} & \longrightarrow & \boxed{H^{2,2}} & \longrightarrow & \boxed{H^{2,3}} & \longrightarrow & \cdots \\
\boxed{H^{1,0}} & \longrightarrow & \boxed{H^{1,1}} & \longrightarrow & \boxed{H^{1,2}} & \longrightarrow & \boxed{H^{1,3}} & \longrightarrow & \boxed{H^{1,4}} & \longrightarrow & \cdots
\end{array}
$$

In the diagram, vertical columns represent the breakup of $HH^n(A, M)$, starting with $n = 1$, and the horizontal arrows display the Hochschild coboundary. The bottom row, $HH^{1,*}(A, M)$, is the Harrison cohomology $Har(A, M)$ which is associated to Barr's idempotent. The idempotent $e_n(0)$ is the skew-symmetrizer $\dfrac{1}{n!}\displaystyle\sum_{\sigma \in S_n}(-1)^\sigma \sigma$ and it follows that the diagonal components, $HH^{n,0}(A, M)$, are the skew multi-derivations, $\bigwedge_A^n \mathrm{Der}(A)$, of A into M. If $A = \mathcal{O}(V)$, the ring of regular functions on a smooth affine variety V, then the celebrated Hochschild–Kostant–Rosenberg Theorem asserts that $HH^n(A, A) = \bigwedge_A^n \mathrm{Der}(A)$, where $\mathrm{Der}(A)$. In terms of the Hodge decomposition, the theorem becomes $HH^n(A, A) = HH^{n,0}(A, A)$. In particular, $Har^2(A, A) = 0$ and these algebras have no commutative deformations. In the case V is not smooth, one expects the components $HH^{r,n-r}(A, A)$ to encode information regarding the singularities. Some interesting results by Fronsdal in this direction can be found in [Fr07].

It is clear that the refinement of $HH^*(A, A)$ provided by the Hodge decomposition can be useful. For example, if $HH^*(A, A)$ is infinite-dimensional, then its Euler–Poincare characteristic is not well-defined. However, its partial Euler–Poincare characteristics (alternating sums of $\dim H^{r,*-r}(A, A)$) may all be defined. Here is an example which illustrates this phenomenon. Let $A = k[\epsilon]/\epsilon^2$ be the ring of dual numbers. It is well-known that $HH^n(A, A)$ has dimension one for all $n \geq 1$. Using the Hodge decomposition, one can show that $HH^n(A, A) = H^{k,n-k}(A, A)$, where $k = \lfloor \dfrac{n+1}{2} \rfloor$. The partial Euler–Poincare characteristics are deformation invariant and as such they can be helpful in detecting whether a given scheme is a deformation of another one.

N. Bergeron and Wolfgang showed that the components $\bigoplus_{r=1}^k HH^{r,n-r}$ (A, A) consist of those classes of cocycles vanishing on $(k + 1)$-shuffles but not on m-shuffles for any $m < k + 1$, see [BW95] for the precise definition and explanation. This generalizes the fact that Harrison cohomology consists of those cocycles vanishing on 2-shuffles. Another fact proved in [BW95] is that $HH^{r,n-r}(A,A)$ behaves well with respect to the filtration $\mathcal{F}_m = \bigoplus_{r\geq m} HH^{*,r}(A,A)$ in the sense that $[\mathcal{F}_p,\mathcal{F}_q]\subset\mathcal{F}_{p+q}$.

Other instances of cohomology decompositions arising from group actions are possible. For example, F. Bergeron and N. Bergeron found in [BB92] a type B decomposition. Specifically, they showed that there are n idempotents in the descent algebra of the Weyl group of type B, the group of signed permutations on n letters. Moreover, if A is an algebra with involution and M is a symmetric A-bimodule, then there is an action of B_n on $A^{\otimes n}$ with the property that the idempotents are compatible with the Hochschild coboundary map. Thus there is a "type B" splitting of the cohomology. This raises the question of whether there are idempotents in the descent algebras of other Coxeter systems (W,S) which decompose $HH^*(A,M)$ for algebras A with a suitable W-action.

7 Bialgebra Deformations

It was clear that, after discovery of quantum groups in the 1980s, there should be a cohomology theory of bialgebras with the usual features related to deformations. In [GS90a] Gerstenhaber and Schack introduced such a theory which we now describe.

The Gerstenhaber–Schack bialgebra cohomology $H^*_{GS}(B,B')$ is defined for certain matched pairs of bialgebras B and B'. For simplicity, we only describe here the case $B'=B$ (any bialgebra is matched with itself). Since B is a bialgebra, any tensor power $B^{\otimes m}$ is both a B-bimodule and a B-bicomodule and thus the Hochschild cohomology $HH^*(B,B^{\otimes m})$ and the coalgebra (Cartier) cohomology $H^*_c(B^{\otimes m},B)$ are well-defined. Set $C^{p,q}(B,B)=\mathrm{Hom}_k(B^{\otimes p},B^{\otimes q})$. The Hochschild coboundary operator provides a map $\delta_h : C^{p,q}(B,B)\to C^{p+1,q}(B,B)$ while the coalgebra coboundary yields $\delta_c : C^{p,q}(B,B)\to C^{p,q+1}(B,B)$. These coboundaries commute giving the Gerstenhaber–Schack complex

$$C^{*,*}_{GS}(B,B) \quad \text{with} \quad C^n_{GS}(B,B)=\bigoplus_{\substack{p+q=n\\ p,q>0}} C^{p,q}(B,B) \quad \text{and} \quad \delta_{GS}=\delta_h+(-1)^q\delta_c.$$

The bialgebra cohomology $H^*_{GS}(B,B)$ is then the homology of this complex.

There are variants of this theory. For example, if one takes $p>0, q\geq 0$ in the definition of $C^n_{GS}(B,B)$, then the resulting cohomology controls the deformations of B to a Drinfel'd (quasi-Hopf) algebra, see [GS90b], [MS96]. Markl has shown in [Mar07] that $H^*_{GS}(B,B)$ carries an intrinsic graded bracket. In fact, Markl's construction shows the existence of a bracket for any type of (bi)algebra over an operad or PROP.

For the rest of this section, B will denote either $\mathcal{O}(G)$ or $U\mathfrak{g}$, where G is a reductive algebraic group and $\mathfrak{g}=\mathrm{Lie}(G)$. In these cases, the bialgebra cohomology is easy to compute since $HH^n(-,\mathcal{O}(G))=0$ and the

$H_c^n(U\mathfrak{g}, -)$ vanish in positive dimensions. Explicitly, if $B = \mathcal{O}(G)$ or $U\mathfrak{g}$, then $H_{GS}^n(B, B) = \bigwedge^n \mathfrak{g}/(\bigwedge^n \mathfrak{g})^{\mathfrak{g}}$, where $(\bigwedge \mathfrak{g})^{\mathfrak{g}}$ is the space of $\mathfrak{g}$-invariants in $\bigwedge^n \mathfrak{g}$ [GS90b]. The Schouten bracket on $\bigwedge^* \mathfrak{g}$ corresponds to the graded Lie algebra structure on $H_{GS}^n(B, B)$. There are no invariants in $\mathfrak{g} \wedge \mathfrak{g}$ and, up to a scalar multiple, there is a unique nonzero invariant in $\bigwedge^3 \mathfrak{g}$. The infinitesimal bialgebra deformations of B are then elements $r \in \mathfrak{g} \wedge \mathfrak{g}$. The condition $[r, r] = 0$ in $H_{GS}^3(B, B)$ means either that r is a solution to the classical Yang–Baxter equation (CYBE) (in the case $[r, r] = 0$) or that it is a solution to the modified CYBE (in the case that $[r, r]$ is a nonzero invariant). Any solution to either of these Yang–Baxter equations gives a Poisson–Lie group structure on G.

The quantization problem for both types of r-matrices is solved. For r a solution to the CYBE, the quantization is given by the UDF associated to r, see Example 3 of Section 5 and [Dr83]. For the solutions to the modified CYBE, the quantization can be deduced from the "dynamical twist" found in [ESS00]. The quantizations of [Dr83] and [ESS00] are universal in the sense that they lie in $(U\mathfrak{g} \otimes U\mathfrak{g})[[t]]$, and so they provide a quantum Yang–Baxter matrix in $\mathrm{End}(V \otimes V)[[t]]$ for any representation V of $\mathfrak{g}$. Computing this R-matrix from the universal quantization can require great effort. However, in [GGS93] a simple explicit "GGS" formula was conjectured to quantize any modified r-matrix for $\mathfrak{g} = \mathfrak{sl}(n)$ and $V = k^n$, the vector representation. After performing computer checks for over ten thousand cases, the GGS formula was proven correct by Schedler in [Sc00]. The proof is far from elementary as it uses intricate combinatorial manipulations to show that the universal solution of [ESS00] coincides with the simple GGS formula. Something is wanting for a simpler proof and real meaning of the GGS formula. It would also be interesting to extend the result to yield elementary quantizations of the modified r-matrices in the symplectic and orthogonal cases.

The bialgebra cohomology of $\mathcal{O}(G)$ also guarantees that any deformation is equivalent to one with a deformed product $*$ which is compatible with the original comultiplication Δ. A deformation of the form $(\mathcal{O}(G), *, \Delta)$ is called *preferred*. Similarly, all bialgebra deformations of $U\mathfrak{g}$ are preferred, although in this case it is the original multiplication which is unchanged. The standard quantization $\mathcal{O}_q(G)$ is equivalent to a preferred deformation but no such presentation has been exhibited – even in the simplest case of $\mathcal{O}_q(\mathrm{SL}(2))$. As in the case of Lie bialgebra quantization, the difficulty in performing explicit computations seems to be that preferred deformations are linked with a choice of Drinfel'd associator. See [BGGS04] for a more complete discussion of deformation quantization as it relates to quantum groups.

Returning to the Yang–Baxter equations, it should be noted that the moduli space of solutions to the MCYBE for a simple Lie algebra has been constructively described by Belavin and Drinfel'd in [BD82]. The solutions fall into a finite disjoint union of components, each of which is determined by an "admissible triple" (certain combinatorial data associated with the root system). In contrast, an explicit classification of solutions to the CYBE is intractable, for it would require as a special case the knowledge of all abelian Lie subalgebras of $\mathfrak{g}$. There is, however, a nonconstructive description of such r-matrices in terms of "quasi-Frobenius" Lie algebras, see [BD82], [Sto91]. A Lie algebra $\mathfrak{q}$ is quasi-Frobenius if there is a nondegenerate function $\phi : \mathfrak{q} \wedge \mathfrak{q} \to k$ which is a two-cocycle in the Chevalley–Eilenberg cohomology. The Lie algebra is Frobenius if the two-cocycle can be taken to be a coboundary, that is, if $\phi(a, b) = F([a, b])$ for some $F \in \mathfrak{q}^*$. If $B = (B_{ij})$ is the matrix of ϕ with respect to some basis $\{x_1, \ldots, x_m\}$ of $\mathfrak{q}$, then $r = \sum B_{ij}^{-1} x_i \wedge x_j$ is a solution to the CYBE. In [GG98] it was shown that some solutions to the CYBE arise as degenerations of solutions to the MCYBE, and others do not. Perhaps it may be feasible to describe all of these "boundary" solutions using the Belavin–Drinfel'd triples.

8 Diagrams of algebras

A "diagram" of algebras is a contravariant functor $\mathbb{A}$ from a small category $\mathcal{C}$ to the category of associative k-algebras, i.e., a presheaf of algebras over $\mathcal{C}$. So for each $i \in \mathrm{Ob}(\mathbb{C})$ there is an algebra $\mathbb{A}^i$ and for each morphism $i \to j$ there is an algebra map $\phi^{ij} : \mathbb{A}^j \to \mathbb{A}^i$. Presheaves of algebras are abundant and surface in a variety of contexts: A single algebra is a diagram over a one-object category with only the identity morphism. A diagram over the category with two objects and one nontrivial morphism $u : 0 \to 1$ is nothing but a homomorphism of algebras $\phi : B \to A$. The structure sheaf $\mathcal{O}_X$ on a quasi-projective variety X is a diagram of commutative algebras over the category $\mathcal{U}$ of open subsets of X. Here $\mathcal{U}$ is a category in which the morphisms correspond to inclusion maps.

In a series of papers, Gerstenhaber and Schack developed natural cohomology and deformation theories for diagrams and proved a number of remarkable results. A description of the theory can be found in the survey [GS88]. The Hochschild cohomology of sheaves of algebras and abelian categories studied [Hi05] and [LVdB06] are closely related to the Gerstenhaber–Schack diagram cohomology.

Perhaps the most useful and difficult result in diagram cohomology theory is the *General Cohomology Comparison Theorem* (see Theorem 6) which asserts, in a sense, that the cohomology and deformation theories of an arbitrary diagram are no more general than that of a single algebra. In order to explain more clearly what this means we give a quick review of the basics of the theory.

An $\mathbb{A}$-bimodule $\mathbb{M}$ is a contravariant functor from $\mathbb{C}$ to the category of abelian groups assigning to every $i \in \mathrm{Ob}(\mathcal{C})$ an $\mathbb{A}^i$-bimodule $\mathbb{M}^i$ and to every morphism $u : i \to j$ in $\mathcal{C}$ a map $T^{ij} : \mathbb{M}^j \to \mathbb{M}^i$ which is required to be an $\mathbb{A}^j$-bimodule map. Here, $\mathbb{M}^i$ becomes an $\mathbb{A}^j$-module by virtue of the algebra homomorphism ϕ^{ij}.

Just as in the case of a single algebra, there are various descriptions of the diagram cohomology $H_d^*(\mathbb{A}, \mathbb{M})$. Once the requisite categorical machinery is laid out, one may define $H_d^*(\mathbb{A}, \mathbb{M}) = \mathrm{Ext}_{\mathbb{A}-\mathbb{A}}(\mathbb{A}, \mathbb{M})$. There is also a cochain description which is quite useful and we present this here. There is a cochain complex $(C_d^*(\mathbb{A}, \mathbb{M}), \delta_d)$ whose homology coincides with $\mathrm{Ext}_{\mathbb{A}-\mathbb{A}}(\mathbb{A}, \mathbb{M})$. The description of $C_d^*(\mathbb{A}, \mathbb{M})$ has both algebraic and simplicial aspects. The nerve Σ of $\mathcal{C}$ is the simplicial complex whose 0-simplices are the objects of $\mathcal{C}$ and the p-simplices are the composable maps $\sigma = (i_0 \to i_i \to \cdots \to i_p)$. For simplicity we write $\sigma = (i_0, \ldots, i_p)$. The boundary of σ is $\partial\sigma = \sum(-1)^j \sigma_j$, where σ_j is the j-th face of σ obtained by omitting i_j.

For a diagram $\mathbb{A}$ and $\mathbb{A}$-bimodule $\mathbb{M}$, the n-cochains are $C_d^n(\mathbb{A}, \mathbb{M}) = \bigoplus_{p+q=n} C_d^{p,q}(\mathbb{A}, \mathbb{M})$, where

$$C_d^{p,q}(\mathbb{A}, \mathbb{M}) = \prod_{\substack{p-\text{simplices}\\ (i_0,\ldots,i_p)}} C^q(\mathbb{A}(i_p), \mathbb{M}(i_0)).$$

Fix $\Gamma \in C_d^{p,q}(\mathbb{A}, \mathbb{M})$. The diagram coboundary will have two components: $\delta_{alg}\Gamma \in C_d^{p,q+1}(\mathbb{A}, \mathbb{M})$ and $\delta_{simp} \in C_d^{p+1,q}(\mathbb{A}, \mathbb{M})$. The algebraic component is defined by $(\delta_{alg}\Gamma)^\sigma = \delta_h(\Gamma^\sigma)$ where $\delta_h : C^q(\mathbb{A}^{i_p}, \mathbb{M}^{i_0}) \to C^{q+1}(\mathbb{A}^{i_p}, \mathbb{M}^{i_0})$ is the ordinary Hochschild coboundary operator. The simplicial component is defined as follows. Let $\sigma = (i_0, \ldots, i_{p+1})$ be a $p+1$-simplex. For faces σ_j with $1 \le j \le p$, we have $\Gamma^{\sigma_j} \in C^q(\mathbb{A}^{i_{p+1}}, \mathbb{M}^{i_0})$, while $\Gamma^{\sigma_0} \in C^q(\mathbb{A}^{i_{p+1}}, \mathbb{M}^{i_1})$ and $\Gamma^{\sigma_{p+1}} \in C^q(\mathbb{A}^{i_p}, \mathbb{M}^{i_0})$. The extreme cases Γ^{σ_0} and $\Gamma^{\sigma_{p+1}}$ lie in different cochain groups than the others, but there are adjustments which correct this. For σ_0 note that the composite $T^{i_0 i_1}\Gamma^{\sigma_0} \in C^q(\mathbb{A}^{i_{p+1}}, \mathbb{M}^{i_0})$. For σ_{p+1} define $\Gamma^{\sigma_{p+1}}\phi^{i_{p+1}i_p} \in C^q(\mathbb{A}^{i_{p+1}}, \mathbb{M}^{i_0})$ by

$$\Gamma^{\sigma_{p+1}}\phi^{i_{p+1}i_p}(a_1, \ldots, a_q) = \Gamma^{\sigma_{p+1}}(\phi^{i_{p+1}i_p}a_1, \ldots, \phi^{i_{p+1}i_p}a_q).$$

Now set

$$(\delta_{simp}\Gamma)^\sigma = \text{“}\Gamma^{\partial\sigma}\text{”} = T^{i_0 i_1}\Gamma^{\sigma_0} - \Gamma^{\sigma_1} + \Gamma^{\sigma_2} - \cdots + (-1)^p \Gamma^{\sigma_p}\phi^{i_p i_p+1}.$$

The full diagram coboundary is then

$$\delta_d = \delta_{alg} + (-1)^p \delta_{simp}$$

and the diagram cohomology $H_d^*(\mathbb{A}, \mathbb{M})$ is defined to be the homology of the complex

$$C_d^*(\mathbb{A}, \mathbb{M}) = \bigoplus_{p+q=n} C_d^{p,q}(\mathbb{A}, \mathbb{M}) \quad \text{with} \quad \delta_d = \delta_{alg} + (-1)^p \delta_{simp}.$$

Note that the cohomology of the bottom row $H_d^{*,0}(\mathbb{A}, \mathbb{M})$ coincides with the simplicial cohomology of $\Sigma(\mathbb{C})$ with local coefficients $\mathbb{M}$.

A deformation of $\mathbb{A}$ is a diagram of $k[[t]]$-algebras whose reduction modulo t is $\mathbb{A}$. The diagram cohomology $H_d^*(\mathbb{A}, \mathbb{A})$ is too large to govern deformations of

$\mathbb{A}$ since the simplicial cohomology of Σ may not be trivial. There are remedies such as using "asimplicial" cochains or adjoining a terminator to $\mathcal{C}$, see [GS88]. Naturally, we would like a graded Lie structure on $H_d^*(\mathbb{A}, \mathbb{A})$ which controls obstructions. It turns out that, unlike the case of a single algebra, the natural bracket on $C_d^*(\mathbb{A}, \mathbb{A})$ gives the structure of only a homotopy graded Lie algebra. Proving that this bracket descends to a graded Lie structure at the cohomology level would be at best a nasty computation using the cochain description. However, the following very difficult and useful result of [GS88] settles this question.

Theorem 6 (General Cohomology Comparison Theorem). *Associated to each diagram $\mathbb{A}$ is a single k-algebra $\mathbb{A}!!$ such that the cohomology and deformation theories of $\mathbb{A}$ are naturally isomorphic to those of $\mathbb{A}!!$. In particular, $H_d^*(\mathbb{A}, \mathbb{A})$ is a Gerstenhaber algebra.*

The *diagram algebra* $\mathbb{A}!!$ is rather complicated and we will not describe it here, although we will see a special case in Section 10. The proof of Theorem 6 relies on the *Special Cohomology Comparison Theorem* which is the case when $\mathbb{A}$ is a poset. To derive the general case, Gerstenhaber and Schack perform a barycentric subdivision of $\mathbb{A}$. It turns out that the second subdivision of an arbitrary diagram is a poset and subdivision preserves the cohomology. Van den Bergh and Lowen have proved Special Cohomology Comparison Theory for prestacks in [LVdB09].

Another important result in diagram cohomology theory is the following theorem which completely reconciles the Kodaira–Spencer manifold deformation theory with the Gerstenhaber–Schack diagram deformation theory.

Theorem 7. *Let X be a smooth compact algebraic variety with tangent bundle T. Suppose $\mathcal{U}$ is an affine open cover of X and let $\mathbb{A}$ be the restriction of $\mathcal{O}_X$ to $\mathcal{U}$. Then there is a Gerstenhaber algebra isomorphism $H_d^*(\mathbb{A}, \mathbb{A}) \simeq H^*(X, \bigwedge^* T)$.*

Using the theorem, one sees that

$$H_d^2(\mathbb{A}, \mathbb{A}) \simeq H^2(X, \mathcal{O}_X) \bigoplus H^1(X, T) \bigoplus H^0(X, \wedge^2 T).$$

The middle term consists of the infinitesimal deformations of X in the Kodaira–Spencer theory. The last term is the space of infinitesimal deformations of X to "noncommutative" spaces; those with vanishing primary obstruction are precisely the Poisson structures on X and, by Theorem 1, these are quantizable. The meaning of the first term of $H_d^2(\mathbb{A}, \mathbb{A})$ is not well-understood.

Besides applications to geometric situations, diagrams naturally arise in other contexts. For example, given an algebra A and an A-module M, one can deform the action of A on M in the evident way, and it is relatively easy in this case to deduce the appropriate deformation cohomology. More generally, one can simultaneously deform A and its action on M in a compatible way. These situations are special cases of diagram deformations. Indeed, the original A-module structure on M is simply an algebra homomorphism $\phi : A \to \mathrm{End}(M)$, and hence is a diagram. Deformations of this diagram yield the various possibilities of deforming A, the action of A on M, or both. The general theory automatically yields appropriate cohomology and obstruction theories. In Section 10 the diagram cohomology theory will be used to cohomologically explain how certain rigid algebras can appear in naturally parametrized families.

9 Deforming relations

Suppose an algebra is given as $A = TX/J$ where X is the k-module spanned by finitely many generators x_i, TX is the tensor algebra, and J is the ideal of relations. If J_t is an ideal of $TX[[t]]$ which reduces to J modulo t, then a natural question is whether $A_t = TX[[t]]/J_t$ is a deformation of A or not. Associativity of A_t is automatic but to be a deformation it must be flat as a $k[[t]]$-module. There is no efficient way in general to determine if the relations in J_t insure flatness. An elementary case where flatness fails is the following: Let $A = k[x, y, z]$ and let J_t be generated by $yx - (1 + t)txy$, $zx - xz - ty^2$ and $yz - zy$. When $t = 0$ all variables commute and the polynomial algebra $k[x, y, z]$ is obtained. For $t \neq 0$, the deformed relations allow for a PBW-type ordering in which every monomial of A_t can be reduced to one of the form $x^i y^k z^k$. However, the element $t(1 + t)y^3$ lies in J_t and so A_t has t-torsion and thus is not flat.

Flatness is relatively easy to check for certain deformations of Koszul algebras, which comprise an important class of quadratic algebras. An algebra A is quadratic if $A = TX/J$, with J generated by relations $R \subset X \otimes X$. Since the relations are homogeneous, such algebras are $\mathbb{N}$-graded, $A = \bigoplus_{i \geq 0} A[i]$ and $\dim A[i] < \infty$ for each i. In particular, $A[0] = k$. A quadratic algebra A is *Koszul* if its dual $A^!$ is isomorphic to the Yoneda algebra $\mathrm{Ext}^*_A(k, k)$. Variations of the following fundamental theorem have appeared in several places in the literature, most notably in the works of Drinfel'd [Dr86] and Braverman–Gaitsgory [BG96].

Theorem 8 (Koszul Deformation Criterion). *Suppose that $A = TX/J$ is Koszul and $A_t = TX[[t]]/J_t$, where J_t is generated by relations $R_t \subset (X \otimes X)[[t]]$ which reduce to J modulo t. Then A_t is a deformation of A if and only if $A_t[3]$ is a flat $k[[t]]$-module.*

The point of the theorem is that in the Koszul case, flatness in dimensions greater than 3 is a consequence of flatness in degree 3. Flatness in the cases of degrees 1 and 2 is automatic.

One of the most interesting and explicit uses of the Koszul deformation criterion has been carried out by Etingof and Ginzburg in the theory of *symplectic reflection algebras*, which are deformations of crossed product algebras $SV \ltimes G$, see [EG02]. One can try to deform $SV \ltimes G$ by imposing additional relations of the form $xy - yx = \kappa(x, y)$ where $x, y \in V$ and $\kappa(x, y) = -\kappa(y, x) \in \mathbb{C}G$. For an arbitrary skew-symmetric function κ, the underlying vector space of the resulting algebra, A_κ, will be smaller than that of $A_0 = SV \ltimes G$ – that is, the deformation will not be flat.

In the case where V is a symplectic vector space and $G \in Sp(V)$, Etingof and Ginzburg have an explicit and remarkable classification of which skew forms κ lead to deformations. To describe these, we first need some notation. Suppose V is a complex vector space equipped with a skew bilinear form $\omega : V \times V \to \mathbb{C}$, and let G be a finite subgroup of $Sp(V)$. An element $s \in G$ is a *symplectic reflection* if the rank of $1 - s$ is 2. The set of all symplectic reflections is denoted S. For each $s \in S$, let ω_s denote the form on V with radical $\mathrm{Ker}(1-s)$ and which coincides with ω on $\mathrm{Im}(1-s)$. The triple (V, ω, G) is *indecomposable* if V cannot be split into a nontrivial direct sum of G-invariant symplectic subspaces.

Theorem 9 (Etingof–Ginzburg [EG02]). *Suppose (V, ω, G) is an indecomposable triple, and let $\kappa : V \times V \to \mathbb{C}G$ be a skew form. Then A_κ is a flat deformation of $SV \ltimes G$ if and only if there exists a G-invariant function $c : S \to \mathbb{C}$, $s \mapsto c_s$ and a constant t, such that*

$$\kappa(x, y) = t\omega(x, y) + \sum_{s \in S} c_s \omega_s(x, y)s.$$

As stated earlier, the applications of symplectic reflection algebras are many. Here is one particularly interesting one. The center of $SV \ltimes G$ is the algebra $(SV)^G$ of G-invariant polynomial functions, which can be viewed as the functions on the orbit space V/G. If $e = \frac{1}{|G|}\sum_{g \in G} g$ is the symmetrizing idempotent in $\mathbb{C}G$, then the *spherical* subalgebra of A_κ is defined to be $eA_\kappa e$. It is known that $eA_0 e \simeq (SV)^G$, and so $eA_\kappa e$ provides a noncommutative deformation of $(SV)^G$. However, if $t = 0$, then the algebra $eA_\kappa e$ is commutative. Thus the symplectic reflection algebras can provide geometric deformations of V/G.

Returning to Theorem 8, there are algebras where, unlike the symplectic reflection algebras, there is no evident ordered or PBW-type basis of A_t. For example, Sklyanin (or elliptic) deformations of polynomial algebras have this property. The simplest case is the algebra with generators $\{x, y, z\}$ and relations

$$ax^2 + byz + czy = 0, \qquad ay^2 + bzx + cxz = 0, \qquad az^2 + bxy + cyx = 0.$$

The triple $(a, b, c) = (0, 1, -1)$ gives the polynomial algebra $k[x, y, z]$, but for generic (a, b, c) the relations are such that there is no PBW-type basis. One way to prove flatness is to associate certain geometric data (an elliptic curve $\mathcal{E}$ and point $\eta \in \mathcal{E}$) to the algebra in question. The geometric information allows one to construct a factor ring of the Sklyanin algebra which can be exploited to establish flatness. A survey of elliptic deformations of polynomial algebras can be found in [Od02].

10 Variation of algebras

As mentioned in Section 4, an algebra with $H^2(A, A) = 0$ may depend essentially on parameters and so the classic deformation theory of A does not detect this dependence. If we instead pass to an appropriate diagram of algebras, it is possible in many cases to detect the dependence of A on parameters from the diagram cohomology and construct the new algebra with the concept of algebra variation.

Suppose that we have k-algebras A, B, B' and monomorphisms $\phi : B \to A$ and $\phi' : B' \to A$ such that A is generated by the images $\phi(B)$ and $\phi'(B')$. If V is the direct sum of the underlying k-modules of B and B', then $A = TV/J$, where J is the ideal of TV generated by relations which we write in the form $R(\phi(b), \phi(b'))$ for $b \in B$ and $b' \in B'$. In this case we have a diagram $\mathbb{A}$ over the poset $\mathcal{C} = \{0, 1, 1'\}$:

$$\begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ 1 & & 1' \end{tikzcd} \quad \xrightarrow{\mathbb{A}} \quad \begin{tikzcd} & A & \\ B \arrow[ur, "\phi"] & & B'. \arrow[ul, "\phi'"'] \end{tikzcd}$$

Now consider a deformation $\mathbb{A}_t$ of $\mathbb{A}$ in which the algebras A, B, and B' remain fixed but the homomorphism ϕ is deformed as $\phi_t = \phi + t\phi_1 + t^2\phi_2 + \cdots$ and similarly assume ϕ' is deformed to ϕ'_t. We can use the *same* relations determining A with deformed inputs to construct a new algebra A_t.

Definition 3. *Suppose $A, B, B', V, R, \mathbb{A}$, and, $\mathbb{A}_t$ are as above. Let J_t be the ideal of $TV[[t]]$ generated by all elements of the form $R(\phi_t(b), \phi'_t(b'))$ for $b \in B$ and $b' \in B'$. The algebra $A_t = TV[[t]]/J_t$ is called a variation of A.*

A variation A_t is certainly associative but there is no guarantee that it is flat, and as noted earlier, there is in general no easy way to determine when such algebras are flat. The concept of variation can clearly be generalized by letting A be generated by more than two subalgebras.

It is important to note that not all algebras of the form $TV[[t]]/J_t$ where J_t is an ideal of $TV[[t]]$ with $J_0 = J$ are variations of A. As an example, take A to be commutative. Then we have in J all relations of the form $\phi(b)\phi'(b') - \phi'(b')\phi(b)$. The ideal J_t defining the variation A_t will therefore have all relations of the form $\phi_t(b)\phi'_t(b') - \phi'_t(b')\phi_t(b)$ and so A_t remains commutative.

Let us return now to the deformation of the diagram $\mathbb{A}$ in the above figure obtained by replacing ϕ with ϕ_t and ϕ' with ϕ'_t. Its infinitesimal lies in $H^2_d(\mathbb{A}, \mathbb{A})$ and is the class of a cocycle of the form $\Gamma = (\Gamma_A, \Gamma_B, \Gamma_{B'}, \Gamma_{BA}, \Gamma_{B'A})$ with

$$\varGamma_A \in HH^2(A,A), \quad \varGamma_B \in HH^2(B,B), \quad \varGamma_{B'} \in HH^2(B',B'),$$
$$\varGamma_{BA} \in HH^1(B,A), \quad \text{and} \quad \varGamma_{B'A} \in HH^1(B',A). \quad (3)$$

The first three components of $\varGamma$ have algebraic dimension 2 and simplicial dimension 0 while the last two have algebraic and simplicial dimension 1 as these correspond to the 1-simplices of the underlying category. The deformation $\mathbb{A}_t$ may be viewed as an *integral* of this cohomology class. We also assign this class to the variation A_t.

Even if the algebras A, B, and B' are absolutely rigid, $H_d^2(\mathbb{A},\mathbb{A})$ may not vanish in general as $HH^1(B,A) = \mathrm{Der}(B,A) \neq 0$ and similarly for $HH^1(B',A)$. In this case, $\varGamma$ obviously can be taken to be of the form $(0,0,0,\varGamma_{BA},\varGamma_{B'A})$. However, if the characteristic of k is zero, then we may further assume that $\varGamma = (0,0,0,0,\varGamma_{B'A})$.

Remark 1. The diagram algebra $\mathbb{A}!!$ associated to $\mathbb{A}$ (see Theorem 6) can be viewed as the algebra of 3×3 matrices of the form

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & b & 0 \\ 0 & 0 & b' \end{bmatrix}, \quad \text{with} \quad a_i \in A,\ b \in B,\ b' \in B',$$

where the multiplication in $\mathbb{A}!!$ uses the convention that $ba = \phi(b)a$ and similarly for $b'a$. Even in this simple case it is difficult to see how to canonically relate the cohomology and deformations of $\mathbb{A}$ with those of $\mathbb{A}!!$.

We end with a reconsideration of the first quantized Weyl algebra $A_q = \mathbb{C}[x,y]$ with relation $qxy - yx = 1$. We have already remarked that A_q is not isomorphic to A_1 for q near 1, yet $HH^2(A_1,A_1) = 0$. Consider now whether A_1 can be varied to A_q. Using our earlier notation, suppose $A = A_1$, $B = \mathbb{C}[x]$, and $B' = \mathbb{C}[y]$ and let $\phi : B \to A$ and $\phi : B' \to A$ be the inclusion maps. All of these algebras are absolutely rigid. Thus, based on the comments above, it suffices to vary the inclusion morphism of $\mathbb{C}[y]$ into A_1. The question becomes whether there exists an element $y' \in A_1[[t]]$ of the form $y + t\eta_1 + t^2\eta_2 + \cdots, \eta_i \in A_1$ such that the relation $[x,y'] = xy' - y'x = 1$ is equivalent to having $[x,y] = 1 - txy$, for this would give $(1+t)xy - yx = 1$, i.e., $A_q[[t]]$ with $q = 1+t$. There are indeed elements y' of the desired form. In [GG08b], it is shown that one may take

$$y' = y + a_1(t)xy^2 + a_2(t)x^2y^3 + \ldots, \qquad \text{where} \ \ a_r(t) = \frac{t^{r+1}}{(1+t)^{r+1}-1}.$$

Thus, A_q is a variation of A_1. It is easy to use the formula for y' to show that the corresponding diagram infinitesimal is $\varGamma = (0,0,0,0,\delta)$, where $\delta \in \mathrm{Der}(\mathbb{C}[y],A_1)$ is the derivation with $\delta(y) = xy^2$. This is a nontrivial cohomology class in $H_d^2(\mathbb{A},\mathbb{A})$ and so the diagram cohomology has detected the variation from A_1 to A_q.

It is instructive to note that in the power series representation of the $a_r(t)$ there could be no value of t for which all the series converge, for each $a_r(t)$ is a rational function with a pole wherever t has the form $\omega - 1$ where ω is an $(r+1)$st root of unity and every neighborhood of 0 in $\mathbb{C}$ contains infinitely many of these. Those A_q with q a root of unity are in some sense "unreachable" from A_1. Nevertheless, y' can actually be evaluated for any complex number t with $1+t$ not a root of unity.

References

[Ba68] Barr, M.: Harrison homology, Hochschild homology, and triples. J. Algebra **8**, 314–323 (1968)

[BFFLS78] Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., Sternheimer, D.: Deformation theory and quantization I: Deformation of symplectic structures, II: Physical applications. Ann. Phys. **111**, 61–110; 111–151 (1978)

[BD82] Belavin, A., Drinfel'd, V.: Solutions of the classical Yang-Baxter equation for simple Lie algebras. (Russian). Funktsional. Anal. i Prilozhen. **16**(3), 1–29 (1982)

[BB92] Bergeron, F., Bergeron, N.: Orthogonal idempotents in the descent algebra of type B_n and applications. J. Pure Appl. Algebra **79**(2), 109–129 (1992)

[BW95] Bergeron, N., Wolfgang, H.L.: The decomposition of Hochschild cohomology and Gerstenhaber operations. J. Pure Appl. Algebra **104**(3), 243–265 (1995)

[BTY07] Bieliavsky, P., Tang, X., Yao, Y.: Rankin-Cohen brackets and formal quantization. Adv. Math. **212**(1), 293–314 (2007)

[BGGS04] Bonneau, P., Gerstenhaber, M., Giaquinto, A., Sternheimer, D.: Quantum groups and deformation quantization: explicit approaches and implicit aspects. J. Math. Phys. **45**(10), 3703–3741 (2004)

[BG96] Braverman, A., Gaitsgory, D.: Poincare-Birkhoff-Witt theorem for quadratic algebras of Koszul type. J. Algebra **181**(2), 315–328 (1996)

[CGW04] Caldararu, A., Giaquinto, A., Witherspoon, S.: Algebraic deformations arising from orbifolds with discrete torsion. J. Pure Appl. Algebra **187**(1), 51–70 (2004)

[CF00] Cattaneo, A.S., Felder, G.: A path integral approach to the Kontsevich quantization formula. Comm. Math. Phys. **212**(3), 591–611 (2000)

[CKTB05] Cattaneo, A., Keller, B., Torossian, C., Bruguieres, A.: Déformation, quantification, théorie de Lie. (French) [Deformation, quantization, Lie theory] Panoramas et Synthèses [Panoramas and Syntheses], 20. Société Mathématique de France, Paris (2005)

[CM04] Connes, A., Moscovici, H.: Rankin-Cohen brackets and the Hopf algebra of transverse geometry. Mosc. Math. J. **4**(1), 111–130 (2004)

[Do60] Douady, A.: Obstruction primaire á la déformation. Séminaire Henri Cartan exposé **4**, 1–19 (1960–1961)

[DMZ07] Doubek, M., Markl, M., Zima, P.: Deformation theory (Lecture Notes). Arch. Math. (Brno) **43**(5), 333–371 (2007)

[Dr83] Drinfel'd, V.G.: On constant, quasiclassical solutions of the classical Yang-Baxter equation. Soviet Math. Dokl. **28**(3), 667–671 (1992)

[Dr86] Drinfel'd, V.G.: On quadratic commutation relations in the quasi-classical case [translation of Mathematical Physics, functional analysis (Russian), 25–34, 143, "Naukova Dumka", Kiev, 1986; MR0906075 (89c:58048)]. Selected translations. Selecta Math. Soviet **11**(4), 317–326 (1992)

[Dr92] Drinfel'd, V.G.: On some unsolved problems in quantum group theory. Lecture Notes in Mathematics, vol. 1510, pp. 1–8 (1992)

[EG02] Etingof, P., Ginzburg, V.: Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism. Invent. Math. **147**(2), 243–348 (2002)

[EK96] Etingof, P., Kazhdan, D.: Quantization of Lie bialgebras. I. Selecta Math. (N.S.) **2**(1), 1–41 (1996)

[ESS00] Etingof, P., Schedler, T., Schiffman, O.: Explicit quantization of dynamical r-matrices for finite dimensional semisimple Lie algebras. J. Am. Math. Soc. **13**(3), 595–609 (2000)

[FN57] Frölicher, A., Nijenhuis, A.: A theorem on stability of complex structures. Proc. Natl. Acad. Sci. USA **43**, 239–241 (1957)

[Fr07] Fronsdal, C.: Quantization on curves. Lett. Math. Phys. (Appendix by Kontsevich, M.) **79**(2), 109–129 (2007)

[Ga90] Garsia, A.: Combinatorics of the free Lie algebra and the symmetric group. Analysis, et Cetera, pp. 309–382. Academic, MA (1990)

[Ge63] Gerstenhaber, M.: The cohomology structure of an associative ring. Ann. Math. **78**, 267–288 (1963)

[Ge64] Gerstenhaber, M.: On the deformation of rings and algebras. Ann. Math. (2) **79**, 59–103 (1964)

[Ge68] Gerstenhaber, M.: On the deformation of rings and algebras III. Ann. Math. (2) **88**(1), 1–34 (1968)

[GG98] Gerstenhaber, M., Giaquinto, A.: Boundary solutions of the quantum Yang-Baxter equation and solutions in three dimensions. Lett. Math. Phys. **44**(2), 131–141 (1998)

[GG08b] Gerstenhaber, M., Giaquinto, A.: Variation of algebras (Preprint, 2008)

[GS86] Gerstenhaber, M., Schack, S.D.: Relative Hochschild cohomology, rigid algebras and the Bockstein. J. Pure Appl. Algebra **43**, 53–74 (1986)

[GS87] Gerstenhaber, M., Schack, S.D.: A Hodge-type decomposition for commutative algebra cohomology. J. Pure Appl. Algebra **48**, 229–247 (1987)

[GS88] Gerstenhaber, M., Schack, S.D.: Algebraic cohomology and deformation theory. Deformation theory of algebras and structures and applications (Il Ciocco, 1986), 11–264, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 247. Kluwer, Dordrecht (1988)

[GS90a] Gerstenhaber, M., Schack, S.D.: Bialgebra cohomology, deformations, and quantum groups. Proc. Natl. Acad. Sci. USA **87**(1), 478–481 (1990)

[GS90b] Gerstenhaber, M., Schack, S.D.: Algebras, bialgebras, quantum groups, and algebraic deformations. Deformation theory and quantum groups with applications to mathematical physics (Amherst, MA, 1990), pp. 51–92, Contemp. Math. vol. 134. Am. Math. Soc., RI (1992)

[GGS93] Gerstenhaber, M., Giaquinto, A., Schack, S.D.: Construction of quantum groups from Belavin-Drinfel'd infinitesimals. Quantum deformations of algebras and their representations. Israel Math. Conf. Proc. vol. 7, pp. 45–64. Bar-Ilan Univ., Ramat Gan (1993)

[GZ98] Giaquinto, A., Zhang, J.: Bialgebra actions, twists, and universal deformation formulas. J. Pure Appl. Algebra **128**(02), 133–151 (1998)

[Hi05] Hinich, V.: Deformations of sheaves of algebras. Adv. Math. **195**(1), 102–164 (2005). (`math.AG/0310116`)

[KS58] Kodaira, K., Spencer, D.C.: On deformations of complex analytic structures. Ann. Math. (2) **67**, 328–466 (1958)

[Ko01] Kontsevich, M.: Deformation quantization of algebraic varieties, EuroConférence Moshé Flato 2000, Part III (Dijon). Lett. Math. Phys. **56**(3), 271–294 (2001)

[Ko97] Kontsevich, M.: Deformation quantization of Poisson manifolds. Lett. Math. Phys. **66**(3), 157–216 (2003), (`q-alg/9709040`)

[KLO01] Kulish, P.P., Lyakhovsky, V.D., del Olmo, M.A.: Chains of twists for classical Lie algebras. J. Phys. A Math. Gen. **32**, 8671–8684 (1999)

[Lo88] Loday, J.-L.: Partition eulérienne et opérations en homologie cyclique. C. R. Acad. Sci. Paris Sér. I Math. **307**(7), 283–286 (1988)

[Lo89] Loday, J.-L.: Opérations sur l'homologie cyclique des algèbres commutatives. Invent. Math. **96**(1), 205–230 (1989)

[LVdB06] Lowen, W., Van den Bergh, M.: Deformation theory of abelian categories. Trans. Am. Math. Soc. **358**, 5441–5483 (2006)

[LVdB09] Lowen, W., Van den Bergh, M.: A Hochschild cohomology comparison theorem for prestacks. Trans. Amer. Math. Soc. **363**, 969–986 (2011)

[LS02] Lyakhovsky, V.D., Samsonov, M.E.: Elementary parabolic twist. J. Algebra Appl. **1**(4), 413–424 (2002)

[Mar07] Markl, M.: Intrinsic brackets and the L_∞-deformation theory of bialgebras. arXiv:math/0411456v6

[MS96] Markl, M., Shnider, S.: Cohomology of Drinfeld algebras: a homological algebra approach. Int. Math. Res. Not. **1996**(9), 431–445 (1996)

[Mat97] Mathieu, O.: Homologies associated with Poisson structures. Deformation theory and symplectic geometry (Ascona, 1996). Math. Phys. Stud. vol. 20, pp. 177–199. Kluwer, Dordrecht (1997)

[Od02] Odesskii, A.V.: Elliptic algebras. (Russian) Uspekhi Mat. Nauk **57**(6(348)), 87–122 (2002); Translation in Russian Math. Surveys **57**(6), 1127–1162 (2002)

[Pi06] Pinczon, G.: On two theorems about symplectic reflection algebras. (`math.QA/0612690`)

[Ri67] Richardson, R.W.: On the rigidity of semi-direct products of Lie algebras. Pac. J. Math. **22**(2), 339–344 (1967)

[Ri57] Riemann, B.: Theorie der Abel'schen Functionen. J. Reine Angew. Math. **54**, 101–155 (1857)

[Sc00] Schedler, T.: Proof of the GGS conjecture. Math. Res. Lett. **7**(5–6), 801–826 (2000)

[Sta63] Stasheff, J.: Homotopy associativity of H-spaces. I, II. Trans. Am. Math. Soc. **108**, 275–292 (1963)

[Sta93] Stasheff, J.: The intrinsic bracket on the deformation complex of an associative algebra, J. Pure Appl. Algebra **89**, 231–235 (1993)

[Ste] Sternheimer, D.: The deformation philosophy, quantization and noncommutative space-time structures. Chapter 3, pp. 39–56, this volume

[Sto91] Stolin, A.: On rational solutions of Yang-Baxter equation for $\mathfrak{sl}(n)$. Math. Scand. **69**, 57–80 (1991)

[VdB07] Van den Bergh, M.: On global deformation quantization in the algebraic case. J. Algebra **315**(1), 326–395 (2007). (`math.AG/0603200`)

[Ye05] Yekutieli A.: Deformation quantization in algebraic geometry. Adv. Math. **198**(1), 383–432 (2005)

Origins and Breadth of the Theory of Higher Homotopies

Johannes Huebschmann

UFR de Mathématiques, Université des Sciences et Technologies de Lille,
CNRS-UMR 8524, 59655 Villeneuve d'Ascq Cedex, France,
Johannes.Huebschmann@math.univ-lille1.fr

To Murray Gerstenhaber and Jim Stasheff

Abstract. Higher homotopies are nowadays playing a prominent role in mathematics as well as in certain branches of theoretical physics. The purpose of the talk is to recall some of the connections between the past and the present developments. Higher homotopies were isolated within algebraic topology at least as far back as the 1940s. Prompted by the failure of the Alexander–Whitney multiplication of cocycles to be commutative, Steenrod developed certain operations which measure this failure in a coherent manner. Dold and Lashof extended Milnor's classifying space construction to associative H-spaces, and a careful examination of this extension led Stasheff to the discovery of A_n-spaces and A_∞-spaces as notions which control the failure of associativity in a coherent way so that the classifying space construction can still be pushed through.

Algebraic versions of higher homotopies have, as we all know, led Kontsevich eventually to the proof of the formality conjecture. Homological perturbation theory (HPT), in a simple form first isolated by Eilenberg and Mac Lane in the early 1950s, has nowadays become a standard tool to handle algebraic incarnations of higher homotopies. A basic observation is that higher homotopy structures behave much better relative to homotopy than strict structures, and HPT enables one to exploit this observation in various concrete situations which, in particular, leads to the effective calculation of various invariants which are otherwise intractable.

Higher homotopies abound but they are rarely recognized explicitly and their significance is hardly understood; at times, their appearance might at first glance even come as a surprise, for example in the Kodaira–Spencer approach to deformations of complex manifolds or in the theory of foliations.

Key words: Higher homotopies, Classifying space, H-space, Homological perturbations, Quantization conjecture, Quantum groups, Operads, Foliations, Maurer–Cartan equation, Deformation theory, Strings, Cohomological physics

2010 Mathematics Subject Classification: 13D10, 16E45, 16T15, 17B37, 17B55, 17B56, 17B65, 17B70, 18G10, 18G55, 55P35, 55P47, 55P48, 55P62, 55R35, 55S30, 70H45, 81T70

A.S. Cattaneo et al. (eds.), *Higher Structures in Geometry and Physics*,
Progress in Mathematics 287, DOI 10.1007/978-0-8176-4735-3_2,

1 Introduction

It gives me great pleasure to join in this celebration of Murray Gerstenhaber's 80th and Jim Stasheff's 70th birthday. I had the good fortune to get into contact with Jim some 25 years ago. In 1981/1982 I spent 6 months at the Swiss Federal Institute of Technology (Zürich) as a Research Scholar. At the time, I received a letter from Jim asking for details concerning my application of *twisting cochains* to the calculation of certain group cohomology groups. What had happened? At Zürich, I had lectured on this topic, and Peter Hilton was among the audience. This was before the advent of the Internet; not even e-mail was available, and people would still write ordinary snail mail letters. Peter Hilton traveled a lot and in this way transmitted information; in particular, he had told Jim about my attempts to do these calculations by means of twisting cochains. By the way, since Peter Hilton was moving around so much, once someone tried to get hold of him, could not manage to do so, and asked a colleague for advice. The answer was: *Stay where you are, and Peter will certainly pass by.*

At that time I knew very little about higher homotopies, but over the years I have, like many of us, learned much from Jim's insight, his habit of bringing his readers, students, and coworkers out from "behind the cloud of unknowing", to quote some of Jim's own prose in his thesis. All of us have benefited from Jim's generosity with ideas.

I cannot reminisce indefinitely, yet I would like to make two more remarks, one related with language and in particular with language skills: For example, I vividly remember, in the fall of 1987, there was a crash at Wall Street. I inquired via e-mail – which was then available – whether this crash created a problem, for Jim or more generally for academic life. His answer sounded somewhat like "Not a problem, but quite a tizzy here". So I had to look up the meaning of "tizzy" in the dictionary. This is just one instance of how I and presumably many others profited from Jim's language skills. Sometimes Jim answers an e-mail message of mine in Yiddish – apparently his grandfather spoke Yiddish to his father. There is no standard Yiddish spelling and, when I receive such a message, to uncover it, I must read it aloud myself to understand the meaning, for example "OY VEH" which, in standard German spelling would be "Oh Weh".

I feel honored by the privilege to have been invited to deliver this tribute talk. I would like to make a few remarks related to Murray Gerstenhaber. I met Murray some 20 years ago when I spent some time at the Institute in Princeton. From my recollections, Murray was then a member of the alumni board of the Institute and was always very busy. We got into real scientific and personal contact only later. In particular, I was involved in reviewing some of the Gerstenhaber–Schack results, and I will never forget that I learnt from Murray about Wigner's approach to the idea of contraction. Also from time to time, beyond talking about mathematics, we talked about history. For example, Ruth Gerstenhaber once observed how people would gather for tea in the Fuld Hall common room in the afternoon as usual around the table, and no-one would say a word but, one after another, would eventually leave the room murmuring "There is no counterexample". The perception of a mathematician through a nonmathematician is sometimes revealing.

Before I go into the mathematical details of my talk, let us wish many more years to Jim and Murray and their wives.

Let me now turn to my talk. There would be much more to say than what I can explain in the remaining time. I shall touch on various topics and make a number of deliberate choices and I will make the attempt to explain some pieces of mathematics. However, my exposition will be far from being complete or systematic and will unavoidably be biased. For example, there are higher homotopies traditions in Russia and in Japan related with Lie loops, Lie triple systems, and the like which I cannot even mention, cf. e.g., [Ki75] and [SaMi88]. There is a good account of Jim Stasheff's contributions up to his 60th birthday, published at the occasion of this event [McCl99]. This was just before the advent of Kontsevich's proof of the formality conjecture. I will try to complement this account and can thereby, perhaps, manage to avoid too many repetitions. Also I will try to do justice to a number of less well known developments.

2 The formality conjecture

Let me run right into modern times and right into our topic: Algebraic versions of higher homotopies have, as we all know, led Kontsevich eventually to the proof of the formality conjecture [Kon97]: Let M be a smooth manifold, let $A = C^\infty(M)$ and $L = \mathrm{Vect}(M)$, and consider the exterior A-algebra $\Lambda_A L$ on L. Let $\mathrm{Hoch}(A)$ denote the Hochschild complex of A, suitably defined, e.g., in the Fréchet sense. Given the vector fields $X_1, \ldots, X_n$ on M, let $\Phi_{X_1,\ldots,X_n}$ be the Hochschild cochain given by

$$\Phi_{X_1,\ldots,X_n}(a_1,\ldots,a_n) = \frac{1}{n!}\sum \mathrm{sign}(\sigma)\prod_{j=1}^{n} X_{\sigma(j)}(a_j), \quad a_1,\ldots,a_n \in A.$$

By a version of a classical result of Hochschild–Kostant–Rosenberg, the obvious map

$$\Lambda_A L \longrightarrow \mathrm{Hoch}(A), \quad X_1 \wedge \ldots \wedge X_n \mapsto \Phi_{X_1,\ldots,X_n}, \tag{1}$$

is an isomorphism on cohomology. That is to say, the Hochschild cohomology of $A = C^\infty(M)$ amounts to the graded algebra $\Lambda_A L$ of multi vector fields on M.

The standard Schouten–Nijenhuis bracket turns the suspension $s(\Lambda_A L)$ of $\Lambda_A L$ – this is $\Lambda_A L$, regraded up by 1, into an ordinary graded Lie algebra. Here the grading convention is the standard one in algebraic topology to the effect that, in particular, a differential *lowers* degree by 1. Likewise, the familiar Gerstenhaber bracket on Hoch(A) turns the suspension $s(\mathrm{Hoch}(A))$ of Hoch(A) into an ordinary differential graded Lie algebra. However, the morphism (1), while certainly being compatible with the differentials, is *not* compatible with the Lie brackets.

For any differential graded Lie algebra $\mathfrak{g}$, the familiar C(artan) C(hevalley) E(ilenberg)-construction $\mathrm{S}^{\mathrm{c}}[\mathfrak{g}]$ furnishes a d(ifferential) g(raded) coalgebra. In fact, given $\mathfrak{g}$, differential graded Lie algebra structures on $\mathfrak{g}$ can be characterized in terms of dg coalgebra structures on the symmetric coalgebra $\mathrm{S}^{\mathrm{c}}[s(\mathfrak{g})]$ on the suspension $s(\mathfrak{g})$ of $\mathfrak{g}$: They correspond precisely to the dg coalgebra structures determined by a linear term, the differential, and a quadratic term, the bracket. This allows for immediate generalization: An sh-Lie algebra is a vector space $\mathfrak{g}$ together with a coalgebra differential on the symmetric coalgebra $\mathrm{S}^{\mathrm{c}}[s(\mathfrak{g})]$ on the suspension $s(\mathfrak{g})$ of $\mathfrak{g}$. The *formality conjecture*, as formulated and established by Kontsevich [Kon97], says that (1) extends to a *Lie algebra twisting cochain*

$$\tau \colon \mathrm{S}^{\mathrm{c}}[s^2(\Lambda_A L)] \longrightarrow s(\mathrm{Hoch}(A)). \tag{2}$$

Here τ being a *twisting cochain* means that τ satisfies the *deformation* or *Maurer–Cartan* equation. Such a Lie algebra twisting cochain furnishes an sh-map from the ordinary (differential) graded Lie algebra $s(\Lambda_A L)$ to the ordinary differential graded Lie algebra $s(\mathrm{Hoch}(A))$.

The twisting cochain τ has homogeneous constituents τ_j, τ_1 being essentially the above morphism (1). The higher terms τ_j $(j \geq 2)$ are an instance of higher homotopies, and τ is an instance of an sh-map, a term created by Jim Stasheff, inspired by terminology introduced by Sugarawa [Su60/61], see Section 4 below; here "sh" stands for "strongly homotopic." Thus, without having the language and notation of *higher homotopies* and that of *deformations* at his disposal – remarkably, both Murray Gerstenhaber and Jim Stasheff are behind the scene at this point and both from 1963 –Kontsevich would not even have been able to *phrase* the formality conjecture. This confirms a variant of an observation which, with a grain of salt, reads thus: *Mathematics consists in continuous and discreet development of language and notation.*[1]

A key observation, advocated by Jim Stasheff from early on, is this: Even though we start with strict objects, an sh-map between them may lead to new insight, not necessarily available from ordinary strict maps. This kind of observation has been successfully exploited in rational homotopy theory for decades. Kontsevich noticed its significance in an area at first independent of rational homotopy and, furthermore, managed to exhibit a particular sh-map which establishes the formality conjecture.

R. Thom had raised the issue of existence of a graded commutative differential graded algebra of cochains on a space [Th54/55]. This prompted the

[1] In the early 1970s, when I was a student at the ETH Zürich, in lectures of Prof. K. Chandrasekharan, I learned the following B. Russell quote: "What is mathematics? Mathematics is a continuous and discreet science."

development of rational homotopy, starting notably with D. Quillen [Qu69] and D. Sullivan [Su78]. A space whose rational (or real) cochain algebra is sh-equivalent to its cohomology algebra is said to be *formal*, the term *formal* referring to the fact that the rational homotopy type is then a *formal* consequence of the structure of the cohomology ring. The term *formality conjecture* derives from this tradition.

The statement of the formality conjecture implies, as we know, that every Poisson bracket on a smooth manifold admits a deformation quantization.

3 Early History

One of the origins of homotopy is Gauß' analytic expression for the linking number of two closed curves (1833). One of the origins of *higher homotopies* is the idea of a classifying space; this idea goes again back to Gauß (1828). Another origin of higher homotopies is the usage of resolutions. It is a common belief, perhaps, that resolutions go back at least to Hilbert's exploration of syzygies [Hi1890]. Hilbert studied syzygies in order to show that the generating function for the number of invariants of each degree is a rational function. He also showed that, for a homogeneous ideal I of a polynomial ring S, the "number of independent linear conditions for a form of degree d in S to lie in I" is a polynomial function of d. However, this is not the entire story. The problem of counting the number of conditions had already been considered for some time; it arose both in projective geometry and in invariant theory. A general statement of the problem, with a clear understanding of the role of syzygies – but without the word, introduced a few years later by Sylvester (1814–1897) [Sy1853] – is given by Cayley (1821–1895) [Ca1848]. In fact, in a sense, Cayley somewhat develops what is nowadays referred to as the *Koszul* resolution [Kos50] more than 100 years before Koszul. The terminology *homotopy* was apparently created by H. Poincaré (1895). Poincaré also introduced the familiar loop composition. Thus we see that, in the historical perspective, Jim Stasheff is in excellent company.

4 Various 20th-century higher homotopies

Prompted by the failure of the Alexander–Whitney multiplication of cocycles to be commutative, Steenrod developed the system of $\cup_i$-products [St47]. These induce the squaring operations which, in turn, measure this failure of commutativity in a coherent manner. The nontriviality of these operations implies in particular that, over the integers, there is no way to introduce a differential graded commutative algebra of cochains on a space. The $\cup_i$-products entailed the development of s(trongly)h(omotopy)c(ommutative) structures as well as that of Steenrod operations.

An A_∞-structure may be described as a system of higher homotopies together with suitable coherence conditions. Massey products [Mas58] may be seen as invariants of certain A_∞-structures. An elementary example arises from the familiar Borromean rings, consisting of three circles which are pairwise unlinked but all together are linked. The name "Borromean" derives from their appearance in the coat of arms of the house of the aristocratic Borromean family in northern Italy. If we regard these rings as situated in the 3-sphere, then the cohomology ring of the complement is a trivial algebra, but there is a Massey product of three variables detecting the simultaneous linking of all three circles.

At the time Massey products were isolated, Jim Stasheff was a graduate student at Princeton. His advisor J. Moore suggested he look at the problem of determining when a cohomology class of a based loop space ΩX was a *suspension* or a *loop class*, i.e., came from a cohomology class of X. In pursuing this question, Stasheff was led to work of Sugawara [Su57], who had a recognition principle for characterizing loop spaces up to homotopy type.

The ordinary loop multiplication on ΩX gives it the structure of an H-space that is associative up to homotopy. Moore's version of the loop space shows that there is a based loop space which is homotopic to the familiar one for which the loop multiplication is *strictly* associative. The conclusion is that associativity is *not* a *homotopy invariant property*; we owe Jim a complete understanding of the homotopy invariance properties of associativity, and his solution furnishes a clean recognition principle for loop spaces and, in fact, for an entire hierarchy of spaces between loop spaces and H-spaces, the loop spaces being spaces which admit a classifying space.

Specifically, Stasheff defined a nested sequence of homotopy associativity conditions and called a space an A_n-space if it satisfies the nth condition. Every space is an A_1-space, an H-space is an A_2-space, and every homotopy associative H-space is A_3. An A_∞-space has the homotopy type of a loop space.

A. Dold and R. Lashof [DoLa59] generalized to associative H-spaces Milnor's construction of a classifying space for a topological group [Mi56]. Jim Stasheff extended the Dold–Lashof construction to A_∞-spaces through his study of homotopy associativity of higher order: an A_∞-structure precisely gives a classifying space. All this was worked out in his thesis, published as [St63]. Sugawara had introduced conditions for a group-like space, see the definition in terms of the conditions 3.1–3.3 on p. 129 of [Su57] to be imposed upon two maps related by what Sugawara had called an iteration of the standard relations. Altering the appropriate part of these conditions to suit the case of associativity more precisely and naturally led Jim Stasheff, apparently prompted by F. Adams, to isolating a now familiar family of polyhedra, that of *associahedra*. We shall see below that these polyhedra actually constitute an *operad*. Moreover, following Sugawara [Su60/61], Stasheff defined maps of A_n-spaces, referred to as A_n-maps, which are special kinds of H-maps [St63] (Def. 4.4 p. 298); these maps are homotopy multiplicative in a strong sense. Via Sugawara's work, A_n-maps are related to the Dold and Lashof construction. When the homotopies defining an A_n-map exist for all n, the corresponding map is *strongly homotopy multiplicative* in the sense of Sugawara [Su60/61] (p. 259). Thus the sh-terminology we are so familiar with nowadays was born.

The algebraic analogue of an A_n-space in the category of algebras is an A_n-algebra, the case $n = \infty$ being included here. The original and motivating example was provided by the singular chains on the based loop space of a space. This notion, and variants thereof, has found many applications. One such variant, L_∞-algebras, has already been mentioned. A key observation here is that A_∞-structures behave correctly with respect to homotopy, which is not the case for strict structures. What corresponds to the classifying space construction in geometry is now the bar tilde construction. Inside the bar tilde construction, Massey products show up which determine the differentials in the resulting bar construction spectral sequence. Stasheff referred to these operations as *Yessam* operations. History relates that once, at the end of a talk of Jim's, S. Mac Lane asked the question: *Who was Yessam?*

Let me recall a warning, one of Jim's *favorite warnings* in this context: When the differential of an A_∞-algebra is zero, the conditions force the algebra to be strictly associative but there may still be nontrivial higher operations encapsulating additional information, as the example of the Borromean rings already shows where the nontriviality of the Massey product reflects the triple linking.

Jim Stasheff continued to work in the realm of fibrations. There is, for example, a notion of *topological parallel transport* developed by him. A recent joint article of J. Stasheff and J. Wirth entitled *Homotopy transition cocycles* [StWi] reworks and extends J. Wirth's thesis written in 1965 under the supervision of J. Stasheff.

5 Homological perturbations

Homological perturbation theory (HPT) has nowadays become a standard tool to construct and handle A_∞-structures. The term "homological perturbation" is apparently due to J. Milgram [GuMi70]. The basic HPT-notion, that of *contraction*, was introduced in Section 12 of [EML53/54]: A *contraction*

$$(X \underset{\pi}{\overset{\nabla}{\rightleftarrows}} Y, h)$$

consists of chain complexes X and Y, chain maps $\nabla\colon X \to Y$ and $\pi\colon Y \to X$, and a degree 1 morphism $h\colon Y \to Y$ such that

$$\pi\nabla = \mathrm{Id},\ \nabla\pi - \mathrm{Id} = dh + hd,\ h\nabla = 0,\ \pi h = 0,\ hh = 0.$$

The notion of "recursive structure of triangular complexes" in Section 5 [He54] is also an example of what was later identified as a perturbation. The "perturbation lemma" [Gu72] is lurking behind the formulas in Chapter II of

Section 1 of [Sh62] and seems to have first been made explicit by M. Barrat (unpublished). The first instance known to us where it appeared in print is [Br64]. Jim Stasheff collaborated with various colleagues on questions related with homological perturbation theory [GuSt86], [GuLaSt90], [GuLaSt91] including myself [HuSt02]. An issue dealt with in these papers, as well as in my joint paper [HuKa91] with T. Kadeishvili, is that of compatibility of the perturbation constructions with algebraic structure. This issue actually shows up when one tries to construct e.g., models for differential graded algebras, cf. also [Hu04b].

A homological algebra and higher homotopies tradition was created as well by Berikashvili and his students in Georgia (at the time part of the USSR). More precise comments about the historical development until the mid eighties may be found in the article [HuKa91], and some specific comments about the Georgian tradition in [Hu99].

In the articles [Hu89a], [Hu89b], [Hu89c], [Hu91], I explored the compatibility of the perturbation constructions with algebraic structure and developed suitable algebraic HPT-constructions to exploit A_∞-modules arising in group cohomology. In this vein, I constructed suitable free resolutions from which I was able to do explicit numerical calculations in group cohomology which until today still cannot be done by other methods. In particular, spectral sequences show up which do not collapse from E_2. These results illustrate a typical phenomenon: Whenever a spectral sequence arises from a certain mathematical structure, there is, perhaps, a certain strong homotopy structure lurking behind, and the spectral sequence is an invariant thereof. The higher homotopy structure is then somewhat finer than the spectral sequence itself. It is also worthwhile noting that the "tensor trick", developed in [Hu86], may be seen as a predecessor of the method of "labelled rooted trees".

6 Quantum groups

The issues of associativity and coassociativity, as clarified by Jim Stasheff, play a major role in the theory of quantum groups and variants thereof, e.g., quasi-Hopf algebras. Suffice it to mention here that Drinfel'd has introduced a notion of quasi-Hopf algebra in which coassociativity of the diagonal is modified in a way in which the pentagon condition plays a dominant role, analogous to the hexagonal Yang–Baxter equation replacement for cocommutativity. Now, given a quasi-Hopf algebra A, the quasi-Hopf structure induces a multiplication $\mathrm{B}\mathcal{C} \times \mathrm{B}\mathcal{C} \longrightarrow \mathrm{B}\mathcal{C}$ on the classifying space $\mathrm{B}\mathcal{C}$ of the category $\mathcal{C}$ of A-modules, and the quasi-Hopficity says that this multiplication is homotopy associative. More details and suitable references may be found in [St90] and [St92].

7 Operads

The notion of A_n-spaces and the clarity they provide for the recognition problem for topological groups became the basis for the development of homotopy invariant algebraic structures. In particular, the recognition problem for infinite loop spaces and the simultaneous interest in coherence properties in categories led to the idea of an operad [MacL65], [May72]. With hindsight we recognize that a space is an A_∞-space if and only if it is an algebra over a suitably defined operad, the nonsymmetric operad $\mathcal{K} = \{K_n\}$ of *associahedra*. In fact, this is the main result of Stasheff's thesis, though not spelled out in this language:

A connected space Y of the homotopy type of a CW-complex has the homotopy type of a loop space if and only if there exist maps $K_n \times Y^n \to Y$ which fit together to make Y an algebra over the operad $\mathcal{K}$. In fact, Y then has the homotopy type of the space ΩX where X is constructed as a quotient of $\coprod K_n \times Y^n$. This brings the generalized classifying space construction to the fore.

Likewise a graded object is an A_∞-algebra if and only if it is an algebra over a suitably defined operad, and an L_∞-algebra can be characterized in the same manner as well. In recent years many more new phenomena and structures and, in particular, applications of operads have been found, in particular in the theory of moduli spaces and in mathematical physics.

8 Deformation theory

There is an obvious formal relationship between homological perturbations and deformation theory but the relationship is actually much more profound: In [HaSt79], Steve Halperin and Jim Stasheff developed a procedure by means of which the classification of rational homotopy equivalences inducing a fixed cohomology algebra isomorphism can be achieved. Moreover, one can explore the rational homotopy types with a fixed cohomology algebra by studying perturbations of a free differential graded commutative model by means of techniques from deformation theory. This was initiated by M. Schlessinger and J. Stasheff [SchlSt]. A related and independent development, phrased in terms of what is called the functor $\mathcal{D}$, is due to N. Berikashvili and his students at Georgia, notably T. Kadeishvili and S. Saneblidze. Some details and references are given in [Hu99]. A third approach in which only the underlying graded vector space was fixed is due to Y. Felix [Fel79].

Prompted by a paper of Barannikov and Kontsevich [BaKo98], in [HuSt02], Jim Stasheff and I developed an approach to constructing solutions of the master equation by means of techniques from HPT. In that paper, we restricted attention to contractions of a differential graded Lie algebra onto its homology. More recently, I extended this approach to the situation of a contraction of a differential graded Lie algebra onto a general chain complex and thereby established the perturbation lemma for differential graded Lie algebras [Hu07a]. Further, I generalized the statement of the perturbation lemma to arbitrary sh-Lie algebras [Hu07b].

9 Strings

Operads and sh-Lie algebras show up naturally in string and conformal field theories, and Jim Stasheff contributed to this area as well. Some details and more references may be found in [St97b].

10 Cohomological physics

One of Jim's long-term interests is physics. Due to his efforts it is, perhaps, no longer a surprise that some structures of interest in physics can be explored by means of tools going back to topology, including graded Lie algebras and homological perturbations. Jim contributed to *anomalies* [St85] and invested time and effort to unravel, for example, the structure behind a field theory construction which originally goes back to *Batalin, Fradkin, and Vilkovisky*. The term "cohomological physics" was created by Jim. See in particular [St96] and [St97a] for details.

11 Higher homotopies, homological perturbations, and the working mathematician

I have already explained how higher homotopies and homological perturbations may be used to solve problems phrased in language entirely different from that of higher homotopies and HPT. Higher homotopies and HPT-constructions occur implicitly in a number of other situations in ordinary mathematics where they are at first not even visible. I can only mention some examples; these are certainly not exhaustive.

- Kodaira–Nirenberg–Spencer: Deformations of complex structures [KNS58];
- Frölicher spectral sequence of a complex manifold [Hu00];
- Toledo–Tong: Parametrix [ToTo76];
- Fedosov: Deformation quantization [Fed94];
- Whitney, Gugenheim: Extension of geometric integration to a contraction [Gu76], [Wh57]. Whitney's geometric integration theory laid some of the groundwork for Sullivan's theory of rational differential forms quoted above. The upshot of Gugenheim's contribution here is that the integration map in de Rham theory is sh-multiplicative, the de Rham algebra being an ordinary graded commutative algebra. This situation is formally the same as that of the formality conjecture explained above;

- Huebschmann: Foliations [Hu04a]; in this paper, the requisite higher homotopies are described in terms of a generalized *Maurer–Cartan* algebra;
- Huebschmann: Equivariant cohomology and Koszul duality [Hu04c], [Hu04d];
- Operads; see e.g., the conference proceedings which contain the article [St97b].

References

[BaKo98] Barannikov, S., Kontsevich, M.: Frobenius manifolds and formality of Lie algebras of polyvector fields. Int. Math. Res. Not. **4**, 201–215 (1998). `Alg-geom/9710032`

[Br64] Brown, R.: The twisted Eilenberg–Zilber theorem. Celebrazioni Archimedee del Secolo XX, In: Simposio di topologia, pp. 33–37 (1964)

[Ca1848] Cayley, A.: On the theory of elimination. Camb. Dublin Math. J. **3**, 116–120 (1848)

[DoLa59] Dold, A., Lashof, R.: Principal quasi-fibrations and fiber homotopy equivalence of bundles. Illinois J. Math. **3**, 285–305 (1959)

[EML53/54] Eilenberg, S., Mac Lane, S.: On the groups $\mathrm{H}(\pi, n)$. I. Ann. Math. **58**, 55–106 (1953). II. Methods of computation. Ann. Math. **60**, 49–139 (1954)

[Fed94] Fedosov, B.V.: A simple geometrical construction of deformation quantization. J. Differ. Geom. **40**, 213–238 (1994)

[Fel79] Felix, Y.: Classification homotopique des espaces rationnels de cohomologie donnée. (French). Bull. Soc. Math. Belg. Sér. **B 31**, 75–86 (1979)

[Gu72] Gugenheim, V.K.A.M.: On the chain complex of a fibration. Illinois J. Math. **16**, 398–414 (1972)

[Gu76] Gugenheim, V.K.A.M.: On the multiplicative structure of the de Rham theory. J. Differ. Geom. **11**, 309–314 (1976)

[GuMi70] Gugenheim, V.K.A.M., Milgram, R.J.: On successive approximations in homological algebra. Trans. Am. Math. Soc. **150**, 157–182 (1970)

[GuSt86] Gugenheim, V.K.A.M., Stasheff, J.D.: On perturbations and A_∞-structures. In: Lemaire, L. (ed.), Festschrift in Honor of G. Hirsch's 60th Birthday. Bull. Soc. Math. Belg. Sér. **A 38**, 237–246 (1986)

[GuLaSt90] Gugenheim, V.K.A.M., Lambe, L., Stasheff, J.D.: Algebraic aspects of Chen's twisting cochains. Illinois J. Math. **34**, 485–502 (1990)

[GuLaSt91] Gugenheim, V.K.A.M., Lambe, L., Stasheff, J.D.: Perturbation theory in differential homological algebra. II. Illinois J. Math. **35**, 357–373 (1991)

[HaSt79] Halperin, S., Stasheff, J.D.: Obstructions to homotopy equivalences. Adv. Math. **32**, 233–278 (1979)

[He54] Heller, A.: Homological resolutions of complexes with operators. Ann. Math. **60**, 283–303 (1954)

[Hi1890] Hilbert, D.: Über die Theorie der algebraischen Formen. Math. Ann. **36**, 473–534 (1890)

[Hu86] Huebschmann, J.: The homotopy type of $F\Psi^q$. The complex and symplectic cases. In: Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory, Part II, Proceedings of a Conference at Boulder, Colorado, 12–18 June 1983. Cont. Math. **55**, 487–518 (1986)

[Hu89b] Huebschmann, J.: Cohomology of nilpotent groups of class 2. J. Algebra **126**, 400–450 (1989)

[Hu89a] Huebschmann, J.: Perturbation theory and free resolutions for nilpotent groups of class 2. J. Algebra **126**, 348–399 (1989)

[Hu89c] Huebschmann, J.: The mod p cohomology rings of metacyclic groups. J. Pure Appl. Algebra **60**, 53–105 (1989)

[Hu91] Huebschmann, J.: Cohomology of metacyclic groups. Trans. Am. Math. Soc. **328**, 1–72 (1991)

[Hu99] Huebschmann, J.: Berikashvili's functor $\mathcal{D}$ and the deformation equation. In: Festschrift in honor of N. Berikashvili's 70th birthday, Proceedings of A. Razmadze Institute, vol. 119, pp. 59–72 (1999). `Math.AT/9906032`

[Hu00] Huebschmann, J.: Differential Batalin-Vilkovisky algebras arising from twilled Lie-Rinehart algebras. Banach Center Publications **51**, 87–102 (2000)

[Hu04a] Huebschmann, J.: Higher homotopies and Maurer-Cartan algebras: quasi-Lie-Rinehart, Gerstenhaber, and Batalin-Vilkovisky algebras. In: Marsden, J., Ratiu, T. (eds.), The Breadth of Symplectic and Poisson Geometry, Festschrift in Honor of Alan Weinstein, Progress in Mathematics, vol. 232, pp. 237–302. Birkhäuser, Basel-Boston-Berlin (2004). `Math.DG/0311294`

[Hu04b] Huebschmann, J.: Minimal free multi models for chain algebras. Georgian Math. J. **11**, 733–752 (2004). `Math.AT/0405172`

[Hu04c] Huebschmann, J.: Homological perturbations, equivariant cohomology, and Koszul duality. Documenta Math. (Accepted for publication). `Math.AT/0401160`

[Hu04d] Huebschmann, J.: Relative homological algebra, homological perturbations, equivariant de Rham theory, and Koszul duality. `Math.AT/0401161`

[Hu07a] Huebschmann, J.: The Lie algebra perturbation lemma. In this volume, pp. 159–179. `Arxiv:0708.3977`

[Hu07b] Huebschmann, J.: The sh-Lie algebra perturbation lemma. Forum Math. (To appear). `Arxiv:0710.2070`

[HuKa91] Huebschmann, J., Kadeishvili, T.: Small models for chain algebras. Math. Z. **207**, 245–280 (1991)

[HuSt02] Huebschmann, J., Stasheff, J.D.: Formal solution of the master equation via HPT and deformation theory. Forum Math. **14**, 847–868 (2002). `Math.AG/9906036`

[Ki75] Kikkawa, M: Geometry of homogeneous Lie loops. Hiroshima Math. J. **5**, 141–179 (1975)

[KNS58] Kodaira, M., Nirenberg, L., Spencer, D.C.: On the existence of deformations of complex analytic structures. Ann. Math. **68**, 450–457 (1958)

[Kon97] Kontsevich, M.: Deformation quantization of Poisson manifolds. Lett. Math. Phys. **66**, 157–216 (2003). `Math.QA/9709040`

[Kos50] Koszul, J.L.: Homologie et cohomologie des algèbres de Lie. Bull. Soc. Math. France **78**, 65–127 (1950)

[MacL65] Mac Lane, S.: Categorical Algebra. Bull. Am. Math. Soc. **75**, 40–106 (1965)

[Mas58] Massey, W.S.: Some higher order cohomology operations. Symposium internacional de topologia algebraica, Universidad Nacional Autónoma de México and UNESCO, Mexico City, pp. 145–154 (1958)

[May72] May, P.J.: The geometry of iterated loop spaces. Lecture Notes in Mathematics, vol. 271. Springer, Berlin (1972)

[McCl99] McCleary, J.: An appreciation of the work of Jim Stasheff. In: Higher Homotopy Structures in Topology and Mathematical Physics, Proceedings of an International Conference, 13–15 June 1996, at Vassar College, Poughkeepsie, New York, to honor the 60th birthday of Jim Stasheff, Cont. Math., vol. 227, pp. 1–16 (1999)

[Mi56] Milnor, J.: Construction of universal bundles. II. Ann. Math. **63**, 430–436 (1956)

[Qu69] Quillen, D.: Rational homotopy theory. Ann. Math. **90**, 205–295 (1969)

[SaMi88] Sabinin, L.V., Mikheev, P.O.: On the infinitesimal theory of local analytic loops. Soviet Math. Dokl. **36**, 545–548 (1988)

[SchlSt] Schlessinger, M., Stasheff, J.D.: Deformation theory and rational homotopy type. Pub. Math. Sci. IHES. New version 13 July 1998 (To appear)

[Sh62] Shih, W.: Homologie des espaces fibrés. Pub. Math. Sci. IHES **13** (1962)

[St63] Stasheff, J.D.: Homotopy associativity of H-spaces. I, II. Trans. Am. Math. Soc. **108**, 275–292; 293–312 (1963)

[St85] Stasheff, J.D.: The de Rham bar construction as a setting for the Zumino, Fadde'ev, etc. descent equations. In: Bardeen, W., White, A. (eds.) Proceedings of the Symposium on Anomalies, Geometry, and Topology, Argonne, Chicago. World Scientific, Singapore (1985)

[St90] Stasheff, J.D.: Differential graded Lie algebras, quasi-Hopf algebras and higher homotopy algebras. In: Quantum groups (Leningrad 1990). Lecture Notes in Mathematics, vol. 1510, pp. 120–137. Springer, Berlin (1992)

[St92] Stasheff, J.D.: Drinfel'd's quasi-Hopf algebras and beyond. In: Gerstenhaber, M., Stasheff, J.D. (eds.) Deformation theory and quantum groups with applications to mathematical physics. AMS-IMS-SIAM 1990 Joint summer research conference, 14–20 June 1990, Amherst, Massachusetts. Cont. Math., vol. 134, pp. 297–307 (1992)

[St96] Stasheff, J.D.: Deformation theory and the Batalin-Vilkovisky master equation. In: Sternheimer, D., Rawnsley, J., Gutt, S. (eds.) Deformation Theory and Symplectic Geometry, Proceedings of the Ascona meeting, June 1996, Mathematical Physics Studies, vol. 20, pp. 271–284. Kluwer, Dordrecht (1997)

[St97b] Stasheff, J.D.: From operads to physically inspired theories. In: Loday, L., Stasheff, J., Voronov, A.A. (eds.), Operads, Proceedings of Renaissance Conferences, 1995, Hartford (Connecticut)/Luminy (France). Cont. Math., vol. 202, pp. 53–81 (1997)

[St97a] Stasheff, J.D.: The (secret?) homological algebra of the Batalin-Vilkovisky approach. In: Henneaux, M. et al. (eds.), Secondary calculus and cohomological physics. Proceedings of a conference, Moscow, Russia, 24–31 August 1997, Providence, RI, AMS. Cont. Math., vol. 219, pp. 195–210 (1998)

[StWi] Stasheff, J.D., Wirth, J.: Homotopy transition cocycles. JHRS **1**, 273–283 (2006). `Math.AT/0609220`

[St47] Steenrod, N.E.: Products of cocycles and extensions of mappings. Ann. Math. **48**, 290–320 (1947)

[Su57] Sugawara, M.: On a condition that a space is group-like. Math. J. Okayama Univ. **7**, 123–149 (1957)

[Su60/61] Sugawara, M.: On the homotopy-commutativity of groups and loop spaces. Mem. Coll. Sci. Univ. Kyoto Ser. A Math. **33**, 257–269 (1960–1961)

[Su78] Sullivan, D.: Infinitesimal computations in topology. Pub. Math. IHES **47**, 269–331 (1978)

[Sy1853] Sylvester, J.J.: On a theory of Syzygetic relations of two rational integral functions, comprising an application of the theory of Sturm's functions, and that of the greatest algebraic common measure. Phil. Trans. Royal Soc. Lond. **143**, 407–548 (1853)

[Th54/55] Thom, R.: Opérations en cohomologie réelle. Séminaire H. Cartan, Exposé **17** (1954–1955)

[ToTo76] Toledo, D., Tong, Y.L.L.: A parametrix for $\overline{\partial}$ and Riemann-Roch in Čech theory. Topology **15**, 273–301 (1976)

[Wh57] Whitney, H.: Geometric integration theory. Princeton University Press, NJ (1957)

The Deformation Philosophy, Quantization and Noncommutative Space-Time Structures

Daniel Sternheimer

Department of Mathematics, Keio University, 3-14-1 Hiyoshi, Kohoku-ku, Yokohama, 223-8522 Japan, Daniel.Sternheimer@u-bourgogne.fr

Abstract. The role of deformations in physics and mathematics, especially the theory of deformations of algebras developed by Gerstenhaber in the 1960s, led to the deformation philosophy promoted in mathematical physics by Flato since the 1970s, exemplified by deformation quantization and its manifold avatars, including quantum groups and the "dual" aspect of quantum spaces. Deforming Minkowski space-time and its symmetry to anti de Sitter has significant physical consequences that we sketch (e.g., singleton physics). We end by presenting an ongoing program in which anti de Sitter would be quantized in some regions, speculating that this could explain baryogenesis in a universe in constant expansion and that higher mathematical structures could provide a unifying framework.

Key words: Deformation theory, Deformation philosophy, Quantization, Quantum groups, Anti de Sitter, Composite elementary particles, Quantized space-time, Cosmology, Baryogenesis

MSC 2010: Primary 53D55; Secondary 00A30, 22E70, 53-02, 81P05, 81S10, 81T75, 81V25, 83C55, 83F05, 53D17

1 Introduction: Deformations

This paper is based on my presentation at the Conference in honor of Murray Gerstenhaber's 80th and Jim Stasheff's 70th birthdays, dedicated to "Higher Structures in Geometry and Physics." The bulk of it will deal with deformations of algebras and their ins and outs. That is a subject in which Gerstenhaber had seminal contributions, starting with his celebrated 1964 paper [Ge64] which Moshé Flato and I discovered in Paris shortly after its publication. It soon became clear to us, and to a number of theoretical physicists in France and elsewhere, that physicists had, unbeknown to them, been speaking the language of deformations like Monsieur Jourdain was speaking prose.

A.S. Cattaneo et al. (eds.), *Higher Structures in Geometry and Physics*, Progress in Mathematics 287, DOI 10.1007/978-0-8176-4735-3_3,

The paradigm which triggered that realization was the passage from the Galilean invariance of Newtonian mechanics to the Poincaré invariance of special relativity. It took another 10 years or so to develop the tools which enabled us to make explicit, rigorous and convincing, what was in the back of the mind of many: quantum mechanics is a deformation of classical mechanics. That developed into what became known as deformation quantization and its manifold avatars and more generally into the realization that quantization is deformation. This stumbling block being removed, the paramount importance of deformations in theoretical physics became clear [Fl82].

It is a pleasure to pay this tribute to Murray's extensive and ongoing contributions. Due mention has to be made to the impact of his (and our) friend Jim, who stressed in his works the importance of higher structures both in mathematics and in physics, where many originated. In fact the two aspects are increasingly imbricated, as exemplified by the use of higher structures in Kontsevich's approach to deformation quantization.

1.1 Deformations in physics, the beginning

The first example of deformations in Physics is probably the realization that the earth is not flat. Without going back to "bereshit" (Hebrew for "in the beginning") from the Bible dear to Murray, where the issue is cautiously avoided, in the antiquity (e.g., Mesopotamia or ancient Greece) it was generally considered that the earth is a flat disk floating on an ocean (wherever the latter may have been) or a plate supported by a giant named Atlas (wherever he may have been placed). Similar assumptions were also present in China.

In the sixth century B.C. that assumption was contradicted by Pythagoras, whom we shall define in this respect as a "theoretical astrophysicist." He is usually considered as the first mathematician. He and his students believed that everything is related to mathematics (which most of us agree with), and on æsthetic grounds he conjectured that *all* celestial bodies are spherical, not only e.g., the moon and the sun. There may also have been some "democratic principle" in the background, which even nowadays might be (sometimes is) considered as heretic by clerics: the idea is that our earth is an "ordinary" part of the universe (and maybe man too).

Around 330 B.C. came Aristotle, a "phenomenologist astronomer" even if nowadays he is more considered as a philosopher, student of Plato and teacher of Alexander the Great. He noted that travelers going South see southern constellations rise higher above the horizon, and that the shadow of the earth on the moon during the partial phase of a lunar eclipse is always circular. That fitted with the spherical hypothesis of Pythagoras and with the observation that, for incoming or outgoing boats, the top of the masts appears first or disappears last, a phenomenon known since at least the seventh century B.C. at the time of Homer.

With ups and downs, the realization that we have to *deform* the Flat Earth model into a spherical model became accepted throughout the Western World since that time (though nowadays some still believe in a flat earth).

Then ca. 240 B.C. came Eratosthenes, for us an "experimentalist" even if he did not perform the experiment by himself, only collected data. He noted that at the summer solstice, the sun is at the vertical of Aswan while its rays have an angle of $\frac{2\pi}{50}$ in Alexandria, about 5,000 "stadions" away. Assuming that the sun is at infinity, he deduced mathematically a circumference of 252,000 "stadions" for the earth, which is within 2% to 20% of the correct value (given the uncertainty as to the value of a stadion in kilometers).

Remarkably a similar observation was used in China around that time to measure the distance to the sun, assuming that the earth is flat – an assumption which remained largely prevalent there until the introduction of European astronomy in the seventeenth century, though celestial bodies were usually considered as spherical since the Han dynasty. I shall refrain from drawing "Pythagorean" conclusions based on that apparent contradiction.

1.2 Riemann and Einstein

The first example of deformations in mathematics dates probably back to Riemann, with his introduction of metric and of Riemann surfaces in the middle of the nineteenth century, a century before the seminal works by Kodaira and Spencer [KS58] and Gerstenhaber [Ge64]. Riemann tackled also questions related to physics already in 1854 in his posthumously published Habilitationsschrift [Ri54] (and in two short papers in 1854 and 1858, see the EMIS site). Quoting[1] from its Section III, §3:

The questions about the infinitely great are for the interpretation of nature useless questions. But this is not the case with the questions about the infinitely small. [...] It seems that the empirical notions on which the metrical determinations of space are founded, the notion of a solid body and of a ray of light, cease to be valid for the infinitely small. We are therefore quite at liberty to suppose that the metric relations of space in the infinitely small do not conform to the hypotheses of geometry; and we ought in fact to suppose it, if we can thereby obtain a simpler explanation of phenomena.

The first part was contradicted by the *paradox* coming from the Michelson and Morley experiment (1887), resolved in 1905 by Einstein with the special theory of relativity which evolved into general relativity where ideas of Riemann proved essential. Here, experimental need triggered the theory and contradicted the physical intuition of one of the greatest mathematicians. The latter part can be considered as prophetic of noncommutative geometry, a major example of deformations.

In modern language one can express the advent of special relativity by saying that the Galilean geometrical symmetry group of Newtonian mechanics

[1] I owe the quotation (in German) to the late Julius Wess.

$(SO(3) \cdot \mathbb{R}^3 \cdot \mathbb{R}^4)$ is deformed, in the Gerstenhaber sense [Ge64], to the Poincaré group $(SO(3,1) \cdot \mathbb{R}^4)$ of special relativity. A deformation parameter comes in, c^{-1} where c is a *new fundamental constant*, the velocity of light in vacuum. Time has to be treated on the same footing as space, expressed mathematically as a purely imaginary dimension.

General relativity is another, more subtle, example of the paramount importance of deformations in physics. In that counterexample to Riemann's conjecture about the infinitely great, one deforms flat Minkowskian space-time by introducing a nonzero pseudo-Riemannian curvature. For instance, one can consider a nonzero constant curvature, de Sitter space (with positive curvature and $SO(4,1)$ symmetry) or anti de Sitter space (AdS, with negative curvature and $SO(3,2)$ symmetry). The latter two symmetry groups are further deformations of the Poincaré group (or Lie algebra) in the sense of Gerstenhaber. Being simple and therefore with zero cohomology (Whitehead's lemma), the Gerstenhaber theory of deformations [Ge64] shows that "the buck stops there" in the category of Lie groups or algebras. However, if one looks at the richer structure of bialgebras or Hopf algebras, one can deform one step further into quantized enveloping algebras, which are intertwined (by the Drinfeld twist) with the undeformed structures but not equivalent to them as deformations of Hopf algebras.

1.3 The deformation philosophy

Physical theories have their domain of applicability defined by the relevant distances, velocities, energies, etc. involved. But the passage from one domain (of distances, etc.) to another does not happen in an uncontrolled way: experimental phenomena appear that cause a paradox and contradict accepted theories. Eventually a new fundamental constant enters and the formalism is modified: the attached structures (symmetries, observables, states, etc.) *deform* the initial structure to a new structure which in the limit, when the new parameter goes to zero, "contracts" to the previous formalism. That constatation is the basis of Flato's deformation philosophy [Fl82], which has been in the background of most of our works since Gerstenhaber's seminal paper [Ge64].

Mathematics and physics are two communities separated by a common language. In mathematics one starts with axioms and uses logical deduction therefrom to obtain results that are absolute truth in that framework. In physics one has to make approximations, depending on the domain of applicability, and often resort to formal calculations "at the physical level of rigor" that, in the good cases (motivated by strong physical intuition), can be considered as heuristic and may eventually be rigorously proved.

The deformation philosophy can be very useful to try to obtain a mathematical framework in which we can develop the new models or theories needed to take into account new physical phenomena or paradoxes. The question is therefore, in which category do we seek for deformations? Usually physics is conservative and if we start e.g., with the category of associative or Lie algebras, we tend to deform in the same category. But there are important generalizations: e.g., quantum groups are deformations of (some commutative) Hopf algebras. We shall make use of that in the present paper.

As in other areas, a *quantitative* change produces a *qualitative change*. Engels [En77] developed that point and gave a series of examples in Science to illustrate the transformation of quantitative change into qualitative change at critical points. That is also a problem in psychoanalysis that was tackled using Thom's catastrophe theory [GL78]. Deformation theory is an algebraic mathematical way to deal with such "catastrophic" situations.

2 Quantization is deformation

2.1 The background

Scientists should keep in mind that there are three questions which they should answer in their research: Why, What, and How. Too often, the "foot soldiers" of the armies of researchers, needed in our modern society, deal only with the "how" question. Even for leading scientists like Gerstenhaber, that is the question which requires the bulk of the work. My mother used to tell me that work is 99% perspiration and 1% inspiration. But the latter is essential. I shall stress that aspect in the present paper. All scientists should have at least some (preferably personal) understanding of why they are doing what they are doing, before tackling the "how" question. The understanding may evolve, as it often happens when qualitative changes occur, and the answers to the same question may be different in the various sciences where the question should be asked. Quantization in an excellent example.

Why Quantization?

In physics the answer is canonical: because there is experimental need for it. That is how it all started, not without hesitations and eventually futile attempts to circumvent it.

In mathematics, a simplistic canonical answer may be: because physicists need it. A more subtle (sometimes subconscious) complementary answer is that it gives nice mathematics. Indeed, very often problems posed by Nature turn out to be more seminal in mathematics than those mathematicians can imagine "out of the blue". That is particularly true of physics, the language of which is traditionally the most mathematical among sciences, even if mathematics plays now an increasing role in all sciences, with mutual benefits.

But physicists and mathematicians speak the mathematical language with different accent and grammar. That is why (cf. the title of [MR74]) we distinguish between three different (even if overlapping) categories:

- Theoretical Physics, in which mathematics is used in an (often much) looser way in order to try and account for difficult physics problems.

- Mathematical Physics, which aims at doing the same thing in as rigorous a mathematical form as possible and at being honest when the conventions of mathematics are stretched.
- Physical Mathematics, which is pure mathematics motivated and inspired by physics. The works of Gerstenhaber and even more Stasheff clearly belong to the latter category.

Deformation quantization, depending on how it is developed and used, belongs to all three categories.

What is quantization?

In (theoretical and mathematical) physics, that is a way to describe new ("quantum") phenomena which appear (usually) in the microworld, on the basis of the ("classical") knowledge which comes from our perception and description of the macroworld. In theoretical physics it often takes the form of effective recipes, while in mathematical physics one aims at a better understanding of the process of quantization itself. That is what we have been doing with deformation quantization [BFFLS].

In (physical) mathematics, one can define quantization in a concise way as a passage from commutative to noncommutative structures. It covers a wide area of mathematics, from more concrete examples such as the quantization of Lie bialgebras to more abstract or involved ones such as quantum cohomology and noncommutative differential geometry.

How do we quantize?

In physics, the traditional way to quantize is via the correspondence principle, in which classical observables, functions on a phase space, are replaced by operators on a Hilbert space following some "quantization rule."

Many mathematicians (Berezin [Be75], Kostant [Kt70], even in a way Weyl [We28], etc.) express that physical idea by saying (in different forms) that quantization is a functor between categories of algebras of "functions" on phase spaces and of operators in Hilbert spaces. They take the physicists' formulation for God's axiom, forgetting that physicists are neither God nor Jesus but humans and that when the best of them "walk over mathematical waters," their physical intuition tells them where, under the surface of the water, lie the stones on which they can walk in relative safety, albeit getting their feet wet.

Deformation quantization is not only an alternative to that formalization of what physicists are doing, it is also (and above all) an autonomous approach to as rigorous as possible a mathematical treatment of quantization, sometimes the only one available for that purpose. The idea, coming from our deformation philosophy, is that quantization is a deformation of the composition laws of physical observables.

We shall now very briefly present the initial object of quantization (classical mechanical systems), the traditional approaches to their quantization, and the most typical example of our deformation philosophy, deformation quantization, surveying some of its manifold avatars. For more details see e.g., [DS02, BGGS] and the (extensive but not exhaustive, and still growing) list of references therein.

2.2 Classical systems and their traditional quantization

Even when we start with the simplest example of phase space, $\mathbb{R}^{2n}$, it often happens that the physical problem considered imposes constraints on the phase space. That reduces it (in the language of Definition 1 below, see e.g., Section V in [FLS76]) to a symplectic submanifold when we have only what Dirac [Di50] called second-class constraints given by some equations, constraining both configuration space and conjugate momenta, and otherwise to a Poisson manifold (in the case of first-class constraints). So we shall recall some basic facts about the mathematical formalism of classical systems in order to make precise what we need to quantize.

Definition 1. *1. A symplectic manifold is a differentiable manifold M endowed with a nondegenerate closed 2-form ω on M.*
2. Given a 2-tensor π on a differentiable manifold M, $\pi = \sum_{i,j} \pi^{ij} \partial_i \wedge \partial_j$ (locally, with obvious notations), a bracket $\{\cdot,\cdot\}\colon C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ defined locally by $\{F, G\} = \pi^{ij} \partial_i F \partial_j G$ is called a Poisson bracket if it is a skew-symmetric ($\{F, G\} = -\{G, F\}$) bilinear map satisfying ($F, G \in C^\infty(M)$):

- *Jacobi identity: $\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0$*
- *Leibniz rule: $\{FG, H\} = \{F, H\}G + F\{G, H\}$.*

3. A Poisson manifold is a differentiable manifold M endowed with a skew-symmetric contravariant 2-tensor π (not necessarily nondegenerate) such that $\{F, G\} = i(\pi)(dF \wedge dG)$ is a Poisson bracket.
4. A classical system is a Poisson manifold (M, π) with a distinguished smooth function, the Hamiltonian $H\colon M \to \mathbb{R}$.

Remark 1. For a symplectic manifold M, necessarily $\dim M = 2n$. We can write locally $\omega = \omega_{ij} dx^i \wedge dx^j$ with $\omega_{ij} = -\omega_{ji}$, $Alt(\partial_i \omega_{jk}) = 0$ and (nondegeneracy) $\det \omega_{ij} \neq 0$. There are (Darboux) coordinates (q_α, p_α) for which ω is constant: $\omega = \sum_{\alpha=1}^{\alpha=n} dq^\alpha \wedge dp^\alpha$. Every Poisson manifold is “foliated” by symplectic manifolds, usually of nonconstant dimension.

Example 1. a. *Symplectic manifolds:*
(1) $\mathbb{R}^{2n}$ with $\omega = dq^\alpha \wedge dp^\alpha$ (summation on repeated indices).
(2) Cotangent bundle T^*N, $\omega = d\lambda$ (the 1-form on T^*N is locally $\lambda = -p_\alpha dq^\alpha$).

b. Poisson manifolds:
(3) Symplectic manifolds ($d\omega = 0 = [\pi, \pi] \equiv$ Jacobi identity, $(\omega_{ij}) = (\pi^{ij})^{-1}$).
(4) Lie algebra with structure constants C_{ij}^k and $\pi^{ij} = \sum_k x^k C_{ij}^k$.
(5) $\pi = X \wedge Y$, where (X, Y) are two commuting vector fields on M.

The beginning of quantum mechanics

The experimental need for quantization became clear when, around 1900, in order to explain the blackbody radiation (cf. any textbook), Planck proposed the quantum hypothesis: the energy of light is not emitted continuously but in quanta proportional to its frequency. He wrote h ($= 2\pi\hbar$) for the proportionality constant which bears his name. That paradoxical situation got a beginning of a theoretical basis when, in 1905, Einstein came with the theory of the photoelectric effect and, in 1913, with Bohr's model for the atom. Reflecting on the photoelectric effect, Louis de Broglie suggested in 1923 that waves and particles are two manifestations of the same physical reality or (as he formulated it in [LdB29]) the concept of the duality of waves and corpuscles in Nature. That brought him to what he called "wave mechanics" in his Thesis, published in 1925, and to Stockholm after (quoting from [LdB29]) experiment which is the final judge of theories, has shown that the phenomenon of electron diffraction by crystals actually exists and that it obeys exactly and quantitatively the laws of wave mechanics.

These laws [LdB29] proved to be identical with a mechanics independently developed (published shortly afterwards) by Schrödinger, Heisenberg and many others, quantum mechanics, which uses the concept of inner product vector space invented by Hilbert in 1916 for purely mathematical reasons. The latter formulation proved extraordinarily effective but led to its probabilistic "Copenhagen" interpretation that neither Einstein nor de Broglie was at ease with, and it obscured the link with classical mechanics.

In the traditional quantization of a classical system $(\mathbb{R}^{2n}, \{\cdot, \cdot\}, H)$ we take a Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n) \ni \psi$ in which acts a "quantized" Hamiltonian $\hat{H}$, the energy levels of which are defined by an eigenvalue equation $\hat{H}\psi = \lambda\psi$. An essential ingredient is the von Neumann representation of the canonical commutation relations (CCR) for which, defining the operators $\hat{q}_\alpha(f)(q) = q_\alpha f(q)$ and $\hat{p}_\beta(f)(q) = -i\hbar\frac{\partial f(q)}{\partial q_\beta}$ for f differentiable in $\mathcal{H}$, we have (CCR) $[\hat{p}_\alpha, \hat{q}_\beta] = i\hbar\delta_{\alpha\beta}I$ $(\alpha, \beta = 1, ..., n)$. We say that the couple $(\hat{q}, \hat{p})$ "quantizes" the coordinates (q, p). A polynomial classical Hamiltonian H is quantized once chosen an operator ordering, e.g., the (Weyl) complete symmetrization of $\hat{p}$ and $\hat{q}$. In general the quantization on $\mathbb{R}^{2n}$ of a function $H(q, p)$ with inverse Fourier transform $\tilde{H}(\xi, \eta)$ can be given by (cf. [We28] where the weight is $\varpi = 1$):

$$H \mapsto \hat{H} = \Omega_\varpi(H) = \int_{\mathbb{R}^{2n}} \tilde{H}(\xi, \eta) \exp(i(\hat{p}.\xi + \hat{q}.\eta)/\hbar)\varpi(\xi, \eta) d^n\xi d^n\eta. \qquad (1)$$

Classical limit, deformation theory and around

In spite of the drastic change in the nature of observables that occurs in passing from classical to quantum mechanics, physicists have almost from the beginning tried to express at least the former as a limit of the latter when $h \to 0$. A posteriori one can see that as a kind of inverse of deformations, in the sense of Gerstenhaber or possibly more general ones [Na98]. The notion is called Wigner–Inönü "contraction" [IW53] for limits of Lie algebras and (in that context) is already present in [Se51].

A first step in that direction was to recover the classical observables from the quantum ones, which was done, shortly after Weyl's 1931 visit to Princeton, by Wigner [Wi32] in the form of a trace, which can be written $H = (2\pi\hbar)^{-n}\mathrm{Tr}[\Omega_1(H)\exp((\xi.\hat{p} + \eta.\hat{q})/i\hbar)]$. The map Ω_1 defined by (1) for $\varpi = 1$ is an isomorphism of Hilbert spaces between $L^2(\mathbb{R}^{2n})$ and the space of Hilbert–Schmidt operators on $L^2(\mathbb{R}^n)$. It took some time before physicists realized that the correspondence $H \mapsto \Omega(H)$ is not an algebra homomorphism, neither for ordinary product of functions nor for the Poisson bracket P ("Van Hove theorem"). In fact, for two functions u_1 and u_2 we have [Gr46, Mo49] $\Omega_1^{-1}(\Omega_1(u_1)\Omega_1(u_2)) = u_1u_2 + \frac{i\hbar}{2}\{u_1, u_2\} + O(\hbar^2)$, and similarly for bracket.

More precisely Ω_1 maps, into product and bracket of operators (resp.), the "functions" defined by the formal series (*Moyal product* and *Moyal bracket*): $u_1 *_M u_2 = \exp(\nu P)(u_1, u_2) = u_1u_2 + \sum_{r=1}^{\infty} \frac{\nu^r}{r!} P^r(u_1, u_2)$ (with $2\nu = i\hbar$), $M(u_1, u_2) = \nu^{-1}\sinh(\nu P)(u_1, u_2) = P(u_1, u_2) + \sum_{r=1}^{\infty} \frac{\nu^{2r}}{(2r+1)!} P^{2r+1}(u_1, u_2)$. We recognize formulas for deformations of algebras in the sense of Gerstenhaber, which can be concisely defined as follows:

Definition 2. *A deformation of an algebra A over a field $\mathbb{K}$ is an algebra $\tilde{A}$ (flat) over $\mathbb{K}[[\nu]]$ such that $\tilde{A}/\nu\tilde{A} \approx A$. Two deformations $\tilde{A}$ and $\tilde{A}'$ are said to be equivalent if they are isomorphic over $\mathbb{K}[[\nu]]$ and a deformation $\tilde{A}$ is said to be trivial if it is isomorphic to the original algebra A considered by base field extension as a $\mathbb{K}[[\nu]]$-algebra.*

Indeed for associative (resp. Lie) algebras, Definition 2 tells us that there exists a new product $*$ (resp. bracket $[\cdot,\cdot]$) such that the new (deformed) algebra is again associative (resp. Lie). Denoting the original composition laws by ordinary product (resp. $\{\cdot,\cdot\}$) this means that for $u_1, u_2 \in A$ (we can extend this to $A[[\nu]]$ by $\mathbb{K}[[\nu]]$-linearity) we have the formal series expansion $u_1 * u_2 = u_1u_2 + \sum_{r=1}^{\infty} \nu^r C_r(u_1, u_2)$, $[u_1, u_2] = \{u_1, u_2\} + \sum_{r=1}^{\infty} \nu^r B_r(u_1, u_2)$ where the C_r are Hochschild 2-cochains and the B_r (skew-symmetric) Chevalley 2-cochains, such that for $u_1, u_2, u_3 \in A$ we have $(u_1 * u_2) * u_3 = u_1 * (u_2 * u_3)$ and $\mathcal{S}[[u_1, u_2], u_3] = 0$, where $\mathcal{S}$ denotes summation over cyclic permutations.

In the early 1970s, at the time when we were "pregnant with" deformation quantization, two related approaches to quantization (see [St06] for a comparison) were initiated by mathematicians who, with different motivations, tried to develop a mathematical framework permitting one to express the correspondence principle in a context more general and intrinsic than e.g., in the canonical or Dirac constraints [Di50] approaches.

The first is *Geometric Quantization* [Kt70, So70]. One looks for a generalized Weyl map from functions on a symplectic manifold M: one starts with "prequantization" on $L^2(M)$ and tries to halve the number of degrees of freedom using (complex, in general) polarizations to get a Lagrangian submanifold $\mathcal{L}$ of dimension half of that of M, and quantized observables as operators in $L^2(\mathcal{L})$. That approach proved very powerful in representation theory (M being a coadjoint orbit, e.g., for solvable Lie groups) but it has turned out that few observables can be so quantized (linear or maybe quadratic).

The second is *Berezin quantization* (see e.g., [Be75]). There quantization is an algorithm by which a quantum system corresponds to a classical dynamical one, i.e., (roughly) is a functor between a category of algebras of classical observables (on phase space) and a category of algebras of operators (in Hilbert space). Several examples (M, π) were treated, e.g., Euclidean and Lobatchevsky planes, cylinder, torus and sphere, Kähler manifolds and duals of Lie algebras. But Hamiltonians H were not considered and the notion of deformation was not present (even if a posteriori one can find it there).

In a visionary 1949 Congress presentation, Dirac [Di49] wrote: One should examine closely even the elementary and the satisfactory features of our Quantum Mechanics and criticize them and try to modify them, because there may still be faults in them. The only way in which one can hope to proceed on those lines is by looking at the basic features of our present Quantum Theory from all possible points of view. Two points of view may be mathematically equivalent and you may think for that reason if you understand one of them you need not bother about the other and can neglect it. But it may be that one point of view may suggest a future development which another point does not suggest, and although in their present state the two points of view are equivalent they may lead to different possibilities for the future. Therefore, I think that we cannot afford to neglect any possible point of view for looking at Quantum Mechanics and in particular its relation to Classical Mechanics. Any point of view which gives us any interesting feature and any novel idea should be closely examined to see whether they suggest any modification or any way of developing the theory along new lines. A point of view which naturally suggests itself is to examine just how close we can make the connection between Classical and Quantum Mechanics. That is essentially a purely mathematical problem – how close can we make the connection between an algebra of non-commutative variables and the ordinary algebra of commutative variables? In both cases we can do addition, multiplication, division . . . .

Dirac's exceptional intuition permitted him, at that time, to only stress that there was a problem in the relation between classical and quantum mechanics. Shortly afterwards [Di50] he introduced his formalism of constraints, probably in an attempt to address the problem. Though we knew the latter, it is only a few years ago that I discovered the above quote, which can be now interpreted as an invitation to develop deformation quantization.

2.3 Deformation quantization

We *forget about the correspondence principle Ω* and work in an *autonomous* manner with "functions" on general phase spaces. The framework is a Poisson manifold (M, π) and, in the spirit of our deformation philosophy, quantization is understood [BFFLS] as a deformation of the usual product of classical observables.

More precisely we start with an algebra $\mathcal{A}_\nu = C^\infty(M)[[\nu]]$ of formal series in a parameter ν with coefficients in $A = C^\infty(M)$.

Definition 3. *A star product is a bilinear map $*_\nu \colon \mathcal{A}_\nu \times \mathcal{A}_\nu \to \mathcal{A}_\nu$ defined by $f *_\nu g = fg + \sum_{r\geq 1} \nu^r C_r(f,g)$ where the C_r are bidifferential operators null on constants, $*_\nu$ is associative and $C_1(f,g) - C_1(g,f) = 2\{f,g\}$.*

Remark 2. 1. Two star products $*_1$ and $*_2$ are *equivalent* (in the sense of Definition 2) if there exists a formal series $T(f) = f + \sum_{r\geq 1} \nu^r T_r(f)$ intertwining them ($T(f *_1 g) = T(f) *_2 T(g)$), the T_r being (necessarily [BFFLS]) differential operators.
2. If we do not require that the unit of the algebra is unchanged by deformation, i.e., that the C_r are null on constants ($1 *_\nu f = f *_\nu 1 = f$), an equivalence can always bring a so-deformed algebra to one with unit unchanged. That was proved by Gerstenhaber [GS88] in the more general context of deformations leaving a subalgebra unchanged.
3. $[f,g]_\nu \equiv \frac{1}{2\nu}(f *_\nu g - g *_\nu f) = \{f,g\} + O(\nu)$ is Lie algebra deformation.

The basic paradigm is the above-mentioned Moyal product on $\mathbb{R}^{2n}$. The choice of a star product fixes a quantization rule. Operator orderings can be implemented by good choices of T (or of the weight ϖ).

If (M, π) is a Poisson manifold, $f \tilde{\star} g = fg + \nu P(f,g)$ does not define an associative product, but $(f \tilde{\star} g) \tilde{\star} h - f \tilde{\star} (g \tilde{\star} h) = O(\nu^2)$. The question is therefore whether it is always possible to modify $\tilde{\star}$ in order to get an associative product. The answer is positive. That was proved in increasing generality over 20 years, from [BFFLS] to [Ko97, Ko99] where it is a consequence of a "formality theorem" in which higher structures play an essential role. See e.g., [DS02] for a presentation of the main steps.

For symplectic manifolds, the equivalence classes of star products are parametrized by the second de Rham cohomology space $H^2_{dR}(M)$ (cf. e.g., [NT95]). In particular, if $H^2_{dR}(\mathbb{R}^{2n})$ is zero, all deformations are equivalent. On $\mathbb{R}^{2n}$, all star products are equivalent to the Moyal product (cf. the von Neumann uniqueness theorem for projective unitary irreducible representations of the CCR). In the general case of Poisson manifolds, the equivalence classes of star products are [Ko97] those of formal Poisson tensors $\pi_\nu = \pi + \nu\pi_1 + \cdots$.

That is quantization: some physical applications

The time evolution of an observable $F \in A$ in a classical system (M, π, H), governed by $\dot{F} \equiv \frac{dF}{dt} = P(H, F)$, keeps the same form after star quantization, the Poisson bracket P being replaced by the deformed bracket $[\cdot,\cdot]_\nu$. When there is a (possibly generalized) Weyl mapping Ω between functions on phase space and operators, what corresponds to the unitary evolution operator is the (autonomously defined) star exponential: $\mathrm{Exp}_*(\frac{tH}{i\hbar}) = \sum_{r\geq 0} \frac{1}{r!}(\frac{t}{i\hbar})^r H^{*r}$ (here we write again $2\nu = i\hbar$). It is a singular object, i.e., does not belong to the quantized algebra $(A[[\nu]], *)$ but to $(A[[\nu,\nu^{-1}]], *)$. *Spectrum and states* are given by a "spectral" (Fourier–Stieltjes in the time t) decomposition of the star exponential.

In order to show that a star product provides an *autonomous* quantization of a manifold M we treated in [BFFLS] a number of examples. For the harmonic oscillator $H = \frac{1}{2}(p^2 + q^2)$, with the Moyal product on $\mathbb{R}^{2n}$, we obtain $\mathrm{Exp}_*(\frac{tH}{i\hbar}) = (\cos(\frac{t}{2}))^{-1} \exp(\frac{2H}{i\hbar}\tan(\frac{t}{2})) = \sum_{k=0}^{\infty} \exp\left(-i(k+\frac{n}{2})t\right)\pi_k^n$ where π_k^n can be expressed as a function of H. As expected the energy levels of H are $E_k = \hbar(k+\frac{n}{2})$. With normal ordering, $E_k = k\hbar$: $E_0 \to \infty$ for $n \to \infty$ in Moyal ordering but $E_0 \equiv 0$ in normal ordering, preferred in Field Theory. Other standard examples can be quantized in an autonomous manner by choosing adapted star products, e.g., the angular momentum with spectrum $k(k+(n-2))\hbar^2$ for the Casimir element of $\mathfrak{so}(n)$ and the hydrogen atom with $H = \frac{1}{2}p^2 - |q|^{-1}$ on $M = T^*S^3$, $E = \frac{1}{2}(k+1)^{-2}\hbar^{-2}$ for the discrete spectrum, and $E \in \mathbb{R}^+$ for the continuous spectrum; etc.

The Feynman path integral is, for Moyal ordering, the Fourier transform in p of the star exponential, and is equal to it (up to multiplicative factor) for the normal ordering [Dt90]. More recently the Kontsevich star product was expressed as a path integral [CF00]. There are now many examples (including in string theory) showing the importance of star products in physics.

It is a matter of practical feasibility of calculations, when there are Weyl and Wigner maps intertwining between both formalisms, to choose to work with operators in Hilbert spaces or with functional analysis methods (distributions, etc.) Dealing e.g., with spectroscopy (where it all started) and matrices on finite-dimensional Hilbert spaces, the operatorial formulation is probably easier, and safer for physicists to use. But when there are no precise Weyl and Wigner maps (e.g., for general phase spaces, possibly singular and/or infinite dimensional which is the case of Field Theory) one does not have much choice but to work (maybe "at the physical level of rigor") with functional analysis.

Some avatars: a very quick overview

2.3.2.1 (Topological) Quantum Groups. See e.g., [BGGS] for more details. One deforms the Hopf algebras of functions (differentiable vectors) on a Poisson–Lie group, and/or their topological duals (as nuclear topological vector space, Fréchet or dual thereof). It is often enough to consider only "preferred deformations" in which one deforms either the product or the coproduct, e.g., for G a semisimple compact Lie group with $A = C^\infty(G)$ (one gets a differential star product) or its dual (compactly supported distributions on G, completion of $\mathcal{U}\mathfrak{g}$, for which we deform the coproduct with a Drinfeld twist); or with $A = \mathcal{H}(G)$, the coefficient functions of finite-dimensional representations of G, or its dual.

2.3.2.2 Algebraic varieties, "manifolds with singularities" and higher structures. A natural question, especially in view of the Dirac constraints formalism where the constraints are often polynomial, is to deal with Gerstenhaber deformations of the commutative product of functions in an algebraic geometry context, on varieties. The subject, initiated by Kontsevich [Ko01] is very active now. Recent references are e.g. [Ye03, Hi05, BGNT, VdB07].

A remarkable property in the case of differentiable manifolds is the vanishing of the Harrison [GS88] (or Tyurina [Pa07]) cohomology which governs commutative deformations, making them trivial. That is why we could not simply deform the product of entries in determinants in our quantization of Nambu brackets (multilinear brackets, the typical example of which is given by a determinant, the Poisson bracket in the bilinear case) and we had to quantize that higher structure in a more complicated way [DFST].

The above vanishing is no more true when singularities are present (e.g., a cone [FK07]), or in the context of complex analytic geometry (cf. e.g., [Pa07], based on the lesser known developments of [KS58] by Grothendieck in the 1960/61 Cartan Seminar) and is related to recent studies (see e.g., [MMS06]) of index theorems for pseudodifferential operators on manifolds with boundary.

Challenging new phenomena occur. Various higher structures (groupoids and algebroids of all kinds, operads, props, gerbes, stacks, etc.) play an important role and constitute a subject of independent interest (see e.g., [BGNT] for deformation quantization of gerbes). A significant presentation of all would, however, exceed both the margin of this survey and the competence of its author. I shall therefore refer for more details to the quoted papers and references therein (or coming soon at an `arXiv` near you).

2.3.2.3 Noncommutative geometry. The Gelfand duality theorem states (roughly speaking) that a commutative topological algebra A can be represented as an algebra of functions on a topological space, its spectrum (the set of maximal ideals). Woronowicz's matrix C^* pseudogroups, another form of quantum groups which came after his earlier works [Wo79] more directly inspired by the idea of finding a noncommutative analogue, can be seen as an attempt in that direction. Gelfand's recent study [GR05] of polynomials in noncommutative variables from a combinatorial point of view is another.

By far the most successful development is Connes' noncommutative geometry, a fast-growing frontier domain of mathematics with many ramifications. Deformed algebras of the type $(A[[\nu]], *_\nu)$ belong [Co94] to that framework. The strategy is to formulate usual differential geometry in an unusual manner, using in particular algebras and related concepts, so as to be able to "plug in" noncommutativity in a natural way. That is now a huge subject in itself with dedicated web sites. It gave precise constructions of quantized Riemannian manifolds and of the Standard Model (see e.g., [CDV05, CM08]).

The last Section will deal with the background and main lines of a recent attempt in that general direction, in which we deform and quantize Minkowski space-time.

3 Deformed symmetries, particle physics and cosmology

Traditionally elementary particles are associated with unitary irreducible representations (UIR) of a symmetry group, in particular the Poincaré group, which is why (following Dirac's suggestion) Wigner studied these, obtaining in 1939 the first example of the unitary dual of a (noncompact) Lie group which developed into a new field of mathematics. Since the background has been reviewed in [St07] we shall here only sketch the main points, referring to that paper (and references therein) for more details, and to [BCSV] for a later development and some new perspectives.

In line with our deformation philosophy, we studied extensively the implications for particle physics of deforming Minkowski space-time by introducing a tiny negative curvature, to AdS_4. It turned out that the UIR which, for many good reasons, represent massless particles in AdS_4, are composite of two "more elementary" particles (massless in 1+2 dimensional Minkowski space-time), the "singletons" that were discovered by Dirac in 1963 [Di63], which for that reason we call Di and Rac. That kinematical description (based on UIR of $SO(2,3)$) was made dynamical for photons as 2-Rac states in [FF88] in a way compatible with quantum electrodynamics. Many more properties were studied in what we call "singleton physics," in particular conformal covariance (giving a prototype of the now extensively studied AdS/CFT correspondence) and BRST. A lot more needs to be done in that spirit, most notably an extension of the new infinite-dimensional Lie algebra defined in [FF88], which is a "square root of the CCR," to a (much more complex, not yet introduced) "square root of a superalgebra" when dealing with both Di and Rac.

About 10 years later, when neutrino oscillations were confirmed, we suggested [FFS99] that they could be interpreted in that framework. Shortly afterwards Frønsdal [Fr00] suggested to interpret the leptons (electron, muon, tau, their antiparticles and neutrinos), none of which is now considered as massless, as initially massless and therefore 2-singleton states in AdS_4, in three flavors and massified by interaction with (five pairs of) Higgs. That predicts two new mesons (the analogs of W and Z for flavor symmetry) which are yet to be observed (as well as the Higgs).

But why deform Minkowski space-time only by introducing a small curvature? Surprising phenomena appear when quantizing the AdS_4 symmetry $SO(2,3)$, e.g., there exist [FHT93] finite-dimensional UIR at even root of unity, hinting that the corresponding quantized AdS spaces should be "q-compact," and (BTZ) black holes occur in $\mathrm{AdS}_{n\geq 3}$. The physical Ansatz is that there may remain after the Big Bang, at the edge of our Universe, "shrapnel" from the initial singularity, playing a role similar to that of stem cells, which in view of our deformation philosophy could be black holes having the form of quantized AdS (deformed Minkowski) spaces, possibly at even root of unity.

A first step is to develop a noncommutative geometry approach to such structures. That was done in [BCSV]. In a nutshell we build a (closed [Co94]) star product using an oscillatory integral, on a 1-dimensional extension $\mathcal{R}_0$ of the Heisenberg group (naturally endowed with a left invariant symplectic structure) and a Dirac operator D on the space $\mathcal{H}$ of a regular representation of $\mathcal{R}_0$. The star product endows the space $\mathcal{A}^\infty$ of smooth vectors in $\mathcal{H}$ with a noncommutative Fréchet algebra structure. We get in this way a noncommutative spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ à la Connes, but in a Lorentzian context, which induces on (an open $\mathcal{R}_0$-orbit $\mathcal{M}_0$ in) AdS space-time a pseudo-Riemannian deformation triple – and raises further questions even for that part.

Another direction is to try to adapt singleton physics, the background of which is nonquantized AdS spaces, in the new quantized context. In particular, to try to develop an analog, on quantized space-times, of the composite electrodynamics of [FF88] and of the incorporation of flavor symmetry [Fr00]. And then possibly to try to extend that to quarks and baryons, which would be created from analogs of singletons emerging from "qAdS black holes" and be massified e.g., by interaction with dark matter or dark energy, which are now believed to constitute 96% of our universe. As we suggest in [BCSV] that (very ambitious) program might provide an explanation of baryogenesis, both as to how and where matter is created in our universe in accelerated expansion, and why there is an imbalance between matter and antimatter.

That approach raises many questions which we can already foresee, and in the course of study more will surge. We can also expect that new higher structures will be needed, even more so if one wants to put it all into a "theory of everything" of a new kind, possibly bundled with strings. In my opinion these problems are definitely worthy of attack. They can be expected to prove their worth by hitting back. Solving them would be a major achievement, solving some should be at least nice mathematics.

References

[BFFLS] Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., Sternheimer, D.: Deformation theory and quantization I: Deformation of symplectic structures; II: Physical applications. Ann. Phys. (NY) **111**, 61–110; 111–151 (1978)

[Be75] Berezin, F.A.: General concept of quantization. Comm. Math. Phys. **40**(2), 153–174 (1975)

[BCSV] Bieliavsky, P., Claessens, L., Sternheimer, D., Voglaire, Y.: Quantized Anti de Sitter spaces and non-formal deformation quantizations of symplectic symmetric spaces. Amer. Math. Soc. Providence, R.I. **450**, 1–24 (2008). `arXiv:0705.4179v1[math.QA]`

[BGGS] Bonneau, P., Gerstenhaber, M., Giaquinto, A., Sternheimer, D.: Quantum groups and deformation quantization: explicit approaches and implicit aspects. J. Math. Phys. **45**(10), 3703–3741 (2004)

[BGNT] Bressler, P., Gorokhovsky, A., Nest, R., Tsygan, B.: Deformation quantization of gerbes. Adv. Math. **214**(1), 230–266 (2007). math.QA/0512136

[LdB29] de Broglie, L.: Nobel lecture, quotations from pp. 255–256 in its translation Nobel Lectures, Physics 1922–1941. Elsevier, Amsterdam (1965). http://nobelprize.org/nobel_prizes/physics/laureates/1929/broglie-lecture.pdf

[CF00] Cattaneo, A.S., Felder, G.: A path integral approach to the Kontsevich quantization formula. Comm. Math. Phys. **212**(3), 591–611 (2000)

[Co94] Connes, A.: Noncommutative geometry. Academic, San Diego (1994)

[CDV05] Connes, A., Dubois-Violette, M.: Noncommutative finite dimensional manifolds II. Moduli space and structure of noncommutative 3-spheres. math.QA/0511337

[CM08] Connes, A., Marcolli M.: Noncommutative geometry, quantum fields and motives. (2008). ftp://ftp.alainconnes.org/bookwebfinal.pdf

[Di50] Dirac, P.A.M.: Lectures on Quantum Mechanics. Belfer Graduate School of Sciences Monograph Series no. 2, Yeshiva University, New York, 1964. Generalized Hamiltonian dynamics, Canad. J. Math. **2**, 129–148 (1950)

[Di49] Dirac, P.A.M.: The relation of classical to quantum mechanics. In: Proc. Second Canadian Math. Congress, Vancouver, 1949, pp. 10–31. University of Toronto Press, ON (1951)

[Di63] Dirac, P.A.M.: A remarkable representation of the $3+2$ de Sitter group. J. Math. Phys. **4**(7), 901–909 (1963)

[Dt90] Dito, J.: Star-product approach to quantum field theory: the free scalar field. Lett. Math. Phys. **20**(2), 125–134 (1990)

[DFST] Dito, G., Flato, M., Sternheimer, D., Takhtajan, L.: Deformation quantization and Nambu mechanics. Comm. Math. Phys. **183**(1), 1–22 (1997). hep-th/9602016

[DS02] Dito, G., Sternheimer, D.: Deformation quantization: genesis, developments and metamorphoses. Deformation quantization (Strasbourg, 2001), pp. 9–54, IRMA Lect. Math. Theor. Phys. vol. 1. de Gruyter, Berlin (2002). math.QA/0201168

[En77] Engels, F.: http://www.marxists.de/science/mcgareng/engels1.htm and/or http://www.marxists.org/archive/marx/works/1877/anti-duhring/index.htm

[Fl82] Flato, M.: Deformation view of physical theories. Czech. J. Phys. **B32**(4), 472–475 (1982)

[FF88] Flato, M., Frønsdal, C.: Composite electrodynamics. J. Geom. Phys. **5**(1), 37–61 (1988)

[FLS76] Flato, M., Lichnerowicz, A., Sternheimer, D.: Deformations of Poisson brackets, Dirac brackets and applications. J. Math. Phys. **17**(9), 1754–1762 (1976)

[FHT93] Flato, M., Hadjiivanov, L.K., Todorov, I.T.: Quantum deformations of singletons and of free zero-mass fields. Found. Phys. **23**(4), 571–586 (1993)

[FFS99] Flato, M., Frønsdal, C., Sternheimer, D.: Singletons, physics in AdS universe and oscillations of composite neutrinos. Lett. Math. Phys. **48**(1), 109–119 (1999)

[Fr00] Frønsdal, C.: Singletons and neutrinos. Conférence Moshé Flato 1999 (Dijon). Lett. Math. Phys. **52**(1), 51–59 (2000)

[FK07] Frønsdal, C.: Quantization on curves. With an appendix by Maxim Kontsevich. Lett. Math. Phys. **79**(2), 109–129 (2007)

[GL78] Galatzer-Levy, R.L.: Qualitative change from quantitative change: mathematical catastrophe theory in relation to psychoanalysis. J. Am. Psychoanal. Assoc. **26**, 921–935 (1978)

[GR05] Gelfand, I., Gelfand, S., Retakh, V., Wilson, R.L.: Factorizations of polynomials over noncommutative algebras and sufficient sets of edges in directed graphs. Lett. Math. Phys. **74**(2), 153–167 (2005)

[Ge64] Gerstenhaber, M.: On the deformation of rings and algebras. Ann. Math. (2) **79**, 59–103 (1964); and (IV), Ann. Math. **99**, 257–276 (1974)

[GS88] Gerstenhaber, M., Schack, S.D.: Algebraic cohomology and deformation theory. Deformation theory of algebras and structures and applications (Il Ciocco, 1986), pp. 11–264, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. vol. 247. Kluwer, Dordrecht (1988)

[Gr46] Groenewold, A.: On the principles of elementary quantum mechanics. Physica **12**, 405–460 (1946)

[Hi05] Hinich, V.: Deformations of sheaves of algebras. Adv. Math. **195**(1), 102–164 (2005). `math.AG/0310116`

[IW53] Inönü, E., Wigner, E.P.: On the contraction of groups and their representations. Proc. Natl. Acad. Sci. USA **39**, 510–524 (1953)

[KS58] Kodaira, K., Spencer, D.C.: On deformations of complex analytic structures. Ann. Math. (2) **67**, 328–466 (1958)

[Ko99] Kontsevich, M.: Operads and motives in deformation quantization. Lett. Math. Phys. **48**(1), 35–72 (1999). `math.QA/9904055`

[Ko01] Kontsevich, M.: Deformation quantization of algebraic varieties. EuroConférence Moshé Flato 2000, Part III (Dijon). Lett. Math. Phys. **56**(3), 271–294 (2001). `math.AG/0106006`

[Ko97] Kontsevich, M.: Deformation quantization of Poisson manifolds. Lett. Math. Phys. **66**(3), 157–216 (2003). `q-alg/9709040`

[Kt70] Kostant, B.: Quantization and unitary representations. I. Prequantization. Lectures in modern analysis and applications, III, pp. 87–208. Lecture Notes in Mathematics, vol. 170, pp. 87–208. Springer, Berlin (1970)

[MMS06] Mathai, V., Melrose, R.B., Singer, I.M.: Fractional analytic index. J. Diff. Geom. **74**(2), 265–292 (2006). `math.DG/0402329,math.DG/0611819`

[MR74] Maurin, K., Rączka, R. (eds.): Mathematical physics and physical mathematics. Proceedings of the International Symposium held in Warsaw, March 25–30, 1974. Mathematical Physics and Applied Mathematics, vol. **2**. Reidel, Dordrecht (1976)

[Mo49] Moyal, J.E.: Quantum mechanics as a statistical theory. Proc. Camb. Phil. Soc. **45**, 99–124 (1949)

[Na98] Nadaud, F.: Generalized deformations, Koszul resolutions, Moyal products. Rev. Math. Phys. **10**(5), 685–704 (1998), *Thèse*, Dijon (janvier 2000)

[NT95] Nest, R., Tsygan, B.: Algebraic index theorem. Comm. Math. Phys. **172**(2), 223–262 (1995)

[Pa07] Palamodov, V.P.: Associative deformations of complex analytic spaces. Lett. Math. Phys. **82**(2–3), 191–217 (2007)

[Ri54] Riemann, B.: Ueber die Hypothesen, welche der Geometrie zu Grunde liegen. 1854 Habilitationsschrift, Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen **13** (1868); translated by William Kingdon Clifford, On the Hypotheses which lie at the Bases of Geometry. Nature **8**, 14–17 (1873) Transcribed and edited in 1998–2000 by David R. Wilkins, see http://www.emis.de/classics/Riemann/

[Se51] Segal, I.E.: A class of operator algebras which are determined by groups. Duke Math. J. **18**, 221–265 (1951)

[So70] Souriau, J.-M.: Structure of dynamical systems. A symplectic view of physics. Translated from the 1970 French edition by C.H. Cushman-de Vries. Translation edited and with a preface by R.H. Cushman and G.M. Tuynman. Progress in Mathematics vol. 149. Birkhäuser, Boston (1997)

[St06] Sternheimer, D.: Some reflections on mathematicians' views of quantization. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **331**, 199–220 (2006); edited in J. Math. Sci. (NY) **141**(4), 1494–1505 (2007)

[St07] Sternheimer, D.: The geometry of space-time and its deformations from a physical perspective. In: From Geometry to Quantum Mechanics. Progr. Math. vol. 252, pp. 287–301. Birkhäuser, Boston (2007)

[VdB07] Van den Bergh, M.: On global deformation quantization in the algebraic case, J. Algebra **315**(1), 326–395 (2007). math.AG/0603200

[We28] Weyl, H.: The theory of groups and quantum mechanics. Dutton/Dover, New York, 1931, translated from the second edition (1931) of *Gruppentheorie und Quantenmechanik*, Hirzel, Leipzig (1928)

[Wi32] Wigner, E.P.: Quantum corrections for thermodynamic equilibrium. Phys. Rev. **40**, 749–759 (1932)

[Wo79] Woronowicz, S.L.: Pseudospaces, pseudogroups and Pontriagin duality. Mathematical problems in theoretical physics. (Proc. Internat. Conf. Math. Phys., Lausanne, 1979). Lecture Notes in Physics, vol. 116, pp. 407–412. Springer, Berlin (1980)

[Ye03] Yekutieli, A.: Deformation quantization in algebraic geometry. Adv. Math. **198**(1), 383–432 (2005). math.AG/0310399,math.AG/07081654

Differential Geometry of Gerbes and Differential Forms

Lawrence Breen

Université Paris 13, CNRS, UMR 7539 LAGA, 99, avenue J.-B. Clément, 93430 Villetaneuse, France, breen@math.univ-paris13.fr

To Murray Gerstenhaber and Jim Stasheff

Abstract. We discuss certain aspects of the combinatorial approach to the differential geometry of non-abelian gerbes due to W. Messing and the author [5], and give a more direct derivation of the associated cocycle equations. This leads us to a more restrictive definition than in [5] of the corresponding coboundary relations. We also show that the diagrammatic proofs of certain local curving and curvature equations may be replaced by computations with differential forms.

Key words: Gerbe, Connective structure, Čech–de Rham cohomology

MSC2010: 53C08

1 Introduction

It is a classical fact[1] that to a principal G-bundle P on a scheme X, endowed with a connection ϵ, is associated a Lie (G)-valued 2-form κ on P, the curvature of the connection, satisfying a certain G-equivariance condition. While κ does not in general descend to a 2-form on X, the equivariance condition may be viewed as a descent condition for κ from a 2-form on P to a 2-form on X, but now with values in the Lie algebra of the gauge group P^{ad} of P. The connection on P also induces a connection μ on the group P^{ad}, and the 2-form κ satisfies the Bianchi equation, an equation which may be expressed in global terms as

$$d\kappa + [\mu, \kappa] = 0 \tag{1}$$

([5] Proposition 1.7, [4] Theorem 3.7). Choosing a local trivialization of the bundle P, on an open cover $\mathcal{U} := \coprod_{i \in I} U_i$ of X, the connection ϵ is described

[1] At least in a differential geometric setting, see [9], but the same construction can be carried out within the context of algebraic geometry.

A.S. Cattaneo et al. (eds.), *Higher Structures in Geometry and Physics*, Progress in Mathematics 287, DOI 10.1007/978-0-8176-4735-3_4,

by a family of Lie (G)-valued connection 1-forms ω_i defined on the open sets U_i, and the associated curvature κ corresponds to a family of Lie (G)-valued 2-forms κ_i defined, according to what is known as the structural equation of Elie Cartan, by the formula[2]

$$\kappa_i = \mathrm{d}\omega_i + \frac{1}{2}\,[\omega_i,\, \omega_i]. \tag{2}$$

Equation (1) then reduces to the classical Bianchi identity

$$\mathrm{d}\kappa_i + [\omega_i, \kappa_i] = 0. \tag{3}$$

J.-L. Brylinski introduced in [7] the notions of connection ϵ and curving K on an abelian G-gerbe $\mathcal{P}$ on a space X (where G was the multiplicative group G_m, or rather in his framework the group $U(1)$), and showed that to such connective data $(\epsilon,\, K)$ is associated a closed G_m-valued 3-form ω on X, the 3-curvature. More recently, W. Messing and the author extended these concepts in [5] from abelian to general, not necessarily abelian, gerbes $\mathcal{P}$ on a scheme X. The coefficients of such a gerbe no longer constitute a sheaf of groups as in the principal bundle situation, but rather a monoidal stack $\mathcal{G}$ on X, as is to be expected in that categorified setting. In particular, when the gerbe is associated to a given non-abelian group G (so that we refer to it as a G-gerbe), the corresponding coefficient stack $\mathcal{G}$ is the monoidal stack associated to the prestack determined by the crossed module $G \longrightarrow \mathrm{Aut}(G)$, where $\mathrm{Aut}(G)$ is the sheaf of local automorphisms of G. It may also be described more invariantly as the monoidal stack of G-bitorsors on X. Once more, to the gerbe $\mathcal{P}$ is associated its gauge stack, a twisted form $\mathcal{P}^{\mathrm{ad}} := \mathcal{E}q(\mathcal{P},\, \mathcal{P})$ of the given monoidal stack $\mathcal{G}$, and the connection on $\mathcal{P}$ induces a connection μ on $\mathcal{P}^{\mathrm{ad}}$. By analogy with the principal bundle case, the corresponding 3-curvature Ω, viewed as a global 3-form on X, now takes its values in the arrows of the stack $\mathcal{P}^{\mathrm{ad}}$.

There now arises a new, and at first sight somewhat surprising feature, but which is simply another facet of the categorification context in which we are operating. The 3-form Ω is accompanied by an auxiliary 2-form κ with values in the objects of the gauge stack $\mathcal{P}^{\mathrm{ad}}$, which we called in [5] the *fake curvature* of the given connective structure $(\epsilon,\, K)$. A first relation between the forms Ω and κ comes from the very definition [5] (4.1.20), (4.1.22) of Ω, and may be stated as in [5](4.3.8) as the categorical equation

$$t\Omega + \mathrm{d}\kappa + [\mu,\, \kappa] = 0 \tag{4}$$

where t stands for "target" of a 1-arrow with source the identity object I in the stack of Lie$(\mathcal{P}^{\mathrm{ad}})$-valued 3-forms on X. On the other hand, the 3-form Ω is no longer closed, even in the μ-twisted sense described for principal bundles

[2] The canonical divided power $1/2[\omega,\, \omega]$ of the 2-form $[\omega,\, \omega]$ is also denoted $\omega \wedge \omega$ or $[\omega]^{(2)}$.

by (1). It satisfies instead the following more complicated analogue [5] (4.1.33) of the Bianchi identity (1):

$$d\Omega + [\mu,\, \Omega] + [\mathcal{K},\, \kappa] = 0\,. \tag{5}$$

While the first two terms in this equation are similar to those of (1), the categorification term $\mathcal{K}$ is an arrow in the stack of 2-forms with values in the monoidal stack $\mathcal{E}q(\mathcal{P}^{\mathrm{ad}},\, \mathcal{P}^{\mathrm{ad}})$ induced by the curving K. The pairing of $\mathcal{K}$ with κ is induced by the evaluation of the natural transformation $\mathcal{K}$ between functors from $\mathcal{P}^{\mathrm{ad}}$ to itself on the object κ of $\mathcal{P}^{\mathrm{ad}}$.

The price to be paid for the compact form in which the global curvature equations (4) and (5) have been stated is their rather abstract nature, and it is of interest to describe them in a more local form in terms of traditional group-valued differential forms, just as was done in (3) for equation (1). Such a local description was already obtained in [5], both for the cocycle conditions (4) and (5), and for the corresponding coboundary equations which arise when alternate local trivializations of the gerbe have been chosen. However, the determination of those local equations was rather indirect, as it required a third description of a gerbe, which we have called the semi-local description [6] §4, and which has also appeared elsewhere in various situations [15], [13], [8].

The present text may be viewed as a companion piece to the author's [6]. Its main purpose is to provide a more transparent construction than in [5] of the cocycle conditions and related equations associated to a gerbe with curving data summarized in [5] Theorem 6.4. We restrict our attention, as in [6], to gerbes which are connected rather than locally connected, as these determine Čech cohomology classes. A cocyclic description in the general case requires hypercovers and could be dealt with along the lines discussed in [3], but would not shed any additional light on the phenomena being investigated here. Our main results are to be found in Sections 4 and 5, while Section 3 reviews for the reader's convenience some aspects of [5] and [6]. Section 2 is a review of some of the formulas in the differential calculus of Lie (G)-valued forms, a few of which do not appear to be well-known.

Another aim of the present work is to revisit the quite complicated coboundary equations of [5] §6.2. The coboundary equations which arise here are simpler, and more consistent than those of [5] with a non-abelian Čech–de Rham interpretation. We refer to remark 5.1 for a specific comparison between the two notions. In order to make this comparison easier, we have chosen the orientations of our arrows consistently with [5]. This accounts for example for the strange choice of orientation of the arrow B_i in diagram (76), or for the change of sign (91) for the arrow γ_{ij}.

A final purpose of this text is to explain how the diagrammatic proofs of some of the local results of [5] can be replaced by more classical computations involving Lie (G)-valued differential forms. For this reason, we have given two separate computations for certain equations, one diagrammatic and the other classical. We do not assert that one of the two methods of proof is always preferable, though one might contend that diagrams provide a better understanding of the situation than the corresponding manipulation of differential forms. As the level of categorification increases, so will the dimension of the diagrams to be considered, and it may not be realistic to expect to tread along the diagrammatic path much beyond the hypercube proof [5] (4.1.33) of the higher Bianchi equation (5). The generality and algebraicity of the formalism of differential forms must then come into its own. In addition, it is our hope that the present approach, which extends to the gerbe context the traditional methods of differential geometry, will provide an accessible point of entry into this topic. A number of other authors have recently described certain aspects of the differential geometry of gerbes in terms of differential forms, particularly [1], [12], and [14], [2].

I wish to thank Bernard Julia and Camille Laurent-Gengoux for enlightening discussions on related topics. The impetus for the present work was provided by my collaboration with Wiliam Messing on our joint papers [4] and [5]. It is a pleasure to thank him here for our instructive and wide-ranging discussions over all these years.

2 Group-valued differential forms

2.1

Let X be an S-scheme. We assume from now on for simplicity that the primes 2 and 3 are invertible in the ring of functions of S (for example $S = \mathrm{Spec}(k)$ where k is a field of characteristic $\neq 2, 3$). A relative differential n-form on an S-scheme X, with values in a sheaf of $\mathcal{O}_S$-Lie algebras $\mathfrak{g}$, is defined as a global section of the sheaf $\mathfrak{g} \otimes_{\mathcal{O}_S} \Omega^n_{X/S}$ on X. When X/S is smooth,

$$\mathfrak{g} \otimes_{\mathcal{O}_S} \Omega^n_{X/S} \simeq \mathrm{Hom}_{\mathcal{O}_X}(T^n_{X/S}, \mathfrak{g}_X) \tag{6}$$

where $\mathfrak{g}_X := \mathfrak{g} \otimes_{\mathcal{O}_S} \mathcal{O}_X$ and $T^n_{X/S}$ is the n-th exterior power $\wedge^n T_{X/S}$ of the relative tangent sheaf $T_{X/S}$, i.e., the sheaf of relative n-vector fields on X. Such an n-form is nothing else than an $\mathcal{O}_X$-linear map

$$T^n_{X/S} \longrightarrow \mathfrak{g}_X \,. \tag{7}$$

In view of this definition, such a map is classically called a $\mathfrak{g}$-valued differential form. A more geometric description of such forms is given in [4], following the ideas of A. Kock in the context of synthetic differential geometry [10], [11]. It is based on the consideration, for any positive integer n, of the scheme $\Delta^n_{X/S}$ of relative infinitesimal n-simplexes on X. For any S-scheme T, a T-valued point of $\Delta^n_{X/S}$ consists of an $(n+1)$-tuple of T-valued points $(x_0, \ldots, x_n)$ of X which are pairwise close to first order in an appropriate sense [4] (1.4.9). We view Δ^n_X as an X-scheme via the projection p_0

of such points to x_0. As n varies, the schemes $\Delta^n_{X/S}$ determine a simplicial X-scheme $\Delta^*_{X/S}$, whose face and degeneracy operations are induced by the usual projection and injection morphisms $X^n \longrightarrow X^{n\pm 1}$.

Let G be a flat S-group scheme, with $\mathcal{O}_S$-Lie algebra $\mathfrak{g}$. A relative $\mathfrak{g}$-valued n-form (7) on X/S may then be identified by [4] Proposition 2.5 with a morphism of S-schemes

$$\Delta^n_{X/S} \xrightarrow{f} G \tag{8}$$

whose restriction to the degenerate subsimplex $s\Delta^n_{X/S}$ of $\Delta^n_{X/S}$ factors through the unit section of G. When differential forms are expressed in this combinatorial language, they deserve to be called G-valued differential forms, even though they actually coincide with the traditional $\mathfrak{g}$-valued differential forms (6), (7). In the combinatorial context, our notation will be multiplicative, and additive when we pass to the traditional language of differential forms.

We will now discuss some of the features of these $\mathfrak{g}$-valued forms, and refer to [4] for further discussion. First of all, let us recall that the action of the symmetric group S_{n+1} on a combinatorial differential n-form $\omega(x_0, \ldots, x_n)$ by permutation of the variables is given by

$$\omega(x_{\sigma(0)}, \ldots, x_{\sigma(n)}) = \omega(x_0, \ldots, x_n)^{\epsilon(\sigma)}$$

where $\epsilon(\sigma)$ is the signature of σ. Also, the commutator pairing

$$[g,\, h] := g\, h\, g^{-1} h^{-1}$$

on the group G determines a bracket pairing on $\mathfrak{g}$-valued forms of degree ≥ 1, defined combinatorially by the rule

$$(\mathfrak{g} \otimes_{\mathcal{O}_S} \Omega^m_{X/S}) \times (\mathfrak{g} \otimes_{\mathcal{O}_S} \Omega^n_{X/S}) \quad \rightarrow \quad (\mathfrak{g} \otimes_{\mathcal{O}_S} \Omega^{m+n}_{X/S}) \tag{9}$$

which sends $(\omega,\, \omega')$ to $[\omega,\, \omega']$, where

$$[\omega,\, \omega'](x_0,\, \ldots, x_{m+n}) := [\omega(x_0,\, \ldots\, , x_m),\, \omega'(x_m,\, \ldots, x_{m+n})]\, .$$

This pairing is defined in classical terms, by

$$[\omega,\, \omega'] := [Y,\, Y'] \otimes (\eta \wedge \eta')$$

for any pair of forms $\omega := Y \otimes \eta$ and $\omega' := Y' \otimes \eta'$ in $\mathfrak{g} \otimes_{\mathcal{O}_S} \Omega^*_{X/S}$. It endows $\mathfrak{g} \otimes_{\mathcal{O}_S} \Omega^*_{X/S}$ with the structure of a graded $\mathcal{O}_S$-Lie algebra. In particular, the bracket satisfies the graded commutativity rule

$$[f,\, g] = (-1)^{|f||g|+1}[g,\, f]\, , \tag{10}$$

where $|f|$ is the degree of the form f, so that

$$[f, f] = 0$$

whenever $|f|$ is even. The graded Jacobi identity is expressed (in additive notation) as

$$(-1)^{|f||h|}[f,\,[g,\,h]] + (-1)^{|f||g|}[g, [h,\,f]] + (-1)^{|g||h|}[h,\,[f,\,g]] = 0\,.$$

In particular,

$$[f,\,[f,\,f]] = 0 \tag{11}$$

and, when $|f| = |g| = 1$,

$$[f,\,\tfrac{1}{2}\,[g,\,g]] = [[f,\,g]\,,g]\,.$$

Let $\mathrm{Aut}(G)$ be the sheaf of local automorphisms of G, whose group of sections above an S-scheme T is the group $\mathrm{Aut}_T(G_T)$ of automorphisms of the T-group $G_T := G \times_S T$. The definition (8) of a combinatorial n-form still makes sense when G is replaced by a sheaf of groups F on S, and the traditional description of such combinatorial n-forms as n-forms with values in the Lie algebra of F remains valid by [4] Proposition 2.3 when $F = \mathrm{Aut}(G)$. The evaluation map

$$\begin{array}{ccc} \mathrm{Aut}(G) \times G & \longrightarrow & G \\ (u,\,g) & \mapsto & u(g) \end{array}$$

induces for all pairs of positive integers a bilinear pairing

$$(\mathrm{Lie}\,(\mathrm{Aut}(G)) \otimes_{\mathcal{O}_S} \Omega^m) \times (\mathfrak{g} \otimes_{\mathcal{O}_S} \Omega^n_{X/S}) \qquad \rightarrow \quad (\mathfrak{g} \otimes_{\mathcal{O}_S} \Omega^{m+n}_{X/S}) \tag{12}$$

which sends $(u,\,g)$ to $[u,\,g]$, where

$$[u,\,g](x_0,\ldots,x_{m+n}) := u(x_0,\ldots,x_m)(g(x_m,\ldots,x_{m+n}))\;g(x_m,\ldots,x_{m+n})^{-1}\,. \tag{13}$$

This pairing is compatible with the pairings (9) associated to the S-groups G and $\mathrm{Aut}(G)$ in the following sense. For any pair of $\mathfrak{g}$-valued forms g, g', and an $\mathrm{Aut}(G)$-valued form u,

$$[i(g),\,g'\,] = [g,\,g'\,] \quad \text{and} \quad i([u,\,g]) = [u,\,i(g)] \tag{14}$$

where $i : G \longrightarrow \mathrm{Aut}(G)$ is the inner conjugation map $i(\gamma)(g) := \gamma\,g\,\gamma^{-1}$. More generally, an isomorphism $r : G \longrightarrow G'$ induces a morphism r from G-valued combinatorial n-forms to G'-valued combinatorial n-forms, compatible with the Lie bracket operation (9), and which corresponds in classical terms to the morphism $\mathrm{Lie}(r) \otimes_{osc} 1 : \mathfrak{g} \otimes_{\mathcal{O}_S} \Omega^n_{X/S} \longrightarrow \mathfrak{g}' \otimes \Omega^n_{X/S}$. The functoriality of the bracket (12) is expressed by the formula

$$r[u,\,g] = [\,^r u,\,r(g)] \tag{15}$$

where $^r u := r\,u\,r^{-1}$.

When u is an $\mathrm{Aut}(G)$-valued form of degree $m \geq 1$ and g is a G-valued function, the definition of a pairing

$$\begin{array}{ccc} (\mathrm{Lie}\ \mathrm{Aut}(G) \otimes_{\mathcal{O}_S} \Omega^m_{X/S}) \times G & \longrightarrow & \mathfrak{g} \otimes_{\mathcal{O}_S} \Omega^m_{X/S} \\ (u,\, g) & \mapsto & [u,\, g] \end{array}$$

is still given by the formula (13), but now with $n = 0$. This pairing is no longer linear in g, but instead satisfies the equation

$$[u,\, g\, g'] = [u,\, g] + {}^g[u,\, g']$$

where for any G-valued form ω and any G-valued function g the adjoint left action ${}^g\omega$ of a function g on a form ω is defined combinatorially by

$$({}^g\omega)(x_0, \ldots, x_n) := g(x_0)\; \omega(x_0, \ldots, x_n)\; g(x_0)^{-1}$$

(and this expression is in fact equal to $g(x_i)\; \omega(x_0, \ldots, x_n)\; g(x_i)^{-1}$ for any $0 \leq i \leq n$). In classical notation this corresponds, for $\omega = Y \otimes \eta\ \in \mathfrak{g} \otimes \Omega^n_{X/S}$, to the formula

$${}^g(Y \otimes \eta) = {}^gY \otimes \eta$$

for the adjoint left action of g on Y. The adjoint right action ω^γ is defined by

$$\omega^g := {}^{(g^{-1})}\omega$$

so that

$$\omega^g(x_0, \ldots, x_n) = g(x_0)^{-1}\, \omega(x_0, \ldots, x_n)\; g(x_0)\,.$$

Similarly, when g is a G-valued and u an $\mathrm{Aut}(G)$-valued form, a pairing $[g,\, u]$ is defined by the combinatorial formula

$$[g,\, u](x_0,\, \ldots, x_{m+n}) := g(x_0, \ldots, x_m)\,(u(x_m, \ldots, x_{m+n})(g(x_0, \ldots, x_m)^{-1}))\,. \tag{16}$$

The pairing (16) satisfies the analogue

$$[g,\, u] = (-1)^{|g||u|+1}[u,\, g]$$

of the graded commutativity rule (10), so that its properties may be deduced from those of the pairing $[u,\, g]$. In particular,

$$[g^{-1},\, u] = -[u,\, g^{-1}] = [u, g]^g\,.$$

We refer to appendix A of [5] for additional properties of these pairings.

2.2

The de Rham differential map

$$\mathfrak{g} \otimes_{\mathcal{O}_S} \Omega^n_{X/S} \xrightarrow{d^n_{X/S}} \mathfrak{g} \otimes_{\mathcal{O}_S} \Omega^{n+1}_{X/S} \tag{17}$$

is defined combinatorially for $n \geq 2$, in Alexander–Spanier fashion, by

$$\mathrm{d}^n_{X/S}\omega(x_0,\, \ldots, x_{n+1}) := \prod_{i=0}^{n+1} \omega(x_0, \ldots, \widehat{x_i}, \ldots, \omega_{n+1})^{(-1)^i} . \tag{18}$$

This definition agrees for $n > 1$ with the classical definition of the G-valued de Rham differential:

$$\mathrm{d}^n_{X/S}\, \omega := \mathrm{d}_{X/S}\ \omega \tag{19}$$

where for $\omega = Y \otimes \eta$ in $\mathfrak{g} \otimes \Omega^n_{X/S}$,

$$\mathrm{d}_{X/S}\, \omega := Y \otimes \mathrm{d}\eta\,. \tag{20}$$

In particular, d^n is an $\mathcal{O}_S$-linear map whenever $n \geq 2$, and it follows from (20) that the composite $d^{n+1}\, d^n$ is trivial. This also follows from the combinatorial definition of d^n, since for $n \geq 2$ the factors in the expression (18) for $d^n\omega$ commute with each other.

For any section g of G, we set

$$\mathrm{d}^0_{X/S}(g)(x_0, x_1) := g(x_0)^{-1} g(x_1)\ . \tag{21}$$

The map

$$\begin{array}{ccc} G_X & \xrightarrow{\mathrm{d}^0_{X/S}} & \mathfrak{g} \otimes_{\mathcal{O}_S} \Omega^1_{X/S} \\ g & \mapsto & g^{-1}\mathrm{d}g \end{array} \tag{22}$$

is a crossed homomorphism, for the adjoint left action of G on $\mathfrak{g}$. Observe that the expression $g^{-1}\mathrm{d}g$ is consistent with the combinatorial definition (21) of $\mathrm{d}^0_{X/S}(g)$. While this traditional expression of $\mathrm{d}^0_{X/S}(g)$ as a product of the two terms g^{-1} and $\mathrm{d}g$ does make sense whenever G is a subgroup scheme of the linear group $GL_{n,S}$, such a decomposition is purely conventional for a general S-group scheme G. A companion to $\mathrm{d}^0_{X/S}$ is the differential $\widetilde{\mathrm{d}}^0 : G \longrightarrow \mathfrak{g} \otimes_{\mathcal{O}_S} \Omega^1_{X/S}$, defined by

$$\widetilde{\mathrm{d}}^0_{X/S}(g)(x_0,\, x_1) := g(x_1) g(x_0)^{-1}\ .$$

The traditional notation for this expression is $dg\, g^{-1}$. This notation is consistent with such formulas (in additive notation) as

$${}^g(g^{-1}dg) = \mathrm{d}g\, g^{-1} \ \text{ and } \ -(g^{-1}\mathrm{d}g) = \mathrm{d}g^{-1}\, g\ .$$

The differential $\mathrm{d}^1_{X/S}$ is defined combinatorially by

$$(\mathrm{d}^1_{X/S}\, \omega)(x, y, z) := \omega(x, y)\, \omega(y, z)\, \omega(z, x). \tag{23}$$

In classical terms, it follows (see [4] Theorem 3.3) that

$$\mathrm{d}^1_{X/S}\, \omega := \mathrm{d}\omega + \frac{1}{2}[\omega,\, \omega]. \tag{24}$$

We will henceforth denote $\mathrm{d}^n_{X/S}$ simply by d^n for all n.

The quadratic term $\frac{1}{2}[\omega, \omega]$ implies that $\mathrm{d}^1_{X/S}$ is not a linear map; in fact, it follows from (24), or the elementary combinatorial calculation of [4] Lemma 3.2, that

$$\mathrm{d}^1(\omega+\omega') = \mathrm{d}^1\omega + \mathrm{d}^1\omega' + [\omega, \omega'] \,.$$

In particular,

$$\mathrm{d}^1(-\omega) = -\mathrm{d}^1(\omega) + [\omega, \omega] \,.$$

It is immediate, from the combinatorial point of view, that

$$\mathrm{d}^1\mathrm{d}^0(g) = \mathrm{d}^1(g^{-1}\,\mathrm{d}g) = 0 \tag{25}$$

for all g in G. The differential d^1 has a companion, which we will denote by $\widetilde{\mathrm{d}}^1$, defined by

$$\widetilde{\mathrm{d}}^1(\omega)(x,\, y,\, z) := \omega(z,\, x)\,\omega(y,\, z)\,\omega(x,\, y)\,.$$

A combinatorial computation implies that

$$\begin{aligned} \widetilde{\mathrm{d}}^1\omega \quad &= \mathrm{d}^1\omega - [\omega, \omega] \\ &= \ \mathrm{d}\omega \ - \tfrac{1}{2}[\omega, \omega]\,, \end{aligned}$$

and the analogue

$$\widetilde{\mathrm{d}}^1(\widetilde{\mathrm{d}}^0(g)) = \widetilde{\mathrm{d}}^1(dg\,g^{-1}) = 0$$

of (25) is satisfied. Finally, it follows from (19) that the d^n satisfy

$$\mathrm{d}^{i+j}[\omega,\, \omega'] = [\mathrm{d}^i\omega,\, \omega'] + (-1)^i[\omega,\, \mathrm{d}^j\omega']$$

whenever $i, j \geq 2$, and the corresponding formula for the pairing $[u,\, g]$ (13) is also valid.

2.3

We now choose, for any S-scheme X and any S-group scheme G, an $\mathrm{Aut}(G)$-valued 1-form m on X. We extend the definition of the de Rham differentials (22), (23) and (17) to the twisted differentials

$$\mathrm{d}^n_{X/S,\,m} : \mathfrak{g} \otimes_{\mathcal{O}_S} \Omega^n_{X/S} \longrightarrow \mathfrak{g} \otimes_{\mathcal{O}_S} \Omega^{n+1}_{X/S} \tag{26}$$

(or simply d^n_m) defined combinatorially by the following formulas:

$$\begin{aligned} \mathrm{d}^1_m\,\omega(x_0, x_1) &:= \omega(x_0, x_1)\ m(x_0, x_1)(\omega(x_1,\, x_2))\, m(x_0, x_1)\, m(x_1, x_2)(\omega(x_2, x_0)) \\ &= \omega(x_0, x_1)\ m(x_0, x_1)(\omega(x_1,\, x_2))\,\omega(x_0,\, x_2)^{-1} \end{aligned}$$

$$\begin{aligned} &\mathrm{d}^n_m\,\omega(x_0, \ldots, x_{n+1}) \\ &\qquad := m(x_0, x_1)(\omega(x_1, \ldots x_{n+1}))\ \prod_{i=1}^{n+1} \omega(x_0, \ldots, \widehat{x_i}, \ldots, x_{n+1})^{(-1)^i} \end{aligned}$$

when $n > 1$. When the $\mathrm{Aut}(G)$-valued form m is the image $i(\eta)$ under inner conjugation of a G-valued form η, the expression $\mathrm{d}^n_{i(\eta)}\omega$ will simply be denoted $\mathrm{d}^n_\eta\,\omega$. The corresponding degree zero map $\mathrm{d}^0_m : G \longrightarrow \mathfrak{g}\otimes_{\mathcal{O}_S}\Omega^1_{X/S}$ is defined by

$$\mathrm{d}^0_m(g)(x_0, x_1) := g(x_0)^{-1}\, m(x_0,\, x_1)(g(x_1))$$

(and $\mathrm{d}^0_m(g)$ will also be denoted $g^{-1}\mathrm{d}_m(g)$, consistently with (21)).

It follows from elementary combinatorial computations that the differentials d^n_m can be defined in classical terms by

$$\mathrm{d}^n_m\omega = \mathrm{d}^n\omega + [m,\, \omega] \tag{27}$$

for all n, so that for any $\mathfrak{g}$-valued 1-form η,

$$\mathrm{d}^n_{m+i_\eta}(\omega) = \mathrm{d}^n_m(\omega) + [\eta,\, \omega]\,. \tag{28}$$

In particular,

$$\mathrm{d}^1_m(\omega) = \mathrm{d}^1\omega + [m,\, \omega] = \mathrm{d}\omega + \frac{1}{2}[\omega,\, \omega] + [m,\, \omega]\,.$$

While the map d^n_m is linear for $n \geq 2$,

$$\mathrm{d}^1_m(\omega + \omega') = \mathrm{d}^1_m\omega + \mathrm{d}^1_m\omega' + [\omega,\, \omega'] \tag{29}$$

so that

$$\mathrm{d}^1_m(-\omega) = -\mathrm{d}^1_m(\omega) - [\omega,\, \omega]. \tag{30}$$

Finally, for any section g of Γ,

$$g^{-1}\mathrm{d}_m g = g^{-1}\mathrm{d}g + [m,\, g]\,.$$

The composite morphism $\mathrm{d}^{n+1}_m\mathrm{d}^n_m$ is in general nontrivial, and the previous classical definitions of d^n_m imply that

$$\mathrm{d}^{n+1}_m\mathrm{d}^n_m\,\omega = [\mathrm{d}^1 m,\, \omega] \tag{31}$$

whenever $n \geq 2$. For $n = 0$, the corresponding formulas are

$$\mathrm{d}^1_m\;\mathrm{d}^0_m g = [g^{-1},\, \mathrm{d}^1 m] \;\text{ and }\; \widetilde{\mathrm{d}}^1_m\;\widetilde{\mathrm{d}}^0_m g = [\mathrm{d}^1 m,\, g] \tag{32}$$

so that, for $n \neq 1$, we recover the well-known assertion that the vanishing of $\mathrm{d}^1 m = 0$ implies that $\mathrm{d}^{n+1}\mathrm{d}^n = 0$. One verifies that for any 1-form ω

$$\mathrm{d}^2_m\,\mathrm{d}^1_m(\omega) = [\mathrm{d}^1 m, \omega] + [\mathrm{d}^1_m\omega,\, \omega] \tag{33}$$

$$= [\mathrm{d}^1 m, \omega] + [\mathrm{d}^1\omega,\, \omega] + [[m,\, \omega],\, \omega]\,. \tag{34}$$

This reduces to the equation

$$\mathrm{d}^2_m\,\mathrm{d}^1_m(\omega) = [\mathrm{d}^1 m,\, \omega]$$

of type (31) whenever $\mathrm{d}^1_m \omega = 0$. For $m = i(\omega)$, equation (33) is equivalent to the classical Bianchi identity [9] II Theorem 5.4:

$$\mathrm{d}^2_\omega \, \mathrm{d}^1 \omega = 0 \, . \tag{35}$$

We now state the functoriality properties of the differential (27) d^n_m for $n \geq 1$. We define the twisted conjugate ${}^{g\,*}\omega$ of a G-valued 1-form ω by

$$\begin{aligned} {}^{g\,*}\omega := (p_0^* g) \; \omega \; (p_1^* g)^{-1} &= {}^{g}\omega + g \, \mathrm{d}g^{-1} \\ &= \; \omega + [g, \, \omega] + g \, \mathrm{d}g^{-1} \, . \end{aligned} \tag{36}$$

It follows from the combinatorial definition (23) of d^1 that

$$ {}^{g}(\mathrm{d}^1 \omega) = \mathrm{d}^1 ({}^{g\,*}\omega) \, . \tag{37}$$

More generally, for any G-valued form ω of degree $n \geq 1$, and any section u of $\mathrm{Aut}(G)$ on X,

$$\begin{aligned} u(\mathrm{d}^n_m(\omega)) &= \mathrm{d}^n_{({}^{u\,*}m)} \, u(\omega) && (38) \\ &= \mathrm{d}^n_{({}^{u}m)} \, u(\omega) + [u \, \mathrm{d}u^{-1}, \, u(\omega)] \\ &= \mathrm{d}^n_m(u(\omega)) + [[u, \, m], \, u(\omega)] + [u \, \mathrm{d}u^{-1}, \, u(\omega)] \, . && (39) \end{aligned}$$

3 Gerbes and their connective structures

3.1

Let $\mathcal{P}$ be a gerbe[3] on an S-scheme X. For simplicity, in discussing gerbes we will make two additional assumptions:

- $\mathcal{P}$ is a G-gerbe, for a given S-group scheme G.
- $\mathcal{P}$ is connected.

The first assumption gives us, for any object x in the fiber category $\mathcal{P}_U$ above an open set $U \subset X$, an isomorphism of sheaves on U

$$G_{|U} \xrightarrow{\sim} \mathrm{Aut}_{\mathcal{P}_U}(x) \, . \tag{40}$$

The second assumption asserts that for any pair of objects $x, y \in \mathrm{ob}(\mathcal{P}_U)$ there exists an arrow $x \longrightarrow y$ in the category $\mathcal{P}_U$. This ensures that the gerbe is described by an element in the degree 2 Čech cohomology of X rather than by degree 2 cohomology with respect to a hypercover of X.

Let us choose a family of local objects $x_i \in \mathcal{P}_{U_i}$, for some open cover $\mathcal{U} = \coprod_i U_i$ of X, and a family of arrows

$$x_j \xrightarrow{\phi_{ij}} x_i \tag{41}$$

[3] We refer to [3] and [6] for the definition of a gerbe, and for additional details regarding the associated cocycle and coboundary equations (46), (53).

in $\mathcal{P}_{U_{ij}}$. Identifying elements of both $\mathrm{Aut}_{\mathcal{P}}(x_i)$ and $\mathrm{Aut}_{\mathcal{P}}(x_j)$ with the corresponding sections of G above U_i and U_j, these arrows determine a family of section $\lambda_{ij} \in \Gamma(U_{ij}, \mathrm{Aut}(G))$, defined by the commutativity of the diagrams

$$\begin{tikzcd} x_j \arrow[r, "\gamma"] \arrow[d, "\phi_{ij}"'] & x_j \arrow[d, "\phi_{ij}"] \\ x_i \arrow[r, "\lambda_{ij}(\gamma)"'] & x_i \end{tikzcd} \tag{42}$$

for every $\gamma \in G_{\,|U_{ij}}$. In addition, the arrows ϕ_{ij} determine a family of elements $g_{ijk} \in G_{|U_{ijk}}$ for all (i,j,k) by the commutativity of the diagrams

$$\begin{tikzcd} x_k \arrow[r, "\phi_{jk}"] \arrow[d, "\phi_{ik}"'] & x_j \arrow[d, "\phi_{ij}"] \\ x_i \arrow[r, "g_{ijk}"'] & x_i \end{tikzcd} \tag{43}$$

above U_{ijk}. By conjugation in the sense made clear by diagram (42), it follows that the λ_{ij} satisfy the cocycle condition

$$\lambda_{ij}\,\lambda_{jk} = i(g_{ijk})\,\lambda_{ik}\,. \tag{44}$$

By [6] Lemma 5.1, the G-valued cochains g_{ijk} also satisfy the cocycle condition

$$\lambda_{ij}(g_{jkl})\,g_{ijl} = g_{ijk}\,g_{ikl}\,. \tag{45}$$

These two cocycle equations may be written more compactly as

$$\begin{cases} \delta^1\lambda_{ij} & = & i(g_{ijk}) \\ \delta^2_{\lambda_{ij}}(g_{ijk}) & = & 1\,, \end{cases} \tag{46}$$

where δ^2_λ is the λ-twisted degree 2 Čech differential determined by equation (45). They may be jointly viewed as the $(G \longrightarrow \mathrm{Aut}(G))$-valued Čech 1-cocycle[4] equations associated to the gerbe $\mathcal{P}$, the open cover $\mathcal{U}$ of X, and the trivializing families of objects x_i and arrows ϕ_{ij} in $\mathcal{P}$.

Let us choose a second family of local objects x'_i in $\mathcal{P}_{U_i}$, and of arrows

$$x'_j \xrightarrow{\phi'_{ij}} x'_i \tag{47}$$

[4] We prefer to emphasize the fact that λ_{ij} is a 1-cochain since this is more consistent with a simplicial definition of the associated cohomology, even though it is more customary to view the pair of equations (46) as a 2-cocycle equation, with (44) an auxiliary condition.

above U_{ij}. To these correspond a new cocycle pair $(\lambda'_{ij}, g'_{ijk})$. In order to compare this set of arrows with the previous one, we choose (after a harmless refinement of the given open cover $\mathcal{U}$ of X) a family of arrows

$$x_i \xrightarrow{\chi_i} x'_i \tag{48}$$

in $\mathcal{P}_{U_i}$ for all i. The arrow χ_i induces by conjugation a section r_i in the group of sections $\Gamma(U_i, \mathrm{Aut}(G))$, characterized by the commutativity of the square

$$\begin{tikzcd} x_i \arrow[r, "u"] \arrow[d, "\chi_i"'] & x_i \arrow[d, "\chi_i"] \\ x'_i \arrow[r, "r_i(u)"'] & x'_i \end{tikzcd} \tag{49}$$

for all $u \in G$. The lack of compatibility between these arrows χ_i and the arrows ϕ_{ij}, ϕ'_{ij} (41), (47) is measured by the family of sections $\vartheta_{ij} \in \Gamma(U_{ij}, G)$ determined by the commutativity of the following diagram:

$$\begin{tikzcd} x_j \arrow[r, "\phi_{ij}"] \arrow[dd, "\chi_j"'] & x_i \arrow[d, "\chi_i"] \\ & x'_i \arrow[d, "\vartheta_{ij}"] \\ x'_j \arrow[r, "\phi'_{ij}"'] & x'_i\,. \end{tikzcd} \tag{50}$$

Under the identifications (40), diagram (50) induces by conjugation, in a sense made clear by the definition (49) of the automorphism r_i, a commutative diagram of group schemes above U_{ij}

$$\begin{tikzcd} G \arrow[r, "\lambda_{ij}"] \arrow[dd, "r_j"'] & G \arrow[d, "r_i"] \\ & G \arrow[d, "i(\vartheta_{ij})"] \\ G \arrow[r, "\lambda'_{ij}"'] & G\,, \end{tikzcd}$$

whose commutativity is expressed by the equation

$$\lambda'_{ij} = i(\vartheta_{ij})\, r_i\, \lambda_{ij}\, r_j^{-1} \tag{51}$$

in $\mathrm{Aut}(G)$.

Consider now the diagram

$$
\begin{tikzcd}
 & x_k \arrow[rr, "\phi_{ik}"] \arrow[ddl, "\phi_{jk}"'] \arrow[ddddd, "\chi_k"] & & x_i \arrow[d, "\chi_i"] \arrow[ddl, "g_{ijk}"'] \\
 & & & x'_i \arrow[ddl, "r_i(g_{ijk})"] \arrow[dddd, "\vartheta_{ik}"] \\
x_j \arrow[rr, "\phi_{ij}"'] \arrow[dd, "\chi_j"'] & & x_i \arrow[d, "\chi_i"'] & \\
 & & x'_i \arrow[d, "\vartheta_{ij}"] & \\
x'_j \arrow[rr, "\phi'_{ij}"] \arrow[dd, "\vartheta_{jk}"'] & & x'_i \arrow[dd, "\lambda'_{ij}(\vartheta_{jk})"] & \\
 & x'_k \arrow[rr, "\phi'_{ik}"] \arrow[dl, "\phi'_{jk}"'] & & x'_i \arrow[dl, "g'_{ijk}"] \\
x'_j \arrow[rr, "\phi'_{ij}"'] & & x'_i &
\end{tikzcd}
\quad . \tag{52}
$$

Both the top and the bottom squares commute, since these squares are of type (43). So do the back, the left and the top front vertical squares, since all three are of type (50). The same is true of the lower front square, and the upper right vertical square, since these two are respectively of the form (42) and (49). It follows that the remaining lower right square in the diagram is also commutative, since all the arrows in diagram (52) are invertible. The commutativity of this final square is expressed algebraically by the equation

$$g'_{ijk}\,\vartheta_{ik} = \lambda'_{ij}(\vartheta_{jk})\,\vartheta_{ij}\,r_i(g_{ijk})\,.$$

We say that two cocycle pairs $(\lambda_{ij},\, g_{ijk})$ and $(\lambda'_{ij},\, g'_{ijk})$ are cohomologous if we are given a pair (r_i, ϑ_{ij}), with $r_i \in \Gamma(U_i,\, \mathrm{Aut}(G))$ and $\vartheta_{ij} \in \Gamma(U_{ij},\, G)$, satisfying those two equations

$$\begin{cases} \lambda'_{ij} & = \; i(\vartheta_{ij})\, r_i\, \lambda_{ij}\, r_j^{-1} \\ g'_{ijk}\,\vartheta_{ik} & = \lambda'_{ij}(\vartheta_{jk})\,\vartheta_{ij}\, r_i(g_{ijk}) \end{cases} \tag{53}$$

and display this as

$$(\lambda_{ij},\, g_{ijk}) \overset{(r_i, \vartheta_{ij})}{\sim} (\lambda'_{ij},\, g'_{ijk})\,. \tag{54}$$

The equivalence class of the cocycle pair $(\lambda_{ij},\, g_{ijk})$ for this relation is independent of the choices of objects x_i and arrows ϕ_{ij} from which it was constructed. By definition, it determines an element in the first non-abelian Čech cohomology set $\check{H}^1(\mathcal{U},\, G \xrightarrow{i} \mathrm{Aut}(G))$ with coefficients in the crossed module $i : G \longrightarrow \mathrm{Aut}(G)$.

3.2

In [5], the combinatorial description of differential forms is used in order to define the concepts of connections and curvings on a gerbe. For any S-group scheme G, a (relative) connection on a principal G-bundle P above the S-scheme X may be defined as a morphism

$$p_1^*P \xrightarrow{\epsilon} p_0^*P \tag{55}$$

between the two pullbacks of P to $\Delta^1_{X/S}$, whose restriction to the diagonal subscheme

$$\Delta : X \hookrightarrow \Delta^1_{X/S}$$

is the identity morphism 1_P.

This type of definition of a connection, as a vehicle for parallel transport, remains valid for other structures than principal bundles. In particular, for any X-group scheme Γ, a connection on Γ is a morphism of group schemes

$$\mu : p_1^*\Gamma \longrightarrow p_0^*\Gamma \tag{56}$$

above $\Delta^1_{X/S}$ whose restriction to the diagonal subscheme $X \hookrightarrow \Delta^1_{X/S}$ is the identity morphism 1_Γ. When Γ is the pullback to X of an S-group scheme G, the inverse images p_1^*G and p_0^*G of G_X above $\Delta^1_{X/S}$ are canonically isomorphic, so that the connection (56) is then described by a Lie(Aut(G))-valued 1-form m.

A connection μ on a group Γ determines de Rham differentials

$$\mathrm{d}^n_{X/S,\,\mu} : \mathrm{Lie}(\Gamma) \otimes_{\mathcal{O}_S} \Omega^n_{X/S} \longrightarrow \mathrm{Lie}(\Gamma) \otimes_{\mathcal{O}_S} \Omega^{n+1}_{X/S}$$

(or simply d^n_μ) defined combinatorially by the formulas [5] (A.1.9)–(A.1.11) and their higher analogues. When Γ is the pullback of an S-group scheme, d^n_μ is described in classical terms as the deformation (27)

$$\mathrm{d}^n_\mu := \mathrm{d}^n_m$$

of the de Rham differential d^n determined by the associated 1-form m. When the curvature $\mathrm{d}^1 m$ of the connection μ is trivial, the connection is said to be flat (or integrable). In that case, it follows from (31) and (32) that the de Rham differentials satisfy the condition $\mathrm{d}^{n+1}_m\, \mathrm{d}^n_m = 0$ for all $n \neq 1$.

The curvature of a connection ϵ (55) on a principal bundle P is the unique arrow

$$\kappa_\epsilon : p_0^*P \longrightarrow p_0^*P$$

such that the following diagram above $\Delta^2_{X/S}$ commutes, with ϵ_{ij} the pullbacks of ϵ under the corresponding projections $p_{ij} : \Delta^2_{X/S} \longrightarrow \Delta^1_{X/S}$:

$$\begin{tikzcd}
p_2^*P \arrow[r, "\epsilon_{12}"] \arrow[d, "\epsilon_{02}"'] & p_1^*P \arrow[d, "\epsilon_{01}"] \\
p_0^*P \arrow[r, "\kappa_\epsilon"'] & p_0^*P
\end{tikzcd}$$

By construction, κ_ϵ is a relative 2-form on X with values in the gauge group $P^{\mathrm{ad}} := \mathrm{Isom}_G(P, P)$ of P.

The connection ϵ on P induces a connection μ_ϵ on the group P^{ad}, determined by the commutativity of the squares

$$\begin{tikzcd} p_1^*P \arrow[r, "u"] \arrow[d, "\epsilon"'] & p_1^*P \arrow[d, "\epsilon"] \\ p_0^*P \arrow[r, "\mu_\epsilon(u)"'] & p_0^*P \end{tikzcd} \tag{57}$$

for all sections u of $p_1^*(P^{\mathrm{ad}})$. By [11], [5] Proposition 1.7, the curvature 2-form κ_ϵ satisfies the Bianchi identity

$$\mathrm{d}^2_{\mu_\epsilon}(\kappa_\epsilon) = 0. \tag{58}$$

For a given family of local sections of P, with associated G-valued 1-cocycles g_{ij}, the connection (55) is described by a family of G-valued 1-forms $\omega_i \in \mathfrak{g} \otimes \Omega^1_{U_i/S}$, satisfying the gluing condition

$$\omega_j = \omega_i^{*\, g_{ij}} = \omega_i^{g_{ij}} + g_{ij}^{-1}\mathrm{d}g_{ij} \tag{59}$$

above U_{ij}, for the action of G on $\mathfrak{g} \otimes_{\mathcal{O}_S} \Omega^1_{U_i/S}$ induced by the adjoint right action of G on $\mathfrak{g}$. A 1-form satisfying this equation is classically known as a connection form. The induced curvature κ is locally described by the family of 2-forms

$$\kappa_i := \mathrm{d}^1\omega_i = \mathrm{d}\omega_i + \frac{1}{2}[\omega_i, \omega_i],$$

and these satisfy the simpler Čech (or gluing) condition

$$\kappa_j = \kappa_i^{g_{ij}} .$$

Equation (58) is reflected at the local level in the equation

$$\mathrm{d}^2_{\omega_i}\kappa_i = 0\,,$$

which is simply the classical Bianchi identity (35) for the 1-form ω_i.

3.3

The notion of a connective structure on a G-gerbe $\mathcal{P}$ is a categorification of the notion of a connection on a principal bundle, as we will now recall, following [5] §4. To $\mathcal{P}$ is associated its gauge stack $\mathcal{P}^{\mathrm{ad}}$. By definition this is the monoidal stack $\mathcal{E}q_X(\mathcal{P}, \mathcal{P})$ of self-equivalences of the stack $\mathcal{P}$, the monoidal structure being defined by the composition of equivalences. A connection on a $\mathcal{P}$ is an equivalence between stacks

$$p_1^*\mathcal{P} \xrightarrow{\ \epsilon\ } p_0^*\mathcal{P} \tag{60}$$

above $\Delta^1_{X/S}$, together with a natural isomorphism between the restriction $\Delta^*\epsilon$ of ϵ to the diagonal subscheme X of $\Delta^1_{X/S}$ and the identity morphism $1_{\mathcal{P}}$. Such a connection ϵ induces as in (57) a connection μ on the gauge stack $\mathcal{P}^{\mathrm{ad}}$.

A curving of $(\mathcal{P}, \epsilon)$ is a natural isomorphism K

$$\begin{tikzcd}
p_2^*\mathcal{P} \arrow[r, "\epsilon_{12}"] \arrow[d, "\epsilon_{02}"'] & p_1^*\mathcal{P} \arrow[d, "\epsilon_{01}"] \\
p_0^*\mathcal{P} \arrow[r, "\kappa"'] \arrow[ur, phantom, "K \Nearrow"] & p_0^*\mathcal{P}\,,
\end{tikzcd} \tag{61}$$

for some morphism

$$\kappa : p_0^*\mathcal{P} \longrightarrow p_0^*\mathcal{P}$$

above $\Delta^2_{X/S}$. It is determined by the choice of some explicit quasi-inverse of the connection ϵ. The arrow κ which arises as part of the definition of K is called the *fake curvature* associated to the connective structure $(\epsilon,\, K)$. It is a global object in the pullback to $\Delta^2_{X/S}$ of the gauge stack $\mathcal{P}^{\mathrm{ad}}$.

The connective structure $(\epsilon,\, K)$ determines a 2-arrow

$$\begin{tikzcd}
p_0^*\mathcal{P} \arrow[r, "\kappa_{013}"] \arrow[d, "\kappa_{023}"'] & p_0^*\mathcal{P} \arrow[d, "\mu_{01}(\kappa_{123})"] \\
p_0^*\mathcal{P} \arrow[r, "\kappa_{012}"'] \arrow[ur, phantom, "\Omega \Swarrow"] & p_0^*\mathcal{P}.
\end{tikzcd}$$

This is the unique 2-arrow which may be inserted in the diagram

$$\begin{tikzcd}
 & p_3^*\mathcal{P} \arrow[rr, "\epsilon_{13}"] \arrow[dl, "\epsilon_{03}"'] \arrow[dd, "\epsilon_{23}"] \arrow[dr, phantom, "\Swarrow K_{123}"] & & p_1^*\mathcal{P} \arrow[dl, "\epsilon_{01}"'] \arrow[dd, "\kappa_{123}"] \arrow[d, phantom, "\Searrow M_{01}(\kappa_{123})"] \\
p_0^*\mathcal{P} \arrow[rr, "\kappa_{013}"] \arrow[dd, "\kappa_{023}"'] \arrow[ur, phantom, "K_{013} \Rightarrow"] \arrow[dr, phantom, "\Searrow K_{023}"] & & p_0^*\mathcal{P} \arrow[dd, "\mu_{01}(\kappa_{123})"] & \\
 & p_2^*\mathcal{P} \arrow[rr, "\epsilon_{12}"] \arrow[dl, "\epsilon_{02}"'] \arrow[d, phantom, "\Nearrow K_{012}"] & & p_1^*\mathcal{P} \arrow[dl, "\epsilon_{01}"] \arrow[dl, phantom, "\Omega \Swarrow"'] \\
p_0^*\mathcal{P} \arrow[rr, "\kappa_{012}"'] & & p_0^*\mathcal{P} & .
\end{tikzcd} \tag{62}$$

so that the two composite 2-arrows

$$\begin{tikzcd} p_3^*\mathcal{P} \arrow[r, bend left, "\mu_{01}(\kappa_{123})\,\kappa_{013}\,\epsilon_{03}"] \arrow[r, bend right, "\epsilon_{01}\,\epsilon_{12}\,\epsilon_{23}"'] \arrow[r, phantom, "\Downarrow"] & p_0^*\mathcal{P} \end{tikzcd}$$

which may be constructed by composition of 2-arrows in (62) coincide.
This 2-arrow Ω may also be viewed as a 1-arrow above $\Delta^3_{X/S}$ in the gauge group $\mathcal{P}^{\mathrm{ad}}$, or even as an arrow in the stack $\mathrm{Lie}(\mathcal{P}^{\mathrm{ad}}) \otimes_{\mathcal{O}_S} \Omega^3_{X/S}$ of relative $\mathrm{Lie}(\mathcal{P}^{\mathrm{ad}})$-valued 3-forms on X. Returning to the combinatorial definition [5] (A.1.10) of the de Rham differential, we may finally view Ω, by horizontal composition with appropriate 1-arrows, as a 1-arrow in $\mathcal{P}^{\mathrm{ad}}$ whose source object is the identity arrow $I_{\mathcal{P}^{\mathrm{ad}}}$:

$$I \xrightarrow{\Omega} \mathrm{d}^2_\mu(\kappa^{-1}). \tag{63}$$

Denoting the twisted differential d^2_μ by the expression $\mathrm{d} + [\mu,\ ]$ to which it reduces when appropriate trivializations have been chosen, the 3-curvature arrow Ω (63) is described by the equation (4). By [5] Theorem 4.4 it satisfies another relation, described by the cubical pasting diagram [5] (4.1.24), and which may be expressed by the higher Bianchi identity[5] (5). The pair of equations (4) and (5) may now be thought of as a categorified version, satisfied by the pair of $\mathcal{P}^{\mathrm{ad}}$-valued forms (κ, Ω), of the classical Bianchi identity (58), and can be written in symbolic form as

$$\mathrm{d}^2_{\mu,\mathcal{K}}(\kappa, \Omega) = 0\,,$$

where $\mathrm{d}^n_{\mu,\mathcal{K}}$ is the twisted de Rham differential on $\mathrm{Lie}(\mathcal{P}^{\mathrm{ad}})$-valued n-forms determined by twisting data $(\mu, \mathcal{K})$ associated to the given connective structure on $\mathcal{P}$.

4 Čech–de Rham cocycles

4.1

We observed in Section 3.1 that a gerbe could be expressed in cocyclic terms, once local trivializations were chosen. We will now show that this is also the case for the connection ϵ. We choose, for each $i \in I$, an arrow

$$\gamma_i : \epsilon p_1^* x_i \longrightarrow p_0^* x_i \tag{64}$$

in $p_0^*\mathcal{P}_{U_i}$ such that $\Delta^*\gamma_i = 1_{x_i}$. The arrow γ_i determines by conjugation a connection

$$m_i : p_1^* G_{|U_i} \longrightarrow p_0^* G_{|U_i}$$

[5] See [5] (4.1.28) for a proof of this identity.

on the pullback $G_{|U_i}$ of the group G above the open set $U_i \subset X$. The arrow m_i is described, for any section $g \in \Gamma(\Delta^1_{X/S_{U_i}}, p_1^* G)$, by the commutativity of the diagram

$$\begin{tikzcd} \epsilon p_1^* x_i \arrow[r, "\epsilon(g)"] \arrow[d, "\gamma_i"'] & \epsilon p_1^* x_i \arrow[d, "\gamma_i"] \\ p_0^* x_i \arrow[r, "m_i(g)"'] & p_0^* x_i \end{tikzcd} \; . \tag{65}$$

The pair (ϕ_{ij}, γ_i) determines a family of arrows γ_{ij} in the pullback $G_{\Delta^1_{U_{ij}}}$ of G, defined by the commutativity of the diagram

$$\begin{tikzcd} \epsilon p_1^* x_j \arrow[r, "\gamma_j"] \arrow[dd, "\epsilon p_1^* \phi_{ij}"'] & p_0^* x_j \arrow[d, "p_0^* \phi_{ij}"] \\ & p_0^* x_i \arrow[d, "\gamma_{ij}"] \\ \epsilon p_1^* x_i \arrow[r, "\gamma_i"'] & p_0^* x_i . \end{tikzcd} \tag{66}$$

By conjugation, this determines a commutative diagram

$$\begin{tikzcd} p_1^* G \arrow[r, "m_j"] \arrow[dd, "p_1^* \lambda_{ij}"'] & p_0^* G \arrow[d, "p_0^* \lambda_{ij}"] \\ & p_0^* G \arrow[d, "i(\gamma_{ij})"] \\ p_1^* G \arrow[r, "m_i"'] & p_0^* G \end{tikzcd} \tag{67}$$

so that the equation

$$i(\gamma_{ij})\,(p_0^* \lambda_{ij})\, m_j\, (p_1^* \lambda_{ij})^{-1} = m_i \tag{68}$$

of [5] (6.1.2) is satisfied.

We may restate (68) as

$$i(\gamma_{ij})\; [(p_0^* \lambda_{ij})\, m_j\, (p_0^* \lambda_{ij})^{-1}] = m_i\; [p_1^* \lambda_{ij}\; (p_0^* \lambda_{ij}^{-1})]\,, \tag{69}$$

an equation all of whose factors are $\mathrm{Aut}(G)$-valued 1-forms on U_{ij} and therefore commute with each other. In the notation introduced in (36), equation (69) can be rewritten as

$${}^{\lambda_{ij}\,*}m_j = m_i - i(\gamma_{ij})\,, \tag{70}$$

or more classically as

$${}^{\lambda_{ij}}m_j = m_i - \lambda_{ij}\; \mathrm{d}\lambda_{ij}^{-1} - i(\gamma_{ij})\,. \tag{71}$$

This is the analogue for the $\mathrm{Aut}(G)$-valued forms m_i and λ_{ij} of the classical expression (59) for a connection form, but now categorified by the insertion of an additional summand $-i(\gamma_{ij})$.

Consider now the following diagram in $\mathcal{P}_{\Delta^1_{U_{ijk}}}$:

$$\begin{tikzcd}
 & \epsilon p_1^* x_k \arrow[rr, "\epsilon p_1^*\phi_{ik}"] \arrow[ddl, "\epsilon p_1^*\phi_{jk}"'] \arrow[ddddd, "\gamma_k"] & & \epsilon p_1^* x_i \arrow[d, "\gamma_i"] \arrow[ddl, "\epsilon p_1^* g_{ijk}"'] \\
 & & & p_0^* x_i \arrow[ddl, "m_i(p_1^* g_{ijk})"] \\
\epsilon p_1^* x_j \arrow[rr, "\epsilon p_1^*\phi_{ij}"] \arrow[dd, "\gamma_j"'] & & \epsilon p_1^* x_i \arrow[d, "\gamma_i"'] & \\
 & & p_0^* x_i & \\
p_0^* x_j \arrow[rr, "p_0^*\phi_{ij}"] & & p_0^* x_i \arrow[u, "\gamma_{ij}"'] & \\
 & p_0^* x_k \arrow[rr, "p_0^*\phi_{ik}"] \arrow[dl, "p_0^*\phi_{jk}"'] & & p_0^* x_i \arrow[uuuu, "\gamma_{ik}"'] \arrow[dl, "p_0^* g_{ijk}"] \\
p_0^* x_j \arrow[uu, "\gamma_{jk}"] \arrow[rr, "p_0^*\phi_{ij}"'] & & p_0^* x_i . \arrow[uu, "\lambda_{ij}(\gamma_{jk})"'] &
\end{tikzcd} \tag{72}$$

Of the eight faces of this cube, seven are known to be commutative. It follows that the remaining lower square on the right vertical side is also commutative. This is the square

$$\begin{tikzcd}
p_0^* x_i \arrow[r, "p_0^* g_{ijk}"] \arrow[dd, "\gamma_{ik}"] & p_0^* x_i \arrow[d, "\lambda_{ij}(\gamma_{jk})"] \\
 & p_0^* x_i \arrow[d, "\gamma_{ij}"] \\
p_0^* x_i \arrow[r, "m_i(p_1^* g_{ijk})"'] & p_0^* x_i \,,
\end{tikzcd} \tag{73}$$

whose commutativity corresponds to the equation

$$\gamma_{ij}\ (p_0^*\lambda_{ij}(\gamma_{jk}))\ =\ m_i(p_1^* g_{ijk})\ \gamma_{ik}\ \ (p_0^* g_{ijk})^{-1},$$

in other words to the equation [5] (6.1.7), all of whose factors are G-valued 1-forms on U_{ijk}. We may rewrite this as

$$\gamma_{ij}\ p_0^*\lambda_{ij}(\gamma_{jk}) = (m_i(p_1^* g_{ijk})\, p_0^* g_{ijk}^{-1})\,(p_0^* g_{ijk}\,\gamma_{ik}\,p_0^* g_{ijk}^{-1})$$

so that, taking into account the equation (44), we finally obtain (in additive notation)

$$\gamma_{ij} + \lambda_{ij}(\gamma_{jk}) - \lambda_{ij}\lambda_{jk}(\lambda_{ik}^{-1}(\gamma_{ik})) = dg_{ijk}\, g_{ijk}^{-1} + [m_i,\, g_{ijk}]\,,$$

with bracket defined by (12) an equation which can be written in abbreviated form as

$$\delta^1_{\lambda_{ij}}(\gamma_{ij}) = \mathrm{d}_{m_i} g_{ijk}\ g_{ijk}^{-1}\,. \tag{74}$$

4.2

We now describe in similar terms the curving K and the fake curvature κ of diagram (61). Just as we associated to the connection ϵ (60) a family of arrows γ_i (64), we now choose, for each $i \in I$, an arrow

$$\kappa p_0^* x_i \xrightarrow{\delta_i} p_0^* x_i \tag{75}$$

in the category $\mathcal{P}_{\Delta^2_{U_i}}$, whose restriction to the degenerate subsimplex $s\Delta^2_{U_i}$ of $\Delta^2_{U_i}$ is the identity. To the curving K is associated a family of "B-field" $\mathfrak{g}$-valued 2-forms $B_i \in \mathfrak{g} \otimes \Omega^2_{U_i}$, characterized by the commutativity of the following diagram[6] in which an expression such as γ_i^{12} is the pullback of γ_i by the corresponding projection $p_{12} : \Delta^2_{X/S} \longrightarrow \Delta^1_{X/S}$:

$$\begin{tikzcd}
\epsilon_{01}\epsilon_{12}(p_2^* x_i) \arrow[r, "K(p_2^* x_i)"] \arrow[d, "\epsilon_{01}\gamma_i^{12}"'] & \kappa\epsilon_{02}(p_2^* x_i) \arrow[d, "\kappa\gamma_i^{02}"] \\
\epsilon_{01}(p_1^* x_i) \arrow[d, "\gamma_i^{01}"'] & \kappa p_0^* x_i \arrow[d, "\delta_i"] \\
p_0^* x_i & p_0^* x_i . \arrow[l, "B_i"]
\end{tikzcd} \tag{76}$$

Let us now define a family of G-valued 2-forms ν_i on U_i by the equations

$$\nu_i := \mathrm{d}^1 m_i - i(B_i) \tag{77}$$

in $\mathrm{Lie}\,\mathrm{Aut}(G) \otimes \Omega^2_{U_i}$, in other words by the commutativity of the diagram

$$\begin{tikzcd}
p_2^* G \arrow[r, equal] \arrow[d, "m_i^{12}"'] & p_2^* G \arrow[d, "m_i^{02}"] \\
p_1^* G \arrow[d, "m_i^{01}"'] & p_0^* G \arrow[d, "\nu_i"] \\
p_0^* G_i & p_0^* G\ . \arrow[l, "i(B_i)"]
\end{tikzcd} \tag{78}$$

By comparing diagram (78) with the conjugate of diagram (76), we see that ν_i is simply the conjugate of the arrow δ_i. It can therefore be described by the commutativity of the diagram

[6] The chosen orientation of the arrow B_i is consistent with that in [5].

$$\begin{CD} \kappa p_0^* x_i @>{\kappa(g)}>> \kappa p_0^* x_i \\ @V{\delta_i}VV @VV{\delta_i}V \\ p_0^* x_i @>>{\nu_i(g)}> p_0^* x_i \end{CD} \tag{79}$$

for all $g \in \Gamma(\Delta^2_{U_i/S}, p_0^* G)$, just as the connection m_i was described by diagram (65).

We also define a family of 2-forms δ_{ij} by the commutativity of the diagram

$$\begin{CD} p_0^* x_i @>{\lambda_{ij}(B_j)}>> p_0^* x_i \\ @V{\delta_{ij}}VV @VV{\gamma_{ij}^{01}}V \\ p_0^* x_i @. p_0^* x_i \\ @V{\gamma_{ij}^{02}}VV @VV{m_i^{01}(\gamma_{ij}^{12})}V \\ p_0^* x_i @>>{B_i}> p_0^* x_i \end{CD}\ , \tag{80}$$

i.e., since all terms commute, by the equation

$$\delta_{ij} := \lambda_{ij}(B_j) - B_i - \mathrm{d}^1_{m_i}(-\gamma_{ij})$$

in $\mathrm{Lie}(G) \otimes \Omega^2_{U_i/S}$. In Čech–de Rham notation, this is

$$\delta_{ij} := \delta^0_{\lambda_{ij}}(B_i) - \mathrm{d}^1_{m_i}(-\gamma_{ij})\ , \tag{81}$$

and in classical notation

$$\delta_{ij} := \lambda_{ij}(B_j) - B_i + \mathrm{d}\gamma_{ij} - \frac{1}{2}[\gamma_{ij}, \gamma_{ij}] + [m_i, \gamma_{ij}]\ .$$

Here is another characterization of δ_{ij}:

Lemma 4.1. *For every pair $(i, j) \in I$, the analogue*

$$\begin{CD} \kappa p_0^* x_j @>{\delta_j}>> p_0^* x_j \\ @. @VV{p_0^*\phi_{ij}}V \\ @V{\kappa p_0^*\phi_{ij}}VV p_0^* x_i \\ @. @VV{\delta_{ij}}V \\ \kappa p_0^* x_i @>>{\delta_i}> p_0^* x_i \end{CD} \tag{82}$$

of diagram (66) is commutative.

Proof: Consider the diagram

$$
\begin{tikzcd}
 & \kappa\epsilon_{02}(\kappa p_2^* x_j) \arrow[rrr, "\kappa\gamma_j^{02}"] \arrow[dddd, "\kappa\epsilon_{02}(p_2^*\phi_{ij})"] & & & \kappa p_0^* x_j \arrow[r, "\delta_j"] \arrow[dd, "\kappa p_0^*\phi_{ij}"] & p_0^* x_j \arrow[d, "p_0^*\phi_{ij}"] \arrow[dll, "B_j"'] \\
\epsilon_{01}\epsilon_{12}(p_2^* x_j) \arrow[ur, "K(p_2^* x_j)"] \arrow[rr, "\gamma_j^{12}"] \arrow[dddd, "\epsilon_{01}\epsilon_{12}(p_2^*\phi_{ij})"] & & \epsilon_{01}(p_1^* x_j) \arrow[r, "\gamma_j^{01}"] \arrow[dd, "\epsilon_{01}(p_1^*\phi_{ij})"] & p_0^* x_j \arrow[d, "p_0^*\phi_{ij}"'] & & p_0^* x_i \arrow[d, "\delta_{ij}"'] \arrow[dll, "\lambda_{ij}(B_j)"'] \\
 & & & p_0^* x_i \arrow[d, "\gamma_{ij}^{01}"] & \kappa p_0^* x_i \arrow[r, "\delta_i"] \arrow[dd, "\kappa\gamma_{ij}^{02}"'] & p_0^* x_i \arrow[dd, "\nu_i(\gamma_{ij}^{02})"'] \\
 & & \epsilon_{01}(p_1^* x_i) \arrow[r, "\gamma_i^{01}"'] \arrow[dd, "\epsilon_{01}(\gamma_{ij}^{12})"'] & p_0^* x_i \arrow[dd, "m_i^{01}(\gamma_{ij}^{12})"'] & & \\
 & \kappa\epsilon_{02}(p_2^* x_i) \arrow[rrr, "\kappa\gamma_i^{02}"] & & & \kappa p_0^* x_i \arrow[r, "\delta_i"] & p_0^* x_i \arrow[dll, "B_i"] \\
\epsilon_{01}\epsilon_{12}(p_2^* x_i) \arrow[ur, "K(p_2^* x_i)"] \arrow[rr, "\epsilon_{01}(\gamma_i^{12})"'] & & \epsilon_{01}(p_1^* x_i) \arrow[r, "\gamma_i^{01}"'] & p_0^* x_i & & .
\end{tikzcd}
\tag{83}
$$

Diagrams (76), (80) and (79) imply that all squares in (83) are commutative[7], except possibly the rear right upper one. This remaining square (82) is therefore also commutative. □

Conjugating diagram (82), we obtain as in (67) a square

$$
\begin{tikzcd}
p_0^* G \arrow[r, "\nu_j"] \arrow[dd, "p_0^*\lambda_{ij}"'] & p_0^* G \arrow[d, "p_0^*\lambda_{ij}"] \\
 & p_0^* G \arrow[d, "i_{\delta_{ij}}"] \\
\kappa p_0^* G \arrow[r, "\nu_i"'] & p_0^* G \,,
\end{tikzcd}
$$

[7] This is true for diagram (80) since $\nu_i(\gamma_{ij}^{02}) = \gamma_{ij}^{02}$.

whose commutativity is expressed algebraically as

$$i(\delta_{ij})\,(p_0^*\lambda_{ij})\,\nu_j \;=\; \nu_i\,(p_0^*\lambda_{ij})\,. \tag{84}$$

In additive notation, this is equation

$$^{\lambda_{ij}}\nu_j = \nu_i - i(\delta_{ij})\,, \tag{85}$$

in other words

$$\delta^0_{\lambda_{ij}}\nu_i = -\,i(\delta_{ij})\,.$$

It is instructive to note that this equation can be derived directly from equation (71) and the definitions (77) and (81) of ν_i and δ_{ij}. First of all, observe that by (37)

$$\mathrm{d}^1(^{\lambda_{ij}*}m_i) = {}^{\lambda_{ij}}(\mathrm{d}^1 m_i)\,. \tag{86}$$

One then computes

$$\begin{aligned} ^{\lambda_{ij}}\nu_j &= {}^{\lambda_{ij}}(\mathrm{d}^1(m_j) - i_{B_j}) \\ &= \mathrm{d}^1(^{\lambda_{ij}*}m_j) - i(\lambda_{ij}(B_j)) \\ &= \mathrm{d}^1(m_i - i(\gamma_{ij})) - i(B_i + \mathrm{d}^1_{m_i}(-\gamma_{ij}) + \delta_{ij}) \\ &= \mathrm{d}^1 m_i - \mathrm{d}^1(i(\gamma_{ij})) - [m_i, \gamma_{ij}] - i(B_i) - i(\mathrm{d}^1 m_i(-\gamma_{ij})) - i(\delta_{ij})\,. \end{aligned}$$

Since the homomorphism i commutes with $\mathrm{d}^1 m$ and $[m_i,\, i(\gamma_{ij})] = i([m_i,\, \gamma_{ij}])$, the summands $i(\mathrm{d}^1 m(-\gamma_{ij}))$ and $\mathrm{d}^1(i(\gamma_{ij})) + [m_i,\, \gamma_{ij}]$ cancel out. The first two remaining summands describe ν_i, so that equation (85) is satisfied.

In the same vein, the analogue for the fake curvature κ of (73) is the following assertion.

Lemma 4.2. *The diagram*

$$\begin{tikzcd} p_0^*x_i \arrow[r, "p_0^*g_{ijk}"] \arrow[dd, "\delta_{ik}"] & p_0^*x_i \arrow[d, "\lambda_{ij}(\delta_{jk})"] \\ & p_0^*x_i \arrow[d, "\delta_{ij}"] \\ p_0^*x_i \arrow[r, "\nu_i(p_0^*g_{ijk})"'] & p_0^*x_i \end{tikzcd} \tag{87}$$

is commutative.

Proof: By (82), (43) and (79), all squares in the diagram

$$\begin{tikzcd}
& \kappa p_0^* x_k \arrow[rr, "\kappa p_0^*\phi_{ik}"] \arrow[ddl, "\kappa p_0^*\phi_{jk}"'] \arrow[ddddd, "\delta_k"] & & \kappa p_0^* x_i \arrow[d, "\delta_i"] \arrow[ddl, "\kappa p_0^* g_{ijk}"'] \\
& & & p_0^* x_i \arrow[ddl, "\nu_i(p_0^* g_{ijk})"] \\
\kappa p_0^* x_j \arrow[rr, "\kappa p_0^*\phi_{ij}"] \arrow[dd, "\delta_j"'] & & \kappa p_0^* x_i \arrow[d, "\delta_i"'] & \\
& & p_0^* x_i & \\
p_0^* x_j \arrow[rr, "p_0^*\phi_{ij}"] & & p_0^* x_i \arrow[u, "\delta_{ij}"'] & \\
& p_0^* x_k \arrow[rr, "p_0^*\phi_{ik}"] \arrow[dl, "p_0^*\phi_{jk}"'] & & p_0^* x_i \arrow[uuuu, "\delta_{ik}"'] \arrow[dl, "p_0^* g_{ijk}"] \\
p_0^* x_j \arrow[uu, "\delta_{jk}"] \arrow[rr, "p_0^*\phi_{ij}"'] & & p_0^* x_i \arrow[uu, "\lambda_{ij}(\delta_{jk})"'] &
\end{tikzcd} \tag{88}$$

are commutative, except possibly the lower right-hand one. It follows that the latter one, which is simply (87), also commutes. □

The commutativity of (87) corresponds to the equation

$$\delta_{ij}\ (p_0^*\lambda_{ij})(\delta_{jk})\ =\ \nu_i(p_0^* g_{ijk})\ \delta_{ik}\,(p_0^* g_{ijk})^{-1}\,,$$

an equation whose terms are G-valued 2-forms on U_{ijk}. By the same reasoning as for (74), this can be written additively as

$$\delta_{ij} + \lambda_{ij}(\delta_{jk}) - \lambda_{ij}\lambda_{jk}(\lambda_{ik}^{-1}(\delta_{ik})) = [\nu_i,\, g_{ijk}]\,,$$

or, in the compact form of [5] (6.1.15), as

$$\delta^1_{\lambda_{ij}}(\delta_{ij}) = [\nu_i,\, g_{ijk}]\,. \tag{89}$$

Just as we were able to derive (85) directly from (71) and the definitions (77) and (81), we now show that it is possible to deduce (89) from (81), (77) and (74). First of all,

$$\begin{aligned}\delta^1_{\lambda_{ij}}(\delta_{ij}) &= \delta^1_{\lambda_{ij}}(\delta^0_{\lambda_{ij}}(B_i) - \mathrm{d}^1_{m_i}(-\gamma_{ij}))\\ &= \delta^1_{\lambda_{ij}}\delta^0_{\lambda_{ij}}(B_i) - \delta^1_{\lambda_{ij}}\mathrm{d}^1_{m_i}(-\gamma_{ij})\,.\end{aligned} \tag{90}$$

We now wish to assert that the Čech differential $\delta^1_{\lambda_{ij}}$ and de Rham differential $\mathrm{d}^1_{m_i}$ in (90) commute with each other, despite the fact that the 1-form

γ_{ij} takes its values in a noncommutative group G, and that $\mathrm{d}^1_{m_i}$ is not a homomorphism. For this we simplify our notation, by setting

$$\widetilde{\gamma}_{ij} := -\gamma_{ij} \in \mathfrak{g} \otimes \Omega^1_{U_{ij}} \tag{91}$$

and

$$\lambda_{ijk} := \lambda_{ij}\, \lambda_{jk}\, \lambda_{ik}^{-1} \in \Gamma(U_{ijk},\, \mathrm{Aut}(G_i)).$$

Equation (74) can be restated as

$$\delta^1_{\lambda_{ij}} \widetilde{\gamma} := \widetilde{\gamma}_{ij} + \lambda_{ij}(\widetilde{\gamma}_{jk}) - \lambda_{ijk}(\widetilde{\gamma}_{ik}) = -\, \mathrm{d}g_{ijk}\, g_{ijk}^{-1} - [m_i,\, g_{ijk}]\,. \tag{92}$$

Lemma 4.3. *The following equality between G-valued 2-forms above U_{ijk} is satisfied:*

$$\mathrm{d}^1_{m_i} \delta^1_{\lambda_{ij}}(\widetilde{\gamma}_{ij}) = \delta^1_{\lambda_{ij}} \mathrm{d}^1_{m_i}(\widetilde{\gamma}_{ij})\,. \tag{93}$$

Proof: We compute the left-hand side of the equation (93), taking into account the quadraticity equation (29)

$$\begin{aligned}
\mathrm{d}^1_{m_i}\, \delta^1_{\lambda_{ij}}(\widetilde{\gamma}_{ij}) =\ & \mathrm{d}_{m_i}(\widetilde{\gamma}_{ij}) + \mathrm{d}^1_{m_i}(\lambda_{ij}(\widetilde{\gamma}_{jk})) + \mathrm{d}^1_{m_i}(-\lambda_{ijk}(\widetilde{\gamma}_{ik}))\\
& + [\widetilde{\gamma}_{ij},\, \lambda_{ij}(\widetilde{\gamma}_{jk})] - [\widetilde{\gamma}_{ij},\, \lambda_{ijk}(\widetilde{\gamma}_{ik})] - [\lambda_{ij}(\widetilde{\gamma}_{jk}),\, \lambda_{ijk}(\widetilde{\gamma}_{ik})]\\
=\ & \mathrm{d}_{m_i}(\widetilde{\gamma}_{ij}) + \mathrm{d}^1_{m_i}(\lambda_{ij}(\widetilde{\gamma}_{jk})) - \mathrm{d}^1_{m_i}(\lambda_{ijk}(\widetilde{\gamma}_{ik}))\\
& + [\lambda_{ijk}(\widetilde{\gamma}_{ik}),\, \lambda_{ijk}(\widetilde{\gamma}_{ik})] + [\widetilde{\gamma}_{ij},\, \lambda_{ij}(\widetilde{\gamma}_{jk})]\\
& - [\widetilde{\gamma}_{ij} + \lambda_{ij}(\widetilde{\gamma}_{jk}),\, \lambda_{ijk}(\widetilde{\gamma}_{ik})]\,.
\end{aligned}$$

We now compute the right-hand side of (93):

$$\delta^1_{\lambda_{ij}}\, \mathrm{d}^1_{m_i}(\widetilde{\gamma}_{ij}) = \mathrm{d}^1_{m_i}(\widetilde{\gamma}_{ij}) + \lambda_{ij}(\mathrm{d}^1_{m_j}(\widetilde{\gamma}_{jk})) - \lambda_{ijk}(\mathrm{d}^1_{m_i}(\widetilde{\gamma}_{ik}))\,. \tag{94}$$

By (70) and by the functoriality property (37), we find that

$$\begin{aligned}
\lambda_{ij}(\mathrm{d}^1_{m_j}(\widetilde{\gamma}_{jk})) &= \mathrm{d}^1{}_{\lambda_{ij} * m_j}(\lambda_{ij}(\widetilde{\gamma}_{jk}))\\
&= \mathrm{d}^1_{m_i}(\lambda_{ij}(\widetilde{\gamma}_{jk})) + [\widetilde{\gamma}_{ij},\, \lambda_{ij}(\widetilde{\gamma}_{jk})]
\end{aligned}$$

and by (39)

$$\begin{aligned}
\lambda_{ijk}(\mathrm{d}^1_{m_i}(\widetilde{\gamma}_{ik})) &= \mathrm{d}^1{}_{\lambda_{ijk} * m_i}(\lambda_{ijk}(\widetilde{\gamma}_{ik}))\\
&= \mathrm{d}^1_{m_i}(\lambda_{ijk}(\widetilde{\gamma}_{ik})) + [[\lambda_{ijk},\, m_i],\, \lambda_{ijk}(\widetilde{\gamma}_{ik})]\\
&\quad + [\lambda_{ijk}\, \mathrm{d}\lambda_{ijk}^{-1},\ \lambda_{ijk}(\widetilde{\gamma}_{ik})]\,.
\end{aligned}$$

Inserting these expressions for $\lambda_{ij}(\mathrm{d}^1_{m_j}(\widetilde{\gamma}_{jk}))$ and $\lambda_{ijk}(\mathrm{d}^1_{m_i}(\widetilde{\gamma}_{ik}))$ into the right-hand side of (94) we find the following expression for $\delta^1_{\lambda_{ij}}\, \mathrm{d}^1_{m_i}(\widetilde{\gamma}_{ij})$:

$$\begin{aligned}
\delta^1_{\lambda_{ij}}\, \mathrm{d}^1_{m_i}(\widetilde{\gamma}_{ij}) = \mathrm{d}^1_{m_i}(\widetilde{\gamma}_{ij}) &+ \mathrm{d}^1_{m_i}(\lambda_{ij}\widetilde{\gamma}_{jk}) + [\widetilde{\gamma}_{ij},\, \lambda_{ij}(\widetilde{\gamma}_{jk})]\\
& - \mathrm{d}^1_{m_i}(\lambda_{ijk})(\widetilde{\gamma}_{ik}) - [[\lambda_{ijk},\, m_i],\, \lambda_{ijk}(\widetilde{\gamma}_{ik})] - [\lambda_{ijk}\, \mathrm{d}\lambda_{ijk}^{-1},\, \lambda_{ijk}(\widetilde{\gamma}_{ik})]\\
& - \mathrm{d}^1_{m_i}(\lambda_{ijk})(\widetilde{\gamma}_{ik}) - [\lambda_{ijk}\, \mathrm{d}\lambda_{ijk}^{-1},\, \lambda_{ijk}(\widetilde{\gamma}_{ik})]\,.
\end{aligned}$$

Comparing this with the expression (94) for $\mathrm{d}^1_{m_i}\,\delta^1_{\lambda_{ij}}(\widetilde{\gamma}_{ij})$, we see that the equation (93) is satisfied if and only if

$$[\widetilde{\gamma}_{ij} + \lambda_{ij}(\widetilde{\gamma}_{jk}) - \lambda_{ijk}(\widetilde{\gamma}_{ik}),\, \lambda_{ijk}(\widetilde{\gamma}_{ik})] = [[\lambda_{ijk},\, m_i],\, \lambda_{ijk}(\widetilde{\gamma}_{ik})] \\ + [\lambda_{ijk}\,\mathrm{d}\lambda^{-1}_{ijk},\, \lambda_{ijk}(\widetilde{\gamma}_{ik})]\,.$$

By (14), this is simply a consequence of (92), since $\lambda_{ijk} = i(g_{ijk})$. □

We now return to our computation (90):

$$\begin{aligned}
\delta^1_{\lambda_{ij}}(\delta_{ij}) &= \delta^1_{\lambda_{ij}}\delta^0_{\lambda_{ij}}(B_i) - \delta^1_{\lambda_{ij}}\mathrm{d}^1_{m_i}(-\gamma_{ij}) \\
&= \delta^1_{\lambda_{ij}}\delta^0_{\lambda_{ij}}(B_i) - d^1_{m_i}\delta^1_{\lambda_{ij}}(-\gamma_{ij}) \\
&= [g_{ijk},\, B_i] - d^1_{m_i}(g_{ijk}\, d_{m_i}(g^{-1}_{ijk})) \\
&= [g_{ijk},\, i_{B_i} - dm_i] && \text{by (32)} \\
&= [\nu_i,\, g_{ijk}]\,.
\end{aligned}$$

This finishes the second proof of equation (89). □

We now set

$$\omega_i := \mathrm{d}^2_{m_i}(B_i)\,. \tag{95}$$

Since the combinatorial definition of the twisted de Rham differential d^2 ([4] (3.3.1)) matches the global geometric definition (62) of the 3-curvature Ω, this 3-curvature Ω is locally described by the G-valued 3-forms ω_i.

It follows from the definitions (77) and (95) of the forms ν_i and ω_i, and from (31), that

$$\begin{aligned}
\mathrm{d}^3_{m_i}(\omega_i) &= \mathrm{d}^3_{m_i}\mathrm{d}^2_{m_i}(B_i) \\
&= [\mathrm{d}^1 m_i,\, B_i] \\
&= [\nu_i,\, B_i] + [B_i,\, B_i]
\end{aligned}$$

so that the local 3-curvature form ω_i satisfies the higher Bianchi identity

$$\mathrm{d}^3_{m_i}(\omega_i) = [\nu_i, B_i]\,. \tag{96}$$

A second relation between the forms ν_i and ω_i follows from their definitions and the Bianchi identity for the $\mathrm{Aut}(G)$-valued 1-form m_i:

$$\begin{aligned}
i(\omega_i) &= \mathrm{d}^2_{m_i} i(B_i) \\
&= \mathrm{d}^2_{m_i}(\mathrm{d}^1 m_i - \nu_i) \\
&= \mathrm{d}^2_{m_i}(-\nu_i)\,,
\end{aligned}$$

in other words

$$\mathrm{d}^2_{m_i}\nu_i + i(\omega_i) = 0\,. \tag{97}$$

This equation is the local form of equation (4), just as (96) was the local form of (5).

We will now show that the equation (81) for the 2-forms B_i, which we write here as

$$\delta^0_{\lambda_{ij}}(B_i) = \mathrm{d}^1_{m_i}(-\gamma_{ij}) + \delta_{ij}\ ,$$

induces the corresponding gluing equation for the local 3-forms ω_i. From the definition of $\lambda_{ij}(\omega_j)$ and (38), it follows that

$$\begin{aligned}\lambda_{ij}(\omega_j) &= \lambda_{ij}(\mathrm{d}^2_{m_j}(B_j))\\ &= \mathrm{d}^2_{\lambda_{ij}\,{}^*m_j}\lambda_{ij}(B_j)\end{aligned}$$

and by the gluing laws (71) and (81) for m_i and B_i, this can be stated as

$$\begin{aligned}\lambda_{ij}(\omega_j) &= \mathrm{d}^2_{m_i - i(\gamma_{ij})}(B_i + \delta_{ij} + \mathrm{d}^1_{m_i}(-\,\gamma_{ij}))\\ &= \mathrm{d}^2_{m_i}(B_i) + \mathrm{d}^2_{m_i}(\delta_{ij}) + \mathrm{d}^2_{m_i}\mathrm{d}^1_{m_i}(-\,\gamma_{ij}) - [\gamma_{ij},\, B_i + \delta_{ij} + \mathrm{d}^1_{m_i}(-\,\gamma_{ij})]\,.\end{aligned}$$

By (33), this last equality can be rewritten as

$$\begin{aligned}\lambda_{ij}(\omega_j) &= \omega_i + \mathrm{d}^2_{m_i}(\delta_{ij}) + [\mathrm{d}^1 m_i,\ -\gamma_{ij}] - [\gamma_{ij},\, B_i] - [\gamma_{ij},\, \delta_{ij}]\\ &= \omega_i + \mathrm{d}^2_{m_i}(\delta_{ij}) + [\gamma_{ij},\, \mathrm{d}^1 m_i - B_i] - [\gamma_{ij},\, \delta_{ij}]\end{aligned}$$

and by (84) this proves the gluing law for the 3-forms ω_i [5] (6.1.23):

$$\lambda_{ij}(\omega_j) = \omega_i + \mathrm{d}^2_{m_i}(\delta_{ij}) + [\gamma_{ij},\, \nu_i] - [\gamma_{ij},\, \delta_{ij}]\,.$$

By combining this with the gluing law (85) for ν_i, we see that (98) can finally be rewritten in the more compact form

$$\lambda_{ij}(\omega_j) + [{}^{\lambda_{ij}}\nu_j,\, \gamma_{ij}] = \omega_i + \mathrm{d}^2_{m_i}(\delta_{ij}). \tag{98}$$

5 Čech–de Rham coboundaries

5.1

We saw in Section 2 how a change in the choice trivializing data $(x_i,\, \phi_{ij})$ in a gerbe $\mathcal{P}$ could be measured by a pair $(r_i,\, \theta_{ij})$ (49),(50) inducing a coboundary relation (54) between the corresponding cocycle pairs $(\lambda_{ij},\, g_{ijk})$. We will now examine how the terms $(m_i,\, \gamma_{ij})$, $(\nu_i,\, \delta_{ij})$ and B_i introduced in Section 4 vary when the arrows γ_i (64) and δ_i (75) which determine them have been modified.

The difference between the arrow γ_i and an analogous arrow γ'_i is measured by a 1-form $e_i \in \mathrm{Lie}\,(G) \otimes \Omega^1_{U_i}$, defined by the commutativity of the following diagram:

$$\begin{tikzcd} \epsilon p_1^* x_i \arrow[rr, "\epsilon p_1^* \chi_i"] \arrow[d, "\gamma_i"'] & & \epsilon p_1^* x'_i \arrow[d, "\gamma'_i"] \\ p_0^* x_i \arrow[r, "p_0^* \chi_i"'] & p_0^* x'_i \arrow[r, "e_i"'] & p_0^* x'_i . \end{tikzcd} \tag{99}$$

This conjugates to a commutative diagram

$$\begin{tikzcd} p_1^* G \arrow[rr, "p_1^* r_i"] \arrow[d, "m_i"'] & & p_1^* G \arrow[d, "m'_i"] \\ p_0^* G \arrow[r, "p_0^* r_i"'] & p_0^* G \arrow[r, "i(e_i)"'] & p_0^* G \end{tikzcd}$$

so that

$$\begin{aligned} m'_i &= i(e_i)\,(p_0^* r_i)\, m_i\ (p_1^* r_i)^{-1} \\ &= i(e_i)\,[p_0^* r_i\, m_i\, p_0^* r_i{}^{-1}]\,[p_0^* r_i\, p_1^* r_i{}^{-1}]. \end{aligned}$$

In classical terms, this is expressed as an equation

$$m'_i = {}^{r_i} m_i + r_i \mathrm{d} r_i^{-1} + i(e_i) \tag{100}$$

$$= {}^{r_i *} m_i + i(e_i), \tag{101}$$

which compares the connections m_i and m'_i induced on the group G by the arrows γ_i and γ'_i.

We now consider the following diagram in $\mathcal{P}_{U_{ij}}$:

$$\begin{tikzcd} p_0^* x'_i \arrow[d, "e_i"'] & p_0^* x'_i \arrow[l, "r_i(\gamma_{ij})"'] \arrow[d, "p_0^* \theta_{ij}"] \\ p_0^* x'_i \arrow[d, "m'_i(p_1^* \theta_{ij})"'] & p_0^* x'_i \arrow[d, "\lambda'_{ij}(e_j)"] \\ p_0^* x'_i & p_0^* x'_i \arrow[l, "\gamma'_{ij}"] . \end{tikzcd} \tag{102}$$

Proposition 5.1. *The diagram* (102) *is commutative.*

Proof: Consider the diagram

$$
\begin{tikzcd}
 & \epsilon p_1^* x_j \arrow[rrr, "\gamma_j"] \arrow[ddl, "\epsilon p_1^* \phi_{ij}"'] \arrow[ddddd, "\epsilon p_1^* \chi_j"] & & & p_0^* x_j \arrow[dl, "p_0^* \phi_{ij}"] \arrow[ddd, "p_0^* \chi_j"] \\
 & & & p_0^* x_i \arrow[dl, "\gamma_{ij}"'] \arrow[dd, "p_0^* \chi_i"] & \\
\epsilon p_1^* x_i \arrow[rr, "\gamma_i"] \arrow[dddd, "\epsilon p_1^* \chi_i"'] & & p_0^* x_i \arrow[dd, "p_0^* \chi_i"'] & & \\
 & & & p_0^* x'_i \arrow[dl, "p_0^* r_i(\gamma_{ij})"'] \arrow[d, "p_0^* \theta_{ij}"] & p_0^* x'_j \arrow[dl, "p_0^* \phi'_{ij}"] \arrow[dd, "e_j"] \\
 & & p_0^* x'_i \arrow[dd, "e_i"] & p_0^* x'_i \arrow[dd, "p_0^* \lambda'_{ij}(e_j)"] & \\
 & \epsilon p_0^* x'_j \arrow[rrr, "\gamma'_j"] \arrow[ddl, "\epsilon p_1^* \phi'_{ij}"'] & & & p_0^* x'_j \arrow[dl, "p_0^* \phi'_{ij}"] \\
\epsilon p_1^* x'_i \arrow[rr, "\gamma'_i"] \arrow[d, "\epsilon p_1^* \theta_{ij}"'] & & p_0^* x'_i \arrow[d, "m'_i(p_1^* \theta_{ij})"'] & p_0^* x'_i \arrow[dl, "\gamma'_{ij}"] & \\
\epsilon p_1^* x'_i \arrow[rr, "\gamma'_i"'] & & p_0^* x'_i & &
\end{tikzcd}
\tag{103}
$$

The lower front square of the right-hand face of this cube is just the square (102). Since we know that all the other squares in this diagram commute, so does the square (102). □

The commutativity of (102) is equivalent to the equation

$$m'_i(p_1^*\theta_{ij})\, e_i\, r_i(\gamma_{ij}) = \gamma'_{ij}\ \lambda'_{ij}(e_j)\, p_0^*\theta_{ij}\ . \tag{104}$$

This may be rewritten in classical notation as

$$(\gamma'_{ij} - {}^{\theta_{ij}} r_i(\gamma_{ij}))\ +\ (\lambda'_{ij}(e_j) - {}^{\theta_{ij}} e_i) = \mathrm{d}_{m'_i}\, \theta_{ij}\ \theta_{ij}^{-1}\ . \tag{105}$$

We now choose a family of arrows $\delta'_i : \kappa p_0^* x'_i \longrightarrow p_0^* x'_i$. The families δ'_i and γ'_i determine as in (76) a family of $\mathfrak{g}$-valued 2-form B'_i above U_i. The latter in turn determines, together with the pair of form $(m'_i,\, \gamma'_{ij})$ (100), (105), a new pair of 2-forms $(\nu'_i,\, \delta'_{ij})$ and a 3-form ω'_i satisfying the corresponding equations (85), (97), (89), (96) and (98). The families δ_i and δ'_i are compared by the following analogue of diagram (99):

$$
\begin{tikzcd}
\kappa p_0^* x_i \arrow[rr, "\kappa p_0^* \chi_i"] \arrow[d, "\delta_i"'] & & \kappa p_0^* x'_i \arrow[d, "\delta'_i"] \\
p_0^* x_i \arrow[r, "p_0^* \chi_i"'] & p_0^* x'_i \arrow[r, "n_i"'] & p_0^* x'_i
\end{tikzcd}
\quad . \tag{106}
$$

We will now compare the 2-forms B_i and B'_i. We consider the diagram

$$
\begin{tikzcd}[column sep=small]
 & & & \epsilon_{01}\epsilon_{12}(p_2^* x_i) \arrow[rrrr, "K(p_2^* x_i)"] \arrow[dlll, "\epsilon_{01}\epsilon_{12}(p_2^* \chi_i)"'] \arrow[dd, "\epsilon_{01}(\gamma_i^{12})"'] & & & & \kappa\epsilon_{02}(p_2^* x_i) \arrow[dlll, "\kappa\, \epsilon_{02}(p_2^* \chi_i)"'] \arrow[dd, "\kappa(\gamma_i^{02})"] \\
\epsilon_{01}\epsilon_{12}(p_2^* x'_i) \arrow[rrrr, "K(p_2^* x'_i)"'] \arrow[ddd, "\epsilon_{01}\, {\gamma'}_i^{12}"] & & & & \kappa\, \epsilon_{02}(p_2^* x'_i) \arrow[ddd, "\kappa({\gamma'}_i^{02})"'] & & & \\
 & & & \epsilon_{01} p_1^* x_i \arrow[dll, "\epsilon_{01} p_1^* \chi_i"'] \arrow[ddd, "\gamma_i^{01}"] & & & & \kappa\, p_0^* x_i \arrow[dll, "\kappa p_0^* \chi_i"] \arrow[ddd, "\delta_i"'] \\
 & \epsilon_{01} p_1^* x'_i \arrow[dl] \arrow[dddd, "{\gamma'}_i^{01}"] & & & & \kappa p_0^* x'_i \arrow[dl, "\kappa(e_i^{02})"] \arrow[dddd, "\delta'_i"] & & \\
\epsilon_{01}\, p_0^*\, x'_i \arrow[dddd, "{\gamma'}_i^{01}"] & & & & \kappa p_0^* x'_i \arrow[dddd, "\delta'_i"] & & & \\
 & & & p_0^* x_i \arrow[dl, "p_0^* \chi_i"'] & & & & p_0^* x_i \arrow[llll, "B_i"'] \arrow[dl, "p_0^* \chi_i"] \\
 & & p_0^* x'_i \arrow[dl, "e_{01}^i"] & & & & p_0^* x'_i \arrow[llll, "r_i(B_i)"'] \arrow[dl, "n_i"] & \\
 & p_0^* x'_i \arrow[dl] & & & & p_0^* x'_i \arrow[dl, "\nu'_i(e_i^{02})"] & & \\
p_0^* x'_i & & & & p_0^* x'_i \arrow[llll, "B'_i"] & & &
\end{tikzcd}
\tag{107}
$$

in which the upper and lower unlabeled arrows are respectively $\epsilon_{01}(p_1^* e_i^{12})$ and ${m'}_i^{01}(e_i^{12})$.

The front square (or rather hexagon) of the bottom face

$$
\begin{tikzcd}
p_0^* x'_i \arrow[d, "e_{01}^i"'] & p_0^* x'_i \arrow[l, "r_i(B_i)"'] \arrow[d, "p_0^* n_i"] \\
p_0^* x'_i \arrow[d, "{m'}_i^{01}(e_i^{12})"'] & p_0^* x'_i \arrow[d, "\nu'_i(e_i^{02})"] \\
p_0^* x'_i & p_0^* x'_i \arrow[l, "B'_i"]
\end{tikzcd}
$$

is commutative, since all other squares in diagram (107) are. Equivalently, since the action of the $\mathrm{Aut}(G)$-valued 2-form ν'_i on e_i^{02} is trivial, this proves that the equation

$$B_i' = r_i(B_i) - \mathrm{d}^1_{m_i'}(-e_i) - n_i \tag{108}$$

is satisfied. In particular for given B_i and e_i, the 2-forms B_i' and n_i actually determine each other.

By conjugation, diagram (106) induces a commutative diagram

$$\begin{tikzcd}
p_0^*G \arrow[rr, "p_0^* r_i"] \arrow[d, "\nu_i"'] & & p_0^*G \arrow[d, "\nu_i'"] \\
p_0^*G \arrow[r, "p_0^* r_i"'] & p_0^*G \arrow[r, "i_{n_i}"'] & p_0^*G
\end{tikzcd}$$

equivalent to the equation

$$i(n_i)\, p_0^* r_i\ \nu_i = \nu_i'\ p_0^* r_i\ .$$

In classical terms, this is the simpler analogue

$$\nu_i' = {}^{r_i}\nu_i + i(n_i) \tag{109}$$

for ν_i of the equation (100) for m_i.

We will now show that this coboundary equation for ν_i can be derived from the definition (77) of ν_i, and the coboundary equations (100) and (108) for m_i and B_i:

$$\begin{aligned}
\nu_i' &= \mathrm{d}^1 m_i' - i(B_i') \\
&= \mathrm{d}^1({}^{r_i *}m_i + i(e_i)) - i(r_i(B_i) + n_i + \mathrm{d}^1_{m_i'}(-e_i)) \\
&= {}^{r_i}\mathrm{d}^1 m_i + i(\mathrm{d}^1 e_i) + [{}^{r_i *}m_i,\, i(e_i)] - i(r_i(B_i)) + i(\mathrm{d}^1_{m_i'}(-e_i)) + i(n_i) \\
&= {}^{r_i}(\mathrm{d}^1 m_i - i(B_i)) + i(n_i) + i(\mathrm{d}^1_{m_i'}(-e_i) + \mathrm{d}^1 e_i + [{}^{r_i *}m_i,\, e_i])
\end{aligned}$$

In order to prove (109), it now suffices to verify that the three terms in the last summand of the final equation cancel each other out:

$$\begin{aligned}
\mathrm{d}^1_{m_i'}(-e_i) + \mathrm{d}^1(e_i) + [{}^{r_i *}m_i,\, e_i] &= \mathrm{d}^1(-e_i) - [m_i',\, e_i] + \mathrm{d}^1 e_i + [{}^{r_i *}m_i,\, e_i] \\
&= \mathrm{d}^1(-e_i) + \mathrm{d}^1 e_i - [e_i,\, e_i] \\
&= 0.
\end{aligned}$$ □

The other equation satisfied by the forms n_i is the counterpart of equation (104). It is obtained by considering the following diagram, analogous to (103):

$$
\begin{tikzcd}
& \kappa p_0^* x_j \arrow[rrr, "\delta_j"] \arrow[ddl, "\kappa p_0^* \phi_{ij}"'] \arrow[ddddd, "\kappa p_0^* \chi_j"] & & & p_0^* x_j \arrow[dl, "p_0^* \phi_{ij}"] \arrow[dd, "p_0^* \chi_j"] \\
& & & p_0^* x_i \arrow[dl, "\delta_{ij}"'] \arrow[dd, "p_0^* \chi_i"] & \\
\kappa p_0^* x_i \arrow[rr, "\delta_i"] \arrow[dddd, "\kappa p_0^* \chi_i"'] & & p_0^* x_i \arrow[dd, "p_0^* \chi_i"'] & & \\
& & & p_0^* x'_i \arrow[dl, "r_i(\delta ij)"'] \arrow[d, "p_0^* \theta_{ij}"] & p_0^* x'_j \arrow[dl, "p_0^* \phi'_{ij}"] \arrow[dd, "n_j"] \\
& & p_0^* x'_i \arrow[dd, "n_i"] & p_0^* x'_i \arrow[dd, "p_0^* \lambda'_{ij}(n_j)"] & \\
& \kappa p_0^* x'_j \arrow[rrr, "\delta'_j"] \arrow[ddl, "\kappa p_0^* \phi'_{ij}"'] & & & p_0^* x'_j \arrow[dl, "p_0^* \phi'_{ij}"] \\
\kappa p_0^* x'_i \arrow[rr, "\delta'_i"] \arrow[d, "\kappa p_0^* \theta_{ij}"'] & & p_0^* x'_i \arrow[d, "\nu'_i(p_0^* \theta_{ij})"'] & p_0^* x'_i \arrow[dl, "\delta'_{ij}"] & \\
\kappa p_0^* x'_i \arrow[rr, "\delta'_i"'] & & p_0^* x'_i & &
\end{tikzcd}
\qquad . \tag{110}
$$

The lower front square on the right-hand face

$$
\begin{tikzcd}
p_0^* x'_i \arrow[r, "p_0^* \theta_{ij}"] \arrow[d, "r_i(\delta_{ij})"'] & p_0^* x'_i \arrow[r, "p_0^* \lambda'_{ij}(n_j)"] & p_0^* x'_i \arrow[d, "\delta'_{ij}"] \\
p_0^* x'_i \arrow[r, "n_i"'] & p_0^* x'_i \arrow[r, "\nu'_i(p_0^* \theta_{ij})"'] & p_0^* x'_i
\end{tikzcd}
$$

of diagram (110) is commutative, since all other squares in this diagram are. This proves that equation

$$\nu'_i(p_0^*\, \theta_{ij})\ n_i\ r_i(\delta_{ij}) = \delta'_{ij}\ p_0^* \lambda'_{ij}(n_j)\ p_0^* \theta_{ij}$$

in $\mathrm{Lie}\,(G) \otimes \Omega^2_{U_i/S}$ is satisfied. Regrouping the various terms in this equation as we did above for equation (104), we find that it is equivalent, in additive notation, to

$$(\delta'_{ij} - r_i(\delta_{ij})) + (\lambda'_{ij}(n_j) - {}^{\theta_{ij}} n_i) = [\nu'_i,\, \theta_{ij}]\ ,$$

an equation for 2-forms very similar to equation (105) for 1-forms.

We will now examine the effect of the chosen transformations

$$(\lambda_{ij},\, g_{ijk},\, m_i,\, \gamma_{ij}) \qquad \longrightarrow \qquad (\lambda'_{ij},\, g'_{ijk},\, m'_i,\, \gamma'_{ij}) \tag{111}$$

and $B_i \longrightarrow B'_i$ (108) on the 3-curvature 3-forms ω_i (95). For this, it will be convenient to set

$$\bar{e}_i := r_i^{-1}(e_i) \qquad \text{and} \qquad \bar{n}_i := r_i^{-1}(n_i)\ .$$

It follows from (28), (15), and the transformation formula (101) that

$$\mathrm{d}^n_{m'_i}(r_i(\eta)) = r_i(\mathrm{d}^n_{m_i}(\eta) + [\bar{e}_i,\, \eta]) \tag{112}$$

for any G-valued n-form η with $n > 1$. In particular,

$$\begin{aligned} \mathrm{d}^1_{m'_i}(-e_i) &= \mathrm{d}^1{}_{r_i{}_* m_i}(-e_i) - [e_i,\, e_i] \\ &= r_i(\mathrm{d}^1_{m_i}(-\bar{e}_i) - [\bar{e}_i,\, \bar{e}_i]) \end{aligned}$$

so that (108) may be expressed as

$$B'_i = r_i(B_i - \mathrm{d}^1_{m_i}(-\bar{e}_i) + [\bar{e}_i,\, \bar{e}_i] - \bar{n}_i)\ .$$

Applying once more the formula (112), we find that

$$\begin{aligned} \omega'_i &= \mathrm{d}^2_{m'_i}(B'_i) \\ &= \mathrm{d}^2_{m'_i}(r_i(B_i - \mathrm{d}^1_{m_i}(-\bar{e}_i) + [\bar{e}_i,\, \bar{e}_i] - \bar{n}_i)) \\ &= r_i(\mathrm{d}^2_{m_i}(B_i - \mathrm{d}^1_{m_i}(-\bar{e}_i) + [\bar{e}_i,\, \bar{e}_i] - \bar{n}_i)) \\ &\qquad + [\bar{e}_i,\, B_i - \mathrm{d}^1_{m_i}(-\bar{e}_i) + [\bar{e}_i,\, \bar{e}_i] - \bar{n}_i]\ . \end{aligned} \tag{113}$$

We now make use of (33) in order to compute the value of the expression $\mathrm{d}^2_{m_i}\mathrm{d}^1_{m_i}(-\bar{e}_i)$ which arises when we expand the first summand of the last equation (113):

$$\begin{aligned} \mathrm{d}^2_{m_i}\mathrm{d}^1_{m_i}(-\,\bar{e}_i) &= [\mathrm{d}^1 m_i\,,\, -\,\bar{e}_i] + [\mathrm{d}^1(-\,\bar{e}_i),\, -\,\bar{e}_i] + [[m_i,\, -\,\bar{e}_i],\, -\,\bar{e}_i] \\ &= -\,[\mathrm{d}^1 m_i\,,\, \bar{e}_i] + [\mathrm{d}^1\bar{e}_i,\, \bar{e}_i] + [[m_i,\, \bar{e}_i],\, \bar{e}_i]\ . \end{aligned}$$

Inserting this expression into (113), we find that

$$\begin{aligned} \omega'_i = r_i(\omega_i &+ [\mathrm{d}^1 m_i,\, \bar{e}_i] - [\mathrm{d}^1\bar{e}_i,\, \bar{e}_i] - [[m_i,\, \bar{e}_i],\, \bar{e}_i] - \mathrm{d}^2_{m_i}(\bar{n}_i) \\ &+ \mathrm{d}^2_{m_i}[\bar{e}_i,\, \bar{e}_i] + [\bar{e}_i,\, B_i] - [\bar{e}_i,\, \mathrm{d}^1_{m_i}(-\bar{e}_i)] - [\bar{e}_i,\, \bar{n}_i])\ . \end{aligned} \tag{114}$$

The four terms

$$-[\mathrm{d}^1\bar{e}_i,\, \bar{e}_i] - [[m_i,\, \bar{e}_i],\, \bar{e}_i] + \mathrm{d}^2_{m_i}[\bar{e}_i,\, \bar{e}_i] - [\bar{e}_i,\, \mathrm{d}^1_{m_i}(-\bar{e}_i)]$$

cancel each other out, so that we are left in (114) with

$$\begin{aligned}\omega_i' &= r_i(\omega_i + [\mathrm{d}^1 m_i,\, \bar{e}_i] - \mathrm{d}^2_{m_i}(\bar{n}_i) + [\bar{e}_i,\, B_i] - [\bar{e}_i,\, \bar{n}_i]) \\ &= r_i(\omega_i + [\mathrm{d}^1 m_i - i(B_i),\, \bar{e}_i] + [\bar{n}_i,\, \bar{e}_i] - \mathrm{d}^2_{m_i}(\bar{n}_i)) \\ &= r_i(\omega_i) + r_i([\nu_i,\, \bar{e}_i]) + r_i([\bar{n}_i,\, \bar{e}_i]) - r_i(\mathrm{d}^2_{m_i}(\bar{n}_i)) \\ &= r_i(\omega_i) + [{}^{r_i}\nu_i,\, e_i] + [n_i,\, e_i] - \mathrm{d}^2_{r_i * m_i}(n_i) \end{aligned} \tag{115}$$

where in the last line we made use of the functoriality property (15) of the bracket operation. Amalgamating the last two summands, we may finally write the coboundary transformation for the 3-curvature form ω_i in the compact form

$$\omega_i' = r_i(\omega_i) + [{}^{r_i}\nu_i,\, e_i] - \mathrm{d}^2_{m_i'}(n_i)\ .$$

If instead we amalgamate the second and third terms in (115), we find the equivalent formulation

$$\omega_i' = r_i(\omega_i) + [\nu_i',\, e_i] - \mathrm{d}^2_{r_i * m_i}(n_i)\ . \tag{116}$$

Remark 5.1 (Comparison with [5]):

The coboundary equation (116) is compatible with equation (6.2.19) of [5], but neither is a special case of the other. Here we allowed both the trivializing data $(x_i,\, \phi_{ij})$ for the gerbe and the expressions $(\gamma_i,\, \delta_i,\, B_i)$ for the curving data to vary, whereas in the coboundary equations of [5] the gerbe data $(x_i,\, \phi_{ij})$ was fixed and only the $(\gamma_i,\, \delta_i,\, B_i)$ varied. This restriction amounted to setting $(r_i,\, \theta_{ij}) = (1,\, 1)$ in our equation (105). On the other hand, a notion of equivalence between cocycles was introduced in [5] which was more extensive than the one considered here. In order for these to be comparable, one must suppose that the arrow h in diagram (4.2.1) of [5] is the identity map, i.e., that the pair of differential forms $(\pi_i,\, \eta_{ij})$ associated to h in *loc. cit.* § 6.2 is trivial. This is a reasonable assumption, since a nontrivial arrow h could really be termed a gauge transformation, rather than a coboundary term. With this additional condition, the last two summands in equation (6.2.19) of [5] vanish, so that this equation reduces to

$$\omega_i' = \omega_i + \delta^2_{m_i}(\alpha_i) - [\nu_i',\, E_i]\ . \tag{117}$$

This simplified equation is compatible with our equation (116) with $r_i = 1$, under the correspondence $e_i := -E_i$ and $n_i := -\,\alpha_i$.

References

1. Aschieri, P., Cantini, L., Jurčo B.: Nonabelian bundle gerbes, their differential geometry and gauge theory. Comm. Math. Phys. **254**, 367–400 (2005)
2. Baez, J.C., Schreiber, U.: Higher gauge theory, in “Categories in algebra, geometry and physics”. Contemp. Math. **431**, 7–30, A.M.S. (2007)

3. Breen, L.: Classification of 2-gerbes and 2-stacks. Astérisque **225**, Société Mathématique de France (1994)
4. Breen, L., Messing, W.: Combinatorial differential forms. Adv. Math. **164**, 203–282 (2001)
5. Breen, L., Messing, W.: Differential geometry of gerbes. Adv. Math. **198**, 732–846 (2005)
6. Breen, L.: Notes on 1- and 2-gerbes. In: Baez, J.C., May, J.P. (eds.) Towards Higher Categories. The IMA Volumes in Mathematics and its Applications, vol. 152, pp. 195–235. Springer, Heidelberg (2009)
7. Brylinski, J.-L.: Loop spaces, characteristic classes and geometric quantization. Progr. Math. **107**, Birkhäuser, Boston (1993)
8. Hitchin, N.: Lectures on special lagrangian submanifolds, in "winter school on mirror symmetry, vector bundles and lagrangian submanifolds (Cambridge, MA, 1999)". AMS/IP Stud. Adv. Math. **23**, 151–182 (2001)
9. Kobayashi, S., Nomizu, K.: Foundations of differential geometry. Intersci. Tracts Pure Appl. Math. **15** (1969)
10. Kock, A.: Differential forms with values in groups. Bull. Aust. Math. Soc. **25**, 357–386 (1982)
11. Kock, A.: Combinatorics of curvature, and the Bianchi identity. Theory Appl. Categories **2**, 69–89 (1996)
12. Laurent-Gengoux, C., Stenion, M., Xu, P.: Non-abelian differential gerbes. Adv. Math. **220**, 1357–1427 (2009)
13. Murray, M.K.: Bundle Gerbes, J. Lond. Math. Soc. **54**, 403–416 (1996)
14. Schreiber, U.: From loop space mechanics to nonabelian strings, University of Duisburg-Essen doctoral dissertation (2005), available as `Arxiv: hep-th/0509163`
15. Ulbrich, K.-H.: On cocycle bitorsors and gerbes over a Grothendieck topos. Math. Proc. Camb. Phil. Soc. **110**, 49–55 (1991)

Symplectic Connections of Ricci Type and Star Products

Michel Cahen[1], Simone Gutt[1,2,*], and Stefan Waldmann[3]

[1] Département de Mathématique, Université Libre de Bruxelles, Campus Plaine, C. P. 218, Boulevard du Triomphe, 1050 Bruxelles, Belgium, mcahen@ulb.ac.be

[2] Département de Mathématique, LMAM, Université Paul Verlaine-Metz, Ile du Saulcy, 57045 Metz Cedex 01, France, gutt@poncelet.univ-metz.fr

[3] Fakultät für Mathematik und Physik, Albert-Ludwigs-Universität Freiburg, Physikalisches Institut, Hermann Herder Strasse 3, 79104 Freiburg, Germany, Stefan.Waldmann@physik.uni-freiburg.de

Abstract. In this article we relate the construction of Ricci-type symplectic connections by reduction to the construction of star products by reduction yielding rather explicit descriptions for the star product on the reduced space.

Key words: Deformation quantization, Symplectic connections, Reduction, Ricci-type

AMS Classification (2010): 53D55, 53C07, 53D20

Introduction

Deformation quantization [4] is a formal deformation – in the sense of Murray Gerstenhaber [18] – of the algebraic structure of the space of smooth functions on a manifold M; it yields at first order in the deformation parameter a Poisson structure on M. When this Poisson structure is nondegenerate, i.e., when the manifold is symplectic, deformation quantization at second-order yields a symplectic connection on M [22].

On a symplectic manifold (M, ω), symplectic connections always exist but are not unique. The curvature R^∇ of such a connection ∇ splits [23] under the action of the symplectic group (when the dimension of M is at least 4), into two irreducible components $R^\nabla = E^\nabla + W^\nabla$ with E^∇ completely determined by the Ricci tensor of the connection. A symplectic connection is said to be of Ricci-type if $R^\nabla = E^\nabla$.

A.S. Cattaneo et al. (eds.), *Higher Structures in Geometry and Physics*, Progress in Mathematics 287, DOI 10.1007/978-0-8176-4735-3_5,

Marsden–Weinstein reduction, see e.g. [1, Sect. 4.3], is a method in symplectic geometry to construct a symplectic manifold (M, ω) – called the reduced space – from a bigger one (P, μ) and the extra data of a coisotropic submanifold C in P.

Under some further assumptions, one can sometimes reduce connections [3, 2], i.e., define a symplectic connection ∇^M on M from a connection ∇^P on P. In particular, any Ricci-type connection on a simply connected $2n(\geq 4)$ dimensional manifold can be obtained [12] by reduction from a flat connection ∇ on a $2n + 2$ dimensional manifold P, with the coisotropic codimension 1 submanifold C defined by the zero set of a function $F : P \to \mathbb{R}$ whose third covariant derivative vanishes.

One way to describe the algebra of functions on the reduced space is the use of BRST methods. Jim Stasheff participated actively in the development of this point of view, see e.g. [17] among many others. Reduction of deformation quantization has been studied by various authors [16, 9, 6, 14, 20, 13]. In particular, a quantized version of BRST methods was introduced in [9] to construct reduction in deformation quantization. We use here those methods to define a star product on any symplectic manifold endowed with a Ricci-type connection.

In this context we rely both on the work of Murray and on the work of Jim and we are very happy and honoured to dedicate this to them.

In Section 1, we recall some basic properties of Ricci-type symplectic connections. Section 2 introduces a natural differential operator of order 2 on a symplectic manifold endowed with a symplectic connection. In Section 3, we recall the expression of the Weyl–Moyal star product on a symplectic manifold with a flat connection. Section 4 explains the construction of reduced star product in our context where the reduced space (M, ω) is a symplectic manifold of dimension $2n \geq 4$, endowed with a Ricci-type connection, and where the big space (P, μ) is of dimension $2n + 2$ with a flat connection and the related Weyl–Moyal star product. In Section 5, we show some properties of the reduced star product, in particular that the connection defined by the reduced star product is the Ricci-type connection.

1 Preliminary Results on Ricci-Type Connections

In this section we recall some basic properties of Ricci-type symplectic connections to explain our notation. We follow essentially [12, 11] and refer for further details to the expository paper [5].

Let (M, ω) be a $2n \geq 4$-dimensional symplectic manifold which allows for a Ricci-type symplectic connection: this is a symplectic connection ∇^M (i.e., a linear torsion-free connection so that the symplectic 2-form ω is parallel) such that the curvature tensor R is entirely determined by its Ricci tensor Ric. Precisely, for $X, Y \in \Gamma^\infty(TM)$ we have

$$R(X,Y) = -\frac{1}{2n+1}\Big(- 2\omega(X,Y)\varrho - \varrho(Y)\otimes X^\flat + \varrho(X)\otimes Y^\flat - X\otimes(\varrho(Y))^\flat + Y\otimes(\varrho(X))^\flat\Big), \tag{1}$$

where $X^\flat = \mathrm{i}_X\,\omega$ as usual and ϱ is the Ricci endomorphism defined by

$$\mathrm{Ric}(X,Y) = \omega(X,\varrho(Y)). \tag{2}$$

It follows that there exists a vector field $\mathsf{U}\in\Gamma^\infty(TM)$, a function $\mathsf{f}\in C^\infty(M)$ and a constant K such that the following identities hold:

$$\nabla^M_X\varrho = -\frac{1}{2n+1}\left(X\otimes\mathsf{U}^\flat + \mathsf{U}\otimes X^\flat\right), \tag{3}$$

$$\nabla^M_X\mathsf{U} = -\frac{2n+1}{2(n+1)}\varrho^2X + \mathsf{f}X, \tag{4}$$

$$\mathsf{tr}(\varrho^2) + \frac{4(n+1)}{2n+1}\mathsf{f} = \mathsf{K}. \tag{5}$$

One of the fundamental properties of such a Ricci-type connection is that (M,ω,∇^M) can be obtained from a Marsden–Weinstein reduction out of a $(2n+2)$-dimensional symplectic manifold (P,μ) which is equipped with a *flat* symplectic torsion-free connection ∇. In fact, we have to assume that M is simply-connected; then by [12] there exists a $(2n+1)$-dimensional manifold Σ with a surjective submersion

$$\pi : \Sigma \longrightarrow M \tag{6}$$

together with a contact one-form $\alpha\in\Gamma^\infty(T^*\Sigma)$, i.e., $\alpha\wedge(\mathrm{d}\alpha)^n$ is nowhere vanishing with $\pi^*\omega = \mathrm{d}\alpha$, whose Reeb vector field $Z\in\Gamma^\infty(T\Sigma)$, defined by $\alpha(Z) = 1$ and $\mathrm{i}_Z\,\mathrm{d}\alpha = 0$, has a flow such that (6) is the quotient onto the orbit space with respect to this flow. This manifold Σ is constructed as the holonomy bundle over M for a connection defined on an extension of the frame bundle. It is an $\mathbb{R}$ or $\mathbb{S}^1$ principal bundle over M with connection one-form α.

Then we consider $P = \Sigma\times\mathbb{R}$ with $\mathrm{pr}_1 : P \longrightarrow \Sigma$ being the canonical projection and $\iota : \Sigma \longrightarrow P$ being the embedding of Σ as $\Sigma\times\{0\}$. The coordinate along $\mathbb{R}$ is denoted by s and we set $S = \frac{\partial}{\partial s}\in\Gamma^\infty(TP)$. On P one has the following exact symplectic form:

$$\mu = \mathrm{d}(\mathrm{e}^s\mathrm{pr}_1^*\alpha) = \mathrm{e}^s\mathrm{d}s\wedge\mathrm{pr}_1^*\alpha + \mathrm{e}^s\mathrm{d}\mathrm{pr}_1^*\alpha. \tag{7}$$

Thanks to the Cartesian product structure we can lift vector fields on Σ canonically to P. In particular, the lift E of the Reeb vector field Z (defined by $ds(E) = 0$ and $Tpr_1(E) = Z$) turns out to be Hamiltonian $E = -X_H$ with

$$H = \mathrm{e}^s \in C^\infty(P). \tag{8}$$

We have

$$\mathfrak{L}_S X_H = 0, \quad \mathfrak{L}_S H = H, \quad \text{and} \quad \mathfrak{L}_S \mu = \mu, \tag{9}$$

whence in particular S is a conformally symplectic vector field. Moreover, $\mu(X_H, S) = H$ and we can rewrite μ as

$$\mu = \mathrm{d}H \wedge \mathrm{pr}_1^* \alpha + H \mathrm{pr}^* \omega, \tag{10}$$

where $\mathrm{pr} = \pi \circ \mathrm{pr}_1 : P \longrightarrow M$. Then (M, ω) is the (Marsden–Weinstein) reduced phase space of (P, μ) with respect to the Hamiltonian flow of H at momentum value $H = 1$, since indeed $\Sigma = H^{-1}(\{1\})$ and $\iota^* \mu = \pi^* \omega$ by (10).

Using the contact form α we can lift vector fields $X \in \Gamma^\infty(TM)$ horizontally to vector fields $\overline{X} \in \Gamma^\infty(T\Sigma)$ by the condition

$$T\pi \circ \overline{X} = X \circ \pi \quad \text{and} \quad \alpha(\overline{X}) = 0. \tag{11}$$

Since $\mathrm{d}\alpha = \pi^* \omega$ we have $[\overline{X}, \overline{Y}] = \overline{[X, Y]} - \pi^*(\omega(X, Y))Z$. Using also the canonical lift to P, we can lift $X \in \Gamma^\infty(TM)$ horizontally to $X^{\mathrm{hor}} \in \Gamma^\infty(TP)$, now subject to the conditions

$$T\mathrm{pr} \circ X^{\mathrm{hor}} = X \circ \mathrm{pr} \quad \text{and} \quad \mathrm{pr}_1^* \alpha(X^{\mathrm{hor}}) = 0 = \mathrm{d}s(X^{\mathrm{hor}}). \tag{12}$$

We have

$$\left[X^{\mathrm{hor}}, Y^{\mathrm{hor}}\right] = [X, Y]^{\mathrm{hor}} + \mathrm{pr}^*(\omega(X, Y)) X_H \tag{13}$$

as well as

$$\left[S, X^{\mathrm{hor}}\right] = 0 = \left[X_H, X^{\mathrm{hor}}\right]. \tag{14}$$

We shall speak of "invariance" always with respect to the flow of X_H (or Z on Σ, respectively) and of "homogeneity" always with respect to the conformally symplectic vector field S, e.g., a differential operator D on P is called homogeneous of degree $k \in \mathbb{Z}$ if $[\mathfrak{L}_S, D] = kD$, etc.

Denote the Poisson tensor on M by Λ_M and the one on P by Λ_P, respectively. We can also extend the horizontal lift to bivectors as usual. Since we have curvature, Λ_M^{hor} is no longer a Poisson bivector, instead one finds

$$\left[\!\left[\Lambda_M^{\mathrm{hor}}, \Lambda_M^{\mathrm{hor}}\right]\!\right] = -2\Lambda_M^{\mathrm{hor}} \wedge X_H. \tag{15}$$

From the Definition (7) of the symplectic 2-form μ on P, we have the relation

$$\Lambda_P = \frac{1}{H}\left(\Lambda_M^{\mathrm{hor}} + S \wedge X_H\right). \tag{16}$$

In particular, for $u, v \in C^\infty(M)$ we find for the Poisson brackets

$$\{\mathrm{pr}^* u, \mathrm{pr}^* v\}_P = \frac{1}{H} \mathrm{pr}^* \{u, v\}_M. \tag{17}$$

We are now in the position to define the flat connection ∇ on P by specifying it on horizontal lifts, on S and on X_H. One defines [12]

$$\nabla_{X^{\mathrm{hor}}} Y^{\mathrm{hor}} = \left(\nabla^M_X Y\right)^{\mathrm{hor}} + \frac{1}{2}\mathrm{pr}^*(\omega(X,Y))X_H + \mathrm{pr}^*(t(X,Y))S, \tag{18}$$

$$\nabla_{X^{\mathrm{hor}}} X_H = \nabla_{X_H} X^{\mathrm{hor}} = (\tau(X))^{\mathrm{hor}} - \mathrm{pr}^*(\omega(X,\mathsf{V}))S, \tag{19}$$

$$\nabla_{X^{\mathrm{hor}}} S = \nabla_S X^{\mathrm{hor}} = \frac{1}{2}X^{\mathrm{hor}}, \tag{20}$$

$$\nabla_{X_H} X_H = \mathrm{pr}^*\phi S - \mathsf{V}^{\mathrm{hor}}, \tag{21}$$

$$\nabla_{X_H} S = \nabla_S X_H = \frac{1}{2}X_H, \tag{22}$$

$$\nabla_S S = \frac{1}{2}S, \tag{23}$$

where we used the abbreviations

$$t = \frac{1}{n+1}\,\mathrm{Ric}, \tag{24}$$

$$\mathsf{V} = \frac{2}{(n+1)(2n+1)}\mathsf{U}, \tag{25}$$

$$\phi = \frac{4}{(n+1)(2n+1)}\mathsf{f}, \tag{26}$$

and τ is the endomorphism corresponding to t analogously to (2). In [12], the following statement was obtained:

Theorem 1. *By (18)–(23) a flat symplectic torsion-free connection ∇ is defined on P and ∇ is invariant under X_H and S. Moreover, the third symmetrized covariant derivative of H vanishes.*

Recall that the operator of symmetrized covariant differentiation as a derivation of the symmetric tensor product $\mathsf{D} : \Gamma^\infty(\mathrm{S}^k T^*P) \longrightarrow \Gamma^\infty(\mathrm{S}^{k+1}T^*P)$ is defined by

$$(\mathsf{D}\gamma)(X_1,\ldots,X_{k+1}) = \sum_{\ell=1}^{k+1} \left(\nabla_{X_\ell}\gamma\right)(X_1,\ldots,\overset{\ell}{\wedge},\ldots,X_{k+1}), \tag{27}$$

where $X_1,\ldots,X_{k+1} \in \Gamma^\infty(TP)$ and $\gamma \in \Gamma^\infty(\mathrm{S}^k T^*P)$. Locally, D can be written as

$$\mathsf{D} = \mathrm{d}x^i \vee \nabla_{\frac{\partial}{\partial x^i}}, \tag{28}$$

where $\vee$ denotes the symmetrized tensor product. Then Theorem 1 means

$$\mathsf{D}^3 H = 0. \tag{29}$$

Remark 1. The Ricci-type connection on M is *symmetric* iff $\mathsf{U} = 0$ in which case f turns out to be constant. This particular case has been studied in detail in [5]. In fact, this situation can be specialized to the case where the big space P is an open subset of $\mathbb{R}^{2n+2}$, the flat connection is the usual one, and the Hamiltonian H is a quadratic function. Then depending on the remaining data f, K and ϱ, one obtains as reduced spaces, e.g., the complex projective space $\mathbb{CP}^n$ with its usual Kähler structure, the cotangent bundle of the sphere $\mathbb{S}^n$ and other examples, see the discussion in [12].

2 General Remarks on the Ricci Operator

Before discussing the star products on P and M, respectively, we introduce the following second-order differential operator on a symplectic manifold with symplectic connection, which is also of independent interest. Let (M, ω) be symplectic with a torsion-free symplectic connection ∇ (not necessarily of Ricci-type). Then the Ricci tensor $\mathrm{Ric} \in \Gamma^\infty(\mathrm{S}^2 T^*M)$ can be used to define a "Laplace"-like operator Δ^{Ric} as follows: We denote by $\mathrm{Ric}^\sharp \in \Gamma^\infty(\mathrm{S}^2 TM)$ the symmetric bivector obtained from Ric under the musical isomorphism $\sharp$ with respect to ω.

Definition 1 (Ricci operator). *The Ricci operator* $\Delta^{\mathrm{Ric}} : C^\infty(M) \longrightarrow C^\infty(M)$ *is defined by*

$$\Delta^{\mathrm{Ric}} u = \frac{1}{2} \left\langle \mathrm{Ric}^\sharp, \mathsf{D}^2 u \right\rangle, \tag{30}$$

where $\langle \cdot, \cdot \rangle$ *denotes the natural pairing.*

If locally we write $\mathrm{Ric}^\sharp = \frac{1}{2} \mathrm{Ric}^{ij} \frac{\partial}{\partial x^i} \vee \frac{\partial}{\partial x^j}$, then

$$\Delta^{\mathrm{Ric}} u = \mathrm{Ric}^{ij} \left(\frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial u}{\partial x^k} \right), \tag{31}$$

where Γ^k_{ij} are the local Christoffel symbols of ∇.

Since on a symplectic manifold we have a canonical volume form, the Liouville form Ω, one can ask whether Δ^{Ric} is a symmetric operator with respect to the L^2-inner product on $C^\infty_0(M)$ induced by Ω: in general, this is not the case. However, there is an easy way to correct this. We need to recall some basic features of the global symbol calculus for differential operators on manifolds with connection; see, e.g., [10, 8].

We denote by (q, p) the canonical coordinates on $T^*U \subseteq T^*M$ induced by a local chart (U, x) of M. Then the standard-ordered quantization of a function $f \in \mathrm{Pol}^\bullet(T^*M)$ on the cotangent bundle $\pi : T^*M \longrightarrow M$, which is polynomial in the momenta, is defined by

$$\varrho_{\mathrm{Std}}(f)u = \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{\hbar}{\mathrm{i}}\right)^r \frac{\partial^r f}{\partial p_{i_1} \cdots \partial p_{i_r}}\Big|_{p=0} \mathrm{i_s}\left(\frac{\partial}{\partial x^{i_1}}\right) \cdots \mathrm{i_s}\left(\frac{\partial}{\partial x^{i_r}}\right) \frac{1}{r!} \mathsf{D}^r u, \tag{32}$$

where $u \in C^\infty(M)$ and $\mathrm{i_s}$ denotes the (symmetric) insertion map into the first argument. Clearly, (32) is globally well-defined and does not depend on the coordinates. The constant $\frac{\hbar}{\mathrm{i}}$ can safely be set to 1 in our context;

however, we have included it for the sake of physical interpretation of ϱ_{Std} as a quantization map. Then (32) gives a linear bijection $\varrho_{\mathrm{Std}} : \mathrm{Pol}^\bullet(T^*M) \longrightarrow \mathrm{Diffop}(M)$.

The symmetric algebra $\mathcal{S}^\bullet(TM) = \bigoplus_{r=0}^{\infty} \Gamma^\infty(\mathrm{S}^r TM)$ is canonically isomorphic to the polynomial functions $\mathrm{Pol}^\bullet(T^*M)$ as graded associative algebra via the "universal momentum map" $\mathcal{J}$, determined by $\mathcal{J}(u) = \pi^* u$ and $(\mathcal{J}(X))(\alpha_q) = \alpha_q(X(_q))$ for $u \in \mathcal{S}^0(TM) = C^\infty(M)$ and $X \in \mathcal{S}^1(TM) = \Gamma^\infty(TM)$. Thus, we can rephrase (30) as

$$\Delta^{\mathrm{Ric}} = -\frac{2}{\hbar^2}\, \varrho_{\mathrm{Std}}(\mathcal{J}(\mathrm{Ric}^\sharp)), \tag{33}$$

whence Δ^{Ric} is the standard-ordered quantization of the quadratic function $\mathcal{J}(\mathrm{Ric}^\sharp)$ on T^*M.

The standard-ordered quantization can be seen as a particular case of the κ-ordered quantization which is obtained as follows. On $C^\infty(T^*M)$ one has a Laplace operator Δ_0 arising from the pseudo-Riemannian metric g_0, which is defined by the natural pairing of the vertical and horizontal (with respect to ∇) subspaces of $T(T^*M)$. Locally, Δ_0 is given by

$$\Delta_0\Big|_{T^*U} = \frac{\partial^2}{\partial q^i \partial p_i} + p_k \pi^*(\Gamma^k_{ij}) \frac{\partial^2}{\partial p_i \partial p_j} + \pi^*(\Gamma^i_{ij}) \frac{\partial}{\partial p_j}. \tag{34}$$

On polynomial functions Δ_0 is just the covariant divergence operator [8, Eq. (111)], i.e.,

$$\Delta_0 \mathcal{J}(T) = \mathcal{J}(\mathrm{div}_\nabla T) \tag{35}$$

for $T \in \mathcal{S}^k(TM)$ where with $\alpha_1, \ldots, \alpha_{k-1} \in \Gamma^\infty(T^*M)$

$$(\mathrm{div}_\nabla T)(\alpha_1, \ldots, \alpha_{k-1}) = \mathsf{tr}\left(X \mapsto (\nabla_X T)(\cdot, \alpha_1, \ldots, \alpha_{k-1})\right). \tag{36}$$

Locally, $\mathrm{div}_\nabla = \mathrm{i_s}(\mathrm{d}x^i)\nabla_{\frac{\partial}{\partial x^i}}$. Using Δ_0, the κ-ordered quantization is defined by [10]

$$\varrho_\kappa(f) = \varrho_{\mathrm{Std}}\left(\mathrm{e}^{-\mathrm{i}\kappa\hbar\Delta_0} f\right), \tag{37}$$

where in particular the *Weyl-ordered case* $\varrho_{\mathrm{Weyl}} = \varrho_{\kappa=1/2}$ is of interest for us. In general, we have for the formal adjoint of $\varrho_\kappa(f)$ with respect to Ω

$$\varrho_\kappa(f)^\dagger = \varrho_\kappa\left(\mathrm{e}^{-\mathrm{i}\hbar(1-2\kappa)\Delta_0}\overline{f}\right) \tag{38}$$

whence $\varrho_{\mathrm{Weyl}}(f)^\dagger = \varrho_{\mathrm{Weyl}}(\overline{f})$. Thus, $\varrho_{\mathrm{Weyl}}(\mathcal{J}(\mathrm{Ric}^\sharp))$ gives a symmetric operator. Explicitly, one finds using (35)

$$-\frac{2}{\hbar^2}\, \varrho_\kappa(\mathcal{J}(\mathrm{Ric}^\sharp)) = \Delta^{\mathrm{Ric}} + 2\kappa\, \mathcal{L}_{\mathrm{div}_\nabla \mathrm{Ric}^\sharp} + \kappa^2\, \mathrm{div}^2_\nabla(\mathrm{Ric}^\sharp), \tag{39}$$

since no higher order terms contribute thanks to $\mathrm{Ric}^\sharp \in \mathcal{S}^2(TM)$. Moreover, $\kappa^2\, \mathrm{div}^2_\nabla(\mathrm{Ric}^\sharp)$ is already a multiplication operator with a *real* function and hence symmetric itself for all κ. For $\kappa = \frac{1}{2}$ we have:

Lemma 1. *The operator* $\Delta^{\mathrm{Ric}} + \mathcal{L}_{\mathrm{div}_\nabla \mathrm{Ric}^\sharp} + \frac{1}{4}\mathrm{div}^2_\nabla \mathrm{Ric}^\sharp$ *as well as the operator* $\Delta^{\mathrm{Ric}} + \mathcal{L}_{\mathrm{div}_\nabla \mathrm{Ric}^\sharp}$ *are symmetric.*

We conclude this section with the computation of the covariant divergences of $\mathrm{Ric}^\sharp$ in the case of a Ricci-type connection. In view of equation (3) and the fact that ∇_X commutes with $\sharp$, we have

$$\nabla_X \mathrm{Ric}^\sharp = (\nabla_X \mathrm{Ric})^\sharp = \frac{1}{2n+1}\left(X^\flat \vee \mathsf{U}^\flat\right)^\sharp = \frac{1}{2n+1} X \vee \mathsf{U} \tag{40}$$

whence

$$\mathrm{div}_\nabla \mathrm{Ric}^\sharp = \mathsf{U}. \tag{41}$$

Moreover, from (4) and (5) we get

$$\mathrm{div}^2_\nabla\left(\mathrm{Ric}^\sharp\right) = \mathrm{div}_\nabla \mathsf{U} = -\frac{2n+1}{2(n+1)}K + 2(n+1)f. \tag{42}$$

We can also obtain from this the symmetric version of the Ricci operator, according to Lemma 1.

3 The Weyl–Moyal Star Product

After this excursion on the Ricci operator, we are now back to the situation of Section 1. On the big space P, the symplectic connection ∇ is flat. We have thus a Weyl–Moyal star product on P, explicitly given by

$$f \star g = \sum_{r=0}^{\infty} \frac{1}{r!}\left(\frac{\nu}{2}\right)^r C_r(f,g) \tag{43}$$

for $f, g \in C^\infty(P)[[\nu]]$, where

$$C_r(f,g) = \left\langle \Lambda_P \otimes \cdots \otimes \Lambda_P, \frac{1}{r!}\mathsf{D}^r f \otimes \frac{1}{r!}\mathsf{D}^r g \right\rangle \tag{44}$$

and the natural pairing is done "over cross." Locally we have

$$\begin{aligned} C_r(f,g) = {}& \Lambda_P^{i_1 j_1} \cdots \Lambda_P^{i_r j_r}\, \mathrm{i_s}\left(\frac{\partial}{\partial x^{i_1}}\right) \cdots \mathrm{i_s}\left(\frac{\partial}{\partial x^{i_r}}\right) \frac{1}{r!}\mathsf{D}^r f \\ & \times \mathrm{i_s}\left(\frac{\partial}{\partial x^{j_1}}\right) \cdots \mathrm{i_s}\left(\frac{\partial}{\partial x^{j_r}}\right) \frac{1}{r!}\mathsf{D}^r g. \end{aligned} \tag{45}$$

Since ∇ is flat $\star$ defines an associative law on the space of formal power series in the parameter ν with coefficients in $C^\infty(P)$ [4]. The C_r are bidifferential operators which are of order at most r in each argument, and satisfy the symmetry condition $C_r(u,v) = (-1)^r C_r(v,u)$. These properties are summarized by saying that $\star$ is a natural star product of Weyl-type.

We have now two derivations of $\star$: first it follows directly from the fact that ∇ is S-invariant and $\mathcal{L}_S \Lambda_P = -\Lambda_P$ that

$$\mathcal{E} = \nu \frac{\partial}{\partial \nu} + \mathcal{L}_S \tag{46}$$

is a *ν-Euler derivation*, i.e., a $\mathbb{C}$-linear derivation of $\star$, see e.g. [21]. Second, we consider the quasi-inner derivation $\frac{1}{\nu}\,\mathrm{ad}(H)$. Since $\mathsf{D}^3 H = 0$, only the terms of order ν^1 and ν^2 contribute. But thanks to the Weyl-type symmetry of $\star$, only odd powers of ν occur in commutators, this is immediate from (45). Thus we have

$$\frac{1}{\nu}\,\mathrm{ad}(H) f = \frac{1}{\nu} \nu \{H, f\} = X_H(f), \tag{47}$$

which shows that X_H is a quasi-inner derivation. With other words, $\star$ is *strongly invariant* with respect to the (classical) momentum map H. This strong invariance will allow us to use the phase space reduction also for $\star$ to obtain a star product on M. To this end, we first note the following:

Lemma 2. *There are unique bidifferential operators $\hat{C}_r^{\mathrm{red}}$ on M of order r in each argument such that*

$$\mathrm{pr}^* u \star \mathrm{pr}^* v = \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{\nu}{2}\right)^r \frac{1}{H^r} \mathrm{pr}^* \hat{C}_r^{\mathrm{red}}(u, v). \tag{48}$$

In particular, we have $\hat{C}_1^{\mathrm{red}}(u, v) = \langle \Lambda_M, \mathrm{d}u \otimes \mathrm{d}v \rangle$.

Proof. Clearly, $H^r C_r(\mathrm{pr}^* u, \mathrm{pr}^* v)$ is invariant under X_H and homogeneous of degree 0 under S, hence a pull-back of a function $\hat{C}_r^{\mathrm{red}}(u, v) \in C^\infty(M)$ via pr. This defines $\hat{C}_r^{\mathrm{red}}$ uniquely. The statement on the order of differentiation is straightforward. Finally, $\hat{C}_1^{\mathrm{red}}$ is obtained from (17). $\square$

Remark 2. Though it seems tempting, the operators $\hat{C}_r^{\mathrm{red}}$ do *not* combine into a star product on M directly: the prefactors H^r spoil the associativity as one can show by a direct computation. Hence we will need a slightly more involved reduction.

We will need the second-order term of $H \star f$ with an arbitrary function $f \in C^\infty(P)$. By the symmetry properties of (45) we know that $C_2(H, f) = C_2(f, H)$.

Proposition 1. *Let $f \in C^\infty(P)$, then*

$$\begin{aligned} C_2(f, H) = \frac{1}{H} \Bigg(& - \frac{1}{n+1} \left(\Delta^{\mathrm{Ric}}\right)^{\mathrm{hor}} f - \frac{1}{2} \mathcal{L}_{\mathsf{V}^{\mathrm{hor}}} f - \mathrm{pr}^*(\phi)\, \mathcal{L}_S^2 f \\ & + \left(\mathcal{L}_{\mathsf{V}^{\mathrm{hor}}} \mathcal{L}_S + \mathcal{L}_S \mathcal{L}_{\mathsf{V}^{\mathrm{hor}}}\right) f + \frac{1}{2} \mathcal{L}_{X_H}^2 f \Bigg), \end{aligned} \tag{49}$$

where $\left(\Delta^{\mathrm{Ric}}\right)^{\mathrm{hor}} f$ *is the pairing of the horizontal lift of the* 2*-tensor* $\mathrm{Ric}^{M\sharp}$ *with the second covariant derivative with respect to the flat connection* ∇ *on* P*. Precisely, in a local chart, it is given by*

$$\mathrm{pr}^*(\mathrm{Ric}^{M\sharp ij})[\mathcal{L}_{\partial_i^{\mathrm{hor}}} \mathcal{L}_{\partial_j^{\mathrm{hor}}} f - \mathrm{pr}^*(\Gamma_{ij}^{Mk}) \mathcal{L}_{\partial_k^{\mathrm{hor}}} f + \frac{1}{n+1} \mathrm{pr}^* \mathrm{Ric}_{ij}^M \mathcal{L}_S f]. \quad (50)$$

Observe that $\mathrm{Ric}^{M\sharp ij} \mathrm{Ric}_{ij}^M = -\mathsf{tr}(\rho^2)$*.*

Proof. First we note that $\mathrm{i_s}(X)\mathsf{D}^2 f = 2\nabla_X \mathrm{d}f$ by the very definition of D. Thus using (16) we can compute $C_2(f,g)$ for $f, g \in C^\infty(P)$ explicitly and get

$$\begin{aligned} C_2(f,g) = \frac{1}{H^2}\Bigg(& \mathrm{pr}^*\left(\Lambda_{\mathrm{red}}^{ij}\Lambda_{\mathrm{red}}^{kl}\right)\left(\nabla_{\partial_k^{\mathrm{hor}}}\mathrm{d}f\right)(\partial_i^{\mathrm{hor}})\left(\nabla_{\partial_l^{\mathrm{hor}}}\mathrm{d}g\right)(\partial_j^{\mathrm{hor}}) \\ & + \mathrm{pr}^*\left(\Lambda_{\mathrm{red}}^{kl}\right)\left(\nabla_{\partial_k^{\mathrm{hor}}}\mathrm{d}f\right)(S)\left(\nabla_{\partial_l^{\mathrm{hor}}}\mathrm{d}g\right)(X_H) \\ & + \mathrm{pr}^*\left(\Lambda_{\mathrm{red}}^{kl}\right)\left(\nabla_{\partial_l^{\mathrm{hor}}}\mathrm{d}f\right)(X_H)\left(\nabla_{\partial_k^{\mathrm{hor}}}\mathrm{d}g\right)(S) \\ & + \mathrm{pr}^*\left(\Lambda_{\mathrm{red}}^{ij}\right)(\nabla_S\mathrm{d}f)(\partial_i^{\mathrm{hor}})(\nabla_{X_H}\mathrm{d}g)(\partial_j^{\mathrm{hor}}) \\ & + \mathrm{pr}^*\left(\Lambda_{\mathrm{red}}^{ij}\right)(\nabla_{X_H}\mathrm{d}f)(\partial_j^{\mathrm{hor}})(\nabla_S\mathrm{d}g)(\partial_i^{\mathrm{hor}}) \\ & + (\nabla_S\mathrm{d}f)(S)(\nabla_{X_H}\mathrm{d}g)(X_H) + (\nabla_{X_H}\mathrm{d}f)(X_H)(\nabla_S\mathrm{d}g)(S) \\ & - (\nabla_S\mathrm{d}f)(X_H)(\nabla_{X_H}\mathrm{d}g)(S) - (\nabla_{X_H}\mathrm{d}f)(S)(\nabla_S\mathrm{d}g)(X_H)\Bigg), \end{aligned} \quad (51)$$

where we have used local coordinates on M as well as the horizontal lift according to (12). Next, one computes the second covariant derivatives of H explicitly. One finds

$$(\nabla_{X^{\mathrm{hor}}}\mathrm{d}H)\left(Y^{\mathrm{hor}}\right) = -\mathrm{pr}^*(t(X,Y))H, \quad (52)$$

$$(\nabla_{X^{\mathrm{hor}}}\mathrm{d}H)(X_H) = \mathrm{pr}^*(\omega(X,\mathsf{V}))H = (\nabla_{X_H}\mathrm{d}H)\left(X^{\mathrm{hor}}\right), \quad (53)$$

$$(\nabla_{X^{\mathrm{hor}}}\mathrm{d}H)(S) = 0 = (\nabla_S\mathrm{d}H)\left(X^{\mathrm{hor}}\right), \quad (54)$$

$$(\nabla_{X_H}\mathrm{d}H)(S) = 0 = (\nabla_S\mathrm{d}H)(X_H), \quad (55)$$

$$(\nabla_{X_H}\mathrm{d}H)(X_H) = -\mathrm{pr}^*(\phi)H \quad \text{and} \quad (\nabla_S\mathrm{d}H)(S) = \frac{1}{2}H, \quad (56)$$

where we used $\mathrm{d}H(X_H) = 0$ and $\mathrm{d}H(S) = H$ as well as $\mathrm{d}H(X^{\mathrm{hor}}) = 0$ together with the explicit formulas (18)–(23). Putting things together gives the result thanks to the local form (31) for Δ^{Ric} and $\mathrm{Ric}^{ij} = \Lambda_M^{ik}\Lambda_M^{jl}\mathrm{Ric}_{kl}$. □

Definition 2. *For $f \in C^\infty(P)$ we define the second-order differential operator Δ by*

$$\Delta f = C_2(f, H). \tag{57}$$

As we shall see in the next section, this operator will be crucial for the construction of the reduced star product.

4 Reduction of the Star Product

We come now to the reduction of $\star$. Here we follow essentially the BRST / Koszul approach advocated in [9] which simplifies drastically thanks to the codimension one reduction. Codimension one reductions have also been discussed by Glößner [20,19] and Fedosov[15], while the general case of reduction for coisotropic constraint manifolds is discussed in [7,6,14,13]. We shall briefly recall those aspects of [9] which are needed here.

We first consider the classical situation: the classical Koszul operator $\partial : C^\infty(P) \longrightarrow C^\infty(P)$ is defined by

$$\partial f = f(H-1). \tag{58}$$

In the general case the Koszul complex is built on $C^\infty(P) \otimes \Lambda^\bullet \mathfrak{g}$ if the reduction procedure is done with respect to a group action of some Lie group G with Lie algebra $\mathfrak{g}$ at momentum level 0. In our case $\mathfrak{g} = \mathbb{R}$ is one-dimensional whence the Koszul complex is just $C^\infty(P) \oplus C^\infty(P)$. Next we define the classical homotopy $h : C^\infty(P) \longrightarrow C^\infty(P)$ by

$$hf = \begin{cases} \frac{1}{H-1}\left(f - \mathrm{pr}_1^* \iota^* f\right) & \text{on } P \setminus \iota(\Sigma) \\ \mathcal{L}_S f & \text{on } \iota(\Sigma). \end{cases} \tag{59}$$

An easy argument shows that hf is actually smooth. Then we have the homotopy formula

$$f = \partial h f + \mathrm{pr}_1^* \iota^* f \tag{60}$$

for $f \in C^\infty(P)$ together with the properties

$$\iota^* \partial = 0 \quad \text{and} \quad h \mathrm{pr}_1^* = 0. \tag{61}$$

Moreover, ∂, h, ι^*, and pr_1^* are equivariant with respect to the action of X_H on P and the Reeb vector field on Σ, respectively. The classical *vanishing ideal* $\mathcal{I}_\Sigma$ of Σ is given by

$$\mathcal{I}_\Sigma = \ker \iota^* = \operatorname{im} \partial \tag{62}$$

by (60), and turns out to be a Poisson subalgebra of $C^\infty(P)$ as Σ is coisotropic. Moreover,

$$\mathcal{B}_\Sigma = \left\{ f \in C^\infty(P) \,\middle|\, \{f, \mathcal{I}_\Sigma\} \subseteq \mathcal{I}_\Sigma \right\} \tag{63}$$

is the largest Poisson subalgebra of $C^\infty(P)$ such that $\mathcal{I}_\Sigma \subseteq \mathcal{B}_\Sigma$ is a Poisson ideal. It is well-known and easy to see that $\mathcal{B}_\Sigma$ can also be characterized by

$$\mathcal{B}_\Sigma = \left\{ f \in C^\infty(P) \;\middle|\; \iota^* f \in C^\infty(\Sigma)^Z = \pi^* C^\infty(M) \right\} \tag{64}$$

from which it easily follows that the Poisson algebra $\mathcal{B}_\Sigma / \mathcal{I}_\Sigma$ is isomorphic to the Poisson algebra $C^\infty(M)$ via ι^* and π^*.

We shall now deform the above picture according to [9] where we only use the "Koszul part" of the BRST complex. First we define the quantum Koszul operator $\boldsymbol{\partial} : C^\infty(P)[[\nu]] \longrightarrow C^\infty(P)[[\nu]]$ by

$$\boldsymbol{\partial} f = f \star (H - 1). \tag{65}$$

We set

$$\boldsymbol{\mathcal{I}}_\Sigma = \operatorname{im} \boldsymbol{\partial} = \left\{ f \in C^\infty(P)[[\nu]] \;\middle|\; f = g \star (H-1) \right\}, \tag{66}$$

which is the left ideal generated by $H - 1$ with respect to $\star$. Next we consider

$$\boldsymbol{\mathcal{B}}_\Sigma = \left\{ f \in C^\infty(P)[[\nu]] \;\middle|\; [f, \boldsymbol{\mathcal{I}}_\Sigma]_\star \subseteq \boldsymbol{\mathcal{I}}_\Sigma \right\}, \tag{67}$$

which is the largest subalgebra of $C^\infty(P)[[\nu]]$ such that $\boldsymbol{\mathcal{I}}_\Sigma \subseteq \boldsymbol{\mathcal{B}}_\Sigma$ is a two-sided ideal, the so-called *idealizer* of $\boldsymbol{\mathcal{I}}_\Sigma$. The following simple algebraic lemma is at the core of Bordemann's interpretation of the reduction procedure [6, 7]:

Lemma 3. *Let $\mathcal{A}$ be a unital $\mathbb{k}$-algebra and $\mathcal{I} \subseteq \mathcal{A}$ a left ideal. Let $\mathcal{B} \subseteq \mathcal{A}$ be the idealizer of $\mathcal{I}$ and $\mathcal{A}_{\mathrm{red}} = \mathcal{B}/\mathcal{I}$. Then the left $\mathcal{A}$-module $\mathcal{A}/\mathcal{I}$ becomes a right $\mathcal{A}_{\mathrm{red}}$-module via $[b] : [a] \mapsto [ab]$ for $[b] \in \mathcal{A}_{\mathrm{red}}$ and $[a] \in \mathcal{A}/\mathcal{I}$, such that*

$$\mathcal{A}_{\mathrm{red}} \cong \mathsf{End}_{\mathcal{A}} \left(\mathcal{A}/\mathcal{I} \right)^{\mathrm{opp}}. \tag{68}$$

This way $\mathcal{A}/\mathcal{I}$ becomes a $(\mathcal{A}, \mathcal{A}_{\mathrm{red}})$-bimodule.

In our situation, we want to show that $C^\infty(P)[[\nu]]/\boldsymbol{\mathcal{I}}_\Sigma$ provides a deformation of $C^\infty(\Sigma)$ and $\boldsymbol{\mathcal{B}}_\Sigma/\boldsymbol{\mathcal{I}}_\Sigma$ induces a star product $\star_{\mathrm{red}}$ on M. This can be done in great generality, in our situation the arguments simplify thanks to the codimension one case.

We define the quantum homotopy

$$\boldsymbol{h} = h \left(\mathsf{id} + (\boldsymbol{\partial} - \partial) h \right)^{-1} \tag{69}$$

and the quantum restriction map

$$\boldsymbol{\iota^*} = \iota^* \left(\mathsf{id} + (\boldsymbol{\partial} - \partial) h \right)^{-1}, \tag{70}$$

which are clearly well-defined since $\boldsymbol{\partial} - \partial$ is at least of order λ and thus $\mathsf{id} + (\boldsymbol{\partial} - \partial) h$ is invertible by a geometric series. From (60) we immediately find

$$f = \boldsymbol{\partial}\, \boldsymbol{h} f + \mathrm{pr}_1^* \boldsymbol{\iota^*} f \tag{71}$$

for all $f \in C^\infty(P)[[\nu]]$. Moreover, we still have the relations

$$\boldsymbol{\iota}^* \mathrm{pr}_1^* = \mathsf{id}_{C^\infty(\Sigma)[[\nu]]} \quad \text{and} \quad \boldsymbol{h} \mathrm{pr}_1^* = 0, \tag{72}$$

as in the classical case. Finally, all maps are still equivariant with respect to the flow of X_H and Z, respectively.

Proposition 2. *The function $C^\infty(\Sigma)[[\nu]]$ becomes a left $\star$-module via*

$$f \bullet \psi = \boldsymbol{\iota}^*(f \star \mathrm{pr}_1^* \psi). \tag{73}$$

This module structure is isomorphic to the module structure of $C^\infty(P)[[\nu]]/\boldsymbol{\mathcal{I}}_\Sigma$ via pr_1^* and $\boldsymbol{\iota}^*$.

Proof. Since $\mathrm{im}\, \boldsymbol{\partial} = \ker \boldsymbol{\iota}^*$ by (71), the quantum restriction map $\boldsymbol{\iota}^*$ induces a linear bijection $C^\infty(P)[[\nu]]/\boldsymbol{\mathcal{I}}_\Sigma \longrightarrow C^\infty(\Sigma)[[\nu]]$ whose inverse is induced by pr_1^* by (72). Then (73) is just the pulled-back module structure. □

From the strong invariance of the star product $\star$ we obtain the following characterization of $\boldsymbol{\mathcal{B}}_\Sigma$:

Lemma 4. $\boldsymbol{\mathcal{B}}_\Sigma = \left\{ f \in C^\infty(P)[[\nu]] \;\middle|\; \boldsymbol{\iota}^* f \in \pi^* C^\infty(M)[[\nu]] \right\}$.

Proof. Indeed, on one hand we have $Z \boldsymbol{\iota}^* f = -\nu \boldsymbol{\iota}^* \,\mathrm{ad}(H) f$ by equivariance of $\boldsymbol{\iota}^*$. Since for $f \in \boldsymbol{\mathcal{B}}_\Sigma$ we have $\mathrm{ad}(H) f \in \boldsymbol{\mathcal{I}}_\Sigma$, this gives $\boldsymbol{\iota}^* f = 0$ by (71). Conversely, $\boldsymbol{\iota}^* f = 0$ implies $\mathrm{ad}(H) f \in \boldsymbol{\mathcal{I}}_\Sigma$ and hence $(H-1) \star f \in \boldsymbol{\mathcal{I}}_\Sigma$ from which $\boldsymbol{\mathcal{I}}_\Sigma \star f \subseteq \boldsymbol{\mathcal{I}}_\Sigma$ follows. But this implies $f \in \boldsymbol{\mathcal{B}}_\Sigma$. □

Theorem 2. *The quotient $\boldsymbol{\mathcal{B}}_\Sigma/\boldsymbol{\mathcal{I}}_\Sigma$ is $\mathbb{C}[[\nu]]$-linearly isomorphic to $C^\infty(M)$ $[[\nu]]$ via*

$$C^\infty(M)[[\nu]] \ni u \;\mapsto\; [\mathrm{pr}^* u] \in \boldsymbol{\mathcal{B}}_\Sigma/\boldsymbol{\mathcal{I}}_\Sigma \tag{74}$$

with inverse induced by

$$\boldsymbol{\mathcal{B}}_\Sigma/\boldsymbol{\mathcal{I}}_\Sigma \ni [f] \;\mapsto\; \boldsymbol{\iota}^* f \in \pi^* C^\infty(M)[[\nu]]. \tag{75}$$

This induces a deformed product $\star_{\mathrm{red}}$ for $C^\infty(M)[[\nu]]$ via

$$\pi^*(u \star_{\mathrm{red}} v) = \boldsymbol{\iota}^*(\mathrm{pr}^* u \star \mathrm{pr}^* v), \tag{76}$$

which turns out to be a differential star product quantizing $\{\cdot,\cdot\}_M$. Finally, the bimodule structure of $C^\infty(\Sigma)[[\nu]]$ according to Lemma 3 and Proposition 2 is given by

$$\psi \bullet u = \boldsymbol{\iota}^*(\mathrm{pr}_1^* \psi \star \mathrm{pr}^* u). \tag{77}$$

Proof. For $u \in C^\infty(M)[[\nu]]$ we clearly have $\boldsymbol{\iota}^* \mathrm{pr}^* u = \boldsymbol{\iota}^* \mathrm{pr}_1^* \pi^* u = \pi^* u \in \pi^* C^\infty(M)[[\nu]]$ whence $\mathrm{pr}^* u \in \boldsymbol{\mathcal{B}}_\Sigma$ by Lemma 4 and (74) is well-defined. Since $\boldsymbol{\mathcal{I}}_\Sigma = \ker \boldsymbol{\iota}^*$, by Lemma 4 it follows that (75) is well-defined and injective. Clearly, (74) and (75) are mutually inverse by (72) whence $\star_{\mathrm{red}}$ is an associative $\mathbb{C}[[\nu]]$-bilinear product for $C^\infty(M)[[\nu]]$. It can be shown [9, Lem. 27] that there exists a formal series $S = \mathsf{id} + \sum_{r=1}^{\infty} \nu^r S_r$ of differential operators S_r on P such that $\boldsymbol{\iota}^* = \iota^* \circ S$, from which it easily follows that $\star_{\mathrm{red}}$ is bidifferential. Finally, computing the first orders of

$$u \star_{\mathrm{red}} v = \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{\nu}{2}\right)^r C_r^{\mathrm{red}}(u, v) \tag{78}$$

explicitly gives $C_0^{\mathrm{red}}(u, v) = uv$ and $C_1^{\mathrm{red}}(u, v) = \{u, v\}_M$, whence $\star_{\mathrm{red}}$ is a star product quantizing the correct Poisson bracket. The last statement is clear by construction. □

Up to now we just followed the general reduction scheme from [9] which simplifies for the codimension one case, see also [20, 19]. Let us now bring our more specific features into the game:

Lemma 5. *Let $f \in C^\infty(P)^{X_H}[[\nu]]$ be invariant. Then*

$$(\boldsymbol{\partial} - \partial) f = \frac{\nu^2}{8} \Delta f. \tag{79}$$

The operator Δ is invariant.

Proof. By invariance of f we only have the second-order term in the right multiplication by $H-1$, which was computed in Proposition 1. The invariance of Δ follows from the strong invariance of $\star$ by

$$\begin{aligned} X_H(\Delta f) &= X_H\left(f \star H - \frac{\nu}{2} C_1(f, H) - fH\right) \\ &= \frac{1}{\nu}\,\mathrm{ad}(H)(f \star H) + \frac{\nu}{2} X_H(X_H(f)) - X_H(f)H \\ &= \frac{1}{\nu}(\mathrm{ad}(H)(f)) \star H - \frac{\nu}{2} C_1(X_H(f), H) - X_H(f)H \\ &= \Delta(X_H(f)). \end{aligned}$$

□

Lemma 6. *Let $f \in C^\infty(P)^{X_H}[[\nu]]$ be invariant. Then*

$$\boldsymbol{\iota}^* f = \iota^* \left(\mathsf{id} + \frac{\nu^2}{8} \Delta h\right)^{-1} f. \tag{80}$$

Proof. Since h and Δ preserve invariance this follows by induction from the last lemma. □

Proposition 3. *Let $u, v \in C^\infty(M)[[\nu]]$. Then $u \star_{\mathrm{red}} v$ is determined by*

$$\pi^*(u \star_{\mathrm{red}} v) = \iota^* \left(\sum_{r=0}^{\infty} \nu^r \sum_{2s+t=r} \frac{1}{(-8)^s 2^t t!} (\Delta h)^s \left(\frac{1}{H^t} \mathrm{pr}^* \hat{C}_t^{\mathrm{red}}(u, v) \right) \right). \tag{81}$$

Proof. This is a simple consequence of Lemma 6 together with Lemma 2. □

From this formula we see that we have to proceed in two steps in order to compute the true bidifferential operators C_r^{red} out of the operators $\hat{C}_r^{\mathrm{red}}$: first we have to control Δ and h applied to functions of the form $\frac{1}{H^t}\mathrm{pr}^* u$. Second, the application of ι^* simply sets $H = 1$ and gives $\iota^* \mathrm{pr}^* = \pi^*$, whence this part can be considered to be trivial.

Lemma 7. *Let $u \in C^\infty(M)$ and $k \in \mathbb{N}$. Then*

$$h\left(\frac{1}{H^k}\mathrm{pr}^* u\right) = -\left(\frac{1}{H} + \cdots + \frac{1}{H^k}\right)\mathrm{pr}^* u. \tag{82}$$

Proof. On $P \setminus \iota(\Sigma)$ we have from (59)

$$\begin{aligned} h\left(\frac{1}{H^k}\mathrm{pr}^* u\right) &= \frac{1}{H-1}\left(\frac{1}{H^k}\mathrm{pr}^* u - \mathrm{pr}_1^* \iota^*\left(\frac{1}{H^k}\mathrm{pr}^* u\right)\right) \\ &= \frac{1}{H-1}\left(\frac{1}{H^k} - 1\right)\mathrm{pr}^* u \\ &= -\left(\frac{1}{H} + \cdots + \frac{1}{H^k}\right)\mathrm{pr}^* u. \end{aligned}$$

By continuity this extends also to $\iota(\Sigma)$. □

In particular, applying the homotopy h to a linear combination of functions of the form $\frac{1}{H^k}\mathrm{pr}^* u$ gives again such a linear combination.

Lemma 8. *Let $u \in C^\infty(M)$ and $k \in \mathbb{N}$. Then*

$$\begin{aligned} \Delta\left(\frac{1}{H^k}\mathrm{pr}^* u\right) = -\frac{1}{n+1}\frac{1}{H^{k+1}}\mathrm{pr}^*\Big(& \Delta^{\mathrm{Ric}}\, u \\ & + \frac{4k+1}{2n+1}\mathcal{L}_{\mathsf{U}}\, u + \frac{k}{n+1}\mathsf{tr}(\varrho^2) u + \frac{4k^2}{2n+1}\mathsf{f} u\Big). \end{aligned} \tag{83}$$

Proof. This follows from $\mathcal{L}_{X_H} H = 0 = \mathcal{L}_{X^{\mathrm{hor}}} H$ and $\mathcal{L}_S H = H$ together with $\mathcal{L}_{X^{\mathrm{hor}}} \mathrm{pr}^* u = \mathrm{pr}^* \mathcal{L}_X u$ and $\mathcal{L}_{X_H} \mathrm{pr}^* u = 0 = \mathcal{L}_S \mathrm{pr}^* u$ as well as Proposition 1. □

Again, we see that applying Δ to this particular class of functions reproduces such linear combinations, though, of course, the combinatorics gets involved. In contrast to (82), the function u is differentiated in (83).

Remark 3. From the above two lemmas the star product $\star_{\mathrm{red}}$ can be computed by Proposition 3 in all orders. However, the explicit evaluation of the iteration $(\Delta h)^s$ seems to be tricky: the combinatorics gets quite involved, even in the case, where ∇^M is symmetric, i.e., $\mathsf{U} = 0$ and $\mathsf{f} = \mathit{const}$, $\mathsf{tr}(\varrho^2) = \mathit{const}$. In this particular case only the Ricci operator Δ^{Ric} has to be applied successively to the $\hat{C}_r^{\mathrm{red}}$ to produce the C_r^{red}, including, of course, still some nontrivial combinatorics.

5 Properties of the Reduced Star Product

In this section we collect some further properties of the reduced star product.

Proposition 4. *The star product $\star_{\mathrm{red}}$ is of Weyl-type, i.e., the bidifferential operators C_r^{red} satisfy*

$$\overline{C_r^{\mathrm{red}}(u,v)} = C_r^{\mathrm{red}}(\overline{u},\overline{v}) \tag{84}$$

and

$$C_r^{\mathrm{red}}(u,v) = (-1)^r C_r^{\mathrm{red}}(v,u). \tag{85}$$

In particular, complex conjugation becomes a *-involution

$$\overline{u \star_{\mathrm{red}} v} = \overline{v} \star_{\mathrm{red}} \overline{u}, \tag{86}$$

where $\overline{\nu} = -\nu$ by definition. Moreover, $\star_{\mathrm{red}}$ is natural in the sense of [22], i.e., C_r^{red} is a bidifferential operator of order r in each argument.

Proof. The operators $\hat{C}_r^{\mathrm{red}}$ clearly satisfy (84) and (85). From (78) it follows that the operators C_r^{red} are real operators, i.e., they satisfy (84), since Δ and h are real. Moreover, by (81), only the operators $\hat{C}_{r-2s}^{\mathrm{red}}$ with $0 \leq 2s \leq r$ contribute to C_r^{red} whence C_r^{red} also satisfy (85) as $2s$ is even. Then (86) follows. Moreover, $\hat{C}_r^{\mathrm{red}}$ is a differential operator of order r in each argument. Since the application of Δh to $\frac{1}{H^k}\mathrm{pr}^* u$ by Lemmas 7 and 8 gives a second-order differential operator on the part $\mathrm{pr}^* u$, we conclude that C_r^{red} is of order r in each argument, too. $\square$

In a next step, we explicitly compute the second-order term C_2^{red} of $\star_{\mathrm{red}}$. According to [22], the second-order term of a natural star product determines uniquely a symplectic connection: in our case, we reproduce the Ricci-type connection ∇^M.

Proposition 5. *Let $u, v \in C^\infty(M)[[\nu]]$. Then*

$$C_2^{\mathrm{red}}(u,v) = \left\langle \Lambda_{\mathrm{red}} \otimes \Lambda_{\mathrm{red}}, \frac{1}{2}\mathsf{D}_M^2 u \otimes \frac{1}{2}\mathsf{D}_M^2 v \right\rangle + \frac{2}{n+1}\left\langle \mathrm{Ric}^\sharp, \mathrm{d}u \otimes \mathrm{d}v \right\rangle. \tag{87}$$

In particular, the symplectic connection determined by $\star_{\mathrm{red}}$ is ∇^M.

Proof. From (81) we see that

$$\begin{aligned}
\pi^* C_2^{\mathrm{red}}(u,v) &= 8\iota^* \left(\sum_{2s+t=2} \frac{1}{(-8)^s 2^t t!} (\Delta h)^s \left(\frac{1}{H^t} \mathrm{pr}^* \hat{C}_2^{\mathrm{red}}(u,v) \right) \right) \\
&= \iota^* \left(\frac{1}{H^2} \mathrm{pr}^* \hat{C}_2^{\mathrm{red}}(u,v) - \Delta h \mathrm{pr}^* \hat{C}_0^{\mathrm{red}}(u,v) \right) \\
&= \pi^* \hat{C}_2^{\mathrm{red}}(u,v),
\end{aligned}$$

since $h\mathrm{pr}^* = 0$ by the very definition (59) of h. Thus $C_2^{\mathrm{red}} = \hat{C}_2^{\mathrm{red}}$ in this order of ν. The corrections to the terms $\hat{C}_r^{\mathrm{red}}$ start only in order $r \geq 3$. To compute $\hat{C}_2^{\mathrm{red}}$ we need the second covariant derivatives $\nabla \mathrm{d}\mathrm{pr}^* u$ of pull-backs $\mathrm{pr}^* u$. Here we obtain

$$\left(\nabla_{X^{\mathrm{hor}}} \mathrm{d}\mathrm{pr}^* u\right)\left(Y^{\mathrm{hor}}\right) = \mathrm{pr}^*\left(\left(\nabla^M_X \mathrm{d}u\right)(Y)\right), \tag{88}$$

$$\left(\nabla_{X^{\mathrm{hor}}} \mathrm{d}\mathrm{pr}^* u\right)(X_H) = -\mathrm{pr}^*\left(\mathrm{d}u(\tau X)\right) = \left(\nabla_{X_H} \mathrm{d}\mathrm{pr}^* u\right)\left(X^{\mathrm{hor}}\right), \tag{89}$$

$$\left(\nabla_{X^{\mathrm{hor}}} \mathrm{d}\mathrm{pr}^* u\right)(S) = -\frac{1}{2}\mathrm{pr}^*\left(\mathrm{d}u(X)\right) = \left(\nabla_S \mathrm{d}\mathrm{pr}^* u\right)\left(X^{\mathrm{hor}}\right), \tag{90}$$

$$\left(\nabla_{X_H} \mathrm{d}\mathrm{pr}^* u\right)(X_H) = \mathrm{pr}^*\left(\mathrm{d}u(\mathsf{V})\right), \tag{91}$$

$$\left(\nabla_{X_H} \mathrm{d}\mathrm{pr}^* u\right)(S) = \left(\nabla_S \mathrm{d}\mathrm{pr}^* u\right)(X_H) = \left(\nabla_S \mathrm{d}\mathrm{pr}^* u\right)(S) = 0. \tag{92}$$

Inserting this into the general expression (51) for C_2 and using $\mathrm{pr}^* \hat{C}_2^{\mathrm{red}}(u, v) = H^2 C_2(\mathrm{pr}^* u, \mathrm{pr}^* v)$ gives the result (87). From this, the last statement follows directly as the star product $\star_{\mathrm{red}}$ is of Weyl type and the only second-order terms in C_2^{red} are described by using ∇^M, see [22, Prop. 3.1]. □

References

1. Abraham, R., Marsden, J.E.: Foundations of Mechanics, 2nd edn. Addison-Wesley, MA (1985)
2. Baguis, P., Cahen, M.: A construction of symplectic connections through reduction. Lett. Math. Phys. **57**, 149–160 (2001)
3. Baguis, P., Cahen, M.: Marsden-Weinstein reduction for symplectic connections. Bull. Belg. Math. Soc. Simon Stevin **10**(1), 91–100 (2003)
4. Bayen, F., Flato, M., Frønsdal, C., Lichnerowicz, A., Sternheimer, D.: Deformation theory and quantization. Ann. Phys. **111**, 61–151 (1978)
5. Bieliavsky, P., Cahen, M., Gutt, S., Rawnsley, J., Schwachhöfer, L.: Symplectic connections. Int. J. Geomet. Meth. Mod. Phys. **3**(3), 375–420 (2006)
6. Bordemann, M.: (Bi)Modules, morphismes et réduction des star-produits: le cas symplectique, feuilletages et obstructions. Preprint `math.QA/0403334`, 135 (2004)
7. Bordemann, M.: (Bi)Modules, morphisms, and reduction of star-products: the symplectic case, foliations, and obstructions. Trav. Math. **16**, 9–40 (2005)
8. Bordemann, M., Neumaier, N., Waldmann, S.: Homogeneous Fedosov star products on cotangent bundles I: Weyl and standard ordering with differential operator representation. Comm. Math. Phys. **198**, 363–396 (1998)
9. Bordemann, M., Herbig, H.-C., Waldmann, S.: BRST cohomology and phase space reduction in deformation quantization. Comm. Math. Phys. **210**, 107–144 (2000)
10. Bordemann, M., Neumaier, N., Pflaum, M.J., Waldmann, S.: On representations of star product algebras over cotangent spaces on Hermitian line bundles. J. Funct. Anal. **199**, 1–47 (2003)

11. Cahen, M., Schwachhöfer, L.J.: Special symplectic connections and Poisson geometry. Lett. Math. Phys. **69**, 115–137 (2004)
12. Cahen, M., Gutt, S., Schwachhöfer, L.: Construction of Ricci-type connections by reduction and induction. In: Marsden, J.E., Ratiu, T.S. (eds.) The Breadth of Symplectic and Poisson Geometry. Progress in Mathematics, vol. 232, pp. 41–57. Birkhäuser Boston, MA (2005); Festschrift in honor of Alan Weinstein
13. Cattaneo, A.S., Felder, G.: Coisotropic submanifolds in Poisson geometry and Branes in the Poisson sigma model. Lett. Math. Phys. **69**, 157–175 (2004)
14. Cattaneo, A.S., Felder, G.: Relative formality theorem and quantisation of coisotropic submanifolds. Adv. Math. **208**(2), 521–548 (2007)
15. Fedosov, B.V.: Reduction and eigenstates in deformation quantization. In: Demuth, M., Schrohe, E., Schulze, B.-W. (eds.) Pseudo-Differential Calculus and Mathematical Physics. Advances in Partial Differential Equations, vol. 5, pp. 277–297. Akademie, Berlin (1994)
16. Fedosov, B.V.: Non-abelian reduction in deformation quantization. Lett. Math. Phys. **43**, 137–154 (1998)
17. Fisch, J., Henneaux, M., Stasheff, J., Teitelboim, C.: Existence, uniqueness and cohomology of the classical BRST charge with ghosts of ghosts. Comm. Math. Phys. **120**, 379–407 (1989)
18. Gerstenhaber, M.: On the deformation of rings and algebras. Ann. Math. **79**, 59–103 (1964)
19. Glößner, P.: Phasenraumreduktion von Sternprodukten für superauflösbare Constraintalgebren. PhD thesis, Fakultät für Physik, Albert-Ludwigs-Universität, Freiburg (1998)
20. Glößner, P.: Star product reduction for coisotropic submanifolds of codimension 1. Preprint Freiburg FR-THEP-98/10 `math.QA/9805049` (1998)
21. Gutt, S., Rawnsley, J.: Equivalence of star products on a symplectic manifold; an introduction to Deligne's Čech cohomology classes. J. Geom. Phys. **29**, 347–392 (1999)
22. Gutt, S., Rawnsley, J.: Natural star products on symplectic manifolds and quantum moment maps. Lett. Math. Phys. **66**, 123–139 (2003)
23. Vaisman, I.: Symplectic curvature tensors. Monatsh. Math. **100**(4), 299–327 (1985)

Effective Batalin–Vilkovisky Theories, Equivariant Configuration Spaces and Cyclic Chains

Alberto S. Cattaneo[1,*] and Giovanni Felder[2]

[1] Institut für Mathematik, Universität Zürich-Irchel, Winterthurerstrasse 190, 8057 Zürich, Switzerland, alberto.cattaneo@math.uzh.ch
[2] Department of Mathematics, ETH Zürich, 8092 Zürich, Switzerland, giovanni.felder@math.ethz.ch

Dedicated to Murray Gerstenhaber and Jim Stasheff

Abstract. Kontsevich's formality theorem states that the differential graded Lie algebra of multidifferential operators on a manifold M is L_∞-quasi-isomorphic to its cohomology. The construction of the L_∞-map is given in terms of integrals of differential forms on configuration spaces of points in the upper half-plane. Here we consider configuration spaces of points in the disk and work equivariantly with respect to the rotation group. This leads to considering the differential graded Lie algebra of multivector fields endowed with a divergence operator. In the case of $\mathbb{R}^d$ with standard volume form, we obtain an L_∞-morphism of modules over this differential graded Lie algebra from cyclic chains of the algebra of functions to multivector fields. As a first application we give a construction of traces on algebras of functions with star-products associated with unimodular Poisson structures. The construction is based on the Batalin–Vilkovisky quantization of the Poisson sigma model on the disk and in particular on the treatment of its zero modes.

Key words: Poisson sigma model, BV quantization, Effective action, L_∞ algebras, Cyclic homology, Unimodular Poisson structures

Mathematics Subject Classification 2010: Primary 53D55; Secondary 81T40, 81T45, 81T70

1 Introduction

The Hochschild complex of any algebra with unit carries a differential graded Lie algebra structure introduced by Gerstenhaber [14]. In the case of the algebra of smooth functions on a manifold, one has a differential graded Lie subalgebra $\mathfrak{g}_G$ of multidifferential operators, whose cohomology is the

A.S. Cattaneo et al. (eds.), *Higher Structures in Geometry and Physics*, Progress in Mathematics 287, DOI 10.1007/978-0-8176-4735-3_6,

graded Lie algebra $\mathfrak{g}_S$ of multivector fields with Schouten–Nijenhuis bracket.[1] Kontsevich [17] showed that $\mathfrak{g}_G$ and $\mathfrak{g}_S$ are quasi-isomorphic as L_∞-algebras, a notion introduced by Stasheff as the Lie version of A_∞-algebras [28], see [24, 20]. A striking application of this result is the classification of formal associative deformations of the product of functions in terms of Poisson structures. Kontsevich's L_∞-quasi-isomorphism is given in terms of integrals over configuration spaces of points in the upper half-plane. As shown in [3], these are Feynman amplitudes of a topological quantum field theory known as the Poisson sigma model [16, 23].

In this paper we consider the case of a manifold M endowed with a volume form Ω. In this case $\mathfrak{g}_S$ comes with a differential, the divergence operator div_Ω of degree -1. One considers then the differential graded Lie algebra $\mathfrak{g}_S^\Omega = \mathfrak{g}_S[v]$ where v is an indeterminate of degree 2, the bracket is extended by v-linearity and the differential is $v\,\mathrm{div}_\Omega$. The relevant topological quantum field theory is a BF theory (or Poisson sigma model with trivial Poisson structure) on a disk whose differential is the Cartan differential on S^1-equivariant differential forms. This theory is described in Section 2. The new feature, compared to the original setting of Kontsevich's formality theorem, is that zero modes are present. We use recent ideas of Losev, Costello and Mnev to treat them in the Batalin–Vilkovisky quantization scheme. This gives the physical setting from which the Feynman amplitudes are derived. In the remaining sections of this paper, which can be read independently of Section 2, we give a purely mathematical treatment of the same objects. The basic result is the construction for $M = \mathbb{R}^d$ of an L_∞-morphism of $\mathfrak{g}_S^\Omega$-modules from the module of negative cyclic chains $(C_{-\bullet}(A)[u], b + uB)$ to the trivial module $(\Gamma(\wedge^{-\bullet} TM), \mathrm{div}_\Omega)$. We also check that this L_∞-morphism has properties needed to extend the result to general manifolds.

As in the case of Kontsevich's theorem, the coefficients of the L_∞-morphism are integrals of differential forms on configuration spaces. Whereas Kontsevich considers the spaces of n-tuples of points in the upper half-plane modulo the action of the group of dilations and horizontal translations, we consider the space of n-tuples of points in the unit disk and work equivariantly with respect to the action of the rotation group. The quadratic identities defining the L_∞-relations are then proved by means of an equivariant version of the Stokes theorem.

As a first application we construct traces in deformation quantization associated with unimodular Poisson structures. Our construction can also be extended to the case of supermanifolds; the trace is then replaced by a nondegenerate cyclic cocycle (Calabi–Yau structure, see [18], Section 10.2, and [10]) for the A_∞-algebra obtained by deformation quantization in [5]. Further applications will be studied in a separate publication [6]. In particular we will derive the existence of an L_∞-quasi-isomorphism of $\mathfrak{g}_S^\Omega$-modules from

[1] We use Tsygan's notation [30]. Kontsevich's notation [17] is $\mathcal{D}_{\mathrm{poly}} = \mathfrak{g}_G$, $\mathcal{T}_{\mathrm{poly}} = \mathfrak{g}_S$.

the complex $\mathfrak{g}_S^{\Omega}$ with the adjoint action to the complex of cyclic cochains with a suitable module structure. This is a module version of the Kontsevich–Shoikhet formality conjecture for cyclic cochains [26].

Notations and conventions

All vector spaces are over $\mathbb{R}$. We denote by S_n the group of permutations of n letters and by $\epsilon\colon S_n \to \{\pm 1\}$ the sign character. We write $|\alpha|$ for the degree of a homogeneous element α of a $\mathbb{Z}$-graded vector space. The sign rules for tensor products of graded vector spaces hold: if f and g are linear maps on graded vector spaces, $(f\otimes g)(v\otimes w) = (-1)^{|g|\cdot|v|} f(v)\otimes g(w)$. The graded vector space $V[n]$ is V shifted by n: $V[n]^i = V^{n+i}$. There is a canonical map (the identity) $s^n\colon V[n] \to V$ of degree n. The graded symmetric algebra $S^\bullet V = \oplus_{n\geq 0} S^n V$ of a graded vector space V is the algebra generated by V with relations $a \cdot b = (-1)^{|a|\cdot|b|} b \cdot a$, $a, b \in V$; the degree of a product of generators is the sum of the degrees. If $\sigma \in S_n$ is a permutation and $a_1, \dots, a_n \in V$, then $a_{\sigma(1)} \cdots a_{\sigma(n)} = \epsilon a_1 \cdots a_n$; we call $\epsilon = \epsilon(\sigma; a_1, \dots, a_n)$ the Koszul sign of σ and a_i. The exterior algebra $\bigwedge V$ is defined by the relations $a\wedge b = -(-1)^{|a|\cdot|b|} b\wedge a$ on generators. We have a linear isomorphism $S^n(V[1]) \to (\wedge^n V)[n]$ given by $v_1 \cdots v_n \mapsto s^{-n}(-1)^{\sum (n-j)(|v_j|-1)} s v_1 \wedge \cdots \wedge s v_n$, $v_j \in V[1]$.

2 BV formalism and zero modes

This section provides the interested reader with some "physical" motivation for the constructions in this paper. It may be safely skipped by the reader who is only interested in the construction and not in its motivation.

The basic idea is to use the Batalin–Vilkovisky (BV) formalism in order to deal with theories with symmetries (like the Poisson sigma model). What is interesting for this paper is the case when "zero modes" are present.

It is well known in algebraic topology that structures may be induced on subcomplexes (in particular, on an embedding of the cohomology) like, e.g., induced differentials in spectral sequences or Massey products. It is also well known in physics that low-energy effective field theories may be induced by integrating out high-energy degrees of freedom. As observed by Losev [21] (and further developed by Mnev [22] and Costello [9]), the two things are actually related in terms of the BV approach to (topological) field theories. We are interested in the limiting case when the low-energy fields are just the zero modes, i.e., the critical points of the action functional modulo its symmetries.

Let $\mathcal{M}$ be an SP-manifold, i.e., a graded manifold endowed with a symplectic form of degree -1 and a compatible Berezinian [25]. Let Δ be the corresponding BV-Laplace operator. The compatibility amounts to saying that Δ squares to zero and that it generates the BV bracket $(\ ,\ )$ (i.e., the Poisson bracket of degree 1 determined by the symplectic structure of degree -1): namely,

$$\Delta(AB) = (\Delta A)B + (-1)^{|A|} A\Delta B - (-1)^{|A|}(A, B). \tag{1}$$

Assume now that $\mathcal{M}$ is actually a product of SP-manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$, with BV-Laplace operators Δ_1 and Δ_2, $\Delta = \Delta_1 + \Delta_2$. The central observation is that for every Lagrangian submanifold $\mathcal{L}$ of $\mathcal{M}_2$ and any function F on $\mathcal{M}$ – for which the integral makes sense – one has

$$\Delta_1 \int_{\mathcal{L}} F = \int_{\mathcal{L}} \Delta F. \tag{2}$$

In infinite dimensions, where we would really like to work, this formula is very formal as both the integration and Δ are ill-defined. In finite dimensions, on the other hand, this is just a simple generalization of the fact that, for any differential form α on the Cartesian product of two manifolds M_1 and M_2 and any closed submanifold S of M_2 on which the integral of α converges, we have

$$\mathrm{d} \int_S \alpha = \pm \int_S \mathrm{d}\alpha,$$

where integration on S yields a differential form on M_1. The correspondence with the BV language is obtained by taking $\mathcal{M}_{1,2} := T^*[-1]M_{1,2}$ and $\mathcal{L} := N^*[-1]S$ (where N^* denotes the conormal bundle). The Berezinian on $\mathcal{M}$ is determined by a volume form $v = v_1 \wedge v_2$ on $M := M_1 \times M_2$, with v_i a volume form on M_i. Finally, Δ is $\phi_v^{-1} \circ \mathrm{d} \circ \phi_v$, with $\phi_v \colon \Gamma(\wedge^\bullet TM) \to \Omega^{\dim M - \bullet}(M)$, $X \mapsto \phi_v(X) := \iota_X v$. The generalization consists in the fact that there are Lagrangian submanifolds of $\mathcal{M}_2$ not of the form of a conormal bundle; however, by a result of Schwarz [25], they can always be brought to this form by a symplectomorphism so that formula (2) holds in general.

In the application we have in mind, $\mathcal{M}_2$ (and so $\mathcal{M}$) is infinite-dimensional, but $\mathcal{M}_1$ is not. Thus, we have a well-defined BV-Laplace operator Δ_1 and try to make sense of Δ by imposing (2), following ideas of [21,22] and, in particular, [9]. More precisely, we consider "BF-like" theories. Namely, let $(\mathcal{V}, \delta)$ and $(\tilde{\mathcal{V}}, \delta)$ be complexes with a nondegenerate pairing $\langle\ ,\ \rangle$ of degree -1 which relates the two differentials:

$$\langle \mathsf{B}, \delta\mathsf{A} \rangle = \langle \delta\mathsf{B}, \mathsf{A} \rangle, \qquad \forall \mathsf{A} \in \mathcal{V},\ \mathsf{B} \in \tilde{\mathcal{V}}. \tag{3}$$

We set $\mathcal{M} = \mathcal{V} \oplus \tilde{\mathcal{V}}$ and define $S \in C^\infty(\mathcal{M})$ as

$$S(\mathsf{A}, \mathsf{B}) := \langle \mathsf{B}, \delta\mathsf{A} \rangle. \tag{4}$$

The pairing defines a symplectic structure of degree -1 on $\mathcal{M}$ and the BV bracket with S is δ. In particular,

$$(S, S) = 0. \tag{5}$$

We denote by $\mathcal{H}$ $(\tilde{\mathcal{H}})$ the δ-cohomology of $\mathcal{V}$ $(\tilde{\mathcal{V}})$. Then we choose an embedding of $\mathcal{M}_1 := \mathcal{H} \oplus \tilde{\mathcal{H}}$ into $\mathcal{M}$ and a complement $\mathcal{M}_2$.

Example 1. Take $\mathcal{V} = \Omega(\Sigma)[1]$ and $\tilde{\mathcal{V}} = \Omega(\Sigma)[s-2]$, with Σ a closed, compact s-manifold, and $\delta = \mathrm{d}$, the de Rham differential, on $\mathcal{V}$; up to a sign, δ on $\tilde{\mathcal{V}}$ is also the de Rham differential if the pairing is defined by integration: $\langle\, \mathsf{B}\,,\, \mathsf{A}\,\rangle := \int_\Sigma \mathsf{B}\wedge\mathsf{A}$, $\mathsf{A}\in\mathcal{V}$, $\mathsf{B}\in\tilde{\mathcal{V}}$. In this case $\mathcal{M}_1 = H(\Sigma)[1]\oplus H(\Sigma)[s-2]$, with $H(\Sigma)$ the usual de Rham cohomology. A slightly more general situation occurs when Σ has a boundary; in this case, appropriate boundary conditions have to be chosen so that δ has an adjoint as in (3). Let $\partial\Sigma = \partial_1\Sigma \sqcup \partial_2\Sigma$ (each of the boundary components $\partial_{1,2}\Sigma$ may be empty). We then choose $\mathcal{V} = \Omega(\Sigma,\partial_1\Sigma)[1]$ and $\tilde{\mathcal{V}} = \Omega(\Sigma,\partial_2\Sigma)[s-2]$, where $\Omega(\Sigma,\partial_i\Sigma)$ denotes differential forms whose restrictions to $\partial_i\Sigma$ vanish. In this case, $\mathcal{M}_1 = H(\Sigma,\partial_1\Sigma)[1]\oplus H(\Sigma,\partial_2\Sigma)[s-2]$.

Example 2. Suppose that S^1 acts on Σ (and that the $\partial_i\Sigma$s are invariant). Let $\mathbf{v}$ denote the vector field on Σ generating the infinitesimal action. Let $\Omega_{S^1}(\Sigma,\partial\Sigma) := \Omega(\Sigma,\partial\Sigma)^{S^1}[u]$ denote the Cartan complex with differential $\mathrm{d}_{S^1} = \mathrm{d} - u\iota_{\mathbf{v}}$, where u is an indeterminate of degree 2. Then we may generalize Example 1 replacing $\Omega(\Sigma,\partial\Sigma)$ with $\Omega_{S^1}(\Sigma,\partial\Sigma)$.

Now suppose that $\mathcal{H}$ (and so $\tilde{\mathcal{H}}$) is finite-dimensional, as in the examples above. In this case it is always possible to choose a BV-Laplacian Δ_1 on $\mathcal{M}_1$. Once and for all we also choose a Lagrangian submanifold $\mathcal{L}$ on which the infinite-dimensional integral makes sense in perturbation theory. Assuming $\Delta S = 0$, the first consequence of (2) and (5) is that the partition function

$$Z_0 = \int_{\mathcal{L}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}$$

is Δ_1-closed. Actually, in the case at hand, Z_0 is constant on $\mathcal{M}_1$.

For every functional $\mathcal{O}$ on $\mathcal{M}$ for which integration on $\mathcal{L}$ makes sense, we define the expectation value

$$\langle\, \mathcal{O}\,\rangle_0 := \frac{\int_{\mathcal{L}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\,\mathcal{O}}{Z_0}.$$

The second consequence of (2), and of the fact that Z_0 is constant on $\mathcal{M}_1$, is the Ward identity

$$\Delta_1\langle\, \mathcal{O}\,\rangle_0 = \left\langle\, \Delta\mathcal{O} - \frac{\mathrm{i}}{\hbar}\delta\mathcal{O}\,\right\rangle_0, \tag{6}$$

where we have also used (1).

To interpret the Ward identity for $\mathcal{O} = \mathsf{B}\otimes\mathsf{A}$, we denote by $\{\theta^\mu\}$ a linear coordinate system on $\mathcal{H}$ and by $\{\zeta_\mu\}$ a linear coordinate system on $\tilde{\mathcal{H}}$, such that their union is a Darboux system for the symplectic form on $\mathcal{M}_1$ with $\Delta_1 = \frac{\partial}{\partial\theta^\mu}\frac{\partial}{\partial\zeta_\mu}$. We next write $\mathsf{A} = \alpha_\mu\theta^\mu + \mathsf{a}$ and $\mathsf{B} = \beta^\mu\zeta_\mu + \mathsf{b}$ with $\mathsf{a}\oplus\mathsf{b}\in\mathcal{M}_2$. The left-hand side of the Ward identity is now simply $\Delta_1\langle\, \mathsf{B}\otimes\mathsf{A}\,\rangle_0 = \sum_\mu (-1)^{|\beta^\mu|}\beta^\mu\otimes\alpha_\mu =: \phi$. On the assumption that the ill-defined BV-Laplacian Δ should be a second-order differential operator, the first term $\langle\, \mathit{\Delta}(\mathsf{B} \otimes \mathsf{A})\,\rangle_0$ on the right-hand side is ill-defined but constant on $\mathcal{M}_1$; we denote it by K. Since δ vanishes in cohomology and, as a differential operator, it can be extracted from the expectation value, (6) yields a constraint for the propagator

$$\omega := \frac{\mathrm{i}}{\hbar}\langle\, \mathsf{b} \otimes \mathsf{a}\,\rangle_0; \tag{7}$$

namely,

$$\delta\omega = K - \phi.$$

From now on we assume that $\mathcal{M}$ is defined in terms of differential forms as in Examples 1 and 2. In this case, ω is a distributional $(s-1)$-form on $\Sigma \times \Sigma$ while ϕ is a representative of the Poincaré dual of the diagonal D_Σ in $\Sigma \times \Sigma$. By the usual naive definition of $\mathit{\Delta}$, K is equal to the delta distribution on D_Σ. Thus, the restriction of ω to the configuration space $C_2(\Sigma) := \Sigma \times \Sigma \setminus D_\Sigma$ is a smooth $(m-1)$-form satisfying $\mathrm{d}\omega = \phi$. If Σ has a boundary, ω satisfies in addition the conditions $\iota_1^*\omega = \iota_2^*\omega = 0$ with ι_1 the inclusion of $\Sigma \times \partial_1\Sigma$ into $\Sigma \times \Sigma$ and ι_2 the inclusion of $\partial_2\Sigma \times \Sigma$ into $\Sigma \times \Sigma$. Denoting by $\pi_{1,2}$ the two projections $\Sigma \times \Sigma \to \Sigma$ and by $\pi_*^{1,2}$ the corresponding fiber-integrations, we may define $P\colon \Omega(\Sigma, \partial_1\Sigma) \to \Omega(\Sigma, \partial_1\Sigma)$ and $\tilde{P}\colon \Omega(\Sigma, \partial_2\Sigma) \to \Omega(\Sigma, \partial_2\Sigma)$ by $P(\sigma) = \pi_*^2(\omega \wedge \pi_1^*\sigma)$ and $\tilde{P}(\sigma) = \pi_*^1(\omega \wedge \pi_2^*\sigma)$. Then the equation for ω implies that P and $\tilde{P}$ are parametrices for the complexes $\Omega(\Sigma, \partial_1\Sigma)$ and $\Omega(\Sigma, \partial_2\Sigma)$; namely, $\mathrm{d}P + P\mathrm{d} = 1 - \varpi$ and $\mathrm{d}\tilde{P} + \tilde{P}\mathrm{d} = 1 - \tilde{\varpi}$, where ϖ and $\tilde{\varpi}$ denote the projections onto cohomology.

This characterization of the propagator of a "BF-like" theory also appears in [9]. Even though not justified in terms of the BV formalism, this choice of propagator was done before in [2] for Chern–Simons theory out of purely topological reasons, and later extended to BF theories in [7]. A propagator with these properties also appears in [13] for the Poisson sigma model on the interior of a polygon.

The quadratic action (4) is usually the starting point for a perturbative expansion. The first singularity that may occur comes from evaluating ω on D_Σ ("tadpole"). A mild form of renormalization consists in removing tadpoles or, in other words, in imposing that ω should vanish on D_Σ. By consistency, one has then to set K equal to the restriction of ϕ to D_Σ. In other words, one has to impose

$$\mathit{\Delta}(\mathsf{B}(x)\mathsf{A}(x)) = \psi(x) := \sum_\mu (-1)^{|\beta^\mu|}\beta^\mu(x)\alpha_\mu(x), \qquad \forall x \in \Sigma. \tag{8}$$

Observe that ψ is a representative of the Euler class of Σ. By (1) and (8) one then obtains a well-defined version of $\mathit{\Delta}$ on the algebra $C^\infty(\mathcal{M})'$ generated by local functionals. This may be regarded as an asymptotic version (for the energy scale going to zero) of Costello's regularized BV-Laplacian [9]. Actually,

Lemma 1. $(C^\infty(\mathcal{M})', \mathit{\Delta})$ *is a BV algebra.*

We now restrict ourselves to the setting of the Poisson sigma model [16,23]. Namely, we assume Σ to be two-dimensional and take $\mathcal{V} = \Omega(\Sigma, \partial_1\Sigma)[1] \otimes (\mathbb{R}^m)^*$ and $\tilde{\mathcal{V}} = \Omega(\Sigma, \partial_2\Sigma) \otimes \mathbb{R}^m$. Here $(\mathbb{R}^m)^* \times \mathbb{R}^m$ is a local patch of the cotangent bundle of an m-dimensional target manifold M. (Whatever we say here and in the following may be globalized by taking $\mathcal{M}$ to be the graded submanifold of $\mathrm{Map}(T[1]\Sigma, T^*[1]M)$ defined by the given boundary conditions.) There is a Lie algebra morphism from the graded Lie algebra $\mathfrak{g}_S = \Gamma(\wedge^{\bullet+1} TM)$ of multivector fields on M to $C^\infty(\mathcal{M})'$ endowed with the BV bracket [4]: to $\gamma \in \Gamma(\wedge^k TM)$ it associates the local functional

$$S_\gamma = \int_\Sigma \gamma^{i_1,\dots,i_k}(\mathsf{B})\,\mathsf{A}_{i_1}\cdots\mathsf{A}_{i_k}.$$

Moreover, for $k > 0$, $(S, S_\gamma) = 0$. With the regularized version of the BV-Laplacian, we get

$$\Delta S_\gamma = \int_\Sigma \psi\,(\mathrm{div}_\Omega\,\gamma)^{i_1,\dots,i_{k-1}}(\mathsf{B})\,\mathsf{A}_{i_1}\cdots\mathsf{A}_{i_{k-1}},$$

where div_Ω is the divergence with respect to the constant volume form Ω on $\mathbb{R}^n$. To account for this systematically, we introduce the differential graded Lie algebra $\mathfrak{g}_S^\Omega := \mathfrak{g}_S[v]$, where v is an indeterminate of degree two and the differential δ_Ω is defined as $v\,\mathrm{div}_\Omega$ (and the Lie bracket is extended by v-linearity). To $\gamma \in \Gamma(\wedge^k TM)\,v^l$ we associate the local functional

$$S_\gamma = (-\mathrm{i}\hbar)^l \int_\Sigma \psi^l\,\gamma^{i_1,\dots,i_k}(\mathsf{B})\,\mathsf{A}_{i_1}\cdots\mathsf{A}_{i_k}.$$

It is now not difficult to prove the following

Lemma 2. *The map $\gamma \mapsto S_\gamma$ is a morphism of differential graded Lie algebras from $(\mathfrak{g}_S^\Omega, [\ ,\ ], \delta_\Omega)$ to $(C^\infty(\mathcal{M})', (\ ,\ ), -\mathrm{i}\hbar\Delta)$. Moreover, for every $\gamma \in \Gamma(\wedge^k TM)\,v^l$ with k or l strictly positive, we have $(S, S_\gamma) = 0$. If $\partial\Sigma = \emptyset$, the last statement holds also for $k = l = 0$.*

Observe that $\psi^2 = 0$ by dimensional reasons. However, in the generalization to the equivariant setting of Example 2, higher powers of ψ survive.

A first application of this formalism is the Poisson sigma model on Σ. If π is a Poisson bivector field (i.e., $\pi \in \Gamma(\wedge^2 TM)$, $[\pi,\pi] = 0$), then $\mathsf{S}_\pi := S + S_\pi$ satisfies the master equation $(\mathsf{S}_\pi, \mathsf{S}_\pi) = 0$ but in general not the quantum master equation $\frac{1}{2}(\mathsf{S}_\pi, \mathsf{S}_\pi) + \mathrm{i}\hbar\Delta\mathsf{S}_\pi = 0$, which by (1) is equivalent to $\Delta \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathsf{S}_\pi} = 0$. Unless ψ is trivial[2] (which is, e.g., the case for Σ the upper half

[2] If the class of ψ is trivial, one may always choose bases in the embedded cohomologies so that $\psi = 0$. If one does not want to make this choice, one observes anyway that for $\psi = \mathrm{d}\tau$ one has $-\mathrm{i}\hbar\Delta S_\gamma = (S, S'_\gamma)$ with $S'_\gamma = (-\mathrm{i}\hbar)^l \int_\Sigma \tau\psi^{l-1}\gamma^{i_1,\dots,i_k}(\mathsf{B})\,\mathsf{A}_{i_1}\cdots\mathsf{A}_{i_k}$, for $\gamma \in \Gamma(\wedge^k TM)\,v^l$, $l > 0$. In the case at hand, one may then define a solution of the quantum master equation as $S + S_\pi + S'_\pi$.

plane, as in [3], or the torus), this actually happens only if π is divergence free. More generally, if π is unimodular [19], by definition we may find a function f such that $\mathrm{div}_\Omega\,\pi = [\pi, f]$. So $\tilde\pi := \pi + vf$ is a Maurer–Cartan element in $\mathfrak{g}_S^\Omega$ (i.e., $\delta_\Omega\tilde\pi - \frac{1}{2}[\tilde\pi,\tilde\pi] = 0$). Hence $\mathsf{S}_{\tilde\pi} := S + S_{\tilde\pi}$ satisfies the quantum master equation. It is not difficult to check that, for ψ nontrivial, the unimodularity of π is a necessary and sufficient condition for having a solution of the quantum master equation of the form $S + S_\pi + O(\hbar)$. For Σ the sphere this was already observed in [1] though using slightly different arguments.

We will now restrict ourselves to the case of interest for the rest of the paper: namely, Σ the disk and $\partial_2\Sigma = \emptyset$. In this case $H(\Sigma)$ is one-dimensional and concentrated in degree 0 while $H(\Sigma, \partial\Sigma)$ is one-dimensional and concentrated in degree two. Thus, $\mathcal{H} = (\mathbb{R}^m)^*[-1]$ and $\mathcal{H} = \mathbb{R}^m$ which implies $\mathcal{M}_1 = T^*[-1]M$. Functions on $\mathcal{M}_1$ are then multivector fields on M but with reversed degree and the operator Δ_1 turns out to be the usual divergence operator div_Ω (which is now of degree $+1$) for the constant volume form. A first simple application is the expectation value

$$\mathrm{tr}\, g := \frac{\int_{\mathcal{L}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathsf{S}_{\tilde\pi}}\, \mathcal{O}_g}{Z_0} = \left\langle\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_{\tilde\pi}}\, \mathcal{O}_g \,\right\rangle_0, \qquad g \in C^\infty(M),$$

where $\tilde\pi$ is a Maurer–Cartan element corresponding to a unimodular Poisson structure and $O_g(\mathsf{A},\mathsf{B}) := g(\mathsf{B}(1))$, with 1 in $\partial\Sigma$ which we identify with the unit circle. Consider now $\mathrm{tr}_2(g,h) := \left\langle\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_{\tilde\pi}}\, \mathcal{O}_{g,h} \,\right\rangle_0$, with $\mathcal{O}_{g,h} := g(\mathsf{B}(1)) \int_{\partial\Sigma\setminus\{1\}} h(\mathsf{B})$. By (1), we then have $\Delta_1\, \mathrm{tr}_2(g,h) = \left\langle\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_{\tilde\pi}}\, \delta\mathcal{O}_{g,h} \,\right\rangle_0$. Arguing as in [3], we see that the right-hand side corresponds to moving the two functions g and h close to each other (in the two possible ways) and by "bubbling" the disk around them; so we get

$$\Delta_1\, \mathrm{tr}_2(g,h) = \mathrm{tr}\, g \star h - \mathrm{tr}\, h \star g,$$

where $\star$ is Kontsevich's star product [17] which corresponds to the Poisson sigma model on the upper half plane [3]. Since Δ_1 is just the divergence operator with respect to the constant volume form Ω, for compactly supported functions we have the trace

$$\mathrm{Tr}\, g := \int_M \mathrm{tr}\, g\ \Omega.$$

More generally, we may work out the Ward identities relative to the quadratic action (4) (there is also an equivariant version for S^1 acting by rotations on Σ). Given $a_0, a_1, \ldots, a_p$ in $C^\infty(M)$ (or in $C^\infty(M)[u]$ for the equivariant version), we define

$$\mathcal{O}_{a_0,\ldots,a_p} := a_0(\mathsf{B}(1)) \int_{t_1<t_2<\cdots<t_p\in\partial\Sigma\setminus\{1\}} a_1(\mathsf{B})\cdots a_p(\mathsf{B})$$

and

$$G_n(\gamma_1,\dots,\gamma_n;a_0,\dots,a_p) := \Big\langle\, S_{\gamma_1}\dots S_{\gamma_n}\,\mathcal{O}_{a_0,\dots,a_p}\,\Big\rangle_0,$$

$\gamma_i \in \mathfrak{g}_S^{\Omega}$, $i=1,\dots,n$. By (6) we then have

$$\begin{aligned}-\mathrm{i}\hbar\Delta_1 G_n(\gamma_1,\dots,\gamma_n;a_0,\dots,a_p) = -\mathrm{i}\hbar\Big\langle\, \Delta(S_{\gamma_1}\dots S_{\gamma_n}\,\mathcal{O}_{a_0,\dots,a_p})\,\Big\rangle_0 + \\ +\Big\langle\, \delta(S_{\gamma_1}\dots S_{\gamma_n}\,\mathcal{O}_{a_0,\dots,a_p})\,\Big\rangle_0.\end{aligned}$$

The left-hand side is just $(-\mathrm{i}\hbar)$ times the divergence operator applied to the multivector field G_n. The first term on the right-hand side may then be computed as

$$\begin{aligned}&-\mathrm{i}\hbar\Big\langle\, \Delta(S_{\gamma_1}\dots S_{\gamma_n}\,\mathcal{O}_{a_0,\dots,a_p})\,\Big\rangle_0 = \\ &\quad=\sum_{i=1}^{n}(-1)^{\sigma_i}G_n(\gamma_1,\dots,\delta_\Omega\gamma_i,\dots,\gamma_n;a_0,\dots,a_p) + \\ &\quad-\mathrm{i}\hbar\sum_{1\le i<j\le n}(-1)^{\sigma_{ij}}G_{n-1}([\gamma_i,\gamma_j],\gamma_1,\dots,\hat{\gamma}_i,\dots,\hat{\gamma}_j,\dots,\gamma_n;a_0,\dots,a_p),\end{aligned}$$

where the caret denotes omission and

$$\begin{aligned}\sigma_i &:= \sum_{c=1}^{i-1}|\gamma_c|,\\ \sigma_{ij} &:= |\gamma_i|\sum_{c=1}^{i-1}|\gamma_c| + |\gamma_j|\sum_{c=1,\,c\neq i}^{j-1}|\gamma_c| + |\gamma_i| + 1,\end{aligned}$$

with $|\gamma| = k$ for $\gamma \in \Gamma(\wedge^k TM)[v]$. The second term on the right-hand side is a boundary contribution (in the equivariant sense if $\delta = \mathrm{d}_{S^1} = \mathrm{d} - u\iota_{\mathbf{v}}$). By bubbling as in [3], some of the γ_is collapse together with some of the consecutive a_ks and the result – which is Kontsevich's formality map – is put back into G. The whole formula can then be interpreted as an L_∞-morphism from the cyclic Hochschild complex to the complex of multivector fields regarded as L_∞-modules over $\mathfrak{g}_S^{\Omega}$, as we are going to explain in the rest of the paper.

The only final remark is that $\mathrm{i}\hbar$ occurs in this formula only as a bookkeeping device. We define F_n by formally setting $\mathrm{i}\hbar = 1$ in G_n.

3 Hochschild chains and cochains of algebras of smooth functions

Kontsevich's theorem states that there is an L_∞-quasi-isomorphism from the graded Lie algebra $\mathfrak{g}_S = \Gamma(\wedge^{\bullet+1} TM)$ of multivector fields on a smooth manifold M, with the Schouten–Nijenhuis bracket and trivial differential, to the differential graded Lie algebra $\mathfrak{g}_G$ of multidifferential operators on M with Gerstenhaber bracket and Hochschild differential. Through Kontsevich's morphism the Hochschild and cyclic chains become a module over $\mathfrak{g}_S$. In this section we review these notions as well as results and conjectures about them.

3.1 Multivector fields and multidifferential operators

Let $\mathfrak{g}_S$ be the graded vector space $\mathfrak{g}_S = \oplus_{j\geq -1}\mathfrak{g}_S^j$ of multivector fields: $\mathfrak{g}_S^{-1} = C^\infty(M), \mathfrak{g}_S^0 = \Gamma(TM), \mathfrak{g}_S^1 = \Gamma(\wedge^2 TM)$, and so on. The Schouten–Nijenhuis bracket of multivector fields is defined to be the usual Lie bracket on vector fields and is extended to arbitrary multivector field by the Leibniz rule: $[\alpha\wedge\beta,\gamma] = \alpha\wedge[\beta,\gamma] + (-1)^{|\gamma|\cdot(|\beta|+1)}[\alpha,\gamma]\wedge\beta,\ \alpha,\beta,\gamma\in\mathfrak{g}_S$. The graded Lie algebra $\mathfrak{g}_S$ is considered here as a differential graded Lie algebra with trivial differential.

The differential graded Lie algebra $\mathfrak{g}_G$ of multidifferential operators is, as a complex, the subcomplex of the shifted Hochschild complex $\mathrm{Hom}(A^{\otimes(\bullet+1)}, A)$ of the algebra $A = C^\infty(M)$ of smooth functions, consisting of multilinear maps that are differential operators in each argument. The Gerstenhaber bracket [14] on $\mathfrak{g}_G$ is the graded Lie bracket $[\phi,\psi] = \phi\bullet_G\psi - (-1)^{|\phi|\cdot|\psi|}\psi\bullet_G\phi$ with Gerstenhaber product[3]

$$\phi\bullet_G\psi = \sum_{k=0}^{n}(-1)^{|\psi|(|\phi|-k)}\phi\circ(\mathrm{id}^{\otimes k}\otimes\psi\otimes\mathrm{id}^{\otimes|\phi|-k}). \tag{9}$$

The Hochschild differential can be written in terms of the bracket as $[\mu,\cdot]$, where $\mu\in\mathfrak{g}_G^1 = \mathrm{Hom}(A\otimes A, A)$ is the multiplication in A.

The Hochschild–Kostant–Rosenberg map $\mathfrak{g}_S^\bullet \to \mathfrak{g}_G^\bullet$ induces an isomorphism of graded Lie algebras on cohomology. It is the identity on $\mathfrak{g}_S^{-1} = C^\infty(M) = \mathfrak{g}_G^{-1}$ and, for any vector fields $\xi_1,\dots,\xi_n$, it sends the multivector field $\xi_1\wedge\dots\wedge\xi_n$ to the multidifferential operator

$$f_1\otimes\dots\otimes f_n \mapsto \frac{1}{n!}\sum_{\sigma\in S_n}\epsilon(\sigma)\xi_{\sigma(1)}(f_1)\cdots\xi_{\sigma(n)}(f_n),\quad f_i\in A.$$

Although the HKR map is a chain map inducing a Lie algebra isomorphism on cohomology, it does not respect the Lie bracket at the level of complexes. The correct point of view on this problem was provided by Kontsevich in his formality conjecture, which he then proved in [17]. The differential graded Lie algebras $\mathfrak{g}_S$, $\mathfrak{g}_G$ should be considered as L_∞-algebras and the HKR map is the first component of an L_∞-morphism. Let us recall the definitions.

3.2 L_∞-algebras

For any graded vector space V let $S^+V = \oplus_{j=1}^\infty S^jV$ be the free coalgebra without counit cogenerated by V. The coproduct is $\Delta(a_1\cdots a_n) = \sum_{p=1}^{n-1}\sum_\sigma \pm a_{\sigma(1)}\cdots a_{\sigma(p)}\otimes a_{\sigma(p+1)}\cdots a_{\sigma(n)}$, with summation over shuffle permutations with Koszul signs. A coderivation of a coalgebra is an

[3] The sign differs by a factor $(-1)^{|\phi|\cdot|\psi|}$ from the sign in [14]. We have chosen the convention making the induced bracket on cohomology equal to the standard Schouten–Nijenhuis bracket on multivector fields.

endomorphism D obeying $\Delta \circ D = (D \otimes \mathrm{id} + \mathrm{id} \otimes D) \circ \Delta$. Coderivations with the commutator bracket form a Lie algebra. What is special about the free coalgebra S^+V is that for any linear map $D\colon S^+V \to V$ there is a unique coderivation $\tilde{D}$ such that $D = \pi \circ \tilde{D}$, where π is the projection onto $V = S^1V$. By definition an L_∞-algebra is a graded vector space $\mathfrak{g}$ together with a coderivation D of degree 1 of $S^+(\mathfrak{g}[1])$ obeying $[D, D] = 0$. A coderivation is thus given by a sequence of maps (the Taylor components) $D_n\colon S^n\mathfrak{g}[1] \to \mathfrak{g}[2]$ (or $\wedge^n\mathfrak{g} \to \mathfrak{g}[2-n]$), $n = 1, 2, \dots$, obeying quadratic relations. In particular D_1 is a differential and D_2 is a chain map obeying the Jacobi identity up to a homotopy D_3. It follows that D_2 induces a Lie bracket on the D_1-cohomology. Differential graded Lie algebras are L_∞-algebras with $D_3 = D_4 = \cdots = 0$. An L_∞-morphism $(\mathfrak{g}, D) \rightsquigarrow (\mathfrak{g}', D')$ is a homomorphism $U\colon S^+\mathfrak{g}[1] \to S^+\mathfrak{g}'[1]$ of graded coalgebras such that $U \circ D = D' \circ U$. Homomorphisms of free coalgebras are uniquely defined by their composition with the projection $\pi'\colon S^+\mathfrak{g}'[1] \to \mathfrak{g}'[1]$; thus U is uniquely determined by its Taylor components $U_n\colon S^n\mathfrak{g}[1] \to \mathfrak{g}'[1]$ (or $\wedge^n\mathfrak{g} \to \mathfrak{g}'[1-n]$): U_n is the restriction to $S^n\mathfrak{g}[1]$ of $\pi' \circ U$. Conversely, any such sequence U_n comes from a coalgebra homomorphism. The first relation between D, D' and U is that U_1 is a chain map.

Theorem 1. *(Kontsevich [17]) There is an L_∞-morphism $\mathfrak{g}_S(M) \rightsquigarrow \mathfrak{g}_G(M)$ whose first Taylor component U_1 is the Hochschild–Kostant–Rosenberg map.*

If M is an open subset of $\mathbb{R}^d$ the formula for the Taylor components U_n is explicitly given in [17] as a sum over Feynman graphs.

3.3 Multivector fields and differential forms

The algebra $\Omega^\bullet(M)$ of differential forms on a manifold M is a module over the differential graded Lie algebra $\mathfrak{g}_S(M)$ of multivector fields: a multivector field $\gamma \in \Gamma(\wedge^{p+1}TM)$ acts on forms as $\mathcal{L}_\gamma\omega = \mathrm{d}\iota_\gamma + (-1)^p\iota_\gamma d$ generalizing Cartan's formula for Lie derivatives of vector fields. Here d is the de Rham differential and the interior multiplication ι_γ is the usual multiplication if γ is a function and is the composition of interior multiplications of vector fields ξ_j if $\gamma = \xi_1 \wedge \cdots \wedge \xi_k$. Moreover the action of $\mathfrak{g}_S(M)$ on $\Omega^\bullet(M)$ commutes with the de Rham differential and induces the trivial action on cohomology.

3.4 Hochschild cochains and cyclic chains

The algebras $\Omega^\bullet(M)$ and $H^\bullet(M)$ are cohomologies of the complexes of the Hochschild and of the periodic cyclic chains of $C^\infty(M)$. The normalized Hochschild chain complex of a unital algebra A is $C_\bullet(A) = A \otimes \bar{A}^{\otimes\bullet}$, where $\bar{A} = A/\mathbb{R}1$. If we denote by $(a_0, a_1, \dots, a_p)$ the class of $a_0 \otimes \cdots \otimes a_p$ in $C_p(A)$, the Hochschild differential is

$$b(a_0,\dots,a_p)=\sum_{i=0}^{p-1}(-1)^i(a_0,\dots,a_ia_{i+1},\dots,a_p)$$
$$+(-1)^p(a_pa_0,a_1,\dots,a_{p-1}).$$

We set $C_p(A)=0$ for $p<0$. There is an HKR map $C_\bullet(A)\to\Omega^\bullet(M)$ given by

$$(a_0,\dots,a_p)\mapsto\frac{1}{p!}a_0\mathrm{d}a_1\cdots\mathrm{d}a_p. \tag{10}$$

It is a chain map if we consider differential forms as a complex with *trivial* differential. The HKR map induces an isomorphism on homology, provided we take a suitable completion of the tensor product $C^\infty(M)^{\otimes(p+1)}$, for example the jets at the diagonal of smooth maps $M^{p+1}\to\mathbb{R}$. On the Hochschild chain complex there is a second differential B of degree 1 and anticommuting with b, see [8]:

$$B(a_0,\dots,a_p)=\sum_{i=0}^{p}(-1)^{ip}(1,a_i,\dots,a_p,a_0,\dots a_{i-1}).$$

The negative cyclic complex, in the formulation of [15], is $CC^-_{-\bullet}(A)=C_{-\bullet}(A)[u]$ with differential $b+uB$, where u is of degree 2. The extension of the HKR map by $\mathbb{R}[u]$-linearity defines a quasi-isomorphism

$$(CC^-_{-\bullet}(A),b+uB)\to(\Omega^{-\bullet}(M)[u],u\,d).$$

Now both $C(A)$ and $CC^-(A)$ are differential graded modules over the Lie algebra $\mathfrak{g}_G$ of multidifferential operators. The action is the restriction of the action of cochains on chains $C^k(A)\otimes C_p(A)\to C_{p-k+1}(A)$, $\phi\otimes a\mapsto\phi\cdot a$, defined for any associative algebra with unit as

$$\begin{aligned}&(-1)^{(k-1)(p+1)}\phi\cdot(a_0,\dots,a_p)\\&=\sum_{i=0}^{p-k+1}(-1)^{i(k-1)}(a_0,\dots,a_{i-1},\phi(a_i,\dots,a_{i+k-1}),a_{i+k},\dots,a_p)\\&\quad+\sum_{i=p-k+2}^{p}(-1)^{ip}(\phi(a_i,\dots,a_p,a_0,\dots,a_{i+k-p-2}),a_{i+k-p-1},\dots,a_{i-1}).\end{aligned}$$

This action extends by $\mathbb{R}[u]$-linearity to an action on the negative cyclic complex.

3.5 L_∞-modules

Let $(\mathfrak{g},D)$ be an L_∞-algebra. The free $S^+\mathfrak{g}[1]$-comodule generated by a vector space V is $\hat V=S\mathfrak{g}[1]\otimes V$ with coaction $\Delta^V:\hat V\to S^+\mathfrak{g}[1]\otimes\hat V$ defined as

$$\Delta^V(\gamma_1\cdots\gamma_n\otimes v)=\sum_{p=1}^{n}\sum_{\sigma\in S_{p,n-p}}\pm\gamma_{\sigma(1)}\cdots\gamma_{\sigma(p)}\otimes(\gamma_{\sigma(p+1)}\cdots\gamma_{\sigma(n)}\otimes v).$$

A coderivation of the L_∞-module V is then an endomorphism D^V of $\hat{V}$ obeying $\Delta^V \circ D^V = (D \otimes \mathrm{id} + \mathrm{id} \otimes D^V)\Delta^V$. An L_∞-module is a coderivation D^V of degree 1 of $\hat{V}$ obeying $D^V \circ D^V = 0$. A coderivation is uniquely determined by its composition with the projection $\hat{V} \to V$ onto the first direct summand and is thus given by its Taylor components $D_n^V : S^n\mathfrak{g}[1] \otimes V \to V[1]$. The lowest component D_0^V is then a differential on V and D_1^V a chain map inducing an honest action of the Lie algebra $H(\mathfrak{g}, D_1)$ on the cohomology $H(V, D_0^V)$. A morphism of L_∞-modules $V_1 \to V_2$ over $\mathfrak{g}$ is a degree 0 morphism of $S^+\mathfrak{g}[1]$-comodules $F\colon \hat{V}_1 \to \hat{V}_2$ intertwining the coderivations. The composition with the projection $\hat{V}_2 \to V_2$ gives rise to Taylor components

$$F_n\colon S^n\mathfrak{g}[1] \otimes V_1 \to V_2, \qquad n = 0, 1, 2, \ldots$$

that determine F completely. The lowest component F_0 is then a chain map inducing a morphism of $H(\mathfrak{g}, D_1)$-modules on cohomology.

3.6 Tsygan and Kontsevich conjectures [30], [26]

Conjecture 1. There exists a quasi-isomorphism of L_∞-modules

$$F\colon C_{-\bullet}(C^\infty(M)) \rightsquigarrow (\Omega^{-\bullet}(M), 0)$$

such that F_0 is the HKR map.

Conjecture 2. There exists a natural $\mathbb{C}[[u]]$-linear quasi-isomorphism of L_∞-modules

$$F\colon CC^-_{-\bullet}(C^\infty(M)) \rightsquigarrow (\Omega^{-\bullet}(M)[[u]], ud)$$

such that F_0 is the Connes quasi-isomorphism [8], given by the u-linear extension of the HKR map (10).

Conjecture 1 is now a theorem. Different proofs for $M = \mathbb{R}^d$ were given in [29] and [27]. Shoikhet's proof [27] gives an explicit formula for the Taylor components of F in terms of integrals over configuration spaces on the disk and extends to general manifolds, as shown in [11].

Let us turn to Kontsevich's formality conjecture for cyclic cochains, as quoted in [26]. Recall that a volume form $\Omega \in \Omega^d(M)$ on a d-dimensional manifold defines an isomorphism $\Gamma(\wedge^k TM) \to \Omega^{d-k}(M)$, $\gamma \mapsto \iota_\gamma \Omega$. The de Rham differential d on $\Omega^\bullet(M)$ translates to a differential div_Ω, the divergence operator of degree -1. The divergence operator is a derivation of the bracket on $\mathfrak{g}_S = \Gamma(\wedge^{\bullet+1} TM)$ of degree -1. Let us introduce the differential graded Lie algebra $\mathfrak{g}_S^\Omega = (\mathfrak{g}_S[v], \delta_\Omega)$, where v is a formal variable of degree 2. The bracket is the Schouten–Nijenhuis bracket and the differential is $\delta_\Omega = v\,\mathrm{div}_\Omega$. The cyclic analogue of $\mathfrak{g}_G$ is the differential graded Lie algebra

$$\mathfrak{g}_G^{\mathrm{cycl}} = \left\{ \varphi \in \mathfrak{g}_G,\ \int_M a_0 \varphi(a_1, \ldots, a_p)\Omega = (-1)^p \int_M a_p \varphi(a_0, \ldots, a_{p-1})\Omega \right\}.$$

Conjecture 3. For each volume form $\Omega \in \Omega^d(M)$ there exists an L_∞-quasi-isomorphism of L_∞-algebras $F\colon \mathfrak{g}_S^\Omega \rightsquigarrow \mathfrak{g}_G^{\mathrm{cycl}}$.

Shoikhet [26] constructed a quasi-isomorphism of complexes $C_1 : \mathfrak{g}_S^\Omega \to \mathfrak{g}_G^{\mathrm{cycl}}$ and conjectural formulae for an L_∞-morphism whose first component is C_1 in terms of integrals over configuration spaces. One consequence of the conjecture is the construction of cyclically-invariant star-products from divergenceless Poisson bivector fields. Such star-products were then constructed independently of the conjecture, see [12].

4 An L_∞-morphism for cyclic chains

4.1 The main results

Let Ω be volume form on a manifold M and $\mathfrak{g}_S^\Omega$ be the differential graded Lie algebra $\mathfrak{g}_S[v]$ with Schouten bracket and differential $\delta_\Omega = v\,\mathrm{div}_\Omega$, see Section 3.6. The Kontsevich L_∞-morphism composed with the canonical projection $\mathfrak{g}_S^\Omega \to \mathfrak{g}_S = \mathfrak{g}_S^\Omega / v\mathfrak{g}_S^\Omega$ is an L_∞-morphism $\mathfrak{g}_S^\Omega \rightsquigarrow \mathfrak{g}_G$. Through this morphism the differential graded $\mathfrak{g}_G$-module $CC_\bullet^-(A)$ of negative cyclic chains of $A = C^\infty(M)$ becomes an L_∞-module over $\mathfrak{g}_S^\Omega$.

Theorem 2. *Let M be an open subset of $\mathbb{R}^d$ with coordinates $x_1, \dots, x_n$ and volume form $\Omega = \mathrm{d}x_1 \cdots \mathrm{d}x_d$. Let $A = C^\infty(M)$. Let $\Gamma(\wedge^{-\bullet} TM)$ be the differential graded module over $\mathfrak{g}_S^\Omega$ with differential div_Ω and trivial $\mathfrak{g}_S^\Omega$-action. Then there exists an $\mathbb{R}[u]$-linear morphism of L_∞-modules over $\mathfrak{g}_S^\Omega$*

$$F\colon CC^-_{-\bullet}(A) \rightsquigarrow \Gamma(\wedge^{-\bullet} TM)[u],$$

such that

(i) The component F_0 of F vanishes on $CC_p(A)$, $p > 0$ and for $f \in A \subset CC_0^-(A)$, $F_0(f) = f$.
(ii) For $\gamma \in \Gamma(\wedge^k TM)$, $\ell = 0, 1, 2, \dots,$ $a = (a_0, \dots, a_p) \in CC_p^-(A)$,

$$F_1(\gamma v^\ell; a) = \begin{cases} (-1)^p u^s \gamma \lrcorner H(a), & \text{if } k \geq p \text{ and } s = k + \ell - p - 1 \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Here $\lrcorner\colon \Gamma(\wedge^k TM) \otimes \Omega^p(M) \to \Gamma(\wedge^{k-p} TM)$ is the contraction map and H is the HKR map (10).
(iii) The maps F_n are equivariant under linear coordinate transformations and $F_n(\gamma_1 \cdots \gamma_n; a) = \gamma_1 \wedge F_{n-1}(\gamma_2 \cdots \gamma_n; a)$ whenever $\gamma_1 = \sum (c_k^i x_k + d^i)\partial_i \in \mathfrak{g}_S \subset \mathfrak{g}_S^\Omega$ is an affine vector field and $\gamma_2, \dots, \gamma_n \in \mathfrak{g}_S^\Omega$.

The proof of this Theorem is deferred to Section 6.3.

In explicit terms, F is given by a sequence of $\mathbb{R}[u]$-linear maps $F_n\colon S^n\mathfrak{g}_S^\Omega[1]\otimes CC^-(A)\to\Gamma(\wedge^n TM)$, $\gamma\otimes a\mapsto F_n(\gamma;a)$, $n\geq 0$, obeying the following relations. For any $\gamma=\gamma_1\cdots\gamma_n\in S^n\mathfrak{g}_S^\Omega[1]$, $a\in CC_p^-(A)$.

$$\begin{aligned} &F_n(\delta_\Omega\gamma;a)+(-1)^{|\gamma|+p}F_n(\gamma;(b+uB)a) \\ &+\sum_{k=0}^{n-1}\sum_{\sigma\in S_{k,n-k}}(-1)^{|\gamma|-1}\epsilon(\sigma;\gamma)F_k(\gamma_{\sigma(1)}\cdots\gamma_{\sigma(k)};U_{n-k}(\bar\gamma_{\sigma(k+1)}\cdots\bar\gamma_{\sigma(n)})\cdot a) \\ &+\sum_{i<j}\epsilon_{ij}F_{n-1}((-1)^{|\gamma_i|-1}[\gamma_i,\gamma_j]\cdot\gamma_1\cdots\hat\gamma_i\cdots\hat\gamma_j\cdots\gamma_n;a)=\mathrm{div}_\Omega\ F_n(\gamma;a). \end{aligned} \tag{11}$$

Here $\bar\gamma_i$ denotes the projection of γ_i to $\mathfrak{g}_S[1]=\mathfrak{g}_S^\Omega[1]/v\mathfrak{g}_S^\Omega[1]$; $S_{p,q}\subset S_{p+q}$ is the set of (p,q)-shuffles and the signs $\epsilon(\sigma;\gamma),\epsilon_{ij}$ are the Koszul signs coming from the permutation of the $\gamma_i\in\mathfrak{g}_S[1]$; $|\gamma|=\sum_i|\gamma_i|$; the differential δ_Ω is extended to a degree 1 derivation of the algebra $S\mathfrak{g}_S^\Omega[1]$; the maps $U_k\colon S^k\mathfrak{g}_S[1]\to\mathfrak{g}_G[1]$ are the Taylor components of the Kontsevich L_∞-morphism of Theorem 1.

We give the explicit expressions of the maps F_n in Section 5. Before that we explore some consequences.

4.2 Maurer–Cartan elements

An element of degree 1 in $\mathfrak{g}_S^\Omega$ has the form $\tilde\pi=\pi+vh$ where π is a bivector field and h is a function. The Maurer–Cartan equation $\delta_\Omega\tilde\pi-\frac{1}{2}[\tilde\pi,\tilde\pi]=0$ translates to

$$[\pi,\pi]=0,\qquad \mathrm{div}_\Omega\ \pi-[h,\pi]=0.$$

Thus π is a Poisson bivector field whose divergence is a Hamiltonian vector field with Hamiltonian h. Such Poisson structures are called unimodular [19]. As explained in [17], Poisson bivector fields in $\epsilon\mathfrak{g}_S[[\epsilon]]$ are mapped to solution of the Maurer–Cartan equations in $\epsilon\mathfrak{g}_G[[\epsilon]]$, which are star-products, i.e., formal associative deformations of the pointwise product in $C^\infty(M)$:

$$f\star g=fg+\sum_{n=1}^\infty\frac{\epsilon^n}{n!}U_n(\pi,\ldots,\pi)(f\otimes g).$$

Here the function part of $\tilde\pi$ does not contribute as it is projected away in the L_∞-morphism $\mathfrak{g}_S^\Omega\rightsquigarrow\mathfrak{g}_G$.

If $\tilde\pi=\pi+vh$ is a Maurer–Cartan element in $\mathfrak{g}_S^\Omega$ then $\tilde\pi_\epsilon=\epsilon\pi+vh$ is a Maurer–Cartan element in $\mathfrak{g}_S^\Omega[[\epsilon]]$. The twist of F by $\tilde\pi$ then gives a chain map from the negative cyclic complex of the algebra $A_\epsilon=(C^\infty(M)[[\epsilon]],\star)$ to $\Gamma(\wedge TM)[u][[\epsilon]]$. In particular, we get a trace

$$f\mapsto\tau(f)=\sum_{n=0}^\infty\frac{1}{n!}\int_M F_n(\tilde\pi_\epsilon,\ldots,\tilde\pi_\epsilon;f)\Omega, \tag{12}$$

on the subalgebra of A_ϵ consisting of functions with compact support. Here there is a question of convergence since there are infinitely many terms contributing to each fixed power of ϵ. The point is that these infinitely many terms combine to exponential functions. More precisely we have the following result.

Proposition 1. *The trace* (12) *can be written as*

$$\tau(f) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \int_M H_n(\pi, h, f) \mathrm{e}^h \Omega = \int_M f \mathrm{e}^h \Omega + O(\epsilon)$$

where H_n is a differential polynomial in π, h, f.

The proof is based on the expression of F_n in terms of graphs. We postpone it to Section 5.5, after we introduce this formalism.

5 Feynman graph expansion of the L_∞-morphism

In this section we construct the morphism of L_∞-modules of Theorem 2. The Taylor components have the form

$$F_n(\gamma; a) = \sum_{\Gamma \in \mathcal{G}_{\mathbf{k},m}} w_\Gamma F_\Gamma(\gamma; a).$$

Here $\gamma = \gamma_1 \cdots \gamma_n$, with $\gamma_i \in \Gamma(\wedge^{k_i} TM)[v]$, $\mathbf{k} = (k_1, \dots, k_n)$ and $a = (a_0, \dots, a_m) \in C_m(A)$. The sum is over a finite set $\mathcal{G}_{\mathbf{k},m}$ of directed graphs with some additional structure. To each graph a weight $w_\Gamma \in \mathbb{R}[u]$, defined as an integral over a configuration space of points in the unit disk, is assigned.

We turn to the descriptions of the graphs and weights.

5.1 Graphs

Let $m, n \in \mathbb{Z}_{\geq 0}$, $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n_{\geq 0}$. We consider directed graphs Γ with $n + m$ vertices with additional data obeying a set of rules. The data are a partition of the vertex set into three totally ordered subsets $V(\Gamma) = V_1(\Gamma) \sqcup V_2(\Gamma) \sqcup V_w(\Gamma)$, a total ordering of the edges originating at each vertex and the assignment of a nonnegative integer, the *degree*, to each vertex in $V_1(\Gamma)$. The rules are:

1. There are n vertices in $V_1(\Gamma)$. There are exactly k_i edges originating at the ith vertex of $V_1(\Gamma)$.
2. There are m vertices in $V_2(\Gamma)$. There are no edges originating at these vertices.
3. There is exactly one edge pointing at each vertex in $V_w(\Gamma)$ and no edge originating from it.

4. There are no edges starting and ending at the same vertex.
5. For each pair of vertices i, j there is at most one edge from i to j.

The last rule is superfluous, but since all graphs with multiple edges will have vanishing weight we may just as well exclude them from the start. This has the notational advantage that we may think of the edge set $E(\Gamma)$ as a subset of $V(\Gamma) \times V(\Gamma)$.

Two graphs are called equivalent if there is a graph isomorphism between them that respects the partition and the orderings. The set of equivalence classes is denoted $\mathcal{G}_{\mathbf{k},m}$.

The vertices in $V_1(\Gamma)$ are called vertices of the first type, those in $V_2(\Gamma)$ of the second type. The vertices in $V_b(\Gamma) = V_1(\Gamma) \cup V_2(\Gamma)$ are called black, those in $V_w(\Gamma)$ are called white. We denote by $E_b(\Gamma)$ the subset of $E(\Gamma)$ consisting of edges whose endpoints are black.

To each $\Gamma \in \mathcal{G}_{\mathbf{k},m}$ there corresponds a multivector field $F_\Gamma(\gamma; a)$ whose coefficients are differential polynomials in the components of γ_i, a_i. The rules are the same as in [17] except for the additional white vertices, representing uncontracted indices and the degrees d_i, that select the power d_i of v in γ_i. Let us consider for example the graph of Fig. 1 and suppose that the degrees of the two vertices of the first type are k and ℓ. The algebra of multivector fields on $M \subset \mathbb{R}^d$ is generated by $C^\infty(M)$ and anticommuting generators $\theta_\nu = \partial/\partial x_\nu$. Thus $\gamma \in \Gamma(\wedge^k TM)$ has the form

$$\gamma = \frac{1}{k!} \sum_{\nu_1, \ldots, \nu_k} \gamma^{\nu_1 \ldots \nu_k} \theta_{\nu_1} \cdots \theta_{\nu_k}.$$

The components $\gamma^{\nu_1 \cdots \nu_k} \in C^\infty(M)$ are skew-symmetric under permutation of the indices ν_i. The graph of Fig. 1, with the convention that the edges originating at each vertex are ordered counterclockwise, gives then

$$F_\Gamma(\gamma_1 v^k, \gamma_2 v^\ell; a_0, a_1, a_2) = \sum \gamma_1^{ij} \partial_j \gamma_2^{pqr} \partial_i a_0 \partial_p a_1 \partial_q a_2 \theta_r,$$

and is zero on other monomials in v.

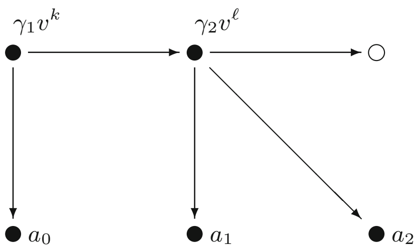


Fig. 1. A graph in $\mathcal{G}_{(2,3),3}$ with two vertices in V_1 of valencies $(2,3)$, three in V_2 and one white vertex. The degrees of the vertices of the first type are k and ℓ

5.2 Equivariant differential forms on configuration spaces

Let Σ be a manifold with an action of the circle $S^1 = \mathbb{R}/\mathbb{Z}$. The infinitesimal action $\mathrm{Lie}(S^1) = \mathbb{R}\frac{\mathrm{d}}{\mathrm{d}t} \to \Gamma(T\Sigma)$ is generated by a vector field $\mathbf{v} \in \Gamma(T\Sigma)$, the image of $\frac{\mathrm{d}}{\mathrm{d}t}$. The Cartan complex of S^1-equivariant forms, computing the equivariant cohomology with real coefficients, is the differential graded algebra

$$\Omega^\bullet_{S^1}(\Sigma) = \Omega^\bullet(\Sigma)^{S^1}[u],$$

of polynomials in an undetermined u of degree 2 with coefficients in the S^1-invariant smooth differential forms. The differential is $\mathrm{d}_{S^1} = \mathrm{d} - u\iota_{\mathbf{v}}$, where d is the de Rham differential and $\iota_{\mathbf{v}}$ denotes interior multiplication by $\mathbf{v}$, extended by $\mathbb{R}[u]$-linearity. If Σ has an S^1-invariant boundary $\partial\Sigma$ and $j\colon \partial\Sigma \to \Sigma$ denotes the inclusion map, then the relative equivariant complex is

$$\Omega^\bullet_{S^1}(\Sigma, \partial\Sigma) = \mathrm{Ker}(j^*\colon \Omega^\bullet_{S^1}(\Sigma) \to \Omega^\bullet_{S^1}(\partial\Sigma)).$$

In the case of the unit disk we have:

Lemma 3. *Let $\bar{D} = \{z \in \mathbb{C}, |z| \leq 1\}$ be the closed unit disk.*

(i) The equivariant cohomology $H^\bullet_{S^1}(\bar{D})$ of $\bar{D}$ is the free $\mathbb{R}[u]$-module generated by the class of $1 \in \Omega^0(\bar{D})$.
(ii) The relative equivariant cohomology $H^\bullet_{S^1}(\bar{D}, \partial\bar{D})$ of $(\bar{D}, \partial\bar{D})$ is the free $\mathbb{R}[u]$-module generated by the class of

$$\phi(z, u) = \frac{\mathrm{i}}{2\pi}\mathrm{d}z \wedge \mathrm{d}\bar{z} + u(1 - |z|^2). \tag{13}$$

5.3 The propagator

The integrals over configuration spaces defining the L_∞-morphism are constructed out of a propagator, a differential 1-form ω on $\bar{D} \times \bar{D} \smallsetminus \Delta$, with a simple pole on the diagonal $\Delta = \{(z, z),\, z \in \bar{D}\}$ and defining the integral kernel of a homotopy contracting equivariant differential forms to a space of representatives of the cohomology. The explicit formula of the propagator associated to the choice of cocycles in Lemma 3 is given by

$$\omega(z, w) = \frac{1}{4\pi\mathrm{i}}\left(\mathrm{d}\ln\frac{(z - w)(1 - z\bar{w})}{(\bar{z} - \bar{w})(1 - \bar{z}w)} + z\,\mathrm{d}\bar{z} - \bar{z}\,\mathrm{d}z\right). \tag{14}$$

Lemma 4. *Let $p_i\colon \bar{D} \times \bar{D} \to \bar{D}$ be the projection to the i-th factor, $i = 1, 2$. The differential form $\omega \in \Omega^1_{S^1}(\bar{D} \times \bar{D} \smallsetminus \Delta)$ has the following properties:*

(i) Let $j\colon \partial\bar{D} \times \bar{D} \to \bar{D} \times D$ be the inclusion map. Then $j^*\omega = 0$.
(ii) $\mathrm{d}_{S^1}\omega = -p_1^*\phi$.
(iii) As $z \to w$, $\omega(z, w) = (2\pi)^{-1}\mathrm{d}\arg(z - w) +$ smooth.

(iv) *As z and w approach a boundary point, $\omega(z,w)$ converges to the Kontsevich propagator $\omega_K(x,y) = (2\pi)^{-1}(\mathrm{d}\arg(x-y) - \mathrm{d}\arg(\bar{x}-y))$ on the upper half-plane H_+ from [17]. More precisely, for small $t>0$ let $\varphi_t(x) = z_0\mathrm{e}^{\mathrm{i}tx}$ be the inclusion of a neighbourhood of $0 \in H_+$ into a neighbourhood of $z_0 \in \partial D$ in D. Then $\lim_{t\to 0}(\varphi_t \times \varphi_t)^*\omega = \omega_K$.*

The proof is a simple computation left to the reader.

5.4 Weights

The weights are integrals of differential forms over configuration spaces $C^0_{n,m}(D)$ of n points in the unit disk $D = \{z \in \mathbb{C},\, |z| < 1\}$ and $m+1$ cyclically ordered points on its boundary $\partial\bar{D}$, the first of which is at 1:

$$C^0_{n,m}(D) = \{(z,t) \in D^n \times (\partial\bar{D})^m\,,\, z_i \neq z_j, (i \neq j),\\ 0 < \arg(t_1) < \cdots < \arg(t_m) < 2\pi\}.$$

The differential forms are obtained from the propagator ω, see (14), and the form ϕ, see (13). Let $\Gamma \in \mathcal{G}_{(k_1,\dots,k_n),m}$. The weight w_Γ of Γ is

$$w_\Gamma = \frac{1}{\prod_{i=1}^n k_i!}\int_{C^0_{n,m}(D)} \omega_\Gamma$$

where $\omega \in \Omega^\bullet(C^0_{n,m}(D))[u]$ is the differential form

$$\omega_\Gamma = \prod_{i\in V_1(\Gamma)}\ \prod_{(i,j)\in E_b(\Gamma)} \omega(z_i,z_j) \prod_{i\in V_1(\Gamma)} \phi(z_i,u)^{r_i}.$$

Here z_i is the coordinate of $z \in C^0_{n,m}(D)$ assigned to the vertex i of Γ: to the vertices of the first type we assign the points in the unit disk and to the vertices of the second type the points on the boundary. The assignment is uniquely specified by the ordering of the vertices in Γ. The number r_i is the degree of the vertex i plus the number of white vertices connected to it. The product over (i,j) is over the edges connecting black vertices to black vertices. For example, a point of $C^0_{2,3}(D)$ is given by coordinates $(z_1,z_2,1,t_1,t_2)$ with $z_i \in D$ and $t_i \in S^1$. The differential form associated to the graph of Fig. 1, with degree assignments k,ℓ, is

$$\pm\omega_\Gamma = \omega(z_1,1)\omega(z_1,z_2)\omega(z_2,t_1)\omega(z_2,t_2)\phi(z_1,u)^k\phi(z_2,u)^{\ell+1}.$$

The signs are tricky. A consistent set of signs may be obtained by the following procedure. View a multivector field $\gamma \in \mathfrak{g}_S[v]$ as a polynomial $\gamma(x,\theta,v)$ whose coefficients are functions on $T^*[1]M = M \times \mathbb{R}^d[1]$. Build a function in $C^\infty((T^*[1]M)^{n+m})[v_1,\dots,v_n]$:

$$g(x^{(1)},\theta^{(1)},v_1,\dots,x^{(\bar{m})})\\ = \gamma_1(x^{(1)},\theta^{(1)},v_1)\cdots\gamma_n(x^{(n)},\theta^{(n)},v_n)a_0(x^{(\bar{0})})\cdots a_m(x^{(\bar{m})}).$$

Then

$$F_n(\gamma;a) = (-1)^{|\gamma|m} \int_{C^0_{n,m}(D)} i^*_\Delta \circ \exp(\Phi_n)(g)|_{v_1=\cdots=v_n=0},$$

$$\Phi_n = \sum_{i\neq k} \omega(z_i,z_k) \sum_{\nu=1}^{d} \frac{\partial^2}{\partial\theta^{(i)}_\nu \partial x^{(k)}_\nu} + \sum_i \phi(z_i,u) \left(\sum_{\nu=1}^{d} \theta_\nu \frac{\partial}{\partial\theta^{(i)}_\nu} + \frac{\partial}{\partial v_i} \right).$$

The sums over i are from 1 to n and the sum over k is over the set $\{1,\dots,n,\bar{1},\dots,\bar{m}\}$, with the understanding that $z_{\bar{j}} = t_j$. The map i^*_Δ is the restriction to the diagonal: its effect is to set all $x^{(i)}$ to be equal to x and all $\theta^{(i)}$ to be equal to θ. The integrand is then an element of the tensor product of graded commutative algebras $\Omega(C^0_{n,m}(D)) \otimes C^\infty(T^*[1]M)[u]$. The integral is defined as $\int(\alpha\otimes\gamma) = (\int \alpha)\gamma$ and the expansion of the exponential functions gives rise to a finite sum over graphs.

5.5 Proof of Proposition 1 on page 126

A vertex of a directed graph is called disconnected if there is no edge originating or ending at it.

Lemma 5. *Let $\tilde{F}_n$ be defined as F_n except that the sum over graphs is restricted to the graphs without disconnected vertices of the first type. Then*

$$F_{k+n}((hv)^k \cdot \pi^n; f) = \sum_{s=0}^{k} \binom{k}{s} h^s \tilde{F}_{k-s+n}((hv)^{k-s} \cdot \pi^n; f).$$

Proof. For each fixed graph Γ_0 without disconnected vertices of the first type, we consider all graphs Γ contributing to F_{k+n} that reduce to Γ_0 after removing all disconnected vertices of the first type. The contribution to F_{k+n} of such a graph Γ is h^s times the contribution of Γ_0, where s is the number of disconnected vertices of the first type. Indeed, each disconnected vertex in a graph Γ gives a factor h to F_Γ and a factor $\int_D \phi = 1$ to the weight w_Γ. The proof of the lemma is complete. □

Let

$$H_n(\pi,h,f) = \sum_{r=0}^{\infty} \frac{1}{r!} \tilde{F}_{n+r}((hv)^r \cdot \pi^n; f).$$

In this sum there are finitely many terms since in the absence of disconnected vertices only derivatives of h can appear and the number of derivatives is bounded (by $2n$). Therefore $H_n(\pi,h,f)$ is a differential polynomial in π,h,f. We conclude that

$$\sum_{n=0}^{\infty} \frac{1}{n!} F_n(\hat{\pi}^n; f) = \sum_{n,k=0}^{\infty} \frac{\epsilon^n}{k!n!} F_{k+n}((hv)^k \pi^n; f)$$

$$= \sum_{n,r,s=0}^{\infty} \frac{\epsilon^n}{r!s!n!} h^s \tilde{F}_{n+r}((hv)^r \pi^n; f) = \mathrm{e}^h \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} H_n(\pi,h,f).$$

This concludes the proof of Proposition 1.

6 Equivariant differential forms on configuration spaces and Stokes theorem

6.1 Configuration spaces and their compactifications

We consider three types of configuration spaces of points, the first two appearing in [17].
(i) *Configuration spaces of points in the plane.* Let $\mathrm{Conf}_n(\mathbb{C}) = \{z \in \mathbb{C}^n\,,\, z_i \neq z_j, (i \neq j)\}$, $n \geq 2$. The three-dimensional real Lie group G_3 of affine transformations $w \mapsto aw + b, a > 0, b \in \mathbb{C}$ acts freely on the manifold $\mathrm{Conf}_n(\mathbb{C})$. We set $C_n(\mathbb{C}) = \mathrm{Conf}_n(\mathbb{C})/G_3$ $(n \geq 2)$. It is a smooth manifold of dimension $2n - 3$. We fix the orientation defined by the volume form $\mathrm{d}\varphi_2 \wedge \bigwedge_{j\geq 3} \mathrm{dRe}(z_j) \wedge \mathrm{dIm}(z_j)$, with the choice of representatives with $z_1 = 0, z_2 = \mathrm{e}^{\mathrm{i}\varphi_2}$.
(ii) *Configuration spaces of points in the upper half-plane.* Let $H_+ = \{z \in \mathbb{C}\,,\, \mathrm{Im}(z) > 0\}$ be the upper half-plane. Let $\mathrm{Conf}_{n,m}(H_+) = \{(z,x) \in H_+^n \times \mathbb{R}^m,\, z_i \neq z_j, (i \neq j), t_1 < \cdots < t_m\}$, $2n + m \geq 2$. The two-dimensional real Lie group G_2 of affine transformations $w \mapsto aw + b, a > 0, b \in \mathbb{R}$ acts freely on the manifold $\mathrm{Conf}_{n,m}(H_+)$. We set $C_{n,m}(H_+) = \mathrm{Conf}_{n,m}(H_+)/G_2$ $(2n+m \geq 2)$. It is a smooth manifold of dimension $2n+m-2$. If $n \geq 1$, we fix the orientation by choosing representatives with $z_1 = i$ and taking the volume form $\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_m \wedge \bigwedge_{j\geq 2} \mathrm{dRe}(z_j) \wedge \mathrm{dIm}(z_j)$. If $m \geq 2$, we fix the orientation defined by the volume form $(-1)^m \mathrm{d}t_2 \wedge \cdots \wedge \mathrm{d}t_{m-1} \wedge \bigwedge_{j\geq 1} \mathrm{dRe}(z_j) \wedge \mathrm{dIm}(z_j)$, with the choice of representatives with $t_1 = 0, t_m = 1$. If $m \geq 2$ and $n \geq 1$, it is easy to check that the two orientations coincide.
(iii) *Configuration spaces of points in the disk.* Let $D = \{z \in \mathbb{C}\,,\, |z| < 1\}$ be the unit disk, $S^1 = \partial\bar{D}$ the unit circle. Let $C_{n,m+1}(D) = \{(z,x) \in D^n \times (S^1)^{m+1},\, z_i \neq z_j, (i \neq j), \arg(t_0) < \cdots < \arg(t_m) < \arg(t_0) + 2\pi\}$, $m \geq 0$. The circle group acts freely on $C_{n,m+1}(D)$ by rotations. We do not take a quotient here, since the differential forms we will introduce are not basic, and work equivariantly instead. Instead of the quotient we consider the section $C^0_{n,m}(D) = \{(z,x) \in C_{n,m+1}(D)\,,\, t_0 = 1\}$, $(m \geq 1)$. It is a smooth manifold of dimension $2n + m$. The orientation of $C_{n,m+1}(D)$ is defined by $\mathrm{d}\arg(t_0) \wedge \cdots \wedge \mathrm{d}\arg(t_m) \wedge \bigwedge_{j=1}^n \mathrm{dRe}(z_j) \wedge \mathrm{dIm}(z_j)$. The orientation of $C^0_{n,m}(D)$ is defined by $\mathrm{d}\arg(t_1) \wedge \cdots \wedge \mathrm{d}\arg(t_m) \wedge \bigwedge_{j=1}^n \mathrm{dRe}(z_j) \wedge \mathrm{dIm}(z_j)$.

As in [17], compactifications $\bar{C}_n(\mathbb{C})$, $\bar{C}_{n,m}(H_+)$, $\bar{C}_{n,m+1}(D)$, $\bar{C}^0_{n,m}(D)$ of these spaces as manifolds with corners are important. Their construction is the same as in [17]. Roughly speaking, one adds strata of codimension 1 corresponding to limiting configurations in which a group of points collapses to a point, possibly on the boundary, in such a way that within the group the relative position after rescaling remains fixed. Higher codimension strata correspond to collapses of several groups of points possibly within each other. The main point is that the Stokes theorem applies for smooth top differential forms on manifold with corners, and for this only codimension 1 strata are important.

Let us describe the codimension 1 strata of $C^0_{n,m}(D)$.
Strata of type I. These are strata where a subset A of $n' \geq 2$ out of n points z_i in the interior of the disk collapse at a point in the interior of the disk, the relative position of the collapsing points is described by a configuration on the plane and the remaining points and the point of collapse are given by a configuration on the disk. This stratum is thus

$$\partial_A \bar{C}^0_{n,m}(D) \simeq \bar{C}_{n'}(\mathbb{C}) \times \bar{C}^0_{n-n'+1,m}(D). \tag{15}$$

Strata of type II. These are strata where a subset A of n' out of n points z_i and a subset B of the m points t_i collapse at a point on the boundary of the disk $(2n'+m' \geq 2)$. The relative position of the collapsing points is described by a configuration on the upper half-plane and the remaining points and the point of collapse are given by a configuration on the disk. This stratum is thus

$$\partial_{A,B} \bar{C}^0_{n,m}(D) \simeq \bar{C}_{n',m'}(H_+) \times \bar{C}^0_{n-n',m-m'+1}(D). \tag{16}$$

6.2 Forgetting the base point and cyclic shifts

Let $j_0\colon C^0_{n,m}(D) \to C_{n,m}(D)$ be the map $(z, 1, t_1, \dots, t_m) \mapsto (z, t_1, \dots, t_m)$ forgetting the base point $t_0 = 1$. It is an orientation preserving open embedding.

The cyclic shift $\lambda\colon C^0_{n,m}(D) \to C^0_{n,m}(D)$ is the map

$$\lambda\colon (z_1, \dots, z_n, 1, t_1, \dots, t_m) \mapsto (z_1, \dots, z_n, 1, t_m, t_1, \dots, t_{m-1}).$$

It is a diffeomorphism preserving the orientation if m is odd and reversing the orientation if m is even. The following fact is then easily checked.

Lemma 6. *The collection of maps $j_k = j_0 \circ \lambda^{\circ k}$, $k = 0, \dots, m-1$, defines an embedding $j\colon C^0_{n,m}(D) \sqcup \cdots \sqcup C^0_{n,m}(D) \to C_{n,m}(D)$ with dense image. The restriction of j to the kth copy of $C_{n,m}(D)$ multiplies the orientation by $(-1)^{(m-1)k}$.*

6.3 Proof of Theorem 2 on page 124

The proof uses the Stokes theorem as in [17]. The new features are: (i) the differential forms in the integrand are not closed and (ii) an equivariant version of the Stokes theorem is used.

We first compute the differential of the differential form associated to a graph Γ.

Lemma 7. *Let $\partial_e\Gamma$ be the graph obtained from Γ by adding a new white vertex $*$ and replacing the black-to-black edge $e \in E_b(\Gamma)$ by an edge originating at the same vertex as e but ending at $*$. Then*

$$\mathrm{d}_{S^1}\omega_\Gamma = \sum_{e\in E_b(\Gamma)} (-1)^{\sharp e}\omega_{\partial_e\Gamma},$$

where $\sharp e = k$ if $e = e_k$ and $e_1, \dots, e_N$ are the edges of Γ in the ordering specified by the ordering of the vertices and of the edges at each vertex.

Proof. This follows from the fact that d_{S^1} is a derivation of degree 1 of the algebra of equivariant forms and Lemma 4, (ii). □

The next lemma is an equivariant version of the Stokes theorem.

Lemma 8. *Let $\omega \in \Omega^\bullet_{S^1}(\bar{C}_{n,m+1}(D))$. Denote also by ω its restriction to $\bar{C}^0_{n,m}(D) \subset \bar{C}_{n,m+1}(D)$ embedded as the subspace where $t_0 = 1$ and to the codimension 1 strata $\partial_i C^0_{n,m}(D)$ of $C^0_{n,m}(D)$. Then*

$$\int_{\bar{C}^0_{n,m}(D)} \mathrm{d}_{S^1}\omega = \sum_i \int_{\partial_i \bar{C}^0_{n,m}(D)} \omega - u\int_{\bar{C}_{n,m+1}(D)} \omega.$$

Proof. Write $\mathrm{d}_{S^1} = \mathrm{d} - u\iota_{\mathbf{v}}$. For $u = 0$ the claim is just the Stokes theorem for manifolds with corners. Let us compare the coefficients of u. The action map restricts to a diffeomorphism $f\colon S^1 \times \bar{C}^0_{n,m}(D) \to \bar{C}_{n,m+1}(D)$. Since ω is S^1-invariant, $\iota_{\mathbf{v}}\omega$ is also invariant and we have $f^*\omega = 1\otimes\omega + \mathrm{d}t\otimes\iota_{\mathbf{v}}\omega \in \Omega(S^1)\otimes\Omega(\bar{C}^0_{n,m}(D)) \subset \Omega(S^1\times\bar{C}^0_{n,m}(D))$, where t is the coordinate on the circle $S^1 = \mathbb{R}/\mathbb{Z}$. Thus

$$\int_{\bar{C}_{n,m+1}(D)} \omega = \int_{S^1\times\bar{C}^0_{n,m}(D)} \mathrm{d}t\otimes\iota_{\mathbf{v}}\omega = \int_{\bar{C}^0_{n,m}(D)} \iota_{\mathbf{v}}\omega. \qquad \square$$

Finally we use Lemma 6 to reduce the integral over $\bar{C}_{n,m+1}(D)$ to integrals over $\bar{C}^0_{n,m+1}(D)$. We obtain:

Lemma 9. *Let $\omega \in \Omega^\bullet_{S^1}(\bar{C}_{n,m+1}(D))$ and let j_k be the maps defined in Lemma 6. Then*

$$\int_{\bar{C}_{n,m+1}(D)} \omega = \sum_{k=0}^m (-1)^{mk} \int_{\bar{C}^0_{n,m+1}(D)} j_k^*\omega.$$

We can now complete the proof of Theorem 2. We first prove the identity (11), starting from the right-hand side. Suppose that $a = (a_0, \dots, a_m) \in C_{-m}(A)$, $\gamma = \gamma_1\cdots\gamma_n$, with $\gamma_i \in \Gamma(\wedge^{k_i} TM)$. It is convenient to identify $\Gamma(\wedge TM)$ with $C^\infty(M)[\theta_1, \dots, \theta_n]$ where θ_i are anticommuting variables, so that $\mathrm{div}_\Omega = \sum \partial^2/\partial t_i\partial\theta_i$. It follows that for any $\Gamma \in \mathcal{G}_{\mathbf{k},m}$, $\mathrm{div}_\Omega F_\Gamma(\gamma; a)$ can be written as a sum (with signs) of terms $F_{\Gamma'}(\gamma; a)$, where Γ' is obtained from

Γ by identifying a white vertex with a black vertex and coloring it black. Some of these graphs Γ' have an edge connecting a vertex to itself and contribute to $F_n(\delta_\Omega\gamma; a)$. The remaining ones yield, in the notation of Lemma 7:

$$\operatorname{div}_\Omega F_n(\gamma; a) - F_n(\delta_\Omega\gamma; a) = \sum_{(\Gamma,e)} (-1)^{\sharp e} w_{\partial_e\Gamma} F_\Gamma(\gamma; a).$$

The summation is over pairs (Γ, e) where $\Gamma \in \mathcal{G}_{\mathbf{k},m}$ and $e \in E_b(\Gamma)$ is a black-to-black edge. By Lemmas 8 and 9,

$$\sum_{e\in E_b(\Gamma)} (-1)^{\sharp e} w_{\partial_e\Gamma} = \sum_i \int_{\partial_i C^0_{n,m}(D)} \omega_\Gamma - u\sum_{k=0}^{m} (-1)^{km} \int_{\bar{C}^0_{n,m+1}} j_k^*\omega_\Gamma.$$

The second term on the right-hand side, containing the sum over cyclic permutations, gives rise to $F_{n+1}(\gamma; Ba)$. The first term is treated as in [17]: the strata of type I (see Section 6.1) give zero by Kontsevich's lemma (see [17], Theorem 6.5) unless the number n' of collapsing interior points is 2. The sum over graphs contributes then to the term with the Schouten bracket $[\gamma_i, \gamma_j]$ in (11). The strata of type II such that $n - k > 0$ interior points approach the boundary give rise to the term containing the components of the Kontsevich L_∞-morphism U_{n-k}. Finally the strata of type II in which only boundary points collapse give the term with Hochschild differential $F_{n-1}(\gamma; ba)$. This proves (11).

Property (i) is clear: F_0 is a sum over graphs with vertices of the second type only. These graphs have no edges. Thus the only case for which the weight does not vanish is when the configuration space is 0-dimensional, namely, when there is only one vertex. Property (ii) is checked by an explicit calculation of the weight. The only graphs with a nontrivial weight have edges connecting the vertex of the first type with white vertices or to vertices of the second type. There must be at least p edges otherwise the weight vanishes for dimensional reasons. In this case, i.e., if $k \geq p$, the integral computing the weight is

$$w_\Gamma = \int \phi(z,u)^{\ell+k-p}\omega(z,t_1)\cdots\omega(z,t_p), \tag{17}$$

with integration over $z \in D$, $t_i \in S^1$, $0 < \arg(t_1) < \cdots < \arg(t_p) < 2\pi$. The integral of the product of the 1-forms ω is a function of z that is independent of z, as is easily checked by differentiating with respect to z, using the Stokes theorem and the boundary conditions of ω. Thus it can be computed for $z = 0$. Since $\omega(0, t_i) = \frac{1}{2\pi}\mathrm{d}\arg(t_i)$ the integral is $1/p!$. The remaining integral over z can then be performed. Set $\ell + k - p = s + 1$. This power must be positive otherwise the integral vanishes for dimensional reasons.

$$\int_D \phi(z,u)^{s+1} = \frac{\mathrm{i}}{2\pi}(s+1)u^s \int_D (1-|z|^2)^s \mathrm{d}z \wedge \mathrm{d}\bar{z} = u^s, \quad s \geq 0,$$

and we obtain $w_\Gamma = u^s/p!$. We turn to Property (iii). The equivariance under linear coordinate transformations is implicit in the construction. The graphs contributing to $F_n(\gamma_1 \cdots ; a)$ for linear γ_1 are of two types: either the vertex associated with γ_1 has exactly one ingoing and one outgoing edge or it has an outgoing edge pointing to a white vertex and there are no incoming edges. The graphs of the second type contribute to $\gamma_1 \wedge F_{n-1}(\cdots ; a)$, since their weight factorize as $1 = \int_D \phi$ times the weight of the graphs obtained by omitting the vertex associated to γ_1 and the white vertex connected to it. The claim then follows from the following vanishing lemma.

Lemma 10. *(i) For all $z, z' \in \bar{D}$, $\int_{w \in D} \omega(z,w)\omega(w,z') = 0$.*
(ii) For all $z \in \bar{D}$, $\int_{w \in D} \omega(z,w)\phi(w,u) = 0$.

Proof. (i) We reduce the first claim to the second: consider the integral

$$I(z,z') = \int_{w_1,w_2 \in D} \mathrm{d}(\omega(z,w_1)\omega(w_1,w_2)\omega(w_2,z')).$$

On the one hand, $I(z,z')$ can be evaluated by using Stokes's theorem, giving three terms all equal up to sign to the integral appearing in (i). On the other hand, the differential can be evaluated explicitly giving

$$I(z,z') = -\int_{w_1,w_2 \in D} \omega(z,w_1)\omega(w_1,w_2)\phi(w_2,0).$$

The integral over w_2 vanishes if (ii) holds. The proof of (ii) is an elementary computation that uses the explicit expression of ω and ϕ. Alternatively, one shows that $\int_{w \in D} \omega(z,w)\phi(w,u)$ is a closed 1-form on the disk that vanishes on the boundary, is invariant under rotations and odd under diameter reflections. Therefore it vanishes. We leave the details to the reader. $\square$

Acknowledgments

We are grateful to Francesco Bonechi, Damien Calaque, Kevin Costello, Florian Schätz, Carlo Rossi, Jim Stasheff, Thomas Willwacher and Marco Zambon for useful comments. This work been partially supported by SNF Grants 20-113439 and 200020-105450, by the European Union through the FP6 Marie Curie RTN ENIGMA (contract number MRTN-CT-2004-5652), and by the European Science Foundation through the MISGAM program. The first author is grateful to the Erwin Schrödinger Institute, where part of this work was done, for hospitality.

References

1. Bonechi, F., Zabzine M.: Poisson sigma model on the sphere. Comm. Math. Phys. **285**, 1033–1063 (2009). `arXiv:0706.3164`
2. Bott, R., Cattaneo, A.S.: Integral invariants of 3-manifolds. J. Differ. Geom. **48**(1), 91–133 (1998)

3. Cattaneo, A.S., Felder, G.: A path integral approach to the Kontsevich quantization formula. Comm. Math. Phys. **212**(3), 591–611 (2000)
4. Cattaneo, A.S., Felder, G.: On the AKSZ formulation of the Poisson sigma model. Lett. Math. Phys. **56**(2), 163–179 (2001) EuroConférence Moshé Flato 2000, Part II (Dijon)
5. Cattaneo, A.S., Felder, G.: Relative formality theorem and quantisation of coisotropic submanifolds. Adv. Math. **208**(2), 521–548 (2007)
6. Cattaneo, A.S., Felder, G., Willwacher, T.: On L-infinity morphisms of cyclic chains. Lett. Math. Phys. **90**, 85–101 (2009). The character map in deformation quantization, `arXiv:0906.3122`
7. Cattaneo, A.S., Rossi, C.A.: Wilson surfaces and higher dimensional knot invariants. Comm. Math. Phys. **256**(3), 513–537 (2005)
8. Connes, A.: Noncommutative differential geometry. Inst. Hautes Études Sci. Publ. Math. **62**, 257–360 (1985)
9. Costello, K.J.: Renormalisation and the Batalin-Vilkovisky formalism. E-print (2007). `arXiv:0706.1533v3`
10. Costello, K.J.: Topological conformal field theories and Calabi-Yau categories. Adv. Math. **210**(1), 165–214 (2007)
11. Dolgushev, V.: A formality theorem for Hochschild chains. Adv. Math. **200**(1), 51–101 (2006)
12. Felder, G., Shoikhet, B.: Deformation quantization with traces. Lett. Math. Phys. **53**(1), 75–86 (2000)
13. Ferrario, A.: Poisson sigma model with branes and hyperelliptic Riemann surfaces. J. Math. Phys. **49**, 092301 (2008). `arXiv:0709.0635v2`
14. Gerstenhaber, M.: The cohomology structure of an associative ring. Ann. Math. (2) **78**, 267–288 (1963)
15. Getzler, E., Jones, J.D.S.: A_∞-algebras and the cyclic bar complex. Illinois J. Math. **34**(2), 256–283 (1990)
16. Ikeda, N.: Two-dimensional gravity and nonlinear gauge theory. Ann. Phys. **235**(2), 435–464 (1994)
17. Kontsevich, M.: Deformation quantization of Poisson manifolds. Lett. Math. Phys. **66**(3), 157–216 (2003)
18. Kontsevich, M., Soibelman, Y.: Notes on A-infinity algebras, A-infinity categories and non-commutative geometry. I. E-print (2006). `arXiv:math/0606241v2 [math.RA]`
19. Koszul, J.-L.: Crochet de Schouten–Nijenhuis et cohomologie. Astérisque. (Numero Hors Série), 257–271 (1985) The mathematical heritage of Élie Cartan (Lyon, 1984)
20. Lada, T., Stasheff, J.: Introduction to SH Lie algebras for physicists. Int. J. Theor. Phys. **32**(7), 1087–1103 (1993)
21. Losev, A.: BV formalism and quantum homotopical structures. Lectures at GAP3, Perugia (2006)
22. Mnëv, P.: Notes on simplicial BF theory. E-print (2006). `arXiv:hep-th/0610326v3`
23. Schaller, P., Strobl, T.: Poisson structure induced (topological) field theories. Mod. Phys. Lett. A **9**(33), 3129–3136 (1994)
24. Schlessinger, M., Stasheff, J.: The Lie algebra structure of tangent cohomology and deformation theory. J. Pure Appl. Algebra **38**(2–3), 313–322 (1985)
25. Schwarz, A.: Geometry of Batalin-Vilkovisky quantization. Comm. Math. Phys. **155**(2), 249–260 (1993)

26. Shoikhet, B.: On the cyclic formality conjecture. E-print. `arXiv:math/9903183v3`
27. Shoikhet, B.: A proof of the Tsygan formality conjecture for chains. Adv. Math. **179**(1), 7–37 (2003)
28. Stasheff, J.D.: Homotopy associativity of H-spaces. I, II. Trans. Am. Math. Soc. **108**, 275–292 (1963); Trans. Am. Math. Soc. **108**, 293–312 (1963)
29. Tamarkin, D., Tsygan, B.: Cyclic formality and index theorems. Lett. Math. Phys. **56**(2), 85–97 (2001)
30. Tsygan, B.: Formality conjectures for chains. In: Differential topology, infinite-dimensional Lie algebras, and applications, Am. Math. Soc. Transl. Ser. 2, vol. 194, pp. 261–274. Am. Math. Soc., Providence, RI (1999)

Noncommutative Calculus and the Gauss–Manin Connection

V.A. Dolgushev[1,2], D.E. Tamarkin[1,2], and B.L. Tsygan[1,2,*]

[1] Department of Mathematics, University of California at Riverside, 900 Big Springs Drive, Riverside, CA 92521, USA, vald@math.ucr.edu
[2] Mathematics Department, Northwestern University, 2033 Sheridan Rd., Evanston, IL 60208, USA, tamarkin@math.northwestern.edu, tsygan@math.northwestern.edu

To Murray Gerstenhaber on his 80th and to Jim Stasheff on his 70th birthday

Abstract. After an overview of noncommutative differential calculus, we construct parts of it explicitly and explain why this construction agrees with a fuller version obtained from the theory of operads.

Key words: Hochschild homology, Cyclic homology, Homotopy algebras, Connections

AMS 2010 Subject Codes: 19D55, 18G55

1 Introduction

In this paper we apply the techniques of brace algebras of Gerstenhaber and of A_∞ and L_∞ algebras of Stasheff to develop a part of what we call noncommutative differential calculus. Noncommutative calculus is a theory that reconstructs basic algebraic structures arising from the calculus on a manifold in terms of the algebra of functions on this manifold, in a way that works for any algebra, commutative or not. This program is being developed in [12], [13], [14], [43], [45]. Let us start by observing that there are several algebraic structures arising from the standard calculus on a manifold:

I. **(Differential graded) Lie algebras and modules over them.** Several key formulas from differential calculus on manifolds, namely, the Cartan formulas, use nothing but commutators (the commutator is always understood in the graded sense) and therefore give rise to graded Lie algebras. It has been emphasized by Dorfman and Gelfand [15] that these graded Lie algebras and their representations are worthy of being studied and generalized. We use the following notation to describe these algebras.

A.S. Cattaneo et al. (eds.), *Higher Structures in Geometry and Physics*, Progress in Mathematics 287, DOI 10.1007/978-0-8176-4735-3_7,

(a) Multivector fields with the Schouten–Nijenhuis–Richardson bracket form a graded Lie algebra that we denote by $\mathfrak{g}^\bullet$; $\mathfrak{g}^\bullet = \wedge^{\bullet+1}T$. This graded Lie algebra acts on the space $\Omega^{-\bullet}$ of forms with reversed grading by the generalized Lie derivative: $L_D = [d, \iota_D]$ where ι_D is the contraction of a form by a multivector.

(b) There is a bigger differential graded Lie algebra (DGLA) $\mathfrak{g}[\epsilon, u]$ with the differential $u\frac{\partial}{\partial\epsilon}$. Here u is a formal parameter of degree 2 and ϵ is a formal parameter of degree 1 such that $\epsilon^2 = 0$. It acts on the complex $\Omega^{-\bullet}[[u]]$ with the differential ud as follows: $X + \epsilon Y$ acts by $L_X + \iota_Y$.

II. **(Differential graded) associative algebras**. There are several:

(a) Forms with wedge multiplication.

(b) Multivectors with wedge multiplication.

(c) Differential operators on functions (and sections of other vector bundles).

(d) In particular, differential operators on differential forms.

The algebras II(a) and II(b) are graded commutative.

III. **Calculi**. The structures from I, as well as from II(b), give rise to an algebraic structure that we call *a calculus*. In particular, multivectors form a *Gerstenhaber algebra*, or simply a *G-algebra*. We recall the definitions in 4.1. They formalize algebraic properties of multivectors with the wedge product and the Schouten bracket, forms with the De Rham differential, and of the former acting on the latter by contraction and by Lie derivative. One reconstructs the algebra II(d) from the calculus as an enveloping algebra of a certain kind.

We would like to generalize all the above constructions to the case when a manifold is replaced by a possibly noncommutative algebra over a field of characteristic zero. First note that all the algebras as in I–III will be replaced by *algebras up to homotopy*, namely, L_∞, A_∞, C_∞, G_∞, and Calc_∞ algebras.

There are two ways to look at such objects. One is to say that a complex $C^\bullet$ is an algebra of certain type up to homotopy if a DG algebra $\mathcal{C}^\bullet$ of this type is given, together with a quasi-isomorphism of complexes $\mathcal{C}^\bullet \to C^\bullet$. (In the case of Calc_∞ algebras one talks rather about pairs of complexes.) An A_∞, etc. morphism $C_1 \to C_2$ is a chains of morphisms $\mathcal{C}_1 \leftarrow \mathcal{C} \to \mathcal{C}_2$ where the arrow on the left is a quasi-isomorphism. Such a morphism is a *quasi-isomorphism* if the map on the right is a quasi-isomorphism, too. There is a natural way of composing such morphisms.

Another way to talk about an algebra up to homotopy is to talk about the complex $C^\bullet$ equipped with a series of higher operations satisfying certain relations. We recall the definition of A_∞ algebras and modules in terms of higher operations in 2.3, and (implicitly) an analogous definition of L_∞ modules, in the beginning of Section 3. The definition of G_∞ algebras in these terms was given in [19], cf. also [39], [21]; an analogous definition of Calc_∞ algebras will be given in a subsequent work. Morphisms are defined in terms of higher operations as well. In all of the cases discussed above, including the BV case, a homotopy algebra structure can be defined as a coderivation of degree one of a free coalgebra of appropriate type (namely, over the cooperad $\mathrm{Lie}^{\mathrm{dual}}$, $\mathrm{Calc}^{\mathrm{dual}}$, $\mathrm{BV}^{\mathrm{dual}}$, etc.); that coderivation satisfies the Maurer–Cartan equation. A morphism of two homotopy algebras can be defined as a morphism of resulting DG coalgebras.

One can move between the two ways of defining algebras up to homotopy: from the first definition to the second by a procedure called *transfer of structure*, from the second to the first by another procedure called *rectification*, cf. [23], [32], [28], [34].

Let $C^\bullet(A)$ be the Hochschild cochain complex and $C_\bullet(A)$ the Hochschild chain complex of an algebra A over a field of characteristic zero. The former will play the role of noncommutative multivectors and the latter of noncommutative forms. We will start with the noncommutative analog of III and work our way back to I.

It was proven in [13], [28] that the pair $C^\bullet(A)$, $C_\bullet(A)$ is a Calc_∞ algebra whose underlying L_∞ structure is the one from Hochschild theory, given by the Gerstenhaber bracket on cochains and by some explicit action of cochains on chains. (Those two Lie operations should be viewed as a noncommutative analog of I(a).)

Noncommutative version of III. The Calc_∞ structure from [13] has two related drawbacks: its construction is highly inexplicit and involved, and it is not canonical. The latter part is due to a fundamental fact about Calc_∞ (as well as G_∞) structures: they know how to generate new such structures from themselves. That is, starting from a Gerstenhaber algebra, one can construct new algebras (deformations of the old one) in a universal way, using only the Gerstenhaber operations of the bracket and the product. Similarly, there are universal moves that produce one G_∞ structure from another (indeed, pass from a G_∞ algebra to a DG G-algebra by rectification, then apply the above construction, and go back by the transfer of structure).

The group generated by these moves is a group of symmetries acting on the set of G_∞ algebra structures on any given space. (Or, which is the same, the operad G_∞ has a nontrivial group of symmetries.) It follows from results of [26], [28], [40] that the Grothendieck–Teichmüller group maps to this group of symmetries. Apparently, all of the above is true for Calc_∞ algebras.

This is a phenomenon that is largely absent from the world of associative or Lie algebras (note, however, that a Lie algebra structure $[,]$ automatically comes in a one-dimensional family $t[,]$). Indeed, higher operations in an L_∞ or an A_∞ algebra are of negative degree, and there is no way to produce a universal formula for such an operation using only the commutator or the product that are of degree zero.

On the other hand, if one has a Gerstenhaber algebra, one can define, say, a new A_∞ structure on it in a universal way, using only the multiplication and the bracket. An example of such a structure over the ring $\mathbb{C}[\hbar]/(\hbar^2)$: $m_3 = 0$; $m_4(a_1, a_2, a_3, a_4) = (-1)^{|a_1|+|a_2|}\hbar[a_1, a_2][a_3, a_4]$; $m_k = 0,\ k > 4$. (This A_∞ structure can be extended to a G_∞ structure; to see this, recall from [19], [39] that a G_∞ structure on C is a collection of operations

$m_{k_1,\ldots,k_n} : C^{k_1+\ldots+k_n} \to C$ subject to some relations; let m_4 be as above, $m_{2,2}(a_1, a_2; a_3, a_4) = \hbar[[a_1, a_2], [a_3, a_4]]$ and let all other higher operations be zero).

Note that all of the above applies to the classical calculus structure on the spaces of multivectors and forms; many other Calc_∞ structures may be generated from it. But, on the one hand, the standard structure is clearly the most natural; on the other hand, one can prove that all Calc_∞ structures that could be written naturally on a smooth manifold are equivalent to the standard one. This can be done using the argument as in [39] and [21], plus some formal differential geometry. (There are nonstandard Calc_∞ structures on multivectors and fields; any 3-cohomology class gives one. All this is of course consistent with the fact that there are no natural odd cohomology classes on manifolds; there are natural even classes, namely, the Chern classes of the tangent bundle).

If we try to look at the noncommutative analogs of the structures I and II that arise from the noncommutative version of III, the situation becomes easier for II and a lot easier for I.

Noncommutative version of II. The structures (a) does not generalize to our version of noncommutative calculus. Indeed, there is a product on the Hochschild homology, but only for a commutative ring A. For example, the degree zero Hochschild homology of A is $A/[A, A]$, the quotient of A by the linear span of commutators; this space does not have any natural multiplication. In comparison, the zero degree Hochschild cohomology of A is the center of A which is always a commutative algebra. (Note, however, that for a deformation quantization of a smooth manifold the space of noncommutative forms, i.e., the Hochschild chain complex, is quasi-isomorphic to the Poisson chain complex; this follows from [11] and [37], cf. also [42]. But the differential in the Poisson chain complex is a BV operator. Therefore, in the case of deformation quantization, the Hochschild chain complex is *a homotopy BV algebra*; that is, there is a natural model for chains that carries a graded commutative product; the differential is not a derivation with respect to this product but rather a BV operator (in particular, a differential operator of order two). As for the structure (c), its generalization to our version of noncommutative calculus is unknown and, in our view, not likely to exist.

The noncommutative version of the DG algebra IId) of differential operators on differential forms was described in [43] (we recall and use it in this paper). It was proven there that, indeed, it is the one coming from the Calc_∞ structure generalizing III.

As for the generalized algebra II(b) of multivectors, the situation is more delicate. It is easy to name one candidate, the DG algebra $C^\bullet(A)$ of Hochschild cochains with the standard differential δ and the cup product $\smile$. All we know is that the C_∞ algebra structure on $C^\bullet(A)$ which is a part of the Calc_∞ structure from [13] and [28] is an A_∞ deformation of the cup product. Equivalently, the cup product is an A_∞ deformation of the C_∞ algebra $C^\bullet(A)$ coming from the Calc_∞ structure. But, as we discussed above, there may be many such deformations. So far we do not even know whether the DGA $(C^\bullet(A), \delta, \smile)$ is A_∞ equivalent to a C_∞ algebra.

Noncommutative version of I. Here the situation becomes much easier: a noncommutative version of I is explicit and canonical. Namely, the negative cyclic complex $\mathrm{CC}^-_{-\bullet}(A) = (C_{-\bullet}(A)[[u]],\, b + uB)$ is an L_∞ module over the DGLA $(\mathfrak{g}^\bullet_A[\epsilon, u], \delta + u\frac{\partial}{\partial\epsilon})$ where $\mathfrak{g}^\bullet_A = (C^{\bullet+1}_A, \delta, [,])$ is the DG Lie algebra of Hochschild cochains with the Gerstenhaber bracket. We give explicit formulas for this L_∞ structure (Theorem 3) and prove that this structure is L_∞ equivalent to the one induced by the Calc_∞ structure on $(C^\bullet(A), C_\bullet(A))$ (Theorem 5).

The reason for the latter statement is the following. Unlike the associative multiplication, the Lie algebra structures that are part of the definition of a calculus live, degreewise, on the very edge of the calculus structure and do not have any room for change. For example, unlike the product, the Lie bracket on a Gerstenhaber algebra cannot be universally deformed, simply because there are no universal operations of needed degrees. Likewise, the DG Lie algebra structure of I(b) cannot be universally deformed: universal operations of needed degrees are too few and easily controlled.

To define the L_∞ structure from Theorem 3, we, following [45], construct it from the noncommutative version of the ring IId) of differential operators on forms, namely, the A_∞ algebra $\mathrm{CC}^-_{-\bullet}(C^\bullet(A))$ that was studied in [43]. It would be interesting, instead of describing the latter by explicit formulas, to interpret it as a part of the A_∞ category of A_∞ functors as described in [28], [25], as well as in [41]. (In a crude form, this was basically the idea in [35].)

We conclude the paper by showing that, given a family $\mathcal{A}$ of algebras on a variety S, assuming that this family admits a connection as a family of vector spaces, there exists a flat superconnection on the family of complexes $s \mapsto \mathrm{CC}^{\mathrm{per}}_{-\bullet}(\mathcal{A}_s)$. This generalizes Getzler's construction of the Gauss–Manin connection [17] from the level of homology to the level of actual complexes. Note that a modified version of Getzler's construction was used in [24].

It would be interesting to compare the results of this paper to the ones from [20] and [44].

Concluding remarks. We see that noncommutative differential calculus has two levels. At the higher level, there is an inexplicit, noncanonical structure of a Calc_∞ algebra on the pair $(C^\bullet(A), C_\bullet(A))$. At the lower level, there are some simpler structures whose existence is implied by the existence of the Calc_∞ structure but they themselves are explicit and canonical. They are:

(1) the A_∞ algebra $\mathrm{CC}^-_{-\bullet}(C^\bullet(A))$ and the A_∞ module $\mathrm{CC}^-_{-\bullet}(A)$ over it (noncommutative differential operators on forms);

(2) the DGLA $\mathfrak{g}^\bullet_A = C^{\bullet+1}(A)$ (noncommutative multivectors);

(3) the L_∞ module structure on $\mathrm{CC}^-_{-\bullet}(A)$ over $(\mathfrak{g}_A[\epsilon, u], \delta + u\frac{\partial}{\partial\epsilon})$ (noncommutative analog of multivectors acting on forms by Lie derivative and by contraction); and also

(4) the calculus $(H^\bullet(A), H_\bullet(A))$, the homology of $(C^\bullet(A), C_\bullet(A))$. The explicit formulas were given in [10].

A large group of symmetries acts on the space of choices for the Calc_∞ structure. Note that the group acts not by automorphisms of any structure but on the space of choices of the structure itself. There is an important case when (some extension of) this group acts by automorphisms of the structures (1)–(4).

Indeed, take for A the sheaf of functions on a smooth manifold (real, complex, or algebraic). A formality theorem is true for the Calc_∞ structure above (cf. [13], [14]); namely, for any choice α of the Calc_∞ structure, there is a Calc_∞ quasi-isomorphism of sheaves of Calc_∞ algebras

$$\Phi_\alpha : (C^\bullet(\mathcal{O}_X), C_\bullet(\mathcal{O}_X))_\alpha \xrightarrow{\sim} (\wedge^\bullet T_X, \Omega_X^\bullet) \tag{1}$$

Here the left-hand side is equipped with the Calc_∞ structure given by α and the right-hand side with the standard calculus structure. The cohomology of the right-hand side can be identified with the standard calculus $(H^\bullet(\mathcal{O}_X), H_\bullet(\mathcal{O}_X))$; comparing two such identifications, for any α and β we get an automorphism of the standard calculus

$$\Phi_{\alpha\beta} : (H^\bullet(X, \wedge^\bullet T_X); H^\bullet(X, \Omega_X^\bullet)) \xrightarrow{\sim} (H^\bullet(X, \wedge^\bullet T_X); H^\bullet(X, \Omega_X^\bullet)) \tag{2}$$

It would be very interesting to compare the above construction to the L_∞ quasi-isomorphisms constructed by Merkulov in [33].

In [26], Kontsevich constructed automorphisms of the cohomology with coefficients in multivector fields that are probably part of the above construction. Similarly, for any α and β one constructs an automorphism of the classical analog of any of the structures (1)–(3): an L_∞ quasi-isomorphism of $\Omega^{0,\bullet}(X, \wedge^{\bullet+1}T_X)$ with itself; a compatible quasi-isomorphism of L_∞ modules over $(\Omega^{0,\bullet}(X, \wedge^{\bullet+1}T_X)[\epsilon, u], \overline{\partial} + u\frac{\partial}{\partial\epsilon})$ between $(\Omega^{0,\bullet}(X, \Omega_X^{-\bullet})[[u]], \overline{\partial} + u\partial)$ and itself, etc. Indeed, one has

$$C^{\bullet+1}(\mathcal{O}_X)_{\mathrm{Gerst}} \xleftarrow{\Psi_\alpha} C^{\bullet+1}(\mathcal{O}_X)_\alpha \xrightarrow{\Phi_\alpha} \wedge^{\bullet+1}T_X$$

The DGLA on the left is the Hochschild cochain complex with the Gerstenhaber bracket. The map on the right is a G_∞ quasi-isomorphism of sheaves of G_∞ algebras (formality); the one on the right is the L_∞ quasi-isomorphism of sheaves of DGLA (rigidity). Comparing two such sequences for α and β, we get an L_∞ quasi-isomorphism at the level of algebras; similarly for modules.

Let us finish by some remarks about the algebraic index theorem. The formality Calc_∞ quasi-isomorphism implies a quasi-isomorphism of complexes

$$\Omega^{0,\bullet}(X, \mathrm{CC}^{\mathrm{per}}_{-\bullet}(\mathcal{O}_X)) \xrightarrow{\sim} \Omega^{0,\bullet}(X, \Omega_X^{-\bullet}((u))) \tag{3}$$

An algebraic index theorem is a statement comparing it to the standard Hochschild–Kostant–Rosenberg map (and an analogous statement for a deformation quantization of $\mathcal{O}_X$; cf. [3], [4] for the symplectic case). Equivalently, it is a statement about the image of the zero-homology class 1. One can show that this image is an expression in the Chern classes of T_X that becomes

$\sqrt{\widehat{A}(T_X)}$ if we send all the odd Chern classes to zero. If the automorphisms in [26] do extend to automorphisms of calculi that come from (2), then it looks like any multiplicative characteristic class with the above property may occur, for an appropriate choice of α. Note also that nothing in our argument implies that the symmetries constructed above are Calc_∞ quasi-isomorphisms of $(\wedge^\bullet T_X, \Omega^\bullet_X)$ with itself. On the other hand, there definitely are some Calc_∞ quasi-isomorphisms, for instance the exponential of the following derivation:

$$\iota_{c_1(TX)} : \Omega^{0,\bullet}(X, \wedge^\bullet T_X) \to \Omega^{0,\bullet+1}(X, \wedge^{\bullet-1} T_X)$$

and

$$c_1(TX)\wedge : \Omega^{0,\bullet}(X, \Omega^\bullet_X) \to \Omega^{0,\bullet+1}(X, \Omega^{\bullet+1}_X)$$

Note also that the Hochschild–Rosenberg map, followed with multiplication by the square root of the Todd class, appears in [31], [5], [6], [7], and [36] and is characterized by preserving another algebraic structure. Namely, it intertwines the Mukai pairing with the standard pairing on the cohomology. This suggests that the correct formality theorem could be formulated for an algebraic structure encompassing both the calculus and the pairing, probably related to (genus zero part of) the TQFT structure from [9] and [28].

2 Operators on forms in noncommutative calculus

2.1 The Hochschild cochain complex

Let A be a graded algebra with unit over a commutative unital ring K of characteristic zero. A Hochschild d-cochain is a linear map $A^{\otimes d} \to A$. Put, for $d \geq 0$,

$$C^d(A) = C^d(A, A) = \mathrm{Hom}_K(\overline{A}^{\otimes d}, A)$$

where $\overline{A} = A/K \cdot 1$. Put $|D| = ($ degree of the linear map $D) + d$.

Put for cochains D and E from $C^\bullet(A, A)$

$$(D \smile E)(a_1, \ldots, a_{d+e}) = (-1)^{|E| \sum_{i\leq d}(|a_i|+1)} D(a_1, \ldots, a_d) E(a_{d+1}, \ldots, a_{d+e});$$

$$(D \circ E)(a_1, \ldots, a_{d+e-1}) = \sum_{j\geq 0} (-1)^{(|E|+1)\sum_{i=1}^{j}(|a_i|+1)} D(a_1, \ldots, a_j, E(a_{j+1}, \ldots, a_{j+e}), \ldots);$$

$$[D,\ E] = D \circ E - (-1)^{(|D|+1)(|E|+1)} E \circ D$$

These operations define the graded associative algebra $(C^\bullet(A, A)\ , \smile)$ and the graded Lie algebra $(C^{\bullet+1}(A, A), [\ ,\ ])$ (cf. [8]; [17]). Let

$$m(a_1, a_2) = (-1)^{\deg a_1}\ a_1 a_2;$$

this is a 2-cochain of A (not in C^2). Put

$$\delta D = [m, D];$$

$$(\delta D)(a_1, \ldots, a_{d+1}) = (-1)^{|a_1||D|+|D|+1} a_1 D(a_2, \ldots, a_{d+1})$$
$$+\sum_{j=1}^{d} (-1)^{|D|+1+\sum_{i=1}^{j}(|a_i|+1)} D(a_1, \ldots, a_j a_{j+1}, \ldots, a_{d+1})$$
$$+(-1)^{|D|\sum_{i=1}^{d}(|a_i|+1)} D(a_1, \ldots, a_d) a_{d+1}$$

One has

$$\delta^2 = 0; \quad \delta(D \smile E) = \delta D \smile E + (-1)^{|degD|} D \smile \delta E$$
$$\delta[D, E] = [\delta D, E] + (-1)^{|D|+1} [D, \delta E]$$

($\delta^2 = 0$ follows from $[m, m] = 0$).

Thus $C^\bullet(A, A)$ becomes a complex; we will denote it also by $C^\bullet(A)$. The cohomology of this complex is $H^\bullet(A, A)$ or the Hochschild cohomology. We denote it also by $H^\bullet(A)$. The $\smile$ product induces the Yoneda product on $H^\bullet(A, A) = Ext^\bullet_{A\otimes A^0}(A, A)$. The operation $[\ ,\ ]$ is the Gerstenhaber bracket [16].

If $(A, \ \partial)$ is a differential graded algebra, then one can define the differential ∂ acting on A by

$$\partial D = [\partial, D]$$

Theorem 1 *[16] The cup product and the Gerstenhaber bracket induce a Gerstenhaber algebra structure on $H^\bullet(A)$.*

2.2 Hochschild chains

Let A be an associative unital DG algebra over a ground ring K. The differential on A is denoted by δ. Recall that by definition

$$\overline{A} = A/K \cdot 1$$

Set

$$C_p(A, A) = C_p(A) = A \otimes \overline{A}^{\otimes p}$$

Define the differentials $\delta : C_\bullet(A) \to C_\bullet(A)$, $b : C_\bullet(A) \to C_{\bullet-1}(A)$, $B : C_\bullet(A) \to C_{\bullet+1}(A)$ as follows:

$$\delta(a_0 \otimes \cdots \otimes a_p) = \sum_{i=1}^{p} (-1)^{\sum_{k<i}(|a_k|+1)+1} (a_0 \otimes \cdots \otimes \delta a_i \otimes \cdots \otimes a_p);$$

$$b(a_0 \otimes \cdots \otimes a_p) = \sum_{k=0}^{p-1} (-1)^{\sum_{i=0}^{k}(|a_i|+1)+1} a_0 \cdots \otimes a_k a_{k+1} \otimes \cdots a_p \tag{4}$$

$$+(-1)^{|a_p|+(|a_p|+1)\sum_{i=0}^{p-1}(|a_i|+1)}a_p a_0 \otimes \cdots \otimes a_{p-1};$$

$$B(a_0\otimes\cdots\otimes a_p) = \sum_{k=0}^{p}(-1)^{\sum_{i\le k}(|a_i|+1)\sum_{i\ge k}(|a_i|+1)}1\otimes a_{k+1}\otimes\cdots a_p\otimes a_0\otimes\cdots\otimes a_k \tag{5}$$

(cf. [30]).

The complex $C_\bullet(A)$ is the total complex of the double complex with the differential $b+\delta$.

Let u be a formal variable of degree two. The complex $(C_\bullet(A)[[u]], b+\delta+uB)$ is called *the negative cyclic complex* of A.

Now put

$$L_D(a_0\otimes\cdots\otimes a_n) = \sum_{k=1}^{n-d}\epsilon_k a_0\otimes\cdots\otimes D(a_{k+1},\ldots,a_{k+d})\otimes\cdots\otimes a_n \tag{6}$$

$$+\sum_{k=n+1-d}^{n}\eta_k D(a_{k+1},\ldots,a_n,a_0,\ldots)\otimes\cdots\otimes a_k$$

(The second sum in the above formula is taken over all cyclic permutations such that a_0 is inside D.) The signs are given by

$$\epsilon_k = (|D|+1)(|a_0|+\sum_{i=1}^{k}(|a_i|+1))$$

and

$$\eta_k = |D| + \sum_{i\le k}(|a_i|+1)\sum_{i\ge k}(|a_i|+1)$$

Proposition 1

$$[L_D, L_E] = L_{[D,E]};\quad [b, L_D] + L_{\delta D} = 0;\quad [L_D, B] = 0$$

2.3 A_∞ algebras and modules

Recall [29], [38] that an A_∞ algebra is a graded vector space $\mathcal{C}$ together with a Hochschild cochain m of total degree 1,

$$m = \sum_{n=1}^{\infty} m_n$$

where $m_n \in C^n(\mathcal{C})$ and

$$[m, m] = 0$$

Recall also the definition of A_∞ modules over A_∞ algebras. First, note that for a graded space $\mathcal{M}$, the Gerstenhaber bracket $[\ ,\ ]$ can be extended to the space

$$\mathrm{Hom}(\overline{\mathcal{C}}^{\otimes\bullet}, \mathcal{C}) \oplus \mathrm{Hom}(\mathcal{M} \otimes \overline{\mathcal{C}}^{\otimes\bullet}, \mathcal{M})$$

For a graded k-module $\mathcal{M}$, a structure of an A_∞ module over an A_∞ algebra $\mathcal{C}$ on $\mathcal{M}$ is a cochain of total degree one

$$\mu = \sum_{n=1}^{\infty} \mu_n$$

$$\mu_n \in Hom(\mathcal{M} \otimes \overline{\mathcal{C}}^{\otimes n-1}, \mathcal{M})$$

such that

$$[m + \mu, m + \mu] = 0$$

2.4 The A_∞ structure on chains of cochains for the definition of the negative cyclic complex, cf. [30]

Theorem 2 *There is a structure $\{\mathbf{m}_n\}$ of an A_∞ algebra on $CC^-_{-\bullet}(C^\bullet(A))$, and a structure $\{\mu_n\}$ of an A_∞ module over this A_∞ algebra on $C_{-\bullet}(A)[[u]]$, such that:*

- *All $\mathbf{m}_n$ and μ_n are $k[[u]]$-linear, (u)-adically continuous.*
- $\mathbf{m}_1 = b + \delta + uB$; $\mu_1 = b + uB$.
- *Modulo u, the space $C_0(C^\bullet(A)) = C^\bullet(A)$ is a subalgebra, with the structure given by the cup product.*
- *For $a \in C_{-\bullet}(A)[[u]]$, $D \in C^\bullet(A)$: $\mu_2(a, 1 \otimes D) = (-1)^{|a||D|} L_D a$.*

Explicit formulas can be found in [43], [45]. They are valid for any brace algebra. In the case of a commutative algebra, i.e. when all the brace operations are zero, they where discovered in [18] and [22]. The proof is given in [45].

3 The L_∞ module structure on the negative cyclic complex

Now introduce the following differential graded algebras. Let $C(\mathfrak{g}^\bullet_A[u,\epsilon])$ be the standard Chevalley–Eilenberg chain complex of the DGLA $\mathfrak{g}^\bullet_A[u,\epsilon]$ over the ring of scalars $K[u]$. It carries the Chevalley–Eilenberg differential ∂ and the differentials δ and ∂_ϵ induced by the corresponding differentials on $\mathfrak{g}^\bullet_A[u,\epsilon]$. Let $C_+(\mathfrak{g}^\bullet_A[u,\epsilon])$ be the augmentation co-ideal, i.e., the sum of all positive exterior powers of our DGLA. The comultiplication defines maps

$$C_+(\mathfrak{g}^\bullet_A[u,\epsilon]) \mapsto C_+(\mathfrak{g}^\bullet_A[u,\epsilon])^{\otimes n};$$

$$c \mapsto \sum c_1^+ \otimes \cdots \otimes c_n^+$$

Definition 1. *Define the associative DGA $B(\mathfrak{g}_A^\bullet[u,\epsilon])$ over $K[[u]]$ as the tensor algebra of $C_+(\mathfrak{g}_A^\bullet[u,\epsilon])$ with the differential d determined by*

$$dc = (\delta+\partial)c - \frac{1}{2}\sum(-1)^{|c_1^+|}c_1^+c_2^+ + u\partial_\epsilon c.$$

Definition 2. *Let the associative DGA $B^{\mathrm{tw}}(\mathfrak{g}_A^\bullet[u,\epsilon])$ over $K[[u]]$ be the tensor algebra of $C_+(\mathfrak{g}_A^\bullet[u,\epsilon])$ with the differential d determined by*

$$dc = (\delta+\partial)c - \frac{1}{2}\sum(-1)^{|c_1^+|}c_1^+c_2^+ + u\sum_{n=1}^{\infty}\partial_\epsilon c_1^+\cdots\partial_\epsilon c_n^+$$

A structure of an L_∞ module over $\mathfrak{g}_A^\bullet[u,\epsilon]$ on a complex $\mathcal{M}$ is by definition a morphism of DGA $B(\mathfrak{g}_A^\bullet[u,\epsilon]) \to \mathrm{End}(\mathcal{M})$. It would be nice to have explicit formulas for such a morphism. What we can do instead is construct an explicit morphism

$$B^{\mathrm{tw}}(\mathfrak{g}_A^\bullet[u,\epsilon]) \to \mathrm{End}\mathrm{CC}_{-\bullet}^{-}(A)$$

together with a quasi-isomorphism of DGAs

$$U(\mathfrak{g}_A^\bullet[u,\epsilon]) \to B^{\mathrm{tw}}(\mathfrak{g}_A^\bullet[u,\epsilon])$$

Let $S(\mathfrak{g}_A^\bullet)^+$ be the augmentation ideal, and let

$$Y \mapsto \sum Y_1^+\otimes\ldots\otimes Y_n^+ \tag{7}$$

denote the map

$$S(\mathfrak{g}_A^\bullet)^+ \to (S(\mathfrak{g}_A^\bullet)^+)^{\otimes n} \tag{8}$$

defined as the n-fold coproduct, followed by the nth power of the projection from $S(\mathfrak{g}_A^\bullet)$ to $S(\mathfrak{g}_A^\bullet)^+$ along $K\cdot 1$.

Definition 3. *For $n \geq 1$, define:*

$$x\cdot(\epsilon E_1\wedge\ldots\wedge\epsilon E_n) = \sum_{n\geq 1}(-1)^{|x|}\mu_{n+1}(x,\overline{Y}_1^+,\ldots,\overline{Y}_n^+);$$

for $n \geq 0$,

$$x\cdot(\epsilon E_1\wedge\ldots\wedge\epsilon E_n\wedge D) = \sum_{n\geq 1}(-1)^{|x|}\mu_{n+2}(x,\overline{Y}_1^+,\ldots,\overline{Y}_n^+,1\otimes D);$$

$$x\cdot(\epsilon E_1\wedge\ldots\wedge\epsilon E_n\wedge D_1\wedge\ldots\wedge D_k) = 0$$

for $k > 1$. Here $D,\ D_i,\ E_j \in \mathfrak{g}_A^\bullet$ and $Y = E_1\cdots E_n \in S(\mathfrak{g}_A^\bullet)^+$.

Proposition 2 *The formulas from Definition 3 above define an action of the DGA $B^{\mathrm{tw}}(\mathfrak{g}_A^\bullet[u,\epsilon])$ on $\mathrm{CC}_{-\bullet}^{-}(A)$.*

The proof is contained in [45].

3.1 The L_∞ action

It remains to pass from $B^{\mathrm{tw}}(\mathfrak{g}_A^\bullet[\epsilon,u])$ to $U(\mathfrak{g}_A^\bullet[\epsilon,u])$.

The following is contained in [45].

Lemma 1 *The formulas*

$$D \to D;$$

$$\epsilon E_1 \wedge \ldots \wedge \epsilon E_n \mapsto \frac{1}{n!}\sum_{\sigma\in S_n}\frac{1}{n!}(\epsilon E_{\sigma_1})E_{\sigma_2}\cdots E_{\sigma_n};$$

$$D_1 \wedge \ldots D_k \wedge \epsilon E_1 \wedge \ldots \wedge \epsilon E_n \mapsto 0$$

for $k>1$ or $k=1$, $n\geq 1$ define a quasi-isomorphism of DGAs

$$B^{\mathrm{tw}}(\mathfrak{g}_A^\bullet[\epsilon,u]) \to U(\mathfrak{g}_A^\bullet[\epsilon,u])$$

Proof. The fact that the above map is a morphism of DGAs follows from an easy direct computation. To show that this is a quasi-isomorphism, consider the increasing filtration by powers of ϵ. At the level of graded quotients, $B^{\mathrm{tw}}(\mathfrak{g}_A^\bullet[\epsilon,u])$ becomes the standard free resolution of $(U(\mathfrak{g}_A^\bullet[\epsilon,u]),\delta)$, and the morphism is the standard map from the resolution to the algebra, therefore a quasi-isomorphism. The statement now follows from the comparison argument for spectral sequences.

To summarize, we have constructed explicitly a DGA $B^{\mathrm{tw}}(\mathfrak{g}_A^\bullet[\epsilon,u])$ and the morphisms of DGAs

$$U(\mathfrak{g}_A^\bullet[\epsilon,u]) \leftarrow B^{\mathrm{tw}}(\mathfrak{g}_A^\bullet[\epsilon,u]) \to \mathrm{End}_{K[[u]]}(\mathrm{CC}^-_{-\bullet}(A)) \tag{9}$$

where the morphism on the left is a quasi-isomorphism. This yields an A_∞ morphism

$$U(\mathfrak{g}_A^\bullet[\epsilon,u]) \to \mathrm{End}_{K[[u]]}(\mathrm{CC}^-_{-\bullet}(A))$$

and therefore an L_∞ morphism

$$\mathfrak{g}_A^\bullet[\epsilon,u] \to \mathrm{End}_{K[[u]]}(\mathrm{CC}^-_{-\bullet}(A))$$

We get

Theorem 3 *The maps (9) define on $\mathrm{CC}^-(A)$ a $K[u]$-linear, (u)-adically continuous structure of an L_∞ module over the DGLA $(\mathfrak{g}_A^\bullet[\epsilon,u],\delta+u\frac{\partial}{\partial\epsilon})$ where $\mathfrak{g}_A^\bullet=(C^{\bullet+1}(A),\delta,[,]_G)$.*

Remark 1 *The property of being $K[u]$-linear is crucial. Indeed, as pointed out by K. Costello, $\mathfrak{g}^\bullet[\epsilon,u]$ is quasi-isomorphic to $\mathfrak{g}^\bullet$, so every $\mathfrak{g}^\bullet$-module is an L_∞ module over $\mathfrak{g}^\bullet[\epsilon,u]$ by transfer of structure.*

4 Relation to the homotopy calculus structure

4.1 Calculi

A *Gerstenhaber algebra* is a graded space $\mathcal{V}^\bullet$ together with

- A graded commutative associative algebra structure on $\mathcal{V}^\bullet$;
- A graded Lie algebra structure on $\mathcal{V}^{\bullet+1}$ such that

$$[a, bc] = [a, b]c + (-1)^{((|a|-1)|b|)} b[a, c]$$

Definition 4. *A precalculus is a pair of a Gerstenhaber algebra $\mathcal{V}^\bullet$ and a graded space $\Omega^\bullet$ together with*

- *A structure of a graded module over the graded commutative algebra $\mathcal{V}^\bullet$ on $\Omega^{-\bullet}$ (corresponding action is denoted by i_a, $a \in \mathcal{V}^\bullet$);*
- *A structure of a graded module over the graded Lie algebra $\mathcal{V}^{\bullet+1}$ on $\Omega^{-\bullet}$ (corresponding action is denoted by L_a, $a \in \mathcal{V}^\bullet$) such that*

$$[i_a, L_b] = i_{[a,b]}$$

and

$$L_{ab} = L_a i_b + (-1)^{|a|} i_a L_b$$

Definition 5. *A calculus is a precalculus together with an operator d of degree 1 on $\Omega^\bullet$ such that $d^2 = 0$ and*

$$[d, i_a] = L_a$$

Example 1. For any manifold one defines a calculus $\mathrm{Calc}(M)$ with $\mathcal{V}^\bullet$ being the algebra of multivector fields, $\Omega^\bullet$ the space of differential forms, and d the De Rham differential. The operator i_a is the contraction of a form by a multivector field.

Definition 6. *A differential graded calculus is a calculus $(\mathcal{V}^\bullet, \Omega^{-\bullet})$ with differentials δ on $\mathcal{V}^\bullet$ and b on $\Omega^\bullet$ that are both of degree one and are derivations with respect to the calculus structure.*

The following construction is motivated by Example 1. For a Gerstenhaber algebra $\mathcal{V}^\bullet$, let $\mathbf{Y}(\mathcal{V}^\bullet)$ be the associative algebra generated by two sets of generators i_a, L_a, $a \in \mathcal{V}^\bullet$, both i and L linear in a,

$$|i_a| = |a|; \ |L_a| = |a| - 1$$

subject to relations

$$i_a i_b = i_{ab}; \ \ [L_a, L_b] = L_{[a,b]};$$

$$[i_a, L_b] = i_{[a,b]}; \ L_{ab} = L_a i_b + (-1)^{|a|} i_a L_b$$

The algebra $\mathbf{Y}(\mathcal{V}^\bullet)$ is equipped with the differential d of degree one which is defined as a derivation sending i_a to L_a and L_a to zero.

For a smooth manifold M one has a homomorphism

$$\mathbf{Y}(\mathcal{T}^{\bullet+1}_{poly}(M)) \to D(\Omega^\bullet(M))$$

The right-hand side is the algebra of differential operators on differential forms on M, and the above homomorphism sends the generators i_a, L_a to corresponding differential operators on forms (cf. Example 1). The above map is in fact an isomorphism, cf. [13], Proposition 11 in Section 6.3.

4.2 Comparing the two L_∞ module structures

For a DG calculus $(\mathcal{V}^\bullet, \Omega^\bullet)$, let $\mathfrak{g}^\bullet$ be the DGLA $(\mathcal{V}^{\bullet+1}, \delta, \{,\})$. One has:

(a) $(\Omega^{-\bullet}[[u]], b + ud)$ is a DG module over $\mathfrak{g}^\bullet$. Moreover:

(b) $(\Omega^{-\bullet}[[u]], b+uB)$ is a DG module over $(\mathfrak{g}^\bullet[\epsilon, u], \delta+u\frac{\partial}{\partial\epsilon})$, $X+\epsilon Y$ acting via $L_X + \iota_Y$.

The same is true for a Calc_∞ algebra $(\mathcal{V}^\bullet, \Omega^\bullet)$ if one replaces DG modules by L_∞ modules. Recall from [13]:

Theorem 4 *$(C^\bullet(A), C_\bullet(A))$ is a Calc_∞ algebra whose underlying L_∞ structure as in (a) above is: $\mathfrak{g}^\bullet = \mathfrak{g}^\bullet_A = (C^{\bullet+1}(A), \delta, [,]_G)$, acting on $\mathrm{CC}^-_{-\bullet}(A)$ via the Lie derivative L_D.*

From this, and from (b) above, we conclude that $\mathrm{CC}^-_{-\bullet}(A)$ is an L_∞ module over $(\mathfrak{g}^\bullet[\epsilon, u], \delta + u\frac{\partial}{\partial\epsilon})$. Indeed, as explained in the introduction, the pair $(C^\bullet, C_\bullet)$ is quasi-isomorphic to another pair of complexes that is actually a DG calculus, with the DGLA structure equivalent to the one given by the Gerstenhaber bracket. Apply (b) to that pair and then get the L_∞ module structure on $C_\bullet(A)[[u]]$ by transfer of structure. Note also that an L_∞ algebra and an L_∞ module are in particular complexes, and the differentials coincide with the Hochschild differentials δ and b.

Theorem 5 *The above L_∞ structure is equivalent to the one given by Theorem 3.*

Proof. First, recall from [28], [27], [43] the notion of a two-colored operad and a chain of quasi-isomorphisms of two-colored operads

$$\mathrm{Calc}_{\mathrm{alg}} \leftarrow \mathrm{Calc}_{\mathrm{geom}} \leftarrow \mathrm{Calc}_\infty \to \mathrm{Calc} \tag{10}$$

Here $\mathrm{Calc}_{\mathrm{alg}}$ is the operad which acts on cochains and chain and which is generated by the cup-product on cochains, the insertions of cochains into a cochain, and insertions of components of a chain into a cochain compatible with the cyclic order on these components. The precise description of this operad can be found in [13] (Section 4.1) or [28] (Sections 11.1–11.3). Note that $\mathrm{Calc}_{\mathrm{alg}}$ was denoted by KS in [13].

The two-colored operad $\mathrm{Calc}_{\mathrm{geom}}$ is the operads of chain complexes of the spaces of configurations of little discs (on a disc and on a cylinder); algebras over Calc are by definition calculi, and Calc_∞ is a cofibrant resolution of Calc; algebras over it are Calc_∞ algebras. The two-colored operad Lie_∞^+ maps to Calc_∞. An algebra over Lie_∞^+ is a pair consisting of an L_∞ algebra and an L_∞ module over it. There is a parallel diagram for precalculi.

Note that for any L_∞ algebra $\mathcal{L}$ and any graded space $\mathcal{M}$, an L_∞ $\mathcal{L}$-module structure on $\mathcal{M}$ is the same as a Maurer–Cartan element of the Chevalley–Eilenberg complex $C^\bullet(\mathcal{L}, \mathrm{End}(\mathcal{M}))$ of cochains of $\mathcal{L}$ with coefficients in $\mathrm{End}(\mathcal{M})$ (viewed as a trivial L_∞ module). The DGLA structure on the Chevalley–Eilenberg complex is given by the commutator on $\mathrm{End}(\mathcal{M})$ combined with the wedge product. Now, assume that we have an algebra $(\mathcal{L},\mathcal{M})$ over a two-colored operad $\mathcal{P}$ to which the L_∞ operad maps. Then L_∞ $\mathcal{L}$-module structures on $\mathcal{M}$ that are given by universal formulas in terms of operations from $\mathcal{P}$ are the same as Maurer–Cartan elements of the complex $C_\mathcal{P}(\mathcal{L}, \mathrm{End}(\mathcal{M}))$ of Lie algebra cochains given by universal operations from $\mathcal{P}$.

It is clear that the L_∞ module structure from Theorem 3 is given by universal formulas in terms of operations from $\mathrm{Calc}_{\mathrm{alg}}$. Modulo u, these formulas involve only the precalculus analog of $\mathrm{Calc}_{\mathrm{alg}}$. Consider three complexes

$$C^\bullet_\mathcal{P}(\mathfrak{g}^\bullet[\epsilon, u], \mathrm{End}_{K[[u]]}(\Omega^{-\bullet}[[u]])) \tag{11}$$

where $\mathcal{P}$ stands for $\mathrm{Calc}_{\mathrm{alg}}$, Calc_∞, or Calc. As explained above, these are complexes of cochains of $\mathfrak{g}^\bullet[\epsilon, u]$ with coefficients in $\mathrm{End}_{K[[u]]}(\Omega^{-\bullet}[[u]])$ that are given by universal operations from $\mathrm{Calc}_{\mathrm{alg}}$, resp. Calc_∞, resp. Calc. Here $(\mathcal{V}^\bullet, \Omega^\bullet)$ is any algebra over one of the three two-colored operads; $\mathfrak{g}^\bullet$ is $\mathcal{V}^{\bullet+1}$ viewed as an L_∞ algebra via the map of Lie_∞^+ to one of these operads. In particular, a cochain in (11) produces a cochain in $C^\bullet(\mathfrak{g}^\bullet[\epsilon, u], \mathrm{End}_{K[[u]]}(\Omega^{-\bullet}[[u]]))$ for any $\mathcal{P}$-algebra.

Recall that a two-colored operad is, in particular, a collection of complexes $\mathcal{O}(n)$ and $\mathcal{M}(n)$; the first stands for operations $\mathcal{V}^{\bullet\otimes n} \to \mathcal{V}^\bullet$; the second for operations $\mathcal{V}^{\bullet\otimes n} \otimes \Omega^\bullet \to \Omega^\bullet$.

First, observe that the three complexes are all quasi-isomorphic. Indeed, they are given by direct sums or products of copies of subspaces of invariants of $\mathcal{M}(n)$ with respect to some subgroups of the symmetric group S_n, with an extra (Chevalley–Eilenberg) differential. Therefore a quasi-isomorphism of operads leads to a quasi-isomorphism of complexes.

The L_∞ module structures that we are looking for are Maurer–Cartan elements of the DGLAs of cochains of the type as above. The above quasi-isomorphisms preserve the Lie algebra structure and therefore induce isomorphisms on the sets of Maurer–Cartan elements up to equivalence. We want to prove that any two Maurer–Cartan cochains as above, defined by universal operations from $\mathrm{Calc}_{\mathrm{alg}}$, are equivalent. We see that we can replace $\mathrm{Calc}_{\mathrm{alg}}$ by Calc. Therefore, it suffices to prove the following. Let an L_∞ module structure be given by universal formulas purely in terms of the calculus operations $[a, b]$, ab, L_a, ι_a, d; moreover, modulo u, it is given by the first four, and it is the original action (b) at the level of homology. We claim that any such L_∞ structure is L_∞ equivalent to the original one.

More precisely, we have to prove the following. Let $(\mathcal{V}^\bullet, \Omega^\bullet)$ be any DG calculus; $\mathfrak{g}^\bullet = (\mathcal{V}^{\bullet+1}, \delta, \{,\})$; consider a Maurer–Cartan element of the DGLA $C^\bullet(\mathfrak{g}^\bullet[\epsilon, u], \mathrm{End}\Omega^{-\bullet}[[u]])$, the cochain complex of $(\mathfrak{g}^\bullet[\epsilon, u], \delta + u\frac{\partial}{\partial\epsilon})$ with coefficients in $\mathrm{End}\Omega^{-\bullet}[[u]]$ on which $\mathfrak{g}^\bullet[\epsilon, u]$ acts trivially. The Maurer–Cartan element, by definition, satisfies

$$(\delta + u\frac{\partial}{\partial\epsilon} + \partial_{\mathrm{Lie}})\lambda + \frac{1}{2}[\lambda, \lambda] = 0$$

where ∂_{Lie} is the Chevalley–Eilenberg differential. The element λ is a cochain defined by universal operations given by formulas involving the five calculus operations; modulo u, it involves the four precalculus operations only. For an n-linear map φ, put

$$\mathrm{Avg}\varphi(a_1, \dots, a_n) = \frac{1}{n!}\sum_{\sigma\in S_n} \pm\varphi(a_{\sigma 1}, \dots, a_{\sigma_n})$$

The only cochains of suitable degree are of the form

$$\lambda = \sum_{n\geq 1} \alpha_n \Phi_n + u\sum_{n\geq 1} \beta_n \Psi_n + \sum_{n\geq 0;\, k} \gamma_n^k \Theta_n^k \tag{12}$$

where

$$\Phi(a_1\epsilon, \dots, a_n\epsilon) = \mathrm{Avg}L_{a_1} \cdots L_{a_{n-1}}\iota_{a_n};$$
$$\Psi(a_1\epsilon, \dots, a_n\epsilon) = \mathrm{Avg}L_{a_1} \cdots L_{a_n} d;$$
$$\Theta_n^k(a_1\epsilon, \dots, a_n\epsilon, c) = \mathrm{Avg}L_{a_1} \cdots L_{a_k}\iota_c L_{a_{k+1}} \dots L_{a_n}$$

First, note that $\gamma_0^0 = 1$ and $\gamma_n^k = 0$ for all $n \geq 1$. Indeed, the component of ∂_{Lie} with values in cochains $\varphi(a_1\epsilon, \dots, a_n\epsilon, c_1, c_2)$ is zero on all Φ_n and Ψ_n; for Θ_n^k, it is equal to $\pm\mathrm{Avg}L_{a_1} \cdots L_{a_k}\iota_{\{c_1,c_2\}}L_{a_{k+1}} \cdots L_{a_n}$. But this component must be zero for a Maurer–Cartan element.

Note that any cochain $\lambda = \sum_{n\geq 1} \alpha_n\Phi_n$ defines an L_∞ action of $(\mathfrak{g}^\bullet[\epsilon], \delta)$; these actions are nonequivalent. For

$$\lambda = \sum_{n\geq 1} \alpha_n \Phi_n + u\sum_{n\geq 1} \beta_n \Psi_n$$

the terms $\delta\lambda$, $u\frac{\partial}{\partial\epsilon}\lambda$, and $\partial_{\mathrm{Lie}}\lambda$ are all zero. Thus we have

$$ud\lambda + \frac{1}{2}[\lambda, \lambda] = 0$$

One has $\beta_1 = 0$ and $\alpha_1 = 1$ because the action on the cohomology is the original one. Now, the gauge transformation

$$\exp(\sum \kappa_n X_n),$$

$$X_n(a_1\epsilon, \dots, a_n\epsilon) = \mathrm{Avg}L_{a_1} \dots L_{a_{n-1}}\iota_{a_n} d,$$

kills the Ψ_n terms, $n \geq 2$. Finally, the Maurer–Cartan equation for the cochain $\sum \alpha_n\Phi_n$ shows that $\alpha_n = 0$, $n \geq 2$.

4.3 Comparison to Section 2

We conclude this section by citing the result from [43] that explains the title of Section 2.

As above, let A be an associative unital algebra over a unital ring K of characteristic zero. Because of Theorem 4, the pair of complexes $(C^\bullet(A), C_\bullet(A))$ is quasi-isomorphic to a pair of complexes $(\mathcal{V}^\bullet(A), \Omega^\bullet(A))$ that has a structure of a DG calculus.

Theorem 6 *There is an A_∞ quasi-isomorphism*

$$(\mathbf{Y}(\mathcal{V}^\bullet(A))[[u]], \delta + ud) \to \mathrm{CC}^-_{-\bullet}(C^\bullet(A))$$

where the left-hand side is an A_∞ algebra as in Theorem 2. There is also an A_∞ quasi-isomorphism of A_∞ modules

$$(\Omega^{-\bullet}(A)[[u]], b + ud) \to \mathrm{CC}^-_{-\bullet}(A)$$

compatible with the A_∞ map above.

5 The Gauss–Manin connection

Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_S$-algebras where S is a manifold (real, complex, or algebraic). We assume that $\mathcal{A}$ carries a connection ∇ (not necessarily compatible with the product). Let $\mathrm{CC}^{\mathrm{per}}_\bullet(\mathcal{A})$ be the sheaf of periodic cyclic complexes of $\mathcal{A}$ over $\mathcal{O}_S$. (If $\mathcal{A}$ is the sheaf of local sections of a bundle of algebras, then $\mathrm{CC}^{\mathrm{per}}_\bullet(\mathcal{A})$ is the sheaf of local sections of the bundle of complexes $s \mapsto \mathrm{CC}^{\mathrm{per}}_\bullet(\mathcal{A}_s)$). We conclude the paper by constructing a *flat superconnection* on $\mathrm{CC}^{\mathrm{per}}_\bullet(\mathcal{A})$, i.e., an operator

$$\nabla_{\mathrm{GM}} : \Omega^\bullet_S \otimes_{\mathcal{O}_S} \mathrm{CC}^{\mathrm{per}}_\bullet(\mathcal{A}) \to \Omega^\bullet_S \otimes_{\mathcal{O}_S} \mathrm{CC}^{\mathrm{per}}_\bullet(\mathcal{A})$$

of degree one such that $\nabla^2_{\mathrm{GM}} = 0$ and $\nabla_{\mathrm{GM}}(fa) = f\nabla_{\mathrm{GM}}(a) + df \cdot a$ for a function f and a local section a.

Let $C^\bullet(\mathcal{A})$ be the sheaf of Hochschild cochain complexes of $\mathcal{A}$ over $\mathcal{O}_S$. The product on $\mathcal{A}$ defines a two-cochain m; then ∇m is a section of $\Omega^1(S, C^2(\mathcal{A}))$. Note also that $\nabla^2 = R \in \Omega^2(S, \mathrm{End}(\mathcal{A})) = \Omega^2(S, C^0(\mathcal{A}))$. Put

$$\alpha = \nabla m + R;$$

one has

$$(\delta + \nabla)^2 = \alpha; \quad (\delta + \nabla)(\alpha) = 0$$

(recall that $\delta = [m, ?]$ is the Hochschild differential). Put

$$\nabla_{\mathrm{GM}} = b + uB + \nabla + \sum_{k,n\geq 1;\, k+n>0} \frac{u^{-n}}{n!}\phi_n(m, \ldots, m\epsilon\alpha, \ldots, \epsilon\alpha)$$

where $\phi_n : S^n(\mathfrak{g}^\bullet_A[\epsilon, u][1]) \to \mathrm{End}(\mathrm{CC}^{\mathrm{per}}_\bullet(\mathcal{A}))$ are the components of the L_∞ module structure given by Theorem 3.

Proposition 3 ∇_{GM} *is a flat superconnection.*

The proof easily follows from the L_∞ identities. The interpretation of a Gauss-Manin connection in terms of teh above differential graded Lie algebra is due to Barannikov [1].

Acknowledgments

D.T. and B.T. are supported by NSF grants. The work of V.D. is partially supported by a Grant for Support of Scientific Schools (NSh-3036.2008.2). We are grateful to Paul Bressler, Kevin Costello, Ezra Getzler, Maxim Kontsevich, and Yan Soibelman for fruitful discussions.

References

1. Barannikov, S.: Quantum periods, I. Semi-infinite variations of Hodge structures, Int. Math. Res. Not. **23**, 1243–1264 (2001)
2. Baranovsky, V.: A universal enveloping of an L_∞ algebra (2007), arxiv:0706.1396
3. Bressler, P., Nest, R., Tsygan, B.: Riemann-Roch theorems via deformation quantization, I, II. Adv. Math. **167**(1), 1–25, 26–73 (2002)
4. Bressler, P., Gorokhovsky, A., Nest, R., Tsygan, B.: Algebraic index theorem for symplectic deformations of gerbes (to appear)
5. Căldăraru, A.: The Mukai pairing, I: the Hochschild structure, Preprint, arXiv:math/0308079
6. Căldăraru, A.: The Mukai pairing, II: the Hochschild-Kostant-Rosenberg isomorphism. Adv. Math. **194**(1), 34–66 (2005), arXiv:math/0308080
7. Căldăraru, A., Willerton, S.: The Mukai pairing, I: a categorical approach. N. Y. J. Math. **16**, 61–98 (2010), arXiv:0707.2052
8. Cartan, H., Eilenberg, S.: Homological Algebra. Princeton University Press, NJ (1956)
9. Costello, K.: Topological conformal field theories and Calabi-Yau categories, Adv. Math. **210**(1), 165–214 (2007)
10. Daletski, Yu., Gelfand, I., Tsygan, B.: On a variant of noncommutative geometry. Soviet Math. Dokl. **40**(2), 422–426 (1990)
11. Dolgushev, V.A.: A Proof of Tsygan's Formality Conjecture for an Arbitrary Smooth Manifold, PhD thesis. MIT, Cambridge, MA, math.QA/0504420
12. Dolgushev, V.A., Tamarkin, D., Tsygan, B.: The homotopy Gerstenhaber algebra of Hochschild cochains of a regular algebra is formal. J. Noncommut. Geom. **1**(1), 1–25 (2007), arXiv:math/0605141
13. Dolgushev, V.A., Tamarkin, D., Tsygan, B.: Formality of the homotopy calculus algebra of Hochschild (co)chains (2008), arXiv:0807.5117
14. Dolgushev, V.A., Tamarkin, D., Tsygan, B.: Formality theorems for Hochschild complexes and their applications. Lett. Math. Phys. **90**(1–3), 103–136 (2009)
15. Dorfman, I.Ya., Gelfand, I.M.: Hamiltonian operators and algebraic structures related to them. Funct. Anal. Appl. **13**, 13–30 (1979)

16. Gerstenhaber, M.: The cohomology structure of an associative ring. Ann. Math. (2) **78**, 267–288 (1963)
17. Getzler, E.: Cartan homotopy formulas and the Gauss-Manin connection in cyclic homology. Quantum deformations of algebras and their representations (Ramat-Gan, 1991/1992; Rehovot, 1991/1992), 65–78, Israel Math. Conf. Proc., 7, Bar-Ilan Univ., Ramat-Gan (1993)
18. Getzler, E., Jones, J.D.S.: A_∞ algebras and the cyclic bar complex. Illinois J. Math. **34**, 256–283 (1990)
19. Getzler, E., Jones, J.D.S.: Operads, homotopy algebra and iterated integrals for double loop spaces. Preprint hep-th9403055
20. Ginzburg, V., Schedler, T.: Free products, cyclic homology, and the Gauss-Manin connection, arXiv:0803.3655
21. Hinich, V.: Tamarkin's proof of Kontsevich formality theorem. Forum Math. **15**(4), 591–614 (2003), math.QA/0003052
22. Hood, C.E., Jones, J.D.S.: Some algebraic properties of cyclic homology groups. K-theory **1**, 361–384 (1987)
23. Kadeishvili, T.V.: On the homology theory of fibered spaces. Uspekhi Mat. Nauk **35**(3), 183–188 (1980)
24. Katzarkov, L., Kontsevich, M., Pantev, T.: Hodge theoretic aspects of mirror symmetry, From Hodge theory to integrability and TQFT tt* geometry. Proc. Sympos. Pure Math. AMS **78**, Providence, RI, (2008), arXiv:0806.0107
25. Keller, B.: A_∞ algebras, modules and functor categories. Trends in representation theory of algebras and related topics. Contemp. Math. AMS **406**, 67–93 (2006)
26. Kontsevich, M.: Operads and motives in deformation quantization. Lett. Math. Phys. **48**, 35–72 (1999)
27. Kontsevich, M., Soibelman, Y.: Deformations of algebras over operads and the Deligne conjecture, Conférence Moshe Flato 1999, vol. 1 (Dijon), pp. 255–307. Math. Phys. Stud., 21. Kluwer, Dordrecht (2000)
28. Kontsevich, M., Soibelman, Y.: Notes on A-infinity algebras, A-infinity categories and non-commutative geometry. I, math.RA/0606241
29. Lada, T., Stasheff, J.: Introduction to SH Lie algebras for physicists. Int. J. Theor. Phys. **32**(7), 1087–1103 (1993)
30. Loday, J.-L.: Cyclic homology. Gründlehren der mathematischen Wissenschaften, 301. Springer, Berlin (1992)
31. Markarian, N.: The Atiyah class, Hochschild cohomology and the Riemann-Roch theorem, J. Lond. Math. Soc. (2) **79**(1), 129–143 (2009), arXiv:math/0610553
32. Merkulov, S.: An L_∞ algebra of an unobstructed deformation functor. Int. Math. Res. Not. **3**, 147–164 (2000)
33. Merkulov, S.: Exotic automorphisms of the Schouten algebra of polyvector fields, arXiv:0809.2385
34. Merkulov, S., Vallette, B.: Deformation theory of representations of prop(erad)s (2007), arXiv:0707.0889
35. Nest, R., Tsygan, B.: On the cohomology ring of an algebra. In: Advances in geometry. Progress in Mathematics, vol. 172, pp. 337–370. Birkhäuser, Boston, MA, (1999)
36. Ramadoss, A.: The Mukai pairing and integral transforms in Hochschild homology, arXiv:0805.1760
37. Shoikhet, B.: A proof of the Tsygan formality conjecture for chains. Adv. Math. **179**(1), 7–37 (2003), math.QA/0010321

38. Stasheff, J.: Homotopy associativity of H-spaces, I and II, Trans. AMS **108**, 275–312 (1963)
39. Tamarkin, D.: Another proof of M. Kontsevich formality theorem, math.QA/9803025
40. Tamarkin, D.: Formality of chain operad of little discs. Lett. Math. Phys. **66**(1–2), 65–72 (2003), math.QA/9809164
41. Tamarkin, D.: What do DG categories form? Compos. Math. **143**(5), 1335–1358 (2007); math.CT/0606553
42. Tamarkin, D., Tsygan, B.: Cyclic formality and index theorems. Talk given at the Moshé Flato Conference. Lett. Math. Phys. **56**(2), 85–97 (2001)
43. Tamarkin, D., Tsygan, B.: The ring of differential operators on forms in noncommutative calculus. Graph patterns in mathematics and theoretical physics. Proc. Symp. Pure Math., vol. 73, pp. 105–131. AMS, RI (2005)
44. Toën, B., Vezzosi, G.: A note on Chern character, loop spaces and derived algebraic geometry. Proc. of the 2007 Abel Symposium, arXiv:0804.1274
45. Tsygan, B.: On the Gauss-Manin connection in cyclic homology. Methods Funkt. Anal. Topol. **13**(1), 83–94 (2007)

The Lie Algebra Perturbation Lemma

Johannes Huebschmann

UFR de Mathématiques, Université des Sciences et Technologies de Lille,
CNRS-UMR 8524, 59655 Villeneuve d'Ascq Cedex, France,
Johannes.Huebschmann@math.univ-lille1.fr

To Murray Gerstenhaber and Jim Stasheff

Abstract. Let R be a commutative ring which contains the rational numbers as a subring. We shall establish the following.

Theorem. Let $(M \underset{\pi}{\overset{\nabla}{\rightleftarrows}} \mathfrak{g}, h)$ be a *contraction* of chain complexes and suppose that $\mathfrak{g}$ *is endowed with a bracket* $[\,\cdot\,,\,\cdot\,]$ *turning it into differential graded Lie algebra. Then the given contraction and the bracket* $[\,\cdot\,,\,\cdot\,]$ *determine an sh-Lie algebra structure on* M*, that is, a coalgebra perturbation* $\mathcal{D}$ *of the coalgebra differential* d^0 *on (the cofree coaugmented differential graded cocommutative coalgebra)* $\mathcal{S}^{\mathrm{c}}[sM]$ *(on the suspension* sM *of* M*), the coalgebra differential* d^0 *being induced by the differential on* M*, a Lie algebra twisting cochain* $\tau\colon \mathcal{S}^{\mathrm{c}}_{\mathcal{D}}[sM] \longrightarrow \mathfrak{g}$ *and, furthermore, a contraction*

$$\left(\mathcal{S}^{\mathrm{c}}_{\mathcal{D}}[sM] \underset{\Pi}{\overset{\overline{\tau}}{\rightleftarrows}} \mathcal{C}[\mathfrak{g}], H\right)$$

of chain complexes which are natural in terms of the data.
Here $\mathcal{C}[\mathfrak{g}]$ refers to the classifying coalgebra of $\mathfrak{g}$.

Key words: Differential graded Lie algebra, Homological perturbation, Lie algebra perturbation, sh-Lie algebra, Cartan-Chevalley-Eilenberg coalgebra, Classifying coalgebra, Twisting cochain, Maurer-Cartan equation

2010 Mathematics Subject Classification: 16E45, 16T15, 17B55, 17B56, 17B65, 17B70, 18G10, 55P62

1 Introduction

The purpose of this paper is to establish the *perturbation lemma* for differential graded Lie algebras. The main technique is H(omological) P(erturbation) T(heory).

A.S. Cattaneo et al. (eds.), *Higher Structures in Geometry and Physics*, Progress in Mathematics 287, DOI 10.1007/978-0-8176-4735-3_8,

A special case of the Lie algebra perturbation lemma has been explored in [HuSt02] (Theorem 2.7), and the details of the proof of that Theorem have been promised to be given elsewhere; the present paper includes these details. The main application of the quoted theorem in [HuSt02] was the construction of solutions of the *master equation* under suitable general circumstances, and the present paper in particular yields solutions of the master equation under circumstances even more general than those in [HuSt02]. In the present paper, we will not elaborate on the master equation, though; suffice it to mention that the master equation amounts to the defining equation of a *Lie algebra twisting cochain* which, in turn, will be reproduced as (10) below. Detailed comments related with the master equation may be found in [HuSt02].

The ordinary perturbation lemma for chain complexes (reproduced below as Lemma 2) has become a standard tool to handle higher homotopies in a constructive manner. This lemma is somehow lurking behind the formulas (1) in Ch. II §1 of [Sh62], seems to have first been made explicit by M. Barratt (unpublished) and, to our knowledge, appeared first in print in [Br64]. Thereafter it has been exploited at various places in the literature, cf. among others [Gu72]–[HuSt02]. The basic reason why HPT works is the old observation that an exact sequence of chain complexes which splits as an exact sequence of graded modules and which has a contractible quotient necessarily splits in the category of chain complexes [Do60] (2.18).

Some more historical comments about HPT may be found e.g. in [Hu07b] and in Section 1 (p. 248) and Section 2 (p. 261) of [HuKa91], which has one of the strongest results in relation to *compatibility with other such as algebra or coalgebra structure*, since it was perhaps first recognized in [Hu89a]. Suitable HPT constructions that are compatible with other algebraic structure enabled us to carry out complete numerical calculations in group cohomology [Hu89b], [Hu89c], [Hu91] which cannot be done by other methods.

In view of the result of Kontsevich that the Hochschild complex of the algebra of smooth functions on a smooth manifold, endowed with the Gerstenhaber bracket, is formal as a differential graded Lie algebra [Ko97], sh-Lie algebras have become a fashionable topic. The attempt to treat the corresponding higher homotopies by means of a suitable version of HPT, relative to the *requisite additional algebraic structure*, that is, to make the perturbations *compatible with Lie brackets or more generally with sh-Lie structures*, led to the paper [HuSt02], but technical complications arise since the *tensor trick*, which was successfully exploited in [GuLaSta91], [Hu86], [Hu89a], [Hu89b], [HuKa91] and elsewhere, breaks down for cocommutative coalgebras; indeed, the notion of *homotopy of morphisms of cocommutative coalgebras is a subtle concept* [SchlSt], and only a special case was handled in [HuSt02], with some of the technical details merely sketched. The present paper provides *all* the necessary details and handles the case of a *general contraction* whereas in [HuSt02] only the case of a contraction of a differential graded Lie algebra onto its *homology* was treated.

In a subsequent paper [Hu07a] we have extended the perturbation lemma to the more general situation of sh-Lie algebras.

I am much indebted to Jim Stasheff for having prodded me on various occasions to pin down the perturbation lemma for Lie algebras as well as for a number of comments on a draft of the paper, and to J. Grabowski and P. Urbanski for discussions about the symmetric coalgebra.

2 The Lie algebra perturbation lemma

To spell out the Lie algebra perturbation lemma, and to illuminate the unexplained terms in the introduction, we need some preparation.

The ground ring is a commutative ring with 1 which is supposed to contain the rational numbers as a subring and will be denoted by R. We will take *chain complex* to mean *differential graded R-module*. A chain complex will not necessarily be concentrated in non-negative or non-positive degrees. The differential of a chain complex will always be supposed to be of degree -1. For a filtered chain complex X, a *perturbation* of the differential d of X is a (homogeneous) morphism ∂ of the same degree as d such that ∂ lowers the filtration and $(d+\partial)^2 = 0$ or, equivalently,

$$[d, \partial] + \partial\partial = 0. \tag{1}$$

Thus, when ∂ is a perturbation on X, the sum $d + \partial$, referred to as the *perturbed differential*, endows X with a new differential. When X has a graded coalgebra structure such that (X, d) is a differential graded coalgebra, and when the *perturbed differential* $d + \partial$ is compatible with the graded coalgebra structure, we refer to ∂ as a *coalgebra perturbation*; the notion of *algebra perturbation* is defined similarly. Given a differential graded coalgebra C and a coalgebra perturbation ∂ of the differential d on C, we will occasionally denote the new or *perturbed* differential graded coalgebra by C_∂.

A *contraction*

$$(N \underset{\pi}{\overset{\nabla}{\rightleftarrows}} M, h) \tag{2}$$

of chain complexes [EML53/54] consists of

– chain complexes N and M,

– chain maps $\pi\colon N \to M$ and $\nabla\colon M \to N$,

– a morphism $h\colon N \to N$ of the underlying graded modules of degree 1;

these data are required to satisfy

$$\pi\nabla = \mathrm{Id}, \tag{3}$$

$$Dh = \mathrm{Id} - \nabla\pi, \tag{4}$$

$$\pi h = 0, \quad h\nabla = 0, \quad hh = 0. \tag{5}$$

The requirements (5) are referred to as *annihilation properties* or *side conditions*.

Let $\mathfrak{g}$ be (at first) a chain complex, the differential being written as the operator $d\colon \mathfrak{g} \to \mathfrak{g}$, and let

$$(M \underset{\pi}{\overset{\nabla}{\rightleftarrows}} \mathfrak{g}, h) \tag{6}$$

be a *contraction* of chain complexes; later we will take $\mathfrak{g}$ to be a differential graded Lie algebra. In the special case where the differential on M is zero, M plainly amounts to the homology $\mathrm{H}(\mathfrak{g})$ of $\mathfrak{g}$; in this case, with the notation $\mathcal{H} = \nabla\mathrm{H}(\mathfrak{g})$, the resulting decomposition

$$\mathfrak{g} = d\mathfrak{g} \oplus \ker(h) = d\mathfrak{g} \oplus \mathcal{H} \oplus h\mathfrak{g}$$

may be viewed as a generalization of the familiar *Hodge* decomposition.

Let C be a *coaugmented* differential graded coalgebra with coaugmentation map $\eta\colon R \to C$ and *coaugmentation* coideal $JC = \mathrm{coker}(\eta)$, the diagonal map being written as $\Delta\colon C \to C \otimes C$ as usual. Recall that the counit $\varepsilon\colon C \to R$ and the coaugmentation map determine a direct sum decomposition $C = R \oplus JC$. The *coaugmentation* filtration $\{\mathrm{F}_n C\}_{n \geq 0}$ is as usual given by

$$\mathrm{F}_n C = \ker(C \longrightarrow (JC)^{\otimes(n+1)}) \ (n \geq 0)$$

where the unlabelled arrow is induced by some iterate of the diagonal Δ of C. This filtration is well known to turn C into a *filtered* coaugmented differential graded coalgebra; thus, in particular, $\mathrm{F}_0 C = R$. We recall that C is said to be *cocomplete* when $C = \cup \mathrm{F}_n C$.

Write s for the *suspension* operator as usual and accordingly s^{-1} for the *desuspension* operator. Thus, given the chain complex X, $(sX)_j = X_{j-1}$, etc., and the differential $d\colon sX \to sX$ on the suspended object sX is defined in the standard manner so that $ds + sd = 0$. Let $\mathcal{S}^{\mathrm{c}} = \mathcal{S}^{\mathrm{c}}[sM]$, the *cofree* coaugmented differential graded *cocommutative* coalgebra or, equivalently, differential graded *symmetric* coalgebra, on the suspension sM of M. This kind of coalgebra is well known to be cocomplete. Further, let $d^0 \colon \mathcal{S}^{\mathrm{c}} \longrightarrow \mathcal{S}^{\mathrm{c}}$ denote the coalgebra differential on $\mathcal{S}^{\mathrm{c}} = \mathcal{S}^{\mathrm{c}}[sM]$ induced by the differential on M. For $b \geq 0$, we will henceforth denote the homogeneous degree b component of $\mathcal{S}^{\mathrm{c}}[sM]$ by $\mathcal{S}^{\mathrm{c}}_b$; thus, as a chain complex, $\mathrm{F}_b \mathcal{S}^{\mathrm{c}} = R \oplus \mathcal{S}^{\mathrm{c}}_1 \oplus \cdots \oplus \mathcal{S}^{\mathrm{c}}_b$. Likewise, as a chain complex, $\mathcal{S}^{\mathrm{c}} = \oplus_{j=0}^{\infty} \mathcal{S}^{\mathrm{c}}_j$. We denote by

$$\tau_M \colon \mathcal{S}^{\mathrm{c}} \longrightarrow M$$

the composite of the canonical projection $\mathrm{proj}\colon \mathcal{S}^{\mathrm{c}} \to sM$ from $\mathcal{S}^{\mathrm{c}} = \mathcal{S}^{\mathrm{c}}[sM]$ to its homogeneous degree 1 constituent sM with the desuspension map s^{-1} from sM to M.

Given two chain complexes X and Y, recall that $\mathrm{Hom}(X, Y)$ inherits the structure of a chain complex by the operator D defined by

$$D\phi = d\phi - (-1)^{|\phi|}\phi d \tag{7}$$

where ϕ is a homogeneous homomorphism from X to Y and where $|\phi|$ refers to the degree of ϕ.

Consider the cofree coaugmented differential graded cocommutative coalgebra $\mathcal{S}^{\mathrm{c}}[s\mathfrak{g}]$ on the suspension $s\mathfrak{g}$ of $\mathfrak{g}$ and, as before, let

$$\tau_{\mathfrak{g}} \colon \mathcal{S}^{\mathrm{c}}[s\mathfrak{g}] \longrightarrow \mathfrak{g}$$

be the composite of the canonical projection to $\mathcal{S}^{\mathrm{c}}_1[s\mathfrak{g}] = s\mathfrak{g}$ with the desuspension map. Suppose that $\mathfrak{g}$ is endowed with a graded skew-symmetric bracket $[\,\cdot\,,\,\cdot\,]$ that is compatible with the differential but not necessarily a graded Lie bracket, i.e. does not necessarily satisfy the graded Jacobi identity. Let C be a coaugmented differential graded cocommutative coalgebra. Given homogeneous morphisms $a, b \colon C \to \mathfrak{g}$, with a slight abuse of the bracket notation $[\,\cdot\,,\,\cdot\,]$, the *cup bracket* $[a, b]$ is given by the composite

$$C \xrightarrow{\ \Delta\ } C \otimes C \xrightarrow{\ a\otimes b\ } \mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\ [\cdot,\cdot]\ } \mathfrak{g}. \tag{8}$$

The cup bracket $[\,\cdot\,,\,\cdot\,]$ is well known to be a graded skew-symmetric bracket on $\mathrm{Hom}(C, \mathfrak{g})$ which is compatible with the differential on $\mathrm{Hom}(C, \mathfrak{g})$. Define the coderivation

$$\partial \colon \mathcal{S}^{\mathrm{c}}[s\mathfrak{g}] \longrightarrow \mathcal{S}^{\mathrm{c}}[s\mathfrak{g}]$$

on $\mathcal{S}^{\mathrm{c}}[s\mathfrak{g}]$ by the requirement

$$\tau_{\mathfrak{g}}\partial = \tfrac{1}{2}[\tau_{\mathfrak{g}}, \tau_{\mathfrak{g}}] \colon \mathcal{S}^{\mathrm{c}}_2[s\mathfrak{g}] \to \mathfrak{g}. \tag{9}$$

Then $D\partial \ (= d\partial + \partial d) = 0$ since the bracket on $\mathfrak{g}$ is supposed to be compatible with the differential d. Moreover, *the bracket on* $\mathfrak{g}$ *satisfies the graded Jacobi identity if and only if* $\partial\partial = 0$, *that is, if and only if* ∂ *is a coalgebra perturbation of the differential* d *on* $\mathcal{S}^{\mathrm{c}}[s\mathfrak{g}]$, cf. e.g. [HuSt02]. The Lie algebra perturbation lemma below will generalize this observation.

We now suppose that the graded bracket $[\,\cdot\,,\,\cdot\,]$ on $\mathfrak{g}$ turns $\mathfrak{g}$ into a differential graded Lie algebra and continue to denote the resulting coalgebra perturbation by ∂, so that $\mathcal{S}^{\mathrm{c}}_{\partial}[s\mathfrak{g}]$ is a coaugmented differential graded cocommutative coalgebra; in fact, $\mathcal{S}^{\mathrm{c}}_{\partial}[s\mathfrak{g}]$ is then precisely the ordinary C(ARTAN)C(HEVALLEY)E(ILENBERG) or *classifying* coalgebra for $\mathfrak{g}$ and, following [Qu69] (p. 291), we denote it by $\mathcal{C}[\mathfrak{g}]$. Furthermore, given a coaugmented differential graded cocommutative coalgebra C, the cup bracket turns $\mathrm{Hom}(C, \mathfrak{g})$ into a differential graded Lie algebra. In particular, $\mathrm{Hom}(\mathcal{S}^{\mathrm{c}}, \mathfrak{g})$ and $\mathrm{Hom}(\mathrm{F}_n\mathcal{S}^{\mathrm{c}}, \mathfrak{g})$ $(n \geq 0)$ acquire differential graded Lie algebra structures.

Given a coaugmented differential graded cocommutative coalgebra C and a differential graded Lie algebra $\mathfrak{h}$, a *Lie algebra twisting cochain* $t \colon C \to \mathfrak{h}$ is a homogeneous morphism of degree -1 whose composite with the coaugmentation map is zero and which satisfies

$$Dt = \tfrac{1}{2}[t, t], \tag{10}$$

cf. [Mo71], [Qu69]. In particular, relative to the graded Lie bracket $[\,\cdot\,,\,\cdot\,]$ on $\mathfrak{g}$, the morphism $\tau_{\mathfrak{g}} \colon \mathcal{C}[\mathfrak{g}] \to \mathfrak{g}$ is a Lie algebra twisting cochain, the

C(ARTAN)C(HEVALLEY) E(ILENBERG) or *universal* Lie algebra twisting cochain for $\mathfrak{g}$. It is, perhaps, worth noting that, when $\mathfrak{g}$ is viewed as an abelian differential graded Lie algebra relative to the zero bracket, $\mathcal{S}^{\mathrm{c}}[s\mathfrak{g}]$ is the corresponding CCE or *classifying* coalgebra and $\tau_{\mathfrak{g}}\colon \mathcal{S}^{\mathrm{c}}[s\mathfrak{g}] \to \mathfrak{g}$ is still the universal Lie algebra twisting cochain. Likewise, when M is viewed as an *abelian* differential graded Lie algebra, $\mathcal{S}^{\mathrm{c}} = \mathcal{S}^{\mathrm{c}}[sM]$ may be viewed as the CCE or *classifying* coalgebra $\mathcal{C}[M]$ for M, and $\tau_M\colon \mathcal{S}^{\mathrm{c}} \to M$ is then the universal differential graded Lie algebra twisting cochain for M.

At the risk of making a mountain out of a molehill, we note that, in (9) and (10) above, the factor $\frac{1}{2}$ is a mere matter of convenience. The correct way of phrasing graded Lie algebras when the prime 2 is not invertible in the ground ring is in terms of an additional operation, the *squaring* operation $\mathrm{Sq}\colon \mathfrak{g}_{\mathrm{odd}} \to \mathfrak{g}_{\mathrm{even}}$ and, by means of this operation, the factor $\frac{1}{2}$ can be avoided. Indeed, in terms of this operation, the equation (10) takes the form

$$Dt = \mathrm{Sq}(t).$$

For intelligibility, we will follow the standard convention, avoid spelling out the squaring operation explicitly, and keep the factor $\frac{1}{2}$.

Given a chain complex $\mathfrak{h}$, an sh-*Lie algebra structure* or L_∞-*structure* on $\mathfrak{h}$ is a *coalgebra perturbation* ∂ of the differential d on the coaugmented differential graded cocommutative coalgebra $\mathcal{S}^{\mathrm{c}}[s\mathfrak{h}]$ on $s\mathfrak{h}$, cf. [HuSt02] (Def. 2.6). Given two sh-Lie algebras $(\mathfrak{h}_1, \partial_1)$ and $(\mathfrak{h}_2, \partial_2)$, an *sh-morphism* or *sh-Lie map* from $(\mathfrak{h}_1, \partial_1)$ to $(\mathfrak{h}_2, \partial_2)$ is a morphism $\mathcal{S}^{\mathrm{c}}_{\partial_1}[s\mathfrak{h}_1] \to \mathcal{S}^{\mathrm{c}}_{\partial_2}[s\mathfrak{h}_2]$ of coaugmented differential graded coalgebras, cf. [HuSt02].

Theorem 1 (Lie algebra perturbation lemma). *Suppose that $\mathfrak{g}$ carries a differential graded Lie algebra structure. Then the contraction (6) and the graded Lie algebra structure on $\mathfrak{g}$ determine an sh-Lie algebra structure on M, that is, a coalgebra perturbation $\mathcal{D}$ of the coalgebra differential d^0 on $\mathcal{S}^{\mathrm{c}}[sM]$, a Lie algebra twisting cochain*

$$\tau\colon \mathcal{S}^{\mathrm{c}}_{\mathcal{D}}[sM] \longrightarrow \mathfrak{g} \tag{11}$$

and, furthermore, a contraction

$$\left(\mathcal{S}^{\mathrm{c}}_{\mathcal{D}}[sM] \underset{\Pi}{\overset{\overline{\tau}}{\rightleftarrows}} \mathcal{C}[\mathfrak{g}], H \right) \tag{12}$$

of chain complexes which are natural in terms of the data so that

$$\pi\tau = \tau_M\colon \mathcal{S}^{\mathrm{c}}[sM] \longrightarrow M, \tag{13}$$
$$h\tau = 0, \tag{14}$$

and so that, since by construction, the injection $\overline{\tau}\colon \mathcal{S}^{\mathrm{c}}_{\mathcal{D}}[sM] \to \mathcal{C}[\mathfrak{g}]$ *of the contraction is the adjoint $\overline{\tau}$ of τ, this injection is then a* morphism *of coaugmented differential graded coalgebras.*

In the statement of this theorem, the perturbation $\mathcal{D}$ then encapsulates the asserted sh-Lie structure on M, and the adjoint $\overline{\tau}$ of (11) is plainly an sh-equivalence in the sense that it induces an isomorphism on homology, including the brackets of all order that are induced on homology.

The proof of Theorem 1 to be given below includes, in particular, a proof of Theorem 2.7 in [HuSt02]; in fact, the statement of that theorem is the special case of Theorem 1 where the differential on M is zero, and the details of the proof of that theorem had been promised to be given elsewhere.

Theorem 1 asserts not only the existence of the Lie algebra twisting cochain (11) and contraction (12) but also includes explicit natural constructions for them. The explicit constructions for the coalgebra perturbation $\mathcal{D}$ and Lie algebra twisting cochain (11) will be spelled out in Complement I below, and explicit constructions of the remaining constituents of the contraction (12) will be given in Complement II. As a notational *road map* for the reader, we note at this stage that Complement II involves an application of the ordinary perturbation lemma which will here yield, as an intermediate step, yet another contraction of chain complexes, of the kind

$$\left(\mathcal{S}^{\mathrm{c}}_{\delta}[sM] \underset{\widetilde{\Pi}}{\overset{\widetilde{\nabla}}{\rightleftarrows}} \mathcal{C}[\mathfrak{g}], \widetilde{H}\right),$$

to be given as (22) below. In particular, δ is yet another perturbation on $\mathcal{S}^{\mathrm{c}}[sM]$ which we distinguish in notation from the perturbation $\mathcal{D}$; apart from trivial cases, the perturbation δ is *not* compatible with the coalgebra structure on $\mathcal{S}^{\mathrm{c}}[sM]$, though, and the injection $\widetilde{\nabla}$ and homotopy $\widetilde{H}$ differ from the ultimate injection $\overline{\tau}$ and homotopy H.

Complement I. *The operator $\mathcal{D}$ and twisting cochain τ are obtained as infinite series by the following recursive procedure:*

$$\tau^1 = \nabla\tau_M : \mathcal{S}^{\mathrm{c}} \to \mathfrak{g}, \tag{15}$$

$$\tau^j = \tfrac{1}{2}h([\tau^1, \tau^{j-1}] + \cdots + [\tau^{j-1}, \tau^1]) \colon \mathcal{S}^{\mathrm{c}} \to \mathfrak{g},\ j \geq 2, \tag{16}$$

$$\tau = \tau^1 + \tau^2 + \ldots : \mathcal{S}^{\mathrm{c}} \to \mathfrak{g}, \tag{17}$$

$$\mathcal{D} = \mathcal{D}^1 + \mathcal{D}^2 + \ldots : \mathcal{S}^{\mathrm{c}} \to \mathcal{S}^{\mathrm{c}} \tag{18}$$

where, for $j \geq 1$, the operator $\mathcal{D}^j$ is the coderivation of $\mathcal{S}^{\mathrm{c}}[sM]$ determined by the identity

$$\tau_M \mathcal{D}^j = \tfrac{1}{2}\pi([\tau^1, \tau^j] + \cdots + [\tau^j, \tau^1]) \colon \mathcal{S}^{\mathrm{c}}_{j+1} \to M. \tag{19}$$

In particular, for $j \geq 1$, the coderivation $\mathcal{D}^j$ is zero on $\mathrm{F}_j\mathcal{S}^{\mathrm{c}}$ and lowers coaugmentation filtration by j.

The sums (17) and (18) are in general infinite. However, applied to a specific element which, since $\mathcal{S}^{\mathrm{c}}$ is cocomplete, necessarily lies in some finite filtration degree subspace, since the operators $\mathcal{D}^j$ ($j \geq 1$) lower coaugmentation filtration by j, only finitely many terms will be non-zero, whence the convergence is naive.

In the special case where the original contraction (6) is the trivial contraction of the kind

$$(\mathfrak{g} \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} \mathfrak{g}, 0), \tag{20}$$

M and $\mathfrak{g}$ coincide as chain complexes, the perturbation $\mathcal{D}$ coincides with the perturbation ∂ determined by the graded Lie bracket $[\,\cdot\,,\,\cdot\,]$ on $\mathfrak{g}$, and $\mathcal{S}^{\mathrm{c}}_{\mathcal{D}}[sM]$ coincides with the ordinary CCE or *classifying* coalgebra $\mathcal{C}[\mathfrak{g}]$ for $\mathfrak{g}$; the Lie algebra twisting cochain τ then comes down to the CCE or *universal* Lie algebra twisting cochain $\tau_{\mathfrak{g}}\colon \mathcal{C}[\mathfrak{g}] \to \mathfrak{g}$ for $\mathfrak{g}$ and in fact coincides with τ_1 (in the present special case) and, furthermore, the new contraction (12) then amounts to the trivial contraction

$$(\mathcal{C}[\mathfrak{g}] \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} \mathcal{C}[\mathfrak{g}], 0).$$

In fact, in this case, the higher terms τ^j and $\mathcal{D}^j$ $(j \geq 2)$ are obviously zero, and the operator $\mathcal{D}^1$ manifestly coincides with the CCE-operator. Likewise, in the special case where the bracket on $\mathfrak{g}$ is trivial or, more generally, when M carries a graded Lie bracket in such a way that ∇ is a morphism of differential graded Lie algebras, the construction plainly stops after the first step, and $\tau = \tau^1$.

Complement II. *Application of the ordinary perturbation lemma (reproduced below as Lemma 2) to the perturbation ∂ on $\mathcal{S}^{\mathrm{c}}[s\mathfrak{g}]$ determined by the graded Lie algebra structure on $\mathfrak{g}$ and the induced* filtered *contraction*

$$\left(\mathcal{S}^{\mathrm{c}}[sM] \underset{\mathcal{S}^{\mathrm{c}}[s\pi]}{\overset{\mathcal{S}^{\mathrm{c}}[s\nabla]}{\rightleftarrows}} \mathcal{S}^{\mathrm{c}}[s\mathfrak{g}], \mathcal{S}^{\mathrm{c}}[sh]\right) \tag{21}$$

of coaugmented differential graded coalgebras, *the filtrations being the ordinary coaugmentation filtrations, yields a perturbation δ of the differential d^0 on $\mathcal{S}^{\mathrm{c}}[sM]$ and a contraction*

$$\left(\mathcal{S}^{\mathrm{c}}_{\delta}[sM] \underset{\widetilde{\Pi}}{\overset{\widetilde{\nabla}}{\rightleftarrows}} \mathcal{C}[\mathfrak{g}], \widetilde{H}\right) \tag{22}$$

of chain complexes. Furthermore, the composite

$$\Phi\colon \mathcal{S}^{\mathrm{c}}_{\mathcal{D}}[sM] \xrightarrow{\;\overline{\tau}\;} \mathcal{C}[\mathfrak{g}] \xrightarrow{\;\widetilde{\Pi}\;} \mathcal{S}^{\mathrm{c}}_{\delta}[sM] \tag{23}$$

is an isomorphism of chain complexes, and the morphisms

$$\Pi = \Phi^{-1}\widetilde{\Pi}\colon \mathcal{C}[\mathfrak{g}] \longrightarrow \mathcal{S}^{\mathrm{c}}_{\mathcal{D}}[sM], \tag{24}$$

$$H = \widetilde{H} - \widetilde{H}\overline{\tau}\,\Pi\colon \mathcal{C}[\mathfrak{g}] \longrightarrow \mathcal{C}[\mathfrak{g}] \tag{25}$$

complete the construction of the contraction (12)*.*

In general, none of the morphisms δ, $\widetilde{\nabla}$, $\widetilde{\Pi}$, Π, $\widetilde{H}$, H is compatible with the coalgebra structures. The isomorphism Φ admits an explicit description in terms of the data as a *perturbation of the identity* and so does its inverse; details will be given in Section 5 below.

3 Some additional technical prerequisites

Let R be a commutative ring which, for the moment, we do not assume to contain the rational numbers as a subring. Let Y be a chain complex. The *cofree* coaugmented differential graded cocommutative coalgebra or *graded symmetric* coalgebra $\mathcal{S}^{\mathrm{c}}[Y]$ on the chain complex Y is characterized by a universal property as usual. To guarantee the existence of a diagonal map for $\mathcal{S}^{\mathrm{c}}[Y]$, some hypothesis is necessary, though: The ordinary tensor coalgebra $\mathrm{T}^{\mathrm{c}}[Y]$, that is, the cofree (coaugmented) coalgebra on Y, decomposes as the direct sum

$$\mathrm{T}^{\mathrm{c}}[Y] = \oplus_{j=0}^{\infty}\mathrm{T}^{\mathrm{c}}_j[Y]$$

of its homogeneous constituents $\mathrm{T}^{\mathrm{c}}_j[Y] = Y^{\otimes j}$ $(j \geq 0)$. For $j \geq 0$, let $\mathcal{S}^{\mathrm{c}}_j[Y] \subseteq \mathrm{T}^{\mathrm{c}}_j[Y]$ be the submodule of invariants in the jth tensor power $\mathrm{T}^{\mathrm{c}}_j[Y]$ relative to the obvious action on $\mathrm{T}^{\mathrm{c}}_j[Y]$ of the symmetric group S_j on j letters, and let $\mathcal{S}^{\mathrm{c}}[Y]$ be the direct sum

$$\mathcal{S}^{\mathrm{c}}[Y] = \oplus_{j=0}^{\infty}\mathcal{S}^{\mathrm{c}}_j[Y]$$

of chain complexes. So far, the construction is completely general, even functorial, and works over any ground ring. In particular, a chain map $\phi\colon Y_1 \to Y_2$ induces a chain map

$$\mathcal{S}^{\mathrm{c}}[\phi]\colon \mathcal{S}^{\mathrm{c}}[Y_1] \longrightarrow \mathcal{S}^{\mathrm{c}}[Y_2].$$

However, some hypothesis is, in general, necessary in order for the homogeneous constituents

$$\mathrm{T}^{\mathrm{c}}_{j+k}[Y] \longrightarrow \mathrm{T}^{\mathrm{c}}_j[Y] \otimes \mathrm{T}^{\mathrm{c}}_k[Y] \ (j,k \geq 0)$$

of the diagonal map $\Delta\colon \mathrm{T}^{\mathrm{c}}[Y] \to \mathrm{T}^{\mathrm{c}}[Y] \otimes \mathrm{T}^{\mathrm{c}}[Y]$ of the graded tensor coalgebra $\mathrm{T}^{\mathrm{c}}[Y]$ to induce a graded diagonal map on $\mathcal{S}^{\mathrm{c}}[Y]$. (I am indebted to P. Urbanski for having prodded me to clarify this point.) Indeed, the diagonal map for Y induces a morphism from $\mathcal{S}^{\mathrm{c}}[Y]$ to $\mathcal{S}^{\mathrm{c}}[Y \oplus Y]$ and the diagonal map for $\mathcal{S}^{\mathrm{c}}[Y]$ is well defined whenever the canonical morphism

$$\oplus_{j_1+j_2=k}\mathcal{S}^{\mathrm{c}}_{j_1}[Y] \otimes \mathcal{S}^{\mathrm{c}}_{j_2}[Y] \longrightarrow \mathcal{S}^{\mathrm{c}}_k[Y \oplus Y] \tag{26}$$

is an isomorphism for every $k \geq 1$.

To explain the basic difficulty, let $k \geq 1$, let Y_1 and Y_2 be two graded R-modules, and consider the kth homogeneous constituent

$$\mathcal{S}^{\mathrm{c}}_k[Y_1 \oplus Y_2] = \left((Y_1 \oplus Y_2)^{\otimes k}\right)^{S_k} \subseteq (Y_1 \oplus Y_2)^{\otimes k}$$

of $\mathcal{S}^{\mathrm{c}}[Y_1 \oplus Y_2]$. For $0 \leq j \leq k$, define $\mathcal{S}_{\binom{k}{j}}[Y_1 \oplus Y_2]$ to be the direct sum

$$Y_1^{\otimes j} \otimes Y_2^{\otimes (k-j)} \oplus Y_1^{\otimes (j-1)} \otimes Y_2 \otimes Y_1 \otimes Y_2^{\otimes (k-j-1)} \oplus \ldots \oplus Y_1^{\otimes (k-j)} \otimes Y_2^{\otimes j}$$

of $\binom{k}{j}$ summands which arises by substituting in the possible choices of j objects out of k objects a tensor factor of Y_1 for each object and filling in the "holes" remaining between the various tensor powers of Y_1 by the appropriate tensor powers of Y_2. Let RS_k denote the group ring of S_k. For $0 \leq j \leq k$, relative to the S_j- and S_{k-j}-actions on $Y_1^{\otimes j}$ and $Y_2^{\otimes(k-j)}$, respectively, there is a canonical isomorphism

$$\left(Y_1^{\otimes j} \otimes Y_2^{\otimes(k-j)}\right) \otimes_{S_j \times S_{k-j}} RS_k \longrightarrow \mathcal{S}_{\binom{k}{j}}[Y_1 \oplus Y_2]$$

of S_k-modules. Consequently, for $0 \leq j \leq k$, the canonical injection

$$\left(Y_1^{\otimes j} \otimes Y_2^{\otimes(k-j)}\right)^{S_j \times S_{k-j}} \longrightarrow \left(\mathcal{S}_{\binom{k}{j}}[Y_1 \oplus Y_2]\right)^{S_k}$$

is an isomorphism. As an S_k-module, $(Y_1 \oplus Y_2)^{\otimes k}$ is well known to decompose as the direct sum

$$(Y_1 \oplus Y_2)^{\otimes k} = \oplus_{j=0}^{k} \mathcal{S}_{\binom{k}{j}}[Y_1 \oplus Y_2]$$

whence $\mathcal{S}_k^{\mathrm{c}}[Y_1 \oplus Y_2]$ decomposes as the direct sum

$$\mathcal{S}_k^{\mathrm{c}}[Y_1 \oplus Y_2] = \oplus_{j=0}^{k} \left(Y_1^{\otimes j} \otimes Y_2^{\otimes(k-j)}\right)^{S_j \times S_{k-j}}.$$

However, some hypothesis is needed in order for the canonical morphisms

$$\mathcal{S}_j^{\mathrm{c}}[Y_1] \otimes \mathcal{S}_{k-j}^{\mathrm{c}}[Y_2] = \left(Y_1^{\otimes j}\right)^{S_j} \otimes \left(Y_2^{\otimes(k-j)}\right)^{S_{k-j}} \longrightarrow \left(Y_1^{\otimes j} \otimes Y_2^{\otimes(k-j)}\right)^{S_j \times S_{k-j}}$$

of graded R-modules to be isomorphisms for $1 \leq j \leq k-1$. This will be so when the ground ring contains the rational numbers as a subring.

The diagonal map $Y \to Y \times Y \cong Y \otimes Y$ of Y induces a diagonal map for the graded symmetric algebra $\mathcal{S}[Y]$ on Y turning that algebra into a differential graded Hopf algebra. We now suppose that the ground ring R contains the rational numbers as a subring. Then the coalgebra which underlies the graded symmetric algebra $\mathcal{S}[Y]$ may be taken as a model for $\mathcal{S}^{\mathrm{c}}[Y]$, cf. the discussion in Appendix B of [Qu69]. Indeed, the map

$$\mathcal{S}[Y] \longrightarrow \mathrm{T}^{\mathrm{c}}[Y],\ x_1 \ldots x_n \longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} \pm x_{\sigma 1} \otimes \ldots \otimes x_{\sigma n},\ x_j \in Y,\ n > 0,$$

induces an explicit isomorphism of $\mathcal{S}[Y]$ onto $\mathcal{S}^{\mathrm{c}}[Y] \subseteq \mathrm{T}^{\mathrm{c}}[Y]$.

As *chain complexes*, not just as graded objects, the Hom-complexes $\mathrm{Hom}(\mathcal{S}^{\mathrm{c}}, \mathfrak{g})$ and $\mathrm{Hom}(\mathrm{F}_n\mathcal{S}^{\mathrm{c}}, \mathfrak{g})$ $(n \geq 0)$ manifestly decompose as direct products

$$\mathrm{Hom}(\mathcal{S}^{\mathrm{c}}, \mathfrak{g}) \cong \prod_{j=0}^{\infty} \mathrm{Hom}(\mathcal{S}_j^{\mathrm{c}}, \mathfrak{g}), \quad \mathrm{Hom}(\mathrm{F}_n\mathcal{S}^{\mathrm{c}}, \mathfrak{g}) \cong \prod_{j=0}^{n} \mathrm{Hom}(\mathcal{S}_j^{\mathrm{c}}, \mathfrak{g}),\ (n \geq 0),$$

and the restriction mappings induce a sequence

$$\ldots \mathrm{Hom}(\mathrm{F}_{n+1}\mathcal{S}^{\mathrm{c}}, \mathfrak{g}) \longrightarrow \mathrm{Hom}(\mathrm{F}_n\mathcal{S}^{\mathrm{c}}, \mathfrak{g}) \longrightarrow \ldots \mathrm{Hom}(\mathrm{F}_1\mathcal{S}^{\mathrm{c}}, \mathfrak{g}) \longrightarrow \mathfrak{g} \tag{27}$$

of surjective morphisms of differential graded Lie algebras. Furthermore, by construction, for each $n \geq 0$, the canonical injection of $\mathrm{F}_n\mathcal{S}^{\mathrm{c}}$ into $\mathcal{S}^{\mathrm{c}}$ is a morphism of coaugmented differential graded coalgebras and hence induces a projection $\mathrm{Hom}(\mathcal{S}^{\mathrm{c}}, \mathfrak{g}) \to \mathrm{Hom}(\mathrm{F}_n\mathcal{S}^{\mathrm{c}}, \mathfrak{g})$ of differential graded Lie algebras, and these projections assemble to an isomorphism from $\mathrm{Hom}(\mathcal{S}^{\mathrm{c}}, \mathfrak{g})$ onto the projective limit $\varprojlim \mathrm{Hom}(\mathrm{F}_n\mathcal{S}^{\mathrm{c}}, \mathfrak{g})$ of (27) in such a way that, in each degree, the limit is attained at a finite stage.

4 The crucial step

For ease of exposition, we introduce the notation $\mathcal{D}^0 = \mathcal{D}_0 = 0$. For $a \geq 1$, let

$$\tau_a = \tau^1 + \tau^2 + \cdots + \tau^a, \tag{28}$$

$$\mathcal{D}_a = \mathcal{D}^1 + \mathcal{D}^2 + \cdots + \mathcal{D}^a, \tag{29}$$

$$\Theta_{a+1} = -D\tau_a - \tau_a\mathcal{D}_{a-1} + \tfrac{1}{2}[\tau_a, \tau_a] \colon \mathcal{S}^{\mathrm{c}} \longrightarrow \mathfrak{g}; \tag{30}$$

$$\vartheta_{a+1} = \Theta_{a+1}|_{\mathcal{S}^{\mathrm{c}}_{a+1}} \colon \mathcal{S}^{\mathrm{c}}_{a+1} \longrightarrow \mathfrak{g}. \tag{31}$$

We note that (30) is equivalent to

$$\Theta_{a+1} = -d\tau_a - \tau_a(d^0 + \mathcal{D}_{a-1}) + \tfrac{1}{2}[\tau_a, \tau_a] \colon \mathcal{S}^{\mathrm{c}} \longrightarrow \mathfrak{g}. \tag{32}$$

The crucial step for the proof of the Lie algebra perturbation lemma, in particular for the statement given as Complement I above, is provided by the following.

Lemma 1. *Let $a \geq 1$.*

$$\pi\tau^{a+1} = 0; \tag{33}$$

$$\vartheta_{a+1} = -\tau^2\mathcal{D}^{a-1} - \ldots - \tau^a\mathcal{D}^1 + \tfrac{1}{2}\left([\tau^1, \tau^a] + \ldots + [\tau^a, \tau^1]\right); \tag{34}$$

$$h\vartheta_{a+1} = \tfrac{1}{2}h\left([\tau^1, \tau^a] + \ldots + [\tau^a, \tau^1]\right) = \tau^{a+1} \colon \mathcal{S}^{\mathrm{c}}_{a+1} \longrightarrow \mathfrak{g}; \tag{35}$$

$$\pi\vartheta_{a+1} = \tfrac{1}{2}\pi\left([\tau^1, \tau^a] + \ldots + [\tau^a, \tau^1]\right) = \tau_M\mathcal{D}^a \colon \mathcal{S}^{\mathrm{c}}_{a+1} \longrightarrow M; \tag{36}$$

$$\Theta_{a+1} \colon \mathcal{S}^{\mathrm{c}} \longrightarrow \mathfrak{g} \textit{ is zero on } \mathrm{F}_a\mathcal{S}^{\mathrm{c}}, \ \textit{i.e. goes to zero in } \mathrm{Hom}(\mathrm{F}_a\mathcal{S}^{\mathrm{c}}, \mathfrak{g}); \tag{37}$$

$$D\vartheta_{a+1} = \tau_1(\mathcal{D}^1\mathcal{D}^{a-1} + \cdots + \mathcal{D}^{a-1}\mathcal{D}^1) \colon \mathcal{S}^{\mathrm{c}}_{a+1} \longrightarrow \mathfrak{g}; \tag{38}$$

$$D\tau^{a+1} = \vartheta_{a+1} - \tau^1\mathcal{D}^a \colon \mathcal{S}^{\mathrm{c}} \longrightarrow \mathfrak{g}; \tag{39}$$

$$0 = d^0\mathcal{D}^a + \mathcal{D}^1\mathcal{D}^{a-1} + \cdots + \mathcal{D}^{a-1}\mathcal{D}^1 + \mathcal{D}^a d^0 \colon \mathcal{S}^{\mathrm{c}} \longrightarrow \mathcal{S}^{\mathrm{c}}. \tag{40}$$

For clarity we note that the morphism (34) is of the kind $\vartheta_{a+1}\colon \mathcal{S}^{\mathrm{c}}_{a+1} \longrightarrow \mathfrak{g}$ and that, for $a=1$, the formula (34) signifies

$$\vartheta_2 = \tfrac{1}{2}[\tau^1, \tau^1]; \tag{41}$$

further, (38) signifies that ϑ_2 is a cycle in this case, i.e. $D\vartheta_2 = 0$.

Let $b \geq 1$. The property (40) implies that, on the differential graded coalgebra $\mathrm{F}_{b+1}\mathcal{S}^{\mathrm{c}}$, the operator $\mathcal{D}_{b-1}$ is a coalgebra perturbation of the differential d^0 and hence $d^0 + \mathcal{D}_{b-1}$ is a coalgebra differential and thence turns the coaugmented graded cocommutative coalgebra $\mathrm{F}_{b+1}\mathcal{S}^{\mathrm{c}}$ into a coaugmented differential graded cocommutative coalgebra which, according to the convention introduced above, we write as $\mathrm{F}_{b+1}\mathcal{S}^{\mathrm{c}}_{\mathcal{D}_{b-1}}$. Furthermore, property (37) implies that the restriction of τ_b to $\mathrm{F}_b\mathcal{S}^{\mathrm{c}}$ is a Lie algebra twisting cochain $\mathrm{F}_b\mathcal{S}^{\mathrm{c}}_{\mathcal{D}_{b-1}} \to \mathfrak{g}$ whence, as b tends to infinity, τ_b tends to a Lie algebra twisting cochain, that is, τ is a Lie algebra twisting cochain. Indeed, in a given degree, the statements of Lemma 1 come down to corresponding statements in a suitable finite stage constituent of the sequence (27).

Proof. The property (33) is an immediate consequence of the annihilation properties (5). Next, let $a \geq 1$. For degree reasons, the restriction of

$$\Theta_{a+1} = -D\tau_a - \tau_a\mathcal{D}_{a-1} + \tfrac{1}{2}[\tau_a, \tau_a] = -d\tau_a - \tau_a(d^0 + \mathcal{D}_{a-1}) + \tfrac{1}{2}[\tau_a, \tau_a]\colon \mathcal{S}^{\mathrm{c}} \longrightarrow \mathfrak{g}$$

to $\mathcal{S}^{\mathrm{c}}_{a+1}$ comes down to

$$-\tau^2\mathcal{D}^{a-1} - \ldots - \tau^a\mathcal{D}^1 + \tfrac{1}{2}\left([\tau^1, \tau^a] + \ldots + [\tau^a, \tau^1]\right) : \mathcal{S}^{\mathrm{c}}_{a+1} \longrightarrow \mathfrak{g},$$

whence (34), being interpreted as (41) for $a=1$. The identity (34), combined with the annihilation properties (5), immediately implies (35) and (36), in view of the definitions (17) and (19) of the terms τ^{j+1} and $\mathcal{D}^j$, respectively, for $j \geq 1$.

Furthermore, the property (39) is a formal consequence of the definitions (19) and (31), combined with (38) and the annihilation properties (5). Indeed,

$$\begin{aligned} D\tau^{a+1} &= D(h\vartheta_{a+1}) = (Dh)\vartheta_{a+1} - hD\vartheta_{a+1} \\ &= \vartheta_{a+1} - \nabla\pi\vartheta_{a+1} - h\tau_1(\mathcal{D}^1\mathcal{D}^{a-1} + \cdots + \mathcal{D}^{a-1}\mathcal{D}^1) \\ &= \vartheta_{a+1} - \nabla\tau_M\mathcal{D}^a \\ &= \vartheta_{a+1} - \tau_1\mathcal{D}^a\colon \mathcal{S}^{\mathrm{c}}_{a+1} \longrightarrow \mathfrak{g}, \end{aligned}$$

whence (39).

By induction on a, we now establish the remaining assertions (37), (38), and (40). To begin with, let $a=1$. Since τ_1 is a cycle in $\mathrm{Hom}(\mathcal{S}^{\mathrm{c}}, \mathfrak{g})$ and since $[\tau_1, \tau_1]$ vanishes on $\mathrm{F}_1\mathcal{S}^{\mathrm{c}}$,

$$\Theta_2 = -d\tau_1 - \tau_1 d^0 + \tfrac{1}{2}[\tau_1, \tau_1] = \tfrac{1}{2}[\tau_1, \tau_1]\colon \mathcal{S}^{\mathrm{c}} \to \mathfrak{g}$$

vanishes on $\mathrm{F}_1\mathcal{S}^{\mathrm{c}}$, whence (37) holds for $a = 1$. Furthermore, since Θ_2 is a cycle, so is ϑ_2 whence (38) is satisfied for $a = 1$. Consequently

$$\begin{aligned} D\tau^2 &= Dh\vartheta_2 = \vartheta_2 - \nabla\pi\vartheta_2 = \tfrac{1}{2}[\tau_1,\tau_1] - \nabla\pi\vartheta_2 \\ &= \tfrac{1}{2}[\tau_1,\tau_1] - \nabla\tau_M\mathcal{D}^1 = \tfrac{1}{2}[\tau_1,\tau_1] - \tau_1\mathcal{D}^1 \end{aligned}$$

whence (39) for $a = 1$. Finally, the identity (40) for $a = 1$ reads

$$d^0\mathcal{D}^1 + \mathcal{D}^1 d^0 = 0. \tag{42}$$

The identity (42), in turn, is a consequence of the bracket on $\mathfrak{g}$ being compatible with the differential d on $\mathfrak{g}$ since this compatibility entails that $[\tau_1,\tau_1]$ is a cycle in $\mathrm{Hom}(\mathcal{S}^{\mathrm{c}},\mathfrak{g})$. Indeed, since

$$d^0\mathcal{D}^1 + \mathcal{D}^1 d^0 = [d^0,\mathcal{D}^1]$$

is a coderivation on $\mathcal{S}^{\mathrm{c}}$, the bracket being the commutator bracket in the *graded Lie algebra of coderivations* of $\mathcal{S}^{\mathrm{c}}$, it suffices to show that

$$\tau_M(d^0\mathcal{D}^1 + \mathcal{D}^1 d^0) = \tau_M[d^0,\mathcal{D}^1]$$

vanishes. However, since $\tau_M\mathcal{D}^1 = \frac{1}{2}\pi[\tau_1,\tau_1]$, cf. (19),

$$\tau_M(d^0\mathcal{D}^1 + \mathcal{D}^1 d^0) = -d\tau_M\mathcal{D}^1 + \tau_M\mathcal{D}^1 d^0 = -\tfrac{1}{2}D(\pi[\tau_1,\tau_1]),$$

and $D(\pi[\tau_1,\tau_1])$ vanishes since $[\tau_1,\tau_1]$ is a cycle in $\mathrm{Hom}(\mathcal{S}^{\mathrm{c}},\mathfrak{g})$ and since π is a chain map. Consequently the identity (40) holds for $a = 1$. Thus the induction starts.

Even though this is not strictly necessary we now explain the case $a = 2$. This case is particularly instructive. Now

$$\begin{aligned} d\tau_2 + \tau_2(d^0 + \mathcal{D}^1) &= d\tau^1 + d\tau^2 + \tau^1 d^0 + \tau^1\mathcal{D}^1 + \tau^2 d^0 + \tau^2\mathcal{D}^1 \\ &= D\tau_1 + D\tau^2 + \tau_1\mathcal{D}^1 + \tau^2\mathcal{D}^1 \\ &= \tfrac{1}{2}[\tau_1,\tau_1] - \tau_1\mathcal{D}^1 + \tau_1\mathcal{D}^1 + \tau^2\mathcal{D}^1 \\ &= \tfrac{1}{2}[\tau_1,\tau_1] + \tau^2\mathcal{D}^1 \end{aligned}$$

whence

$$\Theta_3 = -d\tau_2 - \tau_2(d^0 + \mathcal{D}^1) + \tfrac{1}{2}[\tau_2,\tau_2] = -\tau^2\mathcal{D}^1 + [\tau^1,\tau^2] + \tfrac{1}{2}[\tau^2,\tau^2]$$

which clearly vanishes on $\mathrm{F}_2\mathcal{S}^{\mathrm{c}}$, whence (37) holds for $a = 2$. Furthermore, it is manifest that the restriction ϑ_3 of Θ_3 to $\mathcal{S}^{\mathrm{c}}_3$ takes the form

$$\vartheta_3 = [\tau^1,\tau^2] - \tau^2\mathcal{D}^1 \colon \mathcal{S}^{\mathrm{c}}_3 \longrightarrow \mathfrak{g}, \tag{43}$$

which amounts to (34) for the special case $a = 2$. Hence

$$\begin{aligned} D\vartheta_3 &= -[\tau^1, D\tau^2] - (D\tau^2)\mathcal{D}^1 \\ &= -[\tau^1, -\tau^1\mathcal{D}^1 + \tfrac{1}{2}[\tau^1,\tau^1]] + \tau^1\mathcal{D}^1\mathcal{D}^1 - \tfrac{1}{2}[\tau^1,\tau^1]\mathcal{D}^1 \\ &= [\tau^1, \tau^1\mathcal{D}^1] + \tau^1\mathcal{D}^1\mathcal{D}^1 - [\tau^1, \tau^1\mathcal{D}^1] \\ &= \tau^1\mathcal{D}^1\mathcal{D}^1 \end{aligned}$$

whence (38) at stage $a = 2$. Since

$$D\tau^2 = \tfrac{1}{2}[\tau_1, \tau_1] - \tau_1\mathcal{D}^1$$

and since $[\tau_1, [\tau_1, \tau_1]] = 0$,

$$D\left[\tau_1, \tau^2\right] = \left[\tau_1, -D\tau^2\right] = \left[\tau_1, \tau_1\mathcal{D}_1 - \tfrac{1}{2}\left[\tau_1, \tau_1\right]\right] = \left[\tau_1, \tau_1\mathcal{D}_1\right].$$

Thus, in view of (19), viz.

$$\pi\tau_1\mathcal{D}_1 = \tfrac{1}{2}\pi[\tau_1, \tau_1],$$

we find

$$D(\pi\left[\tau_1, \tau^2\right]) = \pi\tau_1\mathcal{D}^1\mathcal{D}^1, \tag{44}$$

whence

$$D(\pi\left[\tau_1, \tau^2\right]) = \tau_M\mathcal{D}^1\mathcal{D}^1. \tag{45}$$

Since, in view of (43) or (34),

$$\pi\vartheta_3 = \pi[\tau_1, \tau^2] \colon \mathcal{S}^{\mathrm{c}}_3[sM] \longrightarrow M,$$

$\mathcal{D}^2 \colon \mathcal{S}^{\mathrm{c}} \to \mathcal{S}^{\mathrm{c}}$ is the coderivation which is determined by the requirement that the identity

$$\pi\vartheta_3 = \tau_M\mathcal{D}^2 \colon \mathcal{S}^{\mathrm{c}}_3[sM] \longrightarrow M \tag{46}$$

be satisfied. Then

$$\tau_M(d^0\mathcal{D}^2 + \mathcal{D}^1\mathcal{D}^1 + \mathcal{D}^2d^0) = \tau_M(D\mathcal{D}^2 + \mathcal{D}^1\mathcal{D}^1) = 0. \tag{47}$$

Indeed,

$$\begin{aligned} \tau_M D\mathcal{D}^2 &= -D(\tau_M\mathcal{D}^2) \\ &= -D(\pi\vartheta_3) = -D(\pi\left[\tau_1, \tau^2\right]) \\ &= -\tau_M\mathcal{D}^1\mathcal{D}^1. \end{aligned}$$

Consequently

$$d^0\mathcal{D}^2 + \mathcal{D}^1\mathcal{D}^1 + \mathcal{D}^2d^0 = 0 \tag{48}$$

since $d^0\mathcal{D}^2 + \mathcal{D}^1\mathcal{D}^1 + \mathcal{D}^2d^0$ is a coderivation of $\mathcal{S}^{\mathrm{c}}[sM]$. This establishes the identity (40) for $a = 2$.

We pause for the moment; suppose that we are in the special situation where the original contraction (6) is the trivial one of the kind (20) and identify M with the chain complex which underlies $\mathfrak{g}$, endowed with the zero bracket. Then $\mathcal{S}^{\mathrm{c}}$ amounts to the CCE-coalgebra for M (endowed with the zero bracket), the operator $\mathcal{D}^1$ is precisely the ordinary CCE-operator relative to the Lie bracket on $\mathfrak{g}$, the twisting cochain τ_1 is the CCE-twisting cochain relative to the Lie bracket on $\mathfrak{g}$, the term τ^2 is zero (since h is zero), and the construction we are in the process of explaining stops at the present stage. Indeed, τ_1 then coincides with τ_2 and $\Theta_3 = 0$. Moreover, the identity

$$\mathcal{D}^1\mathcal{D}^1 = 0 \tag{49}$$

is then equivalent to the bracket on $\mathfrak{g}$ satisfying the graded Jacobi identity.

Likewise, in the special case where the differential on M is zero so that M amounts to the homology $\mathrm{H}(\mathfrak{g})$ of $\mathfrak{g}$, the identity (48) comes down to

$$\mathcal{D}^1\mathcal{D}^1 = 0. \tag{50}$$

This identity, in turn, is then equivalent to the fact that the induced graded bracket on $\mathrm{H}(\mathfrak{g})$ satisfies the graded Jacobi identity.

We now return to the case of a general contraction (6). Let $b > 2$ and suppose, by induction that, at stage a, $2 \leq a < b$, (37)–(40) have been established. Our aim is to show that (37)–(40) hold at stage b. Now

$$\begin{aligned}
\Theta_{b+1} &= -D\tau_b - \tau_b\mathcal{D}_{b-1} + \tfrac{1}{2}[\tau_b,\tau_b]\colon \mathcal{S}^{\mathrm{c}}[sM] \longrightarrow \mathfrak{g} \\
&= -D\tau_{b-1} - D\tau^b - (\tau_{b-1} + \tau^b)(\mathcal{D}_{b-2} + \mathcal{D}^{b-1}) \\
&\qquad + \tfrac{1}{2}([\tau_{b-1},\tau_{b-1}] + [\tau^b,\tau_{b-1}] + [\tau_{b-1},\tau^b] + [\tau^b,\tau^b]) \\
&= \Theta_b - D\tau^b - \tau_{b-1}\mathcal{D}^{b-1} - \tau^b\mathcal{D}_{b-1} + \tfrac{1}{2}([\tau^b,\tau_{b-1}] + [\tau_{b-1},\tau^b] + [\tau^b,\tau^b]).
\end{aligned}$$

By the inductive hypothesis (37) at stage $b-1$, the operator Θ_b vanishes on $\mathrm{F}_{b-1}\mathcal{S}^{\mathrm{c}}$ whence Θ_{b+1} vanishes on $\mathrm{F}_{b-1}\mathcal{S}^{\mathrm{c}}$ as well since the remaining terms obviously vanish on $\mathrm{F}_{b-1}\mathcal{S}^{\mathrm{c}}$. Moreover,

$$\Theta_{b+1}\big|_{\mathcal{S}^{\mathrm{c}}_b} = \vartheta_b - D\tau^b - \tau_{b-1}\mathcal{D}^{b-1} - \tau^b\mathcal{D}_{b-1} + \tfrac{1}{2}([\tau^b,\tau_{b-1}] + [\tau_{b-1},\tau^b] + [\tau^b,\tau^b]).$$

In view of the inductive hypothesis (39),

$$D\tau^b = \vartheta_b - \tau^1\mathcal{D}^{b-1}$$

and, for degree reasons,

$$\tau^1\mathcal{D}^{b-1} = \tau_{b-1}\mathcal{D}^{b-1}$$

whence

$$\Theta_{b+1}\big|_{\mathcal{S}^{\mathrm{c}}_b} = -\tau^b\mathcal{D}_{b-1} + \tfrac{1}{2}([\tau^b,\tau_{b-1}] + [\tau_{b-1},\tau^b] + [\tau^b,\tau^b])$$

which, for degree reasons, is manifestly zero. Consequently Θ_{b+1} vanishes on $\mathrm{F}_b\mathcal{S}^{\mathrm{c}}$ whence (37) at stage b.

Next we establish the identity (38) at stage b. Recall that, by construction, cf. (30) and (32),

$$\Theta_{b+1} = -D\tau_b - \tau_b\mathcal{D}_{b-1} + \tfrac{1}{2}[\tau_b, \tau_b]$$

whence

$$D\Theta_{b+1} = -D(\tau_b\mathcal{D}_{b-1}) + \tfrac{1}{2}D[\tau_b, \tau_b] = -(D\tau_b)\mathcal{D}_{b-1} + \tau_b D\mathcal{D}_{b-1} + [D\tau_b, \tau_b].$$

However, $b \geq 2$ whence $\mathcal{D}_{b-1}$ lowers filtration, i.e. maps $\mathrm{F}_{b+1}\mathcal{S}^{\mathrm{c}}$ to $\mathrm{F}_b\mathcal{S}^{\mathrm{c}}$. Since (37) has already been established at stage b, restricted to $\mathrm{F}_{b+1}\mathcal{S}^{\mathrm{c}}$,

$$0 = -\Theta_{b+1}\mathcal{D}_{b-1} = D\tau_b\mathcal{D}_{b-1} + \tau_b\mathcal{D}_{b-1}\mathcal{D}_{b-1} - \tfrac{1}{2}[\tau_b, \tau_b]\mathcal{D}_{b-1}$$

whence

$$(D\tau_b)\mathcal{D}_{b-1} + \tau_b\mathcal{D}_{b-1}\mathcal{D}_{b-1} = \tfrac{1}{2}[\tau_b, \tau_b]\mathcal{D}_{b-1} = [\tau_b, \tau_b\mathcal{D}_{b-1}] \in \mathrm{Hom}(\mathrm{F}_{b+1}\mathcal{S}^{\mathrm{c}}, \mathfrak{g}).$$

Consequently

$$\begin{aligned} D\vartheta_{b+1} &= \tau_b\mathcal{D}_{b-1}\mathcal{D}_{b-1} - [\tau_b, \tau_b\mathcal{D}_{b-1}] + \tau_b D\mathcal{D}_{b-1} + [D\tau_b, \tau_b] \in \mathrm{Hom}(\mathrm{F}_{b+1}\mathcal{S}^{\mathrm{c}}, \mathfrak{g}) \\ &= \tau_b(\mathcal{D}_{b-1}\mathcal{D}_{b-1} + D\mathcal{D}_{b-1}) - [\tau_b, \tau_b\mathcal{D}_{b-1} + D\tau_b]. \end{aligned}$$

By induction, in view of (37),

$$\tau_b\mathcal{D}_{b-1} + D\tau_b = \tfrac{1}{2}[\tau_b, \tau_b] \in \mathrm{Hom}(\mathrm{F}_b\mathcal{S}^{\mathrm{c}}, \mathfrak{g}),$$

whence

$$[\tau_b, \tau_b\mathcal{D}_{b-1} + D\tau_b] = \tfrac{1}{2}[\tau_b, [\tau_b, \tau_b]]$$

which, each homogeneous constituent of τ_b being odd, is zero, in view of the graded Jacobi identity in $\mathrm{Hom}(\mathrm{F}_b\mathcal{S}^{\mathrm{c}}, \mathfrak{g})$. Moreover, by induction, in view of (40), for $1 \leq a < b$,

$$\mathcal{D}^1\mathcal{D}^{a-1} + \cdots + \mathcal{D}^{a-1}\mathcal{D}^1 + D\mathcal{D}^a = 0$$

whence

$$\mathcal{D}_{b-1}\mathcal{D}_{b-1} + D\mathcal{D}_{b-1} = \mathcal{D}^1\mathcal{D}^{b-1} + \cdots + \mathcal{D}^{b-1}\mathcal{D}^1.$$

Consequently, on $\mathcal{S}^{\mathrm{c}}_{b+1}$,

$$D\vartheta_{b+1} = \tau_b(\mathcal{D}^1\mathcal{D}^{b-1} + \cdots + \mathcal{D}^{b-1}\mathcal{D}^1) \colon \mathcal{S}^{\mathrm{c}}_{b+1} \to \mathfrak{g}$$

and, for degree reasons,

$$\tau_b(\mathcal{D}^1\mathcal{D}^{b-1} + \cdots + \mathcal{D}^{b-1}\mathcal{D}^1) = \tau_1(\mathcal{D}^1\mathcal{D}^{b-1} + \cdots + \mathcal{D}^{b-1}\mathcal{D}^1) \colon \mathcal{S}^{\mathrm{c}}_{b+1} \to \mathfrak{g}$$

whence

$$D\vartheta_{b+1} = \tau_1(\mathcal{D}^1\mathcal{D}^{b-1} + \cdots + \mathcal{D}^{b-1}\mathcal{D}^1).$$

This establishes the identity (38) at stage b.

Alternatively, in view of what has already been proved, by virtue of (34),

$$\vartheta_{b+1} = -\tau^2\mathcal{D}^{b-1} - \ldots - \tau^b\mathcal{D}^1 + \tfrac{1}{2}\left([\tau^1,\tau^b] + \ldots + [\tau^b,\tau^1]\right) : \mathcal{S}^{\mathrm{c}}_{b+1} \longrightarrow \mathfrak{g},$$

whence, since $D\tau^1 = 0$,

$$\begin{aligned} D\vartheta_{b+1} = &-(D\tau^2)\mathcal{D}^{b-1} + \tau^2 D\mathcal{D}^{b-1} \pm \ldots - (D\tau^{b-1})\mathcal{D}^2 \\ &+ \tau^{b-1}D\mathcal{D}^2 - D\tau^b\mathcal{D}^1 \\ &+ [D\tau^2,\tau^{b-1}] + \ldots + [D\tau^b,\tau^1]. \end{aligned} \tag{51}$$

Thus, using the inductive hypotheses, we can establish (38) at stage b by evaluating the terms on the right-hand side of (51).

Finally, to settle the identity (40) at stage b, we note first that, since

$$d^0\mathcal{D}^b + \mathcal{D}^1\mathcal{D}^{b-1} + \cdots + \mathcal{D}^{b-1}\mathcal{D}^1 + \mathcal{D}^b d^0 = \mathcal{D}^1\mathcal{D}^{b-1} + \cdots + \mathcal{D}^{b-1}\mathcal{D}^1 + D\mathcal{D}^b$$

is a coderivation of $\mathcal{S}^{\mathrm{c}}$, it suffices to prove that

$$\tau_M\left(\mathcal{D}^1\mathcal{D}^{b-1} + \cdots + \mathcal{D}^{b-1}\mathcal{D}^1 + D\mathcal{D}^b\right) = 0.$$

However, we have already observed that the identity (39) at stage b is a formal consequence of (38) and, since the latter has already been established, (39) is now available at stage b, viz.

$$D\tau^{b+1} = \vartheta_{b+1} - \tau^1\mathcal{D}^b \colon \mathcal{S}^{\mathrm{c}} \longrightarrow \mathfrak{g}.$$

Hence

$$0 = D\vartheta_{b+1} - D(\tau^1\mathcal{D}^b) = D\vartheta_{b+1} + \tau^1 D\mathcal{D}^b.$$

Substituting the right-hand side of (38) at stage b for $D\vartheta_{b+1}$, we obtain the identity

$$0 = d^0\mathcal{D}^b + \mathcal{D}^1\mathcal{D}^{b-1} + \cdots + \mathcal{D}^{b-1}\mathcal{D}^1 + \mathcal{D}^b d^0 \colon \mathcal{S}^{\mathrm{c}} \longrightarrow \mathcal{S}^{\mathrm{c}},$$

that is, the identity (40) at stage b. This completes the inductive step.

5 The proof of the Lie algebra perturbation lemma

Lemma 1 entails that the operator $\mathcal{D}$ given by (18) is a coalgebra perturbation and that the morphism τ given by (17) is a Lie algebra twisting cochain. We will now establish Complement II of the Lie algebra perturbation lemma.

The contraction (21) may be obtained in the following way: Any contraction of chain complexes of the kind (2) induces a filtered contraction

$$\left(\mathrm{T}^{\mathrm{c}}[M] \underset{\mathrm{T}^{\mathrm{c}}[\pi]}{\overset{\mathrm{T}^{\mathrm{c}}[\nabla]}{\rightleftarrows}} \mathrm{T}^{\mathrm{c}}[N], \mathrm{T}^{\mathrm{c}}[h]\right) \tag{52}$$

of coaugmented differential graded coalgebras. A version thereof is spelled out as a *contraction of bar constructions* already in Theorem 12.1 of [EML53/54]; the filtered contraction (52) may be found in [GuLaSta91] (2.2) and [HuKa91] $(2.2.0)_*$ (the dual filtered contraction of augmented differential graded algebras being spelled out explicitly in [HuKa91] as $(2.2.0)^*$). The differential graded symmetric coalgebras $\mathcal{S}^{\mathrm{c}}[M]$ and $\mathcal{S}^{\mathrm{c}}[N]$ being differential graded subcoalgebras of $\mathrm{T}^{\mathrm{c}}[M]$ and $\mathrm{T}^{\mathrm{c}}[N]$, respectively, the morphisms $\mathrm{T}^{\mathrm{c}}[\nabla]$ and $\mathrm{T}^{\mathrm{c}}[\pi]$ pass to corresponding morphisms $\mathcal{S}^{\mathrm{c}}[\nabla]$ and $\mathcal{S}^{\mathrm{c}}[\pi]$ respectively, and $\mathcal{S}^{\mathrm{c}}[h]$ arises from $\mathrm{T}^{\mathrm{c}}[h]$ by *symmetrization*, so that

$$\left(\mathcal{S}^{\mathrm{c}}[M] \underset{\mathcal{S}^{\mathrm{c}}[\pi]}{\overset{\mathcal{S}^{\mathrm{c}}[\nabla]}{\rightleftarrows}} \mathcal{S}^{\mathrm{c}}[N], \mathcal{S}^{\mathrm{c}}[h]\right) \tag{53}$$

constitutes a filtered contraction of coaugmented differential graded coalgebras. Alternatively, since $\mathcal{S}^{\mathrm{c}}$ is a functor, application of this functor to (2) yields (53). Here $\mathcal{S}^{\mathrm{c}}[\nabla]$ and $\mathcal{S}^{\mathrm{c}}[\pi]$ are morphisms of differential graded coalgebras but, beware, even though $\mathrm{T}^{\mathrm{c}}[h]$ is compatible with the coalgebra structure in the sense that it is a homotopy of morphisms of differential graded coalgebras, $\mathcal{S}^{\mathrm{c}}[h]$ no longer has such a compatibility property in a naive fashion. Indeed, for differential graded cocommutative coalgebras, the notion of homotopy is a subtle concept, cf. [SchlSt]. To sum up, application of the functor $\mathcal{S}^{\mathrm{c}}$ to the induced contraction

$$\left(sM \underset{s\pi}{\overset{s\nabla}{\rightleftarrows}} s\mathfrak{g}, sh\right)$$

which arises from (6) by suspension yields the contraction (21).

To establish Complement II of the Lie algebra perturbation lemma, we will view the contraction (53) merely as one of filtered chain complexes, that is, we forget about the coalgebra structures. As before, we denote by ∂ the coalgebra perturbation on $\mathcal{S}^{\mathrm{c}}[s\mathfrak{g}]$ which corresponds to the graded Lie bracket on $\mathfrak{g}$, so that the differential on the CCE-coalgebra $\mathcal{C}[\mathfrak{g}]$ (having $\mathcal{S}^{\mathrm{c}}[s\mathfrak{g}]$ as its underlying coaugmented graded coalgebra) is given by $d+\partial$. For intelligibility, we recall the following.

Lemma 2 (Ordinary perturbation lemma). *Let*

$$\left(M \underset{\pi}{\overset{\nabla}{\rightleftarrows}} N, h\right)$$

be a filtered contraction, let ∂ be a perturbation of the differential on N, and let

$$\delta = \sum_{n\geq 0} \pi\partial(-h\partial)^n\nabla = \sum_{n\geq 0} \pi(-\partial h)^n\partial\nabla$$

$$\nabla_\partial = \sum_{n\geq 0} (-h\partial)^n\nabla$$

$$\pi_\partial = \sum_{n\geq 0} \pi(-\partial h)^n$$
$$h_\partial = -\sum_{n\geq 0}(-h\partial)^n h = -\sum_{n\geq 0} h(-\partial h)^n.$$

If the filtrations on M and N are complete, these infinite series converge, δ is a perturbation of the differential on M and, when N_∂ and M_δ refer to the new chain complexes,

$$\left(M_\delta \underset{\pi_\partial}{\overset{\nabla_\partial}{\rightleftarrows}} N_\partial, h_\partial\right) \tag{54}$$

constitute a new filtered contraction that is natural in terms of the given data.

Proof. See [Br64] or [Gu72].

Application of the ordinary perturbation lemma to the contraction (53) and the perturbation ∂ of the differential d on $\mathcal{S}^{\mathrm{c}}[s\mathfrak{g}]$ yields the perturbation δ of the differential d^0 on $\mathcal{S}^{\mathrm{c}}[sM]$ and the contraction (22) where the notation $\widetilde{\nabla}$, $\widetilde{\Pi}$, H in (22) corresponds to, respectively, ∇_∂, π_∂, h_∂ in (54). By construction, the composite

$$\Phi\colon \mathcal{S}^{\mathrm{c}}_{\mathcal{D}}[sM] \xrightarrow{\ \overline{\tau}\ } \mathcal{C}[\mathfrak{g}] \xrightarrow{\ \widetilde{\Pi}\ } \mathcal{S}^{\mathrm{c}}_{\delta}[sM]$$

introduced as (23) above is a morphism of chain complexes and, modulo the filtrations, as a morphism of the underlying graded R-modules, this composite is the identity. More precisely, Φ can be written as an infinite series

$$\Phi = \mathrm{Id} + \Phi^1 + \cdots + \Phi^j + \ldots \tag{55}$$

such that, for $j \geq 1$, the operator Φ^j lowers the coaugmentation filtrations by j. Furthermore, the convergence of the series (55) is naive, that is, in each degree, the limit is achieved after finitely many steps. Consequently Φ is an isomorphism of chain complexes. The inverse Ψ of Φ can be obtained as the infinite series

$$\Psi = \mathrm{Id} + \Psi^1 + \cdots + \Psi^j + \ldots \tag{56}$$

determined by the requirement $\Phi\Psi = \mathrm{Id}$ or, equivalently, Ψ is given by the recursive description

$$\Psi^j + \Phi^1\Psi^{j-1} + \cdots + \Phi^{j-1}\Psi^1 + \Phi^j = 0,\ j \geq 1, \tag{57}$$

with the convention $\Phi^0 = \mathrm{Id}$ and $\Psi^0 = \mathrm{Id}$. Recall from (24) and (25) that, by definition,

$$\Pi = \Psi\widetilde{\Pi}\colon \mathcal{C}[\mathfrak{g}] \longrightarrow \mathcal{S}^{\mathrm{c}}_{\mathcal{D}}[sM],$$
$$H = \widetilde{H} - \widetilde{H}\overline{\tau}\,\Pi\colon \mathcal{C}[\mathfrak{g}] \longrightarrow \mathcal{C}[\mathfrak{g}].$$

By construction,

$$\Pi\,\overline{\tau} = \mathrm{Id}\colon \mathcal{S}^{\mathrm{c}}_{\mathcal{D}}[sM] \longrightarrow \mathcal{S}^{\mathrm{c}}_{\mathcal{D}}[sM]$$

and, since (22) is a contraction of chain complexes,

$$\begin{aligned}
DH = D(\widetilde{H} - \widetilde{H}\overline{\tau}\,\Pi) &= (\mathrm{Id} - \widetilde{\nabla}\widetilde{\Pi})(\mathrm{Id} - \overline{\tau}\,\Pi)\\
&= \mathrm{Id} - \widetilde{\nabla}\widetilde{\Pi} - \overline{\tau}\,\Pi + \widetilde{\nabla}\widetilde{\Pi}\overline{\tau}\,\Pi\\
&= \mathrm{Id} - \widetilde{\nabla}\Phi\Pi - \overline{\tau}\,\Pi + \widetilde{\nabla}\widetilde{\Pi}\overline{\tau}\,\Pi = \mathrm{Id} - \overline{\tau}\,\Pi
\end{aligned}$$

since $\Phi = \widetilde{\Pi}\overline{\tau}$ and $\Phi\Pi = \widetilde{\Pi}$. Consequently

$$(d+\partial)H + H(d+\partial) = \mathrm{Id} - \overline{\tau}\,\Pi,$$

that is, H is a chain homotopy between Id and $\overline{\tau}\,\Pi$. Moreover, since (22) is a contraction of chain complexes, the side conditions (5) hold, that is

$$\Pi H = 0, \quad H\overline{\tau} = 0, \quad HH = 0. \tag{58}$$

Consequently $\overline{\tau}$, Π and H constitute a contraction of chain complexes of the kind (12) as asserted and this contraction is obviously natural in terms of the data. This establishes Complement II and thus completes the proof of the Lie algebra perturbation lemma.

References

[Br64] Brown, R.: The twisted Eilenberg–Zilber theorem. Celebrazioni Archimedee del Secolo XX, Simposio di topologia 33–37 (1964)

[Ca54/55] Cartan, H.: Algèbres d'Eilenberg–Mac Lane et homotopie. Exposés 2–11. Séminaire H. Cartan 1954/55. École Normale Supérieure, Paris (1956)

[Do60] Dold, A.: Zur Homotopietheorie der Kettenkomplexe. Math. Ann. **140**, 278–298 (1960)

[EML53/54] Eilenberg, S., Mac Lane, S.: On the groups $\mathrm{H}(\pi, n)$. I. Ann. Math. **58**, 55–106 (1953) II. Methods of computation. Ann. Math. **60**, 49–139 (1954)

[Gu72] Gugenheim, V.K.A.M.: On the chain complex of a fibration. Illinois J. Math. **16**, 398–414 (1972)

[Gu82] Gugenheim, V.K.A.M.: On a perturbation theory for the homology of the loop space. J. Pure Appl. Algebra **25**, 197–205 (1982)

[GuLaSta91] Gugenheim, V.K.A.M., Lambe, L., Stasheff, J.D.: Perturbation theory in differential homological algebra. II. Illinois J. Math. **35**, 357–373 (1991)

[Hu86] Huebschmann, J.: The homotopy type of $F\Psi^q$. The complex and symplectic cases. In: Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory, Part II, Proceedings of a Conference at Boulder, Colorado, 12–18 June 1983. Cont. Math., vol. 55, pp. 487–518 (1986)

[Hu89a] Huebschmann, J.: Perturbation theory and free resolutions for nilpotent groups of class 2. J. Algebra **126**, 348–399 (1989)

[Hu89b] Huebschmann, J.: Cohomology of nilpotent groups of class 2. J. Algebra **126**, 400–450 (1989)

[Hu89c] Huebschmann, J.: The mod p cohomology rings of metacyclic groups. J. Pure Appl. Algebra **60**, 53–105 (1989)

[Hu91] Huebschmann, J.: Cohomology of metacyclic groups. Trans. Amer. Math. Soc. **328**, 1–72 (1991)

[Hu04a] Huebschmann, J.: Minimal free multi models for chain algebras. Georgian Math. J. **11**, 733–752 (2004). `Math.AT/0405172`

[Hu04b] Huebschmann, J.: Homological perturbations, equivariant cohomology, and Koszul duality. Documenta math. (accepted for publication). `Math.AT/0401160`

[Hu04c] Huebschmann, J.: Relative homological algebra, homological perturbations, equivariant de Rham theory, and Koszul duality. `Math.AT/0401161`

[Hu07a] Huebschmann, J.: The sh-Lie algebra perturbation lemma. Forum math. (accepted for publication). `Arxiv:0710.2070`

[Hu07b] Huebschmann, J.: Origins and breadth of the theory of higher homotopies. In this volume, pp. 25–38. `Arxiv:0710.2645`

[HuKa91] Huebschmann, J., Kadeishvili, T.: Small models for chain algebras. Math. Z. **207**, 245–280 (1991)

[HuSt02] Huebschmann, J., Stasheff, J.D.: Formal solution of the master equation via HPT and deformation theory. Forum Math. **14**, 847–868 (2002). `Math.AG/9906036`

[Ko97] Kontsevich, M.: Deformation quantization of Poisson manifolds. Lett. Math. Phys. **66**, 157–216 (2003). `Math.QA/9709040`

[Mo71] Moore, J.C.: Differential Homological Algebra. Actes, Congrès intern. Math. Nice. Gauthiers-Villars, Paris, 335–339 (1970/1971)

[Qu69] Quillen, D.: Rational homotopy theory. Ann. Math. **90**, 205–295 (1969)

[SchlSt] Schlessinger, M., Stasheff, J.D.: Deformation theory and rational homotopy type. Pub. Math. Sci. IHES. new version July 13 (1998) (to appear)

[Sh62] Shih, W.: Homologie des espaces fibrés. Pub. Math. Sci. IHES **13** (1962)

Twisting Elements in Homotopy G-Algebras

T. Kadeishvili

A. Razmadze Mathematical Institute, 1, M. Alexidze Str., Tbilisi, 0193 Georgia, kade@rmi.acnet.ge

Abstract. We study the notion of twisting elements $da = a \smile_1 a$ with respect to $\smile_1$ product when it is a part of homotopy Gerstenhaber algebra structure. This allows us to bring to one context the two classical concepts, the theory of deformation of algebras of M. Gerstenhaber, and $A(\infty)$-algebras of J. Stasheff.

Key words: A_∞-algebra, C_∞-algebra, Rational Homotopy

AMS Classification (2010): 16E40, 18G55, 55S30

1 Introduction

A *twisting element* in a differential graded algebra (dga) $(A = \{A^i\}, d : A^n \to A^{n+1}, a^m \cdot b^n \in A^{m+n})$ is defined as an element $t \in A^1$ satisfying Brown's condition

$$dt = t \cdot t. \tag{1}$$

Denote the set of all twisting elements by $Tw(A)$. A useful consequence of Brown's condition is the following: let M be a dg module over A, then a twisting element $t \in Tw(A)$ defines on M a new differential $d_t : M \to M$ by $d_t(x) = dx + t \cdot x$, and the condition (1) guarantees that $d_t d_t = 0$.

Twisting elements show up in various problems of algebraic topology and homological algebra. The first appearance was in homology theory of fibre bundles [5]: For a fibre bundle $F \to E \to B$ with a structure group G there exists a twisting element $t \in A = C^*(B, C_*(G))$ such that $(M = C_*(B) \otimes C_*(F), d_t)$ (the twisted tensor product) gives homology of the total space E.

Later N. Berikashvili [4] has introduced in $Tw(A)$ an *equivalence relation* induced by the following group action. Let G be the group of invertible elements in A^0, then for $g \in G$ and $t \in Tw(A)$ let

$$g * t = g \cdot t \cdot g^{-1} + dg \cdot g^{-1}, \tag{2}$$

A.S. Cattaneo et al. (eds.), *Higher Structures in Geometry and Physics*, Progress in Mathematics 287, DOI 10.1007/978-0-8176-4735-3_9,

and it is easy to see that $g * t \in Tw(A)$. The factor set $D(A) = Tw(A)/G$, called Berikashvili's functor D, has nice properties and useful applications. In particular, if $t \sim t'$, then (M, d_t) and $(M, d_{t'})$ are *isomorphic*.

The notion of *homotopy G-algebra* (hGa) was introduced by Gerstenhaber and Voronov in [8] as an additional structure on a dg algebra $(A, d, \cdot)$ that induces a Gerstenhaber algebra structure on homology. The main example is the Hochschild cochain complex of an algebra.

Another point of view is that hGa is a particular case of $B(\infty)$-algebra [10]: this is an additional structure on a dg algebra $(A, d, \cdot)$ that induces a dg bialgebra structure on the bar construction BA.

There is a third aspect of hGa [16]: this is a structure which measures the noncommutativity of A. The Steenrod $\smile_1$ product which is the classical tool which measures the noncommutativity of a dg algebra $(A, d, \cdot)$ satisfies the condition

$$d(a \smile_1 b) = da \smile_1 b + a \smile_1 db + a \cdot b + b \cdot a. \tag{3}$$

The existence of such $\smile_1$ guarantees the commutativity of $H(A)$, but a $\smile_1$ product satisfying just the condition (3) is too poor for some applications. In many constructions some deeper properties of $\smile_1$ are needed, for example the compatibility with the dot product of A (the Hirsch formula)

$$(a \cdot b) \smile_1 c + a \cdot (b \smile_1 c) + (a \smile_1 c) \cdot b = 0. \tag{4}$$

An hGa $(A, d, \cdot, \{E_{1,k}\})$ is a dga $(A, d, \cdot)$ equipped additionally with a sequence of operations (some authors call them *braces*)

$$\{E_{1,k} : A \otimes A^{\otimes k} \to A,\ k = 1, 2, ...\}$$

satisfying some coherence conditions (see bellow). The starting operation $E_{1,1}$ is a kind of $\smile_1$ product: it satisfies the conditions (3) and (4). As for the symmetric expression

$$a \smile_1 (b \cdot c) + b \cdot (a \smile_1 c) + (a \smile_1 b) \cdot c,$$

it is just *homotopical to zero* and the appropriate chain homotopy is the operation $E_{1,2}$. So we can say that an hGa is a dga with a "good" $\smile_1$ product.

There is one more aspect of hGa: the operation $E_{1,1} = \smile_1$ is not associative but the commutator $[a, b] = a \smile_1 b - b \smile_1 a$ satisfies the Jacobi identity, so it forms on the desuspension $s^{-1}A$ the structure of a dg Lie algebra.

Let us present three remarkable examples of homotopy G-algebras.

The first one is the cochain complex of 1-reduced simplicial set $C^*(X)$. The operations $E_{1,k}$ here are dual to cooperations defined by Baues in [2], and the starting operation $E_{1,1}$ is the classical Steenrod's $\smile_1$ product.

The second example is the Hochschild cochain complex $C^*(U, U)$ of an associative algebra U. The operations $E_{1,k}$ here were defined in [14] with the purpose to describe $A(\infty)$-algebras in terms of the Hochschild cochains although the properties of those operations which where used as defining ones for the notion of homotopy G-algebra in [8] did not appear there. These operations were defined also in [9], [10]. Again the starting operation $E_{1,1}$ is the classical Gerstenhaber circle product which is sort of $\smile_1$-product.

The third example is the cobar construction ΩC of a dg *bialgebra* C. The operations $E_{1,k}$ are constructed in [17]. And again the starting operation $E_{1,1}$ is classical: it is Adams's $\smile_1$-product defined for ΩC in [1] using the *multiplication* of C.

The main task of this paper is to introduce the notion of a twisting element and their transformation in an hGa. Shortly a twisting element in an hGa $(A, d, \cdot, \{E_{1,k}\})$ is an element $a \in A$ such that $da = a \smile_1 a$ and two twisting elements $a, \overline{a} \in A$ we call equivalent if there exists $g \in A$ such that

$$\overline{a} = a + dg + g \cdot g + g \smile_1 a + \overline{a} \smile_1 g + E_{1,2}(\overline{a}; g, g) + E_{1,3}(\overline{a}; g, g, g) + \cdots \ .$$

As we see in the definition of a twisting element participates just the operation $E_{1,1} = \smile_1$ but in the definition of equivalence participates the whole hGa structure. We remark that such a twisting element $a \in A$ is a Lie twisting element in the dg Lie algebra $(s^{-1}A, d, [\ ,\ ])$, i.e., satisfies $da = \frac{1}{2}[a, a]$.

In this paper we present the following application of the notion of twisting element in an hGa: it allows us to unify two classical concepts, namely the theory of deformation of algebras of M. Gerstenhaber, and J. Stasheff's $A(\infty)$-algebras.

Namely, a Gerstenhaber's deformation of an associative algebra U (see [7], and below)

$$a \star b = a \cdot b + B_1(a \otimes b)t + B_2(a \otimes b)t^2 + B_3(a \otimes b)t^3 + \cdots \in U[[t]],$$

can be considered as a twisting element $B = B_1 + B_2 + \cdots \in C^2(U, U)$ in the Hochschild cochain complex of U with coefficients in itself: the defining condition of deformation means exactly $dB = B \smile_1 B$. Furthermore, two deformations are equivalent if and only if the corresponding twisting elements are equivalent in the above sense.

On the other side, the same concept of twisting elements in hGa works in the following problem. Suppose $(H, \mu : H \otimes H \to H)$ is a graded algebra. Let us define its *Stasheff deformation* (or minimal $A(\infty$ deformation)) as an $A(\infty)$ algebra structure $(H, \{m_i\})$ with $m_1 = 0$ and $m_2 = \mu$, i.e., which extends the given algebra structure. Then each deformation can be considered as a twisting element $m = m_3 + m_4 + \cdots\ ,\ m_i \in C^i(H, H)$ in the Hochschild cochain complex of H with coefficients in itself: the Stasheff defining condition of $A(\infty)$-algebra means exactly $dm = m \smile_1 m$. Furthermore, to isomorphic (as $A(\infty)$-algebras) deformations correspond equivalent twisting elements in the above sense.

In both cases we present the obstruction theory for the existence of suitable deformations. The obstructions live in suitable Hochschild cohomologies: in $H^2(U, U)$ in the Gerstenhaber deformation case and in $H^i(H, H),\ i = 3, 4, ...$ in the Stasheff deformation case.

Note that the interpretation of *minimal* $A(\infty)$-algebra structure on a given graded algebra (H, μ) as a twisting element in the Hochschild cochain complex $C^*(H, H)$, as well as interpretation of an isomorphism of such $A(\infty)$-algebras as equivalence of corresponding twisting elements, was given in [14].

We hope that more general $A(\infty)$ deformations, known in the literature, also can be treated as certain twisting elements in corresponding hGa or more general brace algebra.

The structure of the paper is the following. In Section 2 necessary definitions are given. In Section 3 the definition of homotopy G-algebra is presented. In Section 4 the notion of twisted element in a homotopy G-algebra is studied. In the last two Sections 5 and 6 the above-mentioned applications of this notion are given.

Acknowledgements. Dedicated to Murray Gerstenhaber's 80th and Jim Stasheff's 70th birthdays.

The author thanks the referee for various suggestions that improved the readability and content of this paper.

2 Notation and Preliminaries

We work over $R = Z_2$. For a graded module M we denote by sM the suspension of M, i.e., $(sM)^i = M^{i-1}$. Respectively, $(s^{-1}M)^i = M^{i+1}$.

2.1 Differential Graded Algebras and Coalgebras

A *differential graded algebra* (dg algebra, or dga) is a graded R-module $C = \{C^i,\ i \in Z\}$ with an associative *multiplication* $\mu : C^i \otimes C^j \to C^{i+j}$ and a *differential* $d : C^i \to C^{i+1}$ satisfying $dd = 0$ and the *derivation condition* $d(x \cdot y) = dx \cdot y + x \cdot dy$, where $x \cdot y = \mu(x \otimes y)$. A dga C is *connected* if $C^{<0} = 0$ and $C^0 = R$. A connected dga C is *n-reduced* if $C^i = 0$ for $1 \leq i \leq n$.

A *differential graded coalgebra* (dg coalgebra, or dgc) is a graded R-module $C = \{C^i,\ i \in Z\}$ with a coassociative *comultiplication* $\Delta : C \to C \otimes C$ and a *differential* $d : C^i \to C^{i+1}$ satisfying $dd = 0$ and the *coderivation condition*$\Delta d = (d \otimes id + id \otimes d)\Delta$. A dgc C is *connected* if $C_{<0} = 0$ and $C_0 = R$. A connected dgc C is *n-reduced* if $C_i = 0$ for $1 \leq i \leq n$.

A *differential graded bialgebra* (dg bialgebra) (C, d, μ, Δ) is a dg coalgebra (C, d, Δ) with a morphism of dg coalgebras $\mu : C \otimes C \to C$ turning (C, d, μ) into a dg algebra.

2.2 Cobar and Bar Constructions

Let M be a graded R-module with $M^{i \leq 0} = 0$ and let $T(M)$ be the tensor algebra of M, i.e., $T(M) = \oplus_{i=0}^{\infty} M^{\otimes i}$. Tensor algebra $T(M)$ is a free graded algebra generated by M: for a graded algebra A and a homomorphism

$\alpha : M \to A$ of degree zero there exists a unique morphism of graded algebras $f_\alpha : T(M) \to A$ (called *the multiplicative extension* of α)such that $f_\alpha(a) = \alpha(a)$. The map f_α is given by $f_\alpha(a_1 \otimes \cdots \otimes a_n) = \alpha(a_1) \cdot \cdots \cdot \alpha(a_n)$.

Dually, let $T^c(M)$ be the tensor coalgebra of M, i.e., $T^c(M) = \oplus_{i=0}^{\infty} M^{\otimes i}$, and the comultiplication $\nabla : T^c(M) \to T^c(M) \otimes T^c(M)$ is given by

$$\nabla(a_1 \otimes \cdots \otimes a_n) = \sum_{k=0}^{n} (a_1 \otimes \cdots \otimes a_k) \otimes (a_{k+1} \otimes \cdots \otimes a_n).$$

The tensor coalgebra $(T^c(M), \nabla)$ is a cofree graded coalgebra: for a graded coalgebra C and a homomorphism $\beta : C \to M$ of degree zero there exists a unique morphism of graded coalgebras $g_\beta : C \to T^c(M)$ (called *the comultiplicative coextension* of β) such that $p_1 g_\beta = \beta$, here $p_1 : T^c(M) \to M$ is the obvious projection. The map g_β is given by

$$g_\beta = \sum_{n=0}^{\infty} (\beta \otimes \cdots \otimes \beta) \Delta^n,$$

where $\Delta^n : C \to C^{\otimes n}$ is the nth iteration of the diagonal $\Delta : C \to C \otimes C$, i.e., $\Delta^1 = id,\ \Delta^2 = \Delta,\ \Delta^n = (\Delta^{n-1} \otimes id)\Delta$.

Let (C, d_C, Δ) be a connected dg coalgebra and $\Delta(c) = c \otimes 1_R + 1_R \otimes c + \Delta'(c)$. The (reduced) *cobar construction* ΩC on C is a dg algebra whose underlying graded algebra is $T(sC^{>0})$. An element $(sc_1 \otimes \cdots \otimes sc_n) \in (sC)^{\otimes n} \subset T(sC^{>0})$ is denoted by $[c_1, ..., c_n] \in \Omega C$. The differential d_Ω of ΩC for a generator $[c] \in \Omega C$ is defined by $d_\Omega[c] = [d_C(c)] + \sum [c', c'']$ where $\Delta'(c) = \sum c' \otimes c''$, and is extended as a derivation.

Let (A, d_A, μ) be a 1-reduced dg algebra. The (reduced) *bar construction* BA on A is a dg coalgebra whose underlying graded coalgebra is $T^c(s^{-1}A^{>0})$. Again an element $(s^{-1}a_1 \otimes \cdots \otimes s^{-1}a_n) \in (s^{-1}A)^{\otimes n} \subset T^c(s^{-1}A^{>0})$ we denote as $[a_1, ..., a_n] \in BA$. The differential d_B of BA is defined by

$$d_B[a_1, ..., a_n] = \textstyle\sum_{i=1}^{n} [a_1, ..., d_A a_i, ..., a_n] + \sum_{i=1}^{n-1} [a_1, ..., a_i \cdot a_{i+1}, ..., a_n].$$

2.3 Twisting Cochains

Let (C, d, Δ) be a dgc and (A, d, μ) be a dga. A twisting cochain [5] is a homomorphism $\tau : C \to A$ of degree $+1$ satisfying Brown's condition

$$d\tau + \tau d = \tau \smile \tau, \tag{5}$$

where $\tau \smile \tau' = \mu_A(\tau \otimes \tau')\Delta$. We denote by $Tw(C, A)$ the set of all twisting cochains $\tau : C \to A$.

There are universal twisting cochains $\tau_C : C \to \Omega C$ and $\tau_A : BA \to A$ being obvious inclusion and projection, respectively.

Here are essential consequences of the condition (5):
(i) The multiplicative extension $f_\tau : \Omega C \to A$ is a map of dg algebras, so there is a bijection $Tw(C,A) \leftrightarrow Hom_{dg-Alg}(\Omega C, A)$;
(ii) The comultiplicative coextension $g_\tau : C \to BA$ is a map of dg coalgebras, so there is a bijection $Tw(C,A) \leftrightarrow Hom_{dg-Coalg}(C, BA)$.

3 Homotopy G-algebras

A *homotopy G-algebra* (hGa) is a dg algebra with "good" $\smile_1$ product. The general notion was introduced in [8], see also [25].

Definition 1. *A homotopy G-algebra is defined as a dg algebra $(A, d, \cdot)$ with a given sequence of operations*

$$E_{1,k} : A \otimes (A^{\otimes k}) \to A, \quad k = 0, 1, 2, 3, ...$$

(the value of the operation $E_{1,k}$ on $a \otimes b_1 \otimes \cdots \otimes b_k \in A \otimes (A \otimes \cdots \otimes A)$ we write as $E_{1,k}(a; b_1, ..., b_k)$) which satisfies the conditions

$$E_{1,0} = id, \tag{6}$$

$$\begin{aligned} &dE_{1,k}(a; b_1, ..., b_k) + E_{1,k}(da; b_1, ..., b_k) + \textstyle\sum_i E_{1,k}(a; b_1, ..., db_i, ..., b_k) \\ &= b_1 \cdot E_{1,k-1}(a; b_2, ..., b_k) + E_{1,k-1}(a; b_1, ..., b_{k-1}) \cdot b_k + \\ &\quad \textstyle\sum_i E_{1,k-1}(a; b_1, ..., b_i \cdot b_{i+1}, ..., b_k), \end{aligned} \tag{7}$$

$$\begin{aligned} &E_{1,k}(a_1 \cdot a_2; b_1, .., b_k) \\ &= a_1 \cdot E_{1,k}(a_2; b_1, ..., b_k) + E_{1,k}(a_1; b_1, ..., b_k) \cdot a_2 + \\ &\quad \textstyle\sum_{p=1}^{k-1} E_{1,p}(a_1; b_1, ..., b_p) \cdot E_{1,m-p}(a_2; b_{p+1}, ..., b_k), \end{aligned} \tag{8}$$

$$\begin{aligned} &E_{1,n}(E_{1,m}(a; b_1, ..., b_m); c_1, ..., c_n) \\ &= \textstyle\sum_{0 \le i_1 \le j_1 \le \cdots \le i_m \le j_m \le n} \\ &\quad E_{1,n-(j_1+\cdots+j_m)+(i_1+\cdots+i_m)+m}(a; c_1, ..., c_{i_1}, E_{1,j_1-i_1}(b_1; c_{i_1+1}, ..., c_{j_1}), \\ &\quad c_{j_1+1}, ..., c_{i_2}, E_{1,j_2-i_2}(b_2; c_{i_2+1}, ..., c_{j_2}), c_{j_2+1}, ..., c_{i_m}, \\ &\quad E_{1,j_m-i_m}(b_m; c_{i_m+1}, ..., c_{j_m}), c_{j_m+1}, ..., c_n). \end{aligned} \tag{9}$$

Let us present these conditions in low dimensions.
The condition (7) for $k = 1$ looks as

$$dE_{1,1}(a; b) + E_{1,1}(da; b) + E_{1,1}(a; db) = a \cdot b + b \cdot a. \tag{10}$$

So the operation $E_{1,1}$ is a sort of $\smile_1$ product: it is the chain homotopy which measures the noncommutativity of A, cf. the condition (3). Below we denote $a \smile_1 b = E_{1,1}(a; b)$.
The condition (8) for $k = 1$ looks as

$$(a \cdot b) \smile_1 c + a \cdot (b \smile_1 c) + (a \smile_1 c) \cdot b = 0, \tag{11}$$

this means that the operation $E_{1,1} = \smile_1$ satisfies the *left Hirsch formula* (4).

The condition (7) for $k = 2$ looks as

$$\begin{aligned} &dE_{1,2}(a;b,c) + E_{1,2}(da;b,c) + E_{1,2}(a;db,c) + E_{1,2}(a;b,dc) \\ &= a \smile_1 (b \cdot c) + (a \smile_1 b) \cdot c + b \cdot (a \smile_1 c), \end{aligned} \tag{12}$$

this means that this $\smile_1$ satisfies the *right Hirsch formula* just up to homotopy and the appropriate homotopy is the operation $E_{1,2}$.

The condition (9) for $n = m = 2$ looks as

$$(a \smile_1 b) \smile_1 c + a \smile_1 (b \smile_1 c) = E_{1,2}(a;b,c) + E_{1,2}(a;c,b), \tag{13}$$

this means that the same operation $E_{1,2}$ measures also the deviation from the associativity of the operation $E_{1,1} = \smile_1$.

3.1 hGa as a $B(\infty)$-algebra

The notion of B_∞-algebra was introduced in [10] as an additional structure on a dg module (A, d) which turns the tensor coalgebra $T^c(s^{-1}A)$ into a dg bialgebra. So it requires a differential

$$\widetilde{d} : T^c(s^{-1}A) \to T^c(s^{-1}A)$$

which is a coderivation (that is an $A(\infty)$-algebra structure on A, see below) and an associative multiplication

$$\widetilde{\mu} : T^c(s^{-1}A) \otimes T^c(s^{-1}A) \to T^c(s^{-1}A)$$

which is a map of dg coalgebras.

Here we show that an hGa structure on A is a particular $B(\infty)$-algebra structure: it induces on $B(A) = (T^c(s^{-1}A), d_B)$ a multiplication but does not change the differential d_B (see [10], [16], [17], [18] for more details).

Let us extend our sequence $\{E_{1,k},\ k = 0,1,2,...\}$ to a sequence $\{E_{p,q} : (A^{\otimes p}) \otimes (A^{\otimes q}) \to A,\ p,q = 0,1,...\}$ adding

$$E_{0,1} = id,\ E_{0,q>1} = 0,\ E_{1,0} = id,\ E_{p>1,0} = 0, \tag{14}$$

and $E_{p>1,q} = 0$.

This sequence defines a map $E : B(A) \otimes B(A) \to A$ by $E([a_1,...,a_m] \otimes [b_1,...,b_n]) = E_{p,q}(a_1,...,a_m;b_1,...,b_n)$. The conditions (7) and (8) mean exactly $dE + E(d_B \otimes id + id \otimes d_B) = E \smile E$, i.e., E is a twisting cochain. Thus according to Section 2.3 its coextension is a dg coalgebra map

$$\mu_E : B(A) \otimes B(A) \to B(A).$$

The condition (9) can be rewritten as $E(\mu_E \otimes id - id \otimes \mu_E) = 0$, so this condition means that the multiplication μ_E is associative. And the condition (14) implies that $[\ ] \in B(A)$ is the unit for this multiplication.

Finally we obtained that $(B(A), d_B, \Delta, \mu_E)$ is a dg bialgebra thus an hGa is a $B(\infty)$-algebra.

Let us mention that a twisting cochain E satisfying just the starting condition (14) was constructed in [19] using acyclic models for $A = C^*(X)$, the singular cochain complex of a topological space. The condition (14) determines this twisting cochain E uniquely up to equivalence of twisting cochains (2).

3.2 Homology of an hGa is a Gerstenhaber algebra

The structure of an hGa on A induces on the homology $H(A)$ the structure of a Gerstenhaber algebra (G-algebra).

Gerstenhaber algebra (see [6], [8], [25]) is defined as a commutative graded algebra $(H, \cdot)$ together with a Lie bracket of degree -1

$$[\ ,\ ] : H^p \otimes H^q \to H^{p+q-1}$$

(i.e., a graded Lie algebra structure on the desuspension $s^{-1}H$) which is a biderivation: $[a, b \cdot c] = [a, b] \cdot c + b \cdot [a, c]$. Main example of Gerstenhaber algebra is the Hochschild cohomology of an associative algebra.

The following argument shows the existence of this structure on the homology $H(A)$ of an hGa.

First, there appears on the desuspension $s^{-1}A$ the structure of a dg Lie algebra: although the $\smile_1 = E_{1,1}$ is not associative, the condition (13) implies the pre-Jacobi identity

$$a \smile_1 (b \smile_1 c) + (a \smile_1 b) \smile_1 c = a \smile_1 (c \smile_1 b) + (a \smile_1 c) \smile_1 b,$$

this condition guarantees that the commutator $[a, b] = a \smile_1 b + b \smile_1 a$ satisfies the Jacobi identity, besides the condition (10) implies that $[\ ,\ ] : s^{-1}A \otimes s^{-1}A \to s^{-1}A$ is a chain map. Consequently there is on $s^{-1}H(A)$ a structure of a graded Lie algebra. The Hirsch formulae (11) and (12) imply that the induced Lie bracket is a biderivation.

3.3 Operadic Description

The operations $E_{1,k}$ forming hGa have a nice description in terms of the *surjection operad*, see [20], [21] [22], [3] for definition. Namely, to the dot product corresponds the element $(1, 2) \in \chi_0(2)$; to $E_{1,1} = \smile_1$ product corresponds $(1, 2, 1) \in \chi_1(2)$, and generally to the operation $E_{1,k}$ corresponds the element

$$E_{1,k} = (1, 2, 1, 3, ..., 1, k, 1, k+1, 1) \in \chi_k(k+1). \tag{15}$$

We remark here that the defining conditions of an hGa (6), (7), (8), (9) can be expressed in terms of the operadic structure (differential, symmetric group action and composition product) and the elements (15) satisfy these conditions *already in the operad* χ.

Note that the elements (15) together with the element $(1, 2)$ generate the suboperad $F_2\chi$ which is equivalent to the little square operad ([20], [22], [3]).

This in particular implies that a cochain complex (A, d) is an hGa if and only if it is an algebra over the operad $F_2\chi$.

This fact and the hGa structure on the Hochschild cochain complex $C^*(U, U)$ of an algebra U [14] were used by some authors to prove the *Deligne conjecture* about the action of the little square operad on the Hochschild cochain complex $C^*(U, U)$.

3.4 Hochschild Cochain Complex as a hGa

Let A be an algebra and M be a two-sided module on A. The Hochschild cochain complex $C^*(A; M)$ is defined as $C^n(A; M) = Hom(A^{\otimes^n}, M)$ with differential $\delta : C^{n-1}(A; M) \to C^n(A; M)$ given by

$$\begin{aligned}\delta f(a_1 \otimes \cdots \otimes a_n) = a_1 \cdot f(a_2 \otimes \cdots \otimes a_n)&\\ +\textstyle\sum_{k=1}^{n-1} f(a_1 \otimes \cdots \otimes a_{k-1} \otimes a_k \cdot a_{k+1} \otimes \cdots \otimes a_n)&\\ +f(a_1 \otimes \cdots \otimes a_{n-1}) \cdot a_n.&\end{aligned}$$

We focus on the case $M = A$.

In this case the Hochschild complex becomes a dg algebra with respect to the $\smile$ product defined in [6] by

$$f \smile g(a_1 \otimes \cdots \otimes a_{n+m}) = f(a_1 \otimes \cdots \otimes a_n) \cdot g(a_{n+1} \otimes \cdots \otimes a_{n+m}).$$

In [14] (see also [9], [10], [8]) there are defined the operations

$$E_{1,i} : C^n(A; A) \otimes C^{n_1}(A; A) \otimes \cdots \otimes C^{n_i}(A; A) \to C^{n+n_1+\cdots+n_i-i}(A; A)$$

given by $E_{1,i}(f; g_1, ..., g_i) = 0$ for $i > n$ and

$$\begin{aligned}&E_{1,i}(f; g_1, ..., g_i)(a_1 \otimes \cdots \otimes a_{n+n_1+\cdots+n_i-i})\\ &= \textstyle\sum_{k_1,...,k_i} f(a_1 \otimes \cdots \otimes a_{k_1} \otimes g_1(a_{k_1+1} \otimes \cdots \otimes a_{k_1+n_1}) \otimes a_{k_1+n_1+1} \otimes \cdots\\ &\quad \otimes a_{k_2} \otimes g_2(a_{k_2+1} \otimes \cdots \otimes a_{k_2+n_2}) \otimes a_{k_2+n_2+1} \otimes \cdots\\ &\quad \otimes a_{k_i} \otimes g_i(a_{k_i+1} \otimes \cdots \otimes a_{k_i+n_i}) \otimes a_{k_i+n_i+1} \otimes \cdots \otimes a_{n+n_1+\cdots+n_i-i}).\end{aligned} \tag{16}$$

The straightforward verification shows that the collection $\{E_{1,k}\}$ satisfies the conditions (6), (7), (8), and (9), thus it forms on the Hochschild complex $C^*(A; A)$ the structure of a homotopy G-algebra.

We note that the operation $E_{1,1}$ coincides with the circle product defined by Gerstenhaber in [6], note also that the operation $E_{1,2}$ satisfying (12) and (13) also is defined there.

4 Twisting Elements

In this section we present an analog of the notion of twisting element (see the introduction) in a homotopy G-algebra replacing in the defining equation $da = a \cdot a$ the dot product by the $\smile_1$ product. The appropriate notion of equivalence also will be introduced.

Let $(C^{*,*}, d, \cdot, \{E_{1,k}\})$ be a *bigraded* homotopy G-algebra. That is, $(C^{*,*}, \cdot)$ is a bigraded algebra $C^{m,n} \cdot C^{p,q} \subset C^{m+p,n+q}$, and we require the existence of a differential (derivation) $d(C^{m,n}) \subset C^{m+1,n}$ and of a sequence of operations

$$E_{1,k} : C^{m,n} \otimes C^{p_1,q_1} \otimes \cdots \otimes C^{p_k,q_k} \to C^{m+p_1+\cdots+p_k-k,n+q_1+\cdots+q_k}$$

so that the *total complex* (the total degree of $C^{p,q}$ is $p+q$) is an hGa.

Below we introduce two versions of the notion of *twisting elements* in a bigraded homotopy G-algebra. Although it is possible to reduce them to each other by changing gradings, we prefer to consider them separately in order to emphasize different areas of their applications. The first one controls Stasheff A_∞-deformation of graded algebras and the second controls Gerstenhaber deformation of associative algebras, see the next two sections.

4.1 Twisting Elements in a Bigraded Homotopy G-algebra (version 1)

A *twisting element* in $C^{*,*}$ we define as

$$m = m^3 + m^4 + \cdots + m^p + \cdots \ , \ m^p \in C^{p,2-p}$$

satisfying the condition $dm = E_{1,1}(m; m)$ or changing the notation $dm = m \smile_1 m$. This condition can be rewritten in terms of the components as

$$dm^p = \sum_{i=3}^{p-1} m^i \smile_1 m^{p-i+2}. \tag{17}$$

Particularly $dm^3 = 0$, $dm^4 = m^3 \smile_1 m^3$, $dm^5 = m^3 \smile_1 m^4 + m^4 \smile_1 m^3, \ldots$. The set of all twisting elements we denote by $Tw(C^{*,*})$.

Consider the set $G = \{g = g^2 + g^3 + \cdots + g^p + \cdots \ ; \ g^p \in C^{p,1-p}\}$, and let us introduce on G the operation

$$\overline{g} * g = \overline{g} + g + \sum_{k=1}^{\infty} E_{1,k}(\overline{g}; g, ..., g), \tag{18}$$

particularly

$$\begin{aligned}
&(\overline{g} * g)^2 = \overline{g} + g^2;\\
&(\overline{g} * g)^3 = \overline{g}^3 + g^3 + \overline{g}^2 \smile_1 g^3;\\
&(\overline{g} * g)^4 = \overline{g}^4 + g^3 + \overline{g}^2 \smile_1 g^3 + \overline{g}^3 \smile_1 g^2 + E_{1,2}(\overline{g}^2; g^2, g^2).
\end{aligned}$$

It is possible to check, using the defining conditions of an hGa (6), (7), (8), (9), that this operation is associative, has the unit $e = 0 + 0 + \cdots$ and the opposite g^{-1} can be solved inductively from the equation $g * g^{-1} = e$. Thus G is a group.

The group G acts on the set $Tw(C^{*,*})$ by the rule $g * m = \overline{m}$ where

$$\overline{m} = m + dg + g \cdot g + E_{1,1}(g; m) + \sum_{k=1}^{\infty} E_{1,k}(\overline{m}; g, ..., g), \tag{19}$$

particularly

$$\begin{aligned}
\overline{m}^3 &= m^3 + dg^2;\\
\overline{m}^4 &= m^4 + dg^3 + g^2 \cdot g^2 + g^2 \smile_1 m^3 + \overline{m^3} \smile_1 g^2;\\
\overline{m}^5 &= m^5 + dg^4 + g^2 \cdot g^3 + g^3 \cdot g^2 + g^2 \smile_1 m^4 + g^3 \smile_1 m^3\\
&\quad + \overline{m}^3 \smile_1 g^3 + \overline{m}^4 \smile_1 g^2 + E_{1,2}(\overline{m}^3; g^2, g^2).
\end{aligned}$$

Note that although on the right-hand side of the formula (19) participates $\overline{m}$ but it has less dimension than the left-hand side $\overline{m}$, thus this action is well defined: the components of $\overline{m}$ can be solved from this equation inductively. It is possible to check that the resulting $\overline{m}$ is a twisting element. By $D(C^{*,*})$ we denote the set of orbits $Tw(C^{*,*})/G$.

This group action (19) allows us to *perturb* twisting elements in the following sense. Let $g^n \in C^{n,1-n}$ be an arbitrary element, then for homogeneous $g = 0 + \cdots + 0 + g^n + 0 + \cdots$ the twisting element $\overline{m} = g * m$ defined by (19) in this case looks as

$$\overline{m} = m^3 + \cdots + m^n + (m^{n+1} + dg^n) + \overline{m}^{n+2} + \overline{m}^{n+3} + \cdots, \tag{20}$$

so the components $m^3, ..., m^n$ remain unchanged and $\overline{m}^{n+1} = m^{n+1} + dg^n$.

The perturbation (20) allows us to consider the following two problems.

Quantization. Let us first mention that for a twisting element $m = \sum m^k$ the first component $m^3 \in C^{3,-1}$ is a cycle and any perturbation does not change its homology class $[m^3] \in H^{3,-1}(C^{*,*})$. Thus we have a well-defined map $\phi : D(C^{*,*}) \to H^{3,-1}(C^{*,*})$.

A *quantization* of a homology class $\alpha \in H^{3,-1}(C^{*,*})$ is defined as a twisting element $m = m^3 + m^4 + \cdots$ such that $[m^3] = \alpha$. Thus α is quantizable if it belongs to the image of ϕ.

The obstructions for quantizability lie in homologies $H^{n,3-n}(C^{*,*}),\ n \geq 5$. Indeed, let $m^3 \in C^{3,-1}$ be a cycle from α. The first step to quantize α is to construct m^4 such that $dm^4 = m^3 \smile_1 m^3$. The necessary and sufficient condition for this is $[m^3 \smile_1 m^3] = 0 \in H^{5,-2}(C^{*,*})$, so this homology class is the first obstruction $O(m^3)$. Suppose it vanishes; so there exists m^4. Then it is easy to see that $m^3 \smile_1 m^4 + m^4 \smile_1 m^3$ is a cycle and its class $O(m^3, m^4) \in H^{6,-3}(C^{*,*})$ is the second obstruction. If $O(m^3, m^4) = 0$, then there exists m^5 such that $dm^5 = m^3 \smile_1 m^4 + m^4 \smile_1 m^3$. If not, then we take another m^4 and try new second obstruction (we remark that changing of m^3 makes no sense). Generally the obstruction is

$$O(m^3, m^4, ..., m^{n-2}) = [\sum_{k=3}^{n-2} m^k \smile_1 m^{n-k+1}] \in H^{n,3-n}(C^{*,*}).$$

Rigidity. A twisting element $m = m^3 + m^4 + \cdots + m^p + \cdots$ is called *trivial* if it is equivalent to 0. A bigraded hGa $C^{*,*}$ is *rigid* if each twisting element is trivial, i.e., if $D(C^{*,*}) = \{0\}$. The obstructions to triviality of a twisting element lie in homologies $H^{n,2-n}(C^{*,*}),\ n \geq 3$. Indeed, for a twisting element $m = m^3 + m^4 + \cdots + m^p + \cdots$ the first component m^3 is a cycle and by (19) each perturbation of m leaves the class $[m^3] \in H^{3,-1}(C^{*,*})$ unchanged and this class is the first obstruction for triviality. If this class is zero, then we choose $g^2 \in C^{2,-1}$ such that $dg^2 = m^3$. Perturbing m by $g = g^2 + 0 + 0 + \cdots$ we kill the first component m^3, i.e., we get a twisting element $\overline{m} \sim m$, which looks as $\overline{m} = 0 + \overline{m}^4 + \overline{m}^5 + \cdots$. Now, because of (17), the component $\overline{m}^4$ becomes a cycle and its homology class is the second obstruction. If this class is zero, then we can kill $\overline{m}^4$. If it is not, then we take another g^2 and try new second obstruction. Generally after killing first components, for $m = 0 + 0 + \cdots + 0 + m^n + m^{n+1} + \cdots$ the obstruction is the homology class $[m^n] \in H^{n,2-n}(C^{*,*})$.

This in particular implies that if for a bigraded homotopy G-algebra $C^{*,*}$ all homology modules $H^{n,2-n}(C^{*,*})$ are trivial for $n \geq 3$, then $D(C^{*,*}) = 0$, thus $C^{*,*}$ is rigid.

4.2 Twisting Elements in a Bigraded Homotopy G-algebra (version 2)

This version can be obtained from the previous one by changing grading: take new bigraded module $\overline{C}^{p,q} = C^{p+q,-q}$. The same operations turn $\overline{C}^{*,*}$ into a bigraded hGa.

A twisting element $m \in C^{*,2-*}$ in this case looks as $b = b_1 + b_2 + \cdots + b_n + \cdots$, $b_n \in \overline{C}^{2,n}$ where $b_k = m^{k-2}$ and satisfies the condition $db = b \smile_1 b$, or equivalently $db_n = \sum_{i=2}^{n-1} b_i \smile_1 b_{n-i}$.

Here we have the group $G' = \{g = g_1 + g_2 + \cdots + g_p + \cdots\ ; g_p \in \overline{C}^{1,p}\}$ with operation $g' * g = g' + g + \sum_{k=1}^{\infty} E_{1,k}(g'; g, .., g)$. This group acts on the set $Tw'(\overline{C}^{*,*})$ by the rule $g * b = b'$ where

$$b' = b + dg + g \cdot g + E_{1,1}(g; b) + \sum_{k=1}^{\infty} E_{1,k}(b'; g, ..., g). \tag{21}$$

By $D'(\overline{C}^{*,*})$ we denote the set of orbits $Tw'(\overline{C}^{*,*})/G'$.

We consider the following two problems.

Quantization. The first component $b_1 \in \overline{C}^{2,1}$ of a twisting element $b = \sum b_i$ is a cycle and any perturbation does not change its homology class $\alpha = [b_1] \in H^{2,1}(\overline{C}^{*,*})$. Thus we have a correct map $\psi : D'(\overline{C}^{*,*}) \to H^{2,1}(\overline{C}^{*,*})$.

A *quantization* of a homology class $\alpha \in H^{2,1}(\overline{C}^{*,*})$ is defined as a twisting element $b = b_1 + b_2 + \cdots$ such that $[b_1] = \alpha$. Thus α is quantizable if $\alpha \in Im\ \psi$.

The argument similar to the above shows that the obstructions to quantizability lie in homologies $H^{3,n}(\overline{C}^{*,*}),\ n \geq 2$.

Rigidity. A twisting element $b = b_1 + b_2 + \cdots$ is called *trivial* if it is equivalent to 0. A bigraded hGa $\overline{C}^{*,*}$ is *rigid* if each twisting element is trivial, i.e., if $D'(\overline{C}^{*,*}) = \{0\}$. The obstructions to triviality of a twisting element lie in homologies $H^{2,n}(\overline{C}^{*,*}),\ n \geq 1$.

This in particular implies that if for a bigraded hGa $\overline{C}^{*,*}$ we have $H^{2,n}(\overline{C}^{*,*}) = 0,\ n \geq 1$, then $D'(\overline{C}^{*,*}) = 0$, thus $\overline{C}^{*,*}$ is rigid.

4.3 Twisting Elements in a dg Lie Algebra corresponding to a hGa

As described above for a homotopy G-algebra $(C, \cdot, d, \{E_{1k}\})$ the desuspension $s^{-1}C$ is a dg Lie algebra with the bracket $[a, b] = a \smile_1 b - b \smile_1 a$. Note that if $C^{*,*}$ is a bigraded homotopy G-algebra, then $L^{*,*} = s^{-1}C^{*,*} = C^{*-1,*}$ is a bigraded dg Lie algebra.

Suppose $m \in C^{*,2-*}$ is a twisting element in $C^{*,*}$. The defining equation $dm = m \smile_1 m$ can be rewritten in terms of the bracket as $dm = \frac{1}{2}[m, m]$ (at this moment we have to switch to the field Q of rationales), so the same m can be regarded as a Lie twisting element in the bigraded dg Lie algebra $L^{*,*}$.

So the notion of a twisting element in an hGa, which involves just the operation $E_{1,1} = \smile_1$, in fact can be expressed in terms of the Lie bracket $[\ ,\ ]$.

But it is unclear whether the group action formulas (19) and (21), which involve all the operations $\{E_{1,k},\ k = 1, 2, ...\}$, can be expressed just in terms of the bracket.

5 Deformation of Algebras

This is just an illustrative application. Using the homotopy G-algebra structure, the notions of twisting element and their transformation, one can obtain the well-known results of Gerstenhaber from [7].

Let $(A, \cdot)$ be an algebra over a field k, $k[[t]]$ be the algebra of formal power series in variable t and $A[[t]] = A \otimes k[[t]]$ be the algebra of formal power series with coefficients from A.

Gerstenhaber deformation of an algebra $(A, \cdot)$ is defined as a sequence of homomorphisms

$$B_i : A \otimes A \to A, \quad i = 0, 1, 2, ...; \quad B_0(a \otimes b) = a \cdot b$$

satisfying the *associativity* condition

$$\sum_{i+j=n} B_i(a \otimes B_j(b \otimes c)) = \sum_{i+j=n} B_i(B_j(a \otimes b) \otimes c). \tag{22}$$

Such a sequence determines the *star product*

$$a \star b = a \cdot b + B_1(a \otimes b)t + B_2(a \otimes b)t^2 + B_3(a \otimes b)t^3 + \cdots \in A[[t]],$$

which can be naturally extended to a $k[[t]]$-bilinear product $\star : A[[t]]\otimes A[[t]] \to A[[t]]$ and the condition (22) guarantees its associativity.

Two deformations $\{B_i\}$ and $\{B'_i\}$ are called *equivalent* if there exists a sequence of homomorphisms $\{G_i : A \to A; i = 0, 1, 2, ...; G_0 = id\}$ such that

$$\sum_{r+s=n} G_r(B_s(a \otimes b)) = \sum_{i+j+k=n} B'_i(G_j(a) \otimes G_k(b)). \tag{23}$$

The sequence $\{G_i\}$ determines homomorphism $G = \sum G_i t^i : A \to A[[t]]$. In its turn this G naturally extends to a $k[[t]]$-linear bijection $(A[[t]], \star) \to (A[[t]], \star')$ and the condition (23) guarantees that this extension is multiplicative.

A deformation $\{B_i\}$ is called *trivial*, if $\{B_i\}$ is equivalent to $\{B_0, 0, 0, ...\}$. An algebra A is called *rigid*, if each of its deformation is trivial.

Now we present the interpretation of deformations and their equivalence in terms of twisting elements of version 2 type and their equivalence in hGa of Hochschild cochains.

As mentioned in Section 3.4 the Hochschild complex $C^*(A, A)$ for an algebra A is a homotopy G-algebra. Then the tensor product $C^{*,*} = C^*(A, A) \otimes k[[t]]$ is a *bigraded* Hirsch algebra with the structure

$$C^{p,q} = C^p(A, A) \cdot t^q,\ \delta(f \cdot t^q) = \delta f \cdot t^q,\quad f \cdot t^p \smile g \cdot t^q = (f \smile g) \cdot t^{p+q},$$
$$E_{1,k}(f \cdot t^p; g_1 \cdot t^{q_1}, ..., g_k \cdot t^{q_k}) = E_{1,k}(f; g_1, ..., g_k) \cdot t^{p+q_1+\cdots+q_k},$$

here we use the notation $f \otimes t^p = f \cdot t^p$.

Then each deformation $\{B_i : A^{\otimes^2} \to A,\ i = 1, 2, 3, ...\}$ can be interpreted as a version 2 type twisting element $b = b_1 + b_2 + \cdots + b_k + \cdots$, $b_k = B_k \cdot t^k \in C^{2,k}$: the associativity condition (22) can be rewritten as

$$\delta B_n \cdot t^n = \sum_{i+j=n} B_i \cdot t^i \smile_1 B_j \cdot t^j.$$

Suppose now two deformations $\{B_i\}$ and $\{B'_i\}$ are equivalent, i.e., there exists $\{G_i\}$ such that the condition (23) is satisfied. In terms of the Hochschild cochains this condition looks as

$$b' = b + \delta g + g \smile g + g \smile_1 b + E_{1,1}(b'; g) + E_{1,2}(b'; g, g),$$

where $g = g_1 + \cdots + g_k + \cdots$, $g_k = G_k \cdot t^k \in C^{1,k}$. This equality slightly differs from (21), but since $g \in C^1(A, A)$ and $b' \in C^2(A, A)$, we have $E_{1,k}(b'; g, ..., g) = 0$ for $k \geq 3$ (see Section 3.4), thus they in fact coincide.

So we obtain that deformations are equivalent if and only if the corresponding Hochschild twisting elements b and b' are equivalent. Consequently the set of equivalence classes of deformations is bijective to $D'(C^{*,*})$.

It is obvious that $H^{p,q}(C^{*,*}) = HH^p(A, A) \cdot t^q$. Then from Section 4.2 follow the classical results of Gerstenhaber: obstructions for quantization of a homomorphism $b_1 : A \otimes A \to A$ lie in $HH^3(A, A)$, and if $HH^3(A, A) = 0$ then each b_1 is quantizable (or *integrable* as it is called in [7]). Furthermore, the obstructions for triviality of a deformation lie in $HH^2(A, A)$, and if $HH^2(A, A) = 0$, then A is rigid.

Remark 1. As we see in the definition of equivalence of deformations participate just the operations $E_{1,1}$ and $E_{1,2}$, the higher operations $E_{1,k},\ k>2$ disappear because of (16). So observing the just deformation problem it is impossible to establish general formula (21) for transformation of twisting elements.

6 $A(\infty)$-deformation of Graded Algebras

In this section we give a similar description of $A(\infty)$-deformation of graded algebras in terms of twisting elements in the hGa of Hochschild cochains. So these two types of deformation will be unified by the notion of twisting element in hGa. Partially these results are given in [14], [15].

6.1 $A(\infty)$-algebras

The notion of $A(\infty)$-algebra was introduced by J.D. Stasheff in [24]. This notion generalizes the notion of dg algebra.

An $A(\infty)$-algebra is a graded module M with a given sequence of operations

$$\{m_i : M^{\otimes i} \to M, \quad i=1,2,..., \quad \deg m_i = 2-i\}$$

which satisfies the conditions

$$\sum_{i+j=n+1}\sum_{k=0}^{n-j} m_i(a_1\otimes\cdots\otimes a_k\otimes m_j(a_{k+1}\otimes\cdots\otimes a_{k+j})\otimes\cdots\otimes a_n)=0. \quad (24)$$

Particularly, for the operation $m_1 : M \to M$ we have $\deg m_1 = 1$ and $m_1m_1=0$, this m_1 can be regarded as a differential on M. The operation $m_2 : M\otimes M\to M$ is of degree 0 and satisfies

$$m_1m_2(a_1\otimes a_2)+m_2(m_1a_1\otimes a_2)+m_2(a_1\otimes m_1a_2)=0,$$

i.e., m_2 can be regarded as a multiplication on M and m_1 is a derivation. Thus (M,m_1,m_2) is a sort of (maybe nonassociative) dg algebra. For the operation m_3 we have $\deg m_3=-1$ and

$$m_1m_3(a_1\otimes a_2\otimes a_3)+m_3(m_1a_1\otimes a_2\otimes a_3)+m_3(a_1\otimes m_1a_2\otimes a_3)$$
$$+m_3(a_1\otimes a_2\otimes m_1a_3)+m_2(m_2(a_1\otimes a_2)\otimes a_3)+m_2(a_1\otimes m_2(a_2\otimes a_3))=0,$$

thus the multiplication m_2 is *homotopy associative* and the appropriate chain homotopy is m_3.

The sequence of operations $\{m_i\}$ determines on the tensor coalgebra

$$T^c(s^{-1}M)=R+s^{-1}M+s^{-1}M\otimes s^{-1}M+s^{-1}M\otimes s^{-1}M\otimes s^{-1}M+\cdots$$

a coderivation

$$d_m(a_1 \otimes \cdots \otimes a_n) = \sum_{k,j} a_1 \otimes \cdots \otimes a_k \otimes m_j(a_{k+1} \otimes \cdots \otimes a_{k+j}) \otimes \cdots \otimes a_n,$$

and the condition (24) is equivalent to $d_m d_m = 0$. The obtained dg coalgebra $(T^c(s^{-1}M), d_m)$ is called *bar construction* and is denoted as $B(M, \{m_i\})$.

A morphism of $A(\infty)$-algebras $(M, \{m_i\}) \to (M', \{m'_i\})$ is defined as a sequence of homomorphisms

$$\{f_i : M^{\otimes^i} \to M', \quad i = 1, 2, ..., \quad \deg f_i = 1 - i\},$$

which satisfy the condition

$$\begin{aligned}
&\textstyle\sum_{i+j=n+1} \sum_{k=0}^{n-j} f_i(a_1 \otimes \cdots \otimes a_k \otimes m_j(a_{k+1} \otimes \cdots \otimes a_{k+j}) \otimes \cdots \otimes a_n) \\
&= \textstyle\sum_{k_1+\cdots+k_t=n} m'_t(f_{k_1}(a_1 \otimes \cdots \otimes a_{k_1}) \otimes f_{k_2}(a_{k_1+1} \otimes \cdots \otimes a_{k_1+k_2}) \\
&\quad \otimes \cdots \otimes f_{k_t}(a_{k_1+\cdots+k_{t-1}+1} \otimes \cdots \otimes a_n)).
\end{aligned} \tag{25}$$

In particular for $n = 1$ this condition gives $f_1 m_1(a) = m'_1 f_1(a)$, i.e., $f_1 : (M, m_1) \to (M', m'_1)$ is a chain map; for $n = 2$ it gives

$$\begin{aligned}
&f_1 m_2(a_1 \otimes a_2) + m'_2(f_1(a_1) \otimes f_1(a_2)) \\
&= m'_1 f_2(a_1 \otimes a_2) + f_2(m_1 a_1 \otimes a_2) + f_2(a_1 \otimes m_1 a_2),
\end{aligned}$$

thus $f_1 : (M, m_1, m_2) \to (M', m'_1, m'_2)$ is multiplicative up to homotopy f_2.

A collection $\{f_i\}$ defines a homomorphism $f : B(M, \{m_1\}) \to M'$. Its *comultiplicative coextension*, see Section 2.2, is a graded coalgebra map of the bar constructions

$$B(f) : B(M, \{m_i\}) \to B(M', \{m'_i\}),$$

and the condition (25) guarantees that $B(f)$ is a chain map, i.e., $B(f)$ is a morphism of dg coalgebras. So B is a functor from the category of $A(\infty)$-algebras to the category of dg coalgebras.

A weak equivalence of $A(\infty)$-algebras is defined as a morphism $f = \{f_i\}$ for which $B(f)$ is a weak equivalence of dg coalgebras. It is possible to show (see for example [15]) that:
(i) f is a weak equivalence of $A(\infty)$-algebras if and only if f_1 is a weak equivalence of dg modules;
(ii) f is an isomorphism of $A(\infty)$-algebras if and only if f_1 is an isomorphism of dg modules.

An $A(\infty)$-algebra $(M, \{m_i\})$ we call *minimal* if $m_1 = 0$. In this case (M, m_2) is *strictly* associative graded algebra.

The following proposition is the immediate consequence of (i) and (ii):

Proposition 1. *Each weak equivalence of minimal $A(\infty)$-algebras is an isomorphism.*

6.2 Stasheff Deformation of Graded Algebras as Twisting Element

Let $(H, \mu : H \otimes H \to H)$ be a graded algebra. A Stasheff (or minimal $A(\infty)$) deformation is defined as a minimal $A(\infty)$-algebra $(H, \{m_i\})$ with $m_2 = \mu$. Two deformations $(H, \{m_i\})$ and $(H, \{m_i'\})$ we call equivalent if there exists an isomorphism of $A(\infty)$-algebras $\{f_i\} : (H, \{m_i\}) \to (H, \{m_i'\})$ with $f_1 = id$.

A deformation $(H, \{m_i\})$ we call trivial if it is equivalent to $(H, \{m_1 = 0, m_2 = \mu, m_{\geq 3} = 0\})$. An algebra (H, μ) we call *rigid* (or *intrinsically formal*, this term is borrowed from rational homotopy theory) if each of its deformations is trivial.

Now we present the interpretation of deformations and their equivalence in terms of twisting elements and their equivalence in hGa of Hochschild cochains.

The Hochschild cochain complex of a graded algebra H with coefficients in itself is bigraded: $C^{m,n}(H, H) = Hom^n(H^{\otimes m}, H)$, here Hom^n denotes degree n homomorphisms. The coboundary operator δ maps $C^{m,n}(H, H)$ to $C^{m+1,n}(H, H)$. Besides, for $f \in C^{m,n}(H, H)$ and $g_i \in C^{p_i,q_i}(H, H)$ one has $f \smile g \in C^{m+p,n+q}(H, H)$, $f \smile_1 g \in C^{m+p-1,n+q}(H, H)$, and

$$E_{1,k}(f; g_1, ..., g_k) \in C^{m+p_1+\cdots+p_k-k,n+q_1+\cdots+q_k}(H, H),$$

thus the Hochschild complex $C^{*,*}(H, H)$ is a *bigraded* homotopy G-algebra in this case. Let us denote the nth homology module of the complex $(C^{*,k}(H, H), \delta)$ by $HH^{n,k}(H, H)$.

Suppose now that $(H, \{m_i\})$ is a Stasheff deformation of H. Each operation $m_i : H^{\otimes^i} \to H$ can be regarded as a Hochschild cochain from $C^{i,2-i}(H, H)$. The condition (24) can be rewritten as

$$\delta m_k = \sum_{i=3}^{k-1} m_i \smile_1 m_{k-i+2},$$

thus $m = m_3 + m_4 + \cdots$ is a *twisting element* (version 1) in $C^{*,*}(H, H)$.

Now let $(H, \{m_i\})$ and $(H, \{m_i'\})$ be two Stasheff deformations of H. Then it follows from (19) that the corresponding twisting elements m and m' are equivalent if and only if these two Stasheff deformations are equivalent: if $m' = p{*}m$, then $\{p_i\} : (H, \{m_i\}) \to (H, \{m_i'\})$ with $p_1 = id$ is an *isomorphism* of $A(\infty)$-algebras. So we obtain the following:

Theorem 1. *The set of isomorphism classes of all Stasheff deformations of a graded algebra (H, μ) is bijective to the set of equivalence classes of twisting elements $D(C^{*,*}(H, \mu))$.*

Moreover, from 4.1 we get the following:

Theorem 2. *If for a graded algebra (H, μ) its Hochschild cohomology modules $HH^{n,2-n}(H, H)$ are trivial for $n \geq 3$, then (H, μ) is intrinsically formal.*

$A(\infty)$-algebra Structure on the Homology of a dg algebra

Let (A, d, μ) be a dg algebra and $(H(A), \mu^*)$ be its homology algebra. Although the product in $H(A)$ is associative, there is a structure of a (generally nondegenerate) minimal $A(\infty)$-algebra, which is a Stasheff deformation of $(H(A), \mu^*)$. Namely, in [12], [13] the following result was proved (see also [23], [11]):

Theorem 3. *If for a dg algebra all homology R-modules $H_i(A)$ are free, then there exist: a structure of a minimal $A(\infty)$-algebra $(H(A), \{m_i\})$ on $H(A)$ and a weak equivalence of $A(\infty)$-algebras*

$$\{f_i\} : (H(A), \{m_i\}) \to (A, \{d, \mu, 0, 0, ...\})$$

such that $m_1 = 0$, $m_2 = \mu^*$, $f_1^* = id_{H(A)}$, such a structure is unique up to isomorphism in the category of $A(\infty)$-algebras.

In particular, we get an $A(\infty)$-algebra structure on cohomology $H^*(X)$ of a topological space X or in the homology $H_*(G)$ of a topological group or H-space G. (Co)homology algebra equipped with this additional structure carries more information than just the (co)homology algebra. Some applications of this structure are given in [13] and [15]. For example, the cohomology $A(\infty)$-algebra $(H^*(X), \{m_i\})$ determines the cohomology of the loop space $H^*(\Omega X)$ when just the algebra $(H^*(X), m_2)$ does not. Similarly, the homology $A(\infty)$-algebra $(H_*(G), \{m_i\})$ determines the homology of the classifying space $H_*(B_G)$ when just the Pontriagin algebra $(H_*(G), m_2)$ does not. Furthermore, the rational cohomology $A(\infty)$-algebra $(H^*(X, Q), \{m_i\})$ (which actually is $C(\infty)$ in this case) determines the rational homotopy type of 1-connected X when just the cohomology algebra $(H^*(X, Q), m_2)$ does not.

Therefore of a particular interest are the cases when this additional structure vanishes, that is, when the $A(\infty)$-algebra $(H(A), \{m_i\})$ is degenerate (in this case a dg algebra A is called *formal*). The above Theorem 2 gives a sufficient condition of formality of A in terms of the Hochschild cohomology of $H(A)$.

References

1. Adams, J.: On the non-existence of elements of Hopf invariant one. Ann. Math. **72**, 20–104 (1960)
2. Baues, H.: The double bar and cobar construction. Compos. Math. **43**, 331–341 (1981)
3. Berger, C., Fresse, B.: Combinatorial operad action on cochains. Math. Proc. Camb. Phil. Soc. **137**, 135–174 (2004)
4. Berikashvili, N.: On the differentials of spectral sequence. Proc. Tbil. Math. Inst. **51**, 1–105 (1976)
5. Brown, E.: Twisted tensor products. Ann. Math. **69**, 223–246 (1959)

6. Gerstenhaber, M.: The cohomology structure of an associative ring. Ann. Math. **78**, 267–288 (1963)
7. Gerstenhaber, M.: On the deformations of rings and algebras. Ann. Math. **79**(1), 59–103 (1964)
8. Gerstenhaber, M., Voronov, A.: Higher operations on Hochschild complex. Funct. Anal. Appl. **29**, 1–6 (1995)
9. Getzler, E.: Cartan homotopy formulas and the Gauss–Manin connection in cyclic homology. Israel Math. Conf. Proc. **7**, 65–78 (1993)
10. Getzler E., Jones, J.D.S.: Operads, homotopy algebra, and iterated integrals for double loop spaces. Preprint (1994) hep-th/9403055
11. Gugenghaim V., Stasheff, J.: On perturbations and A(infty)-structures. Bull. Soc. Math. Belg. **38**, 237–246 (1986)
12. Kadeishvili, T.: On the differentials of spectral sequence of fiber bundle. Bull. Georg. Acad. Sci. **82**, 285–288 (1976)
13. Kadeishvili, T.: On the homology theory of fibrations. Russ. Math. Surv. **35**(3), 231–238 (1980)
14. Kadeishvili, T.: The $A(\infty)$-algebra structure and cohomology of Hochschild and Harrison. Proc. Tbil. Math. Inst. **91**, 19–27 (1988)
15. Kadeishvili, T.: $A(\infty)$-algebra structure in cohomology and rational homotopy type. Proc. Tbil. Math. Inst. **107**, 1–94 (1993)
16. Kadeishvili, T.: Measuring the noncommutativity of DG-algebras. J. Math. Sci. **119**(4), 494–512 (2004)
17. Kadeishvili, T.: On the cobar construction of a bialgebra. Homology, Homotopy and Applications **7**(2), 109–122 (2005)
18. Kadeishvili, T., Saneblidze, S.: A cubical model for a fibration. J. Pure Appl. Algebra **196**(2–3), 203–228 (2005)
19. Khelaia, L.: On some chain operations. Bull. Georg. Acad. Sci. **96**(3), 529–531 (1979)
20. McLure, J., Smith, J.: A solution of Deligne's conjecture. Preprint (1999) math.QA/9910126
21. McLure, J., Smith, J.: A solution of Deligne's Hochschild cohomology conjecture. In: Recent Progress in Homotopy Theory (Baltimore, MD, 2000). Contemporary Mathematics, vol. 293, pp. 153–193. Am. Math. Soc., RI (2002)
22. McLure, J., Smith, J.: Multivariable cochain operations and little n-cubes. J. Am. Math. Soc. **16**(3), 681–704 (electronic) (2003)
23. Smirnov, V.: Homology of fiber spaces, Russ. Math. Surv. **35**(3), 183–188 (1980)
24. Stasheff, J.D.: Homotopy associativity of H-spaces. I, II. Trans. Am. Math. Soc. **108**, 275–312 (1963)
25. Voronov, A.: Homotopy Gerstenhaber algebras. In: Dito, G., Sternheimer, D. (eds.) Conference Moshe Flato 1999, vol. 2, pp. 307–331. Kluwer, Dordrecht (2000), math.QA/9908040

Homological Perturbation Theory and Homological Mirror Symmetry

Hiroshige Kajiura

Department of Mathematics and Informatics, Chiba University, 263-8522 Japan, kajiura@math.s.chiba-u.ac.jp

Abstract. In this article, we discuss an application of homological perturbation theory (HPT) to homological mirror symmetry (HMS) based on Kontsevich and Soibelman's proposal [Kontsevich, M., Soibelman, Y. (2001) Homological mirror symmetry and torus fibrations]. After a brief review of Morse theory, Morse homotopy and the corresponding Fukaya categories, we explain the idea of deriving a Fukaya category from a DG category via HPT, which is expected to give a solution to HMS, and apply it to the cases of $\mathbb{R}^2$ discussed in [Kajiura, H. (2007) An A_∞-structure for lines in a plane] and then T^2. A finite dimensional A_∞-algebra obtained from the Fukaya category on T^2 is also presented.

Key words: A_∞-algebras, Fukaya category, Homological mirror symmetry

Mathematics Subject Classification: 18G55, 53D37

1 Introduction

A strong homotopy associative algebra (= A_∞-algebra) is introduced by Stasheff [46, 47] in the study of H-spaces. Differential graded (DG) algebras, familiar as the algebras of smooth differential forms on manifolds, are the special examples. An A_∞-algebra contains, in addition to a differential and a product, higher products which can be thought of as (higher) Massey products and their generalization. To explore such algebraic properties, homological perturbation theory (HPT) has been developed by Gugenheim, Lambe, Stasheff, Huebschmann, Kadeishvili, etc., [15, 16, 17, 20], which implies that A_∞-algebras are a homotopy invariant notion. The category version of an A_∞-algebra is an A_∞-category introduced by Fukaya [7] to formulate Morse homotopy theory and Floer theory of Lagrangian submanifolds in a symplectic manifold. A category for the latter theory is called a Fukaya category. The Lie version, a strong homotopy *Lie* algebra (= L_∞-algebra) [38], appeared in closed string field theory [52, 32], though it was already implicitly an essential tool in studying deformation theory (see [44]).

A.S. Cattaneo et al. (eds.), *Higher Structures in Geometry and Physics*,
Progress in Mathematics 287, DOI 10.1007/978-0-8176-4735-3_10,

These "infinity-frameworks" became more important via the homological mirror symmetry (HMS) conjecture by Kontsevich [34], which states an equivalence between the Fukaya category on a symplectic manifold X and the category of coherent sheaves on the mirror dual complex manifold $\hat{X}$ for a given mirror pair X and $\hat{X}$. The strongest form of HMS may be to show the equivalence at the level of A_∞-categories up to homotopy equivalence. The deformation of these A_∞-categories should be described by DG Lie algebras of Hochschild complexes of the corresponding A_∞-structure, as a natural generalization of deformation theory of algebras by Gerstenhaber [12, 13] to that for A_∞-structures due to [48]. On the other hand, a more traditional mirror symmetry setup is to show the equivalence of two Frobenius manifold structures associated to X and $\hat{X}$. HMS is expected to give a homological algebraic realization of this traditional mirror symmetry (cf. [34, 1]). From the string theory viewpoint, those Frobenius manifolds and A_∞-categories are associated to certain topological models of tree-level closed strings and tree-level open strings, respectively. A system of tree open-closed strings, i.e., an open-closed homotopy algebra [29, 30], gives a map from the space of closed string states to the space of deformations of an A_∞-category; a typical example is the deformation quantization [35, 4]. Then, the "pullback" of a certain algebraic structure on the space of A_∞ deformations by the map should induce a Frobenius manifold structure (for a different but related approach, see [6]). In any case, to explore the underlying operad structure is important; see [40].

Though such fruitful plans continue to the HMS setup, to formulate and show the HMS itself is still not easy. So, let us concentrate on the HMS. The HMS had been discussed positively, for instance, for two-tori [43, 41, 37] for abelian varieties [8], and for noncommutative tori [21, 42, 22, 25, 23], etc. However, at least for the author, it had been quite mysterious why such equivalences hold even for two-tori.

Kontsevich and Soibelman [36] then proposed a strategy to show the HMS based on Strominger–Yau–Zaslow's torus fibrations [49] (see also [9] for a related approach). The key idea there was to apply HPT to a kind of DG categories of differential forms and "derive" Morse homotopy theory slightly generalized to torus fibration setting. The DG category is related to the category of coherent sheaves on a complex manifold $\hat{X}$. Thus, an equivalence of the category of Morse homotopy with the Fukaya category is supposed to give a solution to the HMS. Physically, those DG categories are related to a kind of Chern–Simons field theory. Applying HPT to the DG categories corresponds to considering perturbation theory of the Chern–Simons theory at tree level. This kind of Chern–Simons theory is thought of as a topological open string field theory [51, 39], where a homotopy operator in HPT corresponds to a gauge fixing for the open string field theory (see [27]). To find a "good" choice of the homotopy operator is the key point in these arguments.

Though Fukaya categories should give interesting geometric examples of A_∞-categories, it is still difficult to define them completely as A_∞-categories because of the problem of *transversality* of Lagrangians (see FOOO [11]). On this point, it is plausible to expect that the Kontsevich–Soibelman argument [36] derives a suitable Fukaya category from the DG category. The aim of this article is to present the setup of deriving Fukaya categories from the DG categories so that it works explicitly at least for the first few simple examples, $X = \mathbb{R}^{2n}$, T^{2n}, etc. For the case $X = \mathbb{R}^2$, the construction is given in detail in [24]. In this article, we rather explain the background of the construction in [24]. Also, finally, we give an explicit example of a finite dimensional A_∞-algebra obtained as a full subcategory of the Fukaya category on T^2.

This article is organized as follows. In Section 2, we recall terminologies for A_∞-categories and HPT. In Section 3, we briefly review Morse theory, Morse homotopy and its relation to a Fukaya category. In particular, a relation of Morse theory to the de Rham theory given by Harvey and Lawson [18] is recalled in Section 3.1, which plays a key role in our discussions. In Section 4, we recall the easy part, *transversal* A_∞-products, in the Fukaya category of lines in $\mathbb{R}^2$ according to [24]. Section 5 is the main part of this article, where we discuss a general approach to HMS via HPT with some insight from Section 3.1. It is then applied to our special example, the Fukaya category of lines in $\mathbb{R}^2$, and we explain how this thought leads to the construction of the A_∞-structure in [24], i.e., how the technical formulation there to determine *nontransversal* A_∞-products is natural. Finally, in Section 6, we present an example of a finite dimensional (cyclic) A_∞-algebra, which should be a full subcategory of the Fukaya A_∞-category of a two-torus and is obtained by applying the argument in the previous section to the case of two-tori.

Throughout this article, by (graded) vector spaces we indicate those over fields $k = \mathbb{R}$. Also, we denote $\mathbf{i} := \sqrt{-1}$, $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ for an integer n, and M stands for an n-dimensional (compact) Riemannian manifold (M, g).

Acknowledgments: This article is based on the talk given at the IHP conference "Higher Structures in Geometry and Physics" in honor of Murray Gerstenhaber's 80th birthday and Jim Stasheff's 70th birthday. I am very grateful to the organizers Albert Cattaneo and Ping Xu for inviting me to such a nice conference. I would like to thank Jim Stasheff for his continuous encouragement and valuable discussions, and M. Akaho, K. Fukaya, T. Kondo and K. Saito, A. Takahashi, Y. Terashima for useful discussions and comments. This work was partly supported by Grant-in-Aid for Scientific Research grant number 19740038 of the Ministry of Education, Science and Culture in Japan. Financial support by the 21st Century COE Program "Formation of an international center of excellence in the frontiers of mathematics and fostering of researchers in future generations" is also acknowledged.

2 A_∞-categories and their homotopical properties

2.1 A_∞-algebras and A_∞-categories

Definition 2.1 **(A_∞-algebra [46, 47])** *An A_∞-algebra $(V, \mathfrak{m})$ consists of a $\mathbb{Z}$-graded vector space $V := \oplus_{r\in\mathbb{Z}} V^r$ with a collection of multilinear maps $\mathfrak{m} := \{m_n : V^{\otimes n} \to V\}_{n\geq 1}$ of degree $(2-n)$ satisfying*

$$0 = \sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^\sigma \; m_k(w_1, \dots, w_j, m_l(w_{j+1}, \dots, w_{j+l}), w_{j+l+1}, \dots, w_n) \tag{1}$$

for $n \geq 1$ with homogeneous elements $w_i \in V^{|w_i|}$, $i = 1, \dots, n$, where $\sigma = (j+1)(l+1) + l(|w_1| + \cdots + |w_j|)$.

That the multilinear map m_k has degree $(2-k)$ indicates the degree of $m_k(w_1, \dots, w_k)$ is $|w_1|+\cdots+|w_k|+(2-k)$. The A_∞-relations include $(m_1)^2 = 0$ for $n = 1$, the Leibniz rule of the differential m_1 with respect to the product m_2 for $n = 2$, and the associativity of m_2 up to homotopy for $n = 3$. In particular, the product m_2 is strictly associative if m_3=0.

Definition 2.2 *An A_∞-algebra $(V, \mathfrak{m})$ with higher products all zero, $m_3 = m_4 = \cdots = 0$, is called a* differential graded algebra (DGA).

Let $s : V^r \to (V[1])^{r-1}$ be the *suspension* and $T^c(V[1]) := \oplus_{k\geq 1}(V[1])^{\otimes k}$ the tensor coalgebra of $V[1]$. By the suspension, the A_∞-structure $\sum_k m_k$ turns out to be a degree one multilinear map $\sum_k m_k \in \mathrm{Hom}(T^c(V[1]), V[1])$, which is lifted to be a coderivation $\mathfrak{m} \in \mathrm{Coder}(T^c(V[1]))$ satisfying $(\mathfrak{m})^2 = 0$. Thus, an A_∞-algebra $(V, \mathfrak{m})$ is equivalent to a DG coalgebra $(T^c(V[1]), \mathfrak{m})$.

Remark 2.3 *By the one-to-one correspondence between* $\mathrm{Hom}(T^c(V[1]), V[1])$ *and* $\mathrm{Coder}(T^c(V[1]))$*, the Gerstenhaber bracket in* $\mathrm{Hom}(T^c(V[1]), V[1])$ *is described by the graded commutator of coderivations in* $\mathrm{Coder}(T^c(V[1]))$*. Then,* $\mathrm{Coder}(T^c(V[1]))$ *with the commutator and the differential $D := [\mathfrak{m}, \ ]$ forms a DG Lie algebra, which controls deformation of the A_∞-structure $\mathfrak{m}$ [48].*

The coalgebra description is simpler in sign, so we define morphisms between A_∞-algebras in this framework.

Definition 2.4 **(A_∞-morphism)** *Given two A_∞-algebras $(V, \mathfrak{m})$ and $(V', \mathfrak{m}')$, a collection of degree-preserving (= degree zero) multilinear maps $\mathcal{G} := \{g_k : (V[1])^{\otimes k} \to V'[1]\}_{k\geq 1}$ is called an A_∞-morphism $\mathcal{G} : (V, \mathfrak{m}) \to (V', \mathfrak{m}')$ if and only if $\mathcal{G}$ gives a map of DG coalgebras:*

$$\mathcal{G} \circ \mathfrak{m} = \mathfrak{m}' \circ \mathcal{G}, \tag{2}$$

where $\mathfrak{m}$ and $\mathfrak{m}'$ are the codifferentials on $T^c(V[1])$ and $T^c(V'[1])$, respectively, and $\mathcal{G} : T^c(V[1]) \to T^c(V'[1])$ is the coalgebra homomorphism whose restriction onto $V'[1] \subset T^c(V'[1])$ is given by $\sum_{k\geq 1} g_k$. (Such a coalgebra homomorphism is determined uniquely.)

The definition, in particular, implies that $g_1 : V[1] \to V'[1]$ forms a chain map $g_1 : (V[1], m_1) \to (V'[1], m'_1)$.

Definition 2.5 *An A_∞-morphism $\mathcal{G} : (V, \mathfrak{m}) \to (V', \mathfrak{m}')$ is called an* A_∞-quasi-isomorphism *if and only if $g_1 : (V[1], m_1) \to (V'[1], m'_1)$ induces an isomorphism between the cohomologies of these two complexes. In this situation, we say* $(V, \mathfrak{m})$ is homotopy equivalent to $(V', \mathfrak{m}')$ *and call the A_∞-quasi-isomorphism $\mathcal{G} : (V, \mathfrak{m}) \to (V', \mathfrak{m}')$ a* homotopy equivalence.

A justification to call it a homotopy equivalence is given by HPT; see the next subsection.

We need the categorical version of these terminologies.

Definition 2.6 (A_∞-category [7]) *An* A_∞-category $\mathcal{C}$ *consists of a set of objects* $\mathrm{Ob}(\mathcal{C}) = \{a, b, \dots\}$*, a $\mathbb{Z}$-graded vector space $V_{ab} :=$* $\mathrm{Hom}_{\mathcal{C}}(a, b)$ *for each two objects $a, b \in$* $\mathrm{Ob}(\mathcal{C})$ *and a collection of multilinear maps*

$$\mathfrak{m} := \{m_n : V_{a_1 a_2} \otimes \cdots \otimes V_{a_n a_{n+1}} \to V_{a_1 a_{n+1}}\}_{n \geq 1}$$

of degree $(2-n)$ satisfying the A_∞-relations (1).

In particular, an A_∞-category $\mathcal{C}$ with higher products all zero, $m_3 = m_4 = \cdots = 0$, is called a DG category.

The suspension $s(\mathcal{C})$ of an A_∞-category $\mathcal{C}$ is defined by the shift

$$s : \mathrm{Hom}_{\mathcal{C}}(a, b) \to s(\mathrm{Hom}_{\mathcal{C}}(a, b)) =: \mathrm{Hom}_{s(\mathcal{C})}(a, b)$$

for any $a, b \in \mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(s(\mathcal{C}))$, where the degree $|m_n|$ of the A_∞-products becomes one for all $n \geq 1$ as in the case of A_∞-algebras.

Definition 2.7 (A_∞-functor) *Given two A_∞-categories $\mathcal{C}$, $\mathcal{C}'$, an A_∞-*functor $\mathcal{G} := \{g; g_1, g_2, \dots\} : s(\mathcal{C}) \to s(\mathcal{C}')$ *is a map $g :$* $\mathrm{Ob}(s(\mathcal{C})) \to \mathrm{Ob}(s(\mathcal{C}'))$ *of objects with degree-preserving multilinear maps*

$$g_k : \mathrm{Hom}_{s(\mathcal{C})}(a_1, a_2) \otimes \cdots \otimes \mathrm{Hom}_{s(\mathcal{C})}(a_k, a_{k+1}) \to \mathrm{Hom}_{s(\mathcal{C}')}(g(a_1), g(a_{k+1}))$$

for $k \geq 1$ satisfying the defining relations of an A_∞-morphism (2).

In particular, if $g :$ $\mathrm{Ob}(s(\mathcal{C})) \to \mathrm{Ob}(s(\mathcal{C}'))$ *is bijective and $g_1 :$* $\mathrm{Hom}_{s(\mathcal{C})}(a, b) \to \mathrm{Hom}_{s(\mathcal{C}')}(g(a), g(b))$ *induces an isomorphism between the cohomologies for any $a, b \in$* $\mathrm{Ob}(s(\mathcal{C}))$*, we call the A_∞-functor a* homotopy equivalence.

2.2 Homological perturbation theory (HPT) for A_∞-structures

For a DG algebra A, HPT starts with what is called *strong deformation retract (SDR)* data [15, 16, 17, 20]:

$$(B \underset{\pi}{\overset{\iota}{\rightleftarrows}} A , h), \tag{3}$$

where $(B, d_B := \pi \circ d_A \circ \iota)$ is a complex with chain maps ι and π so that $\pi \circ \iota = \mathrm{Id}_B$ and $h : A \to A$ is a contracting homotopy satisfying

$$d_A h + h d_A = \mathrm{Id}_A - P, \qquad P := \iota \circ \pi. \tag{4}$$

By definition, P is an idempotent in A which commutes with d_A, $Pd_A = d_A P$. If $dP = 0$, then the SDR (3) gives a Hodge decomposition of the complex (A, d_A), where $P(A) \simeq H(A)$ gives the cohomology.

The next step is the "tensor trick" [16, 17, 20]; the SDR data (3) can be lifted to the one on coalgebras:

$$(\ T^c(B[1]) \ \underset{\pi}{\overset{\iota}{\rightleftarrows}} \ T^c(A[1]) \ , h),$$

where we used the same notation ι, π for those lifted to the coalgebra maps and then h is the contracting homotopy on $T^c(A[1])$. Let $\mathfrak{m}$ be the codifferential on $T^c(A[1])$ obtained by the lift of the DGA structure $m_1 := d_A$ and m_2. The HPT, as in the sense of [16, 17, 20], states that a codifferential, i.e., an A_∞-structure $\mathfrak{m}'$, is induced on $T^c(B[1])$ so that the coalgebra maps ι and π can be perturbed to be DG-coalgebra maps together with a contracting homotopy which is also obtained by perturbing h.

There is a way to construct such an A_∞-structure $\mathfrak{m}'$ on B together with an A_∞-quasi-isomorphism $\mathcal{G} : (T^c(B[1]), \mathfrak{m}') \to (T^c(A[1]), \mathfrak{m})$, the perturbed ι (see [17, 20]). First, a collection of degree zero maps $\mathcal{G} = \{g_l : (B[1])^{\otimes l} \to A[1]\}_{l \geq 1}$ is defined recursively with respect to k as

$$g_k = -h \sum_{k_1+k_2=k} m_2(g_{k_1} \otimes g_{k_2}), \qquad k \geq 2, \tag{5}$$

with $g_1 := \iota : B[1] \to A[1]$ the inclusion. Then, $\mathfrak{m}' = \{m'_k : (B[1])^{\otimes k} \to A[1]\}_{k \geq 1}$ is given recursively by

$$m'_k = \pi \sum_{k_1+k_2=k} m'_2(g_{k_1} \otimes g_{k_2}), \qquad k \geq 2, \tag{6}$$

with $m'_1 := d_B$.

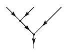
This $\mathcal{G}$ in Eq. (5) is in fact an A_∞-quasi-isomorphism. For the case $B \simeq H(A)$, the induced higher products $\mathfrak{m}'$ can be thought of as a generalization of (higher) Massey products.

The A_∞-structure $\mathfrak{m}'$ can also be described in terms of rooted planar trees as follows (due to [36]).

A *rooted planar tree* is a simply connected planar graph consisting of vertices, internal edges and external edges, where an external edge is distinguished as the *root edge* from the remaining external edges called *leaves* (see [40]). The number of incident edges at a vertex is greater than two. A *k-vertex*, $k \geq 2$, is a vertex at which the number of incident edges is $(k+1)$. By a *k-tree*, we mean a rooted planar tree having k leaves. For $k \geq 2$, the set of the isomorphism classes of k-trees is denoted G_k. The subset consisting of 2-vertices is denoted $G_k^{tri} \subset G_k$. Each edge of a planar rooted tree has a unique orientation so that the orientations form a flow from the leaves to the root edge. We sometimes indicate the orientation by arrows.

For any element $\varGamma_n \in G_n^{tri}$, $n \geq 2$, let us define $m'_{\varGamma_n} : (B[1])^{\otimes n} \to B[1]$ by attaching $\iota : B[1] \to A[1]$ to each leaf, $m_2 : A[1] \otimes A[1] \to A[1]$ to each 2-vertex, $-h : A[1] \to A[1]$ to each internal edge, $\pi : A[1] \to B[1]$ to the root edge and then composing them. For example,

$$\begin{aligned} &m'_{\varGamma_3}(b_1, b_2, b_3) \\ &\quad = \pi m_2(-h m_2(\iota(b_1), \iota(b_2)), \iota(b_3)), \end{aligned} \qquad \varGamma_3 =$$



for $b_1, b_2, b_3 \in B[1]$. Then, $\{m'_n\}_{n\geq 1}$ is given by $m'_1 = d_B$ and

$$m'_n = \sum_{\varGamma_n \in G_n^{tri}} m'_{\varGamma_n} \tag{7}$$

for $n \geq 2$. Thus, m'_n is described as the sum of the value $m'_{\varGamma_n}$ over all the n-trees $\varGamma_n \in G_n^{tri}$. Similarly, $\{g_n\}_{n\geq 1}$ is given by $g_1 = \iota$ and $g_n = \sum_{\varGamma_n \in G_n^{tri}} g_{\varGamma_n}$ for $n \geq 2$, where $g_{\varGamma_n} : (B[1])^{\otimes n} \to A[1]$ is obtained by replacing π by $-h$ in the definition of $m'_{\varGamma_n}$.

If we start from an A_∞-algebra $(A, \mathfrak{m})$, we may simply replace G_k^{tri} by G_n and attach $m_k : (A[1])^{\otimes k} \to A[1]$ to each k-vertex [36]. The generalization of HPT to A_∞-categories is also straightforward. In Section 5, we employ HPT for a DG category with the SDR a Hodge–Kodaira decomposition.

2.3 Cyclic structure

A cyclic A_∞-algebra $(V, \eta, \mathfrak{m})$ is an A_∞-algebra $(V, \mathfrak{m})$ with a nondegenerate symmetric bilinear map $\eta : V \otimes V \to \mathbb{C}$ of a fixed degree $|\eta| \in \mathbb{Z}$ satisfying

$$\eta(m_n(w_1, \ldots, w_n), w_{n+1}) = \pm\eta(m_n(w_2, \ldots, w_{n+1}), w_1), \qquad n \geq 1, \tag{8}$$

for homogeneous elements $w_1, \ldots, w_{n+1} \in V$ (see [41, 22] for the sign).

A cyclic A_∞-category is an A_∞-category $\mathcal{C}$ with a nondegenerate inner product $\eta : \mathrm{Hom}_{\mathcal{C}}(a, b) \otimes \mathrm{Hom}_{\mathcal{C}}(b, a) \to \mathbb{C}$ of a fixed degree $|\eta| \in \mathbb{Z}$ for any $a, b \in \mathrm{Ob}(\mathcal{C})$ which is symmetric, $\eta(w_{ab}, w_{ba}) = (-1)^{|w_{ab}||w_{ba}|}\eta(w_{ba}, w_{ab})$ for homogeneous elements $w_{ab} \in \mathrm{Hom}_{\mathcal{C}}(a, b)$ and $w_{ba} \in \mathrm{Hom}_{\mathcal{C}}(b, a)$, and satisfies cyclicity conditions similar to Eq. (8) (see [41, 22]).

HPT can be applied also for cyclic A_∞-algebras and cyclic A_∞-categories. In particular, the analog of g_k in Eq. (5) gives a homotopy equivalence of cyclic A_∞-categories by starting with an SDR data being compatible with the inner product [27].

3 Morse theory and Morse homotopy

3.1 Morse theory in relation to de Rham theory

A *Morse function* $f \in C^\infty(M)$ is a smooth function whose Hessian at each critical point is nondegenerate. Let $\mathrm{Cr}_\lambda(f)$ be the set of critical points p such that the number of negative eigenvalues of the Hessian at p is λ. For any critical point $p \in \mathrm{Cr}_\lambda(f)$, $\lambda = \lambda(p)$ is called the *Morse index* of p.

A *gradient line* of f from $p \in \mathrm{Cr}_\lambda(f)$ to $q \in \mathrm{Cr}_{\lambda'}(f)$, $\lambda > \lambda'$, is a map $\gamma : \mathbb{R} \to M$ defined by

$$\frac{d\gamma(t)}{dt} = -\mathrm{grad}(f(\gamma(t))), \tag{9}$$

$\lim_{t\to-\infty}\gamma(t) = p$ and $\lim_{t\to\infty}\gamma(t) = q$. The *Morse complex* $(C_\bullet(f), \partial)$ consists of the free abelian group $C_\lambda(f) := \oplus_{p\in\mathrm{Cr}_\lambda(f)}\mathbb{Z}[p]$ generated by the bases $[p]$ of $C_\lambda(f)$ associated to the critical points $p \in \mathrm{Cr}_\lambda(f)$ with a differential $\partial : C_\lambda(f) \to C_{\lambda-1}(f)$ defined by

$$\partial([p]) = \sum_{q\in\mathrm{Cr}_{\lambda(p)-1}(f)} \langle p, q\rangle \cdot [q],$$

where $\langle p, q\rangle \in \mathbb{Z}$ is given by counting the number of gradient lines from p to q with an appropriate sign (cf. [3, 18]). When f is generic (that satisfying the Smale condition), the homology of the Morse complex is known to be isomorphic to the homology of M.

By using Eq. (9), one can define *the gradient flow* $\varphi_t : M \to M$, $t \in \mathbb{R}$, of f by the correspondence $\gamma(0) \mapsto \gamma(t)$. To each $p \in \mathrm{Cr}_\lambda(f)$, there are associated *stable manifolds* S_p and *unstable manifolds* U_p defined by

$$S_p = \{x \in M | \lim_{t\to-\infty}\varphi_t(x) = p\ \}, \qquad U_p = \{x \in M | \lim_{t\to\infty}\varphi_t(x) = p\ \}.$$

They are contractible submanifolds of $\dim(S_p) = \lambda(p)$ and $\dim(U_p) = n-\lambda(p)$. Then, the differential ∂ on $[p]$ is also understood as the boundary operator on the closer $\bar{S}_p$ of S_p.

E. Witten [50] gave an elegant interpretation about the relation between the homology of the Morse complex and the de Rham cohomology of the same smooth manifold M. (See also Bott's survey [3].) Let $\Omega := \oplus_{r=0}^n \Omega^r(M)$ be the space of smooth differential forms on M. For the exterior derivative $d : \Omega^r(M) \to \Omega^{r+1}(M)$ and the adjoint operator $d^* : \Omega^r(M) \to \Omega^{r-1}(M)$, the kernel of the Laplacian $\Delta := dd^* + d^*d : \Omega^r(M) \to \Omega^r(M)$, which is the zero-eigenspace of Δ, gives a representative of the cohomology $H^r(M,\mathbb{R})$. For a Morse function f of M, Witten introduced the operator

$$d_s := e^{-sf}de^{sf}, \qquad s \in \mathbb{R}_{\geq 0},$$

the adjoint $d_s^* := e^{sf}d^*e^{-sf}$ and then the corresponding Laplacian

$$\Delta_s := d_sd_s^* + d_s^*d_s.$$

Since the cohomology of d_s is independent of s, let us consider very large s. One can see that the eigenspaces of Δ_s with very small eigenvalues are generated by differential forms localized at the critical points of f, where the differential d of the Morse complex is interpreted by the instantons between the vacua, i.e., the critical points (see [3]). Remember that, in this large s limit, $d_s^* = e^{sf} d^* e^{-sf} = d^* + s\iota_{\mathrm{grad}(f)}$ "approaches" the inner derivative of the gradient flow up to the constant s.

Another interesting relation of Morse theory to the de Rham theory was discussed by Harvey and Lawson [18]. First, for a chain complex $(A := \oplus_{r\in\mathbb{Z}} A^r, d_A)$, consider a degree minus one linear map $d_A^\dagger : A^r \to A^{r-1}$ and define a map $\psi_t : A^r \to A^r$, $t \in \mathbb{R}_{\geq 0}$, by $\psi_0 := \mathrm{Id}_A$ and

$$\frac{d\psi_t}{dt} = -H_A \psi_t, \qquad H_A := d_A d_A^\dagger + d_A^\dagger d_A. \tag{10}$$

The map ψ_t is formally given by

$$\psi_t = \exp(-tH_A) = 1 - tH_A + \frac{t^2}{2!} H_A^2 - \cdots ,$$

so ψ_t commutes with d_A, $\psi_t d_A = d_A \psi_t$. Integrating Eq. (10) with respect to t yields

$$\mathrm{Id}_A - \psi_t = (d_A h_t + h_t d_A), \qquad h_t := \int_0^t dt' \psi_{t'}\, d_A^\dagger. \tag{11}$$

In particular, if $\varphi_t : M \to M$, $t \in [0, \infty)$, $\varphi_0 = \mathrm{Id}$, is a flow of a vector field $X \in TM$, then differentiating the pullback $\varphi_t^* : \Omega^r(M) \to \Omega^r(M)$ with respect to t gives Eq. (10) with $A = \Omega(M)$, $\psi_t = \varphi_t^*$ and $-H_A = \mathcal{L}_X := d\iota_X + \iota_X d$, the Lie derivative of X. Harvey and Lawson consider the case φ_t is the gradient flow of $-f$. Here we flip the sign compared to Eq. (9) in order to correspond the geometric flow in Eq. (9) to the "algebraic" flow given by its pullback. They asked whether the limit $t \to \infty$ exists or not. The answer is positive if we extend $\Omega(M)$ to the space $D'(M)$ of currents. It is known by the de Rham theorem that the natural inclusion $\mathbf{I} : \Omega^r(M) \to {D'}^r(M)$, $r = 0, ..., n$, induces an isomorphism on the cohomologies.

Theorem 3.1 (Harvey–Lawson [18, Theorem 4.1]) *For generic f, one has*

$$dh + hd = \mathbf{I} - \mathbf{P}, \tag{12}$$

where $h := h_\infty = -\int_0^\infty dt \varphi_t^* \iota_{\mathrm{grad}(f)} : \Omega^r(M) \to {D'}^{r-1}(M)$, $\mathbf{P} := \lim_{t\to\infty} \varphi_t^* : \Omega^r(M) \to {D'}^r(M)$. Furthermore, for any $\alpha \in \Omega^r(M)$, $\mathbf{P}(\alpha)$ is given by

$$\mathbf{P}(\alpha) = \sum_{p\in \mathrm{Cr}(f)} \left(\int_{U_p} \alpha \right) [S_p],$$

where $\int_{U_p} \alpha = 0$ if $r \neq n - \lambda(p)$.

Since by Eq. (12) $d\mathbf{P} = \mathbf{P}d$ holds, the restriction of the operation $d : D'(M) \to D'(M)$ onto $\mathbf{P}(\Omega(M))$ is well-defined, and one has a quasi-isomorphism

$$(\Omega(M), d) \simeq (\mathbf{P}(\Omega(M)), d).$$

In particular, the complex $(\mathbf{P}(\Omega(M)), d)$ is isomorphic to the dual Morse complex $(C^\bullet(f), d)$ defined by

$$C^r(f) := \mathrm{Hom}(C_{n-r}(f), \mathbb{Z}) \ \ (\simeq C_r(-f) \ )$$

with the differential d being induced from the one in $(C_\bullet(f), \partial)$. The correspondence is clear by the realization of the differential ∂ as the boundary operator on $\bar{S}_p$.

Roughly speaking, compared to Witten's approach, this Harvey–Lawson situation corresponds to fixing d, instead of d_s, and considering very large s for $d_s^\dagger$ with a Morse function $-f$. As we explain later, another combination, d_s with $s = 1$ and large s limit for $d_s^\dagger$, provides us with a solution for HMS.

3.2 Morse homotopy and Fukaya category

Morse homotopy [7, 10] deals with many functions on M. This enables us to discuss (higher) products on M in a geometric way as the de Rham theory, an algebraic formulation, has the wedge product and the cup product in its cohomology. These higher products are associated to rooted planar trees, which is the key point in discussing their relation to HPT.

For convenience, in this section, we attach an external (= a leaf or the root) vertex to the free end of each external edge of a k-tree and denote it by the same notation $\Gamma_k \in G_k$. Any k-tree Γ_k can be embedded into a disk D so that all the external vertices are on the boundary $\partial(D)$ of the disk D cyclically. Let $(z_{12}, \dots, z_{k(k+1)}, z_{(k+1)1})$ be the external vertices on $\partial(D)$, where $z_{(k+1)1}$ is the root vertex. Then, the disk is separated into $(k+1)$-regions by the tree. Denote by $\partial_i(D)$ the boundary piece in $\partial(D)$ having endpoints $z_{(i-1)i}$ and $z_{i(i+1)}$, where we identify i and $i + (k+1)$ and then let $i \in \mathbb{Z}_{k+1}$. We attach the number i to the region including $\partial_i(D)$. For any (internal or external) edge e of Γ_k, we denote by $r(e) \in \mathbb{Z}_{k+1}$ (resp. $l(e) \in \mathbb{Z}_{k+1}$) the number attached to the region right (resp. left) to the edge e with respect to the orientation on e.

First, we recall a Morse homotopy. Let $\mathbf{f} := (f_1, \dots, f_{k+1})$ be smooth functions on M. Assume that $f_i - f_{i+1}$ is a Morse function for each $i \in \mathbb{Z}_{k+1}$. We fix a critical point $p_{i(i+1)} \in \mathrm{Cr}(f_i - f_{i+1})$ for each i and denote $\mathbf{p} := (p_{12}, \dots, p_{k(k+1)}, p_{(k+1)1})$. The moduli space $\mathcal{M}_g(M, \mathbf{f}, \mathbf{p})$ consists of pairs (Γ_k, γ), where $\Gamma_k \in G_k$ and $\gamma : \Gamma_k \to M$ is a continuous map such that $\gamma(z_{i(i+1)}) = p_{i(i+1)}$, $i \in \mathbb{Z}_{k+1}$, and, for each edge e and an orientation preserving identification $e \subset \mathbb{R}$, $\gamma|_e$ is included in a gradient line (9) of $f_{r(e)} - f_{l(e)}$. Then, for generic $f_1, ..., f_{k+1} \in C^\infty(M)$, $\mathcal{M}_g(M, \mathbf{f}, \mathbf{p})$ is a C^∞-manifold of dimension

$$n - \sum_{i \in \mathbb{Z}_{k+1}} (n - \lambda(p_{i(i+1)})) + (k-2), \tag{13}$$

where $\lambda(p_{i(i+1)})$ is the Morse index of $p_{i(i+1)} \in \mathrm{Cr}(f_i - f_{i+1})$, and the natural projection $\pi : \mathcal{M}_g(M, \mathbf{f}, \mathbf{p}) \to G_k$ is a smooth map [10].

The category $Ms(M)$ of Fukaya–Oh Morse homotopy [7, 10] is defined as follows. The objects are smooth functions $f \in C^\infty(M)$. If the difference $f_{ab} := f_a - f_b$ of two functions $f_a, f_b \in C^\infty(M)$ is a Morse function, the space $\mathrm{Hom}_{Ms(M)}(a, b)$ of morphisms is set to be the dual Morse complex $C^\bullet(f_{ab})$. The (transversal) A_∞-structure on $Ms(M)$ is defined by

$$m_k([p_{a_1a_2}], \ldots, [p_{a_ka_{k+1}}]) = \sum_{p \in \mathrm{Cr}(f_{a_1a_{k+1}})} \sum_{(\Gamma_k, \gamma) \in \mathcal{M}_g(M, \mathbf{f}, \mathbf{p})} \pm [p], \tag{14}$$

where $[p_{ab}]$ is a base of $C^\bullet(f_{ab})$ associated to a critical point $p_{ab} \in \mathrm{Cr}(f_{ab})$, $\mathbf{f} = (f_{a_1}, \ldots, f_{a_{k+1}})$ and $\mathbf{p} = (p_{a_1a_2}, \ldots, p_{a_ka_{k+1}}, p)$. Here the degree $\deg([p_{a_ia_{i+1}}])$ of $[p_{a_ia_{i+1}}]$ is $n - \lambda(p_{a_ia_{i+1}})$. To $p \in \mathrm{Cr}(f_{a_1a_{k+1}})$ are associated two bases $[p] \in C^\bullet(f_{a_1a_{k+1}})$ and $[p]^* \in C^\bullet(f_{a_{k+1}a_1})$ such that $\deg([p]) + \deg([p]^*) = n$. Recall that m_k is of degree $(2-k)$; by comparing the degree counting with Eq. (13) one sees that $\mathcal{M}_g(M, \mathbf{f}, \mathbf{p})$ in Eq. (14) is zero-dimensional and the summation $\sum_{(\Gamma_k, \gamma)}$ is well-defined. Remember that, for generic $\mathbf{f}$, the A_∞-structure is given by an element $\Gamma_k \in G_k^{tri}$.

Next, we discuss the corresponding Fukaya category. For given generic functions f_i, $i \in \mathbb{Z}_{k+1}$, the graph $df_i \in \Gamma(T^*M)$ defines a Lagrangian submanifold which we denote by L_i. The intersections of two generic Lagrangian submanifolds L_i and L_{i+1} of f_i and f_{i+1} are points. We denote $L_{i(i+1)} := L_i \cap L_{i+1}$.

Let $\mathcal{M}_{k+1}$ be the moduli space of a disk D with cyclic ordered $(k+1)$ points $z_{12}, \ldots, z_{(k+1)1} \in \partial(D)$. For generic f_i, $i \in \mathbb{Z}_{k+1}$, let $\mathbf{v} := (v_{12}, \ldots, v_{k(k+1)}, v_{(k+1)1})$, $v_{i(i+1)} \in L_{i(i+1)}$, be a sequence of intersection points. The moduli space $\mathcal{M}_J(T^*M, \mathbf{f}, \mathbf{v})$ consists of the pairs $(\mathbf{z}, \phi)$, where $\mathbf{z} \in \mathcal{M}_{k+1}$ and $\phi : D \to T^*M$ is a map satisfying the following conditions:

- $\phi(z_{i(i+1)}) = v_{i(i+1)}$ and $\phi(\partial_i(D)) \subset L_i$ for each $i \in \mathbb{Z}_{k+1}$,
- $J \circ T\phi = T\phi \circ J$, where $J : TD \to TD$ is a holomorphic structure on D and $J : T(T^*M) \to T(T^*M)$ is the canonical almost complex structure on T^*M associated to the metric g.

Then, $\mathcal{M}_J(T^*M, \mathbf{f}, \mathbf{v})$ is a C^∞-manifold of dimension

$$n - \sum_{i \in \mathbb{Z}_{k+1}} (n - \lambda(v_{i(i+1)})) + (k-2)$$

and there is a natural map $\mathcal{M}_J(T^*M, \mathbf{f}, \mathbf{p}) \to \mathcal{M}_{k+1}$ [10].

The Fukaya category $Fuk(T^*M)$ is defined as follows. The objects are Lagrangian submanifolds $L_a \subset T^*M$ of graphs df_a. The space of morphisms is set to be $\mathrm{Hom}_{Fuk(T^*M)}(a, b) := \oplus_{v_{ab} \in L_{ab}} \mathbb{Z}[v_{ab}]$, where $[v_{ab}]$ is the base associated to the intersection point $v_{ab} \in L_{ab}$. The projection $x : T^*M \to M$ induces a bijection between L_{ab} and $\mathrm{Cr}(f_{ab})$ and then $\mathrm{Hom}_{Fuk(T^*M)}(a,b) \simeq C^\bullet(f_{ab})$; the degree of $[v_{ab}]$ is defined by this isomorphism.[1] The (transversal) A_∞-structure m_k, $k \geq 1$, is given in a similar way as Eq. (14):

$$m_k([v_{a_1a_2}], \dots, [v_{a_ka_{k+1}}]) = \sum_{v \in L_{a_1a_{k+1}}} \sum_{(\mathbf{z},\phi) \in \mathcal{M}_J(T^*M, \mathbf{f}, \mathbf{v})} \pm [v], \qquad (15)$$

where $\mathbf{v} = (v_{a_1a_2}, \dots, v_{a_ka_{k+1}}, v)$. The equivalence of the category $Ms(M)$ with the Fukaya category $Fuk(T^*M)$ on T^*M, though intuitively being given by the embedding of each k-tree Γ_k into a disk D as we did and the projection $x : T^*M \to M$, follows from the following.

Theorem 3.2 (Fukaya–Oh [10]) *For generic* $\mathbf{f}$*, one has an oriented diffeomorphism*

$$\mathcal{M}_g(M, \mathbf{f}, \mathbf{p}) \simeq \mathcal{M}_J(T^*M, \mathbf{f}, \mathbf{p}).$$

In the above constructions, we considered transversal A_∞-products only, where one sees that the cyclicity in the sense of Section 2.3 is manifest.

The A_∞-structure Eq. (15) was further modified by Kontsevich [34] by multiplying $e^{-\int_D \phi^*\omega}$, where $\int_D \phi^*\omega$ is the symplectic area of the disk $\phi(D)$. The complexes are then defined over $\mathbb{R}$. This modified version is what we shall discuss, and our interest is the corresponding modification for the Morse homotopy side. In the rest of this article, we denote $v_{ab} := [v_{ab}]$.

4 Fukaya category of lines in a plane

For a fixed integer $N \geq 2$, let $\{f_1, \dots, f_N\}$ be a set of polynomial functions on $\mathbb{R}$ of degree equal or less than two. For each $a \in \{1, \dots, N\}$, $y = df_a/dx$ is a line L_a in $\mathbb{R}^2$ with coordinates (x, y) described as

$$L_a : y = \mu_a x + \nu_a, \qquad \mu_a, \nu_a \in \mathbb{R}.$$

Let $\mathfrak{F}_N$ be a collection $\{f_1, \dots, f_N\}$ satisfying the following two conditions:

(i) For any $a \neq b = 1, \dots, N$, the slopes are different from each other: $\mu_a \neq \mu_b$.
(ii) More than two lines do not intersect at the same point in $\mathbb{R}^2$.

We shall construct an A_∞-category $\mathcal{C}(\mathfrak{F}_N)$ with $\mathfrak{F}_N$ the set of objects in the next subsection. In order that $\mathcal{C}(\mathfrak{F}_N)$ would be called a Fukaya category, the following two conditions should be satisfied:

(C1) For any two objects $a \neq b \in \mathfrak{F}_N$, the space $\mathrm{Hom}_{\mathcal{C}(\mathfrak{F}_N)}(a,b) =: V_{ab}$ of morphisms is the following graded vector space of degrees zero and one:

[1] Originally, these degrees in Fukaya categories are defined by the Maslov indices; which consequently coincide with the one defined via the Morse indices [10].

$$V_{ab}^0 = \mathbb{R}\cdot v_{ab}, \quad V_{ab}^1 = 0, \qquad \mu_a < \mu_b,$$
$$V_{ab}^0 = 0, \quad V_{ab}^1 = \mathbb{R}\cdot v_{ab}, \qquad \mu_a > \mu_b.$$

Here, v_{ab} are the bases associated to the intersection points $v_{ab} \in L_{ab}$.

(C2) Let $a_1, \ldots, a_{k+1} \in \mathfrak{F}_N$, $k \geq 1$, be objects such that $a_i \neq a_j$ for any $i \neq j \in \mathbb{Z}_{k+1}$ and $\mathbf{v} := (v_{a_1a_2}, \ldots, v_{a_ka_{k+1}}, v_{a_{k+1}a_1})$ the sequence of points. Then, for $k = 1$, the differential $m_1 : V_{a_1a_2} \to V_{a_1a_2}$ is trivial, $m_1 = 0$. For $k \geq 2$, the structure constant $c(\mathbf{v}) \in \mathbb{R}$ for the higher A_∞-product

$$m_k(v_{a_1a_2}, \ldots, v_{a_ka_{k+1}}) = c(\mathbf{v}) \cdot v_{a_1a_{k+1}} \tag{16}$$

is given by $c(\mathbf{v}) = \pm e^{-Area(\mathbf{v})}$ if $\mathbf{v}$ does not form a clockwise convex (CC) polygon and zero otherwise, where $Area(\mathbf{v})$ is the area of the CC-polygon.

We call an A_∞-product m_k, $k \geq 2$, of the type in (C2) *transversal* and that of the other type *nontransversal*. For a sequence $\mathbf{v} = (v_{a_1a_2}, \ldots, v_{a_ka_{k+1}}, v_{a_{k+1}a_1})$ of intersection points, hereafter the degree of the point $v_{a_ia_{i+1}}$, $i \in \mathbb{Z}_{n+1}$, indicates the degree $|v_{a_ia_{i+1}}|$ of the associated elements $v_{a_ia_{i+1}} \in V_{a_ia_{i+1}}$.

Let us observe briefly how the formula (16) is compatible with the A_∞-constraint (1) and why we need nontransversal A_∞-products. First, the formula $m_n(v_{a_1a_2}, \ldots, v_{a_na_{n+1}})$ for the transversal A_∞-products in Eq. (16) is compatible with that the degree of m_n should be $(2-n)$. Indeed, for the CC

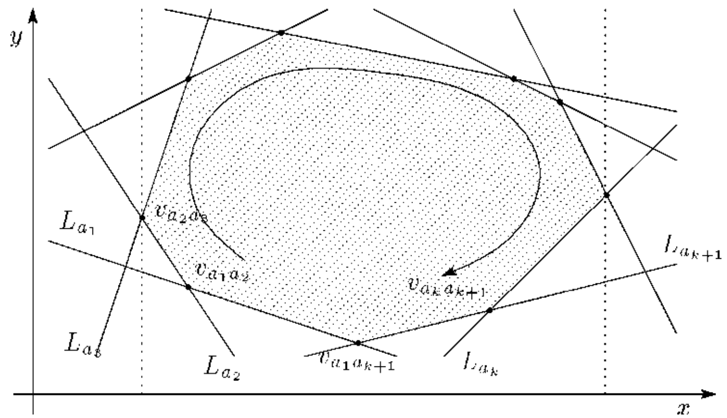


Fig. 1. A clockwise convex polygon (CC-polygon) $\mathbf{v} = (v_{a_1a_2}, \ldots, v_{a_na_{n+1}}, v_{a_{n+1}a_1})$

$(n+1)$-gon $\mathbf{v} := (v_{a_1a_2}, \ldots, v_{a_na_{n+1}}, v_{a_{n+1}a_1})$, if we clockwisely count the degrees of the points $v_{a_ia_{i+1}}$, $i \in \mathbb{Z}_{n+1}$, only those of the left and right extrema have degree zero and the remaining $(n+1)-2$ points have degree one (Fig. 1). Thus, one has

$$\sum_{i\in\mathbb{Z}_{n+1}} |v_{a_ia_{i+1}}| = (n+1) - 2,$$

which implies that the degree of m_n is $(2-n)$, since $|v_{a_1a_{n+1}}| = 1 - |v_{v_{a_{n+1}a_1}}|$.

The A_∞-constraints for transversal A_∞-products have a geometric interpretation in terms of a clockwise polygon having one nonconvex vertex of the polygon. There exist two ways to divide the polygon into two convex polygons. The corresponding terms then appear with opposite signs and cancel each other in the A_∞-constraint. For example, in Fig. 2, consider the polygon

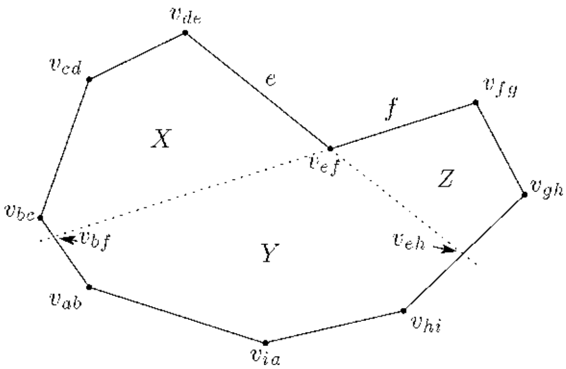


Fig. 2. A clockwise polygon which has one nonconvex vertex v_{ef}

consisting of the sequence $(v_{ab}, v_{bc}, v_{cd}, v_{de}, v_{ef}, v_{fg}, v_{gh}, v_{hi})$ of points. We see that this polygon is nonconvex at v_{ef}. The area $X + Y + Z$ is divided into two by (i) $X + (Y + Z)$ or (ii) $(X + Y) + Z$, to which are associated the compositions of transversal A_∞-products:

$$\begin{aligned}&(\mathrm{i}) \pm m_5(v_{ab}, m_4(v_{bc}, v_{cd}, v_{de}, v_{ef}), v_{fg}, v_{gh}, v_{hi}),\\ &(\mathrm{ii}) \pm m_6(v_{ab}, v_{bc}, v_{cd}, v_{de}, m_3(v_{ef}, v_{fg}, v_{gh}), v_{hi}),\end{aligned}$$

where $m_4(v_{bc}, v_{cd}, v_{de}, v_{ef}) = \pm e^{-X} v_{bf}$, $m_5(v_{ab}, v_{bf}, v_{fg}, v_{gh}, v_{hi}) = \pm e^{-(Y+Z)}$ v_{ai}, etc. Since a transversal A_∞-product can be nonzero only if the corresponding polygon forms a CC-polygon, there does not exist any other composition of A_∞-products, and one has

$$\begin{aligned}&\pm m_5(v_{ab}, m_4(v_{bc}, v_{cd}, v_{de}, v_{ef}), v_{fg}, v_{gh}, v_{hi}) = \pm e^{-X-(Y+Z)} v_{ai}\\ &= \pm e^{-(X+Y)-Z} v_{ai} = \pm m_6(v_{ab}, v_{bc}, v_{cd}, v_{de}, m_3(v_{ef}, v_{fg}, v_{gh}), v_{hi}),\end{aligned}$$

which is just the A_∞-constraint (1) on $(v_{ab}, v_{bc}, v_{cd}, v_{de}, v_{ef}, v_{fg}, v_{gh}, v_{hi})$.

Next, in Fig. 2, let us consider the sequence $(v_{ab}, v_{bf}, v_{fe}, v_{ef}, v_{fg}, v_{gh}, v_{hi})$ and the corresponding A_∞-constraint. There exists a composition

$$m_5(v_{ab}, v_{bf}, v_{fe}, m_3(v_{ef}, v_{fg}, v_{gh}), v_{hi}) = e^{-(Y+Z)} v_{ai}$$

of two transversal A_∞-products. The A_∞-constraint (1) then implies that this composition cancels with other terms. However, there does not exist any more composition of two nonzero transversal A_∞-products on the sequence $(v_{ab}, v_{bf}, v_{fe}, v_{ef}, v_{fg}, v_{gh}, v_{hi})$. This shows the necessity of nonzero nontransversal A_∞-products. For the A_∞-category $\mathcal{C}(\mathfrak{F}_N)$ we shall construct, one has $m_2(v_{fe}, v_{ef}) = \delta_{v_{fe}} \in V^1_{ff}$ and then obtain the A_∞-constraint

$$\begin{aligned} &m_5(v_{ab}, v_{bf}, v_{fe}, m_3(v_{ef}, v_{fg}, v_{gh}), v_{hi}) \\ &\quad = m_6(v_{ab}, v_{bf}, m_2(v_{fe}, v_{ef}), v_{fg}, v_{gh}, v_{hi}). \end{aligned}$$

5 Deriving A_∞-structures on Fukaya categories

We first define a DG category $\mathcal{C}_{DR}(M)$ which is expected to derive a Fukaya A_∞-category on T^*M. We set $\mathrm{Ob}(\mathcal{C}_{DR}(M)) := C^\infty(M)$. For two objects $f_a, f_b \in C^\infty(\mathbb{R})$, the space of morphisms $\mathrm{Hom}_{\mathcal{C}_{DR}}(a,b)$ is set to be the space $\varOmega(M)$ of smooth differential forms, where we set the differential d_{ab} : $\mathrm{Hom}_{\mathcal{C}_{DR}}(a,b) \to \mathrm{Hom}_{\mathcal{C}_{DR}}(a,b)$ as

$$d_{ab} := d - d(f_{ab})\wedge \ = e^{f_{ab}}\, d\, e^{-f_{ab}}.$$

The composition of morphisms is the wedge product of differential forms. Clearly, $\mathcal{C}_{DR}(M)$ forms a DG category.

For $a \neq b \in \mathfrak{F}_N$, assume that f_{ab} is a Morse function, and let us apply Eq. (10) with $A = \varOmega(M)$, $d_A = d_{ab}$ and

$$d_A^\dagger = d_{ab}^\dagger := -\iota_{\mathrm{grad}(f_{ab})}.$$

Since $H := d_{ab}d_{ab}^\dagger + d_{ab}^\dagger d_{ab} = -e^{f_{ab}}\mathcal{L}_{\mathrm{grad}(f_{ab})}e^{-f_{ab}}$, one obtains a modified version of the Harvey–Lawson one in Theorem 3.1 by $e^{f_{ab}}$:

$$h_{ab} = -\int_0^\infty dt e^{f_{ab}}\varphi_t^*(e^{-f_{ab}}\ \iota_{\mathrm{grad}(f_{ab})}), \quad P_{ab} = \lim_{t\to\infty} e^{f_{ab}}\,\varphi_t^*\, e^{-f_{ab}}, \qquad (17)$$

where $\varphi_t : M \to M$ is the gradient flow of $-f_{ab}$. At least for transversal A_∞-products, this is expected to lead to the correct structure constant of the A_∞-structure; via HPT, a modification of the A_∞-category $Ms(M)$ of Morse homotopy given as in [36] is derived, which is further equivalent to a Fukaya category having $e^{-\int_D \phi^*\omega}$ as the coefficients of transversal A_∞-products.

However, as in the case in Theorem 3.1, this P_{ab} in Eq. (17) is a map from $\varOmega(M)$ not to $\varOmega(M)$ itself but to $D'(M)$. Thus, we need some modification of the story. A natural way may be to modify h as h_ϵ with a parameter ϵ such that $d_{DR}h_\epsilon + h_\epsilon d_{DR} = \mathrm{Id} - P_\epsilon$ holds on $\varOmega(M)$ if $\epsilon \neq 0$ and $\lim_{\epsilon\to 0} h_\epsilon = h$. Then, we may apply HPT with contracting homotopy h_ϵ, $\epsilon \neq 0$, construct the induced A_∞-products, and finally take the limit $\epsilon \to 0$.[2] We propose the following modification: define $d^\dagger_{\epsilon;ab} : \varOmega^r_{ab} \to \varOmega^{r-1}_{ab}$, $\epsilon \in (0,1]$, as

[2] A general approach to such a modification h_ϵ is discussed in [36].

$$d^\dagger_{\epsilon;ab} = \epsilon\, d^* - \iota_{\mathrm{grad}(f_{ab})} \quad (= \epsilon \cdot e^{-\frac{f_{ab}}{\epsilon}} d^* e^{\frac{f_{ab}}{\epsilon}})$$

and then $H_A = H_\epsilon := d_{ab} d^\dagger_{\epsilon;ab} + d^\dagger_{\epsilon;ab} d_{ab}$. Since this H_ϵ defines an elliptic operator called a generalized Laplacian, the limit $t \to \infty$ of ψ_t and h_t of Eq. (10) exists [2], which gives an SDR data.

Now, let us discuss the case $M = \mathbb{R}$ in some detail, where we set $g(d/dx, d/dx) = 1$. Let $\Omega(\mathbb{R}) := \Omega^0(\mathbb{R}) \oplus \Omega^1(\mathbb{R})$ be the graded vector space defined by $\Omega^0(\mathbb{R}) := \mathcal{S}(\mathbb{R})$, the space of Schwartz functions, and $\Omega^1(\mathbb{R}) := \mathcal{S}(\mathbb{R}) \cdot dx$, where dx is the base of one-form on $\mathbb{R}$. We consider the set $\mathfrak{F}_N$ of N lines in Section 4, and consider the DG category $\mathcal{C}_{DR}(\mathfrak{F}_N)$ with $\mathfrak{F}_N$ the set of objects, where $\mathrm{Hom}_{\mathcal{C}_{DR}(\mathfrak{F}_N)}(a,b) := \Omega(\mathbb{R})$.

Theorem 5.1 ([24]) *There exists an A_∞-category which satisfies Conditions (C1) and (C2) and is homotopy equivalent to $\mathcal{C}_{DR}(\mathfrak{F}_N)$.*

In the rest of this section, we explain the rough idea on how to construct such a Fukaya A_∞-category $\mathcal{C}(\mathfrak{F}_N)$. For the precise formulation, see [24].

For any $a \in \mathfrak{F}_N$, the graph df_a of f_a is a line L_a in $\mathbb{R}^2 \simeq T^*\mathbb{R}$. Then, for any $a \neq b \in \mathfrak{F}_N$, the intersection point v_{ab} of L_a and L_b is only one and so is the critical point $p_{ab} = x(v_{ab})$ of f_{ab}. Furthermore, $H_\epsilon := d_{ab} d^\dagger_{\epsilon;ab} + d^\dagger_{\epsilon;ab} d_{ab}$ in fact has only nonnegative real eigenvalues. In particular, H_1 gives a Hamiltonian of a harmonic oscillator. For any ϵ, the corresponding Hodge–Kodaira decomposition $d_{ab} h_{\epsilon;ab} + h_{\epsilon;ab} d_{ab} = Id_{\Omega_{ab}} - P_{\epsilon;ab}$ gives

$$P_{\epsilon;ab}\Omega^0_{ab} = \mathrm{Ker}(d_{ab} : \Omega^0_{ab} \to \Omega^1_{ab}), \quad P_{\epsilon;ab}\Omega^1_{ab} = \mathrm{Ker}(d^\dagger_{\epsilon;ab} : \Omega^1_{ab} \to \Omega^0_{ab}).$$

We set bases $\mathbf{e}_{\epsilon;ab}$ of $P_{\epsilon;ab}\Omega^r_{ab}$, $r = 0, 1$, as follows. If $\mu_a < \mu_b$, then it is a Gaussian:

$$\mathbf{e}_{\epsilon;ab} = \mathrm{const} \cdot e^{f_{ab}},$$

where the “const” is normalized so that $\mathbf{e}_{\epsilon;ab}(x(v_{ab})) = 1$. On the other hand, if $\mu_a > \mu_b$, it is a Gaussian one-form:

$$\mathbf{e}_{\epsilon;ab} = \mathrm{const} \cdot e^{-\frac{1}{\epsilon}(f_{ab})} dx$$

normalized so that $\int_{-\infty}^{\infty} \mathbf{e}_{\epsilon;ab} = 1$, which approaches to the delta function one-form localized at the point $x(v_{ab})$ in the limit $\epsilon \to 0$. Thus, in the limit $\epsilon \to 0$, the base $\mathbf{e}_{\epsilon;ab}$ is thought of $[S_{x(v_{ab})}]$ multiplied by the weight $e^{f_{ab}}$, where $S_{x(v_{ab})} = \mathbb{R}$ for $\deg(\mathbf{e}_{\epsilon;ab}) = 0$ and $S_{x(v_{ab})} = x(v_{ab})$ for $\deg(\mathbf{e}_{\epsilon;ab}) = 1$.

Let us observe how, in the limit $\epsilon \to 0$, h_{ab} acts on a delta function one-form. By Eq. (17), it turns out that

$$h_{ab}(\delta(x-p)dx) = -e^{f_{ab}(x) - f_{ab}(p)} \int_{\varphi_0(x)}^{\varphi_\infty(x)} d(\varphi_t(x))\, \delta(\varphi_t(x) - p).$$

This is a step function multiplied by $e^{f_{ab}}$. For instance, for $\mu_a < \mu_b$ and $x(v_{ab}) < p$, one has the one as in Fig. 3.

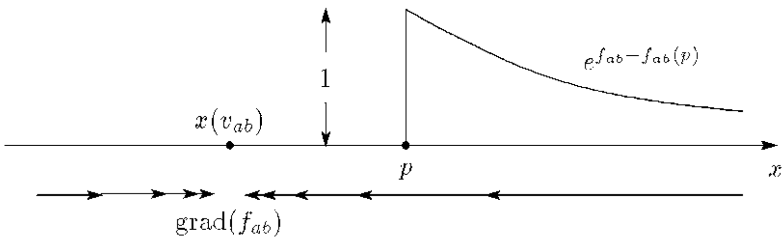


Fig. 3. $h_{ab}(\delta(x-p)dx)$ is a step function multiplied by $e^{f_{ab}}$

Now, let us apply HPT to the DG category $\mathcal{C}_{DR}(\mathfrak{F}_N)$ and derive some examples of the A_∞-products $\{m_n\}$ of $\mathcal{C}(\mathfrak{F}_N)$ under the identifications

$$\begin{aligned} \lim_{\epsilon\to 0} P_{\epsilon;ab}\Omega_{ab} &\longleftrightarrow \mathrm{Hom}_{\mathcal{C}(\mathfrak{F}_N)}(a,b)=V_{ab},\\ \lim_{\epsilon\to 0}\mathbf{e}_{\epsilon;ab}=\mathbf{e}_{ab} &\longleftrightarrow \qquad v_{ab} \end{aligned}$$

for any $a\neq b$. Here, if $a=b$, there is no natural choice of the Hodge decomposition, so we set $h_{aa}=0$. We denote the product in $\mathcal{C}'_{DR}(\mathfrak{F}_N)$ by m.

Example 1 *Let us consider a transversal A_∞-product $m_3(\mathbf{e}_{ab},\mathbf{e}_{bc},\mathbf{e}_{cd})$ such as $|v_{ab}|=|v_{cd}|=0$, $|v_{bc}|=1$, $x(v_{ab})<x(v_{bc})<x(v_{da})<x(v_{cd})$ (Fig. 4). The HPT implies*

$$m_3(\mathbf{e}_{ab},\mathbf{e}_{bc},\mathbf{e}_{cd}) \quad = \quad + \ . \tag{18}$$

On the other hand, one can associate a planar tree graph to the CC-polygon $\mathbf{v}$ *as follows. First, connect two points v_{ab},v_{cd} of* $\mathbf{v}$ *of degree zero with an interval. For each point of* $\mathbf{v}$ *of degree one, draw an interval (external edge = leaf) which is perpendicular to the x-axis, starts from the point, and ends on the interval $(v_{ab}v_{cd})$. Choosing the interval starting from the point v_{da} as the root edge, one obtains a planar rooted tree as in Fig. 4. One can see that the resulting planar rooted tree corresponds to the one in the first term of the right-hand side of Eq. (18). The second term of the right-hand side of Eq. (18) in fact vanishes and the first term derives the area Area$(\mathbf{v})$. The first term is calculated as follows. As in Fig. 4, we divide the CC-polygon* $\mathbf{v}$ *into three by the two lines which are perpendicular to the x-axis and pass through v_{bc} or v_{da}. The areas between $x(v_{ab})$ and $x(v_{bc})$, $x(v_{bc})$ and $x(v_{da})$, $x(v_{da})$ and $x(v_{cd})$ are denoted X, Y, Z, respectively. First, one gets $m(\mathbf{e}_{ab},\mathbf{e}_{bc})=\pm e^{-X}\delta_{v_{bc}}$. We know $h_{ac}\delta_{v_{bc}}=\pm e^{f_{ac}-f_{ac}(x(v_{bc}))}\cdot\vartheta_{v_{bc}}$. Then, $P_{ad}m(-h_{ac}\delta_{v_{bc}},\mathbf{e}_{cd})$ is $\mathbf{e}_{ad}$ times the value of the product of $-h_{ac}\delta_{v_{bc}}$ and $\mathbf{e}_{cd}$ at the point $x(v_{da})\in\mathbb{R}$:*

$$\begin{aligned} P_{ad}m(-h_{ac}\delta_{v_{bc}},\mathbf{e}_{cd}) &= \pm\left(e^{f_{ac}(x(v_{da}))-f_{ac}(x(v_{bc}))}\cdot e^{f_{cd}(x(v_{da}))-f_{cd}(x(v_{cd}))}\right)\cdot\mathbf{e}_{ad}\\ &= \pm\left(e^{-Y}\cdot e^{-Z}\right)\cdot\mathbf{e}_{ad}, \end{aligned}$$

where we note that $f_{ac}(x(v_{bc}))-f_{ac}(x(v_{da}))=Y$ and $f_{cd}(x(v_{cd}))-f_{cd}(x(v_{da}))=Z$. Combining all these together, we obtain the first term on the right-hand side of Eq. (18): $\pm e^{-X-(Y+Z)}\mathbf{e}_{ad}$.

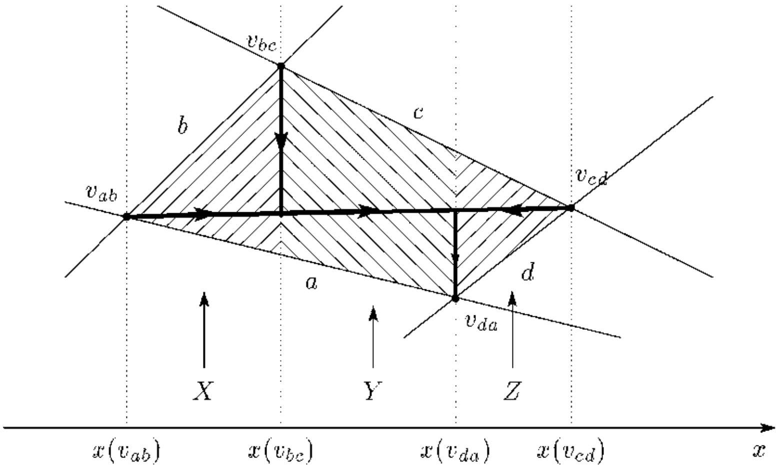


Fig. 4. CC-polygon $\mathbf{v} = (v_{ab}, v_{bc}, v_{cd}, v_{da})$

Thus, HPT machinery defines a higher product m_k in terms of the sum of values associated to planar rooted k-trees over all the k-trees, but only the one compatible with the k-tree associated to the corresponding CC-polygon survives and produces the area of the CC-polygon.

Example 2 *Consider the nontransversal A_∞-product $m_4(\mathbf{e}_{ab}, \mathbf{e}_{bc}, \mathbf{e}_{cd}, \mathbf{e}_{da})$ with $|v_{bc}| = |v_{cd}| = 0$, $|v_{ab}| = 1$, $x(v_{bc}) < x(v_{ab}) < x(v_{da}) < x(v_{cd})$ as in Fig. 5. This is again given as the sum of the values associated to trivalent planar rooted 4-trees via HPT. On the other hand, the 4-tree corresponding to the CC-polygon $\mathbf{v}$ is obtained as follows. Connect the two degree zero points v_{bc} and v_{cd} with an interval. For each degree one point v_{ab} or v_{da}, draw an interval which is perpendicular to the x-axis, starts from the point and ends on the interval $(v_{bc}v_{cd})$. Then, we add the root edge which, perpendicularly to the x-axis, starts from a point on the interval $(v_{bc}v_{cd})$ between $x(v_{ab})$ and $x(v_{da})$ and ends on the interval $(v_{ab}v_{da})$ (Fig. 5).*

In fact, only the multilinear map corresponding to this 4-tree is nonzero and it turns out to be $\pm e^{-(X+Y+Z)}(\vartheta_{v_{ab}} - \vartheta_{v_{da}})$, where ϑ_v, $x(v) \in \mathbb{R}$, denotes the step function on $\mathbb{R}$ such that $\vartheta_v(x) = 0$ for $x < x(v)$ and $\vartheta_v(x) = 1$ for $x > x(v)$.

Now, the step functions ϑ_v, $v \in \{v_{ab}\}_{b \in \mathfrak{F}_N \setminus \{a\}}$, appear, which should be included as elements in V^0_{aa}. Then, $d(\vartheta_v)$ will be a delta function one form with support $x(v)$. Thus, for each $a \in \mathfrak{F}_N$, we introduced a DGA V_{aa} generated by "step functions" and "delta function one forms" in [24, Definition 3.3]. The algebraic structure on V_{aa} is defined so that it is natural as the limit $\delta_{v_{ab}} := \lim_{\epsilon \to 0} \mathbf{e}_{\epsilon;ab}\mathbf{e}_{\epsilon;ba}$ and $\vartheta_{v_{ab}}(x) := \lim_{\epsilon \to 0} \int_{-\infty}^{x} \mathbf{e}_{\epsilon;ab}\mathbf{e}_{\epsilon;ba}$. Then, any other nontransversal A_∞-products can be calculated. The following is a typical example of them.

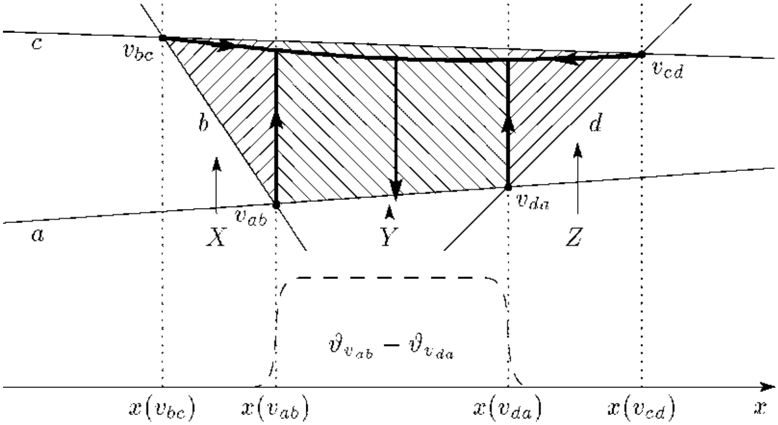


Fig. 5. CC-polygon $\mathbf{v} = (v_{ab}, v_{bc}, v_{cd}, v_{da})$

Example 3 *Consider the nontransversal A_∞-product*

$$m_{1+d+1}(\mathbf{e}_{ab}, (\mathbf{e}_{bc})^{\otimes d}, \mathbf{e}_{cd}) := m_{1+d+1}(\mathbf{e}_{ab}, \overbrace{\delta_{v_{bc}}, \ldots, \delta_{v_{bc}}, \mathbf{e}_{bc}, \delta_{v_{bc}}, \ldots, \delta_{v_{bc}}}^{d}, \mathbf{e}_{cd})$$

in the same situation as Example 1 for $\mathbf{e}_{ab}, \mathbf{e}_{bc}, \mathbf{e}_{cd}$ but we have a d-tuple of elements associated to the point v_{bc}. In fact, the result is independent of the order of $\delta_{v_{bc}}$'s and $\mathbf{e}_{bc}$. We obtain $m_{1+d+1}(\mathbf{e}_{ab}, (\mathbf{e}_{bc})^{\otimes d}, \mathbf{e}_{cd}) = \pm(1/d!)\, e^{-(X+Y+Z)}\mathbf{e}_{ad}$.

In [24], in order to proceed with the construction of $\mathcal{C}(\mathfrak{F}_N)$ as above rigorously, we introduced other DG categories $\mathcal{C}'_{DR}(\mathfrak{F}_N)$ and $\tilde{\mathcal{C}}_{DR}(\mathfrak{F}_N)$ of the same objects so that there exist inclusions $\iota : \mathcal{C}'_{DR}(\mathfrak{F}_N) \to \tilde{\mathcal{C}}_{DR}(\mathfrak{F}_N)$ and $\iota : \mathcal{C}_{DR}(\mathfrak{F}_N) \to \tilde{\mathcal{C}}_{DR}(\mathfrak{F}_N)$ giving homotopy equivalences between them. Here, the space $\mathrm{Hom}_{\mathcal{C}'_{DR}(\mathfrak{F}_N)}(a,b)$ is that generated by step functions and delta-function one-forms multiplied by $e^{f_{ab}}$, and the space $\mathrm{Hom}_{\tilde{\mathcal{C}}_{DR}(\mathfrak{F}_N)}(a,b)$ in addition includes smooth differential forms. Then, an SDR can be given explicitly for $\mathcal{C}'_{DR}(\mathfrak{F}_N)$ and HPT was applied directly there.

$$\mathcal{C}(\mathfrak{F}_N) \xrightarrow{\mathcal{G}} \mathcal{C}'_{DR}(\mathfrak{F}_N) \xrightarrow{\iota} \tilde{\mathcal{C}}_{DR}(\mathfrak{F}_N) \xleftarrow{\iota} \mathcal{C}_{DR}(\mathfrak{F}_N).$$

The intermediate DG category $\mathcal{C}'_{DR}(\mathfrak{F}_N)$ also enabled us to show that all the A_∞-categories $\mathcal{C}(\mathfrak{F}_N)$ are homotopy equivalent, being independent of the slopes of the lines [24, Theorem 3.8]. However, as we see below, if we apply this construction to T^2, the resulting Fukaya A_∞-category on T^2 depends on them.

6 Finite-dimensional A_∞-algebras from two-tori

We can apply the arguments in the previous section to the HMS of two-tori. Namely, applying HPT to a DG category of holomorphic vector bundles (or coherent sheaves in more general) on an elliptic curve (a two-torus with a fixed complex structure $\tau \in H$) yields a Fukaya A_∞-category on the mirror dual symplectic two-torus T^2, where $\mathbb{R}^2$ is the universal cover of T^2. For the DG category, the one in the framework of noncommutative tori (with noncommutativity being set to be zero) [21, 42, 22] fits with our purpose.[3] In Section 6.1, we briefly recall this DG category and explain the relation to $\mathcal{C}_{DG}(\mathfrak{F}_N)$. In Section 6.2, we present a finite-dimensional A_∞-algebra as a subcategory of the Fukaya A_∞-category of finitely many objects.

6.1 DG category for (noncommutative) elliptic curve

A *noncommutative torus* $\mathcal{A}_\theta$, $\theta \in \mathbb{R}$, is an algebra generated by two unitary generators U_1, U_2 satisfying the relation $U_1U_2 = e^{2\pi \mathbf{i}\theta}U_2U_1$. We treat finitely generated projective modules $E_{q,p,\theta}$, called Heisenberg modules, equipped with constant curvature connection, where (p,q) are pairs of relatively prime integers. The numbers p and q are thought of as the noncommutative analogue of the first Chern class and the rank, respectively. For $p \neq 0$, the Heisenberg module $E_{p,q,\theta}$ is given by

$$E_{p,q,\theta} := \mathcal{S}(\mathbb{R} \times \mathbb{Z}_p),$$

the $|p|$ copies of $\mathcal{S}(\mathbb{R})$, where the (right) action of $\mathcal{A}_\theta$ is given in an appropriate way. For $p = 0$, the corresponding vector bundle can be thought of as a trivial one, so we set $E_{0,1,\theta} := \mathcal{A}_\theta$, the free module.

According to A. Schwarz [45], introduce a complex structure $\tau \in H_+$ on $\mathcal{A}_\theta$ and a holomorphic structure $\bar{\nabla}$ on $E_{p,q,\theta}$, which is given by

$$\bar{\nabla} := \left(\nabla_1 - \frac{1}{\tau}\nabla_2\right) d\bar{z} : E_{p,q,\theta} \to E_{p,q,\theta} \otimes d\bar{z},$$

where $\nabla_1, \nabla_2 : E_{p,q,\theta} \to E_{p,q,\theta}$ is a constant curvature connection and $d\bar{z}$ is the formal base of anti-holomorphic one form. Note that any $E_{p,q,\theta}$ equipped with a connection is lifted to be a holomorphic vector bundle in this way in the case of a noncommutative analogue of *two-dimensional* torus. We denote $\mathbb{E} := (E, \bar{\nabla})$.

The space of morphisms between two holomorphic vector bundles $\mathbb{E}_a, \mathbb{E}_b$ is set to be the graded vector space $\mathrm{Hom}^\bullet(\mathbb{E}_a, \mathbb{E}_b) = \mathrm{Hom}_{\mathcal{A}_\theta}(E_a, E_b) \oplus \mathrm{Hom}_{\mathcal{A}_\theta}(E_a, E_b) \cdot d\bar{z}$ with $\mathrm{Hom}_{\mathcal{A}_\theta}(E_a, E_b)$ the space of the module maps of the underlying finitely generated projective modules E_a, E_b, where a differential $d_{ab} : \mathrm{Hom}_{\mathcal{A}_\theta}(E_a, E_b) \to \mathrm{Hom}_{\mathcal{A}_\theta}(E_a, E_b) \cdot d\bar{z}$ is induced naturally by the holomorphic structures of $\mathbb{E}_a$ and $\mathbb{E}_b$. However, when E_a and E_b are Heisenberg modules, the space $\mathrm{Hom}_{\mathcal{A}_\theta}(E_a, E_b)$ is again isomorphic to a Heisenberg module $E_{p_{ab}, q_{ab}, \theta_a}$ with some p_{ab}, q_{ab} and θ_a so that $\mathrm{End}_{\mathcal{A}_\theta}(E_a) \simeq \mathcal{A}_{\theta_a}$, where the differential d_{ab} coincides with the holomorphic structure $\bar{\nabla}_{ab}$ on $E_{p_{ab}, q_{ab}, \theta_a}$

[3] For the relation of the *noncommutative* complex torus description and the usual complex torus description, for instance see [42, 23].

[42, 22] (see [28]). Here, we set $\tau = \mathbf{i}$ for simplicity, and compare the commutative case $\theta = 0$ with the DG category $\mathcal{C}_{DG}(\mathfrak{F}_N)$ in the $\mathbb{R}^2$ case. We concentrate on Heisenberg modules $E_{p,q,0}$ with $q \neq 0$ since otherwise the Heisenberg module (rank zero vector bundle) cannot be defined for $\theta = 0$. We can conclude that, for $p_a/q_a \neq p_b/q_b$, the space $\mathrm{Hom}(\mathbb{E}_a, \mathbb{E}_b)$ is the $|p_{ab}|$ copies of Ω_{ab} in $\mathcal{C}_{DG}(\mathfrak{F}_N)$, where the differentials on both spaces are of the same form under the identification of $d\bar{z}$ with dx. On the other hand, for $p_a/q_a = p_b/q_b$, say $b = a$, one has $\mathrm{Hom}_{\mathcal{A}_{\theta=0}}(E_a, E_a) \simeq \mathrm{End}_{\mathcal{A}_{\theta=0}}(E_a) \simeq \mathcal{A}_{\theta_a}$, which implies that one has the identity morphism $\mathbf{1}_a \in \mathrm{Hom}(\mathbb{E}_a, \mathbb{E}_a)$ and its "dual" degree one element $\mathbf{1}_a \cdot d\bar{z} \in \mathrm{Hom}(\mathbb{E}_a, \mathbb{E}_a)$ as opposed to the case Ω_{aa} of $\mathcal{C}_{DG}(\mathfrak{F}_N)$.

Thus, the structures above form a DG category and, in addition, we have a nondegenerate pairing between $\mathrm{Hom}(\mathbb{E}_a, \mathbb{E}_b)$ and $\mathrm{Hom}(\mathbb{E}_b, \mathbb{E}_a)$ for any $\mathbb{E}_a$ and $\mathbb{E}_b$, which gives a cyclic structure on the DG category. In the case $\mathcal{C}_{DG}(\mathfrak{F}_N)$, clearly the space Ω_{aa} (and the corresponding cohomology $H^\bullet(\Omega_{aa}) \simeq V_{aa}$) cannot have a nondegenerate pairing.

6.2 Examples of finite-dimensional minimal A_∞-algebras

Let us consider a two-torus T^2 whose covering space is $\mathbb{R}^2$ with coordinates $(x, y) \in \mathbb{R}^2$. We have $\pi_{xy}\mathbb{R}^2 = T^2$, $\pi_{xy} := \pi_x\pi_y = \pi_y\pi_x$, where π_x and π_y are the projections associated to the identifications $x \sim x + 1$ and $y \sim y + 1$, respectively.

Let p_a and q_a be relatively prime integers such that $q_a > 0$. Denote $\mu_a := p_a/q_a$, and consider a geodesic cycle $\pi_{xy}(L_a) \in T^2$,

$$L_a : y = \mu_a x + \nu_a, \qquad \nu_a \in \mathbb{R}.$$

One sees that $\pi_{xy}^{-1}\pi_{xy}(L_a)$ is a copy of lines $y = \mu_a x + \nu_a + \mathbb{Z}$ in $\mathbb{R}^2$.[4] We denote by $\mathfrak{F}_N$ the set of N distinct geodesic cycles such that (ii'): more than two objects in $\mathfrak{F}_N$ do not intersect at the same point in T^2.

For two objects $a, b \in \mathfrak{F}_N$, we denote $\mu_{ab} := \mu_b - \mu_a$. If $\mu_{ab} \neq 0$, there exist $|p_a q_b - p_b q_a|$ (transversal) intersection points of $\pi_{xy}(L_a)$ with $\pi_{xy}(L_b)$ in T^2. We denote $L_{ab} := \pi_{xy}(L_a) \cap \pi_{xy}(L_b)$. The degree is attached by $|v_{ab}| = 0$ if $\mu_{ab} > 0$ and $|v_{ab}| = 1$ if $\mu_{ab} < 0$.

Now, let $\mathfrak{a} := (a_1, \ldots, a_{n+1})$, $a_1, \ldots, a_{n+1} \in \mathfrak{F}_N$, be a collection such that $\mu_{a_i a_{i+1}} > 0$ for two of $i \in \{1, \ldots, n+1\}$ and $\mu_{a_i a_{i+1}} < 0$ for other i, where we identify $a_{j+(n+1)}$ with a_j. We call such $\mathfrak{a}$ a *CC-collection*. Consider a sequence $\mathbf{v} := (v_{a_1 a_2}, \ldots, v_{a_n a_{n+1}}, v_{a_{n+1} a_1})$ of intersection points in T^2, where $v_{a_i a_{i+1}} \in L_{a_i a_{i+1}}$, $i \in \mathbb{Z}_{n+1}$. Let $CC'(\mathfrak{a}, \mathbf{v}) = CC'(\mathbf{v})$ be a subset of

$$\pi_{xy}^{-1}(\mathbf{v}) := (\pi_{xy}^{-1}(v_{a_1 a_2}), \ldots, \pi_{xy}^{-1}(v_{a_n a_{n+1}}), \pi_{xy}^{-1}(v_{a_{n+1} a_1}))$$

consisting of all elements $\tilde{v} := (\tilde{v}_{a_1 a_2}, \ldots, \tilde{v}_{a_n a_{n+1}}, v_{a_{n+1} a_1}) \in \pi_{xy}^{-1}(\mathbf{v})$ such that the geodesic interval $[\tilde{v}_{a_{i-1} a_i}, \tilde{v}_{a_i a_{i+1}}]$ is included in $\pi_{xy}^{-1}\pi_{xy}(L_{a_i})$ and

[4] The number ν_a corresponds to α_a/q_a in the general setting [22]. The effect of flat connections β_a there is set to be zero in this article for simplicity.

$0 \leq Angle(\tilde{v}_{a_{i-1}a_i}\tilde{v}_{a_ia_{i+1}}\tilde{v}_{a_{i+1}a_{i+2}}) < \pi$ for any $i \in \mathbb{Z}_{n+1}$. Here, we fix $\tilde{v}_{a_{n+1}a_1} = v_{a_{n+1}a_1} \in \mathbb{R}^2$, that is, we fix an inclusion of the fundamental domain of T^2 to the covering space $\mathbb{R}^2$ and denote the image of $v_{a_{n+1}a_1}$ by the same letter $v_{a_{n+1}a_1}$. We call an element $\tilde{v} \in CC'(\mathbf{v})$ a *CC semi-polygon.* For a CC semi-polygon $\tilde{v} \in CC'(\mathbf{v})$, let $1 \leq i_- < i_+ \leq n+1$ be the pair such that $\mu_{a_ia_{i+1}} > 0$ for $i = i_-$ and $i = i_+$. Then, the left/right extrema of $x(\tilde{v}_{a_ia_{i+1}})$, $i \in \mathbb{Z}_{n+1}$, are given by $i = i_-, i_+$. If $x(\tilde{v}_{a_{i_-}a_{i_-+1}}) = x(\tilde{v}_{a_{i_+}a_{i_++1}})$, we call $\tilde{v}$ a *point.* The set of points in $CC'(\mathbf{v})$ is denoted $P(\mathbf{v})$. If $\tilde{v}$ is not a point, the *sign* $\sigma(\mathbf{v})$ of a CC semi-polygon $\tilde{v}$ is defined by

$$\sigma(\mathbf{v}) := \begin{cases} -1 & x(\tilde{v}_{a_{i_-}a_{i_-+1}}) < x(\tilde{v}_{a_{i_+}a_{i_++1}}) \\ +1 & x(\tilde{v}_{a_{i_+}a_{i_++1}}) < x(\tilde{v}_{a_{i_-}a_{i_-+1}}). \end{cases}$$

Let $L(\mathbf{v}; a_i, a_j)$, $1 \leq i < j \leq n+1$, be the set of all elements $\tilde{v} \in CC'(\mathbf{v})\backslash P(\mathbf{v})$ such that $\tilde{v}_{a_{i-1}a_i} = \tilde{v}_{a_ja_{j+1}}$ and $\tilde{v}_{a_{j-1}a_j} = \tilde{v}_{a_ia_{i+1}}$. If $L(\mathbf{v}; a_i, a_j) \neq \emptyset$, then $a_i = a_j$, $n = 3$ and $j = i + 2$ where we set

$$\zeta^d(\mathbf{v}; a_i, a_j) = \sum_{n \in \mathbb{Z}\backslash\{0\}} \frac{1}{(2\pi\mathbf{i}n)^d} \exp(-2\pi\mathbf{i}nl_i(\tilde{v})), \quad d = 1, 2, \ldots$$

with $l_i(\tilde{v}) := (x(\tilde{v}_{a_ia_{i+1}}) - x(\tilde{v}_{a_{i-1}a_i}))/q_{a_i}$ except that we set $\zeta^d(\mathbf{v}; a_i, a_j) = 0$ when $l_i(\tilde{v}) \in \mathbb{Z}$ and d is odd. If $L(\mathbf{v}; a_i, a_j) = \emptyset$, we set $\zeta^d(\mathbf{v}; a_i, a_j) = 0$. One sees that $\zeta^d(\mathbf{v}; a_i, a_j)$ is independent of the choice of $\tilde{v}$.

Let

$$CC(\mathbf{v}) := CC'(\mathbf{v})\backslash \left(\coprod_{i<j} L(\mathbf{v}; a_i, a_j) \coprod P(\mathbf{v})\right).$$

For a given CC-collection $\mathfrak{a} := (a_1, \ldots, a_{n+1})$, $\mathbf{v} = (v_{a_1a_2}, \ldots, v_{a_na_{n+1}}, v_{a_{n+1}a_1})$ and $\mathbf{b} := (b_1, \ldots, b_{n+1}) \in (\mathbb{Z}_{\geq 0})^{n+1}$, define

$$F^{\mathbf{b}}(\mathbf{v}) := \sum_{\tilde{v} \in CC(\mathbf{v})} (\sigma(\tilde{v}))^{n+b} \left(\prod_{i=1}^{n+1} \frac{|l_i(\tilde{v})|^{b_i}}{(b_i)!}\right) \exp(-Area(\tilde{v})), \qquad (19)$$

where $b := b_1 + \cdots + b_{n+1}$.

Now, we construct a Fukaya A_∞-category $\mathcal{C} = \mathcal{C}(T^2; \mathfrak{F}_N)$. The set $\mathrm{Ob}(\mathcal{C})$ of objects is $\mathfrak{F}_N$. For any $a, b \in \mathfrak{F}_N$, the space of morphisms $\mathrm{Hom}_{\mathcal{C}}(a, b) =: V_{ab}$ is a graded vector space of degree zero and one only. If $\mu_{ab} \neq 0$, we set

$$V_{ab} := \oplus_{v_{ab} \in L_{ab}} \mathbb{R} \cdot v_{ab},$$

where v_{ab}'s are the bases of V_{ab} associated to the intersection points v_{ab}'s. The degree of the bases v_{ab} is zero if $\mu_{ab} > 0$ and one if $\mu_{ab} < 0$. If $\mu_{ab} = 0$, we set $V_{ab} = 0$ for $a \neq b$. For $a = b$, we introduce the base $\mathbf{1}_a$ of V_{aa}^0 and the base $\bar{\mathbf{1}}_a$ of V_{aa}^1 and set

$$V_{aa}^0 := \mathbb{R} \cdot \mathbf{1}_a, \qquad V_{aa}^1 := \mathbb{R} \cdot \bar{\mathbf{1}}_a.$$

Let us define a degree minus one nondegenerate symmetric inner product $\eta : V_{ab} \otimes V_{ba} \to \mathbb{C}$, $a, b \in \mathrm{Ob}(\mathcal{C})$. For $a, b \in \mathrm{Ob}(\mathcal{C})$ such that $\mu_{ab} \neq 0$, we set

$$\eta(v_{ab}, v_{ba}) = 1$$

if $v_{ab} = v_{ba} \in \mathbb{R}^2$ and $\eta(v_{ab}, v_{ba}) = 0$ otherwise. For $a = b \in \mathrm{Ob}(\mathcal{C})$, we set

$$\eta(\mathbf{1}_a, \bar{\mathbf{1}}_a) = \eta(\bar{\mathbf{1}}_a, \mathbf{1}_a) = 1.$$

For any base $v_{ab} \in V_{ab}$, this pairing η defines a dual base which we denote by $(v_{ab})^* \in V_{ba}^{1-|v_{ab}|}$. This means that $(\mathbf{1}_a)^* = \bar{1}_a$ and $(\bar{1}_a)^* = \mathbf{1}_a$ if $a = b$.

Next, for any $n \geq 2$ and $a_1, \ldots, a_{n+1} \in \mathrm{Ob}(\mathcal{C})$, we define a collection $\{\varphi_{n+1}\}_{n\geq 2}$ of multilinear maps

$$\varphi_{n+1} : V_{a_1a_2} \otimes \cdots \otimes V_{a_na_{n+1}} \otimes V_{a_{n+1}a_1} \to \mathbb{R}$$

of degree $(1 - n)$ which satisfies the cyclicity

$$\begin{aligned}&\varphi_{n+1}(w_{a_1a_2}, \ldots, w_{a_na_{n+1}}, w_{a_{n+1}a_1})\\ &\quad = (-1)^{n(|w_{a_{n+1}a_1}|+1)}\varphi_{n+1}(w_{a_{n+1}a_1}, w_{a_1a_2}, \ldots, w_{a_na_{n+1}}, w_{a_{n+1}a_1})\end{aligned}$$

for any homogeneous elements $w_{a_ia_{i+1}} \in V_{a_ia_{i+1}}$, $i \in \mathbb{Z}_{n+1}$.

For $n \geq 3$, if $a_i = a_{i+1}$ and $w_{a_ia_{i+1}} = \mathbf{1}_{a_i}$ for some $i \in \mathbb{Z}_{n+1}$, then we set $\varphi_{n+1}(w_{a_1a_2}, \ldots, w_{a_na_{n+1}}, w_{a_{n+1}a_1}) = 0$. For $n = 2$, for any $a, b \in \mathrm{Ob}(\mathcal{C})$, define a trilinear map $\varphi_3 : V_{ab} \otimes V_{ba} \otimes V_{aa}^0 \to \mathbb{R}$ by

$$\varphi_3(v_{ab}, v_{ba}, \mathbf{1}_a) = \eta(v_{ab}, v_{ba}).$$

Next, for $n \geq 2$ and a given $(b_1, \ldots, b_{n+1}) \in (\mathbb{Z}_{\geq 0})^{\otimes(n+1)}$, assume $\mu_{a_ia_{i+1}} \neq 0$ for $i \in \mathbb{Z}_{n+1}$. We set

$$\varphi_{n+1+b} : (V_{a_1}^1)^{\otimes b_1} \otimes V_{a_1a_2} \otimes (V_{a_2}^1)^{\otimes b_2} \otimes \cdots \otimes (V_{a_{n+1}}^1)^{\otimes b_{n+1}} \otimes V_{a_{n+1}a_1} \to \mathbb{R}$$

by

$$\begin{aligned}&\varphi_{n+1+b}((\bar{\mathbf{1}}_{a_1})^{\otimes b_1}, v_{a_1a_2}, (\bar{\mathbf{1}}_{a_2})^{\otimes b_2}, \ldots, (\bar{\mathbf{1}}_{a_{n+1}})^{\otimes b_{n+1}}, v_{a_{n+1}a_1})\\ &\quad := \sum_{i<j \text{ s.t. } L(\mathbf{v};a_i,a_j)\neq 0} \delta_{b,b_i+b_j} - \frac{b!}{b_i!b_j!}\zeta^{b_i+b_j+1}(\mathbf{v}; a_i, a_j) \ + \ F^{\mathbf{b}}(\mathbf{v})\end{aligned}$$

for $\mathbf{v} = (v_{a_1a_2}, \ldots, v_{a_na_{n+1}}, v_{a_{n+1}a_1})$ if $(a_1, \ldots, a_{n+1})$ is a CC-collection and zero otherwise. These data determine all the cyclic multilinear maps φ_{n+1} for $n \geq 2$.

Define multilinear maps m_n, $n \geq 1$, of degree $(2 - n)$ by $m_1 = 0$ and

$$m_n(w_{a_1a_2}, \ldots, w_{a_na_{n+1}}) := \sum_{v\in L_{a_1a_{n+1}}} \varphi(w_{a_1a_2}, \ldots, w_{a_na_{n+1}}, (v)^*) \cdot v$$

for $n \geq 2$ and $w_{a_ia_{i+1}} \in V_{a_ia_{i+1}}$, $i = 1, \ldots, n$. Let $V := \oplus_{a,b\in\mathfrak{F}_N} V_{ab}$.

Theorem 6.1 *The triple* $(V, \eta, \mathfrak{m})$ *forms a minimal cyclic* A_∞*-algebra.*

Note that the minimal cyclic A_∞-algebra $(V, \eta, \mathfrak{m})$ is unital. The appearance of the terms $|l_i(\tilde{v})|^{b_i}/b_i!$ in Eq. (19) is new, which is due to the effect of $\bar{\mathbf{1}}$ treated as the integration of δ_v over $x(v) \in \mathbb{R}$.

For the construction of $\mathcal{C}(T^2, \mathfrak{F}_N)$, i.e., $(V, \eta, \mathfrak{m})$, we imposed two conditions (i) and (ii') on the configurations of lines. They are just for avoiding additional complicated preparation for the setup; the full version of the Fukaya A_∞-category consisting of all geodesic lines is presented in [26] (see also [28]).

There are other versions of HMS setup. From the physical viewpoint, the HMS we discussed is the duality of tree open strings of A-twisted/B-twisted topological *sigma* models. Other versions deal with topological strings of *Landau–Ginzburg* type [19, 31] also. Recently, for two-tori, the corresponding category of matrix factorizations associated to the B-twisted Landau-Ginzburg model together with an HMS has begun to be studied [14, 33]. See, for instance, [5] for the case of toric Fano varieties.

References

1. Barannikov, S., Kontsevich, M.: Frobenius manifolds and formality of Lie algebras of polyvector fields. Int. Math. Res. Not. **4**, 201–205 (1998) math.AG/9710032
2. Berline, N., Getzler, E., Vergne, M.: Heat kernels and Dirac operators. Grundlehren Text Editions. Springer, Berlin (2004) Corrected reprint of the 1992 original
3. Bott, R.: Morse theory indomitable. Inst. Hautes Études Sci. Publ. Math. 68, 99–114 (1988)
4. Cattaneo, A., Felder, G.: A path integral approach to the Kontsevich quantization formula. Comm. Math. Phys. **212**, 591–611 (2000). math.QA/9902090
5. Cho, C.H.: Products of Floer cohomology of torus fibers in toric Fano manifolds. Comm. Math. Phys. **260**, 613–640 (2005), math.SG/0412414
6. Costello, K.J.: Topological conformal field theories and Calabi-Yau categories. Adv. Math. **210**, 165–214 (2007). math.QA/0412149
7. Fukaya, K.: Morse homotopy, A^∞-category, and Floer homologies. In: Proceedings of GARC Workshop on Geometry and Topology '93 (Seoul, 1993). Lecture Notes in Series, vol. 18, pp. 1–102. Seoul Nat. Univ., Seoul (1993)
8. Fukaya, K.: Mirror symmetry of abelian varieties and multi theta functions. J. Algebr. Geom. **11**, 393–512 (2002)
9. Fukaya, K.: Asymptotic analysis, multivalued Morse theory, and mirror symmetry. In: Graphs and patterns in mathematics and theoretical physics. Proc. Symp. Pure Math., vol. 73, pp. 205–278. Am. Math. Soc., Providence, RI (2005)
10. Fukaya, K., Oh, Y.G.: Zero-loop open strings in the cotangent bundle and Morse homotopy. Asian J. Math. **1**, 96–180 (1997)
11. Fukaya, K., Oh, Y.G., Ohta, H., Ono, K. Lagrangian intersection Floer theory – anomaly and obstruction. AMS/IP Stud. Adv. Math., vol. 46. Am. Math. Soc., Providence, RI (2009)

12. Gerstenhaber, M.: The cohomology structure of an associative ring. Ann. Math. **78**, 267–288 (1963)
13. Gerstenhaber, M.: On the deformation of rings and algebras. Ann. Math. **79**, 59–103 (1964)
14. Govindarajan, S., Jockers, H., Lerche, W., Warner, N.: Tachyon condensation on the elliptic curve. Nucl. Phys. B **765**, 240–286 (2007). hep-th/0512208
15. Gugenheim, V.K.A.M., Stasheff, J.D.: On perturbations and A_∞-structures. Bull. Soc. Math. Belg. **38**, 237–246 (1986)
16. Gugenheim, V.K.A.M., Lambe, L.A., Stasheff, J.D.: Algebraic aspects of Chen's twisting cochain. Illinois J. Math. **34**, 485–502 (1990)
17. Gugenheim, V.K.A.M., Lambe, L.A., Stasheff, J.D.: Perturbation theory in differential homological algebra II. Illinois J. Math. **35**, 357–373 (1991)
18. Harvey, F.R., Lawson, Jr., H.B.: Finite volume flows and Morse theory. Ann. Math. (2) **153**, 1–25 (2001)
19. Hori, K., Vafa, C.: Mirror symmetry. Preprint. hep-th/0002222 (2000)
20. Huebschmann, J., Kadeishvili, T.: Small models for chain algebras. Math. Z. **207**, 245–280 (1991)
21. Kajiura, H.: Kronecker foliation, D1-branes and Morita equivalence of noncommutative two-tori. JHEP **0208**, 050 (2002). hep-th/0207097
22. Kajiura, H.: Homological mirror symmetry on noncommutative two-tori. hep-th/0406233 (2004)
23. Kajiura, H.: Star product formula of theta functions. Lett. Math. Phys. **75**, 279–292 (2006). math.QA/0510307
24. Kajiura, H.: An A_∞-structure for lines in a plane. Int. Math. Res. Not. **20**, 3913–3955 (2009). math.QA/0703164
25. Kajiura, H.: Categories of holomorphic line bundles on higher dimensional noncommutative complex tori. J. Math. Phys. **48**, 053517 (2007). hep-th/0510119
26. Kajiura, H.: Higher theta functions associated to polygons in a torus, in preparation (2007)
27. Kajiura, H.: Noncommutative homotopy algebras associated with open strings. Rev. Math. Phys. **19**(1), 1–99 (2007) Based on doctoral thesis, The Univ. of Tokyo. math.QA/0306332
28. Kajiura, H.: Noncommutative tori and mirror symmetry. Proceedings for the workshop "New development of Operator Algebras", RIMS Kokyuroku 1587: 27–72 (2008)
29. Kajiura, H., Stasheff, J.: Homotopy algebras inspired by classical open-closed string field theory. Comm. Math. Phys. **263**, 553–581 (2006). math.QA/0410291
30. Kajiura, H., Stasheff, J.: Open-closed homotopy algebra in mathematical physics. J. Math. Phys. **47**, 023506 (2006). hep-th/0510118
31. Kapustin, A., Li, Y.: Topological correlators in Landau-Ginzburg models with boundaries. Adv. Theor. Math. Phys. **7**, 727–749 (2004). hep-th/0305136
32. Kimura, T., Stasheff, J., Voronov, A.A.: On operad structures of moduli spaces and string theory. Comm. Math. Phys. **171**, 1–25 (1995). hep-th/9307114
33. Knapp, J., Omer, H.: Matrix factorizations and homological mirror symmetry on the torus. JHEP **03**, 088 (2007). hep-th/0701269
34. Kontsevich, M.: Homological algebra of mirror symmetry. In: Proceedings of the International Congress of Mathematicians. vols. 1, 2 (Zürich, 1994), vol. 184, pp. 120–139. Birkhäuser, MA (1995). math.AG/9411018
35. Kontsevich, M.: Deformation quantization of Poisson manifolds. Lett. Math. Phys. **66**(3), 157–216 (2003)

36. Kontsevich, M., Soibelman, Y.: Homological mirror symmetry and torus fibrations. In: Symplectic geometry and mirror symmetry (Seoul, 2000). World Scientific, River Edge, NJ, pp. 203–263 (2001). math.SG/0011041
37. Kreussler, B.: Homological mirror symmetry in dimension one. In: Advances in algebraic geometry motivated by physics (Lowell, MA, 2000). Contem. Math., vol. 276, pp. 179–198. American Mathematical Society, Providence, RI (2001). math.AG/0012018
38. Lada, T., Stasheff, J.D.: Introduction to sh Lie algebras for physicists. Int. J. Theor. Phys. **32**, 1087–1103 (1993). hep-th/9209099
39. Lazaroiu, C.I.: D-brane categories. Int. J. Mod. Phys. A **18**, 5299 (2003). hep-th/0305095
40. Markl, M., Shnider, S., Stasheff, J.: Operads in Algebra, Topology and Physics. Mathematical Surveys and Monographs, vol. 96. American Mathematical Society, Providence, RI (2002)
41. Polishchuk, A.: A_∞-structures on an elliptic curve. Comm. Math. Phys. **247**, 527 (2004). math.AG/0001048
42. Polishchuk, A., Schwarz, A.: Categories of holomorphic vector bundles on noncommutative two-tori. Comm. Math. Phys. **236**, 135 (2003). math.QA/0211262
43. Polishchuk, A., Zaslow, E.: Categorical mirror symmetry: the elliptic curve. Adv. Theor. Math. Phys. **2**, 443–470 (1998). math.AG/9801119
44. Schlessinger, M., Stasheff, J.: Deformation theory and rational homotopy type. Preprint. University of North Carolina (1979)
45. Schwarz, A.: Theta functions on noncommutative tori. Lett. Math. Phys. **58**, 81–90 (2001). math.QA/0107186
46. Stasheff, J.D.: Homotopy associativity of H-spaces, I. Trans. Am. Math. Soc. **108**, 293–312 (1963)
47. Stasheff, J.D.: Homotopy associativity of H-spaces, II. Trans. Am. Math. Soc. **108**, 313–327 (1963)
48. Stasheff, J.D.: The intrinsic bracket on the deformation complex of an associative algebra. JPAA **89**, 231–235 (1993) Festschrift in Honor of Alex Heller
49. Strominger, A., Yau, A.T., Zaslow, E.: Mirror symmetry is T-duality. Nucl. Phys. B **479**, 243–259 (1996). hep-th/9606040
50. Witten, E.: Supersymmetry and Morse theory. J. Diff. Geom. **17**, 661–692 (1982)
51. Witten, E.: Chern-Simons gauge theory as a string theory. Prog. Math. **133**, 637–678 (1995). hep-th/9207094
52. Zwiebach, B.: Closed string field theory: Quantum action and the Batalin-Vilkovisky master equation. Nucl. Phys. B **390**, 33–152 (1993). hep-th/9206084

Categorification of Acyclic Cluster Algebras: An Introduction

Bernhard Keller

Institut de Mathématiques de Jussieu, UMR 7586 du CNRS, Université Paris 7-Denis Diderot, Case 7012, 2, place Jussieu, 75251 Paris Cedex 05, France, keller@math.jussieu.fr

To Murray Gerstenhaber and Jim Stasheff

Abstract. This is a concise introduction to Fomin–Zelevinsky's cluster algebras and their links with the representation theory of quivers in the acyclic case. We review the definition cluster algebras (geometric, without coefficients), construct the cluster category and present the bijection between cluster variables and rigid indecomposable objects of the cluster category.

Key words: Cluster algebra, Associahedron, Quiver representation, Triangulated category, Hall algebra

MSC 2010 classification: 18E30, 16S99

1 Introduction

1.1 Context

Cluster algebras were invented by S. Fomin and A. Zelevinsky [27] in the spring of 2000 in a project whose aim it was to develop a combinatorial approach to the results obtained by G. Lusztig concerning total positivity in algebraic groups [54] on the one hand and canonical bases in quantum groups [53] on the other hand (let us stress that canonical bases were discovered independently and simultaneously by M. Kashiwara [45]). Despite great progress during the last few years [28], [9], [31], we are still relatively far from these initial aims. Presently, the best results on the link between cluster algebras and canonical bases are probably those of C. Geiss, B. Leclerc and J. Schröer [38], [39], [35] but even they cannot construct canonical bases from cluster variables for the moment. Despite these difficulties, the theory of cluster algebras has witnessed spectacular growth thanks notably to the many links that have been discovered with a wide range of subjects including:

A.S. Cattaneo et al. (eds.), *Higher Structures in Geometry and Physics*, Progress in Mathematics 287, DOI 10.1007/978-0-8176-4735-3_11,

- Poisson geometry [40], [41] ...,
- Integrable systems [30] ...,
- Higher Teichmüller spaces [20], [22], [23] [21] ...,
- Combinatorics and the study of combinatorial polyhedra like the Stasheff associahedra [18], [17], [52], [25], [58], [26] ...,
- Commutative and non-commutative algebraic geometry, in particular the study of stability conditions in the sense of Bridgeland [11], [10], Calabi–Yau algebras [42], Donaldson–Thomas invariants [66], [49], [50], [51] ...,
- And Last but not least the representation theory of quivers and finite-dimensional algebras, cf. for example the surveys [4], [64], [62].

We refer to the introductory papers [68], [29], [69], [70], [71] and to the cluster algebras portal [24] for more information on cluster algebras and their links with other parts of mathematics.

The link between cluster algebras and quiver representations follows the spirit of categorification: One tries to interpret cluster algebras as combinatorial (perhaps K-theoretic) invariants associated with categories of representations. Thanks to the rich structure of these categories, one can then hope to prove results on cluster algebras which seem beyond the scope of the purely combinatorial methods. It turns out that the link becomes especially beautiful if we use a *triangulated category* constructed from the category of quiver representations, the so-called cluster category.

In this brief survey, we will review the definition of cluster algebras and Fomin–Zelevinsky's classification theorem for cluster-finite cluster algebras [28]. We will then recall some basic notions on the representations of a quiver without oriented cycles, introduce the cluster category and describe its link with the cluster algebra.

2 Cluster algebras

The cluster algebras we will be interested in are associated with antisymmetric matrices with integer coefficients. Instead of using matrices, we will use quivers (without loops and 2-cycles), since they are easy to visualize and well-suited to our later purposes.

2.1 Quivers

Let us recall that a *quiver* Q is an oriented graph. Thus, it is a quadruple given by a set Q_0 (the set of vertices), a set Q_1 (the set of arrows) and two maps $s : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$ which take an arrow to its source respectively its target. Our quivers are "abstract graphs" but in practice we draw them as in this example:

$$Q: \quad \begin{tikzcd} & 3 \arrow[dl, "\lambda"'] & & 5 \arrow[loop left, "\alpha"] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & 6 \\ 1 \arrow[rr, "\nu"'] & & 2 \arrow[ul, "\mu"'] \arrow[r, shift left, "\beta"] & 4. \arrow[l, shift left, "\gamma"] & \end{tikzcd}$$

A *loop* in a quiver Q is an arrow α whose source coincides with its target; a 2-*cycle* is a pair of distinct arrows $\beta \neq \gamma$ such that the source of β equals the target of γ and vice versa. It is clear how to define 3-*cycles*, *connected components*, ... A quiver is *finite* if both its set of vertices and its set of arrows are finite.

2.2 Seeds and mutations

Fix an integer $n \geq 1$. A *seed* is a pair (R, u), where

(a) R is a finite quiver without loops or 2-cycles with vertex set $\{1, \ldots, n\}$;
(b) u is a free generating set $\{u_1, \ldots, u_n\}$ of the field $\mathbb{Q}(x_1, \ldots, x_n)$ of fractions of the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$ in n indeterminates.

Notice that in the quiver R of a seed, all arrows between any two given vertices point in the same direction (since R does not have 2-cycles). Let (R, u) be a seed and k a vertex of R. The *mutation* $\mu_k(R, u)$ of (R, u) at k is the seed (R', u'), where:

(a) R' is obtained from R as follows:
  (1) Reverse all arrows incident with k;
  (2) For all vertices $i \neq j$ distinct from k, modify the number of arrows between i and j as follows:

R	R'
$\begin{tikzcd} i \arrow[rr, "r"] \arrow[dr, "s"'] & & j \\ & k \arrow[ur, "t"'] & \end{tikzcd}$	$\begin{tikzcd} i \arrow[rr, "r+st"] & & j \arrow[dl, "t"] \\ & k \arrow[ul, "s"] & \end{tikzcd}$
$\begin{tikzcd} i \arrow[rr, "r"] & & j \arrow[dl, "t"] \\ & k \arrow[ul, "s"] & \end{tikzcd}$	$\begin{tikzcd} i \arrow[rr, "r-st"] \arrow[dr, "s"'] & & j \\ & k \arrow[ur, "t"'] & \end{tikzcd}$

where r, s, t are nonnegative integers, an arrow $i \xrightarrow{l} j$ with $l \geq 0$ means that l arrows go from i to j and an arrow $i \xrightarrow{l} j$ with $l \leq 0$ means that $-l$ arrows go from j to i.

(b) u' is obtained from u by replacing the element u_k with

$$u'_k = \frac{1}{u_k} \left(\prod_{\text{arrows } i \to k} u_i + \prod_{\text{arrows } k \to j} u_j \right). \tag{1}$$

In the *exchange relation* (1), if there are no arrows from i with target k, the product is taken over the empty set and equals 1. It is not hard to see that $\mu_k(R, u)$ is indeed a seed and that μ_k is an involution: we have $\mu_k(\mu_k(R, u)) = (R, u)$.

2.3 Examples of mutations

Let R be the cyclic quiver

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 2 \arrow[ur] & & 3 \arrow[ll] \end{tikzcd} \tag{2}$$

and $u = \{x_1, x_2, x_3\}$. If we mutate at $k = 1$, we obtain the quiver

$$\begin{tikzcd} & 1 \arrow[dl] & \\ 2 & & 3 \arrow[ul] \end{tikzcd}$$

and the set of fractions given by $u_1' = (x_2 + x_3)/x_1$, $u_2' = u_2 = x_2$ and $u_3' = u_3 = x_3$. Now, if we mutate again at 1, we obtain the original seed. This is a general fact: Mutation at k is an involution. If, on the other hand, we mutate (R', u') at 2, we obtain the quiver

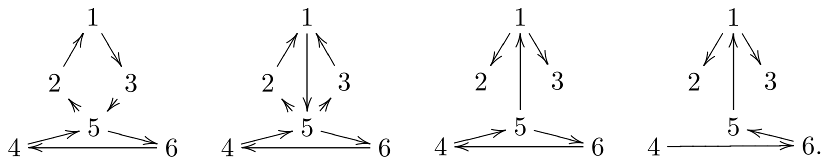
$$\begin{tikzcd} & 1 & \\ 2 \arrow[ur] & & 3 \arrow[ul] \end{tikzcd}$$

and the set u'' given by $u_1'' = u_1' = (x_2 + x_3)/x_1$, $u_2' = \frac{x_1+x_2+x_3}{x_1 x_2}$ and $u_3'' = u_3' = x_3$.

Let us consider the following, more complicated quiver glued together from four 3-cycles:

$$\begin{tikzcd} & & 1 \arrow[dr] & & \\ & 2 \arrow[ur] \arrow[dr] & & 3 \arrow[ll] \arrow[dr] & \\ & & 5 \arrow[ur] \arrow[dll] & & \\ 4 \arrow[uur] & & & & 6. \arrow[ull] \end{tikzcd} \tag{3}$$

If we successively perform mutations at the vertices 5, 3, 1 and 6, we obtain the sequence of quivers (we use [46])

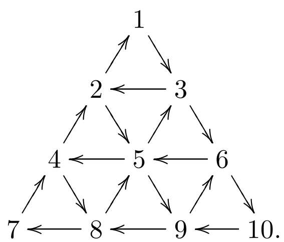


Notice that the last quiver no longer has any oriented cycles and is in fact an orientation of the Dynkin diagram of type D_6. The sequence of new fractions appearing in these steps is

$$u_5' = \frac{x_3x_4 + x_2x_6}{x_5}, \quad u_3' = \frac{x_3x_4 + x_1x_5 + x_2x_6}{x_3x_5},$$
$$u_1' = \frac{x_2x_3x_4 + x_3^2x_4 + x_1x_2x_5 + x_2^2x_6 + x_2x_3x_6}{x_1x_3x_5}, \quad u_6' = \frac{x_3x_4 + x_4x_5 + x_2x_6}{x_5x_6}.$$

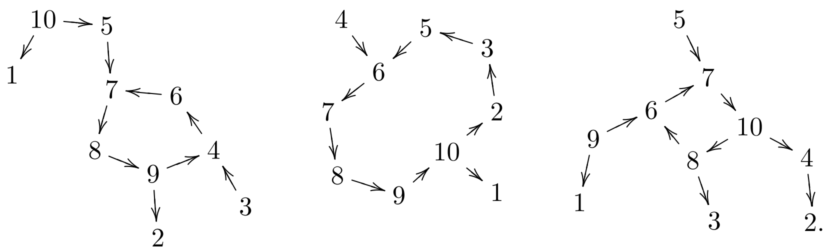
It is remarkable that all the denominators appearing here are monomials and that all the coefficients in the numerators are positive.

Finally, let us consider the quiver

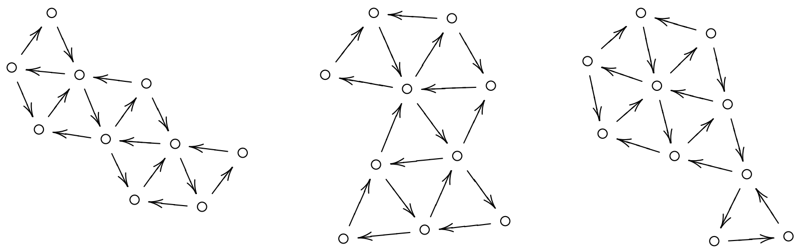


(4)

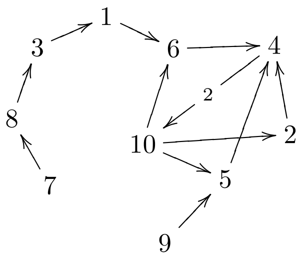
One can show [48] that it is impossible to transform it into a quiver without oriented cycles by a finite sequence of mutations. However, its mutation class (the set of all quivers obtained from it by iterated mutations) contains many quivers with just one oriented cycle, for example



In fact, in this example, the mutation class is finite and it can be completely computed using, for example, [46]: It consists of 5739 quivers up to isomorphism. The above quivers are members of the mutation class containing relatively few arrows. The initial quiver is the unique member of its mutation class with the largest number of arrows. Here are some other quivers in the mutation class with a relatively large number of arrows:



Only 84 among the 5739 quivers in the mutation class contain double arrows (and none contain arrows of multiplicity ≥ 3). Here is a typical example:



The quivers (2), (3) and (4) are part of a family which appears in the study of the cluster algebra structure on the coordinate algebra of the subgroup of upper unitriangular matrices in $SL_n(\mathbb{C})$, cf. [39]. The study of coordinate algebras on varieties associated with reductive algebraic groups (in particular, double Bruhat cells) has provided a major impetus for the development of cluster algebras, cf. [9].

2.4 Definition of cluster algebras

Let Q be a finite quiver without loops or 2-cycles with vertex set $\{1, \dots, n\}$. Consider the seed (Q, x) consisting of Q and the set x formed by the variables $x_1, \dots, x_n$. Following [27] we define:

- The *clusters with respect to* Q to be the sets u appearing in seeds (R, u) obtained from (Q, x) by iterated mutation,
- The *cluster variables* for Q to be the elements of all clusters,
- The *cluster algebra* $\mathcal{A}_Q$ to be the $\mathbb{Q}$-subalgebra of the field $\mathbb{Q}(x_1, \dots, x_n)$ generated by all the cluster variables.

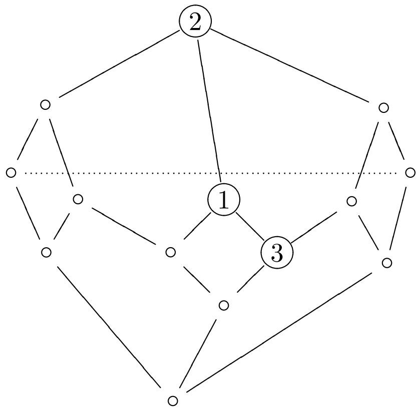
Thus the cluster algebra consists of all $\mathbb{Q}$-linear combinations of monomials in the cluster variables. It is useful to define another combinatorial object associated with this recursive construction: The *exchange graph* associated with Q is the graph whose vertices are the seeds modulo simultaneous renumbering of the vertices and the associated cluster variables and whose edges correspond to mutations.

2.5 The example A_3

Let us consider the quiver

$$Q: \ 1 \longrightarrow 2 \longrightarrow 3$$

obtained by endowing the Dynkin diagram A_3 with a linear orientation. By applying the recursive construction to the initial seed (Q, x) one finds exactly fourteen seeds (modulo simultaneous renumbering of vertices and cluster variables). These are the vertices of the exchange graph, which is isomorphic to the third Stasheff associahedron [65], [18]:



The vertex labeled 1 corresponds to (Q, x), the vertex 2 to $\mu_2(Q, x)$, which is given by

$$\begin{tikzcd} 1 \arrow[rr, bend left] & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd} \ , \ \{x_1, \frac{x_1 + x_3}{x_2}, x_3\} \ ,$$

and the vertex 3 to $\mu_1(Q, x)$, which is given by

$$1 \longleftarrow 2 \longrightarrow 3 \ , \ \{\frac{1 + x_3}{x_1}, x_2, x_3\}.$$

We find a total of 9 cluster variables, namely,

$$x_1 \ , \ x_2 \ , \ x_3, \frac{1 + x_2}{x_1} \ , \ \frac{x_1 + x_3 + x_2 x_3}{x_1 x_2} \ , \ \frac{x_1 + x_1 x_2 + x_3 + x_2 x_3}{x_1 x_2 x_3} \ ,$$
$$\frac{x_1 + x_3}{x_2} \ , \ \frac{x_1 + x_1 x_2 + x_3}{x_2 x_3} \ , \ \frac{1 + x_2}{x_3} \ .$$

Again we observe that all denominators are monomials. Notice also that $9 = 3 + 6$ and that 3 is the rank of the root system associated with A_3 and 6 its number of positive roots. Moreover, if we look at the denominators of the nontrivial cluster variables (those other than x_1, x_2, x_3), we see a natural bijection with the positive roots

$$\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2, \alpha_2 + \alpha_3, \alpha_3$$

of the root system of A_3, where α_1, α_2, α_3 denote the three simple roots.

2.6 Cluster algebras with finitely many cluster variables

The phenomena observed in the above example are explained by the following key theorem:

Theorem 1 (Fomin–Zelevinsky [28]). *Let Q be a finite connected quiver without loops or 2-cycles with vertex set $\{1, \ldots, n\}$. Let $\mathcal{A}_Q$ be the associated cluster algebra:*

(a) All cluster variables are Laurent polynomials, i.e., their denominators are monomials.
(b) The number of cluster variables is finite iff Q is mutation equivalent to an orientation of a simply laced Dynkin diagram Δ. In this case, Δ is unique and the nontrivial cluster variables are in bijection with the positive roots of Δ; namely, if we denote the simple roots by $\alpha_1, \ldots, \alpha_n$, then for each positive root $\sum d_i\alpha_i$, there is a unique nontrivial cluster variable whose denominator is $\prod x_i^{d_i}$.

3 Categorification

We refer to the books [63], [33], [2] and [1] for a wealth of information on the representation theory of quivers and finite-dimensional algebras. Here, we will only need very basic notions.

Let Q be a finite quiver without oriented cycles. For example, Q can be an orientation of a simply laced Dynkin diagram or the quiver

$$\begin{tikzcd} & 2 \arrow[dr, "\beta"] & \\ 1 \arrow[ur, "\alpha"] \arrow[rr, "\gamma"'] & & 3. \end{tikzcd}$$

Let k be an algebraically closed field. Recall that a *representation of Q* is a diagram of finite-dimensional vector spaces of the shape given by Q. Thus, in the above example, a representation of Q is a (not necessarily commutative) diagram

$$\begin{tikzcd} & V_2 \arrow[dr, "V_\beta"] & \\ V_1 \arrow[ur, "V_\alpha"] \arrow[rr, "V_\gamma"'] & & V_3 \end{tikzcd}$$

formed by three finite-dimensional vector spaces and three linear maps. A *morphism of representations* is a morphism of diagrams. We thus obtain the *category of representations* $\mathsf{rep}(Q)$. Notice that it is an abelian category (since it is a category of diagrams in an abelian category, that of finite-dimensional vector spaces): Sums, kernels and cokernels in the category $\mathsf{rep}(Q)$ are computed componentwise. We denote by $\mathcal{D}_Q$ its bounded derived category. Thus, the objects of $\mathcal{D}_Q$ are the bounded complexes

$$\cdots \longrightarrow V^p \xrightarrow{d} V^{p+1} \longrightarrow \cdots$$

of representations and its morphisms are obtained from morphisms of complexes by formally inverting all quasi-isomorphisms (=morphisms inducing isomorphisms in homology). The category $\mathcal{D}_Q$ is still an additive category (direct sums are given by direct sums of complexes) but it is almost never abelian. In fact, it is abelian if and only if Q does not have any arrows. But it is always *triangulated*. This means that $\mathcal{D}_Q$ is additive and endowed with

(a) A *suspension functor* $\Sigma : \mathcal{D}_Q \xrightarrow{\sim} \mathcal{D}_Q$, namely, the functor taking a complex V to $V[1]$, where $V[1]^p = V^{p+1}$ for all $p \in \mathbb{Z}$ and $d_{V[1]} = -d_V$;
(b) A class of *triangles*, namely, the sequences

$$U \longrightarrow V \longrightarrow W \longrightarrow \Sigma U$$

which are "induced" from short exact sequences of complexes.

The triangulated category $\mathcal{D}_Q$ admits a *Serre functor*, i.e., an autoequivalence $S : \mathcal{D}_Q \xrightarrow{\sim} \mathcal{D}_Q$ which makes the Serre duality formula true: We have

$$D\,\mathsf{Hom}(X,Y) \xrightarrow{\sim} \mathsf{Hom}(Y,SX)$$

bifunctorially in X, Y belonging to $\mathcal{D}_Q$, where D denotes the duality functor $\mathsf{Hom}_k(?,k)$ over the ground field k. The *cluster category* is defined as the *orbit category*

$$\mathcal{C}_Q = \mathcal{D}_Q/(S^{-1} \circ \Sigma^2)^{\mathbb{Z}}$$

of $\mathcal{D}_Q$ under the action of the cyclic group generated by the automorphism $S^{-1} \circ \Sigma^2$. Thus, its objects are the same as those of $\mathcal{D}_Q$ and its morphisms are defined by

$$\mathsf{Hom}_{\mathcal{C}_Q}(X,Y) = \bigoplus_{p\in\mathbb{Z}} \mathsf{Hom}_{\mathcal{D}_Q}(X,(S^{-1} \circ \Sigma^2)^p Y).$$

One can show [47] that $\mathcal{C}_Q$ admits a structure of triangulated category such that the projection functor $\mathcal{D}_Q \to \mathcal{C}_Q$ becomes a triangle functor (in general, the orbit category of a triangulated category under the action of an automorphism group is no longer triangulated). It is not hard to see that the cluster category has finite-dimensional morphism spaces, and that it admits a Serre functor induced by that of the derived category. The definition of the cluster category then immediately yields an isomorphism

$$S \xrightarrow{\sim} \Sigma^2$$

and this means that $\mathcal{C}_Q$ is 2-Calabi–Yau: A k-linear triangulated category with finite-dimensional morphism spaces is d-Calabi–Yau if it admits a Serre functor S and if S is isomorphic to Σ^d (the dth power of the suspension functor) as a triangle functor. The definition of the cluster category is due to Buan–Marsh–Reineke–Reiten–Todorov [5] (for arbitrary Q without oriented cycles) and, independently and with a very different, more geometric description, to Caldero–Chapoton–Schiffler [13] (for Q of type A_n).

To state the close relationship between the cluster category $\mathcal{C}_Q$ and the cluster algebra $\mathcal{A}_Q$, we need some notation: For two objects L and M of $\mathcal{C}_Q$, we write

$$\mathsf{Ext}^1(L, M) = \mathsf{Hom}_{\mathcal{C}_Q}(L, \Sigma M).$$

Notice that it follows from the Calabi–Yau property that we have a canonical isomorphism

$$\mathsf{Ext}^1(L, M) \xrightarrow{\sim} D\,\mathsf{Ext}^1(M, L).$$

An object L of $\mathcal{C}_Q$ is *rigid* if we have $\mathsf{Ext}^1(L, L) = 0$. It is *indecomposable* if it is nonzero and in each decomposition $L = L_1 \oplus L_2$, we have $L_1 = 0$ or $L_2 = 0$.

Theorem 2 ([15]). *Let Q be a finite quiver without oriented cycles with vertex set $\{1, \ldots, n\}$.*

(a) There is an explicit bijection $L \mapsto X_L$ from the set of isomorphism classes of rigid indecomposables of the cluster category $\mathcal{C}_Q$ onto the set of cluster variables of the cluster algebra $\mathcal{A}_Q$.

(b) Under this bijection, the clusters correspond exactly to the cluster-tilting subsets, *i.e., the sets $T_1, \ldots, T_n$ of rigid indecomposables such that*

$$Ext^1(T_i, T_j) = 0$$

for all i, j.

(c) If L and M are rigid indecomposables such that the space $\mathsf{Ext}^1(L, M)$ is one-dimensional, then we have the generalized exchange relation

$$X_L = \frac{X_B + X_{B'}}{X_M} \tag{5}$$

where B and B' are the middle terms of "the" nonsplit triangles

$$L \longrightarrow B \longrightarrow M \longrightarrow \Sigma L \quad \text{and} \quad M \longrightarrow B' \longrightarrow L \longrightarrow \Sigma M$$

and we define

$$X_B = \prod_{i=1}^{s} X_{B_i}\,,$$

where $B = B_1 \oplus \cdots \oplus B_s$ is a decomposition into indecomposables.

The relation (5) in part (c) of the theorem can be generalized to the case where the extension group is of higher dimension, cf. [14], [43], [67]. One can show using [6] that relation (5) generalizes the exchange relation (1) which appeared in the definition of the mutation.

The proof of the theorem builds on work by many authors notably Buan–Marsh–Reiten–Todorov [7], Buan–Marsh–Reiten [8], Buan–Marsh–Reineke–Reiten–Todorov [5], Marsh–Reineke–Zelevinsky [57], ... and especially on

Caldero–Chapoton's explicit formula for X_L proved in [12] for orientations of simply laced Dynkin diagrams. We include the formula below. Another crucial ingredient of the proof is the Calabi–Yau property of the cluster category. An alternative proof of part (c) was given by A. Hubery [43] for quivers whose underlying graph is an extended simply laced Dynkin diagram.

The theorem does shed new light on cluster algebras. In particular, we have the following

Corollary 1 (Qin [61], Nak [59]). *Suppose that Q does not have oriented cycles. Then all cluster variables of $\mathcal{A}_Q$ belong to $\mathbb{N}[x_1^{\pm}, \ldots, x_n^{\pm}]$.*

This settles a conjecture of Fomin–Zelevinsky [27] in the case of cluster algebras associated with acyclic quivers. The proof is based on Lusztig's [55] and in this sense it does not quite live up to the hopes that cluster theory ought to explain Lusztig's results. However, it does show that the conjecture is true for this important class of cluster algebras.

4 Caldero–Chapoton's formula

We describe the bijection of part (a) of Theorem 2. Let k be an algebraically closed field and Q a finite quiver without oriented cycles with vertex set $\{1, \ldots, n\}$. Let L be an object of the cluster category $\mathcal{C}_Q$. With L, we will associate an element X_L of the field $\mathbb{Q}(x_1, \ldots, x_n)$. According to [5], the object L decomposes into a sum of indecomposables L_i, $1 \leq i \leq s$, unique up to isomorphism and permutation. By defining

$$X_L = \prod_{i=1}^{s} X_{L_i}$$

we reduce to the case where L is indecomposable. Now again by [5], if L is indecomposable, it is either isomorphic to an object $\pi(V)$, or an object $\Sigma\pi(P_i)$, where $\pi : \mathcal{D}_Q \to \mathcal{C}_Q$ is the canonical projection functor, Σ is the suspension functor of $\mathcal{C}_Q$, V is a representation of Q (identified with a complex of representations concentrated in degree 0) and P_i is the projective representation associated with a vertex i (P_i is characterized by the existence of a functorial isomorphism

$$\mathsf{Hom}(P_i, W) = W_i$$

for each representation W). If L is isomorphic to $\Sigma\pi(P_i)$, we put $X_L = x_i$. If L is isomorphic to $\pi(V)$, we define

$$X_L = X_V = \frac{1}{\prod_{i=1}^{n} x_i^{d_i}} \sum_{0 \leq e \leq d} \chi(\mathrm{Gr}_e(V)) \prod_{i=1}^{n} x_i^{\sum_{j \to i} e_j + \sum_{i \to j} (d_j - e_j)},$$

where $d_i = \dim V_i$, $1 \leq i \leq n$, the sum is taken over all elements $e \in \mathbb{N}^n$ such that $0 \leq e_i \leq d_i$ for all i, the *quiver Grassmannian* $\mathrm{Gr}_e(V)$ is the variety of n-tuples of subspaces $U_i \subset V_i$ such that $\dim U_i = e_i$ and the U_i form a subrepresentation of V, the Euler characteristic χ is taken with respect to étale cohomology (or with respect to singular cohomology with coefficients in a field if $k = \mathbb{C}$) and the sums in the exponent of x_i are taken over all arrows $j \to i$ respectively all arrows $i \to j$. This formula was invented by P. Caldero and F. Chapoton in [12] for the case of a quiver whose underlying graph is a simply laced Dynkin diagram. It is still valid for arbitrary quivers without oriented cycles [15] and further generalizes to arbitrary triangulated 2-Calabi–Yau categories containing a cluster-tilting object [60].

5 Some further developments

The extension of the results presented here to quivers containing oriented cycles is the subject of ongoing research. In a series of papers [38], [34], [39], [35], [36], Geiss–Leclerc–Schröer have obtained remarkable results for a class of quivers which are important in the study of (dual semi-)canonical bases. They use an analogue [37] of the Caldero–Chapoton map due ultimately to Lusztig [56]. The class they consider has been further enlarged by Buan–Iyama–Reiten–Scott [3]. Thanks to their results, an analogue of Caldero–Chapoton's formula and a version of Theorem 2 was proved in [32] for an even larger class.

Building on [57] Derksen–Weyman–Zelevinsky are developing a representation-theoretic model for mutation of general quivers in [19]. Their approach is related to Kontsevich-Soibelman's work [51], where 3-Calabi–Yau categories play an important rôle, as was already the case in [44].

References

1. Assem, I., Simson, D., Skowroński, A.: Elements of the representation theory of associative algebras. Techniques of representation theory, vol. 1, London Mathematical Society Student Texts, vol. 65. Cambridge University Press, Cambridge (2006)
2. Auslander, M., Reiten, I., Smalø, S.: Representation theory of Artin algebras. Cambridge Studies in Advanced Mathematics, vol. 36. Cambridge University Press, Cambridge (1995) (English)
3. Bakke Buan, A., Iyama, O., Reiten, I., Scott, J.: Cluster structures for 2-Calabi-Yau categories and unipotent groups. Compos. Math. **145**(4), 1035–1079 (2009)
4. Bakke Buan, A., Marsh, R.J.: Cluster-tilting theory. Trends in representation theory of algebras and related topics. Contemp. Math., vol. 406, pp. 1–30. American Mathematical Society, Providence, RI (2006)
5. Bakke Buan, A., Marsh, R.J., Reineke, M., Reiten, I., Todorov, G.: Tilting theory and cluster combinatorics. Adv. Math. **204**(2), 572–618 (2006)
6. Bakke Buan, A., Marsh, R.J., Reiten, I.: Cluster-tilted algebras. Trans. Am. Math. Soc. **359**(1), 323–332 (2007) electronic

7. Bakke Buan, A., Marsh, R.J., Reiten, I., Todorov, G.: Clusters and seeds in acyclic cluster algebras. Proc. Amer. Math. Soc. **135**(10), 3049–3060 (2007) (electronic). With an appendix coauthored in addition by P. Caldero and B. Keller
8. Bakke Buan, A., Marsh, R.J., Reiten, I.: Cluster mutation via quiver representations. Comment. Math. Helv. **83**(1), 143–177 (2008)
9. Berenstein, A., Fomin, S., Zelevinsky, A.: Cluster algebras. III. Upper bounds and double Bruhat cells. Duke Math. J. **126**(1), 1–52 (2005)
10. Bridgeland, T.: Stability conditions and Hall algebras. Talk at the meeting Recent developments in Hall algebras. Luminy, November 2006
11. Bridgeland, T.: Stability conditions on triangulated categories. Ann. Math. arXiv:math.AG/0212237 (to appear)
12. Caldero, P., Chapoton, F.: Cluster algebras as Hall algebras of quiver representations. Comment. Math. Helv. **81**(3), 595–616 (2006)
13. Caldero, P., Chapoton, F., Schiffler, R.: Quivers with relations arising from clusters (A_n case). Trans. Am. Math. Soc. **358**(5), 1347–1364 (2006)
14. Caldero, P., Keller, B.: From triangulated categories to cluster algebras. Inv. Math. **172**, 169–211 (2008)
15. Caldero, P., Keller, B.: From triangulated categories to cluster algebras. II. Ann. Sci. École Norm. Sup. (4) **39**(6), 983–1009 (2006)
16. Caldero, P., Reineke, M.: On the quiver Grassmannian in the acyclic case. J. Pure Appl. Algebra **212**(11), 2369–2380 (2008)
17. Chapoton, F.: Enumerative properties of generalized associahedra. Sém. Lothar. Combin. **51**, 16, Art. B51b (2004–2005) (electronic)
18. Chapoton, F., Fomin, S., Zelevinsky, A.: Polytopal realizations of generalized associahedra. Canad. Math. Bull. **45**(4), 537–566 (2002). Dedicated to Robert V. Moody
19. Derksen, H., Weyman, J., Zelevinsky, A.: Quivers with potentials and their representations I: Mutations. Selecta Mathematica **14**, 59–119 (2008)
20. Fock V.V., Goncharov, A.B.: Cluster ensembles, quantization and the dilogarithm. Annales scientifiques de l'ENS **42**(6), 865–930 (2009)
21. Fock V.V., Goncharov, A.B.: Cluster ensembles, quantization and the dilogarithm II: The intertwiner. In Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I, Progr. Math., vol. 269, pp. 655–673. Birkhäuser, Boston (2009)
22. Fock V.V., Goncharov, A.B.: Cluster $\mathcal{X}$-varieties, amalgamation, and Poisson-Lie groups. In: Algebraic geometry and number theory, Progr. Math., vol. 253, pp. 27–68. Birkhäuser, Boston (2006)
23. Fock V.V., Goncharov, A.B.: The quantum dilogarithm and representations of quantum cluster varieties. Invent. Math. **175**(2), 223–286 (2009)
24. Fomin, S.: Cluster algebras portal. `www.math.lsa.umich.edu/~fomin/cluster.html`
25. Fomin, S., Reading, N.: Generalized cluster complexes and Coxeter combinatorics. Int. Math. Res. Not. **44**, 2709–2757 (2005)
26. Fomin, S., Shapiro, M., Thurston, D.: Cluster algebras and triangulated surfaces. Part I: Cluster complexes. arXiv:math/0608367
27. Fomin, S., Zelevinsky, A.: Cluster algebras. I. Foundations, J. Am. Math. Soc. **15**(2), 497–529 (2002) (electronic)
28. Fomin, S., Zelevinsky, A.: Cluster algebras. II. Finite type classification. Invent. Math. **154**(1), 63–121 (2003)

29. Fomin, S., Zelevinsky, A.: Cluster algebras: notes for the CDM-03 conference. Current developments in mathematics, pp. 1–34. Int. Press, Somerville, MA (2003)
30. Fomin, S., Zelevinsky, A.: Y-systems and generalized associahedra, Ann. Math. (2) **158**(3), 977–1018 (2003)
31. Fomin, S., Zelevinsky, A.: Cluster algebras IV: coefficients. Compos. Math. **143**, 112–164 (2007)
32. Fu, C., Keller, B.: On cluster algebras with coefficients and 2-Calabi-Yau categories. arXiv:0710.3152 [math.RT]
33. Gabriel, P., Roiter, A.V.: Representations of finite-dimensional algebras. Encyclopaedia Math. Sci., vol. 73. Springer, Berlin (1992)
34. Geiß, C., Leclerc, B., Schröer, J.: Auslander algebras and initial seeds for cluster algebras. J. LMS. arXiv:math.RT/0506405 (to appear)
35. Geiß, C., Leclerc, B., Schröer, J.: Partial flag varieties and preprojective algebras. Ann. Fourier. arXiv:math.RT/0609138 (to appear)
36. Geiß, C., Leclerc, B., Schröer, J.: Rigid modules over preprojective algebras II: The Kac-Moody case. arXiv:math.RT/0703039
37. Geiß, C., Leclerc, B., Schröer, J.: Semicanonical bases and preprojective algebras II: A multiplication formula. Compos. Math. arXiv:math.RT/0509483 (to appear)
38. Geiß, C., Leclerc, B., Schröer, J.: Semicanonical bases and preprojective algebras. Ann. Sci. École Norm. Sup. (4) **38**(2), 193–253 (2005)
39. Geiß, C.: Leclerc, B., Schröer, J.: Rigid modules over preprojective algebras, Invent. Math. **165**(3), 589–632 (2006)
40. Gekhtman, M., Shapiro, M., Vainshtein, A.: Cluster algebras and Poisson geometry, Mosc. Math. J. **3**(3), 899–934 (2003) 1199 (Dedicated to Vladimir Igorevich Arnold on the occasion of his 65th birthday)
41. Gekhtman, M., Shapiro, M., Vainshtein, A.: Cluster algebras and Weil-Petersson forms. Duke Math. J. **127**(2), 291–311 (2005)
42. Ginzburg, V.: Calabi-Yau algebras. arXiv:math/0612139v3 [math.AG]
43. Hubery, A.: Acyclic cluster algebras via Ringel-Hall algebras. Preprint available at the author's home page
44. Iyama, O., Reiten, I.: Fomin-Zelevinsky mutation and tilting modules over Calabi-Yau algebras. Am. J. Math. arXiv:math.RT/0605136 (to appear)
45. Kashiwara, M.: Bases cristallines. C. R. Acad. Sci. Paris Sér. I Math. **311**(6), 277–280 (1990)
46. Keller, B.: Quiver mutation in java. Interactive computer program, July 2006, available at the author's homepage
47. Keller, B.: On triangulated orbit categories. Doc. Math. **10**, 551–581 (2005)
48. Keller, B., Reiten, I.: Acyclic Calabi-Yau categories are cluster categories. Preprint, with an appendix by Michel Van den Bergh (2006). arXiv:math.RT/0610594
49. Kontsevich, M.: Donaldson-Thomas invariants. Mathematische Arbeitstagung. June 22–28, 2007, MPI Bonn, www.mpim-bonn.mpg.de/preprints/
50. Kontsevich, M.: Donaldson-Thomas invariants, stability conditions and cluster transformations. Report on joint work with Y. Soibelman, talks at the IHES, October 11 and 12 (2007)
51. Kontsevich, M., Soibelman, Y.: Stability structures, Donaldson-Thomas invariants and cluster transformations. arXiv:0811.2435

52. Krattenthaler, C.: The F-triangle of the generalised cluster complex. Topics in discrete mathematics, Algorithms Combin., vol. 26, pp. 93–126. Springer, Berlin (2006)
53. Lusztig, G.: Canonical bases arising from quantized enveloping algebras. J. Am. Math. Soc. **3**(2), 447–498 (1990)
54. Lusztig, G.: Total positivity in reductive groups, Lie theory and geometry. Prog. Math., vol. 123, pp. 531–568. Birkhäuser, Boston (1994)
55. Lusztig, G.: Canonical bases and Hall algebras. Representation theories and algebraic geometry (Montreal, PQ, 1997). NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 514, pp. 365–399. Kluwer, Dordrecht (1998)
56. Lusztig, G.: Semicanonical bases arising from enveloping algebras, Adv. Math. **151**(2), 129–139 (2000)
57. Marsh, R., Reineke, M., Zelevinsky, A.: Generalized associahedra via quiver representations. Trans. Am. Math. Soc. **355**(10), 4171–4186 (2003) (electronic)
58. Musiker, G.: A graph theoretic expansion formula for cluster algebras of type B_n and D_n. arXiv:0710.3574v1 [math.CO], to appear in the Annals of Combinatorics
59. Nakajima, H.: Quiver varieties and cluster algebras. arXiv:0905.0002v3 [math.QA]
60. Palu, Y.: On cluster characters for triangulated categories. arXiv:math/0703540v1 [math.RT]
61. Qin, F.: Quantum cluster variables via Serre polynomials. arXiv:1004.4171 [math.RT]
62. Reiten, I.: Tilting theory and cluster algebras. Preprint available at www.institut.math.jussieu.fr/~keller/ictp2006/lecturenotes/reiten.pdf
63. Ringel, C.M.: Tame algebras and integral quadratic forms. Lecture Notes in Mathematics, vol. 1099. Springer, Berlin (1984)
64. Ringel, C.M.: Some remarks concerning tilting modules and tilted algebras. Origin. Relevance. Future. Handbook of Tilting Theory, LMS Lecture Note Series, vol. 332, pp. 49–104. Cambridge University Press, Cambridge (2007)
65. Stasheff, J.D.: Homotopy associativity of H-spaces. I, II. Trans. Am. Math. Soc. **108**, 275–292 (1963); **108**, 293–312 (1963)
66. Szendröi, B.: Non-commutative Donaldson-Thomas invariants and the conifold. Geom. Topol. **12**(2), 1171–1202 (2008)
67. Xiao, J., Xu, F.: Green's formula with $\mathbf{C}^*$-action and Caldero-Keller's formula for cluster algebras. arXiv:0707.1175
68. Zelevinsky, A.: From Littlewood-Richardson coefficients to cluster algebras in three lectures. Symmetric functions 2001: surveys of developments and perspectives. NATO Sci. Ser. II Math. Phys. Chem., vol. 74, pp. 253–273. Kluwer, Dordrecht (2002)
69. Zelevinsky, A.: Cluster algebras: notes for 2004 IMCC (Chonju, Korea, August 2004). arXiv:math.RT/0407414
70. Zelevinsky, A.: Cluster algebras: origins, results and conjectures. Advances in algebra towards millennium problems, pp. 85–105. SAS Int. Publ., Delhi (2005)
71. Zelevinsky, A.: What is a cluster algebra?. Notices of the A.M.S. **54**(11), 1494–1495 (2007)

Poisson and Symplectic Functions in Lie Algebroid Theory

Yvette Kosmann-Schwarzbach

Centre de Mathématiques Laurent Schwartz, École Polytechnique,
91128 Palaiseau, France, yks@math.polytechnique.fr

For Murray Gerstenhaber and Jim Stasheff, in admiration and respect

Abstract. Emphasizing the role of Gerstenhaber algebras and of higher derived brackets in the theory of Lie algebroids, we show that the several Lie algebroid brackets which have been introduced in the recent literature can all be defined in terms of Poisson and pre-symplectic functions in the sense of Roytenberg and Terashima. We prove that in this very general framework there exists a one-to-one correspondence between nondegenerate Poisson functions and symplectic functions. We also determine the differential associated to a Lie algebroid structure obtained by twisting a structure with background by both a Lie bialgebra action and a Poisson bivector.

Key words: Poisson and symplectic functions in Lie algebroid theory, Gerstenhaber algebra, Graded Poisson bracket, Lie algebroid, Poisson function, Pre-symplectic function, Maurer–Cartan equation, Quasi-Lie bialgebroid, Twisting by a bivector, Dirac structure, Quasi-Poisson space

2000 *Mathematics Subject Classification*: Primary 53D17, 17B70; Secondary 17B62, 17B63, 58A50

1 Introduction

Toward 1958, Ehresmann [14] introduced the idea of differentiable categories, of which the differentiable groupoids, now called Lie groupoids, are an example, and he developed this theory further in the 1960s [15]. At the end of the decade, Pradines introduced the corresponding infinitesimal objects which he called Lie algebroids [45]. The theory of Lie algebroids, which has since been developed by many authors, and in particular by Mackenzie [40, 42], encompasses both differential geometry – because the tangent bundle of a smooth manifold is the prototypical Lie algebroid –, and Lie algebra theory – because the Lie algebras are Lie algebroids whose base manifold is a singleton –,

A.S. Cattaneo et al. (eds.), *Higher Structures in Geometry and Physics*, Progress in Mathematics 287, DOI 10.1007/978-0-8176-4735-3_12,

while other examples of Lie algebroids occur in the theory of foliations (see, e.g., [44]) and in Poisson geometry [9,42]. The corresponding, purely algebraic concept, called pseudo-Lie algebras [41] or Lie–Rinehart algebras [19], among many other names, dates back to Jacobson [21], as has been observed in [29].

While the structure of what is now called a *Gerstenhaber algebra* appeared in the work of Murray Gerstenhaber on the Hochschild cohomology of associative algebras [16], it became clear in the work of Koszul [33] and of many other authors [30,57] that Gerstenhaber algebras play an essential role in the theory of Lie algebroids. Whenever a vector bundle has the structure of a Lie algebroid, the linear space of sections of its exterior algebra bundle is a Gerstenhaber algebra, the prototypical example of which is the linear space of fields of multivectors equipped with the Schouten–Nijenhuis bracket on any smooth manifold. The close relationship between Poisson geometry and Lie algebroid theory appears clearly in the concept of a Lie bialgebroid defined by Mackenzie and Xu [43] as the infinitesimal object of a Poisson groupoid, and characterized in terms of derivations in [24]. For any Poisson manifold M with tangent bundle TM, the pair (TM, T^*M) is a Lie bialgebroid, while the Lie bialgebroids over a point are Drinfeld's Lie bialgebras of Poisson–Lie group theory [12].

When passing from the case of Lie bialgebras to that of the Lie-quasi bialgebras [13], or their dual version, the quasi-Lie bialgebras, or the more general case of proto-bialgebras [23][1], *higher structures*, in the sense of Jim Stasheff [51], appear. The associated algebra is not a Gerstenhaber algebra but only a Gerstenhaber algebra up to homotopy, but with all n-ary brackets beyond the third vanishing (see [20,3,4]). The analogous theory generalizing Lie algebroids was developed by Roytenberg [46] and, more recently, by Terashima [54]. Their articles form the basis of the present exposition[2].

The concept of *twisting* for proto-bialgebroids was defined by Roytenberg [46] as a generalization of the twisting of proto-bialgebras introduced in [23], itself a generalization of the twisting of Lie bialgebras defined by Drinfeld in the theory of the semiclassical limit of the quasi-Hopf algebras [13], while the concept of *Poisson function*, which was already implicit in [46], has now been formally introduced by Terashima in [54], with interesting applications which we review and develop here. Poisson functions generalize both Poisson

[1] In [23,5], Lie-quasi bialgebras were called Jacobian quasi-bialgebras, and quasi-Lie bialgebras were called co-Jacobian quasi-bialgebras. We also point out that, in the translation of Drinfeld's original paper [13], the term "quasi-Lie bialgebra" is used for what we call Lie-quasi bialgebra. Proto-bialgebras were introduced in [23] where they were called proto-Lie-bialgebras, to distinguish them from the associative version of this notion.

[2] There are some changes in the notations. In particular, the notations ϕ and ψ used by Roytenberg in [46] are exchanged in order to return to the conventions of [23,5,27].

structures on manifolds and triangular r-matrices on Lie algebras, and, more generally, Poisson structures on Lie algebroids as well as their twisted versions (see [36, 46, 54]).

The cohomological approach to Lie algebroid theory arose from the viewpoint developed for Lie bialgebras by Lecomte and Roger [34], itself based on the even Poisson bracket introduced by Kostant and Sternberg in [32][3]. In [23], we extended this approach to the Lie-quasi bialgebras defined by Drinfeld [13], and we introduced the dual objects and the more general notion of proto-Lie bialgebra, encompassing both the Lie-quasi bialgebras and their duals. In [46] Roytenberg extended the cohomological approach to Lie bialgebra theory to the "oid" case by combining the supermanifold approach due to Vaintrob [55] and T. Voronov (see [56] citing earlier publications) with the results of [23].

The preprint that Terashima communicated to me in 2006 [54] goes further along the same lines and provides a beautiful unification of results in both recent [6, 8] and not so recent papers [38], showing that they are special cases of a general construction of Lie algebroid structures obtained by twisting certain basic structures.

The main features of this paper are the following. Section 2 deals with the general definition of a *structure* on a vector bundle, V. The basic tool for the study of the properties of "structures" is the *big bracket*, denoted by $\{\ ,\ \}$, the bigraded even Poisson bracket which is the canonical Poisson bracket on the cotangent bundle of the supermanifold ΠV, i.e., V with reversed parity on the fibers, which, on vector-valued forms or 1-form-valued multivectors, coincides with the Nijenhuis–Richardson bracket up to sign. The "structures" are cubic functions on this cotangent bundle whose Poisson square vanishes. Vector bundles equipped with a "structure" generalize the Lie, Lie-quasi and quasi-Lie bialgebroids, in particular the Lie bialgebras.

In Sect. 3, we introduce the dual notions of *twisting by a bivector* and *twisting by a* 2*-form*, and we define the *Poisson functions* and the *pre-symplectic functions* with respect to a given structure. Such bivectors (resp., 2-forms) give rise by twisting to quasi-Lie (resp., Lie-quasi) bialgebroids. We show that the *twist of Lie-quasi bialgebras* in the sense of Drinfeld [13] and the *twisted Poisson structures* on manifolds, introduced by Klimčík and Strobl in [22] (under the name WZW-Poisson structures) and studied by Ševera and Weinstein in [49] (where they are called Poisson structures with background), are both particular cases of the general notion of a twisted structure.

In Sect. 4 we prove that the graphs of Poisson functions and of pre-symplectic functions are Dirac sub-bundles of the Courant algebroid $V \oplus V^*$, which is the "double" of V.

The aim of Sect. 5 is to prove Theorem 5.2, which states that nondegenerate Poisson functions are in one-to-one correspondence with symplectic functions,

[3] Even Poisson brackets had already appeared in the context of the quantization of systems with constraints in the work of Batalin, Fradkin and Vilkovisky. See [50] and references therein.

a generalization of the well-known fact that a nondegenerate bivector on a manifold defines a Poisson structure if and only if its inverse is a closed 2-form. We believe that this theorem had not yet been proved in so general a form.

In Sect. 6, we study the case where a Poisson function involves both a Poisson structure on a manifold M in the ordinary sense and a Lie algebra action on this manifold. In the general case, with nontrivial Lie-quasi bialgebra actions and background 3-forms on the manifold, we determine explicit expressions for the bracket and the differential thus defined. In fact, the twisting of a structure on a vector bundle V by a Poisson function gives rise to a Lie algebroid structure on the dual vector bundle V^* and, dually, to a differential on the sections of $\wedge^\bullet V$, the exterior algebra bundle of V. In particular cases, we recover the brackets on vector bundles of the form $T^*M \times \mathfrak{g}$ which were associated to Poisson actions of Poisson–Lie groups on Poisson manifolds by Lu in [38] and, more generally, to quasi-Poisson G-manifolds in the sense of [2] by Bursztyn and Crainic in [6], and to quasi-Poisson G-spaces in the sense of [1] by Bursztyn, Crainic and Ševera in [8]. This approach gives an immediate proof that these brackets satisfy the Jacobi identity and are indeed Lie algebroid brackets. The formulas for the differential in the general case are, to the best of our knowledge, new.

2 Definition of Structures

2.1 Toward a Unification

It was already clear in the theory of Lie bialgebras that the "big bracket" was the appropriate tool for their study. Roytenberg extended the definition and the use of the big bracket to the case of Lie algebroids [46], and Terashima's article [54] proves additional results, by suitably twisting certain basic structures.

2.2 The Big Bracket

Consider the bigraded supermanifold $X = T^*\Pi V$, where V is a vector bundle over a manifold M, and where Π denotes the change of parity of the fibers. Then X is canonically equipped with an even Poisson bracket [32], the Poisson structure on X actually being symplectic. This Poisson bracket, called the *big bracket*, is here denoted by $\{\ ,\ \}$. The algebra $\mathcal{F}$ of smooth functions on X is bigraded in the following way. If (x^i, ξ^a) are local coordinates on ΠV $(i = 1, \ldots, \dim M,\quad a = 1, \ldots, \operatorname{rank} V)$, we denote by $(x^i, \xi^a, p_i, \theta_a)$ the corresponding local coordinates on $T^*\Pi V$, and we assign them the bidegrees $(0,0)$, $(0,1)$, $(1,1)$ and $(1,0)$, respectively. An element of $\mathcal{F}$ of bidegree (k,ℓ), with $k \geq 0$ and $\ell \geq 0$, is said to be of *shifted bidegree* (p,q) when $p = k-1$ and $q = \ell - 1$ ($p \geq -1$ and $q \geq -1$), whence the table

x^i	ξ^a	p_i	θ_a	
$(0,0)$	$(0,1)$	$(1,1)$	$(1,0)$	bidegree
$(-1,-1)$	$(-1,0)$	$(0,0)$	$(0,-1)$	shifted bidegree

The total degree (resp., total shifted degree) will be called, for short, the degree (resp., shifted degree). The big bracket is of shifted bidegree $(0,0)$, and it satisfies

$$\{x^i, p_j\} = \delta^i_j = -\{p_j, x^i\}\ , \qquad \{\xi^a, \theta_b\} = \delta^a_b = \{\theta_b, \xi^a\}\ .$$

2.3 Definition of Structures

As in [55, 46, 56] (also see [27]) we consider functions on X that define bialgebroid structures or generalizations thereof on (V, V^*). See [23, 5, 46] for proofs of the statements in this section.

Definition 2.1. *A* structure *on V is a homological function on X of degree 3, i.e., an element $S \in \mathcal{F}$ of shifted degree 1 such that $\{S, S\} = 0$.*

Let

$$S = \phi + \gamma + \mu + \psi \tag{1}$$

in the notations of [23] and [5]. Then,

• ϕ, of shifted bidegree $(2,-1)$, is a 3-form on V^*,

$$\phi = \frac{1}{6}\phi^{abc}\theta_a\theta_b\theta_c\ ,$$

• γ, of shifted bidegree $(1,0)$, defines an anchor, $a^* : V^* \to TM$, and a bracket on V^*,

$$\gamma = (a^*)^{ib}p_i\theta_b + \frac{1}{2}\gamma^{bc}_a\theta_b\theta_c\xi^a\ ,$$

• μ, of shifted bidegree $(0,1)$, defines an anchor, $a_* : V \to TM$, and a bracket on V,

$$\mu = (a_*)^i_b p_i\xi^b + \frac{1}{2}\mu^a_{bc}\theta_a\xi^b\xi^c\ ,$$

• ψ, of shifted bidegree $(-1,2)$, is a 3-form on V,

$$\psi = \frac{1}{6}\psi_{abc}\xi^a\xi^b\xi^c\ .$$

Then S is a structure if and only if

$$\begin{cases} \frac{1}{2}\{\mu,\mu\} + \{\gamma,\psi\} = 0\ , \\ \{\mu,\gamma\} + \{\phi,\psi\} = 0\ , \\ \frac{1}{2}\{\gamma,\gamma\} + \{\mu,\phi\} = 0\ , \\ \{\mu,\psi\} = 0\ , \\ \{\gamma,\phi\} = 0\ . \end{cases}$$

By definition, when S is a structure on V, the pair (V, V^*) is a *proto-bialgebroid*. The anchor and bracket of V and of V^* are the following *derived brackets* [25, 26, 27, 46, 56]:

$$\begin{aligned}
&\text{anchor of } V, \quad a_*(X) \cdot f = \{\{X, \mu\}, f\} \, , \\
&\text{bracket of } V, \quad \mu(X, Y) \ = \{\{X, \mu\}, Y\} \, , \\
&\text{anchor of } V^*, \, a^*(\alpha) \cdot f \ = \{\{\alpha, \gamma\}, f\} \, , \\
&\text{bracket of } V^*, \, \gamma(\alpha, \beta) \quad = \{\{\alpha, \gamma\}, \beta\} \, ,
\end{aligned}$$

for $f \in C^\infty(M)$, X and $Y \in \Gamma(V)$, α and $\beta \in \Gamma(V^*)$. The quasi-Gerstenhaber brackets on $\Gamma(\wedge^\bullet V)$, where $\wedge^\bullet V$ is the exterior algebra of V, and on $\Gamma(\wedge^\bullet V^*)$, are expressed by the same formulas. They are denoted by $[\ ,\ ]_\mu$ and $[\ ,\ ]_\gamma$, respectively.

The Lie-quasi bialgebroids, quasi-Lie bialgebroids and Lie bialgebroids are defined as follows:

• (V, V^*) is a *Lie-quasi bialgebroid* if and only if $S = \phi + \gamma + \mu$, i.e., if $\psi = 0$. Then V is a Lie algebroid, $\Gamma(\wedge^\bullet V)$ is a Gerstenhaber algebra, while $\Gamma(\wedge^\bullet V^*)$ is a quasi-Gerstenhaber algebra.

• (V, V^*) is a *quasi-Lie bialgebroid* if and only if $S = \gamma + \mu + \psi$, i.e., if $\phi = 0$. Then V^* is a Lie algebroid, $\Gamma(\wedge^\bullet V^*)$ is a Gerstenhaber algebra, while $\Gamma(\wedge^\bullet V)$ is a quasi-Gerstenhaber algebra.

• (V, V^*) is a *Lie bialgebroid* if and only if $S = \gamma + \mu$, i.e., if $\phi = \psi = 0$. Then both V and V^* are Lie algebroids, and both $\Gamma(\wedge^\bullet V)$ and $\Gamma(\wedge^\bullet V^*)$ are Gerstenhaber algebras.

The quasi-Gerstenhaber algebras (see [46, 20, 3, 4]) are the simplest higher structures beyond the Gerstenhaber algebras themselves; they correspond to the case where all n-ary brackets, ℓ_n, vanish for $n \geq 4$.

On the Poisson manifold $T^*\Pi V$, we can consider the Hamiltonian vector field with Hamiltonian $S \in \mathcal{F}$, which we denote by $d_S = \{S, .\}$. Because $\{S, S\} = 0$, d_S is a *differential* on the space of smooth functions on $T^*\Pi V$, i.e., a derivation of $\mathcal{F}$ of degree 1 and of square zero.

Example 1. When $V = TM$ and $S = \mu = p_i \xi^i$, then $\mu(X, Y)$ is the Lie bracket of vector fields X and Y, the corresponding Gerstenhaber bracket on $\Gamma(\wedge^\bullet TM)$ is the Schouten–Nijenhuis bracket of multivector fields, and the restriction of $d_S = d_\mu$ to the differential forms on M is the de Rham differential.

Example 2. When M is a point, then $V = \mathfrak{g}$ is a vector space and a structure $S = \mu + \gamma$ on V is a Lie bialgebra structure on $(\mathfrak{g}, \mathfrak{g}^*)$, also denoted by $S_{\mathfrak{g}} + S_{\mathfrak{g}^*}$ in Sect. 6, while $d_S = d_\mu + d_\gamma$ is the Chevalley–Eilenberg cohomology operator of the double of the Lie bialgebra. More generally, on $V = \mathfrak{g}$, a structure $S = \mu + \gamma + \phi$, where $\phi \in \wedge^3 V$, is a Lie-quasi bialgebra structure on $(\mathfrak{g}, \mathfrak{g}^*)$.

3 Twisting

We consider a structure S on the vector bundle V that defines a proto-bialgebroid structure on (V, V^*), and we shall now study the *twisting*, $e^{-\sigma}S$, of S by a function σ of shifted bidegree $(1,-1)$ or $(-1,1)$.

3.1 Twisting by Poisson or Pre-Symplectic Functions

Let $\sigma \in \mathcal{F}$ be a function of shifted bidegree $(1,-1)$ or $(-1,1)$. Since the right adjoint action, $\mathrm{ad}_\sigma = \{\,.,\sigma\}$, of an element σ of shifted degree 0 is a derivation of degree 0 of $(\mathcal{F}, \{\ ,\ \})$, and since, for any $a \in \mathcal{F}$, the series $a + \{a,\sigma\} + \frac{1}{2!}\{\{a,\sigma\},\sigma\} + \frac{1}{3!}\{\{\{a,\sigma\},\sigma\},\sigma\} + \ldots$ terminates for reasons of bidegrees, the exponential of ad_σ is well-defined and is an automorphism of $(\mathcal{F}, \{\ ,\ \})$, which, in an abuse of notation, we shall denote by e^σ. It follows that, for any structure S, and for any σ of shifted degree 0, $\{e^\sigma S, e^\sigma S\} = e^\sigma\{S,S\} = 0$, and therefore $e^\sigma S$ is also a structure.

Definition 3.1. *When σ is a function of shifted bidegree $(1,-1)$ or $(-1,1)$, the structure $e^{-\sigma}S$ is called the* twisting *of S by σ.*

A function of shifted bidegree $(1,-1)$ is a bivector σ on V, expressed in local coordinates as

$$\sigma = \frac{1}{2}\sigma^{ab}\theta_a\theta_b \ ,$$

while a function of shifted bidegree $(-1,1)$ is a 2-form τ on V, expressed in local coordinates as

$$\tau = \frac{1}{2}\tau_{ab}\xi^a\xi^b \ .$$

We list the explicit formulas [46] for the homogeneous components of twisted structures.

• For σ of shifted bidegree $(1,-1)$, let $e^{-\sigma}S = \phi_\sigma + \gamma_\sigma + \mu_\sigma + \psi_\sigma$ be the decomposition (1) of $e^{-\sigma}S$ as a sum of terms of homogeneous bidegrees. Then,

$$\begin{cases} \phi_\sigma = \phi - \{\gamma,\sigma\} + \frac{1}{2}\{\{\mu,\sigma\},\sigma\} - \frac{1}{6}\{\{\{\psi,\sigma\},\sigma\},\sigma\} \ , \\ \gamma_\sigma = \gamma - \{\mu,\sigma\} + \frac{1}{2}\{\{\psi,\sigma\},\sigma\} \ , \\ \mu_\sigma = \mu - \{\psi,\sigma\} \ , \\ \psi_\sigma = \psi \ . \end{cases} \tag{2}$$

• For τ of shifted bidegree $(-1,1)$, let $e^{-\tau}S = \phi_\tau + \gamma_\tau + \mu_\tau + \psi_\tau$ be the decomposition (1) of $e^{-\tau}S$ as a sum of terms of homogeneous bidegrees. Then,

$$\begin{cases} \phi_\tau = \phi \ , \\ \gamma_\tau = \gamma - \{\phi,\tau\} \ , \\ \mu_\tau = \mu - \{\gamma,\tau\}, + \frac{1}{2}\{\{\phi,\tau\},\tau\} \ , \\ \psi_\tau = \psi - \{\mu,\tau\} + \frac{1}{2}\{\{\gamma,\tau\},\tau\} - \frac{1}{6}\{\{\{\phi,\tau\},\tau\},\tau\} \ . \end{cases} \tag{3}$$

Definition 3.2. *Let S be a structure on V.*
(i) A function σ of shifted bidegree $(1,-1)$ such that $\phi_\sigma = 0$ is called a Poisson function *with respect to S.*
(ii) A function τ of shifted bidegree $(-1,1)$ such that $\psi_\tau = 0$ is called a pre-symplectic function *with respect to S.*

In view of these definitions, we immediately obtain

Proposition 3.3. *Let S be a structure on V and let σ (resp., τ) be a function of shifted bidegree $(1,-1)$ (resp., $(-1,1)$).*
(i) If σ is a Poisson function, the twisted structure $e^{-\sigma}S$ is a quasi-Lie bialgebroid structure.
(ii) If τ is a pre-symplectic function, the twisted structure $e^{-\tau}S$ is a Lie-quasi bialgebroid structure.

3.2 Twisting by Poisson Functions

It follows from the formula for ϕ_σ in (2) that a section σ of $\wedge^2 V$ is a Poisson function with respect to a structure $S = \phi + \gamma + \mu + \psi$ if and only if

$$\phi - \{\gamma, \sigma\} + \frac{1}{2}\{\{\mu, \sigma\}, \sigma\} - \frac{1}{6}\{\{\{\psi, \sigma\}, \sigma\}, \sigma\} = 0 \ . \tag{4}$$

Equation (4) is called a *generalized twisted Maurer–Cartan equation*, or simply a *Maurer–Cartan equation.*

For any bivector σ, we set $\sigma^\sharp \alpha = i_\alpha \sigma$, for $\alpha \in \Gamma(V^*)$, where i denotes the interior product. Whenever σ is a Poisson function with respect to $S = \phi + \gamma + \mu + \psi$, the term of shifted bidegree $(1,0)$ in $e^{-\sigma}S$,

$$\gamma_\sigma = \gamma - \{\mu, \sigma\} + \frac{1}{2}\{\{\psi, \sigma\}, \sigma\} \ ,$$

defines an anchor $a^* + a_* \circ \sigma^\sharp$ and a Lie bracket on $\Gamma(V^*)$, as well as a Gerstenhaber bracket on $\Gamma(\wedge^\bullet V^*)$, which we denote by $[\ ,\ ]_{\gamma_\sigma}$, and a differential $d_{\gamma_\sigma} = \{\gamma_\sigma, .\}$ on $\Gamma(\wedge^\bullet V)$. There is also a bracket, $[\ ,\ ]_{\mu_\sigma}$, on $\Gamma(\wedge^\bullet V)$ defined by the term of shifted bidegree $(0,1)$, $\mu_\sigma = \mu - \{\psi, \sigma\}$, and a derivation of degree 1, $d_{\mu_\sigma} = \{\mu_\sigma, .\}$, of $\Gamma(\wedge^\bullet V^*)$. Then $\frac{1}{2}\{\mu_\sigma, \mu_\sigma\} + \{\gamma_\sigma, \psi\} = 0$, so that ψ measures the defect in the Jacobi identity for $[\ ,\ ]_{\mu_\sigma}$, and $(d_{\mu_\sigma})^2 = [\psi, \cdot]_{\gamma_\sigma}$.

It appears that the twisting of Lie bialgebras in the sense of Drinfeld [13], as well as its generalizations to proto-bialgebras [23, 5] and to proto-bialgebroids [46], and the twisting of Poisson structures in the sense of Ševera and Weinstein [49], and its generalizations to structures on Lie algebroids [46, 28], all fit into this general framework, although the meaning of the word "twisting" is not quite the same in both instances. In the first instance, one twists a given structure, in the sense of Definition 2.1, on a Lie algebra $\mathfrak{g}$ by an element $\sigma \in \wedge^2 \mathfrak{g}$ (often denoted by t or f), called the "twist" [13, 1]. For any twist, a Lie-quasi bialgebra is twisted into a Lie-quasi bialgebra. In the second case, it would be more appropriate to speak of "Poisson structures with background": the given structure on the vector bundle V is of the form $\mu + \psi$, where ψ is a d_μ-closed 3-form, and equation (4) which reduces to the twisted Poisson condition (6) below is the condition for $\sigma \in \Gamma(\wedge^2 V)$ to twist $\mu + \psi$ into a quasi-Lie bialgebroid structure.

(i) Twist in the sense of Drinfeld. In the case of a twist of a Lie-quasi bialgebra, one twists a structure $S = \phi + \gamma + \mu + 0$ on a Lie algebra $\mathfrak{g}$ by an arbitrary $\sigma \in \wedge^2\mathfrak{g}$ into

$$e^{-\sigma}S = \left(\phi - \{\gamma, \sigma\} + \frac{1}{2}\{\{\mu, \sigma\}, \sigma\}\right) + (\gamma - \{\mu, \sigma\}) + \mu + 0 \ ,$$

and one obtains a "twisted Lie-quasi bialgebra". The resulting object is a Lie bialgebra, with $\mu_\sigma = \mu$ and $\gamma_\sigma = \gamma - \{\mu, \sigma\}$, if and only if σ is a Poisson function, i.e., satisfies the condition

$$\frac{1}{2}[\sigma, \sigma]_\mu + d_\gamma \sigma - \phi = 0 \ .$$

If one twists a Lie bialgebra ($\psi = \phi = 0$), this condition reduces to the usual *Maurer–Cartan equation*,

$$\frac{1}{2}[\sigma, \sigma]_\mu + d_\gamma \sigma = 0 \ . \tag{5}$$

If one twists a trivial Lie bialgebra ($\psi = \phi = \gamma = 0$), the Maurer–Cartan equation reduces to $[\sigma, \sigma]_\mu = 0$, i.e., to the classical Yang–Baxter equation. In fact, for $\sigma = r \in \wedge^2\mathfrak{g}$,

$$-\frac{1}{2}[r, r]_{\mathfrak{g}} = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \ ,$$

and the *classical Yang–Baxter equation* (CYBE) on a Lie algebra $\mathfrak{g}$ is the condition $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$, for $r \in \wedge^2\mathfrak{g}$.

When $S = \mu$, the necessary and sufficient condition for $\mu + \gamma_\sigma$ to be a Lie bialgebra structure on $(\mathfrak{g}, \mathfrak{g}^*)$ is $\{\mu, \{\{\mu, \sigma\}, \sigma\}\} = 0$, the *generalized classical Yang–Baxter equation*, which states that $[\sigma, \sigma]_\mu$ is ad^μ-invariant.

In the same way, a Lie-quasi bialgebroid can be twisted by a bivector, and a Lie bialgebroid is twisted into a Lie bialgebroid if and only if the bivector satisfies the Maurer–Cartan equation (5) (see [36, 46, 27]).

(ii) Twisted Poisson structures. If S is a structure on a vector bundle V such that $\gamma = 0$ and $\phi = 0$, then $\{\mu, \mu\} = 0$, i.e., V is a Lie algebroid, and ψ is a d_μ-closed section of $\wedge^3 V^*$. In this case, one twists $S = 0 + 0 + \mu + \psi$ into

$$\begin{aligned} e^{-\sigma}S = &\left(\frac{1}{2}\{\{\mu, \sigma\}, \sigma\} - \frac{1}{6}\{\{\{\psi, \sigma\}, \sigma\}, \sigma\}\right) \\ &+ \left(-\{\mu, \sigma\} + \frac{1}{2}\{\{\psi, \sigma\}, \sigma\}\right) + (\mu - \{\psi, \sigma\}) + \psi \ . \end{aligned}$$

Thus, σ is a Poisson function if and only if

$$\{\{\mu, \sigma\}, \sigma\} - \frac{1}{3}\{\{\{\psi, \sigma\}, \sigma\}, \sigma\} = 0 \ ,$$

which is the condition

$$\frac{1}{2}[\sigma, \sigma]_\mu = (\wedge^3 \sigma^\sharp)\psi \ , \tag{6}$$

i.e., (σ, ψ) is a *twisted Poisson structure* on the Lie algebroid V. When σ satisfies the twisted Poisson condition (6), the resulting object is a quasi-Lie bialgebroid. In particular, $-\{\mu, \sigma\} + \frac{1}{2}\{\{\psi, \sigma\}, \sigma\}$ is a Lie algebroid bracket on V^*.

If, in addition, $\psi = 0$, then σ is a Poisson function if and only if

$$\{\{\mu, \sigma\}, \sigma\} = 0 \ ,$$

which is the condition

$$[\sigma, \sigma]_\mu = 0 \ ,$$

i.e., σ is a Poisson structure in the usual sense, a section of $\wedge^2 V$ with Schouten–Nijenhuis square zero. The Poisson case is also called the *triangular* case by extension of the terminology used in the theory of Lie bialgebras.

The twisted differential. In the Poisson case ($\gamma = 0$ and $\psi = 0$), the anchor of V^* is $a_* \circ \sigma^\sharp$, and the bracket on $\Gamma(\wedge^\bullet V^*)$ is $\gamma_\sigma = \{\sigma, \mu\}$, the Koszul bracket[4]. The corresponding differential on $\Gamma(\wedge^\bullet V)$ is the *Lichnerowicz–Poisson differential*, [35], $d_\sigma = \{\{\sigma, \mu\}, .\} = [\sigma, .]_\mu$, while the differential on $\Gamma(\wedge^\bullet V^*)$ is the Lie algebroid cohomology operator $d_\mu = \{\mu, .\}$. The pair (V, V^*) is a Lie bialgebroid.

In the twisted Poisson case, $\gamma_\sigma = -\{\mu, \sigma\} + \frac{1}{2}\{\{\psi, \sigma\}, \sigma\}$ restricts to the Lie algebroid bracket on sections of V^* defined by Ševera and Weinstein [49], and the corresponding differential on $\Gamma(\wedge^\bullet V)$ is the *twisted Poisson differential*, $d_\sigma + i_{\psi^{(2)}}$, where $\psi^{(2)} = \frac{1}{2}\{\{\psi, \sigma\}, \sigma\} = (\wedge^2 \sigma^\sharp)\psi$, while the derivation $\{\mu_\sigma, .\}$ is the derivation $d_\mu + i_{\psi^{(1)}}$, where $\psi^{(1)} = \{\psi, \sigma\} = \sigma^\sharp \psi$ (see [49, 46, 28]). The pair (V, V^*) is then a quasi-Lie bialgebroid.

3.3 Twisting by Pre-Symplectic Functions

It follows from formula (3) that a section τ of $\wedge^2 V^*$ is a pre-symplectic function with respect to a structure $S = \phi + \gamma + \mu + \psi$ if and only if

[4] The Koszul bracket [33] restricts to the bracket of sections of $\Gamma(V^*)$ generalizing the well-known bracket of 1-forms on a Poisson manifold. The bracket of 1-forms on symplectic manifolds was introduced in the book of Abraham and Marsden (1967). For Poisson manifolds, it was discovered independently in the 1980s by several authors – Gelfand and Dorfman, Fuchssteiner, Magri and Morosi, Daletskii – and Weinstein (see [9]) has shown that it is a Lie algebroid bracket.

$$\psi - \{\mu, \tau\} + \frac{1}{2}\{\{\gamma, \tau\}, \tau\} - \frac{1}{6}\{\{\{\phi, \tau\}, \tau\}, \tau\} = 0 \ . \tag{7}$$

Equation (7) is dual to (4) and it is also called a *generalized twisted Maurer–Cartan equation* or again simply a *Maurer–Cartan equation.* Pre-symplectic functions generalize pre-symplectic structures on manifolds as well as their twisted versions.

If $\gamma = \phi = 0$, then $\{\mu, \mu\} = 0$, i.e., V is a Lie algebroid, and ψ is a d_μ-closed section of $\wedge^3 V^*$. In this case, τ is pre-symplectic if and only if the pair (τ, ψ) satisfies the twisted pre-symplectic condition,

$$\psi - \{\mu, \tau\} = 0 \ ,$$

which is the condition, $d_\mu \tau = \psi$, i.e., (τ, ψ) is a *twisted pre-symplectic structure* on the Lie algebroid V. (See [49] and see [48] for an example of a twisted symplectic structure arising in the theory of the lattices of Neumann oscillators.)

If, in particular, $\gamma = \phi = \psi = 0$, then $\{\mu, \mu\} = 0$ and V is a Lie algebroid. In this case, τ is pre-symplectic if and only if τ satisfies the pre-symplectic condition,

$$\{\mu, \tau\} = 0 \ ,$$

which is the condition, $d_\mu \tau = 0$, i.e., τ is a d_μ-closed section of $\wedge^2 V^*$, the *pre-symplectic* case.

4 The Graphs of Poisson and Pre-Symplectic Functions

4.1 Courant Algebroids, the Courant Algebroid $V \oplus V^*$

A *Loday algebra* (called *Leibniz algebra* by Loday [37]) is equipped with a bracket (in general non-skew-symmetric) satisfying the Jacobi identity in the form $[u, [v, w]] = [[u, v], w] + [v, [u, w]]$. We give the definition of Courant algebroids in [27] which is equivalent to the original definition of Courant and Weinstein [11, 10].

A *Courant algebroid* is a vector bundle $E \to M$, equipped with a vector bundle morphism, $a_E : E \to TM$, called the *anchor*, a fiber-wise nondegenerate symmetric bilinear form $(\ ,\ )$, and a bracket, $[\ ,\ ] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$, called the *Dorfman–Courant bracket*, such that

- $\Gamma(E)$ is a *Loday algebra*,
- for all x, u, $v \in \Gamma(E)$,

$$a_E(x) \cdot (u, v) = (x\,,\,[u, v] + [v, u]) = ([x, u]\,,\,v) + (u\,,\,[x, v]) \ .$$

A sub-bundle, $F \subset E$, is called a *Dirac sub-bundle* if
- F is maximally isotropic,
- $\Gamma(F)$ is closed under the bracket.

When S is a structure on V, the vector bundle $E = V \oplus V^*$ with the canonical scalar product,
$$(u, v) = \{u, v\} ,$$
and bracket
$$[u, v]_S = \{\{u, S\}, v\} , \tag{8}$$
for u, $v \in \Gamma(V \oplus V^*)$, is a Courant algebroid [46,56,27], called the *double* of V.

Lemma 4.1. *Let S be a structure on V.*
(i) The function $\sigma \in \Gamma(\wedge^2 V)$ is a Poisson function with respect to S if and only if V^* is a Dirac sub-bundle of $(V \oplus V^*, [\ ,\ ]_{e^{-\sigma}S})$.
(ii) The function $\tau \in \Gamma(\wedge^2 V^*)$ is a pre-symplectic function with respect to S if and only if V is a Dirac sub-bundle of $(V \oplus V^*, [\ ,\ ]_{e^{-\tau}S})$.

Proof. Part (i) (resp., (ii)) follows from the computation of the bidegrees of the homogeneous terms in $[u, v]_{e^{-\sigma}S}$ (resp., $[u, v]_{e^{-\tau}S}$) for $u, v \in \Gamma(V)$ (resp., $u, v \in \Gamma(V^*)$). □

4.2 Graphs as Dirac Structures

Theorem 4.2 below generalizes the characterization of the graphs of Poisson, quasi-Poisson and pre-symplectic structures in [36] and [49], and that of twisted pre-symplectic structures in [6] (see also Alekseev and Xu, Derived brackets and Courant algebroids, 2000, Unpublished manuscript). The statement of this theorem can be found in Remark 4.2 in [46] (cf. also Prop. 5 in [8]), and the proof given here is also due to Roytenberg [47]. Both theorems in this section have been proved by Terashima [54].

Theorem 4.2. *Let S be a structure on V.*
(i) A section σ of $\wedge^2 V$ is a Poisson function with respect to S if and only if its graph in the Courant algebroid $(V \oplus V^*, [\ ,\ ]_S)$ is a Dirac sub-bundle.
(ii) A section τ of $\wedge^2 V^*$ is a pre-symplectic function with respect to S if and only if its graph in the Courant algebroid $(V \oplus V^*, [\ ,\ ]_S)$ is a Dirac sub-bundle.

Proof. We need only prove (ii), since the proof of (i) is entirely similar. We shall denote by $\tau^\flat$ the vector bundle morphism from V to V^* induced by $\tau \in \Gamma(\wedge^2 V^*)$, such that $\tau^\flat X = -i_X \tau$, for $X \in V$, as well as the associated map on sections of V. By the graph of τ, we mean the graph of $\tau^\flat$. Since $\tau^\flat(X) = \{X, \tau\}$, for all $X \in \Gamma(V)$, and since, for reasons of bidegree, $e^\tau X = X + \{X, \tau\}$, it follows that
$$\mathrm{Graph}(\tau) = e^\tau V . \tag{9}$$

Since e^τ is an automorphism of $(\mathcal{F}, \{\ ,\ \})$, it is an isomorphism from $(V \oplus V^*, [\ ,\ ]_{e^{-\tau}S})$ to $(V \oplus V^*, [\ ,\ ]_S)$. Thus $e^\tau V$ is a Dirac sub-bundle of $(V \oplus V^*, [\ ,\ ]_S)$ if and only if V is a Dirac sub-bundle of $(V \oplus V^*, [\ ,\ ]_{e^{-\tau}S})$. Thus (ii) follows from (9) and Lemma 4.1 (ii). □

Theorem 4.3. *Let $S = \phi + \gamma + \mu + \psi$ be a structure on V.*
(i) Let σ be a Poisson function with respect to S. The projection $\mathrm{Graph}(\sigma) \to V^*$ *is a morphism of Lie algebroids when* $\Gamma(\mathrm{Graph}(\sigma))$ *is equipped with the Lie bracket induced from the Dorfman–Courant bracket* $[\ ,\ ]_S$ *and* $\Gamma(V^*)$ *is equipped with the Lie bracket* $\gamma_\sigma = \gamma - \{\mu, \sigma\} + \frac{1}{2}\{\{\psi, \sigma\}, \sigma\}$.
(ii) Let τ be a pre-symplectic function with respect to S. The projection $\mathrm{Graph}(\tau) \to V$ *is a morphism of Lie algebroids when* $\Gamma(\mathrm{Graph}(\tau))$ *is equipped with the Lie bracket induced from the Dorfman–Courant bracket* $[\ ,\ ]_S$ *and* $\Gamma(V)$ *is equipped with the Lie bracket* $\mu_\tau = \mu - \{\gamma, \tau\} + \frac{1}{2}\{\{\phi, \tau\}, \tau\}$.

Proof. We need only prove (ii), since the proof of (i) is entirely similar. For any $\tau \in \Gamma(\wedge^2 V^*)$, X and $Y \in \Gamma(V)$, $[e^\tau X, e^\tau Y]_S = e^\tau [X, Y]_{e^{-\tau}S}$. If τ is a pre-symplectic function with respect to S, then $[e^\tau X, e^\tau Y]_S = e^\tau [X, Y]_{\mu_\tau} = [X, Y]_{\mu_\tau} + \{[X, Y]_{\mu_\tau}, \tau\}$, whose V-component is $[X, Y]_{\mu_\tau}$. □

5 Symplectic Functions

Let us now assume that $\sigma \in \Gamma(\wedge^2 V)$ is nondegenerate, i.e., the map $\sigma^\sharp : V^* \to V$ defined by $\sigma^\sharp \alpha = i_\alpha \sigma$, for $\alpha \in \Gamma(V^*)$, is invertible. Set $\tau^\flat = (\sigma^\sharp)^{-1}$, and let $\tau \in \Gamma(\wedge^2 V^*)$ be such that $\tau^\flat X = -i_X \tau$, for $X \in \Gamma(V)$. We say that $\tau \in \Gamma(\wedge^2 V^*)$ and $\sigma \in \Gamma(\wedge^2 V)$ are *inverses* of one another. A nondegenerate pre-symplectic function is called *symplectic.*

5.1 "Nondegenerate Poisson" is Equivalent to "Symplectic"

Many classical results are corollaries of the general theorem which we state and prove in this section. Recall that $\xi^a \theta_b = -\theta_b \xi^a$, $\{\xi^a, \theta_b\} = \delta^a_b = \{\theta_b, \xi^a\}$ and, for $u, v, w \in \mathcal{F}$,

$$\{u, vw\} = \{u, v\}w + (-1)^{|u||v|} v\{u, w\} ,$$

$$\{uv, w\} = u\{v, w\} + (-1)^{|v||w|}\{u, w\}v ,$$

where $|u|$ is the degree of u, and

$$\{u, \{v, w\}\} = \{\{u, v\}, w\} + (-1)^{\|u\|\|v\|}\{v, \{u, w\}\} ,$$

$$\{\{u, v\}, w\} = \{u, \{v, w\}\} + (-1)^{\|v\|\|w\|}\{\{u, w\}, v\} ,$$

where $\|u\|$ is the shifted degree of u. The proof of the theorem depends on the following lemma.

Lemma 5.1. *Assume that $\sigma \in \Gamma(\wedge^2 V)$ is nondegenerate and that its inverse is τ. Then*
(i) $\{\sigma, \tau\} = -\{\tau, \sigma\} = \mathrm{Id}_V$.
(ii) If S is of shifted bidegree (p, q), then

$$\{\{\sigma, \tau\}, S\} = (q - p)S \ . \tag{10}$$

Proof. This lemma is proved by straightforward computations, using the equality $\mathrm{Id}_V = \xi^a \theta_a$. □

Theorem 5.2. *Let S be a structure on V. Let $\sigma \in \Gamma(\wedge^2 V)$ be a nondegenerate bivector with inverse $\tau \in \Gamma(\wedge^2 V^*)$. Then σ is a Poisson function with respect to S if and only if $-\tau$ is a symplectic function with respect to S.*

Proof. Lemma 5.1(ii) applied in the cases $(p, q) = (2, -1)$, $(1, 0)$, $(0, 1)$ and $(-1, 2)$, and repeated applications of the Jacobi identity yield the following computations. Let μ be of shifted bidegree $(0, 1)$. From

$$\{\{\mu, \tau\}, \sigma\} = \{\mu, \{\tau, \sigma\}\} + \{\{\mu, \sigma\}, \tau\} = \mu + \{\{\mu, \sigma\}, \tau\} \ ,$$

we obtain

$$\{\{\{\mu, \tau\}, \sigma\}, \sigma\} = \{\mu, \sigma\} + \{\{\{\mu, \sigma\}, \tau\}, \sigma\}$$

$$= \{\mu, \sigma\} + \{\{\mu, \sigma\}, \{\tau, \sigma\}\} + \{\{\{\mu, \sigma\}, \sigma\}, \tau\} = \{\{\{\mu, \sigma\}, \sigma\}, \tau\} \ .$$

Whence

$$\{\{\{\{\mu, \tau\}, \sigma\}, \sigma\}, \sigma\} = \{\{\{\{\mu, \sigma\}, \sigma\}, \tau\}, \sigma\} = \{\{\{\mu, \sigma\}, \sigma\}, \{\tau, \sigma\}\}$$
$$= -\,3\,\{\{\mu, \sigma\}, \sigma\} \ .$$

Similarly, if γ is of shifted bidegree $(1, 0)$,

$$\{\{\{\{\{\gamma, \tau\}, \tau\}, \sigma\}, \sigma\}, \sigma\} = 12\,\{\gamma, \sigma\} \ .$$

If ϕ is of shifted bidegree $(2, -1)$,

$$\{\{\{\{\{\{\phi, \tau\}, \tau\}, \tau\}, \sigma\}, \sigma\}, \sigma\} = -\,36\,\phi \ .$$

Let $S = \phi + \gamma + \mu + \psi$. The term of shifted bidegree $(-1, 2)$ in $e^{-\tau}S$ is

$$\psi_\tau = \psi - \{\mu, \tau\} + \frac{1}{2}\{\{\gamma, \tau\}, \tau\} - \frac{1}{6}\{\{\{\phi, \tau\}, \tau\}, \tau\} \ ,$$

and the term of shifted bidegree $(2, -1)$ in $e^{-\sigma}S$ is

$$\phi_\sigma = \phi - \{\gamma, \sigma\} + \frac{1}{2}\{\{\mu, \sigma\}, \sigma\} - \frac{1}{6}\{\{\{\psi, \sigma\}, \sigma\}, \sigma\} \ .$$

The preceding equalities and analogous results for other iterated brackets, reversing the roles of σ and τ, yield the equalities:

$$\{\{\{\psi_\tau, \sigma\}, \sigma\}, \sigma\} = 6\,\phi_{-\sigma}$$

and

$$\{\{\{\phi_\sigma, \tau\}, \tau\}, \tau\} = 6\,\psi_{-\tau}\ .$$

Therefore $\psi_\tau = 0$ implies $\phi_{-\sigma} = 0$, and conversely. □

The method of proof used above in the general case can be applied to give one-line proofs of some well-known results.

- For the case of nondegenerate Poisson structures, the proof reduces to $\{\mu, \tau\} = 0$ implies that $\{\{\{\{\mu, \tau\}, \sigma\}, \sigma\}, \sigma\} = 0$, which implies that $\{\{\mu, \sigma\}, \sigma\} = 0$, and a similar argument applies to the converse. This simple argument proves the classical result: nondegenerate closed 2-forms are in one-to-one correspondence with nondegenerate Poisson bivectors.
- For the case of nondegenerate twisted Poisson structures (see Sect. 3.2 (ii)), the proof reduces to $\{\mu, \tau\} = -\psi$ implies that $\{\{\{\{\mu, \tau\}, \sigma\}, \sigma\}, \sigma\} = -\{\{\{\psi, \sigma\}, \sigma\}, \sigma\}$, which implies that $\{\{\mu, \sigma\}, \sigma\} = \frac{1}{3}\{\{\{\psi, \sigma\}, \sigma\}, \sigma\}$, and a similar argument for the converse. Thus $d_\mu \tau = -\,\psi$ implies $\frac{1}{2}[\sigma, \sigma]_\mu = (\wedge^3 \sigma^\sharp)\psi$ and conversely. This constitutes a direct proof of the following corollary of Theorem 5.2 (see [49, 31]).

Corollary 5.3. *(i) A nondegenerate bivector on a Lie algebroid defines a twisted Poisson structure if and only if its inverse is a twisted symplectic 2-form.*
(ii) The leaves of a twisted Poisson manifold are twisted symplectic manifolds.

It follows from this corollary that, in the case of Lie algebras, considered to be Lie algebroids over a point, a nondegenerate $r \in \wedge^2\mathfrak{g}$ is a solution of the *twisted classical Yang–Baxter equation*, generalizing the classical Yang–Baxter equation (see Sect. 3.2),

$$\frac{1}{2}[r, r]_{\mathfrak{g}} = (\wedge^3 r^\sharp)\psi\ ,$$

where ψ is a $d_{\mathfrak{g}}$-closed 3-form on the Lie algebra $\mathfrak{g}$, if and only if its inverse is a nondegenerate 2-form τ satisfying the twisted closure condition, $d_{\mathfrak{g}}\tau = -\psi$. Here $d_{\mathfrak{g}}$ is the Chevalley–Eilenberg cohomology operator of $\mathfrak{g}$ and the bracket, $[\ ,\ ]_{\mathfrak{g}}$, is the algebraic Schouten bracket on $\wedge^\bullet\mathfrak{g}$.

Recall that a Lie algebra is called *quasi-Frobenius* if it possesses a nondegenerate 2-cocycle. Thus, we recover in particular the well-known correspondence [53, 17, 18] between nondegenerate triangular r-matrices, i.e., skew-symmetric solutions of the classical Yang–Baxter equation, and quasi-Frobenius structures.

Corollary 5.4. *A nondegenerate bivector in $\wedge^2\mathfrak{g}$ is a solution of the classical Yang–Baxter equation if and only if its inverse defines a quasi-Frobenius structure on $\mathfrak{g}$.*

5.2 Regular Twisted Poisson Structures

We summarize a result from [31] which can now be considered to be a corollary of Theorem 5.2. Let A be a vector bundle with a bivector $\pi \in \Gamma(\wedge^2 A)$ such that $\pi^\sharp$ is of constant rank. Let B be the image of $\pi^\sharp$. Then B is a Lie subbundle of A and, because π is skew-symmetric, $\pi^\sharp$ defines an isomorphism, $\pi^\sharp_B : B^* \to B$, where $B^* = A^* / \ker \pi^\sharp$ is the dual of B. Then the inverse of $\pi^\sharp_B$ defines a nondegenerate 2-form on B, $\omega_B \in \Gamma(\wedge^2 B^*)$, by $(\pi^\sharp_B)^{-1} X = -i_X \omega_B$, for $X \in \Gamma(B)$.

Assume that the vector bundle, A, is in fact a Lie algebroid. Let ψ be a d_A-closed 3-form on A, and let ψ_B denote the pull-back of ψ under the canonical injection $\iota_B : B \hookrightarrow A$. Then

Proposition 5.5. *Under the preceding assumptions, (A, π, ψ) is a Lie algebroid with a regular twisted Poisson structure if and only if (B, ω_B, ψ_B) is a Lie algebroid with a twisted symplectic structure, i.e., if and only if $d_B \omega = -\psi_B$.*

This proposition constitutes a linearization of the twisted Poisson condition, and can be applied in particular to the case of Lie algebras [31].

6 Another Type of Poisson Function: Lie Algebra Actions on Manifolds

In this section, we consider the twisting of various structures involving the action of a Lie algebra on a manifold.

6.1 Structures on $TM \times \mathfrak{g}^*$

Let $\mathfrak{g}$ be a Lie algebra, and let M be a manifold. We consider the vector bundle $V = TM \times \mathfrak{g}^*$ over M which is, by definition, $TM \underset{M}{\oplus} (M \times \mathfrak{g}^*) \to M$. We introduce local coordinates on $T^*\Pi V$, $(x^i,\ \xi^i,\ e_A,\ p_i,\ \theta_i,\ \epsilon^A)$, where $i = 1, \dots, \dim M$, and $A = 1, \dots, \dim \mathfrak{g}$, with the following bidegrees,

x^i	ξ^i	e_A	p_i	θ_i	ϵ^A	
$(0,0)$	$(0,1)$	$(0,1)$	$(1,1)$	$(1,0)$	$(1,0)$	bidegree
$(-1,-1)$	$(-1,0)$	$(-1,0)$	$(0,0)$	$(0,-1)$	$(0,-1)$	shifted bidegree

satisfying

$$\{x^i, p_j\} = \delta^i_j\,, \quad \{\xi^i, \theta_j\} = \delta^i_j\,, \quad \{e_A, \epsilon^B\} = \delta^B_A\,.$$

Let

$$S_{\mathfrak{g}} = \frac{1}{2} C^D_{AB} \epsilon^A \epsilon^B e_D$$

be the function on $T^*\Pi V$ of shifted bidegree $(1,0)$ defining the Lie bracket of $\mathfrak{g}$, and let

$$S_M = p_i \xi^i$$

be the function on $T^*\Pi V$ of shifted bidegree $(0,1)$ which defines the Schouten–Nijenhuis bracket of multivectors on M. Then

$$[u,v]_{\mathfrak{g}} = \{\{u, S_{\mathfrak{g}}\}, v\} \ , \tag{11}$$

for all u, $v \in \mathfrak{g}$, and

$$[X,Y]_M = \{\{X, S_M\}, Y\} \ , \tag{12}$$

for all X, $Y \in \Gamma(TM)$. It is easy to show that $S_{\mathfrak{g}} + S_M$ is a structure on V.

More generally, consider the following functions on $T^*\Pi V$ of shifted bidegree $(-1,2)$, a 3-form Ψ_M on M,

$$\Psi_M = \frac{1}{6}\Psi_{ijk}\xi^i\xi^j\xi^k \ ,$$

and a 3-form $\Psi_{\mathfrak{g}}$ on $\mathfrak{g}^*$,

$$\Psi_{\mathfrak{g}} = \frac{1}{6}\Psi^{ABC}e_A e_B e_C \ .$$

Then $S_{\mathfrak{g}} + S_M + (\Psi_{\mathfrak{g}} + \Psi_M)$ is a structure on V if and only if
- $\{S_M, \Psi_M\} = 0$, i.e., Ψ_M is a closed 3-form on M, and
- $\{S_{\mathfrak{g}}, \Psi_{\mathfrak{g}}\} = 0$, i.e., $\Psi_{\mathfrak{g}}$ is a 0-cocycle on $\mathfrak{g}$ with values in $\wedge^3\mathfrak{g}$.

More generally still, we can, in addition, introduce a function on $T^*\Pi V$ of shifted bidegree $(0,1)$ which defines a bracket on $\mathfrak{g}^*$,

$$S_{\mathfrak{g}^*} = \frac{1}{2}\Gamma^{AB}_C e_A e_B \epsilon^C \ .$$

Then $S = S_{\mathfrak{g}} + (S_{\mathfrak{g}^*} + S_M) + (\Psi_{\mathfrak{g}} + \Psi_M)$, a sum of terms of shifted bidegrees $(1,0)$, $(0,1)$ and $(-1,2)$, respectively, is a structure on V if and only if
- $\{S_M, \Psi_M\} = 0$, i.e., Ψ_M is a closed 3-form on M, and
- $\{S_{\mathfrak{g}} + S_{\mathfrak{g}^*} + \Psi_{\mathfrak{g}}, S_{\mathfrak{g}} + S_{\mathfrak{g}^*} + \Psi_{\mathfrak{g}}\} = 0$, the condition that $(\mathfrak{g}, \mathfrak{g}^*)$ be a *Lie-quasi bialgebra.*

Let us assume that these conditions are satisfied. By what function can we twist the structure $S_{\mathfrak{g}} + (S_{\mathfrak{g}^*} + S_M) + (\Psi_{\mathfrak{g}} + \Psi_M)$? We can twist it by any function of shifted bidegree $(1,-1)$. Therefore we can choose

$$\rho = \rho^i_A \epsilon^A \theta_i \ ,$$

and twist S by ρ, and/or we can twist S by the bivector

$$\pi = \frac{1}{2}\pi^{ij}\theta_i\theta_j \ .$$

We shall now prove, following Terashima [54], that twisting by $\rho+\pi$ provides a natural and unified way of determining the Lie algebroid structures discovered by Lu [38] and by Bursztyn, Crainic and Ševera [6,8]. This method yields an immediate proof of the fact that these are indeed Lie algebroid structures.

6.2 Twisting by a Lie Algebra Action

Let us first determine the meaning of the condition that ρ be a Poisson function with respect to $S = S_{\mathfrak{g}} + S_M$. We remark that ρ, considered either as a function on $T^*\Pi V$ or as a map from $\mathfrak{g}$ to $\Gamma(TM)$, satisfies, for all $u \in \mathfrak{g}$,

$$\{\rho, u\} = \rho(u) \ .$$

Computing the terms of shifted bidegrees $(2,-1)$, $(1,0)$ and $(0,1)$ of the twisted structure, $e^{-\rho}S$, we obtain

$$e^{-\rho}(S_{\mathfrak{g}} + S_M) = \left(-\{S_{\mathfrak{g}}, \rho\} + \frac{1}{2}\{\{S_M, \rho\}, \rho\}\right) + (S_{\mathfrak{g}} - \{S_M, \rho\}) + S_M \ .$$

Therefore ρ is a Poisson function with respect to $S = S_{\mathfrak{g}} + S_M$ if and only if

$$-\{S_{\mathfrak{g}}, \rho\} + \frac{1}{2}\{\{S_M, \rho\}, \rho\} = 0 \ . \tag{13}$$

Lemma 6.1. *The function ρ is a Poisson function with respect to $S_{\mathfrak{g}} + S_M$ if and only if it is a Lie algebra action of $\mathfrak{g}$ on M.*

Proof. The proof of the fact that relation (13) is equivalent to

$$\rho([u, v]_{\mathfrak{g}}) = [\rho(u), \rho(v)]_M$$

for all u, $v \in \mathfrak{g}$, depends on formulas (11) and (12), the Jacobi identity and the vanishing of all brackets of the form $\{e_A, \theta_i\}$ and $\{\epsilon^A, \theta_i\}$, whence

$$\rho([u, v]_{\mathfrak{g}}) = \{\{\{S_{\mathfrak{g}}, \rho\}, u\}, v\}$$

and

$$[\rho(u), \rho(v)]_M = \frac{1}{2}\{\{\{\{S_M, \rho\}, \rho\}, u\}, v\} \ . \qquad \square$$

6.3 Introducing Additional Twisting by a Bivector

Let us now twist $S = S_{\mathfrak{g}} + (S_{\mathfrak{g}^*} + S_M) + (\Psi_{\mathfrak{g}} + \Psi_M)$ by

$$\sigma = \pi + \rho \ .$$

We first observe that the brackets $\{\pi, \rho\}$, $\{S_{\mathfrak{g}}, \pi\}$, $\{S_{\mathfrak{g}*}, \pi\}$, $\{\{S_{\mathfrak{g}*}, \rho\}, \pi\}$ and $\{\Psi_{\mathfrak{g}}, \pi\}$ vanish. Computing the term of shifted bidegree $(2,-1)$ in $e^{-(\pi+\rho)}S$, we see that $\pi + \rho$ is a Poisson function with respect to S if and only if

$$\begin{aligned} -\{S_{\mathfrak{g}}, \rho\} + \tfrac{1}{2}\{\{S_{\mathfrak{g}*}, \rho\}, \rho\} + \tfrac{1}{2}\{\{S_M, \pi + \rho\}, \pi + \rho\} \\ - \tfrac{1}{6}\{\{\{\Psi_{\mathfrak{g}} + \Psi_M, \pi + \rho\}, \pi + \rho\}, \pi + \rho\} = 0. \end{aligned}$$

The computation of the several terms in this generalized twisted Maurer–Cartan equation yields

Proposition 6.2. *The function $\pi + \rho$ is a Poisson function with respect to $S = S_{\mathfrak{g}} + (S_{\mathfrak{g}^*} + S_M) + (\Psi_{\mathfrak{g}} + \Psi_M)$ if and only if the following four conditions are satisfied:*

(A) $$\{\{\{\Psi_M, \rho\}, \rho\}, \rho\} = 0 \ ,$$

(B) $$-\{S_{\mathfrak{g}}, \rho\} + \frac{1}{2}\{\{S_M, \rho\}, \rho\} - \frac{1}{2}\{\{\{\Psi_M, \rho\}, \rho\}, \pi\} = 0 \ ,$$

(C) $$\{\{S_M, \pi\}, \rho\} + \frac{1}{2}\{\{S_{\mathfrak{g}*}, \rho\}, \rho\} - \frac{1}{2}\{\{\{\Psi_M, \rho\}, \pi\}, \pi\} = 0 \ ,$$

(D) $$\{\{S_M, \pi\}, \pi\} - \frac{1}{3}\{\{\{\Psi_{\mathfrak{g}}, \rho\}, \rho\}, \rho\} - \frac{1}{3}\{\{\{\Psi_M, \pi\}, \pi\}, \pi\} = 0 \ .$$

Condition (A) is the relation $i_{\rho(u)\wedge\rho(v)\wedge\rho(w)}\Psi_M = 0$, for all $u, v, w \in \mathfrak{g}$, which means that Ψ_M is in the kernel of $\wedge^3\rho^*$, where ρ^* is the dual of ρ.

Condition (B) is the relation

$$\rho([u, v]_{\mathfrak{g}}) - [\rho(u), \rho(v)]_M = \pi^\sharp(i_{\rho(u)\wedge\rho(v)}\Psi_M) \ , \tag{14}$$

for all $u, v \in \mathfrak{g}$. This is proved by the same computations as in Lemma 6.1. Thus (B) expresses the fact that ρ is a *twisted action* of $\mathfrak{g}$ on M.

Condition (C) is the relation

$$\mathcal{L}_{\rho(u)}\pi = -(\wedge^2\rho)(\gamma(u)) + (\wedge^2\pi^\sharp)(i_{\rho(u)}\Psi_M) \ , \tag{15}$$

for all $u \in \mathfrak{g}$, where $\gamma : \mathfrak{g} \to \wedge^2\mathfrak{g}$ is $S_{\mathfrak{g}*}$ viewed as a cobracket on $\mathfrak{g}$. In fact,

$$\{\{\{S_M, \pi\}, \rho\}, u\} = \{\{\{\rho, u\}, S_M\}, \pi\} = [\{\rho, u\}, \pi]_M = \mathcal{L}_{\rho(u)}\pi \ ,$$

while

$$\frac{1}{2}\{\{\{S_{\mathfrak{g}*}, \rho\}, \rho\}, u\} = (\wedge^2\rho)(\gamma(u)) \ ,$$

and

$$\frac{1}{2}\{\{\{\{\Psi_M, \rho\}, \pi\}, \pi\}, u\} = (\wedge^2\pi^\sharp)(i_{\rho(u)}\Psi_M) \ .$$

Condition (D) is the relation

$$\frac{1}{2}[\pi, \pi]_M = (\wedge^3\rho)(\Psi_{\mathfrak{g}}) + (\wedge^3\pi^\sharp)(\Psi_M) \ . \tag{16}$$

6.4 Particular Cases

In the light of Proposition 6.2 and formulas (14), (15) and (16), we can interpret several important particular cases of Poisson functions of the type $\pi + \rho$.

• **Case** $\rho = 0$, already studied in Sect. 3.2. Conditions (A), (B) and (C) are identically satisfied and (D) is the condition that M be a *twisted Poisson* *manifold.* If $\rho = 0$ and $\Psi_M = 0$, then (D) is the condition that M be a *Poisson manifold.*

• **Case $\Psi_M = 0$.** While condition (A) is identically satisfied, conditions (B), (C) and (D) express the fact that M is a *quasi-Poisson* $\mathfrak{g}$*-space*, the version of the quasi-Poisson G-spaces in the sense of [1] in which only an infinitesimal Lie algebra action is assumed. When the Lie group G is connected and simply connected, conditions (B), (C) and (D) imply that M is a *quasi-Poisson* G*-space*, and conversely.

• **Case $\Psi_M = 0$ and $S_{\mathfrak{g}*} = 0$.** Conditions (B), (C) and (D) are
(B) M is a $\mathfrak{g}$-manifold,
(C) π is a $\mathfrak{g}$-invariant bivector,
(D) $\frac{1}{2}[\pi, \pi]_M = (\wedge^3 \rho)(\Psi_{\mathfrak{g}})$.

If $\Psi_{\mathfrak{g}}$ is the Cartan 3-vector of the Lie algebra $\mathfrak{g}$ of a connected and simply connected Lie group with a bi-invariant scalar product, conditions (B), (C) and (D) express the fact that M is a *quasi-Poisson* $\mathfrak{g}$*-manifold*, the version of the quasi-Poisson G-manifolds in the sense of [2] in which only an infinitesimal Lie algebra action is assumed. When the Lie group G is connected and simply connected, conditions (B), (C) and (D) imply that M is a *quasi-Poisson* G*-manifold*, and conversely.

• **Case $\Psi_M = 0$ and $\Psi_{\mathfrak{g}} = 0$.** In this case, $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra. Condition (D) expresses the fact that π is a Poisson bivector, and equations (14) and (15) show that conditions (B) and (C) express the fact that ρ is an *infinitesimal Poisson action of the Lie bialgebra* $(\mathfrak{g}, \mathfrak{g}^*)$ *on the Poisson manifold* M in the sense of Lu and Weinstein [39, 38] (which can also be called a *Lie bialgebra action*), corresponding to a Poisson action of the connected and simply connected Poisson–Lie group with Lie algebra $\mathfrak{g}$.

Remark The method described here for the characterization of Poisson and quasi-Poisson structures can be used to recover conditions defining Poisson–Nijenhuis [30] and Poisson quasi-Nijenhuis [52] structures.

6.5 The Lie Algebroid Structure of $V^* = T^*M \times \mathfrak{g}$

Whenever σ is a Poisson function with respect to a structure S on V, with $e^{-\sigma}S$, (V, V^*) becomes a quasi-Lie bialgebroid. Therefore when $\sigma = \pi + \rho$ is a Poisson function with respect to the structure $S = S_{\mathfrak{g}} + (S_{\mathfrak{g}^*} + S_M) + (\Psi_{\mathfrak{g}} + \Psi_M)$ on $V = TM \times \mathfrak{g}^*$, there is a Lie algebroid structure on $V^* = T^*M \times \mathfrak{g}$, with anchor $\pi^\sharp + \rho$ and Lie bracket

$$\gamma_\sigma = S_{\mathfrak{g}} - \{S_{\mathfrak{g}*} + S_M, \pi + \rho\} + \frac{1}{2}\{\{\Psi_{\mathfrak{g}} + \Psi_M, \pi + \rho\}, \pi + \rho\} \ , \tag{17}$$

and $\{\gamma_\sigma, .\}$ is a differential on $\Gamma(\wedge^\bullet(TM \times \mathfrak{g}^*))$. Dually, there is a bracket μ_σ on $TM \times \mathfrak{g}^*$, but the Jacobi identity is not satisfied in general and the derivation $\{\mu_\sigma, .\}$ on $\Gamma(\wedge^\bullet(T^*M \times \mathfrak{g}))$ does not square to zero in general, since (V, V^*) is only a quasi-Lie bialgebroid. From formula (17) and Proposition 6.2, we obtain:

Theorem 6.3. *When conditions (A)–(D) are satisfied, $T^*M \times \mathfrak{g}$ is a Lie algebroid with anchor $\pi^\sharp + \rho$ and Lie bracket*

$$\begin{aligned}\gamma_\sigma &= S_{\mathfrak{g}} - \{S_{\mathfrak{g}^*}, \rho\} - \{S_M, \pi\} - \{S_M, \rho\} \\ &\quad + \tfrac{1}{2}\{\{\Psi_{\mathfrak{g}}, \rho\}, \rho\} + \tfrac{1}{2}\{\{\Psi_M, \pi\}, \pi\} + \{\{\Psi_M, \pi\}, \rho\} + \tfrac{1}{2}\{\{\Psi_M, \rho\}, \rho\}\,. \end{aligned} \tag{18}$$

We shall now show that the preceding general formula yields the brackets of [38], of [6], and of [8] as particular cases.

Case $\rho = 0$. Formula (18) reduces to $\gamma_\sigma = S_{\mathfrak{g}} - \{S_M, \pi\} + \frac{1}{2}\{\{\Psi_M, \pi\}, \pi\}$. The Lie algebroid structure of $V^* = T^*M \times \mathfrak{g}$ is the direct sum of the pointwise Lie bracket of sections of $M \times \mathfrak{g} \to M$ and the Lie algebroid bracket of Ševera and Weinstein [49] on $\Gamma(T^*M)$ for the twisted Poisson manifold (M, π, Ψ_M).

Case $\Psi_M = 0$. Formula (18) reduces to

$$\gamma_\sigma = S_{\mathfrak{g}} - \{S_{\mathfrak{g}*}, \rho\} - \{S_M, \pi\} - \{S_M, \rho\} + \frac{1}{2}\{\{\Psi_{\mathfrak{g}}, \rho\}, \rho\}\ .$$

For u, $v \in \Gamma(M \times \mathfrak{g})$ and α, $\beta \in \Gamma(T^*M)$, we obtain the following expressions entering in the brackets of sections of $T^*M \times \mathfrak{g}$.

$$\begin{cases} \{\{u, S_{\mathfrak{g}} - \{S_M, \rho\}\}, v\} = [u, v]_{\mathfrak{g}} + \mathcal{L}_{\rho(u)} v - \mathcal{L}_{\rho(v)} u\ , \\ \{\{\alpha, \{S_{\mathfrak{g}*}, \rho\}\}, u\} = -i_{\rho^*(\alpha)}\{S_{\mathfrak{g}^*}, u\} = \mathrm{ad}^*_{\rho^*(\alpha)} u\ , \\ \{\{\alpha, \{S_M, \pi\}\}, u\} = -\mathcal{L}_{\pi^\sharp(\alpha)} u\ , \\ \{\{\alpha, \{S_M, \rho\}\}, u\} = \mathcal{L}_{\rho(u)} \alpha\ , \\ \{\{\alpha, \{S_M, \pi\}\}, \beta\} = -[\alpha, \beta]_\pi\ , \\ \frac{1}{2}\{\{\alpha, \{\{\Psi_{\mathfrak{g}}, \rho\}, \rho\}, \beta\} = i_{(\wedge^2\rho^*)(\alpha\wedge\beta)} \Psi_{\mathfrak{g}}\ , \end{cases}$$

where $\mathcal{L}$ denotes the Lie derivation of vector-valued functions and of forms by vectors, and ad^* is defined by means of the bracket of $\mathfrak{g}^*$. The bracket defined by γ_σ is therefore

$$\begin{cases} [u, v] = [u, v]_{\mathfrak{g}} + \mathcal{L}_{\rho(u)} v - \mathcal{L}_{\rho(v)} u\ , \\ [\alpha, u] = \mathcal{L}_{\pi^\sharp(\alpha)} u - \mathcal{L}_{\rho(u)} \alpha - \mathrm{ad}^*_{\rho^*(\alpha)} u\ , \\ [\alpha, \beta] = [\alpha, \beta]_\pi + i_{(\wedge^2\rho^*)(\alpha\wedge\beta)} \Psi_{\mathfrak{g}}\ . \end{cases}$$

The bracket $[u, v]$ is the *transformation Lie algebroid bracket* [40, 42] on $M \times \mathfrak{g} \to M$. Summarizing this discussion, we obtain

Proposition 6.4. *If $\Psi_M = 0$, then M is a quasi-Poisson $\mathfrak{g}$-space in the sense of [1] and the Lie algebroid bracket of $T^*M \times \mathfrak{g}$ is the bracket of Bursztyn, Crainic and Ševera [8]. In particular, if $\Psi_M = 0$ and $S_{\mathfrak{g}^*} = 0$, then M is a quasi-Poisson $\mathfrak{g}$-manifold in the sense of [2], and the Lie algebroid bracket of $T^*M \times \mathfrak{g}$ is the bracket of Bursztyn and Crainic [6].*

Case $\Psi_M = \Psi_{\mathfrak{g}} = 0$. Formula (18) reduces to

$$\gamma_\sigma = S_{\mathfrak{g}} - \{S_{\mathfrak{g}^*}, \rho\} - \{S_M, \pi\} - \{S_M, \rho\} \ .$$

Introducing the notations of Lu [38], the bracket of Bursztyn, Crainic and Ševera reduces to the following expressions, for α, $\beta \in \Gamma(T^*M)$, and constant sections u, v of $M \times \mathfrak{g}$,

$$\begin{cases} [u,v] \ = [u,v]_{\mathfrak{g}}, \\ [\alpha,u] = D_\alpha u - D_u \alpha, \\ [\alpha,\beta] = [\alpha,\beta]_\pi. \end{cases}$$

Proposition 6.5. *If $\Psi_M = 0$ and $\Psi_{\mathfrak{g}} = 0$, then M is a manifold with a Lie bialgebra action and the Lie algebroid bracket of $T^*M \times \mathfrak{g}$ is the bracket of Lu* [38]*, defining a matched pair of Lie algebroids.*

6.6 The Twisted Differential

Let us determine the differential $d_{\gamma_\sigma} = \{\gamma_\sigma, .\}$ on $\Gamma(\wedge^\bullet(TM \times \mathfrak{g}^*))$, where γ_σ is defined by (18). The particular case of the quasi-Poisson $\mathfrak{g}$-spaces was recently treated in [7].

We first prove that the image of a section $X \otimes \eta$ of $\wedge^k TM \otimes \wedge^\ell \mathfrak{g}^*$ is a section of $\sum_{-1 \leq j \leq 2} \wedge^{k+j} TM \otimes \wedge^{\ell+1-j} \mathfrak{g}^*$. We shall write $\Gamma(\mathfrak{g}^*)$ for $\Gamma(M \times \mathfrak{g}^* \to M)$. In fact, for $X \in \Gamma(\wedge^k TM)$,

$$\begin{cases} \{\{S_M, \pi\}, X\} \text{ and } \{\{\{\Psi_M, \pi\}, \pi\}, X\} \in \Gamma(\wedge^{k+1} TM)\,, \\ \{\{S_M, \rho\}, X\} \text{ and } \{\{\{\Psi_M, \pi\}, \rho\}, X\} \in \Gamma(\wedge^k TM \otimes \mathfrak{g}^*)\,, \\ \{\{\{\Psi_M, \rho\}, \rho\}, X\} \in \Gamma(\wedge^{k-1} TM \otimes \wedge^2 \mathfrak{g}^*)\,, \end{cases}$$

and for $\eta \in \Gamma(\wedge^\ell \mathfrak{g}^*)$,

$$\begin{cases} \{S_{\mathfrak{g}}, \eta\} \text{ and } \{\{S_M, \rho\}, \eta\} \in \Gamma(\wedge^{\ell+1} \mathfrak{g}^*)\,, \\ \{\{S_{\mathfrak{g}^*}, \rho\}, \eta\} \in \text{ and } \{\{S_M, \pi\}, \eta\} \in \Gamma(TM \otimes \wedge^\ell \mathfrak{g}^*)\ , \\ \{\{\{\Psi_{\mathfrak{g}}, \rho\}, \rho\}, \eta\} \in \Gamma(\wedge^2 TM \otimes \wedge^{\ell-1} \mathfrak{g}^*)\ , \end{cases}$$

while all other brackets vanish.

Each derivation is determined by its values on the elements of degree 0 and 1. If $f \in C^\infty(M)$,

$$(d_{\gamma_\sigma} f)(\alpha + u) = (\pi^\sharp(\alpha) + \rho(u)) \cdot f\ , \tag{19}$$

for $\alpha \in \Gamma(T^*M)$ and $u \in \Gamma(\mathfrak{g})$. If $X \in \Gamma(TM)$, $d_{\gamma_\sigma}(X)$ is the sum of the following terms,

$$\begin{cases} -\{\{S_M,\pi\},X\}+\frac{1}{2}\{\{\{\Psi_M,\pi\},\pi\},X\}=[\pi,X]_M+(\wedge^2\pi^\sharp)(i_X\Psi_M) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \in\Gamma(\wedge^2TM)\,, \\ -\{\{S_M,\rho\},X\}+\{\{\{\Psi_M,\pi\},\rho\},X\}=[\rho(.),X]_M+(\pi^\sharp\wedge\rho)(i_X\Psi_M) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \in\Gamma(TM\otimes\mathfrak{g}^*)\ , \\ \frac{1}{2}\{\{\{\Psi_M,\rho\},\rho\},X\}=(\wedge^2\rho)(i_X\Psi_M)\in\Gamma(\wedge^2\mathfrak{g}^*)\ , \end{cases}$$

where $[\rho(.),X]_M : u\in\mathfrak{g}\mapsto[\rho(u),X]_M\in\Gamma(TM)$. For $\eta\in\Gamma(\mathfrak{g}^*)$, $d_{\gamma_\sigma}(\eta)$ is the sum of the following terms,

$$\begin{cases} \{S_\mathfrak{g},\eta\}-\{\{S_M,\rho\},\eta\}=d_\mathfrak{g}\eta+\ll\mathcal{L}_{\rho(.)}\eta,.\gg\in\Gamma(\wedge^2\mathfrak{g}^*)\ , \\ -\{\{S_{\mathfrak{g}^*},\rho\},\eta\}-\{\{S_M,\pi\},\eta\}=\rho(\mathrm{ad}^*_\eta(.))+\mathcal{L}_{\pi^\sharp(.)}\eta\in\Gamma(TM\otimes\mathfrak{g}^*), \\ \frac{1}{2}\{\{\{\Psi_\mathfrak{g},\rho\},\rho\},\eta\}=-(\wedge^2\rho)(i_\eta\Psi_\mathfrak{g})\in\Gamma(\wedge^2TM)\ , \end{cases} \tag{20}$$

where $\ll\mathcal{L}_{\rho(.)}\eta,.\gg$: $(u,v)\in\wedge^2\mathfrak{g}\mapsto\langle\mathcal{L}_{\rho(u)}\eta,v\rangle-\langle\mathcal{L}_{\rho(v)}\eta,u\rangle\in C^\infty(M)$, $\rho(\mathrm{ad}^*_\eta(.)) : u\in\mathfrak{g}\to\rho(\mathrm{ad}^*_\eta(u))\in\Gamma(TM)$, and $\mathcal{L}_{\pi^\sharp(.)}\eta : \alpha\in\Gamma(T^*M)\mapsto\mathcal{L}_{\pi^\sharp(\alpha)}\eta\in\Gamma(\mathfrak{g}^*)$. The derivation d_{γ_σ} is then extended to all sections of $\wedge^\bullet(TM\times\mathfrak{g}^*)$ by the graded Leibniz rule. We have thus obtained the following:

Theorem 6.6. *Let $\sigma=\pi+\rho$ be a Poisson function with respect to the structure $S=S_\mathfrak{g}+(S_{\mathfrak{g}^*}+S_M)+(\Psi_\mathfrak{g}+\Psi_M)$.*
(i) For γ_σ defined by (18), $d_{\gamma_\sigma}=\{\gamma_\sigma,.\}$ is a differential on $\Gamma(\wedge^\bullet(TM\times\mathfrak{g}^*))$.
(ii) $d_{\gamma_\sigma}=\sum_{-1\leq j\leq 2}d_{(j,1-j)}$, where

$$d_{(j,1-j)}:\Gamma(\wedge^kTM\otimes\wedge^\ell\mathfrak{g}^*)\to\Gamma(\wedge^{k+j}TM\otimes\wedge^{\ell+1-j}\mathfrak{g}^*)\,,$$

and

$$\begin{aligned} d_{(-1,2)}&=\tfrac{1}{2}\{\{\{\Psi_M,\rho\},\rho\},.\}\,, \\ d_{(0,1)}&=\{-\{S_M,\rho\}+\{\{\Psi_M,\pi\},\rho\}+S_\mathfrak{g},.\}\,, \\ d_{(1,0)}&=\{-\{S_M,\pi\}+\tfrac{1}{2}\{\{\Psi_M,\pi\},\pi\}-\{S_{\mathfrak{g}^*},\rho\},.\}\,, \\ d_{(2,-1)}&=\tfrac{1}{2}\{\{\{\Psi_\mathfrak{g},\rho\},\rho\},.\}\,. \end{aligned}$$

(iii) For $f\in C^\infty(M)$ and $\eta\in\Gamma(M\times\mathfrak{g}^*\to M)$, $d_{\gamma_\sigma}(f)$ and $d_{\gamma_\sigma}(\eta)$ are determined by Equations (19) and (20) while, for $X\in\Gamma(TM)$,

$$d_{\gamma_\sigma}(X)=[\pi,X]_M+[\rho(.),X]_M+(\wedge^2\pi^\sharp+\pi^\sharp\wedge\rho+\wedge^2\rho)(i_X\Psi_M)\ .$$

These formulas simplify in each of the particular cases listed in Sect. 6.4. In the case of the quasi-Poisson $\mathfrak{g}$-spaces, $d_{\gamma_\sigma}(X)=[\pi,X]_M+\mathcal{L}_{\rho(.)}X$. From this formula and from (19), it follows that the restriction of d_{γ_σ} to the space of $\mathfrak{g}$-invariant multivectors on M is the differential of the *quasi-Poisson cohomology* introduced in [2]. This fact was observed in [54].

Remark Throughout this section, the tangent bundle TM can be replaced by an arbitrary Lie algebroid over M, provided that the de Rham differential is replaced by the differential associated with the Lie algebroid in order to yield more general results.

Acknowledgments

The main results of this paper were presented at the international conference "Higher Structures in Geometry and Physics" which was held in honor of Murray Gerstenhaber and Jim Stasheff at the Institut Henri Poincaré in Paris in January 2007. I am very grateful to the organizers, Alberto Cattaneo and Ping Xu, for the invitation to participate in this exciting conference.

I thank Murray Gerstenhaber, Jim Stasheff, and Dmitry Roytenberg for their remarks and useful conversations.

References

1. Alekseev, A., Kosmann-Schwarzbach, Y.: Manin pairs and moment maps. J. Diff. Geom. **56**, 133–165 (2000)
2. Alekseev, A., Kosmann-Schwarzbach, Y., Meinrenken, E.: Quasi-Poisson manifolds. Can. J. Math. **54**, 3–29 (2002)
3. Bangoura, M.: Algèbres quasi-Gerstenhaber différentielles. Travaux Mathématiques (Luxembourg) **16**, 299–314 (2005)
4. Bangoura, M.: Algèbres d'homotopie associées à une proto-bigèbre de Lie. Can. J. Math. **59**, 696–711 (2007)
5. Bangoura, M., Kosmann-Schwarzbach, Y.: The double of a Jacobian quasi-bialgebra. Lett. Math. Phys. **28**, 13–29 (1993)
6. Bursztyn, H., Crainic, M.: Dirac structures, momentum maps, and quasi-Poisson manifolds. In: Marsden, J., Ratiu, T. (eds.) The Breadth of Symplectic and Poisson Geometry. Progress in Mathematics, vol. 232, pp. 1–40. Birkhäuser, Boston (2005)
7. Bursztyn, H., Crainic, M.: Dirac geometry, quasi-Poisson actions and D/G-valued moment maps. J. Diff. Geom. **82**, 501–566 (2009)
8. Bursztyn, H., Crainic, M., Ševera, P.: Quasi-Poisson structures as Dirac structures. Travaux Mathématiques (Luxembourg) **16**, 41–52 (2005)
9. Coste, A., Dazord, P., Weinstein, A.: Groupoïdes symplectiques. Publ. Dép. Math. Univ. Claude Bernard Lyon, Nouvelle Sér. **2**/A, 1–62 (1987)
10. Courant, T.: Dirac manifolds. Trans. Am. Math. Soc. **319**, 631–661 (1990)
11. Courant, T., Weinstein, A.: Beyond Poisson structures. In Actions hamiltoniennes de groupes. Troisième théorème de Lie. Sémin. Sud-Rhodan. Géom. VIII (Lyon, 1986) Travaux en Cours, vol. 27, pp. 39–49, Hermann, Paris (1988)
12. Drinfeld, V.: Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of classical Yang-Baxter equations. Dokl. Akad. Nauk SSSR **268**, 285–287 (1983); translation in Soviet Math. Dokl. **27**, 68–71 (1983)
13. Drinfeld, V.: Quasi-Hopf algebras. Algebra i Analiz **1**, 114–148 (1989); translation in Leningrad Math. J. **1**, 1419–1457 (1990)
14. Ehresmann, C.: Catégories topologiques et catégories différentiables. Centre Belge Rech. Math. Colloque Géom. Différ. Globale (Bruxelles 1958) 137–150 (1959)
15. Ehresmann, C.: Sur les catégories différentiables. Atti Convegno internaz. Geom. Diff. (Bologna 1967) 31–40 (1970)
16. Gerstenhaber, M.: The cohomology structure of an associative ring. Ann. Math. (2) **78**, 267–288 (1963)

17. Gerstenhaber, M., Giaquinto, A.: Boundary solutions of the classical Yang–Baxter equation. Lett. Math. Phys. **40**, 337–353 (1997)
18. Hodges, T.J., Yakimov, M.: Triangular Poisson structures on Lie groups and symplectic reduction. In: Noncommutative Geometry and Representation Theory in Mathematical Physics. Contemporary Mathematics, vol. 391, pp. 123–134. Am. Math. Soc., Providence, RI (2005)
19. Huebschmann, J.: Poisson cohomology and quantization. J. Reine Angew. Math. **408**, 57–113 (1990)
20. Huebschmann, J.: Higher homotopies and Maurer-Cartan algebras: quasi-Lie-Rinehart, Gerstenhaber, and Batalin-Vilkovisky algebras. In: Marsden, J., Ratiu, T. (eds.) The Breadth of Symplectic and Poisson Geometry. Progress in Mathematics, vol. 232, pp. 237–302. Birkhäuser, Boston (2005)
21. Jacobson, N.: On pseudo-linear transformations. Proc. Nat. Acad. Sci. USA **21**, 667–670 (1935)
22. Klimčík, C., Strobl, T.: WZW-Poisson manifolds. J. Geom. Phys. **43**, 341–344 (2002)
23. Kosmann-Schwarzbach, Y.: Jacobian quasi-bialgebras and quasi-Poisson Lie groups. In: Mathematical Aspects of Classical Field Theory (Seattle 1991). Contemporary Mathematics, vol. 132, pp. 459–489. Am. Math. Soc., Providence, RI (1992)
24. Kosmann-Schwarzbach, Y.: Exact Gerstenhaber algebras and Lie bialgebroids. Acta Appl. Math. **41**, 153–165 (1995)
25. Kosmann-Schwarzbach, Y.: From Poisson algebras to Gerstenhaber algebras. Ann. Inst. Fourier (Grenoble) **46**, 1243–1274 (1996)
26. Kosmann-Schwarzbach, Y.: Derived brackets. Lett. Math. Phys. **69**, 61–87 (2004)
27. Kosmann-Schwarzbach, Y.: Quasi, twisted, and all that... in Poisson geometry and Lie algebroid theory. In: Marsden, J., Ratiu, T. (eds.) The Breadth of Symplectic and Poisson Geometry. Progress in Mathematics, vol. 232, pp. 363–389. Birkhäuser, Boston (2005)
28. Kosmann-Schwarzbach, Y., Laurent-Gengoux, C.: The modular class of a twisted Poisson structure. Travaux Mathématiques (Luxembourg) **16**, 315–339 (2005)
29. Kosmann-Schwarzbach, Y., Mackenzie, K.C.H.: Differential operators and actions of Lie algebroids. In: Voronov, T. (ed.) Quantization, Poisson Brackets and Beyond. Contemporary Mathematics, vol. 315, pp. 213–233. Amer. Math. Soc., Providence, RI (2002)
30. Kosmann-Schwarzbach, Y., Magri, F.: Poisson-Nijenhuis structures. Ann. Inst. Henri Poincaré, Série A, **53**, 35–81 (1990)
31. Kosmann-Schwarzbach, Y., Yakimov, M.: Modular classes of regular twisted Poisson structures on Lie algebroids. Lett. Math. Phys. **80**, 183–197 (2007)
32. Kostant, B., Sternberg, S.: Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras. Ann. Phys. **176**, 49–113 (1987)
33. Koszul, J.-L.: Crochet de Schouten–Nijenhuis et cohomologie. The Mathematical heritage of Élie Cartan (Lyon, 1984). Astérisque, numéro hors série, 257–271 (1985)
34. Lecomte, P., Roger, C.: Modules et cohomologies des bigèbres de Lie. C. R. Acad. Sci. Paris Sér. I Math. **310**, 405–410 (1990)
35. Lichnerowicz, A.: Les variétés de Poisson et leurs algèbres de Lie associées. J. Diff. Geom. **12**, 253–300 (1977)

36. Liu, Z.-J., Weinstein, A., Xu, P.: Manin triples for Lie bialgebroids. J. Diff. Geom. **45**, 547–574 (1997)
37. Loday, J.-L.: Une version non commutative des algèbres de Lie: les algèbres de Leibniz. Enseign. Math. **39**, 269–293 (1993)
38. Lu, J.-H.: Poisson homogeneous spaces and Lie algebroids associated to Poisson actions. Duke Math. J. **86**, 261–304 (1997)
39. Lu, J.-H., Weinstein, A.: Poisson Lie groups, dressing transformations, and Bruhat decompositions. J. Diff. Geom. **31**, 501–526 (1990)
40. Mackenzie, K.C.H.: Lie Groupoids and Lie Algebroids in Differential Geometry. London Mathematical Society Lecture Note Series, vol. 124. Cambridge University Press, Cambridge (1987)
41. Mackenzie, K.C.H.: Lie algebroids and Lie pseudoalgebras. Bull. Lond. Math. Soc. **27**, 97–147 (1995)
42. Mackenzie, K.C.H.: General Theory of Lie Groupoids and Lie Algebroids. London Mathematical Society Lecture Note Series, vol. 213. Cambridge University Press, Cambridge (2005)
43. Mackenzie, K.C.H., Xu, P.: Lie bialgebroids and Poisson groupoids. Duke Math. J. **73**, 415–452 (1994)
44. Moerdijk, I., Mrčun, J.: Introduction to Foliations and Lie Groupoids. Cambridge Studies in Advanced Mathematics, vol. 91. Cambridge University Press, Cambridge (2003)
45. Pradines, J.: Théorie de Lie pour les groupoïdes différentiables. Calcul différentiel dans la catégorie des groupoïdes infinitésimaux. C. R. Acad. Sci. Paris Sér. A–B, **264**, A245–A248 (1967)
46. Roytenberg, D.: Quasi-Lie bialgebroids and twisted Poisson manifolds. Lett. Math. Phys. **61**, 123–137 (2002)
47. Roytenberg, D.: e.mail message (2007)
48. Saksida, P.: Lattices of Neumann oscillators and Maxwell-Bloch equations. Nonlinearity **19**, 747–768 (2006)
49. Ševera, P., Weinstein, A.: Poisson geometry with a 3-form background. In: Noncommutative Geometry and String Theory (Yokohama, 2001) Progr. Theoret. Phys. Suppl. **144**, 145–154 (2001)
50. Stasheff, J.: Constrained Hamiltonians, BRS and homological algebra. In: Proceedings of the Conference on Elliptic Curves and Modular Forms in Algebraic Topology (Princeton, 1986) Springer Lecture Notes in Mathematics, vol. 1326, pp. 150–160. Springer, Berlin (1988)
51. Stasheff, J.: Differential graded Lie algebras, quasi-Hopf algebras and higher homotopy algebras. In: Quantum Groups (Leningrad, 1990) Lecture Notes in Mathematics, vol. 1510, pp. 120–137. Springer, Berlin (1992)
52. Stiénon, M., Xu, P.: Poisson quasi-Nijenhuis manifolds. Comm. Math. Phys. **270**, 709–725 (2007)
53. Stolin, A.: On rational solutions of Yang–Baxter equation for $\mathfrak{sl}(n)$. Math. Scand. **69**, 57–80 (1991)
54. Terashima, Y.: On Poisson functions. J. Sympl. Geom. **6**(1), 1–7 (2008)
55. Vaintrob, A.: Lie algebroids and homological vector fields. Uspekhi Mat. Nauk **52**, 2(314), 161–162 (1997); translation in Russ. Math. Surv. **52**, 428–429 (1997)
56. Voronov, T.: Graded manifolds and Drinfeld doubles for Lie bialgebroids. In: Voronov, T. (ed.) Quantization, Poisson Brackets and Beyond. Contemporary Mathematics , vol. 315, pp. 131–168. Amer. Math. Soc., Providence, RI (2002)
57. Xu, P.: Gerstenhaber algebras and BV-algebras in Poisson geometry. Comm. Math. Phys. **200**, 545–560 (1999)

The Diagonal of the Stasheff Polytope

Jean-Louis Loday

Institut de Recherche Mathématique Avancée, CNRS et Université de Strasbourg, 7 rue R. Descartes, 67084 Strasbourg Cedex, France, loday@math.unistra.fr

To Murray Gerstenhaber and Jim Stasheff

Abstract. We construct an A-infinity structure on the tensor product of two A-infinity algebras by using the simplicial decomposition of the Stasheff polytope. The key point is the construction of an operad AA-infinity based on the simplicial Stasheff polytope. The operad AA-infinity admits a coassociative diagonal and the operad A-infinity is a retract by deformation of it. We compare these constructions with analogous constructions due to Saneblidze–Umble and Markl–Shnider based on the Boardman–Vogt cubical decomposition of the Stasheff polytope.

Key words: Stasheff polytope, Associahedron, Operad, Bar–cobar construction, Cobar construction, A-infinity algebra, AA-infinity algebra, Diagonal

Introduction

An associative algebra up to homotopy, or A_∞-algebra, is a chain complex (A, d_A) equipped with an n-ary operation μ_n for each $n \geq 2$ verifying $\mu \circ \mu = 0$. See [15], or, for instance, [5]. Here we put

$$\mu := d_A + \mu_2 + \mu_3 + \cdots : T(A) \to T(A),$$

where μ_n has been extended to the tensor coalgebra $T(A)$ by coderivation. In particular, μ_2 is not associative, but only associative up to homotopy in the following sense:

$$\mu_2 \circ (\mu_2 \otimes \mathrm{id}) - \mu_2 \circ (\mathrm{id} \otimes \mu_2) = d_A \circ \mu_3 + \mu_3 \circ d_{A^{\otimes 3}}.$$

Putting an A_∞-algebra structure on the tensor product of two A_∞-algebras is a longstanding problem, cf. for instance [12, 2]. Recently a solution has been constructed by Saneblidze and Umble, cf. [13, 14], by providing a diagonal

A.S. Cattaneo et al. (eds.), *Higher Structures in Geometry and Physics*, Progress in Mathematics 287, DOI 10.1007/978-0-8176-4735-3_13,

$A_\infty \to A_\infty \otimes A_\infty$ on the operad A_∞ which governs the A_∞-algebras. Recall that, over a field, the operad A_∞ is the minimal model of the operad As governing the associative algebras. The differential graded module $(A_\infty)_n$ of the n-ary operations is the chain complex of the Stasheff polytope. The method of Saneblidze and Umble consists in providing an explicit (i.e., combinatorial) diagonal of the Stasheff polytope considered as a cellular complex. In [11] Markl and Shnider give a construction of the Saneblidze–Umble diagonal by using the Boardman–Vogt model of As. This model is the bar–cobar construction on As, denoted ΩBAs, in the operadic framework. It turns out that there exists a coassociative diagonal on ΩBAs, which is constructed out of the diagonal of the cube. This diagonal, together with the quasi-isomorphisms $q : A_\infty \to \Omega BAs$ and $p : \Omega BAs \to A_\infty$ permit them to construct a diagonal on A_∞ by composition:

$$A_\infty \xrightarrow{q} \Omega BAs \to \Omega BAs \otimes \Omega BAs \xrightarrow{p\otimes p} A_\infty \otimes A_\infty .$$

The aim of this paper is to give an alternative solution to the diagonal problem by relying on the *simplicial decomposition of the Stasheff polytope* described in [8] and using the diagonal of the standard simplex. It leads to a new model AA_∞ of the operad As, whose dg module $(AA_\infty)_n$ is the chain complex of a simplicial decomposition of the Stasheff polytope. Because of its simplicial nature, the operad AA_∞ has a coassociative diagonal (by means of the Alexander–Whitney map) and therefore we get a diagonal on A_∞ by composition:

$$A_\infty \xrightarrow{q'} AA_\infty \to AA_\infty \otimes AA_\infty \xrightarrow{p'\otimes p'} A_\infty \otimes A_\infty .$$

The map $q' : A_\infty \to AA_\infty$ is induced by the simplicial decomposition of the associahedron. The map $p' : AA_\infty \to A_\infty$ is slightly more involved to construct. It is induced by the deformation of the "main simplex" of the associahedron into the big cell of the associahedron. Here the main simplex is defined by the shortest path in the Tamari poset structure of the planar binary trees.

We compute the diagonal map on $(A_\infty)_n$ up to $n = 5$ and we find the same result as the Saneblidze–Umble diagonal. So it is reasonable to conjecture that they coincide.

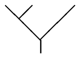
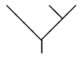
In the last part we give a similar interpretation of the map $p : \Omega BAs \to A_\infty$ constructed in [11] and giving rise to the Saneblidze–Umble diagonal. It is induced by the deformation of the "main cube" into the big cell.

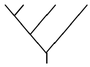
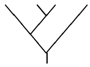
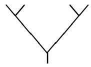
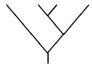
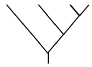
1 Stasheff polytope (associahedron)

We recall briefly the construction of the Stasheff polytope, also called associahedron, and its simplicial realization, which is the key tool of this paper. All chain complexes in this paper are made of free modules over a commutative ring $\mathbb{K}$ (which can be $\mathbb{Z}$ or a field).

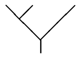
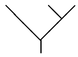
1.1 Planar binary trees

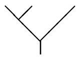
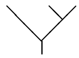
We denote by PBT_n the set of *planar binary trees* having n leaves:

$PBT_1 := \{|\}, PBT_2 := \{$  $\}, PBT_3 := \{$  ,  $\}$,

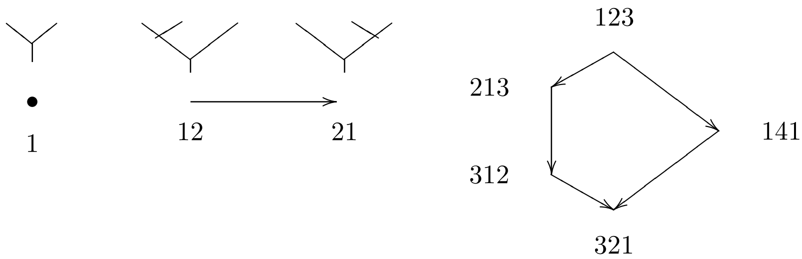
$PBT_4 := \{$  ,  ,  ,  ,  $\}$.

So $t \in PBT_n$ has one root, n leaves, $(n-1)$ internal vertices, $(n-2)$ internal edges. Each vertex is binary (two inputs, one output). The number of elements in PBT_{n+1} is known to be the *Catalan number* $c_n = \frac{(2n)!}{n!\,(n+1)!}$. There is a partial order on PBT_n, called the *Tamari order*, defined as follows. On PBT_3 it is given by

 $\to$  .

More generally, if t and s are two planar binary trees with the same number of leaves, there is a covering relation $t \to s$ if and only if s can be obtained from t by replacing a local pattern like  by  . In other words s is obtained from t by moving a leaf or an internal edge from left to right over a fork.

Examples:

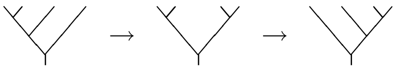


where the elements of PBT_4 (listed above) are denoted $123, 213, 141, 312, 321$, respectively. We recall from [7] how this way of indexing is obtained. First we label the leaves of a tree from left to right by $0, 1, 2, \ldots$. Then we label the vertices by $1, 2, \ldots$ by saying that the label i vertex lies in between the leaves $i-1$ and i (drop a ball). To any binary tree t we associate a sequence of integers $x_1 x_2 \cdots x_{n-1}$ as follows: $x_i = a_i b_i$ where a_i (resp. b_i) is the number of leaves on the left (resp. right) side of the ith vertex.

1.2 Shortest path and long path

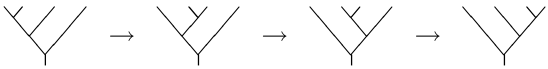
The Tamari poset admits an initial element: the left comb $12\cdots(n-1)$, and a terminal element: the right comb $(n-1)(n-2)\cdots 1$. There is a *shortest path* from the initial element to the terminal element. It is made of the trees which are the grafting of some left comb with a right comb. In PBT_n there are $n-1$ of them. This sequence of planar binary trees will play a significant role in the comparison of different cell realizations of the Stasheff polytope.

Example: the shortest path in PBT_4:

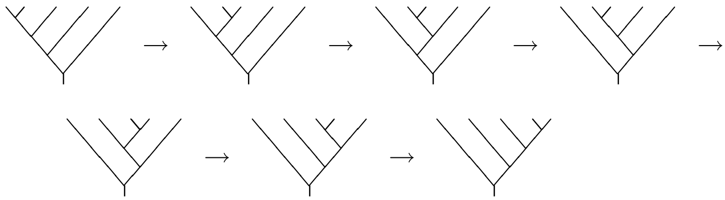


We also define "the long path" as follows. The *long path* from the left comb to the right comb is obtained by taking a covering relation at each step with the following rule: the vertex which is moved is the one with the smallest label (among the movable vertices, of course).

Examples: $n = 2$



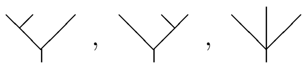
$n = 3$

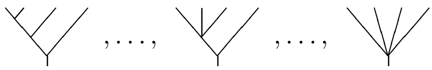


Observe that there are (for $n \geq 3$) other paths with the same length.

1.3 Planar trees

We now consider the planar trees for which an internal vertex has one root and k leaves, where k can be any integer greater than or equal to 2. We denote by PT_n the set of planar trees with n leaves:

$PT_1 := \{|\}, PT_2 := \{$  $\}, PT_3 := \{$  $\},$

$PT_4 := \{$  $\}.$

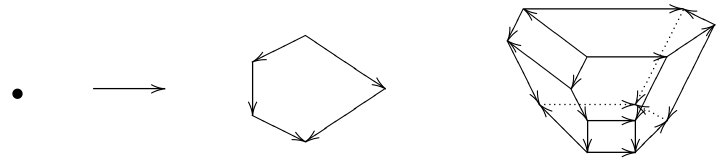
Each set PT_n is graded according to the number of internal vertices, i.e., $PT_n = \bigcup_{p=1}^{p=n} PT_{n,p}$ where $PT_{n,p}$ is the set of planar trees with n leaves and p internal vertices. For instance, $PT_{n,1}$ contains only one element which we call the *n-corolla* (the last element in the above sets). It is clear that $PT_{n,n-1} = PBT_n$.

We order the vertices of a planar tree by using the same procedure as for the planar binary trees.

1.4 The Stasheff polytope, alias associahedron

The *associahedron* is a cellular complex $\mathcal{K}^n$ of dimension n, first constructed by Jim Stasheff [15], which can be realized as a convex polytope whose extremal vertices are in one-to-one correspondence with the planar binary trees in PBT_{n+2}. We showed in [7] that it is the convex hull of the points $M(t) = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$, where the computation of the x_i's has been recalled in Section 1.1. The edges of the polytope are indexed by the covering relations of the Tamari poset.

Examples:



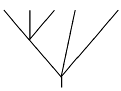
$\mathcal{K}^0$ $\mathcal{K}^1$ $\mathcal{K}^2$ $\mathcal{K}^3$

Its k-cells are in one-to-one correspondence with the planar trees in $PT_{n+2,n+1-k}$. For instance, the 0-cells are indexed by the planar binary trees, and the top cell is indexed by the corolla.

It will prove helpful to adopt the notation $\mathcal{K}^t$ to denote the cell in $\mathcal{K}^n$ indexed by $t \in PT_{n+2}$. For instance, if t is the corolla, then $\mathcal{K}^t = \mathcal{K}^n$. As a space $\mathcal{K}^t$ is the product of p associahedrons (or associahedra, as you like), where p is the number of internal vertices of t:

$$\mathcal{K}^t = \mathcal{K}^{i_1} \times \cdots \times \mathcal{K}^{i_p}$$

where $i_j + 2$ is the number of inputs of the jth internal vertex of t. For instance,

if $t =$  , then $\mathcal{K}^t = \mathcal{K}^1 \times \mathcal{K}^1$.

The shortest path and the long path defined combinatorially in Section 1.1 give rise to concrete paths on the associahedron.

To the cellular complex $\mathcal{K}^n$ we associate its chain complex $C_*(\mathcal{K}^n)$. The module of k-chains admits the set of trees $PT_{n+2,n+1-k}$ as a basis:

$$C_k(\mathcal{K}^n) = \mathbb{K}[PT_{n+2,n+1-k}].$$

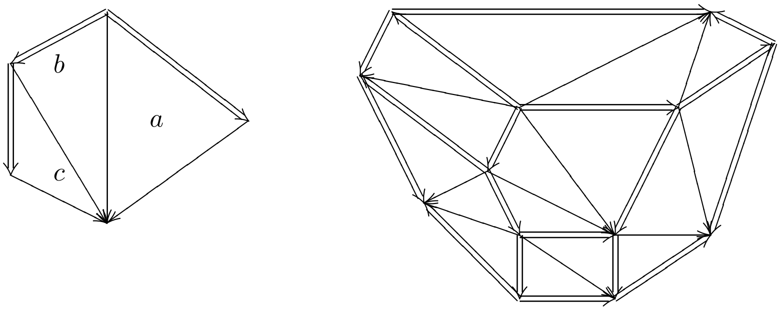
In particular, $C_0(\mathcal{K}^n) = \mathbb{K}[PBT_{n+2}]$ and $C_n(\mathcal{K}^n) = \mathbb{K}\ t_{n+2}$ where t_{n+2} is the corolla.

1.5 The simplicial associahedron

In [8] we constructed a simplicial set $\mathcal{K}^n_{\text{simp}}$ whose geometric realization gives a simplicial decomposition of the associahedron. In other words, the associahedron $\mathcal{K}^n$ is viewed as a union of n-simplices (there are $(n+1)^{n-1}$ of them). This simplicial decomposition is constructed inductively as follows. We fatten the simplicial set $\mathcal{K}^{n-1}_{\text{simp}}$ into a new simplicial set $fat\mathcal{K}^{n-1}_{\text{simp}}$, cf. [8]. Then $\mathcal{K}^n_{\text{simp}}$ is defined as the cone over $fat\mathcal{K}^{n-1}_{\text{simp}}$ (as in the original construction of Stasheff [15]).

For $n = 1$, we have $\mathcal{K}^1_{\text{simp}} = \mathcal{K}^1 = [0, 1]$ (the interval).

Examples: $\mathcal{K}^2_{\text{simp}}$ and $fat\mathcal{K}^3_{\text{simp}}$



Since, in the process of fattenization, the new cells are products of smaller dimensional associahedrons we get the following main property.

Proposition 1.6 *The simplicial decomposition of a face $\mathcal{K}^{i_1} \times \cdots \times \mathcal{K}^{i_k}$ of $\mathcal{K}^n$ is the product of the simplicializations of each component $\mathcal{K}^{i_j}$.*

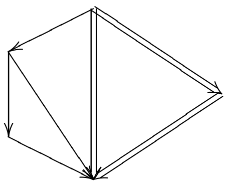
Proof. It is immediate from the inductive procedure which constructs $\mathcal{K}^n$ out of $\mathcal{K}^{n-1}$. □

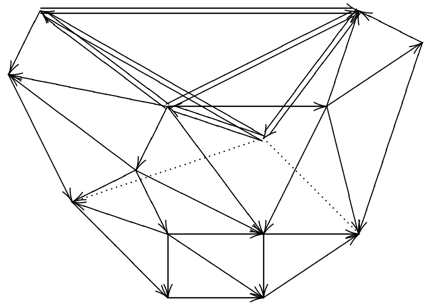
Considered as a cellular complex, still denoted $\mathcal{K}^n_{\text{simp}}$, the simplicialized associahedron gives rise to a chain complex denoted $C_*(\mathcal{K}^n_{\text{simp}})$. This chain complex is the normalized chain complex of the simplicial set. It is the quotient of the chain complex associated to the simplicial set, divided out by the degenerate simplices (cf. for instance [9] Chapter VIII). A basis of $C_0(\mathcal{K}^n_{\text{simp}})$ is given by PBT_{n+2} and a basis of $C_n(\mathcal{K}^n_{\text{simp}})$ is given by the $(n+1)^{n-1}$ top simplices (in bijection with the parking functions, cf. [8]). It is zero higher up.

In the sequel "a simplex of $\mathcal{K}^n_{\text{simp}}$" always means a nondegenerate simplex of $\mathcal{K}^n_{\text{simp}}$.

Among the top simplices there is a particular one which we call the *main simplex*. Its vertices are indexed by the planar binary trees which are part of the shortest path constructed in Section 1.1 (observe that the shortest path has $n+1$ vertices).

Examples (the main simplex is highlighted):





2 The operad AA_∞

We construct the operad AA_∞ and we construct a diagonal on it. A morphism from the operad A_∞ governing the associative algebras up to homotopy to the operad AA_∞ is deduced from the simplicial structure of the associahedron.

2.1 Differential graded nonsymmetric operad [10]

By definition a *differential graded nonsymmetric operad*, dgns operad for short, is a family of chain complexes $\mathcal{P}_n = (\mathcal{P}_n, d)$ equipped with chain complex morphisms

$$\gamma_{i_1\cdots i_n} : \mathcal{P}_n \otimes \mathcal{P}_{i_1} \otimes \cdots \otimes \mathcal{P}_{i_n} \to \mathcal{P}_{i_1+\cdots+i_n},$$

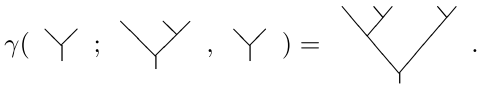
which satisfy the following associativity property. Let $\mathcal{P}$ be the endofunctor of the category of chain complexes over $\mathbb{K}$ defined by $\mathcal{P}(V) := \bigoplus_n \mathcal{P}_n \otimes V^{\otimes n}$. The maps $\gamma_{i_1\cdots i_n}$ give rise to a transformation of functors $\gamma : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$. This transformation of functors γ is supposed to be associative. Moreover we suppose that $\mathcal{P}_0 = 0, \mathcal{P}_1 = \mathbb{K}$ (trivial chain complex concentrated in degree 0). The transformation of functors $\mathrm{Id} \to \mathcal{P}$ determined by $\mathcal{P}_1$ is supposed to be a unit for γ. So we can denote by id the generator of $\mathcal{P}_1$. Since $\mathcal{P}_n$ is a graded module, $\mathcal{P}$ is bigraded. The integer n is called the "arity" in order to differentiate it from the degree of the chain complex.

2.2 The fundamental example A_∞

The operad A_∞ is a dgns operad constructed as follows:

$$A_{\infty,n} := C_*(\mathcal{K}^{n-2}) \text{ (chain complex of the cellular space } \mathcal{K}^{n-2}).$$

Let us denote by As^{i} the family of one-dimensional modules $(As^{\mathsf{i}}_n)_{n\geq 1}$ generated by the corollas (unique top cells). It is easy to check that there is a natural identification of graded (by arity) modules $A_\infty = \mathcal{T}(As^{\mathsf{i}})$, where $\mathcal{T}(As^{\mathsf{i}})$ is the free ns operad over As^{i}. This identification is given by grafting on the leaves as follows. Given trees $t, t_1, \ldots, t_n$ where t has n leaves, the tree $\gamma(t; t_1, \ldots, t_n)$ is obtained by identifying the ith leaf of t with the root of t_i. For instance:



Moreover, under this identification, the composition map γ is a chain map, therefore A_∞ is a dgns operad.

This construction is a particular example of the so-called "cobar construction" Ω, i.e., $A_\infty = \Omega As^{\mathsf{i}}$ where As^{i} is considered as the cooperad governing the coassociative coalgebras (cf. [10]).

For any chain complex A there is a well-defined dgns operad $\mathrm{End}(A)$ given by $\mathrm{End}(A)_n = \mathrm{Hom}(A^{\otimes n}, A)$. An A_∞-algebra is nothing but a morphism of operads $A_\infty \to \mathrm{End}(A)$. The image of the corolla under this isomorphism is the n-ary operation μ_n alluded to in the introduction.

2.3 Hadamard product of operads, operadic diagonal

Given two operads $\mathcal{P}$ and $\mathcal{Q}$, their Hadamard product, also called tensor product, is the operad $\mathcal{P}\otimes\mathcal{Q}$ defined as $(\mathcal{P}\otimes\mathcal{Q})_n := \mathcal{P}_n\otimes\mathcal{Q}_n$. The composition map is simply the tensor product of the two composition maps.

A diagonal on a nonsymmetric operad $\mathcal{P}$ is a morphism of operads $\Delta : \mathcal{P} \to \mathcal{P} \otimes \mathcal{P}$, which is compatible with the unit. Explicitly it is given by chain complex morphisms $\Delta : \mathcal{P}_n \to \mathcal{P}_n \otimes \mathcal{P}_n$ which commute with the composition in $\mathcal{P}$. In other words, the following diagram, where $m := i_1 + \cdots + i_n$, is commutative:

$$\begin{tikzcd}
\mathcal{P}_n \otimes \mathcal{P}_{i_1} \otimes \cdots \arrow[r, "\gamma"] \arrow[d, "\Delta\otimes\Delta\otimes\cdots"] & \mathcal{P}_m \arrow[r, "\Delta"] & \mathcal{P}_m \otimes \mathcal{P}_m \\
(\mathcal{P}_n \otimes \mathcal{P}_n) \otimes (\mathcal{P}_{i_1} \otimes \mathcal{P}_{i_1}) \otimes \cdots \arrow[rr, "\cong"] & & (\mathcal{P}_n \otimes \mathcal{P}_{i_1} \otimes \cdots) \otimes (\mathcal{P}_n \otimes \mathcal{P}_{i_1} \otimes \cdots) \arrow[u, "\gamma\otimes\gamma"']
\end{tikzcd}$$

We do not ask for Δ to be coassociative.

2.4 Tensor product of A_∞-algebras

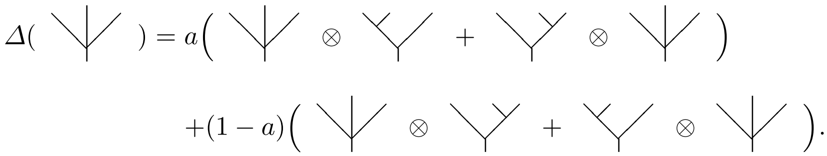
It is a longstanding problem to decide if, given two A_∞-algebras A and B, there is a natural A_∞-structure on their tensor product $A \otimes B$ which extends the natural dg nonassociative algebra structure, cf. [12, 2]. It amounts to construct a diagonal on A_∞, i.e., an operad morphism $\Delta : A_\infty \to A_\infty \otimes A_\infty$, since, by composition, we get an A_∞-structure on $A \otimes B$:

$$A_\infty \to A_\infty \otimes A_\infty \to \mathrm{End}(A) \otimes \mathrm{End}(B) \to \mathrm{End}(A \otimes B) \ .$$

Let us recall that the classical associative structure on the tensor product of two associative algebras can be interpreted operadically as follows. There is a diagonal on the operad As given by

$$As_n \to As_n \otimes As_n, \quad m_n \mapsto m_n \otimes m_n \ ,$$

where m_n is the standard n-ary operation in the associative framework. Since we want the diagonal Δ on A_∞ to be compatible with the diagonal on As ($\mu_2 \mapsto m_2$), there is no choice in arity 2, and we have $\Delta(\mu_2) = \mu_2 \otimes \mu_2$. Observe that these two elements are in degree 0. In arity 3, since μ_3 is of degree 1 and $\mu_3 \otimes \mu_3$ of degree 2, this last element cannot be the answer. In fact, there is already a choice (parameter a) for a solution:



By some tour de force Samson Saneblidze and Ron Umble constructed such a diagonal on A_∞ in [13]. Their construction was reinterpreted in [11] by Markl and Shnider through the Boardman–Vogt construction (see Section 4 below for a brief account of their work). We will use the simplicialization of the associahedron described in [8] to give a solution to the diagonal problem.

2.5 Construction of the operad AA_∞

We define the dgns operad AA_∞ as follows. The chain complex $AA_{\infty,n}$ is the chain complex of the simplicialization of the associahedron considered as a cellular complex (cf. Section 1.5):

$$AA_{\infty,n} := C_*(\mathcal{K}^{n-2}_{\mathrm{simp}}) \ .$$

In low dimension we take $AA_{\infty,0} = 0, AA_{\infty,1} = \mathbb{K}\,\mathrm{id}$. So a basis of $AA_{\infty,n}$ is made of the (nondegenerate) simplices of $\mathcal{K}^{n-2}_{\mathrm{simp}}$. Let us now construct the composition map

$$\gamma = \gamma^{AA_\infty} : AA_{\infty,n} \otimes AA_{\infty,i_1} \otimes \cdots \otimes AA_{\infty,i_n} \to AA_{\infty,i_1+\cdots+i_n} .$$

We denote by Δ^k the standard k-simplex. Let $\iota : \Delta^k \rightarrowtail \mathcal{K}^{n-2}_{\mathrm{simp}}$ be a cell, i.e., a linear generator of $C_k(\mathcal{K}^{n-2}_{\mathrm{simp}})$. Given such cells

$$\iota_0 \in AA_{\infty,n},\ \iota_1 \in AA_{\infty,i_1},\ \ldots,\ \iota_n \in AA_{\infty,i_n}$$

we construct their image $\gamma(\iota_0;\iota_1,\ldots,\iota_n) \in AA_{\infty,m}$, where $m := i_1 + \cdots + i_n$ as follows. We denote by k_i the dimension of the cell ι_i.

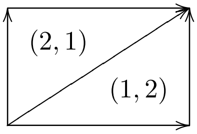
Let t_n be the n-corolla in PT_n and let $s := \gamma(t_n;t_{i_1},\ldots,t_{i_n}) \in PT_m$ be the grafting of the trees $t_{i_1},\ldots,t_{i_n}$ on the leaves of t_n. As noted before this is the composition in the operad A_∞. The tree s indexes a cell $\mathcal{K}^s$ of the space $\mathcal{K}^{m-2}$, which is combinatorially homeomorphic to $\mathcal{K}^{n-2}\times\mathcal{K}^{i_1-2}\times\cdots\times\mathcal{K}^{i_n-2}$. In other words it determines a map

$$s_* : \mathcal{K}^{n-2}\times\mathcal{K}^{i_1-2}\times\cdots\times\mathcal{K}^{i_n-2} = \mathcal{K}^s \rightarrowtail \mathcal{K}^{m-2}.$$

The product of the inclusions $\iota_j, j = 0,\ldots,n$, defines a map

$$\iota_0\times\iota_1\times\cdots\times\iota_n : \Delta^{k_0}\times\Delta^{k_1}\times\cdots\times\Delta^{k_n} \rightarrowtail \mathcal{K}^{n-2}\times\mathcal{K}^{i_1-2}\times\cdots\times\mathcal{K}^{i_n-2}.$$

Let us recall that a product of standard simplices can be decomposed into the union of standard simplices. These pieces are indexed by the multi-shuffles α. Example: $\Delta^1\times\Delta^1 = \Delta^2\cup\Delta^2$:



So, for any multi-shuffle α there is a map

$$f_\alpha : \Delta^l \to \Delta^{k_0}\times\Delta^{k_1}\times\cdots\times\Delta^{k_n},$$

where $l = k_0 + \cdots + k_n$. By composition of maps we get

$$s_*\circ(\iota_0\times\cdots\times\iota_n)\circ f_\alpha : \Delta^l \to \mathcal{K}^{m-2}$$

which is a linear generator of $C_l(\mathcal{K}^{m-2}_{\mathrm{simp}})$ by construction of the triangulation of the associahedron, cf. [7]. By definition $\gamma(\iota_0;\iota_1,\ldots,\iota_n)$ is the algebraic sum of the cells $s_*\circ(\iota_0\times\cdots\times\iota_n)\circ f_\alpha$ over the multi-shuffles.

Proposition 2.6 *The graded chain complex AA_∞ and γ constructed above define a dgns operad, denoted AA_∞. The operad AA_∞ is a model of the operad As.*

Proof. We need to prove associativity for γ. It is an immediate consequence of the associativity for the composition of trees (operadic structure of A_∞) and the associativity property for the decomposition of the product of simplices into simplices.

Since the associahedron is contractible, taking the homology gives a graded linear map $C_*(\mathcal{K}^{n-2}_{\mathrm{simp}}) \to \mathbb{K}\,m_n$, where m_n is in degree 0. This map sends any planar binary tree having n leaves to m_n, and obviously induces an isomorphism on homology. These maps assemble into a dgns operad morphism $AA_\infty \to As$. Since it is a quasi-isomorphism, AA_∞ is a resolution of As, that is, a model of As in the category of dgns operads. □

Proposition 2.7 *The operad AA_∞ admits a coassociative diagonal.*

Proof. This diagonal $\varDelta : AA_\infty \to AA_\infty \otimes AA_\infty$ is determined by its value in arity n for all n, that is, a chain complex morphism

$$C_*(\mathcal{K}^{n-2}_{\text{simp}}) \to C_*(\mathcal{K}^{n-2}_{\text{simp}}) \otimes C_*(\mathcal{K}^{n-2}_{\text{simp}}).$$

This morphism is defined as the composite

$$C_*(\mathcal{K}^{n-2}_{\text{simp}}) \xrightarrow{\varDelta_*} C_*(\mathcal{K}^{n-2}_{\text{simp}} \times \mathcal{K}^{n-2}_{\text{simp}}) \xrightarrow{AW} C_*(\mathcal{K}^{n-2}_{\text{simp}}) \otimes C_*(\mathcal{K}^{n-2}_{\text{simp}}),$$

where $\varDelta_*$ is induced by the diagonal on the simplicial set, and where AW is the Alexander–Whitney map. Observe that under the identification

$$C_*(\mathcal{K}^{n-2}_{\text{simp}}) \otimes C_*(\mathcal{K}^{n-2}_{\text{simp}}) = C_*(\mathcal{K}^{n-2}_{\text{simp}} \times \mathcal{K}^{n-2}_{\text{simp}})$$

the composite morphism maps a k-simplex into a $2k$-simplex. Let us recall from [9], Chapter VIII, the construction of the AW map. Denote by $d_0, \dots, d_k$ the face operators of the simplicial set. If x is a simplex of dimension k, then we define $d_{max}(x) := d_k(x)$. So, for instance, $(d_{max})^2(x) = d_{k-1}d_k(x)$. By definition the AW map on C_k is given by

$$(x, y) \mapsto \sum_{i=0}^{k} \big((d_{max})^{k-i}(x), (d_0)^i(y)\big).$$

We need to check that the diagonal is compatible with the operad structure, that is, the diagram of Section 2.3 for $\mathcal{P} = AA_\infty$ is commutative. First we remark that the diagonal of a product of standard simplices satisfies the following commutativity property:

$$\begin{tikzcd}
C_*(\varDelta^k \times \varDelta^l) \arrow[r, "\varDelta"] \arrow[d, "\varDelta\otimes\varDelta"] & C_*(\varDelta^k \times \varDelta^l) \otimes C_*(\varDelta^k \times \varDelta^l) \arrow[d, "="] \\
C_*(\varDelta^k \times \varDelta^k) \otimes C_*(\varDelta^l \times \varDelta^l) \arrow[r, "\cong"] & C_*(\varDelta^k \times \varDelta^l \times \varDelta^k \times \varDelta^l)
\end{tikzcd}$$

where the isomorphism $\cong$ involves the switching isomorphism $V \otimes V' \cong V' \otimes V$. A similar property holds for a finite product of simplices. Starting with a linear generator

$$(\iota; \iota_1, \dots, \iota_n) \in AA_{\infty,n} \otimes AA_{\infty,i_1} \otimes \cdots \otimes AA_{\infty,i_n}$$

we see that $\varDelta(\gamma(\iota; \iota_1, \dots, \iota_n))$ is made of diagonals of products of simplices. Applying the preceding result we can rewrite this element as the composite of diagonals of simplices. Hence we get

$$\varDelta(\gamma(\iota; \iota_1, \dots, \iota_n)) = \gamma(\varDelta(\iota); \varDelta(\iota_1), \dots, \varDelta(\iota_n))$$

as expected.

The coassociativity property follows from the coassociativity property of the Alexander–Whitney map. $\square$

2.8 Comparing A_∞ to AA_∞

Since $\mathcal{K}^n_{\mathrm{simp}}$ is a decomposition of $\mathcal{K}^n$, there is a chain complex map

$$q' : C_*(\mathcal{K}^n) \to C_*(\mathcal{K}^n_{\mathrm{simp}}),$$

where a cell of $\mathcal{K}^n$ is sent to the algebraic sum of the simplices it is made of.

Proposition 2.9 *The map $q' : A_\infty \to AA_\infty$ induced by the maps $q' : C_*(\mathcal{K}^n) \to C_*(\mathcal{K}^n_{\mathrm{simp}})$ is a quasi-isomorphism of dgns operads.*

Proof. It is sufficient to prove that the maps q' on the chain complexes are compatible with the operadic composition:

$$q'(\gamma^{As}(t; t_1, \ldots, t_n)) = \gamma^{AA_\infty}(q'(t); q'(t_1), \ldots, q'(t_n)).$$

This equality follows from the definition of γ^{AA_∞} given in Section 2.5 and Proposition 1.6. □

Moreover we have commutative diagrams:

$$\begin{tikzcd} C_*(\mathcal{K}^{n-2}) \arrow[rr, "q'"] \arrow[dr, "H_*"'] & & C_*(\mathcal{K}^{n-2}_{\mathrm{simp}}) \arrow[dl, "H_*"] \\ & \mathbb{K}\,\mu_n & \end{tikzcd} \qquad \begin{tikzcd} A_\infty \arrow[rr, "q'"] \arrow[dr, "H_*"'] & & AA_\infty \arrow[dl, "H_*"] \\ & As & \end{tikzcd}$$

3 From AA_∞ to A_∞

The aim of this section is to construct a quasi-inverse to q', that is, a quasi-isomorphism of dgns operads $p' : AA_\infty \to A_\infty$. We first construct chain maps $p' : C_*(\mathcal{K}^n_{\mathrm{simp}}) \to C_*(\mathcal{K}^n)$ by using a deformation of the main simplex to the top cell of the associahedron. This is obtained by an inflating process that we first describe on the cube and on the product of simplices.

3.1 Deformation of the cube

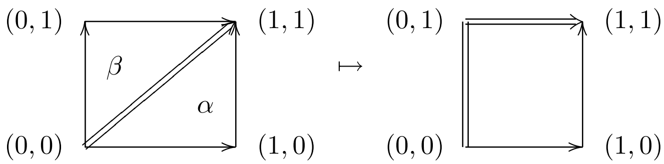
The cube I^n is a polytope whose vertices are indexed by $(x_1, \ldots, x_n)$, where $x_i = 0$ or 1. The long path in I^n is, by definition, the path

$$(0, \ldots, 0, 0) \to (0, \ldots, 0, 1) \to (0, \ldots, 1, 1) \to \cdots \to (1, \ldots, 1, 1).$$

The cube is a cell complex which can be decomposed into $n!$ top simplices, i.e., viewed as the realization of a simplicial set I^n_{simp}. The simplex which corresponds to the identity permutation is called the main simplex of the cube. Let us describe the deformation from the main simplex to the cube, which gives rise to a chain map

$$p' : C_*(I^n_{\mathrm{simp}}) \to C_*(I^n) \ .$$

We work by induction on n. In I^2_{simp} the main simplex, denoted α, is deformed to the square by pushing the diagonal to the long path:

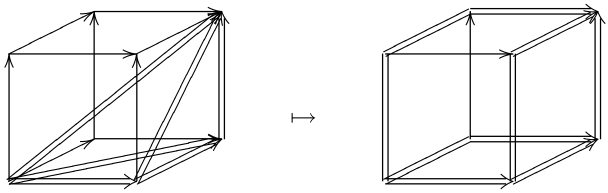


So p' is given by the identity on the boundary and by

$$\begin{array}{ccc} ((0,0),(1,1)) & \mapsto & ((0,0),(0,1)) + ((0,1),(1,1)), \\ \alpha & \mapsto & I^2, \\ \beta & \mapsto & 0, \end{array}$$

on the interior simplices. So, under this inflating process, the main simplex α is mapped to the whole square and the other simplex β is flattened. More generally, the main simplex of I^n_{simp} is deformed into the top cell of I^n by sending the diagonal to the long path. The other edges of the main simplex are deformed according to the lower-dimensional deformation.

Here is the example of the 3-dimensional cube:



3.2 Deformation of a product of simplices

Similarly we define a deformation of the product of simplices $\Delta^r \times \Delta^s$ by inflating the main simplex as follows. Let us denote by $\{\underline{0}, \dots, \underline{r}\}$ the vertices of Δ^r. The *main simplex* of $\Delta^r \times \Delta^s$ is chosen as being the simplex Δ^{r+s} with vertices

$$(\underline{0},\underline{0}), (\underline{1},\underline{0}), \dots, (\underline{r},\underline{0}), (\underline{r},\underline{1}), \dots, (\underline{r},\underline{s}), \ .$$

We deform the main simplex into the whole product by induction on s. So it suffices to give the image of the edge $((\underline{0},\underline{0}),(\underline{r},\underline{s}))$. We send it to the "long path" defined as

$$(\underline{0},\underline{0}), (\underline{0},\underline{1}), \dots, (\underline{0},\underline{s}), (\underline{1},\underline{s}), \dots, (\underline{r},\underline{s}), \ .$$

Under this deformation the main simplex becomes the whole product and all the other simplices are flattened. This deformation defines a chain complex morphism

$$C_*((\Delta^r \times \Delta^s)_{\text{simp}}) \to C_*(\Delta^r \times \Delta^s)$$

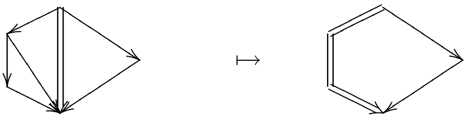
where, on the right side, $\Delta^r \times \Delta^s$ is considered as a cell complex with only one $r+s$ cell.

Observe that some simplices may happen to be deformed into cells of various dimensions. For instance, in $\Delta^2 \times \Delta^1$ the triangle with vertices $(\underline{0},\underline{0}),(\underline{1},\underline{0}),(\underline{2},\underline{1})$ is deformed into the union of a square (with vertices $(\underline{0},\underline{0}),(\underline{1},\underline{0}),(\underline{0},\underline{1}),(\underline{1},\underline{1})$) and an edge (with vertices $(\underline{1},\underline{1}),(\underline{2},\underline{1})$). Its image under the chain morphism is the square.

3.3 Deformation of the associahedron

We construct a topological deformation of the simplicial associahedron by pushing the main simplex to the whole associahedron. All the other simplices are going to be flattened. This topological deformation will induce the chain map p' we are looking for. This inflating process is analogous to what we did for the cube and the product of simplices above. We work by induction on the dimension.

For $n=1$, there is no deformation since $\mathcal{K}^1_{\text{simp}} = \mathcal{K}^1$. For $n=2$, the deformation is the identity on the boundary and the only edge of the main simplex which is not on the boundary is "pushed" to the long path.

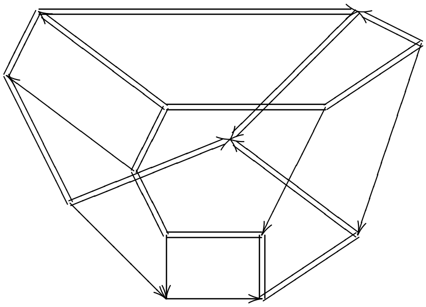


In the meantime, the other interior edge is pushed to the union of two boundary edges and the two other top simplices are flattened.

For higher n we use the inductive construction of $\mathcal{K}^n_{\text{simp}}$ out of $fat\mathcal{K}^{n-1}_{\text{simp}}$. We suppose that the deformation is known for any $i<n$ and we construct it on $fat\mathcal{K}^{n-1}_{\text{simp}}$. The simplicial set $fat\mathcal{K}^{n-1}_{\text{simp}}$ is the union of the simplicial sets of the form $\mathcal{K}^t_{\text{simp}} = \mathcal{K}^i_{\text{simp}} \times \mathcal{K}^j_{\text{simp}}$ indexed by some trees t with one and only one internal edge. The main simplex of this product is the main simplex Δ^{i+j} of $\Delta^i \times \Delta^j$ where Δ^i, resp. Δ^j, is the main simplex of $\mathcal{K}^i_{\text{simp}}$, resp. $\mathcal{K}^j_{\text{simp}}$. The deformation is obtained by, first, deforming the main simplex Δ^{i+j} into $\Delta^i \times \Delta^j$ as described in Section 3.2 and then use the inductive hypothesis (deformation from the main simplex to the associahedron).

The deformation of the interior cells is obtained by pushing the main simplex of $\mathcal{K}^n_{\text{simp}}$ to the top cell. It is determined by the image of the edges of the main simplex. By induction, it suffices to construct the image of the edge which goes from the vertex indexed by the left comb (initial element) to the vertex indexed by the right comb (terminal element). We choose to deform it to the long path of the associahedron as constructed in Section 1.2. Since any simplex of $\mathcal{K}^n_{\mathrm{simp}}$ is either on the boundary, or is a cone (for the last vertex) over a simplex in the boundary, we are done. In particular, the edge going from a 0-simplex labelled by the tree t to the right comb is deformed into a path made of 1-cells of the associahedron, constructed with the same rule as in the construction of the long path.

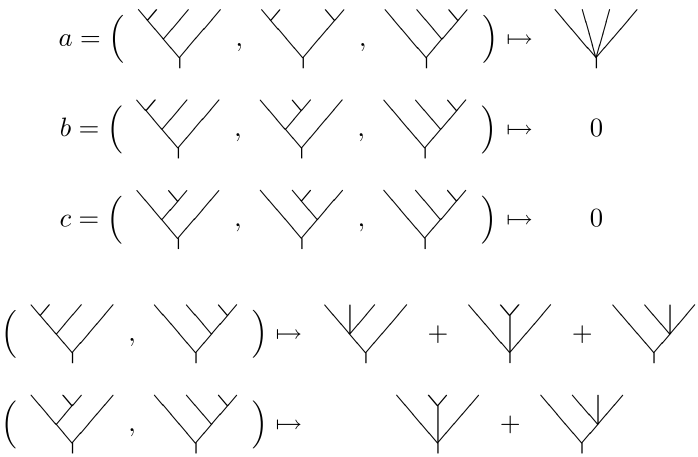
The deformed tetrahedron:



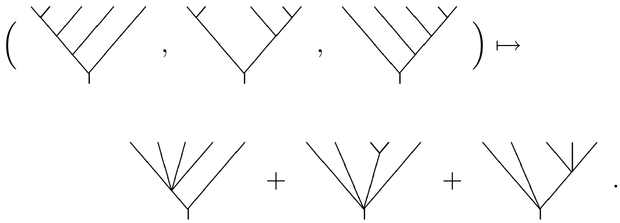
3.4 The map $p' : C_*(\mathcal{K}^n_{\mathrm{simp}}) \to C_*(\mathcal{K}^n)$

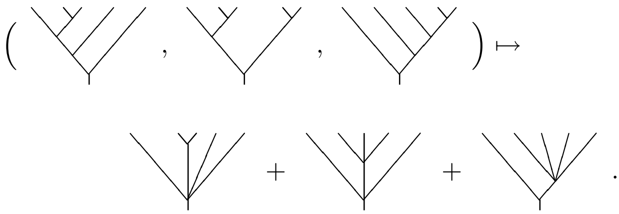
We define the map p' as follows. Under the deformation map any simplex of $\mathcal{K}^n_{\mathrm{simp}}$ is sent to the union of cells of $\mathcal{K}^n$. The image of such a simplex under p' is the algebraic sum of the cells of the same dimension in the union. For instance, the main simplex is sent to the top cell (indexed by the corolla), and all the other top simplices are sent to 0, since under the deformation they are flattened. From its topological nature it follows that p' is a chain complex morphism.

In low dimension we get the following. For $n = 1$, the map p' is the identity. For $n = 2$, the map p' is the identity on the 0-simplices and the 1-simplices of the boundary, and on the interior cells, we get:



Here are examples of the image under p' of some interior 2-dimensional simplices for $n = 3$:





Proposition 3.5 *The chain maps $p' : C_*(\mathcal{K}^n_{\mathrm{simp}}) \to C_*(\mathcal{K}^n)$ assemble into a morphism of dgns operads $p' : AA_\infty \to A_\infty$.*

Proof. We adopt the notation of Section 2.5 where the operadic composition map γ^{AA_∞} is constructed. From this construction it follows that there is a main simplex in $\omega := \gamma^{AA_\infty}(\iota_0; \iota_1, \ldots, \iota_n)$ if and only if all the simplices ι_j are main simplices.

Suppose that one of them, say ι_j, is not a main simplex. Then we have $p'(\iota_j) = 0$, and therefore $\gamma^{A_\infty}(p'(\iota_0); p'(\iota_1), \ldots, p'(\iota_n)) = 0$. But since there is no main simplex in ω, we also get $p'(\omega) = 0$ as expected.

Suppose that all the simplices are main simplices. Then $p'(\iota_j) = t_{i_j}$ for all j and therefore $\gamma^{A_\infty}(p'(\iota_0); p'(\iota_1), \ldots, p'(\iota_n)) = \gamma^{A_\infty}(t_{i_0}; t_{i_1}, \ldots, t_{i_n})$. On the other hand, ω contains the main simplex, therefore $p'(\omega) = \gamma^{A_\infty}(t_{i_0}; t_{i_1}, \ldots, t_{i_n})$ and we are done. □

Corollary 3.6 *The composite*

$$A_\infty \xrightarrow{q'} AA_\infty \xrightarrow{\Delta} AA_\infty \otimes AA_\infty \xrightarrow{p' \otimes p'} A_\infty \otimes A_\infty$$

is a diagonal for the operad A_∞.

Proof. It is immediate to check that this composite of dgns operad morphisms sends μ_2 to $\mu_2 \otimes \mu_2$, since μ_2 corresponds to the 0-cell of $\mathcal{K}^0$. □

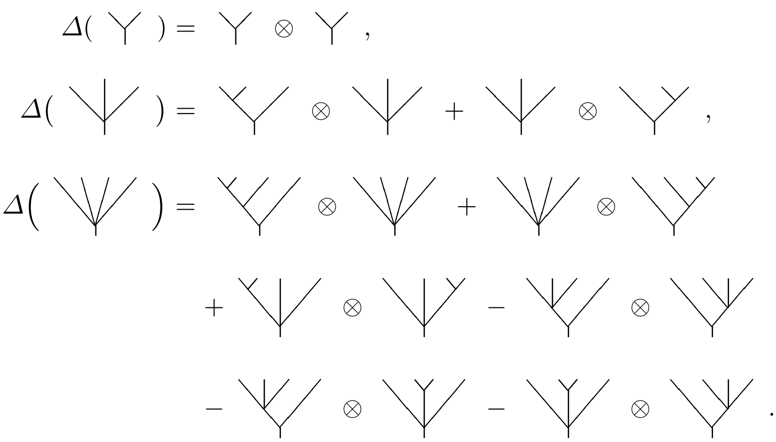
Proposition 3.7 *If A is an associative algebra and B an A_∞-algebra, then the A_∞-structure on $A \otimes B$ is given by*

$$\mu_n(a_1 \otimes b_1, \ldots, a_n \otimes b_n) = a_1 \cdots a_n \otimes \mu_n(b_1, \ldots, b_n).$$

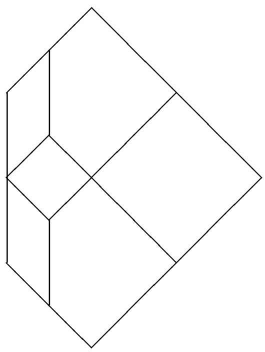
Proof. In the formula for Δ we have $\mu_n = 0$ for all $n \geq 3$, that is, any tree with a k-valent vertex for $k \geq 3$ is 0 on the left side. Hence the only term which is left is $comb \otimes corolla$, whence the assertion. $\square$

3.8 The first formulas

Let us give the explicit form of $\Delta(\mu_n)$ for $n = 2, 3, 4$:



In this last formula the first three summands comes from the triangle $(123, 141, 321)$, the next two summands come from the triangle $(123, 213, 321)$ and the last summand comes from the last triangle $(213, 312, 321)$. It is exactly the same formula as the one obtained by Saneblidze and Umble (cf. [13] Example 1, [11] Exercise 12). Topologically the diagonal of the pentagon is approximated as a union of products of cells as follows:



Each cell of this decomposition corresponds to a summand of the above formula, which indicates where the cell goes in the product $\mathcal{K}^2 \times \mathcal{K}^2$.

3.9 On the non-coassociativity of the diagonal

Though the diagonal of AA_∞ that we constructed is coassociative, the diagonal of A_∞ is not. In fact, it has been shown in [11] that there does not exist any coassociative diagonal on A_∞. The obstruction to coassociativity can be seen topologically on the picture "Iterated diagonal" (see Fig. 1.).

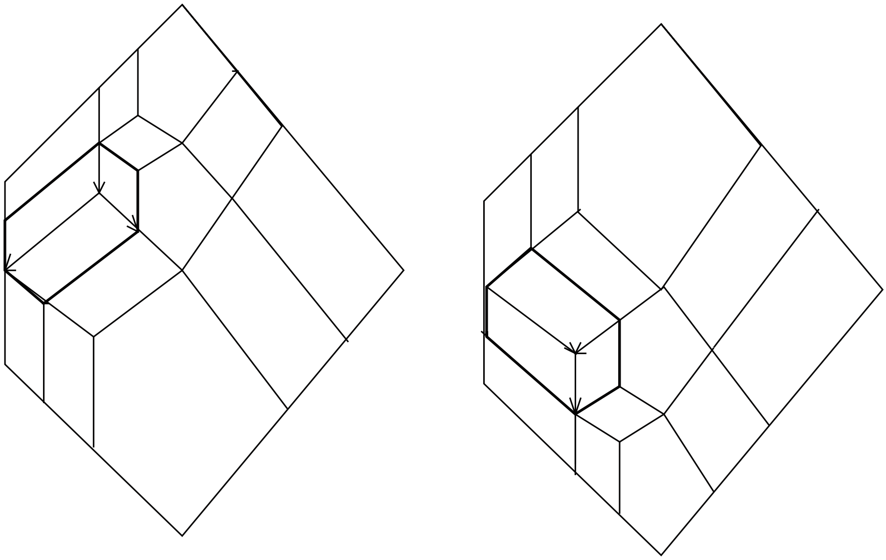
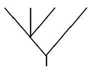
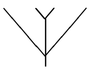
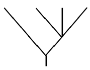


Fig. 1. Iterated diagonal

Both pictures are the same combinatorially, except for a hexagon (highlighted on the pictures), which is the union of three squares one way on the left and the other way on the right. This is the obstruction to coassociativity. Of course there is a way to reconcile these two decompositions via a homotopy which is given by the cube.

Exercise 1. Show that the image of this cube in $\mathcal{K}^2 \times \mathcal{K}^2 \times \mathcal{K}^2$ is indexed by  $\times$  $\times$  .

Exercise 2. Compare the five iterated diagonals of the next step (some nice pictures to draw).

4 Comparing the operads AA_∞ and ΩBAs

We first give a brief account of [11, 13] where a diagonal of the operad A_∞ is constructed by using a coassociative diagonal on the dgns operad ΩBAs. Then we compare the two operads AA_∞ and ΩBAs.

4.1 Cubical decomposition of the associahedron [1]

The associahedron can be decomposed into cubes as follows.

For each tree $t \in PBT_{n+2}$ we take a copy of the cube I^n (where $I = [0,1]$ is the interval) which we denote by I^n_t. Then the associahedron $\mathcal{K}^n$ is the quotient

$$\mathcal{K}^n := \bigsqcup_t I^n_t / \sim$$

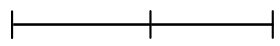
where the equivalence relation is as follows. We think of an element $\tau = (t; \lambda_1, \ldots, \lambda_n) \in I^n_t$ as a tree of type t where the λ_i's are the lengths of the internal edges. If some of the λ_i's are 0, then the geometric tree determined by τ is not binary anymore (since some of its internal edges have been shrunken to a point). We denote the new tree by $\bar{\tau}$. For instance, if none of the λ_i's is zero, then $\bar{\tau} = t$; if all the λ_i's are zero, then the tree $\bar{\tau}$ is the corolla (only one vertex). The equivalence relation $\tau \sim \tau'$ is defined by the following two conditions:

– $\bar{\tau} = \bar{\tau'}$,

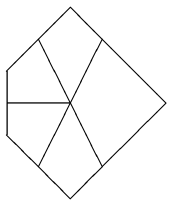
– the lengths of the nonzero-length edges of τ are the same as those of τ'.

Hence $\mathcal{K}^n$ is obtained as a cubical realization denoted $\mathcal{K}^n_{\mathrm{cub}}$.

Examples:



$\mathcal{K}^1$



$\mathcal{K}^2$

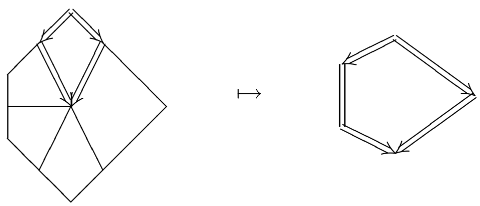
4.2 Markl–Shnider version of Saneblidze–Umble diagonal [11, 13]

In [1] Boardman and Vogt showed that the bar–cobar construction on the operad As is a dgns operad ΩBAs whose chain complex in arity n can be identified with the chain complex of the cubical decomposition of the associahedron:

$$(\Omega BAs)_n = C_*(\mathcal{K}^{n-2}_{\mathrm{cub}}) \ .$$

In [11] (where $\mathcal{K}^{n-2}_{\mathrm{cub}}$ is denoted W_n and $\mathcal{K}^{n-2}$ is denoted K_n) Markl and Shnider use this result to construct a coassociative diagonal on the operad ΩBAs. There is a quasi-isomorphism $q : A_\infty \to \Omega BAs$ induced by the cubical decomposition of the associahedron (the image of the top cell is the algebraic sum of the c_{n-1} cubes). They construct an inverse quasi-isomorphism $p : \Omega BAs \to A_\infty$ by giving explicit algebraic formulas. At the chain level the map $p : C_*(\mathcal{K}^n_{\mathrm{cub}}) \to C_*(\mathcal{K}^n)$ has a topological interpretation using a deformation of the cubical associahedron as follows. The cube indexed by the left comb is called the *main cube* of the decomposition. The deformation sends the main cube to the top cell of the associahedron and flattens all the other ones.

Example:



The exact way the main cube is deformed is best explained by drawing the associahedron on the cube. This is recalled in the Appendix. In [4] Kadeishvili and Saneblidze give a general method for constructing a diagonal on some polytopes admitting a cubical decomposition along the same principle (inflating the main cube).

Markl and Shnider claim that the composite

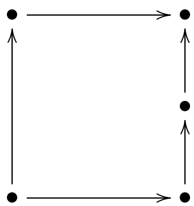
$$A_\infty \xrightarrow{q} \Omega BAs \to \Omega BAs \otimes \Omega BAs \xrightarrow{p\otimes p} A_\infty \otimes A_\infty$$

is the Saneblidze–Umble diagonal.

5 Appendix 1: Drawing a Stasheff polytope on a cube

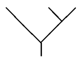
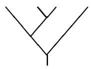
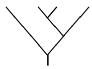
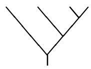
This is an account of some effort to construct the Stasheff polytope that I did in 2002 while visiting Northwestern University. During this visit I had the opportunity to meet Samson Saneblidze and Ron Umble, who were drawing the same kind of figures for different reasons (explained above). It makes the link between Markl and Shnider algebraic description of the map p, the pictures appearing in the Saneblidze and Umble paper, and some algebraic properties of the planar binary trees.

There is a way of constructing an associahedron structure on a cube as follows. For $n = 0$ and $n = 1$ there is nothing to do since $\mathcal{K}^0$ and $\mathcal{K}^1$ are the cubes I^0 and I^1, respectively. For $n = 2$, we simply add one point in the middle of an edge to obtain a pentagon:



Inductively we draw $\mathcal{K}^n$ on I^n out of the drawing of $\mathcal{K}^{n-1}$ on I^{n-1} as follows. Any tree $t \in PBT_{n+1}$ gives rise to an ordered sequence of trees $(t_1, \ldots, t_k)$ in PBT_{n+2} as follows. We consider the edges which are on the right side of t, including the root. The tree t_1 is formed by adding a leaf which starts from the middle of the root and goes rightward (see [6] p. 297). The tree t_2 is formed by adding a leaf which starts from the middle of the next edge and goes rightward. And so forth. Obviously k is the number of vertices lying on the right side of t plus one (so it is always greater than or equal to 2).

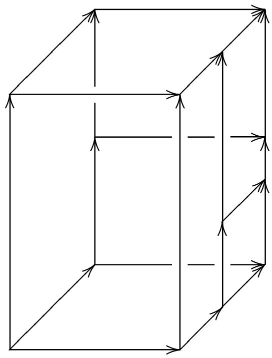
Example:

if $t =$  , then $t_1 =$  , $t_2 =$  , $t_3 =$  .

In $I^n = I^{n-1} \times I$ we label the point $\{t\} \times \{0\}$ by t_1, the point $\{t\} \times \{1\}$ by t_k, and we introduce (in order) the points $t_2, \ldots, t_{k-1}$ on the edge $\{t\} \times I$. For $n = 2$ we obtain (with the coding introduced in Section 1.1):

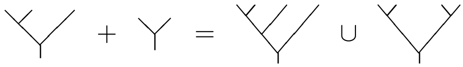
$$\begin{tikzcd}
141 \arrow[r] & 321 \\
 & 312 \arrow[u] \\
123 \arrow[uu] \arrow[r] & 213 \arrow[u]
\end{tikzcd}$$

For $n = 3$ we obtain the following picture:

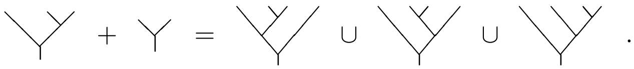


(It is a good exercise to draw the tree at each vertex). Compare with [13], p. 3.) The case $n = 4$ can be found on my home-page. It is important to observe that the order induced on the vertices by the canonical orientation of the cube coincides precisely with the Tamari poset structure. The referee informed me that these pictures already appeared (without any mention of the Stasheff polytope) in [3].

Surprisingly, this way of viewing the associahedron is related to an algebraic structure on the set of planar binary trees $PBT = \bigcup_{n \geq 1} PBT_n$, related to dendriform algebras. Indeed there is a noncommutative monoid structure on the set of homogeneous nonempty subsets of PBT constructed in [6]. It comes from the associative structure of the free dendriform algebra on one generator. This monoid structure is denoted by $+$, the neutral element is the tree $|$. If $t \in PBT_p$ and $s \in PBT_q$, then $s + t$ is a subset of PBT_{p+q-1}. It is proved in [6] that the trees which lie on the edge $\{t\} \times I \subset I^n$ are precisely the trees of $t +$  . For instance:



and

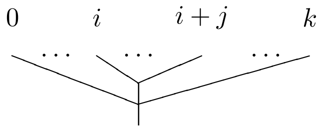


The deformation of the associahedron consisting in inflating the main simplex to the top cell can be performed into two steps by considering a cube inside the associahedron. This cube is determined by the previous construction. First, we inflate the main simplex to the full cube as described in Section 3.1, then we deform the cube into the associahedron as indicated above.

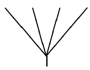
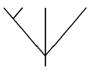
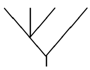
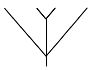
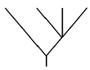
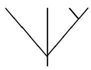
Finally we remark that the deformation described in Section 3.3 permits us to draw the associahedron on the simplex.

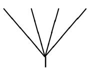
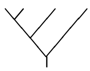
6 Appendix 2: $\Delta(\mu_5)$

In this appendix we give the computation of $\Delta(\mu_5)$ and we show that we get the same result as Saneblidze and Umble. In order to compare with their result we adopt their way of indexing the planar trees, which is as follows. Let t be a tree whose root vertex has $k+1$ inputs, which we label (from left to right) by $0, \ldots, k$. Then, by definition, $d_{ij}(t)$ is the tree obtained by replacing, locally, the root vertex by the following tree with one internal edge:



The operator d_{ij} is well-defined for $0 \le i \le k, 1 \le j \le k-i$ and $(i,j) \neq (0,k)$. So we get:

ij	=	01	02	11	12	21
$d_{ij}($  $)$	=					

and $d_{01}d_{01}($  $) =$  , etc.

Let us index the sixteen 3-simplices forming $\mathcal{K}^3_{\text{simp}}$ by the tree indexing the face in $fat\mathcal{K}^2_{\text{simp}}$ and either a, b, c if this face is a pentagon (cf. Section 1.5) or the shuffle $\alpha = (1,2), \beta = (2,1)$ if this face is a square (cf. Section 2.5). In the following tableau we indicate the image of the 3-simplices under the map $p' \otimes p' \circ \Delta^{AA_\infty}$. In the left column we indicate the information which determines the 3-simplex $(d_{ij}(\mu_5), x)$. In the right column we give its image (up to signs) as a sum of four terms, since in the AW morphism there are four terms.

03	a	$(01)(01)(01) \otimes \mu_5 + (02)(01) \otimes \big(-(21) + (22)\big)$ $+(03) \otimes \big((11)(21) + (12)(21) + (11)(22)\big) + \mu_5 \otimes (11)(21)(31)$
03	b	$0+0+0+0$
03	c	$0+0+0+0$
02	α	$0+0+(02) \otimes \big(-(11)(31) - (12)(31)\big) + 0$
02	β	$0+(01)(02) \otimes \big((11)+(12)+(13)\big) + 0 + 0$
01	a	$0+(01)(01) \otimes (31) + (01) \otimes (21)(31)$
01	b	$0+0+0+0$
01	c	$0+0+0+0$
12	α	$0+0+(12) \otimes \big((12)(21) + (11)(22)\big) + 0$
12	β	$0+0+0+0$
11	a	$0+0-(11) \otimes (12)(31) + 0$
11	b	$0+(02)(11) \otimes \big((13)+(12)\big) + 0 + 0$
11	c	$0+(11)(11) \otimes (13) + 0 + 0$
21	a	$0+(11)(01) \otimes (22) + (21) \otimes (11)(22) + 0$
21	b	$0+0+0+0$
21	c	$0+0+0+0$

As a result $\Delta(\mu_5)$ is the algebraic sum of 22 elements, which are exactly the same as in [13] Example 1. Topologically, it means that $\mathcal{K}^3$ can be realized as the union of 2 copies of $\mathcal{K}^3$ (having only one vertex in common), 6 copies of $\mathcal{K}^1 \times \mathcal{K}^2$, 6 copies of $\mathcal{K}^2 \times \mathcal{K}^1$, 4 copies of $(\mathcal{K}^1 \times \mathcal{K}^1) \times \mathcal{K}^1$ and 4 copies of $\mathcal{K}^1 \times (\mathcal{K}^1 \times \mathcal{K}^1)$.

From this computation it is reasonable to conjecture that the diagonal constructed from the simplicial decomposition of the associahedron is the same as the Saneblidze–Umble diagonal.

Acknowledgments

I thank Bruno Vallette for illuminating discussions on the algebras up to homotopy and Samson Saneblidze for sharing his drawings with me some years ago. Thanks to Emily Burgunder, Martin Markl, Samson Saneblidze, Jim Stasheff and Ron Umble for their comments on the previous versions of this paper. I warmly thank the referee for his careful reading and his precious comments which helped me to improve this text.

This work is partially supported by the French agency ANR.

References

1. Boardman, J.M., Vogt, R.M.: Homotopy invariant algebraic structures on topological spaces. Lecture Notes in Mathematics, vol. 347, p. 257. Springer, Berlin (1973)
2. Gaberdiel, M., Zwiebach, B.: Tensor constructions of open string theories. I. Foundations. Nucl. Phys. B **505**(3), 569–624 (1997)
3. Geyer, W.: On Tamari lattices. Discrete Math. **133**(1–3), 99–122 (1994)
4. Kadeishvili, T., Saneblidze, S.: The twisted Cartesian model for the double path fibration. ArXiv math.AT/0210224
5. Keller, B.: Introduction to A-infinity algebras and modules. Homology Homotopy Appl. **3**(1), 1–35 (2001)
6. Loday, J.-L.: Arithmetree. J. Algebra **258**(1), 275–309 (2002)
7. Loday, J.-L.: Realization of the Stasheff polytope. Arch. Math. (Basel) **83**(3), 267–278 (2004)
8. Loday, J.-L.: Parking functions and triangulation of the associahedron. Proceedings of the Street's fest. Contemp. Math. AMS **431**, 327–340 (2007)
9. MacLane, S.: Homology. Die Grundlehren der mathematischen Wissenschaften, Bd, vol. 114, p. 422. Academic, New York (1963)
10. Markl, M., Shnider, S., Stasheff, J.: Operads in algebra, topology and physics. Mathematical Surveys and Monographs vol. 96, p. 349. American Mathematical Society, Providence, RI (2002)
11. Markl, M., Shnider, S.: Associahedra, cellular W-construction and products of A_∞-algebras. Trans. Am. Math. Soc. **358**(6), 2353–2372 (2006) (electronic)
12. Prouté, A.: A_∞-structures, modèle minimal de Baues-Lemaire et homologie des fibrations. Thèse d'Etat. Université Paris VII (1984)
13. Saneblidze, S., Umble, R.: A Diagonal on the Associahedra, preprint. ArXiv math.AT/0011065
14. Saneblidze, S., Umble, R.: Diagonals on the permutahedra, multiplihedra and associahedra. Homology Homotopy Appl. **6**(1), 363–411 (2004)
15. Stasheff, J.D.: Homotopy associativity of H-spaces. I, II. Trans. Am. Math. Soc. **108**, 275–292; 293–312 (1963)

Permutahedra, HKR Isomorphism and Polydifferential Gerstenhaber–Schack Complex

S.A. Merkulov

Department of Mathematics, Stockholm University, 10691 Stockholm, Sweden, sm@math.su.se

Abstract. This paper aims to give a short but self-contained introduction into the theory of (wheeled) props, properads, dioperads and operads, and illustrate some of its key ideas in terms of a prop(erad)ic interpretation of simplicial and permutahedra cell complexes with subsequent applications to the Hochschild–Kostant–Rosenberg type isomorphisms.

Key words: Operads, Bialgebras, Permutahedra

Mathematics Subject Classifications (2000): 18D50, 16W30

1 Introduction

This paper aims to give a short but self-contained introduction into the theory of (wheeled) props, properads, dioperads and operads, and illustrate some of its key ideas in terms of a prop(erad)ic interpretation of simplicial and permutahedra cell complexes with subsequent applications to the Hochschild–Kostant–Rosenberg type isomorphisms.

Let V be a graded vector space over a field $\mathbb{K}$ and $\mathcal{O}_V := \odot^\bullet V^*$ the free graded commutative algebra generated by the dual vector space $V^* := \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$. One can interpret $\mathcal{O}_V$ as the algebra of polynomial functions on the space V. The classical Hochschild–Kostant–Rosenberg theorem asserts that the Hochschild cohomology of $\mathcal{O}_V$ (with coefficients in $\mathcal{O}_V$) is isomorphic to the space, $\wedge^\bullet \mathcal{T}_V$, of polynomial polyvector fields on V which in turn is isomorphic as a vector space to $\wedge^\bullet V \otimes \odot^\bullet V^*$,

$$HC^\bullet(\mathcal{O}_V) \simeq \wedge^\bullet \mathcal{T}_V \simeq \wedge^\bullet V \otimes \odot^\bullet V^*. \tag{1}$$

The Hochschild complex, $C^\bullet(\mathcal{O}_V) = \oplus_{k\geq 0}\mathrm{Hom}(\mathcal{O}_V^{\otimes k}, \mathcal{O}_V)[1-k]$, of $\mathcal{O}_V$ has a natural subcomplex, $C^\bullet_{diff}(\mathcal{O}_V) \subset C^\bullet(\mathcal{O}_V)$, spanned by polydifferential operators. It was proven in [Ko] (see also [CFL]) that again

A.S. Cattaneo et al. (eds.), *Higher Structures in Geometry and Physics*, Progress in Mathematics 287, DOI 10.1007/978-0-8176-4735-3_14,

$$HC^{\bullet}_{diff}(\mathcal{O}_V) \simeq \wedge^{\bullet}\mathcal{T}_V \simeq \wedge^{\bullet}V \otimes \odot^{\bullet}V^*. \tag{2}$$

The first result (1) actually fails for a ring of smooth functions, $\mathcal{O}_M$, on a generic graded manifold M while the second one (2) stays always true [CFL]. Thus one must, in general, be careful in distinguishing ordinary and polydifferential Hochschild cohomology for smooth functions.

The vector space $\mathcal{O}_V$ has a natural (co)commutative bialgebra structure so that one can also associate to $\mathcal{O}_V$ a Gerstenhaber–Schack complex [GS1], $\mathcal{C}^{\bullet,\bullet}(\mathcal{O}_V) := \bigoplus_{m,n\geq 1} \mathrm{Hom}(\mathcal{O}_V^{\otimes m}, \mathcal{O}_V^{\otimes n})[2-m-n]$. Its cohomology was computed in [GS2, LM],

$$H\mathcal{C}^{\bullet,\bullet}(\mathcal{O}_V) \simeq \wedge^{\bullet\geq 1}V \otimes \wedge^{\bullet\geq 1}V^*. \tag{3}$$

In this paper we introduce (more precisely, *deduce* from the permutahedra cell complex) a relatively nonobvious *polydifferential* subcomplex, $\mathcal{C}^{\bullet,\bullet}_{diff}(\mathcal{O}_V) \subset \mathcal{C}^{\bullet,\bullet}(\mathcal{O}_V)$, such that $\mathcal{C}^{\bullet,\bullet}_{diff}(\mathcal{O}_V) \cap C^{\bullet}(\mathcal{O}_V) = C^{\bullet}_{diff}(\mathcal{O}_V)$, and prove that this inclusion is a quasi-isomorphism,

$$H\mathcal{C}^{\bullet,\bullet}_{diff}(\mathcal{O}_V) \simeq \wedge^{\bullet\geq 1}V \otimes \wedge^{\bullet\geq 1}V^*. \tag{4}$$

In fact, we show in this paper very simple pictorial proofs of all four results, (1)–(4), mentioned above: first we interpret Saneblidze-Umble's [SU] permutahedra cell complex as a differential graded (dg, for short) properad $\mathcal{P}$, then we use tensor powers of $\mathcal{P}$ to create a couple of other dg props, $\mathcal{D}$ and $\mathcal{Q}$, whose cohomology we know immediately by their very constructions, and then, studying representations of $\mathcal{D}$ and $\mathcal{Q}$ in an arbitrary vector space V we *obtain* (rather than *define*) the well-known polydifferential subcomplex of the Hochschild complex for $\mathcal{O}_V$ and, respectively, a new polydifferential subcomplex of the Gerstenhaber–Schack complex whose cohomologies are given, in view of contractibility of the permutahedra, by formulae (2) and (4). Finally, using again the language of props we deduce from (2) and (4) formulae (1) and, respectively, (3). As a corollary to (4) and (3) we show a slight sharpening of the famous Etingof–Kazhdan theorem: *for any Lie bialgebra structure on a vector space V there exists its bialgebra quantization within the class of polydifferential operators from $\mathcal{C}^{\bullet,\bullet}_{poly}(\mathcal{O}_V)$.*

The paper is organized as follows. In §2 we give a short but self-contained introduction into the theory of (wheeled) props, properads, dioperads and operads. In §3 we prove formulae (1)–(4) using properadic interpretation of the permutahedra cell complex. In §4 we study a dg prop, $\mathcal{D}ef\,\mathcal{Q}$, whose representations in a dg space V are in one-to-one correspondence with unital A_∞-structures on $\mathcal{O}_V$, and use it to give a new pictorial proof of another classical result that isomorphisms (1) and (2) extend to isomorphisms of Lie algebras, with $\wedge^{\bullet}\mathcal{T}_M$ assumed to be equipped with Schouten brackets.

We work over a field $\mathbb{K}$ of characteristic zero. If $V = \oplus_{i\in\mathbb{Z}}V^i$ is a graded vector space, then $V[k]$ is a graded vector space with $V[k]^i := V^{i+k}$. We denote $\otimes^{\bullet}V := \bigoplus_{n\geq 0} \otimes^n V$, $\otimes^{\bullet\geq 1}V := \bigoplus_{n\geq 1} \otimes^n V$, and similarly for symmetric and skew-symmetric tensor powers, $\odot^\bullet V$ and $\wedge^\bullet V$. The symbol $[n]$ stands for an ordered set $\{1, 2, \dots, n\}$.

2 An introduction to the theory of (wheeled) props

2.1 An associative algebra as a morphism of graphs. Recall that an *associative algebra* structure on a vector space E is a linear map $E \otimes E \to E$ satisfying the associativity condition, $(a_1 a_2)a_3 = a_1(a_2 a_3)$, for any $a_1, a_2, a_2 \in E$. Let us represent a typical element, $a_1 \otimes a_2 \otimes \cdots \otimes a_n \in \otimes^n E$, of the tensor algebra, $\otimes^\bullet E$, of E as a *decorated directed graph*,

$$G\langle a_1, \dots, a_n\rangle := \begin{array}{l} \bullet\ a_1 \\ \bullet\ a_2 \\ \vdots \\ \bullet\ a_n \end{array},$$

where the adjective *decorated* means that each vertex of the shown graph G is equipped with an element of E and the adjective *directed* means that the graph G is equipped with the flow running by default (unless otherwise is explicitly shown) from the bottom to the top. Let $G\langle E\rangle$ be the vector space spanned by all such decorations, $G\langle a_1, \dots, a_n\rangle$, of the shown chain-like graph G modulo the relations of the type,

$$\begin{array}{l} \bullet\ a_1 \\ \bullet\ \lambda_1 a_2' + \lambda_2 a_2'' \\ \vdots \\ \bullet\ a_n \end{array} = \lambda_1 \begin{array}{l} \bullet\ a_1 \\ \bullet\ a_2' \\ \vdots \\ \bullet\ a_n \end{array} + \ \lambda_2 \begin{array}{l} \bullet\ a_1 \\ \bullet\ a_2'' \\ \vdots \\ \bullet\ a_n \end{array} \quad \forall \lambda_1, \lambda_2 \in \mathbb{K},$$

which identify $G\langle E\rangle$ with $\otimes^n E$. Note that if G has only one internal vertex (we call such graphs (1,1)-*corollas*), then $G\langle E\rangle = E$. The multiplication operation in E gets encoded in this picture as a contraction of an internal edge, e.g.,

$$\begin{array}{l} \bullet\ a_1 \\ \bullet\ a_2 \\ \vdots \\ \bullet\ a_n \end{array} \longrightarrow \begin{array}{l} \bullet\ a_1 a_2 \\ \vdots \\ \bullet\ a_n \end{array}$$

which upon repetition gives a *contraction* map $\mu_G : G\langle E\rangle \to E$. Moreover, the associativity conditions for the multiplication assures us that the map μ_G is canonical, i.e., it does *not* depend on a particular sequence of contractions of the graph G into a corolla and is uniquely determined by the graph G itself.

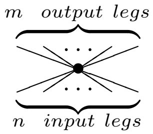
Actually there is no need to be specific about contracting precisely *two* vertices – any connected subset of vertices will do! Denoting the set of all possible directed connected chain-like graphs with one input leg and one output leg by $\mathfrak{G}_1^1$, one can equivalently define an associative algebra structure on a vector space E as a collection of linear maps, $\{\mu_G : G\langle E\rangle \to E\}_{G\in\mathfrak{G}_1^1}$, which satisfy the condition,

$$\mu_G = \mu_{G/H} \circ \mu'_H,$$

for any connected subgraph $H \subset G$. Here $\mu'_H : G\langle E\rangle \to (G/H)\langle E\rangle$ is the map which equals μ_H on the decorated vertices lying in H and which is identity on all other vertices, while $\mu_{G/H} : (G/H)\langle E\rangle \to E$ is the contraction map associated with the graph G/H obtained from G by contracting all vertices lying in the subgraph H into a single corolla.

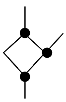
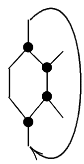
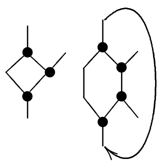
2.2 Families of directed labeled graphs. Thus the notion of an associative algebra can be encoded into the family of graphs $\mathfrak{G}_1^1$ with morphisms of graphs given by contractions along (admissible) subgraphs. This interpretation of an associative algebra structure has a strong potential for generalization leading us directly to the notions of *wheeled props, props, properads, dioperads* and *operads* depending on the way we choose to enlarge the above rather small and primitive family of graphs $\mathfrak{G}_1^1$. There are several natural enlargements of $\mathfrak{G}_1^1$:

(i) $\mathfrak{G}^{\circlearrowright}$ is, by definition, the family of *arbitrary* (not necessarily connected) directed graphs built step-by-step from the so-called (m,n)-*corollas*,



$$, \quad m, n \geq 0, \tag{5}$$

by taking their disjoint unions and/or gluing some output legs of one corolla with the same number of input legs of another corolla. This is the largest possible enlargement of $\mathfrak{G}_1^1$ in the class of *directed* graphs. We have $\mathfrak{G}^{\circlearrowright} = \coprod_{m,n\geq 0} \mathfrak{G}^{\circlearrowright}(m,n)$, where $\mathfrak{G}^{\circlearrowright}(m,n) \subset \mathfrak{G}^{\circlearrowright}$ is the subset of graphs having m output legs and n input legs, e.g.,

 $\in \mathfrak{G}^{\circlearrowright}(2,1)$,  $\in \mathfrak{G}^{\circlearrowright}(1,1)$,  $\in \mathfrak{G}^{\circlearrowright}(3,2)$. (6)

(ii) $\mathfrak{G}_c^{\circlearrowright} = \coprod_{m,n\geq 0} \mathfrak{G}_c^{\circlearrowright}(m,n)$ is a subset of $\mathfrak{G}^{\circlearrowright}$ consisting of *connected* graphs. For example, the first two graphs in (6) belong to $\mathfrak{G}_c^{\circlearrowright}$ while the third one (which is the disjoint union of the first two graphs) does not.

(iii) $\mathfrak{G}^{\uparrow} = \coprod_{m,n\geq 0} \mathfrak{G}^{\uparrow}(m,n)$ is a subset of $\mathfrak{G}^{\circlearrowright}$ consisting of directed graphs with no *closed* directed paths of internal edges which begin and end at the same vertex, e.g., the first graph in (6) belongs to $\mathfrak{G}^{\uparrow}$, while the other two do not.

(iv) $\mathfrak{G}_c^{\uparrow} := \mathfrak{G}^{\uparrow} \cap \mathfrak{G}_c^{\circlearrowright}$.
(v) $\mathfrak{G}_{c,0}^{\uparrow}$ is a subset of $\mathfrak{G}_c^{\uparrow}$ consisting of *trees* (that is, graphs having zero genus when viewed as 1-dimensional CW complexes).
(vi) $\mathfrak{G}^1$ is a subset of $\mathfrak{G}_{c,0}^{\uparrow}$ built from corollas (5) of type $(1,n)$ only, $n \geq 1$. We have $\mathfrak{G}^1 = \coprod_{n\geq 1} \mathfrak{G}^1(1,n)$ and we further abbreviate $\mathfrak{G}^1(n) := \mathfrak{G}^1(1,n)$; thus $\mathfrak{G}^1(n)$ is the subset of $\mathfrak{G}^1$ consisting of graphs with precisely n input legs. All graphs in $\mathfrak{G}^1$ have precisely one output leg.
Let $\mathfrak{G}^{\checkmark}$ be any of the above-mentioned families of graphs. *We assume from now on that input and output legs (if any) of graphs from* $\mathfrak{G}^{\checkmark}(m,n) \subset \mathfrak{G}^{\checkmark}$ *are bijectively* labelled *by elements of the sets* $[n]$ *and* $[m]$*, respectively.* Hence the group $\mathbb{S}_m \times \mathbb{S}_n$ naturally acts on the set $\mathfrak{G}^{\checkmark}(m,n)$ by permuting the labels[1].

2.3 Decorations of directed labeled graphs. Next we have to consider what to use for *decorations* of the vertices of a graph $G \in \mathfrak{G}^{\checkmark}(m,n)$. The presence of the family of the permutation groups $\{\mathbb{S}_m \times \mathbb{S}_n\}_{m,n\geq 0}$ suggests the following notion: an $\mathbb{S}$-*bimodule*, E, is, by definition, a collection of graded vector spaces, $\{E(m,n)\}_{m,n\geq 0}$, equipped with a left action of the group $\mathbb{S}_m$ and with a right action of $\mathbb{S}_n$ which commute with each other. For example, for any graded vector space V the collection, $\mathcal{E}nd\langle V\rangle = \{\mathcal{E}nd\langle V\rangle(m,n) := \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})\}_{m,n\geq 0}$, is naturally an $\mathbb{S}$-bimodule.

Let E be an $\mathbb{S}$-bimodule and $G \in \mathfrak{G}^{\circlearrowright}(m,n)$ an arbitrary graph. The graph G is built by definition from a number of various (p,q)-corollas constituting a set which we denote by $\mathsf{V}(G)$ and call the set of *vertices* of G; the set of output (resp. input) legs of a vertex $v \in \mathsf{V}(G)$ is denoted by Out_v (resp. by In_v). Let $\langle [p] \to \mathsf{Out}_v\rangle$ be the $p!$-dimensional vector space generated over $\mathbb{K}$ by the set of all bijections from $[p]$ to Out_v, i.e., by the set of all possible labeling of Out_v by integers; it is naturally a right $\mathbb{S}_p$-module; we define analogously a left $\mathbb{S}_q$-module $\langle \mathsf{In}_v \to [n]\rangle$ and then define a vector space,

$$E(\mathsf{Out}_v, \mathsf{In}_v) := \langle [p] \to \mathsf{Out}_v\rangle \otimes_{\mathbb{S}_p} E(p,q) \otimes_{\mathbb{S}_q} \langle \mathsf{In}_v \to [q]\rangle.$$

An element of $E(\mathsf{Out}_v, \mathsf{In}_v)$ is called a *decoration of the vertex* $v \in \mathsf{V}(G)$. To define next a space of *decorations of a graph* G we should think of taking the tensor product of the constructed vector spaces $E(\mathsf{Out}_v, \mathsf{In}_v)$ over all vertices $v \in \mathsf{V}(G)$ but face a problem that the set $\mathsf{V}(G)$ is unordered so that the ordinary definition of the tensor product of vector spaces does not immediately apply. The solution is to consider first *all* possible linear orderings, $\gamma : [k] \to \mathsf{V}(G)$, $k := |V(G)|$, of the set $\mathsf{V}(G)$ and then take coinvariants,

$$\otimes_{v\in \mathsf{V}(G)} E(\mathsf{Out}_v, \mathsf{In}_v) := \left(\oplus_\gamma E(\mathsf{Out}_{\gamma(1)}, \mathsf{In}_{\gamma(1)}) \otimes \cdots \otimes E(\mathsf{Out}_{\gamma(k)}, \mathsf{In}_{\gamma(k)})\right)_{\mathbb{S}_k},$$

with respect to the natural action of the group $\mathbb{S}_k$ permuting the orderings. Now we are ready to define the vector space of *decorations* of the graph G as a quotient of the unordered tensor product,

[1] In the case of $\mathfrak{G}^1$ the action of the factor $\mathbb{S}_m$ is, of course, trivial.

$$G\langle E\rangle := (\otimes_{v\in \mathsf{V}(G)} E(\mathsf{Out}_v, \mathsf{In}_v))_{AutG}$$

with respect to the automorphism group of the graph G which is, by definition, the subgroup of the symmetry group of the 1-dimensional CW-complex underlying the graph G which fixes the legs. An element of $G\langle E\rangle$ is called a *graph G with internal vertices decorated by elements of E*, Thus a decorated graph is essentially a pair, $(G, [a_1 \otimes \cdots \otimes a_k])$, consisting of a graph G with $k = |\mathsf{V}(G)|$ and an equivalence class of tensor products of elements $a_\bullet \in E$. Note that if $E = \{E(m,n)\}$ is a *dg* S-bimodule (i.e., each $E(m,n)$ is a complex equipped with an $\mathbb{S}_m \times \mathbb{S}_n$-equivariant differential δ), then $G\langle E\rangle$ is naturally a dg vector space with the differential

$$\delta_G\left(G, \left[a_1 \otimes \cdots \otimes a_{k|}\right]\right) := (G, [\sum_{i=1}^{k=|\mathsf{V}(G)|} (-1)^{a_1+\ldots+a_{i-1}} a_1 \otimes \cdots \otimes \delta a_i \otimes \cdots \otimes a_k]).$$

Note also that if G is an (m,n)-graph with one internal vertex and no edges (i.e., an (m,n)-corolla), then $G\langle E\rangle$ is canonically isomorphic to $E(m,n)$.

2.4 Props, properads, dioperads and operads. Let $\mathfrak{G}^{\checkmark}$ be one of the families of graphs introduced in § 2.2. A subgraph $H \subset G$ of a graph $G \in \mathfrak{G}^{\checkmark}$ is called *admissible* if both H and G/H also belong to $\mathfrak{G}^{\checkmark}$, where G/H is the graph obtained from G by shrinking all vertices and all internal edges of H into a new single vertex.

2.4.1 Definition. A $\mathfrak{G}^{\checkmark}$-*algebra* is an $\mathbb{S}$-bimodule $E = \{E(m,n)\}$ together with a collection of linear maps, $\{\mu_G : G\langle E\rangle \to E\}_{G\in\mathfrak{G}^{\checkmark}}$, satisfying the "associativity" condition,

$$\mu_G = \mu_{G/H} \circ \mu'_H, \tag{7}$$

for any admissible subgraph $H \subset G$, where $\mu'_H : G\langle E\rangle \to (G/H)\langle E\rangle$ is the map which equals μ_H on the decorated vertices lying in H and which is identity on all other vertices of G. If the $\mathbb{S}$-bimodule E underlying a $\mathfrak{G}^{\checkmark}$-algebra has a differential δ satisfying, for any $G \in \mathfrak{G}^{\checkmark}$, the condition $\delta \circ \mu_G = \mu_G \circ \delta_G$, then the $\mathfrak{G}^{\checkmark}$-algebra is called *differential.*

2.4.2 Remarks. (a) For the family of graphs $\mathfrak{G}^1$ the condition (7) is void for elements in $E(m,n)$ with $m \neq 1$. Thus we may assume without loss of generality that a $\mathfrak{G}^1$-algebra E satisfies an extra condition that $E(m,n) = 0$ unless $m = 1$. For the same reason we may assume that a $\mathfrak{G}^1_1$-algebra E satisfies $E(m,n) = 0$ unless $m = n = 1$.

(b) As we have an obvious identity $\mu_G = \mu_{G/G} \circ \mu'_G$, the "associativity" condition (7) can be equivalently reformulated as follows: for any two admissible subgraphs $H_1, H_2 \subset G$ one has

$$\mu_{G/H_1} \circ \mu'_{H_1} = \mu_{G/H_2} \circ \mu'_{H_2}, \tag{8}$$

i.e., the contraction of a decorated graph G into a decorated corolla along a family of admissible subgraphs does not depend on particular choices of these subgraphs (if there are any). This is indeed a natural extension of the notion of associativity from 1 dimension to 3 dimensions, and hence we can omit double commas in the term.

2.4.3 Definitions (see, e.g., [MSS, Va, BM, Me2] and references cited there):
(i) An $\mathfrak{G}^{\circlearrowright}$-algebra E is called a *wheeled prop.*
(ii) An $\mathfrak{G}_c^{\circlearrowright}$-algebra is called a *wheeled properad.*
(iii) An $\mathfrak{G}^{\uparrow}$-algebra is called a *prop.*
(iv) An $\mathfrak{G}_c^{\uparrow}$-algebra is called a *properad.*
(v) An $\mathfrak{G}_{0,c}^{\uparrow}$-algebra is called a *dioperad.*
(vi) An $\mathfrak{G}^{1}$-algebra is called an *operad.*
(vii) An $\mathfrak{G}_1^{1}$-algebra is called an *associative algebra.*

2.4.4 Remarks. (a) There is an obvious chain of forgetful functors between the categories of $\mathfrak{G}^{\checkmark}$-algebras,

$$\begin{array}{ccccccccc} & & (ii) & & & & & & \\ (i) & \nearrow\searrow & & \searrow\nearrow & (iv) \longrightarrow & (v) & \longrightarrow (vi) & \longrightarrow & (vii) \\ & & (iii) & & & & & & \end{array}$$

(b) Note that *every* subgraph of a graph in $\mathfrak{G}^{\circlearrowright}$ is admissible. In this sense wheeled props are the most general and natural algebraic structures associated with the class of *directed* graphs. The set of independent operations in a $\mathfrak{G}^{\circlearrowright}$-algebra is generated by one-vertex graphs with at least one *loop* (that is, an internal edge beginning and ending at the vertex) and by two-vertex graphs without closed directed paths (i.e., the ones belonging to $\mathfrak{G}^{\uparrow}$).

(c) By contrast to wheeled props, the set of operations in an ordinary prop, i.e., in a $\mathfrak{G}^{\uparrow}$-algebra, is generated by the set of two-vertex graphs only, and, as it is not hard to check, if the associativity condition holds for three-vertex graphs, then it holds for arbitrary graphs in $\mathfrak{G}^{\uparrow}$. This is *not* true for $\mathfrak{G}^{\circlearrowright}$-algebras which is a first indication that the homotopy theory for wheeled props should be substantially different from the one for ordinary props.

(d) If we forget orientations (i.e., the flow) on edges and work instead with a family of undirected graphs, $\mathfrak{G}$, built, by definition, from corollas with $m \geq 1$ undirected legs via their gluings, then we get a notion of $\mathfrak{G}$-*algebra* which is closely related to the notion of *modular operad* [GK].

2.5. First basic example: endomorphism $\mathfrak{G}^{\checkmark}$-algebras. For any finite-dimensional vector space V the $\mathbb{S}$-bimodule $\mathcal{E}nd_V = \{\mathrm{Hom}(V^{\otimes n}, V^{\otimes m})\}$ is naturally a $\mathfrak{G}^{\checkmark}$-algebra called the *endomorphism $\mathfrak{G}^{\checkmark}$-algebra of V.*[2] For any two-vertex graph $G \in \mathfrak{G}_c^{\uparrow}$ the associated composition $\mu_G : G\langle \mathcal{E}nd_V \rangle \to \mathcal{E}nd_V$ is the ordinary composition of two linear maps; for a one-vertex graph $G \in \mathfrak{G}^{\circlearrowright}$ with say k loops the associated map μ_G is the ordinary k-fold trace of a linear map; for a two-vertex disconnected graph $G \in \mathfrak{G}^{\uparrow}$ the associated map μ_G is

[2] For $\mathfrak{G}^{1}$-algebras, that is, for operads, it is enough to restrict oneself to the case $m = 1$ only, i.e., set, by default, $\mathcal{E}nd_V := \{\mathrm{Hom}(V^{\otimes n}, V)\}_{n\geq 1}$ (cf. §2.4.2(a)).

the ordinary tensor product of linear maps. It is easy to see that all the axioms are satisfied.

Note that for all $\mathfrak{G}^{\checkmark}$-algebras except $\mathfrak{G}^{\circlearrowright}$ and $\mathfrak{G}_c^{\circlearrowright}$ the basic algebraic operations μ_G do *not* involve traces so that the above assumption on *finite-dimensionality* of V can be dropped for endomorphism props, properads, dioperads and operads. If V is a (finite-dimensional) *dg* vector space, then $\mathcal{E}nd_V$ is naturally a *dg* $\mathfrak{G}^{\checkmark}$-algebra.

2.6 Second basic example: a free $\mathfrak{G}^{\checkmark}$-algebra. For an $\mathbb{S}$-bimodule, $E = \{E(m,n)\}$, one can construct another $\mathbb{S}$-bimodule, $\mathcal{F}^{\checkmark}\langle E\rangle = \{\mathcal{F}^{\checkmark}\langle E\rangle(m,n)\}$ with

$$\mathcal{F}^{\checkmark}\langle E\rangle(m,n) := \bigoplus_{G\in\mathfrak{G}^{\checkmark}(m,n)} G\langle E\rangle.$$

This $\mathbb{S}$-bimodule $\mathcal{F}^{\checkmark}\langle E\rangle$ has a natural $\mathfrak{G}^{\checkmark}$-algebra structure with the contraction maps μ_G being tautological. The $\mathfrak{G}^{\checkmark}$-algebra $\mathcal{F}^{\checkmark}\langle E\rangle$ is called the *free* $\mathfrak{G}^{\checkmark}$*-algebra* (i.e., respectively, *the free wheeled prop, the free prop, the free dioperad,* etc.) *generated by the* $\mathbb{S}$*-bimodule* E.

2.7 Morphisms of $\mathfrak{G}^{\checkmark}$-algebras. A morphism of $\mathfrak{G}^{\checkmark}$-algebras, $\rho : \mathcal{P}_1 \to \mathcal{P}_2$, is a morphism of the underlying $\mathbb{S}$-bimodules such that, for any graph $G \in \mathfrak{G}^{\checkmark}$, one has $\rho\circ\mu_G = \mu_G \circ (\rho^{\otimes G})$, where $\rho^{\otimes G}$ means a map, $G\langle\mathcal{P}_1\rangle \to G\langle\mathcal{P}_2\rangle$, which changes decorations of each vertex in G in accordance with ρ. It is often assumed by default that a morphism ρ is homogeneous which (almost always) implies that ρ has degree 0. Unless otherwise is explicitly stated we do *not* assume in this paper that morphisms of $\mathfrak{G}^{\checkmark}$-algebras are homogeneous so that they can have nontrivial parts in degrees other than zero. A morphism of $\mathfrak{G}^{\checkmark}$-algebras, $\mathcal{P} \to \mathcal{E}nd\langle V\rangle$, is called a *representation* of the $\mathfrak{G}^{\checkmark}$-algebra $\mathcal{P}$ in a graded vector space V. If $\mathcal{P}_1$ is a free $\mathfrak{G}^{\checkmark}$-algebra, $\mathcal{F}^{\checkmark}\langle E\rangle$, generated by some $\mathbb{S}$-bimodule E, then the set of morphisms of $\mathfrak{G}^{\checkmark}$-algebras, $\{\rho : \mathcal{P}_1 \to \mathcal{P}_2\}$, is in one-to-one correspondence with the set of morphisms of $\mathbb{S}$-bimodules, $\{\rho|_E : E \to \mathcal{P}_2\}$, i.e., a $\mathfrak{G}^{\checkmark}$-morphism is uniquely determined by its values on the generators. In particular, the set of morphisms, $\mathcal{F}^{\checkmark}\langle E\rangle \to \mathcal{P}_2$, has a graded vector space structure for any $\mathcal{P}_2$.

A *free resolution* of a dg $\mathfrak{G}^{\checkmark}$-algebra $\mathcal{P}$ is, by definition, a dg free $\mathfrak{G}^{\checkmark}$-algebra, $(\mathcal{F}^{\checkmark}\langle E\rangle, \delta)$, generated by some $\mathbb{S}$-bimodule E together with a degree zero morphism of dg $\mathfrak{G}^{\checkmark}$-algebras, $\pi : (\mathcal{F}^{\checkmark}\langle E\rangle, \delta) \to \mathcal{P}$, which induces a cohomology isomorphism. If the differential δ in $\mathcal{F}^{\checkmark}\langle\mathcal{E}\rangle$ is decomposable with respect to compositions μ_G, then $\pi : (\mathcal{F}^{\checkmark}\langle E\rangle, \delta) \to \mathcal{P}$ is called a *minimal model* of $\mathcal{P}$. In this case the free algebra $\mathcal{F}^{\checkmark}\langle E\rangle$ is often denoted by $\mathcal{P}_\infty$.

2.8 Props and properads. We shall work in this paper only with $\mathfrak{G}^{\uparrow}$- and $\mathfrak{G}_c^{\uparrow}$-algebras, i.e., with props and properads. For later use we mention several useful constructions with these graph-algebras.

(i) There is a functor, Ψ, which associates canonically to an arbitrary dg properad, $\mathcal{P}$, an associated dg prop $\Psi(\mathcal{P})$ [Va]. As we are working over a field of characteristic 0, this functor is, by Künneth theorem, exact, i.e.,

$\Psi(H(\mathcal{P})) = H(\Psi(\mathcal{P}))$. For example, if $\mathcal{P}$ is a dg free properad $(\mathcal{F}_c^{\uparrow}\langle E\rangle, \delta)$, then $\Psi(\mathcal{P})$ is precisely $\mathcal{F}^{\uparrow}\langle E\rangle$ with the same differential (as given on the generators).

(ii) The above-mentioned functor, $\mathcal{F}^{\uparrow} : (E,\delta) \to (\mathcal{F}^{\uparrow}\langle E\rangle, \delta)$, in the category of dg $\mathbb{S}$-bimodules is also exact, $H(\mathcal{F}^{\uparrow}\langle E\rangle) = \mathcal{F}^{\uparrow}\langle H(E)\rangle$. Moreover, if we set in this situation $\mathrm{Rep}_V(\mathcal{F}^{\uparrow}\langle E\rangle)$ for the vector space of all possible representations, $\{\rho : \mathcal{F}^{\uparrow}\langle E\rangle \to \mathcal{E}nd_V\} \simeq \mathrm{Hom}(E, \mathcal{E}nd_V)$, and define a differential δ in $\mathrm{Rep}_V(\mathcal{F}^{\uparrow}\langle E\rangle)$ by the formula $\delta\rho := \rho \circ \delta$, then the resulting functor, $(E,\delta) \to (\mathrm{Rep}_V(\mathcal{F}^{\uparrow}\langle E\rangle), \delta)$, in the category of complexes is exact, i.e.,

$$H(\mathrm{Rep}_V(\mathcal{F}^{\uparrow}\langle E\rangle)) = \mathrm{Rep}_V(H(\mathcal{F}^{\uparrow}\langle E\rangle)) = \mathrm{Rep}_V(\mathcal{F}^{\uparrow}\langle H(E)\rangle). \tag{9}$$

Indeed, as we are working over a field of characteristic zero, we can always choose an equivariant chain homotopy between complexes (E,δ) and $(H(E),0)$. This chain homotopy induces a chain homotopy between complexes $(\mathrm{Rep}_V(\mathcal{F}^{\uparrow}\langle E\rangle), \delta)$ and $(\mathrm{Rep}_V(\mathcal{F}^{\uparrow}\langle H(E)\rangle), 0)$ proving formula (9).

(iii) There is also a natural *parity change* functor, Π, which associates with a dg prop(erad) $\mathcal{P}$ a dg prop(erad) $\Pi\mathcal{P}$ with the following property: every representation of $\Pi\mathcal{P}$ in a graded vector space V is equivalent to a representation of $\mathcal{P}$ in $V[1]$. This functor is also exact. If, for example, $\mathcal{P}$ is a dg free prop $\mathcal{F}^{\uparrow}\langle E\rangle$ generated by an $\mathbb{S}$-bimodule $E = \{E(m,n)\}$, then, as it is not hard to check, $\Pi\mathcal{P} = \mathcal{F}^{\uparrow}\langle \check{E}\rangle$, where $\check{E} := \{sgn_m \otimes E(m,n) \otimes sgn_n[m-n]\}$ and sgn_m stands for the 1-dimensional sign representation of $\mathbb{S}_m$.

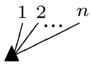
3 Simplicial and permutahedra cell complexes as dg properads

3.1 Simplices as a dg properad. A geometric $(n-1)$-*simplex*, Δ_{n-1}, is, by definition, a subset in $\mathbb{R}^n = \{x^1, \ldots, x^n\}$, $n \geq 1$, satisfying the equation $\sum_{i=1}^n x^i = 1$, $x^i \geq 0$ for all i. To define its cell complex one has to choose an orientation on Δ_{n-1} which is the same as to choose an orientation on the hyperplane $\sum_{i=1}^n x^i = 1$. We induce it from the standard orientation on $\mathbb{R}^{n+1}$ by requiring that the manifold with boundary defined by the equation $\sum_{i=1} x^i \leq 1$ is naturally oriented. Let $(C_\bullet(\Delta_{n-1}) = \oplus_{k=0}^{n-1} C_{-k}(\Delta_{n-1}), \delta)$ be the standard (non-positively graded) cell complex of Δ_{n-1}. By definition, $C_{-k}(\Delta_{n-1})$ is a $\binom{n}{k}$-dimensional vector space spanned by k-dimensional cells,

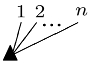
$$\triangle_{n-1}^I := \left\{(x^1, \ldots, x^n) \in \Delta_{n-1} | x^i = 0, i \in I\right\},$$

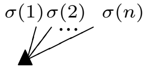
parametrized by all possible subsets I of $[n]$ of cardinality $n-k-1$ and equipped with the natural orientations (which we describe explicitly below).

Note that the action , $(x^1, \ldots, x^n) \to (x^{\sigma(1)}, \ldots, x^{\sigma(n)})$, of the permutation group $\mathbb{S}_n$ on $\mathbb{R}^n$ leaves Δ_{n-1} invariant as a subset but *not* as an oriented manifold with boundary. As an $\mathbb{S}_n$-module, one can obviously identify $C_{1-n}(\Delta_{n-1})$

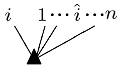
with $sgn_n[n-1]$, and hence one can represent pictorially the oriented cell $\triangle^{\emptyset}_{n-1}$ as a labeled $(0,n)$-corolla, 

, with the symmetry condition

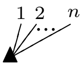


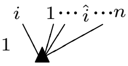
$$= (-1)^{\sigma}$$


$$, \quad \forall \sigma \in \mathbb{S}_n. \qquad (10)$$

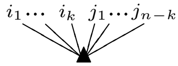
The boundary of $\triangle^{\emptyset}_{n-1}$ is a union of n cells, $\triangle^{i}_{n-1}$, $i = 1, \ldots, n$, of dimension $n-2$. The permutation group $\mathbb{S}_n$ permutes, in general, these cells and changes their natural orientations while keeping their linear span $C_{2-n}(\Delta_{n-1})$ invariant. It is obvious that the subgroup $G_i := \{\sigma \in \mathbb{S}_n \mid \sigma(i) = i\} \simeq \mathbb{S}_{n-1}$ of $\mathbb{S}_n$ is a symmetry group of the cell $\triangle^{i}_{n-1}$ as an *unoriented* manifold with boundary. If we take the orientation into account, then the vector subspace of $C_{2-n}(\Delta_{n-1})$ spanned by $\triangle^{i}_{n-1}$ can be identified as an $\mathbb{S}_{n-1}$-module with sgn_{n-1}, and hence the n-dimensional space $C_{2-n}(\Delta_{n-1})$ itself can be identified as an $\mathbb{S}_n$-module with $\mathbb{K}[\mathbb{S}_n] \otimes_{\mathbb{S}_{n-1}} sgn_{n-1}[n-2]$. Its basis elements, $\triangle^{i}_{n-1}$, can be pictorially represented as $(0,n)$-corollas 

with the legs in the right bunch being skew-symmetric with respect to the change of labelings by an element $\sigma \in G_i$ (cf. (10)). The boundary operator $\delta : C_{1-n}(\Delta_{n-1}) \to C_{2-n}(\Delta_{n-1})$ is equivariant with respect to the $\mathbb{S}_n$-action and is given on the generators by the formula

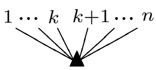
$$\delta$$


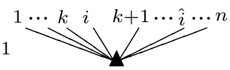
$$= \sum_{i=1}^{n} (-1)^{i+1}$$


$$, \quad \hat{\imath} \text{ omitted},$$

More generally, the symmetry group of, say, a cell $\triangle^{I}_{1-n} \in C_{1-k-n}(\Delta_{n-1})$ with $I = \{i_1 < i_2 < \ldots < i_{n-k}\}$ is $\mathbb{S}_k \times \mathbb{S}_{n-k} \subset \mathbb{S}_n$ with $\mathbb{S}_k \times \mathrm{Id}$ leaving the orientation of $\triangle^{I}_{n-1}$ invariant and $\mathrm{Id} \times \mathbb{S}_{n-k}$ changing the orientation via the sign representation. Thus we can identify the oriented cell $\triangle^{I}_{n-1}$, as an element of the $\mathbb{S}_n$-module $C_{1-k-n}(\Delta_{n-1})$, with a $(0,n)$-corolla, 

, which has "symmetric" output legs in the left bunch and "skew-symmetric" output legs in the right one. Here $\{j_1 < \ldots < j_{n-k}\} := [n] \setminus I$. The $\mathbb{S}_n$-module, $C_{1-k-n}(\Delta_{n-1})$, is then canonically isomorphic to $E_k(n) := \mathbb{K}[\mathbb{S}_n]_{\mathbb{S}_k \times \mathbb{S}_{n-k}}(\mathbb{1}_k \otimes sgn_{n-k})$, where $\mathbb{1}_k$ stands for the trivial 1-dimensional representation of $\mathbb{S}_k$. The boundary operator $\delta : C_{1-k-n}(\Delta_{n-1}) \to C_{2-k-n}(\Delta_{n-1})$ is equivariant with respect to the $\mathbb{S}_n$-action and is given on the generators by the formula

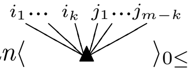
$$\delta$$


$$= \sum_{i=k+1}^{n} (-1)^{i+1}$$


$$(11)$$

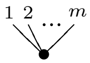
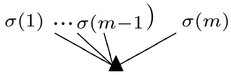
Thus we proved the following:

3.1.1 Proposition. *(i) The standard simplicial cell complex is canonically isomorphic to a dg free properad,* $\mathcal{S} := \mathcal{F}^{\uparrow}_{c}\langle E\rangle$*, generated by an* $\mathbb{S}$*-bimodule,* $E = \{E(m,n)\}$*,*

$$E(m,n) = \begin{cases} \bigoplus_{k=0}^{m-1} E_k(m)[m-k-1] = span\langle \end{cases}$$  $$\rangle_{0\le k\le m-1} \; for\ n=0, \qquad 0 \quad for\ n \geq 1,$$

and equipped with the differential given on the generators by (11).

(ii) The cohomology of $(\mathcal{S}, \delta)$ *is concentrated in degree zero and equals the free properad generated by the following degree zero graphs with "symmetric" legs,*

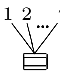
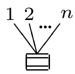
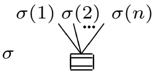
 $$:= \sum_{\sigma\in\mathbb{S}_m}$$  $$, \quad m \geq 1. \tag{12}$$

Claim 3.1.1(ii) follows from the contractibility of simplices, and, for each m, graph (12) represents the sum of all vertices of Δ_{m-1}.

3.1.2 From simplicia to Koszul complex. The vector space, $\mathrm{Rep}_V(\mathcal{S})$, of representations, $\rho : \mathcal{S} \to \mathcal{E}nd_V$, of the simplicial properad in a vector space V can be identified with $\sum_{k=0}^{m-1} \odot^k V \otimes \wedge^{m-k} V[m-k-1]$. We can naturally make the latter into complex by setting $d\rho := \rho \circ \partial$ (cf. §2.8ii). It is easy to see that we get in this way, for each $m \geq 1$,

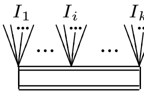
$$\wedge^m V \xrightarrow{d} V \otimes \wedge^{m-1} V \xrightarrow{d} \odot^2 V \otimes \wedge^{m-2} V \xrightarrow{d} \ldots \xrightarrow{d} \odot^{m-1} V \otimes V,$$

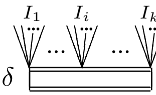
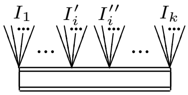
the classical Koszul complex. Hence Proposition 3.1.1(ii) and isomorphism (9) imply the well-known result that its cohomology is concentrated in degree zero and equals $\odot^n V$. Thus the Koszul complex is nothing but a representation of the simplicial cell complex in a particular vector space V.

3.2 Permutahedra as a dg properad. An $(n-1)$-dimensional *permutahedron*, P_{n-1}, is, by definition, the convex hull of $n!$ points $(\sigma(1), \sigma(2), \ldots, \sigma(n))$, $\forall \sigma \in \mathbb{S}_n$, in $\mathbb{R}^n = \{x^1, \ldots, x^n\}$. To define its cell complex one has to choose an orientation on P_{n-1} which is the same as to choose an orientation on the hyperplane $\sum_{i=1}^n x^i = n(n+1)/2$ to which P_{n-1} belongs. We induce it from the standard orientation on $\mathbb{R}^n$ by requiring that the manifold with boundary defined by the equation $\sum_{i=1} x^i \leq n(n+1)/2$ is naturally oriented. Let $(C_\bullet(P_{n-1}) = \oplus_{k=0}^{n-1} C_{-k}(P_{n-1}), \delta)$ stand for the associated (non-positively graded) complex of oriented cells of P_{n-1}. Its $(n-k-1)$-dimensional cells, $P_{n-1}^{I_1,\ldots,I_p}$, are indexed by all possible partitions, $[n] = I_1 \sqcup I_2 \sqcup \ldots \sqcup I_k$, of the ordered set $[n]$ into k disjoint ordered nonempty subsets (see [SU]). The natural action , $(x^1, \ldots, x^n) \to (x^{\sigma(1)}, \ldots, x^{\sigma(n)})$, of the permutation group $\mathbb{S}_n$ on $\mathbb{R}^n$ leaves P_{n-1} invariant, and hence makes the cell complex $C_\bullet(P_{n-1})$ into an $\mathbb{S}_n$-module. We obviously have, for example, $C_{1-n}(P_{n-1}) = sgn_n$, so that we can identify the top cell $P_{n-1}^{[n]}$ *as an element of the* $\mathbb{S}_n$*-module* with the $(n,0)$-corolla  with skew-symmetric output legs,  $= (-1)^\sigma$  , $\forall \sigma \in \mathbb{S}_n$. More generally, a simple analysis (similar to the simplicial case in

§3.1) of how the action of $\mathbb{S}_n$ on $\mathbb{R}^n$ permutes the cells and changes their orientation implies that $C_{1-k-n}(P_{n-1})$ is canonically isomorphic as an $\mathbb{S}_n$-module to

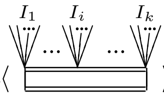
$$W_k(n) := \bigoplus_{[n]=I_1\sqcup I_2\sqcup\ldots\sqcup I_k} \mathbb{K}[\mathbb{S}_n]\otimes_{\mathbb{S}_{I_1}\times\ldots\times S_{I_k}}\left(sgn_{I_1}\otimes\cdots\otimes sgn_{I_k}\right)[k+n-1]$$

and that the cells $P_{n-1}^{I_1,\ldots,I_p}$ can be identified *as elements of the* $\mathbb{S}_n$*-module* with the $(n,0)$-corollas,  , where legs in each I_i-bunch are skew-symmetric and the labels from I_i are assumed to be distributed over them in the increasing order from the left to the right. The boundary operator $\delta : C_{1-k-n}(P_{n-1}) \to C_{2-k-n}(P_{n-1})$ is given on generators by (cf. [SU])

δ  $= \sum_{i=1}^{k} \sum_{\substack{I_i=i'_i\sqcup I''_i\\ |I'_i|,|I''_i|\geq 1}} (-1)^{\varepsilon+\sigma_{I'_i\sqcup I''_i}}$  (13)

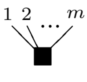
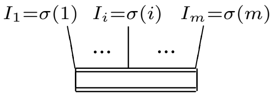
where $\varepsilon := i+1+I_1+\cdots+I_{i-1}+I'_i$ and $(-1)^{\sigma_{I'_i\sqcup I''_i}}$ is the sign of the permutation $[I_i]\to I'_i\sqcup I''_i$. Thus we proved the following:

3.2.1 Proposition. *(i) The Saneblidze-Umble permutahedra cell complex is canonically isomorphic to a dg free properad,* $\mathcal{P}_\bullet := \mathcal{F}_c^{\uparrow}\langle W\rangle$*, generated by an* $\mathbb{S}$*-bimodule,* $W=\{W(m,n)\}$*,*

$W(m,n) := \begin{cases} \bigoplus_{k=1}^m W_k(m) = span\langle \end{cases}$  $\rangle$ *for* $n=0$,
0 *for* $n\geq 1$,

and equipped with the differential given on the generators by (13).

(ii) The cohomology of $(\mathcal{P}_\bullet,\partial)$ *is concentrated in degree zero and equals a free properad generated by the following degree zero graphs:*

 $:= \sum_{\sigma\in\mathbb{S}_m}$  , $m\geq 1$.

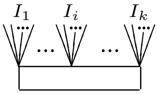
Claim 3.2.1(ii) follows from the contractibility of permutahedra, and, for each n, the above graph represents the sum of all vertices of P_{n-1}.

3.2.2 From permutahedra to a cobar construction. Baranovsky made in [Ba] a remarkable observation that the permutahedra cell complex can be used to compute the cohomology of the cobar construction, $\Omega(\wedge^\bullet V)$, where

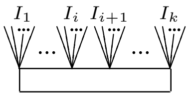
$\wedge^\bullet V \simeq \odot^\bullet(V[1])$ is interpreted as a graded commutative coalgebra generated by a vector space $V[1]$. In our approach this result follows immediately from the following two observations: (i) the graded space $\Omega(\wedge^\bullet V)$ can be identified with $\mathbb{K} \oplus \mathrm{Rep}_V(\mathcal{P}_\bullet)$, where $\mathrm{Rep}_V(\mathcal{P}_\bullet)$ is the space of all possible representations, $\rho : \mathcal{P}_\bullet \to \mathcal{E}nd_V$; the differential in $\Omega(\wedge^\bullet V)$ induced from $\mathcal{P}_\bullet$ by the formula $d\rho := \rho \circ \delta$ is precisely the differential of the cobar construction. Then Proposition 3.2.1(ii) and isomorphism (9) imply that the cohomology, $H(\Omega(\wedge^\bullet V))$, of the cobar construction equals $\mathbb{K} \oplus \odot^{\bullet \geq 1} V = \odot^\bullet V$.

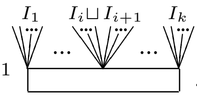
3.2.3 Permutahedra cochain complex. Exactly in the same way as in §3.2 one can construct a dg properad, $(\mathcal{P}^\bullet, \delta)$, out of the permutahedra *cochain* complex, $C^\bullet(P_{n-1}) := \mathrm{Hom}_{\mathbb{K}}(C_\bullet(P_{n-1}), \mathbb{K})$, with the differential δ dual to the one given in (13). Very remarkably, Chapoton has shown in [Ch] that one can make the $\mathbb{S}$-module $\{C^\bullet(P_{n-1})\}$ into a dg *operad*, and that this operad is a quadratic one. We do not use this interesting fact in our paper and continue interpreting instead permutahedra as a dg *properad* from which we shall build below more complicated dg props with nice geometric and/or algebraic meaning. Let us apply first the parity chain functor, Π, to the dg properad $\mathcal{P}^\bullet$ (see §2.8iii). The result, $\mathcal{P} := \Pi\mathcal{P}^\bullet$, is a dg free properad, $\mathcal{F}_c^\uparrow\langle \check{W} \rangle$, generated by an $\mathbb{S}$-bimodule, $\check{W} = \{\check{W}(m,n)\}$ with $\check{W}(m,n) = 0$ for $n \neq 0$ and with $\check{W}(m,0)$ equal to

$$Y(m) := \bigoplus_{k=1}^{m} \bigoplus_{\substack{[m]=I_1 \sqcup I_2 \sqcup \ldots \sqcup I_k \\ |I_\bullet| \geq 1}} \mathbb{K}[\mathbb{S}_n] \otimes_{\mathbb{S}_{I_1} \times \ldots \times S_{I_k}} (1\!\!1_{I_1} \otimes \cdots \otimes 1\!\!1_{I_k})\,[k]. \qquad (14)$$

If we represent the generators of $\mathcal{P}$ by corollas 

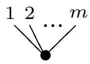
with *symmetric* legs in each I_i-bunch, then the induced differential in $\mathcal{P}$ is given by

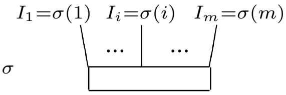
δ 

$= \sum_{i=1}^{k} (-1)^{i+1}$ 

. (15)

3.2.4 Theorem. *The cohomology of the dg properad $\mathcal{P}$ is a free properad generated by the degree $-m$ corollas with skew-symmetric legs,*



$:= \sum_{\sigma \in \mathbb{S}_m} (-1)^\sigma$ 

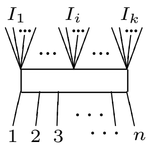
, $m \geq 1$.

Proof. By exactness of the functor Π, the statement follows from Proposition 3.2.1(ii). □

3.2.5 Corollary. *The cohomology of the bar construction, $B(\odot^\bullet V)$, of the graded commutative algebra generated by a vector space V, is equal to $\wedge^\bullet V$.*

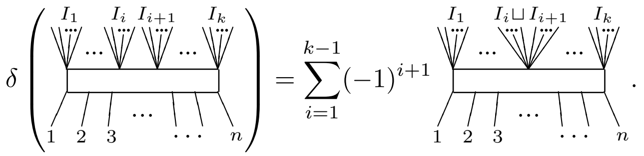
Proof. By definition (see, e.g., [Ba]), $B(\odot^\bullet V)$ is a free tensor coalgebra, $\otimes^\bullet(\odot^{\bullet \geq 1} V[-1])$ with the differential d induced from the ordinary multiplication in $\odot^{\bullet\geq 1}V$. On the other hand, it is easy to see that as a vector space $B(\odot^\bullet V)$ can be identified with $\mathbb{K}\oplus \mathrm{Rep}_V(\mathcal{P})$, where $\mathrm{Rep}_V(\mathcal{P})$ is the space of all possible representations, $\rho : \mathcal{P} \to \mathcal{E}nd_V$, of the parity shifted properad of permutahedra cochains. Moreover, the bar differential d is given precisely by $d\rho := \rho \circ \delta$. Thus isomorphism (9) and Theorem 3.2.4 imply the required result. $\square$

3.3 From permutahedra to polydifferential Hochschild complex. Let us consider a dg $\mathbb{S}$-module, $D = \{D(m,n)\}_{m\geq 1, n\geq 0}$, with $D(m,n) := Y(m) \otimes \mathbb{1}_n[-1]$ and with the differential, $\delta : D(m,n) \to D(m,n)$ being equal to (15) on the tensor factor $Y(m)$ and identity on the factor $\mathbb{1}_n[-1]$. Let $(\mathcal{D} := \mathcal{F}^{\uparrow}\langle D\rangle, \delta)$ be the associated dg free prop. Its generators can be identified with (m,n)-corollas



(16)

of degree $1-k$, one such corolla for every partition $[m] = I_1 \sqcup \ldots \sqcup I_k$. The differential δ in $\mathcal{D}$ is then given by



(17)

3.3.1 Proposition. *The cohomology of the dg prop $\mathcal{D}$ is a free prop, $\mathcal{F}^{\uparrow}\langle X\rangle$, generated by an $\mathbb{S}$-bimodule $X = \{X(m,n)\}_{m\geq 1, n\geq 0}$ with $X(m,n) := sgn_m \otimes \mathbb{1}_n[m-1]$.*

Proof. By §2.8(ii), the functor $\mathcal{F}^{\uparrow}$ is exact so that we have $H(\mathcal{D}) = \mathcal{F}^{\uparrow}\langle H(D)\rangle$. By Künneth's theorem, $H(D(m,n)) = H(Y(m)) \otimes \mathbb{1}_n[-1]$. Finally, by Theorem 3.2.4, $H(Y(m)) = H(\mathcal{P})(m,0) = sgn_m[m]$. $\square$

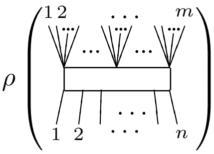
The space of all representations, $\rho : \mathcal{D} \to \mathcal{E}nd_V$, of the prop $\mathcal{D}$ in a (finite-dimensional) vector space V can be obviously identified with

$$Rep_V(\mathcal{D}) := \bigoplus_{k\geq 1} \mathrm{Hom}\left(\odot^\bullet V, (\odot^{\bullet\geq 1}V)^{\otimes k}\right)[1-k] = \bigoplus_{k\geq 1} \mathrm{Hom}(\bar{\mathcal{O}}_V^{\otimes k}, \mathcal{O}_V)[1-k],$$

where $\mathcal{O}_V := \odot^\bullet V^*$ is the graded commutative ring of polynomial functions on the space V, and $\bar{\mathcal{O}}_V := \odot^{\bullet\geq 1}V^*$ is its subring consisting of functions vanishing at 0. Differential (17) in the prop $\mathcal{D}$ induces a differential, δ, in the space $\mathrm{Rep}(\mathcal{D})$ by the formula, $\delta\rho := \rho\circ\delta, \quad \forall \rho \in \mathrm{Rep}(\mathcal{D})$.

3.3.2 Proposition. *The complex $(\mathrm{Rep}(\mathcal{D})_V, \delta)$ is canonically isomorphic to the polydifferential subcomplex, $(C^{\bullet\geq 1}_{diff}(\mathcal{O}_V), d_H)$ of the standard Hochschild complex, $(C^\bullet(\mathcal{O}_V), d_H)$, of the algebra $\mathcal{O}_V$.*

Proof. We shall construct a degree 0 isomorphism of vector spaces, $i : \mathrm{Rep}_V(\mathcal{D}) \to C^{\bullet\geq 1}_{diff}(\mathcal{O}_V)$, such that $i \circ \delta = d_H \circ i$, where d_H stands for the Hochschild differential. Let $\{e_\alpha\}$ be a basis of V, and $\{x^\alpha\}$ the associated dual basis of V^*. Any $\rho \in \mathrm{Rep}_V(\mathcal{D})$ is uniquely determined by its values,



$$=: \sum_{\substack{I_1,\ldots,I_k,J\\ |I_\bullet|>1,|J|\geq 0}} \Gamma_J^{I_1,\ldots,I_k} x^J \otimes e_{I_1} \otimes \cdots \otimes e_{I_k} \in \mathrm{Hom}(\bar{\mathcal{O}}_V^{\otimes k}, \mathcal{O}_V),$$

for some $\Gamma_J^{I_1,\ldots,I_k} \in \mathbb{K}$. Here the summation runs over multi-indices, $I = \alpha_1\alpha_2\cdots\alpha_{|I|}$, $x^I := x^{\alpha_1} \odot \cdots \odot x^{\alpha_{|I|}}$, and $e_I := e_{\alpha_1} \odot \cdots \odot e_{\alpha_{|I|}}$. Then the required map i is given explicitly by

$$i(\rho) := \sum_{I_1,\ldots,I_k,J} \frac{1}{|J|!|I_1|!\cdots|I_k|!} \Gamma_J^{I_1,\ldots,I_k} x^J \frac{\partial^{|I_1|}}{\partial x^{I_1}} \otimes \cdots \otimes \frac{\partial^{|I_k|}}{\partial x^{I_k}}$$

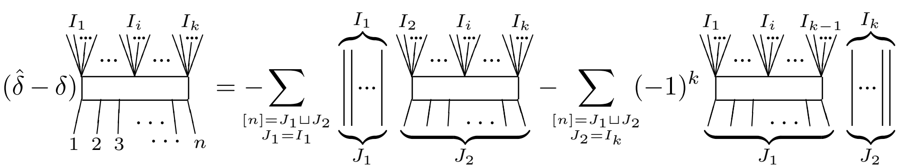
where $\partial^{|I|}/\partial x^I := \partial^{|I|}/\partial x^{\alpha_1}\cdots\partial x^{\alpha_{|I|}}$. Now it is an easy calculation to check (using the definition of the Hochschild differential) that $i \circ \delta = d_H \circ i$. $\square$

3.3.3 Corollary. $H(C^{\bullet\geq 1}_{diff}(\mathcal{O}_V)) = \wedge^{\bullet\geq 1} V \otimes \odot^\bullet V^*$.

Proof. By Proposition 3.3.2, the cohomology $H(C^{\bullet\geq 1}_{diff}(\mathcal{O}_V))$ of the polydifferential Hochschild is equal to the cohomology of the complex $(\mathrm{Rep}_V(\mathcal{D}), \delta)$. The latter is equal, by isomorphism (9), to $\mathrm{Rep}_V(H(\mathcal{D}))$ which in turn is equal, by Proposition 3.3.1, to $\wedge^{\bullet\geq 1} V \otimes \odot^\bullet V^*$. $\square$

The complex $C^\bullet_{diff}(\mathcal{O}_V)$ is a direct sum, $\mathcal{O}_V \oplus C^{\bullet\geq 1}_{diff}(\mathcal{O}_V)$, where $\mathcal{O}_V$ is a trivial subcomplex. Thus Corollary 3.3.3 implies isomorphism (2).

3.4 Hochschild complex. Is there a dg prop whose representation complex is the general (rather than polydifferential) Hochschild complex for polynomial functions? Consider a dg free prop, $\hat{\mathcal{D}}$, which is generated by the same $\mathbb{S}$-bimodule D as the prop $\mathcal{D}$ above, but equipped with a different differential, $\hat{\delta}$, given on the generators by (17) and two extra terms,

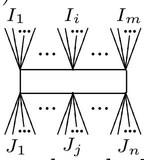


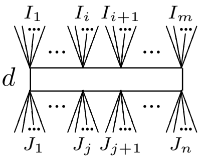
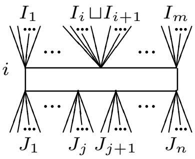
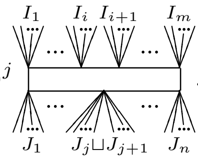
3.4.1 Proposition. $H(\hat{\mathcal{D}}) = H(\mathcal{D}) = \mathcal{F}^\uparrow\langle X\rangle$.

Proof. Define a *weight* of a generating corolla (16) of the prop $\hat{\mathcal{D}}$ to be $\sum_{i=1}^k |I_i|$, and the *weight*, $w(G)$, of a decorated graph G from $\hat{\mathcal{D}}$ to be the sum of weights of all constituent corollas of G. Then $F_p := \{span\langle G\rangle | w(G) \leq p\}_{p\geq 0}$ is a bounded below exhaustive filtration of the complex $(\hat{\mathcal{D}}, \hat{\delta})$. By the classical convergence theorem, the associated spectral sequence $\{E_r, d_r\}_{r\geq 0}$ converges to $H(\hat{\mathcal{D}})$. Its 0th term, (E_0, d_0), is precisely the complex $(\mathcal{D}, \delta)$. Thus E_1, is, by Proposition 3.3.1, the free prop $\mathcal{F}^{\uparrow}\langle X\rangle$ so that d_1 vanishes and the spectral sequence degenerates at the first term completing the proof. □

The complex $(\mathrm{Rep}_V(\hat{\mathcal{D}}), \hat{\delta})$ associated, by §2.8(ii), to the dg prop $(\hat{\mathcal{D}}, \hat{\delta})$ is easily seen to be precisely the standard Hochschild complex, $(C^{\bullet\geq 1}(\bar{\mathcal{O}}_V) = \oplus_{k\geq 1}\mathrm{Hom}(\bar{\mathcal{O}}_V^{\otimes k}, \mathcal{O}_V), d_H)$ of the nonunital algebra $\bar{\mathcal{O}}_V$ with coefficients in the unital algebra $\mathcal{O}_V$. Hence Proposition 3.4.1 and isomorphism (9) immediately imply that $HC^{\bullet\geq 1}(\bar{\mathcal{O}}_V) = \wedge^{\bullet\geq 1}V \otimes \odot^{\bullet}V^*$ which in turn implies, with the help of the theory of simplicial modules (see, e.g., Proposition 1.6.5 in [Lo]), the Hochschild–Kostant–Rosenberg isomorphism (1). Using the language of dg props, we deduced it, therefore, from the permutahedra cell complex.

3.5 From permutahedra to polydifferential Gerstenhaber–Schack complex. Let us consider a dg $\mathbb{S}$-module, $Q = \{Q(m,n)\}_{m\geq 1, n\geq 1}$, with $Q(m,n) := Y(m) \otimes Y(n)^*[-2]$ and the differential, $d = \delta \otimes \mathrm{Id} + \mathrm{Id} \otimes \delta^*$, where δ is given by (15). Let $(\mathcal{Q} := \mathcal{F}^{\uparrow}\langle Q\rangle, d)$ be the associated dg free prop.

Its generators can be identified with corollas  of degree $2 - m - n$ with symmetric legs in each input and output bunch. The differential d is given on the generators by

d  $= \sum_{i=1}^{m-1}(-1)^i$  $+ \sum_{j=1}^{n-1}(-1)^j$  .

3.5.1 Proposition. *$H(\mathcal{Q})$ is a free prop generated by an $\mathbb{S}$-bimodule $\{sgn_m \otimes sgn_n[m+n-2]\}_{m,n\geq 1}$.*

Proof. Use Theorem 3.2.4 and the Künneth theorem. □

The complex of representations, $(\mathrm{Rep}(\mathcal{Q}), d)$, is isomorphic as a graded vector space to $C^{\bullet,\bullet}(\bar{\mathcal{O}}_V) := \bigoplus_{m,n\geq 1}\mathrm{Hom}(\bar{\mathcal{O}}_V^{\otimes m}, \bar{\mathcal{O}}_V^{\otimes n})[m+n-2]$. The latter has a well-known *Gerstenhaber–Schack differential* [GS1],

$$d_{GS} : \mathrm{Hom}(\bar{\mathcal{O}}_V^{\otimes n}, \bar{\mathcal{O}}_V^{\otimes m}) \xrightarrow{d^1_{GS}\oplus d^2_{GS}} \mathrm{Hom}(\bar{\mathcal{O}}_V^{\otimes n+1}, \bar{\mathcal{O}}_V^{\otimes m}) \oplus \mathrm{Hom}(\bar{\mathcal{O}}_V^{\otimes n}, \bar{\mathcal{O}}_V^{\otimes m+1}),$$

with d^1_{GS} given on an arbitrary $\Phi \in \mathrm{Hom}(V^{\otimes n}, V^{\otimes m})$ by

$$(d^1_{GS}\Phi)(f_0, ..., f_n) := -\Delta^{m-1}(f_0)\cdot\Phi(f_1, ..., f_n) + \sum_{i=0}^{n-1}(-1)^i f(f_0, ..., f_i f_{i+1}, ..., f_n)$$
$$+(-1)^n\Phi(f_1, f_2, ..., f_{n-1})\cdot\Delta^{m-1}(f_n), \quad \forall\, f_0, f_1, \ldots, f_n \in \bar{\mathcal{O}}_V,$$

where the multiplication in $\bar{\mathcal{O}}_V$ is denoted by juxtaposition, the induced multiplication in the algebra $\bar{\mathcal{O}}_V^{\otimes m}$ by $\cdot$, the comultiplication in $\bar{\mathcal{O}}_V$ by Δ, and

$$\Delta^{m-1} : (\Delta \otimes \mathrm{Id}^{\otimes m-2}) \circ (\Delta \otimes \mathrm{Id}^{\otimes m-3}) \circ \cdots \circ \Delta : \bar{\mathcal{O}}_V \to \bar{\mathcal{O}}_V^{\otimes m},$$

for $m \geq 2$ while $\Delta^0 := \mathrm{Id}$. The expression for d^2_{GS} is an obvious dual analogue of the one for d^1_{GS}.

It is evident, however, that $(\mathrm{Rep}(\mathcal{Q}), d) \neq (C^{\bullet,\bullet}(\bar{\mathcal{O}}_V), d_{GS})$. What is then the meaning of the *naturally* constructed complex $(\mathrm{Rep}(\mathcal{Q}), d)$?

3.5.2 Definition-proposition. *Let $C^{\bullet,\bullet}_{diff}(\bar{\mathcal{O}}_V)$ be a subspace of $C^{\bullet,\bullet}(\bar{\mathcal{O}}_V)$ spanned by* polydifferential *operators of the form*

$$\begin{array}{rccc} \Phi : & \bar{\mathcal{O}}_V^{\otimes m} & \longrightarrow & \bar{\mathcal{O}}_V^{\otimes n} \\ & f_1 \otimes \cdots \otimes f_m & \longrightarrow & \Gamma(f_1, \ldots, f_m), \end{array}$$

with $\Phi(f_1, \ldots, f_m) = x^{J_1} \otimes \cdots \otimes x^{J_n} \cdot \Delta^{n-1}\left(\frac{\partial^{|I_1|} f_1}{\partial x^{I_1}}\right) \cdot \cdots \cdot \Delta^{n-1}\left(\frac{\partial^{|I_1|} f_m}{\partial x^{I_m}}\right)$ *for some families of nonempty multi-indexes $I_\bullet$ and $J_\bullet$. Then $C^{\bullet,\bullet}_{diff}(\bar{\mathcal{O}}_V)$ is a subcomplex of the Gerstenhaber–Schack complex $(C^{\bullet,\bullet}(\bar{\mathcal{O}}_V), d_{GS})$.*

Proof. Proving that $C^{\bullet,\bullet}_{diff}(\bar{\mathcal{O}}_V)$ is a subcomplex of $(C^{\bullet,\bullet}(\bar{\mathcal{O}}_V), d^1_{GS})$ is very similar to the Hochschild complex case. So we omit these details and concentrate instead on showing that $C^{\bullet,\bullet}_{diff}(\bar{\mathcal{O}}_V)$ is a subcomplex of $(C^{\bullet,\bullet}(\bar{\mathcal{O}}_V), d^2_{GS})$. If, for arbitrary $f \in \bar{\mathcal{O}}_V$, we use Sweedler's notation, $\Delta f = \sum f' \otimes f''$, for the coproduct in $\bar{\mathcal{O}}_V$, then, for an operator Φ as above, one has

$$\begin{aligned} d^2_{GS}\Phi(f_1, ..., f_m) = &-\sum f_1' \cdots f_m' \otimes \Phi(f_1'', ..., f_m'') - \sum_{i=1}^{n} (-1)^i \Delta_i \Phi(f_1, ..., f_m) \\ &+(-1)^n \sum \Phi(f_1', ..., f_m') \otimes f_1'' \cdots f_m'' \end{aligned}$$

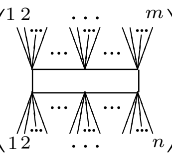
where Δ_i means Δ applied to the i-tensor factor in the space of values, $\bar{\mathcal{O}}_V^{\otimes n}$, of Φ. Taking into account the particular structure of Φ, one can see that $d^2_{GS}\Phi$ is a linear combination of polydifferential operators if and only if an equality holds,

$$\Delta^2 \frac{\partial^{|I|} f}{\partial x^I} = \sum f' \otimes \Delta \frac{\partial^{|I|} f''}{\partial x^I},$$

for arbitrary $f \in \bar{\mathcal{O}}_V$ and arbitrary nonempty multi-index I. As product and coproduct in $\bar{\mathcal{O}}_V$ are consistent, it is enough to check this equality under the assumption that $\dim V = 1$ in which case it is straightforward. □

3.5.3 Proposition. *(i) The complexes $(Rep(\mathcal{Q}), d)$ and $(C^{\bullet,\bullet}_{diff}(\bar{\mathcal{O}}_V), d_{GS})$ are canonically isomorphic. (ii) $HC^{\bullet,\bullet}_{diff}(\bar{\mathcal{O}}_V) = \wedge^{\bullet \geq 1} V \otimes \odot^{\bullet \geq 1} V^*$.*

Proof. (i) Any representation $\rho \in \mathrm{Rep}(\mathcal{Q})$ is uniquely determined by its values on the generators,

$\rho \Bigg($ 

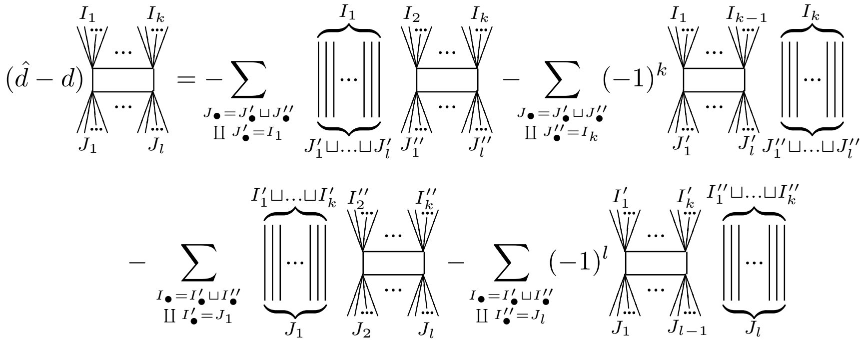
$\Bigg) = \sum \Gamma^{I_1,\ldots,I_k}_{J_1,\ldots,J_l} x^{J_1} \otimes \cdots \otimes x^{J_l} \otimes e_{I_1} \otimes \cdots \otimes e_{I_k},$

for some $\Gamma^{I_1,\ldots,I_k}_{J_1,\ldots,J_l} \in \mathbb{K}$. It is a straightforward calculation to check that the map $i : \mathrm{Rep}(\mathcal{Q}) \to C^{\bullet,\bullet}_{diff}(\bar{\mathcal{O}}_V)$ given by

$$i(\rho) := \sum \frac{\Gamma^{I_1,\ldots,I_k}_{J_1,\ldots,J_l}}{|I_1|!\cdots|J_l|!} x^{J_1}\otimes\cdots\otimes x^{J_l}\cdot \Delta^{l-1}\left(\frac{\partial^{|I_1|}}{\partial x^{I_1}}\right)\cdot\cdots\cdot\Delta^{l-1}\left(\frac{\partial^{|I_l|}}{\partial x^{I_l}}\right)$$

satisfies the condition $\rho\circ d = d_{GS}\circ\rho$. Now 3.5.3(ii) follows immediately from isomorphism (9) and Proposition 3.5.1. □

3.6 Gerstenhaber–Schack complex. It is not hard to guess which dg prop, $(\hat{\mathcal{Q}},\hat{d})$, has the property that its associated dg space of representations, $(\mathrm{Rep}(\hat{\mathcal{Q}}),\hat{d})$, is exactly the Gerstenhaber–Schack complex $(C^{\bullet,\bullet}(\bar{\mathcal{O}}_V), d_{GS})$. As a prop, $\hat{\mathcal{Q}}$ is, by definition, the same as $\mathcal{Q}$ above, but the differential differs from d by the following four groups of terms:



Using a spectral sequence argument very similar to the one used in the proof of Proposition 3.4.1, one easily obtains the following:

3.6.1 Proposition. $H(\hat{\mathcal{Q}}) = H(\mathcal{Q})$.

3.6.2 Corollary. $H(C^{\bullet,\bullet}(\bar{\mathcal{O}}_V)) = H(C^{\bullet,\bullet}_{diff}(\bar{\mathcal{O}}_V)) = \wedge^{\bullet\geq 1}V\otimes\odot^{\bullet\geq 1}V^*$.

The latter result together with the standard results from the theory of simplicial modules [Lo] imply formula (3).

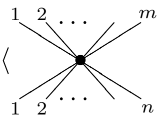
3.7 On the Etingof–Kazhdan quantization. Note that the Gerstenhaber–Schack complex $C^{\bullet,\bullet}(\bar{\mathcal{O}}_V)$ has a structure of prop, the endomorphism prop of $\bar{\mathcal{O}}_V$. Moreover, it is easy to see that $C^{\bullet,\bullet}_{diff}(\bar{\mathcal{O}}_V)$ is also closed under prop compositions so that the natural inclusion, $j : C^{\bullet,\bullet}_{diff}(\bar{\mathcal{O}}_V)\to C^{\bullet,\bullet}(\bar{\mathcal{O}}_V)$, is a morphism of props [Me4]. A choice of a minimal resolution, $\mathcal{A}ss\mathcal{B}_\infty$, of the prop, $\mathcal{A}ss\mathcal{B}$, of associative bialgebras, induces [MV] on $C^{\bullet,\bullet}(\bar{\mathcal{O}}_V)$ (resp. on $C^{\bullet,\bullet}_{diff}(\bar{\mathcal{O}}_V)$) the structure of a filtered L_∞-algebra whose Maurer–Cartan elements describe deformations of the standard bialgebra structure on $\bar{\mathcal{O}}_V$ in the class of (resp. polydifferential) strongly homotopy bialgebra structures. Moreover [MV], the initial term of this induced L_∞-structure is precisely the Gerstenhaber–Schack differential. The inclusion map $j : C^{\bullet,\bullet}_{diff}(\bar{\mathcal{O}}_V)\to C^{\bullet,\bullet}(\bar{\mathcal{O}}_V)$ extends to a morphism of L_∞-algebras which, by isomorphisms (4) and (3), is a quasi-isomorphism. The Etingof–Kazhdan universal quantization [EK] of (possibly, infinite-dimensional) Lie bialgebra structures on V

associates to such a structure, say ν, a Maurer–Cartan element, γ_ν, in the L_∞-algebra $C^{\bullet,\bullet}(\bar{\mathcal{O}}_V)$. As L_∞ quasi-isomorphisms are invertible [Ko], there is always an associated Maurer–Cartan element $j^{-1}(\gamma_\nu^{EK})$ which, for degree reasons, describes an associated to ν *polydifferential* bialgebra structure on $\bar{\mathcal{O}}_V$. Thus we proved that *for any Lie bialgebra structure, ν, on a vector space V there exists its bialgebra quantization, $j^{-1}(\gamma_\nu^{EK})$, within the class of polydifferential operators from $\mathcal{C}^{\bullet,\bullet}_{poly}(\mathcal{O}_V)$.*

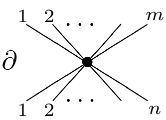
4 Dg prop of unital A_∞-structures

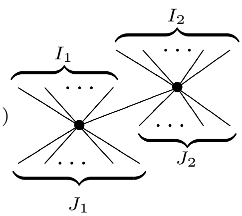
4.1 Differential in a free prop. A differential in a free prop $\mathcal{F}^\uparrow\langle E\rangle$ can be decomposed into a sum, $\delta = \sum_{p\geq 1}\delta_{(p)}$, where $\delta^{(p)}: E \xrightarrow{\delta} \mathcal{F}^\uparrow\langle E\rangle \xrightarrow{pr_p} \mathcal{F}^\uparrow_{(p)}\langle E\rangle$ is the composition of δ with the projection to the subspace, $\mathcal{F}^\uparrow_{(p)}\langle E\rangle \subset \mathcal{F}^\uparrow\langle E\rangle$, spanned by decorated graphs with precisely p vertices. We studied in §3 free props equipped with differentials of the form $\delta = \delta_{(1)}$ which preserve the number of vertices of decorated graphs, and heavily used the fact that δ makes the associated space of representations, $\mathrm{Rep}_V(\mathcal{F}^\uparrow\langle E\rangle) \simeq \mathrm{Hom}(E, \mathcal{E}nd_V)$, into a complex whose cohomology one can easily read from the cohomology of $(\mathcal{F}^\uparrow\langle E\rangle, \delta)$. Remarkably [MV], a generic differential δ in $\mathcal{F}^\uparrow\langle E\rangle$ makes the vector space $\mathrm{Rep}_V(\mathcal{F}^\uparrow\langle E\rangle)[1]$ into an L_∞-*algebra* whose pth homotopy Lie bracket is completely determined by p-th summand, $\delta_{(p)}$, of the differential δ. In particular, if δ has $\delta_{(p)} = 0$ for all $p \geq 3$, then $\mathrm{Rep}_V(\mathcal{F}^\uparrow\langle E\rangle)[1]$ is canonically a dg Lie algebra with the differential determined by $\delta_{(1)}$ and Lie brackets determined by $\delta_{(2)}$. Thus, if we want to extend isomorphisms (1) and (2) into isomorphisms of Lie algebras, we have to look for more complicated (than the ones studied in §3) dg props canonically associated with the (polydifferential) Hochschild complex for $\mathcal{O}_V$.

4.2 Dg prop of polyvector fields. Let $\mathcal{P}oly\mathcal{V}$ be a dg free prop generated by the $\mathbb{S}$-module, $X[-1] = \{X(m,n)[-1]\}_{m\geq 1, n\geq 0}$,

$X(m,n)[-1] = sgn_m \otimes 1\!\!1_n[m-2] = \mathrm{span}\langle$  $\rangle$

which is obtained from the $\mathbb{S}$-module X of Proposition 3.3.1 by a degree shift. The differential in $\mathcal{P}oly\mathcal{V}$ is defined as follows (cf. [Me1]):

∂ 

$= \sum_{\substack{[m]=I_1\sqcup I_2\\ [n]=J_1\sqcup J_2\\ |I_1|\geq 0, |I_2|\geq 1\\ |J_1|\geq 1, |J_2|\geq 0}} (-1)^{\sigma(I_1\sqcup I_2)+|I_1|(|I_2|+1)}$ 

where $\sigma(I_1 \sqcup I_2)$ is the sign of the permutation $[n] \to I_1 \sqcup I_2$. This differential is quadratic, $\partial = \partial_{(2)}$, so that, according to the general theory (see Theorem 60 in [MV]), the space $\mathrm{Rep}(\mathcal{P}oly\mathcal{V})_V[1] = \wedge^{\bullet \geq 1} V \otimes \odot^\bullet V \simeq \wedge^{\bullet\geq 1}\mathcal{T}_V$ comes equipped with a Lie algebra structure which, as it is not hard to check (cf. [Me1]), is precisely the Schouten bracket.

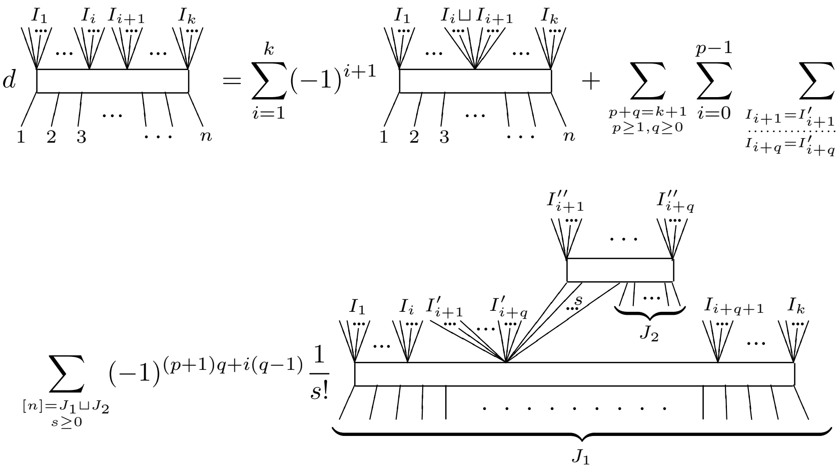
4.3 Unital A_∞-structures on $\mathcal{O}_V$. It is well-known that the vector space $\bar{C}^\bullet(\mathcal{O}_V) := \oplus_{k\geq 1}\mathrm{Hom}(\mathcal{O}_V^{\otimes k}, \mathcal{O}_V)[1-k]$ has a natural graded Lie algebra structure with respect to the Gerstenhaber brackets, $[\ ,\ ]_G$. By definition, an A_∞-*algebra structure* on the space $\mathcal{O}_V$ is a Maurer–Cartan element in this Lie algebra, that is, a total degree 1 element $\Gamma \in \bar{C}^\bullet(\mathcal{O}_V)$ such that $[\Gamma, \Gamma]_G = 0$. Such an element, Γ, is equivalent to a sequence of homogeneous linear maps, $\{\Gamma_k : \mathcal{O}_V^{\otimes k} \to \mathcal{O}_V[2-k]\}$, satisfying a sequence of quadratic equations (cf. [St]). An A_∞-algebra structure is called *unital* if, for every $k \geq 3$, the map Γ_k factors through the composition $\mathcal{O}_V^{\otimes k} \to \bar{\mathcal{O}}_V^{\otimes k} \to \mathcal{O}_V[k-2]$ and $\Gamma_2(1,f) = \Gamma_2(f,1) = f$. The following lemma is obvious.

4.3.1 Lemma. *There is a one-to-one correspondence between unital A_∞-structures on $\mathcal{O}_V$ and Maurer–Cartan elements,*

$$\{\Gamma \in \bar{C}^\bullet(\bar{\mathcal{O}}_V) : \ |\Gamma| = 1 \ \textit{and}\ d_H\Gamma + \frac{1}{2}[\Gamma,\Gamma]_G = 0\},$$

in the Hochschild dg Lie algebra, $\bar{C}^\bullet(\bar{\mathcal{O}}_V)$, for the ring $\bar{\mathcal{O}}_V \subset \mathcal{O}_V$.

4.4 Dg prop of unital A_∞-structures. Consider a dg free prop, $(\mathcal{D}ef\,\mathcal{Q}, d)$, generated by corollas (16) (to which we assign now degree $2-k$) and equipped with the differential given by[3] (cf. [Me2])



(18)

It was shown in [Me2] that there is a one-to-one correspondence between degree 0 representations of the dg prop $(\mathcal{D}ef\,\mathcal{Q}, d)$ in a dg vector space V and Maurer–Cartan elements in the Hochschild dg Lie algebra $(\bar{C}^\bullet(\bar{\mathcal{O}}_V), [\ ,\ ]_G, d_H)$,

[3] We have to assume that $\mathcal{D}ef\,\mathcal{Q}$ is completed with respect to the genus filtration.

i.e., with unital A_∞-structures on $\mathcal{O}_V$. Put another way, the dg Lie algebra induced on $\mathrm{Rep}_V(\mathcal{D}ef\,\mathcal{Q})[1]$ from the above differential d is precisely the Hochschild dg Lie algebra.

Consider now a filtration, $F_{-p} := \{\mathrm{span}\langle G\rangle : \text{number of vertices in } G \geq p\}$, of the complex $(\mathcal{D}ef\,\mathcal{Q}, d)$. It is clear that the 0th term, (E_0, δ), of the associated spectral sequence, $\{E_r, d_r\}_{r\geq 1}$, isomorphic (modulo an inessential shift of degree) to the prop $(\mathcal{D}, \delta)$ introduced in §3.3 so that, by Proposition 3.3.1, we conclude that $E_1 = H(E_0)$ is isomorphic as a free prop to $\mathcal{P}oly\mathcal{V}$ whose shifted representation space, $\mathrm{Rep}_V(\mathcal{P}oly\mathcal{V})[1]$, is $H(\bar{C}^\bullet(\bar{\mathcal{O}}_V)) = \wedge^{\bullet\geq 1}\mathcal{T}_V$. The Lie algebra structure on $H(\bar{C}^\bullet(\bar{\mathcal{O}}_V))$ induced from the Gerstenhaber brackets on $\bar{C}^\bullet(\bar{\mathcal{O}}_V)$ is then given by the differential, d_1, induced on the next term of the spectral sequence, $E_1 = \mathcal{P}oly\mathcal{V}$, from the differential d in $\mathcal{D}ef\,\mathcal{Q}$. A direct inspection of formula (18) implies that d_1 is precisely ∂ which in turn implies by §4.2 that the induced Lie algebra structure on $H(\bar{C}^\bullet(\bar{\mathcal{O}}_V)$ is indeed given by Schouten brackets. It is worth noting in conclusion that L_∞-morphisms (in the sense of Kontsevich [Ko] between dg Lie algebras $\bar{C}^\bullet(\bar{\mathcal{O}}_V)$ and $\wedge^{\bullet\geq 1}\mathcal{T}_V$ can be equivalently understood as morphisms of dg props, $\mathcal{D}ef\,\mathcal{Q} \to \mathcal{P}oly\mathcal{V}^{\circlearrowright}$, where $\mathcal{P}oly\mathcal{V}^{\circlearrowright}$ is the wheeled completion of the prop of polyvector fields (by definition, $\mathcal{P}oly\mathcal{V}^{\circlearrowright}$ is the smallest wheeled prop containing $\mathcal{P}oly\mathcal{V}$ as a subspace). This point of view on quantizations was discussed in more detail in [Me2, Me3].

Acknowledgment. The author is grateful to an anonymous referee for useful comments.

References

[Ba] Baranovsky, V.: A universal enveloping for L_∞-algebras. Math. Res. Lett. **15**(6), 1073–1089 (2008). arXiv:0706.1396

[BM] Borisov, D., Manin, Yu. I.: Generalized operads and their inner cohomomorphisms. In: Geometry and dynamics of groups and spaces, Progr. Math. **265**, 247–308 Birkhäuser, Basel (2008)

[CFL] Cattaneo, A.S., Fiorenza, D., Longoni, R.L.: On the Hochschild-Kostant-Rosenberg map for graded manifolds. Int. Math. Res. Not. **62**, 3899–3918 (2005)

[Ch] Chapoton, F.: Opérades différentielles graduées sur les simplexes et les permutoédres. Bull. Soc. Math. **130**(2), 233–251. France (2002)

[GS1] Gerstenhaber, M., Schack, S.D.: Bialgebra cohomology, deformations, and quantum groups. Proc. Natl. Acad. Sci. USA **87**, 478–481 (1990)

[GS2] Gerstenhaber, M., Schack, S.D.: Algebras, bialgebras, quantum groups, and algebraic deformations. Contemp. Math. **134**, 51–92 (1992)

[GK] Getzler, E., Kapranov, M.M.: Modular operads. Compos. Math. **110**, 65–126 (1998)

[EK] Etingof, P., Kazhdan, D.: Quantization of Lie bialgebras, I. Selecta Math. (N.S.) **2**, 1–41 (1996)

[Ko] Kontsevich, M.: Deformation quantization of Poisson manifolds. Lett. Math. Phys. **66**(3), 157–216 (2003)

[LM] Lazarev, A.Yu., Movshev, M.V.: Deformations of Hopf algebras. Uspekhi Mat. Nauk. **46**, 211–212 (1991)

[Lo] Loday, J.L.: Cyclic homology. Grundlehren der Mathematischen Wissenschaften, 301. Springer-Verlag, Berlin (1998)

[MSS] Markl, M., Shnider, S., Stasheff, J.D.: Operads in algebra, topology and physics. Mathematical surveys and monographs, vol. 96. American Mathematical Society, Providence, RI (2002)

[Me1] Merkulov, S.A.: Prop profile of Poisson geometry. Commun. Math. Phys. **262**(1), 117–135. math.DG/0401034 (2006)

[Me4] Merkulov, S.A.: Quantization of strongly homotopy Lie bialgebras. arXiv:math/0612431 (2006)

[Me2] Merkulov, S.A.: Graph complexes with loops and wheels. In: Tschinkel, Yu. and Zarhin, Yu. (eds.) Algebra, Arithmetic and Geometry – Manin Festschrift, vol. II. Progress in Mathematics. Birkhaüser, MA (2008)

[Me3] Merkulov, S.A.: Lectures on props, Poisson geometry and deformation quantization. In: Dito, G., Lu, J.H., Maeda, Y., Weinstein, A. (eds.) Poisson Geometry in Mathematics and Physics. Contemp. Math., **450**, Amer. Math. Soc., Providence, RI (2008)

[MV] Merkulov, S.A., Vallette, B.: Deformation theory of representations of prop(erad)s. J. Reine Angew. Math. (2007). Preprint, arXiv:0707.0889

[SU] Saneblidze, S., Umble, R.: Diagonals on the permutahedra, multiplihedra and associahedra. Homology Homotopy Appl. **6**, 363–411 (2004)

[St] Stasheff, J.D.: On the homotopy associativity of H-spaces, I II, Trans. Am. Math. Soc. **108**, 272–292; 293–312 (1963)

[Va] Vallette, B.: A Koszul duality for props. Trans. Am. Math. Soc. **359**, 4865–4943 (2007)

Applications de la bi-quantification à la théorie de Lie

Charles Torossian

CNRS-Institut Mathématiques de Jussieu, Université Paris 7, Case 7012,
2 place Jussieu, 75251 Paris Cedex 05, France,
torossian@math.jussieu.fr

Abstract. This article in French, with a large English introduction, is a survey about applications of bi-quantization theory in Lie theory. We focus on a conjecture of M. Duflo. Most of the applications are coming from our article with Alberto Cattaneo [Cattaneo, A.S., Torossian, C.: Quantification pour les paires symétriques et diagrammes de Kontsevich. Ann. Ecole Norm. Sup. 41, 787–852 (2008)] and some extensions are relating discussions with my student [Batakidis, P.: Phd-Thesis. Univ. Paris 7 (2009)]. The end of the article is completely new. We prove that the conjecture $E = 1$ implies the Kashiwara–Vergne conjecture. Our deformation is nongeometric but uses a polynomial deformation of the coefficients.

Key words: Deformation, Invariant differential operators, Poisson structure, Quantization, Duflo's conjecture, Harish–Chandra homomorphism

AMS Classification: 17B, 17B25, 22E, 53C35

Thanks : I thank the organizers of the conference "Higher Structures in Geometry and Physics" which was held at IHP, January 2007 and where our results have been announced. This article is dedicated to Murray Gerstenhaber and Jim Stasheff.

1 English Introduction

1.1 Invariant differential operators on line bundle

Let G be a real Lie group, connected and simply connected. Let $\mathfrak{g}$ be the associated Lie algebra, $U(\mathfrak{g})$ the universal enveloping algebra and $S(\mathfrak{g})$ the symmetric algebra. In this introduction G/H is a homogeneous space with H a connected Lie subgroup. As usual $\mathfrak{h}$ denotes the Lie algebra of H. Fix a character λ of H, it is a group homomorphism from H into $\mathbb{C}^{\times}$. If there is no

A.S. Cattaneo et al. (eds.), *Higher Structures in Geometry and Physics*,
Progress in Mathematics 287, DOI 10.1007/978-0-8176-4735-3_15,

danger of confusion, we will denote by the same letter the differential of the character. So λ is a character of $\mathfrak{h}$, i.e., we have $\lambda[\mathfrak{h},\mathfrak{h}]=0$.

Let $\mathcal{L}_\lambda$ be the line bundle defined by λ. Sections of this bundle, denoted by $\Gamma(\mathcal{L}_\lambda)$, are smooth functions on G such that $\varphi(gh)=\varphi(g)\lambda(h)$. Obviously G acts on the left on $\Gamma(\mathcal{L}_\lambda)$.

Let $\mathcal{D}_\lambda$ be the algebra of invariant differential operators on $\Gamma(\mathcal{L}_\lambda)$.

After Koornwinder [21] we know $\mathcal{D}_\lambda$ is isomorphic to

$$\mathcal{D}_\lambda := \Big(U(\mathfrak{g})_{\mathbb{C}}/U(\mathfrak{g})_{\mathbb{C}}\cdot\mathfrak{h}_{-\lambda}\Big)^{\mathfrak{h}} \tag{1}$$

where $\mathfrak{h}_{-\lambda}=\{H-\lambda(H)\}$. Here are some explanations. For $X\in\mathfrak{g}$, R_X is the left invariant vector field on G associated to X. For $u\in\mathcal{D}_\lambda$, let D_u be the associated differential operator on $\Gamma(\mathcal{L}_\lambda)$ defined by

$$(D_u\varphi)(g)=(R_u\varphi)(g),$$

$\varphi\in\Gamma(\mathcal{L}_\lambda)$. Then D_u is a left invariant differential operator. It is not difficult to verify that we have described all of them[1].

Suppose λ is real. The algebra $\Big(U(\mathfrak{g})/U(\mathfrak{g})\cdot\mathfrak{h}_{-\lambda}\Big)^{\mathfrak{h}}$ is not commutative in general. A conjecture of M. Duflo [14] describes the center of this algebra. I write $\mathcal{S}_\lambda$ for the algebra of H-invariant polynomial functions on $\mathfrak{h}^{\perp}_{-\lambda}:=\{f\in\mathfrak{g}^*, f|_{\mathfrak{h}}=\lambda\}$. We get

$$\mathcal{S}_\lambda := \Big(S(\mathfrak{g})/S(\mathfrak{g})\cdot\mathfrak{h}_{-\lambda}\Big)^{\mathfrak{h}}. \tag{2}$$

This space admits a natural Poisson structure coming from the classical Poisson structure on $\mathfrak{g}^*$. Let $\delta(H)=\frac{1}{2}\mathrm{tr}_{\mathfrak{g}/\mathfrak{h}}\,\mathrm{ad}(H)$ be the character for the half densities.

Duflo's conjecture [14] : The center of $\Big(U(\mathfrak{g})/U(\mathfrak{g})\cdot\mathfrak{h}_{-\lambda-\delta}\Big)^{\mathfrak{h}}$ is isomorphic to the Poisson center of $\mathcal{S}_\lambda$.

This conjecture is far from being solved[2]. Moreover one should probably consider the case of a generic character.

In case G is a nilpotent group, appreciable advances have been achieved in the last few years by Corwin–Greenleaf [12], Fujiwara–Lion–Magneron–Mehdi [15], Baklouti–Fujiwara [5] and Baklouti–Ludwig [6], and Lipsman [23,24,25]. More precisely in the nilpotent case one can prove the following.

Théorème 1 ([15]) *Let G be nilpotent (connected, simply connected) and χ the unitary character of H defined by $\chi(\exp_G(H))=\exp(i\lambda(H))$.*

[1] Consider the local expression around the origin.

[2] Consider the following example; G is reductive $H=U$ the unipotent radical of a Borel. Let $\mathfrak{t}$ be a Cartan subalgebra. Then you get $(U(\mathfrak{g})/U(\mathfrak{g})\cdot\mathfrak{h})^{\mathfrak{h}}=S(\mathfrak{t})=(S(\mathfrak{g})/S(\mathfrak{g})\cdot\mathfrak{h})^{\mathfrak{h}}$. These algebras are commutative. The space G/U is quasi-affine.

$$\begin{aligned}\mathcal{D}_\chi \text{ commutative} &\iff \text{the left representation } L^2(G/H,\chi) \text{ has finite multiplicities}\\ &\Leftrightarrow \text{for } f\in\mathfrak{h}^\perp_{-\lambda} \text{ generic} \quad H\cdot f \text{ is lagrangian in } G\cdot f\\ &\Leftrightarrow \Big(\mathrm{Frac}(S(\mathfrak{g})/S(\mathfrak{g})\cdot\mathfrak{h}_{-\lambda})\Big)^{\mathfrak{h}} \quad \text{Poisson commutative}\\ &\Leftrightarrow \Big(S(\mathfrak{g})/S(\mathfrak{g})\cdot\mathfrak{h}_{-\lambda}\Big)^{\mathfrak{h}} \quad \text{Poisson commutative}\end{aligned}$$

Under these conditions $\mathcal{D}_\chi$ is a subalgebra of $\Big(\mathrm{Frac}(S(\mathfrak{g})/S(\mathfrak{g})\cdot\mathfrak{h}_{-\lambda})\Big)^{\mathfrak{h}}$, the algebra of H-invariant fractions on $\mathfrak{h}^\perp_{-\lambda}$.

In the case where G and H are reductive groups F. Knop [19] gives a satisfying and remarkable answer to the conjecture. In the case H is compact and $G = H \ltimes N$ is a semiproduct of H with a Heisenberg group N, L. Rybnikov [31] makes use of Knop's result to prove Duflo's conjecture. The case of symmetric spaces has been previously studied by myself in [34, 36] and the group case is solved by Duflo [13].

This survey analyzes the Duflo conjecture and other standard problems in Lie theory with the help of Kontsevich's quantization. The main results are generalizations of [10]. Some others come from discussions with my PhD student P. Batakidis [7], and the end of the article is new.

We hope to convince the reader of the value of our methods. In some sense they are a replacement for the orbit method.

1.2 Duflo's conjecture : a review of difficulties

Let us try to list some technical difficulties in Duflo's conjecture. We will see that most of them disappear with Kontsevich's quantization techniques developed in [10].

1 – The algebra $\Big(U(\mathfrak{g})/U(\mathfrak{g})\cdot\mathfrak{h}_{-\lambda}\Big)^{\mathfrak{h}}$ is filtered by the order of differential operators. But it is not obvious to describe the associated graded space. In general we have an injection from $gr\Big(U(\mathfrak{g})/U(\mathfrak{g})\cdot\mathfrak{h}_{-\lambda}\Big)^{\mathfrak{h}}$ into $\Big(S(\mathfrak{g})/S(\mathfrak{g})\cdot\mathfrak{h}\Big)^{\mathfrak{h}}$. Let us remark that the character has disappeared. Regarding the first difficulty, the symbol of a differential operator on a line bundle is just a function on the cotangent space of the underlying space. Except tentative [22] there is no way to keep the character. The next example illustrates the phenomena : consider $\mathfrak{g} =< X, Y, Z >$ with $Z = [X, Y]$ and $\mathfrak{h} =< Z >$.

If $\lambda(Z) = 0$ the algebra $(U(\mathfrak{g})/U(\mathfrak{g})\cdot Z)^Z = S(\mathfrak{g})/S(\mathfrak{g})\cdot Z$ is Poisson commutative.

If $\lambda(Z) \neq 0$, then $\Big(U(\mathfrak{g})/U(\mathfrak{g})\cdot(Z-\lambda(Z))\Big)^Z$ is a Weyl algebra and thus non commutative while the associated graded algebra is Poisson commutative.

There is no link between $\mathcal{D}_\lambda$ and the Poisson algebra $gr(\mathcal{D}_\lambda)$. The quantization procedure developed by Cattaneo–Felder [8, 9] for co-isotropic spaces takes care of the character.

2 – In general the homogeneous space G/H does not admit a G-invariant measure. One has to consider half-densities, essentially to deal with Hilbert spaces. So you have to define the following character of H, $\Delta_{G,H}(h) = (\det_{\mathfrak{g}/\mathfrak{h}} \mathrm{Ad} h)^{1/2}$ and λ is replaced by the shifted character $\lambda + \frac{1}{2}\mathrm{tr}_{\mathfrak{g}/\mathfrak{h}}\mathrm{ad}$. This shift is problematic if you want to construct irreducible representations by induction from a polarization $\mathfrak{b}$. Usually you have to ask for compatibility conditions among $\Delta_{G,H}, \Delta_{G,B}$ [23, 34][3].

3 – In case $\mathfrak{h}$ admits an $\mathfrak{h}$-invariant complement (for the adjoint action), the pair $(\mathfrak{g}, \mathfrak{h})$ is called a *reductive pair* (but $\mathfrak{g}$ is not supposed to be reductive !). If $(\mathfrak{g}, \mathfrak{h})$ is a reductive pair, then an easy consequence of Poincaré–Birkhoff–Witt's theorem indicates $\big(U(\mathfrak{g})/U(\mathfrak{g})\cdot\mathfrak{h}_{-\lambda}\big)^{\mathfrak{h}}$ is isomorphic as a vector space to $\big(S(\mathfrak{g})/S(\mathfrak{g})\cdot\mathfrak{h}_{-\lambda}\big)^{\mathfrak{h}}$. It is not known whether this holds in general, that is, if $\mathcal{D}_\lambda$ is a deformation of $\mathcal{S}_\lambda$. Actually there are no obvious maps from $\big(S(\mathfrak{g})/S(\mathfrak{g})\cdot\mathfrak{h}_{-\lambda}\big)^{\mathfrak{h}}$ into $U(\mathfrak{g})$ or $U(\mathfrak{g})/U(\mathfrak{g})\cdot\mathfrak{h}_{-\lambda}$.

If $(\mathfrak{g}, \mathfrak{h})$ is a reductive pair, then you get the additional equality

$$U(\mathfrak{g})^{\mathfrak{h}}/U(\mathfrak{g})^{\mathfrak{h}} \cap U(\mathfrak{g})\cdot\mathfrak{h}_{-\lambda} = (U(\mathfrak{g})/U(\mathfrak{g})\cdot\mathfrak{h}_{-\lambda})^{\mathfrak{h}}. \tag{3}$$

In general you get just an injection from LHS to RHS, as illustrated by the following example. Consider $\mathfrak{g} = \mathbf{sl}(2)$ with standard basis H, X, Y and take $\mathfrak{h} = <X>$. You get $\big(U(\mathfrak{g})/U(\mathfrak{g})\cdot X\big)^X = \mathbb{R}[X]$ but $U(\mathfrak{g})^X/U(\mathfrak{g})^X \cap U(\mathfrak{g})\cdot X$ is isomorphic to $\mathbb{R}[X^2]$. Our constructions depend on the choice for a complement to $\mathfrak{h}$. In [10] we gave several examples where there are different choices for a complement, the most important being the Iwasawa decomposition and the Cartan decomposition for symmetric pairs. Dependence of our constructions with the complement leads to interesting applications : Harish–Chandra homomorphism for example. At the end of this article, we will explain the group case[4], which leads to the Kashiwara–Vergne conjecture. Invariant complements simplify the calculations but we are able to describe our model even in the general case.

4 – The algebras $\big(U(\mathfrak{g})/U(\mathfrak{g})\cdot\mathfrak{h}_{-\lambda}\big)^{\mathfrak{h}}$ and $\mathcal{S}_\lambda$ should be simultaneously commutative. Even this fundamental question is not solved in general, except for the nilpotent case or for symmetric pairs. If G and H are algebraic and the generic H-orbits in $\mathfrak{h}^\perp_{-\lambda} = \{f \in \mathfrak{g}^*, f|_{\mathfrak{h}} = \lambda\}$ are lagrangian, then $\Big(\mathrm{Frac}(S(\mathfrak{g})/S(\mathfrak{g})\cdot\mathfrak{h}_{-\lambda})\Big)^{\mathfrak{h}}$ is commutative, consequently $\mathcal{S}_\lambda$ is Poisson commutative. Of course $\mathcal{S}_\lambda$ could be commutative, without the lagrangian hypothesis.

[3] A boundary term, which was overlooked in previous publications, has to be added to several of the expressions in Sections 3.4 and 4.1 to 4.4. The determination of this term will be found in a paper by A. Cattaneo, C. Rossi and C. Torossian, in preparation.

[4] This is the double $(\mathfrak{g} \times \mathfrak{g}, \text{diagonal})$.

For example, consider $\mathfrak{g} =< T, X, Y, Z >$ with $< X, Y, Z >$ a Heisenberg Lie algebra and $[T, X] = X, [T, Y] = Y$ and $[T, Z] = 2Z$. Take $\mathfrak{h} =< T >$. Then $\big(U(\mathfrak{g})/U(\mathfrak{g}) \cdot \mathfrak{h}\big)^{\mathfrak{h}} = \mathbb{R}$, $\big(S(\mathfrak{g})/S(\mathfrak{g}) \cdot \mathfrak{h}\big)^{\mathfrak{h}} = \mathbb{R}$ but H-orbits in $\mathfrak{h}^{\perp}$ are not lagrangian because $\mathrm{Frac}(S(\mathfrak{g})/S(\mathfrak{g}) \cdot \mathfrak{h})^{\mathfrak{h}}$ is not commutative.

5 – Fix $\mathfrak{q}$ a complement of $\mathfrak{h}$ in $\mathfrak{g}$ and consider the Exponential map $\mathrm{Exp} : \mathfrak{q} \longrightarrow G/H$ defined by $X \mapsto \exp_G(X)H$. This is a local diffeomorphism and you can write D_u for $u \in \mathcal{D}_\lambda$ in exponential coordinates. Before our work [10] no formulas were known. If you restrict these operators to invariant distributions, you should get interesting simplifications. This is exactly what happens for symmetric pairs [27, 28, 29], especially for the double $G/H = G_1 \times G_1/\mathrm{Diagonal}$. The study of this restriction gives rise to Kashiwara–Vergne's conjecture [17, 1, 4, 37].

6 – Suppose $\chi = i\lambda$ is the differential of a unitary character of H and $\mathcal{D}_\chi$ commutative. How can we associate to $u \in \mathcal{D}_\chi$ a rational function, or a polynomial function on $\mathfrak{h}^{\perp}_{-\lambda}$? The orbit method gives a kind of answer: construct an irreducible representation $(\pi, \mathcal{H})$ of G which admits H-semi-invariant distribution vectors for χ. Most of them are related with orbits $\Omega = G \cdot f$ with $f \in \mathfrak{h}^{\perp}_{-\lambda}$. If we are lucky, these H-semi-invariant distribution vectors are common eigenvectors for all D_u. The eigenvalue is a character for $\mathcal{D}_\lambda$ and should depend on f as a rational function. Usually an irreducible representation $(\pi, \mathcal{H})$ is constructed by induction from a polarization at f (if such a polarization exists !). As we see, there are several analytic difficulties: definition of the distribution vector, L^2 convergence, real structure. All these problems are in some sense far from our starting algebraic problem. We will explain how the bi-quantization gives us a systematic procedure, under the lagrangian hypothesis, to construct this character in a more algebraic (or geometric) way. Of course, if you deal with the spectral decomposition of $L^2(G/H, \chi)$, all these problems are to be considered.

7 – Consider a fundamental remark now : from the point of view of the theory of representations, one should study the algebra $(U(\mathfrak{g})/I)^{\mathfrak{g}}$ where I is a two-sided ideal included in $U(\mathfrak{g}) \cdot \mathfrak{h}_{-\lambda}$ and maximal. This algebra should be smaller than $\Big(U(\mathfrak{g})/U(\mathfrak{g}) \cdot \mathfrak{h}_{-\lambda}\Big)^{\mathfrak{h}}$ and behaves in a much better way. Indeed Schur's lemma proves that the action of any element $u \in (U(\mathfrak{g})/I)^{\mathfrak{g}}$ is scalar on irreducible representations which admit H-semi-invariant distribution vectors. It is not difficult to extend Duflo's arguments [13] in this context. The rational function you should have built by the orbit method (if it exists!) is then a polynomial function.

2 La quantification de Kontsevich

Pour simplifier la compréhension de cet article on rappelle brièvement les constructions de Kontsevich [20] et les extensions dans le cas co-isotrope des à Cattaneo-Felder [8, 9].

2.1 Théorème de Formalité

En 1997, M. Kontsevich a montré que tout variété de Poisson admet une quantification formelle. C'est une conséquence du théorème de formalité qui affirme qu'il existe un quasi-isomorphisme entre l'algèbre de Lie des polychamps de vecteurs munie du crochet de Schouten et l'algèbre de Lie des opérateurs polydifférentiels munie du crochet de Gerstenhaber [16] et de la différentielle de Hochschild.

Théorème 2 ([20]) *Il existe un L_∞-quasi isomorphisme $\mathcal{U} = (U_n)_{n\geq 1}$ entre les algèbres différentielles graduées $\mathfrak{g}_1 = (T_{poly}(\mathbb{R}^d, [\cdot,\cdot]_S, \mathrm{d} = 0)$ et $\mathfrak{g}_2 = D_{poly}(\mathbb{R}^d, [\cdot,\cdot]_G, \mathrm{d}_{Hoch})$. En particulier $\mathcal{U}$ induit une bijection entre les solutions formelles de Maurer-Cartan modulo les groupes de jauge.*

La preuve du théorème utilise une construction explicite en terme de diagrammes, pour décrire les coefficients de Taylor U_n de $\mathcal{U}$. En particulier si π est un bi-vecteur de Poisson vérifiant $[\pi,\pi]_S = 0$ pour le crochet de Schouten, alors

$$\underset{\epsilon}{\star} = m + \sum_{n\geq 1} \frac{\epsilon^n}{n!} U_n(\underbrace{\pi,\ldots,\pi}_{n\text{ fois}}) \tag{4}$$

est une structure associative formelle sur $\mathcal{C}^\infty(\mathbb{R}^d)[[\epsilon]]$. Pour f, g des fonctions de $\mathcal{C}^\infty(\mathbb{R}^d)$ on obtient un produit formel associatif

$$f \underset{Kont}{\star} g = fg + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \sum_{\Gamma\in G_{n,2}} w_\Gamma B_\Gamma(f,g). \tag{5}$$

On explique rapidement la signification de chaque termes de cette formule.

Graphes

Ici $G_{n,2}$ désigne l'ensemble des graphes étiquetés[5] et orientés (les arêtes sont orientées) ayant n sommets de première espèce numérotés $1, 2, \cdots, n$ et deux sommets de deuxième espèce $\overline{1}, \overline{2}$, tels que :

(i) – Les arêtes partent des sommets de première espèce. De chaque sommet de première espèce partent exactement deux arêtes.

(ii) – Le but d'une arête est différent de sa source (il n'y a pas de boucle).

(iii) – Il n'y a pas d'arête multiple.

Remarque importante : Dans le cas linéaire qui nous intéresse, les graphes qui interviennent de manière non triviale (on dira essentiels), sont tels que les sommets de première espèce ne peuvent recevoir qu'au plus une arête. Il en résulte que tout graphe essentiel est superposition de graphes simples de type Lie (graphe ayant une seule racine comme dans Fig. 1) ou de type roue (cf. Fig. 2 pour un exemple). Cela implique que toutes les formules sont des exponentielles.

---

[5] Par graphe étiqueté on entend un graphe Γ muni d'un ordre total sur l'ensemble E_Γ de ses arêtes, compatible avec l'ordre des sommets.

Variétés de configurations

On note $C_{n,m}$ l'espace des configurations de n points distincts dans le demi-plan de Poincaré (points de première espèce ou points aériens) et m points distincts sur la droite réelle (ce sont les points de seconde espèce ou points terrestres), modulo l'action du groupe $az+b$ (pour $a \in \mathbb{R}^{+*}, b \in \mathbb{R}$). Dans son article [20] Kontsevich construit des compactifications de ces variétés notées $\overline{C}_{n,m}$. Ce sont des variétés à coins de dimension $2n-2+m$. Ces variétés ne sont pas connexes pour $m \geq 2$. On notera par $\overline{C}^+_{n,m}$ la composante qui contient les configurations o les points terrestres sont ordonnés dans l'ordre croissant (ie. on a $\overline{1} < \overline{2} < \cdots < \overline{m}$).

Fonctions d'angle et coefficients

On définit la fonction d'angle hyperbolique dans le demi-plan de Poincaré par

$$\phi(p,q) = \arg(p-q) + \arg(p-\overline{q}). \tag{6}$$

C'est une fonction d'angle de $C_{2,0}$ dans $\mathbb{S}^1$ qui s'étend en une fonction régulière à la compactification $\overline{C}_{2,0}$. Si Γ est un graphe dans $G_{n,2}$, alors toute arête e définit par restriction une fonction d'angle notée ϕ_e sur la variété $\overline{C}^+_{n,2}$. On note E_Γ l'ensemble des arêtes du graphe Γ. Le produit ordonné

$$\Omega_\Gamma = \bigwedge_{e \in E_\Gamma} \mathrm{d}\phi_e \tag{7}$$

est donc une $2n$-forme sur $\overline{C}^+_{n,2}$ variété compacte de dimension $2n$. Le poids associé à un graphe Γ est par définition

$$w_\Gamma = \frac{1}{(2\pi)^{2n}} \int_{\overline{C}^+_{n,2}} \Omega_\Gamma. \tag{8}$$

Opérateurs bi-différentiels

Enfin l'opérateur B_Γ est un opérateur bidifférentiel construit à partir de Γ, dont on ne détaille pas la construction. Disons que chaque arête correspond à une dérivée, chaque sommet de première espèce est attaché au bi-vecteur de Poisson et chaque sommet de deuxième espèce est attaché à des fonctions (cf. [20, 11]).

2.2 Formule de Baker-Campbell-Hausdorff

On applique ce théorème pour $\mathbb{R}^d = \mathfrak{g}^*$ et $\pi = \frac{1}{2}\sum_{i,j}[e_i,e_j]\partial_{e_i^*} \wedge \partial_{e_j^*}$ Prenons maintenant $X, Y \in \mathfrak{g}$ et $f = e^X, g = e^Y$. L'équation ci-dessus donne alors une expression nouvelle pour la formule de Baker-Campbell-Hausdorff $Z(X,Y)$; elle utilise tous les crochets possibles[6] [18]

$$Z(X,Y) = X + Y + \sum_{n\geq 1} \sum_{\substack{\Gamma \text{ simple} \\ \text{geometric} \\ \text{Lie type (n,2)}}} w_\Gamma \Gamma(X,Y). \tag{9}$$

Le terme $\Gamma(X,Y)$ est le mot de type Lie que l'on peut fabriquer avec Γ, c'est essentiellement le symbole de l'opérateur B_Γ.

3 La quantification de Cattaneo–Felder[7]

Soit $\mathfrak{g}$ une algèbre de Lie de dimension finie sur $\mathbb{R}$. L'espace dual $\mathfrak{g}^*$ est alors muni d'une structure de Poisson linéaire. On note π le bi-vecteur de Poisson associé à la moitié du crochet de Lie. Supposons donnés $\mathfrak{h} \subset \mathfrak{g}$ sous-algèbre de $\mathfrak{g}$ et λ un caractère réel de $\mathfrak{h}$, c'est à dire une forme linéaire telle que $\lambda[\mathfrak{h},\mathfrak{h}] = 0$. L'orthogonal $\mathfrak{h}^\perp$ de même que $\mathfrak{h}^\perp_{-\lambda} := \{f \in \mathfrak{g}^*, f|_\mathfrak{h} = \lambda\}$ sont des sous-variétés coisotropes de $\mathfrak{g}^*$.

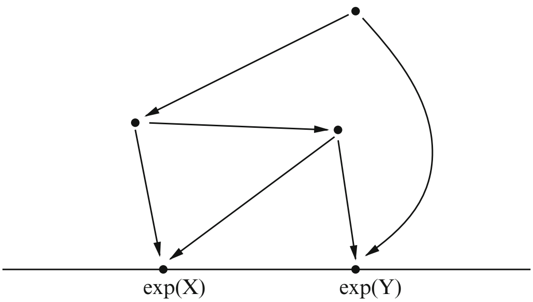


Fig. 1. Graphe simple de type Lie et de symbole $\Gamma(X,Y) = [[X,[X,Y]],Y]$

3.1 Construction par transformée de Fourier impaire [9]

La construction de [8, 9] concerne le cas des variétés co-isotropes en général, mais nous ne nous intéressons ici qu'au cas des sous-algèbres d'une algèbre de Lie. Les constructions sont locales et dépendent donc d'un choix d'un

[6] La formule de Dynkin n'utilise que des crochets itérés.

[7] Un terme de bord, qui aurait dû apparaître dans des publications précédentes, doit être ajouté à plusieurs des expressions ci-dessous et dans les Sections 4.1 à 4.4. La valeur de ce terme sera donnée dans un article de A. Cattaneo, C. Rossi et C. Torossian, en préparation.

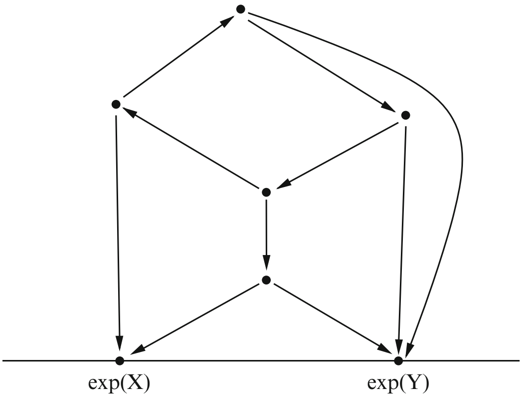


Fig. 2. Graphe de type roue et de symbole $\Gamma(X,Y) = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}X\mathrm{ad}[X,Y]\mathrm{ad}Y\mathrm{ad}Y)$

supplémentaire de $\mathfrak{h}$ dans $\mathfrak{g}$. Notons $\mathfrak{q}$ un tel supplémentaire. On peut alors identifier $\mathfrak{h}^*$ avec $\mathfrak{q}^\perp$. On aura une décomposition (affine)

$$\mathfrak{g}^* = \mathfrak{h}^\perp_{-\lambda} \oplus \mathfrak{q}^\perp = \mathfrak{h}^\perp_{-\lambda} \oplus \mathfrak{h}^*.$$

La variété qui intervient dans cette construction est une super-variété intrinsèque :

$$M := \mathfrak{h}^\perp_{-\lambda} \oplus \Pi\mathfrak{h} \tag{10}$$

o Π désigne le foncteur de changement de parité. L'algèbre des fonctions polynomiales[8] est donc canoniquement

$$\mathcal{A} := \mathrm{Poly}(\mathfrak{h}^\perp_{-\lambda}) \otimes \bigwedge(\mathfrak{g}^*/\mathfrak{h}^\perp) \simeq \Big(S(\mathfrak{g})/S(\mathfrak{g})\cdot\mathfrak{h}_{-\lambda}\Big) \otimes \bigwedge \mathfrak{h}^*.$$

Considérons alors π le bi-vecteur de Poisson linéaire de $\mathfrak{g}^*$ et appliquons la transformée de Fourier impaire [9] dans la direction normale $\mathfrak{h}^* = \mathfrak{q}^\perp$. On obtient un polyvecteur $\widehat{\pi}$ sur M, solution de l'équation de Maurer-Cartan

$$[\widehat{\pi}, \widehat{\pi}]_S = 0.$$

On applique le théorème de formalité pour la super-variété M (voir [9] pour une description complète du théorème de Formalité dans le cas gradué), on obtient alors une solution μ de Maurer-Cartan dans $\mathcal{D}_{poly}(\mathcal{A})$; c'est un opérateur polydifférentiel formel homogène de degré un si l'on tient compte des degrés impairs.

En d'autres termes, comme la graduation tient compte du degré dans les variables impaires, la structure obtenue est en fait une A_∞-structure, définie

[8] Comme on ne considère que des structures de Poisson linéaires, on peut restreindre les constructions aux algèbres de fonctions polynomiales.

par Stasheff [33], sur l'espace $\mathcal{A} = \mathrm{Poly}(\mathfrak{h}^{\perp}_{-\lambda}) \otimes \bigwedge \mathfrak{h}^*$ avec premier terme non nul *a priori* (c'est l'anomalie), c'est à dire une structure

$$\mu = \mu_{-1} + \mu_0 + \mu_1 + \mu_2 + \dots \tag{11}$$

vérifiant[9] $\frac{1}{2}[\mu, \mu]_G = 0$ et μ_i des opérateurs $(i+1)$-polydifférentiels.

Dans le cas linéaire, l'anomalie μ_{-1} est nulle, par conséquent μ_0 est une différentielle et μ_1 un produit associatif modulo des termes contenant μ_0 et μ_2. Le terme de plus bas degré de μ_1 correspond au crochet de Poisson. Ainsi on construit un vrai produit associatif sur l'espace de cohomologie défini par μ_0.

Définition 1 *On notera $H^\bullet(\mu_0, \mathcal{A})$ l'algèbre de cohomologie (graduée) munie de sa loi associative μ_1. On s'intéressera à la sous-algèbre en degré* 0 *que l'on appellera algèbre de réduction et que l'on notera $H^0(\mu_0)$. Le produit μ_1 se restreint en un star-produit noté $\underset{CF}{\star}$.*

3.2 Construction en termes de diagrammes de Feynman [8]

La formule proposée est semblable à celle de Kontsevich [20] dans $\mathbb{R}^n$. Chaque μ_i, opérateur $(i+1)$-polydifférentiel s'exprime sous la forme[10]

$$\mu_i = \sum_{n \geq 0} \frac{\epsilon^n}{n!} \sum_{\Gamma \in G_{n,i+1}} w_\Gamma B_\Gamma \tag{12}$$

o les arêtes des graphes portent deux couleurs[11]. Chaque B_Γ est un opérateur $(i+1)$-polydifférentiel sur $\mathcal{A} = \mathrm{Poly}(\mathfrak{h}^{\perp}_{-\lambda}) \otimes \bigwedge \mathfrak{h}^*$. Il faut donc élargir la notion de graphes admissibles et considérer des graphes avec arêtes colorées par $\mathfrak{h}^*$ issus des points terrestres. Si le bi-vecteur de Poisson π n'est pas linéaire, ces graphes peuvent admettre des arêtes doubles si elles ne portent pas la même couleur. On ne conservera que $2n + i - 1$ arêtes (la dimension de la variété $C_{n,i+1}$) les arêtes restantes seront colorées par $\mathfrak{h}^*$ (on dira que ces arêtes vont à l'infini), elles ne contribuent pas dans le calcul du coefficient w_Γ, mais les arêtes qui partent à l'infini contribuent dans la définition de l'opérateur B_Γ (cf. Fig. 3).

Concernant le coefficient w_Γ, il est obtenu de manière similaire par intégration sur la variété $\overline{C_{n,i+1}}^+$ de la forme Ω_Γ modifiée par la couleur selon les règles suivantes :

– si la couleur est dans $\mathfrak{h}^{\perp}_{-\lambda}$ (variable tangente) la fonction d'angle associée est la même que dans le cas classique

$$\mathrm{d}\phi_+(p,q) := \mathrm{d}\overrightarrow{\phi}(p,q) := \mathrm{d}\arg(p-q) + \mathrm{d}\arg(p-\overline{q}). \tag{13}$$

[9] On note $[\bullet, \bullet]_G$ le crochet de Gerstenhaber.

[10] Il n'y a pas de terme pour $n = 0$ sauf pour μ_1 o on trouve la multiplication m.

[11] Chaque couleur indiquera si la variable de dérivation est dans $\mathfrak{h}^*$ ou $\mathfrak{h}^{\perp}_{-\lambda}$ et précisera la fonction d'angle.

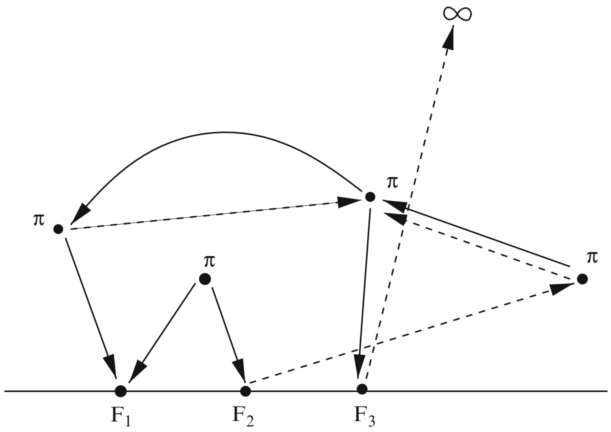


Fig. 3. Graphe type intervenant dans le calcul de $\mathcal{U}_4(\pi,\pi,\pi,\pi)$ pour un bivecteur π non linéaire

– si la couleur est dans $\mathfrak{g}^*/\mathfrak{h}^\perp = \mathfrak{h}^*$ (variable normale[12]) alors la fonction d'angle sera notée $\dashrightarrow$ (en pointillé dans les diagrammes).

$$\mathrm{d}\phi_-(p,q) := \mathrm{d}\overset{\dashrightarrow}{\phi}(p,q) := \mathrm{d}\arg(p-q) - \mathrm{d}\arg(p-\overline{q}). \tag{14}$$

3.3 Description de la différentielle μ_0 et exemples d'algèbres de réduction

Dans [10] on décrit en termes de diagrammes ce que vaut la différentielle μ_0 sur les fonctions polynomiales[13] sur $\mathfrak{h}^\perp_{-\lambda}$.

Proposition 1 ([8, 10]) *La différentielle μ_0 est l'action de tous les graphes de types suivant :*

(i) les graphes de type Bernoulli avec la dernière arête partant à l'infini (cf. Fig. 4)

(ii) les graphes de type roues avec des rayons attachés directement à l'axe réel sauf pour l'un d'entre eux qui est attaché à un graphe de type Bernoulli dont la dernière arête part à l'infini (cf. Fig. 5)

(iii) les graphes de type roues avec des rayons attachés directement à l'axe réel sauf pour l'un d'entre eux qui part à l'infini (cf. Fig. 6).

En particulier on a toujours $\mu_0 = \epsilon \mathrm{d}_{CH} + O(\epsilon^2)$ avec d_{CH} la différentielle de Cartan-Eilenberg.

Donnons quatre exemples d'algèbres de réduction (voir [10] §2 pour les détails).

[12] On a besoin ici de faire un choix d'un supplémentaire de $\mathfrak{h}$, pour identifier $\mathfrak{h}^*$ à un sous-espace de $\mathfrak{g}^*$.

[13] On renvoie à [10] pour l'action sur les éléments de $\mathrm{Poly}(\mathfrak{h}^\perp_{-\lambda}) \otimes \bigwedge \mathfrak{h}^*$.

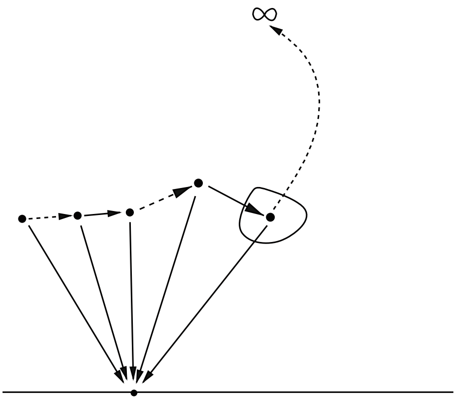


Fig. 4. Graphe de type Bernoulli

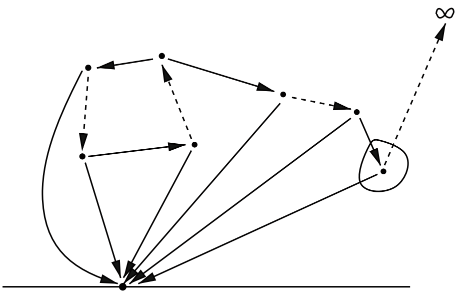


Fig. 5. Graphe de type roue attaché à un Bernoulli

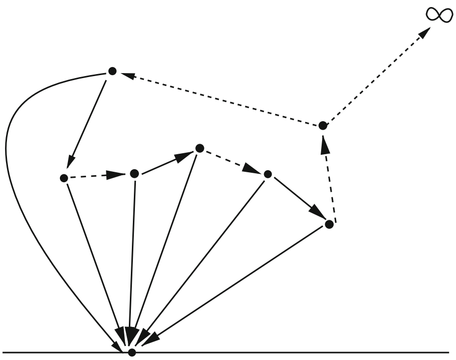


Fig. 6. Graphe de type roue pure

– Soit $0^\perp = \mathfrak{g}^*$. Alors l'algèbre de réduction est $S(\mathfrak{g})$ muni du produit de Kontsevich. En effet il n'y a pas d'arêtes sortantes, donc pas de condition.

– Supposons que $\mathfrak{h}$ admette un supplémentaire stable $\mathfrak{q}$ alors on montre que $\mu_0 = \epsilon \mathrm{d}_{CH}$. Dans ce cas, on en déduit que l'algèbre de réduction s'identifie à $\Big(S(\mathfrak{g})/S(\mathfrak{g})\cdot\mathfrak{h}_{-\lambda}\Big)^{\mathfrak{h}}$. En effet, les graphes avec plus de deux sommets ont tous deux arêtes d'une même couleur qui se suivent (cf. Fig. 4). Le coefficient associé est alors 0.

– Soit $f \in \mathfrak{g}^*$ et soit $\mathfrak{b}$ une polarisation en f, c'est à dire une sous-algèbre subordonnée $f[\mathfrak{b},\mathfrak{b}] = 0$ et lagrangienne pour $B_f(x,y) = f[x,y]$. On prend comme espace affine $f + \mathfrak{b}^{\perp}$, alors l'algèbre de réduction vaut $\mathbb{C}$.

– Soit $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ une décomposition d'Iwasawa d'une algèbre réductive réelle. On note $\mathfrak{m}$ le centralisateur de $\mathfrak{a}$ dans $\mathfrak{k}$ et on prend comme sous-algèbre $\mathfrak{m} \oplus \mathfrak{n}$, alors l'algèbre de réduction vaut $S(\mathfrak{a})$.

Parité :

On montre [10] §2.2 pour $\lambda = 0$, que dans la différentielle μ_0 seuls interviennent les diagrammes avec un nombre impair de sommets de première espèce. En tenant compte du degré de ϵ, on en déduit que l'algèbre de réduction est graduée. Toute fonction F homogène de degré total n dans l'algèbre de réduction s'écrit

$$F = F_n + \epsilon^2 F_{n-2} + \epsilon^4 F_{n-4} + \dots,$$

avec F_i de degré i. On a $F_n \in \Big(S(\mathfrak{g})/S(\mathfrak{g})\cdot\mathfrak{h}\Big)^{\mathfrak{h}}$; c'est l'analogue du symbole. Toutefois on ne peut en déduire que l'algèbre de réduction est une quantification de $\Big(S(\mathfrak{g})/S(\mathfrak{g})\cdot\mathfrak{h}\Big)^{\mathfrak{h}}$ car l'application *limite classique* $LC : F \mapsto F_n$ est injective mais pas forcément surjective. Dans le cas général $\lambda \neq 0$ on peut étendre la construction ci-dessus [7]. On déduit classiquement le corollaire suivant.

Corollaire 1 *Si l'algèbre de réduction $H^0(\mu_0)$ est commutative, alors son image par l'application LC est Poisson commutative.*

3.4 Bi-quantification de Cattaneo-Felder

On suppose données deux sous-algèbres $\mathfrak{h}_1$ et $\mathfrak{h}_2$ de $\mathfrak{g}$. On peut évidemment généraliser la construction en considérant des caractères. La quantification de Cattaneo-Felder définit donc deux algèbres de cohomologie $H^\bullet(\mu_0^{(1)}, \mathcal{A}_1)$ et $H^\bullet(\mu_0^{(2)}, \mathcal{A}_2)$. Dans [8, 9] Cattaneo-Felder définissent une structure de bimodule sur une troisième algèbre de cohomologie.

La construction procède de la façon suivante.

On fixe une décomposition de $\mathfrak{g}$ compatible avec $\mathfrak{h}_1$ et $\mathfrak{h}_2$, c'est à dire que les variables porteront 4 couleurs notées ici $(\pm,\pm)$. Le premier signe (resp. le second) vaut $+$ si la variable est tangente et vaut $-$ si la variable est normale à $\mathfrak{h}_1^{\perp}$ (resp. à $\mathfrak{h}_2^{\perp}$).

On définit alors la fonction d'angle à 4 couleurs dans le premier quadrant $0 \leq \arg(z) \leq \frac{\pi}{2}$ par la formule

$$\phi_{\epsilon_1,\epsilon_2}(p,q) = \arg(p-q) + \epsilon_1 \arg(p-\overline{q}) + \epsilon_2 \arg(p+\overline{q}) + \epsilon_1\epsilon_2 \arg(p+q). \quad (15)$$

La fonction d'angle vérifie la propriété suivante :

– lorsque p, q se concentrent sur l'axe horizontal les fonctions d'angles $d\phi_{\epsilon_1,\epsilon_2}(p,q)$ tendent vers la 1-forme d'angle

$$\mathrm{d}\phi_{\epsilon_1}(p,q) = \mathrm{d}\arg(p-q) + \epsilon_1 \mathrm{d}\arg(p-\overline{q}), \quad (16)$$

– lorsque p, q se concentrent sur l'axe vertical les fonctions d'angles $\mathrm{d}\phi_{\epsilon_1,\epsilon_2}(p,q)$ tendent vers la 1-forme d'angle

$$\mathrm{d}\phi_{\epsilon_2}(p,q) = \mathrm{d}\arg(p-q) + \epsilon_2 \mathrm{d}\arg(p+\overline{q}). \quad (17)$$

On dessine dans le premier quadrant tous les diagrammes Γ de Kontsevich colorés par les 4 couleurs ci-dessus en plaçant les sommets de première espèce dans le quadrant strict et les sommets de deuxième espèce sur les axes.

En considérant les compactifications des configurations de points du premier quadrant modulo l'action du groupe des dilatations, on définit alors des variétés à coins compactes sur lesquelles on pourra intégrer les formes Ω_Γ associées[14].

Chaque graphe coloré Γ va définir un opérateur polydifférentiel, une fois que l'on aura placé aux sommets de deuxième espèce (placés sur les axes) des fonctions[15]. Le résultat est alors restreint à $(\mathfrak{h}_1+\mathfrak{h}_2)^\perp$. Les arêtes colorées $(-,\pm)$ qui arrivent sur l'axe horizontal définissent des formes d'angles triviales, on peut donc placer sur **l'axe horizontal** une fonction de $\mathcal{A}_1$. De manière analogue on placera des fonctions de $\mathcal{A}_2$ sur **l'axe vertical**. Enfin on place à l'origine une fonction de

$$\mathrm{Poly}\Big((\mathfrak{h}_1+\mathfrak{h}_2)^\perp\Big) \otimes \bigwedge(\mathfrak{h}_1^*\cap\mathfrak{h}_2^*).$$

Cette algèbre est munie d'une différentielle $\mu^{(2,1)}$ correspondant aux contributions de tous les graphes colorés avec une arête sortant à l'infini colorée par $\mathfrak{h}_1^*\cap\mathfrak{h}_2^*$, et un seul sommet de deuxième espèce placé à l'origine.

En utilisant la formule de Stokes et en faisant l'inventaire des toutes les contributions, on montre le théorème de compatibilité suivant

Théorème 3 ([8, 10]) *L'espace $H^\bullet(\mu_0^{(2)},\mathcal{A}_2)$ agit par la gauche sur $H^\bullet(\mu_0^{(2,1)},\mathrm{Poly}\Big((\mathfrak{h}_1+\mathfrak{h}_2)^\perp\Big)\otimes\bigwedge(\mathfrak{h}_1^*\cap\mathfrak{h}_2^*))$. L'espace $H^\bullet(\mu_0^{(1)},\mathcal{A}_1)$ agit par la droite. On note $\underset{1}{\star}$ l'action à droite (axe horizontal) et $\underset{2}{\star}$ l'action à gauche (axe vertical).*

[14] A chaque arêtes colorés est associée la différentielle d'une des 4 fonctions d'angle ci-dessus.

[15] J'entends des fonctions avec composantes impaires.

4 Applications en théorie de Lie

On décrit maintenant les applications en théorie de Lie de la bi-quantification et du théorème de compatibilité.

4.1 Description de l'algèbre de réduction

On fixe un supplémentaire $\mathfrak{q}$ de $\mathfrak{h}$. On a donc une décomposition de l'algèbre enveloppante

$$U(\mathfrak{g}) = \beta(S(\mathfrak{q})) \oplus U(\mathfrak{g}) \cdot \mathfrak{h}_{-\lambda}$$

où β désigne la symétrisation. On notera $\beta_{\mathfrak{q}}$ l'application déduite de $S(\mathfrak{q})$ dans $U(\mathfrak{g})/U(\mathfrak{g}) \cdot \mathfrak{h}_{-\lambda}$.

On considère la bi-quantification Cattaneo-Felder pour le couple de variétés co-isotropes $\mathfrak{h}_{-\lambda}^{\perp}$ mis en position horizontale et $0^{\perp} = \mathfrak{g}^*$ mis en position verticale. L'espace de réduction associé à l'origine est tout simplement $S(\mathfrak{g})/S(\mathfrak{g}) \cdot \mathfrak{h}_{-\lambda} = \mathrm{Poly}(\mathfrak{h}_{-\lambda}^{\perp})$. Comme dans [20] on considère pour le bi-vecteur de Poisson, la moitié du crochet de Lie.

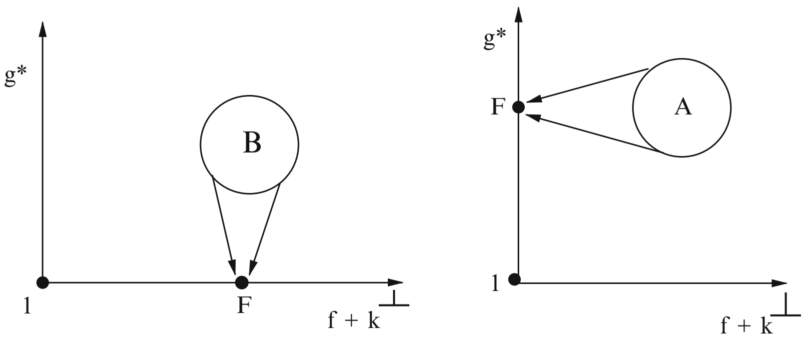


Fig. 7. Contributions des roues pures sur les axes

Pour $F \in S(\mathfrak{g})$ dans l'espace de réduction vertical, $F \underset{2}{\star} 1 \in S(\mathfrak{g})/S(\mathfrak{g}) \cdot \mathfrak{h}_{-\lambda}$ et $F \mapsto F \underset{2}{\star} 1$ est un opérateur donné par diagrammes de Kontsevich à 4-couleurs. Cet opérateur est compliqué et n'est pas à coefficients constants car les diagrammes avec arêtes doubles colorées par $++$ et $+-$ ne sont pas nuls *a priori.*

Lorsque $F \in S(\mathfrak{q})$ alors l'opérateur $A(F) = F \underset{2}{\star} 1$ est plus simple et correspond à l'exponentielle des contributions des roues pures sur l'axe vertical. On notera de même $B(F) = 1 \underset{1}{\star} F$ l'exponentielle des contributions des roues sur l'axe horizontal. On note $A(X)$ et $B(X)$ les symboles associés, c'est à dire pour $X \in \mathfrak{q}$, $A_{\mathfrak{q}}(X) = (e^X \underset{2}{\star} 1)e^{-X}$ et $B_{\mathfrak{q}}(X) = (1 \underset{1}{\star} e^X)e^{-X}$.

Je note pour $X \in \mathfrak{g}$, $j(X) = \det_{\mathfrak{g}}\left(\frac{\sinh \frac{\mathrm{ad} X}{2}}{\frac{\mathrm{ad} X}{2}}\right)$ et on définit $J_{\mathfrak{q}}(X)$ par la formule

$$A_{\mathfrak{q}}(X) J_{\mathfrak{q}}^{1/2}(X) = B_{\mathfrak{q}}(X) j^{1/2}(X). \tag{18}$$

Théorème 4 *L'application $\beta_{\mathfrak{q}} \circ J_{\mathfrak{q}}^{1/2}(\partial)$ définit un isomorphisme d'algèbres de l'algèbre de réduction $H^0(\mu_0)$ muni du produit $\underset{CF}{\star}$ sur l'algèbre $\big(U(\mathfrak{g})/U(\mathfrak{g})\cdot \mathfrak{h}_{-\lambda}\big)^{\mathfrak{h}}$.*

Ce théorème est démontré dans le cas des paires symétriques dans [10] et est étendu au cas général dans la thèse de mon étudiant P. Batakidis [7]. La formule de Stokes est encore l'argument essentiel. Lorsque $\mathfrak{q}$ est invariant par action adjointe de $\mathfrak{h}$ alors $J_{\mathfrak{q}}$ est une fonction $\mathrm{ad}(\mathfrak{h})$-invariante. Dans le cas des paires symétriques on montre [10] §4.1 que $J_{\mathfrak{q}}$ vaut $J(X) = \det_{\mathfrak{q}}\left(\frac{\sinh(\mathrm{ad} X)}{\mathrm{ad} X}\right)$. Dans le cas des paires symétriques on retrouve le produit $\underset{Rou}{\sharp}$ de Rouvière [27, 28, 29, 30].

On dispose donc d'une description complète via la symétrisation des éléments de $\big(U(\mathfrak{g})/U(\mathfrak{g})\cdot \mathfrak{h}_{-\lambda}\big)^{\mathfrak{h}}$.

4.2 Cas des paires symétriques

Supposons dans cette section que $(\mathfrak{g}, \sigma)$ est une paire symétrique, c'est à dire que σ est une involution de Lie. On a donc une décomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ avec $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ et $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. On s'intéresse à l'algèbre $\Big(U(\mathfrak{g})/U(\mathfrak{g})\cdot \mathfrak{k}\Big)^{\mathfrak{k}}$. Le produit μ_1 prend alors une forme remarquable. En effet pour $X, Y \in \mathfrak{p}$ on a

$$\mu_1(e^X, e^Y) = E(X, Y) e^{X+Y}, \tag{19}$$

o $E(X, Y)$ est l'exponentielle des contributions des graphes de types roues[16]. Le produit de $\underset{CF}{\star} = \underset{Rou}{\sharp}$ se résume sur $H^0(\mu_0) = S(\mathfrak{p})^{\mathfrak{k}}$, les éléments $\mathfrak{k}$-invariants, en un opérateur bidifférentiel à coefficient constant de symbole E.

On peut déduire de l'analyse de la fonction E des propriétés non triviales de $\Big(U(\mathfrak{g})/U(\mathfrak{g})\cdot \mathfrak{k}\Big)^{\mathfrak{k}}$. En voici quelques unes (cf. [10] §3) :

Symétrie :

Pour $X, Y \in \mathfrak{p}$, on a $E(X, Y) = E(Y, X)$. On déduit la commutativité de l'algèbre $\Big(U(\mathfrak{g})/U(\mathfrak{g})\cdot \mathfrak{k}\Big)^{\mathfrak{k}}$.

Radical résoluble :

Si X est dans le radical résoluble[17] de $\mathfrak{g}$, alors $E(X, Y) = 1$ pour tout $Y \in \mathfrak{p}$. On en déduit que $F \underset{CF}{\star} G = FG$ si $F \in S(\mathfrak{p} \cap J)^{\mathfrak{k}}$ avec J un idéal résoluble.

[16] C'est à dire des roues colorées, attachées à des graphes de type Lie.

[17] Le plus grand idéal résoluble de $\mathfrak{g}$, il est σ-stable.

Paires symétriques d'Alekseev-Meinrenken:

Supposons que $(\mathfrak{g},\sigma)$ soit munie d'une forme bilinéaire non dégénérée[18] et σ-anti-invariante, alors on a $E(X,Y)=1$.

Double quadratique:

Supposons que $\mathfrak{g}$ soit une algèbre de Lie quadratique, c'est à dire munie d'une forme bilinéaire non dégénérée. Considérons la paire symétrique double $(\mathfrak{g}_{double},\sigma)=(\mathfrak{g}\times\mathfrak{g},\sigma)$ avec $\sigma(X,Y)=(Y,X)$. Alors on a $E_{double}=1$. On conjecture que cette propriété est vraie même si $\mathfrak{g}$ n'est pas quadratique, ce qui a d'intéressantes conséquences (voir la fin de l'article).

4.3 Opérateurs différentiels en coordonnées exponentielles

On reprend dans cette section les notations de l'introduction. On peut raffiner le théorème précédent en donnant l'écriture en coordonnées exponentielles des opérateurs différentiel invariants du fibré $\mathcal{L}_\lambda$. On considère toujours le diagramme de bi-quantification du §4.1. Notons Exp l'application exponentielle de $\mathfrak{q}$ sur G/H. On travaille au voisinage de $0\in\mathfrak{q}$. Soit $\varphi\in\Gamma(\mathcal{L}_\lambda)$, alors φ est une fonction sur G telle que $\varphi(gh)=\varphi(g)\lambda(h)$. Notons $\phi\in\mathcal{C}^\infty(\mathfrak{q})$ définie au voisinage de 0 par

$$\phi(X)=\frac{J_{\mathfrak{q}}^{1/2}(X)}{B(X)}\times\varphi(\exp_G(X)).$$

Le facteur représente une sorte de jacobien. Pour $u\in\mathcal{D}_\lambda$ on note D_u^{Exp} l'opérateur en coordonnées exponentielles défini par

$$D_u^{\mathrm{Exp}}(\phi)(X)=\frac{J_{\mathfrak{q}}^{1/2}(X)}{B(X)}\times D_u(\varphi)(\exp_G(X)).$$

Pour $X,Y\in\mathfrak{q}$ on note $Q(X,Y)\in\mathfrak{q}$ et $H(X,Y)\in\mathfrak{h}$ les composantes exponentielles au voisinage de 0 :

$$\exp_G(X)\exp_G(Y)=\exp_G(Q(X,Y))\exp_G(H(X,Y)). \tag{20}$$

On peut étendre au cas des sous-algèbres la formule de [10] §4.3. Pour R dans l'algèbre de réduction $H^0(\mu_0)$ on a :

$$e^X\underset{1}{\star}R=\frac{J_{\mathfrak{q}}^{1/2}(X)}{B(X)}\times R(\partial_Y)\Big(J_{\mathfrak{q}}^{1/2}(Y)\times\frac{B(Q(X,Y))}{J_{\mathfrak{q}}^{1/2}(Q(X,Y))}e^{Q(X,Y)}e^{\lambda(H(X,Y))}\Big)\,|_{Y=0}\,. \tag{21}$$

[18] On dit que $(\mathfrak{g},\sigma)$ est une paire symétrique quadratique.

Théorème 5 ([10]) *Pour $u = \beta\Big(J_{\mathfrak{q}}^{1/2}(\partial)(R)\Big)$, l'opérateur différentiel D_u^{Exp} en coordonnées exponentielles s'exprime par la formule $e^X \underset{1}{\star} R$.*

Remarquons que l'expression est valable au voisinage de 0 en posant $\epsilon = 1$. La convergence des coefficients se démontre comme dans [3, 4]. Cette expression résout de manière satisfaisante un problème ancien de M. Duflo [14].
Remarque : Ce théorème devrait avoir des conséquences intéressantes dans le cas des espaces symétriques hermitiens. En effet on peut utiliser la réalisation d'Harish–Chandra pour écrire les opérateurs différentiels invariants[19] [32, 26].

4.4 Construction de caractères pour l'algèbre $\big(U(\mathfrak{g})/U(\mathfrak{g}) \cdot \mathfrak{h}_{-\lambda}\big)^{\mathfrak{h}}$

On reprend dans cette section les notations de l'introduction. On cherche des caractères pour l'algèbre $\Big(U(\mathfrak{g})/U(\mathfrak{g}) \cdot \mathfrak{h}_{-\lambda}\Big)^{\mathfrak{h}}$. Afin que ceci soit intéressant il faut que cette dernière soit commutative.

L'hypothèse lagrangienne

Plaçons nous dans l'hypothèse lagrangienne que nous avons décrite dans l'introduction. On suppose que pour $f \in \mathfrak{h}_{-\lambda}^{\perp}$ générique, l'espace $\mathfrak{h} \cdot f$ (action coadjointe) est lagrangien dans $\mathfrak{g} \cdot f$. Rappelons que $\mathfrak{g} \cdot f$ est toujours un espace symplectique muni de la forme de Kostant-Souriau. Il s'identifie à l'espace symplectique $(\mathfrak{g}/\mathfrak{g}(f), B_f)$ avec $\mathfrak{g}(f) = \{X \in \mathfrak{g},\ X \cdot f = 0\ \}$ et $B_f(X, Y) = f[X, Y]$.

Comme on l'a dit l'algèbre de Poisson $\mathcal{S}_\lambda = \Big(S(\mathfrak{g})/S(\mathfrak{g}) \cdot \mathfrak{h}_{-\lambda}\Big)^{\mathfrak{h}}$ est alors Poisson commutative, mais aussi l'algèbre des fractions invariantes[20] $\Big(\mathrm{Frac}(\mathfrak{h}_{-\lambda}^{\perp})\Big)^{\mathfrak{h}}$. En fait la commutativité de cette dernière est équivalente à l'hypothèse lagrangienne.

On suppose de plus que les H-orbites génériques sont polarisables. C'est à dire que pour $f \in \mathfrak{h}_{-\lambda}^{\perp}$ générique, il existe une polarisation $\mathfrak{b}$ en f (sous-algèbre isotrope et de dimension maximale parmi les sous-espaces isotropes). On a $f[\mathfrak{b}, \mathfrak{b}] = 0$ et $\mathfrak{b}/\mathfrak{g}(f)$ lagrangien dans $\mathfrak{g}/\mathfrak{g}(f)$. Cette hypothèse est au cœur de la quantification géométrique.

Dans ces conditions on montre facilement que $(\mathfrak{h} \cap \mathfrak{b}) \cdot f = (\mathfrak{h} + \mathfrak{b})^{\perp}$. Cette hypothèse va être cruciale.

---

[19] Les formules proposées dans [26] ne sont pas correctes.

[20] On notera qu'il faut prendre les invariants du corps des fractions et non pas le corps des fractions des invariants.

Construction d'un caractère

Dans [10] §6, on construit le diagramme de bi-quantification en plaçant sur l'axe horizontal $\mathfrak{h}^{\perp}_{-\lambda}$ et sur l'axe vertical $f + \mathfrak{b}^{\perp}$. Par construction il faut donc choisir une décomposition compatible de $\mathfrak{g}$. En particulier on fixe un supplémentaire $\mathfrak{q}$ de $\mathfrak{h}$ en position d'intersection normale avec $\mathfrak{b}$ c'est à dire que l'on a

$$\mathfrak{b} = \mathfrak{b} \cap \mathfrak{h} \oplus \mathfrak{b} \cap \mathfrak{q}.$$

D'après §3.3 on sait que l'algèbre de réduction verticale est réduite à $\mathbb{C}$ et l'hypothèse $(\mathfrak{h}\cap\mathfrak{b})\cdot f = (\mathfrak{h}+\mathfrak{b})^{\perp}$ implique facilement que l'algèbre de réduction associée à l'origine est aussi $\mathbb{C}$. On en déduit une action à droite de $H^0(\mu_0) \sim \Big(U(\mathfrak{g})/U(\mathfrak{g})\cdot\mathfrak{h}_{-\lambda}\Big)^{\mathfrak{h}}$ sur $\mathbb{C}$. On a donc construit un caractère de cette algèbre.

Dans [10] §6 on développe une théorie des diagrammes à 8-couleurs dans une bande qui nous permet d'interpoler deux situations de bi-quantification et nous permet de calculer le caractère (cf. Fig. 8). On peut déplacer la position de F le long du bord horizontal. Les positions limites aux coins donnent les informations recherchées. Notre méthode utilise la formule de Stokes.

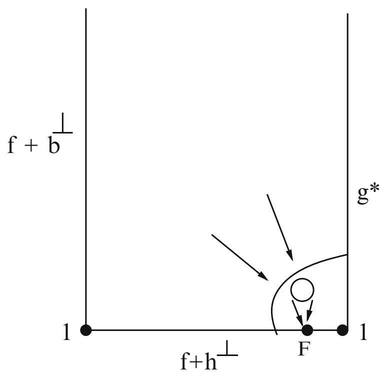


Fig. 8. Calcul du caractère

Théorème 6 *Sous les hypothèses lagrangiennes ci-dessus et l'existence de polarisation, l'application*

$$u \in \Big(U(\mathfrak{g})/U(\mathfrak{g})\cdot\mathfrak{h}_{-\lambda}\Big)^{\mathfrak{h}} \mapsto B_{\mathfrak{q}}(\partial)J_{\mathfrak{q}}^{-1/2}(\partial)(\beta_{\mathfrak{q}}^{-1}(u))(f)$$

est le caractère construit par le diagramme de bi-quantification.

Ce théorème est démontré dans le cas des paires symétriques dans [10] §6 et étendu au cas général dans la thèse de mon étudiant P. Batakidis [7]. Remarquons que dans le cas des paires symétriques résolubles, on retrouve directement la formule de Rouvière, car $B_{\mathfrak{p}}(X) = 1$ et $J_{\mathfrak{p}}(X) = \det_{\mathfrak{q}}\left(\frac{\sinh(\mathrm{ad}X)}{\mathrm{ad}X}\right)$.

On en déduit facilement en regardant le terme dominant le corollaire suivant

Corollaire 2 *Sous les hypothèses lagrangiennes ci-dessus, l'algèbre* $\big(U(\mathfrak{g})/U(\mathfrak{g})\cdot\mathfrak{h}_{-\lambda}\big)^{\mathfrak{h}}$ *est commutative.*

En particulier dans le cas nilpotent, les orbites sont toujours polarisables[21], l'hypothèse lagrangienne est équivalente au fait que la multiplicité de la représentation $L^2(G/H,\chi)$ est finie[22]. On retrouve alors un théorème de Corwin & Greenleaf [12]. On en déduit dans le cas nilpotent, que si l'algèbre S_λ est Poisson commutative alors $\big(U(\mathfrak{g})/U(\mathfrak{g})\cdot\mathfrak{h}_{-\lambda}\big)^{\mathfrak{h}}$ est commutative. Réciproquement d'après le lemme 1, si $\big(U(\mathfrak{g})/U(\mathfrak{g})\cdot\mathfrak{h}_{-\lambda}\big)^{\mathfrak{h}}$ est commutative alors la limite classique $LC(H^0(\mu_0))$ est Poisson commutative. Bien évidemment si cette dernière algèbre de Poisson est assez grosse[23] on pourrait en déduire alors que S_λ est aussi Poisson commutative.

Comparaison avec le vecteur de Penney

Sous les hypothèses lagrangiennes ci-dessus, la méthode des orbites fournit dans les bons cas un vecteur distribution semi-invariant (dit vecteur de Penney [34, 36, 15, 23, 24, 25]). Dans les bons cas ce vecteur est aussi un vecteur propre pour $\mathcal{D}_\lambda$, ce qui fournit aussi un caractère de cette algèbre.

Décrivons le vecteur de Penney dans la situation unimodulaire et $\lambda = 0$ pour simplifier l'exposé. On note B un groupe de Lie connexe d'algèbre de Lie $\mathfrak{b}$. On suppose que $\chi_f(\exp_B(X)) = e^{if(X)}$ définit bien un caractère[24] de B. On note $\mathrm{d}_{G,H}$ et $\mathrm{d}_{B,B\cap H}$ des mesures invariantes[25] sur G/H et $B/B\cap H$.

Le vecteur de Penney est défini par la fonction généralisée

$$\varPhi d_{G,H} \longmapsto j_*(\varPhi d_{G,H}) = \int_{B/B\cap H} \varPhi(b)\chi_f(b)^{-1} d_{B,B\cap K}(b).$$

Sous l'hypothèse lagrangienne c'est une section généralisée propre [34] sous l'action des opérateurs différentiels invariants D_u pour $u \in (U(\mathfrak{g})/U(\mathfrak{g})\cdot\mathfrak{h})^{\mathfrak{h}}$. Ceci fournit donc un caractère de cette algèbre noté $u \longrightarrow \lambda_{f,\mathfrak{b}}(u)$.

En utilisant le théorème 5 on montre que ce caractère vaut aussi la transformée de Fourier du symbole transverse et concide (dans les bons cas) alors avec celui construit par le diagramme de bi-quantification[26].

[21] On peut prendre une polarisation construite par M. Vergne.
[22] Ici χ est le caractère de différentielle $i\lambda$.
[23] Une conjecture raisonnable semble que le corps des fractions de la limite classique contient $\mathcal{S}_\lambda$.
[24] Ces hypothèses d'intégrabilité compliquent la théorie en général.
[25] En général de telle mesure n'existe pas, il faut alors travailler sur des fibrés en droite, cf. Introduction.
[26] Le cas nilpotent sera traité dans la thèse de P. Batakidis [7].

4.5 Dépendance par rapport au supplémentaire et applications

On étudie maintenant la dépendance du produit $\underset{CF}{\star}$ par rapport au choix du supplémentaire $\mathfrak{q}$.

Formule de changement de base

On fixe un supplémentaire $\mathfrak{q}_0$ de $\mathfrak{h}$. On choisit (e_i) une base de $\mathfrak{g}$ adaptée à la décomposition $\mathfrak{h}\oplus\mathfrak{q}_0$, c'est à dire une base $(K_i)_i$ de $\mathfrak{h}$ et une base $(P_a)_a$ de $\mathfrak{q}_o$. Faisons choix d'un autre supplémentaire $\mathfrak{q}_1$ dont on fixe une base $(Q_a)_a$. Sans perte de généralité on peut supposer que la matrice de passage est de la forme

$$\mathbb{M} = \begin{pmatrix} I & \mathbb{D} \\ 0 & I \end{pmatrix}.$$

Notons $\mathbb{D} = [V_1, \dots, V_p]$ les colonnes de la matrice $\mathbb{D}$ et $V_i \in \mathfrak{h}$.

On écrit le bi-vecteur π dans les deux décompositions et on applique la procédure de transformée de Fourier impaire dans les directions normales. On trouve alors deux poly-vecteurs $\widehat{\pi}$ et $\widehat{\pi}^{(1)}$ sur la variété intrinsèque $\mathfrak{h}^\perp \oplus \Pi\mathfrak{h}$. La relation entre les deux poly-vecteurs est la suivante (cf. [10] §1.5). Considérons

$$\pi_{\mathbb{M}} = \mathbb{M}^{-1}[\mathbb{M}e_i, \mathbb{M}e_j]\partial_{e_i^*} \wedge \partial_{e_j^*}.$$

Alors on aura

$$\widehat{\pi}^{(1)} = \widehat{\pi_{\mathbb{M}}}$$

o le membre de droite est la transformée de Fourier partielle impaire pour la première décomposition. Par ailleurs le champ de vecteurs sur $\mathfrak{g}^*$ défini par $v = -V_a\partial_{P_a^*}$ vérifie[27] $[v, v]_S = 0$ et on a la relation

$$\pi_{\mathbb{M}} = e^{\mathrm{ad} v} \cdot \pi = \pi + [v, \pi]_S + \frac{1}{2}[v, [v, \pi]]_S. \tag{22}$$

L'action du champ $-v$ sur le bivecteur π correspond au changement de supplémentaire.

Contrôle de la déformation: l'élément de jauge

Soit t un paramètre réel. Je note

$$\pi_t = e^{t\,\mathrm{ad} v} \cdot \pi = \pi + t[v, \pi] + \frac{t^2}{2}[v, [v, \pi]]$$

et $\widehat{\pi}_t$ sa transformée de Fourier partielle pour la décomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}_0$.

Pour $t = 0$ on trouve π et pour $t = 1$ on trouve $\pi_{\mathbb{M}}$. On applique le L_∞ quasi-isomorphisme du théorème 2 à la super-variété $\mathfrak{h}^\perp \oplus \Pi\mathfrak{h}$. Alors μ_t définie par

[27] On note $[\bullet, \bullet]_S$ le crochet de Schouten-Nijenhuis.

$$\mu_t = \mathcal{U}(e^{\widehat{\pi_t}}) = m + \sum_{n\geq 1} \frac{\epsilon^n}{n!} U_n\left(\widehat{\pi_t}, \dots, \widehat{\pi_t}\right)$$

est une structure A_∞.

La dérivée $\mathrm{d}\mathcal{U}_{\widehat{\pi_t}}$ au point $\widehat{\pi_t}$ est un morphisme de complexes. On a donc en dérivant, une équation différentielle linéaire :

$$\frac{\partial \mu_t}{\partial t} = \epsilon\, \mathrm{d}\mathcal{U}_{\widehat{\pi_t}}\left([\widehat{v}, \widehat{\pi_t}]_{SN}\right) = \left[\mathrm{d}\mathcal{U}_{\widehat{\pi_t}}\left(\widehat{v}\right), \mu_t\right]_G . \tag{23}$$

On peut alors traduire cette formule en terme de diagrammes de Kontsevich colorés comme dans §3.2. On placera aux sommets de première espèce le bi-vecteur π_t et une fois le vecteur v. Les sommets terrestres reçoivent des fonctions de $\left(S(\mathfrak{g})/S(\mathfrak{g})\cdot\mathfrak{h}\right) \otimes \bigwedge \mathfrak{h}^*$.

La différentielle $(\mu_t)_0$, composante de degré 0 de μ_t, vérifie en particulier l'équation différentielle

$$\frac{\partial(\mu_t)_0}{\partial t} = [(DU_{\widehat{\pi_t}}(\widehat{v}))_0\,, (\mu_t)_0]. \tag{24}$$

Cette formule dit que les différentielles $(\mu_t)_0$ sont conjuguées par un élément de type groupe[28]: c'est l'élément de jauge. Pour le décrire, il suffit d'analyser tous les graphes qui interviennent dans $(DU_{\widehat{\pi_t}}(\widehat{v}))_0$. C'est ce que l'on fait dans [10] § 5.5. Ce sont les graphes des figures Fig. 4, Fig. 5 et Fig. 6 o l'arête ∞ dérive le sommet attaché au vecteur v. L'arête issue de v va soit sur la racine du graphe, soit sur le sommet terrestre. Il y a donc 4 types de graphes donnés par les figures suivantes (Fig. 9, Fig. 10, Fig. 11 et Fig. 12).

De même, la composante de degré un $(\mu_t)_1$ vérifie l'équation

$$\frac{\partial(\mu_t)_1}{\partial t} = [(DU_{\widehat{\pi_t}}(\widehat{v}))_0\,, (\mu_t)_1] + [(DU_{\widehat{\pi_t}}(\widehat{v}))_1\,, (\mu_t)_0]. \tag{25}$$

Cette équation exprime alors que l'élément de jauge définit en cohomologie un isomorphisme d'algèbres. En particulier, résoudre l'équation (24) permet de trouver explicitement l'entrelacement des star-produits pour deux choix de supplémentaires.

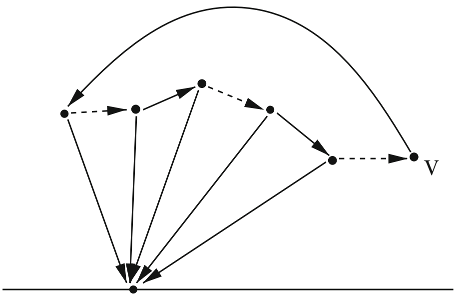


Fig. 9. Bernoulli fermé par v

[28] C'est la résolvante de l'équation différentielle.

Exemple du double

Pour illustrer les conséquences de nos théories, examinons le cas des paires symétriques $(\mathfrak{g}\times\mathfrak{g},\sigma)$ avec $\sigma(X,Y)=(Y,X)$. On a alors $\mathfrak{k}=\{(X,X),\ X\in\mathfrak{g}\}$. Ce sont les doubles.

Déformation du supplémentaire :

On dispose de 3 supplémentaires invariants naturels,

$$\mathfrak{g}_-=\{(0,2X)\,,X\in\mathfrak{g}\}\quad \mathfrak{p}=\{(X,-X)\,,X\in\mathfrak{g}\}\quad \mathfrak{g}_+=\{(2X,0)\,,X\in\mathfrak{g}\}.$$

L'interpolation est donnée par la famille à paramètre de sous-espaces invariants

$$\{((1+t)X,(t-1)X)\,,X\in\mathfrak{g}\}.$$

Le champ de vecteur $v=-V_a\partial_{P_a^*}$ est donc l'application linéaire qui transforme $(X,-X)\in\mathfrak{p}\mapsto-(X,X)\in\mathfrak{k}$. Si on note (e_i) une base de $\mathfrak{g}$, $K_i=(e_i,e_i)$ et $P_j=(e_j,-e_j)$ alors le bi-vecteur π_t prend une forme assez simple

$$\pi_t=2[K_i,P_j]\partial_{K_i^*}\wedge\partial_{P_j^*}+[K_i,K_j]\partial_{K_i^*}\wedge\partial_{K_j^*}+\\ \big([K_i,K_j](1-t^2)+2t[K_i,P_j]\big)\,\partial_{P_i^*}\wedge\partial_{P_j^*}. \quad (26)$$

Élément de jauge :

Comme $(X,-X)$ et (X,X) commutent l'opérateur $(DU_{\widehat{\pi_t}}(\widehat{v}))_0$ va se simplifier. En effet les opérateurs différentiels correspondant aux graphes des Fig. 11 et Fig. 12 seront nuls. Il ne reste alors que les graphes Fig. 9 et Fig. 10. Le coefficient vérifie une condition de symétrie, il est nul si le nombre de sommets attachés à π_t est impair. Au final ces diagrammes correspondent à des opérateurs différentiels de symboles

$$P_{2n}(t)\mathrm{tr}_{\mathfrak{p}}\big(\mathrm{ad}(X,-X)^{2n}\big)=P_{2n}(t)\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}X)^{2n}$$

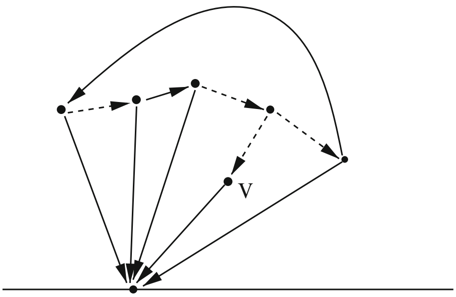


Fig. 10. Roue pure attachée à v

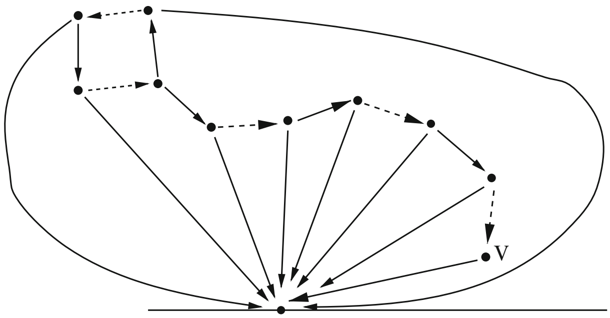


Fig. 11. Roue attachée à un Bernoulli attaché à v

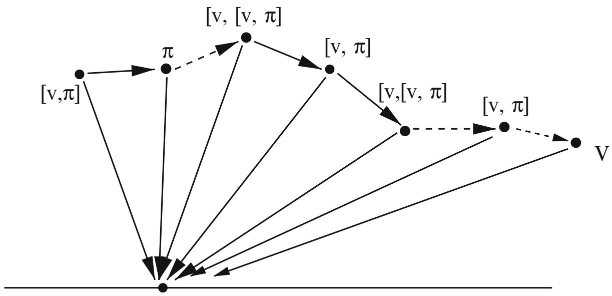


Fig. 12. Graphe de type Bernoulli non fermé

avec $P_{2n}(t)$ un polynme en t de degré inférieur à $2n$. Ce polynme dépend de la couleur dans la roue et des coefficients de Kontsevich associés, il s'écrit sous la forme

$$P_{2n}(t) = \sum w_\Gamma P_\Gamma(t)$$

où Γ décrit tous les graphes Fig. 9 et Fig. 10 avec $2n$ sommets associés π_t. Ces polynmes sont sans doute liés aux polynmes de Bernoulli, mais nous n'avons pas pu le vérifier.

L'opérateur $(DU_{\widehat{\pi_t}}(\widehat{v}))_0$ commute à l'action adjointe, donc l'équation de la différentielle (24) se résout simplement, on trouve $(\mu_t)_0 = \epsilon \mathrm{d}_{CE}$, ce que l'on savait déjà par ailleurs.

Enfin les opérateurs $(DU_{\widehat{\pi_t}}(\widehat{v}))_0$ forment une famille commutative en t donc l'élément de jauge est donné par la résolvante

$$\phi_t := \exp\Big(\int_0^t (DU_{\widehat{\pi_s}}(\widehat{v}))_0 \,\mathrm{d}s\Big).$$

C'est un opérateur universel de symbole

$$\phi_t(X) := \exp\left(\sum_{n>0} Q_{2n+1}(t)\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}X)^{2n}\right).$$

Interpolation du produit :

Regardons le terme de degré 1 pour la structure A_∞. L'espace $\mathfrak{h}^\perp$ s'identifie aux couples $(f,-f)$ avec $f \in \mathfrak{g}^*$. Les formes linéaires

$$(2X,0) \quad (X,-X) \quad (0,2X)$$

sont identiques sur $\mathfrak{h}^\perp$. On la note $\widehat{X}$.

Pour $t=0$, c'est la situation des paires symétriques on a donc d'après § 4.2

$$\mu_{t=0}\left(e^{\widehat{X}}, e^{\widehat{Y}}\right) = E_{double}((X,-X),(Y,-Y))e^{\widehat{X+Y}}.$$

Pour $t=1$, c'est la situation des algèbres de Lie, on a donc [4]

$$\mu_{t=1}\left(e^{\widehat{X}}, e^{\widehat{Y}}\right) = D(2X,2Y)e^{\frac{1}{2}\widehat{Z(2X,2Y)}},$$

avec $Z(X,Y)$ la formule de Campbell-Hausdorff et

$$D(X,Y) = \frac{j_{\mathfrak{g}}^{1/2}(X)j_{\mathfrak{g}}^{1/2}(Y)}{j_{\mathfrak{g}}^{1/2}(Z)}$$

la fonction de densité de Duflo. Rappelons que $j_{\mathfrak{g}}(X) = \det_{\mathfrak{g}}\left(\frac{\sinh\frac{\mathrm{ad}X}{2}}{\frac{\mathrm{ad}X}{2}}\right)$.

Pour $t=1$, il est facile de calculer la fonction $J_{\mathfrak{g}+}$ que nous avons introduite en § 4.1. En effet, les roues A, B n'ont qu'une seule couleur $(+,+)$. Par symétrie on trouve $A=B$. On a donc

$$J_{\mathfrak{g}+}((2X,0)) = j_{\mathfrak{g}}(2X) = \det_{\mathfrak{g}}\left(\frac{\sinh \mathrm{ad}X}{\mathrm{ad}X}\right) = J_{\mathfrak{p}}(X,-X).$$

La fonction J est donc la même pour ces deux choix de supplémentaires.

Proposition 2 *L'élément de jauge ϕ_1 vaut* 1*. Dans le cas du double, le star-produit $\underset{CF}{\star}$ est trivial sur $S(\mathfrak{p})^k$, les éléments $\mathfrak{k}$-invariants de $S(\mathfrak{p})$.*

Preuve: Cet élément est un isomorphisme pour les algèbres de réduction. Pour $t=0$ on trouve $S(\mathfrak{p})^{\mathfrak{k}}$ muni du produit de Rouvière qui vaut $\underset{CF}{\star}$. Pour $t=1$ c'est le produit de Duflo-Kontsevich, c'est à dire la multiplication standard sur les invariants. Or pour le double quadratique la fonction E_{double} vaut 1 [10]. Donc ϕ_1 est un isomorphisme d'algèbres de $S(\mathfrak{g})^{\mathfrak{g}}$ pour toute algèbre quadratique. C'est donc l'identité car $\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}X)^{2n}$ n'agit pas comme une dérivation universelle. Comme ϕ_1 est universel, c'est toujours 1. On en déduit que l'action de E_{double} sur $S(\mathfrak{p})^k$ est triviale, même si $\mathfrak{g}$ n'est pas quadratique. ■

Conjecture de Kashiwara–Vergne :

Plus généralement, comme les formules de quantification dans le cas linéaire sont toujours des exponentielles on aura pour la composante de degré 1 de notre structure A_∞ :

$$\mu_t\left(e^{\widehat{X}}, e^{\widehat{Y}}\right) = E_t(X,Y)e^{\widehat{Z_t(X,Y)}}$$

avec $Z_t(X,Y)$ une série de Lie formelle en X, Y à coefficients polynomiaux en t.

On dispose donc d'une déformation de la formule de Campbell-Hausdorff en la loi additive. On peut alors traduire l'équation (25) sur la déformation. On compense déjà le terme $[(DU_{\widehat{\pi_t}}(\widehat{v}))_0, (\mu_t)_1]$ en conjuguant par $\phi_t(X)$. Posons donc

$$\widehat{E_t(X,Y)} = E_t(X,Y)\frac{\phi_t(X)^{-1}\phi_t(Y)^{-1}}{\phi_t(Z_t)^{-1}}.$$

En examinant ce qu'est le terme $(DU_{\widehat{\pi_t}}(\widehat{v}))_1$ on se convainc sans difficulté, comme dans [35] que l'on est en train de calculer une différentielle en X et en Y de la fonction $Z_t(X,Y)$. On trouve un contrôle à la Kashiwara–Vergne [17] de la déformation $Z_t(X,Y)$ (voir [37] pour un résumé de les méthodes de Kashiwara–Vergne). On a donc montré la théorème suivant.

Théorème 7 *La déformation du supplémentaire produit une déformation de Kashiwara–Vergne, c'est à dire qu'il existe des séries de Lie sans termes constants* $(F_t(X,Y), G_t(X,Y))$ *à coefficients polynomiaux en* t*, telles que*

$$\partial_t Z_t(X,Y) = [X, F_t(X,Y)]\cdot\partial_X Z_t(X,Y) + [Y, G_t(X,Y)]\cdot\partial_Y Z_t(X,Y), \quad (27)$$

$$\partial_t \widehat{E_t(X,Y)} = \Big([X, F_t(X,Y)]\cdot\partial_X + [Y, G_t(X,Y)]\cdot\partial_Y\Big)\widehat{E_t(X,Y)} + \widehat{E_t(X,Y)}\mathrm{tr}_{\mathfrak{g}}\big(\partial_X F_t\circ \mathrm{ad}X + \partial_Y G_t \circ \mathrm{ad}Y\big). \quad (28)$$

Corollaire 3 *Si la conjecture* $E_{double} = 1$ *est vraie, alors la déformation du supplémentaire démontre la conjecture de Kashiwara–Vergne.*

Preuve: En effet, si $E_{double} = 1$ alors le théorème précédent fournit une déformation à la Kashiwara–Vergne qui à les bonnes conditions limites. C'est à dire pour $t = 0$ on a le produit $X + Y$ est pour $t = 1$ le produit de Duflo. Comme dans [1, 2, 37] on construit alors une solution de Kashiwara–Vergne. ■

Remarque finale : Dans [20, 35] les arguments d'homotopie se fondent sur la déformation géométrique réelle des coefficients w_Γ. En regardant les algèbres de Lie comme des paires symétriques, on construit ici une déformation polynomiale des coefficients, ce qui est bien meilleur. On peut donc espérer que notre déformation est rationnelle.

References

1. Alekseev, A., Meinrenken, E.: On the Kashiwara-Vergne conjecture. Inv. Math. **164**(3), 615–634 (2006)
2. Alekseev, A., Torossian, C.: Kashiwara-Vergne conjecture and Drinfeld's associators. arXiv: 0802.4300
3. Andler, M., Dvorsky, A., Sahi, S.: Kontsevich quantization and invariant distributions on Lie groups. Ann. Sci. École Norm. Sup. (4) **35**(3), 371–390 (2002)
4. Andler, M., Sahi, S., Torossian, C., Convolution of invariant distributions: proof of the Kashiwara-Vergne conjecture. Lett. Math. Phys. **69**, 177–203 (2004)
5. Baklouti, A., Fujiwara, H.: Commutativité des opérateurs différentiels sur l'espace des représentations restreintes d'un groupe de Lie nilpotent. J. Pure Math. Appl. (9) **83**(1), 137–161 (2004)
6. Baklouti, A., Ludwig, J.: Invariant differential operators on certain nilpotent homogeneous spaces. Monatsh. Math. **134**(1), 19–37 (2001)
7. Batakidis, P.: Phd-Thesis. Univ. Paris 7 (2009)
8. Cattaneo, A.S., Felder, G.: Coisotropic submanifolds in Poisson geometry and branes in the Poisson sigma model. Lett. Math. Phys. **69**, 157–175 (2004)
9. Cattaneo, A.S., Felder, G.: Relative formality theorem and quantisation of coisotropic submanifolds. Adv. Math. **208**, 521–548 (2007)
10. Cattaneo, A.S., Torossian, C.: Quantification pour les paires symétriques et diagrammes de Kontsevich. arXiv:math/0609693, Ann. Sci. École Norm. Sup. **41**, 787–852 (2008)
11. Cattaneo, A.S., Keller, B., Torossian, C., Bruguières, A.: Déformation, quantification, théorie de Lie. Collection Panoramas et Synthèses no. **20** SMF (2005)
12. Corwin, L.J., Greenleaf, F.P.: Commutativity of invariant differential operators on nilpotent homogeneous spaces with finite multiplicity. Comm. Pure Appl. Math. **45**(6), 681–748 (1992)
13. Duflo, M.: Opérateurs différentiels bi-invariants sur un groupe de Lie. Ann. Sci. École Norm. Sup. **10**, 107–144 (1977)
14. Duflo, M.: *Open problems in representation theory of Lie groups.* In: Oshima, T. (ed.) Conference on Analysis on homogeneous spaces. August 25–30, Kataka, Japan (1986)
15. Fujiwara, H., Lion, G., Magneron, B., Mehdi, S.: A commutativity criterion for certain algebras of invariant differential operators on nilpotent homogeneous spaces. Math. Ann. **327**(3), 513–544 (2003)
16. Gerstenhaber, M.: The cohomology structure of an associative ring. Ann. Math. (2) **78**, 267–288 (1963)
17. Kashiwara, M., Vergne, M.: The Campbell-Hausdorff formula and invariant hyperfunctions. Inv. Math. **47**, 249–272 (1978)
18. Kathotia, V.: Kontsevich's universal formula for deformation quantization and the Campbell-Baker-Hausdorff formula. Int. J. Math. **11**(4), 523–551 (2000)
19. Knop, F.: A Harish-Chandra homomorphism for reductive group actions. Ann. Math. **140**(2), 253–288 (1994)
20. Kontsevich, M.: Deformation quantization of Poisson manifolds, I. Preprint of the IHÉS, October 1997, q-alg/9709040 published in Lett. Math. Phys. **66**(3), 157–216 (2003)
21. Koornwinder, T.H.: Invariant differential operators on nonreductive homogeneous spaces. math.RT/0008116 (preprint 1981, not published)

22. Kostant, B.: Coadjoint orbits and a new symbol calculus for line bundles. Conference on differential geometric methods in theoretical physics (Trieste, 1981), 66–68. World Scientific, Singapore (1983)
23. Lipsman, R.L.: The Plancherel formula for homogeneous spaces with polynomial spectrum. Pac. J. Math. **159**(2):351–377 (1993)
24. Lipsman, R.L.: A unified approach to concrete Plancherel theory of homogeneous spaces. Manuscripta Math. **94**(2), 133–149 (1997)
25. Lipsman, R.L.: Finite multiplicity, invariant differential operators and the orbit method. Indiana Univ. Math. J. **46**(2), 561–574 (1997)
26. Nomura, T.: Algebraically independent generators of invariant differential operators on a bounded symmetric domain. J. Math. Kyoto Univ. **31**, 265–279 (1991)
27. Rouvière, F.: Espaces symétriques et méthode de Kashiwara-Vergne. Ann. Sci. École Norm. Sup. (4) **19**(4), 553–581 (1986)
28. Rouvière, F.: Invariant analysis and contractions of symmetric spaces. I. Compos. Math. **73**(3), 241–270 (1990)
29. Rouvière, F.: Invariant analysis and contractions of symmetric spaces. II. Compos. Math. **80**(2), 111–136 (1991)
30. Rouvière, F.: Fibrés en droites sur un espace symétrique et analyse invariante. J. Funct. Anal. **124**(2), 263–291 (1994)
31. Rybnikov, L.G.: Structure of the center of the algebra of invariant differential operators on certain Riemannian homogeneous spaces. Transform. Groups **9**(4), L381–L397 (2004)
32. Shimura, G.: Invariant differential operators on hermitian symmetric spaces. Ann. Math **132**(2), 237–272 (1991)
33. Stasheff, J.D.: Homotopy associativity of H-spaces. I, II. Trans. Am. Math. Soc. **108**, 275–292, 293–312 (1963)
34. Torossian, C.: Opérateurs différentiels invariants sur les espaces symétriques. I et II . J. Funct. Anal. **117**(1), 118–173 et 174–214 (1993)
35. Torossian, C.: Sur la conjecture combinatoire de Kashiwara-Vergne. J. Lie Theory **12**(2), 597–616 (2002)
36. Torossian, C.: Paires symétriques orthogonales et isomorphisme de Rouvière. J. Lie Theor. **15**(1), 79–87 (2005)
37. Torossian, C.: La conjecture de Kashiwara-Vergne [d'après Alekseev-Meinrenken], séminaire Bourbaki. Juin (2007)

Higher Homotopy Hopf Algebras Found: A Ten-Year Retrospective

Ronold N. Umble

Department of Mathematics, Millersville University of Pennsylvania, Millersville, PA 17551, USA, ron.umble@millersville.edu

To Murray Gerstenhaber and Jim Stasheff

Abstract. At the 1996 conference honoring Jim Stasheff in the year of his 60th birthday, I initiated the search for A_∞-bialgebras in a talk entitled "In Search of Higher Homotopy Hopf Algebras." The idea in that talk was to think of a DG bialgebra as some (unknown) higher homotopy structure with trivial higher order structure and apply a graded version of Gerstenhaber and Schack's bialgebra deformation theory. Indeed, deformation cohomology, which detects some (but not all) A_∞-bialgebra structure, motivated the definition given by S. Saneblidze and myself in 2004.

Key words: Associahedron, $A(n)$-algebra, A_∞-bialgebra, Bialgebra deformation, Biderivative, Deformation cohomology, Matrad, Operad, Permutahedron

AMS 2010 subject codes. 13D10, 55P48, 55P35

1 Introduction

In a preprint dated June 14, 2004, Samson Saneblidze and I announced the definition of A_∞-bialgebras [SU05], marking approximately six years of collaboration that continues to this day. Unknown to us at the time, A_∞-bialgebras are ubiquitous and fundamentally important. Indeed, over a field F, the bialgebra structure on the singular chains of a loop space ΩX pulls back along a quasi-isomorphism $g : H_*(\Omega X; F) \to C_*(\Omega X)$ to an A_∞-bialgebra structure on homology that is unique up to isomorphism [SU11].

Many have tried unsuccessfully to define A_∞-bialgebras. The illusive ingredient in the definition turned out to be an explicit diagonal Δ_P on the permutahedra $P = \sqcup_{n\geq 1} P_n$, the first construction of which was given by S. Saneblidze and myself in [SU04]. This paper is an account of the historical events leading up to the discovery of A_∞-bialgebras and the truly remarkable

A.S. Cattaneo et al. (eds.), *Higher Structures in Geometry and Physics*, Progress in Mathematics 287, DOI 10.1007/978-0-8176-4735-3_16,

role played by Δ_P in this regard. Although the ideas and examples presented here are quite simple, they represent and motivate general theory in [SU04], [SU05], [SU10], and [SU11].

Through their work in the theory of PROPs and the related area of infinity Lie bialgebras, many authors have contributed indirectly to this work, most notably M. Chas and D. Sullivan [CS04], J.-L. Loday [Lod08], M. Markl [Mar06], T. Pirashvili [Pir02], and B. Vallette [Val04]; for extensive bibliographies see [Sul07] and [Mar08].

Several new results spin off of this discussion and are included here: Example 1 in Section 3 introduces the first example of a bialgebra H endowed with an A_∞-algebra structure that is compatible with the comultiplication. Example 2 in Section 4 introduces the first example of a "nonoperadic" A_∞-bialgebra with a nontrivial operation $\omega^{2,2} : H^{\otimes 2} \to H^{\otimes 2}$. And in Section 5 we prove Theorem 1: *Given a DG bialgebra* (H, d, μ, Δ) *and a Gerstenhaber–Schack 2-cocycle* $\mu_1^n \in Hom^{2-n}(H^{\otimes n}, H)$, $n \geq 3$, *let* $H_0 = (H[[t]], d, \mu, \Delta)$. *Then* $(H[[t]], d, \mu, \Delta, t\mu_1^n)$ *is a linear deformation of* H_0 *as a Hopf* $A(n)$-*algebra.*

2 The historical context

Two papers with far-reaching consequences in algebra and topology appeared in 1963. In [Ger63] Murray Gerstenhaber introduced the deformation theory of associative algebras and in [Sta63] Jim Stasheff introduced the notion of an $A(n)$-algebra. Although the notion of what we now call a "non-Σ operad" appears in both papers, this connection went unnoticed until after Jim's visit to the University of Pennsylvania in 1983. Today, Gerstenhaber's deformation theory and Stasheff's higher homotopy algebras are fundamental tools in algebra, topology and physics. An extensive bibliography of applications appears in [MSS02].

By 1990, techniques from deformation theory and higher homotopy structures had been applied by many authors, myself included [Umb89], [LU92], to classify rational homotopy types with a fixed cohomology algebra. And it seemed reasonable to expect that rational homotopy types with a fixed Pontryagin algebra $H_*(\Omega X; \mathbb{Q})$ could be classified in a similar way. Presumably, such a theory would involve deformations of DG bialgebras (DGBs) as some higher homotopy structure with compatible A_∞-algebra and A_∞-coalgebra substructures, but the notion of compatibility was not immediately clear and an appropriate line of attack seemed illusive. But one thing was clear: If we apply a graded version of Gerstenhaber and Schack's (G-S) deformation theory [GS92], [LM91], [LM96], [Umb96] and deform a DGB H as some (unknown) higher homotopy structure, new operations $\omega^{j,i} : H^{\otimes i} \to H^{\otimes j}$ appear and their interactions with the deformed bialgebra operations are partially detected by the differentials. While this is but one small piece of a very large puzzle, it gave us a clue.

During the conference honoring Jim Stasheff in the year of his 60th birthday, held at Vassar College in June 1996, I discussed this particular clue in a talk entitled "*In Search of Higher Homotopy Hopf Algebras*" ([McC98] p. xii). Although G-S deformations of DGBs are less constrained than the A_∞-bialgebras known today, they motivated the definition announced eight years later.

Following the Vassar conference, forward progress halted. Questions of structural compatibility seemed mysterious and inaccessible. Then in 1998, Jim Stasheff ran across some related work by S. Saneblidze [San96], of the A. Razmadze Mathematical Institute in Tbilisi, and suggested that I get in touch with him. Thus began our long and fruitful collaboration. Over the months that followed, Saneblidze applied techniques of homological perturbation theory to solve the aforementioned classification problem [San99], but the higher order structure in the limit is implicit and the structure relations are inaccessible. In retrospect, this is not surprising as explicit structure relations require explicit combinatorial diagonals Δ_P on the permutahedra $P = \sqcup_{n\geq 1} P_n$ and Δ_K on the associahedra $K = \sqcup_{n\geq 2} K_n$. But such diagonals are difficult to construct and were unknown to us at the time. Indeed, one defines the tensor product of A_∞-algebras in terms of Δ_K, and the search for a construction of Δ_K had remained a longstanding problem in the theory of operads. We announced our construction of Δ_K in 2000 [SU00]; our construction of Δ_P followed a year or two later (see [SU04]).

3 Two important roles for Δ_P

The diagonal Δ_P plays two fundamentally important roles in the theory of A_∞-bialgebras. First, one builds the structure relations from components of (co)free extensions of initial maps as higher (co)derivations with respect to Δ_P, and second, Δ_P specifies exactly which of these components to use.

To appreciate the first of these roles, recall the following definition given by Stasheff in his seminal work on A_∞-algebras in 1963 [Sta63]: Let A be a graded module, let $\{\mu^i \in Hom^{i-2}\left(A^{\otimes i}, A\right)\}_{n\geq 1}$ be an arbitrary family of maps, and let d be the cofree extension of $\Sigma\mu^i$ as a coderivation of the tensor coalgebra $T^c A$ (with a shift in dimension). Then (A, μ^i) is an A_∞-*algebra* if $d^2 = 0$; when this occurs, the universal complex $(T^c A, d)$ is called the *tilde-bar construction* and the structure relations in A are the homogeneous components of $d^2 = 0$. Similarly, let H be a graded module and let $\{\omega^{j,i} \in Hom^{3-i-j}\left(H^{\otimes i}, H^{\otimes j}\right)\}_{i,j\geq 1}$, be an arbitrary family of maps. When $\left(H, \omega^{j,i}\right)$ is an A_∞-bialgebra, the map $\omega = \Sigma\omega^{j,i}$ uniquely extends to its *biderivative* $d_\omega \in End\left(TH \oplus T(H^{\otimes 2}) \oplus \cdots\right)$, which is the sum of various (co)free extensions of various subfamilies of $\left\{\omega^{j,i}\right\}$ as Δ_P-(co)derivations ([SU05]). And indeed, the structure relations in H are the homogeneous components of $d_\omega^2 = 0$ with respect to an appropriate composition product.

To demonstrate the spirit of this, consider a free graded module H of finite type and an (arbitrary) map $\omega = \mu + \mu^3 + \Delta$ with components $\mu : H^{\otimes 2} \to H$, $\mu^3 : H^{\otimes 3} \to H$, and $\Delta : H \to H^{\otimes 2}$. Extend Δ as a coalgebra map $\overline{\Delta} : T^c H \to T^c \left(H^{\otimes 2}\right)$, extend $\mu + \mu^3$ as a coderivation $d : T^c H \to T^c H$, and extend μ as an algebra map $\overline{\mu} : T^a \left(H^{\otimes 2}\right) \to T^a H$. Finally, extend $(\mu \otimes 1)\, \mu$ and $(1 \otimes \mu)\, \mu$ as algebra maps $f, g : T^a \left(H^{\otimes 3}\right) \to T^a H$, and extend μ^3 as an (f, g)-derivation $\overline{\mu}^3 : T^a \left(H^{\otimes 3}\right) \to T^a H$. The components of the biderivative in

$$d + \overline{\mu} + \overline{\mu}^3 + \overline{\Delta} \in \bigoplus_{p,q,r,s \geq 1} Hom\left(\left(H^{\otimes p}\right)^{\otimes q}, \left(H^{\otimes r}\right)^{\otimes s}\right)$$

determine the structure relations. Let $\sigma_{r,s} : \left(H^{\otimes r}\right)^{\otimes s} \to \left(H^{\otimes s}\right)^{\otimes r}$ denote the canonical permutation of tensor factors and define a composition product $\circledcirc$ on homogeneous components A and B of $d + \overline{\mu} + \overline{\mu}^3 + \overline{\Delta}$ by

$$A \circledcirc B = \begin{cases} A \circ \sigma_{r,s} \circ B, & \text{if defined} \\ 0, & \text{otherwise.} \end{cases}$$

When $A \circledcirc B$ is defined, $\left(H^{\otimes r}\right)^{\otimes s}$ is the target of B, and $\left(H^{\otimes s}\right)^{\otimes r}$ is the source of A. Then $\left(H, \mu, \mu^3, \Delta\right)$ is an A_∞-infinity bialgebra if $d_\omega \circledcirc d_\omega = 0$. Note that $\Delta\mu$ and $(\mu \otimes \mu)\, \sigma_{2,2}\, (\Delta \otimes \Delta)$ are the homogeneous components of $d_\omega \circledcirc d_\omega$ in $Hom\left(H^{\otimes 2}, H^{\otimes 2}\right)$; consequently, $d_\omega \circledcirc d_\omega = 0$ implies the Hopf relation

$$\Delta\mu = (\mu \otimes \mu)\, \sigma_{2,2}\, (\Delta \otimes \Delta).$$

Now if (H, μ, Δ) is a bialgebra, the operations μ_t, μ_t^3, and Δ_t in a G-S deformation of H satisfy

$$\Delta_t \mu_t^3 = \left[\mu_t\, (\mu_t \otimes 1) \otimes \mu_t^3 + \mu_t^3 \otimes \mu_t\, (1 \otimes \mu_t)\right] \sigma_{2,3} \Delta_t^{\otimes 3} \tag{1}$$

and the homogeneous components of $d_\omega \circledcirc d_\omega = 0$ in $Hom\left(H^{\otimes 3}, H^{\otimes 2}\right)$ are exactly those in (1). So this is encouraging.

Recall that the permutahedron P_1 is a point 0 and P_2 is an interval 01. In these cases Δ_P agrees with the Alexander–Whitney diagonal on the simplex:

$$\Delta_P\,(0) = 0 \otimes 0 \quad \text{and} \quad \Delta_P\,(01) = 0 \otimes 01 + 01 \otimes 1.$$

If X is an n-dimensional cellular complex, let $C_*\,(X)$ denote the cellular chains of X. When X has a single top dimensional cell, we denote it by e^n. An A_∞-algebra structure $\{\mu^n\}_{n \geq 2}$ on H is encoded operadically by a family of chain maps

$$\{\xi : C_*\,(P_{n-1}) \to Hom\,(H^{\otimes n}, H)\},$$

which factor through the map $\theta : C_*\,(P_{n-1}) \to C_*\,(K_n)$ induced by cellular projection $P_{n-1} \to K_n$ given by A. Tonks [Ton97] and satisfy $\xi\left(e^{n-2}\right) = \mu^n$. The fact that

$$(\xi \otimes \xi)\, \Delta_P\left(e^0\right) = \mu \otimes \mu \text{ and}$$

$$(\xi \otimes \xi)\, \Delta_P\left(e^1\right) = \mu\,(\mu \otimes 1) \otimes \mu^3 + \mu^3 \otimes \mu\,(1 \otimes \mu)$$

are components of $\overline{\mu}$ and $\overline{\mu}^3$ suggests that we extend a given μ^n as a higher derivation $\overline{\mu}^n : T^a\left(H^{\otimes n}\right) \to T^a H$ with respect to Δ_P. Indeed, an A_∞-bialgebra of the form $\left(H, \Delta, \mu^n\right)_{n \geq 2}$ is defined in terms of the usual A_∞-algebra relations together with the relations

$$\Delta\mu^n = \left[(\xi \otimes \xi)\, \Delta_P\left(e^{n-2}\right)\right] \sigma_{2,n} \Delta^{\otimes n}, \tag{2}$$

which define the compatibility of μ^n and Δ.

Structure relations in more general A_∞-bialgebras of the form $\left(H, \Delta^m, \mu^n\right)_{m,n \geq 2}$ are similar in spirit and formulated in [Umb08]. Special cases of the form $\left(H, \Delta, \Delta^n, \mu\right)$ with a single Δ^n were studied by H.J. Baues in the case $n = 3$ [Bau98] and by A. Berciano and myself with $n \geq 3$ [BU10]. Indeed, if p is an odd prime and $n \geq 3$, these particular structures appear as tensor factors of the mod p homology of an Eilenberg–Mac Lane space of type $K(\mathbb{Z}, n)$.

Dually, A_∞-bialgebras $\left(H, \Delta, \mu, \mu^n\right)$ with a single μ^n have a coassociative comultiplication Δ, an associative multiplication μ, and $\xi \otimes \xi$ acts exclusively on the primitive terms of Δ_P for lacunary reasons, in which case relation (2) reduces to

$$\Delta\mu^n = \left(f_n \otimes \mu^n + \mu^n \otimes f_n\right) \sigma_{2,n} \Delta^{\otimes n}, \tag{3}$$

where $f_n = \mu\,(\mu \otimes 1) \cdots \left(\mu \otimes 1^{\otimes n-2}\right)$. The first example of this particular structure now follows.

Example 1. Let H be the primitively generated bialgebra $\Lambda\,(x, y)$ with $|x| = 1, |y| = 2$, and

$$\mu^n\left(x^{i_1} y^{p_1} | \cdots | x^{i_n} y^{p_n}\right) = \begin{cases} y^{p_1 + \cdots + p_n + 1}, & i_1 \cdots i_n = 1 \text{ and } p_k \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

One can easily check that H is an A_∞-algebra, and a straightforward calculation together with the identity

$$\binom{p_1 + \cdots + p_n + 1}{i} = \sum_{s_1 + \cdots + s_n = i-1} \binom{p_1}{s_1} \cdots \binom{p_n}{s_n} + \sum_{s_1 + \cdots + s_n = i} \binom{p_1}{s_1} \cdots \binom{p_n}{s_n}$$

verifies relation (3).

The second important role played by Δ_P is evident in A_∞-bialgebras in which $\omega^{n,m}$ is nontrivial for some $m, n > 1$. Just as an A_∞-algebra structure on H is encoded operadically, an A_∞-bialgebra structure on H is encoded *matradically* by a family of chain maps

$$\left\{\varepsilon : C_*\left(KK_{n,m}\right) \to Hom^{3-m-n}\left(H^{\otimes m}, H^{\otimes n}\right)\right\}$$

over contractible polytopes $KK = \sqcup_{m,n\geq 1} KK_{n,m}$, called *biassociahedra*, with single top dimensional cells e^{m+n-3} such that $\varepsilon\left(e^{m+n-3}\right) = \omega^{n,m}$. Note that $KK_{n,1} = KK_{1,n}$ is the associahedron K_n [SU10]. Let $M = \{M_{n,m} = Hom(H^{\otimes m}, H^{\otimes n})\}$ and let $\Theta = \{\theta^n_m = \omega^{n,m}\}$. The A_∞-*bialgebra matrad* $\mathcal{H}_\infty$ is realized by $C_*(KK)$ and is a proper submodule of the free PROP M generated by Θ. The *matrad product* γ on $\mathcal{H}_\infty$ is defined in terms of Δ_P, and a monomial α in the free PROP M is a component of a structure relation if and only if $\alpha \in \mathcal{H}_\infty$.

More precisely, in [Mar06] M. Markl defined the submodule S of *special elements* in PROP M whose additive generators are monomials α expressed as "elementary fractions"

$$\alpha = \frac{\alpha^{y_1}_p \cdots \alpha^{y_q}_p}{\alpha^q_{x_1} \cdots \alpha^q_{x_p}} \tag{4}$$

in which $\alpha^q_{x_i}$ and $\alpha^{y_j}_p$ are additive generators of S and the jth output of $\alpha^q_{x_i}$ is linked to the ith input of $\alpha^{y_j}_p$ (here juxtaposition denotes tensor product). Representing θ^n_m graphically as a double corolla (see Fig. 1), a general decomposable α is represented by a connected nonplanar graph in which the generators appear in order from left-to-right (see Fig. 2). The matrad $\mathcal{H}_\infty$ is a proper submodule of S and the matrad product γ agrees with the restriction of Markl's fraction product to $\mathcal{H}_\infty$.

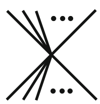
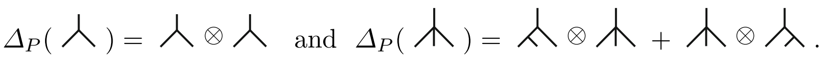
$\theta^n_m =$ 

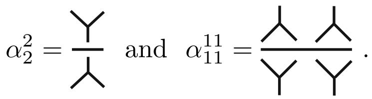
Fig. 1.

The diagonal Δ_P acts as a filter and admits certain elementary fractions as additive generators of $\mathcal{H}_\infty$. In dimensions 0 and 1, the diagonal Δ_P is expressed graphically in terms of up-rooted planar rooted trees (with levels) by

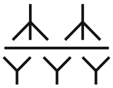


Define $\Delta^{(0)}_P = 1$; for each $k \geq 1$, define $\Delta^{(k)}_P = \left(\Delta_P \otimes 1^{\otimes k-1}\right)\Delta^{(k-1)}_P$ and view each component of $\Delta^{(k)}_P(\theta^1_q)$ as a $(q-2)$-dimensional subcomplex of $(P_{q-1})^{\times k+1}$, and similarly for $\Delta^{(k)}_P(\theta^q_1)$.

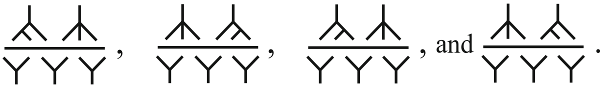
The elements θ^1_1, θ^1_2, and θ^2_1 generate two elementary fractions in $M_{2,2}$ each of dimension zero, namely,



Define $\partial\left(\theta_2^2\right) = \alpha_2^2 + \alpha_{11}^{11}$, and label the edge and vertices of the interval $KK_{2,2}$ by θ_2^2, α_2^2 and α_{11}^{11}, respectively. Continuing inductively, the elements θ_1^1, θ_2^1, θ_1^2, θ_2^2, α_2^2, and α_{11}^{11} generate 18 fractions in $M_{2,3}$ – one in dimension 2, nine in dimension 1, and eight in dimension 0. Of these, 14 label the edges and vertices of the heptagon $KK_{2,3}$. Since the generator θ_3^2 must label the 2-face, we wish to discard the 2-dimensional decomposable

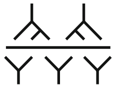
$e =$ 

and the appropriate components of its boundary. Note that e is a square whose boundary is the union of four edges



(5)

Of the five fractions pictured above, only the first two in (5) have numerators and denominators that are components of $\Delta_P^{(k)}(P)$ (numerators are components of $\Delta_P^{(1)}(\theta_3^1)$ and denominators are exactly $\Delta_P^{(2)}(\theta_1^2)$). Our selection rule admits only these two particular fractions, leaving seven 1-dimensional generators to label the edges of $KK_{2,3}$ (see Fig. 2). Now linearly extend the boundary map ∂ to the seven admissible 1-dimensional generators and compute the seven 0-dimensional generators labeling the vertices of $KK_{2,3}$. Since the 0-dimensional generator



is not among them, we discard it.

Subtleties notwithstanding, this process continues indefinitely and produces $\mathcal{H}_\infty$ as an explicit free resolution of the bialgebra matrad $\mathcal{H} = \langle\theta_1^1, \theta_2^1, \theta_1^2\rangle$ in the category of matrads. We note that in [Mar06], M. Markl makes arbitrary choices (independent of our selection rule) to construct the polytopes $B_m^n = KK_{n,m}$ for $m+n \leq 6$. In this range, it is enough to consider components of the diagonal Δ_K on the associahedra.

We conclude this section with a brief review of our diagonals Δ_P and Δ_K (up to sign); for details see [SU04]. Alternative constructions of Δ_K were subsequently given by Markl and Shnider [MS06] and J.-L. Loday [Lod10] (in this volume). Let $\underline{n} = \{1, 2, \ldots, n\}$, $n \geq 1$. A matrix E with entries from $\{0\} \cup \underline{n}$ is a *step matrix* if:

- Each element of $\underline{n}$ appears as an entry of E exactly once.
- The elements of $\underline{n}$ in each row and column of E form an increasing contiguous block.
- Each diagonal parallel to the main diagonal of E contains exactly one element of $\underline{n}$.

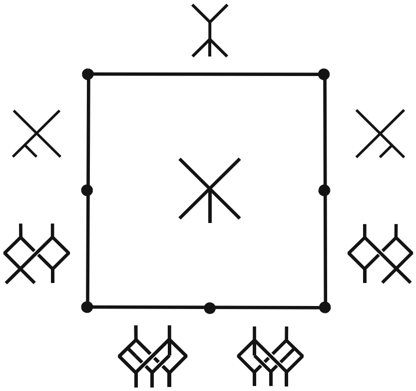


Fig. 2. The biassociahedron $KK_{2,3}$

Right-shift and down-shift matrix transformations, which include the identity (a trivial shift), act on step matrices and produce *derived matrices.*

Given a $q \times p$ integer matrix $M = (m_{ij})$, choose proper subsets $S_i \subset \{\text{non-zero entries in } row\,(i)\}$ and $T_j \subset \{\text{non-zero entries in } col\,(j)\}$, and define *down-shift* and *right-shift* operations D_{S_i} and R_{T_j} on M as follows:

- If $S_i \neq \varnothing$, $\max row(i+1) < \min S_i = m_{ij}$, and $m_{i+1,k} = 0$ for all $k \geq j$, then $D_{S_i}M$ is the matrix obtained from M by interchanging each $m_{ik} \in S_i$ with $m_{i+1,k}$; otherwise $D_{S_i}M = M$.
- If $T_j \neq \varnothing$, $\max col(j+1) < \min T_j = m_{ij}$, and $m_{k,j+1} = 0$ for all $k \geq i$, then $R_{T_j}M$ is the matrix obtained from M by interchanging each $m_{k,j} \in T_j$ with $m_{k,j+1}$; otherwise $R_{Tj}M = M$.

Given a $q \times p$ step matrix E together with subsets $S_1, \ldots, S_q$ and $T_1, \ldots, T_p$ as above, there is the *derived matrix*

$$R_{T_p} \cdots R_{T_2} R_{T_1} D_{S_q} \cdots D_{S_2} D_{S_1} E.$$

In particular, step matrices are derived matrices under the trivial action with $S_i = T_j = \varnothing$ for all i, j.

Let $a = A_1|A_2|\cdots|A_p$ and $b = B_q|B_{q-1}|\cdots|B_1$ be partitions of $\underline{n}$. The pair $a \times b$ is a (p,q)-*complementary pair* (CP) if B_i and A_j are the rows and columns of a $q \times p$ derived matrix. Since faces of P_n are indexed by partitions of $\underline{n}$, and CPs are in one-to-one correspondence with derived matrices, each CP is identified with some product face of $P_n \times P_n$.

Definition 1. *Define $\Delta_P(e^0) = e^0 \otimes e^0$. Inductively, having defined Δ_P on $C_*(P_{k+1})$ for all $0 \leq k \leq n-1$, define Δ_P on $C_n(P_{n+1})$ by*

$$\Delta_P(e^n) = \sum_{\substack{(p,q)\text{-}CPs\ u \times v \\ p+q=n+2}} \pm\, u \otimes v,$$

and extend multiplicatively to all of $C_*(P_{n+1})$.

The diagonal $\varDelta_P$ induces a diagonal $\varDelta_K$ on $C_*(K)$. Recall that faces of P_n in codimension k are indexed by planar rooted trees with $n+1$ leaves and $k+1$ levels (PLTs), and forgetting levels defines the cellular projection $\theta : P_n \to K_{n+1}$ given by A. Tonks [Ton97]. Thus faces of P_n indexed by PLTs with multiple nodes in the same level degenerate under θ, and corresponding generators lie in the kernel of the induced map $\theta : C_*(P_n) \to C_*(K_{n+1})$. The diagonal $\varDelta_K$ is given by $\varDelta_K\theta = (\theta \otimes \theta)\varDelta_P$.

4 Deformations of DG bialgebras as $A(n)$-bialgebras

The discussion above provides the context to appreciate the extent to which G-S deformation theory motivates the notion of an A_∞-bialgebra. We describe this motivation in this section. In retrospect, the bi(co)module structure encoded in the G-S differentials controls some (but not all) of the A_∞-bialgebra structure relations. For example, all structure relations in A_∞-bialgebras of the form $(H, d, \mu, \varDelta, \mu^n)$ are controlled except

$$\sum_{i=0}^{n-1} (-1)^{i(n+1)} \mu^n \left(1^{\otimes i} \otimes \mu^n \otimes 1^{\otimes n-i-1}\right) = 0, \tag{6}$$

which measures the interaction of μ^n with itself. Nevertheless, such structures admit an $A(n)$-algebra substructure and their single higher order operation μ^n is compatible with $\varDelta$. Thus we refer to such structures here as *Hopf $A(n)$-algebras.* General G-S deformations of DGBs, referred to here as *quasi-$A(n)$-bialgebras,* are "partial" $A(n)$-bialgebras in the sense that all structure relations involving multiple higher order operations are out of control.

4.1 $A(n)$-algebras and their duals

The signs in the following definition were given in [SU04] and differ from those given by Stasheff in [Sta63]. We note that either choice of signs induces an oriented combinatorial structure on the associahedra, and these structures are equivalent. Let $n \in \mathbb{N} \cup \{\infty\}$.

Definition 2. *An $A(n)$-**algebra** is a graded module A together with structure maps $\{\mu^k \in Hom^{2-k}\left(A^{\otimes k}, A\right)\}_{1 \leq k < n+1}$ that satisfy the relations*

$$\sum_{j=0}^{k-1} \sum_{i=0}^{k-j-1} (-1)^{j(i+1)} \mu^{k-j} \left(1^{\otimes i} \otimes \mu^{j+1} \otimes 1^{\otimes k-j-1-i}\right) = 0$$

*for each $k < n+1$. Dually, an $A(n)$-***coalgebra** *is a graded module C together with structure maps $\{\Delta^k \in Hom^{2-k}(C, C^{\otimes j})\}_{1\le k<n+1}$ that satisfy the relations*

$$\sum_{j=0}^{k-1} \sum_{i=0}^{k-j-1} (-1)^{j(k+i+1)} \left(1^{\otimes i} \otimes \Delta^{j+1} \otimes 1^{\otimes k-j-1-i}\right) \Delta^{k-j} = 0$$

for each $k < n+1$.

An $A(n)$-algebra is *strict* if $n < \infty$ and $\mu^n = 0$. A *simple* $A(n)$-algebra is a strict $A(n+1)$-algebra of the form (A, d, μ, μ^n); in particular, a simple $A(3)$-algebra is a strict $A(4)$-algebra in which:

(i) d is both a differential and a derivation of μ,
(ii) μ is homotopy associative and μ^3 is an associating homotopy:

$$d\mu^3 + \mu^3 (d \otimes 1 \otimes 1 + 1 \otimes d \otimes 1 + 1 \otimes 1 \otimes d) = \mu(\mu \otimes 1) - \mu(1 \otimes \mu),$$

(iii) μ and μ^3 satisfy a strict pentagon condition:

$$\mu^3 (\mu \otimes 1 \otimes 1 - 1 \otimes \mu \otimes 1 + 1 \otimes 1 \otimes \mu) = \mu\left(1 \otimes \mu^3 + \mu^3 \otimes 1\right).$$

4.2 Deformations of DG bialgebras

In [GS92], M. Gerstenhaber and S. D. Schack defined the cohomology of an *ungraded* bialgebra by joining the dual cohomology theories of G. Hochschild [HKR62] and P. Cartier [Car55]. This construction was given independently by A. Lazarev and M. Movshev in [LM91]. The *G-S cohomology of H* reviewed here is a straightforward extension to the graded case and was constructed in [LM96] and [Umb96].

Let (H, d, μ, Δ) be a connected DGB. We assume $|d| = 1$, although one could assume $|d| = -1$ equally well. For detailed derivations of the formulas that follow see [Umb96]. For each $i \geq 1$, the *i-fold bicomodule tensor power of H* is the H-bicomodule $H^{\underline{\otimes} i} = (H^{\otimes i}, \lambda_i, \rho_i)$ with left and right coactions given by

$$\lambda_i = \left[\mu(\mu \otimes 1) \cdots (\mu \otimes 1^{\otimes i-2}) \otimes 1^{\otimes i}\right] \sigma_{2,i} \Delta^{\otimes i} \quad \text{and}$$

$$\rho_i = \left[1^{\otimes i} \otimes \mu(1 \otimes \mu) \cdots (1^{\otimes i-2} \otimes \mu)\right] \sigma_{2,i} \Delta^{\otimes i}.$$

When $f : H^{\otimes i} \to H^{\otimes *}$, there is the composition

$$(1 \otimes f) \lambda_i = \left[\mu(\mu \otimes 1) \cdots (\mu \otimes 1^{\otimes i-2}) \otimes f\right] \sigma_{2,i} \Delta^{\otimes i}.$$

Dually, for each $j \geq 1$, the *j-fold bimodule tensor power of H* is the H-bimodule
$H^{\overline{\otimes} j} = (H^{\otimes j}, \lambda^j, \rho^j)$ with left and right actions given by

$$\lambda^j = \mu^{\otimes j}\sigma_{j,2}\left[(\Delta \otimes 1^{\otimes j-2}) \cdots (\Delta \otimes 1)\, \Delta \otimes 1^{\otimes j}\right] \text{ and}$$
$$\rho^j = \mu^{\otimes j}\sigma_{j,2}\left[1^{\otimes j} \otimes (1^{\otimes j-2} \otimes \Delta) \cdots (1 \otimes \Delta)\, \Delta\right].$$

When $g : H^{\otimes *} \to H^{\otimes j}$, there is the composition

$$\lambda^j\,(1 \otimes g) = \mu^{\otimes j}\sigma_{j,2}\left[(\Delta \otimes 1^{\otimes j-2}) \cdots (\Delta \otimes 1)\, \Delta \otimes g\right].$$

Let $\mathbf{k}$ be a field. Extend d, μ, and Δ to $\mathbf{k}\,[[t]]$-linear maps and obtain a $\mathbf{k}\,[[t]]$-DGB $H_0 = (H\,[[t]]\,, d, \mu, \Delta)$. We wish to deform H_0 as an $A\,(n)$-structure of the form

$$H_t = \left(H\,[[t]]\,, d_t = \omega_t^{1,1}, \mu_t = \omega_t^{1,2}, \Delta_t = \omega_t^{2,1}, \omega_t^{j,i}\right)_{i+j=n+1} \quad \text{where}$$

$$\omega_t^{j,i} = \sum_{k=0}^{\infty} t^k \omega_k^{j,i} \in Hom^{3-i-j}\left(H^{\underline{\otimes} i}, H^{\overline{\otimes} j}\right),$$

$$\omega_0^{1,1} = d,\ \omega_0^{1,2} = \mu,\ \omega_0^{2,1} = \Delta,\ \texttt{and}\ \omega_0^{j,i} = 0.$$

Deformations of H_0 are controlled by the *G-S n-complex,* which we now review. For $k \geq 1$, let

$$d_{(k)} = \sum_{i=0}^{k-1} 1^{\otimes i} \otimes d \otimes 1^{\otimes k-i-1}$$

$$\partial_{(k)} = \sum_{i=0}^{k-1} (-1)^i 1^{\otimes i} \otimes \mu \otimes 1^{\otimes k-i-1}$$

$$\delta_{(k)} = \sum_{i=0}^{k-1} (-1)^i 1^{\otimes i} \otimes \Delta \otimes 1^{\otimes k-i-1}.$$

These differentials induce strictly commuting differentials d, ∂, and δ on the trigraded module $\{Hom^p(H^{\underline{\otimes} i}, H^{\overline{\otimes} j})\}$, which act on an element f in tridegree (p, i, j) by

$$d(f) = d_{(j)} f - (-1)^p f d_{(i)}$$

$$\partial(f) = \lambda^j (1 \otimes f) - f \partial_{(i)} - (-1)^i \rho^j (f \otimes 1)$$

$$\delta(f) = (1 \otimes f)\lambda_i - \delta_{(j)} f - (-1)^j (f \otimes 1)\rho_i.$$

The submodule of *total G-S r-cochains on H* is

$$C^r_{GS}(H, H) = \bigoplus_{p+i+j=r+1} Hom^p(H^{\underline{\otimes} i}, H^{\overline{\otimes} j})$$

and the total differential D on a cochain f in tridegree (p,i,j) is given by

$$D(f) = \left[(-1)^{i+j} d + \partial + (-1)^i \delta\right](f),$$

where the sign coefficients are chosen so that (1) $D^2 = 0$, (2) structure relations (ii) and (iii) in Definition 2 hold, and (3) the restriction of D to the submodule of r-cochains in degree $p = 0$ agrees with the total (ungraded) G-S differential. The *G-S cohomology of H with coefficients in H* is given by

$$H^*_{GS}(H,H) = H_*\left\{C^r_{GS}(H,H), D\right\}.$$

Identify $Hom^p(H^{\underline{\otimes} i}, H^{\overline{\otimes} j})$ with the point (p,i,j) in $\mathbb{R}^3$. Then the *G-S n-complex* is that portion of the G-S complex in the region $x \geq 2-n$ and the submodule of total r-cochains in the n-complex is

$$C^r_{GS}(H,H;n) = \bigoplus_{p=r-i-j+1\geq 2-n} Hom^p(H^{\underline{\otimes} i}, H^{\overline{\otimes} j})$$

(a 2-cocycle in the 3-complex appears in Fig. 3). The *G-S n-cohomology of H with coefficients in H* is given by

$$H^*_{GS}(H,H;n) = H_*\left\{C^r_{GS}(H,H;n); D\right\}.$$

Note that a general 2-cocycle α has a component of tridegree $(3-i-j,i,j)$ for each i and j in the range $2 \leq i+j \leq n+1$. Thus α has $n(n+1)/2$ components and a standard result in deformation theory tells us that the homogeneous components of α determine an *infinitesimal deformation*, i.e., the component $\omega_1^{j,i}$ in tridegree $(3-i-j,i,j)$ defines the first-order approximation $\omega_0^{j,i} + t\omega_1^{j,i}$ of the structure map $\omega_t^{j,i}$ in H_t.

For simplicity, consider the case $n = 3$. Each of the ten homogeneous components of the deformation equation $D(\alpha) = 0$ produces the infinitesimal

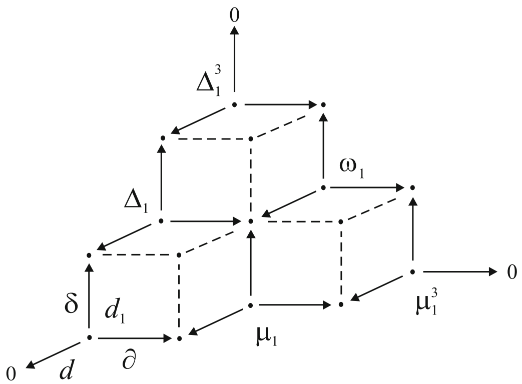


Fig. 3. The 2-cocycle $d_1 + \mu_1 + \Delta_1 + \mu_1^3 + \omega_1 + \Delta_1^3$

form of one structure relation (see below). In particular, a deformation H_t with structure maps $\{\omega_t^{1,i}\}_{1\le i\le 3}$ is a simple $A(3)$-algebra and a deformation H_t with structure maps $\{\omega_t^{j,1}\}_{1\le j\le 3}$ is a simple $A(3)$-coalgebra.

For notational simplicity, let $\mu_t^3 = \omega_t^{1,3}$, $\omega_t = \omega_t^{2,2}$, and $\Delta_t^3 = \omega^{3,1}$, and consider a deformation of (H, d, μ, Δ) as a "quasi-$A(3)$-structure." Then

- $d_t = d + td_1 + t^2 d_2 + \cdots$
- $\mu_t = \mu + t\mu_1 + t^2\mu_2 + \cdots$
- $\Delta_t = \Delta + t\Delta_1 + t^2\Delta_2 + \cdots$
- $\mu_t^3 = t\mu_1^3 + t^2\mu_2^3 + \cdots$
- $\omega_t = t\omega_1 + t^2\omega_2 + \cdots$
- $\Delta_t^3 = t\Delta_1^3 + t^2\Delta_2^3 + \cdots$

and $d_1 + \mu_1 + \Delta_1 + \mu_1^3 + \omega_1 + \Delta_1^3$ is a total 2-cocycle (see Fig. 3). Equating coefficients in $D\left(d_1 + \mu_1 + \Delta_1 + \mu_1^3 + \omega_1 + \Delta_1^3\right) = 0$ gives

1. $d(d_1) = 0$
2. $d(\mu_1) - \partial(d_1) = 0$
3. $d(\Delta_1) + \delta(d_1) = 0$
4. $d\left(\mu_1^3\right) + \partial(\mu_1) = 0$
5. $d\left(\Delta_1^3\right) - \delta(\Delta_1) = 0$
6. $\partial\left(\mu_1^3\right) = 0$
7. $\delta\left(\Delta_1^3\right) = 0$
8. $d(\omega_1) + \partial(\Delta_1) + \delta(\mu_1) = 0$
9. $\partial\left(\Delta_1^3\right) + \delta(\omega_1) = 0$
10. $\partial(\omega_1) - \delta\left(\mu_1^3\right) = 0.$

Requiring $\left(H, d_t, \mu_t, \mu_t^3\right)$ and $\left(H, d_t, \Delta_t, \Delta_t^3\right)$ to be simple $A(3)$-(co)algebras tells us that relations (1)–(7) are linearizations of Stasheff's strict $A(4)$-(co)algebra relations, and relation (8) is the linearization of the Hopf relation relaxed up to homotopy. Since μ_t, ω_t, and Δ_t have no terms of order zero, relations (9) and (10) are the respective linearizations of new relations (9) and (10) below. Thus we obtain the following structure relations in H_t:

1. $d_t^2 = 0$
2. $d_t\mu_t = \mu_t(d_t \otimes 1 + 1 \otimes d_t)$
3. $\Delta_t d_t = (d_t \otimes 1 + 1 \otimes d_t)\Delta_t$
4. $d_t\mu_t^3 + \mu_t^3(d_t\otimes 1\otimes 1 + 1\otimes d_t\otimes 1 + 1\otimes 1\otimes d_t) = \mu_t(1 \otimes \mu_t) - \mu_t(\mu_t \otimes 1)$
5. $(d_t\otimes 1\otimes 1 + 1\otimes d_t\otimes 1 + 1\otimes 1\otimes d_t)\Delta_t^3 + \Delta_t^3 d_t = (\Delta_t \otimes 1)\Delta_t - (1 \otimes \Delta_t)\Delta_t$
6. $\mu_t^3(\mu_t \otimes 1 \otimes 1 - 1 \otimes \mu_t \otimes 1 + 1 \otimes 1 \otimes \mu_t) = \mu_t\left(\mu_t^3 \otimes 1 + 1 \otimes \mu_t^3\right)$

7. $\left(\Delta_t \otimes 1 \otimes 1 - 1 \otimes \Delta_t \otimes 1 + 1 \otimes 1 \otimes \Delta_t\right) \Delta_t^3 = \left(\Delta_t^3 \otimes 1 + 1 \otimes \Delta_t^3\right) \Delta_t$

8. $\left(d_t \otimes 1 + 1 \otimes d_t\right) \omega_t + \omega_t \left(d_t \otimes 1 + 1 \otimes d_t\right) = \Delta_t \mu_t - \left(\mu_t \otimes \mu_t\right) \sigma_{2,2} \left(\Delta_t \otimes \Delta_t\right)$

9. $\left(\mu_t \otimes \omega_t\right) \sigma_{2,2} \left(\Delta_t \otimes \Delta_t\right) - \left(\Delta_t \otimes 1 - 1 \otimes \Delta_t\right) \omega_t - \left(\omega_t \otimes \mu_t\right) \sigma_{2,2} \left(\Delta_t \otimes \Delta_t\right)$

$$= \Delta_t^3 \mu_t - \mu_t^{\otimes 3} \sigma_{3,2} \left[\left(\Delta_t \otimes 1\right) \Delta_t \otimes \Delta_t^3 + \left(\Delta_t^3 \otimes \left(1 \otimes \Delta_t\right) \Delta_t\right)\right]$$

10. $\left(\mu_t \otimes \mu_t\right) \sigma_{2,2} \left(\Delta_t \otimes \omega_t\right) - \omega_t \left(\mu_t \otimes 1 - 1 \otimes \mu_t\right) - \left(\mu_t \otimes \mu_t\right) \sigma_{2,2} \left(\omega_t \otimes \Delta_t\right)$

$$= \left[\mu_t \left(\mu_t \otimes 1\right) \otimes \mu_t^3 + \mu_t^3 \otimes \mu_t \left(1 \otimes \mu_t\right)\right] \sigma_{2,3} \Delta_t^{\otimes 3} - \Delta_t \mu_t^3.$$

By dropping the formal deformation parameter t, we obtain the structure relations in a *quasi-simple* $A(3)$-bialgebra.

The first *nonoperadic* example of an A_∞-bialgebra appears here as a quasi-simple $A(3)$-bialgebra and involves a nontrivial operation $\omega = \omega^{2,2}$. The six additional relations satisfied by A_∞-bialgebras of this particular form will be verified in the next section.

Example 2. Let H be the primitively generated bialgebra $\Lambda(x, y)$ with $|x| = 1$, $|y| = 2$, trivial differential, and $\omega : H^{\otimes 2} \to H^{\otimes 2}$ given by

$$\omega(a|b) = \begin{cases} x|y + y|x, & a|b = y|y \\ x|x, & a|b \in \{x|y, y|x\} \\ 0, & \text{otherwise.} \end{cases}$$

Then $(\Delta \otimes 1 - 1 \otimes \Delta)\, \omega(y|y) = (\Delta \otimes 1 - 1 \otimes \Delta)(x|y + y|x) = 1|x|y + 1|y|x - x|y|1 - y|x|1 = (\mu \otimes \omega - \omega \otimes \mu)(1|1|y|y + y|1|1|y + 1|y|y|1 + y|y|1|1) = (\mu \otimes \omega - \omega \otimes \mu)\, \sigma_{2,2} (\Delta \otimes \Delta)(y|y)$; similar calculations show agreement on $x|y$ and $y|x$ and verifies relation (9). To verify relation (10), note that $\omega(\mu \otimes 1 - 1 \otimes \mu)$ and $(\mu \otimes \mu)\, \sigma_{2,2} (\Delta \otimes \omega - \omega \otimes \Delta)$ are supported on the subspace spanned by

$$B = \{1|y|y, y|y|1, 1|x|y, x|y|1, 1|y|x, y|x|1\},$$

and it is easy to check agreement on B. Finally, note that (H, μ, Δ, ω) can be realized as the linear deformation $\left.(H[[t]], \mu, \Delta, t\omega)\right|_{t=1}$.

5 A_∞-bialgebras in perspective

Although G-S deformation cohomology motivates the notion of an A_∞-bialgebra, G-S deformations of DGBs are less constrained than A_∞-bialgebras and fall short of the mark. To indicate the extent of this shortfall, let us identify those structure relations that fail to appear via deformation cohomology but must be verified to assert that Example 2 is an A_∞-bialgebra.

As mentioned above, structure relations in a general A_∞-bialgebra arise from the homogeneous components of the equation $d_\omega \circledcirc d_\omega = 0$. So to begin, let us construct the components of the biderivative d_ω that determine the structure relations in an A_∞-bialgebra of the form $\left(H, d, \mu, \Delta, \omega^{2,2}\right)$. Given arbitrary maps $d = \omega^{1,1}$, $\mu = \omega^{1,2}$, $\Delta = \omega^{2,1}$, and $\omega^{2,2}$ with $\omega^{j,i} \in Hom^{3-i-j}\left(H^{\otimes i}, H^{\otimes j}\right)$, consider $\omega = \sum \omega^{j,i}$. (Co)freely extend

- d as a linear map $\left(H^{\otimes p}\right)^{\otimes q} \to \left(H^{\otimes p}\right)^{\otimes q}$ for each $p, q \geq 1$,
- $d + \Delta$ as a derivation of $T^a H$,
- $d + \mu$ as a coderivation of $T^c H$,
- $\Delta + \omega^{2,2}$ as a coalgebra map $T^c H \to T^c\left(H^{\otimes 2}\right)$, and
- $\mu + \omega^{2,2}$ as an algebra map $T^a\left(H^{\otimes 2}\right) \to T^a H$.

Note that in this restricted setting, relation (10) in Definition 2 reduces to

$$(\mu \otimes \mu) \circledcirc \left(\Delta \otimes \omega^{2,2} - \omega^{2,2} \otimes \Delta\right) = \omega^{2,2} \circledcirc (\mu \otimes 1 - 1 \otimes \mu).$$

Factors $\mu \otimes 1$ and $1 \otimes \mu$ are components of $\overline{d + \mu}$; factors $\Delta \otimes \omega^{2,2}$ and $\omega^{2,2} \otimes \Delta$ are components of $\overline{\Delta + \omega^{2,2}}$; and the factor $\mu \otimes \mu$ is a component of $\overline{\mu + \omega^{2,2}}$.

To picture this, identify the isomorphic modules $\left(H^{\otimes p}\right)^{\otimes q} \approx \left(H^{\otimes q}\right)^{\otimes p}$ with the point $(p, q) \in \mathbb{N}^2$ and picture the initial map $\omega^{j,i} : H^{\otimes i} \to H^{\otimes j}$ as a "transgressive" arrow from $(i, 1)$ to $(1, j)$ (see Fig. 4).

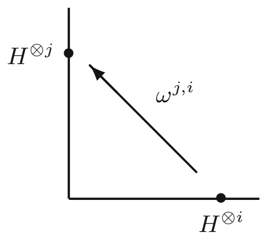


Fig. 4. The initial map $\omega^{j,i}$

Components of the various (co)free extensions above are pictured as arrows that initiate or terminate on the axes. For example, the vertical arrow $\Delta \otimes \Delta$, the short left-leaning arrow $\Delta \otimes \omega^{2,2} - \omega^{2,2} \otimes \Delta$, and the long left-leaning arrow $\omega^{2,2} \otimes \omega^{2,2}$ in Fig. 5 represent components of $\overline{\Delta + \omega^{2,2}}$.

Since we are only interested in transgressive quadratic $\circledcirc$-compositions, it is sufficient to consider the components of d_ω pictured in Fig. 5. Quadratic compositions along the x-axis correspond to relations (1), (2), (4), and (6) in Definition 2; those in the square with its diagonal correspond to relation (8); those in the vertical parallelogram correspond to relation (9); and those in the horizontal parallelogram correspond to relation (10).

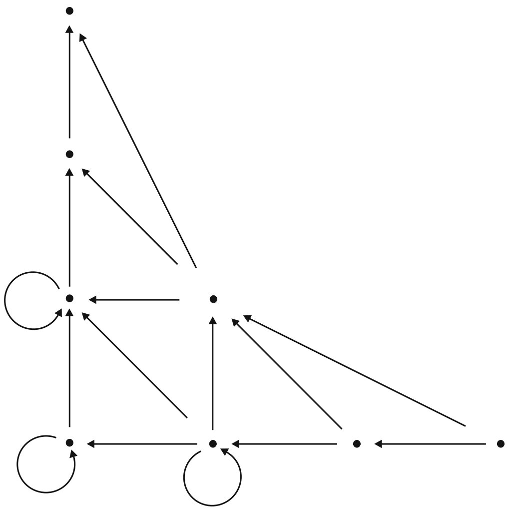


Fig. 5. Components of d_ω when $\omega = d + \mu + \Delta + \omega^{2,2}$

The following six additional relations are not detected by deformation cohomology because the differentials only detect the interactions between ω and (deformations of) d, μ, and Δ induced by the underlying bi(co)module structure:

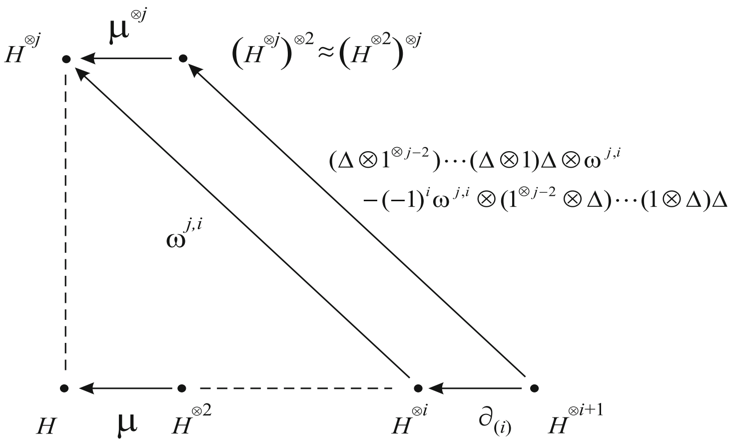
11. $(\mu \otimes \omega - \omega \otimes \mu)\, \sigma_{2,2}\, (\Delta \otimes \omega - \omega \otimes \Delta) = 0$;
12. $(\mu \otimes \mu)\, \sigma_{2,2}\, (\omega \otimes \omega) = 0$;
13. $(\omega \otimes \omega)\, \sigma_{2,2}\, (\Delta \otimes \Delta) = 0$;
14. $(\mu \otimes \omega - \omega \otimes \mu)\, \sigma_{2,2}\, (\omega \otimes \omega) = 0$;
15. $(\omega \otimes \omega)\, \sigma_{2,2}\, (\Delta \otimes \omega - \omega \otimes \Delta) = 0$;
16. $(\omega \otimes \omega)\, \sigma_{2,2}\, (\omega \otimes \omega) = 0$.

Definition 3. *Let H be a* **k***-module together with a family of maps* $\{d = \omega^{1,1}, \mu = \omega^{1,2}, \Delta = \omega^{2,1}, \omega^{2,2}\}$, *where* $\omega^{j,i} \in Hom^{3-i-j}\left(H^{\otimes i}, H^{\otimes j}\right)$, *and let* $\omega = \sum \omega^{j,i}$. *Then* $\left(H, d, \mu, \Delta, \omega^{2,2}\right)$ *is an* A_∞-**bialgebra** *if* $d_\omega \circledcirc d_\omega = 0$.

Example 3. Continuing Example 2, verification of relations (11)–(16) above is straightforward and follows from the fact that $\sigma_{2,2}\,(y|x|x|y) = -y|x|x|y$. Thus (H, μ, Δ, ω) is an A_∞-bialgebra with nonoperadic structure.

Let H be a graded module and let $\left\{\omega^{j,i} : H^{\otimes i} \to H^{\otimes j}\right\}_{i,j\geq 1}$ be an arbitrary family of maps. Given a diagonal Δ_P on the permutahedra and the notion of a Δ_P-(co)derivation, one continues the procedure described above to obtain the general biderivative defined in [SU05]. And as above, the general A_∞-bialgebra structure relations are the homogeneous components of $d_\omega \circledcirc d_\omega = 0$.

For example, consider an A_∞-bialgebra $(H, \mu, \Delta, \omega^{j,i})$ with exactly one higher order operation $\omega^{j,i}$, $i + j \geq 5$. When constructing d_ω, we extend μ as a coderivation, identify the components of this extension in $Hom\left(H^{\otimes i}, H^{\otimes j}\right)$ with the vertices of the permutahedron P_{i+j-2}, and identify $\omega^{j,i}$ with its top dimensional cell. Since μ, Δ, and $\omega^{j,i}$ are the only operations in H, all compositions involving these operations have degree 0 or $3 - i - j$, and k-faces of P_{i+j-2} in the range $0 < k < i + j - 3$ are identified with zero. Thus the extension of $\omega^{j,i}$ as a Δ_P-coderivation only involves the primitive terms of $\Delta(P_{i+j-2})$, and the components of this extension are terms in the expression $\delta\left(\omega^{j,i}\right)$. Indeed, whenever $\omega^{j,i}$ and its extension are compatible with the underlying DGB structure, the relation $\delta\left(\omega^{j,i}\right) = 0$ is satisfied. Dually, we have $\partial\left(\omega^{j,i}\right) = 0$ whenever $\omega^{j,i}$ and its extension as a Δ_P-derivation are compatible with the underlying DGB structure. These structure relations can be expressed as commutative diagrams in the integer lattice $\mathbb{N}^2$ (see Figs. 6 and 7).



$$\partial_{(i)} = \mu \otimes 1^{\otimes i-1} - 1 \otimes \mu \otimes 1^{\otimes i-2} + \cdots + (-1)^{i-1} 1^{\otimes i-1} \otimes \mu$$

Fig. 6. The structure relation $\partial\left(\omega^{j,i}\right) = 0$

Definition 4. *Let* $n \geq 3$. *A* **Hopf** $A(n)$**-algebra** *is a tuple* $(H, d, \mu, \Delta, \mu^n)$ *with the following properties:*

1. (H, d, Δ) *is a coassociative DGC;*
2. (H, d, μ, μ^n) *is an* $A(n)$*-algebra; and*
3. $\Delta\mu^n = [\mu(\mu \otimes 1) \cdots (\mu \otimes 1^{\otimes n-2}) \otimes \mu^n + \mu^n \otimes \mu(1 \otimes \mu) \cdots (1^{\otimes n-2} \otimes \mu)]\sigma_{2,n}\Delta^{\otimes n}$.

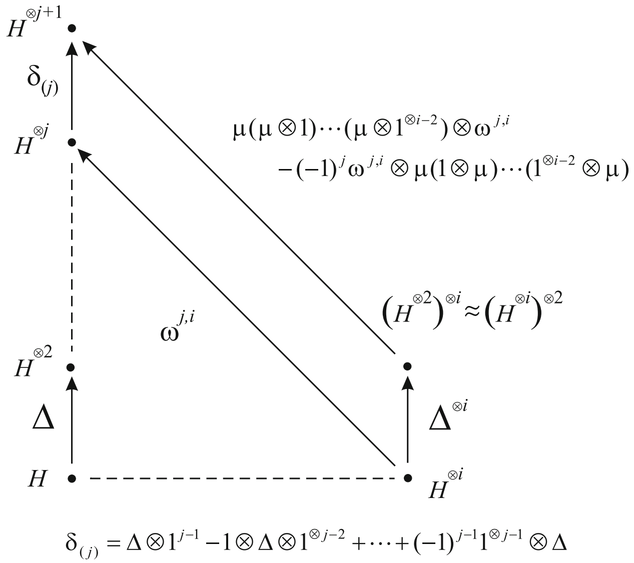


Fig. 7. The structure relation $\delta\left(\omega^{j,i}\right)=0$

A **Hopf** A_∞**-algebra** (H,d,μ,Δ,μ^n) *is a Hopf* $A(n)$*-algebra satisfying the relation in offset (6) above. There are the completely dual notions of a Hopf* $A(n)$*-coalgebra and a Hopf* A_∞*-coalgebra.*

Hopf A_∞-(co)algebras were defined by A. Berciano and this author in [BU10], but with a different choice of signs. A_∞-bialgebras with operations exclusively of the forms $\omega^{j,1}$ and $\omega^{1,i}$, called *special* A_∞*-bialgebras,* were considered by this author in [Umb08].

Hopf $A(n)$-algebras are especially interesting because their structure relations are controlled by G-S deformation cohomology. In fact, if $n\geq 3$ and $H_t=(H[[t]],d_t,\mu_t,\Delta_t,\mu_t^n)$ is a deformation, then $\mu_t^n=t\mu_1^n+t^2\mu_2^n+\cdots$ has no term of order zero. Consequently, if $D(\mu_1^n)=0$, then $t\mu_1^n$ automatically satisfies the required structure relations and $(H[[t]],d,\mu,\Delta,t\mu_1^n)$ is a *linear* deformation of H_0 as a Hopf $A(n)$-algebra. Thus we have proved:

Theorem 1. *If* (H,d,μ,Δ) *is a DGB and* $\mu_1^n\in Hom^{2-n}(H^{\otimes n},H)$, $n\geq 3$, *is a 2-cocycle, then* $(H[[t]],d,\mu,\Delta,t\mu_1^n)$ *is a linear deformation of* H_0 *as a Hopf* $A(n)$*-algebra.*

I am grateful to Samson Saneblidze and Andrey Lazarev for their helpful suggestions on early drafts of this paper, and to the referee, the editors, and Jim Stasheff for their assistance with the final draft. I wish to thank Murray and Jim for their encouragement and support of this project over the years and I wish them both much happiness and continued success.

References

[Bau98] Baues, H.J.: The cobar construction as a Hopf algebra and the Lie differential. Inv. Math. **132**, 467–489 (1998)

[BU10] Berciano, A., Umble, R.: Some naturally occurring examples of A_∞-bialgebras. J. Pure Appl. Algebra (2010 in press)

[Car55] Cartier, P.: Cohomologie des cogèbres. In: Séminaire Sophus Lie, Exposé 5 (1955–1956)

[CS04] Chas, M., Sullivan, D.: Closed string operators in topology leading to Lie bialgebras and higher string algebra. The legacy of Niels Henrik Abel, pp. 771–784. Springer, Berlin (2004)

[Ger63] Gerstenhaber, M.: The cohomology structure of an associative ring. Ann. Math. **78**(2), 267–288 (1963)

[GS92] Gerstenhaber, M., Schack, S.D.: Algebras, bialgebras, quantum groups, and algebraic deformations. In: Contemp. Math. 134. AMS, Providence, RI (1992)

[HKR62] Hochschild, G., Kostant, B., Rosenberg, A.: Differential forms on regular affine varieties. Trans. Am. Math. Soc. **102**, 383–408 (1962)

[LM91] Lazarev, A., Movshev, M.: Deformations of Hopf algebras. Russ. Math. Surv. (translated from Russian) **42**, 253–254 (1991)

[LM96] Lazarev, A., Movshev, M.: On the cohomology and deformations of diffrential graded algebras. J. Pure Appl. Algebra **106**, 141–151 (1996)

[Lod08] Loday, J.-L.: Generalized bialgebras and triples of operads. Astérisque No. 320, x–116 (2008)

[Lod10] Loday, J.-L.: The diagonal of the Stasheff polytope. International Conference in honor of Murray Gerstenhaber and Jim Stasheff (Paris 2007); this volume (2010)

[LU92] Lupton, G., Umble, R.: Rational homotopy types with the rational cohomology of stunted complex projective space. Canadian J. Math. **44**(6), 1241–1261 (1992)

[Mar06] Markl, M.: A resolution (minimal model) of the PROP for bialgebras. J. Pure Appl. Algebra **205**, 341–374 (2006)

[Mar08] Markl, M.: Operads and PROPs. Handbook of algebra, vol. 5, pp. 87–140. Elsevier/North-Holland, Amsterdam (2008)

[MS06] Markl, M., Shnider, S.: Associahedra, cellular W-construction and products of A_∞-algebras. Trans. Am. Math. Soc. **358**(6), 2353–2372 (2006)

[MSS02] Markl, M., Shnider, S., Stasheff, J.: Operads in Algebra, Topology and Physics. Mathematical Surveys and Monographs, 96. AMS, Providence, RI (2002)

[McC98] McCleary, J. (ed.): Higher Homotopy Structures in Topology and Mathematical Physics. Contemp. Math., 227. AMS, Providence, RI (1998)

[Pir02] Pirashvili, T.: On the PROP coresponding to bialgebras. Cah. Topol. Géom. Differ. Catég. **43**(3), 221–239 (2002)

[San96] Saneblidze, S.: The formula determining an A_∞-coalgebra structure on a free algebra. Bull. Georgian Acad. Sci. **154**, 351–352 (1996)

[San99] Saneblidze, S.: On the homotopy classification of spaces by the fixed loop space homology. Proc. A. Razmadze Math. Inst. **119**, 155–164 (1999)

[SU00] Saneblidze, S., Umble, R.: A diagonal on the associahedra (unpublished manuscript 2000). Preprint math.AT/0011065

[SU04] Saneblidze, S., Umble, R.: Diagonals on the permutahedra, multiplihedra and associahedra. J. Homol. Homotopy Appl. **6**(1), 363–411 (2004)

[SU05] Saneblidze, S., Umble, R.: The biderivative and A_∞-bialgebras. J. Homol. Homotopy Appl. **7**(2), 161–177 (2005)

[SU10] Saneblidze, S., Umble, R.: Matrads, biassociahedra and A_∞-bialgebras. J. Homol. Homotopy Appl (in press 2010). Preprint math.AT/0508017

[SU11] Saneblidze, S., Umble, R.: Morphisms of A_∞-bialgebras and Applications (unpublished manuscript)

[Sta63] Stasheff, J.: Homotopy associativity of H-spaces I, II. Trans. Am. Math. Soc. **108**, 275–312 (1963)

[Sul07] Sullivan, D.: String Topology: Background and Present State. Current developments in mathematics, 2005, 41–88, Int. Press, Somerville, MA (2007)

[Ton97] Tonks, A.: Relating the associahedron and the permutohedron. In: Operads: Proceedings of the Renaissance Conferences (Hartford CT / Luminy Fr 1995). Contemp. Math. **202**, 33–36 (1997)

[Umb89] Umble, R.: Homotopy conditions that determine rational homotopy type. J. Pure Appl. Algebra **60**, 205–217 (1989)

[Umb96] Umble, R.: The deformation complex for differential graded Hopf algebras. J. Pure Appl. Algebra **106**, 199–222 (1996)

[Umb08] Umble, R.: Structure relations in special A_∞-bialgebras. Sovremennya Matematika i Ee Prilozheniya (Contemporary Mathematics and its Applications), **43**, Topology and its Applications, 136–143 (2006). Translated from Russian in J. Math. Sci. **152**(3), 443–450 (2008)

[Val04] Vallette, B.: Koszul duality for PROPs. Trans. Am. Math. Soc. **359**, 4865–4943 (2007)