BIRKHÄUSER

Operator Theory
Advances and Applications
Vol. 202

Topics in Operator Theory

Volume 1:
Operators, Matrices and Analytic Functions

A tribute to Israel Gohberg
on the occasion of his 80th birthday

Joseph A. Ball
Vladimir Bolotnikov
J. William Helton
Leiba Rodman
Ilya M. Spitkovsky
Editors

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Founded in 1979 by Israel Gohberg

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Proceedings of the XIXth International Workshop on Operator Theory and its Applications, College of William and Mary, 2008

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2010 Mathematics Subject Classification: 15, 45, 46, 47, 93

Library of Congress Control Number: 2010920057

Bibliographic information published by Die Deutsche Bibliothek.
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie;
detailed bibliographic data is available in the Internet at http://dnb.ddb.de

ISBN 978-3-0346-0157-3

P.O. Box 133, CH-4010 Basel, Switzerland
Part of Springer Science+Business Media
Printed on acid-free paper produced from chlorine-free pulp. TCF∞
Printed in Germany

ISBN 978-3-0346-0157-3 Vol. 1 e-ISBN 978-3-0346-0158-0
ISBN 978-3-0346-0160-3 Vol. 2 e-ISBN 978-3-0346-0161-0
ISBN 978-3-0346-0163-4 Set

9 8 7 6 5 4 3 2 1 www.birkhauser.ch

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The XIXth International Workshop on Operator Theory and its Applications. I

Joseph A. Ball, Vladimir Bolotnikov, J. William Helton, Leiba Rodman and Ilya M. Spitkovsky

Abstract. Information about the workshop and comments about the first volume of proceedings is provided.

Mathematics Subject Classification (2000). 15-06, 47-06.

Keywords. Operator theory, matrix analysis, analytic functions.

The Nineteenth International Workshop on Operator Theory and its Applications – IWOTA 2008 – took place in Williamsburg, Virginia, on the campus of the College of William and Mary, from July 22 till July 26, 2008. It was held in conjunction with the 18th International Symposium on Mathematical Theory of Networks and Systems (MTNS) in Blacksburg, Virginia (Virginia Tech, July 28–August 1, 2008) and the 9th Workshop on Numerical Ranges and Numerical Radii (July 19–July 21, 2008) at the College of William and Mary. The organizing committee of IWOTA 2008 (Ball, Bolotnikov, Helton, Rodman, Spitkovsky) served also as editors of the proceedings.

IWOTA 2008 celebrated the work and career of Israel Gohberg on the occasion of his 80th birthday, which actually fell on August 23, 2008. We are pleased to present this volume as a tribute to Israel Gohberg.

IWOTA 2008 was a comprehensive, inclusive conference covering many aspects of theoretical and applied operator theory. More information about the workshop can be found on its web site

`http://www.math.wm.edu/~vladi/IWOTA/IWOTA2008.htm`

There were 241 participants at IWOTA 2008, representing 30 countries, including 29 students (almost exclusively graduate students), and 20 young researchers (those who received their doctoral degrees in the year 2003 or later). The scientific program included 17 plenary speakers and 7 invited speakers who gave overview of many topics related to operator theory. The special sessions covered

Israel Gohberg at IWOTA 2008, Williamsburg, Virginia

a broad range of topics: Matrix and operator inequalities; hypercomplex operator theory; the Kadison–Singer extension problem; interpolation problems; matrix completions; moment problems; factorizations; Wiener–Hopf and Fredholm operators; structured matrices; Bezoutians, resultants, inertia theorems and spectrum localization; applications of indefinite inner product spaces; linear operators and linear systems; multivariable operator theory; composition operators; matrix polynomials; indefinite linear algebra; direct and inverse scattering transforms for integrable systems; theory, computations, and applications of spectra of operators.

We gratefully acknowledge support of IWOTA 2008 by the National Science Foundation Grant 0757364, as well as by the individual grants of some organizers, and by various entities within the College of William and Mary: Department of Mathematics, the Office of the Dean of the Faculty of Arts and Sciences, the Office of the Vice Provost for Research, and the Reves Center for International Studies. One plenary speaker has been sponsored by the International Linear Algebra Society. The organization and running of IWOTA 2008 was helped tremendously by the Conference Services of the College of William and Mary.

The present volume is the first of two volumes of proceedings of IWOTA 2008. Here, papers on operator theory, linear algebra, and analytic functions are collected. The volume also contains a commemorative article of speeches and reminiscences dedicated to Israel Gohberg. All papers (except the commemorative article) are refereed. The second volume contains papers on systems, differential and difference equations, and mathematical physics.

August 2009

Added on December 14, 2009:

With deep sadness the editors' final act in preparing this volume is to record that Israel Gohberg passed away on October 12, 2009, aged 81. Gohberg was a great research mathematician, educator, and expositor. His visionary ideas inspired many, including the editors and quite a few contributors to the present volume.

Israel Gohberg was the driving force of iwota. He was the first and the only President of the Steering Committee. In iwota, just as in his other endeavors, Gohberg's charisma, warmth, judgement and stature lead to the lively community we have today.

He will be dearly missed.

The Editors: Joseph A. Ball, Vladimir Bolotnikov, J. William Helton, Leiba Rodman, Ilya M. Spitkovsky.

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Operator Theory:
Advances and Applications, Vol. 202, xiii–xxxviii

Speeches and Reminiscences

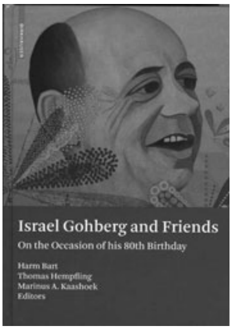
Abstract. This is a collection of speeches given by Israel Gohberg's colleagues and family during the banquet that took place on July 24, 2008 at the Sadler Center, the College of William and Mary, as part of the XIXth International Workshop on Operator Theory and its Applications. The speech by Dan Amir delivered on November 17, 2008 at the meeting of the School of Mathematical Sciences of Tel-Aviv University on the occasion of Israel Gohberg's 80th birthday is also included as well as a note by H. Baumgärtel. The texts by Gohberg's colleagues were revised and approved by speakers. The texts by Gohberg's family were submitted by Gohberg.

Mathematics Subject Classification (2000). 47-06.

Keywords. Israel Gohberg.

1. Presentation of book

Marinus A. Kaashoek



Dear Professor Gohberg, dear Israel, dear Mrs. Gohberg, dear Bella, dear members of the Gohberg family, dear guests, dear colleagues and friends.

Edited by Leiba Rodman.

I am speaking on behalf of Harm Bart and Thomas Hempfling. The three of us are the editors of the book *Gohberg and Friends*, which will be presented to Professor Gohberg shortly.[1]

As you know mathematicians stand a long and time honored tradition. They write papers and sometimes books, they read publications of fellow workers in the field, they meet other mathematicians at conferences all over the world and sometimes in Williamsburg. In this way, in contact with colleagues from far away and nearby, from the past via their writings and the present, mathematical results are obtained which are recognized as valid. In this process, some distinguished individuals play a special and striking role. They assume a position of leadership, they guide people working with them through uncharted territories, thereby making a lasting imprint on the field, something which can only be accomplished through a combination of rare talent, unusually broad knowledge, unfailing intuition, and a certain kind of charisma that binds people together. All this is present in Israel Gohberg, the man to whom this book is dedicated on the occasion of his 80th birthday.

The documents collected here give a fascinating and sometimes moving insight in the human factors that influence the development of mathematics. The focus is not on formal mathematical results but instead on the personal relationships that constitute the underlying propelling power of scientific cooperation. Centered around the remarkable figure of Israel Gohberg, a picture emerges of the development of operator theory and its applications during the last four or five decades.

The above is a quote from the preface, and you can see and hear what an excellent book it is. I want to tell a bit more about the contents of the book. It consists of seven parts, and I will read to you the titles of the parts, adding some additional information.

Part 1. *Mathematical and Philosophical-Mathematical Tales.*

This part begins with *Mathematical Tales* a presentation given by Israel Gohberg at the 1988 Calgary Conference organized to celebrate his 60th birthday. It contains stories from Gohberg's career in mathematics, mostly from the times when he lived in the Soviet Union before immigrating to Israel. The paper is preceded by an introduction by Ralph Phillips. The second contribution, *Philosophical-Mathematical Tales: A personal account*, is a talk given by Gohberg in January 2002 at the University of West Timişoara, where he was awarded the degree of honorary doctor. It contains reflections on the general nature of mathematics and on the way mathematical research is done. About the final article in this part I will speak a little later.

Part 2. *Work and Personalia.*

This part contains Gohberg's curriculum vitae, the list of his publications and a list of his Ph.D students. Also included are a translation of a letter of reference

[1]The present text is an expanded version of my speech given at the banquet. Much of the material is taken from the book.

written by M.G. Krein, when Gohberg was a master student, and translations of letters and telegrams supporting his nomination as a corresponding member of the Academy of Sciences of the Moldavian SSR. The next two documents, written by Rien Kaashoek and by Rien Kaashoek and Leonid Lerer, respectively, present a review of Gohberg's mathematical work. The final document concerns the Nathan and Lily Silver chair of Mathematics of which Israel Gohberg has been the incumbent from 1981 to 1998.

Part 3. *Gohberg Miscellania: Celebrating the 60th birthday in Calgary, Alberta, Canada.*

This part consists of the Gohberg Miscellanea, written on the occasion of his sixtieth birthday. This biographical text was composed by H. Dym, S. Goldberg, M.A. Kaashoek, and P. Lancaster from reminiscences, notes, letters and speeches prepared by Gohberg's former students, colleagues and friends.

Part 4. *Celebrating the 70th Birthday at the IWOTA meeting in Groningen, the Netherlands.*

This part contains the texts of the speeches given by Alek Markus, Hugo Woerdeman, Heinz Langer, Cora Sadosky, Hary Dym, Bill Helton, and Harm Bart at the conference dinner of the IWOTA meeting in Groningen, in the context of a pre-celebration of Israel Gohberg's 70th birthday later in 1998.

Part 5. *About Colleagues and Friends.*

This part presents a collection of sixteen articles that were written or coauthored by Israel Gohberg himself. Some of these have character of a memorial article, paying tribute to a dear colleague who has passed away. Others are recollections or reviews that highlight personality of a friend celebrating a special occasion. These documents together give a fascinating, and sometimes moving, insight into human factors that influenced the development of the field.

Part 6. *Honorary doctorates, laudatios, and replies.*

This part concerns the six honorary doctorates that Israel Gohberg has received. Corresponding documents such as laudatios, acceptance speeches, and other related material are presented here.

Part 7. *Festschrift 2008.*

This final part consists of material comparable to that of Parts 3 and 4, but then from a younger date and written especially for this occasion. In short articles, seventeen friends, colleagues, and co-authors reflect on their experience with Israel Gohberg. All of them have felt his influence. In some cases, it has changed their lives.

Who are the authors of the book *Israel Gohberg and Friends*? From the short description I gave you, you may guess, well, the authors of the book *Israel Gohberg and Friends* are Israel Gohberg and friends. This answer is almost correct. There are two authors who do not fit into these two categories. They are Zvia Faro-Gohberg and Yanina Israeli-Gohberg, the two daughters of Israel and Bella. They wrote a beautiful article which appears in the first part of the book under the title *Dad's Mathematics*. It is a fascinating account on how their father's mathematics came to them in their younger years. At the same time *Dad's Mathematics* gives an impression of Israel Gohberg's talent to convey the beauty of the field even to those lacking elaborate mathematical training. The two Gohberg daughters are present here at this banquet. I ask them to stand up so that we can see them and welcome them with a warm applause.

Dear Israel, my task is completed. I wish you many happy returns. Thomas Hempfling, the mathematics editor of Birkhäuser Verlag, will continue and present the book to you.

Thomas Hempfling

Do not worry, I will make it short. First of all, thanks for your marketing procedures, if you are out of business just apply so that we can do something together.

I would like to congratulate Israel. One reason obviously is that we celebrate his 80th birthday. The second reason is that he has continuous business with us for 30 years, which is really remarkable. And third, I did some computations yesterday afternoon, because I had a guess, concerning the total number of pages Israel is responsible for as an editor. Can you guess? It is close to 100,000 pages [applause], about 37,000 for the journal and more than 60,000 for the books. This is really something remarkable I think.

When the idea came up to do this special commemorative book, we thought that there should be one very special version just for Israel, and here it is. It is a bound version with silver shine on it. I think you deserve it. I congratulate you.

2. Gohberg's colleagues

Joseph A. Ball

My first contact with Israel was as a graduate student at the University of Virginia. There were the books by Gohberg and Krein from which operator theorists were supposed to study Operator Theory. Later we intersected at University of Maryland. He was settling down in a new place at West Hyattsville, Maryland, just a couple of blocks from where I grew up. So I showed him the place I grew up. He said, "Some people are local, but you are an aborigine". I had experience of working with Israel over a period of four or five years on a book, one of many books in which Israel took part. It felt like becoming part of the family. He leaves behind a large legacy in Operator Theory, and I congratulate Israel and wish him the best in the future. Thank you.

Ronald G. Douglas

I first met Israel at the 1966 International Mathematical Congress in Moscow. As far as I know, this is something that most of the people here do not realize, since they are young. If you go back to the 60's, there were two worlds of Operator

Theory: there was one world on one side of the iron curtain, and the other world on the other side of the iron curtain. There were occasionally letters that went back and forth, and a few visits, but not many. The Moscow congress provided people in the rest of the world, the United States, Europe, Australia, Japan and so forth, with an opportunity to actually meet these people that were creating so much of Operator Theory. Otherwise, we would have to learn from translations of books which would occur in one year, two years, three years, or may be never.

Among the people I met there was Israel. At the same time I met both Kreins, M.G. Krein and S.G. Krein, Arov, Adamyan, and I can just keep going. I certainly remember Israel standing out, I do not remember what we talked about but we talked. We both knew who each other was. The "official" translator between the two sides was Ciprian Foiaş. We had some rousing seminars where Ciprian would listen to the Russian and explain it to us in English, and then he would comment in English and then in Russian, and this went on and on and on. In any case, after that meeting I got a letter from Israel, and I also, in a bit of a surprise, started getting these large envelopes with Israel's reprints. And of course you heard a few days ago that there are more than 400. I do not know what the number was there but it was substantial. Of course I was very pleased to get those even though most of them were in Russian; I think all of them at this point were in Russian. I later found that, as one of the family indicated, Israel is methodical about planning and possibly planning for the best or the worst.

After Israel emigrated to Israel, one of his first visits to the United States was to Stony Brook. He came and spent many semesters there. In fact, he reminded me today that his crash course in English was because Stony Brook's Dean or Vice President or somebody told him, "We have money to hire you, but you have to lecture in English, and that classes started almost immediately". So he was telling me that his first month in Stony Brook was a lot of work, a lot of effort.

I'll just say two more things. I remember a couple of more visits, he came to Stony Brook many times around the seventies. But I remember one visit to Amsterdam. I will not go through the whole visit. His grandson talked about berries. The thing I remember, walking back from the Mathematics Department to the apartment, was that Israel spied wild mushrooms all over the place, mushrooms I have never seen before. He picked them up, gathered them, and took them to the apartment. His mother was there, and she cooked wonderful dishes I could not have imagined. I was sure I would be dead next morning. I was somewhat surprised when I woke up. The other comment I am going to make has also to do with Israel's mother. She was very important part of his visits. He brought her to Stony Brook on more than one occasion. My first visit to Israel was in 1977, arranged by Israel, and I came over to their apartment. I was there with my family. Israel's mother fixed a meal that, well, it seemed like it went on and on, and my children had to be excused because they could not even look at the food. It was an absolutely wonderful meal which I still remember. It is clear to me where Israel got his loving nature and optimism for life: that was from his mother. I can say more about mathematics, but probably I have said enough.

Roland Duduchava

Dear Colleagues.

I am very happy to call this outstanding personality my teacher. When I arrived in Kishinev, it was 1968, unexperienced young man not only in mathematics but also in life. I learned a lot from Israel Gohberg. He was a wonderful teacher in mathematics, and in life. When I left Kishinev three years later, I brought with me back to Georgia not only dissertation, but also wife and child. In the process of my marriage Bella Yakovlevna and Israil Tsudikovich played an essential role, and I am very thankful to them for this.

Harry Dym

Sayings of the Fathers is a short extract from the Talmud that includes a number of suggestions for virtuous living. I think that many of you who are here tonight share the sentiments expressed in the following transparency:

> Sayings of the Fathers:
>
> *Acquire a teacher*
> *and*
> *Acquire a friend*
>
> Thanks to Israel on his ~~75th~~
> 80th
> for being both.

As you can see, I have used this transparency on Israel's 75th birthday, I am using it again for his 80th birthday, and I would like to point out that there is a lot of space left for the future.

I do not want to take too much time. Israel's family is a hard act to follow, and there are many more speakers.

You all know about Israel's mathematical accomplishments. I would like instead to tell four stories that illustrate his other dimensions. I have told them before, but I rely on the fact that the audience is largely new, and the hope that the older members of the audience who have heard these stories before have forgotten them.

I met Israel for the first time in Spring of 1975, when he joined the Department of Mathematics at the Weizmann Institute as a part time member. Shortly thereafter we began to work together and continued to do so for close to ten years.

Israel used to spend a day and a half at the Institute. On Sundays he would usually arrive between 9 am and 9:30 am, we worked a little, he would lecture from 11:00 am to 1:00 pm, afterwards we would go for lunch, then work again. One day he arrived rather late. What happened? Well, on the way, while driving from Raanana (where he lives) to the Weizmann Institute, a car crashed into him from behind. I believe it was a police car. Anyway, after he settled in, Israel called Bella, his wife, and explained to her that he had an accident on the way to work, someone bumped into his car from behind, but there was no need to worry, he was not hurt.

Bella: "Why did he do that?"

You can imagine your response to that question. But Israel very calmly says, "Bellechka, this question you have to put to him not to me."

In between our working sessions we would talk about other things, and from time to time would drift into anecdotes. I would like to relate two stories that I heard from Israel from his Russian period, which illustrate Russian humor, maybe Jewish Russian humor, sort of bittersweet.

One story is about a baker. A man goes to a bakery and wants to buy a loaf of bread.

Baker: "Fine, it is two rubles."

Man: "Why is it two rubles? In the store down the street it is only one ruble."

Baker: "Ah, but that baker has no bread. If I had no bread, I would also sell it for one ruble."

Another story is about a collective farm where they raised chickens. An inspector came from the Central Committee, to see how the farm workers were doing. He goes to one worker and asks,

Inspector: "What do you feed your chickens?"

First worker: "Corn."

Inspector: "What! You feed your chickens corn? We do not have enough wheat or grain to feed our children! – Siberia!"

The inspector turns to a second worker and repeats the question.

Second worker: "Chocolate."

Inspector: "What! You feed your chickens chocolate? We do not have candy to feed our children! – Siberia!"

He then turns to a little old Jewish worker with the same question.

Worker: "I do not feed my chickens. I give them each a ruble, and they can buy what they want."

The final story is based on a conversation that took place when Israel met our youngest son Michael for the first time.

Israel: "How many children are there in your class?"

Michael: "Forty three,"

Israel: "Forty three? That's wonderful, so many friends."

Lillian Goldberg

I am pleased to say that I go back before any of these people. I am the wife of Seymour Goldberg who has passed on, and he met Israel in 1964, before the mathematical congress, before anything else. I just tell one funny story, I think it is funny.

Bella and Israel were given permission to have Seymour at their home. This was not easy; the KGB had to know everything about Seymour before he is permitted to enter Russian house, including if you see him mailing a letter tell me what mailbox, if he is talking to somebody tell me who he is talking to. Israel and Bella, being what they are, and Clara, said they are going to make him a special dinner. And they stood on line, both women, cut all these wonderful things, meat, chicken, and everything else which was very hard to get. Shortly before Israel is going to take Seymour home for dinner, Seymour says, "Oh, by the way, I forgot to tell you that I am a vegetarian." Seymour related to me, and to many other people, that Israel called home and told his mother and his wonderful wife. And from across the room Seymour heard in the phone two "*Oy*"'s. That's how loving families got together, and we love them, their children and grandchildren.

J. William Helton

I am honored to be here honoring Israel Gohberg on his 80th birthday. As we are all seeing, this conference honoring Gohberg is in our own self-interest, because it is this occasion which has drawn here this remarkable broad collection of powerful mathematicians from all branches of operator theory.

As we all know Gohberg has had a profound influence and much of what you see around you this week stems from areas he invented, students he trained, and areas where he solved basic problems. Since we all know this, maybe I should say something about his lesser known side.

I will always recall the first time he visited my house in San Diego. We all know Gohberg can fix bad matrices; he adds a column, takes off a row, transposes, permutes and voila! the matrix is beautiful. However, Gohberg can also fix plumbing. Our guest bathroom had drain caps on fancy levers which did not work, so we had rubber stoppers and gaskets. I was afraid he would have trouble with our "system". After his shower, after breakfast, I went in with new towels; but behold all the rubber stoppers were gone. I asked Israel if his shower went OK and he said, "All is fine, I will show you." He took out a coin, unscrewed the plate holding the lever, pulled some rods out of the wall, showed how to clean and unstick them, "that is all there is to it".

My wife was delighted. On the other hand from then on I faced the problem that my wife thought plumbing was easy to fix.

Another example, concerns the IWOTA conference I did not like. This is unusual because I always love IWOTA conferences. However, at this conference a screw fell out of my glasses. They fell apart, so I could not see anything. I could not see the talks, but that probably did not matter much, since they are hard to understand anyway. Unfortunately, the lever on the conference coffee pot was small, so I had trouble using it; clearly the situation was serious. When there are serious situations at IWOTA we go to our president. He looked at the pieces of my spectacles and said: "no problem, such glasses were very common" in the Soviet Union. Immediately he got a paper clip from the registration desk, threaded it through the hole in the glasses frame and in the paper clip tied a knot so strange it is found only in Moldova. Immediately, the conference began to make sense to me.

In conclusion, Israel Gohberg is always welcome in my home, he is of great value at conferences and his mathematics and his friendship are welcome anywhere in the world.

Peter Lancaster

It is a great pleasure and privilege to be here, and to have the opportunity to say a few words. I would like to repeat a number of sentiments that already have been expressed. They each have to do with family.

I remember Israel's mother quite well, although I could not possibly know her intimately. I never cease to wonder at the way that she was able to protect and inspire her children – in spite of the most dreadful times and conditions of war. Israel and I are about the same age, and I can't help contrasting his war years with mine. I had a relatively sheltered and secure existence in England through my first 15 or 16 years. So for me, it is hard to imagine the trauma that Israel and the family went through and, in particular, how much is owed to Israel's mother. I sensed the strength of her personality even though I could not know her very well.

Secondly, it is wonderful to see all the Gohberg family together once more, as we did twenty years ago in Calgary, and I am so delighted for each and every one of you.

The third aspect of Israel's extended family has to do with the mathematical community. How did this come about? Of course, scholarship is at the root of it, but it is unassuming scholarship, a modest scholarship, scholarship that does not intimidate, that welcomes ideas and people on an equal basis. I am privileged to have been one of these friends and colleagues.

The last little twitch on the family theme is to mention my own family, who all have very fond memories of Israel. They join me in wishing you many happy returns of the day.

Henry Landau

When I was a student, we heard a lot about a famous chemist. The problem for chemists in those days was to understand the structure of molecules, and in order to do that they had to crystallize them. This was a difficult art, but this chemist was phenomenal not only for being able to do it seemingly at will in his own lab, but also wherever he went all those waiting chemical solutions would suddenly crystallize. The explanation finally given was that he had a long beard, and that over the years so many different crystals had found their way into this beard that when he leaned over a sample something would drop out that was just the right seed around which everything would coalesce. Now Israel has been doing exactly that for us for about sixty years, going from place to place and wherever he goes ideas crystallize, beautiful structures appear – and he doesn't even have a beard!

Well, Iz'ia, as everybody knows we owe you so much, not only in mathematical ideas – they are precious – but even more precious are he worlds of friendship which you create for us and among us. You take us into your marvellous family: Bella, Feya, Tzvia, Yanina, all of you welcome us, and make us join work with feeling. This is something so extraordinary, as we can see just from our gathering here.

I think that every celebration really unites past and future, folds them into the present. So I think of Mark Grigoryevich Krein, Israel's friend and collaborator, whom he always brings to life on such occasions. And I think too of the dark stories of his early years that terrify even today. Here's one I always remember: when the nazis invaded, Israel's mother miraculously managed to get him and Feya with her on a train heading east away from the fighting. In the course of this voyage, always uncertain, they stopped in a little town and were told that the train would stay all day, so she went to try to find some food, but when she returned a short time later the train was gone! How can one imagine this moment? Someone told her that there was another place where the train might be, so in desperation she gave away her only winter coat to be taken to it, and providentially the train was indeed there and they were reunited. Such things are part of their past, as is the endless hardship of Soviet antisemitism, and yet in all these situations Israel was able to maintain brightness and hope. We saw this so clearly in Kishinev, on the occasion of his honorary doctorate. There was a strange atmosphere at the formal ceremonies, with the officials saying nothing about how he had been treated, but later in more private meetings, when Israel broached the subject, there was such an outpouring of emotion on the part of every one. They remembered details from thirty years ago. It seemed to us that they had always kept him in their minds as their source of joy, learning, and happiness in mathematics.

So as we are here all together, four mathematical generations of your friends and students gathered in this lovely place, with a full moon overhead, it is a wonderful moment in which to thank you. As I picture it, things may get a little dark at times but when you appear the sun comes out and mathematics blossoms. This will always be true, Iz'ia, so the only thing to say is: L'CHAIM!

David C. Lay

I am really pleased to participate in this conference honoring Professor Gohberg. I think my first contact with Israel's work was in 1973, when I spent the first half of a sabbatical at the Free University. Rien Kaashoek and Harm Bart were working on ideas from Gohberg's paper, and Rien invited me to participate. Then I met Israel in 1974 when he came to visit Seymour Goldberg at the University of Maryland.

I have two short stories about Israel and my family that illustrate how generous and kind Israel is on a personal level. You know, I thought people will be talking a lot about his mathematics tonight, and yet I find I am doing the same thing they did, seeing him as a man, a wonderful man.

In the fall of 1979 I began a sabbatical year at the Free University in Amsterdam, and we lived in a house in the same town as Harm and Greetje Bart. My wife Lillian and I had a 14 month old adopted daughter, Christina, and Lillian was pregnant as well. Soon Lillian gave birth to our second daughter in our home. Fortunately, the midwife arrived in time, because I did not have all the preparations ready. After a few weeks, Israel came to visit, to see our new baby. Shortly after he entered our house, Lillian came down with our new baby, Deborah, followed by little Christina who was just 14 months old. When Christina saw Israel she stopped and stood very still. But Israel smiled, held out his arms, and without hesitation Christina ran to him to be held.

After we returned to the University of Maryland, Israel and Bella started to visit Seymour and Lillian Goldberg at our university. They came for about two months each year for twenty years, and Israel visited our family on most of these trips. Israel became like a grandfather or uncle to our children. When our daughter, Deborah, was three and a half years old we had a large playhouse in our main family room. We still have a photograph of this house with Israel down on the carpet playing house with Deborah. He was there a long time that afternoon. You can imagine how the children looked forward to these visits each year.

Jürgen Leiterer

Dear Israel. You know, I too have an anniversary this year. Forty years ago I became a student of yours, exactly forty years ago. In September 1968, I moved to Kishinev and started my active mathematical life as your Ph.D. student. After two years I think I got some qualification after learning the basics, and you proposed to me to come as your collaborator. This was a big aid in my life. After that, I think it was three or four years, we worked together. It was a very good, maybe the best time of my life, it was a pleasure to work with you.

Then this collaboration stopped for political reasons, you moved to the West, I remained in the East. There was an attempt of Rien Kaashoek to join us again inviting us at the same time to Amsterdam. But this was "observed" and prevented. So we have a long way back.

Already when we were working together we observed that several complex variables are important for us, and you encouraged me to study such things. As a result, I became more and more interested in several complex variables. At the end, I changed the field completely. I worked in several complex variables, and forgot about operator theory for almost twenty years.

Then politics changed and we lived again in the same world. (I did not have to move for that, the West came to me to Berlin.) We met again, and you proposed to me to continue our collaboration. In the beginning I was skeptical about this, because I forgot almost everything. Nevertheless, five years ago we have started again, and I am again very happy that we have decided to start. It is again a very pleasant time for me.

How to explain that? There are many remarkable properties in your personality. But one which is most important – you are not only the founder and the head of a mathematical school. What you have is much more – it is a home, a house of hospitality not just for mathematics, but for people doing mathematics.

If I would meet somebody who is looking for a good problem to work on, I would say, “Go go Israel. He will speak to you, you will speak to him, he will speak to you, and at the end you will work on one of his problems. Even more, if you have a good problem, and you approach him, then he will speak to you, you will speak to him . . . , and at the end you will forget about your problem and start to work with him.”

Dear Israel, I am most impressed with your optimism. Meeting you I am always infected by it. It seems to me, there is nothing in the world that could destroy your optimism. I wish you on your birthday, most of all, keep this optimism.

Thank you.

Vadim Olshevsky

I seem to be one of the few people who are not wearing a jacket, but I believe I got a special permission from Gohberg.

Kishinev was already mentioned quite a few times today. I got my Ph.D. degree in Kishinev as well, but this was many years after Israel left. So I did not have a chance to meet him until I moved to Israel in 1989 to start a post doc position at Tel Aviv University.

I remember that Israel immediately told me that we should do something about structured matrices. Today we have a three-day special session on structured matrices at IWOTA which indicates that the topic has garnered a lot of attention. But in 1989 I told him that I do not find structured matrices interesting. Well, he insisted, and we wrote several papers together. When the first joint papers were completed, we submitted a contributed talk for the 1990 ILAS conference in Haifa. By mistake, it was listed as Gohberg’s talk and not mine. It was only a contributed talk, and there were four or five parallel sessions, but many people showed up nonetheless to come hear Israel speak.

There were about a 100 people in the room. Peter Lancaster was the chairman; when he announced Israel’s talk, Gohberg rose up said, “We changed the title of the talk, and we also changed the speaker.”

And somehow I got this large audience which came to listen to Gohberg, but instead received someone completely unknown. This helped me greatly at the beginning of my career.

Now I realize how right was Gohberg in predicting how important this research topic, structured matrices, will be in two decades. Everybody knows that this ability to choose “the right topic” is one of Israel’s many talents. I thank you Israel Tsudicovich for many years of working together [applause].

This was my first remark, and I would like to make a second remark.

I would like to say a few words about the “Gohberg phenomenon,” because I believe Israel is unique in succeeding in every enterprise he starts. One may think why this is, and of course there are many obvious ingredients: talent, hard work, personal charisma. But since this is a dinner speech, here I can tell you the truth. I think that luck that plays a crucial role in Gohberg phenomenon.

How many times today have we heard people quoting Gohberg as saying "Do not worry, everything will be OK?" Somehow there are people with whom everything is always OK: they can eat wild berries, wild mushrooms (as someone mentioned a moment ago), a police car can hit you from behind (as we learnt from another speech today), and yet "Do not worry, everything will be OK."

Let me tell you one more personal story. Seven or eight years ago Israel visited us in Atlanta. He gave a terrific talk, and afterwards we went to dinner together with Mihaly Bakonyi. At that time Israel was two hours late for his dialysis, and Bellochka was very nervous. But Israel said: "Do not worry, everything will be OK."

So, we went to the restaurant, and Israel ordered beer. Bella was again cautious, but Israel said again: "Do not worry, everything will be OK."

A couple of hours later, he was already four hours late for his dialysis. Actually, I needed to go back to the university because I had a class, I believe the time was about 8 pm, and the plan was that Mihaly would drive to highway 75, and Israel in his car would follow Mihaly, and we believed that once Israel was on highway 75, he will find a way to my house (where his dialysis machine was). It was back in pre-GPS era. In accordance with the plan, Mihaly drove to 75, made a gesture to indicate this is it, and took an exit. Israel instead followed him and also took this exit. Mihaly stopped, expecting that Gohberg would also stop and they would talk. Instead, Gohberg continued straight ahead and disappeared.

Now let me describe the topology of Atlanta. The Chattahoochee river divides the city, and one can cross it in only two places. Needless to say, my house was on the other side of the river. Given all this, in the direction which Israel followed it was simply impossible to get to my house. So we were very worried. We tried to call him but his cell phone was off.

About an hour later, I called my house, and Israel picked up the phone. I said, "Israel? Izrail' Tsudikovich?" And he said, "Yes". "How did you make it?" "What do you mean, how did we make it? We just followed the direction you indicated!"

To sum up, "some people" are successful even after initially taking the wrong turn. (As you can see, by successful I mean they come to my house).

3. Gohberg's family

3.1. The young years of Israel Gohberg

Dr. Feya Gohberg

My name is Feya Gochberg–Eidelstein and I have been a surgeon for over 50 years. I am Israel Gochberg's younger sister. My brother Professor Israel Gochberg was born on August, 23, 1928 in the small town Tarutino in the region of Bessarabia, that was then Romania and now belongs to the Ukraine.

He wasn't born an outstanding personality nor even a professor. Like all newborns he was small, helpless and very noisy.

His parents Clara and Tsudick Gochberg adored him and his arrival brought great happiness to their marriage.

His grandparents Izhak and Rivka Gochberg were very observant Jews. They loved their firstborn grandson very much and spoiled the child in every possible way, since he remained their only grandson for quite a long time. Our father was one of 6 children, and he was an educated person, he had graduated in accountancy, opened his own prosperous business – a printing house – and was a very devoted family man.

Our mother was quite a different person: she was a resourceful, very beautiful woman with dark skin, long brown hair and green eyes. Our mother Clara-Haya Gochberg was a midwife. She had lost her parents at a very young age, an orphan since the age of 14, she achieved everything in her life on her own. She graduated from Kishinev's nursing school, got her diploma as a qualified midwife and left for Tarutino where she started her working career. There she met our father. The two fell in love and soon got married. They gave birth to 2 children: my brother Israel and 5 years after his birth – to me.

My brother was brought up in a loving, well-off family, surrounded with warmth and care. From his early childhood his thoughts were filled with logic. When he was only 4 years old, while visiting some relatives, he saw a young couple kissing. When he came back home he asked his mother: "I don't understand, why Leon and Balbina kissed each other all the time?". Mother, worried about his "sexual education" tried to detract his attention and answered: "They probably were going away and saying "good bye" to each other." My brother thought for a while and said: "I don't think so. There were no suitcases around!"

Our parents tried to give him the best of everything: he had a new Mustang bicycle, a real Sima watch and each summer they took him to a sea resort.

When he was 7 years old he began to study in Tarutino primary school and finished with good grades.

At the age of 9 he started smoking and I was honestly fulfilling the role of a guard warning when our mother was approaching. I had a tricycle with 3 wheels. My brother used to let me pedal it and he himself liked to stand behind me on the tricycle and I had to pedal on it forward. Once, while riding like that on a high speed we both fell into a deep hole and almost got killed.

When he was punished for his deeds, I used to cry bitterly and say that it was my fault and I was the one to be punished. He appreciated it and never neither during our childhood nor later did he offend or hit me. Never has there been between us jealousy or envy. Through all of our lives we have always kept a warm and loving relationship and it was all our mother's achievement. It was our mother who taught us to love, honor and take care of each other. During our life in Tarutino our parents did their best to give my brother a good education: he took violin lessons, he was taught the Bible by a private teacher, he liked sports and was the only goalkeeper of the school's football team.

On finishing school in Tarutino, our parents decided to send him to one of the best secondary schools in the region, but he wasn't accepted there because of his low mark in mathematics and only because the school had a lack of students was he enrolled there. He studied there only for one year. The Second World War started – the most dreadful war for the whole world and especially for Jews. But in our family a great disaster had happened before that.

When the Soviet troops occupied Bessarabia our father was arrested in the middle of the night, without any explanation. Our mother was told that in 20 minutes, after answering some questions, father would be back home. We never saw him again. Our father was accused of Zionism and without being even prosecuted, he was sent to Siberia, to a Gulag. There, at the age of 40, he died of hunger. Our mother kept waiting for him all her life! The details of his death became available only few years ago, when the Soviet regime had changed. Our father was rehabilitated due to the lack of any proof of his guilt. In his holy memory my brother's firstborn daughter was called Tsvia. Exactly 15 years later, on the day of the anniversary, of our father's death, I gave birth to my only daughter, whom we naturally also named after him – Tsvia.

During the years of WW II my brother suffered hunger. We were always hungry, we fell asleep being hungry and we woke up being even more hungry. My brother worked in the fields together with our mother in order to get some carrots and potatoes so that we would not starve to death. He was very creative: he learned to make rubber rain-shoes from old tyres and exchanged them for some food. At this period of his life, my brother had already a mature personality and he was our mother's chief adviser and partner. At the age of 14 he decided to change his life and fight starvation. He stopped attending school and started working in a bakery. When our mother found out about his new career – I remember there was a serious scandal at home after which my brother preferred to remain being hungry and went back to school. In spite of his absences he completed his school education during the last year of the war.

He graduated from school with very high marks and at that time his outstanding abilities in mathematics were discovered. His school teacher, Mr. Shumbarsky noticed his talent in math. I think that he was the one who played an important role in forming my brother's interest in math. His teacher was sure that math should become his future. But our mother didn't think so. Working as a midwife, all her life she dreamt to be a doctor, but that was impossible for her to reach. She certainly thought that her talented son should become a physician. There was a lot of disagreement at home and at last mother told my brother that if he didn't apply to medical school she wouldn't support him financially. My brother was scared and told his teacher everything that had happened at home. His teacher paid us a visit and told my mother that if she insisted on my brother's learning medicine – she should know for sure: that on that very day a great talent in mathematics will be not developed and be lost. My mother got scared and gave in. In 1946, at the age of 18 years, my brother became a student at the faculty of mathematics of the Kirghiz State College. A new period in his life started.

All her life our mother lived with my brother's family. She always helped us and the last 10 years of her life she spent in Israel.

Our mother died at the age of 80, she always helped us, until her last day and was full of energy, had a tremendous sense of humor, an outstanding example of dignity, loyalty and love. In memory of our mother Clara-Haya were named our grandchildren: my brother's granddaughter – Keren and my grandson Hannan.

This is our family today: my brother and his wife Bella, his elder daughter Tsvia, her husband Nissim and their children Tali and Jonathan; his younger daughter Yanina, her husband Arie and their children: Keren, Raviv, and Tslil, and I – his sister Feya, my daughter Tsvia, her husband Malcolm and their children: Hannan and Liat.

3.2. My father I.C. Gohberg

Zvia Faro (Gohberg)

As we were growing up Dad's work seemed very mysterious and unclear to us, we considered it his "Dark Side".

There was also the bright side, the Dad, whom we understood, who made us laugh, taught us math, history, science, who was fun to be around and learn from. I want to talk about this side, the side so dear and familiar to us.

Our Dad is a devoted family man, caring son, loving husband and Father, dedicated brother and uncle. He is a wonderful Grandfather, who can do magic tricks like a professional magician.

He is very athletic a good soccer player, skier, swimmer and runner.

His "golden" hands can fix or make anything. Dad is a good cook and can create some elaborate gourmet dishes. He is a well rounded man, and has many other interests in life besides mathematics, being very thorough, he does not accept shortcuts and excels in everything he does.

He had many hobbies, at times it was photography, aquariums, later fishing, agriculture, wine making and many more. When I was born Dad's hobby was photography. At night, when everyone was asleep, our tiny bathroom turned into a dark room and in the morning there were many photos drying on the blanket on the floor.

Another hobby was his bicycle, to which he installed a motor, tied a little pillow to the ramp and often took me for long rides to the country fields.

I still remember the fun, the wind was blowing in our faces, while we were riding and singing. Here I will probably uncover one of the very few things that he is not good at: he does not have an ear-for-music, and I am the same. My Mom and my sister have a perfect ear for music, so we never even dare to sing at home, but on those trips only the wind, the cows and the sheep we passed by were our audience and at that time I thought that they really enjoyed our out-of-tune singing.

On the way we played games, he asked me riddles, logical puzzles and taught addition. As I grew the trips became longer and the problems harder. Our family often vacationed on the Black Sea. I remember how writing with a stick on the sand he explained binary numbers and limit. Limit was hard, I kept asking what does it mean that for every epsilon there is a delta? What if I find a smaller epsilon, I asked, then I will find a smaller delta he replied, drawing another segment on the sand. He was never tired or impatient and could repeat things over and over with new intuitive examples and jokes.

Notwithstanding his busy schedule, there always was time for us. Dad taught us riding bike, skiing, ice skating, swimming, diving. We loved long walks in the woods where we learned survival skills and the difference between good mushrooms and the poisonous ones. When our Mom who is a doctor was on call, he cooked us our favorite dinner, it was the best mashed potatoes I have ever had.

When we decided to immigrate to Israel, we were refused the exit visa and became refuseniks.

I was expelled from the University, Dad stopped going to work, it seemed that my life was over. We have spent a lot of time together, he became my best friend and cheerleader. He was encouraging me, telling about the infinite opportunities that awaited me and my children in the free world. I wanted to hear about our wonderful new life, but he never painted a pink picture, preparing me for difficulties. During those long months we discussed politics, listened to the Voice of America, he allowed me to read forbidden Solzhenytsin SAMIZDAT books. I learned about the world outside the Soviet Union, my heritage, the history of my people and many other subjects that were dangerous even to think about at that time. He taught me to fight and not to give up under any circumstances. I was very impressed by the story of Massada fortress, a source of inspiration and a symbol to everyone, who fights for freedom.

When we came to Israel he could not speak neither English, nor Hebrew and in a very short time with no formal training was fluent and lectured in both languages.

35 years since we left Russia I still admire his courage and confidence, that helped us overcome those difficult times and opened to me and my family new unsurpassed opportunities.

I always looked up to you, you were my role model. Today, when my kids are grown up, I still look up to you, ask for your good advice, for encouragement, you always stands by us and support us in all our endeavors.

When I was little, I looked very much like my Dad. People who did not know me, stopped me on the street and asked if I was Gohberg's daughter. I hope that now when I am all grown up I am at least a little bit like you, and not only look like you.

Many good wishes we are sending your way,
Get younger and younger day after day,
May you live long and happy life
Always together with your wonderful wife
(Always share with us your good advice)
The troubles and misfortunes should pass you by
May your humor and jokes make everyone smile
May your laughter roam like a thunder

And good friends be always around you
Have nakhes from children, grandchildren and family
Happiness and sunshine with Bella sharing
We wish you health – it is needed a lot
May luck always follow you and support
Keep dreaming big and may all your dreams come true
Travel, research, prove new theorems too
And on your 80th birthdays we say right from the start
Accept our best wishes from the bottom of the heart.

3.3. Dad's 80th birthday

Yanina Israeli (Gohberg)

I have always admired my Dad's outstanding personality. He is a man of many talents and excels in everything he does or puts his mind to. There are many contradictions in his character; I often think that these contradictions make him the remarkable person he is.

Dad is an optimist and a believer in good outcomes, but he always plans and prepares himself for the worst.

A person, who had overcome a lot of difficulties in his life, who knew loss and sorrow, but nevertheless loves life and enjoys every minute of it.

He is a wonderful friend with a lot of friends all over the world but on the other hand a very private person, who religiously guards his privacy.

A devoted and loving family man, who spent a lot of time travelling far away from the family and dedicated his life to mathematics.

Humble and modest, does not need much for himself, but very giving and generous to the people he loves.

Flexible, curious, progressive and open-minded, he can be very conservative and stubborn at the same time.

A person who describes himself as not a sentimental one, he is very compassionate and kind-hearted.

Demanding, critical and tough he expects everyone to excel, but at the same time he is the most caring and supportive person, who stands by and encourages in the difficult and most disappointing moments.

Dad has a rare sense of humor and roaming laughter and a joke for every situation, but he knows how to be very serious and with one glance can make serious everyone around him.

He can advise in the most difficult situations, but does not interfere and volunteer his advice, unless he is asked for his opinion.

Although he has very logical and analytical mind and believes in thinking things through, sometimes he tends to rely on his intuition.

He can spend hours concentrating on mathematical research, being disconnected from the world around him, but will drop everything in a second to help his children or grandchildren with their homework.

Can't live without email, loves technology and internet communication, but on the other hand loves nature, enjoys long walks, good swim and camping away from the civilization.

These contradictions in his character make him the most interesting, surprising and creative person. We love you and hope to be together with you and Mom for many years to come, may you be healthy and happy, tell jokes, make us laugh, keep being unexpected and surprise us over and over again.

3.4. Family reminiscences

Bella Gohberg

In 1951 in Bishkek, Middle Asia there were 3 inseparable friends, Nora, Fani and me. We were juniors in college, studying medicine. It was Nora's birthday and we were ready to party. For some reason Fani could not come and send a "delegate", her brother Israel, or Izia, as she called him. The delegate was tall good looking, skinny guy with big green eyes, long eye lashes and full head of hair. He was smart and funny, his laughter was loud and infectious. We liked each other and after the party Israel walked me home, he talked about math with a lot of enthusiasm.

Math was not my strongest subject, I did not believe then that math can be a source of inspiration and disappointment, that it was possible to dedicate one's life to this science. I have learned it much later. Israel was the first mathematician I have ever met.

His vacation was over, he returned to complete his degree in mathematics in Kishinev, my studies were just beginning. After that meeting, we have written each other and met occasionally.

A few month in the beginning of 1954 Israel worked on his Ph.D. thesis in Leningrad, where I was completing my medical studies. He invited me to attend the defence of his Ph.D. thesis. I felt proud and honored, was very impressed how freely, with ease he used mathematical formulas and how attentively everyone was listening. As my husband likes to say: It was wonderful – but not clear at all.

This was my first introduction to Advanced Mathematics.

Later in winter of 1956 Israel came to Leningrad and asked me to become his wife. He stayed for 6 days, everyone told us it was impossible to register in 6 days, but against all odds on February 1, 1956 we have registered our civil marriage. We could not even dream about a traditional Jewish wedding in those difficult times. We have celebrated our Jewish wedding 50 years later in Raanana, Israel.

I joined my husband in the town of Beltsy, Moldova 6 months after our marriage. I was already a licensed medical doctor. 3 months later on November 1, 1956 our first daughter was born. Our friends often joked that those were precise mathematical calculations.

In 1960 we moved to Kishinev, where our younger daughter Yanina was born. Israel's Mother lived with us, she was part of our family and helped us a lot. All five of us shared a 1 bedroom apartment with a tiny kitchen.

After 10:00 pm when everyone was asleep the kitchen became my husband's study. On the kitchen table under the black reading spot lamp he wrote his Habilitation thesis and his books with M.G. Krein.

Every morning we found on the table many new handwritten pages and an ash tray full of cigarette stubs.

Often Israel went to conferences and presented his results, his Mother was impatiently waiting for him. When he returned she would ask: "How did it go? Did people ask you questions?" the answer was "Yes". "So, Did you know the answers to those questions?" "Yes". After the second answer she looked at him with a little skepticism and surprise, but at the same time with great love and admiration. She was very proud of him.

An important part of his life was collaboration with Mark Grigorievich Krein. Israel used every opportunity to work with M.G. and traveled to Odessa, often on the weekends. Trips to Odessa and work with M.G. has inspired Israel and charged him with energy. Even during our vacations on the Black Sea he managed to carve some time for the work with M.G. Krein at his dacha. Professor Krein was very demanding of himself, of his students and his coauthors. There were many revisions of the books and many trips to Odessa.

Usually when he returned from the trips we wanted to know whether the book was completed. Israel's answer was: "Almost, some very little changes remained. One more trip to Odessa". There were anecdotes and legends among his friends about this subject. Josef Semyonovich Iohvidov dedicated the following poem:

(From M.G. Krein's dream, New Year's Eve, 1963)

Around the festive table all our friends
Have come to mark our new book's publication.
The fresh and shiny volume in their hands,
They offer Izia and me congratulations
The long awaited hour is here at last.
The sourest skeptic sees he was mistaken,
And smiling, comes to cheer us like the rest
And I am so delighted … I awaken

(Translated from Russian by Chandler Davis)

I vividly remember an episode, when our daughter Vilia was 4 years old, we moved to Kishinev. I was concerned, that I did not have a job and asked Israel what are we going to do. Always an optimist my husband answered: "We will fight!" Our little girl, heard his answer, understood it literally and said: "I don't want you two to fight". It was very funny and we all laughed then, but on the serious note the "fight" was an important part of Israel's life. He had to fight for his survival during the war. He had to fight for his education and career in a very difficult situation during the time when his Father, was wrongly accused and died in Gulag as a political prisoner.

Israel not once fought diseases, he fought for his life when severely burned from explosion of the gasoline vapors. The doctors considered his recovery a miracle. Israel fought for the immigration to Israel. It was his dream and he initiated this responsible event in our entire family. My husband have won many battles, too many to name here.

His strong will, incredible optimism and intuition helped him in the "dead ends" and most difficult situations. Israel's life was not a rose garden. There were plenty of thorns, that at times hurt leaving deep scars.

I am thankful to G-d for helping us overcome all the difficulties. I am very fortunate and excited to celebrate your 80th birthday, surrounded with colleagues, family and friends. Our life together was never boring, was always interesting and filled with love and understanding. We are blessed with wonderful children and grandchildren, who fortunately were born in the free world in Israel.

I am praying for many more years to be together.

3.5. Congratulations Izinka

Zvia Kavalsky

Good evening ladies and gentlemen. Dear mathematicians, family, friends, and dear Professor Gohberg. My name is Zvia Kavalsky, and my mom Feya Gohberg is Israel's one and only sister and since I am her only daughter, I believe that I just proved to everyone that I am Israel's one and only niece.

I never called you uncle. For me you are Izinka, it's a lovely name we call you only at home and it is reserved only for a very close family.

Today we celebrate your 80th birthday, and I, your sister Feya, and my daughter Liat, travelled from overseas to be able to participate and celebrate this wonderful event. So, Israel, Izinka thank you so much for inviting us and making it possible to share this special moment for our family here together with you.

Ever since I was a little girl, I remember you in my life. Every summer vacation I would go to Kishinev, to my uncle's house and have fun with my cousins. I did it for more than 10 years. More than anything else I love to remember the times that we spent together, the weekends, a lot of good jokes, good food, good laughter and good energy around. It is in your house I was taught to believe that there is a lot of goodness in the world, that one has to work hard to earn wealth and respect, that we should be always honest, very thoughtful and extremely rational, you shared with us your life experience, you taught us to take knowledge and education seriously and your advice through all of my life was always useful, sincere and worthwhile. You and your wife Bella (for me Belluchka), always treated me as your daughter, you both always made me feel welcome and very comfortable, you never made any difference between me and your daughters. Therefore, I will prove now that 2 is equal to 3: what I mean is that everybody knows that you have 2 daughters but really you do have 3, Tzvia, Yanina and me. And if anybody can present a counterexample – I have many other proofs to present.

I would like to conclude with a quotation of a famous scientist, Louis Pasteur, who said: "I am convinced that science will triumph over ignorance, that nations will eventually unite, not to destroy, but to create, and that the future will belong to those who have done the most for the sake of humanity."

I am very proud tonight while I stand here belonging to the family side of this splendid event, I believe that you always managed to gather together a lot of talented scientists that the future belongs to you and to them.

3.6. My grandfather

Jonathan Faro

I am lucky enough to have inherited a lot of traits from Israel Gohberg, my grandfather, my mother's Father. We all grandchildren call him Pappi. I'll start with the most obvious one:

The Bald Gene: As a kid I remember hearing: "Hair is inherited from your mother's father". I knew very early on that the odds were against me having a full head of hair.

An appreciation of sports: I remember as a young boy, Pappi taught me how to play soccer. And, although looking at him you may not be able to tell, he has some serious moves!

A love of nature: Ever since I was a little kid, I remember taking nature walks with Pappi. We'd pass by chicken coops, picked oranges from a grove and threw things at the pecan trees so that we could collect pecans (half for eating and half for my grandmother to bake with). These trips however, were a source of great nervousness for me. As a boy I learned 3 rules of thumb to follow while in nature:

(1) Stay in groups
(2) Avoid dangerous wildlife
(3) Don't eat wild berries

But on these walks Pappi would pick and eat wild berries wherever we went. If he saw me being nervous he would say "Don't worry these are good", he would then point to another bush with IDENTICAL looking berries and say "But don't eat these, they are VERY POISONOUS". To this day I still enjoy our nature walks, But I still don't eat the wild berries.

A Thirst for Knowledge: One of the greatest gifts I got from Pappi was the need to learn and understand things. As a boy, I remember Pappi asking me riddles and giving me challenges. Sometimes I would solve them and sometimes I would not. When I'd ask Pappi for the answer he'd respond "That's not important, It's the road to discovering it that really matters". Ever since, I've looked at problems in a whole new light; I see them as opportunities to think outside the box and learn something new. It is this quality that he instilled in me that has encouraged me to continue my studies after attaining my degree and it motivates me to constantly challenge myself and to learn more.

Pappi, you've always been a role model for me. Every quality I've inherited from you makes me a better person. I hope as I grow older I become even more like you.

I love you and Happy 80th Birthday!

4. To Izia Gohberg on his 80th birthday

Dan Amir

I do not intend to praise Israel Gohberg the mathematician. His mathematical merits, achievements and honors are well known, and other speakers are better qualified to talk about them. Neither am I going to talk about Gohberg the great teacher, I am not one of his lucky students. But I do have my own special point of view on Izia.

It is told in the Mishna that Rabbi Yohanan Ben Zakai, the famous Rabbi who managed to secure the continuation of Judaism after the fall of Jerusalem, asked five of his famous pupils what do they consider as a good course for a man to follow?

Rabbi Eliezer said: *a benevolent eye*. Rabbi Yehoshu'a said: *a good friend.* Rabbi Yossi said: *a good neighbor*. Rabbi Shime'on said: *seeing the forthcoming.* Rabbi El'azar said: *a good heart*. Rabbi Yohanan said he prefers this last answer, because it implies all the others.

One can argue about the logic behind Rabbi Yohanan's statement, and I will not boast about choosing always the best way. Anyhow, I can compliment myself upon following at least three of the advices given by his students:

When I retired and had to give away my single room and share an office with another retired colleague, I had the foresight to choose Israel Gohberg to be my roommate. Thus I gained not only a good neighbor, but also a good friend. As for the other two advices, I got them too, though indirectly: Both the benevolent eye and the good heart, I found them in my roommate Izia.

That decision was not as trivial as it might seem to be. Israel is a very diligent retired mathematician. In fact, besides stopping lecturing, he continued, and he still continues to this day, to work and do mathematics just as he used to do before retirement. It was quite tempting to share office with some other retired fellow who is much less active than Izia, who comes to the office only once a week, and not daily as Izia does, and who has no pupils or collaborators from all over the world who come so often to visit him in the common office, and most important: who does not need so much shelfspace as Izia does: only the so many books he has published, not to mention the huge book series he has edited, fills easily half a room and more.

Yet, I was lucky to overcome all these temptations and even luckier, since Izia was willing to become my roommate. This critical decision cost me at least half of my mathematical books and reprints, but was one of the best decisions I have ever made.

I am afraid Izia had to do the same and reduce his library too, but we manage together beautifully. If you wonder how do we manage, I'll tell you another Jewish story, from the Talmud, about two big rabbis, Rabbi Yishma'el and Rabbi Yossi. "Big" here means literally big – it is told that when they stood together belly to belly, a bull could pass underneath their bellies without touching them. A foreign lady tried to tease them and said: "Your children are not yours" (because of their huge bellies). They answered: “Love squeezes the flesh”, i.e., with good will you can manage even when very cramped. (By the way, there is a also a much nastier answer attributed to them, an answer which lead the same Rabbi Yohanan to wild speculations about the size of Rabbi Yishma’el’s organ).

Anyhow, Izia has proved himself during the years we share office to be a wonderful roommate. He is always patient and good-spirited. He has a great sense of humor and shares with me interesting stories and jokes. We help each other in translation from Hebrew to English and from Russian to Hebrew. He even shares with me the tasty sandwiches, vegetables and fruit that his wonderful loving wife Bella sends with him daily. I don’t believe there is another roommate like him in the all world! I hope we’ll continue to share office for many years to come!

5. Reminiscences of meetings with Israel Cudicovic Gohberg

Hellmut Baumgärtel

The beginning of my meeting and subsequent friendship with Israel Cudicovic Gohberg is a concatenation of several independent events. In 1964 I published a little paper entitled “Zur Störungstheorie beschränkter linearer Operatoren eines Banachschen Raumes” in the *Mathematische Nachrichten* (MN). After that the Editor of the MN invited me to be a referee for this journal. In 1965 I refereed for MN a paper of S.K. Berberian on a theorem of J.v. Neumann with the comment “the proof is too complicated”. After some letter exchange with Berberian he invited me to write a modified paper together with him to publish it in MN. A few months later he became Editor-in-chief of the *Mathematical Reviews* (MR) and he invited me to be a referee of MR. 1967 I received from MR the monograph “Perturbation Theory for Linear Operators” of Tosio Kato. When I overviewed it I was pleased at the positive mention of my paper from MN in this book. It encouraged me to announce this topic under the title “Analytische Störungen isolierter Eigenwerte endlicher algebraischer Vielfachheit” for a talk at the Second Congress of the Bulgarian Mathematicians on September 1967 in Varna/Druzba. (My main interest in that time was directed to the operator theory of wave operators in quantum mechanics.) Professor Gohberg was a participant of this congress and he was even chairing the session with my talk. As Professor Gohberg told me later, at the beginning he was a little bit skeptical and he did not believe that anybody would know more than him about the topic of my talk. As he remembers at the end of the talk he completely changed his opinion. After the talk we have had an exciting discussion on the matter (he spoke Russian, I spoke German but there was a tranlator) with the result that his inquiries could be answered and he invited me for a visit at the Institute of Mathematics of the Academy of Sciences of the MSSR in Kishinev.

The discussion was continued during a visit of Prof. Gohberg at the Institute of Mathematics of the Academy of Sciences of the DDR in Berlin in December

1967. My visit in Kishinev was fixed for 1968 and the encouragement of Prof. Gohberg ("these are good results") was stimulating for the idea to close completely the already detected gap in the analytic perturbation theory. Fortunately there was success in this project and so in June 1968 I could present the final result in two talks entitled "Analytische Störung diskreter Spektren" at Professor Gohberg's Functionalanalytic Seminar of the Institute of Mathematics in Kishinev, i.e., the complete characterization of the behaviour of the Jordan structure for analytic perturbation of an eigenvalue of finite algebraic multiplicity using the theory of vector spaces over suitable function fields. The friendly and helpful atmosphere in this group, into which I was naturally incorporated, did good and it is unforgettable. Moreover, I had the occasion to visit Professor Krein in Odessa, where I got exciting remarks on the structure theory of wave operators which were useful for me later. Finally Prof. Gohberg recommended me to present the now completed theory as a whole in a monograph.

In October 1968 I obtained the qualification "Habilitation" at the Humboldt-University (HU) Berlin with these results together with structural results in scattering theory. (A professorship for Analysis at the HU, supported by Professor Schröder, was not achieved. Probably my activities in the protestant church played a role that I could not get the position. Since the times I was at the university I have been a "black sheep", especially because of June 1953 where I escaped expulsion from the HU only by the invention of the so-called "Neuer Kurs" (New Direction) which turned into the old one soon. The ruling (communist) party forgot nothing.)

In the following time Prof. Gohberg attended the progress of the book by valuable hints and critical remarks (he had much experience how to write Mathematics), for example on the occasion of my second visit in Kishinev in January 1970. We understood then that we may relay one on the other and our discussions touched a much wider list of topics including politics also. Our meeting culminated in the visit of Prof. Gohberg at our Institute of Mathematics in November 1970. At that time the manuscript was finished and found Prof. Gohberg's agreement. It was a great event for our Institute, for example because it was the first visit of a famous mathematician from the SU. It was highly appreciated, especially by the chief of the institute, Prof. Schröder. My last visit in Kishinev took place in October 1971. At that time we discussed already new topics, for example spectral concentration coupled with factorization.

The book appeared in 1972 at the Akademie-Verlag Berlin under the title "Endlichdimensionale analytische Störungstheorie". It is dedicated to Israel Cudicovic Gohberg. In February 1974 I was told that Prof. Gohberg left the SU to emigrate to Israel. Since that time he was "persona non grata" also in the DDR and I have had no further contact with him.

In 1982 I was informed by the Akademie-Verlag that Birkhäuser were interested to publish an English version of the book, i.e., they planned a joint edition with that publisher. The Akademie-Verlag let me know that they would like to have my agreement to omit the dedication but I rejected, and the English version appeared under the title "Perturbation Theory for Matrices and Operators" with the original dedication. In April 1983 Professors Gromov and Lomov from Moscow visited me in our home and told me that they were translating the book to Russian. However a publisher was not yet found. There was a difficulty, the page of dedication. They did many efforts to convince me to agree with omitting this page. They did not succeed and the book did not appear in Russian.

In 1987 I was invited to the conference to be held in 1988 in Calgary on the occasion of the 60th birthday of Prof. Gohberg. In the following months a tug-of-war was developed for preventing this visit: funding problems, limiting of the number of participants from the DDR, missing signatures and wrong dates on visa, to and from between Warsaw and Berlin. Finally, success for me came by mediation of a colleague from the higher staff of the academy, the late Professor Budach. (These discriminations in the eighties were typical for me, in the seventies the situation was better because in that time I was a "single parent with two children" and the children served as hostages for the state, they knew that I would come back anyway. In 1982 I married again.) It was a touching event to meet again after 17 years.

Last but not least I mention with pleasure my private visit in 1992 in Tel Aviv to Israel and his family and the visit of him in December 1993 after his talk at the TU Berlin in our home.

The best way to cement friendship at the occasion of the 80th birthday of the friend seems to be for me to tie together the beginning and the present time by a paper dedicated to the friend. This is the paper [1].

[1] Baumgärtel, Hellmut: "Spectral and Scattering Theory of Friedrichs Models on the positive Half Line with Hilbert-Schmidt Perturbations", Annales Henri Poincaré, 10 (2009), pp. 123–143.

Operator Theory:
Advances and Applications, Vol. 202, 1–10

A Quantitative Estimate for Bounded Point Evaluations in $P^t(\mu)$-spaces

Alexandru Aleman, Stefan Richter and Carl Sundberg

Abstract. In this note we explain how X. Tolsa's work on analytic capacity and an adaptation of Thomson's coloring scheme can be used to obtain a quantitative version of J. Thomson's theorem on bounded point evaluations for $P^t(\mu)$-spaces.

Mathematics Subject Classification (2000). Primary 46E15; Secondary 47B20.

Keywords. Bounded point evaluation, Cauchy transform.

1. Introduction

For $\lambda \in \mathbb{C}$ and $r > 0$ let $B(\lambda, r) = \{z \in \mathbb{C} : |z - \lambda| < r\}$, and let $M_c(\mathbb{C})$ denote the set of all compactly supported complex Borel measures in $\mathbb{C}$. Then for $\nu \in M_c(\mathbb{C})$, $r > 0$, and $\lambda \in \mathbb{C}$ we write

$$U_{|\nu|}(\lambda) = \int \frac{1}{|z - \lambda|} d|\nu|(z)$$

and

$$U_{|\nu|}(\lambda, r) = \int_{B(\lambda,r)} \frac{1}{|z - \lambda|} d|\nu|(z).$$

We will refer to $U_{|\nu|}$ as the potential of ν. It is well known that $U_{|\nu|}(\lambda) < \infty$ for [Area] a.e. $\lambda \in \mathbb{C}$. At every such $\lambda \in \mathbb{C}$ the Cauchy transform

$$C\nu(\lambda) = \int \frac{1}{z - \lambda} d\nu(z)$$

exists and $U_{|\nu|}(\lambda, r) \to 0$ as $r \to 0$. The purpose of this paper is to prove the following theorem.

Work of the second and third authors was supported by the National Science Foundation, grant DMS-055605.

Communicated by J.A. Ball.

Theorem 1.1. *There exists an absolute constant* $C > 0$ *such that for every* $\nu \in M_c(\mathbb{C})$ *and for every* $\lambda \in \mathbb{C}$ *with* $U_{|\nu|}(\lambda) = \int \frac{1}{|z-\lambda|} d|\nu|(z) < \infty$ *there exists* $r_0 > 0$ *such that for all polynomials* p *and for all* $0 < r \leqslant r_0$ *we have*

$$|p(\lambda)C\nu(\lambda)| \leqslant \frac{C}{r^2} \int_{B(\lambda,r)} |p(z)C\nu(z)| dA(z).$$

Here r_0 *depends only on* $|C\nu(\lambda)|$, $U_{|\nu|}(\lambda)$ *and* $U_{|\nu|}(\lambda, r)$ *as* $r \to 0$.

The theorem is nontrivial only at points when $C\nu(\lambda) \neq 0$ and we will see that there is an absolute constant $K_0 > 0$ such that for all such points any r_0 satisfying

$$U_{|\nu|}(\lambda, r_0 + \sqrt{r_0}) + \sqrt{r_0}U_{|\nu|}(\lambda) \leqslant K_0 \; |C\nu(\lambda)|$$

will work.

The insight that such a theorem can be used to establish bounded point evaluations for $P^t(\mu)$-spaces that are proper subspaces of $L^t(\mu)$ is a part of what J. Thomson calls "Brennan's trick", see Theorem 1.1 of [8] and also see Section 2 below. Although as far as we know Theorem 1.1 has never been stated before in full generality, versions of it have been implicitly derived for annihilating measures in [1] and [2]. In fact, we shall see that it follows fairly easily from our paper [1], and it can also be deduced from Brennan's paper [2]. Thus we think of the current paper mostly as an expository note, and we plan to take this opportunity to once more carefully explain how X. Tolsa's theorem on analytic capacity, [9] and an adaptation of Thomson's coloring scheme, [8] come together to prove the current result. In Section 5 we explain how the current approach can also be used to establish that every bounded point evaluation must either arise because of an atom of μ or it must be an analytic bounded point evaluation.

2. Thomson's theorem

Let μ be a positive finite compactly supported measure in the complex plane $\mathbb{C}$, let $1 \leqslant t < \infty$ and let $P^t(\mu)$ denote the closure of the polynomials in $L^t(\mu)$. In 1991 James Thomson proved the following theorem, [8].

Theorem 2.1 (J. Thomson). *If* $P^t(\mu) \neq L^t(\mu)$, *then there are a* $\lambda_0 \in \mathbb{C}$ *and a constant* $c > 0$ *such that*

$$|p(\lambda_0)| \leqslant c \left(\int |p|^t d\mu \right)^{1/t}$$

for every polynomial p.

The point λ_0 is called a bounded point evaluation for $P^t(\mu)$. In fact, Thomson proved that every bounded point evaluation for $P^t(\mu)$ is either a point mass for μ or it is an analytic bounded point evaluation, i.e., the constant c can be chosen so that there is $\varepsilon_0 > 0$ such that $|p(\lambda)| \leqslant c \left(\int |p|^t d\mu\right)^{1/t}$ for every polynomial p and every $\lambda \in \mathbb{C}$ with $|\lambda - \lambda_0| < \varepsilon_0$.

Thomson's proof contains a basic construction, but at its core it is a proof by contradiction and it originally was not clear which points λ_0 occur and how the c depends on μ and λ_0. After the papers [2] and [1] were written we received a note from J. Thomson which showed that a careful analysis of his original proof does show that point evaluations occur at every point where some annihilating measure has finite potential and nonzero Cauchy transform.

The following observation and the realization of its usefulness goes back to Brennan, [4, 2, 5]. It shows that Theorem 1.1 gives some information on how certain changes of the measure would affect the λ_0 and c.

Lemma 2.2 (J. Brennan). *Let μ be a compactly supported positive measure, let $1 \leqslant t < \infty$, and let $1 < t' \leqslant \infty$ satisfy $1/t + 1/t' = 1$. If $G \in L^{t'}(\mu)$ such that with $d\nu = Gd\mu$ we have $\int p d\nu = 0$ for all polynomials p, and if $r, C_0 > 0$ such that*

$$|p(\lambda)| \leqslant \frac{C_0}{r^2} \int_{B(\lambda,r)} |p(z)C\nu(z)|dA(z),$$

then

$$|p(\lambda)| \leqslant \frac{2\pi C_0}{r} \|G\|_{t'} \|p\|_t.$$

Proof. In this paper we shall repeatedly use the inequality

$$\int_{z\in\Delta} \frac{1}{|w-z|} \frac{dA(z)}{\pi} \leqslant 2\sqrt{\frac{A(\Delta)}{\pi}} \tag{2.1}$$

for $w \in \mathbb{C}$, $\Delta \subseteq \mathbb{C}$ (see [7, pages 2–3]). Thus in particular,

$$\int_{B(\lambda,r)} \frac{1}{|w-z|} dA(z) \leqslant 2\pi r,$$

for all $\lambda, w \in \mathbb{C}$. If $\int p d\nu = 0$ for every polynomial p, then $\int \frac{p(w)-p(z)}{w-z} d\nu(w) = 0$ for all $z \in \mathbb{C}$ and hence $p(z)C\nu(z) = C(p\nu)(z)$ for a.e. $z \in \mathbb{C}$. Thus,

$$\begin{aligned}
|p(\lambda)| &\leqslant \frac{C_0}{r^2} \int_{B(\lambda,r)} |p(z)C\nu(z)|dA(z) \\
&= \frac{C_0}{r^2} \int_{B(\lambda,r)} |C(p\nu)(z)|dA(z) \\
&\leqslant \frac{C_0}{r^2} \int_{B(\lambda,r)} \int \frac{|p(w)G(w)|}{|w-z|} d\mu(w)dA(z) \\
&= \frac{C_0}{r^2} \int \int_{B(\lambda,r)} \frac{1}{|w-z|} dA(z)|p(w)G(w)|d\mu(w) \\
&\leqslant \frac{2\pi C_0}{r} \int |pG|d\mu \leqslant \frac{2\pi C_0}{r} \|G\|_{t'} \|p\|_t
\end{aligned}$$

□

Note that in the above setting the largest choice of r as given by Theorem 1.1 will give the best bound for the point evaluation. If one is interested in rational approximation, then there may be an advantage to applying the theorem with smaller values of r. Let $R^t(\mu)$ denote the closure in $L^t(\mu)$ of the rational functions with no poles in the support of μ. It is well known that for $1 \leqslant t \leqslant 2$ there are measures μ such that $R^t(\mu) \neq L^t(\mu)$, but $R^t(\mu)$ does not have any bounded point evaluations, see [3, 6]. Nevertheless the above setup can be used to obtain bounded point evaluations for $R^t(\mu)$ in case the support of μ satisfies an extra condition.

Suppose that $R^t(\mu) \neq L^t(\mu)$ and let $G \in L^{t'}(\mu)$ be such that $d\nu = G d\mu$ annihilates the rational functions with poles outside the support of μ. Let λ, $r_0 > 0$ be as in Theorem 1.1, let $0 < r < r_0$ and let q be a rational function with no poles in $\overline{B(\lambda, r)} = \{z : |z - \lambda| \leqslant r\}$. By Runge's theorem q can be uniformly approximated on $\overline{B(\lambda, r)}$ by polynomials, hence the conclusion of Theorem 1.1 remains valid with q in place of p. If q also has no poles in the support of μ, then the proof of Lemma 2.2 shows that

$$|q(\lambda)| \leqslant \frac{2\pi C}{r} \frac{1}{|C\nu(\lambda)|} \|G\|_{t'} \|q\|_t.$$

Another application of Runge's Theorem now implies that this last inequality remains valid for each rational function q which has no poles in the support of μ, if each component of the complement of the support of μ has a point in $\mathbb{C} \setminus \overline{B(\lambda, r)}$. This implies that if $R^t(\mu) \neq L^t(\mu)$ and if there is $\varepsilon > 0$ such that all components of the complement of the support of μ have diameter $\geqslant \varepsilon$, then $R^t(\mu)$ has bounded point evaluations. This result is due to Brennan, see Theorem 1 of [5].

3. Some auxiliary lemmas

Our argument will make essential use of Xavier Tolsa's work on analytic capacity. For a compact $K \subseteq \mathbb{C}$ we define the *analytic capacity* of K by

$$\gamma(K) = \sup\{|f'(\infty)| : f \in H^\infty(\mathbb{C}_\infty \setminus K), |f(z)| \leqslant 1 \ \forall z \in \mathbb{C}^\infty \setminus K\}$$

where

$$f'(\infty) = \lim_{z \to \infty} z[f(z) - f(\infty)].$$

A good source for basic information about analytic capacity is [7].

A related capacity, γ_+, is defined by

$$\gamma_+(K) = \sup\{\sigma(K) : \sigma \geqslant 0, \text{spt } \sigma \subseteq K, C\sigma \in L^\infty(\mathbb{C}), |C\sigma(z)| \leqslant 1 \text{ for A-a.e. } z \in \mathbb{C}\}.$$

Here $\operatorname{spt} \sigma$ denotes the support of the measure σ. Since $C\sigma$ is analytic in $\mathbb{C}_\infty \setminus \operatorname{spt} \mu$ and $(C\mu)'(\infty) = -\mu(K)$ we have

$$\gamma_+(K) \leqslant \gamma(K)$$

for all compact $K \subseteq \mathbb{C}$. In 2001, Tolsa proved the astounding result that γ_+ and γ are actually equivalent [9]:

Theorem 3.1 (Tolsa). *There is an absolute constant A_T such that*

$$\gamma(K) \leqslant A_T \gamma_+(K)$$

for all compact sets $K \subseteq \mathbb{C}$.

Lemma 3.2. *Suppose ω is a compactly supported bounded function times area measure. We then have the following weak-type inequality for analytic capacity*

$$\gamma([\operatorname{Re} C\omega \geqslant a]) \leqslant \frac{A_T}{a}\|\omega\| \qquad \text{for all } a > 0,$$

where A_T is Tolsa's constant.

For a general compactly supported measure ω, $C\omega$ is only defined A-almost everywhere, so $\gamma([\operatorname{Re} C\omega \geqslant a])$ might not even make sense. The restriction we have put on ω avoids this problem since it implies that $C\omega$ is continuous and the set $[\operatorname{Re} C\omega \geqslant a]$ is compact. A proof of this Lemma can be found in [1], but we note that it is a standard argument that follows easily from the definitions that γ_+ satisfies the weak-type inequality

$$\gamma_+([\operatorname{Re} C\omega \geqslant a]) \leqslant \frac{1}{a}\|\omega\| \qquad \text{for all } a > 0.$$

Thus Lemma 3.2 follows immediately from Tolsa's Theorem.

Lemma 3.3. *There are absolute constants $\epsilon_1 > 0$ and $C_1 < \infty$ with the following property. Let $E \subset \operatorname{clos}\mathbb{D}$ be compact with $\gamma(E) < \epsilon_1$. Then*

$$|p(0)| \leqslant C_1 \int_{(\operatorname{clos}\mathbb{D})\setminus E} |p| \frac{dA}{\pi} \qquad \text{for all } p \in \mathcal{P}.$$

This is Lemma B of [1] and it is proved directly by an adaptation of Thomson's coloring scheme. In fact, using Thomson's terminology for sets E with sufficiently small analytic capacity it turns out that the measure $\chi_{\mathbb{D}\setminus E}dA$ gives rise to a sequence of heavy barriers around 0.

One can use the previous two lemmas to prove the following fact:

Theorem 3.4. *There are constants $\epsilon_0 > 0$ and $C_0 < \infty$ such that the following is true. If ν is a compactly supported measure in $\mathbb{C}$, and $\nu = \nu_1 + \nu_2$ where ν_1 and ν_2 are compactly supported measures in $\mathbb{C}$ with*

$$\operatorname{Re} C\nu_1 \geqslant 1 \qquad \text{a.e. } [A] \text{ in } \operatorname{clos}\mathbb{D}$$

and

$$\|\nu_2\| < \epsilon_0,$$

then

$$|p(0)| \leqslant C_0 \int_{|w|<1} |p(w) C\nu(w)| dA(w) \qquad \text{for all } p \in \mathcal{P}.$$

Proof. Let ν, ν_1, ν_2 satisfy the hypotheses of Theorem 3.4 with $\epsilon_0 = \epsilon_1/2A_T$. By convolving with $\frac{n^2}{\pi}\chi_{B(0,\frac{1}{n})}$ and taking limits as $n \to \infty$, we see that we may assume that the measures ν, ν_1, ν_2 are all compactly supported bounded functions times area measures, so that $C\nu$, $C\nu_1$, $C\nu_2$ are continuous, and the set $E = [-\operatorname{Re} C\nu_2 \geqslant \frac{1}{2}]$ is compact. We apply Lemma 3.2 with $a = \frac{1}{2}$ to $-\nu_2$ to get

$$\gamma(E) \leqslant 2A_T\|\nu_2\| < \epsilon_1. \tag{3.1}$$

For $w \in (\operatorname{clos}\mathbb{D}) \setminus E$ we have

$$|C\nu(w)| \geqslant \operatorname{Re} C\nu(w) > 1 - \frac{1}{2} = \frac{1}{2}. \tag{3.2}$$

By (3.1) E satisfies the hypotheses of Lemma 3.3, hence for $p \in \mathcal{P}$ we can apply that lemma together with (3.2) to obtain

$$\begin{aligned} |p(0)| &\leqslant C_1 \int_{(\operatorname{clos}\mathbb{D})\setminus E} |p| \frac{dA}{\pi} \\ &\leqslant 2C_1 \int_{w \in (\operatorname{clos}\mathbb{D})\setminus E} |p(w) C\nu(w)| \frac{dA(w)}{\pi}. \end{aligned}$$

This proves Theorem 3.4 with $C_0 = 2C_1$. □

4. The proof of Theorem 1.1

Lemma 4.1. *Let $\nu \in M_c(\mathbb{C})$ with $U = \int \frac{1}{|z|} d|\nu|(z) < \infty$, and write $U(r) = \int_{|z|<r} \frac{1}{|z|} d|\nu|(z)$.*

Then for any $r > 0$ we have

$$\frac{1}{r} |\nu|(B(0,r)) \leqslant U(r)$$

and

$$\frac{1}{\pi r^2} \int_{|w|<r} |C\nu(w) - C\nu(0)| dA(w) \leqslant 2U(r + \sqrt{r}) + \frac{2\sqrt{r}}{3} U.$$

Proof. Let $r > 0$. The first inequality is trivial. We will establish the second one. We have

$$\int_{|w|<r} |C\nu(w) - C\nu(0)| dA(w) \leqslant \int_{z\in\mathbb{C}} \left(\int_{|w|<r} \frac{|w|}{|w-z|} dA(w) \right) \frac{1}{|z|} d|\nu|(z).$$

The estimate (2.1) implies that $\int_{|w|<r} \frac{|w|}{|w-z|} dA(w) \leqslant 2\pi r^2$ for all $z \in \mathbb{C}$. Thus

$$\int_{|z|<r+\sqrt{r}} \int_{|w|<r} \frac{|w|}{|w-z|} dA(w) \frac{1}{|z|} d|\nu|(z) \leqslant 2\pi r^2 U(r + \sqrt{r}).$$

If $|z| \geqslant r + \sqrt{r}$, then we use

$$\begin{aligned} \int_{|w|<r} \frac{|w|}{|w-z|} dA(w) &\leqslant \int_{|w|<r} \frac{|w|}{|z| - |w|} dA(w) \\ &\leqslant \frac{1}{\sqrt{r}} \int_{|w|<r} |w| dA(w) = \frac{2\pi r^{5/2}}{3}. \end{aligned}$$

Hence

$$\int_{|z|\geqslant r+\sqrt{r}} \int_{|w|<r} \frac{|w|}{|w-z|} dA(w) \frac{1}{|z|} d|\nu|(z) \leqslant \frac{2\pi r^{5/2}}{3} U.$$

The lemma follows. □

Lemma 4.2. *Let $\varepsilon_0, C_0 > 0$ be as given by Theorem 3.4. Let $\nu \in M_c(\mathbb{C})$ with $\int \frac{1}{|z|} d|\nu|(z) < \infty$ and $C\nu(0) \neq 0$.*

Suppose that $r > 0$ satisfies

$$\int_{|w|<r} |C\nu(w) - C\nu(0)| \frac{dA(w)}{\pi} + 2r|\nu|(B(0,r)) < \frac{9}{32} r^2 |C\nu(0)| \tag{4.1}$$

and

$$\frac{1}{r}|\nu|(B(0,r)) < \frac{1}{4}\varepsilon_0 |C\nu(0)|. \tag{4.2}$$

Then

$$|p(0)C\nu(0)| \leqslant \frac{8C_0}{r^2} \int_{|w|<r} |p(w)C\nu(w)| dA(w)$$

for every polynomial p.

Lemma 4.1 implies that if the potential $U_{|\nu|}(0)$ is finite and if $C\nu(0) \neq 0$, then the hypothesis of this lemma is satisfied for all sufficiently small $r > 0$. Thus it is clear that Lemma 4.2 implies Theorem 1.1.

Proof. Set $C\nu(0) = a \neq 0$. For $r > 0$ satisfying (4.1) and (4.2) set $\nu_1 = \nu|\mathbb{C} \setminus B(0,r)$, $\nu_2 = \nu|B(0,r)$. We have

$$\begin{aligned}
\int_{B(0,r)} |C\nu_2| \frac{dA}{\pi} &= \int_{|z|<r} \left| \int_{|w|<r} \frac{d\nu(w)}{w - z} \right| \frac{dA(z)}{\pi} \\
&\leqslant \int_{|w|<r} \int_{|z|<r} \frac{dA(z)}{\pi|w - z|} d|\nu|(w) \\
&\leqslant 2r|\nu|(B(0,r)),
\end{aligned}$$

where we have used (2.1). Hence by (4.1)

$$\int_{B(0,r)} |C\nu_1 - a| \frac{dA}{\pi} \leqslant \int_{B(0,r)} |C\nu - a| \frac{dA}{\pi} + \int_{B(0,r)} |C\nu_2| \frac{dA}{\pi} \leqslant \frac{9}{32} r^2 |a|. \tag{4.3}$$

The Bergman space estimate ([10])

$$|f(z)| \leqslant \frac{1}{(1 - |z|^2)^2} \int_{\mathbb{D}} |f| \frac{dA}{\pi}$$

valid for f analytic in $\mathbb{D}$ and $z \in \mathbb{D}$, rescales to

$$|f(z)| \leqslant \frac{r^2}{(r^2 - |z|^2)^2} \int_{B(0,r)} |f| \frac{dA}{\pi}$$

for f analytic in $B(0,r)$ and $z \in B(0,r)$. We apply this with $f = C\nu_1 - a$ to get

$$|C\nu_1(z) - a| \leqslant \frac{16}{9r^2} \int_{B(0,r)} |C\nu_1 - a| \frac{dA}{\pi}$$

for $|z| \leqslant \frac{1}{2}r$, and combining this with (4.3) we obtain that

$$|C\nu_1(z) - a| \leqslant \frac{1}{2}|a| \tag{4.4}$$

uniformly in $|z| \leqslant \frac{1}{2}r$.

We now define measures $\hat{\nu}$, $\hat{\nu}_1$, $\hat{\nu}_2$ by the formulas

$$\begin{aligned}
\hat{\nu}(E) &= \frac{4}{ar}\nu\left(\frac{r}{2}E\right) \\
\hat{\nu}_1(E) &= \frac{4}{ar}\nu_1\left(\frac{r}{2}E\right) \\
\hat{\nu}_2(E) &= \frac{4}{ar}\nu_2\left(\frac{r}{2}E\right).
\end{aligned}$$

A calculation shows that

$$C\hat{\nu}_1(z) = \frac{2}{a}C\nu_1\left(\frac{r}{2}z\right).$$

From (4.2) and (4.4) we now see that

$$\|\hat{\nu}_2\| < \epsilon_0 \tag{4.5}$$

and

$$|C\hat{\nu}_1(z) - 2| \leqslant 1 \qquad \text{for } |z| \leqslant 1. \tag{4.6}$$

Clearly $\hat{\nu} = \hat{\nu}_1 + \hat{\nu}_2$. By (4.5) and (4.6) we thus see that $\hat{\nu}$, $\hat{\nu}_1$, $\hat{\nu}_2$ satisfy the hypotheses of Theorem 3.4, so

$$|p(0)| \leqslant C_0 \int_{|w|<1} |p(w)C\hat{\nu}(w)|dA(w)$$

for every polynomial p. It follows that each polynomial p satisfies

$$|p(0)| \leqslant \frac{8C_0}{|a|r^2} \int_{|w|<r} |p(w)C\nu(w)|dA(w)$$

and the Lemma follows. □

5. Analytic bounded point evaluations

Thomson shows in [8] that all bounded point evaluations for $P^t(\mu)$ either come from atoms of the measure μ or they are analytic bounded point evaluations, i.e., if λ_0 is a bounded point evaluation for $P^t(\mu)$ and if $\mu(\{\lambda_0\}) = 0$, then there are $C, \varepsilon > 0$ such that for all $\lambda \in B(\lambda_0, \varepsilon)$ and for all polynomials p we have

$$|p(\lambda)|^t \leq C \int |p|^t d\mu.$$

This fact also follows from our approach.

In fact, by a simple translation and rescaling argument Lemma 3.3 implies

Lemma 5.1. *There are absolute constants* $\epsilon_2 > 0$ *and* $C_2 < \infty$ *with the following property. Let* $E \subset \operatorname{clos}\mathbb{D}$ *with* $\gamma(E) < \epsilon_2$*. Then*

$$|p(\lambda)| \leqslant C_2 \int_{(\operatorname{clos}\mathbb{D})\setminus E} |p| \frac{dA}{\pi} \qquad \text{for all } p \in \mathcal{P} \text{ and all } |\lambda| < 1/2.$$

It is clear that Lemma 5.1 implies that the constants of Theorem 3.4 can be adjusted in such a way that the conclusion will be

$$|p(\lambda)| \leqslant C_0 \int_{|w|<1} |p(w)C\nu(w)| dA(w) \qquad \text{for all } p \in \mathcal{P} \text{ and all } |\lambda| < 1/2.$$

Thus the proof of Lemma 4.2 implies

Theorem 5.2. *There exists an absolute constant* $C > 0$ *such that for every* $\nu \in M_c(\mathbb{C})$ *and for every* $\lambda_0 \in \mathbb{C}$ *with* $U_{|\nu|}(\lambda_0) = \int \frac{1}{|z-\lambda_0|} d|\nu|(z) < \infty$ *there exist* $r_0 > 0$ *such that for all polynomials* p*, for all* $0 < r \leqslant r_0$*, and all* $|\lambda - \lambda_0| < r/2$ *we have*

$$|p(\lambda)C\nu(\lambda_0)| \leqslant \frac{C}{r^2} \int_{B(\lambda_0, r)} |p(z)C\nu(z)| dA(z).$$

Here r_0 *depends only on* $|C\nu(\lambda_0)|$, $U_{|\nu|}(\lambda_0)$ *and* $U_{|\nu|}(\lambda_0, r)$ *as* $r \to 0$.

Theorem 5.2 implies the statement made about analytic bounded point evaluations for $P^t(\mu)$ made at the beginning of this section. In fact, if μ is any compactly supported measure in $\mathbb{C}$, if $1 \leqslant t < \infty$, and if λ_0 is a bounded point evaluations for $P^t(\mu)$ with $\mu(\{\lambda_0\}) = 0$, then there is $h \in L^{t'}(\mu)$ such that

$$p(\lambda_0) = \int p h d\mu$$

for all polynomials p. It then follows that the measure $d\nu(z) = (z - \lambda_0) h d\mu(z)$ satisfies the hypothesis of Theorem 5.2 and $C\nu(\lambda_0) \neq 0$. Thus Theorem 5.2 and Lemma 2.2 prove the desired result. We note that this reasoning together with the explanations near the end of Section 2 also shows that every bounded point evaluation for $R^t(\mu)$ that lies in the interior of the support of μ must either come from an atom of μ or be an analytic bounded point evaluation for $R^t(\mu)$.

References

[1] Alexandru Aleman, Stefan Richter, and Carl Sundberg. Nontangential limits in $P^t(\mu)$-spaces and the index of invariant subspaces. *Ann. of Math.*, 169(2), 2009.

[2] James E. Brennan. Thomson's theorem on mean-square polynomial approximation. *Algebra i Analiz*, 17(2):1–32, 2005.

[3] James E. Brennan. Invariant subspaces and rational approximation. *J. Functional Analysis*, 7:285–310, 1971.

[4] James E. Brennan. Point evaluations, invariant subspaces and approximation in the mean by polynomials. *J. Funct. Anal.*, 34(3):407–420, 1979.

[5] James E. Brennan. The structure of certain spaces of analytic functions. *Comput. Methods Funct. Theory*, 8(1-2):625–640, 2008.

[6] Claes Fernstrøm. Bounded point evaluations and approximation in L^p by analytic functions. In *Spaces of analytic functions* (*Sem. Functional Anal. and Function Theory, Kristiansand,* 1975), pages 65–68. Lecture Notes in Math., Vol. 512. Springer, Berlin, 1976.

[7] John Garnett. *Analytic capacity and measure*. Lecture Notes in Mathematics, Vol. 297. Springer-Verlag, Berlin, 1972.

[8] James E. Thomson. Approximation in the mean by polynomials. *Ann. of Math. (2)*, 133(3):477–507, 1991.

[9] Xavier Tolsa. Painlevé's problem and the semiadditivity of analytic capacity. *Acta Math.*, 190(1):105–149, 2003.

[10] Dragan Vukotić. A sharp estimate for A^p_α functions in $\mathbf{C}^n$. *Proc. Amer. Math. Soc.*, 117(3):753–756, 1993.

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Received: January 15, 2009
Accepted: January 28, 2009

Operator Theory:
Advances and Applications, Vol. 202, 11–37

Weighted Composition Operators on the Bloch Space of a Bounded Homogeneous Domain

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In memory of Israel Gohberg.

Abstract. In this paper, we present the current results in the study of weighted composition operators on the Bloch space of bounded homogeneous domains in $\mathbb{C}^n$ with particular emphasis on the issues of boundedness and compactness. We also discuss the bounded and the compact weighted composition operators from the Bloch space to the Hardy space H^∞.

Mathematics Subject Classification (2000). 47B38, 32A18, 30D45.

Keywords. Weighted composition operators, Bloch space, Homogeneous domains.

1. Introduction

Let X be a Banach space of holomorphic functions on a domain $\Omega \subset \mathbb{C}^n$. For ψ a holomorphic function on Ω and φ a holomorphic self-map of Ω, the linear operator defined by

$$W_{\psi,\varphi}(f) = \psi(f \circ \varphi), \quad f \in X,$$

is called the *weighted composition operator* with symbols ψ and φ. Observe that $W_{\psi,\varphi}(f) = M_\psi C_\varphi(f)$ where $M_\psi(f) = \psi f$ is the *multiplication operator* with symbol ψ and $C_\varphi(f) = f \circ \varphi$ is the *composition operator* with symbol φ. If ψ is identically 1, then $W_{\psi,\varphi} = C_\varphi$, and if φ is the identity, then $W_{\psi,\varphi} = M_\psi$.

The study of weighted composition operators is fundamental in the study of Banach and Hilbert spaces of holomorphic functions. The study of the geometry of a space X is centered on the identification of the isometries on X. The connection between weighted composition operators and isometries can be traced back to Banach himself. In [5], Banach proved that the surjective isometries on $C(Q)$, the

Communicated by V. Bolotnikov.

space of continuous real-valued functions on a compact metric space Q, are of the form $Tf = \psi(f \circ \varphi)$, where $|\psi| \equiv 1$ and φ is a homeomorphism of Q onto itself.

Although the characterization of isometries is an open problem for most Banach spaces of holomorphic functions, there are many spaces for which the isometries are known. In [13], Forelli proved that the isometries on the Hardy space H^p of the open unit disk $\mathbb{D}$ (for $p \neq 2$) are certain weighted composition operators. On the Bergman space A^p of $\mathbb{D}$, Kolaski showed that the surjective isometries are weighted composition operators [17]. El-Gebeily and Wolfe showed that the isometries on the disk algebra are also weighted composition operators [12]. Thus the weighted composition operator plays a central role in the study of the isometries on several spaces of holomorphic functions.

The first study of the isometries on the Bloch space was made by Cima and Wogen in [8]. They analyzed the isometries on the subspace of the Bloch space of the open unit disk whose elements fix the origin. On this space, they showed that the surjective isometries are normalized compressions of weighted composition operators induced by disk automorphisms. In [18], Krantz and Ma extended their results to the Bloch space of the unit ball in $\mathbb{C}^n$. However, in any dimension, a description of all isometries on the entire set of Bloch functions is still unknown.

The study of weighted composition operators is not limited to the study of isometries. The properties of the weighted composition operators are not solely determined by the component operators, namely multiplication and composition operators. Indeed, there exist bounded weighted composition operators on the Bloch space for which the associated multiplication operator is not bounded. Likewise, there are compact weighted composition operators for which neither component operator is compact. Examples of such operators were provided by Ohno and Zhao in [23] in the one-dimensional case. In Sections 5 and 6, we give analogous examples for the unit ball and the unit polydisk in $\mathbb{C}^n$. Thus, the study of weighted composition operators is truly an evolutionary step in the field of composition operators.

1.1. Purpose of the paper

From the previous discussion, it is clear that the study of weighted composition operators is a worthwhile endeavor. A primary purpose of this paper is to bring the current results on the weighted composition operators on the Bloch space to one location. There are still many open questions, and thus opportunities for active research. Thus, our hope is that this exposition will inspire more work in this area. To this end, we add to this paper some new results, accompanied by some conjectures and areas for future research.

1.2. Organization of the paper

In Section 2, we review the notion of the Bloch space on the unit disk $\mathbb{D}$ and on bounded homogeneous domains. In Section 3, we outline the results known on weighted composition operators on the Bloch space and little Bloch space of $\mathbb{D}$.

These include the characterization of the bounded and the compact operators due to Ohno and Zhao and operator norm estimates.

In Section 4, we present the known results on the weighted composition operators on the Bloch space in higher dimensions. For a bounded homogeneous domain D we define quantities which we believe are proper candidates to characterize the bounded and the compact weighted composition operators on the Bloch space of D and on a subspace we refer to as the $*$-*little Bloch space*, which is a higher-dimensional analogue of the little Bloch space. We give sufficient conditions for boundedness and compactness and give operator norm estimates.

In Sections 5 and 6, we prove the conjectures presented in Sections 4 and 5 for the Bloch space on the unit ball and unit polydisk which yield results equivalent to Corollaries 1.4 and 1.6 of [31] and Theorems 1 and 2 of [32].

In Section 7, we characterize the bounded weighted composition operators from the Bloch space and the $*$-little Bloch space into the space of bounded holomorphic functions on a bounded homogeneous domain and determine the norm of such operators. As a special case, we obtain Theorem 1 of [20]. We also prove an extension of Theorem 6.1 of [16] to the unit polydisk.

Finally, in Section 8 we discuss further developments and open problems for the weighted composition operators on the Bloch space of a bounded homogeneous domain.

2. The Bloch space

The Bloch space has been defined on many types of domains in $\mathbb{C}^n$. The first such domain we will consider is the open unit disk $\mathbb{D}$. A complex-valued function f analytic on $\mathbb{D}$ is said to be *Bloch* if

$$\beta_f = \sup_{z\in\mathbb{D}} (1-|z|^2)\,|f'(z)| < \infty.$$

The mapping $f \mapsto \beta_f$ is a semi-norm on the space $\mathcal{B}(\mathbb{D})$ of Bloch functions on $\mathbb{D}$ and $\mathcal{B}(\mathbb{D})$ is a Banach space under the Bloch norm

$$||f||_{\mathcal{B}} = |f(0)| + \beta_f.$$

By the Schwarz-Pick lemma, the space $H^\infty(\mathbb{D})$ of bounded analytic functions on $\mathbb{D}$ is a subset of $\mathcal{B}(\mathbb{D})$ and the containment is proper, since $z \mapsto \mathrm{Log}(1-z)$ is a Bloch function, where Log denotes the principal branch of the logarithm. The *little Bloch space* $\mathcal{B}_0(\mathbb{D})$ on $\mathbb{D}$ is defined as the set of Bloch functions f such that

$$\lim_{|z|\to 1} (1-|z|^2)\,|f'(z)| = 0.$$

The little Bloch space is a separable subspace of $\mathcal{B}(\mathbb{D})$ since the polynomials form a dense subset. Useful references on Bloch functions and the Bloch space on $\mathbb{D}$ include [24], [4] and [7].

As an immediate consequence of the Schwarz-Pick lemma, if $f \in \mathcal{B}(\mathbb{D})$ and φ is an analytic self-map of $\mathbb{D}$, then $f\circ\varphi \in \mathcal{B}(\mathbb{D})$. Furthermore, $\beta_f = \beta_{f\circ\varphi}$ for any conformal automorphism φ of $\mathbb{D}$, that is, the Bloch space is *Möbius invariant.* In fact, it is the largest Möbius invariant Banach space [28].

The notion of Bloch function in higher dimensions was introduced by Hahn in [14]. In [26] and [27] Timoney studied extensively the space of Bloch functions on a bounded homogeneous domain and its subspace known as the little Bloch space on a bounded symmetric domain.

Every bounded domain $D \subset \mathbb{C}^n$ is endowed with a canonical metric called the *Bergman metric*, which is invariant under the action of the group of biholomorphic transformations, which we call *automorphisms* and denote by $\mathrm{Aut}(D)$ [15]. We will focus on a particular class of domains in $\mathbb{C}^n$, the homogeneous domains. A domain D in $\mathbb{C}^n$ is called *homogeneous* if $\mathrm{Aut}(D)$ acts transitively on D, that is, for all $z_1, z_2 \in D$, there exists $\phi \in \mathrm{Aut}(D)$ such that $\phi(z_1) = z_2$.

A domain $D \subset \mathbb{C}^n$ is *symmetric at a point* $z_0 \in D$ if there exists $\phi \in \mathrm{Aut}(D)$ such that $\phi \circ \phi$ is the identity and z_0 is an isolated fixed point of ϕ. A domain is *symmetric* if it is symmetric at each of its points. A symmetric domain is homogeneous and a homogeneous domain that is symmetric at a single point is symmetric. Therefore the unit ball $\mathbb{B}_n$ and the unit polydisk $\mathbb{D}^n$ are symmetric, since they are homogeneous and symmetric at the origin via $z \mapsto -z$.

Let D be a bounded homogeneous domain in $\mathbb{C}^n$. A holomorphic function $f : D \to \mathbb{C}$ is said to be a Bloch function if $\beta_f = \sup_{z \in D} Q_f(z)$ is finite, where

$$Q_f(z) = \sup_{u \in \mathbb{C}^n \setminus \{0\}} \frac{|(\nabla f)(z)u|}{H_z(u, \overline{u})^{1/2}},$$

$(\nabla f)(z)u = \langle \nabla f(z), \overline{u} \rangle = \sum_{k=1}^{n} \frac{\partial f}{\partial z_k}(z) u_k$, and H_z is the Bergman metric on D at z. By fixing a base point $z_0 \in D$, the Bloch space $\mathcal{B}(D)$ is a Banach space under the norm $||f||_{\mathcal{B}} = |f(z_0)| + \beta_f$ [26]. For convenience, we assume the domain D to contain the origin and take $z_0 = 0$.

In [26], Timoney proved that the space $H^\infty(D)$ of bounded holomorphic functions on a bounded homogeneous domain D is a subspace of $\mathcal{B}(D)$ and for each $f \in H^\infty(D)$, $||f||_{\mathcal{B}} \leq c_D \, ||f||_\infty$ where c_D is a constant depending only on the domain D. The precise value of the best bound c_D has been calculated in [9] and [30] when D is a bounded symmetric domain.

In Theorem 3.1 of [2], we showed that the Bloch functions on D are precisely the Lipschitz maps between the metric spaces D and $\mathbb{C}$ under the Bergman metric and Euclidean metric, respectively. Furthermore

$$\beta_f = \sup_{z \neq w} \frac{|f(z) - f(w)|}{\rho(z, w)}, \tag{2.1}$$

where ρ is the Bergman distance. In particular, for all $z, w \in D$,

$$|f(z) - f(w)| \leq ||f||_{\mathcal{B}} \, \rho(z, w). \tag{2.2}$$

In [27] the little Bloch space on the unit ball was defined as

$$\mathcal{B}_0(\mathbb{B}_n) = \left\{ f \in \mathcal{B}(\mathbb{B}_n) : \lim_{||z||\to 1} Q_f(z) = 0 \right\},$$

which is precisely the closure of the polynomials in $\mathcal{B}(\mathbb{B}_n)$. If D is a bounded symmetric domain in $\mathbb{C}^n$ other than $\mathbb{B}_n$, the set of functions f for which $Q_f(z) \to 0$ as z approaches the boundary ∂D of D consists only of the constant functions, so $\mathcal{B}_0(D)$ is defined as the closure of the polynomials in $\mathcal{B}(D)$. The $*$-*little Bloch space* is defined as

$$\mathcal{B}_{0^*}(D) = \left\{ f \in \mathcal{B}(D) : \lim_{z\to\partial^* D} Q_f(z) = 0 \right\},$$

where $\partial^* D$ denotes the distinguished boundary of D. The unit ball is the only bounded symmetric domain D for which $\partial D = \partial^* D$, so that $\mathcal{B}_0(\mathbb{B}_n) = \mathcal{B}_{0^*}(\mathbb{B}_n)$. If $D \neq \mathbb{B}_n$, $\mathcal{B}_0(D)$ is a proper subspace of $\mathcal{B}_{0^*}(D)$ and $\mathcal{B}_{0^*}(D)$ is a non-separable subspace of $\mathcal{B}(D)$.

3. Weighted composition operators on the Bloch space of $\mathbb{D}$

The first results on weighted composition operators on the Bloch space of the unit disk were obtained by Ohno and Zhao in 2001 [23]. For ψ an analytic function on $\mathbb{D}$, φ an analytic self-map of $\mathbb{D}$, and $z \in \mathbb{D}$, define $s_{\psi,\varphi} = \sup_{z\in\mathbb{D}} s_{\psi,\varphi}(z)$ and $\tau_{\psi,\varphi} = \sup_{z\in\mathbb{D}} \tau_{\psi,\varphi}(z)$ where

$$s_{\psi,\varphi}(z) = (1 - |z|^2)\,|\psi'(z)| \log \frac{2}{1 - |\varphi(z)|^2},$$

$$\tau_{\psi,\varphi}(z) = \frac{1 - |z|^2}{1 - |\varphi(z)|^2}\,|\varphi'(z)|\,|\psi(z)|\,.$$

Theorem 3.1. *Let ψ be an analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Then*

(a) ([23], **Theorems 1 and 2).** *$W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{D})$ if and only if $s_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite. Furthermore, the bounded operator $W_{\psi,\varphi}$ is compact on $\mathcal{B}(\mathbb{D})$ if and only if*

$$\lim_{|\varphi(z)|\to 1} s_{\psi,\varphi}(z) = \lim_{|\varphi(z)|\to 1} \tau_{\psi,\varphi}(z) = 0.$$

(b) ([23], **Theorems 3 and 4).** *$W_{\psi,\varphi}$ is bounded on the little Bloch space $\mathcal{B}_0(\mathbb{D})$ if and only if $\psi \in \mathcal{B}_0(\mathbb{D})$, $s_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite, and*

$$\lim_{|z|\to 1} |\psi(z)|\,|\varphi'(z)|\,(1 - |z|^2) = 0.$$

Furthermore, the bounded operator $W_{\psi,\varphi}$ is compact on $\mathcal{B}_0(\mathbb{D})$ if and only if

$$\lim_{|z|\to 1} s_{\psi,\varphi}(z) = \lim_{|z|\to 1} \tau_{\psi,\varphi}(z) = 0.$$

In [1], we established estimates on the norm of the weighted composition operator $W_{\psi,\varphi}$ on $\mathcal{B}(\mathbb{D})$ in terms of $\tau_{\psi,\varphi}$ and the quantity

$$\sigma_{\psi,\varphi} = \sup_{z\in\mathbb{D}} \frac{1}{2}(1-|z|^2)\,|\psi'(z)|\log\frac{1+|\varphi(z)|}{1-|\varphi(z)|},$$

which is closely related to $s_{\psi,\varphi}$ but is more amenable to a higher-dimensional interpretation since the factor $\frac{1}{2}\log\frac{1+|\varphi(z)|}{1-|\varphi(z)|}$ is precisely the Bergman distance between 0 and $\varphi(z)$.

Theorem 3.2. *Let ψ be analytic on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. Then*

(a) *$W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{D})$ if and only if $\psi \in \mathcal{B}(\mathbb{D})$, and $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite. Furthermore,*

$$||W_{\psi,\varphi}|| \geq \max\left\{||\psi||_{\mathcal{B}}, \frac{1}{2}|\psi(0)|\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right\} \tag{3.1}$$

$$||W_{\psi,\varphi}|| \leq \max\left\{||\psi||_{\mathcal{B}}, \frac{1}{2}|\psi(0)|\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|} + \tau_{\psi,\varphi} + \sigma_{\psi,\varphi}\right\}. \tag{3.2}$$

Furthermore, $W_{\psi,\varphi}$ is compact if and only if

$$\lim_{|\varphi(z)|\to 1}\sigma_{\psi,\varphi}(z) = \lim_{|\varphi(z)|\to 1}\tau_{\psi,\varphi}(z) = 0.$$

(b) *$W_{\psi,\varphi}$ is bounded on $\mathcal{B}_0(\mathbb{D})$ if and only if $\psi \in \mathcal{B}_0(\mathbb{D})$, $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite, and*

$$\lim_{|z|\to 1}(1-|z|^2)\,|\psi(z)|\,|\varphi'(z)| = 0.$$

Inequalities (3.1) *and* (3.2) *hold. Furthermore, $W_{\psi,\varphi}$ is compact if and only if*

$$\lim_{|z|\to 1}\sigma_{\psi,\varphi}(z) = \lim_{|z|\to 1}\tau_{\psi,\varphi}(z) = 0.$$

Proof. Assume $W_{\psi,\varphi}$ is bounded. Using as a test function the constant 1, we obtain $\psi \in \mathcal{B}(\mathbb{D})$. Since $\sigma_{\psi,\varphi} \leq s_{\psi,\varphi}$, by Theorem 3.1, it follows that $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite. The estimates (3.1) and (3.2) follow from Theorems 2.1 and 2.2 of [1]. Conversely, assume $\psi \in \mathcal{B}(\mathbb{D})$, and $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite. By the calculation carried out in [1], $W_{\psi,\varphi}$ maps $\mathcal{B}(\mathbb{D})$ into itself and estimates (3.1) and (3.2) hold. Thus $W_{\psi,\varphi}$ is bounded. Observing that for each $z \in \mathbb{D}$, $\sigma_{\psi,\varphi(z)} \leq s_{\psi,\varphi}(z)$ and for $|\varphi(z)| \geq \frac{1}{2}$, $s_{\psi,\varphi}(z) \leq 2\sigma_{\psi,\varphi}(z)$, the characterization of the compactness follows at once from Theorem 3.1. The proof of part (b) is analogous. □

The above estimates agree with the norm estimates for the composition operators on $\mathcal{B}(\mathbb{D})$ in [29] when ψ is taken to be the constant function 1.

4. Weighted composition operators on the Bloch space of a bounded homogeneous domain

Let D be a bounded homogeneous domain in $\mathbb{C}^n$. For $z \in D$, define

$$\omega(z) = \sup\{|f(z)| : f \in \mathcal{B}(D), f(0) = 0 \text{ and } ||f||_{\mathcal{B}} \leq 1\},$$
$$\omega_0(z) = \sup\{|f(z)| : f \in \mathcal{B}_{0^*}(D), f(0) = 0, \text{ and } ||f||_{\mathcal{B}} \leq 1\}.$$

Lemma 4.1. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$. For each $z \in D$, $\omega(z)$ and $\omega_0(z)$ are finite. In fact, $\omega_0(z) \leq \omega(z) \leq \rho(z, 0)$.*

Proof. Let $z \in D$. The inequality $\omega_0(z) \leq \omega(z)$ is obvious. For $f \in \mathcal{B}(D)$, by (2.1), $|f(z) - f(0)| \leq \rho(z, 0)\beta_f$. By taking the supremum over all $f \in \mathcal{B}(D)$ such that $f(0) = 0$ and $||f||_{\mathcal{B}} \leq 1$, we have $\omega(z) \leq \rho(z, 0)$. □

Remark 1. By Theorems 3.9 and 3.14 in [33], it follows immediately that for all $z \in \mathbb{B}_n$, $\omega_0(z) = \omega(z) = \rho(z, 0)$ where

$$\rho(z, 0) = \frac{1}{2} \log \frac{1 + ||z||}{1 - ||z||}.$$

It is unknown whether there are other domains for which either equality holds. The following lemma shows the relationship between point evaluation of Bloch functions (respectively, little Bloch functions) and ω (respectively, ω_0).

Lemma 4.2. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$ and let $f \in \mathcal{B}(D)$ (respectively, $f \in \mathcal{B}_{0^*}(D)$). Then for all $z \in \mathbb{D}$, we have*

$$|f(z)| \leq |f(0)| + \omega(z)\beta_f,$$

(respectively, $|f(z)| \leq |f(0)| + \omega_0(z)\beta_f$).

Proof. Let $f \in \mathcal{B}(D)$. The result is immediate if f is constant. For f non-constant and $z \in D$, the function defined by

$$g(z) = \frac{1}{\beta_f}(f(z) - f(0))$$

is Bloch and satisfies the conditions $g(0) = 0$ and $Q_g(z) = \frac{1}{\beta_f} Q_f(z)$ for all $z \in D$. Thus, $||g||_{\mathcal{B}} = 1$, so $|g(z)| \leq \omega(z)$ for all $z \in D$. Consequently,

$$|f(z)| \leq |f(0)| + |f(z) - f(0)| = |f(0)| + |g(z)|\,\beta_f \leq |f(0)| + \omega(z)\beta_f.$$

The proof for the case $f \in \mathcal{B}_{0^*}(D)$ is analogous. □

For $z \in D$, denote by $J\varphi(z)$ the Jacobian matrix of φ at z (i.e., the matrix whose (j,k)-entry is $\frac{\partial \varphi_j}{\partial z_k}(z)$). Define the *Bergman constant* of φ by $B_\varphi = \sup_{z\in D} B_\varphi(z)$, where for $z \in D$

$$B_\varphi(z) = \sup_{u \in \mathbb{C}^n \setminus \{0\}} \frac{H_{\varphi(z)}(J\varphi(z)u, \overline{J\varphi(z)u})^{1/2}}{H_z(u, \overline{u})^{1/2}}.$$

In [2], the Bergman constant was used for the study of composition operators on the Bloch space. Specifically, for $f \in \mathcal{B}(D)$,

$$Q_{f\circ\varphi}(z) \leq B_\varphi(z) Q_f(\varphi(z)) \tag{4.1}$$

for all $z \in D$. Letting

$$\begin{aligned} T_{0,\varphi}(z) &= \sup\{Q_{f\circ\varphi}(z) : f \in \mathcal{B}_{0^*}(D), \beta_f \leq 1\}, \\ T_\varphi(z) &= \sup\{Q_{f\circ\varphi}(z) : f \in \mathcal{B}(D), \beta_f \leq 1\}, \end{aligned}$$

from (4.1), it follows that

$$T_{0,\varphi}(z) \leq T_\varphi(z) \leq B_\varphi(z) \tag{4.2}$$

for each $z \in D$. Moreover, for each $f \in \mathcal{B}(D)$ (respectively, $\mathcal{B}_{0^*}(D)$) and $z \in D$,

$$\begin{aligned} Q_{f\circ\varphi}(z) &\leq T_\varphi(z)\beta_f \qquad (4.3)\\ \text{(respectively, } Q_{f\circ\varphi}(z) &\leq T_{0,\varphi}(z)\beta_f). \end{aligned}$$

For $z \in \mathbb{D}$, by Remark 1, we have

$$\begin{aligned} \omega(\varphi(z)) &= \frac{1}{2}\log\frac{1+|\varphi(z)|}{1-|\varphi(z)|} \quad \text{and} \\ T_{0,\varphi}(z) &= T_\varphi(z) = \frac{(1-|z|^2)\,|\varphi'(z)|}{1-|\varphi(z)|^2} = B_\varphi(z), \end{aligned}$$

since the right-hand side of the above formula equals $(1-|z|^2)|(f\circ\varphi)'(z)|$ for

$$f(w) = \frac{\varphi(z) - w}{1-\overline{\varphi(z)}w}, \quad w \in \mathbb{D},$$

which is in the little Bloch space.

For a bounded homogeneous domain D in $\mathbb{C}^n$, ψ holomorphic on D, and φ holomorphic self-map of D, we define

$$\begin{aligned} \sigma_{\psi,\varphi} &= \sup_{z\in D}\, \omega(\varphi(z))Q_\psi(z), & \tau_{\psi,\varphi} &= \sup_{z\in D}\, |\psi(z)|\, T_\varphi(z), \\ \sigma_{0,\psi,\varphi} &= \sup_{z\in D}\, \omega_0(\varphi(z))Q_\psi(z), & \tau_{0,\psi,\varphi} &= \sup_{z\in D}\, |\psi(z)|\, T_{0,\varphi(z)}. \end{aligned}$$

In the case of the unit disk, $\sigma_{\psi,\varphi} = \sigma_{0,\psi,\varphi}$, $\tau_{\psi,\varphi} = \tau_{0,\psi,\varphi}$, and these quantities agree with the expressions in the previous section.

Theorem 4.3. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$ and φ a holomorphic self-map of D. If $\psi \in \mathcal{B}(D)$, and $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite, then $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(D)$ and*

$$\max\{\|\psi\|_{\mathcal{B}}, |\psi(0)|\,\omega(\varphi(0))\} \leq \|W_{\psi,\varphi}\| \leq \max\{\|\psi\|_{\mathcal{B}}, |\psi(0)|\,\omega(\varphi(0)) + \tau_{\psi,\varphi} + \sigma_{\psi,\varphi}\}.$$

Proof. We begin by proving the upper estimate. Let $f \in \mathcal{B}(D)$. Then for $z \in D$, by the product rule we have

$$\nabla(\psi(f\circ\varphi))(z) = \psi(z)\nabla(f\circ\varphi)(z) + f(\varphi(z))\nabla(\psi)(z),$$

so for all $u \in \mathbb{C}^n \setminus \{0\}$,

$$|\nabla(\psi(f \circ \varphi))(z)u| \leq |\psi(z)|\, |\nabla(f)(\varphi(z))J\varphi(z)u| + |f(\varphi(z))|\, |\nabla(\psi)(z)u|.$$

By (4.3) and Lemma 4.2, we obtain

$$\sup_{z\in D} Q_{\psi(f\circ\varphi)}(z) \leq \tau_{\psi,\varphi}\beta_f + |f(0)|\, \beta_\psi + \sup_{z\in D} \omega(\varphi(z))Q_\psi(z)\beta_f,$$

which is finite. So $W_{\psi,\varphi}f \in \mathcal{B}(D)$ and again by Lemma 4.2

$$\begin{aligned} ||W_{\psi,\varphi}f||_{\mathcal{B}} &\leq |\psi(0)|\, |f(\varphi(0))| + |f(0)|\, \beta_\psi + (\tau_{\psi,\varphi} + \sigma_{\psi,\varphi})\beta_f \\ &\leq |\psi(0)|\, (|f(0)| + \omega(\varphi(0))\beta_f) + |f(0)|\, \beta_\psi + (\tau_{\psi,\varphi} + \sigma_{\psi,\varphi})\beta_f \\ &= ||\psi||_{\mathcal{B}}\, ||f||_{\mathcal{B}} + (|\psi(0)|\omega(\varphi(0)) + \tau_{\psi,\varphi} + \sigma_{\psi,\varphi} - ||\psi||_{\mathcal{B}})\beta_f. \end{aligned}$$

If $|\psi(0)|\, \omega(\varphi(0)) + \tau_{\psi,\varphi} + \sigma_{\psi,\varphi} \leq ||\psi||_{\mathcal{B}}$, then $||W_{\psi,\varphi}f||_{\mathcal{B}} \leq ||\psi||_{\mathcal{B}}\, ||f||_{\mathcal{B}}$. Otherwise, $||W_{\psi,\varphi}f||_{\mathcal{B}} \leq (|\psi(0)|\, \omega(\varphi(0)) + \tau_{\psi,\varphi} + \sigma_{\psi,\varphi})\, ||f||_{\mathcal{B}}$. Thus, $W_{\psi,\varphi}$ is bounded and

$$||W_{\psi,\varphi}||_{\mathcal{B}} \leq \max\{||\psi||_{\mathcal{B}}\, , |\psi(0)|\, \omega(\varphi(0)) + \tau_{\psi,\varphi} + \sigma_{\psi,\varphi}\}.$$

To prove the lower estimate, observe that by considering as test function the constant function 1, we have $||W_{\psi,\varphi}1||_{\mathcal{B}} = ||\psi||_{\mathcal{B}}$, so that $||W_{\psi,\varphi}|| \geq ||\psi||_{\mathcal{B}}$. Furthermore

$$\begin{aligned} ||W_{\psi,\varphi}|| &= \sup\{||W_{\psi,\varphi}f||_{\mathcal{B}} : f \in \mathcal{B}(D) \text{ and } ||f||_{\mathcal{B}} \leq 1\} \\ &\geq \sup\{||W_{\psi,\varphi}f||_{\mathcal{B}} : f \in \mathcal{B}(D), f(0) = 0, \text{ and } ||f||_{\mathcal{B}} \leq 1\} \\ &\geq \sup\{|\psi(0)|\, |f(\varphi(0))| : f \in \mathcal{B}(D), f(0) = 0, \text{ and } ||f||_{\mathcal{B}} \leq 1\} \\ &= |\psi(0)|\, \omega(\varphi(0)). \end{aligned}$$

Thus $||W_{\psi,\varphi}|| \geq \max\{||\psi||_{\mathcal{B}}\, , |\psi(0)|\, \omega(\varphi(0))\}$. □

Theorem 4.4. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$. If $\psi \in \mathcal{B}_{0^*}(D)$, $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite, and*

$$\lim_{z\to\partial^* D} |\psi(z)|\, T_{0,\varphi}(z) = \lim_{z\to\partial^* D} \omega_0(\varphi(z))Q_\psi(z) = 0,$$

then $W_{\psi,\varphi}$ is bounded on $\mathcal{B}_{0^*}(D)$ and

$$\max\{||\psi||_{\mathcal{B}}\, , |\psi(0)|\, \omega_0(\varphi(0))\} \leq ||W_{\psi,\varphi}|| \leq \max\{||\psi||_{\mathcal{B}}\, , |\psi(0)|\, \omega_0(\varphi(0)) + \tau_{\psi,\varphi} + \sigma_{0,\psi,\varphi}\}.$$

Proof. Arguing as in the proof of Theorem 4.3, it suffices to show that if $\psi \in \mathcal{B}_{0^*}(D)$, $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite, and

$$\lim_{z\to\partial^* D} |\psi(z)|\, T_{0,\varphi}(z) = \lim_{z\to\partial^* D} \omega_0(\varphi(z))Q_\psi(z) = 0,$$

then $W_{\psi,\varphi}$ maps the $*$-little Bloch space into itself. Let $f \in \mathcal{B}_{0^*}(D)$. Without loss of generality, we may assume $||f||_{\mathcal{B}} \leq 1$. For $z \in D$, by Lemma 4.2, we have

$$\begin{aligned} Q_{\psi(f\circ\varphi)}(z) &\leq |\psi(z)|\, Q_{f\circ\varphi}(z) + |f(\varphi(z))|\, Q_\psi(z) \\ &\leq |\psi(z)|\, T_{0,\varphi}(z) + |f(0)|\, Q_\psi(z) + \omega_0(\varphi(z))Q_\psi(z), \end{aligned}$$

which approaches 0 as $z \to \partial^* D$. Thus $\psi(f \circ \varphi) \in \mathcal{B}_{0^*}(D)$. □

Theorem 4.5. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$, ψ a holomorphic function on D, and φ a holomorphic self-map of D. If $W_{\psi,\varphi}$ is bounded on the Bloch space of D, then $\psi \in \mathcal{B}(D)$ and $\sigma_{\psi,\varphi}$ is finite if and only if $\tau_{\psi,\varphi}$ is finite.*

Proof. First observe that $\psi = W_{\psi,\varphi}1 \in \mathcal{B}(D)$. Let $f \in \mathcal{B}(D)$, $z \in D$ and $u \in \mathbb{C}^n \setminus \{0\}$. Then

$$\begin{aligned}\frac{|f(\varphi(z))|\,|\nabla(\psi)(z)u|}{H_z(u,\overline{u})^{1/2}} &= \frac{|\nabla(\psi(f\circ\varphi))(z)u - \psi(z)\nabla(f\circ\varphi)(z)u|}{H_z(u,\overline{u})^{1/2}} \\ &\leq \frac{|\nabla(\psi(f\circ\varphi))(z)u|}{H_z(u,\overline{u})^{1/2}} + \frac{|\psi(z)|\,|\nabla(f\circ\varphi)(z)u|}{H_z(u,\overline{u})^{1/2}}\end{aligned}$$

Taking the supremum over all $u \in \mathbb{C}^n \setminus \{0\}$, and using (4.3) we get

$$\begin{aligned}|f(\varphi(z))|\,Q_\psi(z) &\leq Q_{\psi(f\circ\varphi)}(z) + |\psi(z)|\,Q_{f\circ\varphi}(z) \\ &\leq \beta_{\psi(f\circ\varphi)} + |\psi(z)|\,T_\varphi(z)\beta_f \\ &\leq (||W_{\psi,\varphi}|| + |\psi(z)|\,T_\varphi(z))\,||f||_{\mathcal{B}}\,.\end{aligned}$$

Taking the supremum over all $f \in \mathcal{B}(D)$ with $f(0) = 0$ and $||f||_{\mathcal{B}} \leq 1$, we have

$$\omega(\varphi(z))Q_\psi(z) \leq ||W_{\psi,\varphi}|| + |\psi(z)|\,T_\varphi(z).$$

Thus $\sigma_{\psi,\varphi} \leq ||W_{\psi,\varphi}|| + \tau_{\psi,\varphi}$.

On the other hand, for $g \in \mathcal{B}(D)$, with $g(0) = 0$ and $||g||_{\mathcal{B}} \leq 1$, using Lemma 4.2, we also obtain

$$\begin{aligned}|\psi(z)|\,Q_{g\circ\varphi}(z) &\leq Q_{\psi(g\circ\varphi)}(z) + |g(\varphi(z))|\,Q_\psi(z) \\ &\leq ||W_{\psi,\varphi}g||_{\mathcal{B}} + \omega(\varphi(z))Q_\psi(z) \\ &\leq ||W_{\psi,\varphi}|| + \sigma_{\psi,\varphi}.\end{aligned}$$

More generally, for any non-constant function $f \in \mathcal{B}(D)$, with $\beta_f \leq 1$, letting $g = (f - f(0))/\beta_f$, by the previous case, we obtain

$$|\psi(z)|\,Q_{f\circ\varphi}(z) = |\psi(z)|\,Q_{g\circ\varphi}(z)\beta_f \leq ||W_{\psi,\varphi}|| + \sigma_{\psi,\varphi}.$$

Taking the supremum over all such functions f, we deduce $\tau_{\psi,\varphi} \leq ||W_{\psi,\varphi}|| + \sigma_{\psi,\varphi}$. Consequently, $\sigma_{\psi,\varphi}$ is finite if and only if $\tau_{\psi,\varphi}$ is finite. □

The proof of the following result is analogous.

Proposition 4.6. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$, ψ a holomorphic function on D, and φ a holomorphic self-map of D. If $W_{\psi,\varphi}$ is bounded on the $*$-little Bloch space of D, then $\psi \in \mathcal{B}_{0^*}(D)$ and $\sigma_{0,\psi,\varphi}$ is finite if and only if $\tau_{0,\psi,\varphi}$ is finite.*

We shall now give a sufficient condition for the compactness of $W_{\psi,\varphi}$ which yields Theorem 3 of [25] in the special case when ψ is identically one. We first need the following result.

Lemma 4.7. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$, ψ a holomorphic function on D, and φ a holomorphic self-map of D. Then $W_{\psi,\varphi}$ is compact on $\mathcal{B}(D)$ if and only if for each bounded sequence $\{f_k\}$ in $\mathcal{B}(D)$ converging to* 0 *locally uniformly in D, $||\psi(f_k \circ \varphi)||_{\mathcal{B}} \to 0$, as $k \to \infty$.*

Proof. Assume $W_{\psi,\varphi}$ is compact on $\mathcal{B}(D)$. Let $\{f_k\}$ be a bounded sequence in $\mathcal{B}(D)$ which converges to 0 locally uniformly in D. By rescaling f_k, we may assume $||f_k||_{\mathcal{B}} \leq 1$ for all $k \in \mathbb{N}$. We need to show that $||\psi(f_k \circ \varphi)||_{\mathcal{B}} \to 0$ as $k \to \infty$. Since $W_{\psi,\varphi}$ is compact, the sequence $\{\psi(f_k \circ \varphi)\}$ has a subsequence (which for convenience we reindex as the original sequence) which converges in the Bloch norm to some function $f \in \mathcal{B}(D)$. We are going to show that f is identically 0 by proving that $\psi(f_k \circ \varphi) \to 0$ locally uniformly. Fix $z_0 \in D$ and, without loss of generality, assume $f(z_0) = 0$. Then $\psi(z_0)f_k(\varphi(z_0)) \to 0$ as $k \to \infty$. For $z \in D$, by (2.2), we obtain

$$\begin{aligned}|\psi(z)f_k(\varphi(z)) - f(z)| &\leq |\psi(z)f_k(\varphi(z)) - f(z) - (\psi(z_0)f_k(\varphi(z_0)) - f(z_0))| \\ &\quad + |\psi(z_0)f_k(\varphi(z_0))| \\ &\leq ||\psi(f_k \circ \varphi) - f||_{\mathcal{B}}\, \rho(z, z_0) + |\psi(z_0)f_k(\varphi(z_0))| \to 0\end{aligned}$$

locally uniformly as $k \to \infty$, since $\psi(f_k \circ \varphi) - f \to 0$ in norm. On the other hand, $\psi(f_k \circ \varphi) \to 0$ locally uniformly, so f must be identically 0.

Next, assume $||\psi(g_n \circ \varphi)||_{\mathcal{B}} \to 0$ as $k \to \infty$ for each bounded sequence $\{g_k\}$ in $\mathcal{B}(D)$ converging to 0 locally uniformly in D. To prove the compactness of $W_{\psi,\varphi}$, it suffices to show that if $\{f_k\}$ is a sequence in $\mathcal{B}(D)$ with $||f_k||_{\mathcal{B}} \leq 1$ for all $k \in \mathbb{N}$, there exists a subsequence $\{f_{k_j}\}$ such that $\psi(f_{k_j} \circ \varphi)$ converges in $\mathcal{B}(D)$. Fix $z_0 \in D$. Replacing f_k with $f_k - f_k(z_0)$, we may assume that $f_k(z_0) = 0$ for all $k \in \mathbb{N}$. By (2.1), $|f_k(z)| \leq \rho(z, z_0)$, for each $z \in D$. Thus, on each closed ball centered at z_0 with respect to the Bergman distance, the sequence $\{f_k\}$ is uniformly bounded, and hence also on each compact subset of D. By Montel's theorem, some subsequence $\{f_{k_j}\}$ converges locally uniformly to some function f holomorphic on D. By Theorem 3.3 of [2], f is a Bloch function and $||f||_{\mathcal{B}} \leq 1$. Then, letting $g_{k_j} = f_{k_j} - f$, we obtain a bounded sequence in $\mathcal{B}(D)$ converging to 0 locally uniformly in D. Thus, by the hypothesis, $\left|\left|\psi(g_{k_j} \circ \varphi)\right|\right|_{\mathcal{B}} \to 0$ as $k \to \infty$. Therefore, $\psi(f_{k_j} \circ \varphi)$ converges in norm to $\psi(f \circ \varphi)$, completing the proof. □

Theorem 4.8. *Let D be a bounded homogeneous domain in $\mathbb{C}^n$, ψ a holomorphic function on D, and φ a holomorphic self-map of D. If $\psi \in \mathcal{B}(D)$, then $W_{\psi,\varphi}$ is compact on the Bloch space of D if*

$$\lim_{\varphi(z)\to\partial D} \omega(\varphi(z))Q_\psi(z) = 0 \ \textit{and} \ \lim_{\varphi(z)\to\partial D} |\psi(z)|\, T_\varphi(z) = 0. \tag{4.4}$$

Proof. Assume the conditions in (4.4) hold. By Lemma 4.7, to prove that $W_{\psi,\varphi}$ is compact on $\mathcal{B}(D)$ it suffices to show that for any sequence $\{f_k\}$ in $\mathcal{B}(D)$ converging to 0 locally uniformly in D such that $||f_k||_{\mathcal{B}} \leq 1$, $||\psi(f_k \circ \varphi)||_{\mathcal{B}} \to 0$ as $k \to \infty$. Let $\{f_k\}$ be such a sequence and fix $\epsilon > 0$. Then $|f_k(0)| < \epsilon/(3\,||\psi||_{\mathcal{B}})$ for all k sufficiently large and there exists r such that for all $k \in \mathbb{N}$, $|\psi(z)|\, Q_{f_k\circ\varphi}(z) < \epsilon/3$

and $\omega(\varphi(z))Q_\psi(z) < \epsilon/3$, whenever $\rho(\varphi(z), \partial D) < r$. Thus by Lemma 4.2, if $\rho(\varphi(z), \partial D) < r$, then

$$\begin{aligned} Q_{\psi(f_k\circ\varphi)}(z) &\leq |\psi(z)|\, Q_{f_k\circ\varphi}(z) + |f_k(\varphi(z))|\, Q_\psi(z) \\ &< \frac{\epsilon}{3} + (|f_k(0)| + \omega(\varphi(z)))Q_\psi(z) < \epsilon. \end{aligned}$$

On the other hand, since $f_k \to 0$ locally uniformly in D, $|f_k(\varphi(z))| \to 0$ and $Q_{f_k\circ\varphi} \to 0$ uniformly on the set $\{z \in D : \rho(\varphi(z), \partial D) \geq r\}$. Consequently, for all k sufficiently large, $Q_{\psi(f_k\circ\varphi)}(z) < \epsilon$ for all $z \in D$. Furthermore, $|\psi(0)f_k(\varphi(0))| \to 0$ as $k \to \infty$, so $||\psi(f_k \circ \varphi)||_{\mathcal{B}} \to 0$, completing the proof. □

Remark 2. Even for composition operators, the necessity of the analogue to Theorem 4.8 was established for the unit ball and polydisk [25], but not for general bounded homogeneous domains.

We end the section with the following conjecture.

Conjecture. Let D be a bounded homogeneous domain in $\mathbb{C}^n$, ψ a holomorphic function on D, and φ a holomorphic self-map of D. Then $W_{\psi,\varphi}$ is bounded on the Bloch space of D if and only if $\psi \in \mathcal{B}(D)$, and $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite. Furthermore, the bounded operator $W_{\psi,\varphi}$ is compact on $\mathcal{B}(D)$ if and only if

$$\lim_{\varphi(z)\to\partial D} \omega(\varphi(z))Q_\psi(z) = \lim_{\varphi(z)\to\partial D} |\psi(z)|\, T_\varphi(z) = 0.$$

In the next two sections, we prove the above conjecture when D is the unit ball or the unit polydisk.

5. Special case: The unit ball

In Theorem 3.1 of [33], the following useful formula for calculating the Bloch seminorm of a function $f \in \mathcal{B}(\mathbb{B}_n)$ was given. For $z \in \mathbb{B}_n$

$$Q_f(z) = (1 - ||z||^2)^{1/2} \left(||\nabla(f)(z)||^2 - \left| \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z) \right|^2 \right)^{1/2}. \tag{5.1}$$

Zhou and Chen characterized the bounded and the compact weighted composition operators on the Bloch space of the unit ball under the norm

$$|f(0)| + \sup_{z\in\mathbb{B}_n} (1 - ||z||^2)\, ||\nabla f(z)||\,, \tag{5.2}$$

which is equivalent to the Bloch norm on $\mathbb{B}_n$ [26]. The following theorem is a special case of Corollaries 1.4 and 1.6 of [31]; their results apply to a large set of function spaces which includes the Bloch space.

Theorem 5.1 **([31]).** *Let ψ be a holomorphic function of $\mathbb{B}_n$ and φ a holomorphic self-map of $\mathbb{B}_n$. Then $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{B}_n)$ if and only if*

$$\sup_{z\in\mathbb{B}_n} |\psi(z)|\, B_\varphi(z) < \infty, \ \textit{and} \ \sup_{z\in\mathbb{B}_n} (1-||z||^2)\, ||\nabla\psi(z)|| \log \frac{2}{1-||\varphi(z)||^2} < \infty.$$

Furthermore, $W_{\psi,\varphi}$ is compact if and only if

$$\lim_{||\varphi(z)||\to 1} |\psi(z)|\, B_\varphi(z) = 0, \ \textit{and}$$

$$\lim_{||\varphi(z)||\to 1} (1-||z||^2)\, ||\nabla\psi(z)|| \log \frac{2}{1-||\varphi(z)||^2} = 0. \tag{5.3}$$

We now show that the bounded and the compact weighted composition operators can also be characterized in terms of the quantities $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$.

Theorem 5.2. *Let ψ be a holomorphic function on $\mathbb{B}_n$ and φ a holomorphic self-map of $\mathbb{B}_n$. Then*

(a) *$W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{B}_n)$ if and only if $\psi \in \mathcal{B}(\mathbb{B}_n)$, and $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite.*

(b) *The bounded operator $W_{\psi,\varphi}$ is compact on $\mathcal{B}(\mathbb{B}_n)$ if and only if*

$$\lim_{||\varphi(z)||\to 1} |\psi(z)|\, T_\varphi(z) = 0, \ \textit{and}$$

$$\lim_{||\varphi(z)||\to 1} Q_\psi(z) \log \tfrac{1+||\varphi(z)||}{1-||\varphi(z)||} = 0. \tag{5.4}$$

Remark 3. At first glance it may seem evident that conditions (5.3) and (5.4) are equivalent due to the equivalence between the norm (5.2) and the Bloch norm. However, we have not been able to prove directly that (5.3) implies (5.4) and thus, the proof of (5.4) under the compactness assumption does not make use of (5.3).

Proof. (a) If $\psi \in \mathcal{B}(\mathbb{B}_n)$ and $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite, then $W_{\psi,\varphi}$ is bounded by Theorem 4.3. Conversely, assume $W_{\psi,\varphi}$ is bounded. Then $\psi = W_{\psi,\varphi}1 \in \mathcal{B}(\mathbb{B}_n)$ and by Theorem 5.1, $\sup_{z\in\mathbb{B}_n} |\psi(z)|\, B_\varphi(z)$ is finite. From (4.2), we deduce

$$\tau_{\psi,\varphi} = \sup_{z\in\mathbb{B}_n} |\psi(z)|\, T_\varphi(z) \le \sup_{z\in\mathbb{B}_n} |\psi(z)|\, B_\varphi(z) < \infty.$$

On the other hand, using Theorem 4.5, we see that $\sigma_{\psi,\varphi}$ is also finite, completing the proof of (a).

To prove (b) observe that by Theorem 4.8, if

$$\lim_{||\varphi(z)||\to 1} Q_\psi(z) \log \frac{1+||\varphi(z)||}{1-||\varphi(z)||} = \lim_{||\varphi(z)||\to 1} |\psi(z)|\, T_\varphi(z) = 0,$$

then $W_{\psi,\varphi}$ is compact. Conversely, assume $W_{\psi,\varphi}$ is compact. Then, from Theorem 5.1 we get

$$\lim_{||\varphi(z)||\to 1} |\psi(z)|\, T_\varphi(z) \le \lim_{||\varphi(z)||\to 1} |\psi(z)|\, B_\varphi(z) = 0.$$

Furthermore, $W_{\psi,\varphi}$ is bounded and so

$$\sup_{z\in\mathbb{B}_n} Q_\psi(z)\log\frac{1+||\varphi(z)||}{1-||\varphi(z)||}<\infty.$$

In particular,

$$\lim_{||\varphi(z)||\to 1} Q_\psi(z)=0. \tag{5.5}$$

Let $\{z_k\}$ be a sequence in $\mathbb{B}_n$ such that $||\varphi(z_k)||\to 1$ as $k\to\infty$. For $z\in\mathbb{B}_n$ define

$$f_k(z)=\frac{\left(\mathrm{Log}\dfrac{2}{1-\langle z,\varphi(z_k)\rangle}\right)^2}{\log\dfrac{2}{1-||\varphi(z_k)||^2}}.$$

Then $\{f_k\}$ converges to 0 locally uniformly in $\mathbb{B}_n$. We are now going to show that $\{f_k\}$ is bounded in $\mathcal{B}(\mathbb{B}_n)$. For $z\in\mathbb{B}_n$, set

$$g_k(z)=\mathrm{Log}\frac{2}{1-\langle z,\varphi(z_k)\rangle}.$$

Then by (5.1) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
Q_{g_k}(z) &= (1-||z||^2)^{1/2}\frac{\left(||\varphi(z_k)||^2-|\langle z,\varphi(z_k)\rangle|^2\right)^{1/2}}{|1-\langle z,\varphi(z_k)\rangle|}\\
&\le \sqrt{2}\frac{(1-||z||)^{1/2}}{(1-|\langle z,\varphi(z_k)\rangle|)^{1/2}}(1+|\langle z,\varphi(z_k)\rangle|)^{1/2}\le 2.
\end{aligned}$$

Next, observe that for $z\in\mathbb{B}_n$

$$\nabla(f_k)(z)=\frac{2\mathrm{Log}\frac{2}{1-\langle z,\varphi(z_k)\rangle}}{\log\frac{2}{1-||\varphi(z_k)||^2}}\nabla(g_k)(z).$$

So for $u\in\mathbb{C}^n\setminus\{0\}$

$$\begin{aligned}
\frac{|\nabla(f_k)(z)u|}{H_z(u,\overline{u})^{1/2}} &\le \frac{2\left(\log\frac{2}{1-|\langle z,\varphi(z_k)\rangle|}+\frac{\pi}{2}\right)}{\log\left(\frac{2}{1-||\varphi(z_k)||^2}\right)}\frac{|\nabla(g_k)(z)u|}{H_z(u,\overline{u})^{1/2}}\\
&\le \frac{2\left(\log\left(\frac{4}{1-||\varphi(z_k)||^2}\right)+\frac{\pi}{2}\right)}{\log\frac{2}{1-||\varphi(z_k)||^2}}Q_{g_k}(z)\le 4\left(2+\frac{\pi}{2\log 2}\right).
\end{aligned}$$

Hence $||f_k||_{\mathcal{B}}$ is bounded above by $\log 2+4\left(2+\frac{\pi}{2\log 2}\right)$. By the compactness of $W_{\psi,\varphi}$, $||\psi(f_k\circ\varphi)||_{\mathcal{B}}\to 0$ as $k\to\infty$. Moreover

$$\nabla(f_k)(\varphi(z_k))=\frac{2\overline{\varphi(z_k)}}{1-||\varphi(z_k)||^2},$$

so, for $u \in \mathbb{C}^n \setminus \{0\}$, we have

$$|\nabla(f_k)(\varphi(z_k))J\varphi(z_k)u| = \frac{2\,|\langle J\varphi(z_k)u, \varphi(z_k)\rangle|}{1 - ||\varphi(z_k)||^2}.$$

Hence

$$\begin{aligned} ||\psi(f_k \circ \varphi)||_{\mathcal{B}} &\geq \sup_{z \in \mathbb{B}_n} Q_{\psi(f_k\circ\varphi)}(z) \geq Q_{\psi(f_k\circ\varphi)}(z_k) \\ &\geq \left| Q_\psi(z_k) f_k(\varphi(z_k)) - |\psi(z_k)| \sup_{u \in \mathbb{C}^n\setminus\{0\}} \frac{|\nabla(f_k)(\varphi(z_k))J\varphi(z_k)u|}{H_z(u,\overline{u})^{1/2}} \right| \\ &= \Bigg| Q_\psi(z_k) \log \frac{2}{1 - ||\varphi(z_k)||^2} \\ &\qquad - \frac{2\,|\psi(z_k)|}{1 - ||\varphi(z_k)||^2} \sup_{u \in \mathbb{C}^n\setminus\{0\}} \frac{|\langle J\varphi(z_k)u, \varphi(z_k)\rangle|}{H_z(u,\overline{u})^{1/2}} \Bigg|. \end{aligned}$$

We now show that

$$\lim_{k\to\infty} \frac{|\psi(z_k)|}{1 - ||\varphi(z_k)||^2} \sup_{u \in \mathbb{C}^n\setminus\{0\}} \frac{|\langle J\varphi(z_k)u, \varphi(z_k)\rangle|}{H_z(u,\overline{u})^{1/2}} = 0. \tag{5.6}$$

Once this is proved, it will follow that

$$\lim_{k\to\infty} Q_\psi(z_k) \log \frac{2}{1 - ||\varphi(z_k)||^2} = 0$$

since $||\psi(f_k \circ \varphi)||_{\mathcal{B}} \to 0$ as $k \to \infty$. Noting that

$$\begin{aligned} Q_\psi(z_k) \log \frac{2}{1 - ||\varphi(z_k)||^2} &\geq Q_\psi(z_k) \log \frac{1}{2} \frac{1 + ||\varphi(z_k)||}{1 - ||\varphi(z_k)||} \\ &= Q_\psi(z_k) \log \frac{1 + ||\varphi(z_k)||}{1 - ||\varphi(z_k)||} - Q_\psi(z_k) \log 2 \end{aligned}$$

and that by (5.5), $\lim_{k\to\infty} Q_\psi(z_k) = 0$, we obtain that the limit of the first term of the above difference also goes to 0 as $k \to \infty$, and hence

$$\lim_{||\varphi(z)||\to 1} Q_\psi(z) \log \frac{1 + ||\varphi(z)||}{1 - ||\varphi(z)||} = 0.$$

Let us now proceed with the proof of (5.6). For $k \in \mathbb{N}$ and $z \in \mathbb{B}_n$, let

$$h_k(z) = \frac{1 - ||\varphi(z_k)||^2}{1 - \langle z, \varphi(z_k)\rangle}.$$

Then $h_k \to 0$ uniformly on compact subsets of $\mathbb{B}_n$, and for $j = 1, \dots, n$ and $z \in \mathbb{B}_n$

$$\frac{\partial h_k}{\partial z_j}(z) = \frac{(1 - ||\varphi(z_k)||^2)\overline{\varphi_j(z_k)}}{(1 - \langle z, \varphi(z_k)\rangle)^2}.$$

Thus by (5.1), we obtain

$$\begin{aligned} Q_{h_k}(z) &= \frac{(1-\|z\|^2)^{1/2}(1-\|\varphi(z_k)\|^2)}{|1-\langle z,\varphi(z_k)\rangle|^2}\left(\|\varphi(z_k)\|^2 - |\langle z,\varphi(z_k)\rangle|^2\right)^{1/2} \\ &\le \frac{\sqrt{2}(1-\|z\|^2)^{1/2}(1-\|\varphi(z_k)\|^2)}{(1-|\langle z,\varphi(z_k)\rangle|)^{3/2}} \\ &\le \frac{\sqrt{2}(1-\|z\|^2)^{1/2}(1-\|\varphi(z_k)\|^2)}{(1-\|z\|)^{1/2}(1-\|\varphi(z_k)\|)} \le 4. \end{aligned}$$

So $h_k \in \mathcal{B}(\mathbb{B}_n)$ and $\|h_k\|_{\mathcal{B}} \le 5$. Since $W_{\psi,\varphi}$ is compact, $\|\psi(h_k\circ\varphi)\|_{\mathcal{B}} \to 0$ as $k \to \infty$. Moreover

$$\begin{aligned} \|\psi(h_k\circ\varphi)\|_{\mathcal{B}} &\ge Q_{\psi(h_k\circ\varphi)}(z_k) \\ &\ge \left| Q_\psi(z_k) - |\psi(z_k)| \sup_{u\in\mathbb{C}^n\setminus\{0\}} \frac{|\nabla(h_k)(\varphi(z_k))J\varphi(z_k)u|}{H_z(u,\overline{u})^{1/2}} \right| \\ &= \left| Q_\psi(z_k) - \frac{|\psi(z_k)|}{1-\|\varphi(z_k)\|^2} \sup_{u\in\mathbb{C}^n\setminus\{0\}} \frac{|\langle J\varphi(z_k)u,\varphi(z_k)\rangle|}{H_z(u,\overline{u})^{1/2}} \right|. \end{aligned}$$

Since $\lim_{k\to\infty} Q_\psi(z_k) = 0$, it follows that

$$\lim_{k\to\infty} \frac{|\psi(z_k)|}{1-\|\varphi(z_k)\|^2} \sup_{u\in\mathbb{C}^n\setminus\{0\}} \frac{|\langle J\varphi(z_k)u,\varphi(z_k)\rangle|}{H_z(u,\overline{u})^{1/2}} = 0.$$

The proof is now complete. □

Next, we give an example of a bounded weighted composition operator on the unit ball whose associated component multiplication operator is unbounded and an example of a compact operator on $\mathbb{B}_n$ whose associated component operators are both not compact.

Examples (a) Let $\lambda \in \partial\mathbb{B}_n$ and define the functions $\psi(z) = \frac{1}{2}\mathrm{Log}(1-\langle z,\lambda\rangle)$ and $\varphi(z) = \frac{1}{2}(\lambda - z)$ for $z \in \mathbb{B}_n$. The associated multiplication operator M_ψ is not bounded on $\mathcal{B}(\mathbb{B}_n)$ since $\psi \notin H^\infty(\mathbb{B}_n)$. On the other hand, it is straightforward to verify that $\sup_{z\in\mathbb{B}_n} |\psi(z)|\, B_\varphi(z) < \infty$ and

$$\sup_{z\in\mathbb{B}_n} (1-\|z\|^2)\, \|\nabla\psi(z)\| \log\frac{1}{1-\|\varphi(z)\|^2} < \infty.$$

Therefore, $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{B}_n)$.

(b) Let $\psi(z) = 1 - z_1$ and $\varphi(z) = \frac{\mathbb{1}+z}{2}$, for $z \in \mathbb{B}_n$, where $\mathbb{1} = (1,0,\dots,0)$. The multiplication operator M_ψ is not compact on $\mathcal{B}(\mathbb{B}_n)$, since ψ is not identically zero. Moreover, the composition operator C_φ is not compact on $\mathcal{B}(\mathbb{B}_n)$ [25], since

$$\frac{H_{\varphi(z)}(J\varphi(z)u,\overline{J\varphi(z)u})}{H_z(u,\overline{u})} = \frac{1}{4}\frac{(1-\|\varphi(z)\|^2)\,\|u\|^2 + |\langle\varphi(z),u\rangle|^2}{(1-\|z\|^2)\,\|u\|^2 + |\langle z,u\rangle|^2}\frac{(1-\|z\|^2)^2}{(1-\|\varphi(z)\|^2)^2},$$

which does not go to 0 if $z \to 1$ along the real axis in the first coordinate and $u = (1, 0, \ldots, 0)$. Observe that

$$\lim_{\|\varphi(z)\| \to 1} \psi(z) = \lim_{z_1 \to 1} \psi(z) = 0$$

and $B_\varphi(z)$ is bounded above by a constant independent of φ, so

$$\lim_{\|\varphi(z)\| \to 1} |\psi(z)|\, B_\varphi(z) = 0.$$

Moreover,

$$\lim_{\|\varphi(z)\| \to 1} (1 - \|z\|^2) \log \frac{2}{1 - \left\|\frac{1+z}{2}\right\|^2} = 0.$$

Therefore $W_{\psi,\varphi}$ is compact on $\mathcal{B}(\mathbb{B}_n)$.

6. Special case: The unit polydisk

Theorem 6.1. *Let ψ be a holomorphic function on $\mathbb{D}^n$ and φ a holomorphic self-map of $\mathbb{D}^n$. Then $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{D}^n)$ if and only if $\psi \in \mathcal{B}(\mathbb{D}^n)$, and $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite. Furthermore, the bounded operator $W_{\psi,\varphi}$ is compact on $\mathcal{B}(\mathbb{D}^n)$ if and only if*

$$\lim_{\varphi(z) \to \partial\mathbb{D}^n} \omega(\varphi(z)) Q_\psi(z) = \lim_{\varphi(z) \to \partial\mathbb{D}^n} |\psi(z)|\, T_\varphi(z) = 0. \tag{6.1}$$

To prove this result, we will show that the conditions for the boundedness and compactness of $W_{\psi,\varphi}$ are equivalent to the conditions proven by Zhou and Chen in the following theorem. Their results were obtained by considering on the Bloch space of $\mathbb{D}^n$ the norm

$$\|f\|_* = |f(0)| + \sup_{z \in \mathbb{D}^n} \sum_{j=1}^n (1 - |z_j|^2) \left| \frac{\partial f}{\partial z_j}(z) \right|.$$

In [10], it was shown that for $f \in \mathcal{B}(\mathbb{D}^n)$ and $z \in \mathbb{D}^n$,

$$Q_f(z) = \left\| \left((1 - |z_1|^2) \frac{\partial f}{\partial z_1}(z), \ldots, (1 - |z_n|^2) \frac{\partial f}{\partial z_n}(z) \right) \right\|.$$

Thus $\|\cdot\|_*$ is equivalent to the Bloch norm since

$$\frac{1}{n} \sum_{j=1}^n (1 - |z_j|^2) \left| \frac{\partial f}{\partial z_j}(z) \right| \le Q_f(z) \le \sum_{j=1}^n (1 - |z_j|^2) \left| \frac{\partial f}{\partial z_j}(z) \right|, \tag{6.2}$$

for all $z \in \mathbb{D}^n$.

Theorem 6.2 ([32], Theorems 1 and 2). *Let ψ be a holomorphic function on $\mathbb{D}^n$ and φ a holomorphic self-map of $\mathbb{D}^n$. Then $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{D}^n)$ if and only if*

$$\sup_{z \in \mathbb{D}^n} \sum_{j,k=1}^n (1 - |z_j|^2) \left| \frac{\partial \psi}{\partial z_j}(z) \right| \log \frac{4}{1 - |\varphi_k(z)|^2} < \infty$$

and

$$\sup_{z\in\mathbb{D}^n} |\psi(z)| \sum_{j,k=1}^n \left|\frac{\partial\varphi_k}{\partial z_j}(z)\right| \frac{1-|z_j|^2}{1-|\varphi_k(z)|^2} < \infty.$$

Furthermore, $W_{\psi,\varphi}$ *is compact on* $\mathcal{B}(\mathbb{D}^n)$ *if and only if* $W_{\psi,\varphi}$ *is bounded and*

$$\lim_{\varphi(z)\to\partial\mathbb{D}^n} \sum_{j,k=1}^n (1-|z_j|^2)\left|\frac{\partial\psi}{\partial z_j}(z)\right| \log\frac{4}{1-|\varphi_k(z)|^2} = 0$$

and

$$\lim_{\varphi(z)\to\partial\mathbb{D}^n} |\psi(z)| \sum_{j,k=1}^n \left|\frac{\partial\varphi_k}{\partial z_j}(z)\right| \frac{1-|z_j|^2}{1-|\varphi_k(z)|^2} = 0.$$

Lemma 6.3. *Let* ψ *be a holomorphic function on* $\mathbb{D}^n$ *and* φ *a holomorphic self-map of* $\mathbb{D}^n$*. Then, for* $z \in \mathbb{D}^n$*, the following inequalities hold:*

(a) $\rho(0,z) \leq \sum_{j=1}^n \log\frac{4}{1-|z_j|^2}$;

(b) $\sigma_{\psi,\varphi}(z) \leq \left(\sum_{j=1}^n (1-|z_j|^2)\left|\frac{\partial\psi}{\partial z_j}(z)\right|\right) \sum_{k=1}^n \log\frac{4}{1-|\varphi_k(z)|^2}$;

(c) $T_\varphi(z) \leq \sum_{j,k=1}^n \left|\frac{\partial\varphi_k}{\partial z_j}(z)\right| \frac{1-|z_j|^2}{1-|\varphi_k(z)|^2}$;

(d) $\tau_{\psi,\varphi}(z) \leq |\psi(z)| \sum_{j,k=1}^n \left|\frac{\partial\varphi_k}{\partial z_j}(z)\right| \frac{1-|z_j|^2}{1-|\varphi_k(z)|^2}$;

Proof. Let $z \in D^n$. To prove (a), observe that for $u \in \mathbb{C}^n$,

$$H_z(u,\overline{u}) = \sum_{j=1}^n \frac{|u_j|^2}{(1-|z_j|^2)^2}$$

(e.g., see [26]), and recall that if $\gamma = \gamma(t)$ $(0 \leq t \leq 1)$ is the geodesic from w to z, then

$$\rho(w,z) = \int_0^1 H_{\gamma(t)}(\gamma'(t),\overline{\gamma'(t)})^{1/2}\,dt.$$

Since the geodesic from 0 to $z \in \mathbb{D}^n$ is parametrized by $\gamma(t) = tz$, for $0 \leq t \leq 1$, we obtain

$$\begin{aligned}\rho(0,z) &= \int_0^1 \left(\sum_{j=1}^n \frac{|z_j|^2}{(1-|z_j|^2 t^2)^2}\right)^{1/2} dt \leq \int_0^1 \sum_{j=1}^n \frac{|z_j|}{1-|z_j|^2 t^2}\,dt \qquad (6.3)\\ &= \frac{1}{2}\sum_{j=1}^n \log\frac{1+|z_j|}{1-|z_j|},\end{aligned}$$

proving (a). By the upper estimate of (6.2) and the inequality $\omega(\varphi(z)) \leq \rho(0,z)$, part (b) follows immediately from part (a).

To prove (c), observe that by (1.2) of [10],

$$\begin{aligned} T_\varphi(z) &\leq B_\varphi(z) = \max_{||w||=1} \left(\sum_{k=1}^n \left| \sum_{j=1}^n \frac{\partial \varphi_k}{\partial z_j}(z) \frac{(1-|z_j|^2)w_j}{1-|\varphi_k(z)|^2} \right|^2 \right)^{1/2} \\ &\leq \max_{||w||=1} \left(\sum_{k=1}^n \left(\sum_{j=1}^n \left| \frac{\partial \varphi_k}{\partial z_j}(z) \right| \frac{(1-|z_j|^2)\,|w_j|}{1-|\varphi_k(z)|^2} \right)^2 \right)^{1/2} \\ &\leq \sum_{j,k=1}^n \left| \frac{\partial \varphi_k}{\partial z_j}(z) \right| \frac{1-|z_j|^2}{1-|\varphi_k(z)|^2}. \end{aligned}$$

Part (d) is an immediate consequence of the formula $\tau_{\psi,\varphi}(z) = |\psi(z)|\, T_\varphi(z)$ and part (c). □

Proof of Theorem 6.1. If $W_{\psi,\varphi}$ is bounded, then $\psi = W_{\psi,\varphi}1 \in \mathcal{B}(\mathbb{D}^n)$, and from Theorem 6.2, inequality (d) of Lemma 6.3, and Theorem 4.5, it follows that $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite. Conversely, if $\psi \in \mathcal{B}(\mathbb{D}^n)$, and $\sigma_{\psi,\varphi}$ and $\tau_{\psi,\varphi}$ are finite, then $W_{\psi,\varphi}$ is bounded by Theorem 4.3.

Next, assume $W_{\psi,\varphi}$ is compact. Then, by Theorem 6.2, and inequalities (b) and (d) of Lemma 6.3, the conditions in (6.1) hold. Conversely, if $W_{\psi,\varphi}$ is bounded and the conditions in (6.1) hold, then $W_{\psi,\varphi}$ is compact by Theorem 4.8. □

We conclude the section by giving an example of a bounded weighted composition operator on the polydisk whose corresponding component multiplication operator is not bounded, and an example of a compact weighted composition operator on $\mathbb{D}^n$ whose both component operators are not compact.

Examples. (a) Fix an index $j \in \{1, \ldots, n\}$ and define $\psi(z) = \mathrm{Log}\frac{2}{1-z_j}$, $\varphi_j(z) = \frac{1-z_j}{2}$, and $\varphi_k(z) = 0$ for $k \neq j$, $z \in \mathbb{D}^n$. Since $\psi \notin H^\infty(\mathbb{D}^n)$, the associated multiplication operator M_ψ is unbounded on $\mathcal{B}(\mathbb{D}^n)$. On the other hand

$$\sigma_{\psi,\varphi} \leq \sup_{z \in \mathbb{D}^n} \left(\frac{1-|z_j|^2}{|1-z_j|} \right) \log \frac{4}{1-\left|\frac{1-z_j}{2}\right|^2} < \infty,$$

$$\tau_{\psi,\varphi} \leq \sup_{z \in \mathbb{D}^n} \frac{1-|z_j|^2}{1-\left|\frac{1-z_j}{2}\right|^2} \left| \mathrm{Log}\frac{2}{1-z_j} \right| < \infty,$$

so that $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(\mathbb{D}^n)$.

(b) Fix an index $j \in \{1, \ldots, n\}$, and for $z \in \mathbb{D}^n$ define $\psi(z) = 1 - z_j$, and let $\varphi(z)$ be the vector with kth component 0 for $k \neq j$ and jth component $\frac{1+z_j}{2}$. Clearly, the associated multiplication operator M_ψ is not compact on $\mathcal{B}(\mathbb{D}^n)$. The associated composition operator C_φ is not compact on $\mathcal{B}(\mathbb{D}^n)$ since

$$B_\varphi(z) = \frac{1}{2} \frac{1-|z_j|^2}{1-\left|\frac{1+z_j}{2}\right|^2} \not\to 0$$

for $z_j \to 1$ [25]. Furthermore,

$$\lim_{\varphi(z)\to\partial\mathbb{D}^n} \sigma_{\psi,\varphi}(z) \leq \lim_{z_j\to 1}(1-|z_j|^2)\log\frac{4}{1-\left|\frac{1+z_j}{2}\right|^2} = 0, \text{ and}$$

$$\lim_{\varphi(z)\to\partial\mathbb{D}^n} \tau_{\psi,\varphi}(z) \leq \frac{1}{2}\lim_{z_j\to 1}|1-z_j|\frac{1-|z_j|^2}{1-\left|\frac{1+z_j}{2}\right|^2} = 0.$$

Hence $W_{\psi,\varphi}$ is compact on $\mathcal{B}(\mathbb{D}^n)$.

7. Weighted composition operators from the Bloch spaces into H^∞

In [16], Hosokawa, Izuchi and Ohno characterized the bounded and the compact weighted composition operators from $\mathcal{B}(\mathbb{D})$ and $\mathcal{B}_0(\mathbb{D})$ into $H^\infty(\mathbb{D})$. We now provide a characterization of the bounded operators in the environment of a bounded homogeneous domain and determine the operator norm. We also obtain an extension of their results when the domain is the unit ball and the unit polydisk.

Theorem 7.1. *Let D be a bounded homogeneous domain, ψ a holomorphic function on D, and φ a holomorphic self-map of D. Then*

(a) *$W_{\psi,\varphi} : \mathcal{B}(D) \to H^\infty(D)$ is bounded if and only if $\psi \in H^\infty(D)$ and*

$$\eta_{\psi,\varphi} := \sup_{z\in D}|\psi(z)|\omega(\varphi(z)) < \infty.$$

If $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(D)$, then

$$||W_{\psi,\varphi}|| = \max\{||\psi||_\infty, \eta_{\psi,\varphi}\}. \tag{7.1}$$

(b) *$W_{\psi,\varphi} : \mathcal{B}_{0*}(D) \to H^\infty(D)$ is bounded if and only if $\psi \in H^\infty(D)$ and*

$$\eta_{0,\psi,\varphi} := \sup_{z\in D}|\psi(z)|\omega_0(\varphi(z)) < \infty.$$

If $W_{\psi,\varphi}$ is bounded on $\mathcal{B}_{0*}(D)$, then

$$||W_{\psi,\varphi}|| = \max\{||\psi||_\infty, \eta_{0,\psi,\varphi}\}.$$

Proof. To prove (a), assume $W_{\psi,\varphi}$ is bounded on $\mathcal{B}(D)$. Then $\psi = W_{\psi,\varphi}1 \in H^\infty(D)$, $||\psi||_\infty \leq ||W_{\psi,\varphi}||$, and for each $f \in \mathcal{B}(D)$ with $||f||_{\mathcal{B}} \leq 1$, and for each $z \in D$, we have

$$||W_{\psi,\varphi}|| \geq ||\psi(f\circ\varphi)||_\infty \geq |\psi(z)||f(\varphi(z))|.$$

Taking the supremum over all such functions f such that $f(0) = 0$, and over all $z \in D$, we obtain $||W_{\psi,\varphi}|| \geq \eta_{\psi,\varphi}$, proving that $\eta_{\psi,\varphi} < \infty$ and

$$||W_{\psi,\varphi}|| \geq \max\{||\psi||_\infty, \eta_{\psi,\varphi}\}. \tag{7.2}$$

Conversely, suppose $\psi \in H^\infty(D)$ and $\eta_{\psi,\varphi}$ is finite. Then, by Lemma 4.2, for each $f \in \mathcal{B}(D)$ we have

$$\begin{aligned}\sup_{z\in D} |\psi(z)||f(\varphi(z))| &\leq \sup_{z\in D} |\psi(z)|(|f(0)| + \omega(\varphi(z))\beta_f) \\ &\leq \|\psi\|_\infty (\|f\|_{\mathcal{B}} - \beta_f) + \eta_{\psi,\varphi}\beta_f \\ &\leq \max\{\|\psi\|_\infty, \eta_{\psi,\varphi}\} \|f\|_{\mathcal{B}}, \qquad (7.3)\end{aligned}$$

proving the boundedness of $W_{\psi,\varphi}$. From (7.2) and (7.3) we also obtain (7.1). The proof of (b) is analogous. □

Recalling that for each $z \in \mathbb{B}_n$,

$$\omega_0(z) = \omega(z) = \frac{1}{2}\log\frac{1+\|z\|}{1-\|z\|},$$

we deduce the following extension to the unit ball of Theorem 6.1 of [16], which is equivalent to Theorem 1 in [20]. The evaluation of the operator norm has not appeared before.

Corollary 7.2. *Let ψ be a holomorphic function on $\mathbb{B}_n$ and φ a holomorphic self-map of $\mathbb{B}_n$. Then the following statements are equivalent:*

(a) $W_{\psi,\varphi} : \mathcal{B}(\mathbb{B}_n) \to H^\infty(\mathbb{B}_n)$ *is bounded.*
(b) $W_{\psi,\varphi} : \mathcal{B}_0(\mathbb{B}_n) \to H^\infty(\mathbb{B}_n)$ *is bounded.*
(c) $\psi \in H^\infty(\mathbb{B}_n)$ *and*

$$\sup_{z\in\mathbb{B}_n} |\psi(z)|\log\frac{1+\|\varphi(z)\|}{1-\|\varphi(z)\|} < \infty.$$

Furthermore,

$$\|W_{\psi,\varphi}\| = \max\left\{\|\psi\|_\infty, \sup_{z\in\mathbb{B}_n} \frac{1}{2}|\psi(z)|\log\frac{1+\|\varphi(z)\|}{1-\|\varphi(z)\|}\right\}.$$

In the case where ψ is the constant function one, the condition of the finiteness of

$$\sup_{z\in\mathbb{B}_n} \log\frac{1+\|\varphi(z)\|}{1-\|\varphi(z)\|}$$

implies that $\varphi(z)$ cannot approach the boundary, or else the logarithmic term would tend to infinity. Thus, we have the following corollary.

Corollary 7.3. *Let φ be a holomorphic self-map of $\mathbb{B}_n$. Then the following statements are equivalent:*

(a) $C_\varphi : \mathcal{B}(\mathbb{B}_n) \to H^\infty(\mathbb{B}_n)$ *is bounded.*
(b) $C_\varphi : \mathcal{B}_0(\mathbb{B}_n) \to H^\infty(\mathbb{B}_n)$ *is bounded.*
(c) $\varphi(\mathbb{B}_n)$ *has compact closure in* $\mathbb{B}_n$.

Furthermore, the operator norm of C_φ is the maximum between 1 and the Bergman distance of the boundary of the range of φ from the origin.

We now show that Theorem 6.1 of [16] can be extended to the unit polydisk.

Theorem 7.4. *Let ψ be a holomorphic on $\mathbb{D}^n$ and φ a holomorphic self-map of $\mathbb{D}^n$. Then the following statements are equivalent:*

(a) $W_{\psi,\varphi} : \mathcal{B}(\mathbb{D}^n) \to H^\infty(\mathbb{D}^n)$ *is bounded.*
(b) $W_{\psi,\varphi} : \mathcal{B}_{0*}(\mathbb{D}^n) \to H^\infty(\mathbb{D}^n)$ *is bounded.*
(c) $\psi \in H^\infty(\mathbb{D}^n)$ *and*

$$\sup_{z\in\mathbb{D}^n} |\psi(z)| \sum_{j=1}^{n} \log \frac{1+|\varphi_j(z)|}{1-|\varphi_j(z)|} < \infty. \tag{7.4}$$

Proof. The implication (a) $\Longrightarrow$ (b) is obvious.
(b) $\Longrightarrow$ (c): It is clear that $\psi \in H^\infty(\mathbb{D}^n)$. Fix $j = 1, \dots, n$ and $\lambda \in \mathbb{D}^n$. For $z \in \mathbb{D}^n$ define

$$h(z) = \mathrm{Log}\frac{4}{1 - z_j\overline{\varphi_j(\lambda)}}.$$

Then

$$||h||_{\mathcal{B}} = 2\log 2 + \sup_{|z_j|<1} \frac{(1-|z_j|^2)|\varphi_j(\lambda)|}{|1 - z_j\overline{\varphi_j(\lambda)}|} \le 2\log 2 + 2.$$

Furthermore,

$$Q_h(z) \le \frac{(1-|z_j|^2)|\varphi_j(\lambda)|}{1-|\varphi_j(\lambda)|} \to 0$$

as $|z_j| \to 1$. Thus, $h \in \mathcal{B}_{0*}(\mathbb{D}^n)$. By the boundedness of $W_{\psi,\varphi} : \mathcal{B}_{0*}(\mathbb{D}^n) \to H^\infty(\mathbb{D}^n)$, we obtain

$$\begin{aligned}(2\log 2 + 2)\,||W_{\psi,\varphi}|| \;&\ge\; ||\psi(h\circ\varphi)||_\infty \ge |\psi(\lambda)| \log \frac{4}{1-|\varphi_j(\lambda)|^2} \\ &\ge\; |\psi(\lambda)| \log \frac{1+|\varphi_j(\lambda)|}{1-|\varphi_j(\lambda)|}.\end{aligned}$$

Summing over all $j = 1, \dots, n$ and taking the supremum over all $\lambda \in \mathbb{D}^n$, we get (7.4).
(c) $\Longrightarrow$ (a): Observe that for $z \in \mathbb{D}^n$, by (6.3), we have

$$\omega(\varphi(z)) \le \rho(0, \varphi(z)) \le \frac{1}{2}\sum_{j=1}^{n} \log \frac{1+|\varphi_j(z)|}{1-|\varphi_j(z)|}. \tag{7.5}$$

The result follows at once from Theorem 7.1(a). $\square$

We now give a sufficient condition for compactness which can be proved as Theorem 4.8.

Theorem 7.5. *Let D be a bounded homogeneous domain, ψ a holomorphic function on D, and φ a holomorphic self-map of D. Then $W_{\psi,\varphi} : \mathcal{B}(D) \to H^\infty(D)$ is compact if $\psi \in H^\infty(D)$ and*

$$\lim_{\varphi(z)\to\partial D} |\psi(z)|\omega(\varphi(z)) = 0.$$

This sufficient condition is also necessary when D is the unit ball. Indeed, the following result, proved by Li and Stević in [20] (Theorem 4), is the extension to the unit ball of Theorem 6.2 of [16].

Theorem 7.6 ([20])**.** *Let ψ be a holomorphic function on $\mathbb{B}_n$ and φ a holomorphic self-map of $\mathbb{B}_n$. Then the following statements are equivalent:*

(a) $W_{\psi,\varphi} : \mathcal{B}(\mathbb{B}_n) \to H^\infty(\mathbb{B}_n)$ *is compact.*
(b) $W_{\psi,\varphi} : \mathcal{B}_0(\mathbb{B}_n) \to H^\infty(\mathbb{B}_n)$ *is compact.*
(c) $\psi \in H^\infty(\mathbb{B}_n)$ *and* $\lim_{\|\varphi(z)\|\to 1} |\psi(z)| \log \frac{2}{1-\|\varphi(z)\|} = 0.$

The following is a consequence of Corollary 7.2 and Theorem 7.6. It follows immediately from the finiteness of

$$\sup_{z\in\mathbb{B}_n} |\psi(z)| \log \frac{1+\|z\|}{1-\|z\|},$$

which implies that ψ is identically zero.

Corollary 7.7. *Let ψ be a holomorphic function on $\mathbb{B}_n$. Then the following are equivalent:*

(a) $M_\psi : \mathcal{B}(\mathbb{B}_n) \to H^\infty(\mathbb{B}_n)$ *is bounded.*
(b) $M_\psi : \mathcal{B}_0(\mathbb{B}_n) \to H^\infty(\mathbb{B}_n)$ *is bounded.*
(c) $M_\psi : \mathcal{B}(\mathbb{B}_n) \to H^\infty(\mathbb{B}_n)$ *is compact.*
(d) $M_\psi : \mathcal{B}_0(\mathbb{B}_n) \to H^\infty(\mathbb{B}_n)$ *is compact.*
(e) ψ *is identically zero.*

We now prove the analogue of Theorem 7.6 for the polydisk.

Theorem 7.8. *Let ψ be holomorphic on $\mathbb{D}^n$ and φ a holomorphic self-map of $\mathbb{D}^n$. Then the following statements are equivalent:*

(a) $W_{\psi,\varphi} : \mathcal{B}(\mathbb{D}^n) \to H^\infty(\mathbb{D}^n)$ *is compact.*
(b) $W_{\psi,\varphi} : \mathcal{B}_{0*}(\mathbb{D}^n) \to H^\infty(\mathbb{D}^n)$ *is compact.*
(c) $\psi \in H^\infty(\mathbb{D}^n)$ *and*

$$\lim_{\varphi(z)\to\partial\mathbb{D}^n} |\psi(z)| \sum_{j=1}^n \log \frac{1+|\varphi_j(z)|}{1-|\varphi_j(z)|} = 0.$$

Proof. The implication (a) $\Longrightarrow$ (b) is obvious. We now show (b) $\Longrightarrow$ (c). Since $W_{\psi,\varphi} : \mathcal{B}_{0*}(\mathbb{D}^n) \to H^\infty(\mathbb{D}^n)$ is bounded, by Theorem 7.4, $\psi \in H^\infty(\mathbb{D}^n)$. Suppose there exists a sequence $\{z^{(k)}\}$ in $\mathbb{D}^n$ such that $\varphi(z^{(k)}) \to \partial D^n$ as $k \to \infty$. Then, there is a number $j \in \{1,\ldots,n\}$ such that $|\varphi_j(z^{(k)})| \to 1$ as $k \to \infty$. Since (7.4) holds, it follows that

$$\lim_{k\to\infty} |\psi(z^{(k)})| = 0. \tag{7.6}$$

For any such index j and for $z \in \mathbb{D}^n$, define

$$f_k(z) = \left(\mathrm{Log}\frac{4}{1-z_j\overline{\varphi_j(z^{(k)})}}\right)^2 \Big/ \log\frac{4}{1-|\varphi_j(z^{(k)})|^2}.$$

As shown in the proof of Theorem 5.2 for the case of the ball, the sequence $\{f_k\}$ is bounded in $\mathcal{B}(\mathbb{D}^n)$, converges to 0 locally uniformly in $\mathbb{D}^n$ and each function in the sequence is in $\mathcal{B}_{0*}(\mathbb{D}^n)$, since it is holomorphic on the closure of $\mathbb{D}^n$. By the compactness of $W_{\psi,\varphi} : \mathcal{B}_{0*}(\mathbb{D}^n) \to H^\infty(\mathbb{D}^n)$, we obtain

$$|\psi(z^{(k)})| \log \frac{1 + \left|\varphi_j(z^{(k)})\right|}{1 - \left|\varphi_j(z^{(k)})\right|} \quad \leq \quad |\psi(z^{(k)}) f_k(\varphi(z^{(k)})| \leq ||\psi(f_k \circ \varphi)||_\infty \to 0$$

as $k \to \infty$.

Next, assume $j \in \{1, \ldots, n\}$ is such that $|\varphi_j(z^{(k)})| \not\to 1$ as $k \to \infty$, so that there exists $r \in (0,1)$ such that $|\varphi_j(z^{(k)})| \leq r$ for all $k \in \mathbb{N}$. Then, by (7.6) we obtain

$$|\psi(z^{(k)})| \log \frac{1 + \left|\varphi_j(z^{(k)})\right|}{1 - \left|\varphi_j(z^{(k)})\right|} \leq \log \frac{1+r}{1-r} |\psi(z^{(k)})| \to 0$$

as $k \to \infty$. Hence, combining the cases when $|\varphi_j(z^{(k)})| \to 1$ or $|\varphi_j(z^{(k)})| \not\to 1$ as $k \to \infty$, we deduce

$$\lim_{k\to\infty} |\psi(z^{(k)})| \sum_{j=1}^{n} \log \frac{1 + \left|\varphi_j(z^{(k)})\right|}{1 - \left|\varphi_j(z^{(k)})\right|} = 0,$$

as desired. Lastly, (c) $\Longrightarrow$ (a) follows at once from (7.5) and Theorem 7.5. □

8. Further developments

In this section, we outline other topics of interest for weighted composition operators not considered in the previous sections. This list is certainly not exhaustive. Our intent is to point out some work that has been done in other settings and how it would pertain to the setting of the Bloch space on a bounded homogeneous domain.

8.1. Isometries

A characterization of the isometric weighted composition operators on the Bloch space of the unit disk is not currently known, although the isometric multiplication operators and the isometric composition operators have been described in Theorem 3.1 of [1], and Corollary 2 of [11] (see also [22], Theorem 1.1). These results provide a means by which to construct isometric weighted composition operators.

In higher dimensions, the isometric multiplication operators acting on the Bloch space of a large class of bounded symmetric domains are precisely the constant functions of modulus 1 [3]. Yet, it is not known whether nontrivial isometric multiplication operators exist on a general bounded homogeneous domain. Conditions for a composition operator on the Bloch space of a bounded homogeneous domain to be an isometry were given in [2]. These conditions allow us to generate nontrivial examples of isometric weighted composition operators on the Bloch space for a large class of domains that have the unit disk as a factor.

8.2. Spectrum

The spectrum of the multiplication operator on the Bloch space of the unit disk is known ([1], Theorem 4.1), while the determination of the spectrum of the composition operator on the Bloch space of the unit disk is still an open problem. The authors determined the spectrum of the isometric composition operators on the Bloch space of the unit disk, and in turn, the spectrum of a large class of isometric weighted composition operators on the Bloch space of the unit disk. The spectrum of a non-isometric weighted composition operator has not been determined for a general class of symbols.

In higher dimensions, the spectrum of a class of isometric composition operators on the Bloch space of the unit polydisk has been determined ([2], Theorem 7.1). In [3] (Theorem 5.1), we showed that the spectrum of a multiplication operator on the Bloch space of a bounded homogeneous domain is the closure of the range of its symbol. On the other hand, in [3] (Corollary 3.6), we proved that the only bounded multiplication operators on the Bloch space of the polydisk $\mathbb{D}^n$ (for $n \geq 2$) are those whose symbol is constant. Thus, the only isometric multiplication operators are those induced by constant functions of modulus one and the corresponding spectrum reduces to the value of that constant.

8.3. Essential norm

The *essential norm* of a bounded operator T is the distance from T to the compact operators, i.e., $||T||_e = \inf\{||T - K|| : K \text{ is compact}\}$. In [21], MacCluer and Zhao established estimates on the essential norm of a weighted composition operator acting on the Bloch space of the unit disk. They showed that

$$\max\left\{A_{\psi,\varphi}, \frac{1}{6}B_{\psi,\varphi}\right\} \leq ||W_{\psi,\varphi}||_e \leq A_{\psi,\varphi} + B_{\psi,\varphi},$$

where

$$A_{\psi,\varphi} = \lim_{s\to 1} \sup_{|\varphi(z)|>s} |\psi(z)|\,|\varphi'(z)|\,\frac{1-|z|}{1-|\varphi(z)|}, \quad \text{and}$$

$$B_{\psi,\varphi} = \lim_{s\to 1} \sup_{|\varphi(z)|>s} |\psi'(z)|\,(1-|z|^2)\log\frac{1}{1-|\varphi(z)|^2}.$$

An estimate on the essential norm of the weighted composition operators on the Bloch space of the polydisk has been given by Li in [19]. To date, no results have appeared on the essential norm of a weighted composition operator acting on the Bloch space of the unit ball or other types of bounded homogeneous domains.

Acknowledgment

We wish to thank the referee for the thoughtful comments and helpful suggestions. On July 6, 2009, well after this paper was accepted for publication, Professor Stevo Stević informed us that he had independently determined the norm of the bounded weighted composition operators from $\mathcal{B}(\mathbb{B}_n)$ to $H^\infty(\mathbb{B}_n)$. Furthermore, in joint work with Songxiao Li, he independently obtained, among other results, necessary and sufficient conditions for the weighted composition operator $W_{\psi,\varphi}$

to be bounded and compact from $\mathcal{B}(\mathbb{D}^n)$ to $H^\infty(\mathbb{D}^n)$. We wish to thank him for letting us know about these results, and for supplying us with copies of his papers:

- S. Stević, "Norm of weighted composition operators from Bloch space to H^∞_μ on the unit ball", *Ars. Combin.* **88** (2008), 125–127;
- S. Li and S. Stević, "Weighted composition operators from α-Bloch space to H^∞ on the polydisk", *Numer. Funct. Anal. Optimization* **28** (7) (2007), 911–925.

References

[1] R.F. Allen and F. Colonna, *Isometries and spectra of multiplication operators on the Bloch space*, Bull. Austral. Math. Soc., **79** (2009), 147–160.

[2] ———, *On the isometric composition operators on the Bloch space in* $\mathbb{C}^n$, J. Math. Anal. Appl., **355**(2) (2009), 675–688.

[3] ———, *Multiplication operators on the Bloch space of bounded homogeneous domains*, Comput. Methods and Funct. Theory, **9**(2) (2009), 679–693.

[4] J.M. Anderson, J. Clunie and Ch. Pommerenke, *On Bloch functions and normal functions*, J. Reine Angew. Math. **279** (1974), 12–37.

[5] S. Banach, *Théorie des opérations linéaires*, Chelsea, Warsaw, (1932).

[6] L. Brown and A.L. Shields, *Multipliers and cyclic vectors in the Bloch space*, Michigan Math. J. **38**, (1991), 141–146.

[7] J. Cima, *The basic properties of Bloch functions*, Internat. J. Math. Math. Sci. **3** (1979), 369–413.

[8] J. Cima and W. Wogen, *On isometries of the Bloch space*, Illinois J. Math. **24** (1980), 313–316.

[9] J.M. Cohen and F. Colonna, *Bounded holomorphic functions on bounded symmetric domains*, Trans. Amer. Math. Soc. **343** (1994), 135–156.

[10] ———, *Isometric composition operators on the Bloch space in the polydisk*, Contemp. Math., **454** (2008), 9–21.

[11] F. Colonna, *Characterisation of the isometric composition operators on the Bloch space*, Bull. Austral. Math. Soc. **72** (2005), 283–290.

[12] M. El-Gebeily and J. Wolfe, *Isometries of the disk algebra*, Proc. Amer. Math. Soc. **93** (1985), 697–702.

[13] F. Forelli, *The isometries on* H^p, Canadian J. Math **16** (1964), 721–728.

[14] K.T. Hahn, *Holomorphic mappings of the hyperbolic space in the complex Euclidean space and the Bloch theorem*, Canad. J. Math **27** (1975), 446–458.

[15] S. Helgason, *Differential Geometry and Symmetric Spaces*, Academic Press, New York (1962).

[16] T. Hosokawa, K. Izuchi, S. Ohno, *Topological structure of the space of weighted composition operators on* H^∞, Integr. Equ. Oper. Theory **53** (2005), 509–526.

[17] C.J. Kolaski, *Isometries of weighted Bergman spaces*, Canad. J. Math **34** (1982), 910–915.

[18] S.G. Krantz and D. Ma, *On isometric isomorphisms of the Bloch space on the unit ball of* $\mathbb{C}^n$, Michigan Math. J. **36** (1989), 173–180.

[19] S.X. Li, *Essential norm of the weighted composition operators on the Bloch spaces over the polydisk* (Chinese, English and Chinese summary), Acta Math. Sinica (Chin. Ser.) **51** (2008), no. 4, 655–662.

[20] S. Li and S. Stević, *Weighted composition operators between H^∞ and α-Bloch spaces in the unit ball*, Taiwanese J. Math. **12** (2008), 1625–1639.

[21] B.D. MacCluer and R. Zhao, *Essential norms of weighted composition operators between Bloch-type spaces*, Rocky Mountain J. of Math. **33** (2003), 1437–1457.

[22] M. Martín and D. Vutokić, *Isometries of the Bloch space among the composition operators*, Bull. London Math. Soc. **39** (2007), 151–155.

[23] S. Ohno and R. Zhao, *Weighted composition operators on the Bloch space*, Bull. Austral. Math. Soc. **63** (2001), 177–185.

[24] Ch. Pommerenke, *On Bloch functions*, J. London Math. Soc. **2** (1970), 689–695.

[25] J.H. Shi and L. Luo, *Composition operators on the Bloch space of several complex variables*, Acta Math. Sinica (N.S.) **16** (2000), 85–98.

[26] R.M. Timoney, *Bloch functions in several complex variables I*, Bull. London Math. Soc. **12** (1980), 241–267.

[27] ———, *Bloch functions in several complex variables II*, J. Reine Angew. Math. **319** (1980), 1–22.

[28] M. Tjani, *Compact composition operators on Möbius invariant Banach spaces*, Ph.D. dissertation, Michigan State University (1996).

[29] C. Xiong, *Norm of composition operators on the Bloch space*, Bull. Austral. Math. Soc. **70** (2004), 293–299.

[30] G. Zhang, *Bloch constants of bounded symmetric domains*, Trans. Amer. Math. Soc. **349** (1997), 2941–2949.

[31] Z. Zhou and R. Chen, *Weighted composition operators from $F(p,q,s)$ to Bloch type spaces on the unit ball*, Internat. J. of Math. **19** (2008), 899–926.

[32] ———, *Weighted composition operators between different Bloch-type Spaces in polydisk*, (preprint) (http://arxiv.org/abs/math/0503622v2).

[33] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Springer-Verlag, New York (2004).

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Received: February 23, 2009
Accepted: July 1, 2009

Operator Theory:
Advances and Applications, Vol. 202, 39–51

Images of Minimal-vector Sequences Under Weighted Composition Operators on $L^2(\mathbb{D})$

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Abstract. Let X be either the unit interval in $\mathbb{R}$ or the unit disk $\mathbb{D}$ in $\mathbb{C}$. Chalendar, Flattot, and Partington [2] study weighted composition operators $T_{w,\gamma}$ on $L^2(X)$, where $w \in L^\infty(X)$, $\gamma : X \to X$ is injective, and $T_{w,\gamma} f = w\, f \circ \gamma$ for $f \in L^2(X)$. They introduce a (strict) partial order $\prec$ on X associated with $T_{w,\gamma}$ and use it to obtain a sufficient condition for convergence of the sequence $(T^n_{w,\gamma} y_n)$ where (y_n) is a backward minimal-vector sequence for $T_{w,\gamma}$. For the $L^2(\mathbb{D})$ case, they give a detailed analysis of the situation where γ is linear-fractional. Through further study of the partial order $\prec$, we are able to generalize results from [2] that apply when γ is linear-fractional, replacing the linear-fractional hypotheses with univalence. In particular, our work yields generalizations of an invariant-subspace theorem in [2].

Mathematics Subject Classification (2000). Primary 47A15, 47B33.

Keywords. Minimal-vector, hyperinvariant subspace, weighted composition operator.

1. Introduction and background

Ansari and Enflo [1] introduce the idea of backward (and forward) minimal vectors and explore the roles these vectors may play in establishing the existence of nontrivial hyperinvariant subspaces for certain classes of linear operators, e.g., compact and normal operators. Let T be a bounded linear operator with dense range on the Hilbert space H, let f be a nonzero vector in H, and let ϵ satisfy $\|f\| > \epsilon > 0$. For each positive integer n, the *backward minimal vector* y_n for T, f, and ϵ is the unique vector in H satisfying

$$\|T^n y_n - f\| \le \epsilon \quad \text{and} \quad \|y_n\| = \min\{\|y\| : \|T^n y - f\| \le \epsilon\}.$$

Ansari and Enflo [1, §2] prove the following (a Banach-space version appears in [7]):

Communicated by J.A. Ball.

Theorem AE. *If $T : H \to H$ is quasinilpotent and has a sequence (y_n) of backward minimal points such that $(T^n y_n)$ converges (in norm), then T has a nontrivial hyperinvariant subspace.*

Given the connection between convergence of images of backward minimal-vector sequences and the existence of nontrivial invariant subspaces, the study of such convergence has attracted significant attention. Ansari and Enflo [1, Theorem 7] show that whenever $T : H \to H$ is a cyclic normal operator with dense range, then $(T^n y_n)$ will be convergent for any sequence (y_n) of backward minimal vectors for T. This result has been generalized to noncyclic normal operators [3, Proposition 2.1] and to operators of "normal type" [2, Theorem 2.2]. However, in general, $(T^n y_n)$ need not converge. For example, Wiesner [9, Section 4] shows that there are matrix operators on $\mathbb{C}^2$ that have backward minimal-vector sequences (y_n) for which $(T^n y_n)$ fails to converge, while Chalendar and Partington [3, Theorem 3.1] present a necessary and sufficient condition for a dense-range bilateral weighted shift T on $\ell^2(\mathbb{Z})$ to have the property that $(T^n y_n)$ is convergent for all backward minimal-vector sequences (y_n) for T.

Chalendar, Flattot, and Partington [2, Theorem 3.2] have obtained a sufficient condition for convergence of images of backward minimal-vector sequences for certain weighted composition operators on L^2 spaces (while giving an example [2, p. 96] showing such sequences do not always converge). They work on L^2 of the closed unit interval I or closed unit disk $\overline{\mathbb{D}}$. We will confine our attention to the disk setting and work on the open unit disk $\mathbb{D}$ instead of $\overline{\mathbb{D}}$. (Of course, $L^2(\overline{\mathbb{D}}) = L^2(\mathbb{D})$ because the Lebesgue area measure of $\partial\mathbb{D}$ is zero. None of the results or proofs in [2] are changed by replacing $\overline{\mathbb{D}}$ with $\mathbb{D}$.)

Let $w \in L^\infty(\mathbb{D})$ and let $\gamma : \mathbb{D} \to \mathbb{D}$ be a univalent mapping such that $w/\gamma' \in L^\infty(\mathbb{D})$. Then it is easy to check that the weighted composition operator $T_{w,\gamma}$ defined by

$$T_{w,\gamma} f = w\, f \circ \gamma$$

is bounded and linear on $L^2(\mathbb{D})$ (with $\|T_{w,\gamma}\| \le \|w/\gamma'\|_\infty$). Also easy to check is the following formula for the adjoint $T^*_{w,\gamma}$ of $T_{w,\gamma}$:

$$T^*_{w,\gamma} f = \frac{\overline{w} \circ \gamma^{-1}}{|\gamma' \circ \gamma^{-1}|^2} \chi_{\gamma(\mathbb{D})} f \circ \gamma^{-1}. \tag{1}$$

Throughout this paper, unless we indicate otherwise, the functions w and γ are assumed to satisfy the restrictions described above; in particular, γ is a univalent self-map of $\mathbb{D}$.

We restrict our attention to those operators $T_{w,\gamma}$ that have dense range. Let m denote Lebesgue area measure.

Proposition 1. *The operator $T_{w,\gamma} : L^2(\mathbb{D}) \to L^2(\mathbb{D})$ has dense range if and only if $w \neq 0$ a.e. with respect to m.*

Proof. If w vanished on a subset E of $\mathbb{D}$ for which $m(E) > 0$, then every function in the range of $T_{w,\gamma}$ would also vanish a.e. on E and Range$(T_{w,\gamma})$ would not be dense.

Conversely, suppose that $T_{w,\gamma}$ does not have dense range so that there is nonzero function $f \in L^2(\mathbb{D})$ such that $T^*_{w,\gamma} f \equiv 0$. Let $E \subseteq \mathbb{D}$ be such that $m(E) > 0$ and f is nonzero a.e. on E. Then $m(\gamma(E))$ is also positive (see, e.g., the discussion preceding the statement of Lemma 1 in Section 3). Since we are assuming $T^*_{w,\gamma} f \equiv 0$, the formula (1) now shows that w must vanish a.e. on E, a set of positive area measure. $\square$

Let $T_{w,\gamma} : L^2(\mathbb{D}) \to L^2(\mathbb{D})$ have dense range. In [2], Chalendar, Flattot, and Partington develop a criterion for all backward minimal-vector sequences (y_n) for $T_{w,\gamma}$ to have the property that $(T^n_{w,\gamma} y_n)$ is convergent. The criterion depends on a (strict) partial order $\prec$ on $\mathbb{D}$ related to the symbols w and γ of $T_{w,\gamma}$. For $z, v \in \mathbb{D}$, this partial order is defined by

$$z \prec v \quad \text{if and only if} \quad \limsup_{n\to\infty} \frac{h(\gamma^{[n]}(z))}{h(\gamma^{[n]}(v))} < 1, \tag{2}$$

where $h = |w/\gamma'|^2$ and $\gamma^{[n]}$ denotes the nth iterate of γ. Theorem 3.2 of [2] shows that if $\prec$ has certain regularity properties with respect to m, then $(T^n_{w,\gamma} y_n)$ will be convergent for backward minimal-vector sequences (y_n) for $T_{w,\gamma}$.

Having obtained in their Theorem 3.2 information about the behavior of backward minimal-vector sequences, the authors of [2] turn their attention to invariant subspace theorems, seeking to apply Theorem AE, which means they must develop criteria for quasinilpotence of $T_{w,\gamma}$. Proposition 4.3 of [2] shows that if γ has an attractive fixed point z_0 in $\mathbb{D}$ or in its closure $\overline{\mathbb{D}}$, such that the iterate sequence $(\gamma^{[n]})$ converges uniformly on $\mathbb{D}$ to z_0, and if $h := |w/\gamma'|^2$ extends continuously to z_0, then $T_{w,\gamma}$ is quasinilpotent if and only if $h(z_0) = 0$.

Recall that any (not necessarily univalent) analytic self-map γ of $\mathbb{D}$ that is not an elliptic automorphism has a unique attractive fixed point $\omega_0 \in \overline{\mathbb{D}}$, its Denjoy-Wolff point, and that when ω_0 lies in $\mathbb{D}$, necessarily $|\gamma'(\omega_0)| < 1$ and that when ω_0 lies in $\partial\mathbb{D}$, necessarily $0 < \gamma'(\omega_0) \le 1$, where $\gamma'(\omega_0)$ is the *angular derivative* of γ at ω_0 (see, e.g., [6]). We remark that when γ has an angular derivative at $\zeta \in \partial\mathbb{D}$, then γ' has a continuous extension from $\mathcal{N}_\zeta$ to $\mathcal{N}_\zeta \cup \{\zeta\}$ where $\mathcal{N}_\zeta$ is any nontangential approach region in $\mathbb{D}$ with vertex ζ; however, γ' need not extend continuously from $\mathbb{D}$ to $\mathbb{D} \cup \{\zeta\}$. For further information about angular derivatives, the reader may consult [4] or [6]. When

- $\omega_0 \in \mathbb{D}$ and $0 < |\gamma'(\omega_0)| < 1$, we will say that γ is of *Schröder* type;
- when $\omega_0 \in \partial\mathbb{D}$ and $\gamma'(\omega_0) < 1$, we say that γ is of *hyperbolic type*;
- when $\omega_0 \in \partial\mathbb{D}$ and $\gamma'(\omega_0) = 1$, we say that γ is of *parabolic type*.

Using their Proposition 4.3 and Theorem 3.2 and assuming that $T_{w,\gamma}$ has dense range, Chalendar, Flattot, and Partington obtain the following invariant-subspace theorem for $T_{w,\gamma}$ [2, Theorem 4.5]:

Theorem CFP. *Suppose that $\gamma : \mathbb{D} \to \mathbb{D}$ is a non-automorphic, linear-fractional mapping and one of the following holds:*

(i) *γ has two fixed points, one in $\mathbb{D}$ and the other outside $\overline{\mathbb{D}}$ (possibly at ∞), or*

(ii) *γ has two fixed points, one on $\partial\mathbb{D}$ and the other outside $\overline{\mathbb{D}}$ (possibly at ∞).*

Suppose also that w extends to be (complex) differentiable at the attractive fixed point ω_0 of γ, with $w(\omega_0) = 0$ and $w'(\omega_0) \neq 0$. Then $T_{w,\gamma} : L^2(\mathbb{D}) \to L^2(\mathbb{D})$ has a nontrivial hyperinvariant subspace.

Our interest in the preceding theorem is principally in its proof, where it is shown that when the hypotheses of theorem hold, then the partial order $\prec$ of (2) has the regularity properties required by Theorem 3.2 of [2]; consequently, $(T^n_{w,\gamma} y_n)$ is convergent for each backward minimal-vector sequence (y_n) for $T_{w,\gamma}$. Observe that in case (i) of Theorem CFP, γ is of Schröder type and in case (ii), γ is of hyperbolic type; in particular, in both cases, we have $|\gamma'(\omega_0)| < 1$.

We now summarize the principal results of this paper. Propositions 2, 3, and 5 below combine to show the following: *Let γ be any (not necessarily univalent) analytic self-map of $\mathbb{D}$ that is not an elliptic automorphism and let ω_0 be its Denjoy-Wolff point. Suppose that w satisfies the hypotheses of Theorem CFP and $|\gamma'(\omega_0)| < 1$; then the partial order $\prec$ defined in (2) has the regularity properties needed to apply Theorem 3.2 of [2].* As a consequence, we obtain Theorem 2 below, our main theorem: *if γ is any univalent mapping of Schröder or hyperbolic type, then $(T^n_{w,\gamma} y_n)$ is convergent for each backward minimal-vector sequence (y_n) for $T_{w,\gamma}$.*

We remark that if w is continuous and nonzero at ω_0 (so that it does not satisfy the hypotheses of Theorem CFP), then for γ of any of the types, Schröder, hyperbolic, or parabolic, we have

$$\limsup_{n\to\infty} \frac{h(\gamma^{[n]}(z))}{h(\gamma^{[n]}(v))} = \lim_{n\to\infty} \left|\frac{w(\gamma^{[n]}(z))/\gamma'(\gamma^{[n]}(z))}{w(\gamma^{[n]}(v))/\gamma'(\gamma^{[n]}(v))}\right|^2 = \left|\frac{w(\omega_0)/\gamma'(\omega_0)}{w(\omega_0)/\gamma'(\omega_0)}\right|^2 = 1$$

so that no two points of $\mathbb{D}$ are comparable under definition (2) of $\prec$ and one cannot regard the vacuous partial order $\prec$ as satisfying the regularity properties needed to apply Theorem 3.2 of [2]. Also, when γ is of parabolic type, the discussion on page 101 of [2] shows that even if w satisfies the hypotheses of Theorem CFP, the partial order $\prec$ need not have the regularity properties required by Theorem 3.2 of [2].

Let γ be any (not necessarily univalent) analytic self-map $\mathbb{D}$ that is of Schröder or hyperbolic type and let w satisfy the hypotheses of Theorem CFP. In the following section, we show that for all but countably many points v of $\mathbb{D}$, the limit superior used to define the partial order $\prec$ of (2) may be replaced by a limit; that is,

$$\lim_{n\to\infty} \frac{h(\gamma^{[n]}(z))}{h(\gamma^{[n]}(v))}$$

exists for every $z \in \mathbb{D}$ and $v \in \mathbb{D} \setminus S$, where S is at most a countable set. We show that the limit will be the squared modulus of quotient of classical analytic intertwining maps: Koenigs functions in Case I and Valiron functions in Case II. In Section 3, we describe the regularity properties of $\prec$ needed for Theorem 3.2

of [2] to hold. Then we prove that whenever f is analytic and nonconstant on $\mathbb{D}$, the partial order $\prec$ defined on $\mathbb{D}$ by

$$z \prec v \quad \text{if and only if} \quad |f(z)| < |f(v)|$$

has these regularity properties. In Section 4, we combine the results of our Sections 2 and 3 with Theorem 3.2 of [2] to obtain our main result about convergence of images of backward minimal-vector sequences. We also discuss implications of our work for invariant-subspaces of weighted composition operators, obtaining generalizations of Theorem CFP.

2. Koenigs and Valiron functions

In this section, we assume that $w \in L^\infty(\mathbb{D})$ and that $\gamma : \mathbb{D} \to \mathbb{D}$ is an analytic self-map of $\mathbb{D}$ that is not necessarily univalent.

Suppose that γ is not an elliptic automorphism. Assume γ's Denjoy-Wolff point ω_0 lies in $\mathbb{D}$ and that $\gamma'(\omega_0) \neq 0$. Since γ is not an elliptic automorphism, we know (by Schwarz's Lemma) that $|\gamma'(\omega_0)| < 1$, so that γ is of Schröder type. Koenigs [5] (see also, [6, Chapter 5], e.g.) proved that the sequence

$$\frac{\gamma^{[n]} - \omega_0}{\gamma'(\omega_0)^n} \tag{3}$$

converges uniformly on compact subsets of $\mathbb{D}$ to a nonconstant analytic function σ on $\mathbb{D}$ satisfying Schröder's functional equation

$$\sigma \circ \gamma = \gamma'(\omega_0)\sigma. \tag{4}$$

We call σ the *Koenigs' function* for γ. Note that $\sigma(\omega_0) = 0$ and $\sigma'(\omega_0) = 1$. Koenigs proved that σ is the unique function satisfying $\sigma(\omega_0) = 0$, $\sigma'(\omega_0) = 1$, and the relation (4). Finally, note that Hurwitz's Theorem shows that σ will be univalent whenever γ is univalent.

Recall that associated with our weighted composition operator $T_{w,\gamma}$ is the function $h(z) = |w(z)/\gamma'(z)|^2$ that participates in the definition of the partial order $\prec$ defined by (2). The next result shows when γ satisfies the restrictions holding in the preceding paragraph and w satisfies the hypotheses of Theorem CFP, then the limit superior in (2) can be replaced with a limit for all but at most countably many values of $v \in \mathbb{D}$.

Proposition 2. *Suppose that γ has Denjoy-Wolff point ω_0 lying in $\mathbb{D}$, that $0 < |\gamma'(\omega_0)| < 1$, and that σ is the Koenigs function for γ. Suppose that w is differentiable at ω_0 with $w(\omega_0) = 0$ and $w'(\omega_0) \neq 0$. Then, for every $z \in \mathbb{D}$ and for every $v \in \mathbb{D}$ that is not a zero of σ, we have*

$$\lim_{n\to\infty} \frac{h(\gamma^{[n]}(z))}{h(\gamma^{[n]}(v))} = \left|\frac{\sigma(z)}{\sigma(v)}\right|^2. \tag{5}$$

Proof. We interpret "w is differentiable at ω_0" to include the assumption that w is defined in a neighborhood of ω_0. Let $z \in \mathbb{D}$, and let $v \in \mathbb{D}$ be such that $\sigma(v) \neq 0$. We have

$$\begin{aligned}
\frac{h(\gamma^{[n]}(z))}{h(\gamma^{[n]}(v))} &= \left|\frac{w(\gamma^{[n]}(z))}{w(\gamma^{[n]}(v))}\frac{\gamma'(\gamma^{[n]}(v))}{\gamma'(\gamma^{[n]}(z))}\right|^2 \\
&= \left|\frac{\frac{w(\gamma^{[n]}(z))}{\gamma^{[n]}(z)-\omega_0}\frac{\gamma^{[n]}(z)-\omega_0}{\gamma'(\omega_0)^n}}{\frac{w(\gamma^{[n]}(v))}{\gamma^{[n]}(v)-\omega_0}\frac{\gamma^{[n]}(v)-\omega_0}{\gamma'(\omega_0)^n}}\frac{\gamma'(\gamma^{[n]}(v))}{\gamma'(\gamma^{[n]}(z))}\right|^2 \\
&\to \left|\frac{w'(\omega_0)}{w'(\omega_0)}\frac{\sigma(z)}{\sigma(v)}\frac{\gamma'(\omega_0)}{\gamma'(\omega_0)}\right|^2 \qquad (\text{as } n \to \infty) \\
&= \left|\frac{\sigma(z)}{\sigma(v)}\right|^2 .
\end{aligned}$$

□

Observe that when γ satisfies the hypotheses of the preceding proposition and is also univalent, then its Koenigs function σ, which vanishes at ω_0, is univalent as well, which means that σ vanishes only at ω_0. Thus when γ is univalent the limit fact (5) is valid for all $z \in \mathbb{D}$ and all $v \in \mathbb{D} \setminus \{\omega_0\}$.

We now turn to the case where γ is an analytic self-map of $\mathbb{D}$ that has no fixed point in $\mathbb{D}$, so that its Denjoy-Wolff point ω_0 lies on $\partial\mathbb{D}$. In this case the Julia-Carathéodory Theorem (see, e.g., [6, p. 57] or [4, Theorem 2.44]) shows that

$$\angle \lim_{z \to \omega_0} \gamma'(z) = \gamma'(\omega_0), \tag{6}$$

where $\gamma'(\omega_0)$ is the angular derivative of γ at ω_0 and $\angle \lim$ denotes the nontangential (or angular) limit. Necessarily $\gamma'(\omega_0) \le 1$. If we assume that $\gamma'(\omega_0) < 1$, then for each $z \in \mathbb{D}$, the sequence $(\gamma^{[n]}(z))$ converges nontangentially to ω_0 (see, e.g., [4, Lemma 2.66]). Hence, in view of (6), we see that if $\gamma'(\omega_0) < 1$, then $(\gamma'(\gamma^{[n]}(z)))$ converges to $\gamma'(\omega_0)$ for every $z \in \mathbb{D}$.

Suppose that γ is an analytic self-map of $\mathbb{D}$ whose Denjoy-Wolff point ω_0 lies on $\partial\mathbb{D}$ and suppose that $\gamma'(\omega_0) < 1$, so that γ is of hyperbolic type. Fix $z_0 \in \mathbb{D}$. Valiron [8] proved that the sequence

$$\frac{\gamma^{[n]} - \omega_0}{|\gamma^{[n]}(z_0) - \omega_0|}$$

converges uniformly on compact subsets of $\mathbb{D}$ to a nonconstant analytic map ν satisfying

$$\nu \circ \gamma = \gamma'(\omega_0)\nu.$$

We call ν the *Valiron function* for γ. We remark that the work of [8] is all in the right half-plane, but is easily transferred to the disk and yields the results about ν described above. For example, if $\omega_0 = 1$ is the Denjoy-Wolff point of γ and $F(z) = (1 + z)/(1 - z)$, then $\Gamma = F \circ \gamma \circ F^{-1}$ will be a self-map of the right half-plane with attractive fixed point at ∞.

Proposition 3. *Suppose that γ has Denjoy-Wolff point ω_0 lying in $\partial\mathbb{D}$, that $\gamma'(\omega_0) < 1$, and that ν is the Valiron function for γ. Suppose that w extends to be differentiable at ω_0 with $w(\omega_0) = 0$ and $w'(\omega_0) \neq 0$. Then, for every $z \in \mathbb{D}$ and for every $v \in \mathbb{D}$ that is not a zero of ν, we have*

$$\lim_{n\to\infty} \frac{h(\gamma^{[n]}(z))}{h(\gamma^{[n]}(v))} = \left|\frac{\nu(z)}{\nu(v)}\right|^2. \tag{7}$$

Proof. We interpret "w extends to be differentiable at ω_0" to mean that for some $\epsilon > 0$, w is defined on $\omega_0 \cup \Big(\mathbb{D} \cap \{z : |z - \omega_0| < \epsilon\}\Big)$ and $w(z)/(z - \omega_0)$ converges to $w'(\omega_0)$ as $z \to \omega_0$ from within $\mathbb{D}$. We have for $z \in \mathbb{D}$ and v not a zero of ν,

$$\begin{aligned}
\frac{h(\gamma^{[n]}(z))}{h(\gamma^{[n]}(v))} &= \left|\frac{w(\gamma^{[n]}(z))}{w(\gamma^{[n]}(v))}\frac{\gamma'(\gamma^{[n]}(v))}{\gamma'(\gamma^{[n]}(z))}\right|^2 \\
&= \left|\frac{\frac{w(\gamma^{[n]}(z))}{\gamma^{[n]}(z)-\omega_0}\frac{\gamma^{[n]}(z)-\omega_0}{|\gamma^{[n]}(z_0)-\omega_0|}}{\frac{w(\gamma^{[n]}(v))}{\gamma^{[n]}(v)-\omega_0}\frac{\gamma^{[n]}(v)-\omega_0}{|\gamma^{[n]}(z_0)-\omega_0|}}\frac{\gamma'(\gamma^{[n]}(v))}{\gamma'(\gamma^{[n]}(z))}\right|^2 \\
&\to \left|\frac{w'(\omega_0)}{w'(\omega_0)}\frac{\nu(z)}{\nu(v)}\frac{\gamma'(\omega_0)}{\gamma'(\omega_0)}\right|^2 \qquad (\text{as } n \to \infty) \\
&= \left|\frac{\nu(z)}{\nu(v)}\right|^2.
\end{aligned}$$

□

Observe that when γ satisfies the hypotheses of the preceding proposition and is also univalent, then its Valiron function ν is univalent as well (again by Hurwitz's Theorem), which means that ν can vanish at only one point of $\mathbb{D}$ so that the limit fact (7) is valid for all $z \in \mathbb{D}$ and all but at most one $v \in \mathbb{D}$.

Note that if the hypotheses of either Proposition 2 or Proposition 3 hold, then the conclusions of these propositions show that the partial order (2) used in [2] takes the form

$$z \prec v \quad \text{if and only if} \quad |f(z)| < |f(v)|,$$

where f is a nonconstant analytic function on $\mathbb{D}$ (either a Koenigs function or Valiron function). In the next section we show that any such partial order satisfies the regularity properties required to apply results from [2].

3. Regularity properties of $\prec$

Let f be defined on the open unit disk $\mathbb{D}$. Define the strict partial order $\prec$ on $\mathbb{D}$ by

$$z \prec v \quad \text{iff} \quad |f(z)| < |f(v)|. \tag{8}$$

Call a subset Y of $\mathbb{D}$ *inner-filled* relative to $\prec$ provided that whenever $v \in Y$ and $z \prec v$, then $z \in Y$. For example, if $v \in \mathbb{D}$, then $Y := \{z \in \mathbb{D} : z \prec v\}$ is inner-filled relative to $\prec$. Call a subset Y of $\mathbb{D}$ *outer-filled* relative to $\prec$ provided that whenever

$v \in Y$ and $v \prec z$, then $z \in Y$. For example, if $v \in \mathbb{D}$, then $Y := \{z \in \mathbb{D} : v \prec z\}$ is outer-filled relative to $\prec$.

Let

$$\mu = m/\pi$$

so that μ is normalized Lebesgue area measure for $\mathbb{D}$. We say that $\prec$ is *inner-regular* with respect to μ on $\mathbb{D}$ provided that for each set $Y \subseteq \mathbb{D}$ that is inner-filled relative to $\prec$ and each $\delta > 0$, there is a $v \in Y$ such that $\mu(\{z \in \mathbb{D} : z \prec v\}) > \mu(Y) - \delta$. We say that $\prec$ is *outer-regular* with respect to μ provided that for each set $Y \subseteq \mathbb{D}$ that is outer-filled relative to $\prec$ and each $\delta > 0$, there is a $v \in Y$ such that $\mu(\{z \in \mathbb{D} : v \prec z\}) > \mu(Y) - \delta$. Finally, we say that $\prec$ is *regular* with respect to μ provided that it is both inner- and outer-regular with respect to μ. We remark that our regularity terminology differs from that in [2], where *left-regular* is used instead of inner-regular and *right-regular* is used instead of outer-regular.

Suppose that f is nonconstant and analytic on an open connected set $G \subseteq \mathbb{C}$. Then it's easy to see that f must take a subset of G having positive area measure to a set of positive area measure. This follows immediately from multivalent change-of-variables formulas such as [4, Theorem 2.32]. Here's an alternate argument based on the univalent change-of-variable formula. Suppose that $A \subseteq G$ has positive area; then since f' has at most countable many zeros, there will be a point $z \in A$ that is a Lebesgue-density point of A such that $f'(z) \neq 0$. Because $f'(z) \neq 0$ there is an open disk D_z centered at z of positive radius on which f is univalent. Because z is a Lebesgue-density point of A, $m(D_z \cap A) > 0$. We have

$$0 < \int_{D_z \cap A} |f'|^2 \, dm = \int_{f(D_z \cap A)} dm \leq m(f(A)),$$

Hence we have the following.

Lemma 1. *Let $f : \mathbb{D} \to \mathbb{C}$ be nonconstant and analytic on $\mathbb{D}$. If $E \subseteq \mathbb{C}$ is such that $m(E) = 0$; then $\mu(f^{-1}(E)) = 0$.*

Proposition 4. *Suppose that f is nonconstant and analytic on $\mathbb{D}$ and $\prec$ is defined by* (8)*. Then for every $v \in \mathbb{D}$,*

$$\mu(\{z \in \mathbb{D} : z \prec v \text{ or } v \prec z\}) = 1. \tag{9}$$

Proof. Let $v \in \mathbb{D}$ and let $A = \{z \in \mathbb{D} : z \prec v \text{ or } v \prec z\}$. Note that

$$\mathbb{D} \setminus A = \{z \in \mathbb{D} : |f(z)| = |f(v)|\}.$$

Since $\mathbb{D} \setminus A = f^{-1}\{\zeta|f(v)| : |\zeta| = 1\}$ and $m(\{\zeta|f(v)| : |\zeta| = 1\}) = 0$, Lemma 1 shows that $\mu(\mathbb{D} \setminus A) = 0$ and hence $\mu(A) = 1$, as desired. $\square$

Proposition 5. *Suppose that f is analytic and nonconstant on $\mathbb{D}$; then the partial order defined by* (8) *is regular with respect to μ.*

Proof. Let $Y \subseteq \mathbb{D}$ be inner-filled relative to $\prec$ and let $\delta > 0$. Choose a positive number r with $r < 1$ such that $\mu(Y \cap \{z : |z| > r\}) < \delta/2$. Set $\widetilde{Y} = Y \cap r\overline{\mathbb{D}}$.

Let $s = \sup\{|f(v)| : v \in \widetilde{Y}\}$ and note s is finite since $\widetilde{Y} \subseteq r\overline{\mathbb{D}}$ and f is continuous on $r\overline{\mathbb{D}}$. Let (v_n) be a sequence of points in $\widetilde{Y}$ such that $(|f(v_n)|)$ is an increasing sequence with limit s. Note that

$$f^{-1}(s\overline{\mathbb{D}}) \supseteq \widetilde{Y} \tag{10}$$

Let $n \in \mathbb{N}$ be arbitrary. Observe that

$$\begin{aligned} \mu\left(f^{-1}(s\overline{\mathbb{D}})\right) - \mu\left(f^{-1}(|f(v_n)|\mathbb{D})\right) &= \mu\left(f^{-1}(s\overline{\mathbb{D}}) \setminus \left(f^{-1}(|f(v_n)|\mathbb{D})\right)\right) \\ &= \mu\left(f^{-1}(E_n)\right), \end{aligned} \tag{11}$$

where $E_n = (s\overline{\mathbb{D}}) \setminus (|f(v_n)|\mathbb{D})$. Since $|f(v_n)|$ approaches s as $n \to \infty$, we see that $\mu(E_n) \to 0$ as $n \to \infty$.

We claim that $\mu\left(f^{-1}(E_n)\right)$ must also approach 0 as $n \to \infty$. Note that the sequence $(f^{-1}(E_n))$ of μ-measurable subsets of $\mathbb{D}$ is nested:

$$f^{-1}(E_{n+1}) \subseteq f^{-1}(E_n) \quad \text{for every } n \in \mathbb{N}$$

and thus $F := \cap_{j=1}^{\infty} f^{-1}(E_j)$ satisfies

$$\mu(F) = \lim_{n\to\infty} \mu\left(f^{-1}(E_n)\right).$$

Note that if $z \in F$, then $|f(v_j)| \leq |f(z)| \leq s$ for all j, which implies $|f(z)| = s$. Hence $F \subseteq f^{-1}(\{\zeta s : |\zeta| = 1\})$. Since $m(\{\zeta s : |\zeta| = 1\}) = 0$, Lemma 1 shows that $\mu(F) = 0$ and hence $\lim_{n\to\infty} \mu\left(f^{-1}(E_n)\right) = 0$.

Choosing $n_0 \in \mathbb{N}$ sufficiently large, we have $\mu(f^{-1}(E_{n_0})) < \delta/2$. Now using (10) and the equality from (11), we have

$$\begin{aligned} \mu(Y) - \mu\left(f^{-1}(|f(v_{n_0})|\mathbb{D})\right) &\leq \delta/2 + \mu(\widetilde{Y}) - \mu\left(f^{-1}(|f(v_{n_0})|\mathbb{D})\right) \\ &\leq \delta/2 + \mu\left(f^{-1}(s\overline{\mathbb{D}})\right) - \mu\left(f^{-1}(|f(v_{n_0})|\mathbb{D})\right) \\ &= \delta/2 + \mu\left(f^{-1}(E_{n_0})\right) < \delta, \end{aligned}$$

and it follows that $\prec$ is inner-regular since $f^{-1}(|f(v_{n_0})|\mathbb{D}) = \{z \in Y : z \prec v_{n_0}\}$ and $v_{n_0} \in Y$.

The proof of outer-regularity is quite similar. Let $Y \subseteq \mathbb{D}$ be outer-filled relative to $\prec$ and let $\delta > 0$. Let $v \in Y$ be arbitrary. Observe that

$$\begin{aligned} Y &\supseteq \{z \in \mathbb{D} : v \prec z\} \\ &= \{z \in \mathbb{D} : |f(v)| < |f(z)|\} \\ &= f^{-1}\left(\mathbb{C} \setminus (|f(v)|\overline{\mathbb{D}})\right). \end{aligned}$$

Let $s = \inf\{|f(v)| : v \in Y\}$ and let (v_n) be a sequence of points in Y such that $(|f(v_n)|)$ is decreasing and $\lim_n |f(v_n)| = s$. Let $n \in \mathbb{N}$. We have

$$f^{-1}\left(\mathbb{C} \setminus s\mathbb{D}\right) \supseteq Y \supseteq f^{-1}\left(\mathbb{C} \setminus |f(v_n)|\overline{\mathbb{D}}\right). \tag{12}$$

Now note that

$$\begin{aligned}\mu\Big(f^{-1}(\mathbb{C}\setminus s\mathbb{D})\Big)-\mu\big(f^{-1}(\mathbb{C}\setminus|f(v_n)|\overline{\mathbb{D}})\big)&=\mu\Big(f^{-1}(\mathbb{C}\setminus s\mathbb{D})\setminus\Big(f^{-1}(\mathbb{C}\setminus|f(v_n)|\overline{\mathbb{D}})\Big)\Big)\\&=\mu\big(f^{-1}(E_n)\big),\end{aligned} \tag{13}$$

where $E_n = (C \setminus s\mathbb{D}) \setminus (\mathbb{C} \setminus |f(v_n)|\overline{\mathbb{D}}) = \{z \in \mathbb{C} : s \le z \le |f(v_n)|\}$. Since $|f(v_n)|$ approaches s as $n \to \infty$, we see $\mu(E_n) \to 0$ as $n \to \infty$.

Just as in the proof of inner-regularity, the sequence $(f^{-1}(E_n))$ of μ-measurable subsets of $\mathbb{D}$ is nested:

$$f^{-1}(E_{n+1}) \subseteq f^{-1}(E_n) \quad \text{for every } n \in \mathbb{N}$$

and $F := \cap_{j=1}^{\infty} f^{-1}(E_j)$ satisfies

$$\mu(F) = \lim_{n\to\infty} \mu\left(f^{-1}(E_n)\right).$$

If $z \in F$, then $s \le |f(z)| \le |f(v_j)|$ for all j, which implies $|f(z)| = s$. Hence, $F \subseteq f^{-1}(\{\zeta s : |\zeta| = 1\})$. Since $m(\{\zeta s : |\zeta| = 1\}) = 0$, Lemma 1 shows that $\mu(F) = 0$ and hence $\lim_{n\to\infty} \mu\left(f^{-1}(E_n)\right) = 0$.

Choosing $n_0 \in \mathbb{N}$ sufficiently large, we have $\mu(f^{-1}(E_{n_0})) < \delta$. Now using (12) and the equality from (13), we have

$$\begin{aligned}\mu(Y) - \mu\left(f^{-1}(\mathbb{C}\setminus|f(v_{n_0})|\mathbb{D})\right) &\le \mu\left(f^{-1}(\mathbb{C}\setminus s\mathbb{D})\right) - \mu\left(f^{-1}(\mathbb{C}\setminus|f(v_{n_0})|\overline{\mathbb{D}})\right)\\ &= \mu\left(f^{-1}(E_{n_0})\right) < \delta,\end{aligned}$$

and it follows that $\prec$ is outer-regular since $f^{-1}(\mathbb{C}\setminus|f(v_{n_0})|\overline{\mathbb{D}}) = \{z \in Y : v_{n_0} \prec z\}$ and $v_{n_0} \in Y$. $\square$

4. Main results

For the work of this section, we assume that γ and w have the following properties: $w \in L^\infty(\mathbb{D})$, w is nonzero a.e. with respect to μ, and γ is a univalent self-map of $\mathbb{D}$ such that $w/\gamma' \in L^\infty(\mathbb{D})$. These assumptions ensure that $T_{w,\gamma}$ is a bounded operator on $L^2(\mathbb{D})$ with dense range. As usual, set $h = |w/\gamma'|^2$. Our work up to this point has been directed toward application of the following result from [2], which we state in the context of weighted composition operators on $L^2(\mathbb{D})$.

Theorem 1 (Theorem 3.2 of [2]**).** *Suppose that the partial order* (2) *determined by* h *is regular and such that for every* $v \in \mathbb{D}$

$$\mu(\{z \in \mathbb{D} : z \prec v \text{ or } v \prec z\}) = 1.$$

Then $(T_{w,\gamma}^n y_n)$ *converges in norm for each backward minimal-vector sequence* (y_n) *for* $T_{w,\gamma}$.

The preceding Theorem, together with the work of Sections 2 and 3, yields our main result.

Theorem 2. *Let γ be a univalent self-map of $\mathbb{D}$ of Schröder or hyperbolic type, having Denjoy-Wolff point ω_0; and let w extend to be differentiable at ω_0 with $w(\omega_0) = 0$ and $w'(\omega_0) \neq 0$. Then $(T_{w,\gamma}^n y_n)$ converges in norm for each backward minimal-vector sequence (y_n) for $T_{w,\gamma}$.*

Proof. The work of Section 2 shows that under the hypotheses of this theorem, the partial order $\prec$ defined by (2) is determined by a nonconstant analytic function f on $\mathbb{D}$ as in (8) of Section 3. Thus by Propositions 4 and 5 of Section 3, Theorem 1 applies and Theorem 2 follows. $\square$

Here are two concrete examples to which Theorem 2 applies.

Example 1. Let f be the Koebe function, so that $f(z) = z/(1-z)^2$ and f maps $\mathbb{D}$ univalently onto $\mathbb{C} \setminus (-\infty, -1/4]$. Let $\gamma = f^{-1} \circ (f/2)$ so that γ is a univalent self-map of $\mathbb{D}$ such that $\gamma(\mathbb{D})$ is the slit disk $\mathbb{D} \setminus (-1, -3 + 2\sqrt{2}]$. Note that γ is of Schröder type, with $\omega_0 = 0$ (and $\gamma'(0) = 1/2$). Note also that by the Koebe Distortion Theorem $|\gamma'(z)| \geq c(1 - |z|)$ for some positive constant c. Now let w be defined piecewise by $w(z) = z$ if $|z| \leq 1/2$ and $w(z) = (1 - |z|)$ if $1/2 < |z| < 1$. Then $w \in L^\infty(\mathbb{D})$, $w \neq 0$ a.e. on $\mathbb{D}$, and w is differentiable at 0 with $w(0) = 0$ and $w'(0) \neq 0$. Finally, the continuous function w/γ' is bounded on the compact set $\{z : |z| \leq 1/2\}$; moreover, $|w/\gamma'| \leq 1/c$ on $\{z : 1/2 < |z| < 1\}$. Thus $T_{w,\gamma}$ is a bounded operator on $L^2(\mathbb{D})$ with dense range, and by Theorem 2, $(T_{w,\gamma}^n y_n)$ is convergent for any backward minimal-vector sequence (y_n) for $T_{w,\gamma}$.

Example 2. Take $w(z) = 1 - z$ and

$$\gamma(z) = \frac{z^2 + (2i - 6)z - 3 - 2i}{z^2 + (2i - 2)z - 7 - 2i}.$$

Note $\gamma(1) = 1$ while $\gamma'(1) = 1/2 < 1$. Also γ is analytic on a neighborhood of the closed disk and thus γ' is as well; in particular γ' has continuous extension to 1. Moreover, γ' has no zeros on the closed disk. Thus $w/\gamma' \in L^\infty(\mathbb{D})$ and $T_{w,\gamma}$ is bounded (and has dense range since $w(z) = 1 - z$ is nonzero a.e. with respect to μ). To see that γ is univalent on $\mathbb{D}$, observe that $\gamma(z) = F^{-1} \circ \Gamma \circ F$, where $F(z) = (1+z)/(1-z)$ and $\Gamma(z) = 2z + 1 + i - 1/(z+1)$. The function Γ is a self-map of the right half-plane $\Pi := \{z : \operatorname{Re} z > 0\}$ and $\Gamma'(z) = 2 + 1/(z+1)^2$. Because Γ' has positive real part on Π, we see that $\Gamma : \Pi \to \Pi$ is univalent and that γ is therefore univalent, being a composition of univalent maps. All the hypotheses of Theorem 2 apply and $(T_{w,\gamma}^n y_n)$ is convergent for any backward minimal-vector sequence (y_n) for $T_{w,\gamma}$.

Corollary 1. *Suppose that $\gamma : \mathbb{D} \to \mathbb{D}$ is an analytic self-map of $\mathbb{D}$ that is univalent and that one of the following holds:*

(I) *γ is of Schröder type and $\gamma^{[k]}(\mathbb{D}) \subseteq r\mathbb{D}$ for some $r < 1$ and $k \geq 1$, or*

(II) *γ is of hyperbolic type, γ' extends continuously to $\mathbb{D} \cup \{\omega_0\}$, where ω_0 is the Denjoy-Wolff point of γ, and, for some $k \geq 1$, $\gamma^{[k]}(\mathbb{D})$ is contained in a (proper) subdisk of $\mathbb{D}$ internally tangent to $\partial\mathbb{D}$ at ω_0.*

Suppose also that w extends to be differentiable at the Denjoy-Wolff point ω_0 of γ, with $w(\omega_0) = 0$ and $w'(\omega_0) \neq 0$. Then $T_{w,\gamma} : L^2(\mathbb{D}) \to L^2(\mathbb{D})$ has a nontrivial hyperinvariant subspace.

Proof. Theorem 2 shows that under either (I) or (II), images of backward minimal-vector sequences converge. Moreover, it's not difficult to show that if either (I) or (II) holds, then $(\gamma^{[n]})$ converges uniformly on $\mathbb{D}$ to ω_0. Moreover, because γ' extends continuously to $\mathbb{D} \cup \{\omega_0\}$, $h = w/\gamma'$ extends continuously to ω_0 and $h(\omega_0) = 0$. Thus, by Theorem 4.3 of [2], $T_{w,\gamma}$ is quasinilpotent. Hence the corollary is a consequence of Ansari and Enflo's result, Theorem AE. $\square$

Observe that the preceding corollary is a natural generalization of Theorem CFP, which applies when γ is a non-automorphic linear-fractional mapping of either Schröder or hyperbolic type. However, both of these invariant-subspace theorems are more easily obtained as consequences of a much more general observation, which we state below in the context of subsets of the complex plane, with area measure.

> *Observation: Suppose that $X \subset \mathbb{C}$ has positive area measure, that $\gamma : X \to X$, and that $w \in L^\infty(X)$ is such that $T_{w,\gamma}$ is bounded on $L^2(X)$. If the range of γ omits a subset E of X having positive measure, then the kernel of $T_{w,\gamma}$ is nontrivial, containing, e.g., χ_E, and thus the kernel of $T_{w,\gamma}$ is a nontrivial hyperinvariant subspace of $T_{w,\gamma}$.*

Remarks. (1) If the weighted composition operator $T_{w,\gamma} : L^2(X) \to L^2(X)$ described in the preceding observation has dense range, then w is nonzero a.e. with respect to m; and it's easy to see in this case that $T_{w,\gamma}$ will be injective if and only if $m(X \setminus \gamma(X)) = 0$. (2) Note that the weighted composition operator described in Example 1 above is injective and has the property that images of backward minimal-vector sequences are always convergent. It would be of interest to prove that such weighted composition operators must have nontrivial hyperinvariant subspaces.

Acknowledgment

The authors wish to thank the referee for providing a number of good suggestions leading to improvements in the quality of our exposition.

References

[1] S. Ansari and P. Enflo, *Extremal vectors and invariant spaces*, Trans. Amer. Math. Soc. 350 (1998), 539–558.

[2] Isabelle Chalendar, Antoine Flattot, and Jonathan R. Partington, *The method of minimal vectors applied to weighted composition operators*, Oper. Theory Adv. Appl., 171 (2007), 89–105.

[3] Isabelle Chalendar and Jonathan R. Partington, *Convergence properties of minimal vectors for normal operators and weighted shifts*, Proc. Amer. Math. Soc. 133 (2004), 501–510.

[4] C.C. Cowen and B.D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, 1995.

[5] G. Koenigs, *Recherches sur les intégrales de certaines équations fonctionnelles*, Ann. Sci. Ecole Norm. Sup. (3) 1 (1884), Supplément, 3–41.

[6] J.H. Shapiro, *Composition Operators and Classical Function Theory*, Springer-Verlag, New York, 1993.

[7] V.G. Troitsky, *Minimal vectors in arbitrary Banach spaces*, Proc. Amer. Math. Soc. 132 (2004), 1177–1180.

[8] G. Valiron, *Sur l'itération des fonctions holomorphes dans un demi-plan*, Bull. Sci. Math. (2) 55 (1931), 105–128.

[9] Emilie B. Wiesner, *Backward minimal points for bounded linear operators on finite-dimensional vector spaces*, Linear Algebra Appl. 338 (2001), 251–259.

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Received: October 22, 2008
Accepted: May 15, 2009

Operator Theory:
Advances and Applications, Vol. 202, 53–61

On Extensions of Indefinite Toeplitz-Kreĭn-Cotlar Triplets

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Abstract. We give a definition of κ-indefinite Toeplitz-Kreĭn-Cotlar triplet of Archimedean type, on an interval of an ordered group Γ with an Archimedean point. We show that if a group Γ has the indefinite extension property, then every κ-indefinite Toeplitz-Kreĭn-Cotlar triplet of Archimedean type on an interval of Γ, can be extended to a Toeplitz-Kreĭn-Cotlar triplet on the whole group Γ, with the same number of negative squares κ.

Mathematics Subject Classification (2000). Primary 47B50; Secondary 46C20, 47D03.

Keywords. Operator-valued indefinite functions, ordered group, Archimedean point, Toeplitz kernel.

1. Introduction

The aim of this paper is to introduce a notion of operator-valued κ-indefinite Toeplitz-Kreĭn-Cotlar triplets on an ordered group and to obtain some extension results.

Usually the extension problem for κ-indefinite functions has been considered on an interval of the real line. Gorbachuk [18] proved that every continuous function, with κ negative squares on $(-a, a)$, can be extended to a continuous function on the real line with the same number of negative squares. More information on the extension problem for κ-indefinite scalar-valued continuous functions, on an interval of the real line, can be found in the paper of Kreĭn and Langer [20].

The case of a positive definite function corresponds with $\kappa = 0$. Kreĭn [19] proved that every scalar-valued continuous positive definite function, on an interval of the real line, can be extended to a continuous positive definite function on the real line. Also a scalar-valued positive definite function, defined on an interval of

Both authors were supported in part by the CDCH of the Universidad Central de Venezuela.
Communicated by J.A. Ball.

an ordered group, can be extended to a positive definite function on the whole group (see the book of Sasvári [22, page 105]). Additional information about the extension problem for positive definite functions and κ-indefinite functions can be found in the historical survey [23].

The problem of the extension of an operator-valued κ-indefinite function defined on an interval of an ordered group was studied by the authors in [8], where some extension results were obtained for ordered groups which satisfy an Archimedean condition. The main purpose of this paper is to extend some of the results obtained in [8] to Toeplitz-Kreĭn-Cotlar triplets.

In our previous paper [6] a equivalence between a Naimark-type dilation of a positive definite Toeplitz-Kreĭn-Cotlar triplet and a commutant lifting theorem for contractive representations of the ordered group was shown. Since there is a finite number of negative squares version of the commutant lifting theorem [1], it would be interesting to relate the results of our present paper to obtain a version of the indefinite commutant lifting theorem on the ordered group setting, see also [3].

Also in [7] the case $\kappa = 0$ of the result of the present paper is obtained in the context of the commutant lifting application under the assumption that the ordered group is semi-Archimedean as in the present paper. Later works for the commutant lifting setting [4] and for the Toeplitz-Kreĭn-Cotlar triplet setting [9, 10] showed that, for the $\kappa = 0$ case, this semi-Archimedean hypothesis is removable. We cannot remove the semi-Archimedean hypothesis for the more general case $\kappa > 0$ since we use previous results of our paper [8]; in that paper the Archimedean condition is used, among other things, to guarantee the continuity of some isometric operators on an associated Π_κ space. It is an open problem if the semi-Archimedean hypothesis is removable for the $\kappa > 0$ case.

It should also be pointed out that this ordered group setting leaves out a lot of interesting examples. Thus if one tries to solve the two-dimensional moment problem with moments specified in a general rectangle even with $\kappa = 0$, the obvious necessary conditions are not sufficient, and necessary and sufficient solution criteria are much more complicated see [5, 17] as well as [15, 16, 11].

2. Preliminaries

Let $(\Gamma, +)$ be an Abelian group with neutral element 0_Γ. Γ is an *ordered group* if there exists a set $\Gamma_+ \subset \Gamma$ such that:

$$\Gamma_+ + \Gamma_+ = \Gamma_+, \qquad \Gamma_+ \bigcap (-\Gamma_+) = \{0_\Gamma\}, \qquad \Gamma_+ \bigcup (-\Gamma_+) = \Gamma.$$

In this case if $x, y \in \Gamma$ we write $x \le y$ if $\mathrm{y} - \mathrm{x} \in \Gamma_+$, we also write $x < y$ if $x \le y$ and $x \ne y$, so $\Gamma_+ = \{\gamma \in \Gamma : \gamma \ge 0_\Gamma\}$. If there is not possibility of confusion, we will use 0 instead of 0_Γ. When Γ is a topological group it is supposed that Γ_+ is closed.

If $a, b \in \Gamma$ and $a < b$,

$$(a, b) = \{x \in \Gamma : a < x < b\}, \qquad [a, b] = \{x \in \Gamma : a \le x \le b\}, \qquad \text{etc.}$$

If $\mathcal{H}$ is a Hilbert space, $L(\mathcal{H})$ indicates the space of the continuous linear operators from $\mathcal{H}$ to itself.

Definition 2.1. Let Γ be an ordered group, $a \in \Gamma$, $a > 0$, let $(\mathcal{H}, \langle\, , \,\rangle_{\mathcal{H}})$ be a Hilbert space and let κ be a nonnegative integer.

A function $f : [-2a, 2a] \to L(\mathcal{H})$ is said to be *κ-indefinite* if:

(a) $f(x) = f(-x)^*$ for all $x \in [-2a, 2a]$,

(b) for any finite set of points $x_1, \ldots, x_n \in [-a, a]$ and vectors $h_1, \ldots, h_n \in \mathcal{H}$, the Hermitian matrix

$$(\langle f(x_i - x_j)h_i, h_j\rangle_{\mathcal{H}})_{i,j=1}^n$$

has at most κ negative eigenvalues, counted according to their multiplicities, and at least one such matrix has exactly κ negative eigenvalues.

We will consider a special class of ordered groups, which satisfies an Archimedean condition. For an ordered group Γ the following definitions were given in our previous paper [8].

It is said that $\gamma_0 \in \Gamma$ is an *Archimedean point* if for each $\gamma \in \Gamma$ there exists a positive integer n such that $n\,\gamma_0 \geq \gamma$.

It is said that Γ is *semi-Archimedean* if Γ is an ordered group and if it has an Archimedean point.

Let $a \in \Gamma$, $a > 0$ and $a_o \in (0, a]$. A function $f : [-2a, 2a] \to L(\mathcal{H})$ is said to be *κ-indefinite with respect to a_o* if it is κ-indefinite and if for some choice of $n \in \mathbb{N}$, $x_1, \ldots, x_n \in [-a, a - a_0]$ and $h_1, \ldots, h_n \in \mathcal{H}$, the Hermitian matrix

$$(\langle f(x_i - x_j)h_i, h_j\rangle_{\mathcal{H}})_{i,j=1}^n$$

has exactly κ negative eigenvalues, counted according to their multiplicity. If Γ is semi-Archimedean, it is said that f *is of Archimedean type* if it is κ-indefinite with respect to some Archimedean point.

Definition 2.2. It is said that the ordered group Γ has the *indefinite extension property* if Γ is a locally compact Abelian semi-Archimedean group and the following holds:

If $\mathcal{H}$ is a Hilbert space, $a \in \Gamma$, $a > 0$ and $f : [-2a, 2a] \to L(\mathcal{H})$ is a weakly continuous κ-indefinite function of Archimedean type, then there exists a weakly continuous κ-indefinite function $F : \Gamma \to L(\mathcal{H})$ such that $F|_{[-2a,2a]} = f$.

The groups $\mathbb{Z}$ and $\mathbb{R}$ have the indefinite extension property, see Theorems 5.5 and 5.6 in [8].

Also, in our previous paper [8] it was shown that if a group Γ is semi-Archimedean and it has the indefinite extension property then $\Gamma \times \mathbb{Z}$, with the lexicographic order and the product topology, has the indefinite extension property. As a corollary it was obtained that the groups $\mathbb{Z}^n$ and $\mathbb{R} \times \mathbb{Z}^n$ have the indefinite extension property.

3. Toeplitz-Kreĭn-Cotlar triplets

In the following Γ is an ordered group, $\mathcal{H}_1$,$\mathcal{H}_2$ are Hilbert spaces and $L(\mathcal{H}_\alpha, \mathcal{H}_\beta)$ stands for the space of the continuous linear operators from $\mathcal{H}_\alpha$ to $\mathcal{H}_\beta$ (for $\alpha, \beta = 1, 2$).

Let Q_1 be an interval of the form $[0, d]$, where $d \in \Gamma$, $d > 0$ or $Q_1 = \Gamma_+$.

Definition 3.1. A *Toeplitz-Kreĭn-Cotlar triplet*, $\mathbf{C}$, on $(\Gamma, Q_1, \mathcal{H}_1, \mathcal{H}_2)$ consists of three functions

$$C_{\alpha\beta} : Q_\alpha - Q_\beta \to L(\mathcal{H}_\alpha, \mathcal{H}_\beta) \quad \alpha,\ \beta = 1, 2,\ \alpha \leq \beta,$$

where $Q_2 = -Q_1$.

If $\mathbf{C}$ is a Toeplitz-Kreĭn-Cotlar triplet we define $C_{21}(\gamma) = C_{12}(-\gamma)^*$ for $\gamma \in Q_2 - Q_1$.

Remark 3.2. Toeplitz-Kreĭn-Cotlar triplets were introduced in [10] as a particular case of Toeplitz-Kreĭn-Cotlar forms, according to the definition given in [2] and it is related with the concept of generalized Toeplitz kernels introduced by Cotlar and Sadosky in [14], where a generalization of the Herglotz-Bochner theorem for such kernels and applications to the Helson-Szegö theorem were obtained.

Toeplitz-Kreĭn-Cotlar forms have been usually considered in the positive definite case. We are going to consider the indefinite case.

Definition 3.3. We shall say that the Toeplitz-Kreĭn-Cotlar triplet

$$\mathbf{C} \quad \text{on} \quad (\Gamma, Q_1, \mathcal{H}_1, \mathcal{H}_2)$$

is *κ-indefinite* if for any finite sets of points $x_1^{(1)}, \ldots, x_n^{(1)} \in Q_1$, $x_1^{(2)}, \ldots, x_n^{(2)} \in Q_2$ and vectors $h_1^{(1)}, \ldots, h_n^{(1)} \in \mathcal{H}_1$, $h_1^{(2)}, \ldots, h_n^{(2)} \in \mathcal{H}_2$ the Hermitian matrix

$$\begin{bmatrix} \Big[\langle C_{11}(x_i^{(1)} - x_j^{(1)}) h_i^{(1)}, h_j^{(1)} \rangle_{\mathcal{H}_1} \Big]_{i,j=1}^n & \Big[\langle C_{21}(x_i^{(2)} - x_j^{(1)}) h_i^{(2)}, h_j^{(1)} \rangle_{\mathcal{H}_1} \Big]_{i,j=1}^n \\ \Big[\langle C_{12}(x_i^{(1)} - x_j^{(2)}) h_i^{(1)}, h_j^{(2)} \rangle_{\mathcal{H}_2} \Big]_{i,j=1}^n & \Big[\langle C_{22}(x_i^{(2)} - x_j^{(2)}) h_i^{(2)}, h_j^{(2)} \rangle_{\mathcal{H}_2} \Big]_{i,j=1}^n \end{bmatrix}$$

has at most κ negative eigenvalues, counted according to their multiplicities, and at least one such matrix has exactly κ negative eigenvalues.

As in [8] it will be convenient to consider intervals of the form $[-2a, 2a]$.

Definition 3.4. Let Γ be an ordered group, let $a \in \Gamma$, $a > 0$ and $a_o \in (0, a]$. A Toeplitz-Kreĭn-Cotlar triplet on $(\Gamma, [0, 2a], \mathcal{H}_1, \mathcal{H}_2)$ is said to be *κ-indefinite with respect to a_o* if it is κ-indefinite and if for some choice of $n \in \mathbb{N}$ $x_1^{(1)}, \ldots, x_n^{(1)} \in [0, 2a - a_o]$, $x_1^{(2)}, \ldots, x_n^{(2)} \in [-2a, -a_o]$ and vectors $h_1^{(1)}, \ldots, h_n^{(1)} \in \mathcal{H}_1$, $h_1^{(2)}, \ldots,$

$h_n^{(2)} \in \mathcal{H}_2$ the Hermitian matrix

$$\begin{bmatrix} \left[\langle C_{11}(x_i^{(1)} - x_j^{(1)})h_i^{(1)}, h_j^{(1)}\rangle_{\mathcal{H}_1} \right]_{i,j=1}^n & \left[\langle C_{21}(x_i^{(2)} - x_j^{(1)})h_i^{(2)}, h_j^{(1)}\rangle_{\mathcal{H}_1} \right]_{i,j=1}^n \\ \left[\langle C_{12}(x_i^{(1)} - x_j^{(2)})h_i^{(1)}, h_j^{(2)}\rangle_{\mathcal{H}_2} \right]_{i,j=1}^n & \left[\langle C_{22}(x_i^{(2)} - x_j^{(2)})h_i^{(2)}, h_j^{(2)}\rangle_{\mathcal{H}_2} \right]_{i,j=1}^n \end{bmatrix}$$

has exactly κ negative eigenvalues, counted according to their multiplicity.

If Γ is semi-Archimedean, we will say that $\mathbf{C}$ *is of Archimedean type* if it is κ-indefinite with respect to some Archimedean point.

Lemma 3.5. *Let* $\mathbf{C}$ *be a Toeplitz-Kreĭn-Cotlar triplet on* $(\Gamma, [0, 2a], \mathcal{H}_1, \mathcal{H}_2)$ *and let* $f : [-2a, 2a] \to L(\mathcal{H}_1 \oplus \mathcal{H}_2)$ *be defined by*

$$f(\gamma) = \begin{bmatrix} C_{11}(\gamma) & C_{21}(\gamma - 2a) \\ C_{12}(\gamma + 2a) & C_{22}(\gamma) \end{bmatrix}.$$

Then

(i) *The triplet* $\mathbf{C}$ *is* κ*-indefinite if and only if the function* f *is* κ*-indefinite.*
(ii) *The triplet* $\mathbf{C}$ *is of Archimedean type if and only if the function* f *is of Archimedean type*

Proof.

(i) Consider $n \in \mathbb{N}$, $y_1, \ldots, y_n \in [-a, a]$, $g_1, \ldots, g_n \in \mathcal{H}_1 \oplus \mathcal{H}_2$ and let

$$\mathbf{A} = \left[\langle f(y_p - y_q)g_p, g_q\rangle_{\mathcal{H}_1 \oplus \mathcal{H}_2} \right]_{p,q=1}^n .$$

If $\overrightarrow{\omega} = \left[\omega_1, \ldots, \omega_n\right] \in \mathbb{C}^n$ and $\overrightarrow{\xi} = \left[\omega_1, \ldots, \omega_n, \omega_1, \ldots, \omega_n\right]$, a calculation shows that

$$\langle \mathbf{A}\overrightarrow{\omega}, \overrightarrow{\omega}\rangle_{\mathbb{C}^n} = \langle \mathbf{B}\overrightarrow{\xi}, \overrightarrow{\xi}\rangle_{\mathbb{C}^{2n}}, \tag{3.1}$$

where

$$\mathbf{B} = \begin{bmatrix} \left[\langle C_{11}(x_i^{(1)} - x_j^{(1)})h_i^{(1)}, h_j^{(1)}\rangle_{\mathcal{H}_1} \right]_{i,j=1}^n & \left[\langle C_{21}(x_i^{(2)} - x_j^{(1)})h_i^{(2)}, h_j^{(1)}\rangle_{\mathcal{H}_1} \right]_{i,j=1}^n \\ \left[\langle C_{12}(x_i^{(1)} - x_j^{(2)})h_i^{(1)}, h_j^{(2)}\rangle_{\mathcal{H}_2} \right]_{i,j=1}^n & \left[\langle C_{22}(x_i^{(2)} - x_j^{(2)})h_i^{(2)}, h_j^{(2)}\rangle_{\mathcal{H}_2} \right]_{i,j=1}^n \end{bmatrix},$$

$x_i^{(1)} = y_i + a$, $x_i^{(2)} = y_i - a$ and $h_i^{(1)} \in \mathcal{H}_1$, $h_i^{(2)} \in \mathcal{H}_2$ are such that $g_i = h_i^{(1)} \oplus h_i^{(2)}$ for $1 \leq i \leq n$.

On the other hand for $m \in \mathbb{N}$, $z_1^{(1)}, \ldots, z_m^{(1)} \in [0, 2a]$, $z_1^{(2)}, \ldots, z_m^{(2)} \in [-2a, 0]$, $h_1^{(1)}, \ldots, h_m^{(1)} \in \mathcal{H}_1$, $h_1^{(2)}, \ldots, h_m^{(2)} \in \mathcal{H}_2$ let

$$\mathbf{D} = \begin{bmatrix} \left[\langle C_{11}(z_i^{(1)} - z_j^{(1)})h_i^{(1)}, h_j^{(1)}\rangle_{\mathcal{H}_1} \right]_{i,j=1}^m & \left[\langle C_{21}(z_i^{(2)} - z_j^{(1)})h_i^{(2)}, h_j^{(1)}\rangle_{\mathcal{H}_1} \right]_{i,j=1}^m \\ \left[\langle C_{12}(z_i^{(1)} - z_j^{(2)})h_i^{(1)}, h_j^{(2)}\rangle_{\mathcal{H}_2} \right]_{i,j=1}^m & \left[\langle C_{22}(z_i^{(2)} - z_j^{(2)})h_i^{(2)}, h_j^{(2)}\rangle_{\mathcal{H}_2} \right]_{i,j=1}^m \end{bmatrix}.$$

If $\overrightarrow{\lambda} = \left[\lambda_1, \ldots, \lambda_{2m}\right] \in \mathbb{C}^{2m}$ a calculation shows that

$$\langle \mathbf{D}\overrightarrow{\lambda}, \overrightarrow{\lambda}\rangle_{\mathbb{C}^{2m}} = \langle \mathbf{E}\overrightarrow{\lambda}, \overrightarrow{\lambda}\rangle_{\mathbb{C}^{2n}}, \tag{3.2}$$

where

$$\mathbf{E} = \left[\, \langle f(\zeta_p - \zeta_q)h_p, h_q\rangle_{\mathcal{H}_1\oplus\mathcal{H}_2} \,\right]_{p,q=1}^{2m},$$

$$\zeta_p = z_p^{(1)} - a, \quad h_p = h_p^{(1)} \oplus 0 \quad \text{if } 1 \leq p \leq m$$

and

$$\zeta_p = z_{p-m}^{(2)} + a, \quad h_p = 0 \oplus h_{p-m}^{(2)} \quad \text{if } m+1 \leq p \leq 2m.$$

Equations (3.1) and (3.2) show that $\mathbf{C}$ is κ-indefinite if and only if f is κ-indefinite.

(ii) It is enough to note that

$$y_1, \ldots, y_n \in [-a, a - a_o]$$

if and only if

$$x_1^{(1)}, \ldots, x_n^{(1)} \in [0, 2a - a_o] \quad \text{and} \quad x_1^{(2)}, \ldots, x_n^{(2)} \in [-2a, -a_o]$$

and

$$\zeta_1, \ldots, \zeta_{2m} \in [-a, a - a_o]$$

if and only if $z_1^{(1)}, \ldots, z_m^{(1)} \in [0, 2a - a_o]$, $z_1^{(2)}, \ldots, z_m^{(2)} \in [-2a, -a_o]$. □

4. Extension result

Theorem 4.1. *Let Γ be a group that has the indefinite extension property and let $\mathcal{H}_1$,$\mathcal{H}_2$ be a pair of Hilbert spaces.*

If $\mathbf{C} = (C_{\alpha\beta})$ is a weakly continuous κ-indefinite Toeplitz-Kreĭn-Cotlar triplet on $(\Gamma, [0, 2a], \mathcal{H}_1, \mathcal{H}_2)$ of Archimedean type, then there exist a weakly continuous κ-indefinite Toeplitz-Kreĭn-Cotlar triplet $\mathbf{V} = (V_{\alpha\beta})$ on $(\Gamma, \Gamma_+, \mathcal{H}_1, \mathcal{H}_2)$ such that

$$C_{\alpha\beta}(\gamma) = V_{\alpha\beta}(\gamma) \quad \textit{for } \gamma \in Q_\alpha - Q_\beta,$$

where $Q_1 = [0, 2a]$ and $Q_2 = [-2a, 0]$.

Proof. Let f be as in Lemma 3.5, then f is a weakly continuous κ-indefinite function of Archimedean type. Since Γ has the indefinite extension property there exists a weakly continuous κ-indefinite function $F : \Gamma \to L(\mathcal{H}_1 \oplus \mathcal{H}_2)$ such that $F|_{[-2a,2a]} = f$.

Let

$$F = \begin{bmatrix} F_{11} & F_{21} \\ F_{12} & F_{22} \end{bmatrix}$$

the representation of F with respect to the decomposition $\mathcal{H}_1 \oplus \mathcal{H}_2$ and let $\mathbf{V} = (V_{\alpha\beta})$ the Toeplitz-Kreĭn-Cotlar triplet on $(\Gamma, \Gamma_+, \mathcal{H}_1, \mathcal{H}_2)$ defined by

$$\begin{aligned} V_{11}(\gamma) &= F_{11}(\gamma) \text{ for } \gamma \in \Gamma \\ V_{21}(\gamma) &= F_{21}(\gamma + 2a) \text{ for } \gamma \in -\Gamma_+ \\ V_{12}(\gamma) &= F_{12}(\gamma - 2a) \text{ for } \gamma \in \Gamma_+ \\ V_{22}(\gamma) &= F_{22}(\gamma) \text{ for } \gamma \in \Gamma \end{aligned}$$

We have that F extends f, so $\mathbf{V}$ extends $\mathbf{C}$. Also equation (3.1) holds for F instead of f, for $(V_{\alpha\beta})$ instead of $(C_{\alpha\beta})$ and for $x_1^{(1)}, \ldots, x_n^{(1)} \in \Gamma_+$, $x_1^{(2)}, \ldots, x_n^{(2)} \in -\Gamma_+$. Since F is κ-indefinite we have that the triplet $\mathbf{V} = (V_{\alpha\beta})$ is κ-indefinite. $\square$

5. Generalized Toeplitz kernels with real parameter

Scalar-valued generalized Toeplitz kernels with real parameter were considered and an extension result was given in [12, Theorem 5.1]. Our approach can also be used to obtain an operator-valued extension of this result.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and let a be a positive real number. An *operator-valued generalized Toeplitz kernel on* $I = [-a, a]$ *or* $I = \mathbb{R}$ is a function ψ with domain $I \times I$ such that there exist four functions $\psi_{\alpha\beta} : I_\alpha - I_\beta \to L(\mathcal{H}_\alpha, \mathcal{H}_\beta)$ $\alpha,\ \beta = 1, 2$, such that

$$\psi(x, y) = \psi_{\alpha\beta}(x - y)$$

for every $(x, y) \in I_\alpha \times I_\beta$ for $\alpha, \beta = 1, 2$, where $I_1 = I \cap [0, +\infty)$ and $I_2 = I \cap (-\infty, 0)$.

As usual, it is said that the kernel ψ is *κ-indefinite* if

(a) $\psi(x, y) = \psi(-x, -y)^*$ for all $(x, y) \in I \times I$,

(b) for any finite set of points $x_1, \ldots, x_n \in I$ and vectors $h_1, \ldots, h_n \in \mathcal{H}$, the Hermitian matrix

$$(\langle \psi(x_i, x_j) h_i, h_j \rangle_{\mathcal{H}})_{i,j=1}^n$$

has at most κ negative eigenvalues, counted according to their multiplicities, and at least one such matrix has exactly κ negative eigenvalues.

The generalized Toeplitz kernel ψ is said to be *weakly continuous* if all the functions $\psi_{\alpha\beta}$ are weakly continuous.

Theorem 5.1. *Every operator-valued weakly continuous κ-indefinite generalized Toeplitz kernel on an interval of the form $[-a, a]$ can be extended to a weakly continuous κ-indefinite generalized Toeplitz kernel on the real line $\mathbb{R}$.*

Proof. We will follow the same idea of the proof of Theorem 4.1, with some modifications because 0 is not in the domain of ψ_{12}.

Let $\varphi : (-a, a) \to L(\mathcal{H}_1 \oplus \mathcal{H}_2)$ defined by

$$\varphi(\gamma) = \begin{bmatrix} \psi_{11}(\gamma) & \psi_{21}(\gamma - a) \\ \psi_{12}(\gamma + a) & \psi_{22}(\gamma) \end{bmatrix}.$$

As before it can be proved that φ is a weakly continuous κ-indefinite function on $(-a, a)$. From Theorem 3.5 of [13] it follows that φ can be extended to a weakly continuous κ-indefinite function on the real line $\mathbb{R}$; using the same idea of the proof of Theorem 4.1 the extension result is obtained. □

References

[1] D. Alpay, T. Constantinescu, A. Dijksma and J. Rovnyak, *Notes on interpolation in the generalized Schur class. II. Nudelman's problem.* Trans. Am. Math. Soc. **355** (2003), 813–836.

[2] R. Arocena, *On the Extension Problem for a class of translation invariant positive forms.* J. Oper. Theory **21** (1989), 323–347.

[3] J. Ball and J. Helton, *A Beurling-Lax theorem for the Lie group $U(m;n)$ which contains most classical interpolation theory*, J. Operator Theory **9** (1983), 107–142.

[4] M. Bakonyi and D. Timotin, *The intertwining lifting theorem for ordered groups.* J. Funct. Anal. **199**, No. 2 (2003), 411–426.

[5] M. Bakonyi and G. Nævdal, *The finite subsets of $\mathbb{Z}^2$ having the extension property.* J. Lond. Math. Soc., II. Ser. **62**, No. 3 (2000), 904–916.

[6] R. Bruzual and M. Domínguez, *Equivalence between the dilation and lifting properties of an ordered group through multiplicative families of isometries. A version of the commutant lifting theorem on some lexicographic groups.* Int. Eq. and Op. Theory, **40** No. 1 (2001), 1–15.

[7] R. Bruzual and M. Domínguez, *Extensions of operator valued positive definite functions and commutant lifting on ordered groups.* J. Funct. Anal. **185** No. 2 (2001) 456–473.

[8] R. Bruzual and M. Domínguez, *Extension of locally defined indefinite functions on ordered groups.* Int. Eq. and Op. Theory, **50** (2004), 57–81.

[9] R. Bruzual and M. Domínguez, *A generalization to ordered groups of a Kreĭn theorem.* Operator Theory: Advances and Applications **179**, (2008) 103–109.

[10] R. Bruzual and M. Domínguez, *Dilation of generalized Toeplitz kernels on ordered groups.* Journal of Functional Analysis, **238**, No. 2, (2006), 405–426.

[11] R. Bruzual and M. Domínguez, *On extensions of indefinite functions defined on a rectangle,* Complex Anal. Oper. Theory, in press-available on line.

[12] R. Bruzual and S.A.M. Marcantognini, *Local semigroups of isometries in Π_κ-spaces and related continuation problems for κ-indefinite Toeplitz kernels,* Int. Eq. and Op. Theory, **15** (1992), 527–550.

[13] R. Bruzual and S.A.M. Marcantognini, *The Kreĭn-Langer problem for Hilbert space operator valued functions on the band,* Int. Eq. and Op. Theory, **34** (1999), 396–413.

[14] M. Cotlar, C. Sadosky, *On the Helson-Szegö theorem and a related class of modified Toeplitz kernels.* Proc. Symp. Pure Math. AMS. **35-I** (1979), 383–407.

[15] A. Devinatz, *On the extensions of positive definite functions.* Acta Math. **102**, No. 1-2 (1959), 109–134.

[16] G.I. Eskin, *A sufficient condition for the solvability of the moment problem in several dimensions.* Dokl. Akad. Nauk. SSSR **113** (1960), 540–543.

[17] J. Geronimo and H. Woerdeman, *The operator valued autoregressive filter problem and the suboptimal Nehari problem in two variables.* Int. Eq. and Op. Theory, **53**, (2005), 343–361.

[18] V.I. Gorbachuk (V.I. Plyushčeva), *On the integral representation of hermitian indefinite kernels with a finite number of negative squares.* Dokl. Akad. Nauk. SSRR **145**:3 (1962), 534–537.

[19] M.G. Kreĭn, *Sur le problème du prolongement des fonctions hermitiennes positives et continues.* Dokl. Akad. Nauk. SSSR **26** (1940), 17–22.

[20] M.G. Kreĭn and H. Langer, *On some continuation problems which are closely related to the theory of operators in spaces* Π_κ*. IV. Continuous analogues of orthogonal polynomials on the unit circle with respect to an indefinite weight and related continuation problems for some classes of functions,* J. Operator Theory **13** (1985), 299–417.

[21] W. Rudin, *The extension problem for positive definite functions.* Illinois J. Math. **7** (1963), 532–539.

[22] Z. Sasvári, *Positive definite and definitizable functions.* Akademie Verlag, 1994.

[23] Z. Sasvári, *The Extension Problem for Positive Definite Functions. A Short Historical Survey.* Operator Theory: Advances and Applications, **163**, (2005), 365–379.

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Received: February 27, 2009
Accepted: July 30, 2009

Operator Theory:
Advances and Applications, Vol. 202, 63–85

Multivariable Weighted Composition Operators: Lack of Point Spectrum, and Cyclic Vectors

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Abstract. We study weighted composition operators $T_{\alpha,\omega}$ on $L^2([0,1]^d)$ where $d \geq 1$, defined by

$$T_{\alpha,\omega} f(x_1, \dots, x_d) = \omega(x_1, \dots, x_d) f(\{x_1 + \alpha_1\}, \dots, \{x_d + \alpha_d\}),$$

where $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$ and where $\{.\}$ denotes the fractional part.

In the case where α is an irrational vector, we give a new and larger class of weights ω for which the point spectrum of $T_{\alpha,\omega}$ is empty. In the case of $\alpha \in \mathbb{Q}^d$ and $\omega(x_1, \dots, x_d) = x_1 \dots x_d$, we give a complete characterization of the cyclic vectors of $T_{\alpha,\omega}$.

Mathematics Subject Classification (2000). Primary: 47A15, 47A10, 47A16. Secondary: 47B33, 47A35.

Keywords. Weighted composition operator. Invariant subspace. Point spectrum. Cyclic vector.

1. Introduction

We study weighted composition operators $T_{\alpha,\omega}$ on $L^2([0,1]^d)$ where $d \geq 1$, defined by:

$$T_{\alpha,\omega} f(x_1, \dots, x_d) = \omega(x_1, \dots, x_d) f(\{x_1 + \alpha_1\}, \dots, \{x_d + \alpha_d\}), \tag{1}$$

where $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$ and where $\{.\}$ denotes the fractional part. These are said to be of Bishop type, and in the case of one variable, the $T_{\alpha,\omega}$ where $\omega(x) = x$ and $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ were introduced by Bishop as potential examples of operators with no nontrivial invariant subspace. In 1974 Davie [7] proved two significant results about them (still with $\omega(x) = x$): (1) if α is not a Liouville number, then T_α has nontrivial hyperinvariant subspaces; (2) if α is irrational, then T_α has no point spectrum, and thus the hyperinvariant subspaces are not simply eigenspaces. Since then, there

Communicated by J.A. Ball.

have been several further contributions and generalizations of this result. Blecher and Davie [2] proved that the same conclusion holds if ω is a continuous function with no zeros on $[0,1)$ whose modulus of continuity κ satisfies the condition $\int_0^1 \frac{\kappa(t)}{t}\mathrm{dt} < \infty$. MacDonald [9] considered operators $T_{\alpha,\omega} : L^2([0,1]) \to L^2([0,1])$ of the form $(T_{\alpha,\omega}f)(x) = \omega(x)f(\{x+\alpha\})$ where $\omega \in L^\infty([0,1])$ and $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, and proved the existence of nontrivial hyperinvariant subspaces in the case where ω is a function analytic in some neighbourhood of $[0,1]$. A multivariate version of this result was then proved by MacDonald [10] for operators $T_{\alpha,\omega}$ on $L^2([0,1]^d)$ given by (1).

More recently, Chalendar, Flattot and Guillotin-Plantard [3] obtained an extension of Davie's result on the lack of point spectrum for a general class of multivariate Bishop-type operators, and also characterized the spectrum for the case of a general $d \geq 1$, with α an irrational vector, where the weight ω has the form

$$\omega(x_1, \dots, x_d) = \omega_1(x_1) \dots \omega(x_d),$$

and each ω_j is a positive, convex and increasing function in $L^\infty([0,1])$.

The present paper extends these results in various ways. In Section 2 we show the non-existence of the point spectrum of $T_{\alpha,\omega}$, in the case where $d = 1$, α is in $\mathbb{R}\backslash\mathbb{Q}$ and ω is a positive and an increasing function such that the growth of $\log \omega$ is bounded below, which generalizes the "standard" case of an increasing and convex weight. Then in Section 3, we generalize this result to the general multivariable case ($d \geq 1$) with α equal to an irrational vector. First, we study a weight ω on $[0,1]^d$, $d \geq 2$ having the form $\omega(x_1, \dots, x_d) = \omega_1(x_1)\dots\omega_d(x_d)$, where each ω_j satisfies the same hypothesis as in the one-variable case. Second, we consider ω a positive function on $[0,1]^d$ such that for $(x_1, \dots, x_d), (y_1, \dots, y_d) \in [0,1]^d$, whenever $\omega(x_1, \dots, x_d) = \omega(y_1, \dots, y_d)$, then there exists a permutation $\sigma = \begin{pmatrix} 1 & \cdots & d \\ i_1 & \cdots & i_d \end{pmatrix}$ such that $(x_{i_1}, \dots, x_{i_d}) \leq (y_{i_1}, \dots, y_{i_d})$ for the lexicographic order. Finally, in Section 4, we give a characterization of cyclic vectors of $T_{\alpha,\omega} : L^2([0,1]^d) \to L^2([0,1]^d)$ for $d \geq 1$, where $\alpha \in \mathbb{Q}^d$ and $\omega(x) = x_1 \dots x_d$ on $[0,1]^d$, distinguishing the case where α_i, $i \in \{1, \dots, d\}$ do not have the same denominator and the case where α_i, $i \in \{1, \dots, d\}$ have the same denominator.

The methods employed to study weighted composition operators of Bishop type draw on measure theory, ergodic theory and some number theory; this is in contrast to the study of (weighted) composition operators on spaces of holomorphic functions [6, 11], where tools from complex analysis have been found useful.

We now give some precise definitions and notation. In the sequel, if x is in $[0,1]^d$ where $d \geq 1$, and α is in $\mathbb{R}^d$, we will denote the vector $(\{x_1+\alpha_1\}, \dots, \{x_d+\alpha_d\})$ by $\{x+\alpha\}$. Recall that a vector $\alpha = (\alpha_1, \dots, \alpha_d)$ is an *irrational vector* if $(1, \alpha_1, \dots, \alpha_d)$ is linearly independent over $\mathbb{Q}$. Moreover, for T a bounded linear operator on a complex Banach space X, a vector $f \in X$ is a *cyclic vector for* T if

$$\mathrm{Orb}(T,f) := \{P(T)f : P \in \mathbb{C}[X]\}$$

is dense in X. We shall use μ to denote Lebesgue measure.

2. The single-variable case

Proposition 2.1. *Let ω be a positive function on a sub-interval I of the real line. Suppose that there exists $\delta > 0$ such that for all $c, d \in I$,*

$$\left|\frac{\log(\omega(d)) - \log(\omega(c))}{d-c}\right| \geq \delta, \tag{2}$$

then, for all $\beta \in (0, 1/2)$, we have

$$\mu(\{t \in I : |1-\omega(t)| \leq \beta\}) \leq \frac{4\beta}{\delta}.$$

Proof. First take $s \neq t$ such that $\omega(s), \omega(t) \in [1-\beta, 1+\beta]$. By hypothesis, we have

$$\delta|s-t| \leq |\log\omega(s) - \log\omega(t)| \leq \log\frac{1+\beta}{1-\beta}.$$

Now, note that using the mean value theorem, for all $u > v$, we have

$$\log(1+u) - \log(1+v) \leq \frac{1}{1+v}(u-v).$$

Therefore we get

$$|s-t| \leq \frac{1}{1-\beta}\frac{2\beta}{\delta} \leq \frac{4\beta}{\delta}.$$

□

Corollary 2.2. *Let $\delta > 0$ and suppose that ω is differentiable and satisfies $\frac{|\omega'(t)|}{|\omega(t)|} \geq \delta$ on a real sub-interval I. Then, for all $\beta \in (0, 1/2)$, we have*

$$\mu(\{t \in I : |1-\omega(t)| \leq \beta\}) \leq \frac{4\beta}{\delta}.$$

Proof. The result follows from Proposition 2.1, since we clearly have (2) by the mean value theorem. □

Example 2.3. Let $\mathcal{P} \subset L^\infty([0,1])$ be the class of functions P on $[0,1]$ such that $P(x) = C\prod_{i=0}^{n}(x-x_i)^{s_i}$ for some $s_0, \dots, s_n \in \mathbb{R}_+$ and a constant C, with $x_i \in [0,1]$ for $i = 0, \dots, n$. Operators of Bishop type associated with a weight in $\mathcal{P}$ are studied in [1] and this is called the class of "generalized polynomials". If $\omega \in \mathcal{P}$, then ω satisfies the conditions of Corollary 2.2. In particular, $\omega(t) = \sqrt{t} \in \mathcal{P}$ is not an admissible function in the sense of [3] or [5].

Proposition 2.4. *Let $0 = a_0 < a_1 < \dots < a_N = 1$ and suppose that ω is a positive and increasing function on each interval (a_k, a_{k+1}) for $k = 0, \dots, N-1$, satisfying (2) on each interval. Let $\beta \in (0, 1/2)$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\lambda \in \mathbb{C} \setminus \{0\}$. We write for all $n \geq 1$,*

$$F_n(t) = \frac{1}{\lambda^n}\prod_{j=1}^{n}\omega(\{t-j\alpha\}).$$

Then,

$$\mu(\{t : |1 - |F_n(t)|| \leq \beta\}) \leq \frac{4\beta N}{\delta}.$$

Proof. First, note that we can partition $(0,1)$ into at most nN subintervals such that if c and d belong to the same subinterval, then $\{c - j\alpha\}$ and $\{d - j\alpha\}$ are in the same subinterval of the original partition for each $j = 1, \dots, n$. It follows that

$$\begin{aligned}\frac{\log|F_n(d)| - \log|F_n(c)|}{d-c} &= \frac{\sum_{j=1}^{n} \log(\omega(\{d - j\alpha\})) - \sum_{j=1}^{n} \log(\omega(\{c - j\alpha\}))}{d-c} \\ &= \sum_{j=1}^{n} \frac{\log(\omega(\{d - j\alpha\})) - \log(\omega(\{c - j\alpha\}))}{d-c} \\ &\geq n\delta.\end{aligned}$$

Thus, on each subinterval $|F_n|$ satisfies the hypothesis of Proposition 2.1, with δ replaced by $n\delta$. Using Proposition 2.1, we get

$$\mu(\{t \in (0,1) : |1 - |F_n(t)|| \leq \beta\}) \leq \frac{4\beta}{n\delta} Nn = \frac{4\beta N}{\delta}. \qquad \square$$

The proof of the next result uses ideas from the proof of Theorem 2 in [7].

Theorem 2.5. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and suppose that ω satisfies the hypothesis of the Proposition 2.4. Then,*

$$\sigma_p(T_{\alpha,\omega}) = \varnothing$$

Proof. Suppose that the point spectrum of $T_{\alpha,\omega}$ is not empty. Then, there exist $\lambda \in \mathbb{C}$ and nonzero $f \in L^2([0,1])$ such that $T_{\alpha,\omega} f = \lambda f$.

- If $\lambda = 0$, then, since ω is positive, it follows that $f = 0$ on $[0,1]$, which is impossible.
- Now, suppose that $\lambda \neq 0$. By Dirichlet's theorem, there are two sequences $(p_k)_{k \geq 1}$, $(q_k)_{k \geq 1}$, such that

$$\left| \alpha - \frac{p_k}{q_k} \right| \leq \frac{1}{q_k^2}$$

and $\lim_{k \to \infty} q_k = \infty$.

By Lusin's theorem, for every $\varepsilon > 0$ there is a (uniformly) continuous function g that equals f on the complement of a set of measure at most ε. Since $g(x) - g(\{x - q_k\alpha\}) \xrightarrow[k \to \infty]{} 0$ uniformly, it follows easily that $f(x) - f(\{x - q_k\alpha\}) \xrightarrow[k \to \infty]{} 0$ in measure (similar arguments can be found, for example, in [3, p. 355–356]). Hence

$$\text{for all } \beta, \eta > 0 \quad \mu(\{x \in [0,1] : |f(\{x - q_k\alpha\}) - f(x)| > \beta\}) \leq \eta.$$

for k sufficiently large.

Suppose that $\beta \in (0, \frac{1}{2})$ such that $\beta < \frac{\delta}{5N}$ and $\eta = 1 - \frac{5N\beta}{\delta}$, where δ is as given in condition (2). We can construct a subsequence of $(q_k)_k$, which we continue to call $(q_k)_k$, such that:

$$\text{for all} \quad k \geq 1, \quad \mu\left(\left\{x \in [0,1] : |f(\{x - q_k\alpha\}) - f(x)| \leq \frac{\beta}{k}\right\}\right) > 1 - \eta.$$

By hypothesis, we have:

$$f(\{x - q_k\alpha\}) = \lambda^{-q_k} T_{\alpha,\omega}^{q_k} f(\{x - q_k\alpha\}) = F_k(x) f(x),$$

where $F_k(x) = \lambda^{-q_k} \prod_{j=1}^{q_k} \omega(\{x - j\alpha\})$.

Using Proposition 2.4, we know that $\mu(\{t : |1 - |F_n(t)|| \leq \beta\}) \leq \dfrac{4\beta N}{\delta}$.

Since $|f(x) - f(\{x - q_k\alpha\})| = |1 - F_k(x)|\,|f(x)|$ and $|f(\{x - q_k\alpha\}) - f(x)| \leq \frac{\beta}{k}$ on a set of measure greater than $1 - \eta$, it follows that for all $k \geq 1$, we have $|f(x)| \leq \frac{1}{k}$ on a set of measure greater than $\frac{\beta N}{\delta}$.
The ergodicity of the transformation $x \mapsto \{x + \alpha\}$ implies that $f = 0$ on $[0,1]$. $\square$

Remark 2.6. In [2], the authors study Bishop-type operators whose continuous and positive weight ω satisfies the following condition

$$\int_0^1 \frac{\phi(t)}{t}\, dt < \infty\,, \qquad \text{where} \qquad \phi(t) = \sup_{|x-y| \leq t} |\log \omega(x) - \log \omega(y)|\,. \tag{3}$$

Unfortunately, there is no link between condition (3) and condition (2). Indeed, for $\omega(t) = e^{t^k}$ where $k > 1$, (2) is not satisfied but (3) is satisfied since $\phi(t) \leq kt$. On the other hand, for $\omega(t) = e^{t^k}$ where $0 < k < 1$, (2) is satisfied with $\delta = k$ but (3) is not satisfied since $\phi(t) \geq kt^{k-1}$.

Nevertheless, since (3) is satisfied when $\sup_{a,b \in [0,1] a \neq 0} \frac{|\log \omega(a) - \log \omega(b)|}{|a-b|} < \infty$, it follows that whenever there exist positive numerical constants C_1, C_2 such that

$$C_1 \leq \frac{\log \omega(a) - \log \omega(b)}{a - b} \leq C_2$$

then $\sigma_p(T_{\alpha,\omega}) = \varnothing$ and $T_{\alpha,\omega}$ has nontrivial hyperinvariant subspaces for non-Liouville irrational α.

3. The multivariable case

Theorem 3.1. *Let $\alpha = (\alpha_1, \ldots, \alpha_d)$, where $d \geq 2$, be an irrational vector and suppose that ω is a function of $L^\infty([0,1]^d)$ such that for all $x = (x_1, \ldots, x_d) \in [0,1]^d$, $\omega(x) = \prod_{i=1}^{d} \omega_i(x_i)$ where $\omega_i \in L^\infty([0,1])$, $i \in \{1, \ldots, d\}$ satisfies the conditions of Proposition* 2.4. *Then,*

$$\sigma_p(T_{\alpha,\omega}) = \varnothing.$$

Proof. Suppose that the point spectrum of $T_{\alpha,\omega}$ is nonempty. Then, there exist $\lambda \in \mathbb{C}$ and $f \in L^2([0,1]^d)$ such that $T_{\alpha,\omega} f = \lambda f$.

- If $\lambda = 0$, then, since ω is positive, it follows that $f = 0$ on $[0,1]^d$, which is impossible.
- Now, suppose that $\lambda \neq 0$. By Dirichlet's theorem, for $i \in \{1,\dots,d\}$, there are two sequences $(p_{k,i})_{k\geq 1}$, $(q_k)_{k\geq 1}$ such that
$$\left|\alpha_i - \frac{p_{k,i}}{q_k}\right| \leq \frac{1}{q_k^{1+\frac{1}{d}}}$$
and $\lim\limits_{k\to\infty} q_k = \infty$.

  Using Lusin's theorem again, as in the proof of Theorem 2.5, we have: $f(x) - f(\{x - q_k\alpha\}) \xrightarrow[k\to\infty]{} 0$ in measure. So,
$$\text{for all } \varepsilon, \eta > 0, \quad \mu(\{x \in [0,1]^d : |f(\{x - q_k\alpha\}) - f(x)| > \varepsilon\}) \leq \eta$$
for k sufficiently large.

  Suppose that $\beta \in (0, \frac{1}{2})$ such that $\beta < \frac{\delta}{5N}$ and $\eta = 1 - (\frac{5N\beta}{\delta})^d$, where δ is as in Condition (2). We can construct a subsequence of $(q_k)_k$, which again we call $(q_k)_k$, such that:
$$\text{for all } k \geq 1, \ \mu\left(\left\{x \in [0,1] : |f(\{x - q_k\alpha\}) - f(x)| \leq \frac{(1+\beta)^d - 1}{k}\right\}\right) > 1 - \eta.$$
By hypothesis, we have:
$$f(\{x - q_k\alpha\}) = \lambda^{-q_k} T_{\alpha,\omega}^{q_k} f(\{x - q_k\alpha\}) = F_k(x) f(x),$$
where $F_k(x) = \lambda^{-q_k} \prod\limits_{j=1}^{q_k} \omega(\{x - j\alpha\}) = \lambda^{-q_k} \prod\limits_{j=1}^{q_k} \prod\limits_{i=1}^{d} \omega_i(\{x_i - j\alpha_i\})$.

  Let $f_1(x_1) = \lambda^{-q_k} \prod\limits_{j=1}^{q_k} \omega_1(\{x_1 - j\alpha_1\})$ and $f_i(x_i) = \prod\limits_{j=1}^{q_k} \omega_i(\{x_i - j\alpha_i\})$, $i \in \{2,\dots,d\}$.

  By Proposition 2.4, we have, for $i \in \{1,\dots,d\}$,
$$\mu(\{x_i \in [0,1]^d : |1 - |f_i(x_i)|| \leq \beta\}) \leq \frac{4\beta N}{\delta}.$$

  So, for $i \in \{1,\dots,d\}$, $1 - \beta \leq |f_i(x_i)| \leq 1 + \beta$ on a set of measure less than $\frac{4\beta N}{\delta}$ and $|1 - |F_k(x)|| > (1+\beta)^d - 1$ on a set of measure greater than $1 - (\frac{4\beta N}{\delta})^d$.

  Since $|f(x) - f(\{x - q_k\alpha\})| = |1 - F_k(x)|\,|f(x)|$ and $|f(\{x - q_k\alpha\}) - f(x)| \leq \frac{(1+\beta)^d - 1}{k}$ on a set of measure greater than $1 - \eta = (\frac{5\beta N}{\delta})^d$, it follows that for all $k \geq 1$, $|f(x)| \leq \frac{1}{k}$ on a set of measure greater than $(\frac{\beta N}{\delta})^d (5^d - 4^d)$.

  It follows that $f = 0$ on a set of measure greater than $(\frac{\beta N}{\delta})^d (5^d - 4^d) > 0$; the ergodicity of the transformation $x \mapsto \{x + \alpha\}$ implies that $f = 0$ on $[0,1]^d$, which is impossible. □

Proposition 3.2. *Let ω be a positive function on a domain $D \subset \mathbb{R}^2$. Suppose that there exists $\delta > 0$ such that for all (c_1, c_2) and $(d_1, d_2) \in D$,*

$$\frac{|\log(\omega(c_1,c_2)) - \log(\omega(d_1,d_2))|}{\|(d_1,d_2)-(c_1,c_2)\|_2} \geq \delta, \tag{4}$$

then, for all $\beta \in (0, 1/2)$, we have

$$\mu(\{(s,t) \in D : |1-\omega(s,t)| \leq \beta\}) \leq \frac{8\beta^2}{\delta^2}.$$

Proof. First take $(s_1, s_2) \neq (t_1, t_2)$ such that $\omega(s_1, s_2), w(t_1, t_2) \in [1-\beta, 1+\beta]$. By hypothesis, we have

$$\delta\|(s_1,s_2)-(t_1,t_2)\|_2 \leq |\log\omega(s_1,s_2) - \log\omega(t_1,t_2)| \leq \log\frac{1+\beta}{1-\beta}.$$

Using the mean value theorem as in the single-variable case, we have

$$\|(s_1,s_2)-(t_1,t_2)\|_2 \leq \frac{1}{1-\beta}\frac{2\beta}{\delta} \leq \frac{4\beta}{\delta}$$

and

$$|(s_1-t_1)(s_2-t_2)| \leq \frac{1}{2}\|(s_1,s_2)-(t_1,t_2)\|_2^2 \leq \frac{16\beta^2}{2\delta^2} = \frac{8\beta^2}{\delta^2}.$$

So, we get

$$\mu\left(\{(s,t) \in D : |1-\omega(s,t)| \leq \beta\}\right) \leq \frac{8\beta^2}{\delta^2}. \qquad \square$$

Corollary 3.3. *Let $\delta > 0$ and suppose that ω is differentiable, positive and satisfies $\frac{d\omega_u(h)}{\omega(u)} \geq \delta\|h\|$, for $h \in \mathbb{R}^2$ and $u \in D \subset \mathbb{R}^2$. Then, for all $\beta \in (0, 1/2)$, we have*

$$\mu(\{(s,t) \in D : |1-\omega(s,t)| \leq \beta\}) \leq \frac{8\beta^2}{\delta^2}.$$

Proof. Using Taylor's theorem with the integral remainder term, we have, for $c = (c_1, c_2) \in D, h \in \mathbb{R}^2$,

$$\frac{|\log(\omega(c+h)) - \log(\omega(c))|}{\|h\|_2} = \frac{\int_0^1 d(\log\circ\omega)_{c+th}(h)\,\mathrm{d}t}{\|h\|_2} \geq \delta,$$

so ω satisfies (4). $\square$

Example 3.4. Let $\omega : (x_1, x_2) \mapsto \sqrt{x_1 + x_2} + 2$. Then, ω satisfies (4) on $[0, 1]^2$.

Indeed, let $x = (x_1, x_2), h = (h_1, h_2) \in (0, 1]^2$. We have:

$$d\omega_x(h) = \frac{1}{2}\frac{1}{\sqrt{x_1+x_2}}(h_1+h_2) \Longrightarrow \frac{d\omega_x(h)}{\omega(x)} \geq \frac{\|h\|_1}{2(\sup_{x\in[0,1]^2}\omega(x))^2}.$$

So, ω satisfies the hypothesis of Corollary 3.3 on $(0, 1]^2$ and so, the condition (4) on $(0, 1]^2$. Suppose that $(c_1, c_2) = (0, 0)$ and take $(d_1, d_2) \in (0, 1]^2$.

The function

$$f : x \in (0,4] \mapsto \frac{\log(1+\frac{x}{2})}{x^2}$$

is decreasing. This implies that:

$$\text{for all} \quad (d_1,d_2)\in(0,1]^2, \quad \frac{|\log(2)-\log(\sqrt{d_1+d_2}+2)|}{d_1+d_2} \geq \frac{\log(1+\frac{\sqrt{2}}{2})}{2}.$$

So, ω satisfies (4) on $[0,1]^2$.

Definition 3.5. Let $(\mathbb{R}^2,\preceq)$ and $(\mathbb{R},\leq)$ be ordered sets, where $\leq$ is the usual real order. A function $f:\mathbb{R}^2\mapsto\mathbb{R}$ is *increasing* if

$$\text{for all} \quad x,y\in\mathbb{R}^2, \quad x\preceq y\Rightarrow f(x)\leq f(y).$$

Proposition 3.6. *Suppose that $\omega\in L^\infty([0,1]^2)$ is a positive and increasing function in the sense of Definition 3.5 on each cube $(a_k,a_{k+1})\times(b_l,b_{l+1})$ for $k,l=0,\dots,N-1$, satisfying (4) on each cube. Let $\beta\in(0,\frac{1}{2})$, α an irrational vector and $\lambda\in\mathbb{C}\backslash\{0\}$. For $n\geq 1$, let*

$$F_n(s,t)=\frac{1}{\lambda^n}\prod_{j=1}^{n}\omega(\{s-j\alpha_1\},\{t-j\alpha_2\}).$$

Then

$$\mu(\{(s,t)\in[0,1]^2:|1-|F_n(s,t)||\leq\beta\})\leq\frac{8\beta^2N^2}{\delta^2}.$$

Proof. As in the single-variable case, one can partition $(0,1)^2$ into $(nN)^2$ cubes such that if $c,d\in[0,1]^2$, $c\preceq d$, then, for all $j=1,\dots,n$, $\{c-j\alpha\}\preceq\{d-j\alpha\}$. Then, we have:

$$\begin{aligned}\frac{\log|F_n(d)|-\log|F_n(c)|}{\|d-c\|_2} &= \frac{\sum_{j=1}^{n}\log(w(\{d-j\alpha\}))-\sum_{j=1}^{n}\log(w(\{c-j\alpha\}))}{\|d-c\|_2}\\ &= \sum_{j=1}^{n}\frac{\log(w(\{d-j\alpha\}))-\log(w(\{c-j\alpha\}))}{\|d-c\|_2}\\ &\overset{*}{\geq} n\delta.\end{aligned}$$

$*$: using the condition (2) with $\omega\circ g_j$, where $g_j:x\mapsto\{x+j\alpha\}$.

Then, on each cube, $|F_n|$ satisfies the hypothesis of Proposition 3.2 with $n\delta$ instead of δ. So,

$$\mu(\{(s,t)\in[0,1]^2:|1-|F_n(s,t)||\leq\beta\})\leq\frac{8\beta^2}{(n\delta)^2}(nN)^2=\frac{8\beta^2N^2}{\delta^2}. \qquad \square$$

Example 3.7.

1) Consider the following total order on $[0,1] \times [0,1]$:

$$(x_1, x_2) \preceq_1 (y_1, y_2) \text{ if } \begin{cases} x_1 + x_2 & < & y_1 + y_2 \\ & \text{or} & \\ x_1 + x_2 & = & y_1 + y_2 \quad \text{and} \quad x_1 < y_1 \\ & \text{or} & \\ x_1 = y_1 & \text{and} & x_2 = y_2. \end{cases}$$

Let $\omega(x, y) = \sqrt{x+y} + 2$. Then, ω is positive, increasing with respect to $\preceq_1$ on $[0,1] \times [0,1]$ and satisfies the condition (4) by Example 3.4.

2) Let $C > 0$. One can also consider the total orders on $[0,1] \times [0,1]$ denoted by $\preceq_{p,C}$, $p \geq 1$ and defined by

$$(x_1, x_2) \preceq_{p,C} (y_1, y_2) \text{ if } \begin{cases} (x_1 + C)^p + (x_2 + C)^p \\ \qquad < (y_1 + C)^p + (y_2 + C)^p \\ \qquad \text{or} \\ (x_1 + C)^p + (x_2 + C)^p \\ \qquad = (y_1 + C)^p + (y_2 + C)^p \text{ and } x_1 < y_1 \\ \qquad \text{or} \\ x_1 = y_1 \quad \text{and} \quad x_2 = y_2. \end{cases}$$

The function $\omega : (x_1, x_2) \mapsto (x_1 + C)^p + (x_2 + C)^p$ is increasing relative to the order $\preceq_{p,C}$ and satisfies the hypothesis of Corollary 2.2. Indeed, ω is clearly an increasing function relative to $\preceq_{p,C}$. Letting $(x_1, x_2), (h_1, h_2) \in [0,1]^2$, we have

$$\begin{aligned} \frac{d\omega_{(x_1,x_2)}(h_1, h_2)}{\omega(x_1, x_2)} &= \frac{p(x_1 + C)^{p-1} h_1 + p(x_2 + C)^{p-1} h_2}{\omega(x_1, x_2)} \\ &\geq \frac{p\|(h_1, h_2)\|_1}{2^{p+1}}. \end{aligned}$$

Proposition 3.8. *Suppose that $\omega \in L^\infty([0,1]^2)$ is a positive function such that for all (x_1, x_2) and $(y_1, y_2) \in [0,1]^2$,*

$$\omega(x_1, x_2) = \omega(y_1, y_2) \Longrightarrow \begin{cases} x_1 < y_1 & \textit{or} \quad (x_1 = y_1 \quad \textit{and} \quad x_2 \leq y_2) \\ & \textit{or} \\ x_2 < y_2 & \textit{or} \quad (x_2 = y_2 \quad \textit{and} \quad x_1 \leq y_1) \end{cases} \tag{5}$$

and satisfying the condition (4) on each cube $(a_k, a_{k+1}) \times (b_l, b_{l+1})$ for $k, l = 0, \ldots, N-1$. Let $\beta \in (0, \frac{1}{2})$, α an irrational vector and $\lambda \in \mathbb{C} \backslash \{0\}$. For $n \geq 1$, let

$$F_n(s, t) = \frac{1}{\lambda^n} \prod_{j=1}^{n} \omega(\{s - j\alpha_1\}, \{t - j\alpha_2\}).$$

Then,

$$\mu(\{(s,t)\in[0,1]^2 : |1-|F_n(s,t)||\le\beta\})\le\frac{8\beta^2N^2}{\delta^2}.$$

Proof. Note that if we consider the relation on $[0,1]\times[0,1]$ defined by $(x_1,x_2)\preceq(y_1,y_2)$ if:

$$\begin{cases}\omega(x_1,x_2) & < & \omega(y_1,y_2) \\ & \text{or} & \\ \omega(x_1,x_2) & = & \omega(y_1,y_2) \quad\text{and}\quad \begin{cases} x_1<y_1 \quad \text{or} \quad (x_1=y_1 \quad\text{and}\quad x_2\le y_2) \\ \qquad\qquad \text{or} \\ x_2<y_2 \quad \text{or} \quad (x_2=y_2 \quad\text{and}\quad x_1\le y_1)\end{cases}\end{cases}$$

then by (5), we have that $\preceq$ is a total order on $[0,1]^2$ and $\omega:([0,1]^2,\preceq)\longrightarrow(\mathbb{R},\le)$ is increasing in the sense of Definition 3.5. As in the single-variable case, one can partition $(0,1)^2$ into $(nN)^2$ cubes such that if $c,d\in[0,1]^2$ with $c\preceq d$, then for all $j\in\{1,\dots,n\}$ one has $\{c-j\alpha\}\preceq\{d-j\alpha\}$. Then, we have:

$$\begin{aligned}\frac{\log|F_n(d)|-\log|F_n(c)|}{\|d-c\|_2} &= \frac{\sum_{j=1}^{n}\log(w(\{d-j\alpha\}))-\sum_{j=1}^{n}\log(w(\{c-j\alpha\}))}{\|d-c\|_2} \\ &= \sum_{j=1}^{n}\frac{\log(w(\{d-j\alpha\}))-\log(w(\{c-j\alpha\}))}{\|d-c\|_2} \\ &\overset{*}{\ge} n\delta.\end{aligned}$$

$*$: using the condition (2) with $\omega\circ g_j$, where $g_j : x\mapsto\{x+j\alpha\}$.
Then, on each cube, $|F_n|$ satisfies the hypothesis of Proposition 3.2 with $n\delta$ instead of δ. So,

$$\mu(\{(s,t)\in[0,1]^2 : |1-|F_n(s,t)||\le\beta\})\le\frac{8\beta^2}{(n\delta)^2}(nN)^2=\frac{8\beta^2N^2}{\delta^2}. \qquad \square$$

Proposition 3.9. *Let ω be a positive function on a domain $D\subset\mathbb{R}^d$, $d\ge2$. Suppose that there exists $\delta>0$ such that, for all $(x_1,\dots,x_d)$ and $(y_1,\dots,y_d)\in D$,*

$$\frac{|\log(\omega(x_1,\dots,x_d))-\log(\omega(y_1,\dots,y_d))|}{\|(x_1,\dots,x_d)-(y_1,\dots,y_d)\|_d}\ge\delta, \tag{6}$$

then, for all $\beta\in(0,1/2)$, we have

$$\mu(\{(x_1,\dots,x_d)\in D : |1-\omega(x_1,\dots,x_d)|)\le\beta\}\le\frac{4^d\beta^d}{d\delta^d}.$$

Proof. Take $(x_1,\dots,x_d)\ne(y_1,\dots,y_d)$ such that $\omega(x_1,\dots,x_d)$ and $\omega(y_1,\dots,y_d)$ are in $[1-\beta,1+\beta]$. By hypothesis, we have

$$\delta\|(x_1,\dots,x_d)-(y_1,\dots,y_d)\|_d\le|\log\omega(x_1,\dots,x_d)-\log\omega(y_1,\dots,y_d)|\le\log\frac{1+\beta}{1-\beta}.$$

Using the mean value theorem as in the single-variable case, we have

$$\|(x_1,\dots,x_d)-(y_1,\dots,y_d)\|_d \le \frac{1}{1-\beta}\frac{2\beta}{\delta} \le \frac{4\beta}{\delta}.$$

and using the inequality of arithmetic and geometric means, we have:

$$\left|\prod_{i=1}^{d}(x_i-y_i)\right| \le \frac{1}{d}\|(x_1,\dots,x_d)-(y_1,\dots,y_d)\|_d^d \le \frac{4^d\beta^d}{d\delta^d}.$$

So, we get:

$$\mu\left(\{(x_1,\dots,x_d)\in D : |1-\omega(x_1,\dots,x_d)|\le\beta\}\right) \le \frac{4^d\beta^d}{d\delta^d}.$$ □

We now give an analogous version of Corollary 3.3 in dimension $d\ge 2$.

Corollary 3.10. *Let $\delta>0$ and suppose that ω is differentiable, positive and satisfies $d\omega_u(h)/\omega(u)\ge\delta\|h\|$, $h\in\mathbb{R}^d, u\in D\subset\mathbb{R}^d$. Then, for all $\beta\in(0,1/2)$, we have*

$$\mu(\{(x_1,\dots,x_d)\in D : |1-\omega(x_1,\dots,x_d)|\le\beta\}) \le \frac{4^d\beta^d}{d\delta^d}.$$

Using similar arguments, one can prove the following result in the case of d variables, $d\ge 2$:

Proposition 3.11. *Let $\omega\in L^\infty([0,1]^d)$ be a positive function satisfying (6) on each cube $(a_k,a_{k+1})\times(b_l,b_{l+1})$, with $k,l=0,\dots,N-1$. Suppose that for $(x_1,\dots,x_d)$ and $(y_1,\dots,y_d)$ in $[0,1]^d$, whenever $\omega(x_1,\dots,x_d)=\omega(y_1,\dots,y_d)$, there exists a permutation $\sigma=\begin{pmatrix}1 & \cdots & d\\ i_1 & \cdots & i_d\end{pmatrix}$ such that $(x_{i_1},\dots,x_{i_d})\le(y_{i_1},\dots,y_{i_d})$ for the lexicographic order. Let $\beta\in(0,\frac{1}{2})$, α an irrational vector and $\lambda\in\mathbb{C}\backslash\{0\}$. For $n\ge1$, let*

$$F_n(x_1,\dots,x_d)=\frac{1}{\lambda^n}\prod_{j=1}^{n}\omega(\{x_1-j\alpha_1\},\dots,\{x_d-j\alpha_d\}).$$

Then,

$$\mu(\{(x_1,\dots,x_d)\in[0,1]^d : |1-|F_n(x_1,\dots,x_d)||\le\beta\}) \le \frac{4^d\beta^dN^d}{d\delta^d}.$$

Theorem 3.12. *Let $\alpha=(\alpha_1,\dots,\alpha_d)$ be an irrational vector and suppose that ω is a positive function of $L^\infty([0,1]^d)$ satisfying condition (6) and the hypothesis of Proposition 3.11. Then,*

$$\sigma_p(T_{\alpha,\omega})=\varnothing.$$

Proof. Suppose that the point spectrum of $T_{\alpha,\omega}$ is not empty. Then, there exist $\lambda\in\mathbb{C}$ and $f\in L^2([0,1]^d)$ such that $T_{\alpha,\omega}f=\lambda f$.

- If $\lambda=0$, then, since ω is positive, it follows that $f=0$ on $[0,1]^d$, which is impossible.

- Now, suppose that $\lambda \neq 0$. By Dirichlet's theorem, for $i \in \{1,\dots,d\}$, there are two sequences $(p_{k,i})_{k\geq 1}$, $(q_k)_{k\geq 1}$ such that:
$$\left|\alpha_i - \frac{p_{k,i}}{q_k}\right| \leq \frac{1}{q_k^{1+\frac{1}{d}}},$$
and $\lim_{k\to\infty} q_k = \infty$.
Using Lusin's theorem, we have: $f(x)-f(\{x-q_k\alpha\}) \xrightarrow[k\to\infty]{} 0$ in measure. So,
$$\text{for all} \quad \varepsilon,\eta>0, \quad \mu(\{x\in[0,1]^d : |f(\{x-q_k\alpha\})-f(x)|>\varepsilon\}) \leq \eta.$$
Set $0 < C < \frac{4}{d^{1/d}}$. Suppose that $\beta \in (0,\frac{1}{2})$ such that $\beta < \frac{\delta}{CN}$ and $\eta = 1-\left(\frac{CN\beta}{\delta}\right)^d$ where δ is as in condition (6). We can construct a subsequence of $(q_k)_k$, which will still be called $(q_k)_k$ such that:
$$\text{for all} \quad k\geq 1, \quad \mu\left(\left\{x\in[0,1]^d : |f(\{x-q_k\alpha\})-f(x)| \leq \frac{\beta}{k}\right\}\right) > 1-\eta.$$
By hypothesis, we have:
$$f(\{x-q_k\alpha\}) = \lambda^{-q_k} T_{\alpha,\omega}^{q_k} f(\{x-q_k\alpha\}) = F_k(x)f(x)$$
where $F_k(x) = \lambda^{-q_k}\prod_{j=1}^{q_k}\omega(\{x-j\alpha\})$.
By Proposition 3.11, we have
$$\mu(\{x\in[0,1]^d : |1-|F_n(x)|| \leq \beta\}) \leq \frac{4^d\beta^d N^d}{d\delta^d}.$$
Since $|f(x)-f(\{x-q_k\alpha\})| = |1-F_k(x)|\,|f(x)|$ and $|f(\{x-q_k\alpha\})-f(x)| \leq \frac{\beta}{k}$ on a set of measure greater than $1-\eta = (\frac{C\beta N}{\delta})^d$, it follows that for all $k\geq 1$, $|f(x)| \leq \frac{1}{k}$ on a set of measure greater than $\left(\frac{\beta N}{\delta}\right)^d\left(\frac{4^d}{d}-C^d\right)$.
It follows that $f=0$ on a set of measure greater than $\left(\frac{\beta N}{\delta}\right)^d\left(\frac{4^d}{d}-C^d\right) > 0$; the ergodicity of the transformation $x\mapsto\{x+\alpha\}$ implies that $f=0$ a.e. on $[0,1]^d$, which is impossible. $\square$

4. Cyclic vectors for T_α, $\alpha\in\mathbb{Q}^d$

It is easy to see that, if $\alpha\in\mathbb{Q}^d$, the Bishop operator T_α has many non-trivial invariant subspaces. A full description of the lattice of invariant subspaces for the case $d=1$ was given in the unpublished report [4], and the cyclic vectors were also characterized (this characterization can also be derived from results announced without proof by Lipin [8]). For the reader's convenience we recall the cyclicity result for $d=1$.

Definition 4.1. Let $f \in L^2([0,1])$ and $f_k = T^{k-1} f$, $k \in \{1, \ldots, q\}$. The *determinant of f associated with $\alpha = p/q$* is the $1/q$-periodic function in $L^{2/q}([0,1])$ defined by:

$$\Delta(f, p/q) = \begin{vmatrix} f_1(t) & \cdots & f_q(t) \\ f_1(\{t + p/q\}) & \cdots & f_q(\{t + p/q\}) \\ \vdots & \vdots & \vdots \\ f_1(\{t + (q-1)p/q\}) & \cdots & f_q(\{t + (q-1)p/q\}) \end{vmatrix}.$$

Note that, since p and q have no common divisor, $|\Delta(f, p/q)| = |\Delta(f, 1/q)|$. The cyclicity result for $d = 1$ is then the following.

Theorem 4.2. *Let $T = T_{p/q}$ where $p < q$ and p and q are coprime. A function $f \in L^2([0,1])$ is cyclic for T if and only if $\Delta(f, 1/q)$ is nonzero almost everywhere on $[0,1]$.*

4.1. The case $\alpha \in \mathbb{Q}^2$

In the case $d \geq 2$ the cyclicity results have a similar flavour, but are technically more complicated to derive. We give the case $d = 2$ in detail, since the notation is simpler. The result for the general case is given later, as Theorem 4.15. We also give some examples (Examples 4.10), to show how the condition can be tested.

The operator T_α is defined by

$$\begin{aligned} T_\alpha \,:\, L^2([0,1]^2) &\;\to\; L^2([0,1]^2) \\ f &\;\mapsto\; T_\alpha f : x \in [0,1]^2 \mapsto x_1 x_2 f(\{x_1 + \alpha_1\}, \{x_2 + \alpha_2\}). \end{aligned}$$

Set $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_i = \dfrac{p_i}{q_i}$.

4.1.1. The case $q_1 \neq q_2$.

Suppose that $\mathrm{GCD}(p_i, q_i) = 1, i \in \{1,2\}$. We write $T_\alpha = T$, $q = \mathrm{LCM}(q_1, q_2)$ and for $r \in \{0, \ldots, q-1\}$, $f_{r+1} = T^r f$.

The following formula is easy to derive, and we omit the proof.

Proposition 4.3. *Let $\omega_i(x_i) = x_i\{x_i + \alpha_i\} \ldots \{x_i + (q_i - 1)\alpha_i\}$, $l_i = \dfrac{q}{q_i}$, for $i \in \{1,2\}$, and $(n, r) \in \mathbb{N}^2$, with $r < q$. Take $f \in L^2([0,1]^2)$. Then,*

$$T^{nq+r} f(x_1, x_2) = \omega_1^{nl_1}(x_1)\omega_2^{nl_2}(x_2) f_{r+1}(x_1, x_2).$$

Remark 4.4. Note that for $i \in \{1,2\}$, ω_i is a $\frac{1}{q_i}$-periodic function.

Definition 4.5. Let $f \in L^2([0,1]^2)$. The determinant of f associated with (α_1, α_2) is the determinant

$$\Delta\left(f, \frac{1}{q_1}, \frac{1}{q_2}\right) = \begin{vmatrix} f_1(s,t) & \cdots & f_q(s,t) \\ f_1\left(\left\{s + \frac{1}{q_1}\right\}, \left\{t + \frac{1}{q_2}\right\}\right) & \cdots & f_q\left(\left\{s + \frac{1}{q_1}\right\}, \left\{t + \frac{1}{q_2}\right\}\right) \\ \vdots & \ddots & \vdots \\ f_1\left(\left\{s + \frac{q-1}{q_1}\right\}, \left\{t + \frac{q-1}{q_2}\right\}\right) & \cdots & f_q\left(\left\{s + \frac{q-1}{q_1}\right\}, \left\{t + \frac{q-1}{q_2}\right\}\right) \end{vmatrix}.$$

It is a function in $L^{2/q}([0,1]^2)$.

Lemma 4.6. *Let n be a positive integer and $f \in L^2([0,1]^2)$. Let $h \in L^\infty([0,1]^2)$ be such that $h(s,t) = 0$ for $(s,t) \in \Omega_{n,f}$ where $\Omega_{n,f}$ is the $\left(\frac{1}{q_1}, \frac{1}{q_2}\right)$-periodic set*

$$\begin{aligned}\Omega_{n,f} \quad &= \quad \left\{(s,t) \in [0,1]^2 : \left|\Delta\left(f, \tfrac{1}{q_1}, \tfrac{1}{q_2}\right)\right| < \tfrac{1}{n}\right\} \bigcup \\ &\bigcup_{\substack{0 \le k \le q-1 \\ 1 \le r \le q}} \left\{(s,t) \in [0,1]^2 : \left| f_r\left(\left\{s + \frac{k}{q_1}\right\}, \left\{t + \frac{k}{q_2}\right\}\right)\right| > n\right\}.\end{aligned}$$

Then, there exist $h_1, \dots, h_q \in L^\infty([0,1]^2)$, $\left(\frac{1}{q_1}, \frac{1}{q_2}\right)$-periodic functions such that $h = \sum_{k=1}^{q} h_k f_k$.

Proof. If $h = \sum_{k=1}^{q} h_k f_k$ where $h_1, \dots, h_q \in L^\infty([0,1]^2)$ are $\left(\frac{1}{q_1}, \frac{1}{q_2}\right)$-periodic functions, then, for $(s,t) \in [0,1]^2$,

$$\begin{aligned} h(s,t) &= h_1(s,t) f_1(s,t) + \cdots + h_q(s,t) f_q(s,t), \\ h\left(\left\{s + \tfrac{1}{q_1}\right\}, \left\{t + \tfrac{1}{q_2}\right\}\right) &= h_1(s,t) f_1\left(\left\{s + \tfrac{1}{q_1}\right\}, \left\{t + \tfrac{1}{q_2}\right\}\right) + \cdots \\ &\quad + h_q(s,t) f_q\left(\left\{s + \tfrac{1}{q_1}\right\}, \left\{t + \tfrac{1}{q_2}\right\}\right), \\ &\vdots \\ h\left(\left\{s + \tfrac{q-1}{q_1}\right\}, \left\{t + \tfrac{q-1}{q_2}\right\}\right) &= h_1(s,t) f_1\left(\left\{s + \tfrac{q-1}{q_1}\right\}, \left\{t + \tfrac{q-1}{q_2}\right\}\right) + \cdots \\ &\quad + h_q(s,t) f_q\left(\left\{s + \tfrac{q-1}{q_1}\right\}, \left\{t + \tfrac{q-1}{q_2}\right\}\right). \end{aligned}$$

Using matrices, we have:

$$\begin{pmatrix} h(s,t) \\ h\left(\left\{s + \frac{1}{q_1}\right\}, \left\{t + \frac{1}{q_2}\right\}\right) \\ \vdots \\ h\left(\left\{s + \frac{q-1}{q_1}\right\}, \left\{t + \frac{q-1}{q_2}\right\}\right) \end{pmatrix} = A \begin{pmatrix} h_1(s,t) \\ h_2(s,t) \\ \vdots \\ h_q(s,t) \end{pmatrix},$$

where A is the matrix whose determinant defines $\Delta(f, \frac{1}{q_1}, \frac{1}{q_2})$ in Definition 4.5.

For $(s,t) \in \Omega_{n,f}^c$, we have: $\left|\Delta\left(f, \frac{1}{q_1}, \frac{1}{q_2}\right)\right| \geq \frac{1}{n} > 0$; so, A is an invertible matrix on $\Omega_{n,f}^c$ and

$$\begin{pmatrix} h_1(s,t) \\ h_2(s,t) \\ \vdots \\ h_q(s,t) \end{pmatrix} = A^{-1} \begin{pmatrix} h(s,t) \\ h\left(\left\{s + \frac{1}{q_1}\right\}, \left\{t + \frac{1}{q_2}\right\}\right) \\ \vdots \\ h\left(\left\{s + \frac{q-1}{q_1}\right\}, \left\{t + \frac{q-1}{q_2}\right\}\right) \end{pmatrix}.$$

- On $\Omega_{n,f}^c$, the h_i are combinations of the functions

$$f_r\left(\left\{.+\frac{k}{q_1}\right\},\left\{.+\frac{k}{q_2}\right\}\right) \quad \text{and} \quad h\left(\left\{.+\frac{l}{q_1}\right\},\left\{.+\frac{l}{q_2}\right\}\right)$$

in $L^\infty([0,1]^2)$, for $k,l \in \{0,\dots,q-1\}$ and $r \in \{1,\dots,q\}$. Moreover, for $(s,t) \in \Omega_{n,f}^c$,

$$\left|f_r\left(\left\{s+\frac{k}{q_1}\right\},\left\{t+\frac{k}{q_2}\right\}\right)\right| \leq n,$$

so $f_r(\{.+\frac{k}{q_1}\},\{.+\frac{k}{q_2}\}))$ are bounded. For $(s,t) \in \Omega_{n,f}$, set $h_i(s,t)=0$. Thus, the h_i are functions in $L^\infty([0,1]^2)$.

- One can verify that the h_i are $\left(\frac{1}{q_1},\frac{1}{q_2}\right)$-periodic functions. □

Lemma 4.7. *Let F be a function in $L^2\left(\left[0,\frac{1}{q_1}\right]\times\left[0,\frac{1}{q_2}\right]\right)$ that is not equal to zero almost everywhere. Then,*

$$\{g(\omega_1,\omega_2)F : g \in \mathbb{C}[X,Y]\} \quad \textit{is dense in} \quad L^2\left(\left[0,\frac{1}{q_1}\right]\times\left[0,\frac{1}{q_2}\right]\right).$$

Proof. Let G be a function in $L^2\left(\left[0,\frac{1}{q_1}\right]\times\left[0,\frac{1}{q_2}\right]\right)$ such that

$$G \in \{g(\omega_1,\omega_2)F : g \in \mathbb{C}[X,Y]\}^\perp.$$

Then,

$$\iint_{\left[0,\frac{1}{q_1}\right]\times\left[0,\frac{1}{q_2}\right]} \overline{G(s,t)}F(s,t)g(\omega_1,\omega_2)(s,t)\,\mathrm{d}s\,\mathrm{d}t = 0.$$

For $i \in \{1,2\}$ we write $\widetilde{\omega_i}$ for the continuous function on $\left[0,\frac{1}{q_i}\right]$ obtained by restricting ω_i to $\left[0,\frac{1}{q_i}\right)$ and defining

$$\widetilde{\omega_i}\left(\frac{1}{q_i}\right) = \frac{1}{q_i}\left\{\frac{2}{q_i}\right\}\cdots\left\{\frac{q_i-1}{q_i}\right\}.$$

Now $\mathcal{B} := \{g(\widetilde{\omega_1},\widetilde{\omega_2})F : g \in \mathbb{C}[X,Y]\}$ is a subalgebra of $\mathcal{C}\left(\left[0,\frac{1}{q_1}\right]\times\left[0,\frac{1}{q_2}\right]\right)$ which separates points and contains a non-zero constant function. Moreover,

$$g(\widetilde{\omega_1},\widetilde{\omega_2}) \in \mathcal{B} \Longrightarrow \overline{g(\widetilde{\omega_1},\widetilde{\omega_2})} = \overline{g}(\widetilde{\omega_1},\widetilde{\omega_2}) \in \mathcal{B},$$

therefore $\mathcal{B}$ is a self-adjoint algebra. By the Stone–Weierstrass theorem, we have that $\mathcal{B}$ is dense in $\mathcal{C}\left(\left[0,\frac{1}{q_1}\right]\times\left[0,\frac{1}{q_2}\right]\right)$ for $\|.\|_\infty$.

Let k be an element of $\mathcal{C}\left(\left[0,\frac{1}{q_1}\right]\times\left[0,\frac{1}{q_2}\right]\right)$. Then, there exists a sequence $(k_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{B}$ such that $\lim\limits_{n\to\infty} ||k_n - k||_\infty = 0$.

Now

$$\begin{aligned}&\left|\iint_{\left[0,\frac{1}{q_1}\right]\times\left[0,\frac{1}{q_2}\right]} \overline{G(s,t)}F(s,t)k(s,t)\,\mathrm{d}s\,\mathrm{d}t\right|\\ &\quad=\left|\iint_{\left[0,\frac{1}{q_1}\right]\times\left[0,\frac{1}{q_2}\right]} (-\overline{G(s,t)}F(s,t)k_n(s,t)+\overline{G(s,t)}F(s,t)k(s,t))\,\mathrm{d}s\,\mathrm{d}t\right|\\ &\quad\le\left(\iint_{\left[0,\frac{1}{q_1}\right]\times\left[0,\frac{1}{q_2}\right]} |\overline{G(s,t)}F(s,t)|\,\mathrm{d}s\,\mathrm{d}t\right)\|k_n-k\|_\infty.\end{aligned}$$

Thus, we obtain: $\iint_{\left[0,\frac{1}{q_1}\right]\times\left[0,\frac{1}{q_2}\right]} \overline{G(s,t)}F(s,t)k(s,t)\,\mathrm{d}s\,\mathrm{d}t=0$.

Set

$$\Phi : f\mapsto \iint_{\left[0,\frac{1}{q_1}\right]\times\left[0,\frac{1}{q_2}\right]} f(s,t)\,\mathrm{d}\lambda(s,t)\in\mathcal{C}\left(\left[0,\frac{1}{q_1}\right]\times\left[0,\frac{1}{q_2}\right]\right)^*$$

where $\mathrm{d}\lambda(s,t)=\overline{G(s,t)}F(s,t)\,\mathrm{d}\mu(s,t)$ is an absolutely continuous measure.

The function Φ is null as an element of $\mathcal{C}\left(\left[0,\frac{1}{q_1}\right]\times\left[0,\frac{1}{q_2}\right]\right)^*$, so $\mathrm{d}\lambda=0$.

It follows that for $(s,t)\in\left[0,\frac{1}{q_1}\right]\times\left[0,\frac{1}{q_2}\right]$, $\overline{G(s,t)}F(s,t)=0$ a.e.

Since F is not equal to 0 almost everywhere, we have that $G=0$ a.e.

So, $\{g(\omega_1,\omega_2)F : g\in\mathbb{C}[X,Y]\}$ is dense in $L^2\left(\left[0,\frac{1}{q_1}\right]\times\left[0,\frac{1}{q_2}\right]\right)$. □

We use the above lemma to give a condition guaranteeing that a function is cyclic for T.

Lemma 4.8. *Let $h\in L^\infty([0,1]^2)$ be a $\left(\frac{1}{q_1},\frac{1}{q_2}\right)$-periodic function and $f\in L^2([0,1]^2)$. If:*

(i) $\Delta\left(f,\frac{1}{q_1},\frac{1}{q_2}\right)\neq 0$ *a.e., and*
(ii) *for all $\varepsilon>0$, there exists $g\in\mathbb{C}[X,Y]$ such that $\|g(\omega_1,\omega_2)f_k-hf_k\|_2<\varepsilon$,*

then f is a cyclic vector for T.

Proof. Suppose that $\varepsilon>0$ and $\ell\in L^2([0,1]^2)$. Let us write

$$\mathcal{F}=\bigcup_{n\ge1}\{h\in L^\infty([0,1]^2), h=0 \text{ a.e. on } \Omega_{n,f}\}.$$

Claim: $\mathcal{F}$ is dense in $L^2([0,1]^2)$.

Indeed, suppose that $g\in L^2([0,1]^2)$. Since $L^\infty([0,1]^2)$ is dense in $L^2([0,1]^2)$, there exists $k\in L^\infty([0,1]^2)$ such that $\|k-g\|_2<\frac{\varepsilon}{2}$. For all $n\ge1$, we have

$k\chi_{[0,1]^2\setminus\Omega_{n,f}} \in \mathcal{F}$ and

$$\begin{aligned}\|k\chi_{[0,1]^2\setminus\Omega_{n,f}} - g\|_2 &\leq \|k-g\|_2 + \|k\chi_{[0,1]^2\setminus\Omega_{n,f}} - k\|_2 \\ &\leq \|k-g\|_2 + \|k\|_2 \mu(\Omega_{n,f})^{\frac{1}{2}} \\ &\leq \|k-g\|_2 + \|k\|_\infty \mu(\Omega_{n,f})^{\frac{1}{2}}.\end{aligned}$$

Since $\Delta\left(f, \dfrac{1}{q_1}, \dfrac{1}{q_2}\right) \neq 0$ a.e., then, there exists $n_0 \in \mathbb{N}^*$ such that

$$\mu\left(\left\{(s,t) \in [0,1]^2 : \Delta\left(f, \frac{1}{q_1}, \frac{1}{q_2}\right)(s,t) < \frac{1}{n_0}\right\}\right) < \left(\frac{\varepsilon}{2\|k\|_\infty}\right)^2,$$

and therefore

$$\|k\chi_{[0,1]^2\setminus\Omega_{n,f}} - g\|_2 < \varepsilon.$$

This completes the proof of the claim.
Hence there exists $h \in \mathcal{F}$ such that $\|\ell - h\|_2 < \frac{\varepsilon}{2}$. Since h is an element of $\mathcal{F}$, by Lemma 4.6, $h = \sum_{j=1}^{q} h_k f_k$, where the functions h_j are in $L^\infty([0,1]^2)$ and are $\left(\frac{1}{q_1}, \frac{1}{q_2}\right)$ periodic.
By hypothesis, for all $j \in \{1, \ldots, q\}$, there exists $g_j \in \mathbb{C}[X,Y]$ such that

$$\|g_j(\omega_1,\omega_2)f_j - h_j f_j\|_2 < \frac{\varepsilon}{2q},$$

therefore,

$$\begin{aligned}\left\|\sum_{j=1}^{q} g_j(\omega_1,\omega_2)f_j - h_j f_j\right\|_2 &\leq \sum_{j=1}^{q} ||g_j(\omega_1,\omega_2)f_j - h_j f_j||_2 \\ &\leq \frac{\varepsilon}{2q} q = \varepsilon/2.\end{aligned}$$

By Proposition 4.3, $T^{nq+r}f(x_1,x_2) = \omega_1^{nl_1}(x_1)\omega_2^{nl_2}(x_2)f_{r+1}(x_1,x_2)$; so, for all $P = \sum_i a_i X^i \in \mathbb{C}[X]$,

$$\begin{aligned}P(T)(f) = \sum_i a_i T^i f &= \sum_i a_i \omega_1^{n_i l_1}\omega_2^{n_i l_2} f_{r_i+1}, \;\; i = n_i q + r_i \\ &= \sum_{j=1}^{q} Q_j(\omega_1,\omega_2)f_j, \qquad \text{with} \quad Q_j \in \mathbb{C}[X,Y].\end{aligned}$$

The above equality is deduced by collecting together the functions f_r and noticing that r depends only on the remainder of the division of i by q. So, if $G(T)f =$

$\sum_{j=1}^{q} g_j(\omega_1, \omega_2) f_j$, we have:

$$\begin{aligned} \|G(T)f - \ell\|_2 &\leq \|G(T)f - h\|_2 + \|h - \ell\|_2 \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

which implies that f is a cyclic vector for T. □

We are now ready for the main theorem of this section.

Theorem 4.9. *Suppose that $T = T_{(\alpha_1, \alpha_2)}$ with, for $i \in \{1, 2\}$,*

$$\alpha_i = \frac{p_i}{q_i}, \quad \mathrm{GCD}(p_i, q_i) = 1, \quad f \in L^2([0,1]^2).$$

Then,

$$f \text{ is a cyclic vector for } T \Longleftrightarrow \Delta\left(f, \tfrac{1}{q_1}, \tfrac{1}{q_2}\right) \neq 0 \text{ a.e. on } [0,1]^2.$$

Proof. Suppose that $\Delta\left(f, \frac{1}{q_1}, \frac{1}{q_2}\right) \neq 0$ a.e. on $[0,1]^2$.
Let $k \in \{1, \ldots, q\}$. Set:

$$F : (s,t) \longmapsto \sum_{i=0}^{q_1-1} \sum_{j=0}^{q_2-1} \left| f_k\left(\left\{s + \frac{i}{q_1}\right\}, \left\{t + \frac{j}{q_1}\right\}\right)\right| \in L^2\left(\left[0, \frac{1}{q_1}\right) \times \left[0, \frac{1}{q_2}\right)\right).$$

By Lemma 4.7, there exists $g \in \mathbb{C}[X, Y]$ such that

$$\|g(\omega_1, \omega_2)F - hF\|_{L^2\left(\left[0, \frac{1}{q_1}\right) \times \left[0, \frac{1}{q_2}\right)\right)} < \varepsilon$$

$(s,t) \longmapsto |(g(\omega_1, \omega_2)(s,t) - h(s,t)) f_k(s,t)|^2$ is an integrable function for the product measure. Using Fubini's theorem, we get

$$\begin{aligned} &\iint_{[0,1]^2} |(g(\omega_1, \omega_2)(s,t) - h(s,t)) f_k(s,t)|^2 \mathrm{d}s\, \mathrm{d}t \\ &= \sum_{\substack{0 \leq i \leq q_1 - 1 \\ 0 \leq j \leq q_2 - 1}} \int_0^{\frac{1}{q_1}} \int_0^{\frac{1}{q_2}} \left| \left[g(\omega_1, \omega_2)(s,t) - h\left(\left\{s + \frac{i}{q_1}\right\}, \left\{t + \frac{j}{q_2}\right\}\right)\right] \right. \\ &\qquad\qquad \left. \times f_k\left(\left\{s + \frac{i}{q_1}\right\}, \left\{t + \frac{j}{q_2}\right\}\right)\right|^2 \mathrm{d}s\, \mathrm{d}t \\ &=: \mathcal{I}, \quad \text{say.} \end{aligned}$$

Note that

$$\begin{aligned} |F(s,t)|^2 &= \left(\sum_{i=0}^{q_1-1} \sum_{j=0}^{q_2-1} \left| f_k\left(\left\{s + \frac{i}{q_1}\right\}, \left\{t + \frac{j}{q_2}\right\}\right)\right| \right)^2 \\ &\geq \sum_{i=0}^{q_1-1} \sum_{j=0}^{q_2-1} \left| f_k\left(\left\{s + \frac{i}{q_1}\right\}, \left\{t + \frac{j}{q_2}\right\}\right)\right|^2 \end{aligned}$$

and therefore

$$\begin{aligned}
\mathcal{I} &= \int_0^{\frac{1}{q_1}}\int_0^{\frac{1}{q_2}} \left| g(\omega_1,\omega_2)(s,t) - h\left(\left\{s+\frac{i}{q_1}\right\},\left\{t+\frac{j}{q_2}\right\}\right)\right|^2 \\
&\quad \times \sum_{i=0}^{q_1-1}\sum_{j=0}^{q_2-1}\left| f_k\left(\left\{s+\frac{i}{q_1}\right\},\left\{t+\frac{j}{q_2}\right\}\right)\right|^2 \mathrm{d}s\,\mathrm{d}t \\
&\leq \int_0^{\frac{1}{q_1}}\int_0^{\frac{1}{q_2}} \left| g(\omega_1,\omega_2)(s,t) - h\left(\left\{s+\frac{i}{q_1}\right\},\left\{t+\frac{j}{q_2}\right\}\right)\right|^2 \\
&\qquad \times \left(\sum_{i=0}^{q_1-1}\sum_{j=0}^{q_2-1}\left| f_k\left(\left\{s+\frac{i}{q_1}\right\},\left\{t+\frac{j}{q_2}\right\}\right)\right|\right)^2 \mathrm{d}s\,\mathrm{d}t \\
&= \int_0^{\frac{1}{q_1}}\int_0^{\frac{1}{q_2}} \left| g(\omega_1,\omega_2)(s,t) - h\left(\left\{s+\frac{i}{q_1}\right\},\left\{t+\frac{j}{q_2}\right\}\right)\right|^2 |F(s,t)|^2\,\mathrm{d}s\,\mathrm{d}t,
\end{aligned}$$

implying that

$$\begin{aligned}
\|g(\omega_1,\omega_2)f_k - hf_k\|_2 &< \|g(\omega_1,\omega_2)F - hF\|_{L^2([0,\frac{1}{q_1})[\times[0,\frac{1}{q_2}))} \\
&< \varepsilon.
\end{aligned}$$

Lemma 4.8 implies that f is a cyclic vector for T.

- Now suppose that $\Delta\left(f,\frac{1}{q_1},\frac{1}{q_2}\right) = 0$ a.e. on a set $S\subset[0,1]^2$ of measure $\mu(S)>0$.

Then, the row vectors $(L_1,\ldots,L_q)$ of $\Delta\left(f,\frac{1}{q_1},\frac{1}{q_2}\right)$ are linearly dependent on S. So, there exists a function $\lambda=(\lambda_0,\ldots,\lambda_{q-1}):[0,1]^2\to\mathbb{C}^q\backslash\{0\}$ such that

$$\lambda_0(s,t)L_1(s,t)+\ldots+\lambda_{q-1}(s,t)L_q(s,t)=0 \quad \text{a.e. on } S.$$

Therefore, for all $k\in\{1,\ldots,q\}$,

$$\lambda_0(s,t)f_k(s,t)+\cdots+\lambda_{q-1}(s,t)f_k\left(\left\{s+\frac{q-1}{q_1}\right\},\left\{t+\frac{q-1}{q_2}\right\}\right)=0 \quad \text{a.e. on } S.$$

Set $\Phi=T^lf\in\{T^nf : n\in\mathbb{N}\}$. We can write $\Phi=T^{nq+r}=\omega_1^{nl_1}\omega_2^{nl_2}f_{r+1}$, where $l=nq+r$ and

$$\begin{aligned}
\sum_{j=0}^{q-1}\lambda_j\Phi\left(\left\{s+\frac{j}{q_1}\right\},\left\{t+\frac{j}{q_2}\right\}\right) &= \sum_{j=0}^{q-1}\lambda_j\omega_1^{nl_1}\omega_2^{nl_2}f_{r+1}\left(\left\{s+\frac{j}{q_1}\right\},\left\{t+\frac{j}{q_2}\right\}\right) \\
&= 0 \quad \text{a.e. on } S.
\end{aligned}$$

Therefore, for all $P\in\mathbb{C}[X], P(T)f=\sum_i b_iT^if$, we have

$$\sum_{j=0}^{q-1}\lambda_j(s,t)T^if\left(\left\{s+\frac{j}{q_1}\right\},\left\{t+\frac{j}{q_2}\right\}\right)=0 \quad \text{a.e. on } S.$$

Thus,

$$\begin{aligned}\sum_i b_i \sum_{j=0}^{q-1} \lambda_j(s,t) T^i f \left(\left\{ s + \frac{j}{q_1} \right\}, \left\{ t + \frac{j}{q_2} \right\} \right) \\ &= \sum_{j=0}^{q-1} \lambda_j(s,t) \sum_i b_i T^i f \left(\left\{ s + \frac{j}{q_1} \right\}, \left\{ t + \frac{j}{q_2} \right\} \right) \\ &= \sum_{j=0}^{q-1} \lambda_j(s,t) P(T) f \left(\left\{ s + \frac{j}{q_1} \right\}, \left\{ t + \frac{j}{q_2} \right\} \right) \\ &= 0 \quad \text{a.e. on } S,\end{aligned}$$

and we obtain that for all $\Phi \in \overline{\{P(T)f : P \in \mathbb{C}[X]\}}$, we also have

$$\sum_{j=0}^{q-1} \lambda_j(s,t) \Phi \left(\left\{ s + \frac{j}{q_1} \right\}, \left\{ t + \frac{j}{q_2} \right\} \right) = 0 \quad \text{a.e.} \quad (s,t) \in S. \tag{7}$$

Since $\mu(S) > 0$, there exist $i, j \in \{0, \dots, q-1\}$ such that

$$\mu \left(S \cap \left(\frac{i}{q_1}, \frac{i+1}{q_1} \right) \times \left(\frac{j}{q_2}, \frac{j+1}{q_2} \right) \right) \neq 0.$$

But, $\Delta\left(f, \frac{1}{q_1}, \frac{1}{q_2}\right) = 0$ a.e. on S and $\Delta\left(f, \frac{1}{q_1}, \frac{1}{q_2}\right)$ is a $\left(\frac{1}{q_1}, \frac{1}{q_2}\right)$-periodic function, so necessarily S is a $\left(\frac{1}{q_1}, \frac{1}{q_2}\right)$-periodic set.

So, for all $i, j \in \{0, \dots, q-1\}$,

$$\mu \left(S \cap \left(\frac{i}{q_1}, \frac{i+1}{q_1} \right) \times \left(\frac{j}{q_2}, \frac{j+1}{q_2} \right) \right) \neq 0.$$

If f were cyclic, then, we would have $L^2([0,1]^2) = \overline{\{P(T)f : P \in \mathbb{C}[X]\}}$, and for all $\Phi \in L^2([0,1]^2)$, the relation (7) would hold.
Let $i \in \{1, \dots, q\}$. With the function $\Phi = \chi_{S \cap \left(\frac{i}{q_1}, \frac{i+1}{q_1}\right) \times \left(\frac{i}{q_2}, \frac{i+1}{q_2}\right)}$, if $(s,t) \in S$, we have

$$\Phi(\{s + \tfrac{k}{q_1}\}, \{t + \tfrac{k}{q_2}\}) = \begin{cases} 0 & \text{if } k \neq i, \\ 1 & \text{if } k = i. \end{cases}$$

Using (7), we have:

$$\lambda_i(s,t) \Phi \left(\left\{ s + \frac{i}{q_1} \right\}, \left\{ t + \frac{i}{q_2} \right\} \right) = 0 \text{ a.e. } (s,t) \in S \cap \left(\frac{i}{q_1}, \frac{i+1}{q_1} \right) \times \left(\frac{i}{q_2}, \frac{i+1}{q_2} \right).$$

So, for all $i \in \{1, \dots, q\}$, $\lambda_i(s,t) = 0$ a.e. on S, which is impossible; thus, f is not a cyclic vector for T. $\square$

Example 4.10. Note that the determinant $\Delta\left(f, \frac{1}{q_1}, \frac{1}{q_2}\right)$, defined in Definition 4.5, is a function of s and t that remains unchanged if we either translate s by $1/q_1$

or translate t by $1/q_2$ (modulo 1). Thus checking its vanishing on a set of positive measure can be reduced to checking where it vanishes on $[0,1/q_1] \times [0,1/q_2]$.

The simplest way to produce non-cyclic vectors (and hence proper invariant subspaces) is to stipulate that f vanishes on a non-null subset of $[0,1/q_1]\times[0,1/q_2]$, together with the translates of that set, but there are clearly other more complicated possibilities, found by solving linear relations between the columns of the matrix defining Δ.

Further, the vanishing of the determinant implies the existence of linear relationships between f and its translates over the ring of polynomial functions (at least on a set of positive measure); this gives a way to produce cyclic vectors: for a function such as $s^{1/2} + t^{1/2}$ is necessarily cyclic, as no such relations can exist. Issues such as the density of cyclic vectors can be analysed similarly.

4.1.2. The case $q_1 = q_2$. For completeness, we mention briefly the case of $q_1 = q_2$. With the same notation, one can define the determinant of f associated with $(\frac{p_1}{q}, \frac{p_2}{q})$, denoted by $\Delta\left(f, \frac{1}{q}, \frac{1}{q}\right) \in L^{\frac{2}{q}}([0, \frac{1}{q}]^2)$, which is a $(\frac{1}{q}, \frac{1}{q})$-periodic function.

With the same hypothesis as Lemma 4.6, if $h \in L^\infty([0,1]^2)$ equal to 0 on $\Omega_{n,f}$, there exist periodic functions $h_1, \dots, h_q \in L^\infty([0,1]^2)$, $(\frac{1}{q}, \frac{1}{q})$ such that $h = \sum_{k=1}^{q} h_k f_k$. The following lemma gives us a set dense in $L^2([0, \frac{1}{q}]^2)$ different from that given in Lemma 4.7:

Lemma 4.11. *Let F be a non-trivial function in $L^2([0, \frac{1}{q}]^2)$. Then,*

$$\{g(\omega)F : g \in \mathbb{C}[X]\} \text{ is dense in } L^2\left(\left[0, \tfrac{1}{q}\right]^2\right).$$

Therefore, as in the case $q_1 \neq q_2$, a sufficient condition for cyclicity is given by:

Lemma 4.12. *Let $h \in L^\infty([0,1]^2)$ be a $\left(\frac{1}{q}, \frac{1}{q}\right)$ periodic function and $f \in L^2([0,1]^2)$. If:*

(i) $\Delta\left(f, \frac{1}{q}, \frac{1}{q}\right) \neq 0$ *a.e., and*

(ii) *for all $\varepsilon > 0$, there exists $g \in \mathbb{C}[X]$, such that $\|g(\omega)f_k - hf_k\|_2 < \varepsilon$,*

then, f is a cyclic vector for T.

This implies the following result.

Theorem 4.13. *Suppose that $T = T_{(\alpha_1,\alpha_2)}$ with, for $i \in \{1,2\}$, $\alpha_i = \frac{p_i}{q}$, $\mathrm{GCD}(p_i, q) = 1$, $f \in L^2([0,1]^2)$. Then:*

$$f \text{ is a cyclic vector for } T \iff \Delta\left(f, \tfrac{1}{q}, \tfrac{1}{q}\right) \neq 0 \text{ a.e. on } [0,1]^2.$$

4.2. Cyclic vectors for T_α, $\alpha \in \mathbb{Q}^d$, $d \geq 1$

The general case (including the simpler case $d = 1$) can be treated by similar methods.

The operator T_α is defined by

$$\begin{array}{rcll} T_\alpha : L^2([0,1]^d) & \to & L^2([0,1]^d) \\ f & \mapsto & T_\alpha f : x \in [0,1]^d \mapsto x_1 x_2 \ldots x_d f(\{x_1+\alpha_1\},\ldots,\{x_d+\alpha_d\}). \end{array}$$

Set $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ with $\alpha_i = \dfrac{p_i}{q_i}$, $i \in \{1,\ldots,d\}$. Suppose that $\mathrm{GCD}(p_i, q_i) = 1, i \in \{1,\ldots,d\}$. We write $T_\alpha = T$, $q = \mathrm{LCM}(q_i, i \in \{1,\ldots,d\})$ and $f_{r+1} = T^r f$ for $r \in \{0,\ldots,q-1\}$.

Definition 4.14. Let $f \in L^2([0,1]^d)$. The determinant of f associated with $(\alpha_1, \ldots, \alpha_d)$ is the determinant

$$\Delta\left(f, \frac{1}{q_1}, \ldots, \frac{1}{q_d}\right)$$

$$= \begin{vmatrix} f_1(x_1,\ldots,x_d) & \cdots & f_q(x_1,\ldots,x_d) \\ f_1\left(\left\{x_1+\frac{1}{q_1}\right\},\ldots,\left\{x_d+\frac{1}{q_d}\right\}\right) & \cdots & f_q\left(\left\{x_1+\frac{1}{q_1}\right\},\ldots,\left\{x_d+\frac{1}{q_d}\right\}\right) \\ \vdots & \ddots & \vdots \\ f_1\left(\left\{x_1+\frac{q-1}{q_1}\right\},\ldots,\left\{x_d+\frac{q-1}{q_d}\right\}\right) & \cdots & f_q\left(\left\{x_1+\frac{q-1}{q_1}\right\},\ldots,\left\{x_d+\frac{q-1}{q_d}\right\}\right) \end{vmatrix}.$$

With similar arguments, one can prove the following result:

Theorem 4.15. *Set $f \in L^2([0,1]^d)$. Suppose that $T = T_{(\alpha_1,\ldots,\alpha_d)}$ with $\alpha_i = \frac{p_i}{q_i}$ and $\mathrm{GCD}(p_i, q_i) = 1$ for $i \in \{1,\ldots,d\}$. Then:*

$$f \textit{ is a cyclic vector for } T \Longleftrightarrow \Delta\left(f, \tfrac{1}{q_1}, \ldots, \tfrac{1}{q_d}\right) \neq 0 \textit{ a.e. on } [0,1]^d.$$

Acknowledgements

The first and second authors are grateful for financial support from the EPSRC. The referee is thanked for many detailed and helpful suggestions.

References

[1] G. Androulakis and A. Flattot. Hyperinvariant subspace for weighted composition operator on $L^p([0,1]^d)$. *J. Operator Theory*, to appear.

[2] D.P. Blecher and A.M. Davie. Invariant subspaces for an operator on $L^2(\Pi)$ composed of a multiplication and a translation. *J. Operator Theory*, 23 (1990), no. 1, 115–123.

[3] I. Chalendar, A. Flattot, and N. Guillotin-Plantard. On the spectrum of multivariable weighted composition operators. *Arch. Math.* (*Basel*), 90 (2008), no. 4, 353–359.

[4] I. Chalendar and J.R. Partington. The cyclic vectors and invariant subspaces of rational Bishop operators. Research report, 2008.

[5] I. Chalendar and J.R. Partington. Invariant subspaces for products of Bishop operators. *Acta Sci. Math.* (*Szeged*), 74 (2008), 717–725.

[6] C.C. Cowen and B.D. MacCluer, *Composition operators on spaces of analytic functions.* Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[7] A.M. Davie. Invariant subspaces for Bishop's operators. *Bull. London Math. Soc.*, 6 (1974), 343–348.

[8] A.V. Lipin, Spectral multiplicity of the solutions of polynomial operator equations. *J. Soviet Math.* 44 (1989), no. 6, 856–861

[9] G.W. MacDonald. Invariant subspaces for Bishop-type operators. *J. Funct. Anal.*, 91 (1990), no. 2, 287–311.

[10] G.W. MacDonald, Invariant subspaces for multivariate Bishop-type operators. *J. Operator Theory* 25 (1991), no. 2, 347–366.

[11] J.H. Shapiro, *Composition operators and classical function theory.* Universitext: Tracts in Mathematics. Springer-Verlag, New York, 1993.

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Received: December 9, 2008
Accepted: May 22, 2009

Operator Theory:
Advances and Applications, Vol. 202, 87–109

Factorization Algorithm for Some Special Non-rational Matrix Functions

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Abstract. We construct an algorithm that allows us to determine an effective generalized factorization of a special class of matrix functions. We use the same algorithm to analyze the spectrum of a self-adjoint operator which is related to the obtained generalized factorization.

Mathematics Subject Classification (2000). Primary 47A68; Secondary 47A10.

Keywords. Generalized factorization, inner function, algorithm.

1. Introduction

Let $\mathbb{T}$ denote the unit circle and consider the space $L_2(\mathbb{T})$. As usual,

$$L_2^+(\mathbb{T}) = \mathrm{Im}P_+,\ L_2^{-,0}(\mathbb{T}) = \mathrm{Im}P_-,\ L_2^-(\mathbb{T}) = L_2^{-,0}(\mathbb{T}) \oplus \mathbb{C},$$

where $P_\pm = (I \pm S)/2$ denote the projection operators associated with the singular integral operator S, with Cauchy kernel,

$$S\varphi(t) = \frac{1}{\pi i} \int_{\mathbb{T}} \frac{\varphi(\tau)}{\tau - t}\, d\tau,\ t \in \mathbb{T},$$

and I represents the identity operator.

We say that a matrix-valued function A, such that $A^{\pm 1} \in [L_\infty(\mathbb{T})]_{n,\,n}$, admits a left (right) generalized factorization in $L_2(\mathbb{T})$ if it can be represented as

$$A = A_+ \Lambda A_- \quad (A_- \Lambda A_+),$$

where

$$A_+^{\pm 1} \in [L_2^+(\mathbb{T})]_{n,\,n},\quad A_-^{\pm 1} \in [L_2^-(\mathbb{T})]_{n,\,n},\quad \Lambda = \mathrm{diag}\{t^{\kappa_1}, \ldots, t^{\kappa_n}\},$$

$\kappa_1 \geq \cdots \geq \kappa_n$ are integers, and $A_+ P_+ A_+^{-1} I$ $(A_- P_+ A_-^{-1} I)$ represents a bounded linear operator in $[L_2(\mathbb{T})]_n$.

This research was partially supported by CEAF at Instituto Superior Técnico (Portugal).
Communicated by I.M. Spitkovsky.

If $\kappa_1 = \cdots = \kappa_n = 0$, then A is said to admit a left (right) canonical generalized factorization.

The explicit factorization of matrix-valued functions has applications in different areas, such as the theory of singular integral operators, boundary value problems, scattering theory, the theory of linear and non-linear differential equations (see, for instance, [1, 4, 15, 16]). It is well known that there exist algorithms to determine explicit factorizations for rational matrix functions (see, for instance, [2], [3], and [18]). However, algorithms for obtaining explicit factorizations of non-rational matrix functions exist only for some restricted classes of matrix functions (see, for instance, [6, 9, 10]).

In the following sections we shall be dealing with the class of matrix functions

$$A_\gamma(b) = \begin{pmatrix} e & b \\ b^* & b^*b + \gamma\, e \end{pmatrix} \tag{1.1}$$

where e represents the identity matrix function of order n, b is a matrix function whose entries are essentially bounded functions on the unit circle, b^* is the Hermitian adjoint of b and γ is a non-zero complex constant.

The main objective of this work is the construction of an algorithm for obtaining explicit factorizations for matrix functions of that class. Strong relations between a factorization of (1.1) and the operators

$$N_+(b) = P_+bP_-b^*P_+ \quad \text{and} \quad N_-(b) = P_-b^*P_+bP_-\,, \tag{1.2}$$
$$N_\pm(b) : \, [L_2(\mathbb{T})]_{n,n} \to [L_2(\mathbb{T})]_{n,n}\,,$$

are analyzed.

Some results related with $A_\gamma(b)$ can be seen in [5, 6, 7, 8, 12, 13, 14, 15, 16].

Matrix functions of type (1.1) appeared for the first time related with the generalized Riemann problem (see, for instance, [15, Chap. 4]), and now it is known that a factorization of $A_\gamma(b)$ can also be used in more general cases, as for example, in the generalized Riemann problem with shift (see [12]).

It was discovered, more than thirty years ago, that the factorization problem for matrix functions of type (1.1) is related with the study of singular operators that can be represented as a product of Hankel operators (see [14]).

The paper [13] relates a canonical factorization of a second-order matrix function $A_\gamma(b)$, when $\gamma > 0$, with the resolvent operator of an operator that can be represented through an Hankel operator with symbol b.

In general, it is possible to show, for second-order matrix functions (see [15, Section 15.7] and [16, p. 289]), that the study of the factorization of any Hermitian matrix functions G, with elements belonging to the class of all essentially bounded functions on the unit circle, $L_\infty(\mathbb{T})$, and with (at least) one of the diagonal entries preserving the sign almost everywhere on the unit circle, can be reduced to the study of $A_{-1}(b)$. It is proved in [15, pp. 157–158] that the matrix functions G and $A_{-1}(b)$ admit a generalized factorization in $L_2(\mathbb{T})$ only simultaneously and that their partial indices coincide. It is also proved that the matrix function $A_{-1}(b)$ admits a right generalized factorization in $L_2(\mathbb{T})$ if and only if the unity does not belong to the condensation spectrum (i.e., the set of the accumulation points of the spectrum and of the eigenvalues with infinite multiplicity – see, for instance, [11, p. 59]), σ_l, of the self-adjoint operator $N_-(b^*) = H_b H_b^*$ ($H_b = P_- b P_+$ is a Hankel operator with symbol b) and its partial indices are $\pm l$, where l is the multiplicity of 1 as an eigenvalue of $N_-(b^*)$.

Let us note that, in general, even if we know that

$$\dim\mathrm{Ker}\,(N_-(b^*) - I) < \infty,$$

we do not know if the unity belongs to the condensation spectrum, that is, if $A_{-1}(b)$ admits a generalized factorization.

In [7] we consider the class of matrix-valued functions (1.1). For these matrix-valued functions, when $-\gamma$ belongs to the resolvent set, ρ, of the self-adjoint positive operator $N_+(b) = H_{b^*}^* H_{b^*}$, we obtain that, it is possible to compute a canonical factorization (see Theorem 4.4 in [7]) when the entries of the matrix function b are in a certain decomposing algebra of continuous functions and satisfy some additional conditions. The method used therein was based on the construction of the resolvent of the operator $N_+(b)$.

In [8] we generalize our previous result, simplifying some of the conditions imposed before and obtaining a left canonical factorization of $A_\gamma(b)$ (when $-\gamma \in \rho(N_+(b))$ and b is a scalar function) through the use of the solutions of the non-homogeneous equations,

$$(N_+(b) + \gamma I)u_+ = 1 \tag{1.3}$$

and

$$(N_+(b) + \gamma I)v_+ = b. \tag{1.4}$$

In [5] we generalize our results when b is an essentially bounded function and we also describe a method to solve the equation (1.3) when $A_\gamma(b)$ admits a left canonical generalized factorization and b can be represented as an inner-outer factorization, $b = r\,\theta$, with a rational outer function r, that is, $b \in H_{r,\theta}$.

In [6] we consider second-order matrix functions (1.1) that admit a left non-canonical generalized factorization. In that paper we describe a method to obtain a generalized factorization using the solutions of two related non-homogeneous equations

$$(N_+(b) + \gamma I)x_+ = \gamma\, R_{1,\kappa} \quad \text{and} \quad (N_+(b) + \gamma I)y_+ = \gamma\, b\, R_{2,\kappa}, \tag{1.5}$$

with $R_{i,\kappa}$, $i = 1, 2$, polynomials of degree less than or equal to the dimension of the kernel of the operator $N_+(b) + \gamma I$.

Now we are able to construct an algorithm, [AFact], that allows us to know if a matrix function of the form (1.1), with $b \in H_{r,\theta}$, admits, or not, a generalized factorization. Moreover, if $A_\gamma(b)$ is factorable, the algorithm allows us to determine if the generalized factorization is canonical or non-canonical, and it gives us a left generalized factorization of the matrix function. For that, we construct an algorithm, [AEq], to solve equations of the form

$$(N_+(b) + \gamma I)\omega_+(t) = g_+(t) \tag{1.6}$$

when the function $b \in H_{r,\theta}$.

Using [AEq], we obtain the kernel of the operator $N_+(b) + \gamma I$, by solving the associated homogeneous equation.

Let κ be the dimension of that kernel. Obviously, there are two possibilities: $\kappa = 0$ or $\kappa > 0$.

For the case $\kappa = 0$, we have that $-\gamma \in \rho\,(N_+(b))$ and we use [AEq] to solve the equations (1.3) and (1.4). A left canonical generalized factorization of (1.1) is obtained.

For the case $\kappa > 0$ we have that $\gamma < 0$, and so we can use some results of [5] and the fact that

$$L_2(\mathbb{T}) = \mathrm{Im}\,(N_+(b) + \gamma\, I) \oplus \mathrm{Ker}\,(N_+(b) + \gamma\, I) \tag{1.7}$$

to find out if there exist, or not, two polynomials, $R_{1,\kappa}$ and $R_{2,\kappa}$, such that the non-homogeneous equations (1.5) are solvable. If at least, one of the two polynomials does not exist, then the matrix function (1.1) does not admit a generalized factorization and $-\gamma \in \sigma_l\,(N_+(b))$. Otherwise, we use [AEq] to solve the equations (1.5), and a left non-canonical generalized factorization of (1.1), with partial indices κ and $-\kappa$, is obtained.

For the construction of the algorithms [AEq] and [AFact] we need to use some already published results. So, in Section 2, besides the new results, we repeat the main results on relations between a generalized factorization of (1.1) and the spectrum of the operator $N_+(b)$, and we also generalize some of those results that appear in [5] and [6], for the case when b is a matrix function of order n.

In Section 3, we analyze the solubility of the equation (1.6). We also describe the main steps of the algorithms [AEq] and [AFact]. Using the linear system $S_{\gamma,0}$, that appears at the end of [AEq], we formulate the main results of this paper. In fact, we can relate the spectrum of the operator $N_+(b)$ with the linear system $S_{\gamma,0}$. We can see that, through the solutions of $S_{\gamma,0}$, it is possible to know if $-\gamma$ belongs to the spectrum of $N_+(b)$. And, in that case, using the $\mathrm{Ker}\,(N_+(b) + \gamma\, I)$ and (1.7), it is also possible to know if $-\gamma$ belongs to the condensation spectrum of $N_+(b)$.

Section 4 is dedicated to the description of the algorithms [AEq] and [AFact].

Finally, in the last section, some examples are given for the canonical and non-canonical generalized factorizations.

2. Relations between a generalized factorization of $A_\gamma(b)$ and the spectrum of $N_+(b)$

In this section we describe some strong relations between a generalized factorization of the matrix function (1.1) and the spectrum of the self-adjoint operators (1.2).

Let $\rho\,(N_+(b))$ denote the resolvent set of the operator $N_+(b)$ and $\sigma_T\,(N_+(b))$ its spectrum. Let us consider the set

$$\sigma\,(N_+(b)) = \sigma_T\,(N_+(b))\,\backslash\sigma_l\,(N_+(b))\,,$$

where $\sigma_l\,(N_+(b))$ represents the condensation spectrum of $N_+(b)$.

Using the fact that $A_\gamma(b)$ admits a left canonical generalized factorization in $L_2(\mathbb{T})$ if and only if the singular integral operator $P_+ + A_\gamma(b)P_-$ is an invertible operator in $[L_2(\mathbb{T})]_{n,n}$ and the fact that (see [5])

$$\rho\,(N_+(b)) = \rho\,(N_-(b))\,,$$

we obtain that (see Theorem 2.1 of [5])

Theorem 2.1. *The matrix function $A_\gamma(b)$ admits a left canonical generalized factorization in $L_2(\mathbb{T})$ if and only if $-\gamma \in \rho(N_+(b))$.*

Consequently, and since $N_+(b)$ is a positive operator we can conclude that

Corollary 2.2. *If $\gamma > 0$, or if $\gamma \in \mathbb{C}\backslash\mathbb{R}$, then $A_\gamma(b)$ admits a left canonical generalized factorization in $L_2(\mathbb{T})$.*

For the canonical case, we study the following Riemann boundary value problem

$$\begin{cases} \Phi_+ = A_\gamma(b)(E+\Phi_-) \\ \Phi_-(\infty) = 0 \end{cases}, \tag{2.1}$$

where $b \in [L_\infty(\mathbb{T})]_{n,n}$ and E is the identity matrix function of order $2\,n$.

The objective is to determine matrix functions, $\Phi_\pm \in [L_2^\pm(\mathbb{T})]_{2n,2n}$, solutions of the problem, and, using $\Phi_\pm$, to obtain a canonical generalized factorization of $A_\gamma(b)$. It is possible to show that $\Phi_\pm$ (when (2.1) is solvable) can be represented through the solutions of the non-homogeneous equations

$$(N_+(b)+\gamma I)u_+ = e$$

and

$$(N_+(b)+\gamma I)v_+ = P_+(b).$$

It is known that if $A_\gamma(b)$ admits a left canonical generalized factorization

$$A_\gamma(b) = A_\gamma^+ A_\gamma^-,$$

then the problem (2.1) has the unique solution

$$\Phi_+ = A_\gamma^+,\quad \Phi_- = (A_\gamma^-)^{-1} - E.$$

So, by solving the Riemann boundary value problem (2.1) and relating the existence of a left canonical generalized factorization of the matrix function (1.1) with the fact that $-\gamma$ belongs to the resolvent set of $N_+(b)$, we get the following result about an effective generalized factorization of (1.1).

Theorem 2.3. *If $-\gamma \in \rho(N_+(b))$, then the matrix function $A_\gamma(b)$ admits a left canonical generalized factorization*

$$A_\gamma(b) = A_\gamma^+ A_\gamma^-,$$

where

$$A_\gamma^+ = \gamma \begin{pmatrix} u_+ & v_+ \\ P_+(b^* u_+) & e + P_+(b^* v_+) \end{pmatrix}$$

and

$$A_\gamma^- = \begin{pmatrix} e + P_-[bP_-(b^* u_+)] & -P_- b + P_-[bP_-(b^* v_+)] \\ -P_-(b^* u_+) & e - P_-(b^* v_+) \end{pmatrix}^{-1},$$

with

$$(N_+(b) + \gamma I)\, u_+ = e \quad \text{and} \quad (N_+(b) + \gamma I)\, v_+ = P_+(b).$$

Remark 2.4. If the matrix function b can be represented as $b = b_+ + b_-$, where

$$b_+ \in \left[L_\infty(\mathbb{T}) \cap L_2^+(\mathbb{T})\right]_{n,n} \quad \text{and} \quad b_- \in \left[L_\infty(\mathbb{T}) \cap L_2^{-,0}(\mathbb{T})\right]_{n,n},$$

then

$$A_\gamma(b) = \begin{pmatrix} e & 0 \\ b_-^* & e \end{pmatrix} A_\gamma(b_+) \begin{pmatrix} e & b_- \\ 0 & e \end{pmatrix}.$$

So, we can assume, without any loss of generality, that b has an analytic continuation into the interior of the unit circle.

If $\gamma < 0$, we can always relate $A_\gamma(b_1)$ with $A_{-1}(b)$ through

$$A_\gamma(b_1) = \begin{pmatrix} e & 0 \\ 0 & \sqrt{-\gamma}\, e \end{pmatrix} A_{-1}(b) \begin{pmatrix} e & 0 \\ 0 & \sqrt{-\gamma}\, e \end{pmatrix},$$

where $b_1 = \sqrt{-\gamma}\; b$. So, using a reasoning similar to that used in [15, Chap. 4, Theorem 12] it can be proved that

Theorem 2.5. *The matrix function $A_\gamma(b)$ admits a left generalized factorization in $L_2(\mathbb{T})$ if and only if $-\gamma \notin \sigma_l(N_+(b))$.*

Also, if $A_\gamma(b)$ admits a left generalized factorization in $L_2(\mathbb{T})$, we prove that

Proposition 2.6. *If $\gamma < 0$, then*

$$\dim\mathrm{Ker}(N_+(b) + \gamma I) = \kappa, \tag{2.2}$$

where κ is the sum of the positive partial indices of a left generalized factorization of the matrix (1.1).

Proof. If $\gamma < 0$, then (1.1) is an Hermitian matrix function. In that case, a generalized factorization of $A_\gamma(b)$ has the partial indices

$$\{\kappa_1, \kappa_2, \ldots, \kappa_n, -\kappa_n, \ldots, -\kappa_2, -\kappa_1\}$$

(see [16], p. 258). Since

$$(P_+ + A_\gamma(b)P_-)\begin{pmatrix} I & -b\,P_- \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & \gamma\,P_+ + P_- - P_+b^*P_+bP_- \end{pmatrix}$$
$$= \begin{pmatrix} I & 0 \\ b^*P_- & N_-(b) + \gamma\,I \end{pmatrix},$$

then

$$\kappa_1 + \cdots + \kappa_n = \dim\mathrm{Ker}(N_-(b) + \gamma I).$$ □

So, using Theorems 2.1, 2.5 and Proposition 2.6 we get the following result for a non-canonical generalized factorization of (1.1).

Theorem 2.7. *The matrix function $A_\gamma(b)$ admits a left non-canonical generalized factorization*

$$A_\gamma(b) = A_\gamma^+\,\Lambda\,A_\gamma^- \tag{2.3}$$

if and only if $-\gamma \in \sigma(N_+(b))$. *And, in that case,*

$$\Lambda(t) = \mathrm{diag}\left\{t^{\kappa_1},\,\ldots,\,t^{\kappa_n},\,t^{-\kappa_n},\,\ldots,\,t^{-\kappa_1}\right\}$$

and

$$\kappa_1 + \cdots + \kappa_n = \dim\mathrm{Ker}\,(N_+(b) + \gamma\,I)\,.$$

We consider now the case when $-\gamma \in \sigma(N_+(b))$.

To obtain a left non-canonical generalized factorization of (1.1) we can not consider the Riemann boundary value problem (2.1) because

Proposition 2.8. *If* $-\gamma \in \sigma(N_+(b))$, *then the problem* (2.1) *is not solvable.*

Proof. Since $A_\gamma(b)$ admits a left non-canonical generalized factorization (2.3), we used the fact (see, for instance, [16, Chap. 3, Corollary 3.1]) that for the solvability of the problem (2.1) it is necessary, for all $j = 1,\,\ldots,\,n$ with $\kappa_j < 0$, that

$$\int_{\mathbb{T}} \left(\Lambda(t)A_\gamma^-(t)\right)_j\, t^k dt = 0, \quad k = 0, \ldots, -\kappa_j - 1.$$

□

So, we need to find another Riemann boundary value problem that allows us to obtain a left non-canonical generalized factorization of the matrix function (1.1).

Using Theorem 3.2 of [16, p. 87], we obtain the following result.

Proposition 2.9. *Let $A_\gamma(b)$ admit a left non-canonical generalized factorization* (2.3). *Then there exist two unique matrix functions $R_{1,\kappa}$ and $R_{2,\kappa}$, whose entries are polynomials of degree $\leq \kappa$ such that the Riemann boundary value problem*

$$\begin{cases} \Phi_+ = A_\gamma(b)\left(\Phi_- + \begin{pmatrix} R_{1,\kappa} & 0 \\ 0 & R_{2,\kappa} \end{pmatrix}\right) \\ \Phi_-(\infty) = 0 \end{cases}, \tag{2.4}$$

is solvable.

Using a similar method to the one described in [7], we obtain the solutions of the problem (2.4) (see Theorems 3.6, 3.7, and 3.8 of [6] for the case when b is a scalar function) through the solutions of the non-homogeneous equations (1.5).

Theorem 2.10. *If the problem* (2.4) *is solvable, then the equations* (1.5) *are solvable. In that case, considering the solutions of the equations, ϕ_1^+ and ϕ_2^+, respectively, we have that*

$$\Phi_+ = \begin{pmatrix} \phi_1^+ & \phi_2^+ \\ P_+(b^*\,\phi_1^+) & \gamma R_{2,\kappa} + P_+(b^*\,\phi_2^+) \end{pmatrix}, \tag{2.5}$$

$$\Phi_- = \frac{1}{\gamma} \begin{pmatrix} P_-(bP_-(b^*\,\phi_1^+)) & -P_-b + P_-(bP_-(b^*\,\phi_2^+)) \\ -P_-(b^*\,\phi_1^+) & -P_-(b^*\,\phi_2^+) \end{pmatrix}. \tag{2.6}$$

Although it is possible to prove that, when $-\gamma \in \sigma\left(N_+(b)\right)$, there is a Riemann boundary value problem (2.4) associated to a non-canonical generalized factorization of (1.1), it is not easy to determine the matrix functions $R_{i,\kappa}$, $i = \overline{1,2}$, due to the fact that the matrix function $A_\gamma^-(\infty)$ may assume a lot of different forms. Besides that, since

$$\Phi_- + \begin{pmatrix} R_{1,\kappa} & 0 \\ 0 & R_{2,\kappa} \end{pmatrix} \notin \left[L_2^-(\mathbb{T})\right]_{2n,\,2n}$$

we have to multiply the matrix function Φ_+ by a matrix function G such that

$$A_\gamma^+ = \Phi_+\, G \quad \text{and} \quad A_\gamma^- = \Lambda^{-1}\left(A_\gamma^+\right)^{-1} A_\gamma(b)$$

are the factors of a generalized factorization (2.3) of (1.1). To find the matrix function G we have to consider all the partial indices of the left generalized factorization of (1.1). So, since we do not know how to determine them if b is not a scalar function, we can not obtain yet a left non-canonical generalized factorization of (1.1), for the general case.

For the case when b is a scalar function, we know that the left generalized factorization of (1.1) depends on the behavior of the matrix function $A_\gamma^-(\infty)$, and we have three different cases

(**Case 1**)

$$A_\gamma^-(\infty) = \begin{pmatrix} a_-(\infty) & b_-(\infty) \\ & \\ 0 & d_-(\infty) \end{pmatrix},\ a_-(\infty) \neq 0,\ d_-(\infty) \neq 0,\ b_-(\infty) \text{ arbitrary},$$

(**Case 2**)

$$A_\gamma^-(\infty) = \begin{pmatrix} a_-(\infty) & b_-(\infty) \\ & \\ c_-(\infty) & 0 \end{pmatrix},\ b_-(\infty) \neq 0,\ c_-(\infty) \neq 0,\ a_-(\infty) \text{ arbitrary},$$

(**Case 3**)

$$A_\gamma^-(\infty) = \begin{pmatrix} a_-(\infty) & b_-(\infty) \\ c_-(\infty) & d_-(\infty) \end{pmatrix}, \; c_-(\infty) \neq 0, \quad d_-(\infty) \neq 0,$$

where, in (**Case 3**), $a_-(\infty)$ and $b_-(\infty)$ are not simultaneously equal to zero. However, we have more information on the polynomials $R_{1,\kappa}$ and $R_{2,\kappa}$ (see Propositions 3.2, 3.3 and 3.4 of [6]) and so we can obtain an explicit left generalized non-canonical factorization of the matrix function (1.1), through the solutions of the problem (2.4). We use Theorem 3.2 of [16, p. 87], which describes how to obtain the general solution of a problem of the form

$$\Phi_+ = A_\gamma(b)\Phi_- + g,$$

through the factors of a factorization of the matrix function $A_\gamma(b)$ (see Theorems 3.6, 3.7 and 3.8 of [6]), to obtain the next result (where G depends on the case of $A_\gamma^-(\infty)$).

Let us consider Φ_+ and Φ_- as in (2.5) and (2.6).

Theorem 2.11. *If $-\gamma \in \sigma(N_+(b))$, then the matrix function $A_\gamma(b)$ admits the left non-canonical generalized factorization* (2.3)*, where*

$$A_\gamma^+ = \Phi_+ G, \; \Lambda(t) = \operatorname{diag}\left\{t^\kappa, t^{-\kappa}\right\}, \; A_\gamma^- = \Lambda^{-1}\left(A_\gamma^+\right)^{-1} A_\gamma(b), \tag{2.7}$$

and

$$G = \underbrace{\begin{pmatrix} \frac{1}{\Delta} & 0 \\ 0 & 1 \end{pmatrix}}_{\text{case 1}} \quad or \quad G = \underbrace{\begin{pmatrix} 0 & 1 \\ -\frac{1}{\Delta} & 0 \end{pmatrix}}_{\text{case 2}} \quad or \quad G = \underbrace{\begin{pmatrix} \frac{1}{\Delta} & 0 \\ \frac{\rho}{\Delta} & 1 \end{pmatrix}}_{\text{case 3}}, \tag{2.8}$$

with

$$\Delta = \frac{1}{\gamma} \det \Phi_+ \quad and \quad \rho = -\frac{c_-(\infty)}{d_-(\infty)}.$$

So, to obtain an explicit left generalized non-canonical factorization of (1.1), when it exists, we need to find the polynomials $R_{1,\kappa}(t)$ and $R_{2,\kappa}(t)$ that make the problem (2.4) solvable. For that we use (1.7) and (2.2).

For the case 1:

$$R_{1,\kappa}(t) \equiv 1 \quad \text{and} \quad R_{2,\kappa}(t) = t^\kappa + s_{2,\kappa-1}(t), \tag{2.9}$$

where $s_{2,\kappa-1}$ is a polynomial with degree $\kappa - 1$.

For the case 2:

$$R_{1,\kappa}(t) = t^\kappa + s_{1,\kappa-1}(t) \quad \text{and} \quad R_{2,\kappa}(t) \equiv 1, \tag{2.10}$$

where $s_{1,\kappa-1}$ is a polynomial with degree $\kappa - 1$.

For the case 3:

$$R_{i,\kappa}(t) = t^\kappa + s_{i,\kappa-1}(t), \; i = \overline{1, 2}, \tag{2.11}$$

where $s_{i,\kappa-1}$ is a polynomial with degree $\kappa - 1$.

Now, we need to solve the non-homogeneous equations (1.5). In order to solve this kind of equations, we generalized the method described in [5]. With that generalization we can solve equations of the type (1.6), when the function b can be represented as the product of an inner function θ and a rational outer function.

3. Relations between the spectrum of the operator $N_+(b)$ and a linear system

Let $H_{r,\theta}$ denote the set of all the functions of H_∞ (the class of all bounded and analytic functions in the interior of the unit circle) that can be represented as the product of a rational outer function r and an inner function θ (i.e., θ is a bounded analytic function on the interior of the unit circle such that its modulus is equal to one a.e. on $\mathbb{T}$).

In this section we describe the main results that we need in order to see if (1.1) admits a left generalized factorization and also to obtain a generalized factorization (when it exists) of the matrix function (1.1).

In particular, we describe how we construct the algorithm [AEq] for solving (solvable) equations of the form

$$(N_+(b) + \gamma\, I)\ \omega_{+,g_+}(t) = g_+(t), \tag{3.1}$$

when $b \in H_{r,\theta}$.

It is shown that we can get the solution(s) of equations of the type (3.1), and, consequently, a generalized factorization of a factorable matrix function (1.1), by solving a linear system.

Let us start with the solvability of the equation. Note that if $g_+(t)$ is the null function, then the algorithm [AEq] gives us the kernel of the operator $N_+(b) + \gamma\, I$.

If $-\gamma \in \rho\,(N_+(b))$, then the equation is uniquely solvable,

$$\omega_{+,g_+}(t) = (N_+(b) + \gamma I)^{-1}\, g_+(t).$$

If $-\gamma \in \sigma\,(N_+(b))$, then the equation can or cannot be solvable. Since

$$(N_+(b) + \gamma I)\ \omega_{+,g_+}(t) = g_+(t) \quad \text{is solvable}$$

if and only if

$$g_+(t) \in \operatorname{Im}\,(N_+(b) + \gamma I)\,,$$

we get, using the equality (1.7), the following result.

Proposition 3.1. *Let $\gamma < 0$. The equation* (3.1) *is solvable if and only if*

$$\langle g_+(t), \varphi_j^+(t)\rangle = 0 \ \ for\ all\ \ \varphi_j^+(t) \in \operatorname{Ker}(N_+(b) + \gamma I). \tag{3.2}$$

So, if we are interested in the study of the factorability of a second-order matrix function (1.1), we first solve, with [AEq], the homogeneous equation

$$(N_+(b) + \gamma\, I)\ \omega_{+,g_+}(t) = 0, \tag{3.3}$$

to obtain the kernel of the operator $N_+(b) + \gamma\, I$ and its dimension κ.

If we get $\kappa = 0$ we can conclude that $-\gamma \in \rho\,(N_+(b))$ (see Corollary 2.2 and Proposition 3.1) and a left canonical generalized factorization of (1.1) is obtained using the solutions of the equations (1.3) and (1.4) and Theorem 2.3.

If, on the other hand, we obtain $\kappa > 0$ we can conclude that, if (1.1) is factorable (that is, if $-\gamma \notin \sigma_l\,(N_+(b))$ – see Theorem 2.5), then (1.1) admits a left non-canonical generalized factorization (see Corollary 2.2 and Proposition 2.6).

How can we know if (1.1) admits or not a generalized factorization?

Using Proposition 3.1, we can find if there are two polynomials, $R_{1,\kappa}$ and $R_{2,\kappa}$ of the form (2.9), (2.10), or (2.11), such that the non-homogeneous equations (1.5) are solvable. If at least one of the polynomials does not exist, then the matrix function (1.1) does not admit a generalized factorization, and $-\gamma \in \sigma_l\,(N_+(b))$. Otherwise, $-\gamma \in \sigma\,(N_+(b))$ and we use [AEq] to obtain the solutions of the equations (1.5). A left non-canonical generalized factorization of (1.1) is obtained using Theorem 2.11.

We now describe how to construct [AEq] for solving (solvable) equations of the form (3.1). For that we consider that the function $b \in H_{r,\theta}$. Without any loss of generality we can assume that

$$r(t) = k\,\frac{\prod_{i=1}^{m}(t-\lambda_i)^{\beta_i}}{\prod_{j=1}^{n}(t-\mu_j)^{\alpha_j}}, \quad \text{where } \alpha_i, \beta_i \in \mathbb{N}_0,$$

$k, \lambda_i, \mu_i \in \mathbb{C},$ and $\{\lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_n\}$ has $m+n$ distinct elements, such that $|\lambda_i| > 1$ for all $i = \overline{1,m}$ and $|\mu_j| > 1$ for all $j = \overline{1,n}$.

Let

$$k_0 = \sum_{i=1}^{m} \beta_i - \sum_{j=1}^{n} \alpha_j - 1.$$

Let us consider the Hardy space H_2 and its decomposition

$$H_2 = \theta H_2 \oplus (H_2 \ominus \theta H_2).$$

The following result (see [17, p. 30])) is very important to the construction of [AEq]:

Lemma 3.2. *If θ is an inner function, then*

$$H_2 \ominus \theta H_2 = H_2 \cap \overline{z}\,\theta\,\overline{H_2},$$

where the bar stands for complex conjugation.

We also need to consider the orthogonal projection

$$P_\theta : L_2(\mathbb{T}) \longrightarrow H_2 \ominus \theta H_2, \quad P_\theta = P_{(\theta H_2)^\perp} = P_+ - \theta P_+ \overline{\theta} I.$$

We do not consider the case when b is an inner function, since, in that case, $\sigma_T(N_+(b)) = \{0, 1\}$, and, when the matrix function (1.1) admits a left canonical generalized factorization, such factorization is trivial (see [5]). Besides that,

$$N_+(b) = P_b \quad \text{and} \quad \operatorname{Ker}(N_+(b) - I) = H_2 \ominus b\,H_2.$$

We know that $\dim(H_2 \ominus b\,H_2) < \infty$ if and only if b is a finite Blaschke product (see [17, p. 33]). So, using Theorem 2.7, we conclude that the matrix function $A_{-1}(b)$ admits a left non-canonical generalized factorization if and only if b is a finite Blaschke product. Let b be a finite Blaschke product and φ_+ a function of $H_2 \ominus bH_2$. Considering the factorization of b,

$$b(t) = b_-(t)\, t^{\operatorname{ind} b}\, b_+(t),$$

we get the factorization of $A_{-1}(b)$,

$$A_{-1}(b) = A_{-1}^{+} \Lambda A_{-1}^{-},$$

where

$$A_{-1}^{+} = \begin{pmatrix} -b_+ & (b-\varphi_+)\, b_+^{-1} \\ 0 & b_+^{-1} \end{pmatrix}, \qquad A_{-1}^{-} = \begin{pmatrix} -\varphi_+ \overline{b}^2 b_- & -b_- \\ b_-^{-1} & 0 \end{pmatrix},$$

and

$$\Lambda(t) = \operatorname{diag}\{t^{\operatorname{ind} b}, \ldots, t^{-\operatorname{ind} b}\}.$$

We now show how we can relate the solution(s) of (3.1) to the solution(s) of a linear system.

Applying the substitution

$$\omega_{+,g_+} = \frac{1}{\overline{r}}\, \psi$$

in the equation (3.1), we get

$$\psi = \frac{1}{\gamma}\, \left[\overline{r}\, g_+ - |r|^2 P_\theta\, \psi - \overline{r} P_+ \left(r\psi_-\right)\right],$$

where

$$P_\theta\, \psi = t^{-1}\theta(t)\, \overline{x_+(t)} \quad \text{and} \quad \psi_- = P_- \psi.$$

Using Lemma 3.2 we obtain that

$$t^{-1}\theta(t)\, \overline{x_+(t)}$$

$$= \frac{1}{\gamma} \left(P_\theta \left(\overline{r(t)}\, g_+(t)\right) - P_\theta \left(\frac{|r(t)|^2}{t}\, \theta(t)\, \overline{x_+(t)}\right) - P_\theta \left\{\overline{r(t)}\; P_+ \left[\left(r\,\psi_-\right)(t)\right]\right\}\right). \tag{3.4}$$

So, we need to calculate $\overline{x_+}$ and ψ_- to get the function(s) ψ and, consequently, the solution(s) of (3.1). For that we construct a linear system whose solution(s) gives us the solution(s) of (3.1). We now give a brief description of that construction. The details of going from the equation (3.4) to the linear system can be found (with some adaptations) in [5], where the case $g_+(t) \equiv 1$ was considered.

First, we need to decompose the functions $r(t)$, $\overline{r(t)}$, $|r(t)|^2$ and $\frac{|r(t)|^2}{t}$ in elementary fractions. We obtain different decompositions, depending on the value of the constant k_0:

$$k_0 \geq 0 \;\Rightarrow\; \frac{|r(t)|^2}{t} = \sum_{l=0}^{k_0} a_l\, t^l + \sum_{j=1}^{n} \left\{ \sum_{l=1}^{\alpha_j} \left[\frac{b_{jl}}{(t-\mu_j)^l} + \frac{c_{jl}}{(1-\overline{\mu_j}\, t)^l} \right] \right\} + \sum_{l=1}^{k_0+2} \frac{d_l}{t^l};$$

$$k_0 = -1 \;\Rightarrow\; \frac{|r(t)|^2}{t} = \sum_{j=1}^{n} \left\{ \sum_{l=1}^{\alpha_j} \left[\frac{b_{jl}}{(t-\mu_j)^l} + \frac{c_{jl}}{(1-\overline{\mu_j}\, t)^l} \right] \right\} + \frac{d_1}{t};$$

$$k_0 \leq -2 \;\Rightarrow\; \frac{|r(t)|^2}{t} = \sum_{j=1}^{n} \left\{ \sum_{l=1}^{\alpha_j} \left[\frac{b_{jl}}{(t-\mu_j)^l} + \frac{c_{jl}}{(1-\overline{\mu_j}\, t)^l} \right] \right\};$$

$$k_0 \geq -1 \;\Rightarrow\; r(t) = \sum_{l=0}^{k_0+1} f_l\, t^l + \sum_{j=1}^{n} \sum_{l=1}^{\alpha_j} \frac{g_{jl}}{(t-\mu_j)^l};$$

$$k_0 \leq -2 \;\Rightarrow\; r(t) = \sum_{j=1}^{n} \sum_{l=1}^{\alpha_j} \frac{g_{jl}}{(t-\mu_j)^l};$$

$$k_0 \geq 0 \;\Rightarrow\; \overline{r(t)} = \overline{k}\, \frac{\prod_{i=1}^{m} (-\overline{\lambda_i})^{\beta_i}}{\prod_{j=1}^{n} (-\overline{\mu_j})^{\alpha_j}} + \sum_{j=1}^{n} \sum_{l=1}^{\alpha_j} \frac{r_{jl}}{(1-\overline{\mu_j}\, t)^l} + \sum_{l=1}^{k_0+1} \frac{s_l}{t^l};$$

$$k_0 \leq -1 \;\Rightarrow\; \overline{r(t)} = \overline{k}\, \frac{\prod_{i=1}^{m} (-\overline{\lambda_i})^{\beta_i}}{\prod_{j=1}^{n} (-\overline{\mu_j})^{\alpha_j}} + \sum_{j=1}^{n} \sum_{l=1}^{\alpha_j} \frac{r_{jl}}{(1-\overline{\mu_j}\, t)^l};$$

$$k_0 \geq 0 \;\Rightarrow\; |r(t)|^2 = \sum_{l=0}^{k_0+1} w_l\, t^l + \sum_{j=1}^{n} \left\{ \sum_{l=1}^{\alpha_j} \left[\frac{u_{jl}}{(t-\mu_j)^l} + \frac{s_{jl}}{(1-\overline{\mu_j}\, t)^l} \right] \right\} + \sum_{l=1}^{k_0+1} \frac{z_l}{t^l};$$

$$k_0 = -1 \;\Rightarrow\; |r(t)|^2 = w_0 + \sum_{j=1}^{n} \left\{ \sum_{l=1}^{\alpha_j} \left[\frac{u_{jl}}{(t-\mu_j)^l} + \frac{s_{jl}}{(1-\overline{\mu_j}\, t)^l} \right] \right\};$$

$$k_0 \leq -2 \;\Rightarrow\; |r(t)|^2 = \sum_{j=1}^{n} \left\{ \sum_{l=1}^{\alpha_j} \left[\frac{u_{jl}}{(t-\mu_j)^l} + \frac{s_{jl}}{(1-\overline{\mu_j}\, t)^l} \right] \right\}.$$

Next, using the above decompositions, we define the finite rank operators K_1, K_2, K_3, K_4, K_5 and K_6:

1) $(k_0 \geq 0)$

$$K_1 x_+(t) = \sum_{l=0}^{k_0} a_l \sum_{i=1}^{l+1} A_i\, t^{l-i+1} + \sum_{j=1}^{n} \sum_{l=1}^{\alpha_j} b_{jl} \sum_{i=1}^{l} \frac{B_{ij}}{(t-\mu_j)^{l-i+1}};$$

$$K_2 x_+(t) = \sum_{j=1}^{n} \sum_{l=1}^{\alpha_j} \frac{c_{jl}}{(1-\overline{\mu_j}\, t)^l} \sum_{i=1}^{l} C_{ij} \left(t - \frac{1}{\overline{\mu_j}} \right)^{i-1} + \sum_{l=2}^{k_0+2} \frac{d_l}{t^l} \sum_{i=1}^{l-1} D_i\, t^i;$$

$$K_3\psi_-(t) = P_+\left\{\overline{r(t)}\left[\sum_{l=1}^{k_0+1} f_l \sum_{i=1}^{l} E_i\, t^{l-i} + \sum_{j=1}^{n}\sum_{l=1}^{\alpha_j} g_{jl} \sum_{i=1}^{l} \frac{F_{ij}}{(t-\mu_j)^{l-i+1}}\right]\right\};$$

$$K_4\psi_-(t) = P_+\left\{\overline{\theta(t)}\;\overline{r(t)}\left[\sum_{l=1}^{k_0+1} f_l \sum_{i=1}^{l} E_i\, t^{l-i} + \sum_{j=1}^{n}\sum_{l=1}^{\alpha_j} g_{jl} \sum_{i=1}^{l} \frac{F_{ij}}{(t-\mu_j)^{l-i+1}}\right]\right\};$$

$$K_5x_+(t) = \sum_{j=1}^{n}\sum_{l=1}^{\alpha_j} s_{jl}\left(-\frac{1}{\overline{\mu_j}}\right)^{l}\sum_{i=0}^{l-1}\frac{h_+^{(i)}\left(\frac{1}{\overline{\mu_j}}\right)}{i!\left(t-\frac{1}{\overline{\mu_j}}\right)^{l-i}} + \sum_{l=1}^{k_0+1} w_l \sum_{i=0}^{l-1}\frac{h_+^{(i)}(0)}{i!}t^{i-l};$$

$$K_6\psi_-(t) = P_-\left\{\overline{r(t)}\left[\sum_{l=1}^{k_0+1} f_l \sum_{i=1}^{l} E_i\, t^{l-i} + \sum_{j=1}^{n}\sum_{l=1}^{\alpha_j} g_{jl} \sum_{i=1}^{l} \frac{F_{ij}}{(t-\mu_j)^{l-i+1}}\right]\right\}.$$

2) $(k_0 \leq -1)$

$$K_1x_+(t) = \sum_{j=1}^{n}\sum_{l=1}^{\alpha_j} b_{jl}\sum_{i=1}^{l}\frac{B_{ij}}{(t-\mu_j)^{l-i+1}};$$

$$K_2x_+(t) = \sum_{j=1}^{n}\sum_{l=1}^{\alpha_j}\frac{c_{jl}}{(1-\overline{\mu_j}\,t)^l}\sum_{i=1}^{l} C_{ij}\left(t-\frac{1}{\overline{\mu_j}}\right)^{i-1};$$

$$K_3\psi_-(t) = P_+\left\{\overline{r(t)}\left[\sum_{j=1}^{n}\sum_{l=1}^{\alpha_j} g_{jl}\sum_{i=1}^{l}\frac{F_{ij}}{(t-\mu_j)^{l-i+1}}\right]\right\};$$

$$K_4\psi_-(t) = P_+\left\{\overline{\theta(t)}\;\overline{r(t)}\left[\sum_{j=1}^{n}\sum_{l=1}^{\alpha_j} g_{jl}\sum_{i=1}^{l}\frac{F_{ij}}{(t-\mu_j)^{l-i+1}}\right]\right\};$$

$$K_5x_+(t) = \sum_{j=1}^{n}\sum_{l=1}^{\alpha_j} s_{jl}\left(-\frac{1}{\overline{\mu_j}}\right)^{l}\sum_{i=0}^{l-1}\frac{h_+^{(i)}\left(\frac{1}{\overline{\mu_j}}\right)}{i!\left(t-\frac{1}{\overline{\mu_j}}\right)^{l-i}};$$

$$K_6\psi_-(t) = P_-\left\{\overline{r(t)}\left[\sum_{j=1}^{n}\sum_{l=1}^{\alpha_j} g_{jl}\sum_{i=1}^{l}\frac{F_{ij}}{(t-\mu_j)^{l-i+1}}\right]\right\},$$

through the functionals

$$A_i(x_+) = \frac{\overline{x_+^{(i-1)}(0)}}{(i-1)!};\quad B_{ij}(x_+) = \frac{x_-^{(i-1)}(\mu_j)}{(i-1)!};\quad C_{ij}(x_+) = \frac{(\theta x_-)^{(i-1)}(\frac{1}{\overline{\mu_j}})}{(i-1)!};$$

$$D_i(x_+) = \frac{(\theta x_-)^{(i)}(0)}{i!}; \quad E_i(\psi_-) = \frac{\overline{\varphi_+^{(i)}(0)}}{i!}; \quad F_{ij}(\psi_-) = \frac{\psi_-^{(i-1)}(\mu_j)}{(i-1)!};$$

$$x_-(t) = \overline{x_+(t)}; \quad \varphi_+(t) = \overline{\psi_-(t)}; \quad \text{and} \quad h_+(t) = t^{-1}\theta(t)\, \overline{x_+(t)}.$$

Using the finite rank operators K_i, $i = \overline{1,\, 6}$, we define the functions f_1, f_2, f_3 and f_4:

$$f_1(t) = \theta(t)\; K_1 x_+(t) + K_2 x_+(t) - K_3 \psi_-(t) + \theta(t)\; K_4 \psi_-(t),$$
$$f_2(t) = K_1 x_+(t) + y_-(t) K_2 x_+(t) - y_-(t)\; K_3 \psi_-(t) + K_4 \psi_-(t),$$
$$f_3(t) = -\frac{1}{\gamma}\left[K_5 x_+(t) + K_6 \psi_-(t)\right],$$
$$f_4(t) = \overline{f_3(t)},$$

where $y_-(t) = \overline{\theta(t)}$.

Let $z_{i,+}$, $i = \overline{1, s_+}$, be the zeros, with multiplicity $q_{i,+}$, of $\gamma + |r(t)|^2$, such that $|z_{i,+}| \leq 1$.

Let $z_{i,-}$, $i = \overline{1, s_-}$, be the zeros, with multiplicity $q_{i,-}$, of $\gamma + |r(t)|^2$, such that $|z_{i,-}| > 1$.

We get the linear system, S_{γ, g_+}, that gives us $\overline{x_+}$ and ψ_-:

$$\begin{cases} f_1^{(j)}(z_{i,+}) = -\left(P_\theta\left[\overline{r(t)}\, g_+(t)\right]\right)^{(j)}(z_{i,+}), \; i = \overline{1, s_+}, \; j = \overline{0, q_{i,+} - 1} \\[2ex] f_2^{(j)}(z_{i,-}) = -\left(y_-(t)\, P_\theta\left[\overline{r(t)}\, g_+(t)\right]\right)^{(j)}(z_{i,-}), \; i = \overline{1, s_-}, \; j = \overline{0, q_{i,-} - 1} \\[2ex] \dfrac{f_3^{(i-1)}(\mu_j)}{(i-1)!} - F_{ij} = -\dfrac{1}{\gamma}\, \dfrac{\left(P_-\left[\overline{r(t)} g_+(t)\right]\right)^{(i-1)}(\mu_j)}{(i-1)!}, \; i = \overline{1, \alpha_j}, \; j = \overline{1, n} \\[2ex] \dfrac{\overline{f_4^{(i)}(0)}}{i!} - E_i = -\dfrac{1}{\gamma}\, \dfrac{\left(P_-\left[\overline{r(t)} g_+(t)\right]\right)^{(i)}(0)}{i!}, \; i = \overline{1, k_0 + 1} \;\; (if\ k_0 \geq 0) \end{cases}.$$

Solving this linear system we obtain the solution(s) $\omega_{+,g_+}(t)$ of the equation (3.1):

$$\omega_{+,g_+}(t) = \frac{1}{\gamma}\left\{g_+(t) - r(t)\, t^{-1}\theta(t)\, \overline{x_+(t)} - P_+\left[(r\,\psi_-)(t)\right]\right\},$$

When $g_+(t) \equiv 0$, the solution(s) of $S_{\gamma,0}$ gives us the solution(s) of the homogeneous equation (3.3), that is, we get the kernel of the operator $N_+(b) + \gamma I$ and its dimension κ.

Obviously, $S_{\gamma,0}$ has only the trivial solution if and only if $\kappa = 0$. So, when $S_{\gamma,0}$ has only the trivial solution, we can use Corollary 2.2 and Proposition 3.1 to conclude that, the equations (1.3) and (1.4) are solvable, and, consequently, $-\gamma \in \rho\,(N_+(b))$. This gives the proof of

Theorem 3.3. *Let $b \in H_{r,\theta}$. The system $S_{\gamma,0}$ has only the trivial solution if and only if $-\gamma \in \rho\,(N_+(b))$.*

When the linear system $S_{\gamma,0}$ has no trivial solutions, the kernel of the operator $N_+(b) + \gamma\, I$ is not trivial. So, $-\gamma \in \sigma_T\,(N_+(b))$. We know that if the matrix function (1.1) admits a left generalized factorization, then there exist two polynomials $R_{i,\kappa}(t)$, $i = 1, 2$, such that the problem (2.4) is solvable (see Proposition 2.9). Then, using Proposition 3.1, we can know if the polynomials $R_{i,\kappa}(t)$, $i = 1, 2$, exist. Consequently, we can formulate the following result:

Theorem 3.4. *Let $b \in H_{r,\theta}$ and $S_{\gamma,0}$ a system with no trivial solutions.*

If there exist $R_{i,\kappa}(t)$, $i = 1, 2$ as in (2.9), (2.10), or (2.11) such that

$$g_{+,1}(t) = \gamma\, R_{1,\kappa}(t) \quad \textit{and} \quad g_{+,2}(t) = \gamma\, b\, R_{2,\kappa}(t)$$

satisfy the conditions (3.2), then $-\gamma \in \sigma\,(N_+(b))$. Otherwise, $-\gamma \in \sigma_l\,(N_+(b))$.

So, if the linear system $S_{\gamma,0}$ has only the trivial solution, then the matrix function (1.1) admits a left canonical generalized factorization and we can solve the linear systems $S_{\gamma,1}$ and $S_{\gamma,b}$ to obtain the solutions of the equation (1.3) and (1.4). Using Theorem 2.3, a left canonical generalized factorization of (1.1) can be obtained.

To solve $S_{\gamma,1}$ and $S_{\gamma,b}$ we need to simplify $P_\theta\left[\overline{r(t)}\, g_+(t)\right]$ for

i) $g_+(t) \equiv 1$

$$P_\theta\left[\overline{r(t)} g_+(t)\right] = \overline{r(0)}\left[1 - \overline{\theta(0)}\theta(t)\right].$$

ii) $g_+(t) = r(t)\,\theta(t)$

a) if $k_0 \geq 0$,

$$P_\theta\left[\overline{r(t)} g_+(t)\right] = \sum_{j=1}^{n}\sum_{l=1}^{\alpha_j}\frac{s_{jl}}{(1-\overline{\mu_j}\,t)^l}\left[\theta(t) - \sum_{i=1}^{l} G_{ij}\left(t - \frac{1}{\overline{\mu_j}}\right)^{i-1}\right] + \sum_{l=1}^{k_0+1}\frac{z_l}{t^l}\left(\theta(t) - \sum_{i=0}^{l-1} H_i t^i\right),$$

b) if $k_0 \leq -1$,

$$P_\theta\left[\overline{r(t)} g_+(t)\right] = \sum_{j=1}^{n}\sum_{l=1}^{\alpha_j}\frac{s_{jl}}{(1-\overline{\mu_j}\,t)^l}\left[\theta(t) - \sum_{i=1}^{l} G_{ij}\left(t - \frac{1}{\overline{\mu_j}}\right)^{i-1}\right],$$

where

$$G_{ij} = \frac{\theta^{(i-1)}\left(\frac{1}{\overline{\mu_j}}\right)}{(i-1)!} \quad \text{and} \quad H_i = \frac{\theta^{(i)}(0)}{i!}.$$

If the linear system $S_{\gamma,0}$ has no trivial solution and $-\gamma \in \sigma\left(N_+(b)\right)$, then, using Proposition 3.1, we can determine $R_{i,\kappa}$, $i = 1, 2$, as (2.9), (2.10), or (2.11). We then solve the linear systems $S_{\gamma,\gamma\, R_{1,\kappa}}$ and $S_{\gamma,\gamma\, b\, R_{2,\kappa}}$ to obtain the solutions of the equations (1.5). Using Theorem 2.11, a left non-canonical generalized factorization of (1.1) can be obtained.

To solve $S_{\gamma,\gamma\, R_{1,\kappa}}$ and $S_{\gamma,\gamma\, b\, R_{2,\kappa}}$ we need to simplify $P_\theta \left[\overline{r(t)}\, g_+(t)\right]$ for

i) $g_+ = \gamma\, R_{1,\kappa}$

$$P_\theta \left[\overline{r(t)} g_+(t)\right] = \gamma \left[P_+ - \theta(t) P_+ \overline{\theta(t)}\, I\right] \left[\overline{r(t)} R_{1,\kappa}(t)\right].$$

We have to decompose the function $\overline{r(t)} R_{1,\kappa}(t)$ in elementary fractions. Then, we have to choose the fractions without poles in the interior of the unit circle and use

$$P_+ \left(\overline{\theta(t)}\, t^l\right) = \sum_{i=0}^{l} \frac{\overline{\theta^{(i)}(0)}}{i!}\, t^{l-i}, \quad l \geq 0.$$

ii) $g_+(t) = \gamma\, b\, R_{2,\kappa}$

$$P_\theta \left[\overline{r(t)} g_+(t)\right] = \gamma \left\{P_+ \left[|r(t)|^2\, R_{2,\kappa}(t)\, \theta(t)\right] - \theta(t) P_+ \left[|r(t)|^2\, R_{2,\kappa}(t)\right]\right\}.$$

We have to decompose the function $|r(t)|^2\, R_{2,\kappa}(t)$ in elementary fractions. Then, we have to choose the fractions without poles in the interior of the unit circle and use

$$P_+ \frac{\theta(t)}{\left(1 - \overline{\mu_j}\, t\right)^l} = \frac{1}{\left(1 - \overline{\mu_j}\, t\right)^l} \left[\theta(t) - \sum_{i=1}^{l} \frac{\theta^{(i-1)}\left(\frac{1}{\overline{\mu_j}}\right)}{(i-1)!} \left(t - \frac{1}{\overline{\mu_j}}\right)^{i-1}\right], \quad l \geq 1,$$

and

$$P_+ \frac{\theta(t)}{t^l} = \frac{1}{t^l} \left(\theta(t) - \sum_{i=0}^{l-1} \frac{\theta^{(i)}(0)}{i!} t^i\right), \quad l \geq 1.$$

4. Algorithms [AEq] and [AFact]

This section is dedicated to the algorithms [AEq] and [AFact] [1].

[1]It is possible to implement, on a digital computer, the algorithms [AEq] and [AFact], using the Mathematica 6.0 application. The programming features and the built-in functions of this application can be used to compute the extensive symbolic calculations demanded by the algorithms. As a final result, we can obtain two Mathematica notebooks, one for each algorithm, that automate the factorization process as a whole. The enclosed examples in Section 5 were obtained in such a way. Presently, we are using parts of the implemented notebooks to construct new factorization algorithms. Therefore, we postpone the discloser of the source code.

[AEq]

[[*Input*]]: Insertion of the zeros of $r(t)$ and their algebraic multiplicity. Insertion of the poles of $r(t)$ and their algebraic multiplicity. Insertion of the constants k and γ. Insertion of the function $\theta(t)$.

[[*Initialization*]]: Determination of the constants m, n, and k_0. Definition of the functions $r(t)$, $\overline{r(t)}$, $|r(t)|^2$ and $\dfrac{|r(t)|^2}{t}$. Definition of the auxiliary operator for the decomposition of abstract functions in elementary fractions. Definition of the projection operators P_+ and P_-.

[[*Decomposition in elementary fractions*]]: Decomposition of the functions $r(t)$, $\overline{r(t)}$, $|r(t)|^2$ and $\dfrac{|r(t)|^2}{t}$.

[[*Definition of* f_i *and* K_i]]: Definition of the finite rank operators K_i, $i = \overline{1, 6}$. Definition of the functions f_i, $i = \overline{1, 4}$.

[[*Finding roots of* $\gamma + |r(t)|^2$]]: Resolution of the equation $\gamma + |r(t)|^2 = 0$.

[[*System* S_{γ,g_+}]]: Insertion of $g_+(t)$. Determination of $P_\theta \left[\overline{r(t)}\, g_+(t)\right]$. Determination of $P_- \left[\overline{r(t)}\, g_+(t)\right]$. Resolution of the linear system S_{γ,g_+}.

[[*Output*]]: Determination of the solution(s) $\omega_{+,g_+}(t)$.

[AFact]

[[*Input*]]: Insertion of the zeros of $r(t)$ and their algebraic multiplicity. Insertion of the poles of $r(t)$ and their algebraic multiplicity. Insertion of the constants k and γ. Insertion of the function $\theta(t)$.

[[S_{γ,g_+}]]: Resolution of the linear system S_{γ,g_+}, using [AEq].

[[$\langle g_+(t), \varphi_j^+(t)\rangle = 0$?]]: Analysis of the solubility of the equation (3.1), through the condition (3.2) of Proposition 3.1.

[[$R_{1,\kappa}(t)$]]: Analysis of the existence of a polynomial $R_{1,\kappa}$ satisfying the condition (3.2), for $g_+ = R_{1,\kappa}$.

[[$R_{2,\kappa}(t)$]]: Analysis of the existence of a polynomial $R_{2,\kappa}$ satisfying the condition (3.2), for $g_+ = b\, R_{2,\kappa}$.

[[*No Generalized Factorization*]]: The matrix function (1.1) does not admit a left generalized factorization.

[[*Canonical Generalized Factorization* $A_\gamma = A_\gamma^+ \times A_\gamma^-$]]: The matrix function (1.1) admits a left canonical generalized factorization.

[[*Non-Canonical Generalized Factorization* $A_\gamma = A_\gamma^+ \times \Lambda \times A_\gamma^-$]]: The matrix function (1.1) admits a left non-canonical generalized factorization.

[[*Output* A_γ^+, A_γ^-]]: Determination of the factors A_γ^+ and A_γ^-.

[[*Output* A_γ^+, Λ, A_γ^-]]: Determination of the factors A_γ^+, Λ, and A_γ^-.

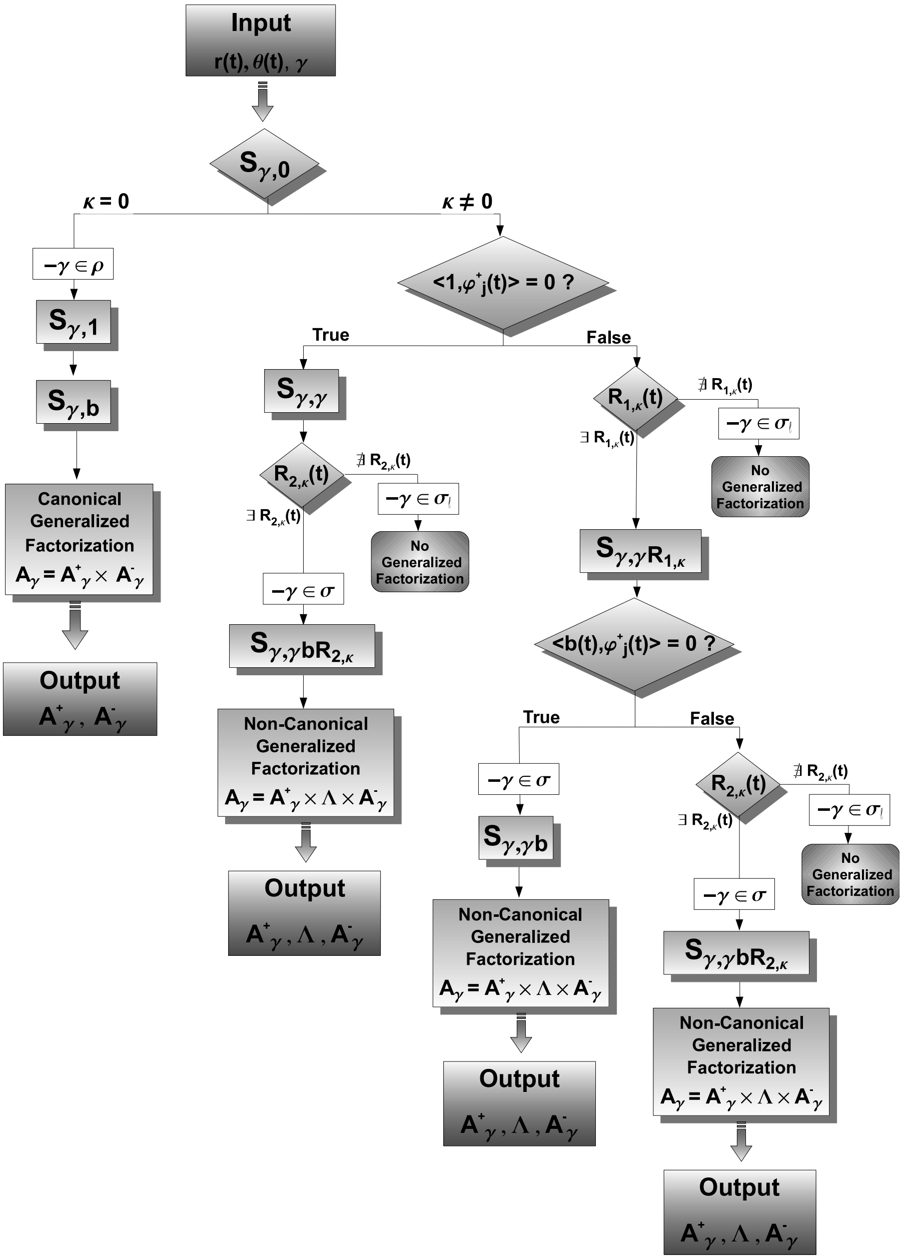

FIGURE 1. Flowchart of [AFact] algorithm

5. Examples

We will now present some examples of the obtained results.

Let us consider the function

$$b(t) = \left(t - \frac{3}{2}\right)\theta(t),$$

where $\theta(t)$ is an inner function.

Using the algorithm [AFact], we obtain a left generalized factorization of the matrix function $A_\gamma(b)$ for two distinct values of γ.

5.1. Canonical factorization

[[*Input*]]: Let $\gamma = \dfrac{7}{4}$. Since $\gamma > 0$, $A_\gamma(b)$ admits a left canonical generalized factorization (see Corollary 2.2).

[[$S_{\gamma,1}$]]:

$$u_+(t) = \frac{2\left(\overline{\theta\left(\frac{1}{3}\right)}\, i_+(t) + 27 - 81\, t\right)}{(7\,|\theta\left(\frac{1}{3}\right)|^2 + 81)(t-3)(3t-1)},$$

where

$$i_+(t) = -7\,(t-3)\theta\left(\frac{1}{3}\right) + 24\, t\,(2t-3)\theta(t).$$

[[$S_{\gamma,b}$]]:

$$v_+(t) = \frac{2}{7}\frac{j_+(t)\theta(t) - 504\,\theta\left(\frac{1}{3}\right)}{\left(7\,|\theta\left(\frac{1}{3}\right)|^2 + 81\right)(t-3)(3t-1)},$$

where

$$j_+(t) = t(2t-3)\left[7(3t-1)|\theta\left(\frac{1}{3}\right)|^2 + 243\,(t-3)\right].$$

[[*Output* A_γ^+, A_γ^-]]: Using Theorem 2.3 we obtain a left canonical generalized factorization of the matrix function $A_{\frac{7}{4}}(b)$,

$$A_{\frac{7}{4}}(b) = A_{\frac{7}{4}}^+\, A_{\frac{7}{4}}^-,$$

where

$$A_{\frac{7}{4}}^+ = \frac{7}{4}\begin{pmatrix} u_+ & v_+ \\ P_+(\bar{b}\, u_+) & 1 + P_+(\bar{b}\, v_+) \end{pmatrix} \quad \text{and} \quad A_{\frac{7}{4}}^- = \left(A_{\frac{7}{4}}^+\right)^{-1} A_{\frac{7}{4}}(b),$$

with

$$P_+\left(\overline{b(t)}u_+(t)\right) = -\frac{1}{7|\theta\left(\frac{1}{3}\right)|^2 + 81}P_+\left[\frac{(3t-2)\overline{\theta(t)}\left(\overline{\theta(\frac{1}{3})}i_+(t) + 27 - 81t\right)}{t\,(t-3)(3t-1)}\right]$$

and

$$P_+\left(\overline{b(t)}v_+(t)\right) = -\frac{1}{7\,|\theta\left(\frac{1}{3}\right)|^2 + 81}P_+\left[\frac{(3t-2)\left(j_+(t) - 504\,\overline{\theta(t)}\theta\left(\frac{1}{3}\right)\right)}{7\,t\,(t-3)(3t-1)}\right].$$

5.2. Non-canonical factorization

[[*Input*]]: Let $\gamma = -1$. Let $\theta(t)$ be a function defined in a neighborhood of $z_{1,+} = \frac{1}{4}(3 - i\sqrt{7})$ and in a neighborhood of $z_{1,-} = \frac{1}{4}(3 + i\sqrt{7})$, such that $\theta(z_{1,+}) = \theta(z_{1,-}) \neq 0$.

[[$S_{\gamma,0}$]]: $\operatorname{Ker}(N_+(b) - I) = \operatorname{span}\left\{\dfrac{2\theta(z_{1,+}) + t(-3+2t)\theta(t)}{\theta(z_{1,+})(2-3t+2t^2)}\right\}$

$\Rightarrow\ 1 \in \sigma_T\,(N_+(b))$.

[[$\langle 1, \varphi_j^+(t)\rangle = 0\,?$]]: $(N_+(b) + \gamma\, I)\, u_+(t) = 1$ is not solvable.

[[$R_{1,\kappa}(t)$]]: $R_{1,\kappa}(t) = t - \frac{3}{2}(1 - \theta(z_{1,+})\overline{\theta(0)})$.

[[$S_{\gamma,\gamma\, R_{1,\kappa}}$]]:

$$\phi_1^+(t) = \frac{-2t(-3+2t)(\theta(z_{1,+}) - \theta(t)) - 3\theta(z_{1,+})t\overline{\theta(0)}h(t) + 3\,A\,h(t)}{3\theta(z_{1,+})(2-3t+2t^2)},$$

where

$$h(t) = 2\theta(z_{1,+}) + t(-3+2t)\theta(t)$$

and A is an arbitrary constant.

[[$\langle b, \varphi_j^+(t)\rangle = 0\,?$]]: $(N_+(b) + \gamma\, I)\, v_+(t) = b(t)$ is not solvable.

[[$R_{2,\kappa}(t)$]]: $R_{2,\kappa}(t) = t$.

[[$S_{\gamma,\gamma\, b\, R_{2,\kappa}}$]]:

$$\phi_2^+(t) = \frac{\theta(z_{1,+})[4\,B + t\theta(t)(6 - 13t + 12t^2 - 4t^3)] + B\,t\theta(t)(4\,t - 6)}{2\,\theta(z_{1,+})(2-3t+2t^2)}$$

where B is an arbitrary constant.

[[*Non-Canonical Generalized Factorization* $A_\gamma = A_\gamma^+ \times \Lambda \times A_\gamma^-$]]: The matrix function $A_{-1}(b)$ admits a left non-canonical generalized factorization **(Case 3)**, see Theorem 2.11,

$$A_{-1}(b) = A_{-1}^+\,\Lambda\,A_{-1}^-,$$

where

$$A_{-1}^+ = \Phi_+\,G,\ \Lambda(t) = \operatorname{diag}\left\{t,\, t^{-1}\right\},\ A_{-1}^- = \Lambda^{-1}\left(A_{-1}^+\right)^{-1} A_{-1}(b),$$

$$\Phi_+ = \begin{pmatrix} \phi_1^+ & \phi_2^+ \\ P_+(\overline{b}\,\phi_1^+) & -R_{2,\kappa} + P_+(\overline{b}\,\phi_2^+) \end{pmatrix},\quad G = \begin{pmatrix} \frac{1}{\Delta} & 0 \\ \frac{\rho}{\Delta} & 1 \end{pmatrix},\quad \Delta = -\det\Phi_+,$$

and the constant ρ depends on the inner function $\theta(t)$.

References

[1] M.J. Ablowitz and P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, London Mathematical Society: Lecture Note Series **149**, Cambridge University Press, 1991.

[2] H. Bart, I. Gohberg, and M.A. Kaashoek, *Minimal Factorization of Matrix and Operator Functions*, Operator Theory: Advances and Applications **1**, Birkhäuser Verlag, Basel-Boston, 1979.

[3] H. Bart, I. Gohberg, M.A. Kaashoek, and A. Ran, *Factorization of Matrix and Operator Functions: the State Space Method*, Operator Theory: Advances and Applications **178**, Birkhäuser Verlag, Basel, 2008.

[4] K. Clancey and I. Gohberg, *Factorization of Matrix Functions and Singular Integral Operators*, Operator Theory: Advances and Applications **3**, Birkhäuser Verlag, Basel-Boston, 1981.

[5] A.C. Conceição and V.G. Kravchenko, *About explicit factorization of some classes of non-rational matrix functions*, Mathematische Nachrichten **280**, No. 9-10, (2007), 1022–1034.

[6] A.C. Conceição and V.G. Kravchenko, *Factorization algorithm for some special matrix functions*, Operator Theory: Advances and Applications, Birkhäuser Verlag **181** (2008), 173–185.

[7] A.C. Conceição, V.G. Kravchenko, and F.S. Teixeira, *Factorization of matrix functions and the resolvents of certain operators*, Operator Theory: Advances and Applications, Birkhäuser Verlag **142** (2003), 91–100.

[8] A.C. Conceição, V.G. Kravchenko, and F.S. Teixeira, *Factorization of some classes of matrix functions and the resolvent of a Hankel operator* in Factorization, Singular Operators and Related Problems, Kluwer Academic Publishers (2003), 101–110.

[9] T. Ehrhardt, and F.-O. Speck, *Transformation techniques towards the factorization of non-rational* 2×2 *matrix functions*, Linear Algebra and its applications **353** (2002), 53–90.

[10] I. Feldman, I. Gohberg, and N. Krupnik, *An explicit factorization algorithm*, Integral Equations and Operator Theory, **49** (2004), 149–164.

[11] I. Gohberg and M.G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Translations of Mathematical Monographs **18**, Providence, RI: American Mathematical Society (AMS), 1969.

[12] V.G. Kravchenko, R.C. Marreiros, and J.S. Rodriguez, *On an estimate for the number of solutions of the generalized Riemann boundary value problem with shift* in Differential and Difference Equations and Applications, Hindawi Publishing Corporation, (2006), 605–615.

[13] V.G. Kravchenko and A.I. Migdal'skii, *A regularization algorithm for some boundary-value problems of linear conjugation*, Dokl. Math. **52** (1995), 319–321.

[14] V.G. Kravchenko and A.M. Nikolaichuk, *On partial indices of the Riemann problem for two pairs of functions*, Soviet Math. Dokl. **15** (1974), 438–442.

[15] G.S. Litvinchuk, *Solvability Theory of Boundary Value Problems and Singular Integral Equations with Shift*, Mathematics and its Applications **523**, Kluwer Academic Publishers, Dordrecht, 2000.

[16] G.S. Litvinchuk and I.M. Spitkovskii, *Factorization of Measurable Matrix Functions*, Operator Theory: Advances and Applications **25**, Birkhäuser Verlag, Basel, 1987.

[17] N.K. Nikol'skiĭ, *Treatise on the Shift Operator. Spectral Function Theory*, Grundlehren der mathematischen Wissenschaften **273**, Springer-Verlag, Berlin, 1986.

[18] S. Prössdorf, *Some Classes of Singular Equations*, North-Holland, Amsterdam, 1978.

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Received: February 28, 2009
Accepted: June 20, 2009

Operator Theory:
Advances and Applications, Vol. 202, 111–141

Structured Primal-dual Interior-point Methods for Banded Semidefinite Programming

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Abstract. For semidefinite programming (SDP) problems, traditional primal-dual interior-point methods based on conventional matrix operations have an upper limit on the problem size that the computer can handle due to memory constraints. But for a special kind of SDP problem, which is called the banded symmetric semidefinite programming (BSDP) problem, a memory-efficient algorithm, called a structured primal-dual interior-point method, can be applied. The method is based on the observation that both banded matrices and their inverses can be represented in sequentially semi-separable (SSS) form with numerical ranks equal to the half bandwidths of the banded matrices. Moreover, all computation can be done sequentially using the SSS form. Experiments of various problem sizes are performed to verify the feasibility of the proposed method.

Mathematics Subject Classification (2000). 65F05, 90C22, 90C51, 65F99, 90C25.

Keywords. Banded matrix, semidefinite program, interior-point method, sequentially semi-separable.

1. Introduction

Let $\mathcal{S}^n$ denote the space of real symmetric $n\times n$ matrices and $\mathcal{B}^n_w$ denote the space of real symmetric $n\times n$ banded matrices with a half bandwidth of w,

$$\begin{aligned}\mathcal{S}^n &= \{X|\ X\in\mathbb{R}^{n\times n},\ X=X^{\mathrm{T}}\}\\ \mathcal{B}^n_w &= \{X|\ X\in\mathcal{S}^n,\ X_{ij}=0 \text{ for } |i-j|>w\}.\end{aligned}$$

The inner product on $\mathcal{S}^n$ is $X\bullet Y=\mathrm{tr}XY$, and we write $X\succeq 0$ ($\succ 0$) to respectively mean that X is positive semidefinite (positive definite). The primal

Communicated by L. Rodman.

form of a banded semidefinite program (BSDP) is

$$\begin{aligned} &\min_X \; C \bullet X \\ &\text{s.t. } A_k \bullet X = b_k, \;\; k = 1, \ldots, m \\ &\;\; X \succeq 0, \end{aligned} \tag{1}$$

where $A_k, C \in \mathcal{B}_w^n$, $b \in \mathbb{R}^m$, and any feasible solution $X \in \mathcal{S}^n$. The dual form is

$$\begin{aligned} &\max_{y,Z} \; b^{\mathrm{T}} y \\ &\text{s.t. } \sum_{k=1}^{m} y_k A_k + Z = C \\ &\;\; Z \succeq 0, \end{aligned} \tag{2}$$

where $y \in \mathbb{R}^m$. It is straightforward that any solution Z also satisfies $Z \in \mathcal{B}_w^n$. The BSDP is just a special case of a semidefinite program (SDP) where the known matrices, A_k and C, are limited to be banded. Therefore, any existing methods that can solve SDP can be directly applied to the BSDP without any changes. In this paper we introduce a revised primal-dual interior-point method that makes use of the banded characteristic and provides fast and memory-efficient iterations.

We assume throughout the paper that there exist $X \succ 0$ satisfying (1) and (y, Z) with $Z \succ 0$ satisfying (2). As is well known [12], these strict feasibility assumptions imply that solutions exist to both (1) and (2), and furthermore that, for all $\mu > 0$, the system

$$\begin{aligned} &A_k \bullet X = b_k \;\; k = 1, \ldots, m \\ &\sum_{k=1}^{m} y_k A_k + Z = C \\ &XZ = \mu I \\ &X \succ 0, \;\; Z \succ 0 \end{aligned} \tag{3}$$

has a unique solution (X_μ, y_μ, Z_μ). The set of such triples is called the central path. As $\mu \to 0$, (X_μ, y_μ, Z_μ) converges to (X_o, y_o, Z_o), where X_o solves (1) and (y_o, Z_o) solves (2).

We consider a standard primal-dual path following algorithm, called the XZ method in [1] and the H..K..M method in [12]. The basic idea is to apply Newton's method to (3), reducing μ as the iteration proceeds. We initialize X and Z to the identity matrix, which satisfies the third equation in (3) with $\mu = 1$, and we initialize $y = 0$. Substituting X, y and Z respectively with $X + \Delta X$, $y + \Delta y$ and $Z + \Delta Z$ in (3), we obtain:

$$\begin{aligned} &A_k \bullet (X + \Delta X) = b_k \;\; k = 1, \ldots, m \\ &\sum_{k=1}^{m} (y_k + \Delta y_k) A_k + (Z + \Delta Z) = C \\ &(X + \Delta X)(Z + \Delta Z) = \mu I \\ &X + \Delta X \succ 0, \;\; Z + \Delta Z \succ 0 \\ &\Delta X = \Delta X^{\mathrm{T}} \\ &\Delta Z = \Delta Z^{\mathrm{T}}. \end{aligned}$$

For now, we neglect the positive definite constraints and the symmetry constraints. The equations for ΔX, Δy, ΔZ become

$$\begin{array}{l} A_k \bullet \Delta X = b_k - A_k \bullet X \;\; k = 1, \ldots, m \\ \sum\limits_{k=1}^{m} \Delta y_k A_k + \Delta Z = C - \sum\limits_{k=1}^{m} y_k A_k - Z \\ X \cdot \Delta Z + \Delta X \cdot Z + \Delta X \cdot \Delta Z = \mu I - XZ. \end{array}$$

Neglecting the second-order term, $\Delta X \cdot \Delta Z$, in the third equation, the equations become linear:

$$\begin{array}{l} A_k \bullet \Delta X = b_k - A_k \bullet X \;\; k = 1, \ldots, m \\ \sum\limits_{k=1}^{m} \Delta y_k A_k + \Delta Z = C - \sum\limits_{k=1}^{m} y_k A_k - Z \\ X \cdot \Delta Z + \Delta X \cdot Z = \mu I - XZ. \end{array}$$

Also, we can convert the matrix-form equations into vector-form equations by applying the matrix stack operator,

$$\begin{aligned} A_k \bullet \Delta X &= \mathrm{vec}(A_k)^{\mathrm{T}} \mathrm{vec}(\Delta X) \\ A_k \bullet X &= \mathrm{vec}(A_k)^{\mathrm{T}} \mathrm{vec}(X), \end{aligned}$$

and applying the Kronecker product operator, [7],

$$\begin{aligned} \mathrm{vec}(X \cdot \Delta Z) &= \mathrm{vec}(X \cdot \Delta Z \cdot I) \\ &= (I \otimes X) \cdot \mathrm{vec}(\Delta Z) \\ \mathrm{vec}(\Delta X \cdot Z) &= \mathrm{vec}(I \cdot \Delta X \cdot Z) \\ &= (Z^{\mathrm{T}} \otimes I) \cdot \mathrm{vec}(\Delta X) \\ &= (Z \otimes I) \cdot \mathrm{vec}(\Delta X). \end{aligned}$$

Therefore, a set of vector-form equations for the unknown vectors $\Delta x \triangleq \mathrm{vec}(\Delta X)$, Δy, $\Delta z \triangleq \mathrm{vec}(\Delta Z)$ can be generated,

$$\mathbf{A} \cdot \Delta x = r_p \tag{4}$$

$$\mathbf{A}^{\mathrm{T}} \cdot \Delta y + \Delta z = r_d \tag{5}$$

$$\mathbf{Z} \cdot \Delta x + \mathbf{X} \cdot \Delta z = r_c, \tag{6}$$

where

$$\begin{aligned} \mathbf{A} &\triangleq [\mathrm{vec}(A_1)\ \mathrm{vec}(A_2)\ \mathrm{vec}(A_3)\ \ldots\ \mathrm{vec}(A_m)]^{\mathrm{T}} \;\in \mathbb{R}^{m \times n^2} \\ \mathbf{X} &\triangleq I \otimes X \;\in \mathbb{R}^{n^2 \times n^2} \\ \mathbf{Z} &\triangleq Z \otimes I \;\in \mathbb{R}^{n^2 \times n^2} \\ x &\triangleq \mathrm{vec}(X) \;\in \mathbb{R}^{n^2} \\ r_p &\triangleq b - \mathbf{A}x \;\in \mathbb{R}^m \\ r_d &\triangleq \mathrm{vec}(C - Z) - \mathbf{A}^{\mathrm{T}} y \;\in \mathbb{R}^{n^2} \\ r_c &\triangleq \mathrm{vec}(\mu I - XZ) \;\in \mathbb{R}^{n^2}. \end{aligned}$$

Since $Z \succ 0$, $\mathbf{Z}$ must be invertible. Block elimination, which we informally write as $(4) + \mathbf{A} \cdot \mathbf{Z}^{-1} \cdot \mathbf{X} \cdot (5) - \mathbf{A} \cdot \mathbf{Z}^{-1} \cdot (6)$, gives

$$\mathbf{M} \cdot \Delta y = r_p + \mathbf{A}\mathbf{Z}^{-1}\mathbf{X} \cdot r_d - \mathbf{A}\mathbf{Z}^{-1} \cdot r_c,$$

where the definition of $\mathbf{M}$ is

$$\mathbf{M} \triangleq \mathbf{A}\mathbf{Z}^{-1}\mathbf{X}\mathbf{A}^{\mathrm{T}} \in \mathbb{R}^{m \times m}. \tag{7}$$

Assuming $\mathbf{A}$ has full row rank, or equivalently all A_k's are independent, $\mathbf{M}$ is non-singular and there is a unique solution for Δx, Δy, Δz for equations (4)-(6),

$$\begin{aligned} \Delta y &= \mathbf{M}^{-1}\left(r_p + \mathbf{A}\mathbf{Z}^{-1}\mathbf{X} \cdot r_d - \mathbf{A}\mathbf{Z}^{-1} \cdot r_c\right) && (8)\\ \Delta z &= r_d - \mathbf{A}^{\mathrm{T}}\Delta y && (9)\\ \Delta x &= \mathbf{Z}^{-1} \cdot (r_c - \mathbf{X} \cdot \Delta z). && (10) \end{aligned}$$

ΔX and ΔZ can be recovered from Δx and Δz,

$$\begin{aligned} \Delta Z &= C - Z - \sum_{k=1}^{m} y_k A_k - \sum_{k=1}^{m} \Delta y_k A_k && (11)\\ \Delta X &= (\mu I - XZ) \cdot Z^{-1} - X \cdot \Delta Z \cdot Z^{-1}. && (12) \end{aligned}$$

However, ΔX, Δy, ΔZ can not be used directly since we still need to check the positive definite constraints and the symmetry constraints. First, we check the symmetry of ΔZ and ΔX. ΔZ is symmetric since

$$\begin{aligned} (\Delta Z)^{\mathrm{T}} &= C^{\mathrm{T}} - Z^{\mathrm{T}} - \sum_{k=1}^{m} y_k A_k^{\mathrm{T}} - \sum_{k=1}^{m} \Delta y_k A_k^{\mathrm{T}}\\ &= C - Z - \sum_{k=1}^{m} y_k A_k - \sum_{k=1}^{m} \Delta y_k A_k\\ &= \Delta Z. \end{aligned}$$

In fact, it is easy to verify that $\Delta Z \in \mathcal{B}_w^n$. But this is not generally true for ΔX since

$$X \cdot \Delta Z \cdot Z^{-1} \neq Z^{-1} \cdot \Delta Z \cdot X.$$

Therefore, we set

$$\Delta X_s = \frac{1}{2}\left(\Delta X + \Delta X^{\mathrm{T}}\right). \tag{13}$$

The symmetric matrix ΔX_s still satisfies (4) but usually does not satisfy (6). Furthermore, we require the new X and Z to be positive definite. We choose a fixed parameter $\tau,\ 0 < \tau < 1$ and define step lengths α and β,

$$\begin{aligned} &\alpha \triangleq \min\{1, \tau\hat{\alpha}\},\ \hat{\alpha} \triangleq \sup\{\bar{\alpha} :\ X + \bar{\alpha}\Delta X_s \succ 0\}\\ &\beta \triangleq \min\{1, \tau\hat{\beta}\},\ \hat{\beta} \triangleq \sup\{\bar{\beta} :\ Z + \bar{\beta}\Delta Z \succ 0\}. \end{aligned} \tag{14}$$

Then X, y and Z can be updated as

$$X_{\text{new}} = X + \alpha \Delta X_s \tag{15}$$

$$y_{\text{new}} = y + \beta \Delta y \tag{16}$$

$$Z_{\text{new}} = Z + \beta \Delta Z. \tag{17}$$

In general, if $\alpha = 1$ and $\beta = 1$, this is an exact feasible solution of the BSDP. If $\alpha < 1$ or $\beta < 1$, the new point does not satisfy the linear constraints. In either case, we call $X_{\text{new}} \bullet Z_{\text{new}}$ the duality gap of the current iteration. For the next iteration, a new parameter μ_{new} can be defined as

$$\mu_{\text{new}} = \theta \cdot \frac{X_{\text{new}} \bullet Z_{\text{new}}}{n},$$

where θ is a parameter with $0 < \theta < 1$.

Now we consider the computational complexity and memory usage in each iteration. The computational work load in each iteration is dominated by the formation and the Cholesky factorization of $\mathbf{M}$. According to (7), the elements of $\mathbf{M}$ must be computed separately,

$$\begin{aligned} M_{ij} &= \text{vec}(A_i)^{\text{T}}(Z^{-1} \otimes X)\text{vec}(A_j) \\ &= \text{vec}(A_i)^{\text{T}}\text{vec}(XA_jZ^{-1}) \\ &= \text{tr}(A_iXA_jZ^{-1}). \end{aligned}$$

Since $X \succ 0$ and $Z \succ 0$, they have Cholesky factorizations, $X = S_X^{\text{T}} S_X$ and $Z = S_Z^{\text{T}} S_Z$, where S_X and S_Z are upper triangular. Therefore,

$$\begin{aligned} M_{ij} &= \text{tr}(A_i S_X^{\text{T}} S_X A_j S_Z^{-1} S_Z^{-\text{T}}) \\ &= \text{tr}(S_Z^{-\text{T}} A_i S_X^{\text{T}} S_X A_j S_Z^{-1}) \\ &= \text{tr}(\tilde{A}_i^{\text{T}} \tilde{A}_j) \qquad (18) \\ &= \tilde{A}_i \bullet \tilde{A}_j \end{aligned}$$

where

$$\tilde{A}_i \triangleq S_X A_i S_Z^{-1}.$$

The computational complexity of one iteration is summarized in Table 1. The memory usage in each iteration is dominated by the memory used to store the $\tilde{A}_i$. The order is $O(mn^2)$.

TABLE 1. Computational complexity of the general primal-dual interior-point method

Computation	Complexity
Computation of all $\tilde{A}_i = S_X A_i S_Z^{-1}$	$O(mn^3)$
Computation of $\mathbf{M}$	$O(m^2n^2)$
Factorization of $\mathbf{M}$	$O(m^3)$
Total	$O(mn^3 + m^2n^2 + m^3)$

From the above analysis, we can see that solving a BSDP by the general primal-dual interior-point method does not have any reduction either in computation complexity or in memory usage compared with solving an SDP, as all matrices except Z are still dense in general. In the following sections, a new method based on a different matrix representation will be developed to make use of the banded characteristic and reduce both computation complexity and memory usage. In the proposed method, not only the banded matrix Z but also other related matrices including S_Z, S_Z^{-1}, X, S_X have compact representations.

This paper is focused specifically on banded SDPs; as far as we know such a structure has not been addressed in the literature previously. For a general survey on exploiting structure in SDP, see [9].

2. Sequentially semi-separable representation (SSS) for banded matrices

In this section, we introduce the sequentially semi-separable(SSS) representation of square matrices and show that all the required matrix computations in the primal-dual interior-point method can be performed in the SSS form.

2.1. Structures of SSS matrices

Let $A \in \mathbb{R}^{n\times n}$ and let $\{n_i, i = 1, \ldots, p\}$ be positive integers satisfying $\sum_{i=1}^{p} n_i = n$. Then A can be partitioned to a compound matrix with sub-block matrices $A_{ij} \in \mathbb{R}^{n_i\times n_j}$, $1 \leqslant i, j \leqslant p$,

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1p} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2p} \\ A_{31} & A_{32} & A_{33} & \cdots & A_{3p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & A_{p3} & \cdots & A_{pp} \end{bmatrix}. \tag{19}$$

All sub-block A_{ij}'s can be expressed in terms of a sequence of matrices $\{D_i, 1 \leqslant i \leqslant p\}$, $\{U_i, 1 \leqslant i \leqslant p-1\}$, $\{V_i, 2 \leqslant i \leqslant p\}$, $\{W_i, 2 \leqslant i \leqslant p-1\}$, $\{P_i, 2 \leqslant i \leqslant p\}$, $\{Q_i, 1 \leqslant i \leqslant p-1\}$, $\{R_i, 2 \leqslant i \leqslant p-1\}$, called sequential matrices, as follows:

$$A_{ij} = \begin{cases} D_i & i = j \\ U_i W_{i+1} \cdots W_{j-1} V_j^{\mathrm{H}} & i < j \\ P_i R_{i-1}^{\mathrm{H}} \cdots R_{j+1}^{\mathrm{H}} Q_j^{\mathrm{H}} & i > j. \end{cases} \tag{20}$$

To make the matrix multiplication operations in (20) valid, dimension constraints must be applied to D_i's, U_i's, V_i's, W_i's, P_i's, Q_i's and R_i's. In fact, we can specify two sequences of positive integers $\{r_i, 1 \leqslant i \leqslant p-1\}$ and $\{l_i, 1 \leqslant i \leqslant p-1\}$, together with $\{n_i\}$, to define their dimensions, as listed in Table 2. The integer set $\{n_i,\ r_i,\ l_i\}$ is called the numerical rank.

TABLE 2. Dimensions of sequential matrices

Matrix	D_i	U_i	V_i	W_i	P_i	Q_i	R_i
Dimension	$n_i \times n_i$	$n_i \times r_i$	$n_i \times r_{i-1}$	$r_{i-1} \times r_i$	$n_i \times l_{i-1}$	$n_i \times l_i$	$l_{i-1} \times l_i$

Consider the memory efficiency of the SSS representation. For simplicity, we assume that all n_i are identical, $n_i \equiv \bar{n}$, all r_i are identical, $r_i \equiv \bar{r}$ and all l_i are identical, $l_i \equiv \bar{l}$. Then the memory size required to store the sequential matrices is

$$p\bar{n}^2 + (p-1)\left(2\bar{n}\bar{r} + 2\bar{n}\bar{l}\right) + (p-2)\left(\bar{r}^2 + \bar{l}^2\right) < p(\bar{n} + \bar{r} + \bar{l})^2.$$

Thus the required memory is at most $O(p(\bar{n} + \bar{r} + \bar{l})^2)$. If there exists a low rank SSS representation so that $\bar{n}, \bar{r}, \bar{l} \ll n$, the SSS representation can be much more memory efficient than the conventional dense matrix representation.

The SSS representation can be applied to any square matrix, as we now show:

Theorem 1. *Let $A \in \mathbb{R}^{n\times n}$. For any specified positive integer sequence $\{n_i, 1 \leqslant i \leqslant p\}$ satisfying $\sum_{i=1}^{p} n_i = n$, we can find sequential matrices so that A is block-partitioned to the form of* (19) *and each block matrix A_{ij} is defined by* (20).

Proof of Theorem 1. The proof is constructive and is similar to that in [3]. The construction of D_i is straightforward,

$$D_i = A_{ii} \ i = 1, \ldots, p.$$

We continue to construct $\{U_i\}$, $\{V_i\}$ and $\{W_i\}$ for the upper triangular part of A. Let H_i be the ith upper off-diagonal block, also known as the ith upper Hankel block following [5],

$$H_i = \begin{bmatrix} A_{1\ i+1} & A_{1\ i+2} & \cdots & A_{1p} \\ \vdots & \vdots & & \vdots \\ A_{i\ i+1} & A_{i\ i+2} & \cdots & A_{ip} \end{bmatrix} \ i = 1, \ldots, p-1.$$

The construction starts from H_1. Let $H_1 = E_1 \Sigma_1 F_1^{\mathrm{H}}$ denote the economic singular value decomposition (SVD) of H_1, so Σ_1 is an invertible diagonal matrix. F_1 can be further partitioned to two sub-blocks,

$$F_1 = \begin{bmatrix} \bar{F}_1 \\ \hat{F}_1 \end{bmatrix} \begin{array}{l} n_2 \text{ rows} \\ \sum_{i=3}^{p} n_i \text{ rows.} \end{array}$$

Now we are ready to define U_1, r_1 and V_2,

$$\begin{aligned} U_1 &= E_1 \\ r_1 &= \text{ number of columns of } U_1 \\ V_2 &= \bar{F}_1 \Sigma_1, \end{aligned}$$

and H_1 has a decomposition

$$H_1 = U_1 \left[V_2^{\mathrm{H}} \ \Sigma_1 \hat{F}_1^{\mathrm{H}} \right].$$

The second step is to look at H_2. According to the decomposition of H_1,

$$\begin{aligned} H_2 &= \begin{bmatrix} A_{13} & \cdots & A_{1p} \\ A_{23} & \cdots & A_{2p} \end{bmatrix} \\ &= \begin{bmatrix} U_1 \Sigma_1 \hat{F}_1^{\mathrm{H}} \\ [A_{23} \ \cdots \ A_{2p}] \end{bmatrix} \\ &= \begin{bmatrix} U_1 & 0 \\ 0 & I \end{bmatrix} \underbrace{\begin{bmatrix} \Sigma_1 \hat{F}_1^{\mathrm{H}} \\ [A_{23} \ \cdots \ A_{2p}] \end{bmatrix}}_{\tilde{H}_2}. \end{aligned}$$

Let $\tilde{H}_2 = E_2 \Sigma_2 F_2^{\mathrm{H}}$ denote the economic SVD of $\tilde{H}_2$, and let

$$E_2 = \begin{bmatrix} \bar{E}_2 \\ \hat{E}_2 \end{bmatrix} \begin{array}{l} r_1 \text{ rows} \\ n_2 \text{ rows} \end{array} \qquad F_2 = \begin{bmatrix} \bar{F}_2 \\ \hat{F}_2 \end{bmatrix} \begin{array}{l} n_3 \text{ rows} \\ \sum_{i=4}^{p} n_i \text{ rows.} \end{array}$$

Therefore we define U_2, r_2, W_2 and V_3,

$$\begin{aligned} U_2 &= \hat{E}_2 \\ r_2 &= \text{ number of columns of } U_2 \\ W_2 &= \bar{E}_2 \\ V_3 &= \bar{F}_2 \Sigma_2, \end{aligned}$$

and the decomposition of H_2 becomes

$$\begin{aligned} H_2 &= \begin{bmatrix} U_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} W_2 \\ U_2 \end{bmatrix} \begin{bmatrix} V_3^{\mathrm{H}} & \Sigma_2 \hat{F}_2^{\mathrm{H}} \end{bmatrix} \\ &= \begin{bmatrix} U_1 W_2 \\ U_2 \end{bmatrix} \begin{bmatrix} V_3^{\mathrm{H}} & \Sigma_2 \hat{F}_2^{\mathrm{H}} \end{bmatrix}. \end{aligned}$$

Now suppose we have accomplished the decomposition of H_{i-1},

$$H_{i-1} = \begin{bmatrix} U_1 W_2 W_3 \cdots W_{i-1} \\ U_2 W_3 \cdots W_{i-1} \\ \vdots \\ U_{i-1} \end{bmatrix} \begin{bmatrix} V_i^{\mathrm{H}} & \Sigma_{i-1} \hat{F}_{i-1}^{\mathrm{H}} \end{bmatrix}.$$

Then for H_i,

$$H_i = \begin{bmatrix} A_{1\ i+1} & \cdots & A_{1p} \\ A_{2\ i+1} & \cdots & A_{2p} \\ \vdots & & \vdots \\ A_{i-1\ i+1} & \cdots & A_{i-1\ p} \\ A_{i\ i+1} & \cdots & A_{ip} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} U_1 W_2 W_3 \cdots W_{i-1} \\ U_2 W_3 \cdots W_{i-1} \\ \vdots \\ U_{i-1} \end{bmatrix} \Sigma_{i-1} \hat{F}_{i-1}^{\mathrm{H}} \\ [A_{i\ i+1} \ \cdots \ A_{ip}] \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} U_1 W_2 W_3 \cdots W_{i-1} \\ U_2 W_3 \cdots W_{i-1} \\ \vdots \\ U_{i-1} \end{bmatrix} & 0 \\ 0 & I \end{bmatrix} \underbrace{\begin{bmatrix} \Sigma_{i-1} \hat{F}_{i-1}^{\mathrm{H}} \\ [A_{i\ i+1} \ \cdots \ A_{ip}] \end{bmatrix}}_{\tilde{H}_i}.$$

Let $\tilde{H}_i = E_i \Sigma_i F_i^{\mathrm{H}}$ denote the economic SVD of $\tilde{H}_i$, with

$$E_i = \begin{bmatrix} \bar{E}_i \\ \hat{E}_i \end{bmatrix} \begin{matrix} r_{i-1} \text{ rows} \\ n_i \text{ rows} \end{matrix} \quad F_i = \begin{bmatrix} \bar{F}_i \\ \hat{F}_i \end{bmatrix} \begin{matrix} n_{i+1} \text{ rows} \\ \sum_{j=i+2}^{p} n_j \text{ rows.} \end{matrix}$$

Therefore we define U_i, r_i, W_i and V_{i+1},

$$\begin{aligned} &U_i = \hat{E}_i \\ &r_i = \text{ number of columns of } U_i \\ &W_i = \bar{E}_i \\ &V_{i+1} = \bar{F}_i \Sigma_i, \end{aligned}$$

and the decomposition of H_i becomes

$$H_i = \begin{bmatrix} \begin{bmatrix} U_1 W_2 W_3 \cdots W_{i-1} \\ U_2 W_3 \cdots W_{i-1} \\ \vdots \\ U_{i-1} \end{bmatrix} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} W_i \\ U_i \end{bmatrix} \begin{bmatrix} V_{i+1}^{\mathrm{H}} & \Sigma_i \hat{F}_i^{\mathrm{H}} \end{bmatrix}$$

$$= \begin{bmatrix} U_1 W_2 W_3 \cdots W_i \\ U_2 W_3 \cdots W_i \\ \vdots \\ U_i \end{bmatrix} \begin{bmatrix} V_{i+1}^{\mathrm{H}} & \Sigma_i \hat{F}_i^{\mathrm{H}} \end{bmatrix}.$$

Repeat this process until H_{p-1} is decomposed. At the last step, $\hat{F}_{p-1}$ is an empty matrix. Then the sequential matrices U_i's, V_i's and W_i's are constructed and r_i is the number of columns of U_i. This algorithm is numerically stable [3]. Substituting $\Sigma_i \hat{F}_i^{\mathrm{H}}$ in the decomposition formula of each H_i, each upper off-diagonal block H_i

has the form

$$H_i = \underbrace{\begin{bmatrix} U_1 W_2 W_3 \cdots W_i \\ U_2 W_3 \cdots W_i \\ \vdots \\ U_i \end{bmatrix}}_{\mathcal{U}_i} \underbrace{\begin{bmatrix} V_{i+1}^{\mathrm{H}} & W_{i+1} V_{i+2}^{\mathrm{H}} & \cdots & W_{i+1} W_{i+2} \cdots W_{p-1} V_p^{\mathrm{H}} \end{bmatrix}}_{\mathcal{V}_{i+1}^{\mathrm{H}}}. \tag{21}$$

Similarly, let G_i denote the ith lower off-diagonal block,

$$G_i = \begin{bmatrix} A_{i+1\ 1} & A_{i+1\ 2} & \cdots & A_{i+1\ i} \\ \vdots & \vdots & & \vdots \\ A_{p\ 1} & A_{p\ 2} & \cdots & A_{p\ i} \end{bmatrix} \quad i = 1, \ldots, p-1.$$

Then P_i, Q_i and V_i can be derived and l_i is the number of columns of Q_i. G_i has the following decomposition formula

$$G_i^{\mathrm{H}} = \underbrace{\begin{bmatrix} Q_1 R_2 R_3 \cdots R_i \\ Q_2 R_3 \cdots R_i \\ \vdots \\ Q_i \end{bmatrix}}_{\mathcal{Q}_i} \underbrace{\begin{bmatrix} P_{i+1}^{\mathrm{H}} & R_{i+1} P_{i+2}^{\mathrm{H}} & \cdots & R_{i+1} R_{i+2} \cdots R_{p-1} P_p^{\mathrm{H}} \end{bmatrix}}_{\mathcal{P}_{i+1}^{\mathrm{H}}}. \tag{22}$$

Therefore a complete SSS representation for A is constructed. □

According to the algorithm in the proof of Theorem 1, representing a matrix in the SSS form requires a lot of computational efforts. However, for banded matrices, if the partition sequence $\{n_i\}$ is properly selected, the SSS representation can be obtained immediately without any computation. Let $A \in \mathcal{B}_w^n$. For simplicity, suppose n and w satisfy the condition $n = p \cdot w$ where p is a positive integer. Then we can assign $n_i = w$, $i = 1, \ldots, p$, and A is partitioned to block matrices $A_{ij}, 1 \leqslant i, j \leqslant p$, satisfying

$$\begin{array}{ll} A_{i\ i+1} \text{ is lower triangular} & \\ A_{i+1\ i} \text{ is upper triangular} & \\ A_{i\ j} = 0 & |i-j| > 1. \end{array}$$

The sequential matrices are

$$\begin{array}{rcl} D_i & = & A_{ii} \\ U_i & = & Q_i = A_{i\ i+1} = A_{i+1\ i}^{\mathrm{T}} \\ V_i & = & P_i = I_{w \times w} \\ W_i & = & R_i = 0. \end{array} \tag{23}$$

The numerical rank is $r_i = l_i = w$ and the order of memory usage is $O(nw)$. A huge memory saving is achieved if $w \ll n$.

2.2. Numerical rank reduction

For a fixed matrix A, the SSS representation is not unique. In fact, there are an infinite number of them. But we are only interested in those which have the minimal numerical rank. Therefore, we need to know what is the optimal SSS representation with minimal numerical rank and how to reduce the numerical rank for any given SSS representation. First, we define the left proper form and the right proper form.

Definition 1. (Left Proper Form and Right Proper Form) The construction manner from the proof of Theorem 1 shows each upper off-diagonal block H_i is separable as (21). The upper triangular part of A is said to be in left proper form if every $\mathcal{U}_i$ has orthogonal columns, that is,

$$\mathcal{U}_i^{\mathrm{H}}\mathcal{U}_i = \text{diagonal}$$

and it is in right proper form if every $\mathcal{V}_{i+1}$ has orthogonal columns, that is,

$$\mathcal{V}_{i+1}^{\mathrm{H}}\mathcal{V}_{i+1} = \text{diagonal.}$$

The same concepts can be applied to the lower triangular part. Each lower off-diagonal block G_i is separable as (22). It is in left proper form if every $\mathcal{Q}_i$ has orthogonal columns and it is in right proper form if every $\mathcal{P}_{i+1}$ has orthogonal columns.

Lemma 1. *Let $A \in \mathbb{R}^{n\times n}$ be represented in the SSS form. Then it can be converted to either the left proper form or the right proper form in a sequential manner.*

Proof of Lemma 1. We prove the theorem by constructing a sequential algorithm. We only consider the conversion of the upper triangular part of A. For the lower triangular part of A, the same algorithm can be applied to the upper triangular part of A^{H}. First, in order to convert the given SSS representation to the left proper form, consider the following recursion to update $\{U_i\}$, $\{V_i\}$, $\{W_i\}$ to $\{\hat{U}_i\}$, $\{\hat{V}_i\}$, $\{\hat{W}_i\}$:

$$\begin{aligned} \begin{bmatrix} \bar{W}_i \\ U_i \end{bmatrix} &= \begin{bmatrix} \hat{W}_i \\ \hat{U}_i \end{bmatrix} \Sigma_i F_i^{\mathrm{H}} \text{ economic SVD} \\ \bar{W}_{i+1} &= \Sigma_i F_i^{\mathrm{H}} W_{i+1} \\ \hat{V}_{i+1} &= V_{i+1} F_i \Sigma_i \end{aligned} \tag{24}$$

with $\bar{W}_1$ and $\hat{W}_1$ being empty matrices. The result is in left proper form. Because $\hat{\mathcal{U}}_1 = \hat{U}_1$,

$$\hat{\mathcal{U}}_1^{\mathrm{H}}\hat{\mathcal{U}}_1 = \hat{U}_1^{\mathrm{H}}\hat{U}_1 = I_{n_1\times n_1}.$$

Furthermore, if $\hat{\mathcal{U}}_{i-1}^{\mathrm{H}}\hat{\mathcal{U}}_{i-1} = I_{n_{i-1}\times n_{i-1}}$,

$$\begin{aligned} \hat{\mathcal{U}}_i^{\mathrm{H}}\hat{\mathcal{U}}_i &= \begin{bmatrix} \hat{W}_i^{\mathrm{H}}\hat{\mathcal{U}}_{i-1}^{\mathrm{H}} & \hat{U}_i^{\mathrm{H}} \end{bmatrix} \begin{bmatrix} \hat{\mathcal{U}}_{i-1}\hat{W}_i \\ \hat{U}_i \end{bmatrix} = \hat{W}_i^{\mathrm{H}}\hat{\mathcal{U}}_{i-1}^{\mathrm{H}}\hat{\mathcal{U}}_{i-1}\hat{W}_i + \hat{U}_i^{\mathrm{H}}\hat{U}_i \\ &= \hat{W}_i^{\mathrm{H}}\hat{W}_i + \hat{U}_i^{\mathrm{H}}\hat{U}_i = I_{n_i\times n_i}. \end{aligned}$$

Therefore, the new SSS representation must be in left proper form.

Second, consider the following recursion to update $\{U_i\}$, $\{V_i\}$, $\{W_i\}$ to $\{\hat{U}_i\}$, $\{\hat{V}_i\}$, $\{\hat{W}_i\}$ so that the result is in right proper form.

$$\begin{array}{rcl} \begin{bmatrix} V_i \\ \bar{W}_i^{\mathrm{H}} \end{bmatrix} & = & \begin{bmatrix} \hat{V}_i \\ \hat{W}_i^{\mathrm{H}} \end{bmatrix} \Sigma_i F_i^{\mathrm{H}} \text{ economic SVD} \\ \bar{W}_{i-1} & = & W_{i+1} F_i \Sigma_i \\ \hat{U}_{i-1} & = & U_{i-1} F_i \Sigma_i \end{array} \tag{25}$$

with $\bar{W}_p$ and $\hat{W}_p$ being empty matrices. The result is in right proper form. Because $\hat{\mathcal{V}}_p = \hat{V}_p$,

$$\hat{\mathcal{V}}_p^{\mathrm{H}} \hat{\mathcal{V}}_p = \hat{V}_p^{\mathrm{H}} \hat{V}_p = I_{n_p \times n_p}.$$

Furthermore, if $\hat{\mathcal{V}}_{i+1}^{\mathrm{H}} \hat{\mathcal{V}}_{i+1} = I_{n_{i+1} \times n_{i+1}}$, then

$$\begin{array}{rcl} \hat{\mathcal{V}}_i^{\mathrm{H}} \hat{\mathcal{V}}_i & = & \begin{bmatrix} \hat{V}_i^{\mathrm{H}} & \hat{W}_i \hat{\mathcal{V}}_{i+1}^{\mathrm{H}} \end{bmatrix} \begin{bmatrix} \hat{V}_i \\ \hat{\mathcal{V}}_{i+1} \hat{W}_i^{\mathrm{H}} \end{bmatrix} \\ & = & \hat{V}_i^{\mathrm{H}} \hat{V}_i + \hat{W}_i \hat{\mathcal{V}}_{i+1}^{\mathrm{H}} \hat{\mathcal{V}}_{i+1} \hat{W}_i^{\mathrm{H}} \\ & = & \hat{V}_i^{\mathrm{H}} \hat{V}_i + \hat{W}_i \hat{W}_i^{\mathrm{H}} \\ & = & I_{n_i \times n_i} \end{array}$$

So, the new SSS representation is in right proper form. □

Lemma 1 states that left proper form and right proper form can be achieved separately. However, we also want to know whether the two proper forms can be achieved at the same time. Lemma 2 addresses this problem.

Lemma 2. *If $A \in \mathbb{R}^{n \times n}$ has been represented in the right proper form, that is,*

$$\mathcal{V}_{i+1}^{\mathrm{H}} \mathcal{V}_{i+1} = \textit{diagonal } i = 1, \ldots, p-1,$$

the new SSS representation of A after the recursion (24) *is still in right proper form. On the other hand, if $A \in \mathbb{R}^{n \times n}$ has been represented in the left proper form, that is,*

$$\mathcal{U}_i^{\mathrm{H}} \mathcal{U}_i = \textit{diagonal } i = 1, \ldots, p-1,$$

the new representation after the recursion (25) *is still in left proper form.*

Proof of Lemma 2. We prove the first statement by induction:

$$\begin{array}{rcl} \hat{\mathcal{V}}_p^{\mathrm{H}} \hat{\mathcal{V}}_p & = & \hat{V}_p^{\mathrm{H}} \hat{V}_p \\ & = & \Sigma_{p-1} F_{p-1}^{\mathrm{H}} V_p^{\mathrm{H}} V_p F_{p-1} \Sigma_{p-1} \\ & = & \Sigma_{p-1} F_{p-1}^{\mathrm{H}} \left(\mathcal{V}_p^{\mathrm{H}} \mathcal{V}_p \right) F_{p-1} \Sigma_{p-1} \\ & = & \Sigma_{p-1} \left(\mathcal{V}_p^{\mathrm{H}} \mathcal{V}_p \right) \Sigma_{p-1} \\ & = & \text{diagonal.} \end{array}$$

Moreover, if $\hat{\mathcal{V}}_{i+1}$ satisfies

$$\hat{\mathcal{V}}_{i+1}^{\mathrm{H}} \hat{\mathcal{V}}_{i+1} = \Sigma_i \left(\mathcal{V}_{i+1}^{\mathrm{H}} \mathcal{V}_{i+1} \right) \Sigma_i = \text{diagonal,}$$

then

$$\begin{aligned}
\hat{\mathcal{V}}_i^{\mathrm{H}}\hat{\mathcal{V}}_i &= \left[\begin{array}{cc} \hat{V}_i^{\mathrm{H}} & \hat{W}_i\hat{\mathcal{V}}_{i+1}^{\mathrm{H}} \end{array}\right]\left[\begin{array}{c} \hat{V}_i \\ \hat{\mathcal{V}}_{i+1}\hat{W}_i^{\mathrm{H}} \end{array}\right] \\
&= \hat{V}_i^{\mathrm{H}}\hat{V}_i + \hat{W}_i\hat{\mathcal{V}}_{i+1}^{\mathrm{H}}\hat{\mathcal{V}}_{i+1}\hat{W}_i^{\mathrm{H}} \\
&= \hat{V}_i^{\mathrm{H}}\hat{V}_i + \hat{W}_i\Sigma_i\left(\mathcal{V}_{i+1}^{\mathrm{H}}\mathcal{V}_{i+1}\right)\Sigma_i\hat{W}_i^{\mathrm{H}} \\
&= \hat{V}_i^{\mathrm{H}}\hat{V}_i + \hat{W}_i\Sigma_iF_i^{\mathrm{H}}\left(\mathcal{V}_{i+1}^{\mathrm{H}}\mathcal{V}_{i+1}\right)F_i\Sigma_i\hat{W}_i^{\mathrm{H}} \\
&= \hat{V}_i^{\mathrm{H}}\hat{V}_i + \bar{W}_i\left(\mathcal{V}_{i+1}^{\mathrm{H}}\mathcal{V}_{i+1}\right)\bar{W}_i^{\mathrm{H}} \\
&= \Sigma_{i-1}F_{i-1}^{\mathrm{H}}\left(V_i^{\mathrm{H}}V_i + W_i\mathcal{V}_{i+1}^{\mathrm{H}}\mathcal{V}_{i+1}W_i^{\mathrm{H}}\right)F_{i-1}\Sigma_{i-1} \\
&= \Sigma_{i-1}F_{i-1}^{\mathrm{H}}\left(\mathcal{V}_i^{\mathrm{H}}\mathcal{V}_i\right)F_{i-1}\Sigma_{i-1} \\
&= \Sigma_{i-1}\left(\mathcal{V}_i^{\mathrm{H}}\mathcal{V}_i\right)\Sigma_{i-1}.
\end{aligned}$$

Therefore, each $\hat{\mathcal{V}}_i$ has orthogonal columns and the right proper form remains. We prove the second statement also by induction. According to recursion (25),

$$\begin{aligned}
\hat{\mathcal{U}}_1^{\mathrm{H}}\hat{\mathcal{U}}_1 &= \hat{U}_1^{\mathrm{H}}\hat{U}_1 \\
&= \Sigma_2F_2^{\mathrm{H}}U_1^{\mathrm{H}}U_1F_2\Sigma_2 \\
&= \Sigma_2F_2^{\mathrm{H}}\left(\mathcal{U}_1^{\mathrm{H}}\mathcal{U}_1\right)F_2\Sigma_2 \\
&= \Sigma_2\left(\mathcal{U}_1^{\mathrm{H}}\mathcal{U}_1\right)\Sigma_2 \\
&= \text{diagonal.}
\end{aligned}$$

Moreover, if $\hat{\mathcal{U}}_{i-1}$ satisfies

$$\hat{\mathcal{U}}_{i-1}^{\mathrm{H}}\hat{\mathcal{U}}_{i-1} = \Sigma_i\left(\mathcal{U}_{i-1}^{\mathrm{H}}\mathcal{U}_{i-1}\right)\Sigma_i,$$

then

$$\begin{aligned}
\hat{\mathcal{U}}_i^{\mathrm{H}}\hat{\mathcal{U}}_i &= \left[\begin{array}{cc} \hat{W}_i^{\mathrm{H}}\hat{\mathcal{U}}_{i-1}^{\mathrm{H}} & \hat{U}_i^{\mathrm{H}} \end{array}\right]\left[\begin{array}{c} \hat{\mathcal{U}}_{i-1}\hat{W}_i \\ \hat{U}_i \end{array}\right] \\
&= \hat{W}_i^{\mathrm{H}}\hat{\mathcal{U}}_{i-1}^{\mathrm{H}}\hat{\mathcal{U}}_{i-1}\hat{W}_i + \hat{U}_i^{\mathrm{H}}\hat{U}_i \\
&= \hat{W}_i^{\mathrm{H}}\Sigma_i\left(\mathcal{U}_{i-1}^{\mathrm{H}}\mathcal{U}_{i-1}\right)\Sigma_i\hat{W}_i + \hat{U}_i^{\mathrm{H}}\hat{U}_i \\
&= \hat{W}_i^{\mathrm{H}}\Sigma_iF_i^{\mathrm{H}}\left(\mathcal{U}_{i-1}^{\mathrm{H}}\mathcal{U}_{i-1}\right)F_i\Sigma_i\hat{W}_i + \hat{U}_i^{\mathrm{H}}\hat{U}_i \\
&= \bar{W}_i^{\mathrm{H}}\left(\mathcal{U}_{i-1}^{\mathrm{H}}\mathcal{U}_{i-1}\right)\bar{W}_i + \hat{U}_i^{\mathrm{H}}\hat{U}_i \\
&= \Sigma_{i+1}F_{i+1}^{\mathrm{H}}\left(W_i^{\mathrm{H}}\mathcal{U}_{i-1}^{\mathrm{H}}\mathcal{U}_{i-1}W_i + U_i^{\mathrm{H}}U_i\right)F_{i+1}\Sigma_{i+1} \\
&= \Sigma_{i+1}F_{i+1}^{\mathrm{H}}\left(\mathcal{U}_i^{\mathrm{H}}\mathcal{U}_i\right)F_{i+1}\Sigma_{i+1} \\
&= \Sigma_{i+1}\left(\mathcal{U}_i^{\mathrm{H}}\mathcal{U}_i\right)\Sigma_{i+1}.
\end{aligned}$$

Therefore, each $\hat{\mathcal{U}}_i$ has orthogonal columns and the left proper form remains. □

Hence, the SSS representation of every matrix can be converted to both the left proper form and the right proper form. And now we are ready to show that an SSS representation in both the left proper form and the right proper form has the minimum numerical rank.

Theorem 2. *For any square matrix $A \in \mathbb{R}^{n\times n}$ and a fixed matrix partition $\{n_i\}$, an SSS representation has the minimum numerical rank if the representation is in the left proper form and the right proper form at the same time. Moreover, the minimum numerical rank satisfies*

$$\begin{aligned} r_i &= \operatorname{rank}(H_i) \ i = 1, \dots, p-1 \\ l_i &= \operatorname{rank}(G_i) \ i = 1, \dots, p-1. \end{aligned}$$

Proof of Theorem 2. According to the construction in the proof of Theorem 1 and the separation formula of H_i in (21),

$$\begin{aligned} r_i &= \text{number of columns of } \mathcal{U}_i \geqslant \operatorname{rank}(\mathcal{U}_i) \geqslant \operatorname{rank}(H_i) \\ r_i &= \text{number of columns of } \mathcal{V}_{i+1} \geqslant \operatorname{rank}(\mathcal{V}_{i+1}) \geqslant \operatorname{rank}(H_i). \end{aligned}$$

If the given SSS representation is in the left proper form and the right proper form at the same time, $\mathcal{U}_i$ and $\mathcal{V}_{i+1}$ have full column rank and all the inequalities in the equations become equalities. Therefore, the given representation must be minimum and $r_i = \operatorname{rank}(H_i)$. The same analysis can be applied to l_i. □

Now we have a lower bound on the numerical rank. In practice, the numerical rank can be further reduced for a given non-zero tolerance. In detail, when we perform economic SVD operations in recursion (24) and (25), we can neglect singular values that are less than a given threshold level δ. Such an SVD operation is called a δ-accurate SVD. The tolerance δ can be any positive number, not necessarily tiny.

2.3. SSS matrix operations

Important matrix operations that can be accomplished in SSS form and related to the proposed structured primal-dual interior-point method are introduced in this section. Computational complexity and the numerical rank of the computational result are analyzed.

Theorem 3. (Inverse of Block Lower Triangular Matrices) *A block lower triangular matrix $L \in \mathbb{C}^{n\times n}$ is represented in SSS form. Then L^{-1} is also block lower triangular with sequential matrices given by*

$$\begin{aligned} D_i(L^{-1}) &= D_i^{-1}(L) \\ P_i(L^{-1}) &= -D_i^{-1}(L)P_i(L) \\ Q_i(L^{-1}) &= D_i^{-\mathrm{H}}(L)Q_i(L) \\ R_i(L^{-1}) &= R_i(L) - P_i^{\mathrm{H}}(L)D_i^{-\mathrm{H}}(L)Q_i(L). \end{aligned}$$

A proof of Theorem 3 can be found in [5]. The numerical rank of the result is the same as the original block lower triangular matrix, $\bar{l}(L^{-1}) = \bar{l}(L)$. The computational complexity is $O(p(\bar{n}+\bar{l})^3)$. For a block upper triangular matrix, we have a similar theorem.

Theorem 4. (Inverse of Block Upper Triangular Matrices) *A block upper triangular matrix $S \in \mathbb{R}^{n\times n}$ is represented in SSS form. Then S^{-1} is also block upper triangular with sequential matrices given by*

$$\begin{array}{rcl} D_i(S^{-1}) & = & D_i^{-1}(S) \\ U_i(S^{-1}) & = & D_i^{-1}(S)U_i(S) \\ V_i(S^{-1}) & = & -D_i^{-\mathrm{H}}(S)V_i(S) \\ W_i(S^{-1}) & = & W_i(S) - V_i^{\mathrm{H}}(S)D_i^{-1}(S)U_i(S). \end{array}$$

Theorem 5. (Cholesky Factorization) *Let $A \in \mathcal{S}^n$ and $A \succ 0$. Let $S = \mathrm{Chol}(A)$ be the unique upper triangular Cholesky factorization of A, with $A = S^{\mathrm{T}}S$ and S upper triangular. If A is in SSS form, then there exists a sequential algorithm to find the block upper triangular matrix S.*

Proof of Theorem 5. We prove the theorem by construction. $A = A^{\mathrm{H}}$ means that A can be represented in a form such that $P_i(A) = V_i(A)$, $Q_i(A) = U_i(A)$, $R_i(A) = W_i(A)$ and $D_i \in \mathcal{S}^n$. Define $\hat{A}_i$ to be a lower-right diagonal block of A,

$$\begin{array}{rcl} \hat{A}_i & = & \begin{bmatrix} A_{i\ i} & A_{i\ i+1} & \cdots & A_{i\ p} \\ A_{i+1\ i} & A_{i+1\ i+1} & \cdots & A_{i+1\ p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p\ i} & A_{p\ i+1} & \cdots & A_{p\ p} \end{bmatrix} \\ & = & \begin{bmatrix} D_i(A) & \cdots & U_i(A)W_{i+1}(A)\cdots V_p^{\mathrm{H}}(A) \\ \vdots & \ddots & \vdots \\ V_p(A)\cdots W_{i+1}^{\mathrm{H}}(A)U_i^{\mathrm{H}}(A) & \cdots & D_p(A) \end{bmatrix} \end{array}$$

and let $\mathcal{V}_{i+1}^{\mathrm{H}}$ be

$$\mathcal{V}_{i+1}^{\mathrm{H}} = \begin{bmatrix} V_{i+1}^{\mathrm{H}} & W_{i+1}V_{i+2}^{\mathrm{H}} & \cdots & W_{i+1}W_{i+2}\cdots V_p^{\mathrm{H}} \end{bmatrix}.$$

Then the factorization process starts from $\hat{A}_1$. We have

$$\begin{array}{rcl} S & = & \mathrm{Chol}(A) = \mathrm{Chol}(\hat{A}_1) \\ & = & \mathrm{Chol}\left(\begin{bmatrix} D_1(A) & U_1(A)\mathcal{V}_2^{\mathrm{H}} \\ \mathcal{V}_2 U_1^{\mathrm{H}}(A) & \hat{A}_2 \end{bmatrix}\right) \\ & = & \begin{bmatrix} \mathrm{Chol}(D_1(A)) & \mathcal{H}_1 \\ 0 & \mathrm{Chol}(\hat{A}_2 - \mathcal{H}_1^{\mathrm{H}}\mathcal{H}_1) \end{bmatrix} \end{array}$$

where

$$\mathcal{H}_1 = \underbrace{[\mathrm{Chol}(D_1(A))]^{-\mathrm{H}}U_1(A)}_{\hat{U}_1}\mathcal{V}_2^{\mathrm{H}}.$$

Let $\hat{U}_1 = E_1\Sigma_1 F_1^{\mathrm{H}}$ denote the economic SVD. Then we can define $D_1(S)$, $U_1(S)$, $V_2(S)$, $\hat{W}_2$:

$$\begin{array}{rcl} D_1(S) & = & \mathrm{Chol}(D_1(A)) \\ U_1(S) & = & E_1 \\ V_2(S) & = & V_2(A)F_1\Sigma_1 \\ \hat{W}_2 & = & \Sigma_1 F_1^{\mathrm{H}} W_2(A) \end{array}$$

The second step is to compute $\mathrm{Chol}(\hat{A}_2 - \mathcal{H}_1^{\mathrm{H}}\mathcal{H}_1)$. $\mathcal{H}_1$ can be expressed as

$$\mathcal{H}_1 = U_1(S)\left[V_2^{\mathrm{H}}(S)\ \hat{W}_2\mathcal{V}_3^{\mathrm{H}}\right].$$

So

$$\begin{array}{rcl} \mathcal{H}_1^{\mathrm{H}}\mathcal{H}_1 & = & \left[\begin{array}{c} V_2(S) \\ \mathcal{V}_3\hat{W}_2^{\mathrm{H}} \end{array}\right] U_1^{\mathrm{H}}(S)U_1(S)\left[\begin{array}{cc} V_2^{\mathrm{H}}(S) & \hat{W}_2\mathcal{V}_3^{\mathrm{H}} \end{array}\right] \\ & = & \left[\begin{array}{cc} V_2(S)V_2^{\mathrm{H}}(S) & V_2(S)\hat{W}_2\mathcal{V}_3^{\mathrm{H}} \\ \mathcal{V}_3\hat{W}_2^{\mathrm{H}}V_2^{\mathrm{H}}(S) & \mathcal{V}_3\hat{W}_2^{\mathrm{H}}\hat{W}_2\mathcal{V}_3^{\mathrm{H}} \end{array}\right] \end{array}$$

$$\begin{array}{cl} & \mathrm{Chol}(\hat{A}_2 - \mathcal{H}_1^{\mathrm{H}}\mathcal{H}_1) \\ = & \mathrm{Chol}\left(\left[\begin{array}{cc} D_2(A) - V_2(S)V_2^{\mathrm{H}}(S) & \left[U_2(A) - V_2(S)\hat{W}_2\right]\mathcal{V}_3^{\mathrm{H}} \\ \mathcal{V}_3\left[U_2^{\mathrm{H}}(A) - \hat{W}_2^{\mathrm{H}}V_2^{\mathrm{H}}(S)\right] & \hat{A}_3 - \mathcal{V}_3\hat{W}_2^{\mathrm{H}}\hat{W}_2\mathcal{V}_3^{\mathrm{H}} \end{array}\right]\right) \\ = & \left[\begin{array}{cc} \mathrm{Chol}(D_2(A) - V_2(S)V_2^{\mathrm{H}}(S)) & \mathcal{H}_2 \\ 0 & \mathrm{Chol}\left(\hat{A}_3 - \left[\mathcal{V}_3\hat{W}_2^{\mathrm{H}}\hat{W}_2\mathcal{V}_3^{\mathrm{H}} + \mathcal{H}_2^{\mathrm{H}}\mathcal{H}_2\right]\right) \end{array}\right] \end{array}$$

where

$$\mathcal{H}_2 = \underbrace{\left[\mathrm{Chol}(D_2(A) - V_2(S)V_2^{\mathrm{H}}(S))\right]^{-\mathrm{H}}\left[U_2(A) - V_2(S)\hat{W}_2\right]}_{\hat{U}_2}\mathcal{V}_3^{\mathrm{H}}.$$

Let

$$\left[\begin{array}{c} \hat{W}_2 \\ \hat{U}_2 \end{array}\right] = E_2\Sigma_2 F_2^{\mathrm{H}}$$

be an economic SVD, and let

$$E_2 = \left[\begin{array}{c} \bar{E}_2 \\ \hat{E}_2 \end{array}\right] \begin{array}{l} r_1(S) \text{ rows} \\ n_2(S) \text{ rows.} \end{array}$$

Then we can define $D_2(S)$, $U_2(S)$, $W_2(S)$, $V_3(S)$, $\hat{W}_3$:

$$\begin{array}{rcl} D_2(S) & = & \mathrm{Chol}(D_2(A) - V_2(S)V_2^{\mathrm{H}}(S)) \\ U_2(S) & = & \hat{E}_2 \\ W_2(S) & = & \bar{E}_2 \\ V_3(S) & = & V_3(A)F_2\Sigma_2 \\ \hat{W}_3 & = & \Sigma_2 F_2^{\mathrm{H}} W_3(A). \end{array}$$

Generally suppose we have finished the $(i-1)$th step. Then $\mathcal{H}_{i-1}$ has the expression

$$\mathcal{H}_{i-1} = \underbrace{\left[\mathrm{Chol}(D_{i-1}(A) - V_{i-1}(S)V_{i-1}^{\mathrm{H}}(S))\right]^{-\mathrm{H}} \left[U_{i-1}(A) - V_{i-1}(S)\hat{W}_{i-1}\right]}_{\hat{U}_{i-1}} \mathcal{V}_i^{\mathrm{H}}.$$

Perform an economic SVD,

$$\begin{bmatrix} \hat{W}_{i-1} \\ \hat{U}_{i-1} \end{bmatrix} = \begin{bmatrix} \bar{E}_{i-1} \\ \hat{E}_{i-1} \end{bmatrix} \Sigma_{i-1} F_{i-1}^{\mathrm{H}}.$$

Then $D_{i-1}(S)$, $U_{i-1}(S)$, $W_{i-1}(S)$, $V_i(S)$, $\hat{W}_i$ are ready to be computed,

$$\begin{array}{rcl} D_{i-1}(S) & = & \mathrm{Chol}(D_{i-1}(A) - V_{i-1}(S)V_{i-1}^{\mathrm{H}}(S)) \\ U_{i-1}(S) & = & \hat{E}_{i-1} \\ W_{i-1}(S) & = & \bar{E}_{i-1} \\ V_i(S) & = & V_i(A)F_{i-1}\Sigma_{i-1} \\ \hat{W}_i & = & \Sigma_{i-1}F_{i-1}^{\mathrm{H}}W_i(A). \end{array}$$

The next step is to compute $\mathrm{Chol}(\hat{A}_i - \left[\mathcal{V}_i\hat{W}_{i-1}^{\mathrm{H}}\hat{W}_{i-1}\mathcal{V}_i^{\mathrm{H}} + \mathcal{H}_{i-1}^{\mathrm{H}}\mathcal{H}_{i-1}\right])$.

$$\begin{array}{rcl} \mathcal{V}_i\hat{W}_{i-1}^{\mathrm{H}}\hat{W}_{i-1}\mathcal{V}_i^{\mathrm{H}} + \mathcal{H}_{i-1}^{\mathrm{H}}\mathcal{H}_{i-1} & = & \mathcal{V}_i\hat{W}_{i-1}^{\mathrm{H}}\hat{W}_{i-1}\mathcal{V}_i^{\mathrm{H}} + \mathcal{V}_i\hat{U}_{i-1}^{\mathrm{H}}\hat{U}_{i-1}\mathcal{V}_i^{\mathrm{H}} \\ & = & \mathcal{V}_i \left[\hat{W}_{i-1}^{\mathrm{H}} \ \hat{U}_{i-1}^{\mathrm{H}}\right] \begin{bmatrix} \hat{W}_{i-1} \\ \hat{U}_{i-1} \end{bmatrix} \mathcal{V}_i^{\mathrm{H}} \\ & = & \begin{bmatrix} V_i(S) \\ \mathcal{V}_{i+1}\hat{W}_i^{\mathrm{H}} \end{bmatrix} \begin{bmatrix} V_i^{\mathrm{H}}(S) & \hat{W}_i\mathcal{V}_{i+1}^{\mathrm{H}} \end{bmatrix} \\ & = & \begin{bmatrix} V_i(S)V_i^{\mathrm{H}}(S) & V_i(S)\hat{W}_i\mathcal{V}_{i+1}^{\mathrm{H}} \\ \mathcal{V}_{i+1}\hat{W}_i^{\mathrm{H}}V_i^{\mathrm{H}}(S) & \mathcal{V}_{i+1}\hat{W}_i^{\mathrm{H}}\hat{W}_i\mathcal{V}_{i+1}^{\mathrm{H}} \end{bmatrix}. \end{array}$$

Therefore

$$\begin{aligned} &\mathrm{Chol}(\hat{A}_i - \left[\mathcal{V}_i\hat{W}_{i-1}^{\mathrm{H}}\hat{W}_{i-1}\mathcal{V}_i^{\mathrm{H}} + \mathcal{H}_{i-1}^{\mathrm{H}}\mathcal{H}_{i-1}\right]) \\ &= \mathrm{Chol}\left(\begin{bmatrix} D_i(A) - V_i(S)V_i^{\mathrm{H}}(S) & \left[U_i(A) - V_i(S)\hat{W}_i\right]\mathcal{V}_{i+1}^{\mathrm{H}} \\ \mathcal{V}_{i+1}\left[U_i^{\mathrm{H}}(A) - \hat{W}_i^{\mathrm{H}}V_i^{\mathrm{H}}(S)\right] & \hat{A}_{i+1} - \mathcal{V}_{i+1}\hat{W}_i^{\mathrm{H}}\hat{W}_i\mathcal{V}_{i+1}^{\mathrm{H}} \end{bmatrix}\right) \\ &= \begin{bmatrix} \mathrm{Chol}\left(D_i(A) - V_i(S)V_i^{\mathrm{H}}(S)\right) & \mathcal{H}_i \\ 0 & \mathrm{Chol}\left(\hat{A}_{i+1} - \left[\mathcal{V}_{i+1}\hat{W}_i^{\mathrm{H}}\hat{W}_i\mathcal{V}_{i+1}^{\mathrm{H}} + \mathcal{H}_i^{\mathrm{H}}\mathcal{H}_i\right]\right) \end{bmatrix} \end{aligned}$$

where

$$\mathcal{H}_i = \underbrace{\left[\mathrm{Chol}(D_i(A) - V_i(S)V_i^{\mathrm{H}}(S))\right]^{-\mathrm{H}} \left[U_i(A) - V_i(S)\hat{W}_i\right]}_{\hat{U}_i} \mathcal{V}_{i+1}^{\mathrm{H}}.$$

Let

$$\begin{bmatrix} \hat{W}_i \\ \hat{U}_i \end{bmatrix} = E_i\Sigma_iF_i^{\mathrm{H}}$$

be an economic SVD, and let

$$E_i = \begin{bmatrix} \bar{E}_i \\ \hat{E}_i \end{bmatrix} \begin{matrix} r_{i-1}(S) \text{ rows} \\ n_i(S) \text{ rows.} \end{matrix}$$

Then we can define $D_i(S)$, $U_i(S)$, $W_i(S)$, $V_i(i+1)$, $\hat{W}_{i+1}$:

$$\begin{array}{rcl} D_i(S) & = & \mathrm{Chol}(D_i(A) - V_i(S)V_i^{\mathrm{H}}(S)) \\ U_i(S) & = & \hat{E}_i \\ W_i(S) & = & \bar{E}_i \\ V_{i+1}(S) & = & V_{i+1}(A)F_i\Sigma_i \\ \hat{W}_{i+1} & = & \Sigma_i F_i^{\mathrm{H}} W_{i+1}(A). \end{array}$$

The ith step is complete. Repeat this process and we can find all the sequential matrices of S.

As a summary, the Cholesky factorization can be accomplished by the following recursion algorithm:

$$\begin{array}{rcl} D_i(S) & = & \mathrm{Chol}\left(D_i(A) - V_i(S)V_i(S)^{\mathrm{H}}\right) \\ \begin{bmatrix} \hat{W}_i \\ D_i(S)^{-\mathrm{H}}[U_i(A) - V_i(S)\hat{W}_i] \end{bmatrix} & = & \begin{bmatrix} W_i(S) \\ U_i(S) \end{bmatrix} \Sigma_i F_i^{\mathrm{H}} \text{ economic SVD} \\ V_{i+1}(S) & = & V_{i+1}(A)F_i\Sigma_i \\ \hat{W}_{i+1} & = & \Sigma_i F_i^{\mathrm{H}} W_{i+1}(A) \end{array}$$

where $V_1(S)$ and $\hat{W}_1$ are empty matrices. $\square$

The computational complexity of the Cholesky factorization is $O(p(\bar{n}+\bar{r})^3)$. The numerical rank of the factorization result is no greater than the original matrix, $\bar{r}(S) \leqslant \bar{r}(A)$.

For $A \in \mathcal{S}^n$ and $A \succ 0$, A^{-1} can be computed sequentially in two steps. First, A can be factorized. Then A^{-1} can be computed by finding the inverses of two block triangular matrices.

Theorem 6. (Addition) *Let $A, B \in \mathbb{R}^{n\times n}$, with both in SSS form and $n_i(A) = n_i(B)$, $i = 1, \dots, p$. Then $A + B$ can be represented in SSS form by sequential matrices given by*

$$\begin{array}{rcl} D_i(A+B) & = & D_i(A) + D_i(B) \\ U_i(A+B) & = & \begin{bmatrix} U_i(A) & U_i(B) \end{bmatrix} \\ V_i(A+B) & = & \begin{bmatrix} V_i(A) & V_i(B) \end{bmatrix} \\ W_i(A+B) & = & \begin{bmatrix} W_i(A) & 0 \\ 0 & W_i(B) \end{bmatrix} \\ P_i(A+B) & = & \begin{bmatrix} P_i(A) & P_i(B) \end{bmatrix} \\ Q_i(A+B) & = & \begin{bmatrix} Q_i(A) & Q_i(B) \end{bmatrix} \\ R_i(A+B) & = & \begin{bmatrix} R_i(A) & 0 \\ 0 & R_i(B) \end{bmatrix}. \end{array}$$

Notice that $n_i(A+B) = n_i(A)$, $r_i(A+B) = r_i(A) + r_i(B)$ and $l_i(A+B) = l_i(A) + l_i(B)$. The numerical rank increases additively which can make the representation of computation result inefficient. The computation complexity is $O(1)$. When the two operands are both banded with the same bandwidth, we know the result will also be banded. Therefore, we do not have to use the algorithm in Theorem 5 to compute their sum. Instead, relying on (23), we have the following algorithm for addition of banded matrices without numerical rank expansion:

$$\begin{array}{rcl} D_i(A+B) & = & D_i(A)+D_i(B) \\ U_i(A+B) & = & U_i(A)+U_i(B) \\ V_i(A+B) & = & I \\ W_i(A+B) & = & 0 \\ P_i(A+B) & = & P_i(A)+P_i(B) \\ Q_i(A+B) & = & I \\ R_i(A+B) & = & 0. \end{array}$$

Theorem 7. (Matrix-Matrix Multiplication) *Let $A, B \in \mathbb{R}^{n\times n}$, with both in SSS form and $n_i(A) = n_i(B)$, $i = 1, \ldots, p$. Then $A \cdot B$ can be represented in SSS form by the following recursions:*

(i) $N_1 = 0,\ N_{i+1} = Q_i^{\mathrm{H}}(A)U_i(B) + R_i^{\mathrm{H}}(A)N_iW_i(B),\ i = 1, \ldots, p-1$

(ii) $M_p = 0,\ M_{i-1} = V_i^{\mathrm{H}}(A)P_i(B) + W_i(A)M_iR_i^{\mathrm{H}}(B),\ i = p, \ldots, 2$

(iii)
$$\begin{aligned} D_i(AB) &= D_i(A)D_i(B) + P_i(A)N_iV_i^{\mathrm{H}}(B) + U_i(A)M_iQ_i^{\mathrm{H}}(B) \\ U_i(AB) &= \begin{bmatrix} D_i(A)U_i(B) + P_i(A)N_iW_i(B) & U_i(A) \end{bmatrix} \\ V_i(AB) &= \begin{bmatrix} V_i(B) & D_i^{\mathrm{H}}(B)V_i(A) + Q_i(B)M_i^{\mathrm{H}}W_i^{\mathrm{H}}(A) \end{bmatrix} \\ W_i(AB) &= \begin{bmatrix} W_i(B) & 0 \\ V_i^{\mathrm{H}}(A)U_i(B) & W_i(A) \end{bmatrix} \\ P_i(AB) &= \begin{bmatrix} D_i(A)P_i(B) + U_i(A)M_iR_i^{\mathrm{H}}(B) & P_i(A) \end{bmatrix} \\ Q_i(AB) &= \begin{bmatrix} Q_i(B) & D_i^{\mathrm{H}}(B)Q_i(A) + V_i(B)N_i^{\mathrm{H}}R_i(A) \end{bmatrix} \\ R_i(AB) &= \begin{bmatrix} R_i(B) & P_i^{\mathrm{H}}(B)Q_i(A) \\ 0 & R_i(A) \end{bmatrix}. \end{aligned}$$

A proof of Theorem 7 can be found in [4]. The result can be inefficient since $n_i(AB) = n_i(A)$, $r_i(AB) = r_i(A) + r_i(B)$, $l_i(AB) = l_i(A) + l_i(B)$. The computational complexity is $O(p(\bar{n} + \bar{r} + \bar{l})^3)$.

3. Structured primal-dual interior-point method

The proposed structured primal-dual interior-point method for BSDP is based on the general primal-dual interior-point method for SDP but all the matrices including intermediate and final results are represented in SSS form. Under the assumption that the banded matrices are narrow-banded, that is $w \ll n$, this method offers the benefits of memory saving and reduced computation complexity.

3.1. Method descriptions

STEP 1. Construct the SSS representations for A_k and C. Since $A_k, C \in \mathcal{B}_w^n$, the conversion can be accomplished easily according to (23) with $n_i = r_i = l_i = w$. The $\{n_i\}$ determines the block partition of all matrices and is kept constant throughout the whole process. The r_i and l_i of A_k and C do not change but those of others, e.g. X and Z, do change. In the following discussions, we only consider r_i due to symmetry. Let $\bar{r}$ denote the average of all r_i. The quantity $\bar{r}$ can be used as a measurement of the numerical rank of an SSS matrix.

Next we specify initial solutions for X, y and Z. Make $X = I_{n\times n}$ and $Z = I_{n\times n}$. Then we convert them to SSS form by the same manner applied to A_k and C. The initial y is set zero.

Note that the initial Cholesky factorizations of X and Z satisfy $S_X = S_Z = I$. Finally, calculate the initial duality gap μ. Choose a fixed parameter θ, $0 < \theta < 1$,

$$\mu_{\text{init}} = \theta \frac{X_{\text{init}} \bullet Z_{\text{init}}}{n} = \theta \frac{I \bullet I}{n} = \theta.$$

STEP 2. Construct $\mathbf{M}$. The formula for $\mathbf{M}$ is defined in (18). The matrices $\tilde{A}_k = S_X A_k S_Z^{-1}$ can be computed in SSS form. The computation includes computing the inverses of a block upper triangular matrix and matrix-matrix multiplications. Each element of $\mathbf{M}$ can be computed by

$$M_{ij} = \operatorname{tr}\left(\tilde{A}_i^{\mathrm{T}} \cdot \tilde{A}_j\right).$$

It is also interesting to examine the numerical rank of $\tilde{A}_k$ since it dominates the memory usage of the algorithm.

$$\begin{aligned} r_i(\tilde{A}_k) &= r_i(S_X) + r_i(A_k) + r_i(S_Z^{-1}) \\ &= r_i(S_X) + r_i(A_k) + r_i(S_Z) \\ &\leqslant r_i(S_X) + r_i(A_k) + r_i(Z) \\ &= r_i(S_X) + 2w. \end{aligned}$$

Here we have used the obvious fact that both A_k and Z in each iteration are banded matrices.

STEP 3. Compute the direction Δy. The computation is based on (8). We need to rewrite the formula to make it suitable for sequential SSS operations. Substitute $\mathbf{M}$, r_p, r_d and r_c in the formula,

$$\begin{aligned} \Delta y &= \mathbf{S_M}^{-1}\mathbf{S_M}^{-\mathrm{T}} \\ &\qquad \times \left[(b - \mathbf{A}x) + \mathbf{A}\mathbf{Z}^{-1}\mathbf{X}\cdot\left(\operatorname{vec}(C - Z) - \mathbf{A}^{\mathrm{T}} y\right) - \mathbf{A}\mathbf{Z}^{-1}\operatorname{vec}(\mu I - XZ)\right] \\ &= \mathbf{S_M}^{-1}\mathbf{S_M}^{-\mathrm{T}} \left(b - \begin{bmatrix} \operatorname{tr}(A_1 X) \\ \operatorname{tr}(A_2 X) \\ \vdots \\ \operatorname{tr}(A_m X) \end{bmatrix} - \begin{bmatrix} \operatorname{tr}(\tilde{A}_1 \tilde{U}) \\ \operatorname{tr}(\tilde{A}_2 \tilde{U}) \\ \vdots \\ \operatorname{tr}(\tilde{A}_m \tilde{U}) \end{bmatrix} \right) = \mathbf{S_M}^{-1}\mathbf{S_M}^{-\mathrm{T}}\tilde{b}, \end{aligned}$$

where $\mathbf{S_M}$ is the Cholesky factorization of $\mathbf{M}$, $\mathbf{M} = \mathbf{S_M}^{\mathrm{T}}\mathbf{S_M}$. Since $\mathbf{M}$ is not structured, this has to be performed by a normal Cholesky factorization algorithm. U and $\tilde{U}$ are defined by

$$U = C - \sum_{k=1}^{m} y_k A_k - \mu S_X^{-1} S_X^{-\mathrm{T}} \text{ and } \tilde{U} = S_X U S_Z^{-1}.$$

The numerical ranks of U and $\tilde{U}$ are

$$\begin{aligned} r_i(U) &= r_i(C - \sum_{k=1}^{m} y_k A_k) + r_i(S_X^{-1}) + r_i(S_X) \\ &= 2r_i(S_X) + w \\ r_i(\tilde{U}) &= r_i(S_X) + r_i(U) + r_i(S_Z^{-1}) \\ &\leqslant r_i(S_X) + [2r_i(S_X) + w] + w \\ &= 3r_i(S_X) + 2w. \end{aligned}$$

The computation of $\tilde{b}$ and Δy can only be performed by normal matrix operations.

Step 4. Direction ΔZ. The computation is based on (11). For convenience we rewrite the formula as

$$\Delta Z = C - Z - \sum_{k=1}^{m} y_k A_k - \sum_{k=1}^{m} \Delta y_k A_k.$$

Since $A_k, C, Z \in \mathcal{B}_w^n$, we have that $\Delta Z \in \mathcal{B}_w^n$. It reflects an important fact that Z will be kept a banded matrix throughout all iterations.

Step 5. Direction ΔX. Substitute ΔZ in (12) by (11),

$$\begin{aligned} \Delta X &= \mu Z^{-1} - X - XCZ^{-1} + X + \sum_{k=1}^{m} y_k X A_k Z^{-1} + \sum_{k=1}^{m} \Delta y_k X A_k Z^{-1} \\ &= \sum_{k=1}^{m} \Delta y_k X A_k Z^{-1} - XUZ^{-1} \\ &= S_X^{\mathrm{T}} S_X \left(\sum_{k=1}^{m} \Delta y_k A_k - U \right) S_Z^{-1} S_Z^{-\mathrm{T}} \qquad (26) \\ &= \sum_{k=1}^{m} \Delta y_k S_X^{\mathrm{T}} \tilde{A}_k S_Z^{-\mathrm{T}} - S_X^{\mathrm{T}} \tilde{U} S_Z^{-\mathrm{T}} \qquad (27) \\ &= S_X^{\mathrm{T}} \left(\sum_{k=1}^{m} \Delta y_k \tilde{A}_k - \tilde{U} \right) S_Z^{-\mathrm{T}}. \qquad (28) \end{aligned}$$

We will use (28) to compute ΔX since it gives better numerical stability and primal feasibility than (26) and (27) according to [1]. The matrices $\tilde{U}$ and $\tilde{A}_k$ have been computed in previous steps. Now consider the numerical rank of ΔX,

$$
\begin{aligned}
r_i(\Delta X) &= r_i(S_X^{\mathrm{T}}) + r_i(\tilde{U}) + \bar{r}\left(\sum_k \Delta y_{<k>}\tilde{A}_k\right) + r_i(S_Z^{-\mathrm{T}}) \\
&\leqslant r_i(S_X) + 3r_i(S_X) + 2w + r_i\left(\sum_{k=1}^m \Delta y_k \tilde{A}_k\right) + r_i(Z) \\
&= 4r_i(S_X) + 3w + r_i\left(\sum_{k=1}^m \Delta y_k \tilde{A}_k\right)
\end{aligned}
$$

A problem arises from the term $r_i(\sum_{k=1}^m \Delta y_k \tilde{A}_k)$. By direct SSS matrix additions, it equals $\sum_{k=1}^m r_i(\tilde{A}_k)$. This is undesirable since it adds a lot of redundancy in the numerical rank. Our strategy is to do matrix addition and rank reduction at the same time to avoid quick growth of the numerical rank. Moreover, $r_i(\sum_{k=1}^m \Delta y_k \tilde{A}_k)$ after rank reduction can be estimated, given that $\sum_{k=1}^m \Delta y_k A_k$ is banded, by

$$
\begin{aligned}
r_i(\sum_{k=1}^m \Delta y_k \tilde{A}_k) &= r_i(S_X\left(\sum_{k=1}^m \Delta y_k A_k\right) S_Z^{-\mathrm{T}}) \\
&= r_i(S_X) + r_i\left(\sum_{k=1}^m \Delta y_k A_k\right) + r_i(S_Z) \\
&\leqslant r_i(S_X) + 2w.
\end{aligned}
$$

So,

$$
r_i(\Delta X) \leqslant 5r_i(S_X) + 5w.
$$

Step 6. Update solutions. Once we have computed search directions ΔX, Δy and ΔZ, we can update our solutions. First we need to symmetrize ΔX by $\Delta X_s = \frac{1}{2}(\Delta X + \Delta X^{\mathrm{T}})$ as in (13). The numerical rank of ΔX_s is

$$
\begin{aligned}
r_i(\Delta X_s) &= 2r_i(\Delta X) \\
&\leqslant 10r_i(S_X) + 10w.
\end{aligned}
$$

Next we need to find step lengths α and β which are defined in (14). In [1], an exact formula relying on an eigenvalue computation can be used to obtain $\hat{\alpha}$ and $\hat{\beta}$. Unfortunately, that does not work here, because there is no sequential algorithm to find the eigenvalues of an SSS matrix. Therefore, we use a bisection search with Cholesky factorization to estimate the values of $\hat{\alpha}$ and $\hat{\beta}$. Then we can define α and β by specifying a fixed parameter τ.

Finally we get updated solutions X_{new}, y_{new} and Z_{new} based on (15)–(17). We still need to estimate the numerical rank of X_{new} since this can affect the sizes of matrices in following iterations. To find $r_i(X_{\text{new}})$, we have the following theorem.

Theorem 8. *Let $X = Z^{-1}$, where both X and Z have been expressed in the simplest SSS form with minimum numerical ranks. Then*

$$\begin{aligned} r_i(X) &= r_i(Z) \\ l_i(X) &= l_i(Z). \end{aligned}$$

Proof of Theorem 8. This conclusion is a direct result of a fact about general matrices that any sub-matrix of a non-singular square matrix X has the same nullity with the complementary sub-matrix of X^{-1}, [11], [2]. Two sub-matrices are "complementary" when the row numbers not used in one are the column numbers used in the other. Therefore, given that $X = Z^{-1}$, any upper off-diagonal matrix of X, $H_i(X)$, and the corresponding upper off-diagonal matrix of Z, $H_i(Z)$ are complementary sub-matrices. They have the same rank:

$$\text{rank}(H_i(X)) = \text{rank}(H_i(Z)) \;\Rightarrow\; r_i(X) = r_i(Z).$$

The second equality is based on Theorem 2. Similarly,

$$\text{rank}(G_i(X)) = \text{rank}(G_i(Z)) \;\Rightarrow\; l_i(X) = l_i(Z). \qquad \square$$

Now if the updated solutions X_{new} and Z_{new} are approximately on the central path, then it is satisfied that $X_{\text{new}} \approx \mu Z_{\text{new}}^{-1}$. We know that Z_{new} is banded, then $r_i(X_{\text{new}}) \approx w$ and $l_i(X_{\text{new}}) \approx w$. The new Cholesky factorization $S_{X\text{new}}$ and $S_{Z\text{new}}$ can also be computed.

At the end, a new target duality gap should be calculated,

$$\mu_{\text{new}} = \frac{\theta}{n}\text{tr}(S_{X\text{new}}^{\text{H}} S_{X\text{new}} S_{Z\text{new}}^{\text{H}} S_{Z\text{new}}).$$

Refresh S_X, S_Z, μ and go back to Step 2 for next the iteration until μ becomes less than a preset threshold.

$$\begin{aligned} S_Z &\Leftarrow S_{Z\text{new}} \\ S_X &\Leftarrow S_{X\text{new}} \\ \mu &\Leftarrow \mu_{\text{new}}. \end{aligned}$$

3.2. Algorithm analysis

We have described each step of the structured primal-dual interior-point method in detail. This method always maintains simple SSS structures since at the end of each iteration, S_X will return to low rank after rank reduction. As a summary, we list the numerical ranks of some critical intermediate matrices in each iteration in Table 3. As in the case for unstructured problems, the memory usage is dominated by the storage of $\tilde{A}_k$. The memory usage is $O(mw^2)$, compared to $O(mn^2)$ for unstructured problems.

It is also interesting to look at the computational complexity of the proposed method, as shown in Table 4. Comparing to the analysis in Table 1, the computational complexity can be reduced everywhere except in the Cholesky factorization

TABLE 3. Numerical ranks of intermediate matrices

Matrix	Numerical Rank
$\tilde{A}_k$	$r_i(S_X) + 2w$
$\tilde{U}$	$3r_i(S_X) + 2w$
ΔZ	w
ΔX	$5r_i(S_X) + 5w$
ΔX_s	$10r_i(S_X) + 10w$
Z_{new}	w
X_{new}	$12r_i(S_X) + 10w$
$S_{Z\text{new}}$	w
$S_{X\text{new}}$ (after rank reduction)	$\approx w$

TABLE 4. Computation complexity of the structured method

Computation	Complexity
Computation of all $\tilde{A}_i = S_X A_i S_Z^{-1}$	$O(mnw^2)$
Computation of $\mathbf{M}$	$O(m^2nw^2)$
Factorization of $\mathbf{M}$	$O(m^3)$
Total	$O(m^2nw^2 + m^3)$

of $\mathbf{M}$. However, this does not dominate the cost of the algorithm. In fact, to guarantee the number of constraints in the BSDP to be less than the number of unknowns, it must be the case that

$$m < \frac{(2n-w)(w+1)}{2}$$

and in the worst scenario

$$m = O(nw).$$

Hence the overall computation complexity of the proposed method is bounded by

$$O(m^2nw^2 + m^3) = O(n^3w^4 + n^3w^3) = O(n^3w^4).$$

Thus the need to factorize $\mathbf{M}$ does not affect the effectiveness of the algorithm.

Another issue concerns the rank reduction of $S_{X\text{new}}$, which is accomplished using δ-accurate SVDs in the SSS computations. There are two possible strategies: one is to fix δ to a small value throughout the algorithm, and the other is to reduce δ adaptively.

3.3. Experiments

In order to test the feasibility of the proposed structured algorithm, several experiments were carried out. We generate large size random BSDP's that have strictly feasible solutions by the following manner:

- Generate random banded matrices A_k, $k = 1, \ldots, n$
- Generate random diagonal matrices X and Z with positive diagonal elements
- Generate a random vector y
- Define $b_k = A_k \bullet X$, $k = 1, \ldots, n$
- Define $C = \sum_{k=1}^{n} y_k A_k + Z$.

Problems with different sizes are tested with different rank reduction strategies.

- $n = 100$, 500, 2000, 5000
- $m = 10$, 20
- $w = 5$
- constant δ, adaptive δ.

We set parameters $\tau = 0.9$ and $\theta = 0.25$ and the convergence condition for the duality gap is $n \cdot 10^{-12}$. In the constant δ strategy, we fix δ to be 10^{-13} while in the adaptive δ strategy, δ is linked to μ in each iteration by $\delta = 0.1\mu$.

Fig. 1–Fig. 8 show experimental results for each iteration including the normalized duality gap μ, the normalized numerical rank $\frac{\bar{r}(X)}{w}$ after rank reduction, the primal residual $\max\{|b_k - A_k \bullet X|\}$ and the dual residual $\max\{|C - Z - \sum_{k=1}^{m} y_k A_k|\}$. We see that for the constant δ strategy, smaller ranks for X, implying less computational cost, are achieved, at the price of increased primal infeasibility, but this infeasibility is steadily reduced as the iteration continues, finishing with the same accuracy as the constant δ strategy. The sharp drop in primal residual at the beginning of the constant δ iterations is explained by the fact that as soon as a primal step $\alpha = 1$ is taken, primal infeasibility drops to zero in exact algorithm. Similarly, the sharp drop in dual residual in later iterations, for both the constant and adaptive δ strategies, is explained by a dual step $\beta = 1$ being taken. The machine used was Dell PowerEdge 2950 with 2×3.0 GHz Dual Core Xeon 5160 processors. The CPU time for different experiments are summarized in Table 5 on page 140.

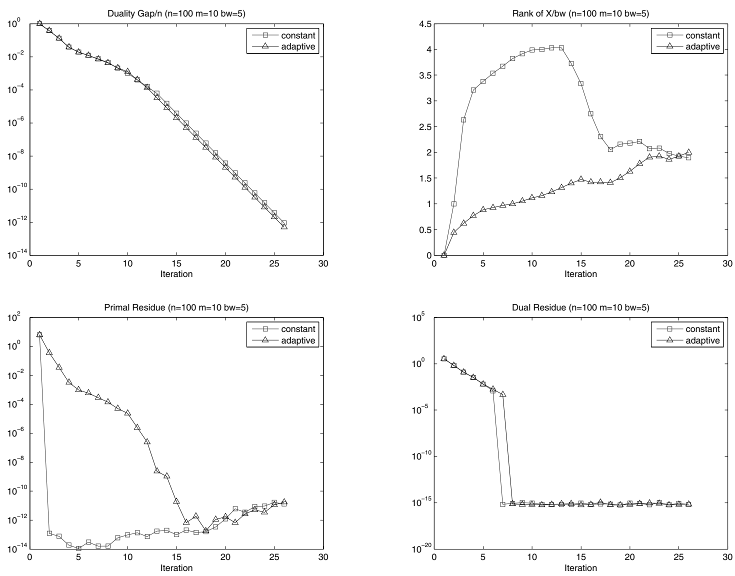

FIGURE 1. Experimental results ($n = 100, m = 10, w = 5$)

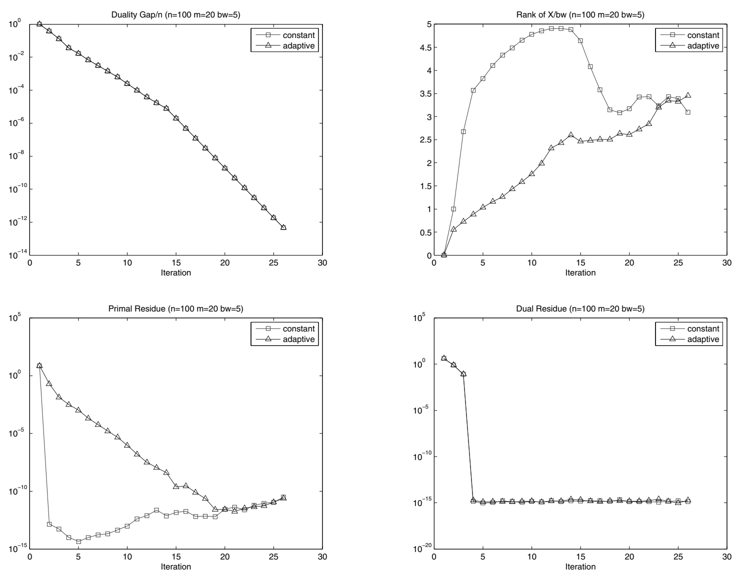

FIGURE 2. Experimental results ($n = 100, m = 20, w = 5$)

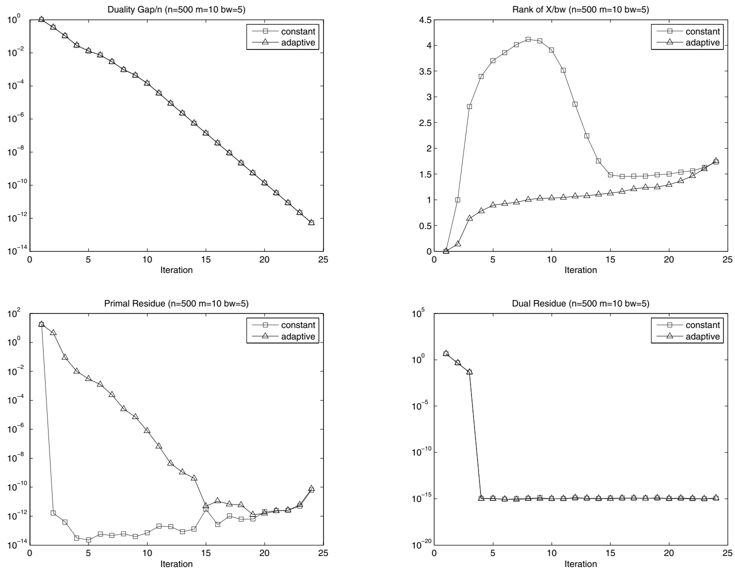

FIGURE 3. Experimental results ($n = 500, m = 10, w = 5$)

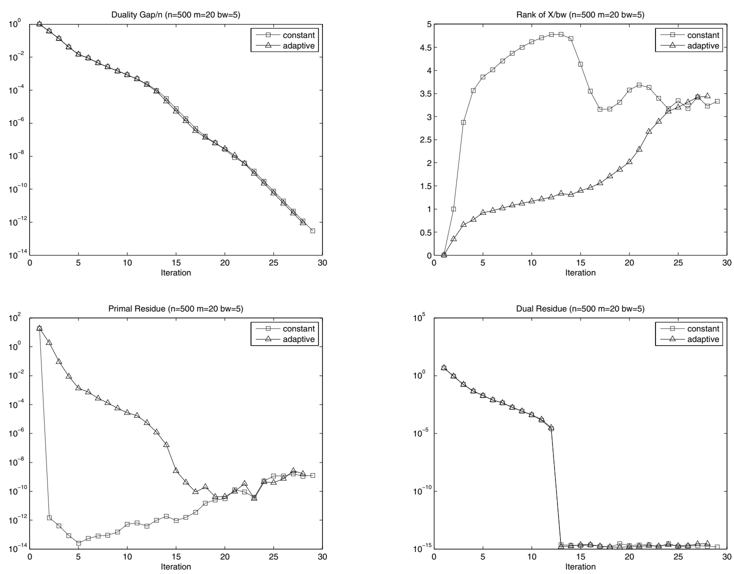

FIGURE 4. Experimental results ($n = 500, m = 20, w = 5$)

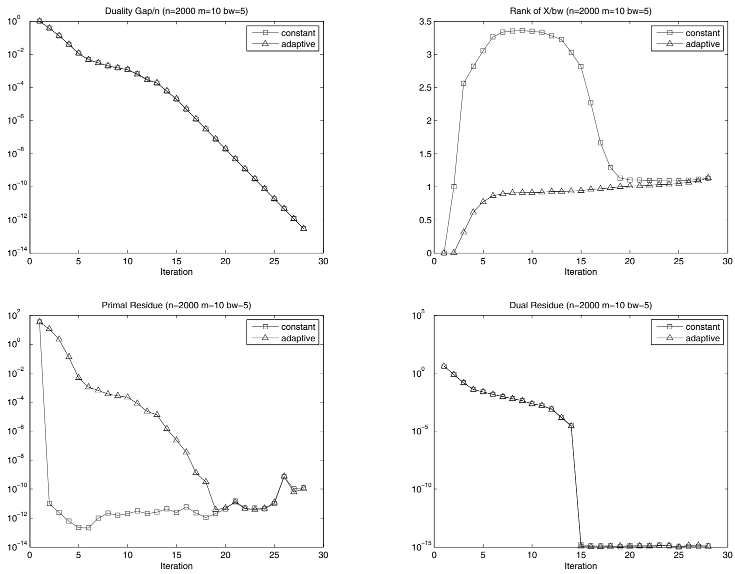

FIGURE 5. Experimental results $(n = 2000, m = 10, w = 5)$

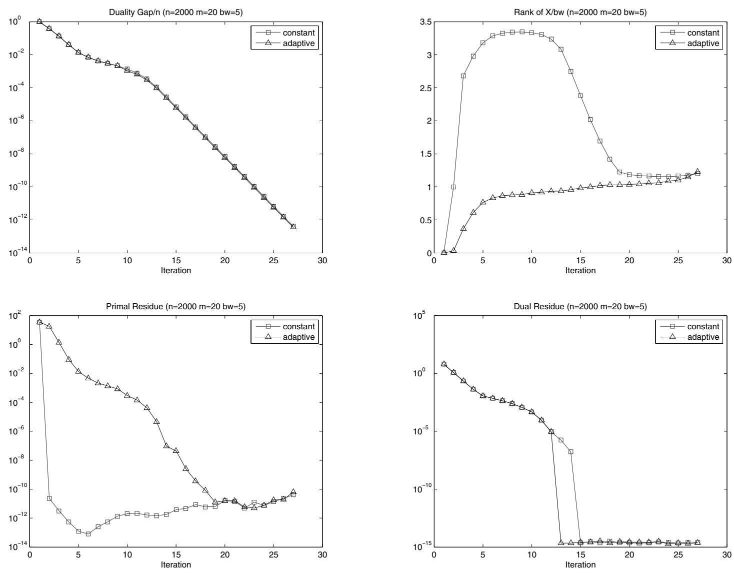

FIGURE 6. Experimental results $(n = 2000, m = 20, w = 5)$

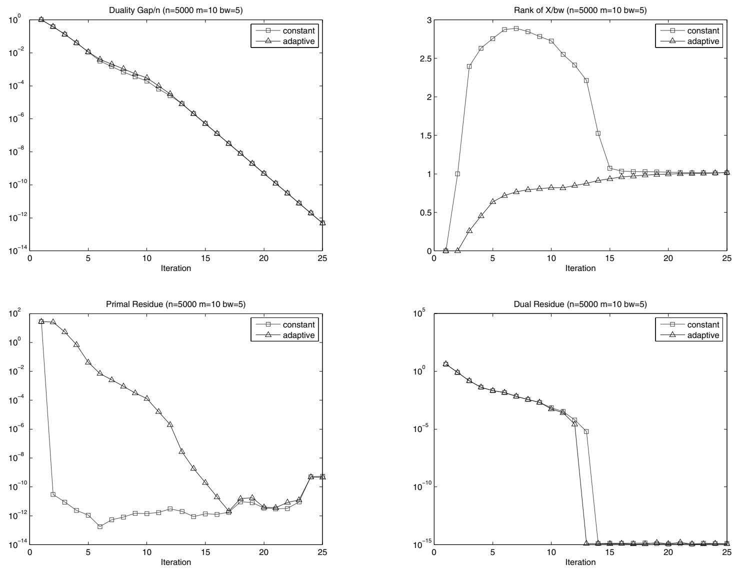

FIGURE 7. Experimental results ($n = 5000, m = 10, w = 5$)

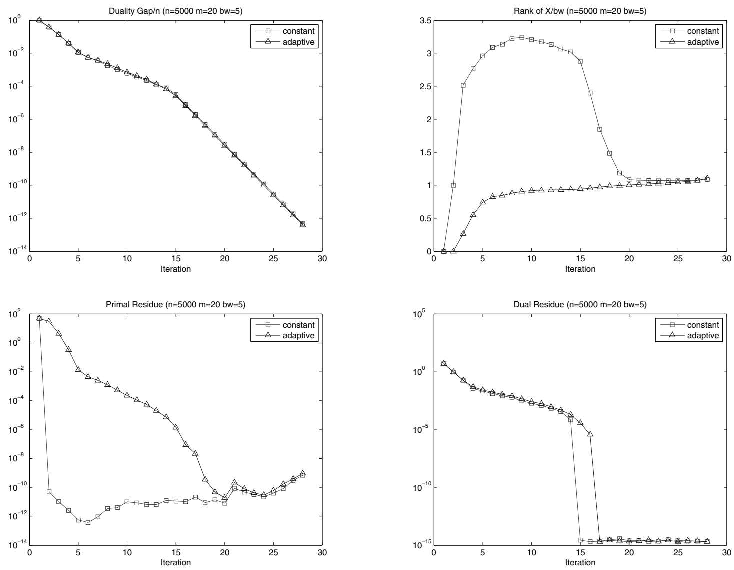

FIGURE 8. Experimental results ($n = 5000, m = 20, w = 5$)

Table 5. CPU time for different experiments

Experiments	Time(constant δ) (seconds)	Time(adaptive δ) (seconds)
1 ($n = 100, m = 10, w = 5$)	4.00×10^1	2.9×10^1
2 ($n = 100, m = 20, w = 5$)	1.19×10^2	9.15×10^1
3 ($n = 500, m = 10, w = 5$)	3.77×10^2	1.91×10^2
4 ($n = 500, m = 20, w = 5$)	7.57×10^2	4.68×10^2
5 ($n = 2000, m = 10, w = 5$)	3.87×10^3	1.61×10^3
6 ($n = 2000, m = 20, w = 5$)	5.59×10^3	2.29×10^3
7 ($n = 5000, m = 10, w = 5$)	1.19×10^4	5.45×10^3
8 ($n = 5000, m = 20, w = 5$)	2.44×10^4	1.16×10^4

4. Conclusion and future work

A structured primal-dual interior-point method has been presented for the BSDP and its feasibility has been tested by solving problems of various sizes. Both theory and experiments show that the application of SSS forms in square matrix representations and operations can save a lot of computation and memory. Therefore solving problems with huge sizes becomes possible by using the proposed structured algorithm.

However, there still exist some open problems. Our experiments demonstrate the importance of the selection of δ. A good δ-selection strategy should reduce numerical ranks dramatically, but the introduced errors must not affect the convergence of solutions. More investigation is needed to find the best strategy.

Acknowledgment

The work of Ming Gu was supported in part by NSF awards CCF-0515034 and CCF-0830764.

The work of M.L. Overton was supported in part by the U.S. National Science Foundation under grant DMS-0714321.

References

[1] F. Alizadeh, J.-P. Haeberly and M. Overton, Primal-Dual Interior-Point Methods For Semidefinite Programming: Convergence Rates, Stability And Numerical Results, *SIAM J. Optim.* p. 746–768, vol. 8, No. 3, Aug 1998.

[2] R.B. Bapat, On generalized inverse of banded matrices, *Electronic J. Line. Alge.* p. 284–290, vol. 16, Sep. 2007.

[3] S. Chandrasekaran, P. Dewilde, M. Gu, T. Pals, X. Sun, A.-J. van der Veen and D. White, Fast Stable Solvers for Sequentially Semi-Separable Linear Systems of Equations and Linear Squares Problems, University of California at Berkeley, 2003.

[4] S. Chandrasekaran, P. Dewilde, M. Gu, T. Pals, X. Sun, A.-J. van der Veen and D. White, Some Fast Algorithms for Sequentially Semiseparable Representations, *SIAM J. Matrix Anal. Appl.*, p. 341–364, vol. 27, No. 2, Sep. 2005.

[5] P. Dewilde and A. van der Veen, *Time-varying systems and computations*, Kluwer Academic Publishers, 1998.

[6] I. Gohberg, T. Kailath and I. Koltracht, Linear complexity algorithms for semiseparable matrices, *Integral Equations Operator Theory*, p. 780–804, vol. 8, 1985.

[7] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[8] E. de Klerk, *Aspects of semidefinite programming: Interior Point Algorithms and Selected Applications*, Kluwer Academic Publishers, 2002.

[9] E. de Klerk, Exploiting special structure in semidefinite programming. *European Journal of Operational Research*, 2009. To appear.

[10] C. Roos, T. Terlaky and J.-P. Vial, *Interior Point Methods For Linear Optimization*, Springer, 2006.

[11] G. Strang and T. Nguyen, The interplay of ranks of submatrices, *SIAM Review*, p. 637–646, vol. 46, No. 4, 2004.

[12] M.J. Todd, Semidefinite Optimization, *Acta Numerica*, p. 515–560, vol. 10, 2001.

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Received: March 31, 2009
Accepted: August 1, 2009

Operator Theory:
Advances and Applications, Vol. 202, 143–150

A Note on Semi-Fredholm Hilbert Modules

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Abstract. A classical problem in operator theory has been to determine the spectrum of Toeplitz-like operators on Hilbert spaces of vector-valued holomorphic functions on the open unit ball in $\mathbb{C}^m$. In this note we obtain necessary conditions for k-tuples of such operators to be Fredholm in the sense of Taylor and show they are sufficient in the case of the unit disk.

Mathematics Subject Classification (2000). 47A13, 46E22, 46M20, 47B32.

Keywords. Hilbert modules, quasi-free Hilbert modules, Fredholm tuple, Corona property.

1. Introduction

A classical problem in operator theory is to determine the invertibility or the spectrum of Toeplitz operators on the Hardy space over the unit disk $\mathbb{D}$. When the symbol or the defining function is continuous, the result is well known and due to Gohberg in the scalar case (see [12]) and Gohberg-Krein in the vector-valued case (see [13]). Generalizations of these results to other Hilbert spaces of holomorphic functions on the disk such as the Bergman space (see [1]) or to the unit ball $\mathbb{B}^m$ (see [16]) or other domains in $\mathbb{C}^m$ (see [2]) have been studied during the past few decades. In the several variables context, the problem is not too interesting unless we start with a matrix-valued symbol or a k-tuple of operators and consider the Taylor spectrum or essential spectrum which involves the Koszul complex (see [14]).

In this note we consider two problems, neither of which is new. However, we believe the results are more general and our methods provide a more constructive approach. Moreover, they identify some questions in multi-variable operator theory (and algebra) indicating their importance in the spectral theory for k-tuples of vector-valued Toeplitz-like operators. Finally, the results suggest lines of investigation for generalizations of the classical Hilbert spaces of holomorphic functions.

This research was partially supported by a grant from the National Science Foundation.
Communicated by J.A. Ball.

All the Hilbert spaces in this note are separable and are over the complex field $\mathbb{C}$. For a Hilbert space $\mathcal{H}$, we denote the Banach space of all bounded linear operators by $\mathcal{L}(\mathcal{H})$.

We begin by recalling the definition of quasi-free Hilbert module over $A(\Omega)$ which was introduced in ([7],[6]) and which generalizes classical functional Hilbert spaces and is related to earlier ideas of Curto–Salinas [4]. Here $A(\Omega)$ is the uniform closure of functions holomorphic on a neighborhood of the closure of Ω, a domain in $\mathbb{C}^m$. The Hilbert space $\mathcal{M}$ is said to be a bounded Hilbert module over $A(\Omega)$ if $\mathcal{M}$ is a unital module over $A(\Omega)$ with module map $A(\Omega) \times \mathcal{M} \to \mathcal{M}$ such that

$$\|\varphi f\|_{\mathcal{M}} \leq C \,\|\varphi\|_{A(\Omega)} \|f\|_{\mathcal{M}}$$

for φ in $A(\Omega)$ and f in $\mathcal{M}$ and some $C \geq 1$. The Hilbert module is said to be contractive in case $C = 1$.

A Hilbert space $\mathcal{R}$ is said to be a bounded quasi-free Hilbert module of rank n over $A(\Omega)$, $1 \leq n \leq \infty$, if it is obtained as the completion of the algebraic tensor product $A(\Omega) \otimes \ell^2_n$ relative to an inner product such that:

(1) $\mathrm{eval}_{\boldsymbol{z}}\colon\ A(\Omega) \otimes l^2_n \to l^2_n$ is bounded for $\boldsymbol{z}$ in Ω and locally uniformly bounded on Ω;
(2) $\|\varphi(\sum \theta_i \otimes x_i)\|_{\mathcal{R}} = \| \sum \varphi\theta_i \otimes x_i\|_{\mathcal{R}} \leq C \,\|\varphi\|_{A(\Omega)}\| \sum \theta_i \otimes x_i\|_{\mathcal{R}}$ for φ, $\{\theta_i\}$ in $A(\Omega)$ and $\{x_i\}$ in ℓ^2_n and some $C \geq 1$; and
(3) For $\{F_i\}$ a sequence in $A(\Omega) \otimes \ell^2_n$ which is Cauchy in the $\mathcal{R}$-norm, it follows that $\mathrm{eval}_{\boldsymbol{z}}(F_i) \to 0$ for all $\boldsymbol{z}$ in Ω if and only if $\|F_i\|_{\mathcal{R}} \to 0$.

If $I_{\boldsymbol{\omega}_0}$ denotes the maximal ideal of polynomials in $\mathbb{C}[\boldsymbol{z}] = \mathbb{C}[z_1, \ldots, z_m]$ which vanish at $\boldsymbol{\omega}_0$ for some $\boldsymbol{\omega}_0$ in Ω, then the Hilbert module $\mathcal{M}$ is said to be semi-Fredholm at $\boldsymbol{\omega}_0$ if $\dim \mathcal{M}/I_{\boldsymbol{\omega}_0} \cdot \mathcal{M} = n$ is finite (cf. [10]). In particular, note that $\mathcal{M}$ semi-Fredholm at $\boldsymbol{\omega}_0$ implies that $I_{\boldsymbol{\omega}_0}\mathcal{M}$ is a closed submodule of $\mathcal{M}$. Note that the notion of semi-Fredholm Hilbert module has been called regular by some authors.

One can show that $\boldsymbol{\omega} \to \mathcal{R}/I_{\boldsymbol{\omega}} \cdot \mathcal{R}$ can be made into a rank n Hermitian holomorphic vector bundle over Ω if $\mathcal{R}$ is semi-Fredholm at $\boldsymbol{\omega}$ in Ω, $\dim \mathcal{R}/I_{\boldsymbol{\omega}} \cdot \mathcal{R}$ is constant n, and $\mathcal{R}$ is quasi-free, $1 \leq n < \infty$. Actually, all we need here is that the bundle obtained is real-analytic which is established in ([4], Theorem 2.2).

A quasi-free Hilbert module of rank n is a reproducing kernel Hilbert space with the kernel

$$K(\boldsymbol{w}, \boldsymbol{z}) = \mathrm{eval}_{\boldsymbol{w}} \mathrm{eval}^*_{\boldsymbol{z}}\colon\ \Omega \times \Omega \to \mathcal{L}(\ell^2_n).$$

2. Necessary conditions

Note that if $\mathcal{R}$ is a bounded quasi-free Hilbert module over $A(\mathbb{B}^m)$ of finite multiplicity, then the module $\mathcal{R}$ over $A(\mathbb{B}^m)$ extends to a bounded Hilbert module over $H^\infty(\mathbb{B}^m)$ (see Proposition 5.2 in [5]). Here $\mathbb{B}^m$ denotes the unit ball in $\mathbb{C}^m$. In particular, the multiplier space of $\mathcal{R}$ is precisely $H^\infty(\mathbb{B}^m) \otimes \mathcal{M}_n(\mathbb{C})$, since $\mathcal{R}$ is the completion of $A(\Omega) \otimes_{\mathrm{alg}} l^2_n$, by definition.

Proposition 1. *Let $\mathcal{R}$ be a contractive quasi-free Hilbert module over $A(\mathbb{B}^m)$ of finite multiplicity n and $\{\varphi_1, \dots, \varphi_k\}$ be a commutator subset of $H^\infty(\mathbb{B}^m) \otimes \mathcal{M}_n(\mathbb{C})$. If $(M_{\varphi_1}, \dots, M_{\varphi_k})$ is a semi-Fredholm tuple, then there exists an $\epsilon > 0$ and $1 > \delta > 0$ such that*

$$\sum_{i=1}^{k} \varphi_i(\boldsymbol{z})\varphi_i(\boldsymbol{z})^* \geq \epsilon I_{\mathbb{C}^n},$$

for all $\boldsymbol{z}$ satisfying $1 > \|\boldsymbol{z}\| \geq 1 - \delta > 0$. In particular, if the multiplicity of $\mathcal{R}$ is one then

$$\sum_{i=1}^{k} |\varphi_i(\boldsymbol{z})|^2 \geq \epsilon,$$

for all $\boldsymbol{z}$ satisfying $1 > \|\boldsymbol{z}\| \geq 1 - \delta$.

Proof. Let $K : \mathbb{B}^m \times \mathbb{B}^m \to \mathcal{M}_n(\mathbb{C})$ be the kernel function for the quasi-free Hilbert module $\mathcal{R}$. By the assumption, the range of the row operator $M_\Phi = (M_{\varphi_1}, \dots, M_{\varphi_k})$ in $\mathcal{L}(\mathcal{R}^k, \mathcal{R})$ has finite co-dimension; that is,

$$\dim[\mathcal{R}/(M_{\varphi_1}\mathcal{R} + \cdots + M_{\varphi_k}\mathcal{R})] < \infty,$$

and, in particular, M_Φ has closed range. Consequently, there is a finite rank projection F such that

$$M_\Phi M_\Phi^* + F = \sum_{i=1}^{k} M_{\varphi_i} M_{\varphi_i}^* + F : \mathcal{R} \to \mathcal{R}$$

is bounded below. Therefore, there exists a $C > 0$ such that

$$\langle F K_{\boldsymbol{z}}, K_{\boldsymbol{z}} \rangle + \left\langle \sum_{i=1}^{k} M_{\varphi_i} M_{\varphi_i}^* K_{\boldsymbol{z}}, K_{\boldsymbol{z}} \right\rangle \geq C \langle K_{\boldsymbol{z}}, K_{\boldsymbol{z}} \rangle,$$

for all $\boldsymbol{z}$ in $\mathbb{B}^m$. Then

$$K_{\boldsymbol{z}}^* \hat{F}(\boldsymbol{z}) K_{\boldsymbol{z}} + \sum_{i=1}^{k} K_{\boldsymbol{z}}^* M_{\varphi_i} M_{\varphi_i}^* K_{\boldsymbol{z}} \geq C K_{\boldsymbol{z}}^* K_{\boldsymbol{z}},$$

and so

$$\hat{F}(z) I_{\mathbb{C}^n} + \sum_{i=1}^{k} \varphi_i(\boldsymbol{z})\varphi_i(\boldsymbol{z})^* \geq C I_{\mathbb{C}^n},$$

for all $\boldsymbol{z}$ in $\mathbb{B}^m$. Here $\hat{F}(\boldsymbol{z})$ denotes the matrix-valued Berezin transform for the operator F defined by $\hat{F}(\boldsymbol{z}) =< F K_{\boldsymbol{z}} |K_{\boldsymbol{z}}|^{-1}, K_{\boldsymbol{z}} |K_{\boldsymbol{z}}|^{-1} >$ (see [5], where the scalar case is discussed). Using the known boundary behavior of the Berezin transform (see Theorem 3.2 in [5]), since F is finite rank we have that $\|\hat{F}(\boldsymbol{z})\| \leq \frac{C}{2}$ for all $\boldsymbol{z}$ such that $1 > \|\boldsymbol{z}\| > 1 - \delta$ for some $1 > \delta > 0$ depending on C. Hence

$$\sum_{i=1}^{k} \varphi_i(\boldsymbol{z})\varphi_i(\boldsymbol{z})^* \geq \frac{C}{2},$$

for all $\boldsymbol{z}$ such that $1 > \|\boldsymbol{z}\| > 1 - \delta > 0$; which completes the proof. $\square$

A k-tuple of matrix-valued functions $(\varphi_1,\dots,\varphi_k)$ in $H^\infty(\mathbb{B}^m)\otimes M_n(\mathbb{C})$ satisfying the conclusion of Proposition 1 will be said to have the *weak Corona property.*

In Theorem 8.2.6 in [11], a version of Proposition 1 is established in case $\mathcal{R}$ is the Bergman module on $\mathbb{B}^m$.

The key step in this proof is the vanishing of the Berezin transform at the boundary of $\mathbb{B}^m$ for a compact operator. The proof of this statement depends on the fact that $K_{\boldsymbol{z}}|K_{\boldsymbol{z}}|^{-1}$ converges weakly to zero as $\boldsymbol{z}$ approaches the boundary which rests on the fact that $\mathcal{R}$ is contractive. This relation holds for many other domains such as ellipsoids Ω with the proof depending on the fact that the algebra $A(\Omega)$ is pointed in the sense of [5].

It is an important question to decide if semi-Fredholm implies Fredholm in the context of Proposition 1. We will discuss this issue more at the end of the paper. However, the converse of this result is known (see Theorem 8.2.4 in [11] and pages 241–242) for the Bergman space for certain domains in $\mathbb{C}^m$.

A necessary condition for the converse to hold for the situation in Proposition 1 is for the essential spectrum of the m-tuple of co-ordinate multiplication operators to have essential spectrum equal to $\partial\mathbb{B}^m$, which is not automatic, but is true for the classical spaces.

3. Sufficient conditions

We will use the following fundamental result of Taylor (see [14], Lemma 1):

Lemma 1. *Let $(T_1,\dots,T_k)$ be in the center of an algebra $\mathcal{A}$ contained in $\mathcal{L}(\mathcal{H})$ such that there exists $(S_1,\dots,S_k)$ in $\mathcal{A}$ satisfying $\sum_{i=1}^k T_iS_i = I_{\mathcal{H}}$. Then the Koszul complex for $(T_1,\dots,T_k)$ is exact.*

Now we specialize to the case when $m=1$ where we can obtain a necessary and sufficient condition. Consider a contractive quasi-free Hilbert module $\mathcal{R}$ over $A(\mathbb{D})$ of multiplicity one, which therefore has $H^\infty(\mathbb{D})$ as the multiplier algebra. It is well known that $H^\infty(\mathbb{D})$ satisfies the *Corona property*; that is, a set $\{\varphi_1,\dots,\varphi_k\}$ in $H^\infty(\mathbb{D})$ satisfies $\sum_{i=1}^k|\varphi_k(z)|\geq\epsilon$ for all z in $\mathbb{D}$ for some $\epsilon>0$ if and only if there exist $\{\psi_1,\dots,\psi_k\}\subset H^\infty(\mathbb{D})$ such that $\sum_{i=1}^k\varphi\psi_i=1$.

The following result is a complement to Proposition 1.

Proposition 2. *Let $\mathcal{R}$ be a contractive quasi-free Hilbert module over $A(\mathbb{D})$ of multiplicity one and $\{\varphi_1,\dots,\varphi_k\}$ be a subset of $H^\infty(\mathbb{D})$. Then the Koszul complex for the k-tuple $(M_{\varphi_1},\dots,M_{\varphi_k})$ on $\mathcal{R}$ is exact if and only if $\{\varphi_1,\dots,\varphi_k\}$ satisfies the Corona property.*

Proof. If $\sum_{i=1}^k\varphi_i\psi_i=1$ for some $\{\psi_1,\dots,\psi_k\}\subset H^\infty(\mathbb{D})$, then the fact that M_Φ is Taylor invertible follows from Lemma 1. On the other hand, the last group of the Koszul complex is $\{0\}$ if and only if the row operator M_φ in $\mathcal{L}(\mathcal{R}^k,\mathcal{R})$ is bounded below which, as before, shows that $\sum_{i=1}^k|\varphi_i(z)|$ is bounded below on $\mathbb{D}$. This completes the proof. $\square$

The missing step to extend the result from $\mathbb{D}$ to the open unit ball $\mathbb{B}^m$ is the fact that it is unknown if the Corona condition for $\{\varphi_1, \ldots, \varphi_k\}$ in $H^\infty(\mathbb{B}^m)$ is equivalent to the Corona property. Other authors have considered this kind of question ([15]) for the case of Hardy-like spaces for the polydisk and ball. See [15] for some recent results and references.

Theorem 1. *Let $\mathcal{R}$ be a contractive quasi-free Hilbert module over $A(\mathbb{D})$ of multiplicity one, which is semi-Fredholm at each point z in $\mathbb{D}$. If $\{\varphi_1, \ldots, \varphi_k\}$ is a subset of $H^\infty(\mathbb{D})$, then the k-tuple $M_\Phi = (M_{\varphi_1}, \ldots, M_{\varphi_k})$ is semi-Fredholm if and only if it is Fredholm if and only if $(\varphi_1, \ldots, \varphi_k)$ satisfies the weak Corona condition.*

Proof. If M_Φ is semi-Fredholm, then by Proposition 1 there exist $\epsilon > 0$ and $1 > \delta > 0$ such that

$$\sum_{i=1}^k |\varphi_i(z)|^2 \geq \epsilon,$$

for all z such that $1 > |z| > 1 - \delta > 0$. Let $\mathcal{Z}$ be the set

$$\mathcal{Z} = \{z \text{ in } \mathbb{D} : \varphi_i(z) = 0 \text{ for all } i = 1, \ldots, k\}.$$

Since the functions $\{\varphi_i\}_{i=1}^k$ can not all vanish for z satisfying $1 > |z| > 1 - \delta$, it follows that the cardinality of the set $\mathcal{Z}$ is finite and we assume that $\operatorname{card}(\mathcal{Z}) = N$. Let

$$\mathcal{Z} = \{z_1, z_2, \ldots, z_N\}$$

and l_j be the smallest order of the zero at z_j for all φ_j and $1 \leq j \leq k$. Let $B(z)$ be the finite Blaschke product with zero set precisely $\mathcal{Z}$ counting the multiplicities. If we define $\xi_i = \frac{\varphi_i}{B}$, then ξ_i is in $H^\infty(\mathbb{D})$ for all $i = 1, \ldots, k$. Since $\{\varphi_1, \ldots, \varphi_k\}$ satisfies the weak Corona property, we obtain

$$\sum_{i=1}^k |\xi_i(z)|^2 \geq \epsilon$$

for all z such that $1 > |z| > 1 - \delta$. Note that $\{\xi_1, \ldots, \xi_n\}$ does not have any common zero and so

$$\sum_{i=1}^k |\xi_i(z)|^2 \geq \epsilon$$

for all z in $\mathbb{D}$. Therefore, $\{\xi_1, \ldots, \xi_k\}$ satisfies the Corona property and hence there exists $\{\eta_1, \ldots, \eta_k\}$, a subset of $H^\infty(\mathbb{D})$, such that $\sum_{i=1}^k \xi_i(z)\eta_i(z) = 1$ for all z in $\mathbb{D}$. Thus, $\sum_{i=1}^k \varphi_i(z)\eta_i(z) = B$ for all z in $\mathbb{D}$. This implies $\sum_{i=1}^k M_{\varphi_i} M_{\eta_i} = M_B$, and consequently,

$$\sum_{i=1}^k \overline{M}_{\varphi_i} \overline{M}_{\eta_i} = \overline{M_B},$$

where $\overline{M}_{\varphi_i}$ is the image of M_{φ_i} in the Calkin algebra, $\mathcal{Q}(\mathcal{R}) = \mathcal{L}(\mathcal{R})/\mathcal{K}(\mathcal{R})$. But the assumption that M_{z-w} is Fredholm for all w in $\mathbb{D}$ yields that M_B is Fredholm. Therefore, $X = \sum_{i=1}^k \overline{M}_{\varphi_i} \overline{M}_{\eta_i}$ is invertible. Moreover, since X commutes with the set $\{\overline{M}_{\varphi_1}, \ldots, \overline{M}_{\varphi_k}, \overline{M}_{\eta_1}, \ldots, \overline{M}_{\eta_k}\}$, it follows that $(M_{\varphi_1}, \ldots, M_{\varphi_k})$ is a Fredholm tuple, which completes the proof. $\square$

Although, the use of a finite Blaschke product allows one to preserve norms, a polynomial with the zeros of $\mathcal{Z}$ to the same multiplicity could be used. This would allow one to extend the Theorem to all domains in $\mathbb{C}$ for which the Corona theorem holds.

Our previous result extends to the case of finite multiplicity quasi-free Hilbert modules.

Theorem 2. *Let $\mathcal{R}$ be a contractive quasi-free Hilbert module over $A(\mathbb{D})$ of multiplicity n, which is semi-Fredholm at each point z in $\mathbb{D}$ and let $\{\varphi_1, \ldots, \varphi_k\}$ be a commutator subset of $H^\infty(\mathbb{D}) \otimes M_n(\mathbb{C})$. Then the k-tuple $M_\Phi = (M_{\varphi_1}, \ldots, M_{\varphi_k})$ is Fredholm if and only if it is semi-Fredholm if and only if $(\varphi_1, \ldots, \varphi_k)$ satisfies the weak Corona condition.*

Proof. As before, the assumption that M_Φ is semi-Fredholm implies that there exists $\epsilon > 0$ and $1 > \delta > 0$ such that

$$\sum_{i=1}^{k} \varphi_i(z)\varphi_i(z)^* \geq \epsilon I_{\mathbb{C}^n},$$

for all z such that $1 > \|z\| > 1 - \delta$. After taking the determinant, this inequality implies

$$\sum_{i=1}^{k} |\det \varphi_i(z)|^2 \geq \epsilon^n.$$

Using the same argument as in Theorem 1, we can find $\eta_1, \ldots, \eta_k$ in $H^\infty(\mathbb{D})$ and a finite Blaschke product B such that

$$\sum_{i=1}^{k} \eta_i(z) \det \varphi_i(z) = B(z),$$

for all z in $\mathbb{D}$. For $1 \leq i \leq k$, let $\hat{\varphi}_i(z)$ be the cofactor matrix function of $\varphi_i(z)$ which is used in Cramer's Theorem. Then

$$\hat{\varphi}_i(z)\phi_i(z) = \phi_i(z)\hat{\varphi}_i(z) = \det \varphi_i(z)\, I_{\mathbb{C}^n},$$

for all z in $\mathbb{D}$ and $1 \leq i \leq k$. Note that this relation implies that the algebra generated by the set $\{M_{\varphi_1}, \ldots, M_{\varphi_k}, M_{\hat{\varphi}_1}, \ldots, M_{\hat{\varphi}_k}\}$ is commutative. Thus we obtain

$$\sum_{i=1}^{k} \phi_i(z)\, \eta_i(z)\, \hat{\varphi}_i(z) = B(z) I_{\mathbb{C}^n}, \quad \text{or} \quad \sum_{i=1}^{k} \phi_i(z)\hat{\eta_i}(z) = B(z) I_{\mathbb{C}^n},$$

where $\hat{\eta_i}(z) = \eta_i(z)\hat{\varphi}_i(z)$ is in $H^\infty(\mathbb{D}) \otimes \mathcal{M}_n(\mathbb{C})$ and $1 \leq i \leq k$. Therefore we have that

$$\sum_{i=1}^{k} M_{\varphi_i} M_{\hat{\eta_i}} = M_B,$$

and consequently, the proof follows immediately from the last part of the proof of Theorem 1. □

4. Further comments

One reason we are able to obtain a converse in the one variable case is that we can represent the zero variety of the ideal generated by the functions in terms of a single function, the finite Blaschke product (or polynomial). This is not surprising since $\mathbb{C}[z]$ is a principal ideal domain. This is, of course, not true for $\mathbb{C}[z_1, \ldots, z_m]$ for $m > 1$ and hence one would need (at least) a finite set of functions to determine the zero variety for the ideal generated by the functions. How to do that in an efficient manner and how to relate the Fredholmness of the k-tuple to that of this generating set is not clear but is the key to answering many such questions.

What is required involves two steps, both in the realm of algebra. The first we have already mentioned but the second is how to relate the generators to the Koszul complex.

Let us consider one example of what might be possible. Consider the case in which the $p_1(\boldsymbol{z}), \ldots, p_k(\boldsymbol{z})$ are polynomials in $\mathbb{C}[z_1, z_2]$ so that $\mathbf{0}$ is the only common zero. Assume that there are sets of polynomials $\{q_1(\boldsymbol{z}), \ldots, q_k(\boldsymbol{z})\}$ and $\{r_1(\boldsymbol{z}), \ldots, r_k(\boldsymbol{z})\}$ such that

$$\sum_{i=1}^{k} p_i(\boldsymbol{z}) q_i(\boldsymbol{z}) = z_1^{k_1} \quad \text{and} \quad \sum_{i=1}^{k} p_i(\boldsymbol{z}) r_i(\boldsymbol{z}) = z_2^{k_2},$$

for some positive integers k_1 and k_2.

Two questions now arise:

(1) Does the assumption that $(M_{p_1}, \ldots, M_{p_k})$ is semi-Fredholm with $\mathcal{Z} = \{\mathbf{0}\}$ imply the existence of the subsets $\{r_1, \ldots, r_k\}$ and $\{q_1, \ldots, q_k\}$ of $\mathbb{C}[z_1, z_2]$? What if the functions $\{p_1, \ldots, p_k\}$ are in $H^\infty(\mathbb{B}^2)$ and we seek $\{r_1, \ldots, r_k\}$ and $\{q_1, \ldots, q_k\}$ in $H^\infty(\mathbb{B}^2)$?

(2) If the functions $\{r_1, \ldots, r_k\}$ and $\{q_1, \ldots, q_k\}$ exist and we assume that $(M_{z_1^{k_1}}, M_{z_2^{k_2}})$ acting on the quasi-free Hilbert module $\mathcal{R}$ is Fredholm, does it follow that $(M_{p_1}, \ldots, M_{p_k})$ is also.

These questions can be generalized to the case where one would need more than two polynomials to determine the zero variety, either because the dimension m is greater than 2 or because $\mathcal{Z}$ contains more than one point. But answering these questions in the simple case discussed above would be good start.

After this note was written, J. Eschmeier informed the authors that both questions have an affirmative answer, at least when the zero variety is a single point.

References

[1] S. Axler, *Multiplication operators on Bergman spaces*, J. Reine Angew. Math. 336 (1982), 26–44.

[2] L. Boutet de Monvel, *On the index of Toeplitz operators of several complex variables*, Invent. Math. 50 (1978/79), no. 3, 249–272.

[3] M.J. Cowen and R.G. Douglas, *Complex geometry and operator theory*, Acta Math. 141 (1978), 187–261;

[4] R. Curto and N. Salinas, *Generalized Bergman kernels and the Cowen-Douglas theory*, Amer. J. Math. 106 (1984), 447–488.

[5] K.R. Davidson and R.G. Douglas, *The generalized Berezin transform and commutator ideals*, Pacific J. Math. 222 (2005), no. 1, 29–56;

[6] R.G. Douglas and G. Misra, *On quasi-free Hilbert modules*, New York J. Math. 11 (2005), 547–561;

[7] R.G. Douglas and G. Misra, *Quasi-free resolutions of Hilbert modules*, Integral Equations Operator Theory 47 (2003), no. 4, 435–456;

[8] R.G. Douglas and V.I. Paulsen, *Hilbert Modules over Function Algebras*, Research Notes in Mathematics Series, 47, Longman, Harlow, 1989;

[9] R.G. Douglas, *Banach algebra techniques in operator theory*, Pure and Applied Math. 49, Academic Press, New York, 1972.

[10] J. Eschmeier, *On the Hilbert-Samuel Multiplicity of Fredholm Tuples*, Indiana Univ. Math. J., 56 (2007), no. 3, 1463–1477.

[11] J. Eschmeier and M. Putinar, *Spectral decompositions and analytic sheaves*, London Mathematical Society Monographs. New Series, 10. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1996.

[12] I.C. Gohberg, *On an application of the theory of normed rings to singular integral equations*, Uspehi Matem. Nauk (N.S.) 7, (1952). no. 2(48), 149–156.

[13] I.C. Gohberg and M.G. Krein, *Fundamental aspects of defect numbers, root numbers and indexes of linear operators*, Uspehi Mat. Nauk (N.S.) 12, (1957) no. 2(74), 43–118.

[14] J.L. Taylor, *A joint spectrum for several commuting operators*, J. Functional Analysis 6 (1970), 172–191;

[15] T.T. Trent and B.D. Wick, *Toeplitz Corona Theorems for the Polydisk and the Unit Ball*, arXiv:0806.3428

[16] U. Venugopalkrishna, *Fredholm operators associated with strongly pseudoconvex domains in* $\mathbb{C}^n$. J. Functional Analysis 9 (1972), 349–373;

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Received: February 27, 2009.
Accepted: June 13, 2009.

Operator Theory:
Advances and Applications, Vol. 202, 151–194

The $\mathcal{S}$-recurrence of Schur Parameters of Non-inner Rational Schur Functions

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Abstract. The main goal of this paper is to investigate the Schur parameter sequences of non-inner rational Schur functions. It is shown that these Schur parameter sequences are characterized by the membership in the space l_2 and a particular recurrence property which is called $\mathcal{S}$-recurrence. Moreover, we indicate a principle of extending a finite sequence of numbers from the open unit disk to an $\mathcal{S}$-recurrence sequence belonging to l_2.

Mathematics Subject Classification (2000). 30D50, 47A48, 47A45.

Keywords. Non-inner rational Schur functions, Schur algorithm, Schur parameter sequences, $\mathcal{S}$-recurrent sequences.

0. Introduction

This paper deals with particular aspects connected to the classical Schur algorithm which was introduced in I. Schur's fundamental paper [9]. Our main goal is to study the Schur parameter sequences of the non-inner rational Schur functions. The inner rational Schur functions are exactly the finite Blaschke products. As was shown by I. Schur in [9], if θ is a Schur function, then the Schur algorithm for θ terminates after a finite number n of steps if and only if θ is a finite Blaschke product of degree n. Thus, the Schur parameter sequence of a finite Blaschke product is finite. Surprisingly, we could not find anything in the mathematical literature about the Schur parameter sequences of non-inner rational Schur functions. The starting point of our investigations is the first author's recent research [4] on the Schur parameter sequences of pseudocontinuable non-inner Schur functions. It will turn out that the machinery developed in [4] can be used to obtain many insights into the structure of the Schur parameter sequences of non-inner rational Schur functions. (What concerns comprehensive expositions of many aspects and facets of the Schur algorithm we refer the reader to the monographs by D. Alpay [1],

Communicated by J.A. Ball.

C. Foias/A. Frazho [6], B. Simon [10], [11], and S.N. Khrushchev [8], and the references therein.)

In order to review the content of this paper in more detail, first we roughly sketch the classical Schur algorithm.

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk of the complex plane $\mathbb{C}$. The symbol $\mathcal{S}$ denotes the set of all Schur functions in $\mathbb{D}$, i.e., the set of all functions $\theta : \mathbb{D} \to \mathbb{C}$ which are holomorphic in $\mathbb{D}$ and satisfy the condition $|\theta(\zeta)| \le 1$ for all $\zeta \in \mathbb{D}$. A function $\theta \in \mathcal{S}$ is called inner if its boundary values are unimodular almost everywhere with respect to the Lebesgue measure on the unit circle. The symbol $\mathcal{RS}$ (resp. J) stands for the subset of $\mathcal{S}$ which consists of all rational (resp. inner) functions belonging to $\mathcal{S}$. The simplest rational inner functions are the elementary Blaschke factors. For $a \in \mathbb{D}$ the elementary Blaschke factor b_a is the rational function given by

$$b_a(\zeta) := \frac{\zeta - a}{1 - \overline{a}\zeta}.$$

A finite product of elementary Blaschke factors multiplied by a unimodular constant is called finite Blaschke product. Thus, the intersection $\mathcal{RS} \cap J$ consists of all finite Blaschke products.

Let $\theta \in \mathcal{S}$. Following I. Schur [9], we set $\theta_0 := \theta$ and $\gamma_0 := \theta_0(0)$. Obviously, $|\gamma_0| \le 1$. If $|\gamma_0| < 1$, then we consider the function $\theta_1 : \mathbb{D} \to \mathbb{C}$ defined by

$$\theta_1(\zeta) := \frac{1}{\zeta} \cdot \frac{\theta_0(\zeta) - \gamma_0}{1 - \overline{\gamma_0}\theta_0(\zeta)}.$$

In view of the Lemma of H.A. Schwarz, we have $\theta_1 \in \mathcal{S}$. As above we set $\gamma_1 := \theta_1(0)$ and if $|\gamma_1| < 1$, we consider the function $\theta_2 : \mathbb{D} \to \mathbb{C}$ defined by

$$\theta_2(\zeta) := \frac{1}{\zeta} \cdot \frac{\theta_1(\zeta) - \gamma_1}{1 - \overline{\gamma_1}\theta_1(\zeta)}.$$

Further, we continue this procedure inductively. Namely, if in the jth step a function θ_j occurs for which $|\gamma_j| < 1$ where $\gamma_j := \theta_j(0)$, we define $\theta_{j+1} : \mathbb{D} \to \mathbb{C}$ by

$$\theta_{j+1}(\zeta) := \frac{1}{\zeta} \cdot \frac{\theta_j(\zeta) - \gamma_j}{1 - \overline{\gamma_j}\theta_j(\zeta)}. \tag{0.1}$$

and continue this procedure in the prescribed way. Then setting $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ two cases are possible:

(1) The procedure can be carried out without end, i.e., $|\gamma_j| < 1$ for each $j \in \mathbb{N}_0$.
(2) There exists a $w \in \mathbb{N}_0$ such that $|\gamma_w| = 1$ and, if $w > 0$, then $|\gamma_j| < 1$ for each $j \in \{0, \ldots, w-1\}$.

Thus, a sequence $(\gamma_j)_{j=0}^{w}$ is associated with each function $\theta \in \mathcal{S}$. Here we have $w = \infty$ (resp. $w = n$) in the first (resp. second) case. From I. Schur's paper [9] it is known that the second case occurs if and only if θ is a finite Blaschke product of degree n. Consequently, condition (2) provides a complete description of all parameter sequences $(\gamma_j)_{j=0}^{w}$ which correspond to functions of the class $\mathcal{RS} \cap J$.

The above procedure is called a Schur algorithm and the sequence $(\gamma_j)_{j=0}^{w}$ obtained here is called the Schur parameter sequence associated with the function θ, whereas for each $j \in \{0, \ldots, w\}$ the function θ_j is called *the jth Schur transform of θ*. The symbol Γ stands for the set of all Schur parameter sequences associated with functions belonging to $\mathcal{S}$.

The following two properties established by I. Schur in [9] determine the particular role which Schur parameters play in the study of functions of class $\mathcal{S}$.

(a) Each sequence $(\gamma_j)_{j=0}^{w}$ of complex numbers, $0 \leq w \leq \infty$, which satisfies one of the conditions (1) or (2) belongs to Γ.
(b) There is a one-to-one correspondence between the sets $\mathcal{S}$ and Γ.

Thus, the Schur parameters are independent parameters which completely determine the functions of class $\mathcal{S}$.

Now we take a look at the class $\mathcal{RS}\backslash J$ from the perspective of the Schur algorithm. Let $\theta \in \mathcal{RS}\backslash J$. and let $(\gamma_j)_{j=0}^{\infty}$ be its Schur parameter sequence. From the shape of formula (0.1) it follows immediately that each member of the sequence $(\theta_j)_{j=0}^{\infty}$ belongs to $\mathcal{RS}\backslash J$, too. Taking into account that for each $j \in \mathbb{N}_0$ the function θ_j has the Schur parameter sequence $(\gamma_{j+k})_{k=0}^{\infty}$, we see that the elimination of an arbitrary first finite section from the Schur parameter sequence $(\gamma_k)_{k=0}^{\infty}$ does not effect the membership of the corresponding function having the reduced sequence as Schur parameter sequence to the class $\mathcal{RS}\backslash J$.

On the other hand, for each $\zeta \in \mathbb{D}$, the relation (0.1) can be rewritten in the form

$$\theta_j(\zeta) = \frac{\zeta\theta_{j+1}(\zeta) + \gamma_j}{1 + \overline{\gamma_j}\zeta\theta_{j+1}(\zeta)}.$$

From this we see that if we replace the sequence $(\gamma_j)_{j=0}^{\infty}$ by the sequence $(\gamma_{-1+j})_{j=0}^{\infty}$ where $|\gamma_{-1}| < 1$, i.e., if we consider the function $\theta_{-1} : \mathbb{D} \to \mathbb{C}$ defined by

$$\theta_{-1}(\zeta) := \frac{\zeta\theta(\zeta) + \gamma_{-1}}{1 + \overline{\gamma_{-1}}\zeta\theta(\zeta)},$$

then we get again a function θ_{-1} belonging to $\mathcal{RS}\backslash J$. Thus, adding a finite number of elements from $\mathbb{D}$ to the sequence $(\gamma_j)_{j=0}^{\infty}$ does not effect the membership of the associated function belonging to the class $\mathcal{RS}\backslash J$.

Let $n \in \mathbb{N}_0$ and let $(\gamma_j)_{j=0}^{n}$ be a sequence from $\mathbb{D}$. Then our previous considerations lead us to the following result which in particular contains a complete description of all functions from $\mathcal{RS}\backslash\ J$ having $(\gamma_j)_{j=0}^{n}$ as the sequence of its first $n+1$ Schur parameters.

Proposition 0.1. *Let $n \in \mathbb{N}_0$ and let $(\gamma_j)_{j=0}^{n}$ be a sequence from $\mathbb{D}$. Further, let $P_{(\gamma_j)_{j=0}^{n}} : \mathbb{C} \to \mathbb{C}^{2\times 2}$ be defined by*

$$P_{(\gamma_j)_{j=0}^{n}}(\zeta) := \prod_{j=0}^{n} \begin{pmatrix} \zeta & \gamma_j \\ \overline{\gamma_j}\zeta & 1 \end{pmatrix} \qquad \text{and let} \qquad P_{(\gamma_j)_{j=0}^{n}} = \begin{pmatrix} a_{(\gamma_j)_{j=0}^{n}} & b_{(\gamma_j)_{j=0}^{n}} \\ c_{(\gamma_j)_{j=0}^{n}} & d_{(\gamma_j)_{j=0}^{n}} \end{pmatrix}.$$

Denote by $[\mathcal{S}\backslash(\mathcal{RS}\cap J)]((\gamma_j)_{j=0}^n)$ *the set of all functions belonging to* $\mathcal{S}\backslash(\mathcal{RS}\cap J)$ *which have* $(\gamma_j)_{j=0}^n$ *as the sequence of their first* $n+1$ *Schur parameters.*

(a) *Let* $\theta \in [\mathcal{S}\backslash(\mathcal{RS}\cap J)]((\gamma_j)_{j=0}^n)$ *and let* θ_{n+1} *be the* $(n+1)$*th Schur transform of* θ. *Then* $\theta_{n+1} \in \mathcal{S}\backslash(\mathcal{RS}\cap J)$ *and*
$$\theta = \frac{a_{(\gamma_j)_{j=0}^n}\cdot\theta_{n+1} + b_{(\gamma_j)_{j=0}^n}}{c_{(\gamma_j)_{j=0}^n}\cdot\theta_{n+1} + d_{(\gamma_j)_{j=0}^n}}.$$
If $\theta \in [\mathcal{S}\backslash(\mathcal{RS}\cap J)]((\gamma_j)_{j=0}^n)\cap(\mathcal{RS}\backslash J)$, *then* $\theta_{n+1}\in\mathcal{RS}\backslash J$.

(b) *Let* $g\in\mathcal{S}\backslash(\mathcal{RS}\cap J)$. *Then*
$$\theta := \frac{a_{(\gamma_j)_{j=0}^n}\cdot g + b_{(\gamma_j)_{j=0}^n}}{c_{(\gamma_j)_{j=0}^n}\cdot g + d_{(\gamma_j)_{j=0}^n}}$$
belongs to $[\mathcal{S}\backslash(\mathcal{RS}\cap J)]((\gamma_j)_{j=0}^n)$ *and* g *coincides with the* $(n+1)$*th Schur transform* θ_{n+1} *of* θ. *If* $g\in\mathcal{RS}\backslash J$, *then* $\theta\in\mathcal{RS}\backslash J$.

(c) *The function*
$$\theta_{(\gamma_j)_{j=0}^n} := \frac{b_{(\gamma_j)_{j=0}^n}}{d_{(\gamma_j)_{j=0}^n}}. \tag{0.2}$$
belongs to $\mathcal{RS}\backslash J$ *and has the Schur parameter sequence* $\gamma_0,\ldots,\gamma_n,0,0,\ldots$.

It should be mentioned that the function defined in (0.2) was already studied by I. Schur in [9]. In the framework of the investigation of the matricial version of the classical Schur problem the matricial generalization of this function was studied with respect to several aspects (see, e.g., the paper [7] where its entropy extremality was proved).

Let $\theta\in\mathcal{S}$ and let
$$\theta(\zeta) = \sum_{j=0}^{\infty} c_j\zeta^j,\quad \zeta\in\mathbb{D}, \tag{0.3}$$
be the Taylor series representation of θ. Moreover, let $(\gamma_j)_{j=0}^w$ be the Schur parameter sequence associated with θ. As it was shown by I. Schur in [9], for each integer n satisfying $0\le n<w$, the identities
$$\gamma_n = \Phi_n(c_0,c_1,\ldots,c_n) \tag{0.4}$$
and
$$c_n = \Psi_n(\gamma_0,\gamma_1,\ldots,\gamma_n) \tag{0.5}$$
hold true. Here, I. Schur presented an explicit description of the function Φ_n. For the function Ψ_n, he obtained the formula
$$\Psi_n(\gamma_0,\gamma_1,\ldots,\gamma_n) = \gamma_n\cdot\prod_{j=0}^{n-1}\left(1-|\gamma_j|^2\right) + \widetilde{\Psi}_{n-1}(\gamma_0\ldots,\gamma_{n-1}) \tag{0.6}$$
where $\widetilde{\Psi}_{n-1}$ is a polynomial of the variables γ_0, $\overline{\gamma_0}$, $\ldots$, γ_{n-1}, $\overline{\gamma_{n-1}}$.

It should be mentioned that the explicit form of the functions $\widetilde{\Psi}_{n-1}$ was described in [3]. Thus, for every integer n satisfying $0 \leq n < w$, the sequences $(c_k)_{k=0}^n$ and $(\gamma_k)_{k=0}^n$ can each be expressed in terms of the other.

We are interested in the rational functions belonging to $\mathcal{S}$. According to a well-known criterion (see, e.g., Proposition 1.1 in [2]), the power series (0.3) corresponds to a rational function if and only if there exist an integer $n_0 \geq 1$ and a sequence $(\alpha_j)_{j=1}^{n_0}$ of complex numbers such that for each $n \geq n_0$ the identity

$$c_{n+1} = \alpha_1 c_n + \alpha_2 c_{n-1} + \ldots + \alpha_{n_0} c_{n-n_0+1} \tag{0.7}$$

is fulfilled. From this, (0.5) and (0.6) it follows that the rationality of a function $\theta \in \mathcal{S}$ can be characterized by relations of the form

$$\gamma_{n+1} = g_n(\gamma_0, \gamma_1, \ldots, \gamma_n), \quad n \geq n_0, \tag{0.8}$$

where $(g_n)_{n \geq n_0}$ is some sequence of given functions. It should be mentioned that the functions $(g_n)_{n \geq n_0}$ obtained in this way do not have such an explicit structure which enables us to perform a detailed analysis of the Schur parameter sequences of functions belonging to the class $\mathcal{RS}\backslash J$.

The main goal of this paper is to present a direct derivation of the relations (0.8) and, in so doing, characterize the Schur parameter sequences associated with functions from $\mathcal{RS}\backslash J$.

Our strategy is based on applying the tools developed in [4]. Our approach is motivated by Theorem 5.9 in [4] (see Theorem 1.11 below), which contains a first characterization of functions belonging to $\mathcal{RS}\backslash J$ in terms of their Schur parameters. This characterization is presented at the beginning of this paper. We want to demonstrate in which way the recurrence properties of the Taylor coefficient sequence of a function from $\mathcal{RS}\backslash J$ are reflected in its Schur parameter sequence.

This paper is organized as follows. In Section 1, we state some preliminary facts and notions. This material is mostly taken from [4].

In Section 2, we indicate the recurrent character of the Schur parameter sequence associated with a function $\theta \in \mathcal{RS}\backslash J$. An important step in realizing this aim is reached by introducing the concept of $\mathcal{S}$-recurrence for sequences $\gamma = (\gamma_j)_{j=0}^{\infty}$. The study of $\mathcal{S}$-recurrence is the central theme of Section 2. The concept of $\mathcal{S}$-recurrence is based on particular vectors which are called $\mathcal{S}$-recurrence vectors associated with γ (see Definition 2.1). It is already known from Theorem 5.9 in [4] that the Schur parameter sequence of a function $\theta \in \mathcal{RS}\backslash J$ belongs to the set Γl_2 of all sequences $(\gamma_j)_{j=0}^{\infty}$ which belong to $\Gamma \cap l_2$. This observation allows us to use more of the tools introduced in [4]. In particular, this concerns various sequences of complex matrices which were associated with a sequence $\gamma \in \Gamma l_2$ in [4]. It will turn out (see Proposition 2.4) that the $\mathcal{S}$-recurrence vectors associated with $\gamma \in \Gamma l_2$ are exactly those vectors from the null space of the matrices $\mathcal{A}_n(\gamma)$ introduced in (1.17) which have a non-zero last entry. This enables us to characterize the Schur parameter sequences of the functions from $\mathcal{RS}\backslash J$ as the sequences $\gamma \in \Gamma l_2$ which are $\mathcal{S}$-recurrent. This is the content of Theorem 2.5 which is one of the main results of this paper. The next central result is Theorem 2.8 which yields essential insights into the intrinsic structure of $\mathcal{S}$-recurrence sequences $\gamma = (\gamma_j)_{j=0}^{\infty}$. In particular, we will see that there is some $r \in \mathbb{N}$ such that for each integer $n \geq r$ the number γ_{n+1} can be expressed in terms of the sequence $(\gamma_j)_{j=0}^{n}$ of the preceding elements.

The main theme of Section 3 is connected to a closer analysis of a sequence $(\mathfrak{L}_n(\gamma))_{n\in\mathbb{N}}$ of complex matrices (see (1.7)) which are associated with a sequence $\gamma \in \Gamma l_2$. It will be shown (see Lemma 3.1) that the matrix $\mathfrak{L}_n(\gamma)$ is completely determined by its first column and the section $(\gamma_j)_{j=0}^{n}$. This leads us to an important notion which is introduced in Definition 3.2. Given a finite sequence $(\gamma_j)_{j=0}^{r}$ from $\mathbb{D}$, we call the data $[\Pi_{1,1}, (L_{j,1})_{j=1}^{r}]$ compatible with $(\gamma_j)_{j=0}^{r}$ if a certain matrix $\mathfrak{L}_{r+1,0}$ (see (3.3)) is contractive, but not strictly contractive. The matrix $\mathcal{A}_{r+1,0} := I_{r+1} - \mathfrak{L}_{r+1,0}\mathfrak{L}_{r+1,0}^*$ is called the information matrix associated with $[(\gamma_j)_{j=0}^{r}, \Pi_{1,1}, (L_{j,1})_{j=1}^{r}]$ because it turns out that this matrix contains essential information on the data under consideration. The study of the structure of $\mathcal{A}_{r+1,0}$ is the central topic of Section 4.

In Section 5, we consider an inverse problem. Starting with suitable data $[(\gamma_j)_{j=0}^{r}, \Pi_{1,1}, (L_{j,1})_{j=1}^{r}]$ we want to construct an $\mathcal{S}$-recurrent sequence which starts with $(\gamma_j)_{j=0}^{r}$. Our strategy is based on a closer analysis of the information matrix $\mathcal{A}_{r+1,0}$. The main result of Section 5 is Theorem 5.2 which contains an explicit recursive construction of an $\mathcal{S}$-recurrent sequence γ with first section $(\gamma_j)_{j=0}^{r}$. This construction is based on the use of vectors from the null space of $\mathcal{A}_{r+1,0}$ having nonzero last element. In the special case $r = 1$ the expressions from Theorem 5.2 can be simplified considerably (see Theorem 5.5).

In subsequent work we plan a closer analysis of the procedure used in the proof of Theorem 5.2 to obtain an $\mathcal{S}$-recurrent extension of a finite sequence $(\gamma_j)_{j=0}^{r}$ from $\mathbb{D}$. More precisely, we are interested in constructing sequences $\gamma = (\gamma_j)_{j=0}^{\infty}$ for which the associated functions $\theta \in \mathcal{RS}\backslash J$ have prescribed properties. In particular, we want to construct outer functions θ which belong to $\mathcal{RS}\backslash J$.

1. Preliminaries

This paper is a direct continuation of [4] where the Schur parameter sequences of pseudocontinuable non-inner Schur functions have been characterized. Keeping in mind that a non-inner rational Schur function is pseudocontinuable it seems to be quite natural to use methods introduced in [4]. In this section, we summarize some notions and results from [4], which we will need later. We continue to work with the notions used in [4].

Let $\theta \in \mathcal{RS}\backslash J$ and let $(\gamma_j)_{j=0}^{w}$ be the associated sequence of Schur parameters. Then from the properties (1) and (2) of Schur parameters listed in the Introduction it follows that $(\gamma_j)_{j=0}^{w}$ is an infinite sequence, i.e., $w = \infty$. From Corollary 4.4 in [4] we get additional essential information relating to the sequence $(\gamma_j)_{j=0}^{\infty}$:

Lemma 1.1. *Let $\theta \in \mathcal{RS}\backslash J$ and denote by $\gamma = (\gamma_j)_{j=0}^{\infty}$ its Schur parameter sequence. Then*

$$\sum_{j=0}^{\infty} |\gamma_j|^2 < +\infty. \tag{1.1}$$

In the following, the symbol l_2 stands for the space of all sequences $(z_j)_{j=0}^{\infty}$ of complex numbers such that $\sum_{j=0}^{\infty} |z_j|^2 < \infty$. Moreover,

$$\Gamma l_2 := \left\{ \gamma = (\gamma_j)_{j=0}^{\infty} \in l_2 : \gamma_j \in \mathbb{D}, j \in \mathbb{N}_0 \right\}.$$

Thus, Γl_2 is the subset of all $\gamma = (\gamma_j)_{j=0}^{\infty} \in \Gamma$, for which the product

$$\prod_{j=0}^{\infty} \left(1 - |\gamma_j|^2\right) \tag{1.2}$$

converges. Hence, if $\theta \in \mathcal{RS}\backslash J$, then Lemma 1.1 implies that its Schur parameter sequence γ belongs to Γl_2.

For functions $\theta \in \mathcal{S}$ with Schur parameter sequence γ belonging to Γl_2, we note that the sequence $(L_n(\gamma))_{n=0}^{\infty}$ introduced in formula (3.12) of [4] via

$$L_0(\gamma) := 1 \text{ and, for each positive integer } n, \text{ via } L_n(\gamma) :=$$
$$\sum_{r=1}^{n} (-1)^r \sum_{s_1+s_2+\ldots+s_r=n} \sum_{j_1=n-s_1}^{\infty} \sum_{j_2=j_1-s_2}^{\infty} \cdots \sum_{j_r=j_{r-1}-s_r}^{\infty} \gamma_{j_1}\overline{\gamma}_{j_1+s_1} \cdots \gamma_{j_r}\overline{\gamma}_{j_r+s_r} \tag{1.3}$$

plays a key role. Here the summation runs over all ordered r-tuples $(s_1, \ldots, s_r)$ of positive integers which satisfy $s_1 + \cdots + s_r = n$. For example,

$$L_1(\gamma) = -\sum_{j=0}^{\infty} \gamma_j \overline{\gamma_{j+1}}$$

and

$$L_2(\gamma) = -\sum_{j=0}^{\infty} \gamma_j \overline{\gamma_{j+2}} + \sum_{j_1=1}^{\infty} \sum_{j_2=j_1-1}^{\infty} \gamma_{j_1} \overline{\gamma_{j_1+1}} \gamma_{j_2} \overline{\gamma_{j_2+1}}.$$

Obviously, if $\gamma \in \Gamma l_2$, then the series (1.3) converges absolutely.

For each $\gamma = (\gamma_j)_{j=0}^{\infty} \in \Gamma l_2$, we set

$$\Pi_k := \prod_{j=k}^{\infty} D_{\gamma_j}, \quad k \in \mathbb{N}_0, \tag{1.4}$$

where

$$D_{\gamma_j} := \sqrt{1 - |\gamma_j|^2}, \quad j \in \mathbb{N}_0. \tag{1.5}$$

In the space l_2 we define the coshift mapping $W : l_2 \to l_2$ via

$$(z_j)_{j=0}^{\infty} \mapsto (z_{j+1})_{j=0}^{\infty}. \tag{1.6}$$

Let $\gamma \in \Gamma l_2$. For each n belonging to the set $\mathbb{N} := \{1, 2, 3, \dots\}$ of all positive integers we set (see formula (5.3) in [4])

$$\mathfrak{L}_n(\gamma) := \begin{pmatrix} \Pi_1 & 0 & 0 & \dots & 0 \\ \Pi_2 L_1(W\gamma) & \Pi_2 & 0 & \dots & 0 \\ \Pi_3 L_2(W\gamma) & \Pi_3 L_1(W^2\gamma) & \Pi_3 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \Pi_n L_{n-1}(W\gamma) & \Pi_n L_{n-2}(W^2\gamma) & \Pi_n L_{n-3}(W^3\gamma) & \dots & \Pi_n \end{pmatrix}. \tag{1.7}$$

The matrices introduced in (1.7) will play an important role in our investigations. Now we turn our attention to some properties of the matrices $\mathfrak{L}_n(\gamma)$, $n \in \mathbb{N}$, which will later be of use. From Corollary 5.2 in [4] we get

Lemma 1.2. *Let $\gamma = (\gamma_j)_{j=0}^{\infty} \in \Gamma l_2$ and let $n \in \mathbb{N}$. Then the matrix $\mathfrak{L}_n(\gamma)$ defined by* (1.7) *is contractive.*

We continue with some asymptotical considerations.

Lemma 1.3. *Let $\gamma = (\gamma_j)_{j=0}^{\infty} \in \Gamma l_2$. Then:*

(a) $\lim_{k\to\infty} \Pi_k = 1$.
(b) *Let $j \in \mathbb{N}$. Then* $\lim_{m\to\infty} L_j(W^m\gamma) = 0$.
(c) *Let $n \in \mathbb{N}$. Then* $\lim_{m\to\infty} \mathfrak{L}_n(W^m\gamma) = I_n$.

Proof. The choice of γ implies that γ is a sequence from $\mathbb{D}$ which satisfies (1.1). From this we infer the convergence of the infinite product $\prod_{k=0}^{\infty} D_{\gamma_k}$. This implies (a). Assertion (b) is an immediate consequence of the definition of the sequence $(L_j(W^m\gamma))_{m=1}^{\infty}$ (see (1.3) and (1.6)). By inspection of the sequence $(\mathfrak{L}_n(W^m\gamma))_{m=1}^{\infty}$ one can immediately see that the combination of (a) and (b) yields the assertion of (c). □

Let $\gamma \in \Gamma l_2$. A closer look at (1.7) yields the block decomposition

$$\mathfrak{L}_{n+1}(\gamma) = \begin{pmatrix} \mathfrak{L}_n(\gamma) & 0_{n\times 1} \\ b_n^*(\gamma) & \Pi_{n+1} \end{pmatrix}, \tag{1.8}$$

where

$$b_n(\gamma) := \Pi_{n+1} \cdot \left(\overline{L_n(W\gamma)}, \overline{L_{n-1}(W^2\gamma)}, \dots, \overline{L_1(W^n\gamma)}\right)^T. \tag{1.9}$$

Analogously, we obtain

$$\mathfrak{L}_{n+1}(\gamma) = \begin{pmatrix} \Pi_1 & 0_{1\times n} \\ B_{n+1}(\gamma) & \mathfrak{L}_n(W\gamma) \end{pmatrix} \tag{1.10}$$

(see [4], formula (5.23)), where

$$B_{n+1}(\gamma) := (\Pi_2 L_1(W\gamma), \Pi_3 L_2(W\gamma), \dots, \Pi_{n+1} L_n(W\gamma))^T. \tag{1.11}$$

The following result is Lemma 5.3 in [4].

Lemma 1.4. *Let $\gamma = (\gamma_j)_{j=0}^{\infty} \in \Gamma l_2$ and let $n \in \mathbb{N}$. Then*

$$\mathfrak{L}_n(\gamma) = \mathfrak{M}_n(\gamma) \cdot \mathfrak{L}_n(W\gamma), \tag{1.12}$$

where

$$\mathfrak{M}_n(\gamma) := \tag{1.13}$$

$$\begin{pmatrix} D_{\gamma_1} & 0 & 0 & \dots & 0 \\ -\gamma_1\overline{\gamma}_2 & D_{\gamma_2} & 0 & \dots & 0 \\ -\gamma_1 D_{\gamma_2}\overline{\gamma}_3 & -\gamma_2\overline{\gamma}_3 & D_{\gamma_3} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -\gamma_1\left(\prod_{j=2}^{n-1} D_{\gamma_j}\right)\overline{\gamma}_n & -\gamma_2\left(\prod_{j=3}^{n-1} D_{\gamma_j}\right)\overline{\gamma}_n & -\gamma_3\left(\prod_{j=4}^{n-1} D_{\gamma_j}\right)\overline{\gamma}_n & \dots & D_{\gamma_n} \end{pmatrix}.$$

Moreover, $\mathfrak{M}_n(\gamma)$ is a nonsingular matrix which fulfills

$$I_n - \mathfrak{M}_n(\gamma)\mathfrak{M}_n^*(\gamma) = \eta_n(\gamma)\eta_n^*(\gamma), \tag{1.14}$$

where

$$\eta_n(\gamma) := \left(\overline{\gamma_1}, \overline{\gamma_2} D_{\gamma_1}, \dots, \overline{\gamma_n}\left(\prod_{j=1}^{n-1} D_{\gamma_j}\right)\right)^T. \tag{1.15}$$

Corollary 1.5. *Let $\gamma = (\gamma_j)_{j=0}^{\infty} \in \Gamma l_2$ and let $n \in \mathbb{N}$. Then the multiplicative decomposition*

$$\mathfrak{L}_n(\gamma) = \overrightarrow{\prod_{k=0}^{\infty}} \mathfrak{M}_n(W^k\gamma) \tag{1.16}$$

holds true.

Proof. Combine part (c) of Lemma 1.3 and (1.12). □

For each $\gamma = (\gamma_j)_{j=0}^{\infty} \in \Gamma l_2$, we introduce the matrices

$$\mathcal{A}_n(\gamma) := I_n - \mathfrak{L}_n(\gamma)\mathfrak{L}_n^*(\gamma), \quad n \in \mathbb{N}. \tag{1.17}$$

Then Lemma 1.2 shows that for each $n \in \mathbb{N}$ the matrix $\mathcal{A}_n(\gamma)$ is nonnegative Hermitian. We will later see that the determinants

$$\sigma_n(\gamma) := \begin{cases} 1 & , \text{ if } n = 0 \\ \det \mathcal{A}_n(\gamma) & , \text{ if } n \in \mathbb{N} \end{cases} \tag{1.18}$$

contain essential information on the behavior of Schur parameters of a function $\theta \in \mathcal{RS}\backslash J$.

The following result is contained in Theorem 5.5 in [4].

Theorem 1.6. *Let $\gamma = (\gamma_j)_{j=0}^{\infty}$ be a sequence from Γl_2. Then*

(a) *For each $n \in \mathbb{N}$, the inequalities $0 \le \sigma_n(\gamma) < 1$ and $\sigma_{n+1}(\gamma) \le \sigma_n(\gamma)$ hold true. Moreover, $\lim_{n\to\infty} \sigma_n(\gamma) = 0$.*

(b) *If there exists some $n_0 \in \mathbb{N}_0$ such that $\sigma_{n_0}(\gamma) > 0$ and $\sigma_{n_0+1}(\gamma) = 0$, then for all integers $n \ge n_0$ the relation $\operatorname{rank} \mathcal{A}_n(\gamma) = n_0$ holds true.*

(c) *For each* $n \in \mathbb{N}$, *the identity*

$$\mathcal{A}_n(\gamma) = \eta_n(\gamma)\eta_n^*(\gamma) + \mathfrak{M}_n(\gamma)\mathcal{A}_n(W\gamma)\mathfrak{M}_n^*(\gamma) \tag{1.19}$$

holds, where $\mathfrak{M}_n(\gamma)$ *and* $\eta_n(\gamma)$ *are defined via* (1.13) *and* (1.15), *respectively.*

Remark 1.7. For each $n \in \mathbb{N}$, the identity (1.19) is an easy consequence of (1.17), (1.12), and (1.14). Indeed, for each $n \in \mathbb{N}$ we have

$$\begin{aligned}
\mathcal{A}_n(\gamma) &= I_n - \mathfrak{L}_n(\gamma)\mathfrak{L}_n^*(\gamma) = I_n - \mathfrak{M}_n(\gamma)\mathfrak{L}_n(W\gamma)\mathfrak{L}_n^*(W\gamma)\mathfrak{M}_n^*(\gamma) \\
&= I_n - \mathfrak{M}_n(\gamma)\mathfrak{M}_n^*(\gamma) + \mathfrak{M}_n(\gamma)\mathcal{A}_n(W\gamma)\mathfrak{M}_n^*(\gamma) \\
&= \eta_n(\gamma)\eta_n^*(\gamma) + \mathfrak{M}_n(\gamma)\mathcal{A}_n(W\gamma)\mathfrak{M}_n^*(\gamma).
\end{aligned}$$

Formula (1.18) and part (b) of Theorem 1.6 lead us to the following notion (see Definition 5.20 in [4])

Definition 1.8. Let $\gamma = (\gamma_j)_{j=0}^\infty \in \Gamma l_2$.

(a) Suppose that there exists some positive integer n such that $\sigma_n(\gamma) = 0$. Then the nonnegative integer n_0 satisfying $\sigma_{n_0}(\gamma) > 0$ and $\sigma_{n_0+1}(\gamma) = 0$ is called the *rank of the sequence* γ. In this case we will write $\operatorname{rank}\gamma = n_0$ to indicate that γ has the finite rank n_0.

(b) If $\sigma_n(\gamma) > 0$ for all $n \in \mathbb{N}_0$, then γ is called *a sequence of infinite rank.*

In the cases (a) and (b), we write $\operatorname{rank}\gamma = n_0$ and $\operatorname{rank}\gamma = \infty$, respectively.

Remark 1.9.

(a) Let $\gamma = (\gamma_j)_{j=0}^\infty \in \Gamma l_2$. Using (1.18), (1.17), and (1.7) we then get

$$\sigma_1(\gamma) = \mathcal{A}_1(\gamma) = 1 - |\mathfrak{L}_1(\gamma)|^2 = 1 - \Pi_1^2.$$

Thus, $\operatorname{rank}\gamma = 0$ if and only if $\gamma_j = 0$ for all $j \in \mathbb{N}$.

(b) Conversely, let $(\gamma_j)_{j=0}^\infty$ be a sequence of complex numbers which satisfies $\gamma_0 \in \mathbb{D}$ and $\gamma_j = 0$ for each $j \in \mathbb{N}$. Then $\gamma \in \Gamma l_2$ and $\operatorname{rank}\gamma = 0$.

Remark 1.10. Let $r \in \mathbb{N}$ and let $\gamma = (\gamma_j)_{j=0}^\infty$ be a sequence from $\mathbb{D}$ which satisfies

$$\gamma_j = 0, \quad j \in \{r+1, r+2, \dots\} \tag{1.20}$$

Then $\gamma \in \Gamma l_2$. From (1.4), (1.5) and (1.20) we conclude $\Pi_{r+1} = 1$. Combining this with (1.8) and Lemma 1.2, we see that

$$\mathfrak{L}_{r+1}(\gamma) = \begin{pmatrix} \mathfrak{L}_r(\gamma) & 0_{r\times 1} \\ 0_{1\times r} & 1 \end{pmatrix}.$$

Thus, (1.17) yields

$$\mathcal{A}_{r+1}(\gamma) = \begin{pmatrix} \mathcal{A}_r(\gamma) & 0_{r\times 1} \\ 0_{1\times r} & 0 \end{pmatrix}. \tag{1.21}$$

Let $e_{r+1} := (0, \dots, 0, 1)^T \in \mathbb{C}^{r+1}$. Then from (1.21) we infer

$$e_{r+1} \in \ker \mathcal{A}_{r+1}(\gamma) \backslash \{0_{(r+1)\times 1}\}.$$

Using (1.18) and (1.21), we get $\sigma_{r+1}(\gamma) = \det \mathcal{A}_{r+1}(\gamma) = 0$. Thus, $\operatorname{rank}\gamma \le r$.

Theorem 5.9 in [4] yields the following characterization of the Schur parameter sequences of rational Schur functions.

Theorem 1.11. *Let $\theta \in \mathcal{S}$ and let by $\gamma = (\gamma_j)_{j=0}^{w}$ denote its Schur parameter sequence. Then the function θ is rational if and only if one of the following two conditions is satisfied:*

(1) $w < \infty$.
(2) $w = \infty$, *the sequence γ belongs to Γl_2, and there exists an $n_0 \in \mathbb{N}$ such that $\sigma_{n_0}(\gamma) = 0$, where $\sigma_{n_0}(\gamma)$ is defined via* (1.18).

If (2) *holds, then $\theta \in \mathcal{RS} \backslash J$.*

Remark 1.12. It should be mentioned that condition (1) in Theorem 1.11 is exactly the well-known criteria by I. Schur for the membership of a function to the class $\mathcal{RS} \cap J$. We have already discussed this fact in detail in the introduction.

2. The $\mathcal{S}$-recurrence property of the Schur parameter sequences associated with non-inner rational Schur functions

It is known (see, e.g., Proposition 1.1 in [2]) that the power series

$$\sum_{j=0}^{\infty} c_j z^j \tag{2.1}$$

can be written as a quotient $\frac{P}{Q}$ of two polynomials P and Q where $Q(z) = 1 - q_1 z - \cdots - q_r z^r$ if and only if there exists some $m \in \mathbb{N}_0$ such that for each integer n with $n \geq m$ the relation

$$c_{n+1} = q_1 c_n + q_2 c_{n-1} + \cdots + q_r c_{n-r+1} \tag{2.2}$$

holds true. In this case the sequence $c = (c_j)_{j=0}^{\infty}$ is said to be a recurrent sequence of rth order and formula (2.2) is called a recurrence formula of order r.

We rewrite equation (2.2) in a different way. Here we consider the vectors

$$q := (-q_r, -q_{r-1}, \ldots, -q_1, 1)^T \tag{2.3}$$

and

$$\mu_{r+1}(c) := (\overline{c_0}, \overline{c_1}, \ldots, \overline{c_{r-1}}, \overline{c_r})^T. \tag{2.4}$$

For each $n \in \{m, m+1, \ldots\}$ we have then

$$\mu_{r+1}(W^{n-r+1}c) = (\overline{c_{n-r+1}}, \overline{c_{n-r+2}}, \ldots, \overline{c_n}, \overline{c_{n+1}})^T,$$

where W is the coshift given by (1.6). Thus, for each integer n with $n \geq m$, the recursion formula (2.2) can be rewritten as an orthogonality condition in the form

$$\big(q, \mu_{r+1}(W^{n-r+1}c)\big)_{\mathbb{C}^{r+1}} = 0, \tag{2.5}$$

where $(\cdot,\cdot)_{\mathbb{C}^{r+1}}$ stands for the usual Euclidean inner product in the space $\mathbb{C}^{r+1}$ (i.e., $(x, y)_{\mathbb{C}^{r+1}} = y^* x$ for all $x, y \in \mathbb{C}^{r+1}$).

Let the series (2.1) be the Taylor series of a function $\theta \in \mathcal{RS}\backslash J$ and let $\gamma = (\gamma_j)_{j=0}^{\infty}$ be the sequence of Schur parameters associated with θ. Then it will turn out that the recurrence property of the Taylor coefficient sequence $(c_j)_{j=0}^{\infty}$ implies some type of recurrence relations for the sequence $\gamma = (\gamma_j)_{j=0}^{\infty}$. With this in mind we introduce the following notion.

Definition 2.1. Let $\gamma = (\gamma_j)_{j=0}^{\infty} \in \Gamma$. Then the sequence γ is called *$\mathcal{S}$-recurrent* if there exist some $r \in \mathbb{N}$ and some vector $p = (p_r, p_{r-1}, \dots, p_0)^T \in \mathbb{C}^{r+1}$ with $p_0 \neq 0$ such that for all integers n with $n \geq r$ the relations

$$\left(p, \left[\overrightarrow{\prod_{k=0}^{n-r-1}} \mathfrak{M}_{r+1}(W^k \gamma) \right] \eta_{r+1}(W^{n-r}\gamma) \right)_{\mathbb{C}^{r+1}} = 0 \tag{2.6}$$

are satisfied, where the matrix $\mathfrak{M}_{r+1}(\gamma)$ and the vector $\eta_{r+1}(\gamma)$ are defined via (1.13) and (1.15), respectively. In this case the vector p is called an *rth order S-recurrence vector associated with γ.*

Remark 2.2. If we compare the vectors $\mu_{r+1}(c)$ and $\eta_{r+1}(\gamma)$ introduced in (2.4) and (1.15), respectively, then we see that the numbers $\overline{\gamma_k}$ in the vector $\eta_{r+1}(\gamma)$ are multiplied with the factor $\prod_{j=1}^{k-1} D_{\gamma_j}$ which can be thought of as a weight factor. Moreover, contrary to (2.5), the vector $\eta_{r+1}(W^{n-r}\gamma)$ is paired in (2.6) with the matrix product

$$\overrightarrow{\prod_{k=0}^{n-r-1}} \mathfrak{M}_{r+1}(W^k \gamma).$$

In the case $n = r$ the latter product has to be interpreted as the unit matrix I_{r+1}.

The following result plays an important role in our subsequent considerations.

Lemma 2.3. *Let $\gamma = (\gamma_j)_{j=0}^{\infty} \in \Gamma l_2$ and let $n \in \mathbb{N}$. Then $\mathcal{A}_n(\gamma)$ defined via* (1.17) *can be represented via*

$$\mathcal{A}_n(\gamma) = \sum_{j=0}^{\infty} \xi_{n,j}(\gamma)\xi_{n,j}^*(\gamma), \tag{2.7}$$

where

$$\xi_{n,j}(\gamma) := \left[\overrightarrow{\prod_{k=0}^{j-1}} \mathfrak{M}_n(W^k \gamma) \right] \eta_n(W^j \gamma), \quad j \in \mathbb{N}_0. \tag{2.8}$$

Proof. Applying (1.19) to $W\gamma$ instead of γ we obtain

$$\mathcal{A}_n(W\gamma) = \eta_n(W\gamma)\eta_n^*(W\gamma) + \mathfrak{M}_n(W\gamma)\mathcal{A}_n(W^2\gamma)\mathfrak{M}_n^*(W\gamma).$$

Inserting this expression into (1.19) we get

$$\begin{aligned} \mathcal{A}_n(\gamma) &= \eta_n(\gamma)\eta_n^*(\gamma) + \mathfrak{M}_n(\gamma)\eta_n(W\gamma)\eta_n^*(W\gamma)\mathfrak{M}_n^*(\gamma) \\ &\quad + \mathfrak{M}_n(\gamma)\mathfrak{M}_n(W\gamma)\mathcal{A}_n(W^2\gamma)\mathfrak{M}_n^*(W\gamma)\mathfrak{M}_n(\gamma). \end{aligned}$$

This procedure will be continued now. Taking into account the contractivity of the matrices $\mathfrak{M}_n(W^j\gamma)$, $j \in \mathbb{N}_0$, and the limit relation $\lim_{m\to\infty} \mathcal{A}_n(W^m\gamma) = 0_{(n+1)\times(n+1)}$, which follows from part (c) of Lemma 1.3 and (1.17), we obtain (2.7). □

Let $r \in \mathbb{N}$. Using (2.8) one can see that condition (2.6), which expresses $\mathcal{S}$-recurrence of rth order, can be rewritten in the form

$$(p, \xi_{r+1,j}(\gamma))_{\mathbb{C}^{r+1}} = 0, \quad j \in \mathbb{N}_0. \tag{2.9}$$

Thus the application of Lemma 2.3 leads us immediately to the following result.

Proposition 2.4. *Let $\gamma = (\gamma_j)_{j=0}^{\infty} \in \Gamma l_2$. Further, let $r \in \mathbb{N}$ and let $p = (p_r, \dots, p_0)^T \in \mathbb{C}^{r+1}$. Then p is an rth order $\mathcal{S}$-recurrence vector associated with γ if and only if $p_0 \neq 0$ and*

$$p \in \ker \mathcal{A}_{r+1}(\gamma). \tag{2.10}$$

Now we are able to prove one of the main results of this paper. It establishes an important connection between the $\mathcal{S}$-recurrence property of a sequence $\gamma \in \Gamma l_2$ and the rationality of the Schur function θ, the Schur parameter sequence of which is γ.

Theorem 2.5. *Let $\gamma = (\gamma_j)_{j=0}^{\infty} \in \Gamma$ and let θ be the Schur function with Schur parameter sequence γ. Then $\theta \in \mathcal{RS}\backslash J$ if and only if γ is an $\mathcal{S}$-recurrent sequence belonging to Γl_2.*

Proof. From Theorem 1.11 it follows that $\theta \in \mathcal{RS}\backslash J$ if and only if γ belongs to Γl_2 and there exists some $r \in \mathbb{N}$ such that $\sigma_{r+1}(\gamma) = 0$. In this case, we infer from part (b) of Theorem 1.6 that there exists an $n_0 \in \mathbb{N}_0$ such that $\sigma_{n_0}(\gamma) > 0$ and $\sigma_{n_0+1}(\gamma) = 0$. If $n_0 = 0$, then

$$0 = \sigma_1(\gamma) = 1 - \prod_{j=1}^{\infty}(1 - |\gamma_j|^2).$$

Thus, $\gamma_j = 0$ for all $j \in \mathbb{N}$. This implies that θ is the constant function in $\mathbb{D}$ with value $\gamma_0 \in \mathbb{D}$. If $n_0 \in \mathbb{N}$, then we have $\det \mathcal{A}_{n_0}(\gamma) > 0$ and $\det \mathcal{A}_{n_0+1}(\gamma) = 0$. The condition $\det \mathcal{A}_{n_0+1}(\gamma) = 0$ is equivalent to the existence of a nontrivial vector $p = (p_{n_0}, \dots, p_0)^T \in \mathbb{C}^{n_0+1}$ which satisfies

$$p \in \ker \mathcal{A}_{n_0+1}(\gamma). \tag{2.11}$$

From (1.8) and (1.17) we obtain the block decomposition

$$\mathcal{A}_{n_0+1}(\gamma) = \begin{pmatrix} \mathcal{A}_{n_0}(\gamma) & -\mathfrak{L}_{n_0}(\gamma) b_{n_0}(\gamma) \\ -b_{n_0}^*(\gamma)\mathfrak{L}_{n_0}^*(\gamma) & 1 - \Pi_{n_0+1}^2 - b_{n_0}^*(\gamma) b_{n_0}(\gamma) \end{pmatrix}. \tag{2.12}$$

From (2.12) it follows that $p_0 \neq 0$. Indeed, if we would have $p_0 = 0$, then from (2.12) we could infer $(p_{n_0}, \dots, p_1)^T \in \ker \mathcal{A}_{n_0}(\gamma)$. In view of $\det \mathcal{A}_{n_0}(\gamma) \neq 0$, this implies $(p_{n_0}, \dots, p_1)^T = 0_{n_0 \times 1}$ which is a contradiction to the choice of p. Now the asserted equivalence follows immediately from Proposition 2.4. □

Proposition 2.6. *Let $n_0 \in \mathbb{N}$, and let $\gamma = (\gamma_j)_{j=0}^{\infty}$ be a sequence which belongs to Γl_2 and satisfies* $\operatorname{rank} \gamma = n_0$. *Then:*

(a) *The sequence γ is $\mathcal{S}$-recurrent and n_0 is the minimal order of $\mathcal{S}$-recurrence vector associated with γ. There is a unique n_0th order $\mathcal{S}$-recurrence vector $p = (p_{n_0}, \dots, p_0)^T$ of γ which satisfies $p_0 = 1$.*

(b) *Let r be an integer with $r \geq n_0$ and let p be an n_0th order $\mathcal{S}$-recurrence vector associated with γ.*

(b1) *Let the sequence $(\widetilde{g}_j)_{j=1}^{r-n_0+1}$ of vectors from $\mathbb{C}^{n_0+1}$ be defined by*

$$\widetilde{g}_1 := p, \quad \widetilde{g}_2 := \mathfrak{M}_{n_0+1}^*(\gamma)p, \quad \dots$$
$$\dots, \quad \widetilde{g}_{r-n_0+1} := \left[\overleftarrow{\prod_{k=0}^{r-n_0+1}} \mathfrak{M}_{n_0+1}^*(W^k\gamma) \right] p. \tag{2.13}$$

Then the $\mathbb{C}^{r+1}$-vectors

$$g_1 := \begin{pmatrix} \widetilde{g}_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad g_2 := \begin{pmatrix} 0 \\ \widetilde{g}_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad g_{r-n_0+1} := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \widetilde{g}_{r-n_0+1} \end{pmatrix} \tag{2.14}$$

form a basis of $\ker \mathcal{A}_{r+1}(\gamma)$.

(b2) *The sequence γ has $\mathcal{S}$-recurrence vectors of rth order and every such vector $\widehat{p}$ has the shape*

$$\widehat{p} = \alpha_1 g_1 + \alpha_2 g_2 + \dots + \alpha_{r-n_0+1} g_{r-n_0+1}, \tag{2.15}$$

where $(\alpha_j)_{j=1}^{r-n_0+1}$ is a sequence of complex numbers satisfying

$$\alpha_{r-n_0+1} \neq 0.$$

Proof. (a) From Definition 1.8 the relation

$$n_0 = \min\{r \in \mathbb{N}_0 : \ker \mathcal{A}_{r+1}(\gamma) \neq \{0_{(r+1)\times 1}\}\} \tag{2.16}$$

follows. The block decomposition (2.12) shows that

$$\dim[\ker \mathcal{A}_{n_0+1}(\gamma)] = 1. \tag{2.17}$$

Let $p = (p_{n_0}, \dots, p_0)^T \in \ker \mathcal{A}_{n_0+1}(\gamma) \setminus \{0_{(n_0+1)\times 1}\}$. As in the proof of Theorem 2.5 it can be shown then that $p_0 \neq 0$. Thus, Proposition 2.4 yields that p is an n_0th order $\mathcal{S}$-recurrence vector associated with γ. Taking into account (2.17) and applying again Proposition 2.4, we see that there is a unique n_0th order $\mathcal{S}$-recurrence vector $p = (p_{n_0}, \dots, p_0)^T$ associated with γ which satisfies $p_0 = 1$. In particular, γ is $\mathcal{S}$-recurrent. In view of (2.16), applying Proposition 2.4 we see that n_0 is the minimal order of $\mathcal{S}$-recurrence vector associated with γ.

(b1) In the case $r = n_0$ the assertion is already proved above. Let $r = n_0 + 1$. Using (1.10) and (1.17), we obtain the block decomposition

$$\mathcal{A}_{r+1}(\gamma) = \begin{pmatrix} 1 - \Pi_1^2 & -\Pi_1 B_{r+1}^*(\gamma) \\ -B_{r+1}^*(\gamma)\Pi_1 & \mathcal{A}_r(W\gamma) - B_{r+1}(\gamma)B_{r+1}^*(\gamma) \end{pmatrix}. \tag{2.18}$$

In view of $p \in \ker \mathcal{A}_{n_0+1}(\gamma)$ the block decomposition (2.12) with $n = n_0 + 1$ implies that $g_1 = \begin{pmatrix} p \\ 0 \end{pmatrix}$ belongs to $\ker \mathcal{A}_{n_0+2}(\gamma)$. Furthermore, using (1.19) with $n = n_0$, we see that

$$\mathfrak{M}_{n_0+1}^*(\gamma)p \in \ker \mathcal{A}_{n_0+1}(W\gamma).$$

Now the block decomposition (2.18) implies that the vector $g_2 = \begin{pmatrix} 0 \\ \widetilde{g}_2 \end{pmatrix}$ also belongs to $\ker \mathcal{A}_{n_0+2}(\gamma)$. In view of $p_0 \neq 0$ and the triangular shape of the matrix $\mathfrak{M}_{n_0+1}^*(\gamma)$ (see (1.13)), we see that the last component of the vector g_2 does not vanish. Thus, the vectors g_1 and g_2 are linearly independent vectors belonging to $\ker \mathcal{A}_{n_0+2}(\gamma)$. Since part (b) of Theorem 1.6 implies that $\dim[\ker \mathcal{A}_{n_0+2}(\gamma)] = 2$, we obtain that g_1 and g_2 form a basis of $\ker \mathcal{A}_{n_0+2}(\gamma)$. One can prove the assertion by induction for arbitrary $r \in \{n_0, n_0 + 1, \ldots\}$.

(b2) This follows immediately by combining Proposition 2.4 with part (b1). □

Proposition 2.6 leads us to the following notion.

Definition 2.7. Let $n_0 \in \mathbb{N}$ and let $\gamma = (\gamma_j)_{j=0}^\infty$ be a sequence which belongs to Γl_2 and satisfies $\operatorname{rank} \gamma = n_0$. Then the unique n_0th order $\mathcal{S}$-recurrence vector $p = (p_{n_0}, \ldots, p_0)^T$ satisfying $p_0 = 1$ is called *the basic $\mathcal{S}$-recurrence vector associated with γ*.

Let $\gamma = (\gamma_j)_{j=0}^\infty \in \Gamma l_2$ be an $\mathcal{S}$-recurrent sequence and let p be the basic $\mathcal{S}$-recurrence vector associated with γ. Then Proposition 2.6 shows that all $\mathcal{S}$-recurrence vectors associated with γ can be obtained from p.

Our next consideration is aimed at working out the recurrent character of formula (2.6). More precisely, we will verify that, for each integer n with $n \geq r$, the element γ_{n+1} can be expressed in terms of the preceding members $\gamma_0, \ldots, \gamma_n$ of the sequence γ. In view of Proposition 2.6, this is the content of the following result.

Theorem 2.8. *Let $\gamma = (\gamma_j)_{j=0}^\infty$ be an $\mathcal{S}$-recurrent sequence which belongs to Γl_2 and let $p = (p_r, p_{r-1}, \ldots, p_0)^T$ be an rth order $\mathcal{S}$-recurrence vector associated with γ. Further, let*

$$\lambda := -\frac{1}{p_0}\left[\prod_{k=1}^{r} D_{\gamma_k}\right] \cdot (p_r, p_{r-1}, \ldots, p_1)^T. \tag{2.19}$$

Then for every integer n *with* $n \geq r$ *the relations*

$$\gamma_{n+1} = \left[\prod_{s=1}^{n} D_{\gamma_s}^{-1}\right] \left(\left[\overleftarrow{\prod_{j=0}^{n-r-1}} \mathfrak{M}_r^{-1}(W^j\gamma)\right] \lambda \ , \ \left[\prod_{k=n-r+1}^{n} D_{\gamma_k}^{-1}\right] \eta_r(W^{n-r}\gamma) \right)_{\mathbb{C}^r} \tag{2.20}$$

hold where D_{γ_j}*,* W*,* $\mathfrak{M}_r(\gamma)$*, and* $\eta_r(\gamma)$ *are defined via* (1.5)*,* (1.6)*,* (1.13)*, and* (1.15)*, respectively.*

Proof. Since p is an rth order $\mathcal{S}$-recurrence vector associated with γ, the relation (2.6) is satisfied. From Definition 2.1 it follows that $p_0 \neq 0$. We rewrite (2.6) in the form

$$\left(\left[\overleftarrow{\prod_{k=0}^{n-r-1}} \mathfrak{M}_{r+1}^*(W^k\gamma)\right] p \ , \ \eta_{r+1}(W^{n-r}\gamma) \right)_{\mathbb{C}^{r+1}} = 0. \tag{2.21}$$

In view of Proposition 2.4, we have $p \in \ker \mathcal{A}_{r+1}(\gamma)$. Applying (1.19) for $n = r+1$, we obtain

$$(p, \eta_{r+1}(\gamma))_{C^{r+1}} = 0 \tag{2.22}$$

and

$$\mathfrak{M}_{r+1}^*(\gamma) \cdot p \in \ker \mathcal{A}_{r+1}(W\gamma). \tag{2.23}$$

Using (1.14) for $n = r+1$, we see that, for all $x \in \mathbb{C}^{r+1}$ which are orthogonal to $\eta_{r+1}(\gamma)$, the identity $\mathfrak{M}_{r+1}^*(\gamma)x = \mathfrak{M}_{r+1}^{-1}(\gamma)x$ holds true. Thus, from (2.22) we infer

$$\mathfrak{M}_{r+1}^*(\gamma)p = \mathfrak{M}_{r+1}^{-1}(\gamma)p. \tag{2.24}$$

Bearing (2.23), (2.24), and Lemma 1.4 in mind, replacing p in these considerations with $\mathfrak{M}_{r+1}^*(\gamma)p$, we obtain

$$\mathfrak{M}_{r+1}^*(W\gamma)\mathfrak{M}_{r+1}^*(\gamma)p \in \ker \mathcal{A}_{r+1}(W^2\gamma)$$

and

$$\mathfrak{M}_{r+1}^*(W\gamma)\mathfrak{M}_{r+1}^*(\gamma)p = \mathfrak{M}_{r+1}^{-1}(W\gamma)\mathfrak{M}_{r+1}^{-1}(\gamma)p.$$

Thus, by induction we get

$$\left[\overleftarrow{\prod_{k=0}^{n-r-1}} \mathfrak{M}_{r+1}^*(W^k\gamma)\right] p = \left[\overleftarrow{\prod_{k=0}^{n-r-1}} \mathfrak{M}_{r+1}^{-1}(W^k\gamma)\right] p. \tag{2.25}$$

From (2.19) we see that the vector p can be written in the form

$$p = p_0 \begin{pmatrix} -\left[\prod_{s=1}^{r} D_{\gamma_s}^{-1}\right] \lambda \\ 1 \end{pmatrix}. \tag{2.26}$$

From (1.13) we infer that the matrix $\mathfrak{M}_{r+1}(\gamma)$ has the block decomposition

$$\mathfrak{M}_{r+1}(\gamma) = \begin{pmatrix} \mathfrak{M}_r(\gamma) & 0_{r\times 1} \\ \star & D_{\gamma_{r+1}} \end{pmatrix}. \tag{2.27}$$

Formula (2.27) implies the block representation

$$\mathfrak{M}_{r+1}(\gamma) = \begin{pmatrix} \mathfrak{M}_r^{-1}(\gamma) & 0_{r\times 1} \\ \star & D_{\gamma_{r+1}}^{-1} \end{pmatrix}. \tag{2.28}$$

Combining (2.28) and (2.26), we conclude that the right-hand side of (2.25) can be rewritten in the form

$$\left[\overleftarrow{\prod_{k=0}^{n-r-1}} \mathfrak{M}_{r+1}^{-1}(W^k\gamma)\right] p = p_0 \cdot \begin{pmatrix} -\left[\prod_{s=1}^{r} D_{\gamma_s}^{-1}\right]\left[\overleftarrow{\prod_{k=0}^{n-r-1}} \mathfrak{M}_r^{-1}(W^k\gamma)\right]\lambda \\ k_{n-r} \end{pmatrix} \tag{2.29}$$

where k_{n-r} is some complex number. On the other hand, taking into account (2.27) and (2.26), we find that the left-hand side of (2.25) can be expressed by

$$\left[\overleftarrow{\prod_{k=0}^{n-r-1}} \mathfrak{M}_{r+1}^{*}(W^k\gamma)\right] p = p_0 \cdot \begin{pmatrix} \star \\ \prod_{k=0}^{n-r-1} D_{\gamma_{r+1+k}} \end{pmatrix}. \tag{2.30}$$

The combination of (2.25), (2.29), and (2.30) yields

$$k_{n-r} = \prod_{k=0}^{n-r-1} D_{\gamma_{r+1+k}}. \tag{2.31}$$

Combining (2.25), (2.29), and (2.31) yields

$$\left[\overleftarrow{\prod_{k=0}^{n-r-1}} \mathfrak{M}_{r+1}^{*}(W^k\gamma)\right] p = p_0 \cdot \begin{pmatrix} -\left[\prod_{s=1}^{r} D_{\gamma_s}^{-1}\right]\left[\overleftarrow{\prod_{k=0}^{n-r-1}} \mathfrak{M}_r^{-1}(W^k\gamma)\right]\lambda \\ \prod_{k=0}^{n-r-1} D_{\gamma_{r+1+k}} \end{pmatrix}. \tag{2.32}$$

From (1.15) we get

$$\eta_{r+1}(\gamma) = \begin{pmatrix} \eta_r(\gamma) \\ \overline{\gamma_{r+1}} \prod_{k=1}^{r} D_{\gamma_k} \end{pmatrix}.$$

Consequently,

$$\eta_{r+1}(W^{n-r}\gamma) = \begin{pmatrix} \eta_r(W^{n-r}\gamma) \\ \overline{\gamma_{n+1}} \prod_{k=1}^{r} D_{\gamma_{k+n-r}} \end{pmatrix}. \tag{2.33}$$

Using (2.32) and (2.33), we infer

$$\begin{aligned}
&\left(\left[\overleftarrow{\prod_{k=0}^{n-r-1}} \mathfrak{M}_{r+1}^{*}(W^k\gamma)\right] p \;,\; \eta_{r+1}(W^{n-r}\gamma)\right)_{\mathbb{C}^{r+1}} \\
&= p_0 \cdot \left[\left(-\left[\prod_{s=1}^{r} D_{\gamma_s}^{-1}\right]\cdot\left[\overleftarrow{\prod_{j=0}^{n-r-1}} \mathfrak{M}_r^{-1}(W^j\gamma)\right]\lambda \;,\; \eta_r(W^{n-r}\gamma)\right)_{\mathbb{C}^r} \right.\\
&\qquad \left. + \left(\prod_{k=0}^{n-r-1} D_{\gamma_{r+1+k}}\right)\gamma_{n+1}\left(\prod_{k=1}^{r} D_{\gamma_{k+n-r}}\right)\right].
\end{aligned} \tag{2.34}$$

Taking into account (2.31), (2.34), and $p_0 \neq 0$, a straightforward computation yields (2.20). Thus, the proof is complete. □

Remark 2.9. Let $\gamma = (\gamma_j)_{j=0}^{\infty}$ be a sequence which satisfies the assumptions of Theorem 2.8. Then it is possible to recover the whole sequence γ via the formulas (2.20) from the section $(\gamma_j)_{j=0}^{r}$ and the vector $\lambda = (\lambda_1, \dots, \lambda_r)^T$. Indeed, for $n = r$ we have

$$\gamma_{r+1} = \left[\prod_{k=1}^{r} D_{\gamma_k}^{-2}\right] \eta_r^*(\gamma)\lambda = \frac{1}{(1-|\gamma_1|^2)\dots(1-|\gamma_r|^2)} \cdot \sum_{k=1}^{r} \lambda_{r-k+1}\gamma_k \left(\prod_{j=1}^{k} D_{\gamma_j}\right).$$

In the case $n = r+2$ the vector λ has to be replaced by $\mathfrak{M}_r^{-1}(W\gamma)$ and the sequence $(\gamma_j)_{j=0}^{r}$ has to be replaced by $(\gamma_{j+1})_{j=0}^{r}$. The matrix $\mathfrak{M}_r(\gamma)$ depends on the section $(\gamma_j)_{j=1}^{r}$. Thus, the matrix $\mathfrak{M}_r^{-1}(W\gamma)$ depends on the section $(\gamma_{j+1})_{j=1}^{r}$. Consequently, formula (2.20) yields an expression for γ_{r+2} in terms of the sequence $(\gamma_j)_{j=1}^{r+1}$. Continuing this procedure inductively we see that, for all integers n with $n \geq r$, formula (2.20) produces an expression for γ_{n+1} which is based on the section $(\gamma_j)_{j=0}^{n}$. Consequently, the sequence $(\gamma_j)_{j=0}^{\infty}$ is completely determined by the section $(\gamma_j)_{j=0}^{r}$ and the vector λ. It should be mentioned that in the case $n_0 = 1$, which corresponds to a sequence of rank 1, for each $n \in \mathbb{N}$ formula (2.20) has the form

$$\gamma_{n+1} = \lambda \cdot \frac{\gamma_n}{\prod_{j=1}^{n}(1-|\gamma_j|^2)}.$$

Observe that, for this particular case $n_0 = 1$, it was derived in Theorem 5.22 in [4].

Our next goal can be described as follows. Let $n_0 \in \mathbb{N}$ and let $\gamma = (\gamma_j)_{j=0}^{\infty} \in \Gamma l_2$ be a sequence which satisfies $\operatorname{rank}\gamma = n_0$. Furthermore, let r be an integer with $r \geq n_0$ and let p be an rth $\mathcal{S}$-recurrence vector associated with γ. Then we will show that the identity

$$\prod_{k=1}^{r}(1-|\gamma_k|^2) - \Pi_1^2 = \lambda^* \left((\mathfrak{L}_r^{-1}(\gamma))^* \mathfrak{L}_r^{-1}(\gamma) - I_r \right) \lambda$$

holds, where Π_1, λ and $\mathfrak{L}_r(\gamma)$ are defined via (1.4), (2.19), and (1.7), respectively. To accomplish this we still need to make some preparations. In this way, we will be led to several results that are, by themselves, of interest.

Let $n \in \mathbb{N}$. Then the symbol $\|.\|_{\mathbb{C}^n}$ stands for the Euclidean norm in the space $\mathbb{C}^n$.

Lemma 2.10. *Let* $\gamma = (\gamma_j)_{j=0}^{\infty}$ *be a sequence from* $\mathbb{D}$. *Let* $n \in \mathbb{N}$ *and let* $\eta_n(\gamma)$ *be defined via* (1.15). *Then*

$$1 - \|\eta_n(\gamma)\|_{\mathbb{C}^n}^2 = \prod_{j=1}^{n} D_{\gamma_j}^2.$$

Proof. For $n = 1$ the asserted equation obviously holds. Now let $n \geq 2$. Then from (1.15) we see the block decomposition

$$\eta_n(\gamma) = \begin{pmatrix} \eta_{n-1}(\gamma) \\ \overline{\gamma_n}\left[\prod_{k=1}^{n-1} D_{\gamma_k}\right] \end{pmatrix}.$$

Thus, taking into account the definition of the Euclidean norm, we get

$$\|\eta_n(\gamma)\|^2_{\mathbb{C}^n} = \|\eta_{n-1}(\gamma)\|^2_{\mathbb{C}^{n-1}} + |\gamma_n|^2 \left[\prod_{k=1}^{n-1} D^2_{\gamma_k}\right].$$

Now, the assertion follows immediately by induction. □

Lemma 2.11. *Let $n \in \mathbb{N}$. Furthermore, let the nonsingular complex $n \times n$ matrix $\mathfrak{M}$ and the vector $\eta \in \mathbb{C}^n$ be chosen such that*

$$I_n - \mathfrak{M}\mathfrak{M}^* = \eta\eta^* \tag{2.35}$$

holds. Then $1 - \|\eta\|^2_{\mathbb{C}^n} > 0$ and the vector

$$\widetilde{\eta} := \frac{1}{\sqrt{1 - \|\eta\|^2_{\mathbb{C}^n}}}\mathfrak{M}^*\eta \tag{2.36}$$

satisfies

$$I_n - \mathfrak{M}^*\mathfrak{M} = \widetilde{\eta}\widetilde{\eta}^*. \tag{2.37}$$

Proof. The case $\eta = 0_{n\times 1}$ is trivial. Now suppose that $\eta \in \mathbb{C}^n \backslash \{0_{n\times 1}\}$. From (2.35) we get

$$(I_n - \mathfrak{M}\mathfrak{M}^*)\eta = \eta\eta^*\eta = \|\eta\|^2_{\mathbb{C}^n} \cdot \eta \tag{2.38}$$

and consequently

$$\mathfrak{M}\mathfrak{M}^*\eta = (1 - \|\eta\|^2_{\mathbb{C}^n}) \cdot \eta. \tag{2.39}$$

Hence $1 - \|\eta\|^2_{\mathbb{C}^n}$ is an eigenvalue of $\mathfrak{M}\mathfrak{M}^*$ with corresponding eigenvector η. Since $\mathfrak{M}$ is nonsingular, the matrix $\mathfrak{M}\mathfrak{M}^*$ is positive Hermitian. Thus, we have $1 - \|\eta\|^2_{\mathbb{C}^n} > 0$. Using (2.38) we infer

$$(I_n - \mathfrak{M}^*\mathfrak{M})\mathfrak{M}^*\eta = \mathfrak{M}^*(I_n - \mathfrak{M}\mathfrak{M}^*)\eta = \|\eta\|^2_{\mathbb{C}^n} \cdot \mathfrak{M}^*\eta. \tag{2.40}$$

Taking into account (2.39) we can conclude

$$\|\mathfrak{M}^*\eta\|^2_{\mathbb{C}^n} = \eta^*\mathfrak{M}\mathfrak{M}^*\eta = \eta^*\left[(1 - \|\eta\|^2_{\mathbb{C}^n}) \cdot \eta\right] = (1 - \|\eta\|^2_{\mathbb{C}^n}) \cdot \|\eta\|^2_{\mathbb{C}^n} \tag{2.41}$$

and therefore from (2.36) we have

$$\|\widetilde{\eta}\|_{\mathbb{C}^n} = \|\eta\|_{\mathbb{C}^n} > 0. \tag{2.42}$$

Formulas (2.40), (2.36) and (2.42) show that $\|\widetilde{\eta}\|^2_{\mathbb{C}^n}$ is an eigenvalue of $I_n - \mathfrak{M}^*\mathfrak{M}$ with corresponding eigenvector $\widetilde{\eta}$. From (2.35) and $\eta \neq 0_{n\times 1}$ we get

$$\operatorname{rank}(I_n - \mathfrak{M}^*\mathfrak{M}) = \operatorname{rank}(I_n - \mathfrak{M}\mathfrak{M}^*) = 1.$$

So for each vector h we can conclude

$$(I_n - \mathfrak{M}^*\mathfrak{M})h = (I_n - \mathfrak{M}^*\mathfrak{M})(h, \frac{\widetilde{\eta}}{\|\widetilde{\eta}\|_{\mathbb{C}^n}})_{\mathbb{C}^n} \frac{\widetilde{\eta}}{\|\widetilde{\eta}\|_{\mathbb{C}^n}} = (h, \widetilde{\eta})_{\mathbb{C}^n} \widetilde{\eta} = \widetilde{\eta}\widetilde{\eta}^* \cdot h. \qquad \square$$

Proposition 2.12. *Let $\gamma = (\gamma_j)_{j=0}^\infty \in \Gamma l_2$ and let $n \in \mathbb{N}$. Then*

$$I_n - \mathfrak{M}_n^*(\gamma)\mathfrak{M}_n(\gamma) = \frac{1}{\prod_{k=1}^n (1-|\gamma_k|^2)} \mathfrak{M}_n^*(\gamma)\eta_n(\gamma)\eta_n^*(\gamma)\mathfrak{M}_n(\gamma)$$

where $\mathfrak{M}_n(\gamma)$ and $\eta_n(\gamma)$ are defined via (1.13) *and* (1.15), *respectively.*

Proof. The combination of Lemma 1.4, Lemma 2.10 and Lemma 2.11 yields the assertion. $\square$

The following result should be compared with Lemma 2.3. Under the assumptions of Lemma 2.3 we will verify that for each $n \in \mathbb{N}$ the right defect matrix $I_n - \mathfrak{L}_n^*(\gamma)\mathfrak{L}_n(\gamma)$ admits a series representation which is similar to the series representation for the left defect matrix $I_n - \mathfrak{L}_n(\gamma)\mathfrak{L}_n^*(\gamma)$.

Proposition 2.13. *Let $\gamma = (\gamma_j)_{j=0}^\infty \in \Gamma l_2$, let $n \in \mathbb{N}$ and let $\mathfrak{L}_n(\gamma)$ be defined via* (1.7). *Then*

$$I_n - \mathfrak{L}_n^*(\gamma)\mathfrak{L}_n(\gamma) = \sum_{j=0}^\infty \tau_{n,j}(\gamma)\tau_{n,j}^*(\gamma) \tag{2.43}$$

where for each $j \in \mathbb{N}_0$ the matrix $\tau_{n,j}(\gamma)$ is defined via

$$\tau_{n,j}(\gamma) := \left(\prod_{k=j+1}^{j+n} D_{\gamma_k}^{-1} \right) \left[\overleftarrow{\prod_{k=j}^{\infty}} \mathfrak{M}_n^*(W^k\gamma) \right] \eta_n(W^j\gamma) \tag{2.44}$$

and where D_{γ_k}, W, $\mathfrak{M}_n(\gamma)$, and $\eta_n(\gamma)$ are given by (1.5), (1.6), (1.13), *and* (1.15), *respectively.*

Proof. From (1.12) we obtain $\mathfrak{L}_n(\gamma) = \mathfrak{M}_n(\gamma) \cdot \mathfrak{L}_n(W\gamma)$. Thus, we get

$$I_n - \mathfrak{L}_n^*(\gamma)\mathfrak{L}_n(\gamma) = \mathfrak{L}_n^*(W\gamma)\left[I_n - \mathfrak{M}_n^*(\gamma)\mathfrak{M}_n(\gamma)\right]\mathfrak{L}_n(W\gamma) + I_n - \mathfrak{L}_n^*(W\gamma)\mathfrak{L}_n(W\gamma).$$

Considering now $\mathfrak{L}_n(W\gamma)$ instead of $\mathfrak{L}_n(\gamma)$ and repeating the above procedure, we obtain, after m-steps, the formula

$$\begin{aligned} I_n - \mathfrak{L}_n^*(\gamma)\mathfrak{L}_n(\gamma) \quad &= \quad \sum_{j=0}^{m-1} \mathfrak{L}_n^*(W^{j+1}\gamma)\left[I_n - \mathfrak{M}_n^*(W^j\gamma)\mathfrak{M}_n(W^j\gamma)\right]\mathfrak{L}_n(W^{j+1}\gamma) \\ &\quad + I_n - \mathfrak{L}_n^*(W^m\gamma)\mathfrak{L}_n(W^m\gamma). \end{aligned} \tag{2.45}$$

Combining (2.45) with part (c) of Lemma 1.3, we get

$$I_n - \mathfrak{L}_n^*(\gamma)\mathfrak{L}_n(\gamma) = \sum_{j=0}^{\infty} \mathfrak{L}_n^*(W^{j+1}\gamma)\left[I_n - \mathfrak{M}_n^*(W^j\gamma)\mathfrak{M}_n(W^j\gamma)\right]\mathfrak{L}_n(W^{j+1}\gamma). \tag{2.46}$$

For each $j \in \mathbb{N}_0$, from (1.16)

$$\mathfrak{L}_n(W^j\gamma) = \overrightarrow{\prod_{k=j}^{\infty}} \mathfrak{M}_n(W^k\gamma) \tag{2.47}$$

follows and Proposition 2.12 implies

$$I_n - \mathfrak{M}_n^*(W^j\gamma)\mathfrak{M}_n(W^j\gamma) = \left(\prod_{k=j+1}^{j+n} D_{\gamma_k}^{-2}\right) \mathfrak{M}_n^*(W^j\gamma)\eta_n(W^j\gamma)\eta_n^*(W^j\gamma)\mathfrak{M}_n(W^j\gamma). \tag{2.48}$$

Now the combination of (2.46)–(2.48) yields (2.43). □

Lemma 2.14. *Let $\gamma = (\gamma_j)_{j=0}^{\infty} \in \Gamma l_2$, let $r \in \mathbb{N}$, and let Π_1 be defined via* (1.4)*. Then*

$$\sum_{n=r}^{\infty} |\gamma_{n+1}|^2 \left[\prod_{k=1}^{n}(1-|\gamma_k|^2)\right] = \prod_{k=1}^{r}(1-|\gamma_k|^2) - \Pi_1^2.$$

Proof. Taking into account (1.4) and (1.5), we obtain

$$\begin{aligned}
\prod_{k=1}^{r}(1-|\gamma_k|^2) &= \left[\prod_{k=1}^{r}(1-|\gamma_k|^2)\right]\left[|\gamma_{r+1}|^2 + (1-|\gamma_{r+1}|^2)\right] \\
&= |\gamma_{r+1}|^2\left[\prod_{k=1}^{r}(1-|\gamma_k|^2)\right] + \prod_{k=1}^{r+1}(1-|\gamma_k|^2) \\
&= |\gamma_{r+1}|^2\left[\prod_{k=1}^{r}(1-|\gamma_k|^2)\right] + |\gamma_{r+2}|^2\left[\prod_{k=1}^{r+1}(1-|\gamma_k|^2)\right] + \prod_{k=1}^{r+2}(1-|\gamma_k|^2).
\end{aligned}$$

Iterating this procedure, for each integer m with $m \geq r$, we get

$$\prod_{k=1}^{r}(1-|\gamma_k|^2) - \Pi_1^2 = \sum_{n=r}^{m} |\gamma_{n+1}|^2\left[\prod_{k=1}^{n}(1-|\gamma_k|^2)\right] + \prod_{k=1}^{m+1}(1-|\gamma_k|^2) - \Pi_1^2.$$

This yields the assertion after passing to the limit $m \to \infty$. □

Theorem 2.15. *Let $n_0 \in \mathbb{N}$ and let $\gamma = (\gamma_j)_{j=0}^{\infty}$ be a sequence which belongs to Γl_2 and satisfies* $\operatorname{rank}\gamma = n_0$*. Further, let r be an integer with $r \geq n_0$, let $p = (p_r, \ldots, p_0)^T$ be an rth order $\mathcal{S}$-recurrence vector associated with γ, and let λ be defined via* (2.19)*. Then*

$$\prod_{k=1}^{r}(1-|\gamma_k|^2) - \Pi_1^2 = \lambda^*\left((\mathfrak{L}_r^{-1}(\gamma))^*\mathfrak{L}_r^{-1}(\gamma) - I_r\right)\lambda, \tag{2.49}$$

where Π_1 and $\mathfrak{L}_r(\gamma)$ are defined by (1.4) *and* (1.7)*, respectively.*

Proof. Let n be an integer with $n \geq r$. Then, from Theorem 2.8 by rewriting formula (2.20) we get the relation

$$\gamma_{n+1}\left(\prod_{k=1}^{n} D_{\gamma_k}\right) = \left(\prod_{k=n-r+1}^{n} D_{\gamma_k}^{-1}\right)\eta_r^*(W^{n-r}\gamma)\left(\overleftarrow{\prod_{k=0}^{n-r-1}}\mathfrak{M}_r^{-1}(W^k\gamma)\right)\lambda. \tag{2.50}$$

From (1.16) it follows that

$$\overleftarrow{\prod_{k=0}^{n-r-1}}\mathfrak{M}_r^{-1}(W^k\gamma) = \left[\overrightarrow{\prod_{k=n-r}^{\infty}}\mathfrak{M}_r(W^k\gamma)\right]\mathfrak{L}_r^{-1}(\gamma).$$

Inserting this into (2.50), we get

$$\gamma_{n+1}\left(\prod_{k=1}^{n} D_{\gamma_k}\right) = \left(\prod_{k=n-r+1}^{n} D_{\gamma_k}^{-1}\right)\eta_r^*(W^{n-r}\gamma)\left[\overrightarrow{\prod_{k=n-r}^{\infty}}\mathfrak{M}_r(W^k\gamma)\right]\mathfrak{L}_r^{-1}(\gamma)\lambda.$$

This implies

$$\begin{aligned}
&\sum_{n=r}^{\infty}|\gamma_{n+1}|^2\left(\prod_{k=1}^{n} D_{\gamma_k}^2\right)\\
&= \sum_{n=r}^{\infty}\left(\prod_{k=n-r+1}^{n} D_{\gamma_k}^{-2}\right)\lambda^*(\mathfrak{L}_r^{-1}(\gamma))^*\left[\overleftarrow{\prod_{k=n-r}^{\infty}}\mathfrak{M}_r^*(W^k\gamma)\right]\eta_r(W^{n-r}\gamma)\\
&\qquad\cdot\eta_r^*(W^{n-r}\gamma)\left[\overrightarrow{\prod_{k=n-r}^{\infty}}\mathfrak{M}_r(W^k\gamma)\right]\mathfrak{L}_r^{-1}(\gamma)\lambda\\
&= \lambda^*(\mathfrak{L}_r^{-1}(\gamma))^*\left(\sum_{n=r}^{\infty}\left(\prod_{k=n-r+1}^{n} D_{\gamma_k}^{-2}\right)\left[\overleftarrow{\prod_{k=n-r}^{\infty}}\mathfrak{M}_r^*(W^k\gamma)\right]\eta_r(W^{n-r}\gamma)\right.\\
&\qquad\left.\cdot\eta_r^*(W^{n-r}\gamma)\left[\overrightarrow{\prod_{k=n-r}^{\infty}}\mathfrak{M}_r(W^k\gamma)\right]\right)\mathfrak{L}_r^{-1}(\gamma)\lambda.
\end{aligned}\tag{2.51}$$

According to Lemma 2.14 the left-hand side of equation (2.51) can be rewritten as

$$\sum_{n=r}^{\infty}|\gamma_{n+1}|^2\left(\prod_{k=1}^{n} D_{\gamma_k}^2\right) = \prod_{k=1}^{r}(1-|\gamma_k|^2) - \Pi_1^2. \tag{2.52}$$

Substituting the summation index $j = n - r$ and taking (2.44) and (2.43) into account, we obtain

$$\begin{aligned}
&\sum_{n=r}^{\infty}\left(\prod_{k=n-r+1}^{n} D_{\gamma_k}^{-2}\right)\left[\overleftarrow{\prod_{k=n-r}^{\infty}} \mathfrak{M}_r^*(W^k\gamma)\right]\eta_r(W^{n-r}\gamma)\\
&\qquad\qquad\cdot\eta_r^*(W^{n-r}\gamma)\left[\overrightarrow{\prod_{k=n-r}^{\infty}} \mathfrak{M}_r(W^k\gamma)\right]\\
&= \sum_{j=0}^{\infty}\left(\prod_{k=j+1}^{j+r} D_{\gamma_k}^{-2}\right)\left[\overleftarrow{\prod_{k=j}^{\infty}} \mathfrak{M}_r^*(W^k\gamma)\right]\eta_r(W^{j}\gamma)\\
&\qquad\qquad\cdot\eta_r^*(W^{j}\gamma)\left[\overrightarrow{\prod_{k=j}^{\infty}} \mathfrak{M}_r(W^k\gamma)\right]\\
&= \sum_{j=0}^{\infty}\tau_{n,j}(\gamma)\tau_{n,j}^*(\gamma) = I_r - \mathfrak{L}_r^*(\gamma)\mathfrak{L}_r(\gamma).
\end{aligned}\tag{2.53}$$

The combination of (2.52), (2.51), and (2.53) yields

$$\begin{aligned}
\prod_{k=1}^{r}(1-|\gamma_k|^2) - \Pi_1^2 &= \lambda^*(\mathfrak{L}_r^{-1}(\gamma))^*\left(I_r - \mathfrak{L}_r^*(\gamma)\mathfrak{L}_r(\gamma)\right)\mathfrak{L}_r^{-1}(\gamma)\lambda\\
&= \lambda^*\left((\mathfrak{L}_r^{-1}(\gamma))^*\mathfrak{L}_r^{-1}(\gamma) - I_r\right)\lambda.
\end{aligned}$$

Thus, the proof is complete. □

Remark 2.16. We reconsider Theorem 2.15 in the particular case that $n_0 = 1$ and $r = 1$ holds. From (1.7) we get $\mathfrak{L}_1(\gamma) = \Pi_1$. Thus, equation (2.49) has the form

$$1 - |\gamma_1|^2 - \Pi_1^2 = |\lambda|^2\left(\frac{1}{\Pi_1^2} - 1\right).$$

Hence,

$$\Pi_1^2\left(1 - |\gamma_1|^2 - \Pi_1^2\right) = |\lambda|^2\left(1 - \Pi_1^2\right).\tag{2.54}$$

Equation (2.54) was obtained in the proof of Theorem 5.22 in [4] (see [4, p. 245]). We note that the method of proving Theorem 2.15 is a generalization to the case of a sequence $\gamma = (\gamma_j)_{j=0}^{\infty} \in \Gamma l_2$ having arbitrary finite rank of the method of proving equation (2.54) in [4, p. 245] which works only for sequences having first rank.

3. Recovering the matrices $\mathfrak{L}_{r+1}(\gamma)$ from its first column and the sequence $(\gamma_j)_{j=0}^{r}$

At the beginning of this section we turn our attention to the sequence of matrices $(\mathfrak{L}_n(\gamma))_{n=1}^{\infty}$ which are associated with a sequence $\gamma = (\gamma_j)_{j=0}^{\infty} \in \Gamma l_2$. We will see that for each $r \in \mathbb{N}$ the matrix $\mathfrak{L}_{r+1}(\gamma)$ can be recovered from its first column and the section $(\gamma_j)_{j=0}^{r}$ of the sequence γ. In this way, we will uncover particular relationships between the columns of the matrix $\mathfrak{L}_{r+1}(\gamma)$.

Lemma 3.1. *Let $\gamma = (\gamma_j)_{j=0}^{\infty} \in \Gamma l_2$. Further, let $r \in \mathbb{N}$ and let the matrix $\mathfrak{L}_{r+1}(\gamma)$ be defined via* (1.7). *Then the sequence $(\gamma_j)_{j=0}^{r}$ and the elements of the first column*

$$(\Pi_1, \Pi_2 L_1(W\gamma), \Pi_3 L_2(W\gamma), \ldots, \Pi_{r+1} L_r(W\gamma))^T \tag{3.1}$$

of $\mathfrak{L}_{r+1}(\gamma)$ uniquely determine all the remaining elements of $\mathfrak{L}_{r+1}(\gamma)$.

Proof. Given the sequence $(\gamma_j)_{j=0}^{r}$ and Π_1 we first obtain, successively

$$\Pi_2 = D_{\gamma_1}^{-1}\Pi_1,\ \Pi_3 = D_{\gamma_2}^{-1}\Pi_2, \ldots,\ \Pi_{r+1} = D_{\gamma_r}^{-1}\Pi_r,$$

where D_{γ_j} is defined in (1.5). Thus, using (3.1) we now compute the numbers $L_1(W\gamma), \ldots, L_r(W\gamma)$. According to Corollary 3.9, in [4] we have for $m \in \{1, 2, \ldots, r-1\}$ and $k \in \{1, 2, \ldots, r+1-m\}$ the recurrence formulas

$$L_m(W^{k+1}\gamma) = L_m(W^k\gamma) + \overline{\gamma_{m+k}} \sum_{j=k}^{m+k-1} \gamma_j L_j(W^k\gamma). \tag{3.2}$$

From (3.2) we see that, for each $j \in \{1, 2, \ldots, r\}$, the elements of the $(j+1)$th column of $\mathfrak{L}_{r+1}(\gamma)$ can be expressed in terms of the elements of the jth column of $\mathfrak{L}_{r+1}(\gamma)$. Iterating this procedure, we get that the elements of all columns of $\mathfrak{L}_{r+1}(\gamma)$ can be expressed in terms of the sequence $(\gamma_j)_{j=0}^{r}$ and the first column (3.1) of $\mathfrak{L}_{r+1}(\gamma)$. □

Lemma 3.1 leads us to the following considerations. Let $\gamma = (\gamma_j)_{j=0}^{\infty} \in \Gamma l_2$. Further, let $r \in \mathbb{N}$ and let the matrix $\mathfrak{L}_{r+1}(\gamma)$ be defined via (1.7). Then Lemma 3.1 tells us that given the sequence $(\gamma_j)_{j=0}^{r}$ and the elements of the first column (3.1) of $\mathfrak{L}_{r+1}(\gamma)$ all remaining elements of $\mathfrak{L}_{r+1}(\gamma)$ can be computed. More precisely, the proof of Lemma 3.1 shows in which way the remaining elements can be calculated. In our following investigations we suppose that some $r \in \mathbb{N}$ and some sequence $(\gamma_j)_{j=0}^{r}$ from $\mathbb{D}$ are given. Then we are looking for a positive number $\Pi_{1,1}$ and a sequence $(L_{j,1})_{j=1}^{r}$ such that if we construct the complex $(r+1) \times (r+1)$ matrix $\mathfrak{L}_{r+1,0}$ as we did the matrix $\mathfrak{L}_{r+1}(\gamma)$ in the proof of Lemma 3.1, then the corresponding defect matrix $I_{r+1} - \mathfrak{L}_{r+1,0}\mathfrak{L}_{r+1,0}^*$ is nonnegative Hermitian and singular. This leads us to the following notion.

Definition 3.2. Let $r \in \mathbb{N}$ and let $(\gamma_j)_{j=0}^r$ be a sequence from $\mathbb{D}$. Let $\Pi_{1,1} \in (0,\infty)$ and let $(L_{j,1})_{j=1}^r$ be a sequence of complex numbers. Let

$$\mathfrak{L}_{r+1,0} := \begin{pmatrix} \Pi_{1,1} & 0 & 0 & \dots & 0 \\ \Pi_{1,2}L_{1,1} & \Pi_{1,2} & 0 & \dots & 0 \\ \Pi_{1,3}L_{2,1} & \Pi_{1,3}L_{1,2} & \Pi_{1,3} & \dots & 0 \\ \vdots & \vdots & & \vdots & \\ \Pi_{1,r+1}L_{r,1} & \Pi_{1,r+1}L_{r-1,2} & \Pi_{1,r+1}L_{r-2,3} & \dots & \Pi_{1,r+1} \end{pmatrix} \tag{3.3}$$

where

$$\Pi_{1,2} := D_{\gamma_1}^{-1}\Pi_{1,1},\ \Pi_{1,3} := D_{\gamma_2}^{-1}\Pi_{1,2}, \dots,\ \Pi_{1,r+1} := D_{\gamma_r}^{-1}\Pi_{1,r}, \tag{3.4}$$

and where the numbers $(L_{m,k})_{\substack{m=1,\dots,r-1 \\ k=2,\dots,r+1-m}}$ are defined by the recurrent formulas

$$L_{m,k+1} := L_{m,k} + \overline{\gamma_{m+k}} \sum_{j=k}^{m+k-1} \gamma_j L_{j,k}. \tag{3.5}$$

Then $[\Pi_{1,1}, (L_{j,1})_{j=1}^r]$ is called *compatible* with $(\gamma_j)_{j=0}^r$ if

$$I_{r+1} - \mathfrak{L}_{r+1,0}\mathfrak{L}_{r+1,0}^* \geq 0_{(r+1)\times(r+1)} \text{ and } \det\left(I_{r+1} - \mathfrak{L}_{r+1,0}\mathfrak{L}_{r+1,0}^*\right) = 0$$

hold. In this case, the matrix

$$\mathcal{A}_{r+1,0} := I_{r+1} - \mathfrak{L}_{r+1,0}\mathfrak{L}_{r+1,0}^*$$

is called *the information matrix associated with* $[(\gamma_j)_{j=0}^r, \Pi_{1,1}, (L_{j,1})_{j=1}^r]$.

Lemma 3.3. *Let $r \in \mathbb{N}$ and let $(\gamma_j)_{j=0}^r$ be a sequence from $\mathbb{D}$. Further, let $[\Pi_{1,1}, (L_{j,1})_{j=1}^r]$ be compatible with $(\gamma_j)_{j=0}^r$. Then:*

(a) *Let the sequence $(\Pi_{1,j})_{j=2}^{r+1}$ be defined via* (3.4)*. Then $(\Pi_{1,j})_{j=1}^{r+1}$ is a monotonically increasing sequence from $(0,1]$.*

(b) *Let $s \in \{1,2,\dots,r\}$. Then the following statements are equivalent:*
   (i) *$\Pi_{1,s} = 1$.*
   (ii) *For all $j \in \{s, s+1, \dots, r+1\}$, the relation it holds $\Pi_{1,j} = 1$ holds.*
   (iii) *For all $j \in \{s, s+1, \dots, r\}$ the relation $\gamma_j = 0$ is valid.*

   If (i) *is satisfied, then $L_{j,1} = 0$ for each $j \in \{s, s+1, \dots, r\}$.*

(c) *If $\Pi_{1,1} = 1$, then the matrix $\mathfrak{L}_{r+1,0}$ defined via* (3.3)–(3.5) *coincides with the unit matrix I_{r+1}. In particular, the matrix $\mathcal{A}_{r+1,0} := I_{r+1} - \mathfrak{L}_{r+1,0}\mathfrak{L}_{r+1,0}^*$ fulfills $\mathcal{A}_{r+1,0} = 0_{(r+1)\times(r+1)}$.*

Proof. (a) From the construction of the sequence $(\Pi_{1,j})_{j=1}^{r+1}$ it is immediately obvious that this is a monotonically increasing sequence from $(0,\infty)$. Since by assumption the matrix $\mathfrak{L}_{r+1,0}$ is contractive and since the sequence $(\Pi_{1,j})_{j=1}^{r+1}$ forms the main diagonal of $\mathfrak{L}_{r+1,0}$, we obtain that it is a sequence from $(0,1]$.

(b) The equivalence of (i), (ii), and (iii) is an immediate consequence of (a). Let (i) be satisfied and let $j \in \{s, s+1, \dots, r\}$. In view of (ii), then $\Pi_{1,s+1} = 1$. Since $\Pi_{1,s+1}$ is the $(s+1)$th diagonal element of the contractive matrix $\mathfrak{L}_{r+1,0}$ all remaining elements of the $(s+1)$th row of $\mathfrak{L}_{r+1,0}$ have to be 0. Since in view of

(3.3) and $\Pi_{1,s+1} = 1$ the first element of the $(s+1)$th row of $\mathfrak{L}_{r+1,0}$ is $L_{s,1}$, we get $L_{s,1} = 0$.

(c) Taking into account $\Pi_{1,1} = 1$ we infer from (b) that $\Pi_{1,j} = 1$ for each $j \in \{1, 2, \dots, r+1\}$. Thus, all diagonal elements of the contractive matrix $\mathfrak{L}_{r+1,0}$ are equal to 1. This forces $\mathfrak{L}_{r+1,0} = I_{r+1}$ and consequently $\mathcal{A}_{r+1,0} = 0_{(r+1)\times(r+1)}$. □

Remark 3.4. Let $r \in \mathbb{N}$ and let $(\gamma_j)_{j=0}^r$ be a sequence from $\mathbb{D}$. Let $\Pi_{1,1} := \prod_{j=1}^r D_{\gamma_j}$ and let $(L_{j,1})_{j=1}^r$ be a sequence of complex numbers such that $[\Pi_{1,1}, (L_{j,1})_{j=1}^r]$ is compatible with $(\gamma_j)_{j=0}^r$. From (3.4) we see then that

$$\Pi_{1,r+1} = 1. \tag{3.6}$$

Let $\mathfrak{L}_{r+1,0}$ be defined via (3.3)–(3.5). Then by assumption $\mathfrak{L}_{r+1,0}$ is contractive. Combining this with (3.6), we see that the last row of $\mathfrak{L}_{r+1,0}$ is $(0, \dots, 0, 1)$. Thus the information matrix $\mathcal{A}_{r+1,0}$ associated with $[(\gamma_j)_{j=0}^r, \Pi_{1,1}, (L_{j,1})_{j=1}^r]$ has the shape

$$\mathcal{A}_{r+1,0} = \begin{pmatrix} \star & 0_{r\times 1} \\ 0_{1\times r} & 0 \end{pmatrix}.$$

Hence, the $(r+1)\times 1$ matrix $e_{r+1} := (0, \dots, 0, 1)^T$ belongs to $\ker \mathcal{A}_{r+1,0} \setminus \{0_{(r+1)\times 1}\}$.

Now we turn our attention to the special case $r = 1$.

Remark 3.5. Let $\gamma_1 \in \mathbb{D}$, $\Pi_{1,1} \in (0, \infty)$ and $L_{1,1} \in \mathbb{C}$. Furthermore, let $\Pi_{1,2} := D_{\gamma_1}^{-1} \Pi_{1,1}$,

$$\mathfrak{L}_{2,0} := \begin{pmatrix} \Pi_{1,1} & 0 \\ \Pi_{1,2} L_{1,1} & \Pi_{1,2} \end{pmatrix}, \tag{3.7}$$

and

$$\mathcal{A}_{2,0} := I_2 - \mathfrak{L}_{2,0} \mathfrak{L}_{2,0}^*. \tag{3.8}$$

Then

$$\mathcal{A}_{2,0} = \begin{pmatrix} 1 - \Pi_{1,1}^2 & -\Pi_{1,1} \Pi_{1,2} \overline{L_{1,1}} \\ -\Pi_{1,1} \Pi_{1,2} L_{1,1} & 1 - \Pi_{1,2}^2 (1 + |L_{1,1}|^2) \end{pmatrix} \tag{3.9}$$

and

$$\det \mathcal{A}_{2,0} = \left(1 - \Pi_{1,1}^2\right) \frac{1 - |\gamma_1|^2 - \Pi_{1,1}^2}{1 - |\gamma_1|^2} - \frac{\Pi_{1,1}^2 |L_{1,1}|^2}{1 - |\gamma_1|^2}. \tag{3.10}$$

Lemma 3.6. *Let $(\gamma_j)_{j=0}^1$ be a sequence from $\mathbb{D}$ and let $[\Pi_{1,1}, (L_{j,1})_{j=1}^1]$ be compatible with $(\gamma_j)_{j=0}^1$. Suppose that $\Pi_{1,1} < 1$. Then:*

(a) *The relation*

$$|L_{1,1}|^2 = \frac{1 - \Pi_{1,1}^2}{\Pi_{1,1}^2} \left(1 - |\gamma_1|^2 - \Pi_{1,1}^2\right)$$

holds true.

(b) *The inequality $\Pi_{1,1} \le D_{\gamma_1}$ holds.*

(c) *The null space $\ker \mathcal{A}_{2,0}$ is the linear hull of the vector $(p_1, 1)^T$ where*

$$p_1 := \frac{\Pi_{1,1}^2 \overline{L_{1,1}}}{D_{\gamma_1}(1 - \Pi_{1,1}^2)}. \tag{3.11}$$

(d) *The number* $\lambda := -D_{\gamma_1} p_1$ *fulfills*

$$\Pi_{1,1}^4 - (1 - |\gamma_1|^2 + |\lambda|^2)\Pi_{1,1}^2 + |\lambda|^2 = 0.$$

Proof. (a) Let $\mathcal{A}_{2,0}$ be defined by (3.7) and (3.8). By assumption, we have

$$\det \mathcal{A}_{2,0} = 0. \tag{3.12}$$

Now the combination of (3.12) and (3.10) yields (a).

(b) Since $1 - \Pi_{1,1}^2 > 0$, the assertion of (b) is an immediate consequence of (a).

(c) In view of $1 - \Pi_{1,1}^2 \neq 0$, we see from (3.9) that there is a unique $p_1 \in \mathbb{C}$ such that $\mathcal{A}_{2,0} \cdot (p_1, 1)^T = (0,0)^T$. From (3.9) we get

$$\left(1 - \Pi_{1,1}^2, -\Pi_{1,1}\Pi_{1,2}\overline{L_{1,1}}\right)(p_1, 1)^T = 0.$$

This implies (3.11).

(d) Using (3.10) and the identity

$$\overline{L_{1,1}} = -\frac{(1 - \Pi_{1,1})^2}{\Pi_{1,1}^2}\lambda,$$

we obtain

$$\det \mathcal{A}_{2,0} = -\frac{1 - \Pi_{1,1}^2}{1 - |\gamma_1|^2}\left[\Pi_{1,1}^4 - (1 - |\gamma_1|^2 + |\lambda|^2)\Pi_{1,1}^2 + |\lambda|^2\right]. \tag{3.13}$$

Taking into account $\Pi_{1,1} < 1$, we obtain part (d) from (3.12) and (3.13). □

The combination of Definition 3.2, Definition 1.8, and the proof of Lemma 3.1 provides the following result.

Proposition 3.7. *Let* $n_0 \in \mathbb{N}$, *let* $\gamma = (\gamma_j)_{j=0}^{\infty} \in \Gamma l_2$ *be such that* $\operatorname{rank}\gamma = n_0$, *and let* r *be an integer with* $r \geq n_0$. *Let* Π_1 *and let the sequence* $(L_j(W\gamma))_{j=1}^{r}$ *be defined by* (1.4) *and* (1.3), *respectively. Then* $(\gamma_j)_{j=0}^{r}$ *is a sequence from* $\mathbb{D}$ *and* $[\Pi_1, (L_j(W\gamma))_{j=1}^{r}]$ *is compatible with* $(\gamma_j)_{j=0}^{r}$. *Moreover, the matrix*

$$\mathcal{A}_{r+1}(\gamma) = I_{r+1} - \mathfrak{L}_{r+1}(\gamma)\mathfrak{L}_{r+1}^*(\gamma)$$

is the information matrix associated with $[(\gamma_j)_{j=0}^{r}, \Pi_1, (L_j(W\gamma))_{j=1}^{r}]$.

Now let $r \in \mathbb{N}$ and let $(\gamma_j)_{j=0}^{r}$ be a sequence from $\mathbb{D}$. Using Proposition 3.7 we will show then that in a simple way one can always find data compatible with $(\gamma_j)_{j=0}^{r}$.

Remark 3.8. Let $r \in \mathbb{N}$ and let $(\gamma_j)_{j=0}^{r}$ be a sequence from $\mathbb{D}$. For each $j \in \{r+1, r+2, \ldots\}$ we set $\gamma_j := 0$. Then Remark 1.10 shows that γ belongs to Γl_2 and that $\operatorname{rank}\gamma \leq r$. Thus Proposition 3.7 implies that $[\Pi_1, (L_j(W\gamma))_{j=1}^{r}]$ is compatible with $(\gamma_j)_{j=0}^{r}$. Moreover, $\Pi_1 = \prod_{j=1}^{r} D_{\gamma_j}$.

Remark 3.8 leads us to the following notion.

Definition 3.9. Let $r \in \mathbb{N}$, let $(\gamma_j)_{j=0}^r$ be a sequence from $\mathbb{D}$, and let $\Pi_{1,1} := \prod_{j=1}^r D_{\gamma_j}$. For each $k \in \{1,2,\dots,r\}$ let $L_{k,1} := L_k(W\gamma)$, where the sequence $\gamma = (\gamma_j)_{j=0}^\infty$ is that extension of $(\gamma_j)_{j=0}^r$ which is given by $\gamma_j = 0$,for each integer j with $j \geq r+1$. Then $[\Pi_{1,1}, (L_{j,1})_{j=1}^r]$ are called *the canonical data compatible with* $(\gamma_j)_{j=0}^r$.

4. On the structure of the information matrix $\mathcal{A}_{r+1,0}$

Let $r \in \mathbb{N}$ and let $(\gamma_j)_{j=0}^r$ be a sequence from $\mathbb{D}$. Furthermore, let $[\Pi_{1,1}, (L_{j,1})_{j=1}^r]$ be compatible with $(\gamma_j)_{j=0}^r$. Let the matrix $\mathfrak{L}_{r+1,0}$ be built from this data as in (3.3)–(3.5) and let (in generalization of (3.8))

$$\mathcal{A}_{r+1,0} := I_{r+1} - \mathfrak{L}_{r+1,0}\mathfrak{L}_{r+1,0}^* \tag{4.1}$$

be the information matrix associated with $[(\gamma_j)_{j=0}^r, \Pi_{1,1}, (L_{j,1})_{j=1}^r]$.

To analyse the structure of the information matrix $\mathcal{A}_{r+1,0}$, for all $m \in \{1,2,\dots,r\}$ and $k \in \{1,2,\dots,r+2-m\}$, we introduce the matrices

$$\mathfrak{L}_{m,k-1} := \begin{pmatrix} \Pi_{1,k} & 0 & 0 & \dots & 0 \\ \Pi_{1,k+1}L_{1,k} & \Pi_{1,k+1} & 0 & \dots & 0 \\ \Pi_{1,k+2}L_{2,k} & \Pi_{1,k+2}L_{1,k+1} & \Pi_{1,k+2} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \Pi_{1,k+m-1}L_{k+m-1,k} & \Pi_{1,k+m-1}L_{k+m-2,k+1} & \Pi_{1,k+m-1}L_{k+m-3,k+2} & \dots & \Pi_{1,k+m-1} \end{pmatrix} \tag{4.2}$$

and

$$\mathcal{A}_{m,k-1} := I_m - \mathfrak{L}_{m,k-1}\mathfrak{L}_{m,k-1}^*. \tag{4.3}$$

If we compare the matrices introduced in (3.3)–(3.5) and (4.2)–(4.3) with the matrices defined in (1.7), then we observe that now the numbers $\Pi_{1,k}$ and $L_{m,k}$ play the role of the numbers Π_k and $L_m(W^k\gamma)$ in (1.7). Thus, the matrices $\mathfrak{L}_{m,k}$ and $\mathcal{A}_{m,k}$ play the role of the matrices $\mathfrak{L}_m(W^k\gamma)$ and $\mathcal{A}_m(W^k\gamma)$, respectively.

The recurrence formulas (3.5) are modelled after the pattern of the recurrence formulas (3.2). It can be immediately checked that the formulas (3.2) are equivalent to (1.12). Let $m \in \{1,2,\dots,r\}$ and $k \in \{1,2,\dots,r-m\}$. Starting with the sequence $(\gamma_j)_{j=0}^r$, we introduce the matrix

$$\mathfrak{M}_{m,k} := \begin{pmatrix} D_{\gamma_{1+k}} & 0 & 0 & \dots & 0 \\ -\gamma_{1+k}\overline{\gamma_{2+k}} & D_{\gamma_{2+k}} & 0 & \dots & 0 \\ -\gamma_{1+k}D_{\gamma_{2+k}}\overline{\gamma_{3+k}} & -\gamma_{2+k}\overline{\gamma_{3+k}} & D_{\gamma_{3+k}} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -\gamma_{1+k}\left(\prod_{j=2+k}^{m-1} D_{\gamma_j}\right)\overline{\gamma_{m+k}} & -\gamma_{2+k}\left(\prod_{j=3+k}^{m-1} D_{\gamma_j}\right)\overline{\gamma_{m+k}} & -\gamma_{3+k}\left(\prod_{j=4+k}^{m-1} D_{\gamma_j}\right)\overline{\gamma_{m+k}} & \dots & D_{\gamma_{m+k}} \end{pmatrix}. \tag{4.4}$$

Obviously, $\mathfrak{M}_{m,k}$ coincides with the matrix $\mathfrak{M}_m(W^k\gamma)$ introduced in (1.13). However, the notations are different because now we only know the finite section $(\gamma_j)_{j=0}^r$ of the first $r+1$ elements of the sequence $\gamma = (\gamma_j)_{j=0}^\infty$. Consequently, we have

$$I_m - \mathfrak{M}_{m,k}\mathfrak{M}_{m,k}^* = \eta_{m,k}\eta_{m,k}^*, \tag{4.5}$$

where

$$\eta_{m,k} := \left(\overline{\gamma_{1+k}}, \overline{\gamma_{2+k}}D_{\gamma_{1+k}}, \dots, \overline{\gamma_{m+k}}\left(\prod_{j=1+k}^{m+k-1} D_{\gamma_j}\right)\right)^T. \tag{4.6}$$

Obviously, the vector $\eta_{m,k}$ coincides with the vector $\eta_m(W^k\gamma)$ defined in (1.15). Moreover, we get

$$\mathfrak{L}_{m,k} = \mathfrak{M}_{m,k}\mathfrak{L}_{m,k+1} \tag{4.7}$$

which resembles

$$\mathfrak{L}_m(W^k\gamma) = \mathfrak{M}_m(W^k\gamma)\mathfrak{L}_m(W^{k+1}\gamma).$$

In the case $m = r$ and $k = 0$ identity (4.7) has the form

$$\mathfrak{L}_{r,0} = \mathfrak{M}_{r,0}\mathfrak{L}_{r,1}. \tag{4.8}$$

Using (4.7) we obtain, in the same way we did (1.19), the identity

$$\mathcal{A}_{m,k} = \eta_{m,k}\eta_{m,k}^* + \mathfrak{M}_{m,k}\mathcal{A}_{m,k+1}\mathfrak{M}_{m,k}^*. \tag{4.9}$$

In particular, in the case $n = r$ and $k = 0$ we have

$$\mathcal{A}_{r,0} = \eta_{r,0}\eta_{r,0}^* + \mathfrak{M}_{r,0}\mathcal{A}_{r,1}\mathfrak{M}_{r,0}^*. \tag{4.10}$$

From (4.2) we obtain block decompositions for the matrix $\mathfrak{L}_{m,k-1}$ which are analogous to the block decompositions (1.8) and (1.10), namely

$$\mathfrak{L}_{m,k-1} = \begin{pmatrix} \mathfrak{L}_{m-1,k-1} & 0 \\ b_{m-1,k-1}^* & \Pi_{1,m} \end{pmatrix} \tag{4.11}$$

and

$$\mathfrak{L}_{m,k-1} = \begin{pmatrix} \Pi_{1,k} & 0 \\ B_{m,k-1} & \mathfrak{L}_{m-1,k+1} \end{pmatrix}, \tag{4.12}$$

where

$$b_{m-1,k-1} := \Pi_{1,m} \cdot \left(\overline{L_{m-1,k}}, \overline{L_{m-2,k+1}}, \ldots, \overline{L_{1,k+m-2}}\right)^T \tag{4.13}$$

and

$$B_{m,k-1}(\gamma) := \left(\Pi_{1,k+1}L_{1,k}, \Pi_{1,k+2}L_{2,k}, \ldots, \Pi_{1,m}L_{m,k}\right)^T, \tag{4.14}$$

respectively. From (4.11) and (4.12) we obtain block decompositions for the matrix $\mathcal{A}_{m,k-1}$ which are similar to the block decompositions (2.12) and (2.18), namely

$$\mathcal{A}_{m,k-1} = \begin{pmatrix} \mathcal{A}_{m-1,k-1} & -\mathfrak{L}_{m-1,k-1}b_{m-1,k-1} \\ -b^*_{m-1,k-1}\mathfrak{L}^*_{m-1,k-1} & 1-\Pi^2_{1,m} - b^*_{m-1,k-1}b_{m-1,k-1} \end{pmatrix} \tag{4.15}$$

and

$$\mathcal{A}_{m,k-1} = \begin{pmatrix} 1-\Pi^2_{1,k} & -\Pi_{1,k}B^*_{m,k-1} \\ -B_{m,k-1}\Pi_{1,k} & \mathcal{A}_{m-1,k+1} - B_{m,k-1}B^*_{m,k-1} \end{pmatrix}, \tag{4.16}$$

respectively. Formulas (4.7)–(4.16) show that the information matrix $\mathcal{A}_{r+1,0}$ has the same structure as the matrix $\mathcal{A}_{r+1}(\gamma)$ introduced in (1.17). Thus, imitating the proof of Proposition 2.6, we obtain the following result.

Proposition 4.1. *Let $r \in \mathbb{N}$, let $(\gamma_j)_{j=0}^r$ be a sequence from $\mathbb{D}$, and let $[\Pi_{1,1}, (L_{j,1})_{j=1}^r]$ be compatible with $(\gamma_j)_{j=0}^r$. Let the matrix $\mathfrak{L}_{r+1,0}$ be built from these data as in* (3.3)–(3.5) *and let $\mathcal{A}_{r+1,0}$ be the associated information matrix introduced in* (4.1). *Let $\mathcal{A}_{0,0} := 1$ and for each $n \in \{1, 2, \ldots, r\}$ let $\mathcal{A}_{n,0}$ be the $n \times n$-matrix in the left upper corner of $\mathcal{A}_{r+1,0}$. Then:*

(a) *There exists some $\widetilde{n}_0 \in \{0, 1, \ldots, r\}$ such that $\det \mathcal{A}_{\widetilde{n}_0,0} > 0$ and $\operatorname{rank} \mathcal{A}_{m,0} = \widetilde{n}_0$ for each $m \in \{\widetilde{n}_0+1, \widetilde{n}_0+2, \ldots, r+1\}$.*

(b) *for each $\widetilde{p} = (\widetilde{p}_{\widetilde{n}_0}, \ldots, \widetilde{p}_0)^T \in \ker \mathcal{A}_{\widetilde{n}_0+1,0} \setminus \{0_{(\widetilde{n}_0+1)\times 1}\}$ the inequality $\widetilde{p}_0 \neq 0$ is true.*

(c) *Let $\widetilde{p} \in \ker \mathcal{A}_{\widetilde{n}_0+1,0} \setminus \{0_{(\widetilde{n}_0+1)\times 1}\}$ and let the sequence $(\widetilde{g}_j)_{j=1}^{r-\widetilde{n}_0+1}$ be defined by*

$$\widetilde{g}_1 := \widetilde{p}, \quad \widetilde{g}_2 := \mathfrak{M}^*_{\widetilde{n}_0+1,0}\widetilde{p}, \quad \ldots, \quad \widetilde{g}_{r-\widetilde{n}_0+1} := \left[\overleftarrow{\prod_{k=0}^{r-\widetilde{n}_0+1}} \mathfrak{M}^*_{\widetilde{n}_0+1,k}\right]\widetilde{p}.$$

Then the $(r+1)\times 1$ matrices

$$g_1 := \begin{pmatrix} \widetilde{g}_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad g_2 := \begin{pmatrix} 0 \\ \widetilde{g}_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad g_{r-\widetilde{n}_0+1} := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \widetilde{g}_{r-\widetilde{n}_0+1} \end{pmatrix}$$

form a basis of $\ker \mathcal{A}_{r+1,0}$.

Corollary 4.2. *Under the assumptions of Proposition* 4.1 *there exists some vector* $p = (p_r, \dots, p_0)^T \in \ker \mathcal{A}_{r+1,0}$ *which satisfies* $p_0 \neq 0$.

5. Constructing a sequence belonging to Γl_2 and having finite rank $n_0 \le r$ from a section $(\gamma_j)_{j=0}^r$ and compatible data $[\Pi_{1,1}, (L_{j,1})_{j=1}^r]$

The main goal of this section can be described as follows. Let $r \in \mathbb{N}$, let $(\gamma_j)_{j=0}^r$ be a sequence from $\mathbb{D}$, and let $[\Pi_{1,1}, (L_{j,1})_{j=1}^r]$ be compatible with $(\gamma_j)_{j=0}^r$. Then we will demonstrate a method for extending $(\gamma_j)_{j=0}^r$ to an infinite sequence $\gamma = (\gamma_j)_{j=0}^\infty \in \Gamma l_2$ which satisfies $\operatorname{rank} \gamma \le r$. Our method of extending the sequence $(\gamma_j)_{j=0}^r$ is of recurrent type. More precisely, it is suggested by the recurrence formulas which were obtained in Theorem 2.8.

Let $p, q \in \mathbb{N}$ and let $A \in \mathbb{C}^{p \times q}$. Then $\|A\|_S$ stands for the operator norm of A.

Lemma 5.1. *Let* $n \in \mathbb{N}$. *Further, let* $A \in \mathbb{C}^{n \times n}$ *and* $\eta \in \mathbb{C}^n$ *be such that* $A - \eta\eta^* \ge 0_{n\times n}$. *Then* $\|\eta\|_{\mathbb{C}^n}^2 \le \|A\|_S$.

Proof. In view of $A - \eta\eta^* \ge 0_{n\times n}$ and $\eta\eta^* \ge 0_{n\times n}$ we obtain

$$\|A\|_S \ge \|\eta\eta^*\|_S = \|\eta\|_S^2 = \|\eta\|_{\mathbb{C}^n}^2. \qquad \square$$

The following theorem is one of the central results of this paper.

Theorem 5.2. *Let* $r \in \mathbb{N}$, *let* $(\gamma_j)_{j=0}^r$ *be a sequence from* $\mathbb{D}$, *and let* $[\Pi_{1,1}, (L_{j,1})_{j=1}^r]$ *be compatible with* $(\gamma_j)_{j=0}^r$. *Let the matrix* $\mathfrak{L}_{r+1,0}$ *be built from these data as in* (3.3)–(3.5) *and let* $\mathcal{A}_{r+1,0}$ *be the associated information matrix introduced in* (4.1). *According to Corollary* 4.2, *let* $p = (p_r, \dots, p_0)^T$ *be a vector from* $\ker \mathcal{A}_{r+1,0}$ *which satisfies* $p_0 \neq 0$ *and let*

$$\lambda := -\frac{1}{p_0}\left[\prod_{k=1}^{r} D_{\gamma_k}\right] \cdot (p_r, p_{r-1}, \dots, p_1)^T,$$

where D_{γ_j} *is defined via* (1.5). *Let the sequence* $(\gamma_j)_{j=0}^r$ *be extended to an infinite sequence* $\gamma = (\gamma_j)_{j=0}^\infty$ *by defining recursively for* $n \in \{r, r+1, \dots\}$ *the elements* γ_{n+1} *via*

$$\gamma_{n+1} := \left[\prod_{s=1}^{n} D_{\gamma_s}^{-1}\right]\left(\left[\overleftarrow{\prod_{k=0}^{n-r-1}} \mathfrak{M}_{r,k}^{-1}\right]\lambda, \left[\prod_{k=n-r+1}^{n} D_{\gamma_k}^{-1}\right]\eta_{r,n-r}\right)_{\mathbb{C}^r} \tag{5.1}$$

where $\mathfrak{M}_{r,k}$ *and* $\eta_{r,n-r}$ *are defined via* (4.4) *and* (4.6), *respectively. Then* γ *belongs to* Γl_2 *and satisfies* $\operatorname{rank} \gamma \le r$. *Moreover,* $\mathcal{A}_{r+1,0} \ge \mathcal{A}_{r+1}(\gamma)$ *and* p *is an* r*th order* $\mathcal{S}$*-recurrence vector associated with* γ.

Proof. Let $\mathcal{A}_{0,0} := 1$ and, for each $n \in \{1, 2, \dots, r\}$, let $\mathcal{A}_{n,0}$ be the $n \times n$-matrix in the left upper corner of $\mathcal{A}_{r+1,0}$. In view of part (a) of Proposition 4.1, there exists some nonnegative integer $\widetilde{n}_0$ with $\widetilde{n}_0 \leq r$ such that $\det \mathcal{A}_{\widetilde{n}_0,0} > 0$ and $\operatorname{rank} \mathcal{A}_{m,0} = \widetilde{n}_0$ for each $m \in \{\widetilde{n}_0+1, \widetilde{n}_0+2, \dots, r+1\}$. Thus

$$\operatorname{rank} \mathcal{A}_{r+1,0} = \operatorname{rank} \mathcal{A}_{r,0} = \widetilde{n}_0. \tag{5.2}$$

By assumption (see Definition 3.2) we have

$$\mathcal{A}_{r+1,0} \geq 0_{(r+1)\times(r+1)}. \tag{5.3}$$

Taking into account (4.15), (5.2), and (5.3), we obtain by a standard argument (see, e.g., Lemma 1.1.7 in [5])

$$\mathcal{A}_{r+1,0} = F_r \begin{pmatrix} \mathcal{A}_{r,0} & 0_{r\times 1} \\ 0_{1\times r} & 0 \end{pmatrix} F_r^* \tag{5.4}$$

where

$$F_r = \begin{pmatrix} I_r & 0_{r\times 1} \\ -b_{r,0}^* \mathfrak{L}_{r,0}^* \mathcal{A}_{r,0}^+ & 1 \end{pmatrix} \tag{5.5}$$

and where the symbol $\mathcal{A}_{r,0}^+$ stands for the Moore-Penrose inverse of the matrix $\mathcal{A}_{r,0}$. Denote by M_r the orthogonal complement of the linear subspace of $\mathbb{C}^{r+1}$ which is generated by the columns of the matrix $\widehat{F}_r := F_r \begin{pmatrix} I_r & 0_{r\times 1} \\ 0_{1\times r} & 0 \end{pmatrix}$. From (5.5) we see that $\operatorname{rank} \widehat{F}_r = r$. Hence $\dim M_r = 1$. Let

$$\widetilde{p} = (\widetilde{p}_r, \dots, \widetilde{p}_0)^T \in M_r \backslash \{0_{(r+1)\times 1}\}. \tag{5.6}$$

From (5.3) and the choice of M_r we have

$$\widetilde{p}^* \widehat{F}_r = 0_{1\times(r+1)}. \tag{5.7}$$

Using (5.4) and (5.7), we get

$$\widetilde{p}^* \mathcal{A}_{r+1,0} \widetilde{p} = \widetilde{p}^* F_r \begin{pmatrix} \mathcal{A}_{r,0} & 0_{r\times 1} \\ 0_{1\times r} & 0 \end{pmatrix} F_r^* \widetilde{p} = \widetilde{p}^* \widehat{F}_r \begin{pmatrix} \mathcal{A}_{r,0} & 0_{r\times 1} \\ 0_{1\times r} & 0 \end{pmatrix} F_r^* \widetilde{p} = 0. \tag{5.8}$$

From (5.3) and (5.8) we infer

$$\widetilde{p} \in \ker \mathcal{A}_{r+1,0}. \tag{5.9}$$

Taking into account (5.6) and (5.7) we see that

$$\widetilde{p}_0 \neq 0. \tag{5.10}$$

Our next step is to define the element γ_{r+1}. This will be done as follows. In accordance with (4.6) let

$$\eta_{r,0} := \left(\overline{\gamma_1}, \overline{\gamma_2} D_{\gamma_1}, \dots, \overline{\gamma_r} \left(\prod_{j=1}^{r-1} D_{\gamma_j} \right) \right)^T \tag{5.11}$$

and let

$$\widetilde{q} := (\widetilde{p}_r, \widetilde{p}_{r-1}, \dots, \widetilde{p}_1)^T. \tag{5.12}$$

Taking into account (5.10)–(5.12) we define

$$\gamma_{r+1} := -\frac{1}{\widetilde{p}_0 \left(\prod_{j=1}^{r} D_{\gamma_j}\right)} \cdot (\widetilde{q}, \eta_{r,0})_{\mathbb{C}^r} . \tag{5.13}$$

In view of (5.11), the vector

$$\eta_{r+1,0} := \left(\overline{\gamma_1}, \overline{\gamma_2} D_{\gamma_1}, \dots, \overline{\gamma_{r+1}} \left(\prod_{j=1}^{r} D_{\gamma_j}\right)\right)^T$$

has the block decomposition

$$\eta_{r+1,0} = \begin{pmatrix} \eta_{r,0} \\ \overline{\gamma_{r+1}} \left(\prod_{j=1}^{r} D_{\gamma_j}\right) \end{pmatrix} . \tag{5.14}$$

Using (5.6) and (5.12)–(5.14), we get

$$\begin{aligned} (\widetilde{p}, \eta_{r+1,0})_{\mathbb{C}^{r+1}} &= \left(\begin{pmatrix} \widetilde{q} \\ \widetilde{p}_0 \end{pmatrix}, \begin{pmatrix} \eta_{r,0} \\ \overline{\gamma_{r+1}} \left(\prod_{j=1}^{r} D_{\gamma_j}\right) \end{pmatrix} \right)_{\mathbb{C}^{r+1}} \\ &= (\widetilde{q}, \eta_{r,0})_{\mathbb{C}^r} + \widetilde{p}_0 \gamma_{r+1} \left(\prod_{j=1}^{r} D_{\gamma_j} \right) = 0. \end{aligned} \tag{5.15}$$

Now we are going to show that

$$\mathcal{A}_{r+1,0} - \eta_{r+1,0} \eta_{r+1,0}^* \geq 0_{(r+1)\times(r+1)}. \tag{5.16}$$

In view of (5.15) and the construction of the space M_r, we see that the vector $\eta_{r+1,0}$ belongs to the linear subspace of $\mathbb{C}^{r+1}$ which is generated by the columns of the matrix $\widehat{F}_r$. Thus, there is some $u \in \mathbb{C}^{r+1}$ such that $\eta_{r+1,0} = \widehat{F}_r u$. Choosing, the block decomposition $u = \begin{pmatrix} v \\ u_{r+1} \end{pmatrix}$, where $v \in \mathbb{C}^r$, we obtain

$$\eta_{r+1,0} = F_r \begin{pmatrix} I_r & 0_{r\times 1} \\ 0_{1\times r} & 0 \end{pmatrix} \begin{pmatrix} v \\ u_{r+1} \end{pmatrix} = F_r \begin{pmatrix} v \\ 0 \end{pmatrix} . \tag{5.17}$$

Combining (5.14), (5.5), and (5.17) we get

$$\eta_{r+1,0} = F_r \begin{pmatrix} \eta_{r,0} \\ 0 \end{pmatrix} . \tag{5.18}$$

By virtue of (5.4) and (5.18) we infer

$$\mathcal{A}_{r+1,0} - \eta_{r+1,0} \eta_{r+1,0}^* = F_r \begin{pmatrix} \mathcal{A}_{r,0} - \eta_{r,0}\eta_{r,0}^* & 0_{r\times 1} \\ 0_{1\times r} & 0 \end{pmatrix} F_r^* . \tag{5.19}$$

In view of (4.10) we have

$$\mathcal{A}_{r,0} - \eta_{r,0} \eta_{r,0}^* = \mathfrak{M}_{r,0} \mathcal{A}_{r,1} \mathfrak{M}_{r,0}^* . \tag{5.20}$$

Using (4.16) for $m = r+1$ and $k = 1$ it follows

$$\mathcal{A}_{r+1,0} = \begin{pmatrix} \star & \star \\ \star & \mathcal{A}_{r,1} - B_{r+1,k-1} B^*_{r+1,k-1} \end{pmatrix}. \tag{5.21}$$

From (5.3) and (5.21) we can conclude $\mathcal{A}_{r,1} - B_{r+1,k-1} B^*_{r+1,k-1} \geq 0_{r\times r}$. Thus,

$$\mathcal{A}_{r,1} \geq 0_{r\times r}. \tag{5.22}$$

Now the combination of (5.19), (5.20), and (5.22) yields (5.16). From (5.16) and Lemma 5.1 we get

$$\|\eta_{r+1,0}\|^2_{\mathbb{C}^{r+1}} \leq \|\mathcal{A}_{r+1,0}\|_S. \tag{5.23}$$

From (3.3) it is obvious that $\det \mathfrak{L}_{r+1,0} \neq 0$. Consequently, the matrix $\mathcal{A}_{r+1,0}$ is strictly contractive. This implies

$$\|\mathcal{A}_{r+1,0}\|_S < 1. \tag{5.24}$$

Taking into account (1.15) and (5.11), and applying Lemma 2.10 yields

$$\|\eta_{r,0}\|_{\mathbb{C}^r} = 1 - \prod_{j=1}^{r} (1 - |\gamma_j|^2). \tag{5.25}$$

Because of (5.14), (5.25), and (1.5) we get

$$\begin{aligned} \|\eta_{r+1,0}\|^2_{\mathbb{C}^{r+1}} &= \left\| \begin{pmatrix} \eta_{r,0} \\ \overline{\gamma_{r+1}} \left(\prod_{j=1}^r D_{\gamma_j}\right) \end{pmatrix} \right\|^2_{\mathbb{C}^{r+1}} = \|\eta_{r,0}\|^2_{\mathbb{C}^r} + |\gamma_{r+1}|^2 \left(\prod_{j=1}^{r} D^2_{\gamma_j} \right) \\ &= 1 - \prod_{j=1}^{r} (1 - |\gamma_j|^2) + |\gamma_{r+1}|^2 \left(\prod_{j=1}^{r} (1 - |\gamma_j|^2) \right) = 1 - \prod_{j=1}^{r+1} (1 - |\gamma_j|^2). \end{aligned}$$

Hence

$$\prod_{j=1}^{r+1} (1 - |\gamma_j|^2) = 1 - \|\eta_{r+1,0}\|^2_{\mathbb{C}^{r+1}}. \tag{5.26}$$

Since $(\gamma_j)_{j=0}^r$ is a sequence from $\mathbb{D}$ we have

$$\prod_{j=1}^{r} (1 - |\gamma_j|^2) > 0. \tag{5.27}$$

From (5.26) and (5.27) we have

$$1 - |\gamma_{r+1}|^2 = \frac{1 - \|\eta_{r+1,0}\|^2_{\mathbb{C}^{r+1}}}{\prod_{j=1}^r (1 - |\gamma_j|^2)}. \tag{5.28}$$

The combination of (5.28), (5.23), (5.24), and (5.27) yields $|\gamma_{r+1}| < 1$. Consequently, we now have a sequence $(\gamma_j)_{j=0}^{r+1}$ from $\mathbb{D}$. Starting with $(\gamma_j)_{j=0}^{r+1}$ we introduce the matrix $\mathfrak{M}_{r+1,0}$ via (4.4). From (4.4) it is obvious that

$$\det \mathfrak{M}_{r+1,0} \neq 0. \tag{5.29}$$

Corresponding to (4.8) we define

$$\mathfrak{L}_{r+1,1} := \mathfrak{M}_{r+1,0}^{-1}\mathfrak{L}_{r+1,0}. \tag{5.30}$$

Bearing in mind (4.3) we then obtain

$$\mathcal{A}_{r+1,1} := I_{r+1} - \mathfrak{L}_{r+1,1}\mathfrak{L}_{r+1,1}^{*}. \tag{5.31}$$

Using (4.1), (5.30), formula (4.5) for $m = r+1$ and $k = 0$, and (5.31) we get

$$\begin{aligned}\mathcal{A}_{r+1,0} &= I_{r+1} - \mathfrak{L}_{r+1,0}\mathfrak{L}_{r+1,0}^{*} = I_{r+1} - \mathfrak{M}_{r+1,0}\mathfrak{L}_{r+1,1}\mathfrak{L}_{r+1,1}^{*}\mathfrak{M}_{r+1,0}^{*}\\ &= I_{r+1} - \mathfrak{M}_{r+1,0}\mathfrak{M}_{r+1,0}^{*} + \mathfrak{M}_{r+1,0}\left(I_{r+1} - \mathfrak{L}_{r+1,1}\mathfrak{L}_{r+1,1}^{*}\right)\mathfrak{M}_{r+1,0}^{*}\\ &= \eta_{r+1,0}\eta_{r+1,0}^{*} + \mathfrak{M}_{r+1,0}\mathcal{A}_{r+1,1}\mathfrak{M}_{r+1,0}^{*}.\end{aligned} \tag{5.32}$$

In view of (5.32) and (5.29), we conclude

$$\mathcal{A}_{r+1,1} = \mathfrak{M}_{r+1,0}^{-1}\left(\mathcal{A}_{r+1,0} - \eta_{r+1,0}\eta_{r+1,0}^{*}\right)\mathfrak{M}_{r+1,0}^{-*}. \tag{5.33}$$

The combination of (5.16) and (5.33) yields

$$\mathcal{A}_{r+1,1} \geq 0_{(r+1)\times(r+1)}. \tag{5.34}$$

From (5.29) and (5.33) we see that

$$\operatorname{rank}\mathcal{A}_{r+1,1} = \operatorname{rank}\left(\mathcal{A}_{r+1,0} - \eta_{r+1,0}\eta_{r+1,0}^{*}\right). \tag{5.35}$$

By assumption, we have $\det \mathcal{A}_{r+1,0} = 0$. Combining this with (5.16) yields

$$\det\left(\mathcal{A}_{r+1,0} - \eta_{r+1,0}\eta_{r+1,0}^{*}\right) = 0. \tag{5.36}$$

Applying (5.35) and (5.36) we get

$$\det \mathcal{A}_{r+1,1} = 0. \tag{5.37}$$

Obviously (5.30) implies

$$\mathfrak{L}_{r+1,0} = \mathfrak{M}_{r+1,0}\mathfrak{L}_{r+1,1}. \tag{5.38}$$

This means that the matrix $\mathfrak{L}_{r+1,1}$ is built from $(\gamma_{j+1})_{j=0}^{r}$ and $[\Pi_{1,2}, (L_{j,2})_{j=1}^{r}]$ in the same way as the matrix $\mathfrak{L}_{r+1,0}$ is built from $(\gamma_j)_{j=0}^{r}$ and $[\Pi_{1,1}, (L_{j,1})_{j=1}^{r}]$. Taking into account that $(\gamma_{j+1})_{j=0}^{r}$ is a sequence from $\mathbb{D}$, we conclude from (5.34) and (5.37) that $[\Pi_{1,2}, (L_{j,2})_{j=1}^{r}]$ is compatible with $(\gamma_{j+1})_{j=0}^{r}$. Thus, it is possible to repeat the above procedure and to define the number $\gamma_{r+2} \in \mathbb{D}$. In this second step one has to increase the corresponding index associated with the γ's by one unit. For instance, instead of (5.32) we get

$$\mathcal{A}_{r+1,1} = \eta_{r+1,1}\eta_{r+1,1}^{*} + \mathfrak{M}_{r+1,1}\mathcal{A}_{r+1,2}\mathfrak{M}_{r+1,1}^{*}.$$

Inserting this into (5.32) provides us

$$\begin{aligned}\mathcal{A}_{r+1,0} &= \eta_{r+1,0}\eta_{r+1,0}^{*} + \mathfrak{M}_{r+1,0}\eta_{r+1,1}\eta_{r+1,1}^{*}\mathfrak{M}_{r+1,0}^{*}\\ &\quad + \mathfrak{M}_{r+1,0}\mathfrak{M}_{r+1,1}\mathcal{A}_{r+1,2}\mathfrak{M}_{r+1,1}^{*}\mathfrak{M}_{r+1,0}^{*}.\end{aligned} \tag{5.39}$$

After the second step formula (5.38) has the shape

$$\mathfrak{L}_{r+1,0} = \mathfrak{M}_{r+1,0}\mathfrak{M}_{r+1,1}\mathfrak{L}_{r+1,2}. \tag{5.40}$$

Analogously to (5.34) and (5.37) we obtain

$$\mathcal{A}_{r+1,2} \geq 0_{(r+1)\times(r+1)} \tag{5.41}$$

and

$$\det \mathcal{A}_{r+1,2} = 0. \tag{5.42}$$

By induction we now continue the above procedure to obtain an infinite sequence $\gamma = (\gamma_j)_{j=0}^{\infty}$ from $\mathbb{D}$. Let us consider an arbitrary $n \in \mathbb{N}$. After the nth step formula (5.40) has the shape

$$\mathfrak{L}_{r+1,0} = \mathfrak{M}_{r+1,0}\mathfrak{M}_{r+1,1} \cdot \quad \cdots \quad \cdot \mathfrak{M}_{r+1,n-1}\mathfrak{L}_{r+1,n}. \tag{5.43}$$

Instead of (5.41) and (5.42) the matrix

$$\mathcal{A}_{r+1,n} := I_{r+1} - \mathfrak{L}_{r+1,n}\mathfrak{L}^*_{r+1,n} \tag{5.44}$$

satisfies the relations

$$\mathcal{A}_{r+1,n} \geq 0_{(r+1)\times(r+1)} \tag{5.45}$$

and

$$\det \mathcal{A}_{r+1,n} = 0. \tag{5.46}$$

Now we compare the elements in the left upper corner of the matrices on both sides of equation (5.43). Taking into account (3.3), (4.4), and (4.2) we obtain

$$\Pi_{1,1} = D_{\gamma_1} D_{\gamma_2} \cdot \quad \cdots \quad \cdot D_{\gamma_n} \Pi_{1,n+1}. \tag{5.47}$$

In view of (5.45) the matrix $\mathfrak{L}_{r+1,n}$ is contractive. Since $\Pi_{1,n+1}$ is the element in the left upper corner of $\mathfrak{L}_{r+1,n}$ we obtain

$$\Pi_{1,n+1} \leq 1. \tag{5.48}$$

By assumption we have $\Pi_{1,1} > 0$ and $D_{\gamma_1} D_{\gamma_2} \cdot \ \cdots \ \cdot D_{\gamma_n} > 0$. Thus, (5.47) implies

$$\Pi_{1,n+1} > 0. \tag{5.49}$$

Combining (5.47)–(5.49) we obtain

$$D_{\gamma_1} D_{\gamma_2} \cdot \quad \cdots \quad \cdot D_{\gamma_n} \geq \Pi_{1,1} > 0. \tag{5.50}$$

After the nth step the analogue of formula (5.39) is

$$\begin{aligned} \mathcal{A}_{r+1,0} \quad = \quad & \sum_{k=0}^{n-1} \left(\overrightarrow{\prod_{j=0}^{k-1}} \mathfrak{M}_{r+1,j} \right) \eta_{r+1,k}\eta^*_{r+1,k} \left(\overleftarrow{\prod_{j=0}^{k-1}} \mathfrak{M}^*_{r+1,j} \right) \\ & + \left(\overrightarrow{\prod_{j=0}^{n-1}} \mathfrak{M}_{r+1,j} \right) \mathcal{A}_{r+1,n} \left(\overleftarrow{\prod_{j=0}^{k-1}} \mathfrak{M}^*_{r+1,j} \right) \end{aligned} \tag{5.51}$$

where as above we have used the convention

$$\overrightarrow{\prod_{j=0}^{-1}} \cdots = \overleftarrow{\prod_{j=0}^{-1}} \cdots = I_{r+1}.$$

Let $m \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$. Comparing (1.13) and (4.4) we get

$$\mathfrak{M}_{m,k} = \mathfrak{M}_m(W^k\gamma) \tag{5.52}$$

whereas from (1.15) and (4.6) we see

$$\eta_{m,k} = \eta_m(W^k\gamma). \tag{5.53}$$

Taking into account (5.52) and (5.53), an application of (5.43) and (5.51) yields

$$\mathfrak{L}_{r+1,0} = \left(\overrightarrow{\prod_{j=0}^{n-1}} \mathfrak{M}_{r+1}(W^j\gamma)\right) \mathfrak{L}_{r+1,n} \tag{5.54}$$

and

$$\begin{aligned} \mathcal{A}_{r+1,0} \quad = \quad & \sum_{k=0}^{n-1} \left(\overrightarrow{\prod_{j=0}^{k-1}} \mathfrak{M}_{r+1}(W^j\gamma)\right) \eta_{r+1}(W^k\gamma)\eta^*_{r+1}(W^k\gamma) \left(\overleftarrow{\prod_{j=0}^{k-1}} \mathfrak{M}^*_{r+1}(W^j\gamma)\right) \\ & + \left(\overrightarrow{\prod_{j=0}^{n-1}} \mathfrak{M}_{r+1}(W^j\gamma)\right) \mathcal{A}_{r+1,n} \left(\overleftarrow{\prod_{j=0}^{k-1}} \mathfrak{M}^*_{r+1}(W^j\gamma)\right). \end{aligned} \tag{5.55}$$

Now we use the previous observations connected to the nth step of our procedure to derive several properties of the sequence γ. Since γ is a sequence from $\mathbb{D}$ we conclude from (5.50) that the infinite product $\prod_{j=0}^{\infty} D_{\gamma_j}$ converges. This implies

$$\gamma \in \Gamma l_2. \tag{5.56}$$

In view of (5.56), applying (1.16) yields

$$\overrightarrow{\prod_{k=0}^{\infty}} \mathfrak{M}_{r+1}(W^k\gamma) = \mathfrak{L}_{r+1}(\gamma), \tag{5.57}$$

where $\mathfrak{L}_{r+1}(\gamma)$ is defined via (1.7). Because of (5.57) and $\det \mathfrak{L}_{r+1}(\gamma) \neq 0$ we infer from (5.54) that the sequence $(\mathfrak{L}_{r+1,n})_{n=1}^{\infty}$ converges and that its limit

$$\mathfrak{L}_{r+1,\infty} := \lim_{n\to\infty} \mathfrak{L}_{r+1,n} \tag{5.58}$$

satisfies the equation

$$\mathfrak{L}_{r+1,0} = \mathfrak{L}_{r+1}(\gamma)\mathfrak{L}_{r+1,\infty}. \tag{5.59}$$

Let

$$\mathcal{A}_{r+1,\infty} := I_{r+1} - \mathfrak{L}_{r+1,\infty}\mathfrak{L}^*_{r+1,\infty}. \tag{5.60}$$

Using (5.44), (5.58), and (5.60), we get

$$\mathcal{A}_{r+1,\infty} := \lim_{n\to\infty} \mathcal{A}_{r+1,n}. \tag{5.61}$$

Applying (5.45) and (5.60) we obtain

$$\mathcal{A}_{r+1,\infty} \geq 0_{(r+1)\times(r+1)}. \tag{5.62}$$

Now we pass to the limit $n \to \infty$ in formula (5.55). Using (5.56), Lemma 2.3, (5.57), and (5.61), we obtain

$$\mathcal{A}_{r+1,0} = \mathcal{A}_{r+1}(\gamma) + \mathfrak{L}_{r+1}(\gamma)\mathcal{A}_{r+1,\infty}\mathfrak{L}^*_{r+1}(\gamma). \tag{5.63}$$

From (5.56) and Lemma 1.2 we infer

$$\mathcal{A}_{r+1}(\gamma) \geq 0_{(r+1)\times(r+1)}. \tag{5.64}$$

According to Corollary 4.2 let

$$p = (p_r, \dots, p_0)^T \in \ker \mathcal{A}_{r+1,0} \tag{5.65}$$

be such that

$$p_0 \neq 0. \tag{5.66}$$

Using (5.62)–(5.65) we see that

$$p \in \ker \mathcal{A}_{r+1}(\gamma). \tag{5.67}$$

Taking into account (5.56), (5.66), (5.67), and applying Proposition 2.4 yields that p is an rth order $\mathcal{S}$-recurrence vector associated with γ. Having this in mind and paying attention to (5.52)–(5.53), from Theorem 2.8 we know that (5.1) holds for each integer n with $n \geq r$. Taking into account (5.62) and (5.63), we infer $\mathcal{A}_{r+1,0} \geq \mathcal{A}_{r+1}(\gamma)$. Thus, the proof is complete. □

Corollary 5.3. *Let $r \in \mathbb{N}$ and let $(\gamma_j)_{j=0}^r$ be a sequence from $\mathbb{D}$. Further, let $[\Pi_{1,1}, (L_{j,1})_{j=1}^r]$ be the canonical data compatible with $(\gamma_j)_{j=0}^r$ and let $\mathcal{A}_{r+1,0}$ be the associated information matrix. Then:*

(a) *The $(r+1)\times 1$ matrix $e_{r+1} := (0,\dots,0,1)^T$ belongs to $\ker \mathcal{A}_{r+1,0} \backslash \{0_{(r+1)\times 1}\}$.*
(b) *Denote by $(\gamma_j)_{j=r+1}^\infty$ that sequence which is constructed by using $p = e_{r+1}$ as in Theorem 5.2. Then $\gamma_{n+1} = 0$ for each integer n with $n \geq r$.*

Proof. (a) From Definition 3.9 and Remark 3.8 we have $\Pi_{1,1} = \prod_{j=1}^r D_{\gamma_j}$. Thus (a) follows from Remark 3.4.

(b) Taking into account (a) we choose $p = e_{r+1}$ in Theorem 5.2. Then, if λ is chosen as in Theorem 5.2, we get $\lambda = 0_{r\times 1}$. Hence, from (5.1) we conclude (b). □

Now we consider the situation of Theorem 5.2 for the particular case $r = 1$. We will then obtain a considerable simplification of the recurrence formulas (5.1). First we state an auxiliary result.

Lemma 5.4. *Let $\gamma_1 \in \mathbb{D}\backslash\{0\}$ and let $\lambda \in \mathbb{C}$ be such that $0 < \lambda \leq 1$. Then there exists some $u \in (0,1)$ which satisfies*

$$u^2 - (1 - |\gamma_1|^2 + |\lambda|^2)u + |\lambda|^2 = 0 \tag{5.68}$$

if and only if $|\lambda| \leq 1 - |\gamma_1|$.

Proof. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := x^2 - (1 - |\gamma_1|^2 + |\lambda|^2)x + |\lambda|^2. \tag{5.69}$$

Let

$$x_0 := \frac{1}{2}(1 - |\gamma_1|^2 + |\lambda|^2). \tag{5.70}$$

From (5.70) and the choice of γ_1 and λ we get $x_0 \in (0, 1)$. Moreover,

$$f(0) = |\lambda|^2 > 0 \text{ and } f(1) = |\gamma_1|^2 > 0. \tag{5.71}$$

From (5.69) and (5.70) we obtain $f(x) \geq f(x_0)$ for each $x \in \mathbb{R}$. Thus, taking into account (5.71), we see that there exists some $u \in (0, 1)$ satisfying (5.68) if and only if

$$f(x_0) \leq 0. \tag{5.72}$$

In view of

$$f(x_0) = -\frac{1}{4}(1 - |\gamma_1|^2 + |\lambda|^2)^2 + |\lambda|^2$$

we infer that (5.72) is equivalent to

$$4|\lambda|^2 \leq (1 - |\gamma_1|^2 + |\lambda|^2)^2.$$

Because of $\gamma_1 < 1$ this is equivalent to $2|\lambda| \leq 1 - |\gamma_1|^2 + |\lambda|^2$, i.e, to $|\gamma_1|^2 \leq 1 - 2|\lambda| + |\lambda|^2 = (1 - |\lambda|)^2$. Thus, because of $|\lambda| \leq 1$ we obtain that this is equivalent to $|\gamma_1| \leq 1 - |\lambda|$ and the proof is complete. □

Theorem 5.5. *Let $(\gamma_j)_{j=0}^1$ be a sequence from $\mathbb{D}$ and let $[\Pi_{1,1}, (L_{j,1})_{j=1}^1]$ be compatible with $(\gamma_j)_{j=0}^1$. Denote by $\gamma = (\gamma_j)_{j=0}^\infty$ that extension of $(\gamma_j)_{j=0}^1$ which is constructed as in Theorem* 5.2. *Taking into account* $0 < \Pi_{1,1} \leq 1$, *we set*

$$\lambda := \begin{cases} -\frac{\Pi_{1,1}^2 \overline{L_{1,1}}}{1-\Pi_{1,1}^2} & \text{, if } \Pi_{1,1} \in (0,1) \\ 0 & \text{, if } \Pi_{1,1} = 1. \end{cases}$$

Then:

(a) *For each $n \in \mathbb{N}$, the number γ_{n+1} can be represented via*

$$\gamma_{n+1} = \lambda \frac{\gamma_n}{\prod_{j=1}^n (1 - |\gamma_j|^2)}.$$

(b) *The relations $\gamma \in \Gamma l_2$ and* $\operatorname{rank} \gamma \leq 1$ *hold true. The vector $(-D_{\gamma_1}^{-1}\lambda, 1)^T$ is an $\mathcal{S}$-recurrence vector associated with γ.*

(c) *The following statements are equivalent:*
   (i) $\lambda = 0$.
   (ii) *For all $n \in \mathbb{N}$, $\gamma_{n+1} = 0$.*
   (iii) *There exists an $m \in \mathbb{N}$ such that $\gamma_{m+1} = 0$.*

(d) *The inequality $|\lambda| \leq 1 - |\gamma_1|$ holds.*

Proof. (a) Let $n \in \mathbb{N}$ and let $\mathcal{A}_{2,0}$ be the information matrix associated with $[(\gamma_j)_{j=0}^1, \Pi_{1,1}, (L_{j,1})_{j=1}^1]$. Then using part (b) of Lemma 3.6 in the case $\Pi_{1,1} \in (0,1)$ and part (c) of Lemma 3.3 in the case $\Pi_{1,1} = 1$, we get

$$\begin{pmatrix} -\lambda D_{\gamma_1}^{-1} \\ 1 \end{pmatrix} \in \ker \mathcal{A}_{2,0}. \tag{5.73}$$

Taking into account (5.73), (4.4), and (4.6), we conclude from (5.1) the relation

$$\begin{aligned} \gamma_{n+1} &= \left[\prod_{s=1}^{n} D_{\gamma_s}^{-1}\right] \left(\left[\overleftarrow{\prod_{k=0}^{n-2}} \mathfrak{M}_{1,k}^{-1}\right] \lambda, \left[\prod_{k=n}^{n} D_{\gamma_k}^{-1}\right] \eta_{1,n-1} \right)_{\mathbb{C}^1} \\ &= \left[\prod_{s=1}^{n} D_{\gamma_s}^{-1}\right] \left(\left[\overleftarrow{\prod_{k=0}^{n-2}} D_{\gamma_{1+k}}^{-1}\right] \lambda, D_{\gamma_n}^{-1}\overline{\gamma_n} \right)_{\mathbb{C}^1} \\ &= \lambda \frac{\gamma_n}{\prod_{j=1}^{n}(1-|\gamma_j|^2)}. \end{aligned}$$

(b) In view of (5.73), part (b) is an immediate consequence of Theorem 5.2.

(c) The implications "(i)⇒(ii)" and "(ii)⇒(iii)" are obvious. Now suppose (iii). Taking into account (iii), let k be the smallest positive integer m such that $\gamma_m = 0$. If $k = 1$, then $\gamma_1 = 0$ and part (b) of Lemma 3.3 implies $\Pi_{1,1} = 1$ and hence, $\lambda = 0$. Let us consider the case $k \geq 2$. Then $\gamma_{k-1} \neq 0$ and $\gamma_k = 0$. Thus, using (b) we get

$$\lambda = \left[\prod_{j=1}^{k-1}(1-|\gamma_j|^2)\right] \frac{\gamma_k}{\gamma_{k-1}} = 0.$$

Consequently, in any case condition (i) is satisfied.

(d) For $\lambda = 0$ the assertion is trivial. Now suppose

$$\lambda \neq 0. \tag{5.74}$$

Then from (c) and the definition of λ we infer

$$\gamma_n \neq 0, \quad \text{for each } n \in \mathbb{N}. \tag{5.75}$$

According to (5.75) and (a), for each $n \in \mathbb{N}$, we conclude

$$\frac{|\gamma_{n+1}|}{|\gamma_n|} = \frac{|\lambda|}{\prod_{j=1}^{n}(1-|\gamma_j|^2)}.$$

From this we get

$$\lim_{n\to\infty} \frac{|\gamma_{n+1}|}{|\gamma_n|} = \frac{|\lambda|}{\Pi_1^2}, \tag{5.76}$$

where Π_1 is defined in (1.4). Because of $\gamma \in \Gamma l_2$ the series $\sum_{j=0}^{\infty} |\gamma_j|^2$ converges. Combining this with (5.76) the quotient criterion for the convergence of infinite series yields

$$\frac{|\lambda|^2}{\Pi_1^4} = \lim_{n\to\infty} \frac{|\gamma_{n+1}|^2}{|\gamma_n|^2} \le 1$$

and, consequently,

$$|\lambda| \le \Pi_1^2 \le 1. \tag{5.77}$$

In view of (5.75), parts (a) and (b) of Lemma 3.3 provide us with

$$\Pi_{1,1} \in (0,1). \tag{5.78}$$

Taking into account (5.73) and (b), we apply part (c) of Lemma 3.6 and obtain

$$\Pi_{1,1}^4 - (1 - |\gamma_1|^2 + |\lambda|^2)\Pi_{1,1}^2 + |\lambda|^2 = 0. \tag{5.79}$$

Because of (5.74) and (5.77)–(5.79), we obtain from Lemma 5.4 the inequality $|\lambda| \le 1 - |\gamma_1|$. □

Lemma 5.6. *Let $(\gamma_j)_{j=0}^1$ be a sequence from $\mathbb{D}$ where $\gamma_1 \neq 0$. Further, let $\lambda \in \mathbb{C}$ satisfy $0 < |\lambda| \le 1 - |\gamma_1|$. Then:*

(a) *There exists some $\Pi_{1,1} \in (0,1)$ such that*

$$\Pi_{1,1}^4 - (1 - |\gamma_1|^2 + |\lambda|^2)\Pi_{1,1}^2 + |\lambda|^2 = 0.$$

(b) *Let $\Pi_{1,1}$ be chosen as in* (a) *and let*

$$L_{1,1} := -\frac{1 - \Pi_{1,1}^2}{\Pi_{1,1}^2}\overline{\lambda}.$$

Then $[\Pi_{1,1}, (L_{j,1})_{j=1}^1]$ is compatible with $(\gamma_j)_{j=0}^1$.

(c) *Denote by $\mathcal{A}_{2,0}$ the information matrix associated with*

$$[(\gamma_j)_{j=0}^1, \Pi_{1,1}, (L_{j,1})_{j=1}^1].$$

Then

$$\begin{pmatrix} -D_{\gamma_1}^{-1}\lambda \\ 1 \end{pmatrix} \in \ker \mathcal{A}_{2,0}.$$

Proof. (a) This follows from Lemma 5.4.

(b) Let the matrix $\mathcal{A}_{2,0}$ be defined by (3.8). In view of (a), we obtain from (3.13) then $\det \mathcal{A}_{2,0} = 0$. Thus, taking into account $1 - \Pi_{1,1} > 0$ and (3.9), we see that $\mathcal{A}_{2,0} \ge 0_{2\times 2}$. Hence, (b) is proved.

(c) Because of (b) and the definition of $L_{1,1}$ the assertion of (c) follows from part (c) of Lemma 3.6. □

Theorem 5.7. *Let $(\gamma_j)_{j=0}^1$ be a sequence from $\mathbb{D}$ where $\gamma_1 \neq 0$. Furthermore, let $\lambda \in \mathbb{C}$ satisfy $0 < |\lambda| \le 1 - |\gamma_1|$. For each $n \in \mathbb{N}$, let*

$$\gamma_{n+1} := \lambda \frac{\gamma_n}{\prod_{j=1}^n (1 - |\gamma_j|^2)}.$$

Then the sequence $\gamma = (\gamma_j)_{j=0}^\infty$ belongs to Γl_2 and satisfies $\operatorname{rank}\gamma \le 1$. Moreover, the vector $(-D_{\gamma_1}^{-1}\lambda, 1)^T$ is an $\mathcal{S}$-recurrence vector associated with γ.

Proof. Let $[\Pi_{1,1}, (L_{j,1})_{j=1}^{1}]$ be the data compatible with $(\gamma_j)_{j=0}^{1}$ which were constructed in Lemma 5.6. Then from the definition of $L_{1,1}$ we get

$$\lambda = -\frac{\Pi_{1,1}^2 \overline{L_{1,1}}}{1 - \Pi_{1,1}^2}.$$

Thus, part (a) of Theorem 5.5 shows that $(\gamma_j)_{j=0}^{\infty}$ is that extension of $(\gamma_j)_{j=0}^{1}$ which was constructed in Theorem 5.2. Now all assertions follow from part (b) of Theorem 5.5. $\square$

It should be mentioned that Theorem 5.5 and Theorem 5.7 contain reformulations of statements which were obtained in Theorem 5.22 and Corollary 5.23 of [4] by other methods. The difference is that Theorem 5.5 is formulated in terms of compatible data. Moreover, Theorem 5.5 is a consequence of Theorem 5.2 which describes the situation for an arbitrary positive integer n.

Finally, we note that in Section 5.4 of [4], for the case $r = 1$, several concrete examples were considered, which we do not discuss here.

The last assertion in Theorem 5.2 leads us to the following notion.

Definition 5.8. Let $r \in \mathbb{N}$, let $(\gamma_j)_{j=0}^{r}$ be a sequence from $\mathbb{D}$, and let $[\Pi_{1,1}, (L_{j,1})_{j=1}^{r}]$ be compatible with $(\gamma_j)_{j=0}^{r}$. Furthermore, let $\mathcal{A}_{r+1,0}$ be the associated information matrix and let $p = (p_r, \dots, p_0)^T$ be a vector from $\ker \mathcal{A}_{r+1,0}$ which satisfies $p_0 \neq 0$. Denote by γ the extension of $(\gamma_j)_{j=0}^{r}$ to a sequence belonging to Γl_2 which was constructed in Theorem 5.2 outgoing from the quadruple $[(\gamma_j)_{j=0}^{r}, \Pi_{1,1}, (L_{j,1})_{j=1}^{r}, p]$. Then the triple $[\Pi_{1,1}, (L_{j,1})_{j=1}^{r}, p]$ is called *minimal with respect to* $(\gamma_j)_{j=0}^{r}$ if $\mathcal{A}_{r+1,0} = \mathcal{A}_{r+1}(\gamma)$ where $\mathcal{A}_{r+1}(\gamma)$ is given by (1.17).

Let $n_0 \in \mathbb{N}$ and let $\gamma = (\gamma_j)_{j=0}^{\infty}$ be a sequence which belongs to Γl_2 and satisfies $\operatorname{rank} \gamma = n_0$. Furthermore, let r be an integer with $r \geq n_0$. Then according to Proposition 3.7 there is a natural choice of data $[\Pi_{1,1}, (L_{j,1})_{j=1}^{r}]$ compatible with $(\gamma_j)_{j=0}^{r}$. Now we verify that the procedure of Theorem 5.2 applied to these data provides exactly the original sequence γ and, moreover, it produces a triple which is minimal with respect to $(\gamma_j)_{j=0}^{r}$.

Proposition 5.9. *Let $n_0 \in \mathbb{N}$ and let $\gamma = (\gamma_j)_{j=0}^{\infty}$ be a sequence which belongs to Γl_2 and satisfies $\operatorname{rank} \gamma = n_0$. Furthermore, let r be an integer with $r \geq n_0$ and let $p = (p_r, \dots, p_0)^T$ be an rth order $\mathcal{S}$-recurrence vector associated with γ. Let Π_1 and the sequence $(L_j(W\gamma))_{j=1}^{r}$ be defined by* (1.4) *and* (1.3), *respectively.*

(a) *It is $(\gamma_j)_{j=0}^{r}$ a sequence from $\mathbb{D}$ and $[\Pi_1, (L_j(W\gamma))_{j=1}^{r}]$ is compatible with $(\gamma_j)_{j=0}^{r}$. Moreover, the matrix $\mathcal{A}_{r+1}(\gamma)$ defined in* (1.17) *is the information matrix associated with $[(\gamma_j)_{j=0}^{r}, \Pi_1, (L_j(W\gamma))_{j=1}^{r}]$.*
(b) *The relations $p \in \ker \mathcal{A}_{r+1}(\gamma)$ and $p_0 \neq 0$ hold true.*
(c) *Taking into account* (a) *and* (b) *and given $(\gamma_j)_{j=0}^{r}$,$[\Pi_1, (L_j(W\gamma))_{j=1}^{r}]$, and p, let $\widetilde{\gamma} = (\widetilde{\gamma}_j)_{j=0}^{\infty}$ be the sequence from Γl_2 which is constructed via Theorem* 5.2. *Then $\widetilde{\gamma} = \gamma$.*
(d) *The triple $[\Pi_1, (L_j(W\gamma))_{j=1}^{r}, p]$ is minimal with respect to $(\gamma_j)_{j=0}^{r}$.*

Proof. (a) This follows from Proposition 3.7.

(b) This follows from Proposition 2.4.

(c) Taking into account (5.52) and (5.53), the assertion of (c) follows inductively by combining Theorem 2.8 and Theorem 5.2. Indeed, one has only to compare formulas (2.20) and (5.1).

(d) This follows from (a). □

Corollary 5.10. *Let $r \in \mathbb{N}$ and let $(\gamma_j)_{j=0}^{r}$ be a sequence from $\mathbb{D}$. Furthermore, let $[\Pi_{1,1}, (L_{j,1})_{j=1}^{r}]$ be the canonical data compatible with $(\gamma_j)_{j=0}^{r}$ and let $e_{r+1} := (0, \ldots, 0, 1)^T \in \mathbb{C}^{r+1}$. Then the triple $[\Pi_{1,1}, (L_{j,1})_{j=1}^{r}, e_{r+1}]$ is minimal with respect to $(\gamma_j)_{j=0}^{r}$.*

Proof. Let $\gamma = (\gamma_j)_{j=0}^{\infty}$ be that extension of $(\gamma_j)_{j=0}^{r}$ which is given for each $j \in \{r+1, r+2, \ldots\}$ by $\gamma_j = 0$. Then from Remark 1.9 we infer that $\gamma \in \Gamma l_2$ and $\operatorname{rank} \gamma \leq r$. Moreover, taking into account Definition 3.9, we get $\Pi_{1,1} = \Pi_1$ and $L_{j,1} = L_j(W\gamma)$ for each $j \in \{1, 2, \ldots, r\}$, The combination of Remark 1.9 and Proposition 2.6 shows that e_{r+1} is an rth order $\mathcal{S}$-recurrence vector associated with γ. Thus, part (d) of Proposition 5.9 yields the assertion. □

Acknowledgement

The main part of this paper was written during the first author's research stay at the Mathematical Institute of Leipzig University in autumn 2008. The first author would like to thank the DAAD for financial support of his visit in Leipzig.

References

[1] D. Alpay: *The Schur Algorithm, Reproducing Kernel Spaces and System Theory*, SMF/AMS Texts and Monographs, Vol. 5, Amer. Math. Soc., Providence, R.I. 2001.

[2] M.J. Bertin, A. Decomps-Guilloux, M. Grandet-Hugot, M. Pathiaux-Delefosse, J.P. Schreiber: *Pisot and Salem numbers*, Birkhäuser, Basel-Boston-Berlin 1992.

[3] V.K. Dubovoy: *Schur's parameters and pseudocontinuation of contractive holomorphic functions in the unit disk*, Dopovidi NAN Ukr., Vol. 2 (1998), 24–29.

[4] V.K. Dubovoy: *Shift operators contained in contractions, Schur parameters and pseudocontinuable Schur functions*, in: Interpolation, Schur Functions and Moment Problems (Eds.: D. Alpay, I. Gohberg), OT Series, Vol. 165, Birkhäuser, Basel-Boston-Berlin 2006, pp. 175–250.

[5] V.K. Dubovoj, B. Fritzsche, B. Kirstein: *Matricial Version of the Classical Schur Problem*, Teubner-Texte zur Mathematik, Band 129, B.G. Teubner, Stuttgart – Leipzig 1992.

[6] C. Foias, A. Frazho: *The Commutant Lifting Approach to Interpolation Problems*, OT Series, Vol. 44, Birkhäuser, Basel-Boston-Berlin 1990.

[7] B. Fritzsche, B. Kirstein: *A Schur type matrix extension problem II*, Math. Nachr. 138 (1988), 195–216.

[8] S.N. Khrushchev: *Orthogonal Polynomials and Continued Fractions – From Euler's Point of View*, Encyclopedia of Mathematics, Vol. 121, Cambridge University Press, Cambridge 2008.

[9] I. Schur: *Über Potenzreihen, die im Inneren des Einheitskreises beschränkt sind*, J. reine u. angew. Math., Part I: 147 (1917), 205–232; Part II: 148 (1918), 122–145.

[10] B. Simon: *Orthogonal Polynomials on the Unit Circle. Part 1: Classical Theory*, Amer. Math. Soc., Colloquium Publications, Vol. 54, Part 1, Providence, R.I. 2004.

[11] B. Simon: *Orthogonal Polynomials on the Unit Circle. Part 2: Spectral Theory*, Amer. Math. Soc., Colloquium Publications, Vol. 54, Part 2, Providence, R.I. 2004.

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Received: February 23, 2009
Accepted: March 26, 2009

Operator Theory:
Advances and Applications, Vol. 202, 195–222

Curvature of Universal Bundles of Banach Algebras

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For Professor I. Gohberg

Abstract. Given a Banach algebra we construct a principal bundle with connection over the similarity class of projections in the algebra and compute the curvature of the connection. The associated vector bundle and the connection are a universal bundle with attendant connection. When the algebra is the linear operators over a Hilbert module, we establish an analytic diffeomorphism between the similarity class and the space of polarizations of the Hilbert module. Likewise, the geometry of the universal bundle over the latter is studied. Instrumental is an explicit description of the transition maps in each case which leads to the construction of certain functions. These functions are in a sense pre-determinants for the universal bundles in question.

Mathematics Subject Classification (2000). Primary 46M20 37K20; Secondary 58B99 58B25.

Keywords. Hilbert module, Banach Grassmannian, similarity class, polarization, universal bundle, connection, curvature.

1. Introduction

The book of Helton et al. [22] outlined a program of operator-analytic techniques using flag manifold models, the theorems of Beurling-Lax-Halmos, Wiener-Hopf factorization and $\mathcal{M} \times \mathcal{M}$-theory, which could be applied to the study of integrable systems (such as the Sato-Segal-Wilson theory [33, 32, 34]) and Lax-Phillips scattering (cf. work of Ball and Vinnikov [2, 3]). Several of the fundamental techniques implemented in this scheme of ideas can be traced back to the remarkable accomplishments of Professor I. Gohberg and his co-workers spanning a period of many years.

Communicated by J.A. Ball.

Our interest in this general subject arose from two directions. Initially, the first two authors (with Evard) studied the problem of smooth as well as analytic parametrization of subspaces of a Banach space using global techniques. The work on this problem had been significantly motivated by that of Gohberg and Leiterer [18, 19]. The general results that were obtained appear in [17, 11, 12]. From another direction [14, 15, 16] we have developed an operator-theoretic, Banach algebra approach to the Sato-Segal-Wilson theory, in the setting of Hilbert modules with the extension of the classical Baker and Tau(τ)-functions to types of operator-valued functions. One aspect of this latter work involved looking at the geometry of the space of polarizations of a Hilbert module using a Grassmannian over the Banach algebra A in question, a topic which is developed in this paper. We consider the techniques and results as presented here to be also of independent interest in related areas of operator theory.

If $P(A)$ denotes the space of projections in A, then we consider the geometry of the space $\Lambda = \mathrm{Sim}(p, A)$, namely the similarity class of a given projection $p \in P(A)$. We construct a principal bundle with connection over Λ and compute the curvature of the connection. The transition map for this bundle leads to the construction of a function which we refer to as the $\mathcal{T}$*-function.* If $\mathfrak{P}$ denotes the space of polarizations of a Hilbert module $H_{\boldsymbol{\mathcal{A}}}$ (where $\boldsymbol{\mathcal{A}}$ is a unital C*-algebra), we show that Λ and $\mathfrak{P}$ are analytically diffeomorphic (Theorem 4.1). Related (in the case $\boldsymbol{\mathcal{A}} = \mathbb{C}$) is the $\mathfrak{T}$-function of [28, 39] obtained over $\mathfrak{P}$ via a cross-ratio approach.

To be more specific, let us point out that the $\mathcal{T}$-function is effectively the co-cycle for the universal bundle over the space of restricted polarizations, relating essentially the same two underlying sections, but initially this is viewed in terms of the corresponding principal bundle. Hence the interest is in the calculation of the geometry, connection, and curvature of the principal bundle of the universal bundle using two sections which are each covariantly constant over two complementary subbundles of the tangent bundle of the space of restricted polarizations. Our approach is justified by the fact that, technically, one only needs a single section to trivialize a principal bundle over the domain of the section and hence knowledge of the covariant derivative of that section allows the computation of the horizontal subspace over points of the image of the section, which can then be transferred to any fiber passing through the image of that section using the action of the structure group of the principal bundle. However, if one can find sections known to have zero covariant derivative along certain subbundles of the base tangent bundle, then the computation is certainly simplified, and in the case at hand we have two which suffice.

One main task we describe in this paper is to use the restricted algebra directly. Since the analysis only depends on the fact that the restricted algebra is a Banach algebra, our treatment presents, for any Banach algebra, a representation of the manifolds in question, as those naturally embedded in Banach spaces which provide a natural geometry recovering the exact same geometry that arises in [28, 39] thus leading to the well-known Tau(τ)-function [33, 34]. In particular, we are able to obtain simple expressions for the $\mathcal{T}$-function, the connection form, and the curvature (see, e.g., Theorem 8.1). As observed in [39] one can calculate in coordinates, but here we have natural embeddings which give the geometry. Using coordinates we can calculate, but we cannot visualize, whereas using the natural embeddings we can both visualize and simplify the final formulas. This means the determination of the Tau-function is reduced purely to analytic questions concerning the existence of determinants of the operator values in the particular subgroup of the algebra which forms the group of the principal bundle. This, along with other related issues, is taken up in [16].

2. Algebraic preliminaries

2.1. The Grassmannian over a semigroup

To commence, let A be a (multiplicative) semigroup with group of units denoted by $G(A)$, if A has an identity. Let

$$P(A) := \{p \in A \ : \ p^2 = p\}, \tag{2.1}$$

that is, $P(A)$ is the set of idempotent elements in A (for suitable A, we can regard elements of $P(A)$ as projections). Recall that the right Green's relation is $p\mathcal{R}q$, if and only if $pA = qA$ for $p, q \in A$.

Let $\mathrm{Gr}(A) = P(A)/\mathcal{R}$ be the set of equivalence classes in $P(A)$ under $\mathcal{R}$. As the set of such equivalence classes, $\mathrm{Gr}(A)$ will be called *the Grassmannian of* A. Note that as the equivalence classes partition A, elements of $\mathrm{Gr}(A)$ are in fact subsets of $P(A)$. Relative to a given topology on A, $\mathrm{Gr}(A)$ is a space with the quotient topology resulting from the natural quotient map

$$\Pi \ : \ P(A) \longrightarrow \mathrm{Gr}(A). \tag{2.2}$$

In fact if A is a Banach algebra, it follows that $P(A)$ is an analytic submanifold of A, and that $\mathrm{Gr}(A)$ has a unique analytic manifold structure (holomorphic, if A is a complex algebra) such that Π is an analytic open map having local analytic sections passing through each point of $P(A)$ (see [11, § 4], cf. [30]).

Let $h : A \longrightarrow B$ be a semigroup homomorphism. Then it is straightforward to see that the diagram below is commutative:

$$\begin{array}{ccc} P(A) & \xrightarrow{P(h)} & P(B) \\ {\scriptstyle\Pi}\big\downarrow & & \big\downarrow{\scriptstyle\Pi} \\ \mathrm{Gr}(A) & \xrightarrow{\mathrm{Gr}(h)} & \mathrm{Gr}(B) \end{array} \tag{2.3}$$

Clearly, if A is a semigroup of linear transformations of a vector space E, then we have $\Pi(r) = \Pi(s)$, if and only if $r(E) = s(E)$ as subspaces of E. Notice that $r^{-1}(0)$ is a complement for $r(E)$, so if E is a topological vector space and all members of A are continuous, then $r(E)$ is closed with a closed complement, that is, $r(E)$ is a *splitting subspace.*

If we reverse the multiplication of A, we obtain the opposite semigroup A^{op} and consequently, the right Green's relation in A^{op} is the left Green's relation in A. But $P(A) = P(A^{\mathrm{op}})$, and so this construction gives $\Pi^{\mathrm{op}} : P(A) \longrightarrow \mathrm{Gr}^{\mathrm{op}}(A)$, where by definition $\mathrm{Gr}^{\mathrm{op}}(A) = \mathrm{Gr}(A^{\mathrm{op}})$.

In the case where A is a semigroup of linear transformations of a vector space E, we see immediately that $\Pi^{\mathrm{op}}(r) = \Pi^{\mathrm{op}}(s)$, if and only if $r^{-1}(0) = s^{-1}(0)$ as subspaces of E. Because of this we sometimes denote $\Pi(r) = \mathrm{Im}(r)$, and $\Pi^{\mathrm{op}}(r) = \mathrm{Ker}(r)$, for $r \in P(A)$ with A now taken to be an arbitrary semigroup. Clearly, if $h : A \longrightarrow B$ is a semigroup homomorphism, then so too is $h : A^{\mathrm{op}} \longrightarrow B^{\mathrm{op}}$. Thus Gr^{op} and Π^{op} produce an analogous commutative diagram to (2.3). We observe that $\Pi(r) = \Pi(s)$ if and only if both $rs = s$ and $sr = r$, so in the dual sense, $\Pi^{\mathrm{op}}(r) = \Pi^{\mathrm{op}}(s)$, if and only if both $rs = r$ and $sr = s$. Consequently, if both $\mathrm{Im}(r) = \mathrm{Im}(s)$ and $\mathrm{Ker}(r) = \mathrm{Ker}(s)$, then $r = s$, and thus the map

$$(\mathrm{Im}, \mathrm{Ker}) : P(A) \longrightarrow \mathrm{Gr}(A) \times \mathrm{Gr}^{\mathrm{op}}(A), \tag{2.4}$$

is an injective map which, in the case A is a Banach algebra, we later show to be an analytic embedding of manifolds whose image is open in the righthand side product.

Remark 2.1. Notice that if A is commutative, then $A^{\mathrm{op}} = A$, so $\mathrm{Im}(r) = \mathrm{Im}(s)$, if and only if $\mathrm{Ker}(r) = \mathrm{Ker}(s)$ and therefore by (2.4), $\Pi = \Pi^{\mathrm{op}}$ is injective and thus bijective.

2.2. The canonical section

As in the case where A is a Banachable algebra, we know that Π is a continuous open map [11]. Then it follows that if A is a commutative Banach algebra, then Π is a homeomorphism. Because of (2.4), we see that if $K \in \mathrm{Gr}^{\mathrm{op}}(A)$, then $\mathrm{Im}|K : K \longrightarrow \mathrm{Im}(K) \subset \mathrm{Gr}(A)$ is a bijection whose inverse, we refer to as the *canonical section* over $\mathrm{Im}(K)$. If $p \in K$, then we denote this canonical section by S_p. We set $U_p = \mathrm{Im}(K) \subset \mathrm{Gr}(A)$ and $W_p = \mathrm{Im}^{-1}(U_p) \subset P(A)$. Thus, we have $S_p : U_p \longrightarrow W_p \subset P(A)$ is a section of $\mathrm{Im} = \Pi$ for $p \in W_p$, and $S_p(\mathrm{Im}(p)) = p$. In this situation we refer to S_p as *the canonical section through* p. In fact, from the results of [11], we know that if A is a Banach algebra, then U_p is open in $\mathrm{Gr}(A)$ and S_p is a local analytic section of $\mathrm{Im} = \Pi$.

2.3. Partial isomorphisms and relative inverses

Definition 2.1. We say that $u \in A$ is a *partial isomorphism* if there exists a $v \in A$ such that $uvu = u$, or equivalently, if $u \in uAu$. If also $vuv = v$, we call v a *relative inverse* (or *pseudoinverse*) for u. In general, such a relative inverse always exists, but it is not unique. Effectively, if $u = uwu$, then $w = wuw$ is a relative inverse for u. We take $W(A)$ to denote the set (or space, if A has a topology) of all partial isomorphisms of A.

Notice that $W(A^{\mathrm{op}}) = W(A)$ and $P(A) \subset W(A)$. If u and v are *mutually* (*relative*) *inverse* partial isomorphisms, then $r = vu$ and $s = uv$ are in $P(A)$. In this latter case, we will find it useful to simply write $u : r \longrightarrow s$ and $v : s \longrightarrow r$. Thus we can say u *maps* r to s, regarding the latter as a specified map of idempotents in $P(A)$. Moreover, v is now uniquely determined by the triple (u, r, s), meaning that if w is also a relative inverse for u and both $wu = r$ and $uw = s$ hold, then it follows that $v = w$. Because of this fact, it is also useful to denote this dependence symbolically as

$$v = u^{-(r,s)}, \tag{2.5}$$

which of course means that $u = v^{-(s,r)}$. If $u, v \in W(A)$ with $u : p \longrightarrow r$ and $v : r \longrightarrow s$, then $vu : p \longrightarrow s$. Thus we have

$$(vu)^{-(p,s)} = u^{-(p,r)}v^{-(r,s)}. \tag{2.6}$$

In particular, the map $u : r \longrightarrow r$ implies that $u \in G(rAr)$ and $u^{-(r,r)}$ is now the inverse of u in this group. Thus $G(rAr) \subset W(A)$, for each $r \in P(A)$. For $u \in G(rAr)$, we write $u^{-r} = u^{-(r,r)}$, for short. It is a trivial, but useful observation that if $r, s \in P(A) \subset W(A)$, and if $\mathrm{Im}(r) = \mathrm{Im}(s)$, then $r : r \longrightarrow s$ and $s : s \longrightarrow r$, are mutually inverse partial isomorphisms. Likewise working in A^{op}, and translating the result to A, we have that if $\mathrm{Ker}(r) = \mathrm{Ker}(s)$, then $r : s \longrightarrow r$ and $s : r \longrightarrow s$, are mutually inverse partial isomorphisms. Therefore, if $u : q \longrightarrow r$, if $p, s \in P(A)$ with $\mathrm{Ker}(p) = \mathrm{Ker}(q)$ and $\mathrm{Im}(r) = \mathrm{Im}(s)$, then on applying (2.6), it follows that $u = ruq : p \longrightarrow s$ has a relative inverse

$$u^{-(p,s)} = pu^{-(q,r)}s : s \longrightarrow p. \tag{2.7}$$

Thus the relative inverse is changed (in general) by changing q and r for fixed u, and (2.7) is a useful device for calculating such a change.

Now it is easy to see [11] that the map Π has an extension $\Pi = \mathrm{Im} : W(A) \longrightarrow \mathrm{Gr}(A)$, which is well defined by setting $\Pi(u) = \Pi(s)$, whenever $u \in W(A)$ maps to s. Again, working in A^{op}, we have $\Pi^{\mathrm{op}} = \mathrm{Ker} : W(A) \longrightarrow \mathrm{Gr}^{\mathrm{op}}(A)$, and because $u : r \longrightarrow s$ in A, is the same as $u : s \longrightarrow r$ in A^{op}, this means that $\mathrm{Ker}(u) = \mathrm{Ker}(r)$ if $u : r \longrightarrow s$. More precisely, observe that if $p, q, r, s \in P(A)$, if $u \in W(A)$ satisfies both $u : p \longrightarrow q$ and $u : r \longrightarrow s$, then it follows that $\mathrm{Ker}(p) = \mathrm{Ker}(r)$ and $\mathrm{Im}(q) = \mathrm{Im}(s)$. In fact, if $v = u^{-(p,q)}$ and $w = u^{-(r,s)}$, then we have

$$rp = (wu)(vu) = w(uv)u = wqu = wu = r, \tag{2.8}$$

so $rp = r$ and symmetrically, $pr = p$,which implies $\mathrm{Ker}(p) = \mathrm{Ker}(r)$. Applying this in A^{op}, yields $\mathrm{Im}(q) = \mathrm{Im}(s)$.

Remark 2.2. Of course the commutative diagram (2.3) for Π extends to the same diagram with $W(\ )$ replacing $P(\ )$ and likewise, in the dual sense, for $\Pi^{\mathrm{op}} = \mathrm{Ker}$, on replacing A by A^{op}.

2.4. Proper partial isomorphisms

If $p \in P(A)$, then we take $W(p, A) \subset W(A)$ to denote the subspace of all partial isomorphisms u in A having a relative inverse v satisfying $vu = p$. Likewise, $W(A, q)$ denotes the subspace of all partial isomorphisms u in A having a relative inverse v satisfying $uv = q$. So it follows that $W(A, q) = W(q, A^{\mathrm{op}})$. Now for

$p, q \in P(A)$, we set

$$\begin{aligned} W(p,A,q) &= W(p,A) \cap W(A,q) \\ &= \{u \in W(A) : u : p \longrightarrow q\} \\ &= \{\ u \in qAp : \exists\ v \in pAq\ ,\ vu = p \text{ and } uv = q\ \}. \end{aligned} \tag{2.9}$$

Recall that two elements $x, y \in A$ are *similar* if x and y are in the same orbit under the inner automorphic action $*$ of $G(A)$ on A. For $p \in P(A)$, we say that the orbit of p under the inner automorphic action is *the similarity class of* p and denote the latter by $\mathrm{Sim}(p, A)$. Hence it follows that $\mathrm{Sim}(p, A) = G(A) * p$.

Definition 2.2. Let $u \in W(A)$. We call u a *proper partial isomorphism* if for some $W(p, A, q)$, we have $u \in W(p, A, q)$, where p and q are similar.

We let $V(A)$ denote the space of all proper partial isomorphisms of A. Observe that $G(A)V(A)$ and $V(A)G(A)$ are both subsets of $V(A)$. In the following we set $G(p) = G(pAp)$.

2.5. The spaces $V(p, A)$ and $\mathrm{Gr}(p, A)$

If $p \in P(A)$, then we denote by $V(p, A)$ the space of all proper partial isomorphisms of A having a relative inverse $v \in W(q, A, p)$, for some $q \in \mathrm{Sim}(p, A)$. With reference to (2.9) this condition is expressed by

$$V(p,A) := \bigcup_{q \in \mathrm{Sim}(p,A)} W(p,A,q). \tag{2.10}$$

Observe that $V(p, A) \subset V(A) \cap W(p, A)$, but equality may not hold in general, since for $u \in V(A)$, it may be the case that $\mathrm{Ker}(p) \subset P(A)$ intersects more than one similarity class and that $u \in V(A)$ by virtue of having $u : r \longrightarrow s$ where r and s are similar. But $u : p \longrightarrow q$ only for $q \notin \mathrm{Sim}\,(p, A)$. However, we shall see that if A is a ring with identity, then each class in $\mathrm{Gr}(A)$ is contained in a similarity class and thus also for $\mathrm{Gr}^{\mathrm{op}}(A)$. Further, as Π and Π^{op} are extended to $W(A)$, this means that as soon as we have $u : p \longrightarrow q$, with p and q belonging to the same similarity class, then $u : r \longrightarrow s$ implies that r and s are in the same similarity class.

Clearly, we have $G(A) \cdot p \subset V(p, A)$ and just as in [11], it can be shown that equality holds if A is a ring. The image of $\mathrm{Sim}(p, A)$ under the map Π defines the space $\mathrm{Gr}(p, A)$ viewed as the Grassmannian naturally associated to $V(p, A)$.

For a given unital semigroup homomorphism $h : A \longrightarrow B$, there is a restriction of (2.3) to a commutative diagram:

$$\begin{array}{ccc} V(p,A) & \xrightarrow{V(p,h)} & V(q,B) \\ \Pi_A \big\downarrow & & \big\downarrow \Pi_B \\ \mathrm{Gr}(p,A) & \xrightarrow{\mathrm{Gr}(p,h)} & \mathrm{Gr}(q,B) \end{array} \tag{2.11}$$

where for $p \in P(A)$, we have set $q = h(p) \in P(B)$. Observe that in the general semigroup setting, $V(p, A)$ properly contains $G(A)p$. In fact, if $p \in P(A)$, then $V(p, A) = G(A)G(pAp)$ (see [13] Lemma 2.3.1).

Henceforth we shall restrict mainly to the case where A and B are Banach(able) algebras or suitable multiplicative subsemigroups of Banachable algebras. In this case, as shown in [11], the vertical maps of the diagram (2.11) are right principal bundles, the group for $V(p, A)$ being $G(pAp)$. Moreover, $G(A)$ acts $G(pAp)$-equivariantly on the left of $V(p, A)$ simply by left multiplication, the equivariance being nothing more than the associative law.

Let $H(p)$ denote the isotropy subgroup for this left-multiplication. We have then (see [11]) the analytically equivalent coset space representation

$$\mathrm{Gr}(p, A) = G(A)/G(\Pi(p)), \tag{2.12}$$

where $G(\Pi(p))$ denotes the isotropy subgroup of $\Pi(p)$. Then there is the inclusion of subgroups $H(p) \subset G(\Pi(p)) \subset G(A)$, resulting in a fibering $V(p, A) \longrightarrow \mathrm{Gr}(p, A)$ given by the exact sequence

$$G(\Pi(p))/H(p) \hookrightarrow V(p, A) = G(A)/H(p) \longrightarrow \mathrm{Gr}(p, A) = G(A)/G(\Pi(p)), \tag{2.13}$$

generalizing the well-known *Stiefel bundle* construction in finite dimensions.

In general, if A is a semigroup, we say that the multiplication is *left trivial* provided that always $xy = x$, whereas we call it *right trivial* if $xy = y$. In either case, we have $P(A) = A$. If the multiplication is right trivial, then obviously $\Pi = \mathrm{Im}$ is constant and $\Pi^{\mathrm{op}} = \mathrm{Ker}$ is bijective. Whereas if the multiplication is left trivial, then Ker is constant and $\mathrm{Im} = \Pi$ is bijective.

Remark 2.3. For the 'restricted algebra' to be considered in § 3.2, we recover the 'restricted Grassmannians' as studied in [29, 32, 34] (cf. [21]). Spaces such as $V(p, A)$ and $\mathrm{Gr}(p, A)$ are infinite-dimensional Banach homogeneous spaces of the type studied in, e.g., [4, 8, 9, 36] in which different methods are employed. Emphasis on the case where A is a C*-algebra, can be found in, e.g., [5, 25, 26, 27, 37], again using different methods. Other approaches involving representations and conditional expectations are treated in [1, 5, 6, 31].

2.6. The role of the canonical section

Suppose that R is any ring with identity. Now for $x \in R$, we define $\hat{x} = 1 - x$. The 'hat' operation is then an involution of R leaving $P(R)$ invariant. Further, it is easy to check that for $r, s \in P(R)$, we have $\mathrm{Im}(\hat{r}) = \mathrm{Im}(\hat{s})$, if and only if $\mathrm{Ker}(r) = \mathrm{Ker}(s)$. This means that there is a natural identification of $\mathrm{Gr}^{\mathrm{op}}(R)$ with $\mathrm{Gr}(R)$ unique such that $\mathrm{Ker}(r) = \mathrm{Im}(\hat{r})$, for all $r \in P(R)$. For instance, if $r \in P(R)$, then $rR\hat{r}$ and $\hat{r}Rr$ are subrings with zero multiplication. On the other hand, $r + \hat{r}Rr$ is a subsemigroup with left trivial multiplication and $r + rR\hat{r}$ is a subsemigroup with right trivial multiplication. Thus $\mathrm{Im}|(r + \hat{r}Rr)$ is injective and $\mathrm{Ker}|(r + \hat{r}Rr)$ is constant, whereas $\mathrm{Im}|(r + rR\hat{r})$ is constant and $\mathrm{Ker}|(r + rR\hat{r})$ is injective. In fact, we can now easily check that (e.g., see [11])

$$\mathrm{Im}^{-1}(\mathrm{Im}(r)) = r + rA\hat{r}, \tag{2.14}$$

and

$$\mathrm{Ker}^{-1}(\mathrm{Ker}(r)) = r + \hat{r}Ar = P(A) \cap V(p, A). \tag{2.15}$$

Thus this section is again none other than the canonical section through r. From (2.15), it now follows immediately that when $\mathrm{Ker}(r) = \mathrm{Ker}(s)$, we have

$$r + \hat{r}Ar = s + \hat{s}As, \tag{2.16}$$

and from the symmetry here, one easily deduces that

$$\hat{r}Ar = \hat{s}As. \tag{2.17}$$

This means that the sub-ring $\hat{s}As$ is constantly the same as $\hat{r}Ar$ along the points of the image of the canonical section through r which is $r+\hat{r}Ar = P(A)\cap V(p, A)$, by (2.15). But this also means that $sA\hat{s}$ is constantly the same as $rA\hat{r}$ at all points of $\hat{r} + rA\hat{r}$. If $s \in r + rA\hat{r}$, then

$$\hat{s} \in \hat{r} - rA\hat{r} = \hat{r} + rA\hat{r}, \tag{2.18}$$

and consequently we obtain again $sA\hat{s} = rA\hat{r}$. Thus $P(A)$ in effect contains a 'flat X-shaped subset' through any $r \in P(A)$, namely

$$\mathsf{X} = (r + \hat{r}Ar) \cup (r + rA\hat{r}). \tag{2.19}$$

This suggests that $P(A)$ is everywhere 'saddle-shaped'.

Now, as in [11], we observe here that if $\mathrm{Im}(r) = \mathrm{Im}(s)$, then r and s are in the same similarity class. For there is $y \in rA\hat{r}$ with $s = r + y$. But the multiplication in $rA\hat{r}$ is zero, so $e^y = 1 + y \in G(A)$ with inverse $e^{-y} = 1 - y$, and

$$s = rs = re^y = e^{-y}re^y. \tag{2.20}$$

As $r : r \longrightarrow s$, this means that $r \in V(r, A, s)$, and so each class in $\mathrm{Gr}(A)$ is contained in a similarity class. In the dual sense then, each class of $\mathrm{Gr}^{\mathrm{op}}(A)$ is also contained in a similarity class, as is easily checked directly by the same technique and (2.15). In particular, we now see that for each $p \in P(A)$, we have $V(p, A) = V(A) \cap W(p, A)$, and if $u : r \longrightarrow s$ belongs to $W(A)$, and also $u \in V(A)$, then r and s belong to the same similarity class.

Recalling the canonical section S_p (through p) let us take $p, r \in P(A)$ with $r \in W_p$, and therefore $\mathrm{Im}(r) = \mathrm{Im}(S_p(\mathrm{Im}(r)))$. We have of course $\mathrm{Ker}(S_p(\mathrm{Im}(r))) = \mathrm{Ker}(p)$, by definition of S_p, and hence r and p are in the same similarity class. Set $r_p = S_p(\mathrm{Im}(r))$. Thus $\mathrm{Im}(r) = \mathrm{Im}(r_p)$ and $\mathrm{Ker}(r_p) = \mathrm{Ker}(p)$. We can find $x \in \hat{p}Ap$ so that $r_p = p + x$, and then we have $pr_p = p = pr_pp$ and $r_ppr_p = r_pp = r_p$. This shows that

$$S_p(\mathrm{Im}(r)) = p^{-(S_p(\mathrm{Im}(r)),p)} \tag{2.21}$$

and

$$(S_p(\mathrm{Im}(r)))^{-(p,S_p(\mathrm{Im}(r))} = p. \tag{2.22}$$

Proposition 2.1.

(1) *We have the equation*

$$(S_p(\mathrm{Im}(r))^{-(r,p)} = pr : r \longrightarrow p. \tag{2.23}$$

(2) *The canonical section is a local section of* $\Pi|V(p, A) : V(p, A) \longrightarrow \mathrm{Gr}(p, A)$.

Proof. Part (1) follows from (2.7) and (2.22). For part (2), observe that since $\mathrm{Ker}(S_p(\mathrm{Im}(r))) = \mathrm{Ker}(p)$, we have $S_p(\mathrm{Im}(r))$ and p in the same similarity class and thus the canonical section is actually simultaneously a local section of $\Pi|V(p, A) : V(p, A) \longrightarrow \mathrm{Gr}(p, A)$. □

If A is any semigroup and $u : r \longrightarrow s$ is in $W(A)$ and $k \in P(A)$, we say that u *projects along* k provided that $ku = kr$. Thus, if A is a semigroup of linear transformations of a vector space E, then this condition guarantees that $u(h) - h$ belongs to $k^{-1}(0)$, for every $h \in r(E)$.

Remark 2.4. Clearly this last statement has no content unless $k^{-1}(0)$ is close to being complementary to $r(E)$ and $s(E)$, but in applications this is not a problem.

If $m \in P(A)$ with $\mathrm{Ker}(m) = \mathrm{Ker}(k)$, then $mk = m$ and $km = k$, so $u \in W(A)$ projects along k if and only if it projects along m. Thus we can say u projects along $K \in \mathrm{Gr}^{\mathrm{op}}(A)$ provided that it projects along k, for some and hence all $k \in K$. We can now easily check that if $u : r \longrightarrow s$ in $W(A)$ projects along K, then so too does $u^{-(r,s)}$. It will be important to observe this when later we consider the $\mathcal{T}$-function.

If $r, s \in P(A)$ and it happens that $rs : s \longrightarrow r$, then it is the case that rs projects along $\mathrm{Ker}(r)$, and hence $(rs)^{-(s,r)}$ does also. Thus even though $\mathrm{Ker}(rs) = \mathrm{Ker}(s)$, we have rs projecting along $\mathrm{Ker}(r)$. In particular, by (2.23), if $r \in W_p$, then $S_p(\mathrm{Im}(r))$ and its inverse pr both project along $\mathrm{Ker}(p)$, and therefore, if also $p \in W_r$, then $S_r(\mathrm{Im}(p))$ and its inverse rp both project along $\mathrm{Ker}(r)$. If we consider the case of a semigroup of linear transformations of a vector space E, then we see that for rs to be in $W(A)$ requires that $r^{-1}(0)$ has zero intersection with $s(E)$. Thus, if $rs \in W(A)$, then we should think of r as *close to* s. For instance, if A is any ring with identity and $r, p \in P(A)$ with $rp + \hat{r}\hat{p} \in G(A)$, then, for $g = rp + \hat{r}\hat{p}$, we have

$$rg = rp = gp. \tag{2.24}$$

Therefore, $rp = gp$, so $rp : p \longrightarrow r$ must project along $\mathrm{Ker}(r)$. Moreover as $r = gpg^{-1}$, we have $rp : p \longrightarrow r$ is a proper partial isomorphism and $rp \in V(p, A)$ such that $(rp)^{-(p,r)} = pg^{-1} = g^{-1}r$. Note that for A a Banach algebra, the group of units is open in A, and therefore the set of idempotents $r \in P(A)$ for which $rp + \hat{r}\hat{p} \in G(A)$, is itself an open subset of $P(A)$.

2.7. The spatial correspondence

If $\boldsymbol{\mathcal{A}}$ is a given topological algebra and E is some $\boldsymbol{\mathcal{A}}$-module, then $A = \mathcal{L}_{\boldsymbol{\mathcal{A}}}(E)$ may be taken as the ring of $\boldsymbol{\mathcal{A}}$-linear transformations of E. An example is when E is a complex Banach space and $A = \mathcal{L}(E)$ is the Banach algebra of bounded linear operators on E. In order to understand the relationship between spaces such as $\mathrm{Gr}(p, A)$ and the usual Grassmannians of subspaces (of a vector space E), we will describe a 'spatial correspondence'.

Given a topological algebra $\mathcal{A}$, suppose E is an $\mathcal{A}$-module admitting a decomposition

$$E = F \oplus F^c, \qquad F \cap F^c = \{0\}, \tag{2.25}$$

where F, F^c are fixed closed subspaces of E. We have already noted $A = \mathcal{L}(E)$ as the ring of linear transformations of E. Here $p \in P(E) = P(\mathcal{L}(E))$ is chosen such that $F = p(E)$, and consequently $\mathrm{Gr}(A)$ consists of all such closed splitting subspaces. The assignment of pairs $(p, \mathcal{L}(E)) \mapsto (F, E)$, is called a *spatial correspondence*, and so leads to a commutative diagram

$$\begin{tikzcd} V(p,\mathcal{L}(E)) \arrow[r, "\varphi"] \arrow[d, "\Pi"'] & V(p,E) \arrow[d, "\Pi"] \\ \mathrm{Gr}(p,\mathcal{L}(E)) \arrow[r, "="] & \mathrm{Gr}(F,E) \end{tikzcd} \tag{2.26}$$

where $V(p,E)$ consists of linear homomorphisms of $F = p(E)$ onto a closed splitting subspace of E similar to F. If $u \in V(p, \mathcal{L}(E))$, then $\varphi(u) = u|F$ and if $T : F \longrightarrow E$ is a linear homeomorphism onto a closed complemented subspace of E similar to F, then $\varphi^{-1}(T) = Tp : E \longrightarrow E$. In particular, the points of $\mathrm{Gr}(p, \mathcal{L}(E))$ are in a bijective correspondence with those of $\mathrm{Gr}(F,E)$.

Suppose E is a complex Banach space admitting a decomposition of the type (2.25). We will be considering a 'restricted' group of units from a class of Banach Lie groups of the type

$$\widehat{G}(E) \subset \{\begin{bmatrix} T_1 & S_1 \\ S_2 & T_2 \end{bmatrix} : T_1 \in \mathrm{Fred}(F),\ T_2 \in \mathrm{Fred}(F^c),\ S_1, S_2 \in \mathcal{K}(E)\}, \tag{2.27}$$

that generates a Banach algebra A acting on E, but with possibly a different norm. Here we mention that both compact and Fredholm operators are well defined in the general category of complex Banach spaces; reference [38] provides the necessary details.

3. The restricted Banach *-algebra A_{res} and the space of polarizations

3.1. Hilbert modules and their polarizations

Let $\mathcal{A}$ be a unital C*-algebra. We may consider the standard (free countable dimensional) Hilbert module $H_{\mathcal{A}}$ over $\mathcal{A}$ as defined by

$$H_{\mathcal{A}} = \{\{\zeta_i\}\ ,\ \zeta_i \in \mathcal{A}\ ,\ i \geq 1\ : \sum_{i=1}^{\infty} \zeta_i \zeta_i^* \in \mathcal{A}\ \} \cong \oplus \mathcal{A}_i, \tag{3.1}$$

where each $\mathcal{A}_i$ represents a copy of $\mathcal{A}$. Let H be a separable Hilbert space (separability is henceforth assumed). We can form the algebraic tensor product $H \otimes_{\mathrm{alg}} \mathcal{A}$ on which there is an $\mathcal{A}$-valued inner product

$$\langle x \otimes \zeta\ ,\ y \otimes \eta \rangle\ =\ \langle x, y\rangle\ \zeta^* \eta\ , \qquad x, y \in H\ ,\ \zeta, \eta \in \mathcal{A}. \tag{3.2}$$

Thus $H \otimes_{\text{alg}} \boldsymbol{\mathcal{A}}$ becomes an inner product $\boldsymbol{\mathcal{A}}$-module whose completion is denoted by $H \otimes \boldsymbol{\mathcal{A}}$. Given an orthonormal basis for H, we have the following identification (unitary equivalence) given by $H \otimes \boldsymbol{\mathcal{A}} \approx H_{\boldsymbol{\mathcal{A}}}$ (see, e.g., [23]).

3.2. The restricted Banach *-algebra A_{res}

Suppose now that $H_{\boldsymbol{\mathcal{A}}}$ is *polarizable*, meaning that we have a pair of submodules $(\mathsf{H}_+, \mathsf{H}_-)$, such that $H_{\boldsymbol{\mathcal{A}}} = \mathsf{H}_+ \oplus \mathsf{H}_-$ and $\mathsf{H}_+ \cap \mathsf{H}_- = \{0\}$ (cf., e.g., [24]). Thus we call the pair $(\mathsf{H}_+, \mathsf{H}_-)$ a *polarization of* $H_{\boldsymbol{\mathcal{A}}}$. If we have a unitary $\boldsymbol{\mathcal{A}}$-module map J satisfying $J^2 = 1$, there is an induced eigenspace decomposition $H_{\boldsymbol{\mathcal{A}}} = H_+ \oplus H_-$, for which $H_\pm \cong H_{\boldsymbol{\mathcal{A}}}$. This leads to the Banach algebra $A_{\text{res}} = \mathcal{L}_J(H_{\boldsymbol{\mathcal{A}}})$ as described in [14] (generalizing that of $\boldsymbol{\mathcal{A}} = \mathbb{C}$ in [32]). Specifically, we define

$$A_{\text{res}} := \mathcal{L}_J(H_{\boldsymbol{\mathcal{A}}}) = \{T \in \mathcal{L}_{\boldsymbol{\mathcal{A}}}(H_{\boldsymbol{\mathcal{A}}}) : [J, T] \text{ is Hilbert-Schmidt}\}, \tag{3.3}$$

for which the norm is $\|T\|_J = \|T\| + \|[J, T]\|_2$, for $T \in A_{\text{res}}$.

• Once this restriction is understood, we shall simply write $A = A_{\text{res}} := \mathcal{L}_J(H_{\boldsymbol{\mathcal{A}}})$ until otherwise stated, and let $G(A)$ denote its group of units.

Remark 3.1. Note that A is actually a (complex) Banach *-algebra. The spaces $\mathrm{Gr}(p, A)$ are thus generalized 'restricted Grassmannians' [14, 15], which for the case $\boldsymbol{\mathcal{A}} = \mathbb{C}$, reduce to the usual restricted Grassmannians of [32, 34]. In this case, $V(p, A)$ is regarded as the Stiefel bundle of 'admissible bases' (loosely, those for which a 'determinant' is definable).

The space $\mathrm{Gr}(p, A)$ may be realized more specifically in the following way. Suppose that a fixed $p \in P(A)$ acts as the projection of $H_{\boldsymbol{\mathcal{A}}}$ on H_+ along H_-. Therefore $\mathrm{Gr}(p, A)$ is the Grassmannian consisting of subspaces $W = r(H_{\boldsymbol{\mathcal{A}}})$, for $r \in P(A)$, such that:

(1) the projection $p_+ = pr : W \longrightarrow \mathsf{H}_+$ is in $\mathrm{Fred}(H_{\boldsymbol{\mathcal{A}}})$, and
(2) the projection $p_- = (1 - p)r : W \longrightarrow \mathsf{H}_-$ is in $\mathcal{L}_2(\mathsf{H}_+, \mathsf{H}_-)$ (Hilbert-Schmidt operators).

Alternatively, for (2) we may take projections $q \in P(A)$ such that for the fixed $p \in P(A)$, the difference $q - p \in \mathcal{L}_2(\mathsf{H}_+, \mathsf{H}_-)$. Further, there is the *big cell* $C_{\text{b}} = C_{\text{b}}(p_1, A) \subset \mathrm{Gr}(p, A)$ as the collection of all subspaces $W \in \mathrm{Gr}(p, A)$, such that the projection $p_+ \in \mathrm{Fred}(H_{\boldsymbol{\mathcal{A}}})$ is an isomorphism.

3.3. The space $\mathfrak{P}$ of polarizations

Let us define $p_\pm \in A$ by

$$p_\pm = \frac{1 \pm J}{2}. \tag{3.4}$$

Then $p_\pm \in P(A)$ can be seen to be the spectral projection of J with eigenvalue ± 1. Clearly $p_- + p_+ = 1$, so $p_- = 1 - p_+ = \hat{p}_+$. Thus,

$$(\mathsf{H}_+, \mathsf{H}_-) = (p_+(H_{\boldsymbol{\mathcal{A}}}), p_-(H_{\boldsymbol{\mathcal{A}}})), \tag{3.5}$$

is a polarization. Notice that if $H_{\boldsymbol{\mathcal{A}}}$ is infinite-dimensional, then members of the group of units $G = G(\mathcal{L}(H_{\boldsymbol{\mathcal{A}}}))$ of the unrestricted algebra, are clearly not Hilbert-Schmidt in general. If $g \in G$ with $g(p_+)g^{-1} = p_-$, then using (3.4), we find

$gJ+Jg=0$, which means that $[g,J]=2gJ \in G$. This means that in the restricted algebra $A=A_{\mathrm{res}}$, the projections p_+ and p_- must be in different similarity classes. For this reason, when dealing with the Grassmannian $\mathrm{Gr}(p_+,A)$ and the Stiefel bundle $V(p_+,A)$ over it, the map Ker will take values in $\mathrm{Gr}(p_-,A)$ which is an entirely different space referred to as the *dual Grassmannian of* $\mathrm{Gr}(p_+,A)$. Thus for any $p \in P(A)$, let

$$\mathrm{Gr}^*(p,A)=\mathrm{Gr}(\hat{p},A)=\mathrm{Gr}^{\mathrm{op}}(p,A). \tag{3.6}$$

We also note that by (3.4), we have $[T,J]=2[T,p_+]$, for any operator in $\mathcal{L}(H_{\boldsymbol{\mathcal{A}}})$. So the definition of the restricted algebra is equally well given as the set of operators $T \in \mathcal{L}(H_{\boldsymbol{\mathcal{A}}})$ for which $[T,p_+]$ is Hilbert-Schmidt.

Now let $(\mathsf{H}_+,\mathsf{H}_-)$ be the fixed polarization defined by p_+ and $(\mathsf{K}_+,\mathsf{K}_-)$ another polarization, so that $H_{\boldsymbol{\mathcal{A}}}=\mathsf{H}_+\oplus\mathsf{H}_-=\mathsf{K}_+\oplus\mathsf{K}_-$, whereby the projections parallel to H_- and K_- are isomorphisms of the spaces H_+ and K_+ respectively. Further, when restricting $\mathsf{K}_\pm$ to be in $\mathrm{Gr}(p_\pm,A)$, then under these specified conditions, the Grassmannian $\mathrm{Gr}(p_-,A)$ is the 'dual Grassmannian' of $\mathrm{Gr}(p_+,A)$. Let us denote this dual Grassmannian by $\mathrm{Gr}^*(p_+,A)$. Then, on setting $p=p_+$, the space $\mathfrak{P}$ of such polarizations can be regarded as a subspace

$$\mathfrak{P} \subset \mathrm{Gr}(p,A)\times\mathrm{Gr}^*(p,A). \tag{3.7}$$

3.4. The case where $\boldsymbol{\mathcal{A}}$ is commutative

Here we address the case where $\boldsymbol{\mathcal{A}}$ is a commutative separable C*-algebra. The Gelfand transform implies there exists a compact metric space Y such that $Y=\mathrm{Spec}(\boldsymbol{\mathcal{A}})$ and $\boldsymbol{\mathcal{A}}\cong C(Y)$. Setting $B=\mathcal{L}_J(H)$, we can now express the Banach *-algebra A in the form

$$A \cong B\otimes\boldsymbol{\mathcal{A}} \cong \{\text{continuous functions } Y \longrightarrow B\}, \tag{3.8}$$

for which the $\|\ \|_2$-trace in the norm of A is regarded as continuous as a function of Y. The Banach algebra $B=\mathcal{L}_J(H)$ corresponds to taking $\boldsymbol{\mathcal{A}}=\mathbb{C}$, and as mentioned in Remark 3.1, with respect to the polarization $H=H_+\oplus H_-$, we recover the usual restricted Grassmannians $\mathrm{Gr}(H_+,H)$. Given our formulation, and in view of the spatial correspondence, it will sometimes be convenient to set $\mathrm{Gr}(q,B)=\mathrm{Gr}(H_+,H)$, for suitable $q\in P(A)$. In fact, there is a natural inclusion $\mathrm{Gr}(q,B)\subset\mathrm{Gr}(p,A)$ as deduced in [15].

4. Constructions for the submanifold geometry and bundle theory

4.1. Some preliminaries

In this section we will compute in various bundles where the manifolds involved are submanifolds of Banach spaces, and in this context, adopt some notation which will facilitate the calculations. If $\xi=(\pi,B,X)$ denotes a bundle, meaning simply that we start with a map $\pi: B\longrightarrow X$, and denote by $\xi_x=B_x=\pi^{-1}(x)$, the fiber of ξ over $x\in X$. We write $\pi=\pi_\xi$ for the projection of this bundle and $B=B_\xi$ for its total space. When no confusion results, we will simply write B for the bundle

ξ. If $\psi = (h; f) : \xi \longrightarrow \zeta$, meaning that $\pi_\zeta h = f\pi_\xi$, then $\psi_x = h_x$ denotes the restriction of h to a map of ξ_x into $\zeta_{f(x)}$. By the same token we shall simply write h in place of ψ. As usual, by a section of ξ, we simply mean a map $s : X \longrightarrow B$ satisfying $\pi s = \mathrm{id}_X$.

If ξ is a vector bundle over X, then we take z_ξ to denote the zero section of ξ. We denote by $\epsilon(X, F)$ the trivial bundle $X \times F$ over X with fiber F. If M is a manifold (of some order of differentiability), then we will need to distinguish between the tangent bundle $\mathsf{T}(M)$ of M and the total space TM of the former. We let $z_M = z_{\mathsf{T}(M)}$. Thus, z_M is a standard embedding of M into TM.

When ξ is a subbundle of the trivial bundle $\epsilon = \epsilon(X, F)$, then π_ϵ is the first factor projection and the second factor projection, π_2 assigns each $b \in X \times F$ its principal part. Thus we have a subset $F_x = \pi_2(B_x) \subset F$, so that $B_x = \{x\} \times F_x$. Moreover, if s is here a section of $\xi \subset \epsilon$, then we call $\pi_2 s$ *the principal part of* s. Consequently, $s = (\mathrm{id}_X, f)$, where $f = \pi_2 s : X \longrightarrow F$, must have the property that $f(x) \in F_x$ for each $x \in X$, and any $f : X \longrightarrow F$ having this property is the principal part of a section. In particular, if M is a submanifold of a Banach space F, then $\mathsf{T}(M)$ is a vector subbundle of $\epsilon(M, F)$, and we define $T_xM = F_x$. Thus $\mathsf{T}_x(M) = \{x\} \times T_xM$. If H is another Banach space, N a submanifold of H, and $f : M \longrightarrow N$ is smooth, then $T_xf : T_xM \longrightarrow T_f(x)N$, is the principal part of the tangent map, so that we have $Tf_x = \mathrm{id}_x \times T_xf$.

Locally, we can assume that M is a smooth retract in F which means any smooth map on M can be assumed to have at each point, a local smooth extension to some open set in F containing that point. So if $v \in T_xM$, then $T_xf(v) = D_vf(x) = f'(x)v$, this last term being computed with any local smooth extension. In our applications, the maps will be defined by simple formulas which usually have obvious extensions, as both F and H will be at most products of a fixed Banach algebra A and the formulas defined using operations in A.

4.2. The tangential extension

If $\varphi : M \times N \longrightarrow Q$ is a smooth map, then we have the associated tangent map $T\varphi : TM \times TN \longrightarrow TQ$. If we write $\varphi(a, b) = ab$, then we also have $T\varphi(x, y) = xy$, if $(x, y) \in TM \times TN$. Employing the zero sections, we shall write ay in place of $z_M(a)y$ and xb in place of $xz_N(b)$. Thus it follows that $ab = z_M(a)z_N(b)$ is again identified with $\varphi(a, b)$; that is, we regard $T\varphi$ as an extension of φ which we refer to as *the tangential extension* (of φ).

Since $T(M \times N) = TM \times TN$, which is fiberwise the direct sum of vector spaces, we readily obtain for $(x, y) \in T_aM \times T_bN$, the relation

$$xy = ay + xb = ay +_{ab} xb, \tag{4.1}$$

where for emphasis, we denote by $+_{ab}$ the addition map in the vector space $T_{ab}Q$ (recall that $\varphi(a, b) = ab$).

4.3. Tangential isomorphisms

In the following, we will have to be particularly careful in distinguishing between the *algebraic commutator* '$[\ ,\ ]_{\mathrm{alg}}$' and the *Lie bracket* '$[\ ,\ ]_{\mathfrak{L}}$' (of vector fields), when dealing with functions taking values in a Banach algebra. Specifically, we let $[x,y]_{\mathrm{alg}}$ denote the algebraic commutator which can be taken pointwise if x, y are algebra-valued functions, and $[x,y]_{\mathfrak{L}}$ to denote the Lie bracket of vector fields or principal parts of vector fields which may also be algebra-valued functions.

Relative to the restricted algebra A_{res} in (3.3), let us recall that the space of polarizations is the space $\mathfrak{P}$ of complementary pairs in the product

$$\mathfrak{P} \subset \mathrm{Gr}(p, A_{\mathrm{res}}) \times \mathrm{Gr}^{op}(p, A_{\mathrm{res}}). \tag{4.2}$$

A significant observation, is that as a set, $\mathfrak{P}$ can be identified with the similarity class $\mathrm{Sim}(p, A_{\mathrm{res}})$ of A_{res}. In fact (see below),

$$\mathfrak{P} \cong \mathrm{Sim}(p, A_{\mathrm{res}}) \subset P(A_{\mathrm{res}}). \tag{4.3}$$

Now from [11], we know that $\Pi = \mathrm{Im}$ and $\Pi^{\mathrm{op}} = \mathrm{Ker}$ are analytic open maps. In fact, the calculations are valid in any Banach algebra, *so henceforth, A can be taken to be any Banach algebra with identity.* Thus, we can begin by observing from (2.4) that for any Banach algebra A, the map $\phi = (\Pi, \Pi^{\mathrm{op}}) = (\mathrm{Im}, \mathrm{Ker})$ is an embedding of the space of idempotents $P(A)$ as an open subset of $\mathrm{Gr}(A) \times \mathrm{Gr}(A)$.

Theorem 4.1. *Let $\phi = (\Pi, \Pi^{\mathrm{op}}) = (\mathrm{Im}, \mathrm{Ker}) : P(A) \longrightarrow \mathrm{Gr}(A) \times \mathrm{Gr}(A)$, be as above and let $r \in P(A)$.*

(1) *We have an isomorphism*

$$T\Pi_r|[\{r\} \times (\hat{r}Ar)] : \{r\} \times (\hat{r}Ar) \xrightarrow{\cong} T_{\Pi(r)}\mathrm{Gr}(A), \tag{4.4}$$

and

$$\mathrm{Ker}\ T\Pi_r = \{r\} \times (rA\hat{r}). \tag{4.5}$$

(2) *In the dual sense, we also have an isomorphism*

$$T\Pi_r^{\mathrm{op}}|[\{r\} \times (rA\hat{r})] : \{r\} \times (rA\hat{r}) \xrightarrow{\cong} T_{\Pi(\hat{r})}\mathrm{Gr}(A), \tag{4.6}$$

and

$$\mathrm{Ker}\ T\Pi_r^{\mathrm{op}} = \{r\} \times (\hat{r}Ar). \tag{4.7}$$

(3) *The map ϕ is an injective open map and an analytic diffeomorphism onto its image. Hence $\mathfrak{P}$ is analytically diffeomorphic to* $\mathrm{Sim}(p, A)$.

Proof. As we already know, since the map ϕ is injective, it suffices to apply the Inverse Function Theorem (see, e.g., [20]) when noting that the tangent map $T\phi$ is an isomorphism on fibers of the tangent bundles. To do this, we apply the formulation of [11]. Firstly, from [11], we know that

$$T_r P(A) = \hat{r}Ar + rA\hat{r}. \tag{4.8}$$

If $r \in P(A)$, then we deduce from [11] the canonical section $S_r : U_r \longrightarrow P(A)$ whose image is $P(A) \cap V(r, A) = r + \hat{r}Ar$, which is analytic on its domain $U_r \subset \mathrm{Gr}(A)$. Specifically, we know from [11] that S_r is the inverse of the analytic diffeomorphism $\Pi|(r + \hat{r}Ar)$, which maps onto U_r and that U_r is an open subset of $\mathrm{Gr}(A)$ containing r. This shows that $T_r\Pi|\{r\} \times (\hat{r}Ar)$ is an isomorphism onto $T_{\Pi(r)}\mathrm{Gr}(A)$. On the other hand, Π is constant on $r + rA\hat{r} = \Pi^{-1}(\Pi(r)) \subset P(A)$. Thus, we see that $\mathrm{Ker}\ T_r\Pi = \{r\} \times (rA\hat{r})$. This establishes part (1).

Likewise for part (2), $\mathrm{Ker}|(r + \hat{r}Ar)$ is constant and $\mathrm{Ker}|(r + rA\hat{r})$ is an analytic diffeomorphism onto an open subset of $\mathrm{Gr}(\hat{r}, A)$ which of course is an open subset of $\mathrm{Gr}(A)$ as Π is an open map and $\mathrm{Sim}(q, A)$ is open in $P(A)$. Thus (2) follows.

For part (3), note that since $\hat{r}Ar$ and $rA\hat{r}$ are complementary subspaces of $T_rP(A)$, it follows that $T_r\phi_r = T_r(\Pi, \Pi^{\mathrm{op}})$ is an isomorphism onto $T_{\phi(r)}[\mathrm{Gr}(A) \times \mathrm{Gr}(A)]$. Thus ϕ is indeed an injective open map and an analytic diffeomorphism onto its image. Now $\mathrm{Gr}^{\mathrm{op}}(p, A) = \mathrm{Gr}^*(p, A) = \mathrm{Gr}(\hat{p}, A)$, and clearly ϕ carries $\mathrm{Sim}(p, A)$ onto this sub-product, namely the space of polarizations $\mathfrak{P}$. □

5. The space V_Λ and its geometry

5.1. Transversality and the transition map

We now fix any idempotent $p \in P(A)$, and for ease of notation in the following, we set

$$\begin{aligned} \Lambda &= \mathrm{Sim}(p, A),\ \mathrm{Gr}(p) = \mathrm{Gr}(p, A),\ V = V(p, A) \\ \pi_\Lambda &= \Pi|\Lambda,\ \text{and}\ \pi_V = \Pi|V. \end{aligned} \tag{5.1}$$

Note that from Theorem 4.1(3), we have the analytic diffeomorphism $\Lambda \cong \mathfrak{P}$.

From [11, § 7] we know that $(\pi_V, V, \mathrm{Gr}(p))$ is an analytic right principal $G(pAp)$-bundle whose transition map

$$t_V : V \times_\pi V \longrightarrow G(pAp), \tag{5.2}$$

is the analytic map such that if $u, v \in V$, and $r \in \Lambda$, with $\pi_V(u) = \pi_V(v) = \pi_\Lambda(r)$, then (recalling the notation of (2.5)) we have

$$t_V(u, v) = u^{-(p,r)}v. \tag{5.3}$$

Define $V_\Lambda = \pi_\Lambda^*(V)$, so then $V_\Lambda \subset \Lambda \times V$ is an analytic principal right $G(pAp)$-bundle over Λ, and clearly

$$V_\Lambda = \{(r, u) \in \Lambda \times V : \pi_\Lambda(r) = \pi_V(u)\}. \tag{5.4}$$

The fact that V_Λ is an analytic submanifold of $\Lambda \times V$ and hence of $A \times A$, follows from the fact that by (4.4) any smooth map to $\mathrm{Gr}(p)$ is *transversal* over π_Λ.

Likewise, we denote by t_Λ the transition map for V_Λ, as the analytic map given by the formula:

$$t_\Lambda((r, u), (r, v)) = t_V(u, v) = u^{-(p,r)}v. \tag{5.5}$$

We keep in mind that if $(r, u) \in V_\Lambda$, then as $\pi_\Lambda(r) = \pi_V(u)$, it follows that $u : p \longrightarrow r$ and therefore $u^{-(p,r)}$ is defined.

The next step is to uncover the geometry natural to V_Λ coming from the fact that we can calculate $T_{(r,u)}V_\Lambda \subset A \times A$. Since π_Λ and π_V are transversal as maps to $\mathrm{Gr}(p)$, it follows that

$$T_{(r,u)}V_\Lambda = \{(x,y) \in T_r\Lambda \times T_uV_\Lambda : [T\pi_\Lambda]_r(r,x) = [T\pi_V]_u(u,y)\} \subset A \times A. \quad (5.6)$$

Lemma 5.1. *We have $T_uV = Ap$, and rAp is the vertical tangent space of V over $\pi_E(r) = \pi_V(u)$. Further, $\hat{r}Ap$ and rAp are complementary subspaces of $Ap = T_uV$.*

Proof. It is straightforward to see that $T_r\Lambda = \hat{r}Ar + rA\hat{r} \subset A$, and from [11], we know that $V = G(A)p$ is open in Ap. It follows that $T_uV = Ap$. As π_V is a principal bundle projection, we know that Ker $T_u\pi_V = T_u[uG(pAp)]$, the tangent space to the fiber over $u \in V$, is the kernel of $T\pi_V$. As there is a $g \in G(A)$ with $u = gp$, and as left multiplication by g is $G(pAp)$-equivariant (simply by the associative law for multiplication in A), it follows that

$$T_u[uG(pAp)] = gT_pG(pAp) = gpAp = uAp. \quad (5.7)$$

Since $ru = u$, and $uu^{-(p,r)} = r$, it follows that $uAp = rAp$. Thus rAp is the vertical tangent space of V over $\pi_E(r) = \pi_V(u)$, so $\hat{r}Ap$ and rAp are complementary subspaces of $Ap = T_uV$. □

On the other hand, from [11], we know that $\Lambda \cap V = S_p(U_p)$ is the image of the canonical section and both π_Λ, π_V coincide on $\Lambda \cap V$. This means that by (4.4), we know $[T\pi_v]_p$ carries $\{p\} \times \hat{p}Ap$ isomorphically onto $T_{\pi(p)}\mathrm{Gr}(p)$ and agrees with the isomorphism (4.4), so we see easily that

$$T_{(p,p)}V_\Lambda = \{(x,y) \in [\hat{p}Ap + pA\hat{p}] : xp = \hat{p}y\}. \quad (5.8)$$

Differentiating the equation $ru = u$, we see that any $(x,y) \in T_{(r,u)}V_\Lambda$ must satisfy $xu + ry = y$ which is equivalent to the equation $xu = \hat{r}y$. Notice this is exactly the equation for the tangent space at (p,p), so we claim

$$T_{(r,u)}V_\Lambda = \{(x,y) \in T_r\Lambda \times Ap : xu = \hat{r}y\}. \quad (5.9)$$

Effectively, a straightforward calculation using (5.8) and the fact that $G(A)$ acts $G(pAp)$-equivariantly on V on the left by ordinary multiplication to translate the result in (5.8) over to the point (r,u), establishes (5.9).

5.2. The connection map $\mathcal{V}$

Now the projection $\pi^* = \pi_{V_\Lambda}$ of V_Λ is a restriction of the first factor projection of $A \times A$ onto A which is linear. Thus $T_{(r,u)}\pi^*(x,y) = x$, and therefore the vertical subspace of $T_{(r,u)}V_\Lambda$ is the set $\{0\} \times rAp$. The projection of the tangent bundle TV_Λ onto this vertical subbundle is clear, and we define

$$\begin{gathered} \mathcal{V} : TV_\Lambda \longrightarrow TV_\Lambda \\ \mathcal{V}((r,u),(x,y)) = ((r,u),(0,ry)), \end{gathered} \quad (5.10)$$

for any $(x,y) \in T_{(r,u)}V_\Lambda$, and for any $(r,u) \in V_\Lambda$. For convenience, let $\mathcal{V}_{(r,u)}$ be the action of $\mathcal{V}$ on principal parts of tangent vectors, so that we obtain

$$\mathcal{V}_{(r,u)}(x,y) = (0, ry). \quad (5.11)$$

It is obvious that $\mathcal{V}$ is a vector bundle map covering the identity on V_Λ and that $\mathcal{V} \circ \mathcal{V} = \mathcal{V}$. Thus we call $\mathcal{V}$ *the connection map.*

Since the right action of $G(pAp)$ on V is defined by just restricting the multiplication map on $A \times A$, it follows that the tangential extension of the action of $G(pAp)$ to act on TV is also just multiplication on the right, that is, yg is just the ordinary product in A. This means that in TV_Λ we have $(x,y)g = (x,yg)$ as the tangential extension of the right action of $G(pAp)$ on $T_{(r,u)}V_\Lambda$. From this, the fact that $\mathcal{V}$ is $G(pAp)$-equivariant, is clear. Thus the map $\mathcal{V}$ defines a connection on V_Λ.

Let $\mathcal{H} = (\mathrm{id}_{TV} - \mathcal{V})$, so $\mathcal{H}$ is the resulting horizontal projection in each fiber. Then clearly for $(x,y) \in T_{(r,u)}V_\Lambda$, we have on principal parts of tangent vectors

$$\mathcal{H}_{(r,u)}(x,y) = (x,y) - (0,ry) = (x,\hat{r}y) = (x,xu). \tag{5.12}$$

Moreover, this clarifies that $(x,xu) \in \mathcal{H}(T_{(r,u)}V_\Lambda)$ is (the principal part of) the horizontal lift of $x \in T_r\Lambda$.

If σ is any smooth local section of V_Λ, then for a vector field χ on Λ it follows that the covariant derivative is just the composition

$$\nabla_\chi \sigma = \mathcal{V}[T\sigma]\chi, \tag{5.13}$$

which is a map of Λ to $\mathcal{V}(TV_\Lambda)$ lifting σ. Because the differentiation here is essentially applied to the principal part of the vector field, if f is the principal part of σ and w is the principal part of χ, then for the purpose of calculations, we can also write $\nabla_w f = \mathcal{V}[f'w] = \mathcal{V}D_w f$, where the meaning is clear.

6. The connection form and its curvature

6.1. The connection form ω_Λ

The right action of $G(pAp)$ on V_Λ in (5.4), when tangentially extended, gives $(r,u)y \in T_{(r,u)}V_\Lambda$ when $y \in T_pG(pAp) = pAp$. As the right action of $G(pAp)$ on V_Λ is defined by $(r,u)g = (r,ug)$, it follows that $(r,u)y = (0,uw)$, for any $w \in T_pG(pAp) = pAp$. The connection 1-form $\omega = \omega_\Lambda$ can then be determined because it is the unique 1-form such that, in terms of the connection map $\mathcal{V}$, we have

$$(r,u)\omega_{(r,u)}(x,y) = \mathcal{V}_{(r,u)}(x,y). \tag{6.1}$$

Notice that if $(x,y) \in T_{(r,u)}V_\Lambda$, then we have $y \in Ap$, and so $u^{-(p,r)}y \in pAp = T_pG(pAp)$. We therefore have both

$$(r,u)\omega_{(r,u)}(x,y) = (0,ry) \text{ and } (r,u)u^{-(p,r)}y = (0,ry), \tag{6.2}$$

which by comparison expresses the connection form as

$$\omega_{(r,u)}(x,y) = u^{-(p,r)}y \in T_pG(pAp) = pAp. \tag{6.3}$$

6.2. The curvature form Ω_Λ

To find the curvature 2-form Ω_Λ of ω_Λ, we simply take the covariant exterior derivative of ω_Λ:

$$\Omega_\Lambda = \nabla\omega_\Lambda = \mathcal{H}^* d\omega_\Lambda. \tag{6.4}$$

Notice that by (5.12), as $r\hat{r} = 0$, we have $\omega_\Lambda(\mathcal{H}v) = 0$, for any $v \in TV_\Lambda$, as should be the case, and therefore if w_1 and w_2 are local smooth tangent vector fields on V_Λ, then, on setting $\Omega = \Omega_\Lambda$ for ease of notation, we have

$$\Omega(w_1, w_2) = -\omega([\mathcal{H}(w_1), \mathcal{H}(w_2)]_{\mathfrak{L}}). \tag{6.5}$$

This means that the curvature calculation is reduced to calculating the Lie bracket of two vector fields on V_Λ. Since $V_\Lambda \subset A \times A$ is an analytic submanifold, it is a local smooth retract in $A \times A$.

In order to facilitate the calculation, let

$$(\tilde{r}, \tilde{u}) : W \longrightarrow W \cap V_\Lambda, \qquad (W \subset A \times A), \tag{6.6}$$

be an analytic local retraction of an open set W in $A \times A$, onto the open subset $W \cap V_\Lambda$ of V_Λ. We can then use $(\tilde{r}, \tilde{u})$ to extend all functions on $W \cap V_\Lambda$ to be functions on W. As w_1 and w_2 are tangent vector fields, assumed analytic on $W \cap V_\Lambda$, their principal parts can be expressed in the form $a_1 = (x_1, y_1)$ and $a_2 = (x_2, y_2)$, and we can therefore assume that as functions, they all are defined on W. We then have pointwise on $W \cap V_\Lambda$,

$$x_i\tilde{u} = \hat{\tilde{r}} y_i = (1 - \tilde{r})y_i, \qquad \text{for } i = 1, 2. \tag{6.7}$$

But then $\mathcal{H}_{(r,u)}(x_i, y_i) = (x_i, x_i u)$ on $W \cap V_\Lambda$, meaning that the principal part of $[\mathcal{H}(w_1), \mathcal{H}(w_2)]_{\mathfrak{L}}$ is just $[(x_1, x_1\tilde{u}), (x_2, x_2\tilde{u})]_{\mathfrak{L}}|(W \cap V_\Lambda)$.

The next simplification is to notice that on $W \cap V_\Lambda$, the function $\tilde{u}$ is just the same as the second factor projection $A \times A \longrightarrow A$. On differentiating, this simplifies the application of the product rule. The result is that the principal part of $[\mathcal{H}(w_1), \mathcal{H}(w_2)]_{\mathfrak{L}}$ evaluated at $(r, u) \in V_\Lambda$, has the form

$$(c, cu + [x_2, x_1]_{\mathrm{alg}} u), \tag{6.8}$$

for suitable c, and where x_i is now just the value of the preceding function of the same symbol at (r, u).

Proposition 6.1. *For $w_1, w_2 \in (TV_\Lambda)_{(r,u)}$ having principal parts (x_1, y_1) and (x_2, y_2) respectively, we have the curvature formula*

$$\Omega_\Lambda(w_1, w_2) = u^{-(p,r)}[x_1, x_2]_{\mathrm{alg}} u. \tag{6.9}$$

Proof. As the Lie bracket of a pair of vector fields tangent to a submanifold, again remains tangent to that submanifold, this means that $(c, cu + [x_2, x_1]_{\mathrm{alg}} u)$ in (6.8), is tangent to V_Λ. Hence, we must also have

$$cu = \hat{r}(cu + [x_2, x_1]_{alg} u), \tag{6.10}$$

and therefore,

$$rcu = \hat{r}[x_2, x_1]_{\mathrm{alg}}. \tag{6.11}$$

Applying (6.3) and (6.5), we now obtain

$$\omega([\mathcal{H}(w_1), \mathcal{H}(w_2)]_{\mathfrak{L}})_{(r,u)} = u^{-(p,r)}(cu + [x_2, x_1]_{\text{alg}}u). \tag{6.12}$$

In view of the fact that $u^{-(p,r)}r = u^{-(p,r)}$ and (6.12) above, we deduce that

$$\omega([\mathcal{H}(w_1), \mathcal{H}(w_2)]_{\mathfrak{L}})_{(r,u)} = u^{-(p,r)}[x_2, x_1]_{\text{alg}}u. \tag{6.13}$$

Thus by (6.5), we finally arrive at

$$\Omega(w_1, w_2) = u^{-(p,r)}[x_1, x_2]_{\text{alg}}u, \tag{6.14}$$

where now $w_1, w_2 \in (TV_\Lambda)_{(r,u)}$ have principal parts (x_1, y_1) and (x_2, y_2) respectively. □

This of course means that $x_1, x_2 \in T_r\Lambda = \hat{r}Ar + rA\hat{r}$, that $y_1, y_2 \in T_uV_\Lambda = Ap$, and thus $x_iu = \hat{r}y_i$, for $i = 1, 2$. But, $V_\Lambda = G(A)u$, so there is $g \in G(A)$ with $u = gp$. It then follows that $u^{-(p,r)} = pg^{-1}$, and therefore we can also write, when $u = gp$,

$$\Omega(w_1, w_2) = [g^{-1}x_1g, g^{-1}x_2g]_{\text{alg}}. \tag{6.15}$$

In this way we can simply transfer the computation to the Lie algebra of $G(pAp)$. We make the following observations:

(1) Because $ru = u$ and $u^{-(p,r)}r = u^{-(p,r)}$, when $(r, u) \in V_\Lambda$, it follows that (6.14) can also be written as

$$\Omega(w_1, w_2) = u^{-(p,r)}r[x_1, x_2]_{\text{alg}}ru, \tag{6.16}$$

and the factor $r[x_1, x_2]_{\text{alg}}r$ simplifies greatly because $x_1, x_2 \in rA\hat{r} + \hat{r}Ar$.

(2) If x_1 and x_2 both belong to $rA\hat{r}$, or both belong to $\hat{r}Ar$, then the result is $\Omega(w_1, w_2) = 0$.

(3) If $x_1 \in rA\hat{r}$ and $x_2 \in \hat{r}Ar$, the result is

$$\Omega(w_1, w_2) = u^{-(p,r)}x_1x_2u. \tag{6.17}$$

Whereas if the reverse is the case, that is $x_1 \in \hat{r}Ar$ and $x_2 \in rA\hat{r}$, the result is

$$\Omega(w_1, w_2) = -u^{-(p,r)}x_2x_1u. \tag{6.18}$$

Remark 6.1. Again, by Theorem 4.1(3), since $\Lambda \cong \mathfrak{P}$, the construction of the principal bundle with connection $(V_\Lambda, \omega_\Lambda) \longrightarrow \Lambda$, may be seen to recover that of the principal bundle with connection $(V_{\mathfrak{P}}, \omega_{\mathfrak{P}}) \longrightarrow \mathfrak{P}$ as in [39, § 3]. We will elaborate on matters when we come to describe the $\mathcal{T}$-function in § 8.1. This principal bundle has for its associated vector bundle (with connection) the universal bundle $(\gamma_{\mathfrak{P}}, \nabla_{\mathfrak{P}}) \longrightarrow \mathfrak{P}$. In the following section, the latter will be recovered when we construct the universal bundle (with connection) $(\gamma_\Lambda, \nabla_\Lambda) \longrightarrow \Lambda$ associated to $(V_\Lambda, \omega_\Lambda) \longrightarrow \Lambda$.

7. The universal bundle over Λ

7.1. The Koszul connection

Next we relate the geometry of V_Λ to the geometrical context of [39] (cf. [28]). First we must show that V_Λ is the principal bundle of the universal bundle in an appropriate sense. In fact, if E is a Banach A-module, then we can form an obvious universal vector bundle, denoted γ_Λ over Λ, as defined by

$$\gamma_\Lambda = \{(r, m) \in \Lambda \times E : rm = m\}, \tag{7.1}$$

and whose projection π_γ is just the restriction of first factor projection. Thus the principal part of a section is here simply a map $f : \Lambda \longrightarrow E$ with the property that $f(r) \in rE$, for every $r \in \Lambda$.

In this case, a natural Koszul connection ∇_Λ arises. Effectively, we have a covariant differentiation operator, given by its operation on principal parts of sections of γ_Λ, via the formula

$$\nabla_x f(r) = rD_x f(r) = rT_r f(x),\ \ x \in T_r\Lambda. \tag{7.2}$$

If x is the principal part of a tangent vector field on Λ, then it follows that

$$\nabla_x f = \mathrm{id}_\Lambda D_x f = \mathrm{id}_\Lambda T_{\mathrm{id}_\Lambda} f(x). \tag{7.3}$$

If $(r, m) \in \gamma_\Lambda$, then the principal part of the tangent space to γ_Λ at the point (r, m) is just

$$T_{(r,m)}\gamma_\Lambda = \{(x, w) \in T_r\Lambda \times E :\ rw + xm = w\}, \tag{7.4}$$

which can also be written as

$$T_{(r,m)}\gamma_\Lambda = \{(x, w) \in T_r\Lambda \times E : xm = \hat{r}w\}. \tag{7.5}$$

Since π_γ is simply the restriction of first factor projection which is linear, it follows that the vertical subspace is

$$VT_{(r,m)}\gamma_\Lambda = \mathrm{Ker}\ T_{(r,m)}\pi_\gamma = \{(0, w) \in T_r\Lambda \times E : rw = w\}, \tag{7.6}$$

so the vertical projection

$$\mathcal{V}_\gamma : \mathsf{T}\gamma_\Lambda \longrightarrow \mathsf{T}\gamma_\Lambda, \tag{7.7}$$

as a vector bundle map covering $\mathrm{id}_{\gamma_\Lambda}$, is given by

$$\mathcal{V}_\gamma((r, m), (x, w)) = ((r, m), (0, rw)). \tag{7.8}$$

This of course means that the horizontal projection $\mathcal{H}_\gamma$ is given by

$$\mathcal{H}_\gamma((r, m), (x, w)) = ((r, m), (x, \hat{r}w)) = ((r, m), (x, xm)), \tag{7.9}$$

which makes it clear that the horizontal lift to $(r, m) \in \gamma_\Lambda$ of $(r, x) \in T\Lambda$ is just $((r, x), (x, xm))$.

Thus, the geometry of the universal bundle γ_Λ turns out to be very natural and straightforward. In order to see that γ_Λ is the associated vector bundle to the principal bundle V_Λ, we first note that the principal part of the fiber of γ_Λ over $p \in \Lambda$ is pE and we can define the *principal map*

$$Q : V_\Lambda \times pE \longrightarrow \gamma_\Lambda, \tag{7.10}$$

by

$$Q((r,u),m) = (r,um), \ ((r,u),m) \in V_\Lambda \times pE. \tag{7.11}$$

Proposition 7.1. *The map Q in* (7.11) *is the analytic principal bundle map for which the universal bundle $(\gamma_\Lambda = V_\Lambda[pE], \nabla_\Lambda)$ is an analytic vector bundle with connection associated to the principal bundle with connection $(V_\Lambda, \omega_\Lambda)$.*

Proof. Clearly $V_\Lambda \times pE$ has a principal right $G(pAp)$-action given by

$$((r,u),m))g = ((r,u)g, g^{-p}m) = ((r,ug), g^{-p}m), \tag{7.12}$$

with transition map

$$t(((r,u),m),((r,v),n)) = t_\Lambda((r,u),(r,v)), \tag{7.13}$$

and Q establishes a bijection with the orbit space of this action. To conclude that Q is the actual principal map making $\gamma_\Lambda = V_\Lambda[pE]$ the associated bundle to V_Λ with fiber pE, it suffices to show that Q has analytic local sections, because Q itself is clearly analytic.

To that end, observe that if σ is a local section of V_Λ over the open subset $U \subset \Lambda$, then $\sigma = (\mathrm{id}_\Lambda, u)$ where $u : U \longrightarrow V = V(p,A)$, such that for every $r \in U$, we have $u(r) : p \longrightarrow r$ is a proper partial isomorphism. We then define λ, the corresponding local analytic cross section of Q by

$$\lambda(r,m) = ((r,u(r)), u(r)^{-(p,r)}m). \tag{7.14}$$

Following [11] we know that $u^{-(p,r)}$ as a function of $r \in U$, is analytic as a map to $V(A)$. Indeed, Q is the principal map and $\gamma_\Lambda = V_\Lambda[pM]$. It is now a routine calculation to see that the connection on γ_Λ defined above is the same as the connection derived from the connection ω_Λ already defined on V_Λ. □

For instance, if $f : V_\Lambda \longrightarrow pE$ is an equivariant smooth map, and x is any section of $T\Lambda$, then f defines a smooth section s of γ_Λ whose covariant derivative $\nabla_x s$ is the same as the section defined by the derivative of f in the direction of the horizontal lift of x. As Q is the principal map, it is the projection of a principal bundle and therefore TQ is vector bundle map covering Q which is surjective on the fibers. We have

$$TQ(((r,u),m),((x,y),w)) = ((r,um),(x,ym+uw)), \tag{7.15}$$

and

$$\mathcal{V}_\gamma TQ((r,u),m),((x,y),w)) = ((r,um),(0,r[ym+uw])), \tag{7.16}$$

along with

$$\begin{aligned} TQ(\mathcal{V}^*(((r,u),m),((x,y),w)) &= TQ(((r,u),m),((0,ry),w)) \\ &= ((r,um),(0,rym+uw)). \end{aligned} \tag{7.17}$$

But $ru = u$ for $(r,u) \in V_\Lambda$. Hence from (7.16) and (7.17), we have $\mathcal{V}_\gamma TQ = TQ\mathcal{V}^*$, where $\mathcal{V}^*$ denotes the connection map of the vertical projection on $V_\Lambda \times pE$ pulled back from V_Λ by the first factor projection map of $V_\Lambda \times pE \longrightarrow V_\Lambda$, which being equivariant, defines a pullback square. This shows that the vertical projection on

γ_Λ is that defined by the vertical projection on V_Λ. Thus we have constructed V_Λ to be the principal bundle for any universal bundle defined by any left Banach A-module such as E. In particular, we could take $E = A$ for the existence of one, but for the $\mathcal{T}$-function construction we would take $E = H_{\boldsymbol{\mathcal{A}}}$. In other words, we would take E to be the underlying Banach space of $H_{\boldsymbol{\mathcal{A}}}$ so $\boldsymbol{\mathcal{A}}$ would act as a subalgebra of the commutant of A in the algebra of bounded operators.

8. The $\mathcal{T}$-function

8.1. Definition of the $\mathcal{T}$-function

From our constructions so far, even though they are quite general, it should be clear that we have all the ingredients for the construction of a function, denoted by $\mathcal{T}$, that generalizes the function, denoted by $\mathfrak{T}$ and defined via cross-ratio in [28, 39] as a pre-determinant, thus providing the Tau (τ)-function studied in [28, 34]. Similar to [39], we will define two local sections α_p and β_p over W_p^0, the latter taken to be an open neighborhood of $p \in P(A)$, which is our reference projection. For W_p^0 we take the set of $r \in W_p = \pi_\Lambda^{-1}(p + pA\hat{p})$ such that $\phi_p(r) = rp + \hat{r}\hat{p} \in G(A)$. As $G(A)$ is open in A, and as $\phi_p(p) = 1 \in G(A)$, it follows that W_p^0 is indeed open in Λ and contains p.

Next we describe the sections α_p and β_p:

(1) For α_p we take the restriction of the pullback by π_Λ of the canonical section S_p which is defined over $\pi_\Lambda(W_p) \subset \mathrm{Gr}(p, A)$. Thus, as in the pullback, α_p becomes a composition with π_Λ. It follows from (4.5) that if $w = (r, x) \in T\Lambda$ with $x \in rA\hat{r}$, then $\nabla_w \alpha_p = 0$.

(2) For β_p, with $g = \phi_p(r)$ and $r \in W_p^0$, we have $g \in G(A)$ and $rp : p \longrightarrow r$ is a proper partial isomorphism which projects along $\mathrm{Ker}(r)$, so we define $\beta_p(r) = (r, rp)$.

As $S_p(\mathrm{Im}(r))$ projects along $\mathrm{Ker}(p)$, we generalize the $\mathfrak{T}$-function of [39] by the function $\mathcal{T}$ by recalling the transition map t_Λ in (5.5), and then defining

$$\mathcal{T}(r) = t_\Lambda(\alpha_p(r), \beta_p(r)). \tag{8.1}$$

Hence we may express the latter by $\mathcal{T} = t_\Lambda(\alpha_p, \beta_p)$.

In [39], the function $\mathfrak{T}$ constructed via cross-ratio is used to define the connection form $\omega_{\mathfrak{P}}$ on the principal bundle $V_{\mathfrak{P}} \longrightarrow \mathfrak{P}$, where the corresponding curvature 2-form $\Omega_{\mathfrak{P}}$ can be computed in coordinates on the product of Grassmannians. In order to see that the geometry here is essentially the same as that of [39], we show that α_p and β_p are parallel (covariantly constant) sections. Specifically, it suffices to show that $\nabla_w \alpha_p = 0$, if $w = (r, x)$ with $x \in rA\hat{r}$, and that $\nabla_w \beta_p = 0$ if $w = (r, x)$ with $x \in \hat{r}Ar$. The first of these has already been observed in (1) above. As for the second, since $\beta_p(r) = (r, rp)$, it follows that $T_r\beta_p(x) = (x, xp)$, for any $x \in T_r\Lambda$, and therefore

$$\nabla_w \beta_p = \mathcal{V}((r, rp), (x, xp)) = ((r, rp), (0, rxp)). \tag{8.2}$$

As $x \in \hat{r}Ar$ implies $rxp = 0$, we also have $\nabla_w \beta_p = 0$, for $w = (r, x)$ with $x \in \hat{r}Ar$. We therefore know that the geometry is the same as in [39] and we can now apply our formulas to calculate $\mathcal{T}$. But, we know from the definition of the transition function t_Λ in (5.5), that we have

$$t_\Lambda((r, u), (r, v)) = u^{-(p,r)}v, \tag{8.3}$$

and we know that the relative inverse for the canonical section is p itself, independent of r. Hence, we finally have $\mathcal{T}(r) = prp$.

8.2. Curvature formulas

Returning to the universal bundle (with connection) $(\gamma_\Lambda, \nabla_\Lambda) \longrightarrow \Lambda$, we can easily calculate the curvature form using the Koszul connection of the connection ∇_Λ operating on principal parts of sections of γ_Λ. If x and y are principal parts of local smooth tangent vector fields to Λ, and if f is an E-valued smooth function on the same domain, then we can consider that ordinary differentiation D acting on functions, is the Koszul connection of the flat connection on $\epsilon(\Lambda, E)$. So the curvature operator $\mathcal{R}_\nabla$ can be computed keeping in mind that $\mathcal{R}_D = 0$. Thus, letting $L : \Lambda \longrightarrow \mathcal{L}(E, E)$ be the action of left multiplication of Λ on E, noting that $L(r)m = em$, we then have

$$\mathcal{R}_\nabla(x, y)f = [\nabla_x, \nabla_y]f - \nabla_{[x,y]_{\mathfrak{L}}} f. \tag{8.4}$$

Theorem 8.1. *With respect to the above action $L : \Lambda \longrightarrow \mathcal{L}(E, E)$ of left multiplication of Λ on E, we have the following formulas for the curvature operator $\mathcal{R}_\nabla$, for $x, y \in T_r\Lambda$:*

(1)

$$\mathcal{R}_\nabla(x, y) = L[(D_xL)D_y - (D_yL)D_x]. \tag{8.5}$$

(2)

$$\mathcal{R}_\nabla(x, y) = L[x, y]_{\text{alg}}. \tag{8.6}$$

Proof. Firstly, observe that notationally $\nabla_x f = LD_x f$. Since the pointwise product is $LL = L$, it follows that

$$\nabla_x \nabla_y f = LD_x(LD_y f) = L[D_xL][D_y f] + LD_xD_y f, \tag{8.7}$$

and therefore (8.4) becomes

$$\mathcal{R}_\nabla(x, y)f = L[D_xL][D_y f] + LD_xD_y f - (L[D_yL][D_x f] + LD_yD_x f) - LD_{[x,y]_{\mathfrak{L}}} f. \tag{8.8}$$

Consequently, we have

$$\mathcal{R}_\nabla(x, y)f = (L[D_xL]D_y - [D_yL]D_x)f + L\mathcal{R}_D(x, y)f, \tag{8.9}$$

and therefore, as $\mathcal{R}_D = 0$, it follows that

$$\mathcal{R}_\nabla(x, y)f = L[(D_xL)D_y - (D_yL)D_x]f. \tag{8.10}$$

Thus we may write

$$\mathcal{R}_\nabla(x, y) = L[(D_xL)D_y - (D_yL)D_x], \tag{8.11}$$

which establishes (1).

On the other hand, we note that L is the restriction of the linear map defined by the left regular representation L_A of A on E, defined by the module action of A on M. So we have $D_x L = L_A(x)$, the composition of L_A with x. This means that

$$[(D_x L)(r)]m = L_A(x(r))m = [x(r)]m = (xm)(r), \tag{8.12}$$

for $r \in \Lambda$ and $m \in rE$. Therefore, we have for f, that

$$\mathcal{R}_\nabla(x,y)f = L[(L_A(x))D_y - (L_A(y))D_x]f = L[xD_y - yD_x]f. \tag{8.13}$$

For the curvature operator at a specific point, we can take any $m \in E$, and define $f_m = Lm$, so that we have $f_m(r) = L(r)m = rm$. Then f is given by the module action of A on E which is linear, for fixed $m \in E$. Thus, $D_x f = L_A(x)m = xm$ and (8.13) becomes

$$\mathcal{R}_\nabla(x,y)f = L[x,y]_{\mathrm{alg}} m, \tag{8.14}$$

which means that we finally arrive at (2):

$$\mathcal{R}_\nabla(x,y) = L[x,y]_{\mathrm{alg}}. \tag{8.15}$$

□

8.3. Remarks on the operator cross ratio

Returning to the case $A = \mathcal{L}_J(H_{\mathcal{A}})$, let us now mention some examples (to be further developed in [16]). Firstly, we recall the $\mathfrak{T}$ function of [39] defined via cross-ratio. Consider a pair of polarizations $(\mathsf{H}_+, \mathsf{H}_-), (\mathsf{K}_+, \mathsf{K}_-) \in \mathfrak{P}$. Let $\mathsf{H}_\pm$ and $\mathsf{K}_\pm$ be 'coordinatized' via maps $P_\pm : \mathsf{H}_\pm \longrightarrow \mathsf{H}_\mp$, and $Q_\mp : \mathsf{K}_\pm \longrightarrow \mathsf{K}_\mp$, respectively. Following [39] (Proposition 2), we can consider the composite map

$$\mathsf{H}_+ \xrightarrow{\mathsf{K}_-} \mathsf{K}_+ \xrightarrow{\mathsf{H}_-} \mathsf{H}_+, \tag{8.16}$$

as represented by the operator cross-ratio (cf. [39]):

$$\mathfrak{T}(\mathsf{H}_+, \mathsf{H}_-, \mathsf{K}_+, \mathsf{K}_-) = (P_-P_+ - 1)^{-1}(P_-Q_+ - 1)(Q_-Q_+ - 1)^{-1}(Q_-P_+ - 1). \tag{8.17}$$

For this construction there is no essential algebraic change in generalizing from polarized Hilbert spaces to polarized Hilbert modules. The principle here is that the transition between charts define endomorphisms of $W \in \mathrm{Gr}(p, A)$ that will become the transition functions of the universal bundle $\gamma_{\mathfrak{P}} \longrightarrow \mathfrak{P}$. These transition functions are defined via the cross ratio as above and thus lead to $\mathrm{End}(\gamma_{\mathfrak{P}})$-valued 1-cocyles, in other words, elements of the cohomology group $H^1(\mathrm{Gr}(p, A), \mathrm{End}(\gamma_{\mathfrak{P}}))$.

Regarding the universal bundle $\gamma_\Lambda \longrightarrow \Lambda$, the transition between charts is already achieved by means of the $\mathcal{T}$-function on Λ. From Theorem 4.1 (3) we have an analytic diffeomorphism $\tilde{\phi} : \mathfrak{P} \longrightarrow \Lambda$ (where $\tilde{\phi} = \phi^{-1}$), and effectively, $\tilde{\phi}^* \mathcal{T} = \mathfrak{T}$ in this case.

8.4. The connection and curvature forms on $V_{\mathfrak{P}}$

In view of § 8.1, we will exemplify the construction of [39, § 3] for the connection form $\omega_{\mathfrak{P}}$ on the principal bundle $V_{\mathfrak{P}} \longrightarrow \mathfrak{P}$, and the curvature form $\Omega_{\mathfrak{P}}$. We start by fixing a point $\mathcal{P} = (\mathsf{H}_+, \mathsf{H}_-) \in \mathfrak{P}$, and consider a pair of local sections α, β of $V_{\mathfrak{P}}$, which are related as follows:

$$\alpha = \beta\mathfrak{T} \ , \qquad \beta = \alpha\mathfrak{T}^{-1}. \tag{8.18}$$

Next let $\nabla_{\pm}$ denote covariant differentiation with respect to the direction $\mathsf{H}_{\pm}$. The local sections α, β have the property that:

(a) α is covariantly constant along $\{\mathsf{H}_+\} \times \mathrm{Gr}^*(p, A)$, with respect to fixed H_+.

(b) β is covariantly constant along $\mathrm{Gr}(p, A) \times \{\mathsf{H}_-\}$ with respect to fixed H_-.

(c) Properties (a) and (b) imply the equations $\nabla_-\alpha = 0, \nabla_+\beta = 0$, along with $\nabla_+\alpha = \beta\nabla_+\mathfrak{T} = \alpha\mathfrak{T}^{-1}\nabla_+\mathfrak{T}$.

We obtain the connection $\omega_{\mathfrak{P}}$ on the principal bundle $V_{\mathfrak{P}}$ by setting $\omega_{\mathfrak{P}} = \omega_+ = \mathfrak{T}^{-1}\nabla_+\mathfrak{T}$. We have the exterior covariant derivative $d = \partial_+ + \partial_-$, where $\partial_{\pm}$ denotes the covariant derivative along $\mathsf{H}_{\pm}$. Straightforward calculations as in [39, § 3] yield the following:

$$\begin{aligned} &\partial_+\omega_+ = 0 \ , \\ &\partial_-\omega_+ = (Q_-Q_+ - 1)^{-1}dQ_-Q_+(Q_-Q_+ - 1)^{-1}Q_-dQ_+ - (Q_-Q_+ - 1)^{-1}dQ_-dQ_+. \end{aligned} \tag{8.19}$$

The curvature form $\Omega_{\mathfrak{P}}$ relative to $\omega_{\mathfrak{P}}$ is then given by

$$\Omega_{\mathfrak{P}} = (Q_-Q_+ - 1)^{-1}dQ_-Q_+(Q_-Q_+ - 1)^{-1}Q_-dQ_+ - (Q_-Q_+ - 1)^{-1}dQ_-dQ_+. \tag{8.20}$$

8.5. Trace class operators and the determinant

An alternative, but equivalent, operator description leading to $\mathfrak{T}$ above can be obtained following [28]. Suppose $(\mathsf{H}_+, \mathsf{H}_-), (\mathsf{K}_+, \mathsf{K}_-) \in \mathfrak{P}$ are such that H_+ is the graph of a linear map $S : \mathsf{K}_+ \longrightarrow \mathsf{K}_-$ and H_- is the graph of a linear map $T : \mathsf{K}_- \longrightarrow \mathsf{K}_+$. Then on $H_{\boldsymbol{\mathcal{A}}}$ we consider the identity map $\mathsf{H}_+ \oplus \mathsf{H}_- \longrightarrow \mathsf{K}_+ \oplus \mathsf{K}_-$, as represented in the block form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{8.21}$$

where $a : \mathsf{H}_+ \longrightarrow \mathsf{K}_+$, $d : \mathsf{H}_+ \longrightarrow \mathsf{K}_-$ are zero-index Fredholm operators, and $b : \mathsf{H}_+ \longrightarrow \mathsf{K}_+$, $c : \mathsf{H}_+ \longrightarrow \mathsf{K}_-$ are in $\mathcal{K}(H_{\boldsymbol{\mathcal{A}}})$ (the compact operators), such that $S = ca^{-1}$ and $T = bd^{-1}$.

The next thing is to consider the operator $1 - ST = 1 - ca^{-1}bd^{-1}$. In particular, with a view to defining a generalized determinant leading to *an operator-valued Tau-function*, we need to consider cases where ST is assuredly of *trace class.*

(a) When $\boldsymbol{\mathcal{A}} = \mathbb{C}$ as in [28, 34, 39], we take b, c to be Hilbert-Schmidt operators. Then ST is of trace-class, the operator $(1 - ST)$ is essentially

$$\mathfrak{T}(\mathsf{H}_+, \mathsf{H}_-, \mathsf{K}_+, \mathsf{K}_-)$$

above, and the Tau (τ)-function is defined as

$$\tau(\mathsf{H}_+, \mathsf{H}_-, \mathsf{K}_+, \mathsf{K}_-) = \mathrm{Det}\ \mathfrak{T}(\mathsf{H}_+, \mathsf{H}_-, \mathsf{K}_+, \mathsf{K}_-) = \mathrm{Det}(1 - ca^{-1}bd^{-1}). \qquad (8.22)$$

Starting from the universal bundle $\gamma_{\mathcal{E}} \longrightarrow \mathrm{Gr}(p, A)$, then with respect to an admissible basis in $V(p, A)$, the Tau function in (8.22) is equivalently derived from the canonical section of $\mathrm{Det}(\gamma_{\mathcal{E}})^* \longrightarrow \mathrm{Gr}(p, A)$.

(b) The case where $\boldsymbol{\mathcal{A}}$ is a commutative C*-algebra is relevant to von Neumann algebras (see, e.g., [7]), and we may deal with a continuous trace algebra. In particular, for Hilbert *-algebras in general, we have the nested sequence of Schatten ideals in the compact operators [35]. Thus if we take the operators b, c as belonging to the Hilbert-Schmidt class, then ST is of trace class [35], and $\tau(\mathsf{H}_+, \mathsf{K}_-, \mathsf{K}_+, \mathsf{K}_-)$ is definable when the operator $(1 - ST)$ admits a determinant in a suitable sense.

Acknowledgment

We wish to thank the referees for their respective comments. E.P. very gratefully acknowledges partial research support under grant NSF-DMS-0808708.

References

[1] E. Andruchow, A. Larotonda, L. Recht and D. Stojanoff, Infinite dimensional homogeneous reductive spaces and finite index conditional expectations. *Illinois J. Math.* **41** (1997), 54–76.

[2] J. Ball and V. Vinnikov, Hardy spaces on a finite bordered Riemann surface, multiplicative operator model theory and Fourier anlaysis along a unimodular curve, in 'Systems, approximation, singular integral operators, and related topics (Bordeaux, 2000)' *Operator Theory: Advances and Applications* **129** (pp. 37–56), Birkhäuser Verlag, Basel, 2001.

[3] J. Ball and V. Vinnikov, Lax-Phillips scattering and conservative linear systems: a Cuntz-algebra multidimensional setting. *Mem. Amer. Math. Soc.* **178** (2005), no. 837, iv+101 pp.

[4] D. Beltiţă, *Smooth Homogeneous Structures in Operator Theory.* Monographs and Surveys in Pure and Appl. Math. **137**, Chapman and Hall/CRC, Boca Raton FL, 2006.

[5] D. Beltiţă and J.E. Galé, Holomorphic geometric models for representations of C*-algebras. *J. Funct. Anal.* **255** (2008), no. 10, 2888–2932.

[6] D. Beltiţă and T.S. Ratiu, Geometric representation theory for unitary groups of operator algebras. *Adv. in Math.* **208**(1) (2007), 299–317.

[7] A. Connes, *Noncommutative Geometry.* Academic Press, 1994.

[8] G. Corach, H. Porta and L. Recht, Differential geometry of systems of projections in Banach algebras. *Pacific J. Math.* **143**(2) (1990), 209–228.

[9] G. Corach, H. Porta and L. Recht, Differential geometry of spaces of relatively regular operators. *Integral Equations Operator Theory* **13**(1990), 773–794.

[10] R.G. Douglas, *Banach Algebra Techniques in Operator Theory* (2nd Ed.). Graduate Texts in Mathematics **179**, Springer Verlag, New York – Berlin, 1998.

[11] M.J. Dupré and J.F. Glazebrook, The Stiefel bundle of a Banach algebra. *Integral Equations Operator Theory* **41** No. 3 (2001), 264–287.

[12] M.J. Dupré and J.F. Glazebrook, Holomorphic framings for projections in a Banach algebra. *Georgian Math. J.* **9**(3) (2002), 481–494.

[13] M.J. Dupré and J.F. Glazebrook, Relative inversion and embeddings. *Georgian Math. J.* **11**(3) (2004), 425–448.

[14] M.J. Dupré, J.F. Glazebrook and E. Previato, A Banach algebra version of the Sato Grassmannian and commutative rings of differential operators. *Acta Applicandae Math.* **92** (3) (2006), 241–267.

[15] M.J. Dupré, J.F. Glazebrook and E. Previato, Differential algebras with Banach-algebra coefficients I: From C*-algebras to the K-theory of the spectral curve. *Max Planck Institut für Mathematik*, Preprint Series **73** (2008).

[16] M.J. Dupré, J.F. Glazebrook and E. Previato, Differential algebras with Banach-algebra coefficients II: the operator cross-ratio Tau-function and the Schwartzian derivative, in preparation.

[17] M.J. Dupré, J.-Cl. Evard and J.F. Glazebrook, Smooth parametrization of subspaces of a Banach space. *Rev. Un. Mat. Argentina* **41** No. 2 (1998), 1–13.

[18] I. Gohberg and J. Leiterer, On cocycles, operator functions and families of subspaces. *Mat. Issled* (*Kishinev*) **8** (1973) 23–56.

[19] I. Gohberg and J. Leiterer, Über Algebren stetiger Operatorfunktionen. *Studia Mathematica* LVII (1976), 1–26.

[20] R. Hamilton, The inverse function theorem of Nash and Moser. *Bull. Amer. Math. Soc.* **7** (1982), 65–222.

[21] G.F. Helminck and A.G. Helminck, Hilbert flag varieties and their Kähler structure. *J. Phys. A: Math. Gen.* **35** (2002), 8531–8550.

[22] J.W. Helton, *Operator Theory, Analytic Functions, Matrices and Electrical Engineering.* CBMS Reg. Conf. Ser. **68**, Amer. Math. Soc., Providence RI, 1987.

[23] E.C. Lance, *Hilbert C*-Modules.* London Math. Soc. Lect. Notes **210**, Cambridge Univ. Press, 1995.

[24] V.M. Manuilov and E.V. Troitsky, *Hilbert C*-modules.* Trans. Math. Monographs **226**, Amer. Math. Soc., 2005.

[25] M. Martin and N. Salinas, Flag manifolds and the Cowen–Douglas theory. *Journal of Operator Theory* **39** (1997), 329–365.

[26] M. Martin and N. Salinas, Differential geometry of generalized Grassmann manifolds in C*-algebras. In, *Operator Theory: Advances and Applications* **80** (pp. 206–243) Birkhäuser Verlag, Basel, 1995.

[27] M. Martin and N. Salinas, The canonical complex structure of flag manifolds in a C*-algebra. In, *Operator Theory: Advances and Applications* **104** (pp. 173–187) Birkhäuser Verlag, Basel, 1998.

[28] L.J. Mason, M.A. Singer and N.M.J. Woodhouse, Tau functions and the twistor theory of integrable systems. *J. Geom. and Phys.* **32** (2000), 397–430.

[29] J. Mickelsson, *Current Algebras and Groups.* Plenum Press, New York – London, 1989.

[30] H. Porta and L. Recht, Spaces of projections in a Banach algebra. *Acta Cient. Venez.* **38** (1987), 408–426.

[31] H. Porta and L. Recht, Conditional expectations and operator decompositions. *Ann. Global Anal. Geom* **12**(4) (1994), 335–339.
[32] A. Pressley and G. Segal, *Loop Groups and their Representations*. Oxford Univ. Press, 1986.
[33] M. Sato, The KP hierarchy and infinite dimensional Grassmann manifolds. *Proc. Sympos. Pure Math.* **49** Part 1. (pp. 51–66) Amer. Math. Soc., Providence, RI, 1989.
[34] G. Segal and G. Wilson, Loop groups and equations of KdV type. *Publ. Math. IHES* **61** (1985), 5–65.
[35] J.F. Smith, The p-classes of a Hilbert module. *Proc. Amer. Math. Soc.* **36** (2) (1972), 428–434.
[36] H. Upmeier, *Symmetric Banach Manifolds and Jordan C*-Algebras*, Math. Studies **104**, North Holland, Amsterdam – New York – Oxford, 1985.
[37] D.R. Wilkins, The Grassmann manifold of a C*-algebra. *Proc. Royal Irish Acad.* **90** A No. 1 (1990), 99–116.
[38] M.G. Zaidenberg, S.G. Krein, P.A. Kushment and A.A. Pankov, Banach bundles and linear operators. *Russian Math. Surveys* **30**(5) (1975), 115–175.
[39] M.I. Zelikin, Geometry of operator cross ratio. *Math. Sbornik* **197** (1) (2006), 39–54.

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Received: February 16, 2009
Accepted: July 22, 2009

Operator Theory:
Advances and Applications, Vol. 202, 223–252

A Contractive Operator View on an Inversion Formula of Gohberg-Heinig

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Dedicated to Israel Gohberg, on the occasion of his 80th birthday, in friendship, with gratitude and admiration.

Abstract. This paper presents a contractive operator view on the inversion formula for finite Toeplitz operator matrices due to Gohberg-Heinig. The general setting that will be used involves a Hilbert space operator T and a contraction A such that the compression of $T - A^*TA$ to the orthogonal complement of the defect space of A is the zero operator. For such an operator T the analogue of the Gohberg-Heinig inversion formula is obtained. The main results are illustrated on various special cases, including Toeplitz plus Hankel operators and model operators.

Mathematics Subject Classification (2000). Primary 47A45, 47A50, 47B35; Secondary 15A09, 47A20, 65F05.

Keywords. Gohberg-Heinig inversion formula, Toeplitz operator matrices, contractive operators, Toeplitz plus Hankel operators, compression of Toeplitz operators, model operators, Stein equation.

1. Introduction

Let T be an operator acting on a direct sum $\mathcal{E}^n$ of n copies of a Hilbert space $\mathcal{E}$, and let T be generated by an $n \times n$ Toeplitz operator matrix, that is,

$$T = \begin{bmatrix} R_0 & R_{-1} & \cdots & R_{-n+1} \\ R_1 & R_0 & \cdots & R_{-n+2} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n-1} & R_{n-2} & \cdots & R_0 \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{E} \\ \mathcal{E} \\ \vdots \\ \mathcal{E} \end{bmatrix}. \tag{1.1}$$

Communicated by V. Bolotnikov.

To state the Gohberg-Heinig inversion theorem for T we need to consider the following four equations:

$$TX = T \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad TZ = T \begin{bmatrix} z_{-n+1} \\ \vdots \\ z_{-1} \\ z_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix},$$

$$YT = \begin{bmatrix} y_0 & y_{-1} & \cdots & y_{-n+1} \end{bmatrix} T = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix},$$

$$WT = \begin{bmatrix} w_{n-1} & \cdots & w_1 & w_0 \end{bmatrix} T = \begin{bmatrix} 0 & \cdots & 0 & I \end{bmatrix}.$$

The entries in these matrices are operators on $\mathcal{E}$, and I denotes the identity operator on $\mathcal{E}$. In the case when T is invertible, X and Z are, respectively, equal to the first and last column in the $n \times n$ operator matrix representation of T^{-1}, and Y and W are, respectively, equal to the first and last row in the $n \times n$ operator matrix representation of T^{-1}. If the above equations are solvable, then it is straightforward to check (see [12]) that $x_0 = y_0$ and $z_0 = w_0$.

Now assume that these four equations are solvable, and that one of the operators x_0 and z_0 is invertible. Then the Gohberg-Heinig theorem from [12] (see also [11]) tells us that T is invertible, that both operators x_0 and z_0 are invertible, and that T^{-1} is given by the operator matrix

$$T^{-1} = \begin{bmatrix} \gamma_{0,0} & \cdots & \gamma_{0,n-1} \\ \vdots & & \vdots \\ \gamma_{n-1,0} & \cdots & \gamma_{n-1,n-1} \end{bmatrix} \quad \text{on} \quad \begin{bmatrix} \mathcal{E} \\ \vdots \\ \mathcal{E} \end{bmatrix},$$

where

$$\gamma_{j,k} = \sum_{\nu=0}^{\min\{j,k\}} x_{j-\nu} x_0^{-1} y_{\nu-k} - \sum_{\nu=1}^{\min\{j,k\}} z_{n-1+j-\nu} z_0^{-1} w_{n-1-k+\nu} \quad (j, k \geq 0).$$

For the scalar case, i.e., when $\mathcal{E} = \mathbb{C}$, this result is due to Gohberg-Semencul [18].

Solving the four equations does not require the full inverse of the operator T. In fact, in the positive definite case, one only needs two of the four equations and these can be solved recursively by using Levinson type algorithms. This is a great advantage, and the Gohberg-Semencul/Heinig inversion formula has inspired the development of fast algorithms for inversion of Toeplitz matrices, of block Toeplitz matrices, of block Toeplitz like matrices and, more generally, of structured matrices of different classes. Such algorithms are now widely used in numerical computations. The literature on this subject is extensive; here we only mention [21], [14], [16, 17], and the books [19], [24] and [25].

In the present paper we present a generalization of the Gohberg-Heinig inversion formula to a contractive operator setting. This contractive operator version will allow us to view a number of different inversion formulas from one point of view and to extend them to a somewhat more general setting.

To put the Gohberg-Heinig inversion formula in a contractive operator perspective, we first observe that in closed form the above formula for T^{-1} can be rewritten as

$$T^{-1} = \sum_{\nu=0}^{n-1} N^{\nu} \left(X x_0^{-1} Y - N Z z_0^{-1} W N^* \right) N^{*\,\nu}, \tag{1.2}$$

where N is the block lower shift on $\mathcal{E}^n$ given by the $n \times n$ operator matrix

$$N = \begin{bmatrix} 0 & & & \\ I & 0 & & \\ & \ddots & \ddots & \\ & & I & 0 \end{bmatrix}, \quad N \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ e_0 \\ \vdots \\ e_{n-2} \end{bmatrix}. \tag{1.3}$$

Moreover, an operator T on a direct sum $\mathcal{E}^n$ admits an operator matrix representation as (1.1) if and only if

$$T - N^*TN = \begin{bmatrix} 0 & \cdots & 0 & \star \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & \star \\ \star & \cdots & \star & \star \end{bmatrix},$$

where the $\star$ denotes unspecified entries. In other words, T is of the form (1.1) if and only if the compression of the operator $T - N^*TN$ to the first $n-1$ coordinate spaces is zero. The operator $T - N^*TN$ is usually referred to as a displacement operator (cf., [22], [3], and the review article [23]).

Next, note that the block lower shift N is a contraction on $\mathcal{E}^n$, and that the subspace of $\mathcal{E}^n$ spanned by the first $n-1$ coordinate spaces is precisely the orthogonal complement of the defect space $\mathcal{D}_N$ of N, and the subspace of $\mathcal{E}^n$ spanned by the last $n-1$ coordinate spaces is precisely the orthogonal complement of defect space $\mathcal{D}_{N^*}$ of N^* (see the final paragraph of the present section for the used terminology and notation). Thus the fact that T is given by a Toeplitz operator matrix is just equivalent to the requirement that the compression of the displacement operator $T - N^*TN$ to the orthogonal complement of the defect space $\mathcal{D}_N$ is zero. Furthermore, the operators X, Z, Y, W appearing in (1.2) are solutions of the following equations

$$TX = \Pi^*_{\mathcal{D}_{N^*}}, \quad TZ = \Pi^*_{\mathcal{D}_N}, \quad YT = \Pi_{\mathcal{D}_{N^*}}, \quad WT = \Pi_{\mathcal{D}_N},$$

and x_0 and z_0 in (1.2) are given by

$$x_0 = \Pi_{\mathcal{D}_{N^*}} X, \quad z_0 = \Pi_{\mathcal{D}_N} Z.$$

Here and in the sequel we use the convention that for a subspace $\mathcal{F}$ of a Hilbert space $\mathcal{H}$, the symbol $\Pi_{\mathcal{F}}$ denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{F}$, viewed as an operator from $\mathcal{H}$ onto $\mathcal{F}$, and hence $\Pi^*_{\mathcal{F}}$ is just the canonical embedding of $\mathcal{F}$ into $\mathcal{H}$.

Finally, recall that an operator A is called *exponentially stable* if $A^\nu \to 0$ in the operator norm for $\nu \to \infty$ or, equivalently, the spectrum of A is contained in the open unit disc. Since $N^n = 0$, the operator N is trivially exponentially stable.

Using these connections one sees that the following theorem, which is the first main result of this paper, is the natural analogue of the Gohberg-Heinig inversion formula in a contractive operator setting.

Theorem 1.1. *Let T be an operator on $\mathcal{X}$, and let A be an exponentially stable contraction on $\mathcal{X}$ such that the compression of the displacement operator $T - A^*TA$ to the orthogonal complement of the defect space $\mathcal{D}_A$ of A is zero. Assume that there exist operators*

$$X : \mathcal{D}_{A^*} \to \mathcal{X}, \quad Z : \mathcal{D}_A \to \mathcal{X}, \quad Y : \mathcal{X} \to \mathcal{D}_{A^*}, \quad W : \mathcal{X} \to \mathcal{D}_A \tag{1.4}$$

satisfying the equations

$$TX = \Pi^*_{\mathcal{D}_{A^*}}, \quad TZ = \Pi^*_{\mathcal{D}_A}, \quad YT = \Pi_{\mathcal{D}_{A^*}}, \quad WT = \Pi_{\mathcal{D}_A}, \tag{1.5}$$

and put $x_0 = \Pi_{\mathcal{D}_{A^*}} X$ and $z_0 = \Pi_{\mathcal{D}_A} Z$. If, in addition, one of the operators x_0 or z_0 is invertible, then the operator T is invertible, both operators x_0 and z_0 are invertible, and the inverse of T is given by

$$T^{-1} = \sum_{\nu=0}^{\infty} A^\nu \left(X x_0^{-1} Y - A Z z_0^{-1} W A^* \right) A^{*\,\nu}, \tag{1.6}$$

$$T^{-1} = \sum_{\nu=0}^{\infty} A^{*\,\nu} \left(Z z_0^{-1} W - A^* X x_0^{-1} Y A \right) A^{\nu}. \tag{1.7}$$

In general, without the exponential stability condition on A, the hypotheses in the above theorem do not yield the invertibility of the operator T, not even in the case when the underlying space $\mathcal{X}$ is finite dimensional. A counter example is given in Section 3 below. On the other hand, assuming T to be invertible, a large part of the above theorem holds true. In fact, we shall prove the following result.

Theorem 1.2. *Let T be an invertible operator on $\mathcal{X}$, and let A be a contraction on $\mathcal{X}$ such that the compression of the displacement operator $T - A^*TA$ to the orthogonal complement of the defect space $\mathcal{D}_A$ of A is zero. Consider the operators*

$$X = T^{-1}\Pi^*_{\mathcal{D}_{A^*}} : \mathcal{D}_{A^*} \to \mathcal{X}, \quad Z = T^{-1}\Pi^*_{\mathcal{D}_A} : \mathcal{D}_A \to \mathcal{X}, \tag{1.8}$$

$$Y = \Pi_{\mathcal{D}_{A^*}} T^{-1} : \mathcal{X} \to \mathcal{D}_{A^*}, \quad W = \Pi_{\mathcal{D}_A} T^{-1} : \mathcal{X} \to \mathcal{D}_A, \tag{1.9}$$

and put $x_0 = \Pi_{\mathcal{D}_{A^*}} X$ and $z_0 = \Pi_{\mathcal{D}_A} Z$. Then x_0 is invertible if and only if z_0 is invertible, and in this case the inverse of T is determined by

$$T^{-1} - AT^{-1}A^* = X x_0^{-1} Y - A Z z_0^{-1} W A^*, \tag{1.10}$$

$$T^{-1} - A^*T^{-1}A = Z z_0^{-1} W - A^* X x_0^{-1} Y A. \tag{1.11}$$

In particular,

$$T^{-1}h = \sum_{\nu=0}^{\infty} A^{\nu}\left(Xx_0^{-1}Y - AZz_0^{-1}WA^*\right)A^{*\,\nu}h,\ h\in\mathcal{X},$$

$$\textit{whenever } A^* \textit{ is pointwise stable,} \tag{1.12}$$

$$T^{-1}h = \sum_{\nu=0}^{\infty} A^{*\,\nu}\left(Zz_0^{-1}W - A^*Xx_0^{-1}YA\right)A^{\nu}h,\ h\in\mathcal{X},$$

$$\textit{whenever } A \textit{ is pointwise stable.} \tag{1.13}$$

Recall that an operator A on $\mathcal{X}$ is called *pointwise stable* if for each x in $\mathcal{X}$ the vector $A^{\nu}x \to 0$ as $\nu \to \infty$. Exponential stability implies pointwise stability, but the converse is not true. If $\mathcal{X}$ is finite dimensional, the two notions coincide, and in that case we simply say that A is *stable*. Notice that the two theorems above are of interest only when $\mathcal{D}_A$ is not the full space. In fact, the "smaller" the space $\mathcal{D}_A$ the better it is.

We shall also show (see Corollary 3.2 below) that T will be one-to-one whenever T satisfies the conditions of Theorem 1.1 with A being pointwise stable. In particular, in that case T will be invertible if T is the sum of an invertible operator and a compact operator.

Theorem 1.2 will be proved in the next section in a somewhat more general setting. The proof we shall give is inspired by the proof of the Gohberg-Heinig inversion formula as given in Section 1 of [13].

We shall illustrate our main theorems by deriving some known inversion formulas as corollaries, including a somewhat modified version of Arov's generalization of the Gohberg-Heinig formula for the model operator given in [1]. A new application will be an inversion formula for operators that are of the form block Toeplitz plus block Hankel, which have been considered in the book [4].

The paper consists of seven sections, including the present introduction. In Section 2 we prove Theorem 1.2. Section 3 contains the counter example referred to above and the proof of Theorem 1.1. In the remaining sections we illustrate our main theorems. In Section 4 we show that Theorem 1.2 covers the classical formula for the inverse of a block Toeplitz operator from [15]. Section 5 specifies our results for operators that are of block Toeplitz plus block Hankel type. In particular, we present a generalization to the non-selfadjoint case of Theorem 11.1.2 in the Ellis-Gohberg book [4]. In Section 6 we deal with model operators and Arov's generalization [1] of the Gohberg-Heinig inversion formula. In the final section we apply Theorem 1.1 to obtain an inversion formula for certain structured operators, namely for operators that satisfy Stein (discrete Lyapunov) equations appearing in metric constrained interpolation problems.

We conclude this introduction with some notation and terminology used in this paper. Throughout $\mathcal{X}$ is a Hilbert space. We write $I_{\mathcal{X}}$ (or simply I when the underlying space is clear) for the identity operator on $\mathcal{X}$. Given a subspace $\mathcal{F}$ of $\mathcal{X}$, the symbol $\mathcal{F}^{\perp}$ denotes the orthogonal complement of $\mathcal{F}$ in $\mathcal{X}$, that is,

$\mathcal{F}^\perp = \mathcal{X} \ominus \mathcal{F}$. As mentioned before, we write $\Pi_\mathcal{F}$ for the orthogonal projection operator of $\mathcal{X}$ onto $\mathcal{F}$ viewed as an operator from $\mathcal{X}$ onto $\mathcal{F}$. The operator $\Pi_\mathcal{F}^*$, the adjoint of $\Pi_\mathcal{F}$, is the canonical embedding of $\mathcal{F}$ into $\mathcal{X}$, that is $\Pi_\mathcal{F}^* = I_\mathcal{X}|_\mathcal{F}$. By definition, for an operator R on $\mathcal{X}$, the *compression* of R to the subspace $\mathcal{F}$ is the operator $\Pi_\mathcal{F}^* R \Pi_\mathcal{F}$. Finally, recall that for a contraction A on $\mathcal{X}$, the *defect operator* D_A is the positive square root of $I - A^*A$ and the *defect space* $\mathcal{D}_A$ is the closure of the range of D_A.

2. Proof of Theorem 1.2

It will be convenient first to prove a somewhat more general theorem. Assume that the Hilbert space $\mathcal{X}$ admits two orthogonal direct sum decomposition

$$\mathcal{X} = \mathcal{U}_1 \oplus \mathcal{Y}_1 = \mathcal{U}_2 \oplus \mathcal{Y}_2, \tag{2.1}$$

and let A be an operator on $\mathcal{X}$ such that relative to these decompositions A is of the form:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} \mathcal{U}_1 \\ \mathcal{Y}_1 \end{bmatrix} \to \begin{bmatrix} \mathcal{U}_2 \\ \mathcal{Y}_2 \end{bmatrix} \quad \text{where } A_2 \text{ is invertible.} \tag{2.2}$$

Next, let K be another operator on $\mathcal{X}$ of the form:

$$K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} : \begin{bmatrix} \mathcal{U}_2 \\ \mathcal{Y}_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{U}_1 \\ \mathcal{Y}_1 \end{bmatrix} \quad \text{where} \quad K_2 = A_2^{-1}. \tag{2.3}$$

It is emphasized that both A_2 and K_2 are invertible, and $K_2 = A_2^{-1}$. As before, $\Pi_\mathcal{H}$ stands for the orthogonal projection of $\mathcal{X}$ onto the subspace $\mathcal{H}$, viewed as an operator from $\mathcal{X}$ onto $\mathcal{H}$.

The next theorem contains Theorem 1.2 as a special case.

Theorem 2.1. *Let T be an invertible operator on $\mathcal{X} = \mathcal{U}_1 \oplus \mathcal{Y}_1 = \mathcal{U}_2 \oplus \mathcal{Y}_2$, and let A and K be as in* (2.2) *and* (2.3), *respectively. Assume that*

$$\Pi_{\mathcal{Y}_1} (T - KTA) \Pi_{\mathcal{Y}_1}^* = 0. \tag{2.4}$$

Consider the operators defined by

$$\begin{aligned} X &= T^{-1}\Pi_{\mathcal{U}_2}^* : \mathcal{U}_2 \to \mathcal{X}, \quad Z = T^{-1}\Pi_{\mathcal{U}_1}^* : \mathcal{U}_1 \to \mathcal{X}, & (2.5)\\ Y &= \Pi_{\mathcal{U}_2} T^{-1} : \mathcal{X} \to \mathcal{U}_2, \quad W = \Pi_{\mathcal{U}_1} T^{-1} : \mathcal{X} \to \mathcal{U}_1. & (2.6) \end{aligned}$$

Furthermore, put $x_0 = \Pi_{\mathcal{U}_2} X$ and $z_0 = \Pi_{\mathcal{U}_1} Z$. Then x_0 is invertible if and only if z_0 is invertible, and in this case the inverse of T satisfies the identities

$$\begin{aligned} T^{-1} - AT^{-1}K &= Xx_0^{-1}Y - AZz_0^{-1}WK, & (2.7)\\ T^{-1} - KT^{-1}A &= Zz_0^{-1}W - KXx_0^{-1}YA. & (2.8) \end{aligned}$$

In particular, if A and K are contractions, then

$$T^{-1}h = \sum_{n=0}^{\infty} A^n \left(Xx_0^{-1}Y - AZz_0^{-1}WK\right) K^n h, \ h \in \mathcal{X}, \quad \text{whenever } K \text{ is pointwise stable,} \tag{2.9}$$

$$T^{-1}h = \sum_{n=0}^{\infty} K^n \left(Zz_0^{-1}W - KXx_0^{-1}YA\right) A^n h, \ h \in \mathcal{X}, \quad \text{whenever } A \text{ is pointwise stable.} \tag{2.10}$$

Proof. Consider the following two operator matrix representations of T:

$$T = \begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{U}_1 \\ \mathcal{Y}_1 \end{bmatrix}, \quad T = \begin{bmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{U}_2 \\ \mathcal{Y}_2 \end{bmatrix}. \tag{2.11}$$

A simple calculation shows that

$$T - KTA = \begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{bmatrix} - \begin{bmatrix} K_1\alpha_2 A_1 & K_1\beta_2 A_2 \\ K_2\gamma_2 A_1 & K_2\delta_2 A_2 \end{bmatrix}.$$

Thus

$$\Pi_{\mathcal{Y}_1}(T - KTA)\Pi_{\mathcal{Y}_1}^* = 0 \iff \delta_1 = K_2\delta_2 A_2, \tag{2.12}$$

Next we apply Lemma 2.2 below. According to the definitions of x_0 and z_0 we have

$$T^{-1} = \begin{bmatrix} z_0 & \star \\ \star & \star \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{U}_1 \\ \mathcal{Y}_1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} x_0 & \star \\ \star & \star \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{U}_2 \\ \mathcal{Y}_2 \end{bmatrix}. \tag{2.13}$$

Here $\star$ denotes unspecified entries. By comparing the first representation of T in (2.11) with the first representation of T^{-1} in (2.13), we see that Lemma 2.2 below implies that the operator z_0 is invertible if and only if δ_1 is invertible. Analogously, using the second parts of (2.11) and (2.13), we see that x_0 is invertible if and only if δ_2 is invertible.

In what follows we assume that the hypotheses of the theorem are fulfilled, that is, (i) the compression of $T - KTA$ to $\mathcal{Y}_1$ is zero and (ii) x_0 or z_0 is invertible.

According to (2.12) assumption (i) implies that $\delta_1 = K_2\delta_2 A_2$. Note that the identity $\delta_1 = K_2\delta_2 A_2$, together with the fact that K_2 and A_2 are invertible, implies that δ_1 is invertible if and only if δ_2 is invertible. But then the result of the previous paragraph, together with assumption (ii), yields that the operators x_0 and z_0 are both invertible and that the same holds true for δ_1 and δ_2.

Since δ_1 and δ_2 are both invertible, the operator T admits the following factorizations:

$$T = \begin{bmatrix} I & \beta_k \\ 0 & \delta_k \end{bmatrix} \begin{bmatrix} \Xi_k & 0 \\ 0 & \delta_k^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ \gamma_k & \delta_k \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{U}_k \\ \mathcal{Y}_k \end{bmatrix} \quad (k = 1, 2). \tag{2.14}$$

Here $\Xi_k = \alpha_k - \beta_k\delta_k^{-1}\gamma_k$ is the Schur complement of α_k in T for $k = 1, 2$. Note that

$$\Xi_1^{-1} = \Pi_{\mathcal{U}_1} T^{-1} \Pi_{\mathcal{U}_1}^* = z_0, \quad \Xi_2^{-1} = \Pi_{\mathcal{U}_2} T^{-1} \Pi_{\mathcal{U}_2}^* = x_0. \tag{2.15}$$

Observe that the matrix factorization for T in (2.14) can also be expressed as

$$T = \begin{bmatrix} \Pi_{\mathcal{U}_k}^* & T\Pi_{\mathcal{Y}_k}^* \end{bmatrix} \begin{bmatrix} \Xi_k & 0 \\ 0 & \delta_k^{-1} \end{bmatrix} \begin{bmatrix} \Pi_{\mathcal{U}_k} \\ \Pi_{\mathcal{Y}_k} T \end{bmatrix}$$
$$= \Pi_{\mathcal{U}_k}^* \Xi_k \Pi_{\mathcal{U}_k} + T\Pi_{\mathcal{Y}_k}^* \delta_k^{-1} \Pi_{\mathcal{Y}_k} T.$$

This implies that

$$T = \Pi_{\mathcal{U}_k}^* \Xi_k \Pi_{\mathcal{U}_k} + T\Pi_{\mathcal{Y}_k}^* \delta_k^{-1} \Pi_{\mathcal{Y}_k} T \qquad (\text{for } k = 1, 2). \tag{2.16}$$

Multiplying by T^{-1} on the left and right of this equation and using the identities in (2.15), yields

$$T^{-1} = X x_0^{-1} Y + \Pi_{\mathcal{Y}_2}^* \delta_2^{-1} \Pi_{\mathcal{Y}_2} \qquad (k = 2) \tag{2.17}$$
$$T^{-1} = Z z_0^{-1} W + \Pi_{\mathcal{Y}_1}^* \delta_1^{-1} \Pi_{\mathcal{Y}_1} \qquad (k = 1). \tag{2.18}$$

Recall that $\delta_1 = K_2 \delta_2 A_2$, and thus $\delta_2^{-1} = A_2 \delta_1^{-1} K_2$. Using

$$\Pi_{\mathcal{Y}_2}^* A_2 = A\Pi_{\mathcal{Y}_1}^*, \quad K_2 \Pi_{\mathcal{Y}_2} = \Pi_{\mathcal{Y}_1} K$$

this yields $\Pi_{\mathcal{Y}_2}^* \delta_2^{-1} \Pi_{\mathcal{Y}_2} = A\Pi_{\mathcal{Y}_1}^* \delta_1^{-1} \Pi_{\mathcal{Y}_1} K$. Thus by multiplying (2.18) by A on the left and K on the right, and then subtracting the resulting identity from (2.17) we obtain the identity (2.7). Analogously, using $\delta_1^{-1} = K_2 \delta_2^{-1} A_2$, we have $\Pi_{\mathcal{Y}_1}^* \delta_1^{-1} \Pi_{\mathcal{Y}_1} = K\Pi_{\mathcal{Y}_2}^* \delta_2^{-1} \Pi_{\mathcal{Y}_2} A$. Thus by multiplying (2.17) by K on the left and A on the right, and then subtracting the resulting identity from (2.18) we arrive at the identity (2.8).

To prove the final formulas for T^{-1}, note that (2.7) and (2.8) imply that for each $n = 0, 1, 2 \ldots$ we have

$$T^{-1} - A^{n+1} T^{-1} K^{n+1} = \sum_{\nu=0}^{n} A^\nu (X x_0^{-1} Y - A Z z_0^{-1} W K) K^\nu, \tag{2.19}$$
$$T^{-1} - K^{n+1} T^{-1} A^{n+1} = \sum_{\nu=0}^{n} K^\nu (Z z_0^{-1} W - K X x_0^{-1} Y A) A^\nu. \tag{2.20}$$

By assumption A and K are contractions. Then for each $h \in \mathcal{X}$ and for n going to infinity the term $A^{n+1} T^{-1} K^{n+1} h$ tends to zero whenever K is pointwise stable and $K^{n+1} T^{-1} A^{n+1} h$ tends to zero whenever A is pointwise stable. This yields the desired formulas for the inverse of T. □

In the above proof we used the following lemma. The result is standard; see, e.g., Theorem III.4.1 in [8].

Lemma 2.2. *Let T be an invertible operator on $\mathcal{X} = \mathcal{U} \oplus \mathcal{Y}$. Then $\Pi_{\mathcal{U}} T^{-1} \Pi_{\mathcal{U}}^*$ is invertible if and only if $\Pi_{\mathcal{Y}} T \Pi_{\mathcal{Y}}^*$ is invertible.*

Proof of Theorem 1.2. Recall that D_A is the positive square root of $I - A^*A$ and $\mathcal{D}_A$ is the closure of the range of D_A. It is well known that $AD_A = D_{A^*}A$ and

$A^*D_{A^*} = D_A A^*$. Hence A maps $\mathcal{D}_A$ into $\mathcal{D}_{A^*}$ and A^* maps $\mathcal{D}_{A^*}$ into $\mathcal{D}_A$. It follows that A admits a matrix representation of the form

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} \mathcal{D}_A \\ \mathcal{D}_A^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{D}_{A^*} \\ \mathcal{D}_{A^*}^\perp \end{bmatrix}.$$

The operator A_2 is a unitary operator mapping $\mathcal{D}_A^\perp$ onto $\mathcal{D}_{A^*}^\perp$; see Lemma V.2.1 in [5]. Indeed, for $y \in \mathcal{D}_A^\perp$ we have $(I - A^*A)y = D_A D_A y = 0$, and hence

$$\|y\| = \|A^*Ay\| \leq \|A^*\|\,\|Ay\| \leq \|Ay\| \leq \|y\|,$$

because A and A^* are contractions. Thus $\|Ay\| = \|y\|$. Hence A_2 is an isometry. As $(A_2)^* = A^*|_{\mathcal{D}_{A^*}^\perp}$ an analogous reasoning shows that $(A_2)^*$ is also an isometry. Thus A_2 is unitary.

Now consider the spaces

$$\mathcal{U}_1 = \mathcal{D}_A, \quad \mathcal{Y}_1 = \mathcal{D}_A^\perp, \quad \mathcal{U}_2 = \mathcal{D}_{A^*}, \quad \mathcal{Y}_2 = \mathcal{D}_{A^*}^\perp. \tag{2.21}$$

In this setting, we take $K = A^*$. In other words, K admits a matrix representation of the form

$$K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} = \begin{bmatrix} A_1^* & 0 \\ 0 & A_2^* \end{bmatrix} : \begin{bmatrix} \mathcal{D}_{A^*} \\ \mathcal{D}_{A^*}^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{D}_A \\ \mathcal{D}_A^\perp \end{bmatrix}.$$

Since A_2 is a unitary operator, $K_2 = A_2^*$ is the inverse of A_2. By consulting Theorem 2.1, we obtain the desired formulas (1.10), (1.11), (1.12), and (1.13). □

3. Invertibility and proof of Theorem 1.1

Let A be a contraction on $\mathcal{X}$, and let T be an operator on $\mathcal{X}$ such that the compression of $T - A^*TA$ to $\mathcal{D}_A^\perp$ is zero. Assume that there exist operators

$$X : \mathcal{D}_{A^*} \to \mathcal{X}, \quad Z : \mathcal{D}_A \to \mathcal{X}, \quad Y : \mathcal{X} \to \mathcal{D}_{A^*}, \quad W : \mathcal{X} \to \mathcal{D}_A \tag{3.1}$$

such that

$$TX = \Pi^*_{\mathcal{D}_{A^*}}, \quad TZ = \Pi^*_{\mathcal{D}_A}, \quad YT = \Pi_{\mathcal{D}_{A^*}}, \quad WT = \Pi_{\mathcal{D}_A}. \tag{3.2}$$

Furthermore, assume one of the operators $x_0 = \Pi_{\mathcal{D}_{A^*}} X$ and $z_0 = \Pi_{\mathcal{D}_{A^*}} Z$ to be invertible. First we present an example with $\mathcal{X}$ finite-dimensional showing that the above assumptions do not imply that T is invertible.

Counter example. Take $\mathcal{X} = \mathbb{C}^3$, and let T and A on $\mathcal{X} = \mathbb{C}^3$ be defined by

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then A is a contraction,

$$D_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D_{A^*} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T - A^*TA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

With e_1, e_2, e_3 being the standard basis vectors of $\mathbb{C}^3$, we have

$$\mathcal{D}_A = \mathrm{span}\{e_1\}, \qquad \mathcal{D}_A^{\perp} = \mathrm{span}\{e_2, e_3\},$$

$$\mathcal{D}_{A^*} = \mathrm{span}\{e_3\}, \qquad \mathcal{D}_{A^*}^{\perp} = \mathrm{span}\{e_1, e_2\}.$$

It follows that the compression of $T - A^*TA$ to the orthogonal complement of $\mathcal{D}_A$ is equal to zero.

Next, consider the operators

$$X = \Pi_{\mathcal{D}_{A^*}}^*, \quad Z = \Pi_{\mathcal{D}_A}^*, \quad Y = \Pi_{\mathcal{D}_{A^*}}, \quad W = \Pi_{\mathcal{D}_A}.$$

Then X, Z, Y, W satisfy (3.1) and (3.2). Moreover, $x_0 := \Pi_{\mathcal{D}_{A^*}} X = I_{\mathcal{D}_{A^*}}$ and $z_0 := \Pi_{\mathcal{D}_A} Z = I_{\mathcal{D}_A}$, and thus x_0 and z_0 are both invertible. Nevertheless T is not invertible. Notice that for this example

$$\begin{aligned} Xx_0^{-1}Y - AZz_0^{-1}WA^* &= \Pi_{\mathcal{D}_{A^*}}^* \Pi_{\mathcal{D}_{A^*}}, \\ A\left(Xx_0^{-1}Y - Az_0^{-1}WA^*\right)A^* &= \Pi_{\mathcal{D}_A}^* \Pi_{\mathcal{D}_A}, \\ A^2\left(Xx_0^{-1}Y - AZz_0^{-1}WA^*\right)A^{*2} &= 0. \end{aligned}$$

Thus, the expression $\sum_{\nu=0}^{\infty} A^{\nu}\left(Xx_0^{-1}Y - AZz_0^{-1}WA^*\right)A^{*\nu}$ makes sense, although A is not pointwise stable and T is not invertible.

Next we prove Theorem 1.1. In fact, we shall prove a somewhat more general version of Theorem 1.1 by using the setting of Section 2. In other words, we have

$$\mathcal{X} = \mathcal{U}_1 \oplus \mathcal{Y}_1 = \mathcal{U}_2 \oplus \mathcal{Y}_2,$$

and A and K are operators on $\mathcal{X}$ admitting the following partitionings:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} \mathcal{U}_1 \\ \mathcal{Y}_1 \end{bmatrix} \to \begin{bmatrix} \mathcal{U}_2 \\ \mathcal{Y}_2 \end{bmatrix}, \tag{3.3}$$

$$K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} : \begin{bmatrix} \mathcal{U}_2 \\ \mathcal{Y}_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{U}_1 \\ \mathcal{Y}_1 \end{bmatrix}. \tag{3.4}$$

Furthermore, we assume that A_2 is invertible and $K_2 = A_2^{-1}$.

Theorem 3.1. *Let T be an operator on $\mathcal{X}$ such that the compression of $T - KTA$ to $\mathcal{Y}_1$ is the zero operator. Assume that there exist operators*

$$X : \mathcal{U}_2 \to \mathcal{X}, \quad Z : \mathcal{U}_1 \to \mathcal{X}, \quad Y : \mathcal{X} \to \mathcal{U}_2, \quad W : \mathcal{X} \to \mathcal{U}_1 \tag{3.5}$$

such that

$$TX = \Pi_{\mathcal{U}_2}^*, \quad TZ = \Pi_{\mathcal{U}_1}^*, \quad YT = \Pi_{\mathcal{U}_2}, \quad WT = \Pi_{\mathcal{U}_1}. \tag{3.6}$$

Furthermore, assume one of the operators $x_0 = \Pi_{\mathcal{U}_2} X$ and $z_0 = \Pi_{\mathcal{U}_1} Z$ to be invertible. Then

$$\mathrm{Ker}\, T \subset \cap_{n\geq 0} \mathrm{Ker}\, \Pi_{\mathcal{U}_1} A^n, \quad \mathrm{Im}\, T \supset \mathrm{span}_{n\geq 0}\, \mathrm{Im}\, K^n \Pi_{\mathcal{U}_1}^*. \tag{3.7}$$

In particular, the operator T is invertible if, in addition, the following identities hold:

$$\cap_{n\geq 0} \mathrm{Ker}\, \Pi_{\mathcal{U}_1} A^n = \{0\} \quad \text{and} \quad \mathrm{span}_{n\geq 0}\, \mathrm{Im}\, K^n \Pi_{\mathcal{U}_1}^* = \mathcal{X}. \tag{3.8}$$

We emphasize that in the second parts of (3.7) and (3.8) we do not take the closure but just the algebraic linear span. Let us show how Theorem 1.1 follows from Theorem 3.1.

Proof of Theorem 1.1. The fact that A is assumed to be an exponentially stable contraction implies that for some positive integer N we have

$$\mathcal{X} = \mathcal{D}_A + A^*\mathcal{D}_A + \cdots + A^{*\,N-1}\mathcal{D}_A. \tag{3.9}$$

To see this let W_n be the operator acting on $\mathcal{D}_A \oplus \cdots \oplus \mathcal{D}_A$, the Hilbert space direct sum of n copies of $\mathcal{D}_A$, defined by

$$W_n = \begin{bmatrix} D_A & A^*D_A & A^{*2}D_A & \cdots & A^{*n-1}D_A \end{bmatrix}.$$

Here n is an arbitrary positive integer. Multiplying the identity $I - A^*A = D_A^2$ form the left by A^* and from the right by A yields

$$W_nW_n^* = \sum_{j=0}^{n-1} A^{*\,j}D_A^2A^j = I - A^{*n}A^n.$$

Because A is exponentially stable, A^n converges to zero in the operator norm, and thus, $A^{*n}A^n$ also converges to zero in the operator norm. Using $W_nW_n^* = I - A^{*n}A^n$, we see that there exists an integer N such that $W_NW_N^*$ is invertible, or equivalently, W_N is onto the whole space $\mathcal{X}$. In other words, for this N the identity (3.9) holds.

Next we apply Theorem 3.1 with $K = A^*$ and with

$$\mathcal{U}_1 = \mathcal{D}_A, \quad \mathcal{Y}_1 = \mathcal{D}_A^\perp, \quad \mathcal{U}_2 = \mathcal{D}_{A^*}, \quad \mathcal{Y}_2 = \mathcal{D}_{A^*}^\perp.$$

Note that (3.9) implies that $\operatorname{span}_{n\geq 0} \operatorname{Im} A^{*\,n}\Pi_{\mathcal{U}_1}^* = \mathcal{X}$. With $K = A^*$ the latter identity is just the second part of (3.8). By taking adjoints in $\operatorname{span}_{n\geq 0} \operatorname{Im} A^{*\,n}\Pi_{\mathcal{U}_1}^* = \mathcal{X}$, we see that the first part of (3.8) is also fulfilled. Hence, according to the final statement in Theorem 3.1, the operator T is invertible. To finish the proof we just apply Theorem 1.2. □

Proof of Theorem 3.1. Throughout we assume that x_0 is invertible. The proof with z_0 invertible follows an analogous line of reasoning. Since the final statement in Theorem 3.1 is an immediate corollary of (3.7), it suffices to prove (3.7). We split the proof into two parts. In the first part we establish the first inclusion in (3.7), and in the second part we prove the second inclusion in (3.7).

Part 1. In this part we first show that $A^n\operatorname{Ker} T \subset \operatorname{Ker} T \subset \mathcal{Y}_1$. Take $\varphi \in \operatorname{Ker} T$, that is, $T\varphi = 0$. It follows that $WT\varphi = 0$. But $WT = \Pi_{\mathcal{U}_1}$. Hence $\Pi_{\mathcal{U}_1}\varphi = 0$. In other words, $\varphi \in \mathcal{Y}_1$. Thus $\operatorname{Ker} T \subset \mathcal{Y}_1$.

Again take $\varphi \in \operatorname{Ker} T$. Since $\varphi \in \mathcal{Y}_1$, the fact that $\Pi_{\mathcal{Y}_1}(T - KTA)\Pi_{\mathcal{Y}_1}^*$ is zero implies that $KTA\varphi = -(T - KTA)\varphi \in \mathcal{U}_1$. Write $TA\varphi$ as

$$TA\varphi = \Pi_{\mathcal{U}_2}^*u_2 + \Pi_{\mathcal{Y}_2}^*y_2, \quad \text{where } u_2 \in \mathcal{U}_2 \text{ and } y_2 \in \mathcal{Y}_2.$$

Then $KTA\varphi = \Pi_{\mathcal{U}_1}^*K_1u_2 + \Pi_{\mathcal{Y}_1}^*K_2y_2$, where $K_1u_2 \in \mathcal{U}_1$ and $K_2y_2 \in \mathcal{Y}_1$. But $KTA\varphi \in \mathcal{U}_1$. Thus $K_2y_2 = 0$. Recall that K_2 is invertible. It follows that $y_2 = 0$, and hence $TA\varphi = \Pi_{\mathcal{U}_2}^* u_2$. As $\varphi \in \operatorname{Ker} T$ and $\operatorname{Ker} T \subset \mathcal{Y}_1$, we have $A\varphi \in \mathcal{Y}_2$, and thus $\Pi_{\mathcal{U}_2} A\varphi = 0$. Next observe that

$$YTA\varphi = (YT)A\varphi = \Pi_{\mathcal{U}_2} A\varphi = 0,$$

$$YTA\varphi = Y(TA\varphi) = Y\Pi_{\mathcal{U}_2}^* u_2 = YTXu_2 = \Pi_{\mathcal{U}_2} Xu_2 = x_0 u_2.$$

We conclude $x_0 u_2 = 0$. But x_0 is assumed to be invertible, and therefore $u_2 = 0$. In other words, $TA\varphi = 0$.

Repeating the argument with $A\varphi$ in place of φ we see that $TA^2\varphi = 0$. Continuing in this way one proves by induction that $TA^n\varphi = 0$ for each $n \geq 0$. Hence $A^n \operatorname{Ker} T \subset \operatorname{Ker} T \subset \mathcal{Y}_1$.

From the inclusions proved so far we see that

$$\Pi_{\mathcal{U}_1} A^n \varphi = 0 \quad \text{for each } \varphi \in \operatorname{Ker} T \text{ and each } n \geq 0.$$

In other words, $\operatorname{Ker} T \subset \cap_{n\geq 0} \operatorname{Ker} \Pi_{\mathcal{U}_1} A^n$, which is the first part of (3.7).

Part 2. Let f be a linear functional on $\mathcal{X}$ such that f annihilates the range of T, that is, $fT = 0$. Note that we do not require f to be continuous. We first prove that $fKT = 0$.

From $fT = 0$ it follows that $fTZ = 0$. But $TZ = \Pi_{\mathcal{U}_1}^*$. Hence the map $f\Pi_{\mathcal{U}_1}^* = 0$. In other words, $f|_{\mathcal{U}_1} = 0$. Next, using $\Pi_{\mathcal{Y}_1}(T - KTA)\Pi_{\mathcal{Y}_1}^* = 0$, we obtain

$$\begin{aligned} f(T - KTA)\Pi_{\mathcal{Y}_1}^* &= f(P_{\mathcal{U}_1} + P_{\mathcal{Y}_1})(T - KTA)\Pi_{\mathcal{Y}_1}^* \\ &= f\Pi_{\mathcal{Y}_1}^* \Pi_{\mathcal{Y}_1}(T - KTA)\Pi_{\mathcal{Y}_1}^* = 0. \end{aligned}$$

(Here $P_{\mathcal{H}}$ denotes the orthogonal projection onto the subspace $\mathcal{H}$.) Since $fT = 0$, we conclude that $fKTA|_{\mathcal{Y}_1} = 0$. But $A\mathcal{Y}_1 = \mathcal{Y}_2$, and therefore $fKT|_{\mathcal{Y}_2} = 0$.

Next note that

$$\begin{aligned} fKTX &= fK(TX) = fK\Pi_{\mathcal{U}_2}^* = 0 \quad \text{because } K\mathcal{U}_2 \subset \mathcal{U}_1, \\ fKTX &= (fKT)X = fKT(P_{\mathcal{U}_2} + P_{\mathcal{Y}_2})X = fKTP_{\mathcal{U}_2}X \\ &= fKT\Pi_{\mathcal{U}_2}^* \Pi_{\mathcal{U}_2} X = fKT\Pi_{\mathcal{U}_2}^* x_0. \end{aligned}$$

Recall that x_0 is invertible. Hence $fKT|_{\mathcal{U}_2} = 0$. By combining this with the result of the previous paragraph we obtain $fKT = 0$.

Repeating the argument with fK in place of f we obtain $fK^2T = 0$. Continuing in this way we see by induction that $fK^nT = 0$ for each $n \geq 0$. It follows (see the beginning of the second paragraph of this part of the proof) that $fK^n\Pi_{\mathcal{U}_1}^* = 0$. Thus $fT = 0$ implies $fK^n\Pi_{\mathcal{U}_1}^* = 0$ for each $n \geq 0$.

Let us now prove the second inclusion in (3.7). Since $\operatorname{Im} T$ is a linear space, it suffices to show that $\operatorname{Im} K^n\Pi_{\mathcal{U}_1}^*$ is contained in $\operatorname{Im} T$ for each $n \geq 0$. Suppose this inclusion does not hold for some n, $n = n_\circ$, say. In that case there exists a vector $x_\circ \in \operatorname{Im} K^{n_\circ}\Pi_{\mathcal{U}_1}^*$ such that $x_\circ \notin \operatorname{Im} T$. But then (see, e.g., Theorem 2.4.2 in [27]) there exists a linear functional f on $\mathcal{X}$ such that $f(x_\circ)$ is non-zero and $fTx = 0$ for each $x \in \mathcal{X}$. However, this contradicts the conclusion from the previous paragraph.

Thus $\operatorname{Im} K^n \Pi_{\mathcal{U}_1}^*$ is contained in $\operatorname{Im} T$ for each $n \geq 0$, and the second part of (3.7) is proved. □

Corollary 3.2. *Let T be an operator on $\mathcal{X}$, and let A be a pointwise stable contraction on $\mathcal{X}$ such that the compression of the displacement operator $T - A^*TA$ to the orthogonal complement of the defect space $\mathcal{D}_A$ of A is zero. Assume that there exist operators*

$$X : \mathcal{D}_{A^*} \to \mathcal{X}, \quad Z : \mathcal{D}_A \to \mathcal{X}, \quad Y : \mathcal{X} \to \mathcal{D}_{A^*}, \quad W : \mathcal{X} \to \mathcal{D}_A$$

satisfying the equations (3.2), *and let one of the operators $x_0 = \Pi_{\mathcal{D}_{A^*}} X$ and $z_0 = \Pi_{\mathcal{D}_A} Z$ be invertible. Then the operator T is injective. Furthermore, T is invertible if, in addition, T is the sum of an invertible operator and a compact operator.*

Proof. The fact that A is a pointwise stable contraction implies that

$$h - \sum_{j=0}^{n-1} A^{*\,j} D_A^2 A^j h = A^{*n} A^n h \to 0 \quad (n \to \infty)$$

for each $h \in \mathcal{X}$. It follows that $\operatorname{span}_{n \geq 0} \operatorname{Im} A^{*\,n} \Pi_{\mathcal{D}_A}^*$ is dense in $\mathcal{X}$. In other words, $\bigcap_{n \geq 0} \operatorname{Ker} \Pi_{\mathcal{D}_A} A^n = \{0\}$. According to the first part of (3.7) in Theorem 3.1, the latter identity implies that T is injective.

Finally, if T is of the form invertible plus compact, then T is invertible if and only if T is injective. Indeed, if T is of the form invertible plus compact, then $\operatorname{Im} T$ is closed and $\dim \operatorname{Ker} T = \dim X/\operatorname{Im} T$; see Theorem 15.4.1 in [10]. This together with the result of the previous paragraph yields the final statement of this corollary. □

To conclude this section let us mention that conditions (3.8) can be understood as controllability and observability conditions. The fact that such conditions appear is expected from the theory of Toeplitz like operators as developed in the book [19]; see, e.g., Propositions 1.3 and 1.3′ in the second part of [19]. Note that in [19] displacement operators of the form $UT - TV$ are used. Here we work with displacement operators of the form $T - KTA$.

4. Toeplitz operators

Theorem 1.2 covers the classical formula for the inverse of a block Toeplitz operator from [15]. To see this let us consider the case when $A = S$ is the forward shift on $\ell_+^2(\mathcal{E})$, the Hilbert space of all square summable unilateral sequences with entries from the Hilbert space $\mathcal{E}$. In this case A is an isometry, thus D_A is the zero operator, and hence $\mathcal{D}_A^\perp$ is the full space. Thus with $A = S$ the compression of $T - A^*TA$ to $\mathcal{D}_A^\perp$ is the zero operator if and only if $T - S^*TS = 0$, that is, if and only if T is a block Toeplitz operator.

Now assume additionally that T is invertible. Since D_A is the zero operator and D_{A^*} is the orthogonal projection of $\ell^2_+(\mathcal{E})$ onto its first coordinate space, we see that we only have to consider the operators

$$X = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} = T^{-1} \begin{bmatrix} I \\ 0 \\ 0 \\ \vdots \end{bmatrix},$$

$$Y = \begin{bmatrix} y_0 & y_{-1} & y_{-2} & \cdots \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & \cdots \end{bmatrix} T^{-1}.$$

Obviously, $x_0 = y_0$. Since T is invertible, a usual Schur complement argument shows that x_0 is invertible. In this case the identity (1.10) reduces to

$$T^{-1} - AT^{-1}A^* = Xx_0^{-1}Y.$$

Since $S^* = A^*$ is a pointwise stable contraction, we get

$$T^{-1}h = \sum_{\nu=0}^{\infty} S^\nu X x_0^{-1} Y S^{*\nu} h, \quad h \in \ell^2_+(\mathcal{E}).$$

Thus

$$T^{-1} = \begin{bmatrix} \gamma_{0,0} & \gamma_{0,1} & \gamma_{0,2} & \cdots \\ \gamma_{1,0} & \gamma_{1,1} & \gamma_{1,2} & \cdots \\ \gamma_{2,0} & \gamma_{2,1} & \gamma_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad \gamma_{j,k} = \sum_{\nu=0}^{\min\{j,k\}} x_{j-\nu} x_0^{-1} y_{\nu-k} \quad (j,k \geq 0),$$

which is the classical formula for the inverse of a block Toeplitz operator from [15].

5. Toeplitz plus Hankel

In this section we will use our techniques to invert the Toeplitz plus Hankel operators occurring in Chapter 11 of the book Ellis-Gohberg [4]. Such operators act on $\ell^2_+(\mathcal{E}) \oplus \ell^2_-(\mathcal{E})$, where $\ell^2_+(\mathcal{E})$ is defined as in the previous section and $\ell^2_-(\mathcal{E})$ is a copy of $\ell^2_+(\mathcal{E})$ with the sequences ordered in the reverse direction.

Let R on $\ell^2_+(\mathcal{E})$ and V on $\ell^2_-(\mathcal{E})$ be the operators defined by the following Toeplitz operator matrices:

$$R = \begin{bmatrix} R_0 & R_{-1} & R_{-2} & \cdots \\ R_1 & R_0 & R_{-1} & \cdots \\ R_2 & R_1 & R_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad V = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots \\ \cdots & V_0 & V_{-1} & V_{-2} \\ \cdots & V_1 & V_0 & V_{-1} \\ \cdots & V_2 & V_1 & V_0 \end{bmatrix}. \tag{5.1}$$

Let G be the Hankel operator from $\ell^2_-(\mathcal{E})$ into $\ell^2_+(\mathcal{E})$ given by the operator matrix:

$$G = \begin{bmatrix} \cdots & G_2 & G_1 & G_0 \\ \cdots & G_3 & G_2 & G_1 \\ \cdots & G_4 & G_3 & G_2 \\ \cdots & \vdots & \vdots & \vdots \end{bmatrix} : \ell^2_-(\mathcal{E}) \to \ell^2_+(\mathcal{E}). \tag{5.2}$$

Notice that G starts with G_0 in the upper right-hand corner. Let H be the Hankel operator from $\ell^2_+(\mathcal{E})$ into $\ell^2_-(\mathcal{E})$ given by the operator matrix:

$$H = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ H_2 & H_3 & H_4 & \cdots \\ H_1 & H_2 & H_3 & \cdots \\ H_0 & H_1 & H_2 & \cdots \end{bmatrix} : \ell^2_+(\mathcal{E}) \to \ell^2_-(\mathcal{E}). \tag{5.3}$$

Thus H starts with H_0 in the lower left-hand corner. Now consider the operator T on $\mathcal{X}$ defined by

$$T = \begin{bmatrix} R & G \\ H & V \end{bmatrix} \text{ on } \mathcal{X} = \begin{bmatrix} \ell^2_+(\mathcal{E}) \\ \ell^2_-(\mathcal{E}) \end{bmatrix}. \tag{5.4}$$

We refer to T as a *Toeplitz plus Hankel operator*. Finally, let S_+ be the forward shift on $\ell^2_+(\mathcal{E})$, and S_- the forward shift on $\ell^2_-(\mathcal{E})$, that is,

$$S_+ = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ I & 0 & 0 & \cdots \\ 0 & I & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \text{ on } \ell^2_+(\mathcal{E}), \quad S_- = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots \\ \cdots & 0 & I & 0 \\ \cdots & 0 & 0 & I \\ \cdots & 0 & 0 & 0 \end{bmatrix} \text{ on } \ell^2_-(\mathcal{E}). \tag{5.5}$$

To present an inversion formula of Gohberg-Heinig type for Toeplitz plus Hankel operators we need some additional notation.

We define $\Pi_{\mathcal{U}_1}$ to be the operator which picks out the last component of $\ell^2_+(\mathcal{E}) \oplus \ell^2_-(\mathcal{E})$, and $\Pi_{\mathcal{U}_2}$ will be the operator which picks out the first component of $\ell^2_+(\mathcal{E}) \oplus \ell^2_-(\mathcal{E})$, that is,

$$\Pi_{\mathcal{U}_1} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & I \end{bmatrix} : \ell^2_+(\mathcal{E}) \oplus \ell^2_-(\mathcal{E}) \to \mathcal{E}, \tag{5.6}$$

$$\Pi_{\mathcal{U}_2} = \begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} : \ell^2_+(\mathcal{E}) \oplus \ell^2_-(\mathcal{E}) \to \mathcal{E}. \tag{5.7}$$

We shall need operators X, Y, Z and W of the form

$$X = \begin{bmatrix} X_+ \\ X_- \end{bmatrix} : \mathcal{U}_2 \to \begin{bmatrix} \ell^2_+(\mathcal{E}) \\ \ell^2_-(\mathcal{E}) \end{bmatrix}, \tag{5.8}$$

$$Z = \begin{bmatrix} Z_+ \\ Z_- \end{bmatrix} : \mathcal{U}_1 \to \begin{bmatrix} \ell^2_+(\mathcal{E}) \\ \ell^2_-(\mathcal{E}) \end{bmatrix}, \tag{5.9}$$

$$Y = \begin{bmatrix} Y_+ & Y_- \end{bmatrix} : \begin{bmatrix} \ell^2_+(\mathcal{E}) \\ \ell^2_-(\mathcal{E}) \end{bmatrix} \to \mathcal{U}_2, \tag{5.10}$$

$$W = \begin{bmatrix} W_+ & W_- \end{bmatrix} : \begin{bmatrix} \ell^2_+(\mathcal{E}) \\ \ell^2_-(\mathcal{E}) \end{bmatrix} \to \mathcal{U}_1. \tag{5.11}$$

Theorem 5.1. *Let T be an invertible Toeplitz plus Hankel operator of the form* (5.4), *and let the operators X, Y, Z, W in* (5.8)–(5.11) *be solutions of the following four equations*

$$TX = \Pi_{\mathcal{U}_2}^*, \quad TZ = \Pi_{\mathcal{U}_1}^*, \quad YT = \Pi_{\mathcal{U}_2}, \quad WT = \Pi_{\mathcal{U}_1}. \tag{5.12}$$

Furthermore, put

$$x_0 = \Pi_{\mathcal{U}_2} X \quad \text{and} \quad z_0 = \Pi_{\mathcal{U}_1} Z. \tag{5.13}$$

Then x_0 is invertible if and only if z_0 is invertible, and in this case a formula for T^{-1} can be obtained in the following way. Write

$$T^{-1} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \text{ on } \mathcal{X} = \begin{bmatrix} \ell_+^2(\mathcal{E}) \\ \ell_-^2(\mathcal{E}) \end{bmatrix}. \tag{5.14}$$

Then α, β, γ and δ are determined by

$$\begin{aligned} \alpha - S_+\alpha S_+^* &= X_+x_0^{-1}Y_+ - S_+Z_+z_0^{-1}W_+S_+^* \\ \beta - S_+\beta S_- &= X_+x_0^{-1}Y_- - S_+Z_+z_0^{-1}W_-S_- \\ \gamma - S_-^*\gamma S_+^* &= X_-x_0^{-1}Y_+ - S_-^*Z_-z_0^{-1}W_+S_+^* \\ \delta - S_-^*\delta S_- &= X_-x_0^{-1}Y_- - S_-^*Z_-z_0^{-1}W_-S_- \end{aligned} \tag{5.15}$$

and

$$\begin{aligned} \alpha - S_+^*\alpha S_+ &= Z_+z_0^{-1}W_+ - S_+^*X_+x_0^{-1}Y_+S_+ \\ \beta - S_+^*\beta S_-^* &= Z_+z_0^{-1}W_- - S_+^*X_+x_0^{-1}Y_-S_-^* \\ \gamma - S_-\gamma S_+ &= Z_-z_0^{-1}W_+ - S_-X_-x_0^{-1}Y_+S_+ \\ \delta - S_-\delta S_-^* &= Z_-z_0^{-1}W_- - S_-X_-x_0^{-1}Y_-S_-^* \end{aligned} \tag{5.16}$$

Because S_+^* is pointwise stable, α and γ are given by (*see* (5.15)):

$$\begin{aligned} \alpha h &= \sum_{\nu=0}^{\infty} (S_+)^\nu \left(X_+x_0^{-1}Y_+ - S_+Z_+z_0^{-1}W_+S_+^* \right) (S_+^*)^\nu h \\ \gamma h &= \sum_{\nu=0}^{\infty} (S_-^*)^\nu \left(X_-x_0^{-1}Y_+ - S_-^*Z_-z_0^{-1}W_+S_+^* \right) (S_+^*)^\nu h \end{aligned}$$

Since S_+^* is pointwise stable, β and δ are given by (*see* (5.16)):

$$\begin{aligned} \beta k &= \sum_{\nu=0}^{\infty} (S_+^*)^\nu \left(Z_+z_0^{-1}W_- - S_+^*X_+x_0^{-1}Y_-S_-^* \right) (S_-^*)^\nu k \\ \delta k &= \sum_{\nu=0}^{\infty} (S_-)^\nu \left(Z_-z_0^{-1}W_- - S_-X_-x_0^{-1}Y_-S_-^* \right) (S_-^*)^\nu k \end{aligned}$$

Here h is an arbitrary vector in $\ell_+^2(\mathcal{E})$ and k is an arbitrary vector in $\ell_-^2(\mathcal{E})$.

Proof. A simple calculation shows that the following holds

$$R = S_+^*RS_+, \quad V = S_-^*VS_-, \quad S_+^*G = GS_- \quad S_-^*H = HS_+. \tag{5.17}$$

Let A be the operator on $\mathcal{X}$ defined by

$$A = \begin{bmatrix} S_+ & 0 \\ 0 & S_-^* \end{bmatrix} \text{ on } \begin{bmatrix} \ell_+^2(\mathcal{E}) \\ \ell_-^2(\mathcal{E}) \end{bmatrix}. \tag{5.18}$$

Set $K = A^*$. Consider the subspaces

$$\mathcal{U}_1 = \operatorname{Ker} A, \quad \mathcal{Y}_1 = (\operatorname{Ker} A)^\perp, \quad \mathcal{U}_2 = \operatorname{Ker} A^* \quad \mathcal{Y}_2 = (\operatorname{Ker} A^*)^\perp.$$

Notice that $\mathcal{U}_1$ is the subspace of $\mathcal{X}$ obtained by embedding $\mathcal{E}$ in the last component of $\mathcal{X}$, while $\mathcal{U}_2$ is the subspace of $\mathcal{X}$ obtained by embedding $\mathcal{E}$ in the first component of $\mathcal{X}$. Moreover, $\Pi_{\mathcal{U}_1}$ and $\Pi_{\mathcal{U}_2}$ are given by (5.6) and (5.7). So $\Pi_{\mathcal{U}_1}^*$ embeds $\mathcal{E}$ into the last component of $\mathcal{X} = \ell_+^2(\mathcal{E}) \oplus \ell_-^2(\mathcal{E})$, while $\Pi_{\mathcal{U}_2}^*$ embeds $\mathcal{E}$ into the first component of $\mathcal{X}$. Observe that

$$\Pi_{\mathcal{Y}_1} = \begin{bmatrix} I & 0 \\ 0 & S_-^* \end{bmatrix} : \begin{bmatrix} \ell_+^2(\mathcal{E}) \\ \ell_-^2(\mathcal{E}) \end{bmatrix} \to \begin{bmatrix} \ell_+^2(\mathcal{E}) \\ \ell_-^2(\mathcal{E}) \end{bmatrix}.$$

Since A is a partial isometry, $A_2 = A|_{\mathcal{Y}_1}$ is a unitary operator mapping $(\operatorname{Ker} A)^\perp$ onto $\operatorname{Im} A$. In particular, A_2 is invertible. This allows us to apply Theorem 2.1.

Notice that $GS_- = \widetilde{G}$, where $\widetilde{G}$ is the Hankel operator matrix determined by replacing G_j with G_{j+1} in (5.2). Observe that G_1 in the upper right-hand corner of $\widetilde{G}$. Moreover, $HS_+ = \widetilde{H}$ where $\widetilde{H}$ is the Hankel operator matrix determined by replacing H_j with H_{j+1} in (5.3). The operator H_1 appears in the lower left-hand corner of $\widetilde{H}$. Using $V = S_-^* V S_-$, we arrive at

$$\begin{aligned} \Pi_{\mathcal{Y}_1} T \Pi_{\mathcal{Y}_1}^* &= \begin{bmatrix} I & 0 \\ 0 & S_-^* \end{bmatrix} \begin{bmatrix} R & G \\ H & V \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & S_- \end{bmatrix} \\ &= \begin{bmatrix} R & GS_- \\ HS_+ & S_-^* V S_- \end{bmatrix} = \begin{bmatrix} R & \widetilde{G} \\ \widetilde{H} & V \end{bmatrix} =: \widetilde{T}. \end{aligned}$$

The operator $\widetilde{T}$ is the Toeplitz plus Hankel operator defined by the last equality. Furthermore, we have

$$\begin{aligned} A^* T A &= \begin{bmatrix} S_+^* & 0 \\ 0 & S_- \end{bmatrix} \begin{bmatrix} R & G \\ H & V \end{bmatrix} \begin{bmatrix} S_+ & 0 \\ 0 & S_-^* \end{bmatrix} \\ &= \begin{bmatrix} S_+^* R S_+ & S_+^* G S_-^* \\ S_- H S_+ & S_- V S_-^* \end{bmatrix} = \begin{bmatrix} R & GS_- S_-^* \\ S_- H S_+ & S_- V S_-^* \end{bmatrix} \\ &= \begin{bmatrix} R & \widetilde{G} S_-^* \\ S_- \widetilde{H} & S_- V S_-^* \end{bmatrix} = \Pi_{\mathcal{Y}_1}^* \widetilde{T} \Pi_{\mathcal{Y}_1}. \end{aligned}$$

This readily shows that

$$A^* T A = \Pi_{\mathcal{Y}_1}^* \widetilde{T} \Pi_{\mathcal{Y}_1}. \tag{5.19}$$

Using the fact that $\widetilde{T} = \Pi_{\mathcal{Y}_1} T \Pi_{\mathcal{Y}_1}^*$ with $I = \Pi_{\mathcal{Y}_1} \Pi_{\mathcal{Y}_1}^*$, we obtain

$$\Pi_{\mathcal{Y}_1} (T - A^* T A) \Pi_{\mathcal{Y}_1}^* = \Pi_{\mathcal{Y}_1} T \Pi_{\mathcal{Y}_1}^* - \Pi_{\mathcal{Y}_1} A^* T A \Pi_{\mathcal{Y}_1}^* = \widetilde{T} - \Pi_{\mathcal{Y}_1} \Pi_{\mathcal{Y}_1}^* \widetilde{T} \Pi_{\mathcal{Y}_1} \Pi_{\mathcal{Y}_1}^* = 0.$$

Thus $\Pi_{\mathcal{Y}_1} (T - A^* T A) \Pi_{\mathcal{Y}_1}^* = 0$.

Finally, note that x_0 and z_0 are the compressions of T^{-1} to $\mathcal{U}_2$ and $\mathcal{U}_1$, respectively. Since one of the operators x_0 and z_0 is assumed to be invertible, Theorem 2.1 shows that both are invertible and gives the desired inversion formulas. To see this one uses the block matrix representations in (5.8)–(5.11) for the operators X, Y, Z and W defined by (5.12). Then (5.15) and (5.16) follow from equations (1.10) and (1.11) in Theorem 1.2. □

The next proposition extends Theorem 11.1.2 in [4] to the non-selfadjoint setting.

Proposition 5.2. *Let T be a Toeplitz plus Hankel operator of the form* (5.4)*, and assume that the Toeplitz operators R and V are invertible and that the Hankel operators G and H are compact. Furthermore, assume there exist operators X, Y, Z, W as in* (5.8)–(5.11) *satisfying the equations*

$$TX = \Pi_{\mathcal{U}_2}^*, \quad TZ = \Pi_{\mathcal{U}_1}^*, \quad YT = \Pi_{\mathcal{U}_2}, \quad WT = \Pi_{\mathcal{U}_1}. \tag{5.20}$$

If, in addition, one of the operators $x_0 = \Pi_{\mathcal{U}_2} X$ and $z_0 = \Pi_{\mathcal{U}_1} Z$ is invertible. Then T is invertible.

Proof. In what follows we use freely the notations introduced in the first paragraph of the proof of Theorem 5.1. First note that

$$\cap_{n\geq 0}\mathrm{Ker}\, \Pi_{\mathcal{U}_1} A^n = \begin{bmatrix} \ell_+^2(\mathcal{E}) \\ 0 \end{bmatrix} \subset \begin{bmatrix} \ell_+^2(\mathcal{E}) \\ \ell_-^2(\mathcal{E}) \end{bmatrix}.$$

According to the first part of (3.7) we have $\mathrm{Ker}\, T \subset \cap_{n\geq 0}\mathrm{Ker}\, \Pi_{\mathcal{U}_1} A^n$. Thus

$$\phi = \begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix} \in \mathrm{Ker}\, T \ \Rightarrow\ \phi_- = 0, \text{ and hence } \begin{bmatrix} R\phi_+ \\ H\phi_+ \end{bmatrix} = T\phi = 0.$$

Since R is assumed to be invertible, we conclude that $\phi_+ = 0$. But then $\phi = 0$. Thus T is injective.

Next note that R, V invertible and G, H compact imply that T is the sum of an invertible operator and a compact operator. Hence T is injective yields T is invertible. □

Theorem 5.1 and Proposition 5.2 have natural analogues for the case when R and V in (5.4) are finite block Toeplitz matrices and G and H in (5.4) are finite block Hankel matrices. For this case Theorem 1.1 yields a result of the type appearing in Section II.2.2 of [19]; we omit the details. See [14] for related numerical aspects.

6. Compressions of a Toeplitz operator

In this section we show that operators T of the type appearing in Theorems 1.1 and 1.2 naturally occur when a (block) Toeplitz operator is compressed to a subspace invariant under the backward shift. In the first subsection, this idea is presented in the abstract setting of isometric liftings. In the second subsection, we treat a special model case studied by Arov [1] and extend it to a non-selfadjoint setting.

6.1. The isometric lifting setting

Let A be a contraction on a Hilbert space $\mathcal{X}$, and let V on $\mathcal{K} = \mathcal{X} \oplus \mathcal{H}$ be an isometric lifting of A, that is, V is an isometry and the operator matrix of V relative to the decomposition $\mathcal{K} = \mathcal{X} \oplus \mathcal{H}$ is of the form

$$V = \begin{bmatrix} A & 0 \\ C & F \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{X} \\ \mathcal{H} \end{bmatrix}. \tag{6.1}$$

Here C is an operator mapping $\mathcal{X}$ into $\mathcal{H}$, and F is an operator on $\mathcal{H}$. The fact that V is an isometry implies that the same holds true for F. We say that B is a *Toeplitz operator with respect to* V if B is an operator on $\mathcal{K} = \mathcal{X} \oplus \mathcal{H}$ satisfying $B = V^*BV$.

Proposition 6.1. *Let A be a contraction on $\mathcal{X}$, and let T on $\mathcal{X}$ be the compression of a Toeplitz operator B with respect to V, where V is the isometric lifting of A in* (6.1). *Then*

$$\Pi_{\mathcal{D}_A^\perp} (T - A^*TA)\, \Pi^*_{\mathcal{D}_A^\perp} = 0. \tag{6.2}$$

If, in addition, T is invertible, then its inverse may be obtained by the formulas in Theorem 1.2.

Proof. Since T is the compression of B to $\mathcal{X}$, the operator B admits a matrix representation of the form:

$$B = \begin{bmatrix} T & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{X} \\ \mathcal{H} \end{bmatrix}. \tag{6.3}$$

Using the fact that $B = V^*BV$, we have

$$\begin{aligned}
\begin{bmatrix} T & B_{12} \\ B_{21} & B_{22} \end{bmatrix} &= V^* \begin{bmatrix} T & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} A & 0 \\ C & F \end{bmatrix} \\
&= \begin{bmatrix} A^* & C^* \\ 0 & F^* \end{bmatrix} \begin{bmatrix} TA + B_{12}C & B_{12}F \\ B_{21}A + B_{22}C & B_{22}F \end{bmatrix} \\
&= \begin{bmatrix} A^*TA + A^*B_{12}C + C^*B_{21}A + C^*B_{22}C & A^*B_{12}F + C^*B_{22}F \\ F^*B_{21}A + F^*B_{22}C & F^*B_{22}F \end{bmatrix}.
\end{aligned}$$

By matching the $(1,1)$-entry of these 2×2 matrices, we obtain

$$T = A^*TA + A^*B_{12}C + C^*B_{21}A + C^*B_{22}C. \tag{6.4}$$

As V is an isometry, we have $A^*A + C^*C = I_{\mathcal{X}}$, and hence $C^*C = D_A^2$. It follows that $\mathcal{D}_A^\perp = \operatorname{Ker} C$. By consulting (6.4), we obtain (6.2). □

In the sequel, given a Hilbert space $\mathcal{E}$, the symbol $H^2(\mathcal{E})$ denotes the Hardy space of $\mathcal{E}$-valued analytic functions on the open unite disc $\mathbb{D}$ with square summable $\mathcal{E}$-valued Taylor coefficients. Furthermore, $\mathcal{L}(\mathcal{E}, \mathcal{E})$ stands for the space of all bounded linear operators on $\mathcal{E}$.

6.2. The model case

In this subsection, $\mathcal{X} = H^2(\mathcal{E}) \ominus mH^2(\mathcal{E})$, where m is a scalar-valued inner function and $\mathcal{E}$ is a Hilbert space. Let A on $\mathcal{X}$ be the compression of the unilateral shift S on $H^2(\mathcal{E})$ to $\mathcal{X}$, that is, $A = \Pi_{\mathcal{X}} S|_{\mathcal{X}}$. Notice that S is an isometric lifting of A. Since S^* is pointwise stable and $A^* = S^*|_{\mathcal{X}}$, we see that A^* is also pointwise stable.

Now let B be any Toeplitz operator on $H^2(\mathcal{E})$ and T on $\mathcal{X}$ the compression of B to $\mathcal{X}$, that is $T = \Pi_{\mathcal{X}} B|_{\mathcal{X}}$. By Proposition 6.1, the compression T of B satisfies the identity

$$\Pi_{\mathcal{D}_A^\perp} (T - A^*TA)\, \Pi_{\mathcal{D}_A^\perp}^* = 0.$$

When T is invertible, we can apply Theorem 1.2.

It is well known that $\mathcal{D}_A$, $\mathcal{D}_{A^*}$ and $\mathcal{E}$ are unitarily equivalent. This fact allows us to rewrite the solutions X, Y, Z and W of the four equations in (1.5) as analytic functions with values in $\mathcal{E}$. To be more specific, there exists an isometry φ from $\mathcal{E}$ into $\mathcal{X}$ mapping $\mathcal{E}$ onto $\mathcal{D}_A$ and an isometry ϕ from $\mathcal{E}$ into $\mathcal{X}$ mapping $\mathcal{E}$ onto $\mathcal{D}_{A^*}$. In fact, two such isometries are given by

$$\begin{aligned} (\varphi a)(\lambda) &= \frac{1 - m(\lambda)m(0)}{\sqrt{1 - |m(0)|^2}}\, a \qquad (a \in \mathcal{E}) \\ (\phi a)(\lambda) &= \frac{m(\lambda) - m(0)}{\lambda\sqrt{1 - |m(0)|^2}}\, a \qquad (a \in \mathcal{E}); \end{aligned}$$

see Section XIV.8 in [5]. Assume that $X, Z : \mathcal{E} \to \mathcal{X}$ and $Y, W : \mathcal{X} \to \mathcal{E}$ are operators satisfying the equations

$$TX = \phi, \quad TZ = \varphi, \quad YT = \phi^*, \quad WT = \varphi^*. \tag{6.5}$$

Furthermore, put

$$x_0 = \phi^* X, \quad \text{and} \quad z_0 = \varphi^* Z. \tag{6.6}$$

Notice that the operators X, Z, Y, W, x_0, z_0 in (6.5) and (6.6) are unitarily equivalent to the corresponding operators in Theorems 1.1 and 1.2.

In order to restate Theorems 1.1 and 1.2 for the present setting we need some additional notation. Since $\mathcal{X} = H^2(\mathcal{E}) \ominus mH^2(\mathcal{E})$ is a subspace of $H^2(\mathcal{E})$, any operator F from $\mathcal{E}$ into $\mathcal{X}$ can be identified in a canonical way with a function $F(\cdot)$, analytic on the open unit disc, with values in $\mathcal{L}(\mathcal{E}, \mathcal{E})$, via the formula

$$F(\lambda)a = (Fa)(\lambda), \quad (\lambda \in \mathbb{D},\ a \in \mathcal{E}). \tag{6.7}$$

We say that the operator F from $\mathcal{E}$ into $\mathcal{X}$ *generates an H^∞-function* if this associate function $F(\cdot)$ is uniformly bounded in the operator norm on the open unit disc. In that case M_F will the operator of multiplication by $F(\cdot)$, acting on $H^2(\mathcal{E})$, that is, $(M_F h)(\lambda) = F(\lambda)h(\lambda)$ for each h in $H^2(\mathcal{E})$. The compression of this operator to $\mathcal{X}$ will be denoted by Λ_F. Thus $\Lambda_F = \Pi_{\mathcal{X}} M_F|_{\mathcal{X}}$. The fact that the inner function m is assumed to be scalar implies that the space $mH^2(\mathcal{E})$ is invariant under M_F, and hence

$$\Pi_{\mathcal{X}} M_F = \Lambda_F \Pi_{\mathcal{X}}. \tag{6.8}$$

Next, let E_0 be the canonical embedding of $\mathcal{E}$ into $H^2(\mathcal{E})$ defined by $(E_0a)(\cdot) \equiv a$ for each $a \in \mathcal{E}$, and let u be any operator on $\mathcal{E}$. Then E_0u is an operator from $\mathcal{E}$ into $H^2(\mathcal{E})$ and, trivially, E_0u generates an H^∞-function. The corresponding operator of multiplication M_{E_0u} acts as a block diagonal operator. The compression of M_{E_0u} to $\mathcal{X}$ will be denoted by $\Delta(u)$, that is, $\Delta(u) = \Lambda_{E_0u}$. If u is an invertible operator on $\mathcal{E}$, then $\Delta(u)$ is also invertible and $\Delta(u)^{-1} = \Delta(u^{-1})$.

We are now ready to state the analogue of Theorem 1.2 for the model case.

Proposition 6.2. *Let $\mathcal{X} = H^2(\mathcal{E}) \ominus mH^2(\mathcal{E})$, where m is a scalar-valued inner function. Let T on $\mathcal{X}$ be the compression of a Toeplitz operator B on $H^2(\mathcal{E})$ to $\mathcal{X}$, and assume that T is invertible. Furthermore, assume that X, Z, Y^* and W^*, where X, Z, Y and W are the operators given by* (6.5)*, all generate H^∞-functions. If, in addition, x_0 or z_0 is invertible, then both x_0 and z_0 are invertible, and the inverse of T is given by*

$$T^{-1} = \Lambda_X \Delta(x_0)^{-1} \Lambda^*_{Y^*} - A\Lambda_Z \Delta(z_0)^{-1} \Lambda^*_{W^*} A^*. \tag{6.9}$$

Here A is the compression of the unilateral shift S on $H^2(\mathcal{E})$ to $\mathcal{X}$.

The above result is a mild generalization of the Gohberg-Heinig type inversion formula in Arov [1]. Note that in Arov's paper [1] the operator T is assumed to be strictly positive. On the other hand, in [1] there is an interesting additional condition on T that allows one to work with H^∞-functions. See also Proposition 6.3 below.

Proof. Due to the unitary equivalence between the operators in (6.5) and the corresponding operators in Theorem 1.2, induced by ϕ and φ, we only have to derive (6.9). Note that in the present setting equation (1.13) becomes

$$T^{-1} - AT^{-1}A^* = Xx_0^{-1}Y - AZz_0^{-1}WA^*. \tag{6.10}$$

Since A^* is pointwise stable, we have

$$T^{-1}h = \sum_{k=0}^{\infty} A^k \left(Xx_0^{-1}Y - AZz_0^{-1}WA^*\right) A^{*\,k}h, \quad h \in \mathcal{X}. \tag{6.11}$$

To write T^{-1} in the desired form (6.9), we use the fact that X, Z, Y^*, and W^* generate H^∞-functions. In what follows F is one of these operators, and we use freely the notations introduced in the second paragraph preceding the present proposition. Thus F maps $\mathcal{E}$ into $\mathcal{X}$ and F generates an H^∞-function. Recall that E_0 is the canonical embedding of $\mathcal{E}$ into $H^2(\mathcal{E})$ defined by $(E_0a)(\cdot) \equiv a$ for each $a \in \mathcal{E}$. It follows that $F = \Pi_{\mathcal{X}} M_F E_0$, and hence (6.8) yields $F = \Lambda_F \Pi_{\mathcal{X}} E_0$. Thus we have

$$X = \Lambda_X \Pi_{\mathcal{X}} E_0, \quad Z = \Lambda_Z \Pi_{\mathcal{X}} E_0, \quad Y = E_0^* \Pi_{\mathcal{X}}^* \Lambda^*_{Y^*}, \quad W = E_0^* \Pi_{\mathcal{X}}^* \Lambda^*_{W^*},$$

and the right-hand side of (6.10) can be rewritten as

$$\begin{aligned} &Xx_0^{-1}Y - AZz_0^{-1}WA^* \\ &\quad = \Lambda_X \Pi_{\mathcal{X}} E_0 x_0^{-1} E_0^* \Pi_{\mathcal{X}}^* \Lambda^*_{Y^*} - A\Lambda_Z \Pi_{\mathcal{X}} E_0 z_0^{-1} E_0^* \Pi_{\mathcal{X}}^* \Lambda^*_{W^*} A^*. \end{aligned} \tag{6.12}$$

Next we use that $A = \Pi_{\mathcal{X}} S \Pi_{\mathcal{X}}^*$, where S is the unilateral shift on $H^2(\mathcal{E})$. Since S leaves $mH^2(\mathcal{E})$ invariant, $A\Pi_{\mathcal{X}} = \Pi_{\mathcal{X}} S$. This implies that

$$A\Lambda_F \Pi_{\mathcal{X}} = \Lambda_F \Pi_{\mathcal{X}} S. \tag{6.13}$$

Indeed, $A\Lambda_F\Pi_{\mathcal{X}} = A\Pi_{\mathcal{X}} M_F = \Pi_{\mathcal{X}} S M_F = \Pi_{\mathcal{X}} M_F S = \Lambda_F \Pi_{\mathcal{X}} S$. Using (6.13) with X, Z, Y^*, or W^* in place of F we obtain

$$\begin{aligned} &A^k \left(\Lambda_X \Pi_{\mathcal{X}} E_0 x_0^{-1} E_0^* \Pi_{\mathcal{X}}^* \Lambda_{Y^*}^*\right) A^{*\,k} \\ &\qquad = \Lambda_X \Pi_{\mathcal{X}} \left(S^k E_0 x_0^{-1} E_0^* S^{*\,k}\right) \Pi_{\mathcal{X}}^* \Lambda_{Y^*}^*, \qquad (6.14) \\ &A^k \left(A\Lambda_Z \Pi_{\mathcal{X}} E_0 z_0^{-1} E_0^* \Pi_{\mathcal{X}}^* \Lambda_{W^*}^* A^*\right) A^{*\,k} \\ &\qquad = A\Lambda_Z \Pi_{\mathcal{X}} \left(S^k E_0 z_0^{-1} E_0^* S^{*\,k}\right) \Pi_{\mathcal{X}}^* \Lambda_{W^*}^* A^*, \qquad (6.15) \end{aligned}$$

for $k = 0, 1, 2, \ldots$. Finally, note that

$$\sum_{k=0}^{\infty} \Pi_{\mathcal{X}} S^k E_0 u E_0^* S^{*\,k} \Pi_{\mathcal{X}}^* = \Delta(u) \quad \text{for } u = x_0^{-1} \text{ or } u = z_0^{-1}, \tag{6.16}$$

with pointwise convergence. Using the identities (6.12), (6.14), (6.15) and (6.16) in (6.11) we obtain the desired formula (6.9). □

Proposition 6.3. *Let $\mathcal{X} = H^2(\mathcal{E}) \ominus mH^2(\mathcal{E})$, where m is a scalar finite Blaschke product. Let T on $\mathcal{X}$ be the compression of a Toeplitz operator B on $H^2(\mathcal{E})$ to $\mathcal{X}$, and assume that there exist operators $X, Z : \mathcal{E} \to \mathcal{X}$ and $Y, W : \mathcal{X} \to \mathcal{E}$ satisfying the equations* (6.5)*. Furthermore, let one the operators $x_0 = \phi^* X$ or $z_0 = \varphi^* Z$ be invertible. Then the operator T is invertible and the operators X, Z, Y^* and W^* generate H^∞-functions.*

Proof. Recall that m is the minimal function for A; see Sz.-Nagy-Foias [26]. In particular, $m(A) = 0$. If p is the polynomial formed by the numerator for m, then all the zeros of p are contained in the open unit disc and $p(A) = 0$. Because $p(A) = 0$, the spectral mapping theorem (cf., Exercise 4 to Part I in [8]) implies that the spectrum of A consist of eigenvalues contained in the zeros of p. In particular, the spectrum of A is in the open unit disc. Hence A is exponentially stable, and we can apply Theorem 1.1 to show that T is invertible.

Since m is a scalar finite Blaschke product, there exists $r > 1$ such that the space $\mathcal{X} = H^2(\mathcal{E}) \ominus mH^2(\mathcal{E})$ consists of $\mathcal{E}$-valued rational functions that are analytic on the disc $|\lambda| < r$; see Section X.1 in [5]. It follows that for each operator $F : \mathcal{E} \to \mathcal{X}$ the $\mathcal{L}(\mathcal{E}, \mathcal{E})$-valued function $F(\cdot\,)$ defined by (6.7) is analytic on $|\lambda| < r$. In particular, such a function $F(\cdot\,)$ is uniformly bounded on $\mathbb{D}$, and hence F generates an H^∞-function. It follows the operators X, Z, Y^* and W^* generate H^∞-functions. □

To conclude this section we note that for $m(\lambda) = \lambda^n$, Propositions 6.2 and Proposition 6.3 yield the classical Gohberg-Heinig inversion result discussed in Section 1.

7. Inverting solutions of Stein equations

In this section we use Theorem 1.1 to derive the inverse of an operator R satisfying the following Stein equation (discrete Lyapunov equation):

$$R - A^* RA = \Psi C + C^* \Upsilon. \tag{7.1}$$

Here A is an exponentially stable operator on a Hilbert space $\mathcal{X}$, and C is an operator mapping $\mathcal{X}$ into a Hilbert space $\mathcal{Y}$. Furthermore, Υ and Ψ are operators mapping $\mathcal{X}$ into $\mathcal{Y}$ and $\mathcal{Y}$ into $\mathcal{X}$, respectively. Without loss of generality we shall assume that the range of C is dense in $\mathcal{Y}$, that is, C^* is one-to-one.

Operator equations of the form (7.1) appear naturally when solving interpolation problems of Nevanlinna-Pick and Carathéodory-Toeplitz type; see, e.g., Chapters 18 and 22 in [2], where the spaces $\mathcal{X}$ and $\mathcal{Y}$ are finite dimensional, or Chapter 1 of [6], where $\mathcal{X}$ and $\mathcal{Y}$ are allowed to be infinite dimensional (see also [20] and [7]). In the interpolation setting the operator R represents the Carathéodory-Pick operator. When $\Upsilon = \Psi^*$, equation (7.1) is usually referred to as a symmetric Stein equation (see [2], page 578). Notice that (6.4) is also an equation of the form (7.1).

The identity (7.1) implies that the compression of $R - A^*RA$ to $\operatorname{Ker} C$ is the zero operator. Conversely, if the latter holds true and $\operatorname{Im} C = \mathcal{Y}$, then (7.1) is satisfied for a suitable choice of Ψ and Υ. In what follows the particular choice of Ψ and Υ does not play a role.

We do not assume that A is contractive. However, we require the operator $Q = \sum_{\nu=0}^{\infty} A^{*\,\nu} C^* C A^{\nu}$ to be strictly positive. Since A is exponentially stable, the operator Q is well defined and is the unique solution to the Stein equation

$$Q - A^* QA = C^* C. \tag{7.2}$$

In the case when the space $\mathcal{X}$ is finite dimensional and the operator A is stable, the existence of a strictly positive solution Q to (7.2) is equivalent to the requirement that the pair (C, A) is observable.

The condition that Q is strictly positive and satisfies (7.2) is equivalent to the requirement that the operator $Q^{1/2} A Q^{-1/2}$ is a contraction. In other words, the operator A is assumed to be a contraction with respect to the inner product $[x, x'] = \langle Qx, x' \rangle$, where $\langle x, x' \rangle$ is the original inner product on $\mathcal{X}$. Note that the two inner products $[\cdot,\cdot]$ and $\langle \cdot,\cdot \rangle$ are equivalent.

Since the adjoint of a contraction is again a contraction, it follows that $Q^{-1/2} A^* Q^{1/2}$ is a contraction and thus the operator $Q^{-1} - AQ^{-1}A^*$ is non-negative. So there exists a one-to-one operator B mapping $\mathcal{U}$ into $\mathcal{X}$ such that

$$Q^{-1} - AQ^{-1}A^* = BB^*. \tag{7.3}$$

In the sequel we assume additionally that the operator C maps the space $\mathcal{X}$ onto $\mathcal{Y}$. Since the range of C is assumed to be dense in $\mathcal{Y}$, this condition is automatically fulfilled in the case when $\mathcal{Y}$ is finite dimensional. The condition $\operatorname{Im} C = \mathcal{Y}$ implies that the operator B in (7.3) has closed range. To see this, note that $\operatorname{Im} C = \mathcal{Y}$ implies that the defect operator of the contraction $Q^{1/2}AQ^{-1/2}$

has closed range. But then the defect operator of the adjoint of $Q^{-1} - AQ^{-1}A^*$ has closed range too. Thus the range of the operator $Q^{-1} - AQ^{-1}A^*$ is closed, and hence the range of B is closed. Therefore in what follows we have

$$Q - A^*QA = C^*C \text{ where } C : \mathcal{X} \to \mathcal{Y} \text{ is onto,} \tag{7.4}$$

$$Q^{-1} - AQ^{-1}A^* = BB^* \text{ where } B : \mathcal{U} \to \mathcal{X} \text{ is one-to-one and has closed range.} \tag{7.5}$$

The following result is the analogue of Theorem 1.1 for the case considered here.

Theorem 7.1. *Let Q be a strictly positive operator on $\mathcal{X}$, and let C and B be such that* (7.4) *and* (7.5) *hold. Assume that R is a solution to the Stein equation* (7.1) *with A being exponentially stable, and assume that there exist operators*

$$F : \mathcal{U} \to \mathcal{X}, \quad H : \mathcal{Y} \to \mathcal{X}, \quad G : \mathcal{X} \to \mathcal{U}, \quad K : \mathcal{X} \to \mathcal{Y} \tag{7.6}$$

satisfying the equations

$$RF = QB, \quad RH = C^*, \quad GR = B^*Q, \quad KR = C. \tag{7.7}$$

Then $B^*QF = GQB$ and $CH = KC^*$. If, in addition, one of the operators B^*QF and CH is invertible, then R is invertible, both B^*QF and CH are invertible, and the inverse of R is given by

$$R^{-1} = \sum_{n=0}^{\infty} A^n \left(F\,(B^*QF)^{-1}\,G - AH\,(CH)^{-1}\,KA^* \right) A^{*n}. \tag{7.8}$$

Proof. We split the proof into two parts. First we prove the theorem for the special case when Q is the identity operator on $\mathcal{X}$. In the second part we reduce the general case to this special case.

Part 1. Assume the hypotheses of the theorem are fulfilled for $Q = I_{\mathcal{X}}$. From (7.4) with $Q = I_{\mathcal{X}}$ it follows that $D_A^2 = C^*C$. Since the range of C is equal to $\mathcal{Y}$, we have $\operatorname{Im} D_A^2 = \operatorname{Im} C^*$. This implies that

$$\operatorname{Ker} D_A = \operatorname{Ker} D_A^2 = (\operatorname{Im} D_A^2)^\perp = (\operatorname{Im} C^*)^\perp = \operatorname{Ker} C.$$

Hence $\mathcal{D}_A^\perp = \operatorname{Ker} C$. But then the identity (7.1) shows that the compression of $R - A^*RA$ to $\mathcal{D}_A^\perp$ is the zero operator. Thus Theorem 1.1 is applicable with $T = R$.

Since C is onto and B is one-to-one with closed range, the operators CC^* and B^*B are invertible. This allows us to introduce the following auxiliary operators:

$$E : \mathcal{Y} \to \mathcal{X}, \qquad E = C^*(CC^*)^{-1/2}; \tag{7.9}$$

$$E_\star : \mathcal{U} \to \mathcal{X}, \quad E_\star = B(B^*B)^{-1/2}. \tag{7.10}$$

From the properties of C and B it follows that both E and $E_\star$ are isometries, the range of E is equal to $\mathcal{D}_A$ and the range of $E_\star$ is equal to $\mathcal{D}_{A^*}$. In particular, EE^* and $E_\star E_\star^*$ are the orthogonal projections on $\mathcal{D}_A$ and $\mathcal{D}_{A^*}$, respectively. Now, define

$$X : \mathcal{D}_{A^*} \to \mathcal{X}, \quad Z : \mathcal{D}_A \to \mathcal{X}, \quad Y : \mathcal{X} \to \mathcal{D}_{A^*}, \quad W : \mathcal{X} \to \mathcal{D}_A$$

by setting

$$X = F(B^*B)^{-1/2}E_\star^*\Pi_{\mathcal{D}_{A^*}}^*, \qquad Z = H(CC^*)^{-1/2}E^*\Pi_{\mathcal{D}_A}^*,$$
$$Y = \Pi_{\mathcal{D}_{A^*}}E_\star(B^*B)^{-1/2}G, \quad W = \Pi_{\mathcal{D}_A}E(CC^*)^{-1/2}K.$$

Here R, H, G, and K are assumed to satisfy (7.7) with $Q = I_{\mathcal{X}}$. Since EE^* and $E_\star E_\star^*$ are the orthogonal projections on $\mathcal{D}_A$ and $\mathcal{D}_{A^*}$, respectively, it is straightforward to check that

$$RX = \Pi_{\mathcal{D}_{A^*}}^*, \quad RZ = \Pi_{\mathcal{D}_A}^*, \quad YR = \Pi_{\mathcal{D}_{A^*}}, \quad WR = \Pi_{\mathcal{D}_A}.$$

Thus the identities in (1.5) are satisfied with R in place of T.

Next, put $x_0 = \Pi_{\mathcal{D}_{A^*}}X$ and $z_0 = \Pi_{\mathcal{D}_A}Z$. Using $\operatorname{Ker} B^* = \mathcal{D}_{A*}^\perp$ and $\operatorname{Ker} C = \mathcal{D}_A^\perp$ one computes that

$$(E_\star^*\Pi_{\mathcal{D}_{A^*}}^*)x_0 = (B^*B)^{-1/2}(B^*F)(B^*B)^{-1/2}(E_\star^*\Pi_{\mathcal{D}_{A^*}}^*),$$
$$(E^*\Pi_{\mathcal{D}_A}^*)z_0 = (CC^*)^{-1/2}(CH)(CC^*)^{-1/2}(E^*\Pi_{\mathcal{D}_A}^*).$$

Notice that $E_\star^*\Pi_{\mathcal{D}_{A^*}}^*$ is a unitary operator from $\mathcal{D}_{A^*}$ onto $\mathcal{U}$ and $E^*\Pi_{\mathcal{D}_A}^*$ is a unitary operator from $\mathcal{D}_A$ onto $\mathcal{Y}$. It follows that x_0 is invertible if and only if B^*F is invertible, and z_0 is invertible if and only if CH is invertible. According to our hypotheses (with $Q = I_{\mathcal{X}}$) one of the operators B^*F and CH is invertible, and hence the same holds true for one of the operators x_0 and z_0. Thus we can apply Theorem 1.1 (with R in place of T) to show that R is invertible. Moreover in this case (1.6) transforms into (7.8). Thus Theorem 7.1 is proved for the case when $Q = I_{\mathcal{X}}$.

Part 2. In this part we prove Theorem 7.1 by reduction to the case when $Q = I_{\mathcal{X}}$. Put

$$\tilde{A} = Q^{1/2}AQ^{-1/2}, \qquad \tilde{B} = Q^{1/2}B, \qquad \tilde{C} = CQ^{-1/2},$$
$$\tilde{R} = Q^{-1/2}RQ^{-1/2}, \quad \tilde{\Psi} = Q^{-1/2}\Psi, \quad \tilde{\Upsilon} = \Upsilon Q^{-1/2}.$$

Then $\tilde{A}$ is exponentially stable and $\tilde{R}$ satisfies the Stein equation

$$\tilde{R} - \tilde{A}^*\tilde{R}\tilde{A} = \tilde{\Psi}\tilde{C} + \tilde{C}^*\tilde{\Upsilon}.$$

Moreover

$$I - \tilde{A}^*\tilde{A} = \tilde{C}^*\tilde{C} \text{ where } \tilde{C} : \mathcal{X} \to \mathcal{Y} \text{ is onto,} \tag{7.11}$$

$$I - \tilde{A}\tilde{A}^* = \tilde{B}\tilde{B}^* \text{ where } \tilde{B} : \mathcal{U} \to \mathcal{X} \text{ is one-to-one and has closed range.} \tag{7.12}$$

Thus we are in the setting of the previous part. Put

$$\tilde{F} = Q^{1/2}F, \quad \tilde{G} = GQ^{1/2}, \quad \tilde{H} = Q^{1/2}H, \quad \tilde{K} = KQ^{1/2}.$$

Then

$$\tilde{R}\tilde{F} = \tilde{B}, \quad \tilde{R}\tilde{H} = \tilde{C}^*, \quad \tilde{G}\tilde{R} = \tilde{B}^*, \quad \tilde{K}\tilde{R} = \tilde{C},$$
$$\tilde{B}^*\tilde{F} = B^*QF, \quad \tilde{G}\tilde{B} = GQB, \quad \tilde{C}\tilde{H} = CH, \quad \tilde{K}\tilde{C}^* = KC^*.$$

From these identities and the result of the previous part, it follows that B^*QF is invertible if and only if CH is invertible. Now assume that one of the operators B^*QF and CH is invertible. Then one of operators $\tilde{B}^*\tilde{F}$ and $\tilde{C}\tilde{H}$ is invertible, and from what has been proved in the previous part we know that $\tilde{R}$ is invertible and

$$\tilde{R}^{-1} = \sum_{n=0}^{\infty} \tilde{A}^n \left(\tilde{F}(\tilde{B}^*\tilde{F})^{-1}\tilde{G} - \tilde{A}\tilde{H}(\tilde{C}\tilde{H})^{-1}\tilde{K}\tilde{A}^* \right) \tilde{A}^{*n}.$$

It is then clear that R is invertible and that R^{-1} is given by (7.8). □

Notice that apart from the given operators A and C, Theorem 7.1 also requires the operator B which involves the inverse of Q. In some cases, for instance when the spaces $\mathcal{X}$ and $\mathcal{Y}$ are finite dimensional, one can construct a B satisfying (7.3) without inverting Q. This fact will be illustrated by the next example, which is also presented to illustrate Theorem 7.1.

Example. Consider the $n \times n$ matrix

$$R = \begin{bmatrix} \dfrac{\psi_1 c_1 + \overline{c}_1 \upsilon_1}{1 - \overline{\alpha}_1 \alpha_1} & \cdots & \dfrac{\psi_1 c_n + \overline{c}_1 \upsilon_n}{1 - \overline{\alpha}_1 \alpha_n} \\ \vdots & \cdots & \vdots \\ \dfrac{\psi_n c_1 + \overline{c}_n \upsilon_1}{1 - \overline{\alpha}_n \alpha_1} & \cdots & \dfrac{\psi_n c_n + \overline{c}_n \upsilon_n}{1 - \overline{\alpha}_n \alpha_n} \end{bmatrix}.$$

Here $\alpha_1, \ldots, \alpha_n$ are distinct complex numbers in the open unit disc $\mathbb{D}$, while $c_1, \ldots, c_n$ are non-zero complex numbers, and $\psi_1, \ldots, \psi_n$ and $\upsilon_1, \ldots, \upsilon_n$ are arbitrary complex numbers. We shall use Theorem 7.1 to show that R is invertible whenever certain equations are solvable and to compute its inverse.

First we show that R satisfies the Stein equation

$$R - A^*RA = \Psi C + C^*\Upsilon,$$

with A, C, Ψ, and Υ being given by

$$\begin{aligned} A &= \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \alpha_n \end{bmatrix}, \quad \Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{bmatrix}, \\ C &= \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}, \\ \Upsilon &= \begin{bmatrix} \upsilon_1 & \upsilon_2 & \cdots & \upsilon_n \end{bmatrix}. \end{aligned}$$

In this setting $\mathcal{X} = \mathbb{C}^n$ and $\mathcal{Y} = \mathbb{C}$. Note A is stable because all α_j are inside $\mathbb{D}$. In this case

$$Q = \sum_{\nu=0}^{\infty} A^{*\,\nu} C^* C A^{\nu} = \begin{bmatrix} \dfrac{\overline{c}_1 c_1}{1 - \overline{\alpha}_1 \alpha_1} & \cdots & \dfrac{\overline{c}_1 c_n}{1 - \overline{\alpha}_1 \alpha_n} \\ \vdots & \cdots & \vdots \\ \dfrac{\overline{c}_n c_1}{1 - \overline{\alpha}_n \alpha_1} & \cdots & \dfrac{\overline{c}_n c_n}{1 - \overline{\alpha}_n \alpha_n} \end{bmatrix}.$$

The fact that $\alpha_1, \dots, \alpha_n$ are distinct numbers in $\mathbb{D}$ and the fact that all numbers $c_1, \dots, c_n$ are non-zero together imply that Q is strictly positive.

Lemma 7.2. *A matrix B of size $n \times 1$ satisfying $Q^{-1} - AQ^{-1}A^* = BB^*$ is given by*

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad b_j = \frac{1 - |\alpha_j|^2}{c_j m_j}, \quad m_j = \prod_{k \neq j} \frac{\alpha_j - \alpha_k}{1 - \bar{\alpha}_k \alpha_j} \quad (1 \leq j \leq n).$$

Proof. Let m be the Blaschke product whose zero's are given by the distinct numbers $\bar{\alpha}_1, \dots, \bar{\alpha}_n$, that is,

$$m(\lambda) = \prod_{k=1}^{n} \frac{\lambda - \bar{\alpha}_k}{1 - \alpha_k \lambda}. \tag{7.13}$$

Notice that m admits a partial series expansion of the form:

$$m(\lambda) = m(0) + \lambda \sum_{j=1}^{n} \frac{(1 - |\alpha_j|)^2}{m_j(1 - \alpha_j \lambda)}, \tag{7.14}$$

where m_j is the complex number defined by

$$m_j = \prod_{k \neq j} \frac{\alpha_j - \alpha_k}{1 - \bar{\alpha}_k \alpha_j} \qquad (j = 1, 2, \dots, n). \tag{7.15}$$

Using our definition of $b_1, \dots, b_n$, we see that

$$m(\lambda) = m(0) + \lambda \sum_{j=1}^{n} \frac{c_j b_j}{(1 - \alpha_j \lambda)}. \tag{7.16}$$

Set $D = m(0) = (-1)^n \prod_{j=1}^{n} \bar{\alpha}_j$. Then using the partial fraction expansion in (7.16), it is easy to verify that $\{A\,, B\,, C\,, D\}$ is a realization of m, that is,

$$m(\lambda) = D + \lambda C(I - \lambda A)^{-1} B\,. \tag{7.17}$$

Since the dimension of the state equals n, the degree of the Blaschke product, $\{A, B, C, D\}$ is a minimal realization. Hence $\{A, B, C, D\}$ is a controllable and observable realization of m. Because m is an inner function and $\{A, B, C, D\}$ is minimal, it follows from the theory of unitary realizations (see, e.g., Sections XXVIII.2 and XXVIII.2 in [9]) that Q is the observability Gramian for $\{C, A\}$ if and only if Q^{-1} is the controllability Gramian for $\{A, B\}$. Therefore B satisfies the equation $Q^{-1} = AQ^{-1}A^* + BB^*$. $\square$

We are now ready to apply Theorem 7.1. Assume that there exist matrices

$$F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}, \quad H = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}, \quad G = \begin{bmatrix} g_1 & g_2 & \cdots & g_n \end{bmatrix},$$
$$K = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix},$$

satisfying the equations

$$RF = QB, \quad RH = C^*, \quad GR = B^*Q, \quad \text{and} \quad KR = C. \tag{7.18}$$

Then, according to Theorem 7.1, the matrix R is invertible if and only if

$$\gamma_1 = B^*QF = \sum_{p,q=1}^{n} \frac{\bar{b}_p \bar{c}_p c_q f_q}{1 - \bar{\alpha}_p \alpha_q} \neq 0 \quad \text{and} \quad \gamma_2 = CH = \sum_{\nu=1}^{n} c_\nu h_\nu \neq 0.$$

In this case, by (7.8), the inverse of R is given by

$$R^{-1} = \frac{1}{\gamma_1} M_1 - \frac{1}{\gamma_2} M_2,$$

where

$$M_1 = \sum_{\nu=0}^{\infty} A^\nu F G A^{*\,\nu} = \begin{bmatrix} \dfrac{f_1 g_1}{1 - \alpha_1 \overline{\alpha}_1} & \cdots & \dfrac{f_1 g_n}{1 - \alpha_1 \overline{\alpha}_n} \\ \vdots & \cdots & \vdots \\ \dfrac{f_n g_1}{1 - \alpha_n \overline{\alpha}_1} & \cdots & \dfrac{f_n g_n}{1 - \alpha_n \overline{\alpha}_n} \end{bmatrix},$$

$$M_2 = \sum_{\nu=1}^{\infty} A^\nu H K A^{*\,\nu} = \begin{bmatrix} \dfrac{\alpha_1 h_1 k_1 \overline{\alpha}_1}{1 - \alpha_1 \overline{\alpha}_1} & \cdots & \dfrac{\alpha_1 h_1 k_n \overline{\alpha}_n}{1 - \alpha_1 \overline{\alpha}_n} \\ \vdots & \cdots & \vdots \\ \dfrac{\alpha_n h_n k_1 \overline{\alpha}_1}{1 - \alpha_n \overline{\alpha}_1} & \cdots & \dfrac{\alpha_n h_n k_n \overline{\alpha}_n}{1 - \alpha_n \overline{\alpha}_n} \end{bmatrix}.$$

Acknowledgment

We thank Freek van Schagen for a discussion on the paper and for pointing out a few typos.

References

[1] D.Z. Arov, On strictly positive compressions of block Toeplitz operators, *Indiana Univ. Math. J.* **46** (1997), 357–373.

[2] J.A. Ball, I. Gohberg, and L. Rodman, *Interpolation of rational matrix functions*, OT **45**, Birkhäuser Verlag, Basel, 1990.

[3] T. Constantinescu, A.H. Sayed, and T. Kailath, Displacement structure and completion problems, *Siam J. Matrix Anal. Appl.* **16** (1995), 58–78.

[4] R.L. Ellis and I. Gohberg, *Orthogonal systems and convolution operators*, OTAA 140, Birkhäuser Verlag, Basel, 2003.

[5] C. Foias and A.E. Frazho, *The Commutant Lifting Approach to Interpolation Problems,* Operator Theory: Advances and Applications, **44**, Birkhäuser Verlag, Basel, 1990.

[6] F. Foias, A.E. Frazho, I. Gohberg, and M.A. Kaashoek, *Metric constrained interpolation, commutant lifting and systems*, Birkhäuser Verlag, Basel, 1998.

[7] A.E. Frazho and M.A. Kaashoek, A Naimark perspective of Nevanlinna-Pick interpolation, *Integral Equations and Operator Theory* **49** (2004), 323–378.

[8] I. Gohberg, S. Goldberg, and M.A. Kaashoek, *Classes of Linear Operators*, Volume I, Birkhäuser Verlag, Basel, 1990.

[9] I. Gohberg, S. Goldberg, and M.A. Kaashoek, *Classes of Linear Operators*, Volume II, Birkhäuser Verlag, Basel, 1993.

[10] I. Gohberg, S. Goldberg, and M.A. Kaashoek, *Basic Classes of Linear Operators*, Birkhäuser Verlag, Basel, 2003.

[11] I.C. Gohberg and G. Heinig, On the inversion of finite Toeplitz matrices, *Math. Issled* **8** (3) (1973), 151–155 (in Russian).

[12] I.C. Gohberg and G. Heinig, The inversion of finite Toeplitz matrices consisting of elements of a non-commutative algebra, *Rev. Roum. Math. Pures et Appl.* **20** (1974), 623–663 (in Russian).

[13] I. Gohberg, M.A. Kaashoek, and F. van Schagen, On inversion of finite Toeplitz matrices with elements in an algebraic ring, *Lin. Alg. Appl.* **385** (2004), 381–389.

[14] I. Gohberg and I. Koltracht, Efficient algorithm for Toeplitz plus Hankel matrices, *Integral Equations and Operator Theory* **12** (1989), 136–142.

[15] I.C. Gohberg and M.G. Krein, Systems of integral equations with kernels depending on the difference of arguments, *Uspekhi Math. Nauk* **13** 2(80) (1958), 3–72 (Russian); English Transl., *Amer. Math. Soc. Transl.* (*Series* 2) **14** (1960), 217–287.

[16] I. Gohberg and V. Olshevsky, Circulants, displacements and decompositions of matrices, *Integral Equations and Operator Theory* **15** (1992), 730–743.

[17] I. Gohberg and V. Olshevsky, Complexity of multiplication with vectors for structured matrices, *Lin. Alg. Appl.* **202** (1994), 163–192.

[18] I.C. Gohberg and A.A. Semencul, On the invertibility of finite Toeplitz matrices and their continuous analogues, *Math. Issled* **7** (2) (1972), 151–155 (in Russian).

[19] G. Heinig and K. Rost, *Algebraic methods for Toeplitz-like matrices and operators*, Akademie-Verlag, Berlin, 1984.

[20] M.A. Kaashoek and C.G. Zeinstra, The band and generalized Carathéodory-Toeplitz interpolation at operator points, *Integral Equations and Operator Theory* **33** (1999), 175–210.

[21] T. Kailath and J. Chun, Generalized Gohberg-Semencul formulas, in: *The Gohberg Anniversary Collection, Vol. I*, OT **49**, Birkhäuser Verlag, Basel, 2003; pp. 231–246

[22] T. Kailath, S.-Y. Kung, and M. Morf, Displacement ranks of matrices and linear equations, *J. Math. Anal. Appl.* **68** (1979), 395–407.

[23] T. Kailath and A.H. Sayed, Displacement structure: Theory and applications, *SIAM Rev.* **37** (1995), 297–386.

[24] T. Kailath and A.H. Sayed (eds.), *Fast Reliable Algorithms for Matrices with Structure*, SIAM, Philadelphia, 1999.

[25] V.Y. Pan, *Structured matrices and polynomials*, Birkhäuser Boston, 2001.

[26] B. Sz.-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space,* North Holland Publishing Co., Amsterdam-Budapest, 1970.

[27] A. Wilansky, *Functional Analysis*, Blaisdell Publ. Co., New York, 1964.

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Received: January 3, 2009
Accepted: May 6, 2009

Operator Theory:
Advances and Applications, Vol. 202, 253–264

A Spectral Weight Matrix for a Discrete Version of Walsh's Spider

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Abstract. We consider a discrete space version of Walsh's spider, see [W] as well as [ES] and its references. This process can be seen as an instance of a quasi-birth-and-death process, a class of random walks for which the classical theory of Karlin and McGregor can be nicely adapted as in [DRSZ], [G1, G2] and [GdI]. We give here a simple expression for a family of weight matrices that make the corresponding matrix-valued orthogonal polynomials orthogonal to each other.

Mathematics Subject Classification (2000). 33C45, 22E45.

Keywords. Matrix-valued orthogonal polynomials, Karlin–McGregor representation.

1. Birth-and-death processes and orthogonal polynomials

If $\mathbb{P}$ denotes the one-step transition probability matrix for a birth and death process on the non-negative integers

$$\mathbb{P} = \begin{pmatrix} r_0 & p_0 & 0 & 0 & \\ q_1 & r_1 & p_1 & 0 & \\ 0 & q_2 & r_2 & p_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

there is a powerful tool to analyze the random walk in question, [KMcG] as well as [vD, ILMV].

The author was supported in part by NSF Grant DMS-0603901.
Communicated by L. Rodman.

If one introduces the polynomials $Q_j(x)$ by the conditions $Q_{-1}(0) = 0$, $Q_0(x) = 1$ and using the notation

$$Q(x) = \begin{pmatrix} Q_0(x) \\ Q_1(x) \\ \vdots \end{pmatrix}$$

one insists on the recursion relation

$$\mathbb{P}Q(x) = xQ(x)$$

one proves the existence of a unique measure $\psi(dx)$ supported in $[-1,1]$ such that

$$\pi_j \int_{-1}^{1} Q_i(x)Q_j(x)\psi(dx) = \delta_{ij}$$

and one gets the Karlin–McGregor representation formula

$$(\mathbb{P}^n)_{ij} = \pi_j \int_{-1}^{1} x^n Q_i(x)Q_j(x)\psi(dx).$$

Many probabilistic properties of the walk are reflected in the measure that appears above. For instance the process is recurrent exactly when the integral

$$\int_{-1}^{1} \frac{\psi(dx)}{1-x}$$

diverges. The process returns to the origin in a finite expected time when the measure has a mass at $x = 1$. The existence of

$$\lim_{n\to\infty} (\mathbb{P}^n)_{ij}$$

is equivalent to $\psi(dx)$ having no mass at $x = -1$.

In some cases all these quantities can be computed explicitly.

As an example suppose that we have $r_1 = r_2 = \cdots = 0$, $q_1 = q_2 = \cdots = q$ and $p_1 = p_2 = \cdots = p$, with $0 \le p \le 1$ and $q = 1 - p$.

One can show that

$$Q_j(x) = \left(\frac{q}{p}\right)^{j/2} \left[2(p_0 - p)/p_0 T_j(x^*) + (2p - p_0)/p_0 U_j(x^*) - r_0/p_0 (p/q)^{1/2} U_{j-1}(x^*)\right]$$

where T_j and U_j are the Chebyshev polynomials of the first and second kind, and $x^* = x/(2\sqrt{pq})$. The polynomials $Q_j(x)$ are orthogonal with respect to a spectral measure in the interval $[-1,1]$ which can also be determined explicitly.

In the very special case when $r_0 = 0$, $p_0 = 1$ (i.e., a reflecting boundary condition) one has the following dichotomy, illustrating the relation mentioned earlier.

If $p \ge 1/2$ we have

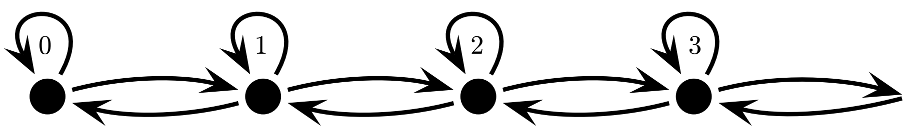
$$\left(\frac{p}{1-p}\right)^n \int_{-\sqrt{4pq}}^{\sqrt{4pq}} Q_n(x)Q_m(x)\frac{\sqrt{4pq - x^2}}{1-x^2}dx = \delta_{nm} \begin{cases} 2(1-p)\pi, & n = 0 \\ 2p(1-p)\pi, & n \ge 1 \end{cases}$$

while if $p \leq 1/2$ we get a new phenomenon, namely the presence of point masses in the spectral measure

$$\left(\frac{p}{1-p}\right)^n \Bigg[\int_{-\sqrt{4pq}}^{\sqrt{4pq}} Q_n(x)Q_m(x)\frac{\sqrt{4pq-x^2}}{1-x^2}dx \\ + (2-4p)\pi[Q_n(1)Q_m(1)+Q_n(-1)Q_m(-1)]\Bigg] \\ = \delta_{nm}\begin{cases} 2(1-p)\pi, & n=0 \\ 2p(1-p)\pi, & n\geq 1. \end{cases}$$

The result above is due to S. Karlin and McGregor, who also study a few more cases.

In the case of a birth and death process it is, of course, useful to think of a graph like



The nodes here represent the states $0, 1, 2, \ldots$ and the arrows go along with the one step transition probabilities. One should imagine that the graph extends all the way to the right.

The ideas behind the Karlin–McGregor formula seen earlier can be used to study these more complicated random walks. This is the point of the next section.

2. Recall M.G. Krein

Given a positive definite matrix-valued smooth weight function $W(x)$ with finite moments, consider the skew symmetric bilinear form defined for any pair of matrix-valued polynomial functions $P(x)$ and $Q(x)$ by the numerical matrix

$$(P,Q) = (P,Q)_W = \int_{\mathbb{R}} P(x)W(x)Q^*(x)dx,$$

where $Q^*(x)$ denotes the conjugate transpose of $Q(x)$.

It is clear that one can replace a matrix-valued measure with a nice density, as above, by a more general measure which could contain discrete as well as continuous pieces. This is indeed the case of some of the examples discussed below, but we retain the notation used by M.G. Krein.

By the usual Gram–Schmidt construction this leads to the existence of a sequence of matrix-valued orthogonal polynomials with non-singular leading coefficient, $P_n(x) = M_n x^n + M_{n-1}x^{n-1} + \cdots$.

Given an orthogonal sequence $\{P_n(x)\}_{n\geq 0}$ of matrix-valued orthogonal polynomials one gets by the usual argument a three term recursion relation

$$xP_n(x) = A_nP_{n-1}(x) + B_nP_n(x) + C_nP_{n+1}(x), \tag{1}$$

where A_n, B_n and C_n are matrices and the last one is non-singular. All of this is due to M.G. Krein, see [K1, K2].

It is convenient to introduce the block tridiagonal matrix L

$$L = \begin{pmatrix} B_0 & C_0 & & \\ A_1 & B_1 & C_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix}.$$

This matrix will also be denoted by $\mathbb{P}$ specially when we have a situation when its (scalar) entries are non-negative and possibly add up to one.

If $\mathbb{P}_{i,j}$ denotes the i,j block of $\mathbb{P}$ we can generate a sequence of 2×2 matrix-valued polynomials $Q_i(t)$ by imposing the three term recursion given above. By using the notation of the scalar case, we would have

$$\mathbb{P}Q(x) = xQ(x)$$

where the entries of the column vector $Q(x)$ are now 2×2 matrices.

Proceeding as in the scalar case, this relation can be iterated to give

$$\mathbb{P}^nQ(x) = x^nQ(x)$$

and if we assume the existence of a weight matrix $W(x)$ as in Krein's theory, with the property

$$(Q_j, Q_j)\delta_{i,j} = \int_{\mathbb{R}} Q_i(x)W(x)Q_j^*(x)dx,$$

it is then clear that one can get an expression for the (i,j) entry of the block matrix $\mathbb{P}^n$ that would look exactly as in the scalar case, namely

$$(\mathbb{P}^n)_{ij}(Q_j, Q_j) = \int x^nQ_i(x)W(x)Q_j^*(x)dx.$$

These expressions were given first in [DRSZ] and then (independently) in [G1].

Just as in the scalar case, this expression becomes useful when we can get our hands on the matrix-valued polynomials $Q_i(x)$ and the orthogonality measure $W(x)$.

Notice that we have not discussed conditions on the matrix $\mathbb{P}$ to give rise to such a measure. One can see that this is just the condition that the matrix $\mathbb{P}$ should be block-symmetrizable, i.e., a matrix version of the old reversibility condition, but with the positive scalars π_i being replaced by positive matrices.

3. The first example

The spectral theory of a scalar double-infinite tridiagonal matrix leads naturally to a 2×2 semi-infinite matrix, as was already recognized by Karlin and McGregor.

The example of random walk on the integers, in Karlin–McGregor. The probabilities of going right or left are p and q, $p+q=1$.

$$L=\begin{pmatrix} 0 & q & p & 0 & 0 & 0 & \dots & \dots & \dots \\ p & 0 & 0 & q & 0 & 0 & \dots & \dots & \dots \\ q & 0 & 0 & 0 & p & 0 & \dots & \dots & \dots \\ 0 & p & 0 & 0 & 0 & q & \dots & \dots & \dots \\ 0 & 0 & q & 0 & 0 & 0 & p & 0 & \dots \\ 0 & 0 & 0 & p & 0 & 0 & 0 & q & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

In this case, as already pointed out by [KMcG], see also [G1, DRSZ], the weight matrix is given as follows: In the interval $|x|\le\sqrt{4pq}$ one has for its density $W(x)$ the expression

$$\frac{1}{\sqrt{4pq-x^2}}\begin{pmatrix} 1 & x/2q \\ x/2q & p/q \end{pmatrix}.$$

One can use the method described above to obtain a complete description of the orthogonality measure for situations obtained by modifying simpler ones.

An interesting collection of such examples is given in the original paper of Karlin and McGregor.

The following example is not included in their paper. Other examples that can be obtained by modifying simpler ones are given below.

4. Random walk with an attractive force

A modification of the example in Karlin–McGregor with probabilities p of going away from the center (one should imagine it located at $1/2$) and q of going towards the center, $p+q=1$.

$$L=\begin{pmatrix} 0 & q & p & 0 & \dots & \dots & \dots & \dots & \dots \\ q & 0 & 0 & p & \dots & \dots & \dots & \dots & \dots \\ q & 0 & 0 & 0 & p & 0 & \dots & \dots & \dots \\ 0 & q & 0 & 0 & 0 & p & \dots & \dots & \dots \\ 0 & 0 & q & 0 & 0 & 0 & p & 0 & \dots \\ 0 & 0 & 0 & q & 0 & 0 & 0 & p & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

In this case, the weight matrix is given as follows: In the interval $|x| \leq \sqrt{4pq}$ one has for its density $W(x)$ the expression

$$\frac{\sqrt{4pq - x^2}}{1 - x^2} \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix}$$

and if $p < 1/2$ one adds the "point masses"

$$(1 - 2p)\pi \left[\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \delta_{-1} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \delta_1 \right].$$

It is reasonable to consider this example as a discrete analog of the problem for a one-dimensional Schrödinger equation with a potential given by a scalar multiple of the absolute value of x. We have not seen this problem discussed in the literature, although a complete solution should be around in terms of appropriate Airy functions.

5. Allowing for a "defect" at the origin

We consider a modification of the previous example where at the origin, the probabilities of a right or left transitions are given by the non-negative quantities x_1 and x_2, such that $x_1 + x_2 = 1$.

The corresponding block tridiagonal transition probability matrix is given by

$$L = \begin{pmatrix} 0 & x_2 & x_1 & 0 & \dots & \dots & \dots & \dots & \dots \\ q & 0 & 0 & p & \dots & \dots & \dots & \dots & \dots \\ q & 0 & 0 & 0 & p & 0 & \dots & \dots & \dots \\ 0 & q & 0 & 0 & 0 & p & \dots & \dots & \dots \\ 0 & 0 & q & 0 & 0 & 0 & p & 0 & \dots \\ 0 & 0 & 0 & q & 0 & 0 & 0 & p & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

In this case the weight matrix has a density supported in the interval $|t| \leq \sqrt{4pq}$ and given by

$$W = \frac{\sqrt{4pq - x^2}}{1 - x^2} \begin{pmatrix} p(1 - x_1) & p(1 - x_1)x \\ p(1 - x_1)x & (1 - p)x_1 + (p - x_1)x^2 \end{pmatrix}.$$

If $p < 1/2$ one needs to add "point masses" as given below

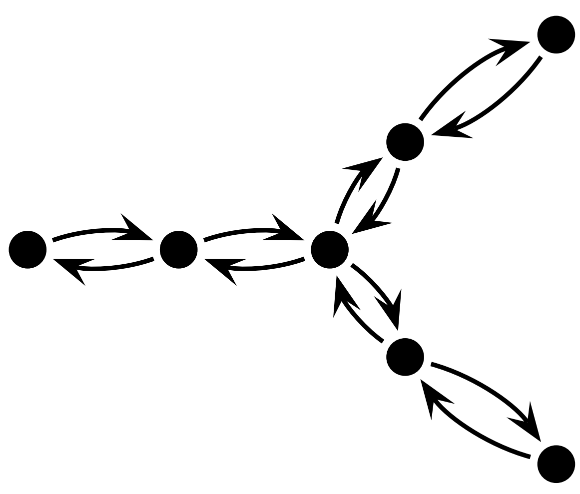
$$p(1 - x_1)(1 - 2p)\pi \left[\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \delta_{-1} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \delta_1 \right].$$

6. An assortment of graphs

In queueing theory people consider discrete time Markov chains where the sate space is given by pairs of integers (n, i) with $n = 0, 1, 2, \ldots$ and i between 1 and a fixed m. The first coordinate is called the level and the second one the phase of the sate (n, j). One is then led to consider a block tridiagonal matrix as the one step transition probability of the Markov chain if transitions in one step are restricted to states in the same level or in the two adjacent levels. This is a natural area to look for useful applications of the theory of matrix-valued orthogonal polynomials.

For a good reference to the queueing models, where these models are called **Quasi-birth-and-death process, QBD**, see [BB, LR, N, LPT, G2].

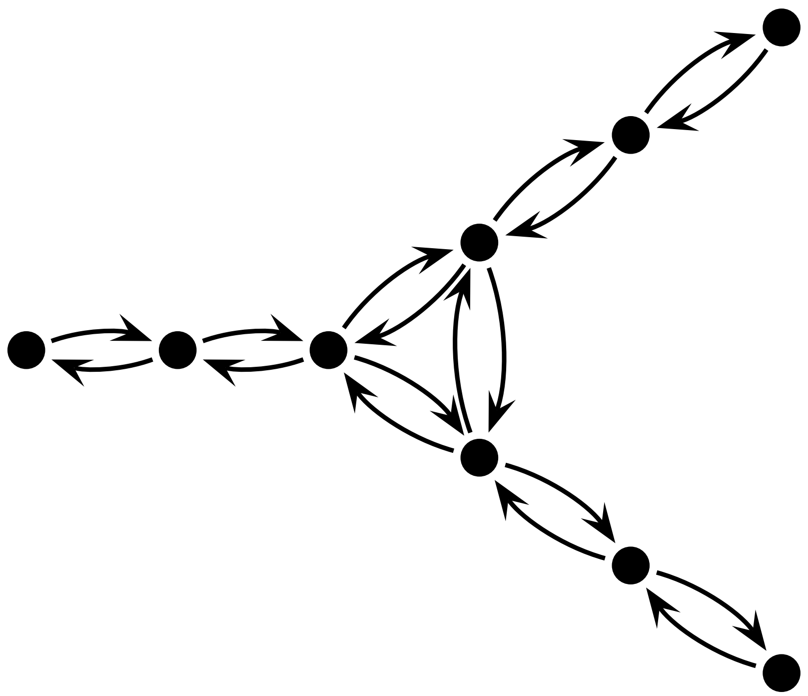
I have recently studied a rather complicated QBD process, see [GdI], where we manage to find the orthogonal (matrix-valued) polynomials, the orthogonality matrix-valued measure, and most surprisingly we find explicitly the invariant measure for the process in question. Another paper where a similar type of study is made is [DRSZ]. In this paper the authors study a number of previously known examples and they consider a new one, depicted by the network of the type given below, where each of the arms extends to infinity



I will analyze this network in the next section. The results that I get are different from those reported in [DRSZ].

It is not obvious how to make such a network into one that can be analyzed by means of matrix-valued orthogonal polynomials, but this can be done, as shown nicely in [DRSZ], and in this fashion one can start to analyze many complicated direct as well as inverse problems for large classes of networks.

A good example of a network that one would like to analyze with these methods is given below, where once again the external arms extend to infinity.



7. Spider or star graphs

For a spider or star graph as considered in [DRSZ] and depicted earlier on in this paper (in the case of $N = 3$) one has a center node and N legs that extend to infinity. If drawn properly the nodes on these legs form a sequence of concentric circles centered at the center node. This is clearly a discrete version of the spiders considered by J.B. Walsh, [W, ES].

It is convenient to label the center node as 1 and the N nodes on the first circle as $2, 3, \ldots, N+1$ in a counter-clockwise fashion. The N nodes in the second circle centered at the origin are labelled as $N+2, N+3, \ldots, 2N+1$, etc. ... The transition probabilities from the center node to each one of the nodes on the first circle are denoted by $x_2, x_3, \ldots, x_N, x_1$.

For each node that is not the center node, and therefore lies on one of the legs of the graph, the probability of getting closer to the center node, while remaining on the same leg, is given by the common value q while the probability of a transition moving one step away from the center node, while staying on the same leg, is given by the common value $p = 1 - q$.

It is now convenient to consider $N \times N$ matrix-valued orthogonal polynomials resulting from a block tridiagonal matrix with blocks given by the following expressions

$$B_0 = \begin{pmatrix} 0 & x_2 & x_3 & \dots & x_N \\ q & 0 & 0 & & 0 \\ q & & & & \\ \vdots & & & & \\ q & & & & \end{pmatrix}, \quad C_0 = \begin{pmatrix} x_1 & 0 & 0 & \dots & 0 \\ 0 & p & & & \\ & & p & & \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & p \end{pmatrix}$$

and

$$A_I \equiv qI, \ i \geq 1; \ B_i = 0I, \ i \geq 1; \ C_i = pI, \ i \geq 1.$$

We move now to the task of finding an $N \times N$ matrix-valued weight that makes the corresponding polynomials, obtained from the recursion relation mentioned at the beginning of the paper, orthogonal. For this purpose it is convenient to introduce a certain collection of matrices. For simplicity assume from now on that $N > 3$. This will guarantee that all the matrices introduced below will enter as ingredients in the weight matrix we are about to construct. The case $N = 2$ was considered earlier on, and the case $N = 3$ fits the treatment below, with obvious adjustments.

For a spider graph with N legs we need to consider the following $N \times N$ matrix

$$M \equiv \frac{\sqrt{4pq - x^2}}{1 - x^2} \left\{ \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & 1 & \dots & 1 \end{pmatrix} + x \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix} \right\}$$

$$+ \sqrt{4pq - x^2} \frac{(x_1 - p)}{p x_2^2} \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 2x_2 + x_1 - 1 & x_2 & \dots & x_2 \\ 0 & x_2 & 0 & \dots & 0 \\ 0 & x_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & x_2 & 0 & \dots & 0 \end{pmatrix}$$

$$+ (1 - 2p)\pi \left\{ \begin{pmatrix} 1 & -1 & -1 & \dots & -1 \\ -1 & 1 & 1 & \dots & 1 \\ -1 & 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ -1 & 1 & 1 & \dots & 1 \end{pmatrix} \delta_{-1} + \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix} \delta_1 \right\}.$$

We need to consider for each $i \in \{3,\dots,N\}$ the matrices M_i obtained by multiplying the scalar quantity $\sqrt{4pq - x^2}$ by the matrix M_i given by

$$M_i \equiv \begin{array}{cc} & \begin{array}{cccccccc} & 2 & & & & i & & \\ & \downarrow & & & & \downarrow & & \end{array} \\ \begin{array}{c} \\ 2 \rightarrow \\ \\ \\ i \rightarrow \\ \\ \\ \\ \end{array} & \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ 0 & \frac{x_i^2}{x_2^2} & 0 & \dots & 0 & -\frac{x_i}{x_2} & 0 & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ 0 & -\frac{x_i}{x_2} & 0 & \dots & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots \end{pmatrix} \end{array}.$$

Finally for each pair (i,j) where both i and j $(i < j)$ come from the set $\{3,\dots,N\}$ we need to consider the matrix obtained by multiplying the scalar quantity $\sqrt{4pq - x^2}$ by the matrix M_{ij} given by

$$M_{ij} \equiv \begin{array}{cc} & \begin{array}{cccccccc} & 2 & & & i & & j & \\ & \downarrow & & & \downarrow & & \downarrow & \end{array} \\ \begin{array}{c} \\ 2 \rightarrow \\ \\ \\ i \rightarrow \\ \\ \\ j \rightarrow \\ \\ \\ \end{array} & \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \dots & 0 & \dots \\ 0 & \frac{2x_i x_j}{x_2^2} & 0 & \dots & -\frac{x_j}{x_2} & \dots & -\frac{x_i}{x_2} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & -\frac{x_j}{x_2} & 0 & \dots & 0 & \dots & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & -\frac{x_i}{x_2} & 0 & \dots & 1 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \end{array}.$$

If, using standard practice, we denote by E_{ij} the $N \times N$ matrix with a one in the (i,j) entry and all zeros everywhere else, we can write

$$M_i = \frac{x_i^2}{x_2^2} E_{22} - \frac{x_i}{x_2}(E_{2i} + E_{i2}) + E_{ii}$$

for $i \in \{3,\dots,N\}$ and

$$M_{ij} = 2\frac{x_i x_j}{x_2^2} E_{22} - \frac{x_j}{x_2}(E_{2i} + E_{i2}) - \frac{x_i}{x_2}(E_{2j} + E_{j2}) + (E_{ij} + E_{ji})$$

for distinct (i,j) in $\{3,\dots,N\}$.

The matrix M is now written as

$$M = \frac{\sqrt{4pq - x^2}}{1 - x^2} \left\{ \left(E_{11} + \sum_{\{i,j\} \in \{2,\dots,N\}} E_{ij} \right) + x \sum_{i=1}^{N} (E_{1i} + E_{i1}) \right\}$$
$$+ \sqrt{4pq - x^2} \frac{(x_1 - p)}{p x_2^2} \left((2x_2 + x_1 - 1) E_{22} + \sum_{i=3}^{N} (E_{2i} + E_{i2}) \right)$$
$$+ (1 - 2p)\pi \left\{ \left(E_{11} - \sum_{i=2}^{N} (E_{1i} + E_{i1}) + \sum_{\{i,j\} \in \{2,\dots,N\}} E_{ij} \right) \delta_{-1} \right.$$
$$\left. + \left(\sum_{\{i,j\} \in \{1,2,\dots,N\}} E_{ij} \right) \delta_1 \right\}.$$

In terms of these matrices the weight of interest is

$$M + \sqrt{4pq - x^2} \sum_{i=3}^{N} c_i M_i + \sqrt{4pq - x^2} \sum_{i<j \in \{3,\dots,N\}} c_{ij} M_{ij}.$$

where the coefficients c_i and c_{ij} are picked according to the recipe

$$\begin{array}{lll} c_i & = \left(\frac{1}{p} - 1 \right) \frac{1}{x_i} - \frac{1}{p} & i \in \{3, \dots, N\} \\ c_{ij} & \equiv -\frac{1}{p} & i, j \in \{3, \dots, n\} \text{ and } i < j. \end{array}$$

In the case when $p \geq \frac{1}{2}$ we leave out the last term in the matrix M, dealing with point masses at $x = \pm 1$.

8. The invariant measure

We observe that the matrix-valued squared norms of the successive polynomials with respect to this measure happen to be diagonal. This, as seen for instance in [GdI], allows one to come up with a very good candidate for an invariant measure for the process.

References

[BB] Basu, S., and Bose, N.K., *Matrix Stieltjes series and network models*, SIAM J. Matrix Anal. Applic. **14**, No. 2 (1983), 209–222.

[DRSZ] Dette, H., Reuther, B., Studden, W. and Zygmunt, M., *Matrix measures and random walks with a block tridiagonal transition matrix*, SIAM J. Matrix Anal. Applic. **29**, No. 1 (2006), 117–142.

[vD] van Doorn, E., *On associated polynomials and decay rates for birth-death processes*, J. Math. Anal. Appl. **278**, No. 2 (2003), 500–511.

[ES] Evans, S. and Sowers, R., *Pinching and twisting Markov processes*, Ann. Probab. 31(2003), no. 1, pp. 486–527.

[G1] Grünbaum, F.A., *Random walks and orthogonal polynomials: some challenges*, in Probability, Geometry and Integrable systems, Mark Pinsky and Bjorn Birnir editors, MSRI publication vol 55, 2007, pp. 241–260, see also arXiv math PR/0703375.

[G2] Grünbaum, F.A., *QBD processes and matrix-valued orthogonal polynomials: some new explicit examples*, Dagstuhl Seminar Proceedings 07461, Numerical Methods in structured Markov Chains, 2007, D. Bini editor.

[GdI] Grünbaum, F.A. and de la Iglesia, M.D., *Matrix-valued orthogonal polynomials arising from group representation theory and a family of quasi-birth-and-death processes*, Siam J. Matrix Anal. Appl. Vol. 30, 2, pp. 741–763, 2008.

[ILMV] Ismail, M.E.H., Letessier, J., Masson, D., and Valent, G., *Birth and death processes and orthogonal polynomials*, in Orthogonal Polynomials, P. Nevai (editor), Kluwer Acad. Publishers (1990), 229–255.

[KMcG] Karlin, S., and McGregor, J., *Random walks*, Illinois J. Math. **3** (1959), 66–81.

[K1] Krein, M.G., *Fundamental aspects of the representation theory of hermitian operators with deficiency index* (m, m), AMS Translations, Series 2, vol. 97, Providence, Rhode Island (1971), 75–143.

[K2] Krein, M.G., *Infinite J-matrices and a matrix moment problem*, Dokl. Akad. Nauk SSSR **69**, nr. 2 (1949), 125–128.

[LR] Latouche, G., and Ramaswami, V., *Introduction to Matrix Analytic Methods in Stochastic Modeling*, ASA-SIAM Series on Statistics and Applied Probability, 1999.

[LPT] Latouche, G., Pearce, C.E.M., and Taylor, P.G., *Invariant measures for quasi-birth-and-death processes*, Comm. Statist. Stochastic Models, **14** (1998), 443–460.

[N] Neuts, M.F., *Structured Stochastic Matrices of $M/G/1$ Type and Their Applications*, Marcel Dekker, NY, 1989.

[W] Walsh, J.B., *A diffusion with discontinuous local time*, in Temps Locaux, Astérisque, vol. 52-53 Soc. Math. de France, Paris, 1978.

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Received: March 4, 2009
Accepted: June 16, 2009

Operator Theory:
Advances and Applications, Vol. 202, 265–272

Norm Inequalities for Composition Operators on Hardy and Weighted Bergman Spaces

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Abstract. Any analytic self-map of the open unit disk induces a bounded composition operator on the Hardy space H^2 and on the standard weighted Bergman spaces A^2_α. For a particular self-map, it is reasonable to wonder whether there is any meaningful relationship between the norms of the corresponding operators acting on each of these spaces. In this paper, we demonstrate an inequality which, at least to a certain degree, provides an answer to this question.

Mathematics Subject Classification (2000). 47B33.

Keywords. Composition operator, operator norm, Hardy space, weighted Bergman spaces, Schur product.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane and let φ be an analytic self-map of $\mathbb{D}$. If $\mathcal{H}$ is a Hilbert space of analytic functions on $\mathbb{D}$, the *composition operator* C_φ on $\mathcal{H}$ is defined by the rule $C_\varphi(f) = f \circ \varphi$. While there are some Hilbert spaces (the Dirichlet space, for example) on which there are unbounded composition operators, every analytic $\varphi : \mathbb{D} \to \mathbb{D}$ induces a bounded operator on all of the spaces we will be considering in this paper. Our main goal is to develop a better sense of the relationship between the operator norms of C_φ acting on different spaces.

The Hilbert spaces of primary interest to us will be the Hardy space H^2 and the weighted Bergman spaces A^2_α. The *Hardy space* consists of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|^2_{H^2} = \sup_{0<r<1} \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^2 \frac{d\theta}{2\pi} < \infty.$$

Communicated by J.A. Ball.

This space is a Hilbert space, with inner product

$$\langle f,g\rangle_{H^2} = \lim_{r\uparrow 1}\int_0^{2\pi} f\big(re^{i\theta}\big)\overline{g\big(re^{i\theta}\big)}\,\frac{d\theta}{2\pi}.$$

The Hardy space can be described as a *reproducing kernel Hilbert space*, since for every point λ in $\mathbb{D}$ there is a unique function K_λ in H^2 (known as a *reproducing kernel function*) such that $\langle f, K_\lambda\rangle_{H^2} = f(\lambda)$ for all f in H^2. In the case of the Hardy space, it is not difficult to see that $K_\lambda(z) = 1/(1-\overline{\lambda}z)$ (see Corollary 2.11 in [8]).

For $\alpha > -1$, the *weighted Bergman space* A^2_α consists of all analytic f on $\mathbb{D}$ such that

$$\|f\|^2_{A^2_\alpha} = \int_{\mathbb{D}} |f(z)|^2(\alpha+1)(1-|z|^2)^\alpha\, dA(z) < \infty,$$

where dA signifies normalized area measure on $\mathbb{D}$. The case where $\alpha = 0$ is known as the (unweighted) Bergman space, and is often denoted simply A^2. For any α, we write $\langle\cdot,\cdot\rangle_{A^2_\alpha}$ to denote the obvious inner product on A^2_α. These spaces are all reproducing kernel Hilbert spaces, with kernel functions $K^\alpha_\lambda(z) = 1/(1-\overline{\lambda}z)^{\alpha+2}$ (see Corollary 2.12 in [8] and Proposition 1.4 in [11]).

There is an obvious likeness between the reproducing kernels for H^2 and the analogous functions for A^2_α. For the sake of efficiency, it will often behoove us to write A^2_{-1} to denote the Hardy space H^2, with $K^{-1}_\lambda = K_\lambda$ and $\langle\cdot,\cdot\rangle_{A^2_{-1}} = \langle\cdot,\cdot\rangle_{H^2}$. We will state many of our results in these terms, with the understanding that the $\alpha = -1$ "weighted Bergman space" always signifies the Hardy space.

For any analytic $\varphi : \mathbb{D} \to \mathbb{D}$, we will write $\|C_\varphi\|_{\mathcal{H}}$ to denote the norm of C_φ acting on a Hilbert space $\mathcal{H}$. While it is generally not easy to calculate $\|C_\varphi\|_{A^2_\alpha}$ explicitly, some concrete results are known – most notably in the case of the Hardy space H^2 (see [2], [3], [9], and [10]). Fortunately, it is not difficult to obtain sharp upper and lower bounds for the norm of C_φ. In particular, it is well known that

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{\alpha+2} \le \|C_\varphi\|^2_{A^2_\alpha} \le \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\alpha+2} \tag{1}$$

for any $\alpha \ge -1$ (see Corollary 3.7 in [8] and Lemma 2.3 in [16]).

Reflecting on Equation (1), one might wonder whether there is some relationship between the quantities $\|C_\varphi\|_{A^2_\alpha}$ for different values of α. For example, considering $\alpha = 0$ and $\alpha = -1$, one might ask whether it is always the case that $\|C_\varphi\|_{A^2} = \|C_\varphi\|^2_{H^2}$. While this equality does hold for some maps, it is not true in general (see Section 4 of [4]). In this paper, we shall prove that $\|C_\varphi\|_{A^2} \le \|C_\varphi\|^2_{H^2}$ for all φ (see Corollary 5), answering a question posed by the authors of [4], and derive a collection of inequalities relating to the norms of C_φ on different spaces (see Theorem 4).

Before proceeding to our main results, we should mention a helpful fact relating to composition operators and reproducing kernel functions. Let C^*_φ denote the adjoint of C_φ on a particular space A^2_α; it is a simple exercise to show that $C^*_\varphi\big(K^\alpha_\lambda\big) = K^\alpha_{\varphi(\lambda)}$ for any λ in $\mathbb{D}$ (see Theorem 1.4 in [8]). This observation will provide exactly the information we need to compare the action of C_φ on different spaces.

2. Positive semidefinite matrices

Let $\Lambda = \{\lambda_m\}_{m=1}^{\infty}$, a sequence of distinct points in $\mathbb{D}$, be a set of uniqueness for the collection of analytic functions on $\mathbb{D}$. In other words, the zero function is the only analytic f with $f(\lambda_m) = 0$ for all m. (For example, Λ could be any sequence with a limit point inside $\mathbb{D}$.) The span of the kernel functions $\left\{K_{\lambda_m}^{\alpha}\right\}_{m=1}^{\infty}$ is dense in every space A_{α}^2, since any function orthogonal to each $K_{\lambda_m}^{\alpha}$ must be identically 0. For the duration of this paper, we will assume that such a sequence Λ has been fixed.

Consider an analytic map $\varphi : \mathbb{D} \to \mathbb{D}$. For a positive constant ν, a natural number n, and real number $\alpha \geq -1$, define the $n \times n$ matrix

$$M(\nu, n, \alpha) = \left[\frac{\nu^2}{(1-\overline{\lambda_j}\lambda_i)^{\alpha+2}} - \frac{1}{(1-\overline{\varphi(\lambda_j)}\varphi(\lambda_i))^{\alpha+2}}\right]_{i,j=1}^{n}.$$

Recall that an $n \times n$ matrix A is called *positive semidefinite* if $\langle Ac, c\rangle \geq 0$ for all c in $\mathbb{C}^n$, where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product. Any such matrix must necessarily be self-adjoint. We often write $A \geq 0$ to denote A being positive semidefinite; for self-adjoint matrices A and B, we write $A \geq B$ to denote $A - B$ being positive semidefinite. The following proposition relates $\|C_{\varphi}\|_{A_{\alpha}^2}$ to the positive semidefiniteness of $M(\nu, n, \alpha)$.

Proposition 1. *Let φ be an analytic self-map of $\mathbb{D}$ and ν be a positive constant. Then, for any $\alpha \geq -1$, the matrix $M(\nu, n, \alpha)$ is positive semidefinite for all natural numbers n if and only if $\|C_{\varphi}\|_{A_{\alpha}^2} \leq \nu$.*

Proof. Assume first that $\|C_{\varphi}\|_{A_{\alpha}^2} \leq \nu$, from which it follows that $\|C_{\varphi}^*\|_{A_{\alpha}^2} \leq \nu$. In other words,

$$\|C_{\varphi}^*(f)\|_{A_{\alpha}^2}^2 \leq \nu^2 \|f\|_{A_{\alpha}^2}^2 \tag{2}$$

for all f in A_{α}^2. Let n be any natural number and $c_1, \ldots, c_n$ be complex numbers, and take $f = c_1 K_{\lambda_1}^{\alpha} + \cdots + c_n K_{\lambda_n}^{\alpha}$. If we substitute this function into inequality (2), remembering that $C_{\varphi}^*(K_{\lambda}^{\alpha}) = K_{\varphi(\lambda)}^{\alpha}$, we obtain

$$\|c_1 K_{\varphi(\lambda_1)}^{\alpha} + \cdots + c_n K_{\varphi(\lambda_n)}^{\alpha}\|_{A_{\alpha}^2}^2 \leq \nu^2 \|c_1 K_{\lambda_1}^{\alpha} + \ldots + c_n K_{\lambda_n}^{\alpha}\|_{A_{\alpha}^2}^2,$$

from which it follows that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{c_i} c_j \left\langle K_{\varphi(\lambda_j)}^{\alpha}, K_{\varphi(\lambda_i)}^{\alpha}\right\rangle_{A_{\alpha}^2} \leq \sum_{i=1}^{n}\sum_{j=1}^{n} \nu^2 \overline{c_i} c_j \left\langle K_{\lambda_j}^{\alpha}, K_{\lambda_i}^{\alpha}\right\rangle_{A_{\alpha}^2},$$

and thus

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{c_i} c_j \left(\frac{\nu^2}{(1-\overline{\lambda_j}\lambda_i)^{\alpha+2}} - \frac{1}{(1-\overline{\varphi(\lambda_j)}\varphi(\lambda_i))^{\alpha+2}}\right) \geq 0. \tag{3}$$

Inequality (3) is exactly the statement that $M(\nu, n, \alpha)$ is positive semidefinite.

For the converse, assume that $M(\nu, n, \alpha)$ is positive semidefinite for all natural numbers n. Hence inequality (3) holds for all n, which in turn implies that

$$\|c_1 K^\alpha_{\varphi(\lambda_1)} + \cdots + c_n K^\alpha_{\varphi(\lambda_n)}\|^2_{A^2_\alpha} \le \nu^2 \|c_1 K^\alpha_{\lambda_1} + \cdots + c_n K^\alpha_{\lambda_n}\|^2_{A^2_\alpha} \tag{4}$$

for any n and any complex constants $c_1, \ldots, c_n$. Now let f be an arbitrary element of A^2_α. Since Λ is a set of uniqueness, the span of $\left\{K^\alpha_{\lambda_n}\right\}_{n=1}^\infty$ is dense in A^2_α. Hence there exists a sequence $\{f_m\}_{m=1}^\infty$ that converges to f in norm, where each f_m is a finite linear combination of these kernel functions. Line (4) implies that $\|C^*_\varphi(f_m)\|^2_{A^2_\alpha} \le \nu^2 \|f_m\|^2_{A^2_\alpha}$ for all m. Letting m go to infinity, we see that $\|C^*_\varphi(f)\|^2_{A^2_\alpha} \le \nu^2 \|f\|^2_{A^2_\alpha}$, from which it follows that $\|C_\varphi\|_{A^2_\alpha} = \|C^*_\varphi\|_{A^2_\alpha} \le \nu$. □

In other words, Proposition 1 states that $\|C_\varphi\|_{A^2_\alpha} \le \nu$ exactly when

$$\kappa(\lambda, z) = \frac{\nu^2}{(1 - \overline{\lambda} z)^{\alpha+2}} - \frac{1}{(1 - \overline{\varphi(\lambda)}\varphi(z))^{\alpha+2}}$$

is a positive semidefinite kernel on the unit disk.

Before proceeding to our main results, we need the following lemma relating to positive semidefinite matrices.

Lemma 2. *Let n be a natural number and $\lambda_1, \ldots, \lambda_n$ be a finite collection of (not necessarily distinct) points in $\mathbb{D}$. Any matrix of the form*

$$M = \left[\frac{1}{(1 - \overline{\lambda_j}\lambda_i)^\rho}\right]_{i,j=1}^n,$$

for any real number $\rho \ge 1$, must be positive semidefinite.

Proof. Let $\alpha = \rho - 2$, so that $\alpha \ge -1$. Taking $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$, we see that

$$\langle Mc, c\rangle = \sum_{i=1}^n \sum_{j=1}^n \frac{\overline{c_i} c_j}{(1 - \overline{\lambda_j}\lambda_i)^{\alpha+2}} = \left\langle \sum_{j=1}^n c_j K^\alpha_{\lambda_j}, \sum_{i=1}^n c_i K^\alpha_{\lambda_i} \right\rangle_{A^2_\alpha} \ge 0,$$

from which our assertion follows. □

As a consequence of Lemma 2, we see that any matrix of the form

$$\left[\frac{1}{(1 - \overline{\varphi(\lambda_j)}\varphi(\lambda_i))^\rho}\right]_{i,j=1}^n,$$

where φ is a self-map of $\mathbb{D}$, must also be positive semidefinite.

3. Norm inequalities

The proof of our major theorem relies heavily on the use of Schur products. Recall that, for any two $n \times n$ matrices $A = [a_{i,j}]_{i,j=1}^n$ and $B = [b_{i,j}]_{i,j=1}^n$, the *Schur* (or *Hadamard*) *product* $A \circ B$ is defined by the rule $A \circ B = [a_{i,j} b_{i,j}]_{i,j=1}^n$. In other words, the Schur product is obtained by entrywise multiplication. A proof of the following result appears in Section 7.5 of [12].

Proposition 3 (Schur Product Theorem). *If A and B are $n\times n$ positive semidefinite matrices, then $A\circ B$ is also positive semidefinite.*

We are now in position to state our main result, a theorem that allows us to compare the norms of C_φ on certain spaces.

Theorem 4. *Take $\beta \geq \alpha \geq -1$ and let φ be an analytic self-map of $\mathbb{D}$. Then*

$$\|C_\varphi\|_{A^2_\beta} \leq \|C_\varphi\|^\gamma_{A^2_\alpha} \tag{5}$$

whenever the quantity $\gamma = (\beta+2)/(\alpha+2)$ is an integer.

Proof. Assume that $\gamma = (\beta+2)/(\alpha+2)$ is an integer. Fix a natural number n and let $i, j \in \{1, \dots, n\}$. A difference of powers factorization shows that

$$\begin{aligned}
&\frac{\|C_\varphi\|^{2\gamma}_{A^2_\alpha}}{(1-\overline{\lambda_j}\lambda_i)^{\beta+2}} - \frac{1}{(1-\overline{\varphi(\lambda_j)}\varphi(\lambda_i))^{\beta+2}} \\
&\qquad = \left(\frac{\|C_\varphi\|^{2}_{A^2_\alpha}}{(1-\overline{\lambda_j}\lambda_i)^{\alpha+2}} - \frac{1}{(1-\overline{\varphi(\lambda_j)}\varphi(\lambda_i))^{\alpha+2}}\right) \\
&\qquad\qquad \cdot \left(\sum_{k=0}^{\gamma-1} \frac{\|C_\varphi\|^{2k}_{A^2_\alpha}}{(1-\overline{\lambda_j}\lambda_i)^{(\alpha+2)k}(1-\overline{\varphi(\lambda_j)}\varphi(\lambda_i))^{(\alpha+2)(\gamma-1-k)}}\right).
\end{aligned}$$

Since the preceding equation holds for all i and j, we obtain the following matrix equation:

$$\begin{aligned}
&M(\|C_\varphi\|^\gamma_{A^2_\alpha}, n, \beta) = \\
&M(\|C_\varphi\|_{A^2_\alpha}, n, \alpha) \circ \sum_{k=0}^{\gamma-1}\left[\frac{\|C_\varphi\|^{2k}_{A^2_\alpha}}{(1-\overline{\lambda_j}\lambda_i)^{(\alpha+2)k}(1-\overline{\varphi(\lambda_j)}\varphi(\lambda_i))^{(\alpha+2)(\gamma-1-k)}}\right]^n_{i,j=1}
\end{aligned} \tag{6}$$

where $\circ$ denotes the Schur product. The matrix $M(\|C_\varphi\|_{A^2_\alpha}, n, \alpha)$ is positive semidefinite by Proposition 1. Lemma 2, together with the Schur Product Theorem, dictates that every term in the matrix sum on the right-hand side of (6) is positive semidefinite, so the sum itself is positive semidefinite. Therefore the Schur Product Theorem shows that $M(\|C_\varphi\|^\gamma_{A^2_\alpha}, n, \beta)$ must also be positive semidefinite. Since this assertion holds for every natural number n, Proposition 1 shows that $\|C_\varphi\|_{A^2_\beta} \leq \|C_\varphi\|^\gamma_{A^2_\alpha}$. □

Taking $\alpha = -1$ and $\alpha = 0$, we obtain the following corollaries.

Corollary 5. *Let φ be an analytic self-map of $\mathbb{D}$. Then*

$$\|C_\varphi\|_{A^2_\beta} \leq \|C_\varphi\|^{\beta+2}_{H^2}$$

whenever β is a non-negative integer. In particular, $\|C_\varphi\|_{A^2} \leq \|C_\varphi\|^2_{H^2}$.

Corollary 6. *Let φ be an analytic self-map of $\mathbb{D}$. Then*

$$\|C_\varphi\|_{A^2_\beta} \leq \|C_\varphi\|_{A^2}^{(\beta+2)/2}$$

whenever β is a positive even integer.

Corollary 5 is particularly useful since, as we have already mentioned, more is known about the norm of C_φ on H^2 than on any other space. Hence any result pertaining to $\|C_\varphi\|_{H^2}$ can be translated into an upper bound for $\|C_\varphi\|_{A^2_\beta}$. The significance of Corollary 6 will become apparent in the next section.

There are certainly instances of analytic $\varphi : \mathbb{D} \to \mathbb{D}$ for which there is equality in line (5) for all α and β. If $\varphi(0) = 0$, for example, then line (1) shows that $\|C_\varphi\|_{A^2_\alpha} = 1$ for all α. A slightly less trivial class of examples are the maps $\varphi(z) = sz + t$, where s and t are complex numbers with $|s| + |t| \leq 1$. Combining results of Cowen [5] and Hurst [14], we see that

$$\|C_\varphi\|_{A^2_\alpha} = \left(\frac{2}{1 + |s|^2 - |t|^2 + \sqrt{(1 - |s|^2 + |t|^2)^2 - 4|t|^2}} \right)^{(\alpha+2)/2}$$

for any $\alpha \geq -1$. On the other hand, as noted in [4], there are cases where the inequality in (5) is strict, at least for some choices of α and β. For example, if φ is a non-univalent inner function that does not fix the origin, Theorem 3.3 in [4] shows that $\|C_\varphi\|_{A^2_\beta} < \|C_\varphi\|_{H^2}^{\beta+2}$ for all $\beta > -1$.

4. Open questions

The major unanswered question, of course, is whether the conclusion of Theorem 4 still holds when the quantity γ is not an integer. In particular, one might wonder whether Corollary 6 can be extended to odd values of β.

The proof of Theorem 4 cannot be automatically extended to non-integer values of γ, since the Schur Product Theorem cannot be generalized to non-integer entrywise powers. If $A = [a_{i,j}]_{i,j=1}^n$ is self-adjoint, the *entrywise* (or *Hadamard*) *power* $A^{\circ,\gamma}$ is defined by the rule $A^{\circ,\gamma} = \left[a_{i,j}^\gamma\right]_{i,j=1}^n$, where the arguments of the entries of A are chosen consistently so that all of the matrix powers are self-adjoint. It turns out that the condition $A \geq 0$ does not imply that $A^{\circ,\gamma} \geq 0$ for non-integer values of γ. (If a matrix A does have the special property that $A^{\circ,\gamma} \geq 0$ for all $\gamma \geq 0$, then A is called *infinitely divisible.* A necessary and sufficient condition for this property is discussed in Section 6.3 of [13].) The proof of Theorem 4 essentially involves using the Schur Product Theorem to show that $A \geq B \geq 0$ implies $A^{\circ,k} \geq B^{\circ,k}$ whenever k is a positive integer. Little seems to be known, however, about conditions on A and B which would guarantee that $A \geq B \geq 0$ implies $A^{\circ,\gamma} \geq B^{\circ,\gamma}$ for all $\gamma \geq 1$. Such conditions could help determine to what extent Theorem 4 can be generalized.

Taking a different point of view, one might try to "fill in the gaps" of Theorem 4 using some sort of interpolation argument (such as Theorem 1.1 in [15]). While such techniques initially appear promising, they generally involve working with Hilbert spaces that have equivalent norms to the spaces in which we are interested. Hence such an approach cannot be applied to any question that deals with the precise value of an operator norm.

It might be helpful to recast this question in terms of the relationship between the norm of a composition operator and the property of cosubnormality (that is, the adjoint of the operator being subnormal). Based on the scant evidence we have (see [2] and [3]), one might conjecture that, for any univalent φ with Denjoy–Wolff point on $\partial\mathbb{D}$, the norm of C_φ equals its spectral radius on A^2_α if and only if C_φ is cosubnormal on that space. If that conjecture were accurate, then Corollary 6 would not hold for odd values of β.

In particular, consider the maps of the form

$$\varphi(z) = \frac{(r+s)z + 1 - s}{r(1-s)z + 1 + sr} \tag{7}$$

for $-1 \le r \le 1$ and $0 < s < 1$, a class introduced by Cowen and Kriete [7]. Richman [16] showed that C_φ is cosubnormal on A^2 precisely when $-1/7 \le r \le 1$. On the other hand, he showed in [17] that C_φ is cosubnormal on A^2_1 if and only if $0 \le r \le 1$. Take, for example,

$$\varphi(z) = \frac{7}{8-z},$$

which corresponds to (7) with $r = -1/7$ and $s = 1/7$. We know that C_φ is cosubnormal on A^2, which means that its norm on A^2 is equal to its spectral radius, which is $\varphi'(1)^{-1} = 7$. On the other hand, C_φ is not cosubnormal on A^2_1, so it is *possible* that its norm on that space might exceed its spectral radius, which is $7^{3/2}$. If that were the case, then Corollary 6 – and hence Theorem 4 – would not be valid for intermediate spaces. We have attempted (in the spirit of [1]) to show that $\|C_\varphi\|_{A^2_1} > 7^{3/2}$ through a variety of numerical calculations, all of which have been inconclusive.

The following result, a sort of "cousin" to our Theorem 4, may also be relevant to the question at hand:

Theorem 7 (Cowen [6]**).** *Take $\beta \ge \alpha \ge -1$ and let φ be an analytic self-map of $\mathbb{D}$. Suppose that $\gamma = (\beta+2)/(\alpha+2)$ is an integer. If C_φ is cosubnormal on A^2_α, then it is also cosubnormal on A^2_β.*

Cowen only stated this result for $\alpha = -1$, but an identical argument works for $\alpha > -1$. The proof makes use of the Schur Product Theorem in a similar fashion to that of Theorem 4. Moreover, we know that the result does *not* hold for intermediate spaces. For example,

$$\varphi(z) = \frac{7}{8-z},$$

induces a cosubnormal composition operator on A^2, and hence on A^2_2, but not on the space A^2_1.

References

[1] M.J. Appel, P.S. Bourdon, and J.J. Thrall, Norms of composition operators on the Hardy space, *Experiment. Math.* **5** (1996), 111–117.
[2] E.L. Basor and D.Q. Retsek, Extremal non-compactness of composition operators with linear fractional symbol, *J. Math. Anal. Appl.* **322** (2006), 749–763.
[3] P.S. Bourdon, E.E. Fry, C. Hammond, and C.H. Spofford, Norms of linear-fractional composition operators, *Trans. Amer. Math. Soc.* **356** (2004), 2459–2480.
[4] B.J. Carswell and C. Hammond, Composition operators with maximal norm on weighted Bergman spaces, *Proc. Amer. Math Soc.* **134** (2006), 2599–2605.
[5] C.C. Cowen, Linear fractional composition operators on H^2, *Integral Equations Operator Theory* **11** (1988), 151–160.
[6] C.C. Cowen, Transferring subnormality of adjoint composition operators, *Integral Equations Operator Theory* **15** (1992), 167–171.
[7] C.C. Cowen and T.L. Kriete, Subnormality and composition operators on H^2, *J. Funct. Anal.* **81** (1988), 298–319.
[8] C.C. Cowen and B.D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, 1995.
[9] C. Hammond, On the norm of a composition operator with linear fractional symbol, *Acta Sci. Math.* (*Szeged*) **69** (2003), 813–829.
[10] C. Hammond, Zeros of hypergeometric functions and the norm of a composition operator, *Comput. Methods Funct. Theory* **6** (2006), 37–50.
[11] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman Spaces*, Springer-Verlag, New York, 2000.
[12] R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.
[13] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.
[14] P.R. Hurst, Relating composition operators on different weighted Hardy spaces, *Arch. Math.* (*Basel*) **68** (1997), 503–513.
[15] J.E. McCarthy, Geometric interpolation between Hilbert spaces, *Ark. Mat.* **30** (1992), 321–330.
[16] A.E. Richman, Subnormality and composition operators on the Bergman space, *Integral Equations Operator Theory* **45** (2003), 105–124.
[17] A.E. Richman, Composition operators with complex symbol having subnormal adjoint, *Houston J. Math.* **29** (2003), 371–384.

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Received: January 20, 2009
Accepted: February 2, 2009

Operator Theory:
Advances and Applications, Vol. 202, 273–284

Theory vs. Experiment: Multiplicative Inequalities for the Numerical Radius of Commuting Matrices

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Dedicated to Leiba Rodman on the occasion of his 60th birthday

Abstract. Under what conditions does the inequality $w(TS) \leq w(T)\|S\|$, or the stronger $w(TS) \leq w(T)w(S)$, hold? Here $w(T)$ denotes the numerical radius $\max\{|(Tu, u)| : \|u\| = 1\}$ of the matrix T and $\|S\|$ is the operator norm; we assume that T and S are commuting $n \times n$ matrices. The questions posed above have a long history in matrix analysis and this paper provides new information, combining theoretical and experimental approaches. We study a class of matrices with simple structure to reveal a variety of new counterexamples to the first inequality. By means of carefully designed computer experiments we show that the first inequality may fail even for 3×3 matrices. We also obtain bounds on the constant that must be inserted in the second inequality when the matrices are 3×3. Among other results, we obtain new instances of the phenomenon discovered by Chkliar: for certain contractions C we may have $w(C^{m+1}) > w(C^m)$.

Mathematics Subject Classification (2000). 15A60, 15A04, 47A49.

Keywords. Numerical radius, matrix norm inequalities.

1. Introduction

The numerical range $W(T)$ and the numerical radius $w(T)$ (where T is a matrix) have a long history in matrix analysis. Among the older references we may mention the wide-ranging book [GR1997] by Gustafson and Rao, which includes an

Several results are taken from Schoch's doctoral thesis [Sch2002]. Holbrook's work was supported in part by NSERC of Canada. The authors also thank David Kribs for helpful discussions. This work was partially supported by CFI, OIT, and other funding agencies.
Communicated by V. Bolotnikov.

account of the importance of $w(T)$ estimates for Lax–Wendroff schemes in numerical analysis. Recently numerical ranges of a new type have proved useful in quantum information theory (see, for example, the work of Li and Sze [LS2008]).

In this paper we provide answers, several of which are definitive at least in terms of dimension, to long-standing questions about multiplicative inequalities for the numerical radius. Some of the results depend on a structure theory for the matrices involved while others seem to require carefully designed computer experiments.

The power inequality of C. Berger says that

$$w(T^n) \le w^n(T) \;\; (n = 1, 2, 3, \dots) \tag{1}$$

for any Hilbert space operator T, where $w(T)$ denotes the numerical radius of T:

$$w(T) = \sup\{|(Tu, u)| : \|u\| = 1\}. \tag{2}$$

Berger's original proof (see [B1965]) used a "strange" unitary dilation of T in the case where $w(T) \le 1$. It was natural to combine the Berger dilation of T with the Nagy dilation of a contraction S to obtain the related inequality

$$w(TS) \le w(T)\|S\| \tag{3}$$

under appropriate conditions. In [H1969], for example, (3) is shown to hold when T and S "double commute", i.e., when $TS = ST$ and $TS^* = S^*T$; [H1969] also provides dilation-free proofs of such inequalities.

In view of the Ando dilation theorem, stating that any two commuting contractions have a simultaneous unitary dilation (see [A1963]), it is reasonable to ask whether (3) holds when T and S merely commute. This possibility was raised in [H1969] and later a number of results in this direction were obtained. For example, Bouldin showed that (3) holds whenever S is an isometry commuting with T (see [Bo1971]) and Ando and Okubo showed that

$$w(TS) \le (1/2)\sqrt{2 + 2\sqrt{3}}\,w(T)\|S\| < 1.169w(T)\|S\| \tag{4}$$

for arbitrary commuting T and S (see [OA1976], where parts of the argument are credited to Crabb).

It was not until 1988 that the first "counterexamples" to (3) were found: Müller, following clues in the proof of (4), devised computer experiments to find 12×12 commuting matrices T and S such that $w(TS) > 1.02w(T)\|S\|$ (see [M1988]). Soon after, Davidson and Holbrook (see [DH1988]) found certain commuting 0–1 matrices for which numerical radii could be computed explicitly, so that 9×9 examples with $w(TS) = (1/\cos(\pi/10))w(T)\|S\|$ ($\approx 1.05w(T)\|S\|$) were obtained. In Section 2 we refine the Davidson–Holbrook technique to reveal a variety of new counterexamples, some with size as small as 7×7. It is known (see below) that (3) holds for all commuting 2×2 matrices, so that 3×3 counterexamples are, in a sense, best possible. It turns out that these do exist but are surprisingly elusive. While we have no structure theory for the matrices, we report on carefully designed computer routines that reliably produce 3×3 counterexamples. In

Section 2 we also comment on counterexamples found by Chkliar (see [C1997]) for which $w(C^{m+1}) > w(C^m)$ while $\|C\| = 1$.

Berger's power inequality (1) leads one to wonder under which conditions the numerical radius itself is submultiplicative:

$$w(TS) \le w(T)w(S). \tag{5}$$

Such questions are treated in Section 3. It has long been known that (5) holds for all commuting 2×2 matrices T and S. Strangely, we do not know of an elementary proof of this "elementary" fact, although a proof is included in [H1992]. That argument is extended to certain other situations in Proposition 3.1. It is convenient to introduce the constants

$$b_n = \max\{w(TS) : T, S \text{ are } n \times n, TS = ST, \text{ and } w(T), w(S) = 1\}. \tag{6}$$

Thus $b_2 = 1$. On the other hand, Brown and Shields (as reported by Pearcy, see [P1966]) had noted that $b_4 > 1$. In fact, a modified example shows that $b_4 \ge 2$ and this is best (worst?) possible, so that $b_n = 2$ for all $n \ge 4$ (see [H1969]). Relatively straightforward computer experiments reveal that $b_3 > 1$, but no structure theory for the corresponding matrices is apparent. On the other hand, we can show that $b_3 < 2$ (see Corollary 3.6). Perhaps b_3 is an "interesting" constant but at present we can only say that $1.19 < b_3 < 2$.

2. Multiplicative inequalities relative to $w(T)\|S\|$

Although several of the inequalities we have introduced (e.g., (1) and (4)) hold also in an infinite-dimensional setting, we are concerned in what follows with operators (linear maps) on finite-dimensional complex Hilbert spaces; if the dimension is n we represent the space via column vectors in $\mathbb{C}^n$ and the operators via matrices, i.e., elements of M_n, the algebra of complex $n \times n$ matrices.

A basic tool in [DH1988] is the following identification of $w(T)$ for certain 0–1 matrices T. We'll call a matrix $T \in M_n$ a DH matrix if the entries are in $\{0, 1\}$, the diagonal has only 0's, and for each $k = 1, 2, \dots, n$ the cross-shaped region consisting of the union of the kth row and the kth column contains at most two ones. The corresponding graph $G(T)$, with vertices $1, 2, \dots, n$ and with an edge from i to j iff $t_{ij} = 1$, consists of disjoint chains and cycles. By the length of a chain we mean the number of vertices in the chain.

Proposition 2.1. *Let T be a DH matrix. If $G(T)$ includes a cycle, then $w(T) = 1$. If $G(T)$ has only chains, then $w(T) = \cos(\pi/(m+1))$ where m is the length of the longest chain.*

A proof of this simple proposition may be found in [DH1988]; for other versions of this technique see [MS1979], [HH1992] and [GHJ1989].

In [DH1988] this technique led to examples of commuting $T, S \in M_n$ with $w(TS) = (1/\cos(\pi/(m+1)))w(T)\|S\|$, i.e., violations of (3) having a well-understood structure. The largest value of the constant obtained was $1/\cos(\pi/9) \approx$

1.064, while the smallest dimension n obtained was 9. Here we refine this technique via the following observation.

Proposition 2.2. *Let S be the backward shift (nilpotent Jordan block) on $\mathbb{C}^n$ and let $T(x) = S^k + xS^j$ where $1 < k < j < n$ and $0 < x$. Let $p(\lambda, x)$ be the characteristic polynomial of $A(x) = (T(x) + T^t(x))/2$. Then the derivative of $w(T(x))$ at $x = 1$ is given by*

$$\frac{d}{dx} w(T(x))\big|_{x=1} = -D_2 p(w(T(1), 1)/D_1 p(w(T(1)), 1), \tag{7}$$

provided that $D_1 p(w(T(1)), 1) \neq 0$. If, on the other hand, $D_1 p(w(T(1)), 1) = 0$, then $s = \frac{d}{dx} w(T(x))\big|_{x=1}$ may be obtained as a root of the quadratic

$$D_{11} p(w(T(1)), 1)s^2 + 2D_{12} p(w(T(1)), 1)s + D_{22}(w(T(1)), 1) = 0. \tag{8}$$

Proof. Since the elements of $T(x)$ are nonnegative,

$$w(T(x)) = \max\{(T(x)u, u) : \|u\| = 1 \text{ and all } u_i \geq 0\}.$$

Since $(T(x)u, u)$ is real, this is also

$$\max\{((T(x)u, u) + (T^t(x)u, u))/2 : \|u\| = 1, u_i \geq 0\},$$

i.e., the largest eigenvalue $\lambda_1(x)$ of the real symmetric matrix $A(x)$. In case $D_1 p(w(T(1)), 1) \neq 0$, $\lambda_1(1)$ is a simple eigenvalue of $A(1)$ and, in a neighborhood of 1, $\lambda_1(x)$ is a differentiable function of x. Since $p(\lambda_1(x), x) = 0$, implicit differentiation gives (7). Differentiating the relation $p(\lambda_1(x), x) = 0$ twice with respect to x yields (8) in those cases where $D_1 p(w(T(1)), 1) = 0$. $\square$

With notation as in Proposition 2.2, it is easy to see that $T(1) = S^k + S^j$ is a DH matrix provided $k + j \geq n$. In this case Propositions 2.1 and 2.2 combine (with the help of computer algebra software to manipulate the polynomials $p(\lambda, x)$!) to display or explain a number of "violations" of (3). We have, for example, 7×7 counterexamples T and S as follows.

Proposition 2.3. *Let S be the shift on $\mathbb{C}^7$ and let $T(x) = S^2 + xS^5$. Then for some $x > 1$ we have $w(T(x)S) > w(T(x))\|S\|$.*

Proof. Both $T(1)$ and $T(1)S = S^3 + S^6$ are DH matrices. The corresponding graphs $G(T(1))$ and $G(T(1)S)$ are easily constructed. In each case the graph includes a cycle and Proposition 2.1 tells us that $w(T(1)) = 1 = w(T(1)S)$. Computing the characteristic polynomial $p(\lambda, x)$ of $(T(x) + T^t(x))/2$ we find that

$$\begin{aligned} p(\lambda, x) = \lambda^7 - \frac{1}{2}\lambda^5 x^2 - \frac{5}{4}\lambda^5 + \frac{1}{16}\lambda^3 x^4 + \frac{3}{8}\lambda^3 x^2 + \frac{7}{16}\lambda^3 \\ - \frac{1}{16}\lambda x^2 - \frac{1}{32}\lambda - \frac{1}{64}\lambda x^4 - \frac{1}{64}x^2. \end{aligned}$$

Since $D_1 p(1, 1) = 49/64$ ($\neq 0$) and $D_2 p(1, 1) = -7/32$, Proposition 2.2 tells us that $\frac{d}{dx} w(T(x))|_{x=1} = 2/7$.

On the other hand, the characteristic polynomial

$$q(\lambda, x) \quad \text{of} \quad (T(x)S + (T(x)S)^t)/2$$

satisfies

$$\begin{aligned} q(\lambda, x) = \lambda^7 - \lambda^5 + \frac{5}{16}\lambda^3 - \frac{1}{32}\lambda - \frac{1}{4}\lambda^5 x^2 \\ - \frac{1}{4}\lambda^4 x + \frac{1}{8}\lambda^3 x^2 + \frac{1}{8}\lambda^2 x - \frac{1}{64}\lambda x^2 - \frac{1}{64}x, \end{aligned}$$

so that $D_1 q(1,1) = 81/64$ ($\neq 0$) and $D_2 q(1,1) = -27/64$. Proposition 2.2 tells us that $\frac{d}{dx} w(T(x)S)|_{x=1} = 1/3$. Since $\frac{1}{3} > \frac{2}{7}$, $w(T(x)S) > w(T(x))$ for some values of $x > 1$. Since $\|S\| = 1$, the proposition follows. $\square$

In fact, numerical experiments reveal that, with $T(x)$ and S as in the proposition above, the ratio $w(T(x)S)/w(T(x))\|S\|$ is maximized when $x \approx 2.34$ with a value close to 1.022.

Essentially the same argument as that used in Proposition 2.3 yields a number of additional new counterexamples, "new" in the sense that the original Davidson–Holbrook technique fails in those cases. Using the notation of Proposition 2.2, such examples occur when

$$(n, k, j) = (10, 2, 8), (10, 3, 7), (13, 2, 11), (14, 3, 11), (16, 2, 14), \text{ and } (16, 5, 11).$$

In each case $w(T(1)) = w(T(1)S) = 1$ but the argument of Proposition 2.3 can be adapted to show that $w(T(x)S) > w(T(x))$ for some values of $x > 1$.

In [DH1988] (see Corollary 5 and the Remark following it) it was observed that with $(n, k, j) = (16, 4, 14)$ we have $w(T(1)) = \cos(\pi/9)$ and $w(T(1)S) = 1$, yielding the ratio $1/\cos(\pi/9) \approx 1.064$, but that numerical experiments with $x \approx 1.22$ yield a larger ratio: $w(((x)S)/w(T(x)) \approx 1.066$. Proposition 2.2 (this time including the exceptional case where $D_1 p(w(T(1)), 1) = 0$) explains this earlier observation, as follows.

Proposition 2.4. *Let S be the shift on $\mathbb{C}^{16}$ and let $T(x) = S^4 + xS^{14}$. For some $x > 1$ we have*

$$\frac{w(T(x)S)}{w(T(x))} > \frac{w(T(1)S)}{w(T(1))}. \tag{9}$$

Proof. The graph $G(T(1))$ consists of two chains, each of length 8. Thus, by Proposition 2.1, $w(T(1)) = \cos(\pi/9)$. On the other hand, $G(T(1)S)$ includes a cycle (vertices 1,6,11,16) so that $w(T(1)S) = 1$.

Computing the characteristic polynomial $p(\lambda, x)$ of $(T(x) + T^t(x))/2$, we find that $D_1 p(\cos(\pi/9), 1) = 0$, so we use (8) to compute $s = \frac{d}{dx} w(T(x))|_{x=1}$. These calculations yield 0.2155 as the approximate value of s.

It turns out that $D_1 q(1,1) \neq 0$, where $q(\lambda, x)$ is the characteristic polynomial of $(T(x)S + (T(x)S)^t)/2$. Thus we can use the appropriate version of (7) to discover that

$$\frac{d}{dx} w(T(x)S)|_{x=1} = \frac{1}{4}.$$

Since $1/4 > 0.2155/\cos(\pi/9)$, the inequality (9) follows for (small enough) $x > 1$. $\square$

Remark 2.5. The ratio of approximately 1.066, obtained by optimizing with respect to x in the proposition above, is the largest known to us, regardless of the techniques used to find the commuting T, S.

We now turn to questions that seem to require an experimental approach. Consider first an $n \times n$ matrix C with $\|C\| = 1$; a special case of (3) (taking $T = C^m$ and $S = C$) would imply that

$$w(C^{m+1}) \leq w(C^m). \tag{10}$$

Indeed, Berger's inequality (1) implies that $w(C^2) \leq (w(C))^2 \leq w(C)$, since $w(C) \leq \|C\| = 1$. Chkliar showed, however, that (10) may fail for $m = 3$ (see [C1997]). He modified an example from [DH1988], using the shift S on $\mathbb{C}^9$: with $S' = S + \frac{1}{4}S^5$ we have $(S')^4 = S^4 + S^8$ and $(S')^3 = S^3 + \frac{3}{4}S^7$. With $C = S'/\|S'\|$, he noted that $w(C^4) \geq 1.0056 w(C^3)$. Numerical experiments reported in [Sch2002] showed that an "improvement" results from taking $S' = S + 0.144S^5$ (and C as above): then $w(C^4) \geq 1.0118 w(C^3)$. More recently, working with contractions C having no special structure and numerical experiments involving appropriate optimization techniques (see below), we have found that the Chkliar phenomenon can occur in dimensions lower than 9. For example, there exist $C \in M_7$ such that $\|C\| = 1$ and $w(C^4) \geq 1.018 w(C^3)$. Two natural questions remain unanswered at this time, but may be clarified by further experimental work:
(i) What is the minimal dimension at which the Chkliar phenomenon can occur?
(ii) Can (10) fail also with $m = 2$?

Next, returning to the general form of (3), we report that failure can occur even for commuting $T, S \in M_3$, the lowest possible dimension (in view of Corollary 3.2, for example). We note that the structured examples considered above were 7×7 or larger (see Proposition 2.3), but that 4×4 examples (with no special structure) were found in [Sch2002]. More recently we have had success in experiments with 3×3 matrices. Two optimization techniques have proved the most useful (used individually or in conjunction): simulated annealing and particle swarm optimization (see [SA], [PSO], and the references cited there). Implementation of our algorithms is coded in FORTRAN 95; detailed programming information is available from the authors. We find that optimized commuting $T, S \in M_3$ can be as good (bad?) as

$$\frac{w(TS)}{w(T)\|S\|} \geq 1.017.$$

Remark 2.6. Given that such examples exist, it would be possible in principle to find some of them by means of a simple random search. In practice, it appears that a more sophisticated approach is required.

3. Multiplicative inequalities relative to $w(T)w(S)$

If the $n \times n$ matrix T has n distinct eigenvalues λ_k and $TS = ST$ then the corresponding eigenvectors v_k are also eigenvectors for S: $Sv_k = \mu_k v_k$. With this understanding we shall say that the μ_k are *matching* eigenvalues of S. The following proposition gives a useful sufficient condition for $w(TS) \leq w(T)w(S)$.

PROPOSITION 3.1 Suppose that T has distinct eigenvalues λ_k and that $w(T) > \max|\lambda_k|$ (generic conditions). If $TS = ST$ and μ_k are eigenvalues of S matching the λ_k, then $w(TS) \leq w(T)w(S)$ provided that

$$\left[\frac{w^2(S) - \overline{\mu_i}\mu_j}{w^2(T) - \overline{\lambda_i}\lambda_j}\right] \geq 0,$$

i.e., the matrix is positive semidefinite.

Proof. Equivalently, we show that $w(AB) \leq 1$ where $A = T/w(T)$ and $B = S/w(S)$. The eigenvalues $\alpha_k = \lambda_k/w(T)$ of A lie in the open unit disc $\mathbb{D}$ and those of B, i.e., $\beta_k = \mu_k/w(S)$, satisfy the Pick interpolation condition:

$$\left[\frac{1 - \overline{\beta_i}\beta_j}{1 - \overline{\alpha_i}\alpha_j}\right] \geq 0.$$

Thus (see, for example, [Ma1974]) there exists analytic $f : \mathbb{D} \to \overline{\mathbb{D}}$ such that $f(\alpha_k) = \beta_k$ for each k, i.e., $f(A) = B$. A theorem of Berger and Stämpfli (see [BS1967]) says that if $w(A) \leq 1$ and analytic $g : \mathbb{D} \to \overline{\mathbb{D}}$ has $g(0) = 0$ then also $w(g(A)) \leq 1$ (see also Kato [K1965]). Hence, setting $g(z) = zf(z)$ we see that $w(AB) \leq 1$. □

This general result provides one proof (see below) of the fact that (using the notation of (6)) $b_2 = 1$; this curiosity is perhaps "folklore" among numerical radius enthusiasts, but we do not know of an elementary proof.

Corollary 3.2. *For all commuting* 2×2 *matrices* T, S *we have* $w(TS) \leq w(T)w(S)$.

Proof. It is easy to see that T and S may be approximated arbitrarily well by commuting matrices with distinct eigenvalues (this is true for two commuting matrices of arbitrary dimension, but that is harder to prove; for some of the history of such results, and the surprising failure of the corresponding result for commuting *triples*, see [HO2001]). Thus we assume that T, S have distinct eigenvalues. By homogeneity, we may assume also that $w(T) = w(S) = 1$. It is well known that a nonnormal 2×2 has as numerical range a nondegenerate filled ellipse with the eigenvalues as foci. Thus we may assume that the eigenvalues $\lambda_1, \lambda_2, \mu_1, \mu_2$ of T and S lie in $\mathbb{D}$, except when one is normal; but then both are normal and our inequality is trivial. It remains to check the Pick condition: the determinant of the Pick matrix is nonnegative iff

$$\frac{|1-\overline{\lambda_1}\lambda_2|^2}{(1-|\lambda_1|^2)(1-|\lambda_2|^2)} \geq \frac{|1-\overline{\mu_1}\mu_2|^2}{(1-|\mu_1|^2)(1-|\mu_2|^2)}.$$

Thus we may either apply Proposition 3.1 to T and S or to the matrices with the roles of T and S exchanged. In either case we obtain $w(TS) \leq 1$. □

Straightforward computer experiments reveal that $b_3 > 1$, i.e., that there exist commuting 3×3 matrices T, S such that $w(TS) > w(T)w(S)$. Efforts to optimize such examples lead us to the estimate $b_3 > 1.19$, but it is difficult to know how accurate such lower bounds are. We do know, however, (see Corollary 3.6, below) that $b_3 < 2$ whereas, as explained in Section 1, $b_n = 2$ for all $n \geq 4$.

Recall that the numerical range of a matrix T is the set $W(T) = \{(Tu, u) : \|u\| = 1\}$, so that $w(T) = \max\{|z| : z \in W(T)\}$; a compactness argument shows that $W(T)$ is closed in the finite-dimensional (matrix) setting.

Proposition 3.3. *Suppose that $n \times n$ matrices T and S commute and satisfy the relations $w(T) = w(S) = 1$ and $w(TS) = 2$. Then $W(T) = W(S) = \overline{\mathbb{D}}$, the closed unit disc.*

Proof. This approach involves an induction on n. We know that the hypotheses are never satisfied when $n = 1$ or $n = 2$; these facts provide the base for the induction.

Let θ be any real; we show that $e^{i\theta} \in W(T)$ and hence, by symmetry, $e^{i\theta} \in W(S)$ as well. We have $w(e^{-i\theta}TS) = 2$ so that for some real φ and unit vector u we have $(e^{-i\theta}Te^{i\varphi}Su, u) = 2$. Let $A = e^{-i\theta}T$ and $B = e^{i\varphi}S$ so that $w(A) = w(B) = 1$, $AB = BA$, and $(ABu, u) = 2$. It remains to show that $1 \in W(A)$.

Note that $((A+B)^2u, u) - ((A-B)^2u, u) = 8$. Using Berger's inequality (1) we have

$$w((A \pm B)^2) \leq (w(A \pm B))^2 \leq (w(A) + w(B))^2 = 4.$$

Thus we must have $((A \pm B)^2u, u) = \pm 4$. Let $v_\pm = (2I \pm (A+B))u$ so that
$((2I - (A+B))v_+, u) = 0$; since $u = (v_+ + v_-)/4$ we also have
$((2I - (A+B))v_+, v_+ + v_-) = 0$, i.e.,

$$((2I - (A+B))v_+, v_+) = -((2I - (A+B))v_+, v_-) = -((2I + (A+B))v_-, v_-).$$

Now $w(A+B) \leq 2$ so that $W(2I - (A+B)) \subseteq 2 + 2\overline{\mathbb{D}}$ and

$$((2I - (A+B))v_+, v_+) \in \|v_+\|^2(2 + 2\overline{\mathbb{D}}) = Q_+.$$

Similarly

$$-((2I + (A+B))v_-, v_-) \in \|v_-\|^2(-2 + 2\overline{\mathbb{D}}) = Q_-.$$

Since $Q_+ \cap Q_- = \{0\}$, we must have $((2I - (A+B))v_+, v_+) = 0$. If $v_+ \neq \vec{0}$ we have $((A+B)u_+, u_+) = 2$, where $u_+ = v_+/\|v_+\|$. Since $W(A), W(B) \subseteq \overline{\mathbb{D}}$, we must have $(Au_+, u_+) = 1$ so that $1 \in W(A)$.

If $v_+ = \vec{0}$, then $(A+B)u = -2u$. Since A and B commute with $A+B$ there is a common unit eigenvector w with $(A+B)w = -2w$, $Aw = \lambda w$, and $Bw = \mu w$. The eigenvalues of A and B lie within their numerical ranges, so that $|\lambda|, |\mu| \leq 1$. It follows that $\lambda = \mu = -1$, since $\lambda+\mu = -2$. Thus $(Aw, w) = -1 = \min\{\mathrm{Re}(Ah, h) : \|h\| = 1\}$, i.e.,

$$\left(\frac{A+A^*}{2}w, w\right) = -1 = \min\left\{\left(\frac{A+A^*}{2}h, h\right) : \|h\| = 1\right\},$$

so that w is an eigenvector for the Hermitian $(A+A^*)/2$ with $(A+A^*)w = -2w$; it follows that $A^*w = -w$ also.
(This argument illustrates the more general fact that an eigenvalue lying on the boundary of the numerical range of a matrix must be a reducing eigenvalue; see, for example, Theorem 5.1-9 in [GR1997].)

With respect to the decomposition $\mathrm{span}\{w\} \oplus w^\perp$, $A = -1 \oplus A_0$ and $B = -1 \oplus B_0$. Now $W(A) = \mathrm{conv}\{-1, W(A_0)\}$ and similarly for B and AB. It follows that the commuting A_0, B_0 satisfy the relations $w(A_0), w(B_0) \leq 1$ and $w(A_0B_0) = 2$ so that, by induction, $W(A_0) = W(B_0) = \overline{\mathbb{D}}$ and, finally, $W(A) = W(B) = \overline{\mathbb{D}}$. $\square$

Remark 3.4. One sees Proposition 3.3 in action through the examples that show $b_n = 2$ for $n \geq 4$. In particular, the 4×4 commuting matrices

$$T = 2(I_2 \otimes J_2) \quad \text{and} \quad S = 2(J_2 \otimes I_2),$$

where $J_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, satisfy

$$w(T) = w(S) = 1 \quad \text{and} \quad w(TS) = 2,$$

and it is easy to check that, indeed, $W(T) = W(S) = \overline{\mathbb{D}}$.

Remark 3.5. The connoisseur will detect (in part of our proof of Proposition 3.3) some echoes of the technique introduced by Carl Pearcy in his elementary proof of Berger's inequality (1); see [P1966].

Corollary 3.6. *If T and S are commuting 3×3 matrices then $w(TS) < 2w(T)w(S)$; hence the constant b_3 (defined in (6)) satisfies $1.19 < b_3 < 2$.*

Proof. The lower bound 1.19 comes from the numerical experiments discussed above. Compactness arguments ensure that b_3 is attained, so to show $b_3 < 2$ it is sufficient to argue that commuting 3×3 matrices T, S with $w(T) = w(S) = 1$ and $w(TS) = 2$ cannot occur.

By Proposition 3.3, such hypothetical T, S would satisfy $W(T) = W(S) = \overline{\mathbb{D}}$. Fortunately, the possible geometry of $W(X)$ is well understood when X is 3×3. For example, Keeler, Rodman, and Spitkovsky explore this matter in [KRS1997]; we shall use a characterization of the case $W(X) = \overline{\mathbb{D}}$ due to Chien and Tam (see [CT1994]), who refer also to earlier results of N.K. Tsing. Theorem 2 from [CT1994] says that if a 3×3 matrix X has $W(X) = \overline{\mathbb{D}}$ and is in uppertriangular form, then at least two of the diagonal entries (eigenvalues) are 0, and the strictly uppertriangular entries x, y, z satisfy $|x|^2 + |y|^2 + |z|^2 = 4$; there is an additional relation but we do not need it here (it completes the list of necessary and sufficient conditions).

We consider several cases:

(i) *T and S are nilpotent.* Since they commute we may put them simultaneously in uppertriangular form (by a unitary similarity); say

$$T = \begin{bmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then

$$TS = \begin{bmatrix} 0 & 0 & x_1 z_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and $x_1 z_2 = x_2 z_1 = (x_1 z_2 + x_2 z_1)/2$ so that

$$|x_1 z_2| \leq ((|x_1|^2 + |z_2|^2)/2 + (|x_2|^2 + |z_1|^2)/2)/2 \leq (4+4)/4 = 2;$$

it follows easily from the form of TS that $w(TS) \leq 1$, a contradiction.

(ii) *Among the eigenvalues of T and S the largest in modulus is $a_1 \neq 0$.* We may assume that a_1 belongs to T and put T and S simultaneously in uppertriangular form with

$$T = \begin{bmatrix} a_1 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{bmatrix}$$

and S in one of three forms (iiA), (iiB), or (iiC):

$$\begin{bmatrix} a_2 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & x_2 & y_2 \\ 0 & b_2 & z_2 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & c_2 \end{bmatrix}.$$

In case (iiA), we have $a_2 = ra_1$ with $|r| \leq 1$. Since $TS = ST$ we have $a_1 x_2 = a_2 x_1$, so that $x_2 = rx_1$. Thus $S = rT + R$ where R has the form

$$\begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus $TR = RT = 0_3$, the 3×3 zero matrix. It follows that $TS = rT^2$ and using once again Berger's inequality (1) we have $w(TS) \leq r \leq 1$, a contradiction.

Finally, in cases (iiB) and (iiC), comparing TS and ST and recalling that $a_1 \neq 0$, we see by elementary arguments that $TS = ST \implies TS = 0_3$, i.e., $w(TS) = 0$, a contradiction. $\square$

References

[A1963] T. Ando, *On a pair of commutative contractions.* Acta Sci. Math. (Szeged) **24**, pp. 88–90 (1963)

[B1965] C.A. Berger, *A strange dilation theorem.* Notices A.M.S. **12**, p. 590 (1965)

[Bo1971] R. Bouldin, *The numerical range of a product II.* J. Math, Anal. Appl. **33**, pp. 212–219 (1971)

[BS1967] C.A. Berger and J.G. Stämpfli, *Mapping theorems for the numerical radius.* Amer. J. Math. **89**, pp. 1047–1055 (1967)

[C1997] V. Chkliar, *Numerical radii of simple powers.* Linear Algebra Appl. **265**, pp. 119–121 (1997)

[CT1994] M.-T. Chien and B.-S. Tam, *Circularity of the numerical range.* Linear Algebra Appl. **201**, pp. 113–133 (1994)

[DH1988] K.R. Davidson and J. Holbrook, *Numerical radii for zero-one matrices.* Michigan Math. J. **35**, pp. 261–267 (1988)

[GHJ1989] F.M. Goodman, P. de la Harpe, and V.F.R. Jones, *Coxeter Graphs and Towers of Algebras.* Springer-Verlag (1989)

[GR1997] K.E. Gustafson and D.K.M. Rao, *Numerical Range.* Springer-Verlag (1997)

[H1969] J. Holbrook, *Multiplicative properties of the numerical radius in operator theory.* J. Reine Angew. Math. **237**, pp. 166–174 (1969)

[H1992] J. Holbrook, *Inequalities of von Neumann type for small matrices.* Function Spaces (ed. K. Jarosz), pp. 189–193 (1992)

[HH1992] U. Haagerup and P. de la Harpe, *The numerical radius of a nilpotent operator on a Hilbert space.* Proceedings A.M.S. **115**, pp. 371–379 (1992)

[HO2001] J. Holbrook and M. Omladič, *Approximating commuting operators.* Linear Algebra Appl. **327**, pp. 131–149 (2001)

[K1965] T. Kato, *Some mapping theorems for the numerical range.* Proc. Japan Acad. **41**, pp. 652–655 (1965)

[KRS1997] D.S. Keeler, L. Rodman, and I.M. Spitkovsky, *The numerical range of* 3×3 *matrices.* Linear Algebra Appl. **252**, pp. 115–139 (1997)

[LS2008] C.-K. Li and N.-S. Sze, *Canonical forms, higher rank numerical ranges, totally isotropic subspaces, and matrix equations.* Proc. Amer. Math. Soc. **136**, pp. 3013–3023 (2008)

[M1988] V. Müller, *The numerical radius of a commuting product.* Michigan Math. J. **35**, pp. 255–260 (1988)

[Ma1974] D.E.D. Marshall, *An elementary proof of the Pick–Nevanlinna interpolation theorem.* Michigan Math. J. **21**, pp. 219–223 (1974)

[MS1979] M. Marcus and B.N. Shure, *The numerical range of certain* $(0, 1)$*-matrices.* Linear and Multilinear Algebra **7**, no. 2, 111–120 (1979)

[OA1976] K. Okubo and T. Ando, *Operator radii of commuting products.* Proc. A.M.S. **56**, pp. 203–210 (1976)

[P1966] C. Pearcy, *An elementary proof of the power inequality for the numerical radius.* Michigan Math. J. **13**, pp. 289–291 (1966)

[PSO] http://en.wikipedia.org/wiki/Particle_swarm_optimization

[SA] http://en.wikipedia.org/wiki/Simulated_annealing

[Sch2002] J.-P. Schoch, *Theory vs Experiment in Matrix Analysis.* PhD thesis, University of Guelph, Canada (2002)

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Received: November 8, 2008
Accepted: June 3, 2009

Operator Theory:
Advances and Applications, Vol. 202, 285–295

Best Constant Inequalities Involving the Analytic and Co-Analytic Projection

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In memory of Professor Israel Gohberg

Abstract. Let P_+ denote the Riesz projection and $P_- = I - P_+$ denote the co-analytic projection where I is the identity operator. We prove

$$\| \max(|P_+ f|,\, |P_- f|)\|_{L^p(\mathbf{T})} \le \csc\frac{\pi}{p}\, \|f\|_{L^p(\mathbf{T})}, \qquad 1 < p < \infty,$$

where $f \in L^p(\mathbf{T})$ is a complex-valued function, and the constant $\csc\frac{\pi}{p}$ is sharp. Our proof is based on an explicit construction of a plurisubharmonic minorant for the function $F(w,z) = \csc^p \frac{\pi}{p}\, |w + \bar{z}|^p - \max(|w|,\, |z|)^p$ on $\mathbf{C}^2$.

More generally, we discuss the best constant problem for the inequality

$$\|(|P_+ f|^s + |P_- f|^s)^{\frac{1}{s}}\|_{L^p(\mathbf{T})} \le C(p,s)\, \|f\|_{L^p(\mathbf{T})}, \qquad 1 < p < \infty,$$

where $0 < s < \infty$, which may serve as a model problem for some vector-valued inequalities, where the method of plurisubharmonic minorants seems to be promising.

Mathematics Subject Classification (2000). Primary 42A50, 47B35; Secondary 31C10, 32A35.

Keywords. Analytic projection, Hilbert transform, best constants, plurisubharmonic functions.

1. Introduction

Let $\mathbf{T}$ represent the unit circle, and $\mathbf{D}$ the unit disc in the complex plane. A function f, analytic in $\mathbf{D}$, is in the Hardy space H^p $(0 < p < \infty)$ if

$$\|f\|_{H^p} = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^p \, dt \right)^{\frac{1}{p}} < \infty.$$

The second author was supported in part by NSF Grant DMS-0556309.
Communicated by I.M. Spitkovsky.

As usual, boundary values of functions $f \in H^p$ will be identified with a subspace of $L^p(\mathbf{T})$. By P_+ we denote the Riesz projection from $L^p(\mathbf{T})$ onto H^p, where $1 < p < \infty$. In other words, if $f \in L^p(\mathbf{T})$ $(1 < p < \infty)$ is of the form, $f(e^{it}) = \sum_{n\in\mathbf{Z}} \hat{f}(n)e^{\mathrm{int}}$, then

$$P_+ f(e^{it}) = \sum_{n\geq 0} \hat{f}(n)e^{\mathrm{int}} \in H^p.$$

Similarly, we can define the co-analytic projection, $P_- f(e^{it}) = \sum_{n<0} \hat{f}(n)e^{\mathrm{int}}$. In [HV] it was shown

$$\| \max(|P_+ f|, |P_- f|)\|_{L^p(\mathbf{T})} \leq \csc \frac{\pi}{p} \, \|f\|_{L^p(\mathbf{T})}, \qquad 1 < p \leq 2, \tag{1.1}$$

where the constant $\csc \frac{\pi}{p}$ is sharp. In Section 2 we will extend (1.1) to $2 < p < \infty$. We remark that this result answers a question raised by the late Matts Essén in correspondence with the second author in January, 2000.

It should be mentioned that a consequence of (1.1) is the following inequality:

$$\|P_\pm f\|_{L^p(\mathbf{T})} \leq \csc \frac{\pi}{p} \, \|f\|_{L^p(\mathbf{T})}, \qquad 1 < p < \infty. \tag{1.2}$$

Inequality (1.2) verifies a conjecture made by Gohberg and Krupnik in 1968 [GKr1] and solves a problem stated later by Pełczyński ([Pe], Problem 3). The constant $\csc \frac{\pi}{p}$ in (1.2) is sharp, as was already shown in [GKr1] (see also [GKr2], [KrV]).

The proof of (1.2) along with other related results was given in [HV] where plurisubharmonic functions on $\mathbf{C}^2$ were used in this context for the first time. Subharmonic minorants for best constant inequalities involving the Hilbert transform were used earlier in [P], [E], [V]. This method was developed systematically and applied to a variety of best constant inequalities in [HKV].

A recent survey of best constant inequalities for one-dimensional singular integral operators is given in [Kr]. Some related open problems, including the best constants for vector-valued analogues of (1.1), (1.2) are discussed in Section 3 below.

2. A best constant inequality involving P_+ and P_-

To begin, we need an analogue of subharmonic functions that is valid for functions of two complex variables. These are known as *plurisubharmonic* functions. A function is plurisubharmonic if its restrictions to complex lines are subharmonic [Ra].

Let us assume the following lemma.

Lemma 2.1. *Let $1 < p < \infty$. Then there exists a plurisubharmonic function, $F(w, z)$, with the property $F(w, 0) = 0$ for every $w \in \mathbf{C}$, such that for every $(w, z) \in \mathbf{C}^2$*

$$\max\left(|w|^p, \, |z|^p\right) \leq a_p \, |w + \bar{z}|^p - F(w, z), \tag{2.1}$$

where $a_p = \csc^p \frac{\pi}{p}$.

Given Lemma 2.1, we immediately prove the main result of this section.

Theorem 2.2. *Let* $1 < p < \infty$ *and* $f \in L^p(\mathbf{T})$ *be a complex-valued function. Then*

$$\|\max(|P_+f|,\, |P_-f|)\|_{L^p(\mathbf{T})} \le \csc\frac{\pi}{p}\,\|f\|_{L^p(\mathbf{T})}, \tag{2.2}$$

and the constant $\csc\frac{\pi}{p}$ *is sharp.*

Proof. Since the trigonometric polynomials are dense in $L^p(\mathbf{T})$, we may assume $f(e^{it}) = \sum_{n=-k}^{m} \hat{f}(n)\, e^{\mathrm{int}}$, $e^{it} \in \mathbf{T}$. Hence we define

$$f_+(e^{it}) = P_+f(e^{it}) = \sum_{n=0}^{m} \hat{f}(n)\, e^{\mathrm{int}}, \quad f_-(e^{it}) = \overline{P_-f}(e^{it}) = \sum_{n=-k}^{-1} \overline{\hat{f}(n)}\, e^{-int}.$$

Notice that f_+ and f_- are analytic trigonometric polynomials on $\mathbf{T}$ and they can be extended to polynomials defined on $\mathbf{C}$. So we can replace z and w in (2.1) with f_+ and f_- to obtain

$$\max(|f_+(e^{it})|^p,\, |f_-(e^{it})|^p) \le a_p|f(e^{it})|^p - F(f_+(e^{it}),\, f_-(e^{it})). \tag{2.3}$$

Therefore, integrating both sides of (2.3) over $\mathbf{T}$ yields

$$\int_{\mathbf{T}} \max(|f_+(e^{it})|^p,\, |f_-(e^{it})|^p)\, dt \le a_p \int_{\mathbf{T}} |f|^p\, dt - \int_{\mathbf{T}} F(f_+(e^{it}), f_-(e^{it}))\, dt.$$

Since $f_-(z)$ and $f_+(z)$ are analytic functions, their composition with the plurisubharmonic function F yields a subharmonic function $F(f_+(z), f_-(z))$ on $\mathbf{C}$ ([Ra], Theorem 4.13). Also, since f_- involves only exponentials e^{ikt} with negative k, it follows that $f_-(0) = 0$ and thus $F(f_+(0), f_-(0)) = F(w, 0) = 0$ by the definition of F. So we can use the sub-mean-value property to conclude

$$\frac{1}{2\pi}\int_{\mathbf{T}} F(f_+(e^{it}), f_-(e^{it}))\, dt \ge F(f_+(0),\, f_-(0)) = 0. \tag{2.4}$$

Thus we have

$$\int_{\mathbf{T}} \max(|f_+(e^{it})|^p,\, |f_-(e^{it})|^p)\, dt \le \csc^p\frac{\pi}{p} \int_{\mathbf{T}} |f|^p\, dt.$$

Dividing each side by 2π and then taking the pth root, we arrive at (2.2).

Since it has already been shown in [GKr1] that $\|P_\pm\|_{L^p(\mathbf{T})} \ge \csc\frac{\pi}{p}$ for $1 < p < \infty$, it follows that the constant is sharp. □

We now prove Lemma 2.1.

Proof. For $1 < p \le 2$ the proof can be found in [HV] (Lemma 2.2). Thus we need only verify (2.1) for $2 < p < \infty$. If $z = re^{it}$ and $w = \rho e^{i\theta}$ with $0 \le r, \rho < \infty$ and $-\pi \le t, \theta \le \pi$, then let us define $F(w, z) = b_p\Phi(\sqrt{wz})$ where $b_p = 2|\cos\frac{\pi}{p}|^{1-\frac{p}{2}}\csc\frac{\pi}{p}$ and $\Phi(z) = r^p\phi(t)$ is a subharmonic function. Here $\sqrt{z} = r^{\frac{1}{2}}e^{i\frac{t}{2}}$ if $z = re^{it}$, $r \ge 0$, $-\pi < t \le \pi$.

If $\Phi(z)$ is subharmonic then we automatically have that $F(w,z)$ is plurisubharmonic ([Ra], Theorem 4.13) and we write it in the following form: $F(w,z) = (r\rho)^{\frac{p}{2}}\phi(\frac{t+\theta}{2})$. To define $\phi(t)$ we use a formula that was originally defined in [V] as

$$\phi(t) = \begin{cases} -\cos(p(\frac{\pi}{2}-|t|)), & \frac{\pi}{2}-\frac{\pi}{p} \le |t| \le \frac{\pi}{2}, \\ \max\{|\cos(p(\frac{\pi}{2}-t))|, |\cos(p(\frac{\pi}{2}+t))|\}, & |t| \le \frac{\pi}{2}-\frac{\pi}{p}, \end{cases} \tag{2.5}$$

and $\phi(t) = \phi(\pi - |t|)$ if $\frac{\pi}{2} \le |t| \le \pi$.
Note that ϕ is continuous on $[-\pi,\pi]$, and therefore $\Phi(z) = r^p\phi(t)$ is a subharmonic function ([V], Lemma 3).

To verify (2.1) we note that by symmetry it suffices to show for $2 < p < \infty$ that

$$|w|^p \le a_p\,|w+\bar{z}|^p - F(w,z), \tag{2.6}$$

for every $(w,z) \in \mathbf{C}^2$. We next observe that inequality (2.6) is invariant under the transformation

$$(w,z) \to (\bar{\zeta}\,w,\,\zeta\,z), \qquad \zeta \in \mathbf{C}, \quad \zeta \ne 0.$$

Letting $\zeta = \frac{w}{|w|^2}$ so that $\bar{\zeta}\,w = 1$ and using the above transformation, we reduce (2.6), for any $w \in \mathbf{C}$, $w \ne 0$, to the inequality

$$1 \le a_p|1+\zeta\,z|^p - F(1,\,\zeta\,z). \tag{2.7}$$

Since (2.6) is obviously true for $w = 0$, it suffices to prove for $2 < p < \infty$:

$$1 \le a_p|1+z|^p - b_p\,\Phi(\sqrt{z}), \qquad z \in \mathbf{C}. \tag{2.8}$$

Letting $z = re^{it}$, we can express (2.8) as

$$1 \le a_p\,(1+2r\cos t + r^2)^{\frac{p}{2}} - b_p\,r^{\frac{p}{2}}\,\phi\left(\frac{t}{2}\right), \tag{2.9}$$

where $0 \le r < \infty$ and $|t| \le \pi$.

To verify (2.9), let us define a function $G(t,r)$ for $t \in [-\pi,\pi]$ and $r \in [0,\infty)$ such that

$$G(t,r) = a_p(1+2r\cos t + r^2)^{\frac{p}{2}} - b_p r^{\frac{p}{2}}\phi\left(\frac{t}{2}\right) - 1,$$

where $\phi(t)$ is given by (2.5) and the constants a_p and b_p are as before.

So we need to show $G(t,r) \ge 0$ for all $(t,r) \in [-\pi,\pi]\times[0,\infty)$. Since $G(t,r)$ is even in t, it is enough to consider only non-negative t. In fact, it is enough to show

$$G(t,r) \ge 0, \qquad (t,r) \in W = [(p-2)\frac{\pi}{p},\pi] \times [0,\infty). \tag{2.10}$$

Let us assume (2.10) holds and suppose $t \in [0,(p-2)\frac{\pi}{p}]$ and $r \ge 0$.

Obviously, $0 \le \phi\left(\frac{t}{2}\right) \le 1$ and $-\cos\left(\frac{p}{2}(\pi-\theta)\right)$ decreases from 1 to -1 for $\theta \in [(p-2)\frac{\pi}{p},\pi]$. Therefore, for the given t, there exists $\theta \in [(p-2)\frac{\pi}{p},\pi]$ such that

$\phi\left(\frac{t}{2}\right) = -\cos\left(\frac{p}{2}(\pi-\theta)\right) = \phi\left(\frac{\theta}{2}\right)$. Since $\cos t > \cos\theta$, we obtain

$$\begin{aligned} a_p(1+2r\cos t+r^2)^{\frac{p}{2}} &> a_p(1+2r\cos\theta+r^2)^{\frac{p}{2}} \\ &\geq 1+b_p r^{\frac{p}{2}}\phi\left(\frac{\theta}{2}\right) \\ &= 1+b_p r^{\frac{p}{2}}\phi\left(\frac{t}{2}\right). \end{aligned}$$

So we conclude $G(t,r) \geq 0$ for $t \in [0,(p-2)\frac{\pi}{p}]$.

Therefore, it remains to show that the function $G(t,r)$ attains a minimum value of zero in W. We will do this in three steps, and require the following partial derivatives, valid for all $(t,r) \in W$:

$$\begin{aligned} \frac{\partial G}{\partial t} &= -pra_p(1+2r\cos t+r^2)^{\frac{p}{2}-1}\sin t+\frac{p}{2}r^{\frac{p}{2}}b_p\sin\left(\frac{p}{2}(\pi-t)\right), \\ \frac{\partial G}{\partial r} &= pa_p(r+\cos t)(1+2r\cos t+r^2)^{\frac{p}{2}-1}+\frac{p}{2}r^{\frac{p}{2}-1}b_p\cos\left(\frac{p}{2}(\pi-t)\right). \end{aligned}$$

Step 1*:* We show that the minimum of $G(t,r)$ is attained in the interior of W. First note that if $r=0$, then $G(t,0)=a_p-1>0$. Similarly, if $r\to\infty$, then

$$G(t,r) \geq a_p\,(r-1)^p - b_p\,r^{\frac{p}{2}} - 1 \asymp \frac{r^p}{\sin^p\frac{\pi}{p}} > 0.$$

For the case when $t_0=(p-2)\frac{\pi}{p}$, we evaluate $\frac{\partial G}{\partial t}$ at $t=t_0$:

$$\frac{\partial G}{\partial t}(t_0,r) = -pa_p r\left(1-2r\cos\left(\frac{2\pi}{p}\right)+r^2\right)^{\frac{p}{2}-1}\sin\left(\frac{2\pi}{p}\right) < 0,$$

so a minimum does not occur in this case.

Finally, assume $t=\pi$. For this case, we will show there exists a single saddle point when $0<r<1$ and therefore no minimum can occur on the boundary of W.

First note that $\frac{\partial G}{\partial t}(\pi,r)=0$ for all r and so a critical point can only occur if $\frac{\partial G}{\partial r}(\pi,r)=0$, i.e., if

$$pa_p(r-1)|1-r|^{p-2}+\frac{p}{2}r^{\frac{p}{2}-1}b_p = 0. \tag{2.11}$$

Notice (2.11) can only hold if $0<r<1$. Indeed, if $r\geq 1$, then (2.11) is equivalent to

$$(r-1)^{p-1}r^{1-\frac{p}{2}} = -\frac{b_p}{2a_p}$$

which is impossible since both b_p and a_p are positive constants.

However, if $0<r<1$, then a critical point will occur if

$$(1-r)^{p-1}r^{1-\frac{p}{2}} = \frac{b_p}{2a_p}. \tag{2.12}$$

One can quickly note that the left-hand side of (2.12) is a decreasing function in r taking on all values on $(0,\infty)$. Since the right-hand side does not depend on r, we conclude that (2.12) has a unique solution, r_0, where $0 < r_0 < 1$. To show (π, r_0) is a saddle point, we evaluate the second partial derivatives:

$$\begin{aligned}
\frac{\partial^2 G}{\partial r^2}(\pi, r_0) &= \frac{b_p}{4}(p-2)pr_0^{\frac{p}{2}-2} + a_p p(p-1)(1-r_0)^{p-2} > 0, \\
\frac{\partial^2 G}{\partial r \partial t}(\pi, r_0) &= 0, \\
\frac{\partial^2 G}{\partial t^2}(\pi, r_0) &= a_p p r_0 (1-r_0)^{p-2} - \frac{b_p}{4} p^2 r_0^{\frac{p}{2}}.
\end{aligned} \tag{2.13}$$

We only need to show that $\frac{\partial^2 G}{\partial t^2}(\pi, r_0) < 0$ to conclude (π, r_0) is a saddle point. By (2.12) and (2.13), this is equivalent to showing

$$a_p p r_0 (1-r_0)^{p-1} \left(\frac{1}{1-r_0} - \frac{p}{2} \right) < 0. \tag{2.14}$$

In other words, we wish to show $r_0 < 1 - \frac{2}{p}$. Let $s = 1 - \frac{2}{p}$ and note that $0 < s < 1$. So $r_0 < s$ if and only if $(1-r_0)^{1+s} r_0^{-s} > (1-s)^{1+s} s^{-s}$. However, from (2.12), we know

$$(1-r_0)^{1+s} r_0^{-s} = \left(\sin \frac{s\pi}{2} \right)^{-s} \left(\cos \frac{s\pi}{2} \right)^{1+s}.$$

Therefore we need to show

$$\left(\sin \frac{s\pi}{2} \right)^{-s} \left(\cos \frac{s\pi}{2} \right)^{1+s} > (1-s)^{1+s} s^{-s}. \tag{2.15}$$

Proving (2.15) requires two cases.

Case I: Assume $\frac{1}{2} \leq s < 1$.
Note that (2.15) is equivalent to $X^{1+s} Y^{-s} > 1$ where

$$X = \frac{\cos(\frac{s\pi}{2})}{1-s}, \qquad Y = \frac{\sin(\frac{s\pi}{2})}{s}.$$

Consequently, we will be done if we can prove $X > 1$ and $X \geq Y$. But $X > 1$ if

$$g(s) = \cos \frac{s\pi}{2} + s - 1 > 0.$$

Clearly, $g(s)$ is a decreasing function for $\frac{1}{2} \leq s < 1$, therefore $g(s) > g(1) = 0$. Similarly, $X \geq Y$ is equivalent to

$$h(s) = 1 - \frac{1}{s} + \cot \frac{s\pi}{2} \geq 0.$$

Note that $h(\frac{1}{2}) = h(1) = 0$. Also, $h(s)$ has a critical point when $\sqrt{2} \sin \frac{s\pi}{2} = \sqrt{\pi} s$, which has a single solution when $\frac{1}{2} \leq s < 1$. Since we can find points where $h(s) > 0$, (for instance, $s = \frac{2}{3}$), we conclude $h(s) \geq 0$. Case I is proved.

Case II: $0 < s \le \frac{1}{2}$.
By substituting $u = 1 - s$, we can apply Case I and obtain

$$\left(\frac{\cos\frac{s\pi}{2}}{1-s}\right)^{1+s}\left(\frac{\sin\frac{s\pi}{2}}{s}\right)^{-s} = \left(\frac{\sin\frac{u\pi}{2}}{u}\right)^{2-u}\left(\frac{\cos\frac{u\pi}{2}}{1-u}\right)^{u-1}$$
$$= \left(\tan\frac{u\pi}{2}\right)^{2}\left(\frac{\sin\frac{u\pi}{2}}{u}\right)^{-u}\left(\frac{\cos\frac{u\pi}{2}}{1-u}\right)^{u+1} > \left(\tan\frac{u\pi}{2}\right)^{2} > 1.$$

Case II is proved and thus we have proved the minimum of $G(t, r)$ is attained in the interior of W.

Step 2. We show there exists a single critical point of $G(t, r)$ in W.
For a particular (t, r) to be a critical point of $G(t, r)$, we know that both partial derivatives must equal zero at that point; that is

$$\frac{2a_p(1 + 2r\cos t + r^2)^{\frac{p}{2}-1}}{b_p r^{\frac{p}{2}-1}} = \frac{\sin T}{\sin t}$$

and

$$\frac{2a_p(1 + 2r\cos t + r^2)^{\frac{p}{2}-1}}{b_p r^{\frac{p}{2}-1}} = \frac{\cos T}{r + \cos t},$$

where $T = \frac{p(\pi - t)}{2}$. Equating the right sides, and solving for r, we have

$$r + \cos t = -\frac{\cos T \sin t}{\sin T}. \tag{2.16}$$

From this it follows

$$r = -\frac{\sin(t + T)}{\sin T}, \tag{2.17}$$

where $r > 0$ because $\sin T > 0$ and $\sin(t + T) < 0$ for all $t \in ((p - 2)\frac{\pi}{p}, \pi)$. Note that by squaring both sides, we can also express (2.16) as

$$r^2 + 2r\cos t + 1 = \frac{\sin^2 t}{\sin^2 T}. \tag{2.18}$$

Using (2.17) and (2.18), we reduce $\frac{\partial G}{\partial t} = 0$ to the following

$$\frac{(\sin t)^{p-1}}{(-\sin(t + T))^{\frac{p}{2}-1}(\sin T)^{\frac{p}{2}}} = \frac{b_p}{2a_p}.$$

For fixed p, the right-hand side of the above is constant, while the left-hand side is a function of t. Thus, to conclude there is a unique critical point, we need only show the left-hand side is a decreasing function in t. We therefore consider the logarithm of the left-hand side:

$$f(t) = (p - 1)\ln\sin t - \frac{p - 2}{2}\ln(-\sin(t + T)) - \frac{p}{2}\ln\sin T.$$

We wish to prove for $t \in ((p - 2)\frac{\pi}{p}, \pi)$

$$f'(t) = (p - 1)\cot t + \frac{(p - 2)^2}{4}\cot(t + T) + \frac{p^2}{4}\cot T < 0. \tag{2.19}$$

Using the identity

$$\frac{(p-2)^2}{p^2} = 1 - \frac{4(p-1)}{p^2}, \tag{2.20}$$

we rewrite (2.19) as

$$\begin{aligned} f'(t) &= \frac{p^2}{4}\left[\frac{4(p-1)}{p^2}(\cot t - \cot(t+T)) + \cot t + \cot(t+T)\right] \\ &= \frac{p^2}{4\sin(t+T)}\left[\frac{4(p-1)}{p^2}\frac{\sin T}{\sin t} + \frac{\sin(t+2T)}{\sin T}\right]. \end{aligned} \tag{2.21}$$

Recalling that $\sin(t+T) < 0$ we need to show the term inside in the brackets in (2.21) is positive. Equivalently, we would like to show

$$1 - \frac{4(p-1)}{p^2} < 1 + \frac{\sin(t+2T)\sin t}{\sin^2 T}. \tag{2.22}$$

But the right-hand side of (2.22) can be simplified to

$$\frac{2\sin^2 T + \cos 2T - \cos(2t+2T)}{2\sin^2 T} = \frac{1-\cos(2t+2T)}{2\sin^2 T} = \frac{\sin^2(t+T)}{\sin^2 T}.$$

Referring to (2.20), we conclude that (2.22) is equivalent to

$$\left|\frac{p-2}{p}\right| < \left|\frac{\sin(t+T)}{\sin T}\right|.$$

Thus we need to show

$$g(t) = \frac{\sin T}{p} + \frac{\sin(t+T)}{p-2} < 0, \qquad t \in ((p-2)\frac{\pi}{p}, \pi). \tag{2.23}$$

Since $g(\pi) = 0$, we need only show $g'(t) > 0$ on $((p-2)\frac{\pi}{p}, \pi)$. A simple calculation shows

$$g'(t) = -\frac{1}{2}(\cos T + \cos(t+T)) = -\cos\left(\frac{t}{2}+T\right)\cos\left(\frac{t}{2}\right). \tag{2.24}$$

Since $t \in ((p-2)\frac{\pi}{p}, \pi)$ and $2 < p < \infty$, we can see that the first factor in (2.24) is always negative and the second factor is always positive. Thus, $g'(t)$ is positive and we have verified (2.23). So we have shown there is exactly one critical point in the interior of W.

Step 3. We show that $(\frac{(p-1)\pi}{p}, \cos\frac{\pi}{p})$ is a critical point of $G(t,r)$.

Evaluating $\frac{\partial G}{\partial t}$ at $(\frac{(p-1)\pi}{p}, \cos\frac{\pi}{p})$ yields (with $\gamma = \frac{\pi}{p}$)

$$-p\cos\gamma(\sin\gamma)^{-2}\sin(\pi-\gamma) + p\cot\gamma = -p\cot\gamma + p\cot\gamma = 0.$$

Similarly, for $\frac{\partial G}{\partial r}$,

$$pa_p(\cos\gamma + \cos(\pi-\gamma))(1+\cos^2\gamma + 2\cos\gamma\cos(\pi-\gamma))^{\frac{p}{2}-1} = 0$$

since $\cos\gamma = -\cos(\pi-\gamma)$.

We therefore have a critical point and evaluating the function there gives

$$\begin{aligned} -1 + a_p(1+\cos^2\gamma + 2\cos\gamma\cos(\pi-\gamma))^{\frac{p}{2}-1} &= -1 + a_p(1-\cos^2\gamma)^{\frac{p}{2}} \\ &= -1 + \left(\frac{\sin\gamma}{\sin\gamma}\right)^p = 0. \end{aligned}$$

Finally, we can show $(\frac{(p-1)\pi}{p}, \cos\frac{\pi}{p})$ is in fact the absolute minimum by confirming the Hessian of G is positive definite there. Notice that the Hessian of G at the critical point is given by the matrix

$$\begin{pmatrix} \frac{p}{\sin^2\frac{\pi}{p}} & \frac{-p(2-p)}{2\sin\frac{\pi}{p}} \\ \frac{-p(2-p)}{2\sin\frac{\pi}{p}} & \frac{p(p-1)\cos^2\frac{\pi}{p}}{\sin^2\frac{\pi}{p}} \end{pmatrix}.$$

By letting $s = 1-\frac{2}{p}$, $(0 < s < 1)$, we conclude the determinant of the Hessian is given by

$$\frac{4}{(1-s)^4\cos^4(\frac{s\pi}{2})}[\sin^2(\frac{s\pi}{2}) - s^2] > 0.$$

Thus we have verified (2.9) for all $1 < p < \infty$ and Lemma 2.1 is proved. □

3. Some open problems

We note that (2.2) is a special case of the more general question of finding the best constant, $A_{p,s}$, in the inequality:

$$\left\|(|P_+f|^s + |P_-f|^s)^{\frac{1}{s}}\right\|_{L^p(\mathbf{T})} \leq A_{p,s}\|f\|_{L^p(\mathbf{T})} \tag{3.1}$$

where $f \in L^p(\mathbf{T})$, f is complex-valued, and $1 < p < \infty$, $0 < s \leq \infty$.

Indeed, we have already shown that

$$A_{p,\infty} = \frac{1}{\sin\frac{\pi}{p}}, \quad 1 < p < \infty.$$

We can arrive at a conjectured best value for $A_{p,s}$ for $0 < s < \infty$ by looking at an "extremal" function $f = \alpha\,\mathrm{Re}\,\mathrm{g} + i\beta\,\mathrm{Im}\,\mathrm{g}$ where $\alpha, \beta \in \mathbf{R}$ and $g(z) = \left(\frac{1+z}{1-z}\right)^{\frac{2\gamma}{\pi}}$ with $\gamma \to \frac{\pi}{2p}$, assuming $1 < p \leq 2$. We conjecture that the value for $A_{p,s}$ will be the maximum of the following function

$$F_{p,s}(x) = \frac{\left(\left|x+\tan\frac{\pi}{2p}\right|^s + \left|x-\tan\frac{\pi}{2p}\right|^s\right)^{\frac{1}{s}}}{2\sin\frac{\pi}{2p}\sqrt{x^2+1}}, \qquad x \in \mathbf{R}.$$

It is easy to see using the above extremal function that $A_{p,s} \geq \max_{x\in\mathbf{R}} F_{p,s}(x)$, and hence these estimates would be sharp.

Since $F_{p,s}(x)$ is even, we expect a local extremum to occur at $x = 0$. We can use a computer algebra system such as *Mathematica* to analyze $F_{p,s}(x)$. By fixing p, and letting s vary, it appears that the constant $2^{-\frac{1}{s}}A_{p,s}$ remains unchanged while $x = 0$ is a maximum locally. When it becomes a local minimum, $2^{-\frac{1}{s}} A_{p,s}$ begins to increase. To find the value of s where this transition occurs, we need only calculate the value of s where the concavity of $F_{p,s}(x)$ at $x = 0$ changes. We have verified using *Mathematica* that this happens when $s = \sec^2 \frac{\pi}{2p}$.

Thus we conjecture that

$$A_{p,s} = \frac{2^{\frac{1}{s}}}{2\cos\frac{\pi}{2p}}, \quad 1 < p < 2, \quad 0 < s \le \sec^2 \frac{\pi}{2p},$$

and that $A_{p,s}$ will tend to the limiting value of $A_{p,\infty}$ as $s \to \infty$. Similarly, we conjecture that

$$A_{p,s} = \frac{2^{\frac{1}{s}}}{2\sin\frac{\pi}{2p}}, \quad 2 < p < \infty, \quad 0 < s \le \csc^2 \frac{\pi}{2p}.$$

It is not hard to see that $A_{2,s} = \max(1, 2^{\frac{1}{s}-\frac{1}{2}})$.

A similar phenomenon was observed in [HKV] for the best constant in the inequality

$$\left\| \left((Hf)^2 + s^2 f^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbf{T})} \le B_{p,s} \, ||f||_{L^p(\mathbf{T})}, \tag{3.2}$$

where $f \in L^p(\mathbf{T})$ is a real-valued function, Hf is the conjugate harmonic function (the Hilbert transform of f; see [Z], Ch. IV.3), and s is a positive constant. For certain values of s, the extremal function g used above (with $f =$ Re g and $Hf =$ Im g) leads to the best constant $B_{p,s}$ determined in [HKV], Theorem 5.5. (For $s = 1$ the best constant was found earlier in [E] and [V].) However, for other values of s this extremal function is no longer adequate, and the best constant in (3.2) remains unknown.

Best constant inequalities (3.1) and (3.2) may serve as model problems for some vector-valued inequalities. In various applications, of particular interest are the norms of the Hilbert transform and the Riesz projection on the mixed-norm space $L^p(l^s)$ where $1 < p < \infty$ and $1 < s < \infty$, i.e., best constants in the inequalities

$$\|\Big(\sum_{k=1}^{\infty} |Hf_k|^s\Big)^{\frac{1}{s}}\|_{L^p(\mathbf{T})} \le C_{p,s} \, \|\Big(\sum_{k=1}^{\infty} |f_k|^s\Big)^{\frac{1}{s}}\|_{L^p(\mathbf{T})}, \tag{3.3}$$

$$\|\Big(\sum_{k=1}^{\infty} |P_\pm f_k|^s\Big)^{\frac{1}{s}}\|_{L^p(\mathbf{T})} \le C_{p,s} \, \|\Big(\sum_{k=1}^{\infty} |f_k|^s\Big)^{\frac{1}{s}}\|_{L^p(\mathbf{T})}, \tag{3.4}$$

As was shown in [HKV], Theorem 2.3, for a finite number of $\{f_k\}_{k=1}^n$, these problems are equivalent to the existence of plurisubharmonic minorants on $\mathbf{C}^n$ for certain functions of n complex variables associated with (3.3) and (3.4) respectively.

We note that when $2 \le s \le p$ or $1 < p \le s \le 2$, the best constants $C_{p,s}$ are known to be the same as in the scalar case. This is obvious if $s = p$ and classical if $s = 2$ (due to Marcinkiewicz and Zygmund; see [Z], Ch. XV.2); for other s it follows by interpolation.

References

[E] M. Essén, *A superharmonic proof of the M. Riesz conjugate function theorem.* Ark. för Mat., **22** (1984), 241–249.

[GKr1] I. Gohberg and N. Krupnik, *Norm of the Hilbert transformation in the L_p space.* Funct. Anal. Pril. **2** (1968), 91–92 (Russian); English transl. in Funct. Anal. Appl. **2** (1968), 180–181.

[GKr2] I. Gohberg and N. Krupnik, *One-Dimensional Linear Singular Integral Equations.* Vols. I, II. Operator Theory: Advances Appl. **53, 54**. Birkhäuser, Basel – Boston – Berlin, 1992.

[HKV] B. Hollenbeck, N.J. Kalton and I.E. Verbitsky, *Best constants for some operators associated with the Fourier and Hilbert transforms.* Studia Math. **157** (3) (2003), 237–278.

[HV] B. Hollenbeck and I. Verbitsky, *Best constants for the Riesz projection.* J. Funct. Anal. **175** (2000), 370–392.

[Kr] N. Krupnik, *Survey on the best constants in the theory of one-dimensional Singular Integral Operators.* Operator Theory: Advances and Appl. **202** (2010), 365–393 (this volume).

[KrV] N. Krupnik and I. Verbitsky, *The norm of the Riesz projection.* Linear and Complex Analysis Problem Book, Eds. V.P. Havin, S.V. Hruschev, and N.K. Nikolski. Springer-Verlag, Lecture Notes Math. **1043** (1984), 325–327.

[Pe] A. Pełczyński, *Norms of classical operators in function spaces.* Colloque en l'honneur de Laurent Schwartz, Vol. I. Astérisque **131** (1985), 137–162.

[P] S.K. Pichorides, *On the best values of the constants in the theorems of M. Riesz, Zygmund and Kolmogorov.* Studia Math. **44** (1972), 165–179.

[Ra] R.M. Range, *Holomorphic Functions and Integral Representations in Several Complex Variables.* Graduate Texts in Math. **108**, Springer-Verlag, New York, 1986.

[V] I. Verbitsky, *Estimate of the norm of a function in a Hardy space in terms of the norms of its real and imaginary part.* Mat. Issled. **54** (1980), 16–20 (Russian); English transl. in Amer. Math. Soc. Transl. (2), **124** (1984), 11–15.

[Z] A. Zygmund, *Trigonometric Series.* Vols. I, II, 2nd edition. Cambridge University Press, London, 1968.

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Received: February 27, 2009
Accepted: April 8, 2009

Operator Theory:
Advances and Applications, Vol. 202, 297–314

Quasi Commutativity of Regular Matrix Polynomials: Resultant and Bezoutian

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To Israel Gohberg, an outstanding mathematician, an inspiring teacher and a wonderful friend, on the occasion of his 80th birthday.

Abstract. In a recent paper of I. Gohberg and the authors necessary and sufficient conditions are obtained in order that for two regular matrix polynomials L and M the dimension of the null space of the associate square resultant matrix is equal to the sum of the multiplicities of the common zeros of L and M, infinity included. The conditions are stated in terms of quasi commutativity. In the case of commuting matrix polynomials, in particular, in the scalar case, these conditions are automatically fulfilled. The proofs in the above paper are heavily based on the spectral theory of matrix polynomials. In the present paper a new proof is given of the sufficiency part of the result mentioned above. Here we use the connections between the Bezout and resultant matrices and a general abstract scheme for determining the null space of the Bezoutian of matrix polynomials which is based on a state space analysis of Bezoutians.

Mathematics Subject Classification (2000). Primary 47A56, 15A18; secondary 47B35, 47B99.

Keywords. Matrix polynomials, common spectral data, quasi commutativity, block resultant matrices of square size, Bezoutian, state space analysis.

0. Introduction

Let L and M be $n \times n$ matrix polynomials of degrees ℓ and m, respectively. Thus

$$L(\lambda) = L_0 + \lambda L_1 + \cdots + \lambda^\ell L_\ell \quad \text{and} \quad L_\ell \neq 0, \tag{0.1}$$

$$M(\lambda) = M_0 + \lambda M_1 + \cdots + \lambda^m M_m \quad \text{and} \quad M_m \neq 0. \tag{0.2}$$

The research of the second author is supported by ISF – Israel Science Foundation (grant no. 121/09) and by the Fund for Promotion of Research at the Technion, Haifa.
Comminucated by J.A. Ball.

In this paper we assume both ℓ and m to be nonzero, and we deal with the $(\ell+m)\times(\ell+m)$ block matrix $\mathbf{R}(L,M)$ given by

$$\mathbf{R}(L,M)=\underbrace{\begin{bmatrix} L_0 & \cdots & \cdots & L_\ell & & & \\ & L_0 & \cdots & \cdots & L_\ell & & \\ & & \ddots & & & \ddots & \\ & & & L_0 & \cdots & \cdots & L_\ell \\ M_0 & \cdots & \cdots & M_{m-1} & M_m & & \\ & \ddots & & & & \ddots & \\ & & M_0 & \cdots & \cdots & \cdots & M_m \end{bmatrix}}_{\ell+m} \begin{array}{l} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} m \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} \ell \end{array}$$

Here the blocks are matrices of size $n\times n$, and the unspecified entries are zero matrices. In the scalar case ($n=1$) the determinant of the matrix $\mathbf{R}(L,M)$ is the classical Sylvester resultant (see [19], also [16] or Section 27 in [20]). As is common nowadays we use the term *resultant* for the matrix $\mathbf{R}(L,M)$ rather than for its determinant.

The key property of the classical Sylvester resultant matrix is that its null space provides a complete description of the common zeros of the polynomials involved. In particular, in the scalar case the number of common zeros of the polynomials L and M, multiplicities taken into account, is equal to the dimension of the null space of $\mathbf{R}(L,M)$.

This property does not carry over to matrix polynomials, not even if L and M are *regular*, that is, if $\det L(\lambda)$ and $\det M(\lambda)$ do not vanish identically, which we shall assume throughout this paper. In [3] (see also [4]) it has been shown that

$$\dim\operatorname{Ker}\mathbf{R}(L,M)\geq\hat{\nu}(L,M). \tag{0.3}$$

Here $\hat{\nu}(L,M)$ denotes the total common multiplicity of the common eigenvalues of L and M (see Section 1 below for the precise definition of $\hat{\nu}(L,M)$). The hat refers to the fact that the point infinity is taking into account too. Simple examples show that the inequality (0.3) can be strict. Thus the question arises under what additional conditions on L and M do we have equality in (0.3)? This question has recently been answered in [9] where the following theorem is proved.

Theorem 0.1. ([9]) *Let L and M be the regular $n\times n$ matrix polynomials in* (0.1) *and* (0.2). *Then* $\dim\operatorname{Ker}\mathbf{R}(L,M)=\hat{\nu}(L,M)$ *if and only if there exist regular $n\times n$ matrix polynomials P and Q of degrees at most m and ℓ, respectively, such that*

$$P(\lambda)L(\lambda)=Q(\lambda)M(\lambda),\quad \lambda\in\mathbb{C}. \tag{0.4}$$

Let L and M be regular $n\times n$ matrix polynomials of degrees ℓ and m, respectively. We call L and M *quasi commutative* whenever there exist regular $n\times n$ matrix polynomials P and Q of degrees at most m and ℓ, respectively, such that (0.4) holds. In that case we also say that the quadruple $\{L, M; P, Q\}$ has the *quasi commutativity property.* Thus Theorem 0.1 tells us that

$$\dim \operatorname{Ker} \mathbf{R}(L, M) = \hat{\nu}(L, M) \tag{0.5}$$

holds if and only if the polynomials L and M are quasi commutative.

The resultant matrix appears in a natural way in the study of the matrix polynomial equations of the form

$$X(\lambda)L(\lambda) + Y(\lambda)M(\lambda) = G(\lambda). \tag{0.6}$$

Here L and M are as in (0.1) and (0.2), and the right-hand side is an $n \times n$ matrix polynomial $G(\lambda) = \sum_{j=0}^{\ell+m-1} \lambda^j G_j$ of degree at most $\ell + m - 1$. Then (see, e.g., Section 3 of [9]) equation (0.6) has solutions X and Y,

$$X(\lambda) = \sum_{j=0}^{m-1} \lambda^j X_j, \quad Y(\lambda) = \sum_{j=0}^{\ell-1} \lambda^j Y_j,$$

if and only if

$$\sum_{j=0}^{\ell+m-1} G_j y_j = 0 \text{ for each } y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{\ell+m-1} \end{bmatrix} \in \operatorname{Ker} \mathbf{R}(L, M).$$

The necessity of the quasi commutativity condition in Theorem 0.1 is proved in [9] using the spectral theory of regular matrix polynomials developed in the eighties, in the papers [5], [6], [10] and the book [11], together with the connection between the resultant matrix and matrix polynomial equations mentioned in the previous paragraph. The sufficiency of the condition is covered by Proposition 3.2 in [12].

The aim of the present paper is to give an alternative proof of the sufficiency part using the analogue of the classical Bezoutian for matrix polynomials. Recall that for each quasi commutative quadruple $\{L, M; P, Q\}$ we have $P(\lambda)L(\mu) - Q(\lambda)M(\mu)$ is zero at $\lambda = \mu$. Hence we can consider the following series expansion

$$\frac{P(\lambda)L(\mu) - Q(\lambda)M(\mu)}{\lambda - \mu} = \sum_{i,j=0}^{r-1} T_{i,j} \lambda^i \mu^j, \quad \text{where } r = \max\{\ell, m\}.$$

The $r \times r$ block matrix $T = [T_{i,j}]_{i,j=0}^{r-1}$ is the analogue of the classical Bezoutian we shall be working with. This notion (without the restriction on the degree of the polynomials P and Q) was introduced in [1], and its null space has been described in [18], again using heavily the results of the spectral theory of matrix polynomials (in particular, those from [5] and [6]). In the present paper we shall describe (see (2.13)) the null space of T following the general abstract scheme developed in [2] which is based on a state space analysis of the Bezoutian and its properties (earlier results in this direction can be found in [13], [17], and [15]).

The formula for the dimension of the null space of the Bezoutian T in Theorem 2.1 below together with the relation between the resultant and the Bezout matrix T (Theorem 3.1 below) will allows us to give a new proof of the sufficiency part in Theorem 0.1.

The paper consists of four sections, not counting the present introduction. In the first section we define the total common multiplicity of two regular matrix polynomials. In the second section we prove the formula for the dimension of the null space of the Bezout matrix. The third section establishes the relation between resultant and the Bezout matrix, and the final section contains the proof of the sufficiency part in Theorem 0.1.

Finally, we mention that the approach followed in the present paper was inspired by the recent papers [2], [7], and [8], where we proved (in co-authorship with I. Gohberg) results similar in nature to Theorems 0.1, 2.1, and 3.1 for certain entire matrix functions.

1. Definition of total common multiplicity

Let L and M be regular $n \times n$ matrix polynomials as in (0.1) and (0.2). In this section we introduce the quantity $\hat{\nu}(L, M)$. The fact that $\operatorname{degree} L = \ell$ and $\operatorname{degree} M = m$ will only play a role in the final paragraph of this section.

Let λ_0 be a point in $\mathbb{C}$. We say that λ_0 is a *common eigenvalue* of L and M if there exists a vector $x_0 \neq 0$ such that $L(\lambda_0)x_0 = M(\lambda_0)x_0 = 0$. In this case we refer to x_0 as a *common eigenvector* of L and M at λ_0. Note that x_0 is a common eigenvector of L and M at λ_0 if and only if x_0 is a non-zero vector in

$$\operatorname{Ker} L(\lambda_0) \cap \operatorname{Ker} M(\lambda_0) = \operatorname{Ker} \begin{bmatrix} L(\lambda_0) \\ M(\lambda_0) \end{bmatrix}.$$

For L and M to have a common eigenvalue at λ_0 it is necessary that $\det L(\lambda_0) = 0$ and $\det M(\lambda_0) = 0$ but the converse is not true. To see this, take

$$L(\lambda) = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \quad M(\lambda) = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}.$$

Then $\det L(\lambda)$ and $\det M(\lambda)$ both vanish at zero but $\lambda_0 = 0$ is not a common eigenvalue of L and M because L and M do not have a common eigenvector at λ_0.

An ordered sequence of vectors $x_0, x_1, \ldots, x_{\nu-1}$ is called a *common Jordan chain* for L and M at λ_0 if $x_0 \neq 0$ and

$$\sum_{j=0}^{k} \frac{1}{j!} L^{(j)}(\lambda_0) x_{k-j} = 0, \quad k = 0, \ldots, \nu - 1,$$

$$\sum_{j=0}^{k} \frac{1}{j!} M^{(j)}(\lambda_0) x_{k-j} = 0, \quad k = 0, \ldots, \nu - 1.$$

Note that in that case x_0 is a common eigenvector of L and M at λ_0.

Let x_0 be a common eigenvector of L and M at λ_0. Since L and M are regular, the lengths of the common Jordan chains of L and M at λ_0 with initial vector x_0 have a finite supremum, which we shall call the *common rank* of the common eigenvector x_0. This fact allows us to define the notion of a canonical set of common Jordan chains. The first step is to choose a common eigenvector $x_{1,0}$ in $\operatorname{Ker} L(\lambda_0) \cap \operatorname{Ker} M(\lambda_0)$ such that the common rank ν_1 of $x_{1,0}$ is maximal, and let $x_{1,0}, \dots, x_{1,\nu_1-1}$ be a corresponding common Jordan chain. Next, we choose among all vectors x in $\operatorname{Ker} L(\lambda_0) \cap \operatorname{Ker} M(\lambda_0)$, with x not a multiple of $x_{1,0}$, a vector $x_{2,0}$ of maximal common rank, ν_2 say, and we choose a corresponding common Jordan chain $x_{2,0}, \dots, x_{2,\nu_2-1}$. We proceed by induction. Assume

$$x_{1,0}, \dots, x_{1,\nu_1-1}, \dots, x_{k,0}, \dots, x_{k,\nu_k-1}$$

have been chosen. Then we choose $x_{k+1,0}$ to be a vector in the space $\operatorname{Ker} L(\lambda_0) \cap \operatorname{Ker} M(\lambda_0)$ that does not belong to $\operatorname{span}\{x_{1,0}, \dots, x_{k,0}\}$ and such that $x_{k+1,0}$ is of maximal common rank among all vectors in the space $\operatorname{Ker} L(\lambda_0) \cap \operatorname{Ker} M(\lambda_0)$ not belonging to $\operatorname{span}\{x_{1,0}, \dots, x_{k,0}\}$. In this way, in a finite number of steps, we obtain a basis $x_{1,0}, x_{2,0}, \dots, x_{p,0}$ of $\operatorname{Ker} L(\lambda_0) \cap \operatorname{Ker} M(\lambda_0)$ and corresponding common Jordan chains

$$x_{1,0}, \dots, x_{1,\nu_1-1}, x_{2,0}, \dots, x_{2,\nu_2-1}, \dots, x_{p,0}, \dots, x_{p,\nu_p-1}. \tag{1.1}$$

The system of vectors (1.1) is called a *canonical set of common Jordan chains* of L and M at λ_0.

From the construction it follows that $p = \dim \operatorname{Ker} L(\lambda_0) \cap \operatorname{Ker} M(\lambda_0)$. Furthermore, the numbers $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_p$ are uniquely determined by L and M, and they do not depend on the particular choices made above. The number

$$\nu(L, M; \lambda_0) := \sum_{j=1}^{p} \nu_j$$

is called the *common multiplicity of* λ_0 as a common eigenvalue of the matrix polynomials L and M.

The fact that L and M are regular also implies that L and M have a finite number of common eigenvalues in $\mathbb{C}$. We define the *total common multiplicity of L and M in* $\mathbb{C}$ to be the number $\nu(L, M)$ given by

$$\nu(L, M; \mathbb{C}) = \sum_{\lambda \in \mathbb{C}} \nu(L, M; \lambda).$$

Next we introduce the common multiplicity at infinity. For this purpose we consider the so-called reversed polynomials:

$$L^{\#}(\lambda) = \lambda^{\ell} L(\lambda^{-1}) = L_{\ell} + \lambda L_{\ell-1} + \cdots + \lambda^{\ell} L_0,$$

$$M^{\#}(\lambda) = \lambda^{m} M(\lambda^{-1}) = M_{m} + \lambda M_{m-1} + \cdots + \lambda^{m} M_0.$$

The matrix polynomials $L^{\#}$ and $M^{\#}$ are again regular. We define the *common multiplicity of L and M at infinity* to be the number $\nu(L, M; \infty) = \nu(L^{\#}, M^{\#}; 0)$.

The *total common multiplicity of L and M* is then defined as the number $\hat{\nu}(L,M)$ given by

$$\hat{\nu}(L,M) = \nu(L,M;\mathbb{C}) + \nu(L,M;\infty).$$

In the next two sections reduction to comonic and/or monic polynomials will play a role in the proofs. Recall that an $n \times n$ matrix polynomial L is called *comonic* whenever its value at zero is the identity matrix I_n, and L is said to be *monic* when its leading coefficient is equal to I_n.

2. The Bezout matrix for regular matrix polynomials

Throughout this section we deal with four regular $n \times n$ matrix polynomials:

$$L(\lambda) = L_0 + \lambda L_1 + \cdots + \lambda^r L_r, \quad M(\lambda) = M_0 + \lambda M_1 + \cdots + \lambda^r M_r, \tag{2.1}$$

$$P(\lambda) = P_0 + \lambda P_1 + \cdots + \lambda^r P_r, \quad Q(\lambda) = Q_0 + \lambda Q_1 + \cdots + \lambda^r Q_r. \tag{2.2}$$

We assume $r > 0$. The polynomials in (2.1) and (2.2) are of degree at most r; in other words, the leading coefficients L_r, M_r, P_r, Q_r are not required to be non-zero. We also assume that

$$P(\lambda)L(\lambda) = Q(\lambda)M(\lambda). \tag{2.3}$$

Then $P(\lambda)L(\mu) - Q(\lambda)M(\mu)$ is zero at $\lambda = \mu$, and hence we can consider the following series expansion

$$\frac{P(\lambda)L(\mu) - Q(\lambda)M(\mu)}{\lambda - \mu} = \sum_{i,j=0}^{r-1} T_{i,j}\lambda^i \mu^j. \tag{2.4}$$

The $r \times r$ block matrix $[T_{i,j}]_{i,j=0}^{r-1}$ is called the *Bezout matrix* associated with the matrix polynomials (2.1) and (2.2), for short $T = \mathbf{B}_r(L,M;P,Q)$.

To state the main theorem of this section we have to reconsider the common multiplicity at infinity. Let L and M be regular $n \times n$ matrix polynomials as in (2.1). Put

$$L^\dagger(\lambda) = \lambda^r L(\lambda^{-1}) \; = L_r + \lambda L_{r-1} + \cdots + \lambda^r L_0,$$
$$M^\dagger(\lambda) = \lambda^r M(\lambda^{-1}) = M_r + \lambda M_{r-1} + \cdots + \lambda^r M_0.$$

Note that $L^\dagger$ and $M^\dagger$ are again regular $n \times n$ matrix polynomials. Since the degrees of L and M can be strictly less than r, the polynomials $L^\dagger$ and $M^\dagger$ are generally not equal to the respective reversed polynomials $L^\#$ and $M^\#$, which we used in the final paragraph of Section 1. Furthermore, note that the definition of $L^\dagger$ and $M^\dagger$ depends on the choice of r. We define the *common multiplicity of L and M at infinity relative to r* to be the number

$$\nu_r(L,M;\infty) = \nu(L^\dagger, M^\dagger; 0).$$

Finally, the *total common multiplicity of L and M relative to r* is the number

$$\hat{\nu}_r(L,M) = \nu(L,M;\mathbb{C}) + \nu_r(L,M;\infty).$$

Example. To see the difference between $\hat{\nu}(L, M)$ and $\hat{\nu}_r(L, M)$ consider the polynomials $L(\lambda) = I + \lambda I$ and $M(\lambda) = I + \lambda^2 R$. Here I is the 2×2 identity matrix and R is an arbitrary 2×2 non-zero matrix. Take $r = 2$, and put $L^\dagger(\lambda) = \lambda^2 L(\lambda^{-1})$ and $M^\dagger(\lambda) = \lambda^2 M(\lambda^{-1})$. Then $M^\dagger = M^\#$ but $L^\dagger$ does not coincide with $L^\#$. Since, L is monic, $L^\#$ is comonic, and hence $\nu(L^\#, M^\#; 0) = 0$. On the other hand, $L^\dagger$ has the value zero at zero, and one computes that $\nu(L^\dagger, M^\dagger; 0) = \dim \operatorname{Ker} R$. It follows that

$$\begin{aligned}\hat{\nu}(L, M) &= \nu(L, M; -1) = \dim \operatorname{Ker} (I + R),\\ \hat{\nu}_2(L, M) &= \nu(L, M; -1) + \nu(L^\dagger, M^\dagger; \infty) =\\ &\qquad\qquad = \dim \operatorname{Ker} (I + R) + \dim \operatorname{Ker} R.\end{aligned}$$

Note that in this example L and M commute, and thus we may consider $\mathbf{B}_2(L, M; M, L)$. Let us compare this Bezout matrix with the resultant of L and M. We have

$$\mathbf{R}(L, M) = \begin{bmatrix} I & I & 0\\ 0 & I & I\\ I & 0 & R\end{bmatrix}, \quad \mathbf{B}_2(L, M; M, L) = \begin{bmatrix} -I & R\\ R & R\end{bmatrix}.$$

One checks that

$$\begin{aligned}\dim \operatorname{Ker} \mathbf{R}(L, M) &= \dim \operatorname{Ker} (I + R),\\ \dim \operatorname{Ker} \mathbf{B}_2(L, M; M, L) &= \dim \operatorname{Ker} (I + R) + \dim \operatorname{Ker} R.\end{aligned}$$

Hence,

$$\dim \operatorname{Ker} \mathbf{R}(L, M) = \hat{\nu}(L, M), \quad \dim \operatorname{Ker} \mathbf{B}_2(L, M; M, L) = \hat{\nu}_2(L, M).$$

Since L and M commute, the polynomials L and M are quasi commutative, and thus the first identity in the above formula also follows from Theorem 0.1. The second identity can be seen as a corollary of the following result which is the main theorem of this section.

Theorem 2.1. *Let L, M, P, Q be regular $n \times n$ matrix polynomials of degree at most r, and assume that* (2.3) *is satisfied. Then*

$$\dim \operatorname{Ker} \mathbf{B}_r(L, M; P, Q) = \hat{\nu}_r(L, M). \tag{2.5}$$

Theorem 2.1 has a long history; see [1], [13], and the references therein. We shall need the above theorem only for the case when the two matrix polynomials L an M in (2.1) are monic of degree r. For this case the theorem can be found in [18]. When L an M in (2.1) are monic of degree r, then $\hat{\nu}_r(L, M)$ is just equal to $\nu(L, M; \mathbb{C})$.

We shall give a proof of Theorem 2.1 using the abstract scheme for Bezout matrices and Bezout operators given in Section 4 of [2], which originates from [13]. In particular, we shall need the description of the kernel of the Bezoutian in Theorem 4.3 of [2] which does not appear in [13]. It will be convenient to use the following lemma.

Lemma 2.2. *In order to prove Theorem* 2.1 *it suffices to prove the result for the case when the four polynomials L, M, P, Q are comonic, each of degree r and each with a non-singular leading coefficient.*

Proof. We shall divide the proof into four parts.

Part 1. Choose $\alpha \in \mathbb{C}$ such that for all four polynomials the value at α is non-singular. This allows us to consider the polynomials

$$L_\alpha(\lambda) = L(\lambda + \alpha), \quad M_\alpha(\lambda) = M(\lambda + \alpha),$$
$$P_\alpha(\lambda) = P(\lambda + \alpha), \quad Q_\alpha(\lambda) = Q(\lambda + \alpha).$$

The polynomials L_α, M_α, P_α, Q_α have a non-singular constant term, are of degree at most r, and (2.3) remains true with L, M, P, Q being replace by L_α, M_α, P_α, Q_α in this order. We claim that

$$\dim \operatorname{Ker} \mathbf{B}_r(L, M; P, Q) = \dim \operatorname{Ker} \mathbf{B}_r(L_\alpha, M_\alpha; P_\alpha, Q_\alpha), \tag{2.6}$$
$$\hat{\nu}_r(L, M) = \hat{\nu}_r(L_\alpha, M_\alpha). \tag{2.7}$$

The identity (2.7) is simple to check, we omit the details. The identity (2.6) follows from the fact that $\mathbf{B}_r(L, M; P, Q)$ and $\mathbf{B}_r(L_\alpha, M_\alpha; P_\alpha, Q_\alpha)$ are equivalent matrices. In fact,

$$\mathbf{B}_r(L_\alpha, M_\alpha; P_\alpha, Q_\alpha) = \mathbf{F}(\alpha)^{\mathrm{T}} \mathbf{B}_r(L, M; P, Q) \mathbf{F}(\alpha), \tag{2.8}$$

where $\mathbf{F}(\alpha)^{\mathrm{T}}$ is the block transpose of $\mathbf{F}(\alpha)$, while $\mathbf{F}(\alpha)$ is the $r \times r$ block matrix of which the (j, k)th entry is the $n \times n$ matrix given by

$$F_{jk}(\alpha) = \begin{cases} \begin{pmatrix} j \\ k \end{pmatrix} \alpha^{j-k} I_n, & \text{for } j \geq k, \\ 0, & \text{otherwise}. \end{cases} \tag{2.9}$$

Clearly, $\mathbf{F}(\alpha)$ is block lower triangular with the $n \times n$ identity matrix on the main diagonal. Thus $\mathbf{F}(\alpha)$ is non-singular, and the identity (2.8) shows that (2.6) holds.

Thus in order to prove (2.5) we may assume the constant terms in (2.1) and (2.2) to be non-singular.

Part 2. Assume that the constant terms in (2.1) and (2.2) are non-singular. Put

$$\tilde{L}(\lambda) = L_0^{-1} L(\lambda), \qquad \tilde{M}(\lambda) = M_0^{-1} M(\lambda),$$
$$\tilde{P}(\lambda) = (P_0 L_0)^{-1} P(\lambda) L_0, \quad \tilde{Q}(\lambda) = (Q_0 M_0)^{-1} Q(\lambda) M_0.$$

The four polynomials $\tilde{L}$, $\tilde{M}$, $\tilde{P}$, $\tilde{Q}$ are comonic of degree at most r. Note that (2.3) implies that $P_0 L_0 = Q_0 M_0$, and hence

$$\begin{aligned} \tilde{P}(\lambda)\tilde{L}(\lambda) &= (P_0 L_0)^{-1} P(\lambda) L_0 L_0^{-1} L(\lambda) \\ &= (P_0 L_0)^{-1} P(\lambda) L(\lambda) = (Q_0 M_0)^{-1} Q(\lambda) M(\lambda) \\ &= \tilde{Q}(\lambda) \tilde{M}(\lambda). \end{aligned}$$

Thus (2.3) holds for $\tilde{L}$, $\tilde{M}$, $\tilde{P}$, $\tilde{Q}$ in place of L, M, P, Q, respectively. It is straightforward to check that

$$\begin{aligned}&\dim\operatorname{Ker}\mathbf{B}_r(L,M;P,Q)=\dim\operatorname{Ker}\mathbf{B}_r(\tilde{L},\tilde{M};\tilde{P},\tilde{Q}),\\&\hat{\nu}_r(L,M)=\hat{\nu}_r(\tilde{L},\tilde{M}).\end{aligned}$$

Thus in order to prove (2.5) we may assume that the polynomials in (2.1) and (2.2) are comonic.

Part 3. Assume that the polynomials in (2.1) and (2.2) are comonic. Put

$$\begin{aligned}L^\dagger(\lambda)=\lambda^r L(\lambda^{-1}),\quad M^\dagger(\lambda)=\lambda^r M(\lambda^{-1}),\\P^\dagger(\lambda)=\lambda^r P(\lambda^{-1}),\quad Q^\dagger(\lambda)=\lambda^r Q(\lambda^{-1}).\end{aligned}$$

Then $L^\dagger$, $M^\dagger$, $P^\dagger$, $Q^\dagger$ are monic matrix polynomials, each of degree r. Furthermore

$$P^\dagger(\lambda)L^\dagger(\lambda)=\lambda^{2r}P(\lambda^{-1})L(\lambda^{-1})=\lambda^{2r}Q(\lambda^{-1})M(\lambda^{-1})=Q^\dagger(\lambda)M^\dagger(\lambda),$$

and hence (2.3) holds for $L^\dagger$, $M^\dagger$, $P^\dagger$, $Q^\dagger$ in place of L, M, P, Q, respectively. One checks that

$$\begin{aligned}&\dim\operatorname{Ker}\mathbf{B}_r(L,M;P,Q)=\dim\operatorname{Ker}\mathbf{B}_r(L^\dagger,M^\dagger;P^\dagger,Q^\dagger),\\&\hat{\nu}_r(L,M)=\hat{\nu}_r(L^\dagger,M^\dagger).\end{aligned}$$

The second identity is easy to check, and the first identity follows from the equivalence relation

$$\mathbf{B}_r(L^\dagger,M^\dagger;P^\dagger,Q^\dagger)=-E\mathbf{B}_r(L,M;P,Q)E,$$

where E is the $r\times r$ block permutation matrix whose entries are zero except those on the main skew diagonal which are all equal to the $n\times n$ identity matrix.

Thus in order to prove (2.5) we may assume that the polynomials in (2.1) and (2.2) are all monic and of degree r.

Part 4. Assume that the polynomials in (2.1) and (2.2) are all monic and of degree r. Since the polynomials L, M, P, Q are monic, they are regular, and hence we can find $\beta\in\mathbb{C}$ such that the values of L, M, P, Q at β are non-singular. Now repeat the arguments of the first two parts with β in place of α. Let $L^\diamondsuit$, $M^\diamondsuit$, $P^\diamondsuit$, $Q^\diamondsuit$ be the resulting polynomials. Then $L^\diamondsuit$, $M^\diamondsuit$, $P^\diamondsuit$, $Q^\diamondsuit$ are comonic polynomials, each of degree r and each with a non-singular leading coefficient. Furthermore,

$$\begin{aligned}&\dim\operatorname{Ker}\mathbf{B}_r(L,M;P,Q)=\dim\operatorname{Ker}\mathbf{B}_r(L^\diamondsuit,M^\diamondsuit;P^\diamondsuit,Q^\diamondsuit),\\&\hat{\nu}_r(L,M)=\hat{\nu}_r(L^\diamondsuit,M^\diamondsuit).\end{aligned}$$

Hence in order to prove (2.5) we may assume that the matrix polynomials in (2.1) and (2.2) are all comonic, of degree precisely r, and have non-singular leading coefficients. $\square$

Next we show that $\mathbf{B}_r(L,M;P,Q)$ is an abstract Bezout matrix in the sense of [13]; cf., Section 4 in [2]. For this purpose we introduce the following block matrices. Throughout I is the $n\times n$ identity matrix.

$$B=\begin{bmatrix} I\\ 0\\ \vdots\\ 0\end{bmatrix},\qquad C=\begin{bmatrix} I & 0 & \cdots & 0\end{bmatrix},\tag{2.10}$$

$$N=\begin{bmatrix} 0 & & & 0\\ I & & & 0\\ & \ddots & & \\ & & I & 0\end{bmatrix},\qquad V=\begin{bmatrix} 0 & I & & \\ & & \ddots & 0\\ & & & I\\ 0 & 0 & & 0\end{bmatrix}.\tag{2.11}$$

Note that the block matrices

$$\begin{bmatrix} B & NB & \cdots & N_{r-1}B\end{bmatrix}\quad\text{and}\quad\begin{bmatrix} C\\ CV\\ \vdots\\ CV^{r-1}\end{bmatrix}$$

are both equal to the $nr\times nr$ identity matrix. In particular, the pair (N,B) is controllable and the pair (C,V) is observable (see [14] for an explanation of this terminology). Now let $T=[T_{i,j}]_{i,j}^{r-1}$ be the Bezout matrix $\mathbf{B}_r(L,M;P,Q)$. It is straightforward to check that

$$C(I-\lambda V)^{-1}T(I-\mu N)^{-1}B=\sum_{i,j=0}^{r-1}T_{i,j}\lambda^i\mu^j.$$

Thus (2.4) can be rewritten as

$$\frac{P(\lambda)L(\mu)-Q(\lambda)M(\mu)}{\lambda-\mu}=C(I-\lambda V)^{-1}T(I-\mu N)^{-1}B.\tag{2.12}$$

Hence $T=\mathbf{B}_r(L,M;P,Q)$ is an abstract Bezout matrix in the sense of [13] (see also Section 4.3 in [2]).

Proof of Theorem 2.1. Let L, M, P, Q be regular $n\times n$ matrix polynomials of degree at most r satisfying (2.3). According to Lemma 2.2, without loss of generality we may assume that the polynomials L and M are comonic, that each of these polynomials is of degree r, and that the leading coefficient of M is non-singular. In that case

$$\hat{\nu}_r(L,M)=\nu(L,M)=\sum_{\lambda\in\mathbb{C}}\nu(L,M;\lambda).$$

Next we write the comonic polynomials L, M, P, Q in realized form, as follows

$$L(\lambda) = I_r + \lambda C_L(I - \lambda N)^{-1}B, \quad M(\lambda) = I_r + \lambda C_M(I - \lambda N)^{-1}B,$$
$$P(\lambda) = I_r + \lambda C(I - \lambda V)^{-1}B_P, \quad Q(\lambda) = I_r + \lambda C(I - \lambda V)^{-1}B_Q.$$

Here B and C are the block matrices defined by (2.10), the block matrices V and N are defined by (2.11), and

$$B_P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_r \end{bmatrix}, \quad B_Q = \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_r \end{bmatrix},$$
$$C_L = \begin{bmatrix} L_1 & L_2 & \cdots & L_r \end{bmatrix}, \quad C_M = \begin{bmatrix} M_1 & M_2 & \cdots & M_r \end{bmatrix}.$$

Since the pair (N, B) is controllable and the pair (C, V) is observable, the above state space representations in combination with (2.12) imply that

$$TN - VT = B_P C_L - B_Q C_M, \quad TB = B_P - B_Q, \quad CT = C_M - C_L.$$

But then we can use Theorem 4.3 in [2] (see also Theorem 2 in [15]) to show that

$$\operatorname{Ker} B_r(L, M; P, Q) = \mathcal{M}, \tag{2.13}$$

where $\mathcal{M}$ is the largest subspace contained in $\operatorname{Ker}(C_L - C_M)$ that is invariant under $N - BC_L$. Note that the operators $N - BC_L$ and $N - BC_M$ coincide on the space $\operatorname{Ker}(C_L - C_M)$. Thus $\mathcal{M}$ is also equal to the largest subspace contained in $\operatorname{Ker}(C_L - C_M)$ that is invariant under $N - BC_M$.

Recall that N is a nilpotent matrix. This allows to apply Theorem 2.6 in [2]. Note that

$$N - BC_M = \begin{bmatrix} M_1 & \cdots & M_{r-1} & M_r \\ I & & & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{bmatrix}.$$

Thus the fact that M_r is non-singular implies that $N - BC_M$ is invertible. But then Theorem 2.6 in [2] tells us that $\nu(L, M) = \dim \mathcal{M}$. By combining this result with that of the preceding paragraph we obtain the desired equality (2.5). □

3. The resultant in relation to the Bezout matrix

It is well known that the resultant and the Bezoutian are closely related. We shall need the following theorem.

Theorem 3.1. *Let L and M be regular $n \times n$ matrix polynomials of degrees ℓ and m, respectively,*

$$L(\lambda) = L_0 + \lambda L_1 + \cdots + \lambda^\ell L_\ell, \quad M(\lambda) = M_0 + \lambda M_1 + \cdots + \lambda^m M_m.$$

Assume that $m \geq \ell > 0$ *and that* M_m *is non-singular. Furthermore, assume that there exist* $n \times n$ *matrix polynomials,*

$$P(\lambda) = P_0 + \lambda P_1 + \cdots + \lambda^m P_m, \quad Q(\lambda) = Q_0 + \lambda Q_1 + \cdots + \lambda^\ell Q_\ell,$$

such that P_m *is non-singular and quasi commutativity condition* (2.3) *is satisfied. Then the matrices*

$$\begin{bmatrix} \mathbf{B}_m(L, M; P, Q) & 0 \\ 0 & I_{mn} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{R}(L, M) & 0 \\ 0 & I_{(m-\ell)n} \end{bmatrix} \tag{3.1}$$

are equivalent. In particular, in that case,

$$\dim \operatorname{Ker} \mathbf{B}_m(L, M; P, Q) = \dim \operatorname{Ker} \mathbf{R}(L, M).$$

The above theorem may be derived as a corollary from Proposition 1.2 in [18]. See also Section 3.4 in [9] for a continuous time version of the result. For the sake of completeness, we shall present a proof of Theorem 3.1.

It will be convenient to first rewrite the quasi commutativity condition (2.3) in matrix form. For this purpose we need some additional notation. Let $A(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^r A_r$ be any $n \times n$ matrix polynomial of degree at most r. With A we associate two $r \times r$ block Toeplitz matrices, namely

$$\mathcal{L}_A = \begin{bmatrix} A_0 & & & \\ A_1 & A_0 & & \\ \vdots & \vdots & \ddots & \\ A_{r-1} & A_{r-2} & \cdots & A_0 \end{bmatrix}, \ \mathcal{U}_A = \begin{bmatrix} A_r & A_{r-1} & \cdots & A_1 \\ & A_r & \cdots & A_2 \\ & & \ddots & \vdots \\ & & & A_r \end{bmatrix} \tag{3.2}$$

The unspecified entries in the strictly upper triangular part of $\mathcal{L}_A$ stand for zero $n \times n$ matrices, and thus $\mathcal{L}_A$ is block lower triangular. Similarly, $\mathcal{U}_A$ is block upper triangular.

Proposition 3.2. *Let* L, M, P, Q *be* $n \times n$ *matrix polynomials of degree at most* r. *Then* (2.3) *holds if and only if the following three conditions are satisfied:*

$$P_r L_r = Q_r M_r, \quad \mathcal{L}_P \mathcal{L}_L = \mathcal{L}_Q \mathcal{L}_M, \quad \mathcal{U}_P \mathcal{L}_L + \mathcal{L}_P \mathcal{U}_L = \mathcal{U}_Q \mathcal{L}_M + \mathcal{L}_Q \mathcal{U}_M. \tag{3.3}$$

Moreover, in that case $\mathcal{U}_P \mathcal{U}_L = \mathcal{U}_Q \mathcal{U}_M$.

Proof. Let $A(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^r A_r$ be any $n \times n$ matrix polynomial $A(\lambda)$ of degree at most r. With $A(\lambda)$ we associate the $2r \times 2r$ block lower triangular Toeplitz matrix

$$\mathcal{T}_A = \begin{bmatrix} A_0 & & & & & \\ \vdots & \ddots & & & & \\ A_{r-1} & \cdots & A_0 & & & \\ A_r & \cdots & A_1 & A_0 & & \\ \vdots & & \vdots & \vdots & \ddots & \\ A_{2r-1} & \cdots & A_r & A_{r-1} & \cdots & A_0 \end{bmatrix}.$$

Here $A_j = 0$ for $j = r+1, \dots, 2r-1$. Using the block matrices in (3.2) we see that $\mathcal{T}_A$ can be partitioned as

$$\mathcal{T}_A = \begin{bmatrix} \mathcal{L}_A & 0 \\ \mathcal{U}_A & \mathcal{L}_A \end{bmatrix}. \tag{3.4}$$

In terms of the above notation condition (2.3) is equivalent to

$$P_r L_r = Q_r M_r, \quad \text{and} \quad \mathcal{T}_P \mathcal{T}_L = \mathcal{T}_Q \mathcal{T}_M. \tag{3.5}$$

The first equality in (3.5) is just the first equality in (3.3). Using (3.4) with P, L, Q, M in place of A it is straightforward to show that the second equality in (3.5) is equivalent to the combination of the second and third equality in (3.3).

To prove the final statement, let $A(\lambda)$ be as in the first paragraph, and define

$$A^\dagger(\lambda) = A_r + \lambda A_{r-1} + \cdots + \lambda^r A_0 = \lambda^r A(\lambda^{-1}).$$

A simple computation shows that

$$\mathcal{L}_{A^\dagger} = E\mathcal{U}_A E, \quad \mathcal{U}_{A^\dagger} = E\mathcal{L}_A E. \tag{3.6}$$

Here the matrix E is the $r \times r$ block permutation matrix all whose entries are zero except those on the main skew diagonal which are all equal to the $n \times n$ identity matrix.

Since our four polynomials in (2.1) and (2.2) are of degree at most r we can consider the polynomials $L^\dagger(\lambda)$, $M^\dagger(\lambda)$, $P^\dagger(\lambda)$, $Q^\dagger(\lambda)$. Obviously, we have

$$P(\lambda)L(\lambda) = Q(\lambda)M(\lambda) \;\Leftrightarrow\; P^\dagger(\lambda)L^\dagger(\lambda) = Q^\dagger(\lambda)M^\dagger(\lambda).$$

The second identity in (3.3) applied to the polynomials $L^\dagger(\lambda)$, $M^\dagger(\lambda)$, $P^\dagger(\lambda)$, $Q^\dagger(\lambda)$ now yields $\mathcal{L}_{P^\dagger}\mathcal{L}_{L^\dagger} = \mathcal{L}_{Q^\dagger}\mathcal{L}_{M^\dagger}$. But then we can use (3.6) and the fact that $E^2 = I$ to derive $\mathcal{U}_P\mathcal{U}_L = \mathcal{U}_Q\mathcal{U}_M$. □

The following proposition appears in a somewhat different form in [18], Section 1.2.

Proposition 3.3. *Let L, M, P, Q be $n \times n$ matrix polynomials of degree at most r satisfying* (2.3). *Then*

$$\mathbf{B}_r(L, M; P, Q) = (\mathcal{U}_P\mathcal{L}_L - \mathcal{U}_Q\mathcal{L}_M)\,E = (\mathcal{L}_Q\mathcal{U}_M - \mathcal{L}_P\mathcal{U}_L)\,E. \tag{3.7}$$

Here E is the $r \times r$ block permutation matrix whose entries are zero except those on the second main diagonal which are all equal to the $n \times n$ identity matrix.

Proof of Theorem 3.1. Without further explanation we use the notation introduced in the preceding paragraphs with $r = m$. Consider the 2×2 block matrix

$$\tilde{\mathbf{R}}(L, M) = \begin{bmatrix} E\mathcal{L}_L E & E\mathcal{U}_L E \\ E\mathcal{L}_M E & E\mathcal{U}_M E \end{bmatrix}.$$

A straightforward calculation (using $r = m$, the first identity in (3.7) and the equality $\mathcal{U}_P\mathcal{U}_L = \mathcal{U}_Q\mathcal{U}_M$) shows that

$$\begin{bmatrix} 0 & E \\ \mathcal{U}_P E & -\mathcal{U}_Q E \end{bmatrix} \tilde{\mathbf{R}}(L, M) = \begin{bmatrix} I & 0 \\ 0 & \mathbf{B}_m(L, M; P, Q) \end{bmatrix} \begin{bmatrix} \mathcal{L}_M E & \mathcal{U}_M E \\ I & 0 \end{bmatrix}.$$

The fact that $r = m$ and the matrices M_m and P_m are non-singular implies that the block matrices

$$\begin{bmatrix} 0 & E \\ \mathcal{U}_P E & -\mathcal{U}_Q E \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathcal{L}_M E & \mathcal{U}_M E \\ I & 0 \end{bmatrix}$$

are both non-singular. It follows that the matrix $\tilde{\mathbf{R}}(L, M)$ is equivalent to the first matrix in (3.1).

It remains to show that $\tilde{\mathbf{R}}(L, M)$ is also equivalent to the second matrix in (3.1). Since $r = m$, we have

$$E\mathcal{L}_L E = \begin{bmatrix} L_0 & \cdots & L_{m-1} \\ & \ddots & \vdots \\ & & L_0 \end{bmatrix}, \qquad E\mathcal{U}_L E = \begin{bmatrix} L_m & & \\ \vdots & \ddots & \\ L_1 & \cdots & L_m \end{bmatrix},$$

$$E\mathcal{L}_M E = \begin{bmatrix} M_0 & \cdots & M_{m-1} \\ & \ddots & \vdots \\ & & M_0 \end{bmatrix}, \qquad E\mathcal{U}_M E = \begin{bmatrix} M_m & & \\ \vdots & \ddots & \\ M_0 & \cdots & M_m \end{bmatrix}.$$

Recall that $m \geq \ell$. Put $s = m - \ell$. Then we see from the above identities that $\tilde{\mathbf{R}}(L, M)$ can be written as a 2×2 block matrix as follows:

$$\tilde{\mathbf{R}}(L, M) = \begin{bmatrix} \mathbf{R}(L, M) & 0 \\ X & Y \end{bmatrix}$$

Here 0 is a zero matrix of size $(\ell + m) \times s$. Furthermore, X is a block matrix of size $s \times (\ell + m)$ whose entries we do not need to specify further, and Y is a block lower triangular matrix of size $s \times s$ which has M_m as its main diagonal entries. In particular Y is invertible. It follows that $\tilde{\mathbf{R}}(L, M)$ is equivalent to the second matrix in (3.1), which completes the proof. $\square$

4. Proof of the sufficiency part of Theorem 0.1

Throughout we assume that there exist regular $n \times n$ matrix polynomials P and Q of degrees at most m and ℓ, respectively, such that (0.4) holds. Our aim is to prove (0.5). This will be done in two steps.

Part 1. In this part we assume additionally that L, M, P, and Q are comonic, that is, the matrices $L(0)$, $M(0)$, $P(0)$, and $Q(0)$ are all equal to the $n \times n$ identity matrix.

To prove (0.5), let $L^{\#}$ and $M^{\#}$ be the reversed polynomials associated with L and M, and put

$$P^{\dagger}(\lambda) = \lambda^m P(\lambda^{-1}), \quad Q^{\dagger}(\lambda) = \lambda^{\ell} Q(\lambda^{-1}).$$

Then all four polynomials $L^{\#}$, $M^{\#}$, $P^{\dagger}$, $Q^{\dagger}$ are monic, $L^{\#}$ and $Q^{\dagger}$ have degree ℓ, and $M^{\#}$ and $P^{\dagger}$ have degree m. Moreover, (0.4) yields

$$P^{\dagger}(\lambda)L^{\#}(\lambda) = Q^{\dagger}(\lambda)M^{\#}(\lambda), \quad \lambda \in \mathbb{C}.$$

Next, we set $r = \max\{\ell, m\}$ and apply Theorems 2.1 and 3.1 to the polynomials $L^{\#}$, $M^{\#}$, $P^{\dagger}$, $Q^{\dagger}$. This yields the following two identities

$$\dim \operatorname{Ker} \mathbf{B}_r(L^{\#}, M^{\#}; P^{\dagger}, Q^{\dagger}) = \hat{\nu}_r(L^{\#}, M^{\#}),$$

$$\dim \operatorname{Ker} \mathbf{B}_r(L^{\#}, M^{\#}; P^{\dagger}, Q^{\dagger}) = \dim \operatorname{Ker} \mathbf{R}(L^{\#}, M^{\#}).$$

Thus $\dim \operatorname{Ker} \mathbf{R}(L^{\#}, M^{\#}) = \hat{\nu}_r(L^{\#}, M^{\#})$. Since $L^{\#}$ and $M^{\#}$ are monic and $r = \max\{\ell, m\}$, we see that $\hat{\nu}_r(L^{\#}, M^{\#}) = \hat{\nu}(L^{\#}, M^{\#})$. Hence

$$\dim \operatorname{Ker} \mathbf{R}(L^{\#}, M^{\#}) = \hat{\nu}(L^{\#}, M^{\#}).$$

To get (0.5) it remains to show that

$$\dim \operatorname{Ker} \mathbf{R}(L^{\#}, M^{\#}) = \dim \operatorname{Ker} \mathbf{R}(L, M), \quad \hat{\nu}(L^{\#}, M^{\#}) = \hat{\nu}(L, M). \tag{4.1}$$

To prove the first identity in (4.1) note that $L^{\#}$ has the same degree as L, and that $M^{\#}$ has the same degree as M. Thus the resultants $\mathbf{R}(L, M)$ and $\mathbf{R}(L^{\#}, M^{\#})$ are block matrices of the same size, and the block entries are of the same size too. We shall see that the two resultants $\mathbf{R}(L, M)$ and $\mathbf{R}(L^{\#}, M^{\#})$ are equivalent. Indeed, given a positive integer q we define $\mathbf{E}_q$ to be the $q \times q$ block matrix

$$\mathbf{E}_q = \left[\delta_{q+1-i,j} I_n\right]_{i,j=1}^{q}.$$

Here I_n is the $n \times n$ identity matrix and $\delta_{p,q}$ is the Kronecker delta. A straightforward calculation shows that

$$\begin{bmatrix} \mathbf{E}_m & 0 \\ 0 & \mathbf{E}_{\ell} \end{bmatrix} \mathbf{R}(L, M)\mathbf{E}_{\ell+m} = \mathbf{R}(L^{\#}, M^{\#}).$$

The fact that for each q the matrix $\mathbf{E}_q$ is invertible implies that $\mathbf{R}(L, M)$ and $\mathbf{R}(L^{\#}, M^{\#})$ are equivalent. Thus the null spaces $\operatorname{Ker} \mathbf{R}(L^{\#}, M^{\#})$ and $\operatorname{Ker} \mathbf{R}(L, M)$ have the same dimension, which proves the first identity in (4.1).

Next, we prove the second identity in (4.1). The fact that L and $L^{\#}$ have the same degree implies that the reversed polynomial of $L^{\#}$ is again L, that is, $(L^{\#})^{\#} = L$. Similarly, $(M^{\#})^{\#} = M$. It follows that

$$\nu(L, M; \infty) = \nu(L^{\#}, M^{\#}; 0), \quad \nu(L, M; 0) = \nu(L^{\#}, M^{\#}; \infty).$$

Furthermore,

$$\nu(L, M; \lambda_0) = \nu(L^{\#}, M^{\#}; \lambda_0)^{-1}, \quad \lambda_0 \in \mathbb{C}.$$

From these identities the second part of (4.1) is clear. Thus (4.1) is proved, and for comonic matrix polynomials the sufficiency part of the proof of Theorem 0.1 is established.

Part 2. In this part we deal with the general case. Since L, M, P, and Q are all regular, we can choose $\alpha \in \mathbb{C}$ such that the matrices $L(\alpha)$, $M(\alpha)$, $P(\alpha)$ and $Q(\alpha)$ are non-singular. This allows us to define

$$\tilde{L}(\lambda) = L(\alpha)^{-1}L(\lambda + a), \quad \tilde{M}(\lambda) = M(\alpha)^{-1}M(\lambda + a),$$
$$\tilde{P}(\lambda) = P(\lambda + a)L(\alpha), \qquad \tilde{Q}(\lambda) = Q(\lambda + a)M(\alpha).$$

The polynomials $\tilde{L}$ and $\tilde{M}$ are comonic and have the same degrees as L and M, respectively, that is, degree $\tilde{L} = \ell$ and degree $\tilde{M} = m$. The matrix polynomials $\tilde{P}$ and $\tilde{Q}$ are also comonic with degree $\tilde{P}$ = degree P and degree $\tilde{Q}$ = degree Q. In particular, the degrees of $\tilde{P}$ and $\tilde{Q}$ are at most m and ℓ, respectively. Moreover, since (0.4) holds, we have

$$\tilde{P}(\lambda)\tilde{L}(\lambda) = \tilde{Q}(\lambda)\tilde{M}(\lambda), \quad \lambda \in \mathbb{C}.$$

By the result of the previous part, it follows that

$$\dim \operatorname{Ker} \mathbf{R}(\tilde{L}, \tilde{M}) = \hat{\nu}(\tilde{L}, \tilde{M}).$$

Thus to complete the proof it remains to show that

$$\dim \operatorname{Ker} \mathbf{R}(\tilde{L}, \tilde{M}) = \dim \operatorname{Ker} \mathbf{R}(L, M), \quad \hat{\nu}(\tilde{L}, \tilde{M}) = \hat{\nu}(L, M). \tag{4.2}$$

Since $L(\alpha)$ and $M(\alpha)$ are invertible, we have $\nu(\tilde{L}, \tilde{M}; \lambda) = \nu(L, M; \lambda + \alpha)$ for each $\lambda \in \mathbb{C}$ and $\nu(\tilde{L}, \tilde{M}; \infty) = \nu(L, M; \infty)$. Hence the second identity in (4.2) holds. To prove the first identity in (4.2), we first observe that degree $\tilde{L}$ = degree $L = \ell$ and degree $\tilde{M}$ = degree $M = m$. Thus the resultants $\mathbf{R}(\tilde{L}, \tilde{M})$ and $\mathbf{R}(L, M)$ are block matrices of block size $(\ell + m) \times (\ell + m)$. Moreover, each block entry is a matrix of size $n \times n$. We shall see that $\mathbf{R}(\tilde{L}, \tilde{M})$ and $\mathbf{R}(L, M)$ are equivalent.

For $j, k = 0, 1, 2, \ldots$ let $F_{jk}(\alpha)$ be the $n \times n$ matrix defined by (2.9). For $q = 1, 2, \ldots$ let $\mathbf{F}_q(\alpha)$ be the $q \times q$ block matrix given by

$$\mathbf{F}_q(\alpha) = \left[F_{jk}(\alpha)\right]_{j,k=0}^{q-1},$$

where $F_{jk}(\alpha)$ be the $n \times n$ matrix defined by (2.9). Furthermore, let $\Lambda_{L(\alpha)}$ and $\Lambda_{M(\alpha)}$ be block diagonal matrices with $L(\alpha)^{-1}$ and $M(\alpha)^{-1}$, respectively, on the main diagonal. We require $\Lambda_{L(\alpha)}$ to be of block size $m \times m$ and $\Lambda_{M(\alpha)}$ is of block size $\ell \times \ell$. One checks that

$$\begin{bmatrix} \mathbf{F}_m(\alpha)\Lambda_{L(\alpha)} & 0 \\ 0 & \mathbf{F}_\ell(\alpha)\Lambda_{M(\alpha)} \end{bmatrix} \mathbf{R}(L, M) = \mathbf{R}(\tilde{L}, \tilde{M})\mathbf{F}_{\ell+m}(\alpha). \tag{4.3}$$

This identity provides the desired equivalence. Indeed, for each q the block matrix $\mathbf{F}_q(\alpha)$ is block lower triangular with the $n \times n$ identity matrix on the main diagonal. Thus $\mathbf{F}_q(\alpha)$ is non-singular for each q. Since $L(\alpha)$ and $M(\alpha)$ are also invertible, it follows that the first and fourth factor in (4.3) are non-singular, and hence $\mathbf{R}(\tilde{L}, \tilde{M})$ and $\mathbf{R}(L, M)$ are equivalent, which proves the first part of (4.2). Thus (4.2) holds. □

References

[1] B.D.O. Anderson and E.I. Jury, Generalized Bezoutian and Sylvester matrices in multivariable linear control, *IEEE Trans. Automatic Control*, AC-**21** (1976), 551–556.

[2] I. Gohberg, I. Haimovic, M.A. Kaashoek, and L. Lerer, The Bezout integral operator: main property and underlying abstract scheme, in: *The state space method. Generalizations and applications*, OT **161**, Birkhäuser Verlag, Basel, 2005, pp. 225–270.

[3] I. Gohberg and G. Heinig, The resultant matrix and its generalizations, I. The resultant operator for matrix polynomials, *Acta Sci. Math.* (*Szeged*) **37** (1975), 41–61 [in Russian].

[4] I. Gohberg and G. Heinig, The resultant matrix and its generalizations, II. Continual analog of resultant matrix, *Acta Math. Acad. Sci. Hungar* **28** (1976), 198–209 [in Russian].

[5] I. Gohberg, M.A. Kaashoek, L. Lerer, and L. Rodman, Common multiples and common divisors of matrix polynomials, I. Spectral method, *Indiana J.Math.* **30** (1981), 321–356.

[6] I. Gohberg, M.A. Kaashoek, L. Lerer, and L. Rodman, Common multiples and common divisors of matrix polynomials, II. Vandermonde and resultant matrices, *Linear and Multilinear Algebra* **12** (1982), 159–203.

[7] I. Gohberg, M.A. Kaashoek, and L. Lerer, Quasi-commutativity of entire matrix functions and the continuous analogue of the resultant, in: *Modern operator theory and applications. The Igor Borisovich Simonenko Anniversary Volume,* OT **170**, Birkhäuser Verlag, Basel, 2007, pp. 101–106.

[8] I. Gohberg, M.A. Kaashoek, and L. Lerer, The continuous analogue of the resultant and related convolution operators, in: *The extended field of operator theory* (M.A. Dritschel, ed.), OT **171**, Birkhäuser Verlag, Basel, 2007, pp. 107–127.

[9] I. Gohberg, M.A. Kaashoek, and L. Lerer, The resultant for regular matrix polynomials and quasi commutativity, *Indiana University Mathematics Journal*, **57** (2008), 2783–2813.

[10] I. Gohberg, M.A. Kaashoek and L. Rodman, Spectral analysis of families of operator polynomials and a generalized Vandermonde matrix, I. The finite dimensional case, in: *Topics in Functional Analysis.* Advances in Mathematics, Supplementary Studies, vol. 3, Academic Press, London 1978; pp. 91–128.

[11] I. Gohberg, P. Lancaster, and L. Rodman, *Matrix polynomials*, Academic Press, New York, 1982.

[12] I. Gohberg and L. Lerer, Matrix generalizations of M.G. Krein theorems on orthogonal polynomials. OT **34** Birkhäuser Verlag, Basel, 1988, pp. 137–202.

[13] I. Haimovici, and L. Lerer, Bezout operators for analytic operator functions, I. A general concept of Bezout operator, *Integral Equations Oper. Theory* **21** (1995), 33–70.

[14] T. Kailath, *Linear systems.* Prentice-Hall, Inc., Englewood Cliffs, N.J., 1980.

[15] I. Karelin and L. Lerer, Matrix quadratic equations, column/row reduced factorizations and an inertia theorem for matrix polynomials, *Int. J. Appl. Math. Comput. Sci.* **11** (2001), 1285–1310.

[16] M.G. Krein and M.A. Naimark, The method of symmetric and hermitian forms in theory of separation of the roots of algebraic equations, GNTI, Kharkov, 1936 [Russian]; *Linear Multilinear Algbra* **10** (1981), 265–308.

[17] L. Lerer and L. Rodman, Bezoutians of rational matrix functions, *J. Funct. Anal.* **141** (1996), 1–36.

[18] L. Lerer and M. Tismenetsky, The eigenvalue separation problem for matrix polynomials, *Integral Equations Oper. Theory* **5** (1982), 386–445.

[19] I. Sylvester, On a theory of syzygetic relations of two rational integral functions, comprising an application to the theory of Sturm's functions, and that of the greatest algebraical common measure, *Philos. Trans. Roy. Soc. London* **143** (1853), 407–548.

[20] B.L. van der Waerden, *Modern Algbra*, I and II (English translation), Frederick Ungar Publ. Co., New York, 1949 and 1950.

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Received: March 2, 2009
Accepted: March 30, 2009

Operator Theory:
Advances and Applications, Vol. 202, 315–319

Quasidiagonal Extensions of the Reduced Group C*-algebras of Certain Discrete Groups

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Abstract. Let G be countable group containing a free subgroup F of finite index. We show that the reduced group C^*-algebra $C^*_{\text{red}}(G)$ has a quasidiagonal extension. Our proof is based on a result of Haagerup and Thorbjørnsen [HT] asserting the existence of such an extension of $C^*_{\text{red}}(F)$ when F is a free group of rank greater than one. A consequence of our result is that if Γ is a free product of finitely many (non-trivial) cyclic groups and $\Gamma \neq Z_2 \star \mathbb{Z}_2$, then $\text{Ext}(C^*_{\text{red}}(\Gamma))$ is not a group.

Mathematics Subject Classification (2000). Primary 46L05; Secondary 46L45, 46L35.

Keywords. Quasi diagonal extension, reduced group C^*-algebra, free product of cyclic groups.

1. Introduction

1.1 Let $\mathcal{H}$ be a separable Hilbert space. Consider a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots$, with $\dim(\mathcal{H}_i) < \infty$ $(i = 1, \dots)$. A bounded linear operator $T \in \mathcal{B}(\mathcal{H})$ is *block-diagonal* with respect to the decomposition $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots$, of $\mathcal{H}$ if $T = B_1 \oplus B_2 \oplus \cdots$, for some $B_i \in \mathcal{B}(\mathcal{H}_i)$. T is *block-diagonal* if it is block-diagonal with respect to some decomposition $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots$ of $\mathcal{H}$ with $\dim(\mathcal{H}_i) < \infty$ $(i = 1, \dots)$. T is *quasidiagonal* if it is a compact perturbation of a block-diagonal operator on $\mathcal{H}$, that is if $T = D + K$ for some block-diagonal operator D and some compact operator K on $\mathcal{H}$. Similarly, a set $\mathcal{S}$ of operators on $\mathcal{H}$ is *quasidiagonal* if there is a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots$, with $\dim(\mathcal{H}_i) < \infty$ $(i = 1, \dots)$, such that each operator in $\mathcal{S}$ is a compact perturbation of some operator that is block-diagonal with respect to this decomposition of $\mathcal{H}$. For a norm-separable set $\mathcal{S}$ quasidiagonality is equivalent to the existence of an increasing sequence of finite rank projections $P_1 \leq P_2 \leq P_3 \leq \cdots$ converging strongly to $I_{\mathcal{H}}$, such that $\lim_n \| [T, P_n] \| = 0$ for any $T \in \mathcal{S}$. The latter is, in turn, equivalent to quasidiagonality of the operator C^*-algebra

We are indebted to the referees for a number of helpful suggestions.
Communicated by J.A. Ball.

$C^*(\mathcal{S}) + \mathcal{K}(\mathcal{H}) + \mathbb{C}I_{\mathcal{H}}$, where $\mathcal{K}(\mathcal{H})$ is the algebra of compact operators on $\mathcal{H}$. A separable (abstract) C^*-algebra $\mathcal{A}$ is called quasidiagonal if it has a faithful representation $\phi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ such that the set $\phi(\mathcal{A})$ is quasidiagonal.

By the Weyl-von Neumann-Berg theorem, any normal operator on a separable Hilbert space is quasidiagonal. On the other hand, a non-unitary isometry is not quasidiagonal. Rosenberg [Ro, Theorem A1] (see also [Bl, V.4.2.13]) showed that the reduced C^*-algebras $C^*_{\text{red}}(G)$ of a discrete countable non-amenable group G is not quasidiagonal. $C^*_{\text{red}}(G)$ is the C^*-algebra generated by the left regular representation of G on $\ell^2(G)$. Excellent sources for information on quasidiagonality and related notions are Blackadar's book [Bl] and the survey articles by Brown [Br] and Voiculescu [Vo].

One of the peculiar aspects of quasidiagonality is the presence of separable quasidiagonal C^*-algebras of operators $\mathcal{E}$ whose quotients $A \cong \mathcal{E}/\mathcal{K}$ by the ideal of compact operators $\mathcal{K}$ are not quasidiagonal ([Wa1], [Wa2], [HT]). In terms of the C^*-algebra extension theory this amounts to the existence of essential unital *quasidiagonal extensions* $0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E} \longrightarrow \mathcal{A} \longrightarrow 0$ of certain separable unital non-quasidiagonal C^*-algebras $\mathcal{A}$ by the C^*-algebra of compact operators $\mathcal{K}$, where the extension C^*-algebra $\mathcal{E}$ is quasidiagonal.

1.2 Any essential unital extension of a separable unital C^*-algebra $\mathcal{A}$ by $\mathcal{K}$ defines a unital $*$-monomorphism $\tau : \mathcal{A} \longrightarrow \mathcal{C}(\mathcal{H})$, where $\mathcal{C}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the Calkin algebra on $\mathcal{H}$, and thus determines an element $[\tau]$ of the invariant $\text{Ext}(\mathcal{A})$ consisting of the unitary equivalence classes of all such $*$-monomorphisms (see [Ar], [Bl], [BDF]). Using the isomorphisms $\mathcal{H} \cong \mathcal{H} \oplus \mathcal{H}$ and $\mathcal{C}(\mathcal{H} \oplus \mathcal{H}) \cong M_2(\mathcal{C}(\mathcal{H}))$ the assignment $[\tau_1] + [\tau_2] = [\tau_1 \oplus \tau_2]$ defines the (commutative) addition operation on $\text{Ext}(\mathcal{A})$. By Voiculescu's theorem, $\text{Ext}(\mathcal{A})$ is a unital semigroup. The identity element is the class defined by the "trivial" extension with extension algebra $\mathcal{E} = \phi(\mathcal{A}) + \mathcal{K}$, where ϕ is any faithful unital representation of $\mathcal{A}$ on a separable infinite-dimensional Hilbert space such that $\phi(\mathcal{A}) \cap \mathcal{K} = \emptyset$. From the preceding it follows that an element $[\tau]$ has the inverse in $\text{Ext}(\mathcal{A})$ precisely when τ lifts to a completely positive unital map $\varphi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ such that $\tau = \rho \circ \varphi$, where ρ is the quotient map of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{C}(\mathcal{H})$. A quasidiagonal extension of a non-quasidiagonal separable C^*-algebra $\mathcal{A}$ is not liftable (so that $\text{Ext}(\mathcal{A})$ is not a group), and that the corresponding extension algebra $\mathcal{E}$ is not exact (in particular $\mathcal{E}$ is not nuclearly embeddable) (cf. [Br, Corollary 13.5], [EH, Corollary 5.6], [Ki, Corollary 1.4]).

1.3 While the existence of quasidiagonal extensions of some non-quasidiagonal C^*-algebras, were realized long ago, relatively few examples are known. In particular, a problem of considerable interest is:

If G is a discrete countable non-amenable group must $C^*_{\text{red}}(G)$ have a quasidiagonal extension?

In [HT] Haagerup and Thorbjørnsen answered this question in the affirmative for countable free groups of rank greater than 1. In this note we show that $C^*_{\text{red}}(G)$ has a quasidiagonal extension for groups G containing a free subgroup of finite index. Thus we obtain the following:

Theorem. *Let G be a countable discrete group containing a free subgroup of finite index. Then there exists a quasidiagonal unital extension*

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E} \longrightarrow C^*_{\mathrm{red}}(G) \longrightarrow 0.$$

If G is non-amenable (i.e., the free group has rank greater than 1), then this extension defines a non-invertible element of the semigroup $\mathrm{Ext}(C^*_{\mathrm{red}}(G))$ *and the extension algebra $\mathcal{E}$ is not exact.*

Let G, H be groups. A *reduced word* in G and H is an alternating product of elements of G and elements of H, e.g., $g_1h_1g_2h_2\cdots g_kh_k$ where g_1 or h_k may be the identity. The *free product* $G \star H$ is the group whose elements are the reduced words in G and H, under the operation of concatenation followed by reduction. The class of non-amenable groups considered in the Theorem contains all finite free products of cyclic groups except the infinite dihedral group $\mathbb{Z}_2 \star \mathbb{Z}_2$ (which contains a subgroup isomorphic to $\mathbb{Z}$ of index two).

Corollary. *If Γ is a finite free product of non-trivial cyclic groups and $\Gamma \neq Z_2 \star \mathbb{Z}_2$, then* $\mathrm{Ext}(C^*_{\mathrm{red}}(\Gamma))$ *is not a group.*

2. Proof of the theorem

Let G be a countable discrete group containing a free subgroup F of rank greater than 1 and of finite index $[G : F] = m$. If α is a unitary representation of G, we let $C^*_\alpha(G)$ denote the C^*-algebra generated by $\alpha(G)$. We will not distinguish in notation between α and its canonical extension $C^*(G) \longrightarrow C^*_\alpha(G)$ to the (full) group C^*-algebra $C^*(G)$, which is the universal C^*-algebra for unitary representations of G. By restricting the unitary representations of G to F the group algebra $C^*(F)$ can be identified with a subalgebra of $C^*(G)$ (cf. [Rie, Proposition 1.2]). Let $\{g_1, \ldots, g_m\}$ be a set of left coset representatives of G modulo F, and let $u_1, \ldots, u_m$ be the unitaries of $C^*(G)$ corresponding to $g_1, \ldots, g_m$.

Let E denote the conditional expectation of $C^*(G)$ onto $C^*(F)$ which is the extension to $C^*(G)$ of the projection map of $\ell^1(G)$ onto $\ell^1(F)$ obtained by restricting functions from G to F. Then for each $x \in C^*(G)$ we have

$$\sum_{i=1}^{m} u_i E(u_i^* x) = x = \sum_{i=1}^{m} E(xu_i)u_i^*,$$

which follows by continuity of E from the similar identity holding for each x in the group algebra $\mathbb{C}[G]$ (cf. [Wat, Example 1.2.3]).

Suppose π is a representation of $C^*(F)$, and let α be the representation of $C^*(G)$ induced from π. Following [Rie], the representation space $\mathcal{H}_\alpha$ is the completion of the quotient of the vector space $C^*(G) \otimes_{C^*(F)} \mathcal{H}_\pi$ (the tensor product of the natural right $C^*(F)$-module $C^*(G)$ and the left $C^*(F)$-module $\mathcal{H}_\pi$) by the subspace of vectors of length zero with respect to the inner product given by

$$\langle x \otimes \xi,\, y \otimes \eta\rangle = \langle \pi(E(y^*x)\xi,\, \eta\rangle \qquad (x, y \in C^*(G);\ \xi, \eta \in \mathcal{H}_\pi).$$

For simplicity of exposition we will not distinguish in notation between elements of $C^*(G)\otimes_{C^*(F)}\mathcal{H}_\pi$ and their images in $\mathcal{H}_\alpha$. As $E(u_i^*u_j)=0$ when $i\neq j$, it follows that $\mathcal{H}_\alpha$ is the direct sum of m subspaces $u_i\otimes\mathcal{H}_\pi$ $(i=1,\dots,m)$.

For each $x\in C^*(G)$ the action of $\alpha(x)$ is (up to a unitary equivalence) defined by

$$\begin{aligned}\alpha(x)\Big(\sum_{i=1}^m u_i\otimes\zeta_i\Big) &= \sum_{i=1}^m u_i x^*\otimes\zeta_i = \sum_{i=1}^m\Big(\sum_{j=1}^m u_j E(u_j^*u_i x^*)\Big)\otimes\zeta_i\\ &= \sum_{i=1}^m\sum_{j=1}^m u_i E(u_i^*u_j x^*)\otimes\zeta_j = \sum_{i=1}^m u_i\otimes\sum_{j=1}^m \pi(E(u_i^*u_j x^*))\zeta_j.\end{aligned}$$

Consequently, each operator $\alpha(x)$ is represented, relative to this direct sum decomposition of $\mathcal{H}_\alpha$, by the $m\times m$ matrix $(\pi(E(u_i^*u_jx^*)))$ in $M_m(C^*_\pi(F))$. It is easily seen that the restriction of α to $C^*(F)$ is unitarily equivalent to the m-fold amplification of π. In particular, if $\pi=\lambda_F$, the left regular representation, then α is unitarily equivalent to λ_G; so that $C^*_{\mathrm{red}}(G)$ can be identified with a unital C^*-subalgebra of $M_m(C^*_{\mathrm{red}}(F))$. Noting that F can be embedded in F_2 (the free group on two generators), F_2 embeds in $Z_2\star\mathbb{Z}_3$ and $C^*_{\mathrm{red}}(Z_2\star\mathbb{Z}_3)$ has a $*$-isomorphic embedding in the Cuntz algebra $\mathcal{O}_2$ (by a result of Choi [Ch]), it follows that $C^*_{\mathrm{red}}(G)$ is $*$-isomorphic to a C^*-subalgebra of the nuclear C^*-algebra $M_m(\mathcal{O}_2)$. Hence $C^*_{\mathrm{red}}(G)$ is exact.

In the course of the proof of [HT, Theorem 8.2] Haagerup and Thorbjørnsen discovered a sequence of finite-dimensional unitary representations π_k $(k=1,\dots)$ of F, such that $\lim_{k\to\infty}\|\pi_k(f)\|=\|\lambda_F(f)\|$ for any f in the group algebra $\mathbb{C}[F]$. Letting $\pi=\sum_{k=1}^\infty\oplus\pi_k$ the latter implies that the quotient map of $\mathcal{B}(\mathcal{H}_\pi)$ onto the Calkin algebra carries $C^*_\pi(F)$ onto a copy of $C^*_{\mathrm{red}}(F)$ (cf. [Pe, Lemma 1.5.4]).

Let α be the representation of $C^*(G)$ induced from π. By the preceding part, $C^*_\alpha(G)$ can be identified with a unital C^*-subalgebra of $M_m(C^*_\pi(F))$. Since $\mathcal{K}(\mathcal{H}_\alpha)\cong M_m(\mathcal{K}(\mathcal{H}_\pi))$ and $C^*_\pi(F)/C^*_\pi(F)\cap\mathcal{K}(\mathcal{H}_\pi)\cong C^*_{\mathrm{red}}(F)$, it follows that the quotient map ρ of $\mathcal{B}(\mathcal{H}_\alpha)$ onto $\mathcal{C}(\mathcal{H}_\alpha)$ carries $C^*_\alpha(G)$ onto the C^*-subalgebra of $M_m(C^*_{\mathrm{red}}(F))$ corresponding to $C^*_{\mathrm{red}}(G)$. Consequently $C^*_\alpha(G)/C^*_\alpha(G)\cap\mathcal{K}(\mathcal{H}_\alpha)\cong C^*_{\mathrm{red}}(G)$. As $C^*_\pi(F))$ is quasidiagonal, so are the C^*-algebras $M_m(C^*_\pi(F))$ and $C^*_\alpha(G)+\mathcal{K}(\mathcal{H}_\alpha)$. We thus obtain the quasidiagonal extension

$$0\longrightarrow\mathcal{K}(\mathcal{H}_\alpha)\xrightarrow{\ \iota\ }C^*_\alpha(G)+\mathcal{K}(\mathcal{H}_\alpha)\xrightarrow{\ \rho\ }C^*_{\mathrm{red}}(G)\longrightarrow 0.$$

Since $C^*_{\mathrm{red}}(G)$ is not quasidiagonal, this extension is not liftable and the extension algebra $C^*_\alpha(G)+\mathcal{K}(\mathcal{H}_\alpha)$ is not exact. Another, perhaps easier, way to see the latter, which was pointed out to us by one of the referees, is to note that $C^*_\alpha(G)$ contains a $*$-isomorphic copy of $C^*_\pi(F)$, which is not exact by [HT, Remark 8.7]. This implies that $C^*_\alpha(F)$ and $C^*_\alpha(G)+\mathcal{K}(\mathcal{H}_\alpha)$ are not exact, since exactness passes to C^*-subalgebras and to quotients, by results of Kirchberg [Ki]. Since $C^*_{\mathrm{red}}(G)$ is exact, it follows by results of Effros and Haagerup [EH] that the above extension is not liftable.

The existence of a quasidiagonal extension of $C^*_{\mathrm{red}}(G)$ is also obviously true in the amenable case $F = \mathbb{Z}$ using any trivial unital extension of $C^*(G)$ $(= C^*_{\mathrm{red}}(G))$ (for instance, one may use the above representation α induced from any faithful block-diagonal representation π of $C(S^1)$ $(\cong C^*(\mathbb{Z}))$ such that $\pi(C(S^1))$ does not contain compact operators).

References

[Ar] W. Arveson, *Notes on extensions of C^*-algebras*, Duke Math. J. **44** (1977), 329–355.

[Bl] B. Blackadar, *Operator Algebras: Theory of C^*-algebras and von Neumann Algebras*, Encyclopaedia of Mathematical Sciences Volume **122**, Operator Algebras and Non-Commutative Geometry III, Springer, 2006.

[BDF] L.G. Brown, R.G. Douglas and P.A. Fillmore, *Extensions of C^*-algebra and K-homology*, Annals Math. **105** (1977), 265–324.

[Br] N. Brown, *On quasidiagonal C^*-algebras*, Advanced Studies in Pure Mathematics Volume **38**, Operator algebras and applications, pp. 19–64, Math. Soc. Japan, Tokyo 2004.

[Ch] M.D. Choi, *A simple C^*-algebra generated by two finite-order unitaries*, Canad. J. Math. **31** (1979), 867–880.

[EH] E.G. Effros and U. Haagerup, *Lifting problems and local reflexivity of C^*-algebras*, Duke Math. J. **52** (1985), 103–128.

[HT] U. Haagerup and S. Thorbjørnsen, *A new application of random matrices: $\mathrm{Ext}(C^*_{\mathrm{red}}(F_2))$ is not a group*, Annals Math. **162** (2005), 711–775.

[Ki] E. Kirchberg, *On subalgebras of the CAR-algebra*, J. Funct. Anal. **129** (1995), 35–63.

[Pe] G.K. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, 1979.

[Rie] M.A. Rieffel, *Induced representations of C^*-algebras*, Adv. in Math. **13** (1974), 176–257.

[Ro] J. Rosenberg, *Appendix to "Strongly quasidiagonal C^*-algebras" by D. Hadwin*, Journ. Operator Theory **18** (1987), 3–18.

[Vo] D. Voiculescu, *Around quasidiagonal operators*, Integral Equat. Operator Theory **17** (1993), 137–149.

[Wa1] S. Wassermann, *C^*-algebras associated with groups with Kazhdan's property T*, Annals. Math. **134** (1991), 423–431.

[Wa2] S. Wassermann, *A separable quasidiagonal C^*-algebra with a nonquasidiagonal quotient by the compact operators*, Math. Proc. Cambridge Philos. Soc. **110** (1991), 143–145.

[Wat] Y. Watatani, *Index for C^*-subalgebras*, Memoirs Amer. Math. Soc. **424**, 1990.

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Received: February 27, 2009
Accepted: June 25, 2009

Operator Theory:
Advances and Applications, Vol. 202, 321–336

Singular Integral Operators on Variable Lebesgue Spaces over Arbitrary Carleson Curves

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To Professor Israel Gohberg on the occasion of his eightieth birthday

Abstract. In 1968, Israel Gohberg and Naum Krupnik discovered that local spectra of singular integral operators with piecewise continuous coefficients on Lebesgue spaces $L^p(\Gamma)$ over Lyapunov curves have the shape of circular arcs. About 25 years later, Albrecht Böttcher and Yuri Karlovich realized that these circular arcs metamorphose to so-called logarithmic leaves with a median separating point when Lyapunov curves metamorphose to arbitrary Carleson curves. We show that this result remains valid in a more general setting of variable Lebesgue spaces $L^{p(\cdot)}(\Gamma)$ where $p : \Gamma \to (1, \infty)$ satisfies the Dini-Lipschitz condition. One of the main ingredients of the proof is a new condition for the boundedness of the Cauchy singular integral operator on variable Lebesgue spaces with weights related to oscillations of Carleson curves.

Mathematics Subject Classification (2000). Primary 47B35; Secondary 45E05, 46E30, 47A68.

Keywords. Fredholmness, variable Lebesgue space, Dini-Lipschitz condition, Carleson curve, singular integral operator, piecewise continuous coefficient, spirality indices.

1. Introduction

About forty years ago I. Gohberg and N. Krupnik [11] constructed an elegant Fredholm theory for singular integral operators with piecewise continuous coefficients on Lebesgue spaces $L^p(\Gamma)$ over Lyapunov curves. Their result says that the local spectra at discontinuity points of the coefficients have the shape of circular arcs depending on p. That paper was the starting point for generalizations and exten-

Communicated by I.M. Spitkovsky.

sions of those results to the case of power weights, matrix coefficients, and Banach algebras generated by singular integral operators (see [12, 13]). I. Spitkovsky [37] discovered that circular arcs metamorphose to massive horns if one replaces power weights by general Muckenhoupt weights. A. Böttcher and Yu. Karlovich [2] observed that local spectra of singular integral operators with piecewise continuous coefficients can be massive even on $L^2(\Gamma)$ when Γ is an arbitrary Carleson curve. The Fredholm theory for the Banach algebra generated by matrix singular integral operators on $L^p(\Gamma, w)$ under the most general conditions on the curve Γ and the weight w is constructed by A. Böttcher and Yu. Karlovich and is presented in the monograph [3] (although, we advise to start the study of this theory from the nice survey [4]).

I. Gohberg and N. Krupnik [11] also obtained some sufficient conditions for the Fredholmness of singular integral operators with piecewise continuous coefficients on so-called symmetric spaces (see [28] for the definition) known also as rearrangement-invariant spaces (see [1]). These spaces include classical Lebesgue, Orlicz, and Lorentz spaces. The author [15, 16] proved a criterion for the Fredholmness of singular integral operators on rearrangement-invariant spaces and observed that a "complicated" space may also cause massiveness of local spectra.

Another natural generalization of the standard Lebesgue space $L^p(\Gamma)$ is a so-called variable Lebesgue space $L^{p(\cdot)}$ defined in terms of the integral

$$\int_\Gamma |f(\tau)|^{p(\tau)}\,|d\tau|$$

(see the next section for the definition). Here the exponent p is a continuous function on Γ. Notice that variable Lebesgue spaces are not rearrangement-invariant. V. Kokilashvili and S. Samko [25] extended the results of [11] to the setting of variable Lebesgue spaces over Lyapunov curves. In this setting, the circular arc depends on the value of the exponent $p(t)$ at a discontinuity point $t \in \Gamma$. Later on, the author gradually extended results known for singular integral operators with piecewise continuous coefficients on weighted standard Lebesgue spaces (see [3, 13]) to the case of weighted variable Lebesgue spaces (see [17, 19] for power weights and Lyapunov curves; [18] for power weights and so-called logarithmic Carleson curves; [20] for radial oscillating weights and logarithmic Carleson curves).

In this paper we construct a symbol calculus for the Banach algebra of singular integral operators with matrix piecewise continuous coefficients on (unweighted) variable Lebesgue space over arbitrary Carleson curves. We suppose that the variable exponent is little bit better than continuous and, roughly speaking, show that local spectra at the points t of discontinuities of coefficients are so-called logarithmic leaves (with a median separating point) [3, Section 7.5] depending on the spirality indices δ_t^-, δ_t^+ of the curve at t and the value $p(t)$. So we replace the constant exponent p in the results for $L^p(\Gamma)$ [2] by the value $p(t)$ at each point. Let us explain why this is not as easy as it sounds. The only known method for studying singular integral operators with piecewise continuous coefficients over arbitrary Carleson curves is based on the Wiener-Hopf factorization techniques, which in turn requires information on the boundedness of the Cauchy singular integral operator on spaces with special weights related to oscillations of Carleson curves. For logarithmic Carleson curves this boundedness problem is reduced to the case of power weights treated in [22]. However, for arbitrary Carleson curves this is not the case, a more general boundedness result was needed. This need is satisfied in the present paper by a combination of two very recent results by V. Kokilashvili and S. Samko [26] and the author [21].

Let us also note that for standard Lebesgue spaces over slowly oscillating Carleson curves (in particular, logarithmic Carleson curves) there exists another method for studying singular integral operators based on the technique of Mellin pseudodifferential operators and limit operators (see, e.g., [32, 5, 6] and the references therein). It allows one to study not only piecewise continuous coefficients but also coefficients admitting discontinuities of slowly oscillating type. In this connection note that very recently V. Rabinovich and S. Samko [33] have started to study pseudodifferential operators in the setting of variable Lebesgue spaces. However, it seems that the method based on the Mellin technique does not allow one to consider the case of arbitrary Carleson curves.

The paper is organized as follows. In Section 2 we give necessary definitions and formulate the main results: 1) the above-mentioned condition for the boundedness of the Cauchy singular integral operator on a variable Lebesgue space with a weight related to oscillations of an arbitrary Carleson curve; 2) a Fredholm criterion for an individual singular integral operator with piecewise continuous coefficients in the spirit of [11] and [2]; 3) a symbol calculus for the Banach algebra of singular integral operators with matrix piecewise continuous coefficients. Sections 3–5 contain the proofs of the results 1)–3), respectively.

2. Preliminaries and main results

2.1. Carleson curves

By a Jordan curve Γ we will understand throughout this paper a curve homeomorphic to a circle. We suppose that Γ is rectifiable. We equip Γ with Lebesgue length measure $|d\tau|$ and the counter-clockwise orientation. The *Cauchy singular integral* of $f \in L^1(\Gamma)$ is defined by

$$(Sf)(t) := \lim_{R\to 0} \frac{1}{\pi i} \int_{\Gamma\setminus\Gamma(t,R)} \frac{f(\tau)}{\tau - t} d\tau \quad (t \in \Gamma),$$

where $\Gamma(t,R) := \{\tau \in \Gamma : |\tau - t| < R\}$ for $R > 0$. David [9] (see also [3, Theorem 4.17]) proved that the Cauchy singular integral generates the bounded operator S on the Lebesgue space $L^p(\Gamma)$, $1 < p < \infty$, if and only if Γ is a *Carleson* (*Ahlfors-David regular*) *curve*, that is,

$$\sup_{t\in\Gamma} \sup_{R>0} \frac{|\Gamma(t,R)|}{R} < \infty,$$

where $|\Omega|$ denotes the measure of a set $\Omega \subset \Gamma$.

2.2. Variable Lebesgue spaces with weights

A measurable function $w : \Gamma \to [0, \infty]$ is referred to as a *weight function* or simply a *weight* if $0 < w(\tau) < \infty$ for almost all $\tau \in \Gamma$. Suppose $p : \Gamma \to (1, \infty)$ is a continuous function. Denote by $L^{p(\cdot)}(\Gamma, w)$ the set of all measurable complex-valued functions f on Γ such that

$$\int_\Gamma |f(\tau)w(\tau)/\lambda|^{p(\tau)} |d\tau| < \infty$$

for some $\lambda = \lambda(f) > 0$. This set becomes a Banach space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{p(\cdot),w} := \inf\left\{\lambda > 0 : \int_\Gamma |f(\tau)w(\tau)/\lambda|^{p(\tau)} |d\tau| \le 1\right\}.$$

If p is constant, then $L^{p(\cdot)}(\Gamma, w)$ is nothing but the weighted Lebesgue space. Therefore, it is natural to refer to $L^{p(\cdot)}(\Gamma, w)$ as a *weighted generalized Lebesgue space with variable exponent* or simply as a *weighted variable Lebesgue space*. This is a special case of Musielak-Orlicz spaces [30] (see also [27]). Nakano [31] considered these spaces (without weights) as examples of so-called modular spaces, and sometimes the spaces $L^{p(\cdot)}(\Gamma, w)$ are referred to as weighted Nakano spaces. In the case $w \equiv 1$ we will simply write $L^{p(\cdot)}(\Gamma)$.

2.3. Boundedness of the Cauchy singular integral operator

Let us define the weight we are interested in. Fix $t \in \Gamma$ and consider the function $\eta_t : \Gamma \setminus \{t\} \to (0, \infty)$ defined by

$$\eta_t(\tau) := e^{-\arg(\tau - t)},$$

where $\arg(\tau - t)$ denotes any continuous branch of the argument on $\Gamma \setminus \{t\}$. For every $\gamma \in \mathbb{C}$, put

$$\varphi_{t,\gamma}(\tau) := |(\tau - t)^\gamma| = |\tau - t|^{\operatorname{Re}\gamma} \eta_t(\tau)^{\operatorname{Im}\gamma} \quad (\tau \in \Gamma \setminus \{t\}).$$

A. Böttcher and Yu. Karlovich [2] (see also [3, Chap. 1]) proved that if Γ is a Carleson Jordan curve, then at each point $t \in \Gamma$, the following limits exist:

$$\delta_t^- := \lim_{x \to 0} \frac{\log(W_t^0 \eta_t)(x)}{\log x}, \quad \delta_t^+ := \lim_{x \to \infty} \frac{\log(W_t^0 \eta_t)(x)}{\log x},$$

where

$$(W_t^0 \eta_t)(x) = \limsup_{R \to 0} \left(\max_{\{\tau \in \Gamma : |\tau - t| = xR\}} \eta_t(\tau) \Big/ \min_{\{\tau \in \Gamma : |\tau - t| = R\}} \eta_t(\tau) \right).$$

Moreover,

$$-\infty < \delta_t^- \le \delta_t^+ < +\infty.$$

These numbers are called the lower and upper spirality indices of the curve Γ at t. For piecewise smooth curves $\delta_t^- \equiv \delta_t^+ \equiv 0$, for curves behaving like a logarithmic spiral in a neighborhood of t, one has $\delta_t^- = \delta_t^+ \neq 0$. However, the class of Carleson curves is much larger: for all real numbers $-\infty < \alpha < \beta < +\infty$ there is a Carleson curve Γ such that $\delta_t^- = \alpha$ and $\delta_t^+ = \beta$ at some point $t \in \Gamma$ (see [3, Proposition 1.21]). Put

$$\alpha_t^0(x) := \min\{\delta_t^- x, \delta_t^+ x\}, \quad \beta_t^0(x) := \max\{\delta_t^- x, \delta_t^+ x\} \quad (x \in \mathbb{R}).$$

We will always suppose that $p : \Gamma \to (1, \infty)$ is a continuous function satisfying the Dini-Lipschitz condition on Γ, that is, there exists a constant $C_p > 0$ such that

$$|p(\tau) - p(t)| \leq \frac{C_p}{-\log|\tau - t|}$$

for all $\tau, t \in \Gamma$ such that $|\tau - t| \leq 1/2$.

Our first main result is the following theorem.

Theorem 2.1. *Let Γ be a Carleson Jordan curve and $p : \Gamma \to (1, \infty)$ be a continuous function satisfying the Dini-Lipschitz condition. Suppose $t \in \Gamma$ and $\gamma \in \mathbb{C}$. Then the operator S is bounded on $L^{p(\cdot)}(\Gamma, \varphi_{t,\gamma})$ if and only if*

$$0 < \frac{1}{p(t)} + \operatorname{Re}\gamma + \alpha_t^0(\operatorname{Im}\gamma), \quad \frac{1}{p(t)} + \operatorname{Re}\gamma + \beta_t^0(\operatorname{Im}\gamma) < 1. \tag{2.1}$$

For constant $p \in (1, \infty)$ this result is actually proved by A. Böttcher and Yu. Karlovich [2], see also [3, Section 3.1]. For a variable Lebesgue space with a power weight, that is, in the case when $\operatorname{Im}\gamma = 0$, this result is due to V. Kokilashvili, V. Paatashvili, and S. Samko [22]. Note that V. Kokilashvili, N. Samko, and S. Samko [24] generalized the sufficiency portion of that result also to the case of so-called radial oscillating weights $w_t(\tau) = f(|\tau - t|)$ (and their products), where f is an oscillating function at zero. Obviously, η_t is not of this type, in general. Further, the necessity of their conditions has been proved in [20, Theorem 1.1].

The proof of Theorem 2.1 will be given in Section 3.

2.4. Fredholm criterion

Let X be a Banach space and $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators on X. An operator $A \in \mathcal{B}(X)$ is said to be Fredholm if its image $\operatorname{Im} A$ is closed in X and the defect numbers $\dim \operatorname{Ker} A$, $\dim \operatorname{Ker} A^*$ are finite. By $PC(\Gamma)$ we denote the set of all $a \in L^\infty(\Gamma)$ for which the one-sided limits

$$a(t \pm 0) := \lim_{\tau \to t \pm 0} a(\tau)$$

exist and are finite at each point $t \in \Gamma$; here $\tau \to t - 0$ means that τ approaches t following the orientation of Γ, while $\tau \to t + 0$ means that τ goes to t in the opposite direction. Functions in $PC(\Gamma)$ are called *piecewise continuous* functions. Put

$$P := (I + S)/2, \quad Q := (I - S)/2.$$

By using Theorem 2.1 and the machinery developed in [17] (see also [3]), we will prove our second main result.

Theorem 2.2. *Let Γ be a Carleson Jordan curve and $p : \Gamma \to (1, \infty)$ be a continuous function satisfying the Dini-Lipschitz condition. Suppose $a \in PC(\Gamma)$. The operator $aP + Q$ is Fredholm on $L^{p(\cdot)}(\Gamma)$ if and only if $a(t \pm 0) \neq 0$ and*

$$\frac{1}{p(t)} - \frac{1}{2\pi} \arg \frac{a(t-0)}{a(t+0)} + \theta\alpha_t^0 \left(\frac{1}{2\pi} \log \left| \frac{a(t-0)}{a(t+0)} \right| \right) + (1-\theta)\beta_t^0 \left(\frac{1}{2\pi} \log \left| \frac{a(t-0)}{a(t+0)} \right| \right)$$

is not an integer number for all $t \in \Gamma$ and all $\theta \in [0, 1]$.

It is well known that $\alpha_t^0(x) \equiv \beta_t^0(x) \equiv 0$ if Γ is piecewise smooth. For Lyapunov curves and constant p, Theorem 2.2 was obtained by I. Gohberg and N. Krupnik [11] (see also [13, Chap. 9]), it was extended to variable Lebesgue spaces over Lyapunov curves or Radon curves without cusps by V. Kokilashvili and S. Samko [25]. For arbitrary Carleson curves and constant p, Theorem 2.2 is due to A. Böttcher and Yu. Karlovich [2] (see also [3, Chap. 7]).

The proof of Theorem 2.2 is presented in Section 4. It is developed following the well-known scheme (see [37], [3, Chap. 7], and also [17, 18, 20]).

2.5. Leaves with a median separating point

Let $p \in (0, 1)$ and $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ be continuous functions such that α is concave, β is convex, $\alpha(x) \leq \beta(x)$ for all $x \in \mathbb{R}$, and $0 < 1/p + \alpha(0) \leq 1/p + \beta(0) < 1$. Put

$$Y(p, \alpha, \beta) := \left\{ \gamma = x + iy \in \mathbb{C} : \frac{1}{p} + \alpha(x) \leq y \leq \frac{1}{p} + \beta(x) \right\}.$$

Given $z_1, z_2 \in \mathbb{C}$, let

$$\mathcal{L}(z_1, z_2; p, \alpha, \beta) := \left\{ M_{z_1,z_2}(e^{2\pi\gamma}) : \gamma \in Y(p, \alpha, \beta) \right\} \cup \{z_1, z_2\},$$

where

$$M_{z_1,z_2}(\zeta) := (z_2\zeta - z_1)/(\zeta - 1)$$

is the Möbius transform. The set $\mathcal{L}(z_1, z_2; p, \alpha, \beta)$ is referred to as the *leaf* about (or between) z_1 and z_2 determined by p, α, β.

If $\alpha(x) = \beta(x) = 0$ for all $x \in \mathbb{R}$, then $\mathcal{L}(z_1, z_2; p, \alpha, \beta)$ is nothing but the circular arc

$$\mathcal{A}(z_1, z_2; p) := \left\{ z \in \mathbb{C} \setminus \{z_1, z_2\} : \arg \frac{z - z_1}{z - z_2} \in \frac{2\pi}{p} + 2\pi\mathbb{Z} \right\} \cup \{z_1, z_2\}.$$

H. Widom [38] was the first who understood the importance of these arcs in the spectral theory of singular integral operators (in the setting of $L^p(\mathbb{R})$). These arcs play a very important role in the Gohberg-Krupnik Fredholm theory for singular integral operators with piecewise continuous coefficients over Lyapunov curves (see [12, 13]).

Suppose that $\alpha(x) = \beta(x) = \delta x$ for all $x \in \mathbb{R}$, where $\delta \in \mathbb{R}$. Then the leaf $\mathcal{L}(z_1, z_2; p, \alpha, \beta)$ is nothing but the logarithmic double spiral

$$\mathcal{S}(z_1, z_2; p, \delta) := \left\{ z \in \mathbb{C} \setminus \{z_1, z_2\} : \arg \frac{z - z_1}{z - z_2} - \delta \log \left| \frac{z - z_1}{z - z_2} \right| \in \frac{2\pi}{p} + 2\pi\mathbb{Z} \right\} \cup \{z_1, z_2\}.$$

These logarithmic double spirals appear in the Fredholm theory for singular integral operators over logarithmic Carleson curves, that is, when the spirality indices δ_t^- and δ_t^+ coincide at every point $t \in \Gamma$ (see [3] and also [18]).

Now let δ^-, δ^+ be real numbers such that $\delta^- \le \delta^+$. Suppose that

$$\alpha(x) = \min\{\delta^- x, \delta^+ x\}, \quad \beta(x) = \max\{\delta^- x, \delta^+ x\} \quad (x \in \mathbb{R}). \tag{2.2}$$

Then it is not difficult to show that

$$\mathcal{L}(z_1, z_2; p, \alpha, \beta) = \bigcup_{\delta \in [\delta^-, \delta^+]} \mathcal{S}(z_1, z_2; p, \delta).$$

This set is always bounded by pieces of at most four logarithmic double spirals. The point $m := M_{z_1,z_2}(e^{2\pi i/p})$ has two interesting properties: m disconnects (separates) the leaf, that is, $\mathcal{L}(z_1, z_2; p, \alpha, \beta)$ is connected, while $\mathcal{L}(z_1, z_2; p, \alpha, \beta) \setminus \{m\}$ is a disconnected set; and m is a median point, that is, $|m - z_1| = |m - z_2|$ (see [3, Example 7.10]). Leaves generated by functions α and β of the form (2.2) are called logarithmic leaves with a median separating point. We refer to [3, Chap. 7] for many nice plots of leaves (not only generated by (2.2), but also more general).

2.6. Symbol calculus for the Banach algebra of singular integral operators

Let N be a positive integer. We denote by $L_N^{p(\cdot)}$ the direct sum of N copies of $L^{p(\cdot)}(\Gamma)$ with the norm

$$\|f\| = \|(f_1, \dots, f_N)\| := (\|f_1\|_{p(\cdot)}^2 + \dots + \|f_N\|_{p(\cdot)}^2)^{1/2}.$$

The operator S is defined on $L_N^{p(\cdot)}(\Gamma)$ elementwise. We let $PC_{N\times N}(\Gamma)$ stand for the algebra of all $N \times N$ matrix functions with entries in $PC(\Gamma)$. Writing the elements of $L_N^{p(\cdot)}(\Gamma)$ as columns, we can define the multiplication operator aI for $a \in PC_{N\times N}(\Gamma)$ as multiplication by the matrix function a. Let $\mathcal{B} := \mathcal{B}(L_N^{p(\cdot)}(\Gamma))$ be the Banach algebra of all bounded linear operators on $L_N^{p(\cdot)}(\Gamma)$ and $\mathcal{K} := \mathcal{K}(L_N^{p(\cdot)}(\Gamma))$ be its two-sided ideal consisting of all compact operators on $L_N^{p(\cdot)}(\Gamma)$. By $\mathcal{A}$ we denote the smallest closed subalgebra of $\mathcal{B}$ containing the operator S and the set $\{aI : a \in PC_{N\times N}(\Gamma)\}$.

Our last main result is the following.

Theorem 2.3. *Suppose Γ is a Carleson Jordan curve and $p : \Gamma \to (1, \infty)$ is a continuous function satisfying the Dini-Lipschitz condition. Define the "bundle" of logarithmic leaves with a median separating point by*

$$\mathcal{M} := \bigcup_{t\in\Gamma} \big(\{t\} \times \mathcal{L}(0, 1; p(t), \alpha_t^0, \beta_t^0)\big).$$

(a) *We have $\mathcal{K} \subset \mathcal{A}$.*
(b) *For each point $(t, z) \in \mathcal{M}$, the map*

$$\sigma_{t,z} : \{S\} \cup \{aI \ : \ a \in PC_{N\times N}(\Gamma)\} \to \mathbb{C}^{2N\times 2N}$$

given by

$$\sigma_{t,z}(S) = \begin{bmatrix} E & O \\ O & -E \end{bmatrix},$$

$$\sigma_{t,z}(aI)$$

$$= \begin{bmatrix} a(t+0)z + a(t-0)(1-z) & (a(t+0)-a(t-0))\sqrt{z(1-z)} \\ (a(t+0)-a(t-0))\sqrt{z(1-z)} & a(t+0)(1-z)+a(t-0)z \end{bmatrix},$$

where E and O denote the $N \times N$ identity and zero matrices, respectively, and $\sqrt{z(1-z)}$ denotes any complex number whose square is $z(1-z)$, extends to a Banach algebra homomorphism

$$\sigma_{t,z} : \mathcal{A} \to \mathbb{C}^{2N \times 2N}$$

with the property that $\sigma_{t,z}(K)$ is the $2N \times 2N$ zero matrix whenever K is a compact operator on $L^{p(\cdot)}_N(\Gamma)$.

(c) *An operator $A \in \mathcal{A}$ is Fredholm on $L^{p(\cdot)}_N(\Gamma)$ if and only if*

$$\det \sigma_{t,z}(A) \neq 0 \quad \textit{for all} \quad (t,z) \in \mathcal{M}.$$

(d) *The quotient algebra $\mathcal{A}/\mathcal{K}$ is inverse closed in the Calkin algebra $\mathcal{B}/\mathcal{K}$, that is, if a coset $A + \mathcal{K} \in \mathcal{A}/\mathcal{K}$ is invertible in $\mathcal{B}/\mathcal{K}$, then $(A+\mathcal{K})^{-1} \in \mathcal{A}/\mathcal{K}$.*

This theorem was proved by I. Gohberg and N. Krupnik for constant p and Lyapunov curves (and power weights) in [12], and extended to the setting of variable Lebesgue spaces over Lyapunov curves (again with power weights) by the author [19]. The case of constant p and arbitrary Carleson curves was treated by A. Böttcher and Yu. Karlovich [2] and Theorem 2.3 is a direct generalization of their result to the setting of variable Lebesgue spaces.

We will present a sketch of the proof of Theorem 2.3 in Section 5.

3. Proof of the boundedness result

3.1. Main ingredients

It is well known that the boundedness of the Cauchy singular integral operator S is closely related to the boundedness of the following maximal function

$$(Mf)(t) := \sup_{\varepsilon>0} \frac{1}{|\Gamma(t,\varepsilon)|} \int_{\Gamma(t,\varepsilon)} |f(\tau)|\,|d\tau| \quad (t \in \Gamma)$$

defined for (locally) integrable functions f on Γ. In particular, both operators are bounded on weighted standard Lebesgue spaces $L^p(\Gamma, w)$ $(1 < p < \infty)$ simultaneously and this happen if and only if w is a Muckenhoupt weight. For weighted variable Lebesgue spaces a characterization of this sort is unknown.

One of the main ingredients of the proof of Theorem 2.1 is the following very recent result by V. Kokilashvili and S. Samko.

Theorem 3.1 ([26, Theorem 4.21]). *Let* Γ *be a Carleson Jordan curve. Suppose that* $p : \Gamma \to (1, \infty)$ *is a continuous function satisfying the Dini-Lipschitz condition and* $w : \Gamma \to [1, \infty]$ *is a weight. If there exists a number* p_0 *such that*

$$1 < p_0 < \min_{\tau \in \Gamma} p(\tau)$$

and M *is bounded on* $L^{p(\cdot)/(p(\cdot)-p_0)}(\Gamma, w^{-p_0})$, *then* S *is bounded on* $L^{p(\cdot)}(\Gamma, w)$.

The above conditional result allows us to derive sufficient conditions for the boundedness of the Cauchy singular integral operator on weighted variable Lebesgue spaces when some conditions for the boundedness of the maximal operator in weighted variable Lebesgue spaces are already known.

Very recently, sufficient conditions for the boundedness of the maximal operator that fit our needs were obtained by the author [21] following the approach of [23].

Theorem 3.2 ([21, Theorem 4]). *Let* Γ *be a Carleson Jordan curve. Suppose that* $p : \Gamma \to (1, \infty)$ *is a continuous function satisfying the Dini-Lipschitz condition. If* $t \in \Gamma$, $\gamma \in \mathbb{C}$, *and* (2.1) *is fulfilled, then* M *is bounded on* $L^{p(\cdot)}(\Gamma, \varphi_{t,\gamma})$.

3.2. Proof of Theorem 2.1

Necessity portion follows from [20, Corollary 4.2] because $\operatorname{Re}\gamma + \alpha_t^0(\operatorname{Im}\gamma)$ and $\operatorname{Re}\gamma + \beta_t^0(\operatorname{Im}\gamma)$ coincide with $\alpha(V_t^0\varphi_{t,\gamma})$ and $\beta(V_t^0\varphi_{t,\gamma})$ from [20] (see also [3, Chap. 3]), respectively. The latter fact is a corollary of [17, Lemma 5.15(a)].

Sufficiency. Since $p : \Gamma \to (1, \infty)$ is continuous and Γ is compact, we deduce that $\min_{\tau \in \Gamma} p(\tau) > 1$. If the inequality

$$\frac{1}{p(t)} + \operatorname{Re}\gamma + \beta_t^0(\operatorname{Im}\gamma) < 1$$

is fulfilled, then there exists a number p_0 such that

$$1 < p_0 < \min_{\tau \in \Gamma} p(\tau), \qquad \frac{1}{p(t)} + \operatorname{Re}\gamma + \beta_t^0(\operatorname{Im}\gamma) < \frac{1}{p_0}.$$

The latter inequality is equivalent to

$$0 < 1 - \frac{p_0}{p(t)} - p_0\big(\operatorname{Re}\gamma + \beta_t^0(\operatorname{Im}\gamma)\big) = \frac{p(t) - p_0}{p(t)} - p_0 \operatorname{Re}\gamma + \alpha_t^0(-p_0 \operatorname{Im}\gamma). \quad (3.1)$$

Analogously, the inequality

$$0 < \frac{1}{p(t)} + \operatorname{Re}\gamma + \alpha_t^0(\operatorname{Im}\gamma)$$

is equivalent to

$$1 > 1 - \frac{p_0}{p(t)} - p_0\big(\operatorname{Re}\gamma + \alpha_t^0(\operatorname{Im}\gamma)\big) = \frac{p(t) - p_0}{p(t)} - p_0 \operatorname{Re}\gamma + \beta_t^0(-p_0 \operatorname{Im}\gamma). \quad (3.2)$$

From the equality $\varphi_{t,-p_0\gamma} = \varphi_{t,\gamma}^{-p_0}$, inequalities (3.1)–(3.2), and Theorem 3.2 it follows that the maximal operator M is bounded on $L^{p(\cdot)/(p(\cdot)-p_0)}(\Gamma, \varphi_{t,\gamma}^{-p_0})$. To finish the proof, it remains to apply Theorem 3.1. $\square$

4. Proof of the Fredholm criterion for the operator $aP + Q$

4.1. Local representatives

In this section we suppose that Γ is a Carleson Jordan curve and $p : \Gamma \to (1, \infty)$ is a continuous functions satisfying the Dini-Lipschitz condition. Under these assumptions, the operator S is bounded on $L^{p(\cdot)}(\Gamma)$ by Theorem 2.1.

Functions $a, b \in L^\infty(\Gamma)$ are said to be locally equivalent at a point $t \in \Gamma$ if

$$\inf\left\{\|(a-b)c\|_\infty \ : \ c \in C(\Gamma),\ c(t) = 1\right\} = 0.$$

Theorem 4.1. *Suppose $a \in L^\infty(\Gamma)$ and for each $t \in \Gamma$ there exists a function $a_t \in L^\infty(\Gamma)$ which is locally equivalent to a at t. If the operators $a_t P + Q$ are Fredholm on $L^{p(\cdot)}(\Gamma)$ for all $t \in \Gamma$, then $aP + Q$ is Fredholm on $L^{p(\cdot)}(\Gamma)$.*

For weighted Lebesgue spaces this theorem is known as Simonenko's local principle [35]. It follows from [17, Theorem 6.13].

The curve Γ divides the complex plane $\mathbb{C}$ into the bounded simply connected domain D^+ and the unbounded domain D^-. Without loss of generality we assume that $0 \in D^+$. Fix $t \in \Gamma$. For a function $a \in PC(\Gamma)$ such that $a^{-1} \in L^\infty(\Gamma)$, we construct a "canonical" function $g_{t,\gamma}$ which is locally equivalent to a at the point $t \in \Gamma$. The interior and the exterior of the unit circle can be conformally mapped onto D^+ and D^- of Γ, respectively, so that the point 1 is mapped to t, and the points $0 \in D^+$ and $\infty \in D^-$ remain fixed. Let Λ_0 and Λ_∞ denote the images of $[0,1]$ and $[1,\infty) \cup \{\infty\}$ under this map. The curve $\Lambda_0 \cup \Lambda_\infty$ joins 0 to ∞ and meets Γ at exactly one point, namely t. Let $\arg z$ be a continuous branch of argument in $\mathbb{C} \setminus (\Lambda_0 \cup \Lambda_\infty)$. For $\gamma \in \mathbb{C}$, define the function $z^\gamma := |z|^\gamma e^{i\gamma \arg z}$, where $z \in \mathbb{C} \setminus (\Lambda_0 \cup \Lambda_\infty)$. Clearly, z^γ is an analytic function in $\mathbb{C} \setminus (\Lambda_0 \cup \Lambda_\infty)$. The restriction of z^γ to $\Gamma \setminus \{t\}$ will be denoted by $g_{t,\gamma}$. Obviously, $g_{t,\gamma}$ is continuous and nonzero on $\Gamma \setminus \{t\}$. Since $a(t \pm 0) \neq 0$, we can define $\gamma_t = \gamma \in \mathbb{C}$ by the formulas

$$\operatorname{Re}\gamma_t := \frac{1}{2\pi} \arg \frac{a(t-0)}{a(t+0)}, \quad \operatorname{Im}\gamma_t := -\frac{1}{2\pi} \log \left| \frac{a(t-0)}{a(t+0)} \right|, \tag{4.1}$$

where we can take any value of $\arg(a(t-0)/a(t+0))$, which implies that any two choices of $\operatorname{Re}\gamma_t$ differ by an integer only. Clearly, there is a constant $c_t \in \mathbb{C} \setminus \{0\}$ such that $a(t \pm 0) = c_t g_{t,\gamma_t}(t \pm 0)$, which means that a is locally equivalent to $c_t g_{t,\gamma_t}$ at the point $t \in \Gamma$.

4.2. Wiener-Hopf factorization of local representatives

We say that a function $a \in L^\infty(\Gamma)$ admits a *Wiener-Hopf factorization on* $L^{p(\cdot)}(\Gamma)$ if $a^{-1} \in L^\infty(\Gamma)$ and a can be written in the form

$$a(t) = a_-(t) t^\kappa a_+(t) \quad \text{a.e. on } \Gamma, \tag{4.2}$$

where $\kappa \in \mathbb{Z}$, the factors $a_\pm$ enjoy the following properties:

$$a_- \in QL^{p(\cdot)}(\Gamma) \dotplus \mathbb{C},\ a_-^{-1} \in QL^{q(\cdot)}(\Gamma) \dotplus \mathbb{C},\ a_+ \in PL^{q(\cdot)}(\Gamma),\ a_+^{-1} \in PL^{p(\cdot)}(\Gamma),$$

where $1/p(t) + 1/q(t) = 1$ for all $t \in \Gamma$, and the operator S is bounded on the space $L^{p(\cdot)}(\Gamma, |a_+^{-1}|)$. One can prove that the number κ is uniquely determined.

Theorem 4.2. *A function $a \in L^\infty(\Gamma)$ admits a Wiener-Hopf factorization* (4.2) *on $L^{p(\cdot)}(\Gamma)$ if and only if the operator $aP + Q$ is Fredholm on $L^{p(\cdot)}(\Gamma)$.*

This theorem goes back to Simonenko [34, 36] for constant p. For more information about this topic we refer to [3, Section 6.12], [7, Section 5.5], [13, Section 8.3] and also to [8, 29] in the case of weighted Lebesgue spaces. Theorem 4.2 follows from [17, Theorem 6.14].

From [17, Lemma 7.1] and the theorem on the boundedness of the Cauchy singular integral operator on arbitrary Carleson curves (see [22] or Theorem 2.1) we get the following.

Lemma 4.3. *If, for some $k \in \mathbb{Z}$ and $\gamma \in \mathbb{C}$, the operator S is bounded on the space $L^{p(\cdot)}(\Gamma, \varphi_{t,k-\gamma})$, then the function $g_{t,\gamma}$ defined in Section* 4.1 *admits a Wiener-Hopf factorization on the space $L^{p(\cdot)}(\Gamma)$.*

Combination of the above lemma and Theorem 2.1 is the key to the proof of the sufficiency portion of Theorem 2.2.

4.3. Proof of Theorem 2.2

Necessity. If Γ is a Carleson Jordan curve, then S is bounded on $L^{p(\cdot)}(\Gamma)$ (see [22] or Theorem 2.1). This implies that the assumptions of [17, Theorem 8.1] are satisfied. Note that the indicator functions α_t and β_t considered in [17, Theorem 8.1] (see also [3, Chap. 3]) coincide with α_t^0 and β_t^0, respectively, whenever we are in the unweighted situation (see, e.g., [3, Proposition 3.23] or [17, Lemma 5.15(a)]). Therefore, the necessity portion of Theorem 2.2 follows from [17, Theorem 8.1].

Sufficiency. If $aP + Q$ is Fredholm on $L^{p(\cdot)}(\Gamma)$, then $a^{-1} \in L^\infty(\Gamma)$ in view of [17, Theorem 6.11]. Therefore $a(t \pm 0) \neq 0$ for all $t \in \Gamma$. Fix an arbitrary $t \in \Gamma$ and choose $\gamma = \gamma_t \in \mathbb{C}$ as in (4.1). Then a is locally equivalent to $c_t g_{t,\gamma_t}$ at the point t, where c_t is a nonzero constant and the hypotheses of the theorem read as follows:

$$\frac{1}{p(t)} - \operatorname{Re}\gamma_t + \theta\alpha_t^0(-\operatorname{Im}\gamma_t) + (1-\theta)\beta_t^0(-\operatorname{Im}\gamma_t) \notin \mathbb{Z} \text{ for all } \theta \in [0,1].$$

Then there exists a number $k_t \in \mathbb{Z}$ such that

$$0 < \frac{1}{p(t)} + k_t - \operatorname{Re}\gamma_t + \theta\alpha_t^0(-\operatorname{Im}\gamma_t) + (1-\theta)\beta_t^0(-\operatorname{Im}\gamma_t) < 1$$

for all $\theta \in [0,1]$. In particular, if $\theta = 1$, then

$$0 < \frac{1}{p(t)} + \operatorname{Re}(k_t - \gamma_t) + \alpha_t^0(\operatorname{Im}(k_t - \gamma_t)); \tag{4.3}$$

if $\theta = 0$, then

$$\frac{1}{p(t)} + \operatorname{Re}(k_t - \gamma_t) + \beta_t^0(\operatorname{Im}(k_t - \gamma_t)) < 1. \tag{4.4}$$

From (4.3)–(4.4) and Theorem 2.1 it follows that the operator S is bounded on $L^{p(\cdot)}(\Gamma, \varphi_{t,k_t-\gamma_t})$. By Lemma 4.3, the function g_{t,γ_t} admits a Wiener-Hopf factorization on $L^{p(\cdot)}(\Gamma)$. Then, in view of Theorem 4.2, the operator $g_{t,\gamma_t}P + Q$ is Fredholm on $L^{p(\cdot)}(\Gamma)$. It is easy to see that in this case the operator $c_t g_{t,\gamma_t}P + Q$ is also Fredholm. Thus, for all local representatives $c_t g_{t,\gamma_t}$, the operators $c_t g_{t\gamma_t} P + Q$ are Fredholm. To finish the proof of the sufficiency part, it remains to apply the local principle (Theorem 4.1). □

5. Construction of the symbol calculus

5.1. Allan-Douglas local principle

In this section we present a sketch of the proof of Theorem 2.3 based on the Allan-Douglas local principle and the two projections theorem following the scheme of [3, Chap. 8] (see also [18, 19, 20]).

Let B be a Banach algebra with identity. A subalgebra Z of B is said to be a *central subalgebra* if $zb = bz$ for all $z \in Z$ and all $b \in B$.

Theorem 5.1 (see [7]**, Theorem 1.35(a)).** *Let B be a Banach algebra with identity e and let Z be a closed central subalgebra of B containing e. Let $M(Z)$ be the maximal ideal space of Z, and for $\omega \in M(Z)$, let J_ω refer to the smallest closed two-sided ideal of B containing the ideal ω. Then an element b is invertible in B if and only if $b + J_\omega$ is invertible in the quotient algebra B/J_ω for all $\omega \in M(Z)$.*

The algebra B/J_ω is referred to as the *local algebra* of B at $\omega \in M(Z)$ and the spectrum of $b + J_\omega$ in B/J_ω is called the *local spectrum* of b at $\omega \in M(Z)$.

5.2. Localization

An operator $A \in \mathcal{B}$ is said to be of local type if its commutator with the operator of multiplication by the diagonal matrix function $\operatorname{diag}\{c, \dots, c\}$ is compact for every continuous function c on Γ. The set $\mathcal{L}$ of all operators of local type forms a Banach subalgebra of $\mathcal{B}$. By analogy with [19, Lemma 5.1] one can prove that $\mathcal{K} \subset \mathcal{L}$. From [17, Lemma 6.5] it follows that the operator S is of local type. Thus,

$$\mathcal{K} \subset \mathcal{A} \subset \mathcal{L}.$$

It is easy to see that $A \in \mathcal{L}$ is Fredholm if and only if the coset $A + \mathcal{K}$ is invertible in $\mathcal{L}/\mathcal{K}$. We will study the invertibility of a coset $A + \mathcal{K}$ of $\mathcal{A}/\mathcal{K}$ in the larger algebra $\mathcal{L}/\mathcal{K}$ by using the Allan-Douglas local principle. Consider

$$\mathcal{Z}/\mathcal{K} := \{\operatorname{diag}\{c, \dots, c\}I + \mathcal{K} : c \in C(\Gamma)\}.$$

Every element of this subalgebra commutes with all elements of $\mathcal{L}/\mathcal{K}$. The maximal ideal spaces $M(\mathcal{Z}/\mathcal{K})$ of $\mathcal{Z}/\mathcal{K}$ may be identified with the curve Γ via the Gelfand map

$$\mathcal{G} : \mathcal{Z}/\mathcal{K} \to C(\Gamma), \quad \big(\mathcal{G}(\operatorname{diag}\{c, \dots, c\}I + \mathcal{K})\big)(t) = c(t) \quad (t \in \Gamma).$$

For every $t \in \Gamma$ we define $\mathcal{J}_t \subset \mathcal{L}/\mathcal{K}$ as the smallest closed two-sided ideal of $\mathcal{L}/\mathcal{K}$ containing the set

$$\big\{ \operatorname{diag}\{c, \dots, c\}I + \mathcal{K} : \; c \in C(\Gamma), \; c(t) = 0\big\}.$$

Let χ_t be the characteristic function of a proper arc of Γ starting at $t \in \Gamma$. For a matrix function $a \in PC_{N\times N}(\Gamma)$, let

$$a_t := a(t-0)(1-\chi_t) + a(t+0)\chi_t.$$

It is easy to see that $aI - a_tI + \mathcal{K} \in \mathcal{J}_t$. This implies that for any operator $A \in \mathcal{A}$, the coset $A + \mathcal{K} + \mathcal{J}_t$ belongs to the smallest closed subalgebra $\mathcal{A}_t$ of the algebra $\mathcal{L}_t := (\mathcal{L}/\mathcal{K})/\mathcal{J}_t$ that contains the cosets

$$p := P + \mathcal{K} + \mathcal{J}_t, \quad q := \operatorname{diag}\{\chi_t, \dots, \chi_t\}I + \mathcal{K} + \mathcal{J}_t \tag{5.1}$$

and the algebra

$$\mathcal{C} := \{cI + \mathcal{K} + \mathcal{J}_t : \ c \in \mathbb{C}^{N\times N}\}. \tag{5.2}$$

Thus, by the Allan-Douglas local principle, for every $A \in \mathcal{A}$, the problem of invertibility of $A + \mathcal{K}$ in the algebra $\mathcal{L}/\mathcal{K}$ is reduced to the problem of invertibility of $A + \mathcal{K} + \mathcal{J}_t \in \mathcal{A}_t$ in the local algebra $\mathcal{L}_t$ for every $t \in \Gamma$.

5.3. The two projections theorem

Recall that an element r of a Banach algebra is called an *idempotent* (or, somewhat loosely, also a *projection*), if $r^2 = r$.

The following two projections theorem was obtained by T. Finck, S. Roch, and B. Silbermann [10] and in a slightly different form by I. Gohberg and N. Krupnik [14] (see also [3, Section 8.3]).

Theorem 5.2. *Let B be a Banach algebra with identity e, let $\mathcal{C}$ be a Banach subalgebra of B which contains e and is isomorphic to $\mathbb{C}^{N\times N}$, and let r and s be two idempotent elements in B such that $cr = rc$ and $cs = sc$ for all $c \in \mathcal{C}$. Let $A = \operatorname{alg}(\mathcal{C}, r, s)$ be the smallest closed subalgebra of B containing $\mathcal{C}, r, s$. Put*

$$x = rsr + (e-r)(e-s)(e-r),$$

denote by $\operatorname{sp} x$ the spectrum of x in B, and suppose the points 0 and 1 are not isolated points of $\operatorname{sp} x$. Then

(a) *for each $z \in \operatorname{sp} x$ the map σ_z of $\mathcal{C} \cup \{r, s\}$ into the algebra $\mathbb{C}^{2N\times 2N}$ of all complex $2N \times 2N$ matrices defined by*

$$\sigma_z c = \begin{bmatrix} c & O \\ O & c \end{bmatrix}, \sigma_z r = \begin{bmatrix} E & O \\ O & O \end{bmatrix}, \sigma_z s = \begin{bmatrix} zE & \sqrt{z(1-z)}E \\ \sqrt{z(1-z)}E & (1-z)E \end{bmatrix},$$

where $c \in \mathcal{C}$, E and O denote the $N \times N$ identity and zero matrices, respectively, and $\sqrt{z(1-z)}$ denotes any complex number whose square is $z(1-z)$, extends to a Banach algebra homomorphism

$$\sigma_z : A \to \mathbb{C}^{2N\times 2N};$$

(b) *every element a of the algebra A is invertible in the algebra B if and only if*

$$\det \sigma_z a \neq 0 \quad \text{for all} \quad z \in \operatorname{sp} x;$$

(c) *the algebra A is inverse closed in B if and only if the spectrum of x in A coincides with the spectrum of x in B.*

5.4. Local algebras $\mathcal{A}_t$ and $\mathcal{L}_t$ are subject to the two projections theorem

In this subsection we verify that the algebras $\mathcal{A}_t$ and $\mathcal{L}_t$ defined in Section 5.2 satisfy the assumptions of the two projections theorem (Theorem 5.2). It is obvious that the algebra $\mathcal{C}$ defined by (5.2) is isomorphic to the algebra $\mathbb{C}^{N\times N}$. It is easy to see also that

$$p^2 = p, \quad q^2 = q, \quad pc = cp, \quad qc = cq$$

for all $c \in \mathcal{C}$.

From Theorem 2.2 by analogy with [3, Theorem 8.19] one can derive the following.

Theorem 5.3. *Let $t \in \Gamma$ and the elements $p, q \in \mathcal{A}_t$ be given by* (5.1). *The spectrum of the element*

$$x := pqp + (e-p)(e-q)(e-p)$$

in the algebra $\mathcal{L}_t$ coincides with the logarithmic leaf with a median separating point $\mathcal{L}(0, 1; p(t), \alpha_t^0, \beta_t^0)$.

Notice that 0 and 1 are not isolated points of the leaf $\mathcal{L}(0, 1; p(t), \alpha_t^0, \beta_t^0)$.

We have shown that $\mathcal{A}_t$ and $\mathcal{L}_t$ satisfy all the assumptions of the two projections theorem. Thus, our last main result (Theorem 2.3) is obtained by localizing as above and then by applying the two projections theorem to the local algebras $\mathcal{A}_t$ and $\mathcal{L}_t$ (see [3] and also [18, 19, 20] for more details). We only note that the mapping $\sigma_{t,z}$ in Theorem 2.3 is constructed by the formula

$$\sigma_{t,z} = \sigma_z \circ \pi_t,$$

where σ_z is the mapping from Theorem 5.2 and π_t acts by the rule $A \mapsto A + \mathcal{K} + \mathcal{J}_t$.

References

[1] C. Bennett and R. Sharpley, *Interpolation of Operators.* Academic Press, Boston, 1988.

[2] A. Böttcher and Yu.I. Karlovich, *Toeplitz and singular integral operators on general Carleson Jordan curves.* In: "Singular integral operators and related topics (Tel Aviv, 1995)". Operator Theory: Advances and Applications **90** (1996), 119–152.

[3] A. Böttcher and Yu.I. Karlovich, *Carleson Curves, Muckenhoupt Weights, and Toeplitz Operators.* Birkhäuser, Basel, 1997.

[4] A. Böttcher and Yu.I. Karlovich, *Cauchy's singular integral operator and its beautiful spectrum.* In: "Systems, approximation, singular integral operators, and related topics". Operator Theory: Advances and Applications **129** (2001), 109–142.

[5] A. Böttcher, Yu.I. Karlovich, and V.S. Rabinovich, *Emergence, persistence, and disappearance of logarithmic spirals in the spectra of singular integral operators.* Integral Equations Operator Theory **25** (1996), 406–444.

[6] A. Böttcher, Yu.I. Karlovich, and V.S. Rabinovich, *The method of limit operators for one-dimensional singular integrals with slowly oscillating data.* J. Operator Theory **43** (2000), 171–198.

[7] A. Böttcher and B. Silbermann, *Analysis of Toeplitz Operators*. 2nd edition. Springer-Verlag, Berlin, 2006.

[8] K.P. Clancey and I. Gohberg, *Factorization of Matrix Functions and Singular Integral Operators.* Operator Theory: Advances and Applications **3**. Birkhäuser, Basel, 1981.

[9] G. David, *Operáteurs intégraux singuliers sur certaines courbes du plan complexe.* Ann. Sci. École Norm. Super. **17** (1984), 157–189.

[10] T. Finck, S. Roch, and B. Silbermann, *Two projections theorems and symbol calculus for operators with massive local spectra.* Math. Nachr. **162** (1993), 167–185.

[11] I.C. Gohberg and N.Ya. Krupnik, *The spectrum of singular integral operators in L_p spaces.* Studia Math. **31** (1968), 347–362 (in Russian).

[12] I. Gohberg and N. Krupnik, *Singular integral operators with piecewise continuous coefficients and their symbols.* Math. USSR Izvestiya **5** (1971), 955–979.

[13] I. Gohberg and N. Krupnik, *One-Dimensional Linear Singular Integral Equations.* Vols. 1 and 2. Operator Theory: Advances and Applications **53–54**. Birkhäuser, Basel, 1992.

[14] I. Gohberg and N. Krupnik, *Extension theorems for Fredholm and invertibility symbols.* Integral Equations Operator Theory **16** (1993), 514–529.

[15] A.Yu. Karlovich, *Singular integral operators with piecewise continuous coefficients in reflexive rearrangement-invariant spaces.* Integral Equations Operator Theory **32** (1998), 436–481.

[16] A.Yu. Karlovich, *Algebras of singular integral operators with PC coefficients in rearrangement-invariant spaces with Muckenhoupt weights.* J. Operator Theory **47** (2002), 303–323.

[17] A.Yu. Karlovich, *Fredholmness of singular integral operators with piecewise continuous coefficients on weighted Banach function spaces.* J. Integr. Equat. Appl. **15** (2003), 263–320.

[18] A.Yu. Karlovich, *Algebras of singular integral operators on Nakano spaces with Khvedelidze weights over Carleson curves with logarithmic whirl points.* In: "Pseudodifferential Equations and Some Problems of Mathematical Physics", Rostov-on-Don, 2005, 135–142. Preprint is available at arXiv:math/0507312.

[19] A.Yu. Karlovich, *Algebras of singular integral operators with piecewise continuous coefficients on weighted Nakano spaces.* In: "The Extended Field of Operator Theory". Operator Theory: Advances and Applications **171** (2006), 171–188.

[20] A.Yu. Karlovich, *Singular integral operators on variable Lebesgue spaces with radial oscillating weights.* In: Operator Algebras, Operator Theory and Applications. Operator Theory: Advances and Applications **195** (2009), 185–212.

[21] A.Yu. Karlovich, *Maximal operators on variable Lebesgue spaces with weights related to oscillations of Carleson curves.* Math. Nachr., to appear. Preprint is available at arXiv:0808.0258.

[22] V. Kokilashvili, V. Paatashvili, and S. Samko, *Boundedness in Lebesgue spaces with variable exponent of the Cauchy singular operator on Carleson curves.* In: "Modern Operator Theory and Applications. The Igor Borisovich Simonenko Anniversary Volume". Operator Theory: Advances and Applications **170** (2006), 167–186.

[23] V. Kokilashvili, N. Samko, and S. Samko, *The maximal operator in weighted variable spaces* $L^{p(\cdot)}$. J. Funct. Spaces Appl. **5** (2007), 299–317.

[24] V. Kokilashvili, N. Samko, and S. Samko, *Singular operators in variable spaces* $L^{p(\cdot)}(\Omega, \rho)$ *with oscillating weights.* Math. Nachr. **280** (2007), 1145–1156.

[25] V. Kokilashvili and S. Samko, *Singular integral equations in the Lebesgue spaces with variable exponent.* Proc. A. Razmadze Math. Inst. **131** (2003), 61–78.

[26] V. Kokilashvili and S. Samko, *Operators of harmonic analysis in weighted spaces with non-standard growth.* J. Math. Anal. Appl. **352** (2009), 15–34.

[27] O. Kováčik and J. Rákosník, *On spaces* $L^{p(x)}$ *and* $W^{k,p(x)}$. Czechoslovak Math. J. **41(116)** (1991), 592–618.

[28] S.G. Krein, Ju.I. Petunin, and E.M. Semenov, *Interpolation of Linear Operators.* AMS Translations of Mathematical Monographs **54**, Providence, RI, 1982.

[29] G.S. Litvinchuk and I.M. Spitkovsky, *Factorization of Measurable Matrix Functions.* Operator Theory: Advances and Applications **25**. Birkhäuser, Basel, 1987.

[30] J. Musielak, *Orlicz Spaces and Modular Spaces.* Lecture Notes in Mathematics **1034**. Springer-Verlag, Berlin, 1983.

[31] H. Nakano, *Modulared Semi-Ordered Linear Spaces.* Maruzen Co., Ltd., Tokyo, 1950.

[32] V.S. Rabinovich, *Algebras of singular integral operators on composed contours with nodes that are logarithmic whirl points.* Izvestiya Mathematics **60** (1996), 1261–1292.

[33] V. Rabinovich and S. Samko, *Boundedness and Fredholmness of pseudodifferential operators in variable exponent spaces.* Integral Equations Operator Theory **60** (2008), 507–537.

[34] I.B. Simonenko, *The Riemann boundary value problem for* n *pairs functions with measurable coefficients and its application to the investigation of singular integral operators in the spaces* L^p *with weight.* Izv. Akad. Nauk SSSR, Ser. Matem. **28** (1964), 277–306 (in Russian).

[35] I.B. Simonenko, *A new general method of investigating linear operator equations of singular integral equations type.* Izv. Akad. Nauk SSSR, Ser. Matem. **29** (1965), part I: 567–586,

[36] I.B. Simonenko, *Some general questions in the theory of the Riemann boundary value problem.* Math. USSR Izvestiya **2** (1968), 1091–1099.

[37] I.M. Spitkovsky, *Singular integral operators with PC symbols on the spaces with general weights.* J. Funct. Anal. **105** (1992), 129–143.

[38] H. Widom, *Singular integral equations on* L^p. Trans. Amer. Math. Soc. **97** (1960), 131–160.

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Received: February 27, 2009
Accepted: April 4, 2009

Operator Theory:
Advances and Applications, Vol. 202, 337–354

Almost Periodic Polynomial Factorization of Some Triangular Matrix Functions

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Abstract. Explicit factorization formulas are established for triangular almost periodic matrix functions with trinomial off diagonal terms in the so-called borderline cases. An application to a more general configuration via the Portuguese transformation also is given.

Mathematics Subject Classification (2000). Primary 47A68; Secondary 42A75.

Keywords. Almost periodic matrix functions, factorization, Portuguese transformation.

1. Introduction

The following standard notation is used throughout: $\mathbb{C}$ ($\mathbb{R}$, $\mathbb{Q}$, $\mathbb{N}$) – the set of complex (resp. real, rational, natural) numbers: $\mathbb{R}_+$ ($\mathbb{R}_-$) – the set of nonnegative (resp., nonpositive) real numbers; $\mathbb{Z}_+$ – the set of nonnegative integers: $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

Denote by APP the set of all *almost periodic polynomials*, that is, finite linear combinations of the exponential functions $e_\mu(x) =: e^{i\mu x}$ with real parameters μ:

$$f \in APP \Longleftrightarrow f = \sum_j c_j e_{\mu_j} \colon c_j \in \mathbb{C},\ \mu_j \in \mathbb{R}. \tag{1.1}$$

The set of all μ_j for which in (1.1) $c_j \neq 0$ is denoted $\Omega(f)$, and we let

$$APP_\pm = \{f \in APP \colon \Omega(f) \subset \mathbb{R}_\pm\}.$$

We are concerned in this paper with the factorization problem for matrix functions of the form

$$G = \begin{pmatrix} e_\lambda & 0 \\ f & e_{-\lambda} \end{pmatrix} \tag{1.2}$$

The work was partially supported by the SEP-CONACYT Project 25564 (Yuri Karlovich) and NSF grant DMS-0456625 (Ilya Spitkovsky).

Communicated by L. Rodman.

with $\lambda > 0$ and $f \in APP$, referring to [2] for motivation and necessary background. Since another paper [9] in this volume is devoted to related issues, we will not give here the exact definitions of AP and APW factorizations (more general than the APP factorization), and only note that the *APP factorization* of (1.2), when exists, can be written as

$$G = G_+ \operatorname{diag}[e_{-\kappa}, e_\kappa] G_-^{-1}. \tag{1.3}$$

Here $\kappa(\in [0,\lambda])$ and $-\kappa$ are the so-called *partial AP indices* of G, the entries of $G_\pm$ are in $APP_\pm$, and $\det G_+ = \det G_-$ is a (non-zero) constant. Factorization (1.3) is *canonical* if $\kappa = 0$.

Not every matrix function of the form (1.2) admits an APP factorization: for

$$f = c_{-1}e_{-\nu} - c_0 + c_1 e_\delta \text{ with } c_{-1}c_0c_1 \neq 0,\ \delta, \nu > 0,\ \delta + \nu = \lambda \tag{1.4}$$

and δ/ν irrational the matrix (1.2) either admits a canonical AP factorization with non-polynomial $G_\pm$ or no AP factorization at all, depending on whether or not the condition

$$|c_{-1}|^\delta\, |c_1|^\nu \neq |c_0|^\lambda$$

holds (see Sections 15.1, 23.3 of [2] and the respective historical Notes for the exact statements and original references).

However, if $\Omega(f) \subset -\nu + h\mathbb{Z}_+$ for some $\nu \in \mathbb{R}$, $h > 0$ (the *commensurable*, or *regular* case), then G is APP factorable, as follows from the explicit factorization formulas obtained in [7], see also [2, Section 14.4]. This, of course, covers the *binomial* case, when f consists of at most two terms. The situation persists in the *big gap* case, when $\Omega(f) \cap (\alpha - \lambda, \alpha) = \emptyset$ for some $\alpha \in [0, \lambda]$. This follows by inspection from the factorization formulas obtained in [4, Section 2]. Finally, if f is a *trinomial*

$$f = c_{-1}e_{-\nu} - c_0 e_\mu + c_1 e_\delta,\ 0 < \delta, \nu < \lambda, \tag{1.5}$$

where, in contrast with (1.4), $\nu + |\mu| + \delta > \lambda$, then again G is APP factorable – see Section 15 of [2]. The justification of the latter result (beyond the regular setting, that is, when $(\delta+\nu)/(\mu+\nu)$ is irrational) is constructive but recursive, so that the explicit formulas for the factorization, and the partial AP indices in particular, are hard to extract. This issue in its full generality will be addressed elsewhere, while here we revisit the "borderline" cases $\delta + \nu > \lambda$, $\mu = 0$ and $\delta + \nu = \lambda$, $\mu \neq 0$. This is done in Section 2, where we show in particular that the factorization in these cases is always canonical. In Section 3 we consider the case of $\Omega(f)$ lying in the union of two shifted grids, which can be reduced to one of the borderline cases via the Portuguese transformation. The latter, originally introduced in [1], is used repeatedly throughout the paper. We refer to [2] for its detailed exposition, and to [9] for a brief description.

2. Borderline trinomials

We start with the first borderline case. With a slight change of notation,

$$f = c_{-1}e_{-\alpha} - c_0 + c_1 e_\beta, \text{ where } 0 < \alpha, \beta < \lambda < \alpha+\beta, \text{ and } \alpha/\beta \notin \mathbb{Q}. \tag{2.1}$$

Theorem 2.1. *Suppose the off diagonal term of the matrix function* (1.2) *satisfies* (2.1)*. Then* (1.2) *admits an APP factorization, and this factorization is canonical if and only if* $c_0 \neq 0$*. If the latter condition holds, then the factorization in question is given by*

$$G = \begin{pmatrix} g_+ & \widetilde{g}_+ \\ \varphi_+ & \widetilde{\varphi}_+ \end{pmatrix} \cdot \begin{pmatrix} \widetilde{\varphi}_- & -\widetilde{g}_- \\ -\varphi_- & g_- \end{pmatrix} c_0^{-1},$$

where

$$\begin{aligned}
g_- &= 1 + \sum_{j=0}^{m} \sum_{k=\lfloor \frac{j\beta}{\alpha} \rfloor +1}^{\lceil \frac{j\beta+\lambda}{\alpha} \rceil -1} \left(\frac{c_1}{c_0}\right)^j \left(\frac{c_{-1}}{c_0}\right)^k e_{j\beta-k\alpha}, \\
\varphi_- &= -c_0 \sum_{j=0}^{m} \left(\frac{c_1}{c_0}\right)^j \left(\frac{c_{-1}}{c_0}\right)^{\lceil \frac{j\beta+\lambda}{\alpha} \rceil} e_{j\beta+\lambda-\lceil \frac{j\beta+\lambda}{\alpha} \rceil \alpha}, \\
\widetilde{g}_- &= e_{-\lambda} + \sum_{j=1}^{n} \sum_{k=\lfloor \frac{j\beta-\lambda}{\alpha} \rfloor +1}^{\lfloor \frac{j\beta}{\alpha} \rfloor} \left(\frac{c_1}{c_0}\right)^j \left(\frac{c_{-1}}{c_0}\right)^k e_{j\beta-k\alpha-\lambda}, \\
\widetilde{\varphi}_- &= c_0 - c_{-1}\, e_{-\alpha} - c_0 \sum_{j=1}^{n} \left(\frac{c_1}{c_0}\right)^j \left(\frac{c_{-1}}{c_0}\right)^{\lfloor \frac{j\beta}{\alpha} \rfloor +1} e_{j\beta-(\lfloor \frac{j\beta}{\alpha} \rfloor+1)\alpha}, \\
g_+ &= e_\lambda + \sum_{j=0}^{m} \sum_{k=\lfloor \frac{j\beta}{\alpha} \rfloor +1}^{\lceil \frac{j\beta+\lambda}{\alpha} \rceil -1} \left(\frac{c_1}{c_0}\right)^j \left(\frac{c_{-1}}{c_0}\right)^k e_{j\beta-k\alpha+\lambda}, \\
\varphi_+ &= -c_0 + c_1\, e_\beta + c_0 \sum_{j=1}^{m} \sum_{k=\lfloor \frac{(j-1)\beta}{\alpha} \rfloor +1}^{\lfloor \frac{j\beta}{\alpha} \rfloor} \left(\frac{c_1}{c_0}\right)^j \left(\frac{c_{-1}}{c_0}\right)^k e_{j\beta-k\alpha} \\
&\quad + c_0 \sum_{k=\lfloor \frac{m\beta}{\alpha} \rfloor +1}^{\lceil \frac{m\beta+\lambda}{\alpha} \rceil -1} \left(\frac{c_1}{c_0}\right)^{m+1} \left(\frac{c_{-1}}{c_0}\right)^k e_{(m+1)\beta-k\alpha}, \\
\widetilde{g}_+ &= 1 + \sum_{j=1}^{n} \sum_{k=\lfloor \frac{j\beta-\lambda}{\alpha} \rfloor +1}^{\lfloor \frac{j\beta}{\alpha} \rfloor} \left(\frac{c_1}{c_0}\right)^j \left(\frac{c_{-1}}{c_0}\right)^k e_{j\beta-k\alpha}, \\
\widetilde{\varphi}_+ &= c_0 \sum_{j=2}^{n} \sum_{k=\lfloor \frac{(j-1)\beta-\lambda}{\alpha} \rfloor +1}^{\lfloor \frac{j\beta-\lambda}{\alpha} \rfloor} \left(\frac{c_1}{c_0}\right)^j \left(\frac{c_{-1}}{c_0}\right)^k e_{j\beta-k\alpha-\lambda} \\
&\quad + c_0 \sum_{k=\lfloor \frac{n\beta-\lambda}{\alpha} \rfloor +1}^{\lfloor \frac{n\beta}{\alpha} \rfloor} \left(\frac{c_1}{c_0}\right)^{n+1} \left(\frac{c_{-1}}{c_0}\right)^k e_{(n+1)\beta-k\alpha-\lambda}.
\end{aligned} \tag{2.2}$$

Here and below, we use the standard notation $\lfloor x \rfloor$ for the largest integer not exceeding $x \in \mathbb{R}$, and $\lceil x \rceil$ for the smallest integer not exceeded by $x \in \mathbb{R}$. In (2.2), m denotes the minimal number in $\mathbb{Z}_+$ such that

$$\left\lceil \frac{(m+1)\beta}{\alpha} \right\rceil = \left\lceil \frac{m\beta+\lambda}{\alpha} \right\rceil,$$

and n stands for the minimal number in $\mathbb{N}$ such that

$$\left\lfloor \frac{(n+1)\beta-\lambda}{\alpha} \right\rfloor = \left\lfloor \frac{n\beta}{\alpha} \right\rfloor.$$

Note that such numbers exist because the set of fractional parts of $\{j\gamma\colon j \in \mathbb{N}\}$ is dense in $[0,1]$ for any fixed irrational γ (see, e.g., [5, Chapter 7, § 2, Theorem 1]).

Proof. If $c_0 = 0$, then f is (at most) a binomial, and therefore the matrix function (1.2) admits an APP factorization. Its partial AP indices equal $\pm(\alpha+\beta-\lambda)$ if $c_{-1}c_1 \neq 0$, $\pm\alpha$ if the only non-zero coefficient is c_{-1}, $\pm\beta$ if the only non-zero coefficient is c_1, and $\pm\lambda$ if $f = 0$ (see [2, Theorem 14.5]). So, in this case the factorization is never canonical.

Let now $c_0 \neq 0$. The existence of APP factorization follows from [6, Theorem 6.1] (see also Theorem 15.7 in [2]). The fact that the factorization is canonical can also be derived from there, if one observes that condition (1) of this theorem actually cannot materialize when α/β is irrational. Naturally, the explicit factorization formulas make this reasoning obsolete. The formulas themselves can be checked directly, but of course this is not the way they were established. To derive them constructively, one may consider the Riemann-Hilbert problems

$$G\begin{pmatrix} g_- \\ \varphi_- \end{pmatrix} = \begin{pmatrix} g_+ \\ \varphi_+ \end{pmatrix}, \quad G\begin{pmatrix} \widetilde{g}_- \\ \widetilde{\varphi}_- \end{pmatrix} = \begin{pmatrix} \widetilde{g}_+ \\ \widetilde{\varphi}_+ \end{pmatrix},$$

seeking g_- and $\widetilde{g}_-$ in the form

$$g_- = 1 + \sum\nolimits_{j=0}^{m} \sum\nolimits_{k=\lfloor \frac{j\beta}{\alpha} \rfloor + 1}^{\lceil \frac{j\beta+\lambda}{\alpha} \rceil - 1} a_{j,k}\, e_{j\beta - k\alpha}.$$

and

$$\widetilde{g}_- = e_{-\lambda} + \sum\nolimits_{j=1}^{n} \sum\nolimits_{k=\lfloor \frac{j\beta-\lambda}{\alpha} \rfloor + 1}^{\lfloor \frac{j\beta}{\alpha} \rfloor} b_{j,k}\, e_{j\beta - k\alpha - \lambda}.$$

Formulas (2.2) emerge while solving systems of linear equations reflecting the requirement that the exponents of fg_-, $f\widetilde{g}_-$ lie outside $(-\lambda, 0)$. □

Note that the explicit canonical APP factorization in the setting of Theorem 2.1 (in a slightly different form) can be extracted from [3, Theorem 5.1].

Moving to the second borderline case, we introduce the notation

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Theorem 2.2. *Let the off diagonal term f of the matrix function* (1.2) *be given by* (1.5) *with $c_{-1}c_1 \neq 0$, $\delta + \nu = \lambda$, $\mu \neq 0$, and the ratio $\beta = \frac{\delta-\mu}{\mu+\nu}$ is irrational. Then G admits a canonical APP factorization.*

Proof. If $c_0 = 0$ or $\mu \notin (-\nu, \delta)$, then $-\nu, \delta \in \Omega(f)$ while the interval $(-\nu, \delta)$ of length λ is disjoint with $\Omega(f)$. This is a particular realization of the big gap case in which the canonical APP factorization exists (see, e.g., [2, Theorem 14.5] or [4, Theorem 2.1]).

It remains to consider the situation $c_0 \neq 0$, $-\nu < \mu < \delta$. Passing from G to JG^*J if necessary, we may without loss of generality suppose that $\mu > 0$. In one step of the Portuguese transformation, the matrix under consideration is then reduced either to the case covered by Theorem 2.1, or to the case of at most a binomial f with a non-zero constant term (compare with the way in which Theorem 4.3 is derived from Theorem 4.2 in [8] or Theorem 15.8 is derived from Theorem 15.7 in [2]). In all these cases, the resulting matrix function admits a canonical APP factorization. Therefore, so does the original matrix G. $\square$

We will now implement the last part of the proof (thus making it more self-contained) in order to actually construct the canonical APP factorization in the setting of Theorem 2.2 with $0 < \mu < \delta$.

Applying the Portuguese transformation, we can find explicitly the functions $g_1^+, g_2^+ \in APP_+$ such that

$$g_1^+ e_{\lambda+\nu} + g_2^+(e_\nu f) = 1$$

where

$$e_\nu f := c_{-1} - c_0 e_{\mu+\nu} + c_1 e_{\delta+\nu}.$$

Then, setting $f_1 := g_2^+ e_{-\lambda}$ and

$$X^+ := \begin{pmatrix} -e_\nu f & e_{\lambda+\nu} \\ g_1^+ & g_2^+ \end{pmatrix}, \quad G_1 := \begin{pmatrix} e_\nu & 0 \\ f_1 & e_{-\nu} \end{pmatrix},$$

we obtain

$$G_1 = X^+ G J. \tag{2.3}$$

Assuming below that $n_1, n_2 \in \mathbb{Z}_+$, by [2, (13.42)] we have

$$g_2^+ = \sum_{-\lambda \le n_1(\mu+\nu)+(n_2-1)\lambda < \nu} c_{-1}^{-1} \frac{(n_1+n_2)!}{n_1! n_2!} \left(\frac{c_0}{c_{-1}}\right)^{n_1} \left(-\frac{c_1}{c_{-1}}\right)^{n_2} e_{n_1(\mu+\nu)+n_2\lambda}. \tag{2.4}$$

Set

$$\omega_- = \sum_{0 \le n_1(\mu+\nu)+n_2\lambda \le \delta} c_{-1}^{-1} \frac{(n_1+n_2)!}{n_1! n_2!} \left(\frac{c_0}{c_{-1}}\right)^{n_1} \left(-\frac{c_1}{c_{-1}}\right)^{n_2} e_{n_1(\mu+\nu)+n_2\lambda-\delta}.$$

Then, by [2, Proposition 13.4], we get

$$G_2 := G_1 \begin{pmatrix} 1 & 0 \\ -\omega_- & 1 \end{pmatrix} = \begin{pmatrix} e_\nu & 0 \\ f_2 & e_{-\nu} \end{pmatrix} \tag{2.5}$$

where

$$\begin{aligned} f_2 &= g_2^+ e_{-\lambda} - e_{-\nu}\omega_- \\ &= \sum_{-\nu < n_1(\mu+\nu)+(n_2-1)\lambda < \nu} c_{-1}^{-1} \frac{(n_1+n_2)!}{n_1! n_2!} \left(\frac{c_0}{c_{-1}}\right)^{n_1} \left(-\frac{c_1}{c_{-1}}\right)^{n_2} e_{n_1(\mu+\nu)+(n_2-1)\lambda}. \end{aligned}$$

Let $g_\pm$, $\varphi_\pm$, $\widetilde{g}_\pm$, $\widetilde{\varphi}_\pm$ be linearly independent solutions of the Riemann-Hilbert problems

$$\begin{pmatrix} e_\nu & 0 \\ f_2 & e_{-\nu} \end{pmatrix} \begin{pmatrix} g_- \\ \varphi_- \end{pmatrix} = \begin{pmatrix} g_+ \\ \varphi_+ \end{pmatrix}, \quad \begin{pmatrix} e_\nu & 0 \\ f_2 & e_{-\nu} \end{pmatrix} \begin{pmatrix} \widetilde{g}_- \\ \widetilde{\varphi}_- \end{pmatrix} = \begin{pmatrix} \widetilde{g}_+ \\ \widetilde{\varphi}_+ \end{pmatrix}. \tag{2.6}$$

Then we infer from (2.3), (2.5) and (2.6) that

$$X^+ \begin{pmatrix} e_\nu & 0 \\ f_2 & e_{-\nu} \end{pmatrix} \begin{pmatrix} g_- & \widetilde{g}_- \\ \varphi_- & \widetilde{\varphi}_- \end{pmatrix} = GJ \begin{pmatrix} 1 & 0 \\ -\omega_- & 1 \end{pmatrix} \begin{pmatrix} g_- & \widetilde{g}_- \\ \varphi_- & \widetilde{\varphi}_- \end{pmatrix}. \tag{2.7}$$

Clearly, by (2.6),

$$X^+ \begin{pmatrix} e_\nu & 0 \\ f_2 & e_{-\nu} \end{pmatrix} \begin{pmatrix} g_- & \widetilde{g}_- \\ \varphi_- & \widetilde{\varphi}_- \end{pmatrix} = X^+ \begin{pmatrix} g_+ & \widetilde{g}_+ \\ \varphi_+ & \widetilde{\varphi}_+ \end{pmatrix} =: G^+ \tag{2.8}$$

belongs entry-wise to APP_+. On the other hand, the matrix function

$$G^- := J \begin{pmatrix} 1 & 0 \\ -\omega_- & 1 \end{pmatrix} \begin{pmatrix} g_- & \widetilde{g}_- \\ \varphi_- & \widetilde{\varphi}_- \end{pmatrix} = \begin{pmatrix} \varphi_- - \omega_- g_- & \widetilde{\varphi}_- - \omega_- \widetilde{g}_- \\ g_- & \widetilde{g}_- \end{pmatrix} \tag{2.9}$$

belongs entry-wise to APP_-. Finally, (2.7)–(2.9) imply that

$$G = G^+(G^-)^{-1}$$

is a canonical APP factorization of the matrix function

$$G = \begin{pmatrix} e_\lambda & 0 \\ c_{-1}e_{-\nu} - c_0 e_\mu + c_1 e_\delta & e_{-\lambda} \end{pmatrix}.$$

Thus, if we know ω_-, g_-, $\widetilde{g}_-$, φ_-, $\widetilde{\varphi}_-$, we can obtain G^-. In that case $G^+ = GG^-$ gives the second factor of a canonical APP factorization of G. Hence,

$$G^+ = \begin{pmatrix} e_\lambda(\varphi_- - \omega_- g_-) & e_\lambda(\widetilde{\varphi}_- - \omega_- \widetilde{g}_-) \\ e_{-\lambda} g_- + f(\varphi_- - \omega_- g_-) & e_{-\lambda}\widetilde{g}_- + f(\widetilde{\varphi}_- - \omega_- \widetilde{g}_-) \end{pmatrix}.$$

On the other hand, since by construction $\det G^\pm = -C$, where $C := \det \begin{pmatrix} g_- & \widetilde{g}_- \\ \varphi_- & \widetilde{\varphi}_- \end{pmatrix}$ is a non-zero constant, we conclude that

$$(G^-)^{-1} = \begin{pmatrix} -\widetilde{g}_- & \widetilde{\varphi}_- - \omega_- \widetilde{g}_- \\ g_- & -\varphi_- + \omega_- g_- \end{pmatrix} C^{-1}.$$

Let

$$N := \left\lfloor \frac{\lambda+\mu}{\mu+\nu} \right\rfloor, \quad \widetilde{N} := \left\lceil \frac{\lambda-\mu}{\mu+\nu} \right\rceil.$$

Following [2, Section 15.3], we can rewrite (2.4) in the form

$$g_2^+ = -\frac{c_1}{c_{-1}^2}\, e_\lambda + \sum\nolimits_{n=0}^{\widetilde{N}} \frac{c_0^n}{c_{-1}^{n+1}}\, e_{n(\mu+\nu)}.$$

Consequently,

$$\omega_- = \sum\nolimits_{s=0}^{N-1} \frac{c_0^s}{c_{-1}^{s+1}}\, e_{s(\mu+\nu)-\delta}. \tag{2.10}$$

The following relations between N and $\widetilde{N}$ are possible:

$$\text{(i) } \widetilde{N} = N-1, \quad \text{(ii) } \widetilde{N} = N, \quad \text{(iii) } \widetilde{N} = N+1.$$

Depending on which of them takes place,

$$f_2 = \begin{cases} -\dfrac{c_1}{c_{-1}^2} & \text{in case (i)}, \\ -\dfrac{c_1}{c_{-1}^2} + \dfrac{c_0^N}{c_{-1}^{N+1}}\, e_{N(\mu+\nu)-\lambda} & \text{in case (ii)}, \\ \dfrac{c_0^N}{c_{-1}^{N+1}}\, e_{N(\mu+\nu)-\lambda} - \dfrac{c_1}{c_{-1}^2} + \dfrac{c_0^{N+1}}{c_{-1}^{N+2}}\, e_{(N+1)(\mu+\nu)-\lambda} & \text{in case (iii)}. \end{cases}$$

It remains to find linearly independent solutions of the Riemann-Hilbert problems (2.6).

In case (i) we obtain

$$\begin{pmatrix} g_- & \widetilde{g}_- \\ \varphi_- & \widetilde{\varphi}_- \end{pmatrix} = \begin{pmatrix} 1 & \frac{c_{-1}^2}{c_1}\, e_{-\nu} \\ 0 & 1 \end{pmatrix} \tag{2.11}$$

because

$$\begin{pmatrix} e_\nu & 0 \\ -\frac{c_1}{c_{-1}^2} & e_{-\nu} \end{pmatrix} \begin{pmatrix} 1 & \frac{c_{-1}^2}{c_1}\, e_{-\nu} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e_\nu & \frac{c_{-1}^2}{c_1} \\ -\frac{c_1}{c_{-1}^2} & 0 \end{pmatrix}.$$

Hence, by (2.9), (2.10) and (2.11), we infer that G^- is given by

$$G^- = \begin{pmatrix} -\sum_{s=0}^{N-1} \frac{c_0^s}{c_{-1}^{s+1}}\, e_{s(\mu+\nu)-\delta} & 1 - \sum_{s=0}^{N-1} \frac{c_0^s}{c_1 c_{-1}^{s-1}}\, e_{s(\mu+\nu)-\lambda} \\ 1 & \frac{c_{-1}^2}{c_1}\, e_{-\nu} \end{pmatrix},$$

and therefore

$$(G^-)^{-1} = \begin{pmatrix} -\frac{c_{-1}^2}{c_1}\, e_{-\nu} & 1 - \sum_{s=0}^{N-1} \frac{c_0^s}{c_1 c_{-1}^{s-1}}\, e_{s(\mu+\nu)-\lambda} \\ 1 & \sum_{s=0}^{N-1} \frac{c_0^s}{c_{-1}^{s+1}}\, e_{s(\mu+\nu)-\delta} \end{pmatrix}.$$

Then the matrix $G^+ = GG^-$ is given by

$$G^+ = \begin{pmatrix} -\sum_{s=0}^{N-1} \frac{c_0^s}{c_{-1}^{s+1}}\, e_{s(\mu+\nu)+\nu} & e_\lambda - \sum_{s=0}^{N-1} \frac{c_0^s}{c_1 c_{-1}^{s-1}}\, e_{s(\mu+\nu)} \\ \left(\frac{c_0}{c_{-1}}\right)^N e_{N(\mu+\nu)-\lambda} - \sum_{s=0}^{N-1} \frac{c_1 c_0^s}{c_{-1}^{s+1}}\, e_{s(\mu+\nu)} & g_{22}^+ \end{pmatrix},$$

where

$$g_{22}^+ = \frac{c_0^N}{c_1 c_{-1}^{N-2}}\, e_{N(\mu+\nu)-\lambda-\nu} - \sum_{s=2}^{N-1} \frac{c_0^s}{c_{-1}^{s-1}}\, e_{s(\mu+\nu)-\nu} - 2c_0\, e_\mu + c_1\, e_\delta.$$

Consider now case (ii). Setting

$$b_0 := \frac{c_1}{c_{-1}^2}, \quad b_1 := \frac{c_0^N}{c_{-1}^{N+1}}, \quad \gamma := N(\mu+\nu) - \lambda,$$

we conclude that $f_2 = -b_0 + b_1\, e_\gamma$. If $\gamma > 0$, then

$$f_2 g_- = \varphi_+ - e_{-\nu}\, \varphi_-, \quad f_2 \widetilde{g}_- = \widetilde{\varphi}_+ - e_{-\nu}\, \widetilde{\varphi}_-, \tag{2.12}$$

where

$$g_- = 1, \quad \varphi_- = 0, \quad \varphi_+ = -b_0 + b_1 e_\gamma;$$

$$\widetilde{g}_- = \sum\nolimits_{k=0}^{\lfloor \frac{\nu}{\gamma} \rfloor} b_0^{-1} \left(\frac{b_1}{b_0}\right)^k e_{k\gamma-\nu}, \quad \widetilde{\varphi}_- = 1, \quad \widetilde{\varphi}_+ = \left(\frac{b_1}{b_0}\right)^{\lfloor \frac{\nu}{\gamma} \rfloor + 1} e_{(\lfloor \frac{\nu}{\gamma} \rfloor + 1)\gamma - \nu}.$$

Hence, if $N(\mu+\nu) - \lambda > 0$, then

$$\begin{pmatrix} g_- & \widetilde{g}_- \\ \varphi_- & \widetilde{\varphi}_- \end{pmatrix} = \begin{pmatrix} 1 & \frac{c_{-1}^2}{c_1} \sum_{k=0}^{\lfloor \frac{\nu}{N(\mu+\nu)-\lambda} \rfloor} \left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^k e_{k(N(\mu+\nu)-\lambda)-\nu} \\ 0 & 1 \end{pmatrix}. \tag{2.13}$$

If $\gamma < 0$, then (2.12) holds with

$$g_- = -b_0^{-1} e_{-\nu}, \quad \varphi_- = -1 + \frac{b_1}{b_0} e_\gamma, \quad \varphi_+ = 0;$$

$$\widetilde{g}_- = \sum\nolimits_{k=0}^{\lfloor -\frac{\nu}{\gamma} \rfloor} \left(\frac{b_1}{b_0}\right)^k e_{k\gamma}, \quad \widetilde{\varphi}_- = -\frac{b_1^{\lfloor -\frac{\nu}{\gamma} \rfloor + 1}}{b_0^{\lfloor -\frac{\nu}{\gamma} \rfloor}} e_{(\lfloor -\frac{\nu}{\gamma} \rfloor + 1)\gamma + \nu}, \quad \widetilde{\varphi}_+ = -b_0.$$

Hence, if $\gamma = N(\mu+\nu) - \lambda < 0$, then

$$\begin{pmatrix} g_- & \widetilde{g}_- \\ \varphi_- & \widetilde{\varphi}_- \end{pmatrix} = \begin{pmatrix} -\frac{c_{-1}^2}{c_1} e_{-\nu} & \sum_{k=0}^{\lfloor -\frac{\nu}{\gamma} \rfloor} \left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^k e_{k\gamma} \\ -1 + \frac{c_0^N}{c_1 c_{-1}^{N-1}} e_\gamma & -\frac{c_1}{c_{-1}^2} \left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^{\lfloor -\frac{\nu}{\gamma} \rfloor + 1} e_{(\lfloor -\frac{\nu}{\gamma} \rfloor + 1)\gamma + \nu} \end{pmatrix}. \tag{2.14}$$

Thus, in case (ii) the matrix function G^- is given by

$$G^- := \begin{pmatrix} -\sum_{s=0}^{N-1} \frac{c_0^s}{c_{-1}^{s+1}} e_{s(\mu+\nu)-\delta} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} g_- & \widetilde{g}_- \\ \varphi_- & \widetilde{\varphi}_- \end{pmatrix} \tag{2.15}$$

where the matrix $\begin{pmatrix} g_- & \widetilde{g}_- \\ \varphi_- & \widetilde{\varphi}_- \end{pmatrix}$ is given by (2.13) if $N(\mu+\nu) - \lambda > 0$ and by (2.14) if $N(\mu+\nu) - \lambda < 0$. Hence,

$$G^- := \begin{pmatrix} g_{11}^- & g_{12}^- \\ g_{21}^- & g_{22}^- \end{pmatrix}, \quad (G^-)^{-1} = \begin{pmatrix} -g_{22}^- & g_{12}^- \\ g_{21}^- & -g_{11}^- \end{pmatrix}, \tag{2.16}$$

where for $N(\mu+\nu) - \lambda > 0$,

$$g_{11}^- = -\sum\nolimits_{s=0}^{N-1} \frac{c_0^s}{c_{-1}^{s+1}} e_{s(\mu+\nu)-\delta},$$

$$g_{12}^- = 1 - \sum\nolimits_{s=0}^{N-1} \sum\nolimits_{k=0}^{\lfloor \frac{\nu}{N(\mu+\nu)-\lambda} \rfloor} \frac{c_0^s}{c_1 c_{-1}^{s-1}} \left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^k e_{(s+kN)(\mu+\nu)-(k+1)\lambda},$$

$$g_{21}^- = 1,$$

$$g_{22}^- = \sum\nolimits_{k=0}^{\lfloor \frac{\nu}{N(\mu+\nu)-\lambda} \rfloor} \frac{c_{-1}^2}{c_1} \left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^k e_{k(N(\mu+\nu)-\lambda)-\nu}$$

and for $N(\mu+\nu)-\lambda<0$,

$$
\begin{aligned}
g_{11}^{-} &= -1+\sum\nolimits_{s=0}^{N} \frac{c_0^s}{c_1 c_{-1}^{s-1}}\, e_{s(\mu+\nu)-\lambda},\\
g_{12}^{-} &= -\frac{c_1}{c_{-1}^2}\left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^{\lfloor\frac{\nu}{\lambda-N(\mu+\nu)}\rfloor+1} e_{(\lfloor\frac{\nu}{\lambda-N(\mu+\nu)}\rfloor+1)(N(\mu+\nu)-\lambda)+\nu}\\
&\quad -\sum\nolimits_{s=0}^{N-1}\sum\nolimits_{k=0}^{\lfloor\frac{\nu}{\lambda-N(\mu+\nu)}\rfloor} \frac{c_0^s}{c_{-1}^{s+1}}\left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^k e_{(s+kN)(\mu+\nu)-(k+1)\lambda+\nu},\\
g_{21}^{-} &= -\frac{c_{-1}^2}{c_1}\, e_{-\nu},\\
g_{22}^{-} &= \sum\nolimits_{k=0}^{\lfloor\frac{\nu}{\lambda-N(\mu+\nu)}\rfloor}\left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^k e_{k(N(\mu+\nu)-\lambda)}.
\end{aligned}
$$

In that case the entries of the matrix

$$
G^{+}=GG^{-}=\begin{pmatrix} g_{11}^{+} & g_{12}^{+}\\ g_{21}^{+} & g_{22}^{+}\end{pmatrix}
$$

are given by

$$
\begin{aligned}
g_{11}^{+} &= -\sum\nolimits_{s=0}^{N-1}\frac{c_0^s}{c_{-1}^{s+1}}\, e_{s(\mu+\nu)+\nu},\\
g_{12}^{+} &= e_\lambda-\sum\nolimits_{s=0}^{N-1}\sum\nolimits_{k=0}^{\lfloor\frac{\nu}{N(\mu+\nu)-\lambda}\rfloor}\frac{c_0^s}{c_1 c_{-1}^{s-1}}\left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^k e_{(s+kN)(\mu+\nu)-k\lambda},\\
g_{21}^{+} &= \frac{c_0^N}{c_{-1}^N}\, e_{N(\mu+\nu)-\lambda}-\sum\nolimits_{s=0}^{N-1}\frac{c_1 c_0^s}{c_{-1}^{s+1}}\, e_{s(\mu+\nu)},\\
g_{22}^{+} &= -c_0\, e_\mu+c_1\, e_\delta+c_{-1}\left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^{\lfloor\frac{\nu}{N(\mu+\nu)-\lambda}\rfloor+1} e_{(\lfloor\frac{\nu}{N(\mu+\nu)-\lambda}\rfloor+1)(N(\mu+\nu)-\lambda)-\nu}\\
&\quad -\sum\nolimits_{s=1}^{N-1}\sum\nolimits_{k=0}^{\lfloor\frac{\nu}{N(\mu+\nu)-\lambda}\rfloor}\frac{c_0^{s+1}}{c_{-1}^s}\left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^k e_{(s+kN)(\mu+\nu)-k\lambda-\nu}
\end{aligned}
$$

if $N(\mu+\nu)-\lambda>0$, and by

$$
\begin{aligned}
g_{11}^{+} &= -e_\lambda+\sum\nolimits_{s=0}^{N}\frac{c_0^s}{c_1 c_{-1}^{s-1}}\, e_{s(\mu+\nu)},\\
g_{12}^{+} &= -\frac{c_1}{c_{-1}^2}\left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^{\lfloor\frac{\nu}{\lambda-N(\mu+\nu)}\rfloor+1} e_{(\lfloor\frac{\nu}{\lambda-N(\mu+\nu)}\rfloor+1)(N(\mu+\nu)-\lambda)+\nu+\lambda}\\
&\quad -\sum\nolimits_{s=0}^{N-1}\sum\nolimits_{k=0}^{\lfloor\frac{\nu}{\lambda-N(\mu+\nu)}\rfloor}\frac{c_0^s}{c_{-1}^{s+1}}\left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^k e_{(s+kN)(\mu+\nu)-k\lambda+\nu},
\end{aligned}
$$

$$
\begin{aligned}
g_{21}^{+} &= c_0\, e_\mu - c_1\, e_\delta - \frac{c_0^{N+1}}{c_1 c_{-1}^{N-1}}\, e_{(N+1)(\mu+\nu)-\lambda-\nu} + \sum_{s=1}^{N} \frac{c_0^s}{c_{-1}^{s-1}}\, e_{s(\mu+\nu)-\nu},\\
g_{22}^{+} &= -\frac{c_1}{c_{-1}} + \frac{c_0 c_1}{c_{-1}^2}\left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^{\lfloor \frac{\nu}{\lambda-N(\mu+\nu)} \lfloor +1} e_{(\lfloor \frac{\nu}{\lambda-N(\mu+\nu)}\rfloor+1)(N(\mu+\nu)-\lambda)+\mu+\nu}\\
&\quad - \frac{c_1^2}{c_{-1}^2}\left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^{\lfloor \frac{\nu}{\lambda-N(\mu+\nu)}\rfloor+1} e_{(\lfloor \frac{\nu}{\lambda-N(\mu+\nu)}\rfloor+1)(N(\mu+\nu)-\lambda)+\lambda}\\
&\quad - \sum_{s=1}^{N-1} \sum_{k=0}^{\lfloor \frac{\nu}{\lambda-N(\mu+\nu)}\rfloor} \frac{c_1 c_0^s}{c_{-1}^{s+1}}\left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^{k} e_{(s+kN)(\mu+\nu)-k\lambda}
\end{aligned}
$$

if $N(\mu+\nu)-\lambda<0$.

Note that

$$
\begin{aligned}
&\left(\left\lfloor \frac{\nu}{\lambda-N(\mu+\nu)}\right\rfloor+1\right)(N(\mu+\nu)-\lambda)+\mu+\nu\\
&\geq \left(\frac{\nu}{\lambda-N(\mu+\nu)}+1\right)(N(\mu+\nu)-\lambda)+\mu+\nu\\
&= (N+1)(\mu+\nu)-(\lambda+\nu)>0,
\end{aligned}
$$

$$
\begin{aligned}
&\left(\left\lfloor \frac{\nu}{\lambda-N(\mu+\nu)}\right\rfloor+1\right)(N(\mu+\nu)-\lambda)+\lambda\\
&\geq \left(\frac{\nu}{\lambda-N(\mu+\nu)}+1\right)(N(\mu+\nu)-\lambda)+\lambda\\
&= (N-1)(\mu+\nu)+\mu>0 \qquad \left(N=1+\left\lfloor \frac{\lambda-\nu}{\mu+\nu}\right\rfloor\right),
\end{aligned}
$$

$$
\begin{aligned}
&\left\lfloor \frac{\nu}{\lambda-N(\mu+\nu)}\right\rfloor(N(\mu+\nu)-\lambda)+\mu+\nu\\
&\geq \frac{\nu}{\lambda-N(\mu+\nu)}\,(N(\mu+\nu)-\lambda)+\mu+\nu=\mu>0.
\end{aligned}
$$

In case (iii) we have the following:

$$
G^{+} = \begin{pmatrix} e_\lambda & 0\\ c_{-1}e_{-\nu}-c_0 e_\mu + c_1 e_\delta & e_{-\lambda}\end{pmatrix} G^{-},
$$

where

$$
G^{-} = \begin{pmatrix} -\sum_{s=0}^{N-1} \frac{c_0^s}{c_{-1}^{s+1}}\, e_{s(\mu+\nu)-\delta} & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} g_- & \widetilde{g}_-\\ \varphi_- & \widetilde{\varphi}_-\end{pmatrix} \tag{2.17}
$$

and the functions $g_\pm, \widetilde{g}_\pm, \varphi_\pm, \widetilde{\varphi}_\pm \in APP_\pm$ are given by

$$g_- = 1 + \sum\nolimits_{j=0}^{m} \sum\nolimits_{k=\lfloor \frac{j\beta}{\alpha} \rfloor+1}^{\lceil \frac{j\beta+\nu}{\alpha} \rceil -1} \left(\frac{c_0^{N+1}}{c_1 c_{-1}^N}\right)^j \left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^k e_{j\beta-k\alpha},$$

$$\varphi_- = -\frac{c_1}{c_{-1}^2} \sum\nolimits_{j=0}^{m} \left(\frac{c_0^{N+1}}{c_1 c_{-1}^N}\right)^j \left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^{\lceil \frac{j\beta+\nu}{\alpha} \rceil} e_{j\beta+\nu-\lceil \frac{j\beta+\nu}{\alpha} \rceil\alpha},$$

$$\widetilde{g}_- = e_{-\nu} + \sum\nolimits_{j=1}^{n} \sum\nolimits_{k=\lfloor \frac{j\beta-\nu}{\alpha} \rfloor+1}^{\lfloor \frac{j\beta}{\alpha} \rfloor} \left(\frac{c_0^{N+1}}{c_1 c_{-1}^N}\right)^j \left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^k e_{j\beta-k\alpha-\nu},$$

$$\widetilde{\varphi}_- = \frac{c_1}{c_{-1}^2} - \frac{c_0^N}{c_{-1}^{N+1}} e_{-\alpha} - \frac{c_1}{c_{-1}^2} \sum\nolimits_{j=1}^{n} \left(\frac{c_0^{N+1}}{c_1 c_{-1}^N}\right)^j \left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^{\lfloor \frac{j\beta}{\alpha} \rfloor+1} e_{j\beta-(\lfloor \frac{j\beta}{\alpha} \rfloor+1)\alpha},$$

with

$$\alpha := \lambda - N(\mu+\nu), \quad \beta := (N+1)(\mu+\nu) - \lambda,$$

and m, n defined as minimal numbers (in $\mathbb{Z}_+$ and $\mathbb{N}$, respectively) for which

$$\left\lceil \frac{(m+1)((N+1)(\mu+\nu)-\lambda)}{\lambda - N(\mu+\nu)} \right\rceil = \left\lceil \frac{m((N+1)(\mu+\nu)-\lambda)+\nu}{\lambda - N(\mu+\nu)} \right\rceil,$$
$$\left\lfloor \frac{(n+1)((N+1)(\mu+\nu)-\lambda)-\nu}{\lambda - N(\mu+\nu)} \right\rfloor = \left\lfloor \frac{n((N+1)(\mu+\nu)-\lambda)}{\lambda - N(\mu+\nu)} \right\rfloor.$$

Hence, applying (2.17), we conclude that $\det G^\pm = -\dfrac{c_1}{c_{-1}^2}$, and therefore the matrix function G^- is given by (2.16) and $(G^-)^{-1} = \dfrac{c_{-1}^2}{c_1} \begin{pmatrix} -g_{22}^- & g_{12}^- \\ g_{21}^- & -g_{11}^- \end{pmatrix}$, where

$$g_{11}^- = -\frac{c_1}{c_{-1}^2} \sum\nolimits_{j=0}^{m} \left(\frac{c_0^{N+1}}{c_1 c_{-1}^N}\right)^j \left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^{\lceil \frac{j\beta+\nu}{\alpha} \rceil} e_{j\beta+\nu-\lceil \frac{j\beta+\nu}{\alpha} \rceil\alpha}$$
$$-\sum\nolimits_{s=0}^{N-1} \frac{c_0^s}{c_{-1}^{s+1}} e_{s(\mu+\nu)-\delta} \left(1 + \sum\nolimits_{j=0}^{m} \sum\nolimits_{k=\lfloor \frac{j\beta}{\alpha} \rfloor+1}^{\lceil \frac{j\beta+\nu}{\alpha} \rceil -1} \left(\frac{c_0^{N+1}}{c_1 c_{-1}^N}\right)^j \left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^k e_{j\beta-k\alpha}\right),$$

$$g_{12}^- = \frac{c_1}{c_{-1}^2} - \frac{c_0^N}{c_{-1}^{N+1}} e_{-\alpha} - \frac{c_1}{c_{-1}^2} \sum\nolimits_{j=1}^{n} \left(\frac{c_0^{N+1}}{c_1 c_{-1}^N}\right)^j \left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^{\lfloor \frac{j\beta}{\alpha} \rfloor+1} e_{j\beta-(\lfloor \frac{j\beta}{\alpha} \rfloor+1)\alpha}$$
$$-\sum\nolimits_{s=0}^{N-1} \frac{c_0^s}{c_{-1}^{s+1}} e_{s(\mu+\nu)-\lambda} \left(1 + \sum\nolimits_{j=1}^{n} \sum\nolimits_{k=\lfloor \frac{j\beta-\nu}{\alpha} \rfloor+1}^{\lfloor \frac{j\beta}{\alpha} \rfloor} \left(\frac{c_0^{N+1}}{c_1 c_{-1}^N}\right)^j \left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^k e_{j\beta-k\alpha}\right),$$

$$g_{21}^- = 1 + \sum\nolimits_{j=0}^{m} \sum\nolimits_{k=\lfloor \frac{j\beta}{\alpha} \rfloor+1}^{\lceil \frac{j\beta+\nu}{\alpha} \rceil -1} \left(\frac{c_0^{N+1}}{c_1 c_{-1}^N}\right)^j \left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^k e_{j\beta-k\alpha},$$

$$g_{22}^- = e_{-\nu} + \sum\nolimits_{j=1}^{n} \sum\nolimits_{k=\lfloor \frac{j\beta-\nu}{\alpha} \rfloor+1}^{\lfloor \frac{j\beta}{\alpha} \rfloor} \left(\frac{c_0^{N+1}}{c_1 c_{-1}^N}\right)^j \left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^k e_{j\beta-k\alpha-\nu}$$

and

$$g_{11}^{+} = -\frac{c_1}{c_{-1}^2}\sum_{j=0}^{m}\left(\frac{c_0^{N+1}}{c_1 c_{-1}^N}\right)^j\left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^{\left\lceil\frac{j\beta+\nu}{\alpha}\right\rceil} e_{j\beta+\nu-\left\lceil\frac{j\beta+\nu}{\alpha}\right\rceil\alpha+\lambda}$$

$$-\sum_{s=0}^{N-1}\frac{c_0^s}{c_{-1}^{s+1}}\, e_{s(\mu+\nu)+\nu}\left(1+\sum_{j=0}^{m}\sum_{k=\left\lfloor\frac{j\beta}{\alpha}\right\rfloor+1}^{\left\lceil\frac{j\beta+\nu}{\alpha}\right\rceil-1}\left(\frac{c_0^{N+1}}{c_1 c_{-1}^N}\right)^j\left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^k e_{j\beta-k\alpha}\right),$$

$$g_{12}^{+} = \frac{c_1}{c_{-1}^2}\, e_\lambda - \frac{c_0^N}{c_{-1}^{N+1}}\, e_{N(\mu+\nu)} - \sum_{s=0}^{N-1}\frac{c_0^s}{c_{-1}^{s+1}}\, e_{s(\mu+\nu)}$$

$$\times\left(1+\sum_{j=1}^{n}\sum_{k=\left\lfloor\frac{j\beta-\nu}{\alpha}\right\rfloor+1}^{\left\lfloor\frac{j\beta}{\alpha}\right\rfloor}\left(\frac{c_0^{N+1}}{c_1 c_{-1}^N}\right)^j\left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^k e_{j\beta-k\alpha}\right)$$

$$-\frac{c_1}{c_{-1}^2}\sum_{j=1}^{n}\left(\frac{c_0^{N+1}}{c_1 c_{-1}^N}\right)^j\left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^{\left\lfloor\frac{j\beta}{\alpha}\right\rfloor+1} e_{j\beta-\left(\left\lfloor\frac{j\beta}{\alpha}\right\rfloor+1\right)\alpha+\lambda},$$

$$g_{21}^{+} = \frac{c_1}{c_{-1}}\left(\frac{c_0^{N+1}}{c_1 c_{-1}^N}\right)^{m+1}\left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^{\left\lfloor\frac{(m+1)\beta}{\alpha}\right\rfloor} e_{(m+1)\beta-\left\lfloor\frac{(m+1)\beta}{\alpha}\right\rfloor\alpha}$$

$$-\sum_{s=1}^{N-1}\sum_{j=0}^{m}\sum_{k=\left\lfloor\frac{j\beta}{\alpha}\right\rfloor+1}^{\left\lceil\frac{j\beta+\nu}{\alpha}\right\rceil-1}\frac{c_1 c_0^s}{c_{-1}^{s+1}}\left(\frac{c_0^{N+1}}{c_1 c_{-1}^N}\right)^j\left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^k e_{j\beta-k\alpha+s(\mu+\nu)}$$

$$-\frac{c_1^2}{c_{-1}^2}\sum_{j=0}^{m}\left(\frac{c_0^{N+1}}{c_1 c_{-1}^N}\right)^j\left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^{\left\lceil\frac{j\beta+\nu}{\alpha}\right\rceil} e_{j\beta+\lambda-\left\lceil\frac{j\beta+\nu}{\alpha}\right\rceil\alpha}-\sum_{s=0}^{N-1}\frac{c_1 c_0^s}{c_{-1}^{s+1}}\, e_{s(\mu+\nu)},$$

$$g_{22}^{+} = -\frac{c_0 c_1}{c_{-1}^2}\, e_\mu + \frac{c_1^2}{c_{-1}^2}\, e_\delta - \sum_{s=1}^{N-1}\frac{c_1 c_0^s}{c_{-1}^{s+1}}\, e_{s(\mu+\nu)-\nu}$$

$$+\frac{c_1}{c_{-1}}\left(\frac{c_0^{N+1}}{c_1 c_{-1}^N}\right)^{n+1}\left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^{\left\lfloor\frac{n\beta}{\alpha}\right\rfloor} e_{(n+1)\beta-\nu-\left\lfloor\frac{n\beta}{\alpha}\right\rfloor\alpha}$$

$$-\frac{c_1^2}{c_{-1}^2}\sum_{j=0}^{n}\left(\frac{c_0^{N+1}}{c_1 c_{-1}^N}\right)^j\left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^{\left\lfloor\frac{j\beta}{\alpha}\right\rfloor+1} e_{j\beta-\left(\left\lfloor\frac{j\beta}{\alpha}\right\rfloor+1\right)\alpha+\delta}$$

$$-\sum_{s=1}^{N-1}\sum_{j=1}^{n}\sum_{k=\left\lfloor\frac{j\beta-\nu}{\alpha}\right\rfloor+1}^{\left\lfloor\frac{j\beta}{\alpha}\right\rfloor}\frac{c_1 c_0^s}{c_{-1}^{s+1}}\left(\frac{c_0^{N+1}}{c_1 c_{-1}^N}\right)^j\left(\frac{c_0^N}{c_1 c_{-1}^{N-1}}\right)^k e_{j\beta-k\alpha-\nu+s(\mu+\nu)}.$$

To check the non-negativity of the exponents in g_{21}^{+} and g_{22}^{+}, we need to take into account the following relations. First, for $k=\left\lfloor\frac{j\beta}{\alpha}\right\rfloor+1,\dots,\left\lceil\frac{j\beta+\nu}{\alpha}\right\rceil-1$ and $s\geq 1$, we obtain

$$j\beta - k\alpha + s(\mu+\nu) \geq j\beta + \nu - \left(\left\lceil \frac{j\beta+\nu}{\alpha} \right\rceil - 1\right)\alpha + \mu > 0.$$

Since $\lceil \frac{j\beta+\nu}{\alpha} \rceil = \lceil \frac{(j+1)\beta}{\alpha} \rceil + 1 = \lfloor \frac{(j+1)\beta}{\alpha} \rfloor + 2$ for $j = 0, 1, \dots, m-1$ and since $N = \lfloor \frac{\lambda+\mu}{\mu+\nu} \rfloor \geq 1$, we conclude that

$$\lambda - \alpha = N(\mu+\nu) \geq \mu + \nu = \alpha + \beta$$

and hence, for $j = 0, 1, \dots, m-1$,

$$\begin{aligned} j\beta + \lambda - \left\lceil \frac{j\beta+\nu}{\alpha} \right\rceil \alpha &\geq j\beta + 2\alpha + \beta - \left(\left\lfloor \frac{(j+1)\beta}{\alpha} \right\rfloor + 2\right)\alpha \\ &= (j+1)\beta - \left\lfloor \frac{(j+1)\beta}{\alpha} \right\rfloor \alpha > 0. \end{aligned}$$

If $j = m$, then

$$\left\lceil \frac{m\beta+\nu}{\alpha} \right\rceil = \left\lceil \frac{(m+1)\beta}{\alpha} \right\rceil = \left\lfloor \frac{(m+1)\beta}{\alpha} \right\rfloor + 1,$$
$$\lambda - \alpha - \beta = \lambda - \mu - \nu = \delta - \mu > 0,$$

and therefore

$$\begin{aligned} m\beta + \lambda - \left\lceil \frac{m\beta+\nu}{\alpha} \right\rceil \alpha &\geq m\beta + \alpha + \beta + \delta - \mu - \left(\left\lfloor \frac{(j+1)\beta}{\alpha} \right\rfloor + 1\right)\alpha \\ &= (m+1)\beta - \left\lfloor \frac{(m+1)\beta}{\alpha} \right\rfloor \alpha + \delta - \mu > 0. \end{aligned}$$

On the other hand, $\delta - \alpha = \delta - \lambda + N(\mu+\nu) = -\nu + N(\mu+\nu)$ and therefore

$$j\beta - \left(\left\lfloor \frac{j\beta}{\alpha} \right\rfloor + 1\right)\alpha + \delta = j\beta - \left\lfloor \frac{j\beta}{\alpha} \right\rfloor \alpha - \nu + N(\mu+\nu) \geq 0.$$

Finally,

$$(n+1)\beta - \nu - \left\lfloor \frac{n\beta}{\alpha} \right\rfloor \alpha \geq \left\lfloor \frac{(n+1)\beta - \nu}{\alpha} \right\rfloor \alpha - \left\lfloor \frac{n\beta}{\alpha} \right\rfloor \alpha = 0.$$

3. Beyond trinomials

In this section we consider the case of matrix functions (1.2) with the off diagonal term f satisfying

$$\Omega(f) \subset (-\nu + h\mathbb{Z}_+) \cup (\alpha + h\mathbb{Z}_+) \tag{3.1}$$

with some $\nu, \alpha \in (0, \lambda)$ and $h > 0$. Only the case of irrational $(\alpha+\nu)/h$ is of interest, since otherwise the distances between the points of $\Omega(f)$ are commensurable, and APP factorability of G then follows from [2, Section 14.4].

Theorem 3.1. *Suppose that in* (3.1)

$$h > \nu,\ 2\alpha + \nu \geq \lambda \text{ and } \alpha + h \geq \lambda. \tag{3.2}$$

Then the matrix function (1.2) *is APP factorable.*

Observe that the APP factorability of matrix functions (1.2) satisfying (3.1) with $h = \nu$ and $2\alpha + \nu \geq \lambda$ was considered in [9]; a more restrictive (under the circumstances) condition $\alpha + h \geq \lambda$ was not needed there. Since we have to impose it now, and since only the points of $\Omega(f)$ lying in $(-\lambda, \lambda)$ are relevant, condition (3.1) effectively means that

$$\Omega(f) \subset (-\nu + h\mathbb{Z}_+) \cup \{\alpha\}.$$

Proof. If $-\nu \notin \Omega(f)$, then all the exponents of f are non-negative, and APP factorability of G follows from [2, Section 14.1]. So, a non-trivial case is when $-\nu \in \Omega(f)$. Applying the Portuguese transformation, we can then substitute (1.2) by the matrix function

$$G_1 = \begin{pmatrix} e_\nu & 0 \\ f_1 & e_{-\nu} \end{pmatrix},$$

having the same factorability properties. From the description of the Portuguese transformation [2, Section 13] and the pattern (3.1) it follows that

$$\Omega(f_1) \subset \{n_1 h + n_2(\alpha + \nu) - \lambda \colon n_1, n_2 \in \mathbb{Z}_+\} \cap (-\nu, \nu).$$

Due to (3.2), however, the only pairs $(n_1, n_2) \in \mathbb{Z}_+^2$ for which $n_1 h + n_2(\alpha + \nu) - \lambda$ possibly falls between $-\nu$ and ν are $(0, 1)$, $(k, 0)$ and $(k + 1, 0)$, where

$$k = \left\lfloor \frac{\lambda - \nu}{h} \right\rfloor + 1.$$

Consequently, f_1 is at most a trinomial, with

$$\Omega(f_1) \subset \{kh - \lambda,\ \alpha + \nu - \lambda,\ (k + 1)h - \lambda\}. \tag{3.3}$$

If in fact f_1 contains less than three terms, the APP factorability of G_1 (and therefore G) follows from [2, Section 14.3]. On the other hand, if all three terms are actually present, then the term $\alpha + \nu - \lambda$ can lie either inside or outside the interval $(kh - \lambda, (k+1)h - \lambda)$. In the former case, we are dealing with the trinomial pattern in which the distance h between the endpoints of $\Omega(f_1)$ is strictly bigger than the diagonal exponent ν. In the latter case, one endpoint of $\Omega(f_1)$ is at a distance h bigger than ν (the diagonal exponent) from the rest of $\Omega(f_1)$ – the big gap case. Either way, the matrix function G_1 is APP factorable (according to [2], Sections 15.2, 15.4 or 14.2, respectively). Therefore, so is G. □

In principle, in the setting of Theorem 3.1 it is possible to construct the APP factorization of G explicitly, in particular to compute its partial AP indices. We will not provide these formulas in their full generality, because of a large number of cases this would require. Let us concentrate on a particular situation, when $\alpha + \nu = \lambda$. Of course, then the first inequality in (3.2) implies the other two, which therefore become redundant.

Theorem 3.2. *Suppose that in* (3.1) $h > \nu$ *and* $\alpha + \nu = \lambda$. *Then the APP factorization of the matrix function* (1.2) (*which exists due to Theorem* 3.1) *is canonical if and only if* $-\nu, \alpha \in \Omega(f)$.

Proof. Condition $-\nu \in \Omega(f)$ is necessary, because otherwise $\Omega(f)$ consists of positive numbers only. Denoting the smallest of them by μ, according to [2, Theorem 14.1] we see that the partial AP indices of G are $\pm\mu \neq 0$.

Supposing that the condition $-\nu \in \Omega(f)$ holds, denote by $c_{-1} (\neq 0)$ the respective coefficient of f, and by c_1 the coefficient corresponding to e_α. Using formulas for the Portuguese transformation (see, e.g., [2, Section 13]), we find that the constant term of f_1 equals $-c_1/c_{-1}^2$, while according to (3.3) the only other possible points in $\Omega(f_1)$ are $kh - \lambda$ and $(k+1)h - \lambda$.

If $c_1 = 0$, then f_1 is either a binomial with the exponents at a distance $h > \nu$ or a monomial with a non-zero exponent (mh cannot equal λ for any integer m due to irrationality of $(\alpha + \nu)/h$ which in our setting is the same as λ/h). In any of these cases, the factorization of G_1 (and therefore of G as well) is not canonical (see [2, Theorem 14.10]). This concludes the proof of necessity.

To take care of sufficiency, observe that $0 \in \Omega(f_1)$, since $c_1 \neq 0$. If the points $kh - \lambda$, $(k+1)h - \lambda$ also belong to $\Omega(f_1)$ and in addition have different signs, then G_1 satisfies (after an obvious change of notation) conditions of Theorem 2.1, and therefore its APP factorization is canonical. In all other cases $\Omega(f_1)$ is either non-negative or non-positive. Since $0 \in \Omega(f_1)$, the AP factorization of G_1 is canonical by [2, Theorem 14.1]. Along with G_1, the matrix function G also admits a canonical AP factorization. □

As in Section 2, we will now construct the factorization explicitly. Write the matrix (1.2) satisfying conditions of Theorem 3.2 as

$$G = \begin{pmatrix} e_\lambda & 0 \\ c_{-1}e_{-\nu} - \sum_{j=1}^{s} c_j\, e_{-\nu+jh} + c_{s+1}e_\alpha & e_{-\lambda} \end{pmatrix}. \tag{3.4}$$

Here $c_j \neq 0$ for all $j = 1, 2, \dots, s$ ($s := \lceil \frac{\lambda+\nu}{h} \rceil - 1$), $-\lambda < -\nu < 0 < \alpha < \lambda$, $\lambda = \alpha + \nu$, $h > \nu$, and the number $\frac{\lambda}{h} = \frac{\alpha+\nu}{h}$ is irrational.

Applying the Portuguese transformation and letting below $n_1, \dots, n_{s+1} \in \mathbb{Z}_+$, we obtain

$$g_2^+ = \sum_{-\lambda \leq \sum_{j=1}^{s} n_j(jh) + n_{s+1}(\alpha+\nu) - \lambda < \nu} c_{-1}^{-1} \frac{(n_1 + \cdots + n_{s+1})!}{n_1! \dots n_{s+1}!} \left(-\frac{c_{s+1}}{c_{-1}} \right)^{n_{s+1}}$$
$$\times \prod_{j=1}^{s} \left(\frac{c_j}{c_{-1}} \right)^{n_j} e_{\sum_{j=1}^{s} jn_jh + n_{s+1}(\alpha+\nu)},$$

$$\omega_- = \sum_{0 \leq \sum_{j=1}^{s} jn_j \leq \lfloor \frac{\lambda-\nu}{h} \rfloor} c_{-1}^{-1} \frac{(n_1 + \cdots + n_s)!}{n_1! \dots n_s!} \prod_{j=1}^{s} \left(\frac{c_j}{c_{-1}} \right)^{n_j} e_{\sum_{j=1}^{s} jn_jh - \lambda + \nu}. \tag{3.5}$$

Let

$$k := \left\lfloor \frac{\lambda-\nu}{h} \right\rfloor + 1, \quad s := \left\lceil \frac{\lambda+\nu}{h} \right\rceil - 1\,.$$

We again have three possibilities:

$$\text{(i) } s = k-1, \quad \text{(ii) } s = k, \quad \text{(iii) } s = k+1.$$

Respectively,

$$f_2 = \begin{cases} -\dfrac{c_1}{c_{-1}^2} & \text{in case (i),} \\ -\dfrac{c_1}{c_{-1}^2} + a_{-1}e_{kh-\lambda} & \text{in case (ii),} \\ a_{-1}e_{kh-\lambda} - a_0 + a_1 e_{(k+1)h-\lambda} & \text{in case (iii).} \end{cases}$$

Here

$$\begin{aligned} a_{-1} &= \sum_{\sum_{j=1}^s jn_j = k} c_{-1}^{-1}\frac{(n_1+\cdots+n_s)!}{n_1!\dots n_s!}\prod_{j=1}^s \left(\frac{c_j}{c_{-1}}\right)^{n_j}, \\ a_0 &= \frac{c_1}{c_{-1}^2}, \\ a_1 &= \sum_{\sum_{j=1}^s jn_j = k+1} c_{-1}^{-1}\frac{(n_1+\cdots+n_s)!}{n_1!\dots n_s!}\prod_{j=1}^s \left(\frac{c_j}{c_{-1}}\right)^{n_j}. \end{aligned}$$

Then

$$G^+ = \begin{pmatrix} e_\lambda & 0 \\ c_{-1}e_{-\nu} - \sum_{j=1}^s c_j\, e_{-\nu+jh} + c_{s+1}e_\alpha & e_{-\lambda} \end{pmatrix} G^- \tag{3.6}$$

and

$$G^- = \begin{pmatrix} \varphi_- - \omega_- g_- & \widetilde{\varphi}_- - \omega_- \widetilde{g}_- \\ g_- & \widetilde{g}_- \end{pmatrix}, \tag{3.7}$$

where ω_- is given by (3.5) and

$$\begin{pmatrix} g_- & \widetilde{g}_- \\ \varphi_- & \widetilde{\varphi}_- \end{pmatrix} = \begin{pmatrix} 1 & \frac{c_{-1}^2}{c_1}\, e_{-\nu} \\ 0 & 1 \end{pmatrix} \tag{3.8}$$

in case (i),

$$\begin{pmatrix} g_- & \widetilde{g}_- \\ \varphi_- & \widetilde{\varphi}_- \end{pmatrix} = \begin{pmatrix} 1 & \frac{c_{-1}^2}{c_1} \sum_{r=0}^{\lfloor \frac{\nu}{kh-\lambda} \rfloor} \left(\frac{a_{-1}c_{-1}^2}{c_1}\right)^r e_{r(kh-\lambda)-\nu} \\ 0 & 1 \end{pmatrix} \tag{3.9}$$

in case (ii) with $kh-\lambda > 0$,

$$\begin{pmatrix} g_- & \widetilde{g}_- \\ \varphi_- & \widetilde{\varphi}_- \end{pmatrix} = \begin{pmatrix} -\frac{c_{-1}^2}{c_1}\, e_{-\nu} & \sum_{r=0}^{\lfloor \frac{\nu}{\lambda-kh} \rfloor} \left(\frac{a_{-1}c_{-1}^2}{c_1}\right)^r e_{r(kh-\lambda)} \\ -1 + \frac{a_{-1}c_{-1}^2}{c_1}\, e_{kh-\lambda} & -\frac{c_1}{c_{-1}^2}\left(\frac{a_{-1}c_{-1}^2}{c_1}\right)^{\lfloor \frac{\nu}{\lambda-kh} \rfloor + 1} e_{(\lfloor \frac{\nu}{\lambda-kh} \rfloor+1)(kh-\lambda)+\nu} \end{pmatrix} \tag{3.10}$$

in case (ii) with $kh-\lambda<0$. Finally, in case (iii)

$$\begin{aligned}
g_- &= 1+\sum_{j=0}^{m}\sum_{r=\lfloor\frac{j\beta}{\alpha}\rfloor+1}^{\lceil\frac{j\beta+\nu}{\alpha}\rceil-1}\left(\frac{a_1c_{-1}^2}{c_1}\right)^j\left(\frac{a_{-1}c_{-1}^2}{c_1}\right)^r e_{j\beta-r\alpha},\\
\varphi_- &= -\frac{c_1}{c_{-1}^2}\sum_{j=0}^{m}\left(\frac{a_1c_{-1}^2}{c_1}\right)^j\left(\frac{a_{-1}c_{-1}^2}{c_1}\right)^{\lceil\frac{j\beta+\nu}{\alpha}\rceil} e_{j\beta+\nu-\lceil\frac{j\beta+\nu}{\alpha}\rceil\alpha},\\
\widetilde{g}_- &= e_{-\nu}+\sum_{j=1}^{n}\sum_{r=\lfloor\frac{j\beta-\nu}{\alpha}\rfloor+1}^{\lfloor\frac{j\beta}{\alpha}\rfloor}\left(\frac{a_1c_{-1}^2}{c_1}\right)^j\left(\frac{a_{-1}c_{-1}^2}{c_1}\right)^r e_{j\beta-r\alpha-\nu},\\
\widetilde{\varphi}_- &= \frac{c_1}{c_{-1}^2}-a_{-1}\,e_{-\alpha}-\frac{c_1}{c_{-1}^2}\sum_{j=1}^{n}\left(\frac{a_1c_{-1}^2}{c_1}\right)^j\left(\frac{a_{-1}c_{-1}^2}{c_1}\right)^{\lfloor\frac{j\beta}{\alpha}\rfloor+1} e_{j\beta-(\lfloor\frac{j\beta}{\alpha}\rfloor+1)\alpha},
\end{aligned}\tag{3.11}$$

with

$$\alpha:=\lambda-kh,\quad \beta:=(k+1)h-\lambda,\tag{3.12}$$

and with the minimal numbers $m\in\mathbb{Z}_+$ and $n\in\mathbb{N}$ such that

$$\begin{aligned}
\left\lceil\frac{(m+1)((k+1)h-\lambda)}{\lambda-kh}\right\rceil &= \left\lceil\frac{m((k+1)h-\lambda)+\nu}{\lambda-kh}\right\rceil,\\
\left\lfloor\frac{(n+1)((k+1)h-\lambda)-\nu}{\lambda-kh}\right\rfloor &= \left\lfloor\frac{n((k+1)h-\lambda)}{\lambda-kh}\right\rfloor.
\end{aligned}\tag{3.13}$$

Thus, we have proved the following result.

Theorem 3.3. *If $c_j\neq 0$ for all $j=1,2,\dots,\lceil\frac{\lambda+\nu}{h}\rceil-1$, $-\lambda<-\nu<0<\alpha<\lambda$, $\lambda=\alpha+\nu$, $h>\nu$, and the number $\frac{\lambda}{h}$ is irrational, then the matrix function* (3.4) *admits the canonical APP factorization $G=G^+(G^-)^{-1}$, where G^+ and G^- are given by* (3.6) *and* (3.7) *and the functions ω_- and $g_-,\varphi_-,\widetilde{g}_-,\widetilde{\varphi}_-\in APP_-$ are defined by* (3.5) *and* (3.8)–(3.13), *respectively.*

References

[1] M.A. Bastos, Yu.I. Karlovich, I.M. Spitkovsky, and P.M. Tishin, *On a new algorithm for almost periodic factorization*, Recent progress in operator theory (Regensburg, 1995) (I. Gohberg, R. Mennicken, and C. Tretter, eds.), Operator Theory: Advances and Applications, vol. 103, Birkhäuser Verlag, Basel and Boston, 1998, pp. 53–74.

[2] A. Böttcher, Yu.I. Karlovich, and I.M. Spitkovsky, *Convolution operators and factorization of almost periodic matrix functions*, Operator Theory: Advances and Applications, vol. 131, Birkhäuser Verlag, Basel and Boston, 2002.

[3] M.C. Câmara, A.F. dos Santos, and M.C. Martins, *A new approach to factorization of a class of almost-periodic triangular symbols and related Riemann-Hilbert problems*, J. Funct. Anal. **235** (2006), no. 2, 559–592.

[4] M.C. Câmara, Yu.I. Karlovich, and I.M. Spitkovsky, *Almost periodic factorization of some triangular matrix functions*, Operator Theory: Advances and Applications, vol. 190, Birkhäuser Verlag, Basel and Boston, 2009, pp. 171–190.

[5] I.P. Cornfeld, S.V. Fomin, and Ya.G. Sinaĭ, *Ergodic theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 245, Springer-Verlag, New York, 1982, Translated from the Russian by A. B. Sosinskiĭ.

[6] Yu.I. Karlovich and I.M. Spitkovsky, *Factorization of almost periodic matrix functions*, J. Math. Anal. Appl. **193** (1995), 209–232.

[7] Yu.I. Karlovich, I.M. Spitkovsky, and R.A. Walker, *Almost periodic factorization of block triangular matrix functions revisited*, Linear Algebra Appl. **293** (1999), 199–232.

[8] D. Quint, L. Rodman, and I.M. Spitkovsky, *New cases of almost periodic factorization of triangular matrix functions*, Michigan Math. J. **45** (1998), 73–102.

[9] A. Rastogi, L. Rodman, and I.M. Spitkovsky, *Almost periodic factorization of* 2×2 *matrix functions: New cases of off diagonal spectrum*, Operator Theory: Advances and Applications **202**, Birkhäuser Verlag, Basel and Boston, 2010, pp. 469–487 (this volume).

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Received: March 28, 2009
Accepted: July 21, 2009

Operator Theory:
Advances and Applications, Vol. 202, 355–363

Revisit to a Theorem of Wogen

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Abstract. In this note we provide a new proof of a theorem of Wogen on the boundedness criterion for composition operators on Hardy space $H^2(U^n)$ induced by holomorphic self-maps of the unit ball U^n, and then generalize it to more general inducing self-maps.

Mathematics Subject Classification (2000). Primary 47B33, Secondary 30D55, 46E15.

Keywords. Composition operator, smooth map, Bergman space, Hardy space, Carleson measure, boundedness, Wogen criterion.

1. Introduction

Let U^n be the open unit ball centered at origin in $\mathbf{C}^n$ and write $H(U^n)$ for the space of all holomorphic functions on U^n. For $0 < p < \infty$ and $\alpha > -1$, we denote by $L^p_\alpha(U^n)$ the set of all measurable functions f with

$$||f||^p_{L^p_\alpha(U^n)} = \int_{U^n} |f(z)|^p \, dV_\alpha(z) < \infty,$$

where $dV_\alpha(z) = (1 - |z|^2)^\alpha \, dV(z)$ and dV is the normalized Lebesgue volume measure on U^n. The weighted Bergman space $A^p_\alpha(U^n)$ is the space of all $f \in H(U^n)$ for which $||f||_{L^p_\alpha(U^n)} < \infty$. For the case $\alpha = 0$, we will often write $A^p_0(U^n) = A^p(U^n)$. For $0 < p < \infty$, the Hardy space $H^p(U^n)$ is the space of all $g \in H(U^n)$ for which

$$||g||^p_{H^p(U^n)} = \sup_{0<r<1} \int_{\partial U^n} |g(r\zeta)|^p d\sigma(\zeta) < \infty,$$

where $d\sigma$ is the normalized Lebesgue surface measure on the unit sphere ∂U^n. We will often use the following notations to allow unified statements:

$$A^p_{-1}(U^n) = H^p(U^n).$$

Koo is partially supported by the KRF-2008-314-C00012 and Wang is partially supported by the NSF-10671147 of China.
Communicated by J.A. Ball.

Let Φ be a map from U^n into itself, then Φ induces the composition operator C_Φ, defined by

$$C_\Phi f = f \circ \Phi.$$

When Φ is a holomorphic self-map of U^n with $n \geq 2$, W. Wogen [11] has given a necessary and sufficient condition for a smooth map $\Phi \in C^3(\overline{U^n})$ to induce a bounded composition operator C_Φ on the Hardy spaces $H^p(U^n)$ for $0 < p < \infty$, which recently has been generalized to weighted Bergman spaces $A^p_\alpha(U^n)$ in [7].

To state Wogen's criterion, we need to introduce some notations. For a fixed integer $n \geq 2$, $\eta \in \partial U^n$ and a smooth map $\Phi : U^n \to U^n$, let

$$\Phi_\eta(z) = \langle \Phi(z), \eta \rangle\,, \quad z \in U^n,$$

where $\langle z, w \rangle$ is the usual Hermitian inner product of $z, w \in \mathbf{C}^n$. Let $D_\zeta = \sum_{j=1}^n \zeta_j \frac{\partial}{\partial z_j}$ be the complex direction derivative in the $\zeta = (\zeta_1, \cdots, \zeta_n) \in \partial U^n$ direction. The following is the Wogen's result and its generalization to the weighted Bergman spaces duo to Koo and Smith.

Theorem 1.1. *Let $\alpha \geq -1$, $0 < p < \infty$ and Φ be a holomorphic self-map of U^n with $\Phi \in C^3(\overline{U^n})$. Then, C_Φ is bounded on $A^p_\alpha(U^n)$ if and only if*

$$D_\zeta \Phi_\eta(\zeta) > |D_{\tau\tau}\Phi_\eta(\zeta)| \tag{1.1}$$

for all $\zeta, \eta, \tau \in \partial U^n$ with $\Phi(\zeta) = \eta$ and $\langle \zeta, \tau \rangle = 0$.

We say Φ satisfies "Wogen's condition " if (1.1) holds. The proof of the necessity of Theorem 1.1 is standard by some local analysis of the mapping properties of Φ at points of ∂U^n which map to ∂U^n. The sufficiency is hard and based on an inequality that gives a local Lipschitz invertibility condition at certain points of ∂U^n. (See [11] and [7].) In this note we use the compactness argument to reprove the sufficiency of Theorem 1.1 and then apply it to non-holomorphic inducing symbols. Since (1.1) is strictly inequality at $\zeta \in \Phi^{-1}(\partial U^n)$ and $\Phi^{-1}(\partial U^n)$ is compact, similar strictly inequality continues to hold in a neighborhood of $\Phi^{-1}(\partial U^n)$. Using this we prove the key lemma, Lemma 3.2 and then reprove the sufficiency of Theorem 1.1.

In the rest of the paper we often use the letters C and c, depending only on the allowed parameters, to denote various positive constants which may change at each occurrence. For nonnegative quantities X and Y, we often write $X \lesssim Y$ or $Y \gtrsim X$ if X is dominated by Y times some inessential positive constant. Also, we write $X \approx Y$ if $X \lesssim Y \lesssim X$.

2. Carleson measures

For every $\zeta \in \partial U^n$ and $0 < \delta < 1$, let $S(\zeta, \delta)$ and $\widehat{S}(\zeta, \delta)$ be the Carleson boxes on U^n and $\overline{U^n}$ defined by

$$S(\zeta, \delta) = \{z \in U^n : |1 - \langle z, \zeta \rangle\,| < \delta\},$$

and

$$\widehat{S}(\zeta,\delta) = \{z \in \overline{U^n} : |1 - \langle z,\zeta\rangle| < \delta\},$$

respectively. For a vector-valued function $\Phi : U^n \to U^n$, which is continuous on $\overline{U^n}$, we have the following change of variables formula [6]:

$$\int_{U^n} |f \circ \Phi|^p \, dV_\alpha = \int_{U^n} |f|^p d\mu_\alpha, \quad \forall f \in L^p_\alpha(U^n),$$
$$\int_{\partial U^n} |f \circ \Phi|^p \, d\sigma = \int_{\overline{U^n}} |f|^p d\mu^*, \quad \forall f \in L^p_\alpha(\overline{U^n}),$$

where the Borel measures μ_α on U^n and μ^* on $\overline{U^n}$ are defined by $\mu_\alpha(E) = V_\alpha(\Phi^{-1}(E))$ and $\mu^*(F) = \sigma(\Phi^{-1}(F))$, respectively.

Therefore, the usual Carleson measure type characterizations also hold for the non-holomorphic map $\Phi : U^n \to U^n$ which is continuous on $\overline{U^n}$. Notice that every $f \in H^p(U^n)$ has the radial limit almost everywhere on ∂U^n and the radial limit function belongs to $L^p(\partial U^n)$ (refer to [10]). We use the notation $f(\zeta)$ for the radial limit of f at ζ if it exists.

Proposition 2.1. *Let $0 < p < \infty$ and $\alpha, \beta > -1$. Suppose that $\Phi : U^n \to U^n$ is a map which is continuous on $\overline{U^n}$. Define the Borel measure μ_β on U^n by $\mu_\beta(E) = V_\beta(\Phi^{-1}(E))$ and the Borel measure μ^* on $\overline{U^n}$ by $\mu^*(F) = \sigma(\Phi^{-1}(F))$. Then*

(1) $||f \circ \Phi||_{L^p(\partial U^n)} \le C||f||_{H^p(U^n)}$ *for all* $f \in H^p(U^n)$ *and some* $C > 0$ *if and only if there is some* $C_1 > 0$ *such that*

$$\mu^*\left(\widehat{S}(\zeta,\delta)\right) \le C_1\delta^n \text{ for all } \zeta \in \partial U^n \text{ and } 0 < \delta < 1.$$

(2) $||f \circ \Phi||_{L^p_\beta(U^n)} \le C||f||_{H^p(U^n)}$ *for all* $f \in H^p(U^n)$ *and some* $C > 0$ *if and only if there is some* $C_1 > 0$ *such that*

$$\mu_\beta\left(S(\zeta,\delta)\right) \le C_1\delta^n \text{ for all } \zeta \in \partial U^n \text{ and } 0 < \delta < 1.$$

(3) $||f \circ \Phi||_{L^p_\beta(U^n)} \le C||f||_{A^p_\alpha(U^n)}$ *for all* $f \in A^p_\alpha(U^n)$ *and some* $C > 0$ *if and only if there is some* $C_1 > 0$ *such that*

$$\mu_\beta\left(S(\zeta,\delta)\right) \le C_1\delta^{n+1+\alpha} \text{ for all } \zeta \in \partial U^n \text{ and } 0 < \delta < 1.$$

These types of embedding characterizations are called Carleson measure criteria and well-known for the holomorphic self-maps Φ. But due to the change of variables formula above, the same proof works for the non-holomorphic map Φ which is continuous on $\overline{U^n}$. Carleson measure criterion was first proved by Carleson in [2] for a general Borel measure μ when $n = 1$ and $\alpha = \beta = 0$. For a proof of Proposition 2.1 we refer to [4]. Part (1) of Proposition 2.1 is [4, Theorem 3.35], and the proof there also works to prove part (2). When $\alpha = \beta$, part (3) is [4, Theorem 3.37] and the same proof again works for $\alpha \neq \beta$.

3. A new proof for the sufficiency of Wogen's theorem

For $\delta > 0$ and a smooth map $\Phi \in C^3(\overline{U^n})$ with $\Phi(U^n) \subset U^n$, let

$$\begin{aligned} K &= \Phi^{-1}(\partial U^n), \\ K_\delta &= \cup_{\zeta \in K}\{(z,w) \in \overline{U^n} \times \overline{U^n} : |z-\zeta| < \delta,\ |w - \Phi(\zeta)| < \delta\}, \\ U_\delta &= \{z \in \overline{U^n} : d(z,K) < \delta\}, \\ W_\delta &= \{z \in \overline{U^n} : d(z, \Phi(K)) < \delta\}. \end{aligned} \tag{3.1}$$

Here $d(z,K) = \min_{\zeta \in K} d(z,\zeta)$ and $d(z,\zeta) = |1 - \langle z, \zeta\rangle|$.

Lemma 3.1. *Let $\Phi : U^n \to U^n$ with $\Phi \in C^3(\overline{U^n})$ and satisfy Wogen's condition. Then, there exits $\delta_0 > 0$ such that Wogen's condition also holds for every pair $(\zeta, \eta) \in K_{\delta_0}$.*

Proof. For a fixed $w \in \overline{U^n}$ and any $x \in \partial U^n$, we define g_w and h_w by

$$\begin{aligned} g_w(x, \cdot) &= |D_x \langle \Phi(\cdot),\ w\rangle|, \\ h_w(x, \cdot) &= \sup_{\xi \in x^\perp \cap \partial U^n} |D_{\xi\xi} \langle \Phi(\cdot),\ w\rangle|, \end{aligned}$$

respectively. Here $x^\perp$ is the set of all vector $v \in \mathbf{C}^n$ satisfying $< v, x >= 0$. Let $\hat{z} = \Phi(z)$. Then Wogen's condition is equivalent to

$$g_{\hat{\zeta}}(\zeta,\zeta) > h_{\hat{\zeta}}(\zeta,\zeta) \quad \text{for all } \ \zeta \in K.$$

Note that $g_w(x,z)$ and $h_w(x,z)$ are continuous in x and z since $\Phi \in C^3(\overline{U^n})$, and since K is compact subset of ∂U^n, the above condition also holds in some neighborhood O of $K \times \Phi(K)$ in $U_\delta \times W_\delta$, i.e.,

$$g_w(x,z) > h_w(x,z) \qquad \text{for all } \ (x,z) \in O.$$

Now choose $\delta_0 > 0$ sufficiently small so that $K_{\delta_0} \subset O$, which immediately completes the proof. ∎

Lemma 3.2. *Let $\Phi : U^n \to U^n$ with $\Phi \in C^3(\overline{U^n})$ and satisfy Wogen's condition. Then, there exist $\delta_0 > 0$ and $C > 0$ such that, if $\eta \in W_{\delta_0} \cap \partial U^n$ and $\zeta \in U_{\delta_0}$ is a local minimum point for $|1 - \Phi_\eta(z)|$ with $|\eta - \Phi(\zeta)| < \delta_0$, then $\zeta \in \partial U^n$ and for all $0 < \delta < \delta_0$*

$$\Phi[S(\zeta, \delta_0) \setminus S(\zeta, C\delta)] \cap S(\eta, \delta) = \emptyset.$$

Moreover, for $|z - \zeta| < \delta_0$

$$|1 - \Phi_\eta(z)| \approx |1 - \Phi_\eta(\zeta)| + |1 - \langle z, \zeta\rangle|.$$

Proof. We will choose δ_0 small enough so that our local Taylor polynomial expansion of Φ_η holds with the uniform control over the coefficients up to the second-order terms and the remainder terms, which is possible since $\Phi \in C^3(\overline{U^n})$. Using Lemma 3.1, further choose $\delta_0 > 0$ sufficiently small such that the condition (1.1) holds for all $(\zeta, \eta) \in K_{\delta_0}$.

Fix $\eta \in W_{\delta_0} \cap \partial U^n$ and let $\zeta \in U_{\delta_0}$ be a local minimum point for $|1 - \Phi_\eta(z)|$ with $|\eta - \Phi(\zeta)| < \delta_0$. Then, $(\zeta, \eta) \in K_{\delta_0}$ and so (1.1) holds for (ζ, η). Since $D_\zeta \Phi_\eta(\zeta) \neq 0$, we see that Φ_η is an open map near ζ which implies that $\zeta \in \partial U^n$.

We may assume that $\zeta = e_1 =: (1, 0, \ldots, 0)$ by some unitary transformations. Since e_1 is a local minimum point for $|1 - \Phi_\eta(z)|$, it is easy to see that $\frac{\partial \Phi_\eta(e_1)}{\partial z_j} = 0$ for $j = 2, \ldots, n$. By Taylor expansion, we have

$$\Phi_\eta(z) = \Phi_\eta(e_1) + a_1(z_1 - 1) + \sum_{i=2}^{n} \sum_{j=2}^{n} a_{ij} z_i z_j / 2 + O(|1 - z_1|^{3/2}).$$

Then after another unitary transformation about $(z_2, \ldots, z_n)$, we have

$$\Phi_\eta(z) = \Phi_\eta(e_1) + a_1(z_1 - 1) + \sum_{j=2}^{n} a'_{jj} z_j^2 / 2 + O(|1 - z_1|^{3/2}). \tag{3.2}$$

Since $|z_2|^2 + \cdots + |z_n|^2 < 2|1 - z_1|$, it is straightforward from (3.2) to see that

$$|1 - \Phi_\eta(z)| \lesssim |1 - z_1| + |1 - \Phi_\eta(e_1)|. \tag{3.3}$$

Since $\Phi \in C^3(\overline{U^n})$ and Wogen's condition holds, by choosing δ_0 sufficiently small, there exists $\epsilon > 0$ independent of η and ζ such that

$$a := \max_{2 \le j \le n} \{|a'_{jj}|\} < a_1 - \epsilon.$$

Thus we have

$$\begin{aligned}
|1 - \Phi_\eta(z)| &= |1 - \Phi_\eta(e_1) - a_1(z_1 - 1) - \sum_{j=2}^{n} a'_{jj} z_j^2 / 2 - O(|1 - z_1|^{3/2})| \\
&\ge |a_1(1 - z_1)| - |1 - \Phi_\eta(e_1)| - a/2 \sum_{j=2}^{n} |z_j|^2 - O(|1 - z_1|^{3/2}) \\
&\ge a_1|1 - z_1| - a/2(1 - |z_1|^2) - |1 - \Phi_\eta(e_1)| - O(|1 - z_1|^{3/2}) \\
&\ge \epsilon|1 - z_1| - |1 - \Phi_\eta(e_1)| - O(|1 - z_1|^{3/2}) \\
&\ge c|1 - z_1| - |1 - \Phi_\eta(e_1)| \\
&= c[|1 - z_1| + |1 - \Phi_\eta(e_1)|] - (c+1)|1 - \Phi_\eta(e_1)|.
\end{aligned}$$

Since e_1 is a local minimum point for $|1 - \Phi_\eta(z)|$, $|1 - \Phi_\eta(z)| \ge |1 - \Phi_\eta(e_1)|$ when z is near z_1. Then

$$|1 - \Phi_\eta(z)| \gtrsim |1 - z_1| + |1 - \Phi_\eta(e_1)|. \tag{3.4}$$

From (3.4) it is easy to see that $\Phi[S(\zeta, \delta_0) \setminus S(\zeta, C\delta)] \cap S(\eta, \delta) = \emptyset$ for all $0 < \delta < \delta_0$ and some $C > 0$. More precisely, suppose $z \in S(\zeta, \delta_0) \setminus S(\zeta, C\delta)$, then

$$|1 - \Phi_\eta(z)| \gtrsim |1 - \Phi_\eta(\zeta)| + |1 - \langle z, \zeta \rangle| \ge |1 - \langle z, \zeta \rangle| \ge C\delta.$$

Therefore, we can choose $C > 0$ such that $\Phi(z) \notin S(\eta, \delta)$ when $z \in S(\zeta, \delta_0) \setminus S(\zeta, C\delta)$. $\square$

Lemma 3.2 is a key lemma for the sufficiency of Theorem 1.1. We now reconstruct the sufficiency proof of Wogen's result(Theorem 1.1) using Lemma 3.2.

Proof of the sufficiency of Theorem 1.1. We complete the proof by verifying the Carleson condition

$$\mu_\alpha(S(\eta,\delta)) = O(\delta^{n+\alpha+1})$$

for all $\eta \in \partial U^n$ and $0 < \delta < 1$ when $\alpha > -1$, and

$$\mu^*\left(\widehat{S}(\eta,\delta)\right) \le C_1\delta^n$$

for all $\eta \in \partial U^n$ and $0 < \delta < 1$ when $\alpha = -1$ by Proposition 2.1. Here we only check it for $\alpha > -1$, the other case can be checked similarly. Clearly it is enough to check this for all $0 < \delta < \delta_0$ and $\eta \in \partial U^n$ which is close to $\Phi(K)$, because the other case is trivial. Here δ_0 is the number which satisfies Lemma 3.2. Let K, U_δ and W_δ be as in (3.1). Suppose $\eta \in W_{\delta_0} \cap \partial U^n$, and let O_j be one of the components of $\Phi^{-1}(S(\eta,\delta_0))$ which also intersects with $\Phi^{-1}(S(\eta,\delta_0/2))$. Let ζ_j satisfy

$$|1 - \Phi_\eta(\zeta_j)| = \min_{z\in\overline{O_j}}\{|1-\Phi_\eta(z)|\}.$$

By shrinking δ_0 if necessary we may assume condition (1) holds at ζ_j, and thus Φ_η is an open mapping near ζ_j if $\zeta_j \in U^n$. Since ζ_j is a local minimum point for $|1-\Phi_\eta(z)|$, we have $\zeta_j \in \partial U^n$.

Next, we show that there is a finite upper bound M,which is independent of $\eta \in W_{\delta_0}$, on the number of the components of $\Phi^{-1}(S(\eta,\delta_0))$ which also intersect with $\Phi^{-1}(S(\eta,\delta_0/2))$. To see this, note that by Lemma 3.2, there is $c > 0$ independent of $\eta \in W_{\delta_0}$ such that $\Phi(S(\zeta_j, c\delta_0)) \subset S(\eta,\delta_0)$, then $S(\zeta_j, c\delta_0) \subset O_j$ by the connectivity of O_j. Therefore, the number of components has a finite upper bound $M < \infty$ since $\sum_{j=1}^M V_\alpha(S(\zeta_j, c\delta)) \approx M\delta^{n+1+\alpha} \lesssim 1$.

Now fix such a component O_j as above. Then, by Lemma 3.2, there is $C > 0$ independent of η such that

$$V_j := O_j \cap \Phi^{-1}(S(\eta,\delta)) \subset S(\zeta_j, C\delta). \tag{3.5}$$

Since $V_\alpha(S(\zeta_j,\delta)) \approx \delta^{n+1+\alpha}$, we can easily verify the Carleson measure condition $V_\alpha(\Phi^{-1}(S(\eta,\delta))) \lesssim \delta^{n+1+\alpha}$ since the number of the components O_j has a finite upper bound M which is independent of $\eta \in W_{\delta_0}$.

4. General inducing maps

Let B^n be the open unit ball centered at origin in $\mathbf{R}^n$ and write $h(B^n)$ for the space of all harmonic functions on B^n. For $0 < p < \infty$ and $\alpha > -1$, with Ω either U^n or B^n, we let $L^p_\alpha(\Omega)$ be the space of all measurable functions f on Ω such that

$$||f||^p_{L^p_\alpha(\Omega)} = \int_\Omega |f(z)|^p\, dV_\alpha(z) < \infty,$$

where $dV_\alpha(z) = (1-|z|^2)^\alpha\, dV(z)$ and dV is the normalized Lebesgue volume measure on Ω. The weighted harmonic Bergman space $b^p_\alpha(\Omega)$ is the space of all $f \in$

$h(\Omega)$ for which $||f||_{L^p_\alpha(\Omega)} < \infty$, here we identify $h(U^n)$ with $h(B^{2n})$ in the natural way. For the case $\alpha = 0$, we will often write $b^p(\Omega) = b^p_0(\Omega)$ for simplification.

With Ω either U^n or B^n, let Ψ be a map from Ω into itself, then, Ψ also induces the composition operator C_Ψ, defined by

$$C_\Psi f = f \circ \Psi.$$

Here, we assume Ψ is a smooth but not necessarily holomorphic self-map of Ω and find a necessary and sufficient condition on Ψ such that

$$\int_\Omega |C_\Psi f(z)|^p \, dV_\alpha(z) \leq C \int_\Omega |f(z)|^p \, dV_\alpha(z), \tag{4.1}$$

for some constant $C > 0$ and all $f \in X$, where $X = A^p_\alpha(U^n)$ when $\Omega = U^n$, and $X = b^p_\alpha(B^n)$ when $\Omega = B^n$.

This is motivated by the map $\Psi(z_1, z_2) = (z_1, 0)$ because it follows from Wogen's Theorem that this map induces a bounded operator C_Ψ on $A^p_\alpha(U^2)$. But for the same map Ψ, the harmonic counterpart does not hold. In fact, if we take $f_k(z) = \frac{1}{|(z_1,z_2)-(1+1/k,0)|^2}$, then, by a direct calculation, we can show that $f_k(z)$ is harmonic and belongs to $b^2(U^2)$, but

$$\lim_{k\to\infty} \frac{||C_\Psi f_k||_{L^2(U^2)}}{||f_k||_{b^2(U^2)}} = \infty.$$

This raises a natural question:

What is the condition for (4.1) *to hold with* $X = b^p_\alpha(B^n)$ *?*

If Ψ is not holomorphic, we can not expect $C_\Psi f$ to be holomorphic even if f is, and $C_\Psi f$ may not be harmonic even Ψ and f are harmonic. Therefore, if we do not impose the analyticity condition of the symbol map $\Psi : \Omega \to \Omega$, then we lose the analyticity or the harmonicity of $C_\Psi f$, but we have much more flexibility for the choice of the symbol map Ψ.

In [9], we provide the following characterizations for smooth self-map of Ω to induce a bounded composition operator on the defined spaces. See [9] for details.

The first is for the harmonic spaces.

Theorem 4.1. *Let* $0 < p < \infty$, $\alpha > -1$ *and* $\Psi : B^n \to B^n$ *be a map with* $\Psi \in C^2(\overline{B^n})$. *Then there exists a constant* $C > 0$ *such that*

$$||f \circ \Psi||_{L^p_\alpha(B^n)} \leq C||f||_{b^p_\alpha(B^n)}$$

for all $f \in b^p_\alpha(B^n)$ *if and only if*

$$J_\Psi(\zeta) \neq 0 \quad \textit{for all} \quad \zeta \in \Psi^{-1}(\partial B^n).$$

Here $J_\Psi(\zeta)$ *is the Jacobian of* Ψ *at* ζ.

The second is for the holomorphic spaces.

Theorem 4.2. *Let* $0 < p < \infty$*,* $\alpha > -1$ *and* $\Phi : U^n \to U^n$ *be a map with* $\Phi \in C^4(\overline{U^n})$*. Then there exists a constant* $C > 0$ *such that*

$$||f \circ \Phi||_{L^p_\alpha(U^n)} \leq C||f||_{A^p_\alpha(U^n)}$$

for all $f \in A^p_\alpha(U^n)$ *if and only if* Φ *satisfies:*

(1) Rank $M_{\Phi_\eta}(\zeta) = 2$ *for all* $\zeta, \eta \in \partial U^n$ *with* $\Phi(\zeta) = \eta$ *and*

(2) $\widetilde{D}_\zeta \Re \Phi_\eta(\zeta) > \widetilde{D}_{\tau\tau} \Re \Phi_\eta(\zeta)$ *for all* $\zeta, \eta, \tau \in \partial U^n$ *with* $\Phi(\zeta) = \eta$ *and* $\tau \in \zeta^\perp$*.*

Here, $\Phi_\eta(\cdot) = \langle \Phi(\cdot), \eta\rangle$ is the Hermitian inner product of $\Phi(\cdot)$ and η, $M_{\Phi_\eta}(\zeta)$ is the real Jacobi matrix of this map at ζ, $\zeta^\perp$ is the subspace of $\mathbf{R}^{2n}$ which is orthogonal to $(x_1, x_2, \ldots, x_{2n})$ with $(x_1 + ix_2, \ldots, x_{2n-1} + ix_{2n}) = \zeta$, and $\widetilde{D}_\zeta$ is the real directional derivative in the ζ direction considered as a real vector in ∂B^{2n}.

As shown in [9], our necessary and sufficient condition in Theorem 4.2 is equivalent to Wogen's original condition in Theorem 1.1 when Φ is a holomorphic self-map of U^n and of class $C^4(\overline{U^n})$.

Moreover, in [9] we also show that there are jump phenomena in the optimal target spaces of Theorems 4.1 and 4.2: if the inducing self-map Ψ is smooth enough and C_Ψ does not map $A^p_\alpha(\Omega)$ (resp. $b^p_\alpha(\Omega)$) into $L^p_\alpha(\Omega)$, then it does not map $A^p_\alpha(\Omega)$ (resp. $b^p_\alpha(\Omega)$) into some larger spaces $L^p_\beta(\Omega)$ for all $\alpha < \beta < \alpha + \epsilon_0$. Where $\epsilon_0 = \min\{1/4, \alpha + 1\}$ for the homomorphic spaces, and $\epsilon_0 = \min\{1/2, \alpha + 1\}$ for the harmonic spaces, which are all are sharp. This contrasts with the case of holomorphic spaces with holomorphic inducing symbol Φ, where the jumps is always $1/4$ (refer to [7]).

[8] is adapted for the proof of Theorem 4.1. Here, we outline the proof of Theorem 4.2. The necessity is routine by local analysis of Φ on the boundary ∂U^n. See [9] for details. For the sufficiency, unfortunately, Lemma 3.2 does not hold for a non-holomorphic map. We will need a corresponding version of Lemma 3.2 which may be applied to non-holomorphic symbols. To this end, we replace $|1 - \Phi_\eta(z)|$ by $\Re(1 - \Phi_\eta(z))$. Then we can have some similar mapping properties, which are included in the following lemma. With the following lemma, one can complete the proof following the routine scheme of Wogen's theorem.

Lemma 4.3. *Let* $\Phi : U^n \to U^n$ *with* $\Phi \in C^4(\overline{U^n})$ *and satisfy the condition in Theorem* 4.2*. Then, there exist* $\delta_0 > 0$ *and* $C > 0$ *such if* $\eta \in W_{\delta_0} \cap \partial U^n$ *and* $\zeta \in U_{\delta_0}$ *is a local minimum point for* $\Re(1 - \Phi_\eta(z))$ *with* $|\eta - \Phi(\zeta)| < \delta_0$*, then* $\zeta \in \partial U^n$ *and for all* $0 < \delta < \delta_0$*,*

$$\Phi[\widetilde{S}(\zeta, \delta_0) \setminus \widetilde{S}(\zeta, C\delta)] \cap S(\eta, \delta) = \emptyset.$$

Moreover, for $|z - \zeta| < \delta_0$*,*

$$|1 - \Phi_\eta(z)| \leq C[d(\Phi(\zeta), \eta) + \widetilde{d}(z, \zeta)].$$

Here, $\widetilde{S}(\zeta, \delta)$ is a "twisted" Carleson box and $\widetilde{d}$ is a "twisted" distance (refer to [9] for the details).

References

[1] S. Axler, P. Bourdon and W. Ramey, *Harmonic function theory*, Springer-Verlag, New York, 2001.

[2] L. Carleson, *Interpolation by bounded analytic functions and the corona problem*, Ann. of Math., 76 (1962), 547–559.

[3] J. Cima and W. Wogen, *Unbounded composition operators on* $H^2(B_2)$, Proc. Amer. Math. Soc., 99(1987), 477–483.

[4] C. Cowen and B. MacCluer, *Composition operators on spaces of analytic functions*, CRC Press, Boca Raton, 1995.

[5] P. Duren, *Theory of* H^p *spaces*, Academic Press, New York, 1970.

[6] P. Halmos, *Measure Theory*, Springer-Verlag, New York, 1974.

[7] H. Koo and W. Smith, *Composition operators induced by smooth self maps of the unit ball in* $\mathbf{C}^N$, J. Math. Anal. Appl. 329(2007), 617–633.

[8] H. Koo, M. Stessin and K. Zhu, *Composition operators on the polydisc induced by smooth symbols*, J. Funt. Anal., 254(2008), 2911–2925.

[9] H. Koo and M. Wang, *Composition operators induced by smooth self-maps on the real or the complex unit balls*, J. Funct. Anal., 256(2009), 2747–2767.

[10] W. Rudin, *Function theory in the unit ball of* $\mathbf{C}^n$, Springer-Verlag, New York, 1980.

[11] W. Wogen, *The smooth mappings which preserve the Hardy space* $H^2_{B_n}$, Operator Theory: Advances Appl. 35(1988), 249–267.

[12] W. Wogen, *On geometric properties of smooth maps which preserve* $H^2_{B_n}$, Michigan Math. J., 54(2006), 301–306.

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Received: September 28, 2008.

Accepted: June 9, 2009.

Operator Theory:
Advances and Applications, Vol. 202, 365–393

Survey on the Best Constants in the Theory of One-dimensional Singular Integral Operators

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To blessed memory of my dear teacher Israel Gohberg

Abstract. A survey on the best constants in the theory of one-dimensional singular integral operators is given. Some open questions are formulated.

Mathematics Subject Classification (2000). Primary 47G10; Secondary 47A30.

Keywords. Norm, singular integral operators, local principle, matrix symbol.

1. Introduction

By singular integral operators (*sio* for short) we mean the operators $A = aI + bS_\Gamma + T$ acting on weighted spaces $L_p(\Gamma, \rho)$, where Γ is an appropriate contour in the complex plane $\mathbb{C}$, $a, b \in L_\infty(\Gamma)$, I is the unit operator, T a compact operator and S_Γ is the simplest sio

$$S_\Gamma f(t) = \frac{1}{\pi i} \int_\Gamma \frac{f(\tau)d\tau}{\tau - t} \ (t \in \Gamma). \tag{1.1}$$

We assume that S_Γ is bounded on $L_p(\Gamma, \rho)$ $(1 < p < \infty)$ with the norm

$$\|f\|_{p,\rho}^p := \int_\Gamma |f(t)|^p \rho(t)|dt|. \tag{1.2}$$

By *best constants* we mean the norm $\|A\|$ of the operator A and its essential norm

$$|A| = \inf_{K \in \mathcal{K}} \|A + K\|, \tag{1.3}$$

where $\mathcal{K}$ is the ideal of all compact operators.

The best constants play an important role in the theory and applications of Volterra operators (see [GoKr], Ch3); in scattering by unbounded obstacles in the

The work was partially supported by Retalon Inc., Toronto, ON, Canada.
Communicated by I.M. Spitkovsky.

plane (see [Bi] and [HKV]), in the spectral theory of sio (see [GK6,7], [K2,3], [Sp], [VK1,2]) and in many other areas.

Let Γ be *a circle* or *a straight line*, and $S := S_\Gamma$. The boundedness of the operator S (and hence of operators $A = aI + bS + T$) on the spaces $L_p(\Gamma)$ was first proved by M. Riesz [R], and on some weighted L_p spaces by Hardy and Littlewood [HL]. Since S acts as a unitary operator on the Hilbert space $L_2(\Gamma)$, of course $\|S\|_2 = 1$.

In [T] (see also [Z, Ch. VII, Problem 2]) and [GoKr, Sec. III,6] the following estimates were obtained for $\|S\|_p$: two positive constants N and M were found such that $Np \le \|S\|_p < Mp$ for all $p \in [2, \infty)$, and $\|S\|_q = \|S\|_p$ for $p^{-1} + q^{-1} = 1$.

The exact values of the norms $\|S\|$ for a sequence of the values of $p \in (1, \infty)$ were first obtain 40 years ago by I. Gohberg and the author [GK1,2]. Namely, it was proved that

$$\|S\|_p = \begin{cases} \cot \frac{\pi}{2p} & \text{if } \ p = 2^n \\ \tan \frac{\pi}{2p} & \text{if } \ p = 2^n/(2^n - 1) \end{cases} \quad (n = 1, 2, \dots). \tag{1.4}$$

Also in [GK1,2] the following estimates were obtained:

$$\|Q\| \ge |Q| \ge \frac{1}{\sin \pi/p}, \quad \|P\| \ge |P| \ge \frac{1}{\sin \pi/p} \quad \text{and } \|S\|_p \ge |S| \ge \gamma_p, \tag{1.5}$$

where $P := (I + S)/2;\ Q := (I - S)/2; p \in (1, \infty)$ and $\gamma_p = \cot(\pi/2p^*)$, where $p^* = \max(p, p/(p - 1))$.

It was conjectured in [GK1,2] that

1.1. Inequalities in (1.5) can be replaced by equalities.

These results gave rise to a large number of publications dedicated to the best constants and such publications continue to appear. Almost all new results related to best constants required new ideas and methods for their proofs. Some problems turned out to be very complicated. For example, it took more than 30 years of attempts of many authors to justify Conjecture 1.1 for the analytical projections P and Q (see Subsection 2.3 below). Also, it took almost 20 years to answer the question, stated by M. S. Birman as Problem 1 in [Bi], on the exact value of the norms of operators $I - A_\pm$ on $L_p(\mathbb{R}_+)$. Here the so-called *re-expansion operators* can be represented as

$$A_+ f(x) = \frac{1}{\pi} \int_{\mathbb{R}_+} \frac{2xf(t)dt}{x^2 - t^2}, \quad A_- f(x) = \frac{1}{\pi} \int_{\mathbb{R}_+} \frac{2tf(t)dt}{t^2 - x^2}. \tag{1.6}$$

(see Section 3 below). An important role in the computation of the norms of various sio is played by the *matrix symbols* for sio with *scalar* piece-wise continuous coefficients, introduced by I. Gohberg and the author (see Section 8 below). The exact constant in Simonenko's theorem on the envelope of a family of operators of local type was figured out due to a new special theorem on the covering of abstract topological spaces (see [K4]).

In this survey we summarize the main results obtained up to date, and formulate some open problems which should be interesting to solve. Some results, which were just included in the books [GK5,7] and [K3], we formulate here not only for completeness. In these books we restricted our considerations to the case of piece-wise Lyapunov contours and power weights. Here we show that the class of piece-wise Lyapunov contours can be always replaced with a larger class of piece-wise smooth contours and we do not always restrict ourselves to the power weights. For example, the local principle of computation of the essential norms of sio is formulated here for piece-wise smooth contour and general Muckenhoupt weight. The main results are described in Sections 2–6. Some interesting inequalities obtained and used for these results are discussed in Section 7. A brief information about the symmetric matrix symbols (which are used in computation of the norms of sio) is given in Section 8.

The following definitions and notation will be used in this paper.

- By a *simple contour* we mean a connected piece-wise smooth bounded curve without self-intersections. It may have a finite number of knots.
- The union of finite number of simple contours is called a *composed* contour.
- A contour Γ is called *closed* if it admits an orientation such that $\dot{\mathbb{C}} \setminus \Gamma$ is divided into two domains, D_+ and D_-, lying to the left and to the right of Γ respectively. Here $\dot{\mathbb{C}}$ denotes the *extended complex plane* $\mathbb{C} \cup \{\infty\}$.

If Γ is a closed contour, then $P_\Gamma := (I + S_\Gamma)/2$ as well as $Q_\Gamma = I - P_\Gamma$ are (analytical) projections. A weight ρ of the form

$$\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k}, \quad 1 < \beta < \infty, \quad t_j \in \Gamma \quad \text{and} \quad t_j \neq t_k \quad \text{for} \quad j \neq k \tag{1.7}$$

is called a *power* weight. If $\rho(t) = |t - t_0|^\beta$, then the norm of operator A is denoted by $\|A\|_\beta$ or, if the value of p is not clear from the context, by $\|A\|_{p,\beta}$.

Let Γ be a composed contour, and $L_p(\Gamma, \rho)$ the weighted Lebesgue space with the norm defined by (1.2). The weight ρ is a *Muckenhoupt weight* (denoted $\rho \in A_p$) if

$$\sup_{t\in\Gamma} \sup_{\epsilon>0} \frac{1}{\epsilon} \left(\int_{\Gamma_{t,\epsilon}} w(\tau)^p \, |d\tau| \right)^{1/p} \left(\int_{\Gamma_{t,\epsilon}} w(\tau)^{-q} \, |d\tau| \right)^{1/q} < \infty, \tag{1.8}$$

where $w := \rho^{1/p}$, $\Gamma_{t,\epsilon} = \{\tau \in \Gamma \colon |\tau - t| < \epsilon\}$. Condition (1.8) is necessary and sufficient for the operator S_Γ to be bounded on $L_p(\Gamma, \rho)$.

It is my pleasure to thank Ilya Spitkovsky and Igor Verbitsky for useful remarks and comments. I also thank A.Yu. Karlovich who carefully read the manuscript of this paper and made several interesting observations.

2. Γ is the unit circle

In this section we denote the unit circle by Γ_0, and write S_0, P_0, Q_0 in place of $S_{\Gamma_0}, P_{\Gamma_0}, Q_{\Gamma_0}$, respectively.

2.1. An important result which allowed to prove Conjecture 1.1 for the operator S was obtained by S.K. Pichorides [Pi], who figured out the norm of the operator

$$Cf(x) = \frac{1}{2\pi}\int_0^{2\pi} \cot\frac{x-y}{2} f(y)dy \tag{2.1}$$

on $L_p[0,2\pi]$:

$$\|C\|_p = \tan\frac{\pi}{2p} \quad (1 < p \le 2). \tag{2.2}$$

A brief description of the method of the proof of this equality is presented below in Subsection 7.2.

The following corollary follows from (2.2):

$$\|S_0 - K\|_p = \gamma_p, \text{ where } Kf(t) = \frac{1}{2\pi i}\int_{\Gamma_0} f(t)|dt| \tag{2.3}$$

Finally, in [KP] equalities

$$\|S_0\|_p = |S_0|_p = \gamma_p \tag{2.4}$$

were proved. There we used equality (2.2) and the following statement:

Lemma 2.1. *Let $L(\mathcal{B})$ be the algebra of all linear bounded operators on a Banach space $\mathcal{B}$, and let $\{R_n\} \subset L(\mathcal{B})$ be a sequence of operators such that $\|R_n f\| = \|f\|$ for all $f \in \mathcal{B}$ and R_n converges weakly to zero. If $A \in L(\mathcal{B})$ and $AR_n = R_n A$ for all n, then $|A| = \|A\|$.*

For $A = S_0$ the following sequence $\{R_n\}$ was used in [KP]:

$$R_n f(t) := t^n f(t^{2n}).$$

2.2. Using Lemma 2.1 with the same sequence R_n as above one can obtain equalities

$$\|aP_0 + bQ_0\|_p = |aP_0 + bQ_0|_p, \ (Q_0 := I - P_0) \tag{2.5}$$

for any p $(1 < p < \infty)$ and $a, b \in \mathbb{C}$. In particular these equalities hold for operators P_0, Q_0.

2.3. Conjecture $||P_0|| = 1/\sin(\pi/p)$ for the analytical projection P_0 has a long history (1968–2000). This conjecture had been included in two problem books “99 unsolved problems” and “199 unsolved problems” (see [VK3]) and in several other publications. Since 1968 many intermediate results related to this conjecture had been obtained (see, for example, [Ba], [GK7], [Pa], [Pe], [V] and the references in these publications). It was only in 2000 that B. Hollenbeck and I. Verbitsky [HV] proved that the conjecture is true, i.e.,

$$\|P_0\|_p = \|Q_0\|_p = |P_0|_p = |Q_0|_p = \frac{1}{\sin(\pi/p)} \quad (1 < p < \infty). \tag{2.6}$$

A brief description of the method of the proof of these equalities is presented below in Subsection 7.3.

2.4. Let $p \geq 2$ and $-1 < \beta < p-1$. Denote

$$\gamma(p,\beta) := \begin{cases} \cot \frac{\pi(1+\beta)}{2p} & \text{if} \quad -1 < \beta < 0 \\ \cot \frac{\pi}{2p} & \text{if} \quad 0 \leq \beta \leq p-2 \\ \tan \frac{\pi(1+\beta)}{2p} & \text{if} \quad p-2 < \beta < p-1 \end{cases} \tag{2.7}$$

and $\gamma(p,\beta) := \gamma(q, \beta(1-q))$ $(p^{-1}+q^{-1}=1)$ if $1 < p < 2$.

Also denote

$$\delta(p,\beta) = \max\left[\left(\sin\frac{\pi}{p}\right)^{-1}, \left(\sin\frac{\pi(1+\beta)}{p}\right)^{-1}\right]$$

$$= \begin{cases} \left(\sin\frac{\pi}{p}\right)^{-1} & \text{if} \quad \beta \in [\min(0,p-2), \max(0,p-2)] \\ \left(\sin\frac{\pi(1+\beta)}{p}\right)^{-1} & \text{if} \quad \beta \notin [\min(0,p-2), \max(0,p-2)] \end{cases}. \tag{2.8}$$

Theorem 2.4. *Let t_0 be an arbitrary point on the unit circle Γ_0. Then*

$$\|S_0\|_{p,\beta} = |S_0|_{p,\beta} = \gamma(p,\beta) \tag{2.9}$$

and

$$\|P_0\|_{p,\beta} = |P_0|_{p,\beta} = \|Q_0\|_{p,\beta} = |Q_0|_{p,\beta} = \delta(p,\beta). \tag{2.10}$$

The estimates "$\geq$" in (2.9), (2.10) have been obtained in [GK5, Chapter XIII, Section 3] with the use of results from [GK4]. For $p = 2$ equalities (2.9), (2.10) have been proved in [K1]. For $\min(0,p-2) \leq \beta \leq \max(0,p-2)$ equality (2.9) have been proved by N. Krupnik and V. Neaga [KN]. For general p and β equalities (2.9) have been proved by I. Verbitsky and N. Krupnik [VK1,2]. Finally, equalities (2.10) have been proved by B. Hollenbeck and I. Verbitsky [HV].

2.5. Let

$$\rho(t) = \prod_{k=1}^{n} |t-t_k|^{\beta_k}, \tag{2.11}$$

where $t_1, t_2, \ldots, t_n$ are different points on Γ_0 and $\beta_k \in (-1, p-1)$. Then (see [K1] for $p = 2$ and [VK1,2] for arbitrary p)

$$|S_0|_{p,\rho} = \max_{k=1,\ldots,n} \gamma(p,\beta_k), \tag{2.12}.$$

where $\gamma(p,\beta)$ is defined by (2.7). Equality (2.12) shows that the essential norm $|S_\Gamma|_{p,\rho}$ does not depend on the location of the (distinct) points t_k on the contour Γ_0.

2.6. In contrast with essential norms (see equality (2.12)), the norm $\|S_0\|_{p,\rho}$ depends on the location of the points $t_1, t_2, \ldots, t_n$, $n \geq 2$ on the contour Γ.

Suppose that the numbers $\beta_1, \ldots, \beta_n$ satisfy the inequalities

$$-1 < \beta_j < 0 \ (j = 1, \ldots, m); \quad 0 \leq \beta_j < p-1 \ (j = m+1, \ldots, n) \quad (m \leq n). \tag{2.13}$$

We denote

$$\mu_1 = \sum_{j=1}^{m} \beta_j, \quad \mu_2 = \sum_{j=m+1}^{n} \beta_j$$

and

$$\mu = \begin{cases} \mu_1 & \text{if } \mu_1 + \mu_2 < p-2, \\ \mu 2 & \text{if } \mu_1 + \mu_2 \geq p-2 \end{cases}$$

Theorem 2.6 *Let* $\rho(t)$ *be defined by* (2.11). *If* $\mu \in (-1, p-1)$ *then*

$$\sup_{t_1,\dots,t_n \in \Gamma_0} \|S_0\|_{p,\rho} = \gamma(p,\mu), \tag{2.14}$$

whereas if $\mu \notin (-1, p-1)$ *then*

$$\sup_{t_1,\dots,t_n \in \Gamma_0} \|S_0\|_{p,\rho} = \infty. \tag{2.15}$$

Corollary 2.6.1. *Let* $p \geq 2$. *Suppose one of the following conditions is satisfied:*

$$\beta_1 + \cdots + \beta_n \leq p-2, \quad \beta_k \geq 0 \ (k = 1, \dots, n);$$
$$\beta_1 + \cdots + \beta_n \leq p-2, \ \beta_1 \leq 0, \ \beta_k \geq 0 \ (k = 2, \dots, n);$$
$$\beta_1 + \cdots + \beta_n \leq p-2, \quad \beta_k \geq 0 \ (k = 1, \dots, n);$$
$$\beta_1 + \cdots + \beta_n \geq p-2, \ \beta_1 \geq p-2, \beta_k \leq 0 \ (k = 2, \dots, n). \tag{2.16}$$

Then

$$\|S_0\|_{p,\rho} = \max_{1 \leq k \leq n} \gamma(p, \beta_k). \tag{2.17}$$

The case $1 < p < 2$ *reduces to the case* $p \geq 2$ *upon replacing* p *by* q $(1/p + 1/q = 1)$ *and* β_k *by* $\beta_k(1-q)$.

We also mention

Theorem 2.6.2. *Let* $-1 < \alpha, \beta < 1$, *and* $\rho(t) = |t-1|^\alpha |t+1|^\beta$, *then*

$$\|S_0\|_{2,\rho} = \max(\gamma(2,\alpha), \gamma(2,\beta)). \tag{2.18}$$

The results of this subsection were obtained in [VK2] (see also [K3, Section 8]).

2.7. Here we provide a formula for $\|S_0\|_{2,\rho}$ for a general Muckenhoupt weight (without any additional restrictions) ρ which was obtained in [FKS, Section 5]. Notice that $\rho^{\pm 1} \in L_1(\Gamma_0)$ (see, for example, [GK6, page 35]).

Theorem 2.7. *Let* ρ_+ *be an outer function such that* $|\rho_+(t)| = \rho(t)$, *and* $\omega = \overline{\rho_+}/\rho_+$. *Then*

$$\|S_0\|_{2,\rho}^2 = \frac{1 + \|H_\omega\|}{1 - \|H_\omega\|}, \tag{2.19}$$

where H_ω *denotes the Hankel operator* $Q_0 \omega P_0$.

Note also that according to Nehari's theorem [Ni, p. 181],

$$\|H_\omega\| = \mathrm{dist}_{L_\infty}(\omega, H^\infty). \tag{2.20}$$

Example 2.7.1. Let $\rho(t) := |t - t_1|/|t - t_2|$ $(t_j \notin S_0)$. Then

$$\rho_+(t) = \frac{1 - t\overline{t_1}}{1 - t\overline{t_2}}, \ \omega(t) = \frac{(t - t_1)(1 - t\overline{t_2})}{(t - t_2)(1 - t\overline{t_1})}$$

and the distance from ω to H^∞ in this case equals

$$\operatorname{dist}_{L_\infty}(\omega, H^\infty) = \frac{|\operatorname{Res}(\omega; t_2)|}{1-|t^2|^2} = \frac{|t_2 - t_1|}{|1 - t_2\overline{t_1}|}. \tag{2.21}$$

(Here $\operatorname{Res}(f, z)$ denotes the residue of the function f at the point z.) It follows from (2.19)–(2.21) that

$$\|S_0\|_{2,\rho} = \sqrt{\frac{|1 - t_1\overline{t}_2| + |t_1 - t_2|}{\big||1 - t_1\overline{t}_2| - |t_1 - t_2|\big|}}. \tag{2.22}$$

Remark 2.7.2. Setting t_1 or t_2 at zero we arrive to a particular case of (2.22):

$$\|S_0\|_{2,|t-t_0|^{\pm 1}} = \sqrt{\frac{1+|t_0|}{1-|t_0|}}. \tag{2.23}$$

The function ω is rational for all weights of the form $\rho(t) = \Pi|t - t_j|^{n_j}$, $n_j \in Z, t_j \notin \Gamma_0$, more general than considered in Example 2.7.1. According to Kronecker's lemma ([Ni, p. 183]), for such weights ρ the Hankel operator H_ω is finite dimensional. Its norm (and therefore the norm of S_0) can be found in terms of the eigenvalues of a finite matrix corresponding to integral operator with degenerate kernel

$$\frac{1}{\tau - t}\left(\overline{\rho_+(\tau)/\rho_+(t)} - \rho_+(t)/\rho_+(\tau)\right). \tag{2.24}$$

In practice, however, these computations lead to quite unpleasant formulas.

Question 1. It would be interesting to compute the norm $\|S_0\|_{p,\rho}$ for some weight (2.11) in the case when $\|S_0\|_{p,\rho} \neq |S_0|_{p,\rho}$.

The following corollary[1] follows from Theorem 2.7: if $\|S_0\|_{2,\rho} = 1$, then $\rho =$ const. This statement is not true for essential norms. In the next subsection we give a full description of all weights for which equality $|S_0|_{2,\rho} = 1$ holds.

2.8. Recall that VMO, the class of the functions of vanishing mean oscillation, consists of all ϕ such that

$$M_\delta(\phi) = \sup_{|I|<\delta} \frac{1}{|I|}\int_I |\phi - \phi_I|dt < \infty \quad \text{and} \lim_{\delta\to 0} M_\delta(\phi) = 0.$$

Here, as usual, $I \subset S_0$ is an arbitrary arc of length $|I|$ and $\phi_I := \frac{1}{|I|}\int_I \phi dt$.

According to Sarason's theorem (see[Ni, p. 376]), $VMO = C + \tilde{C}$, that is, VMO consists exactly of the sums $u + \tilde{v}$ where u, v are continuous functions and $\sim$ stands for the harmonic conjugation.

Theorem 2.8.1. *Formula* $|S_0|_{2,\rho} = 1$ *holds if and only if* $\log\rho \in VMO$.

Note that for non-singular continuous weights ρ the function $\log\rho$ is a continuous function too. In this case, obviously, $\log\rho \in VMO$, and therefore $|S_0|_{2,\rho} = 1$. This result follows also from the local principle (see Section 4 below).

[1] see Theorem 5.6 below for a more general statement

The "if" part of Theorem 2.8.1 admits a following generalization:

Theorem 2.8.2. *Let* $1 < p < \infty$ *and let* ρ *be a Muckenhoupt weight. If* $\log \rho \in VMO$ *then*

$$|S_0|_{p,\rho} = |S_0|_p = \gamma_p.$$

This observation was made by A.Yu. Karlovich [private communication] by using the results from [GD]. It is not known at the moment whether the condition $\log \rho \in VMO$ in Theorem 2.8.2 is also necessary, as was the case in Theorem 2.8.1.

2.9. Spaces of vector-valued functions. Let H be a separable Hilbert space and $\mathcal{B} = L_p(\Gamma_0, H, \rho)$ $(1 < p < \infty)$ the Banach space of weakly-measurable vector-valued functions $f\colon \Gamma_0 \to H$, for which the norm

$$\|f\|_{\mathcal{B}}^p := \int_\Gamma \|f(t)\|_H^p \rho(t)|dt| \tag{2.25}$$

is finite. Here the weight is defined by (2.11). See, e.g., [Z, Vol. II, Lemma 5.18] for the boundedness of the operator S_0 on $\mathcal{B}$.

Theorem 2.9. *Let* $\rho(t) = |t - t_0|^\beta$, *where* $t_0 \in \Gamma_0$. *Then*

$$\|S_0\|_{\mathcal{B}} = \|S_0\|_{p,\beta} = \gamma(p, \beta). \tag{2.26}$$

For details see [VK2] and [K3, Section 8].

2.10. If $\dim H < \infty$ then equality (2.12) can be extended to the spaces of vector-valued functions. If $\dim H = \infty$, then $|S_0|_{\mathcal{B}}$ depends on the location of the points t_k on Γ_0.

Question 2. It would be interesting to obtain analogs of the results from Subsections 2.9, 2.10 for operators P_0 and Q_0.

3. Γ coinciding with $\mathbb{R}$ or with its connected subset

3.1. We start with the following

Theorem 3.1. *Let* $x_1, \dots, x_n$ *be distinct points of* $\mathbb{R}$,

$$t_k := (x_k + i)(x_k - i)^{-1} \ (k = 1, \dots, n), \quad t_0 := 1, \tag{3.1}$$

and let $p, \beta, \beta_1, \dots, \beta_n$ *be real numbers satisfying conditions*

$$1 < p < \infty, \ -1 < \beta_k < p - 1 \ (k = 1, \dots, n), \ -1 < \beta + \sum_{k=1}^n \beta_k < p - 1. \tag{3.2}$$

Set

$$\rho_1(x) = (x^2 + 1)^{\beta/2} \prod_{k=1}^n |x - x_k|^{\beta_k} \quad \textit{and} \ \rho_2(t) = \prod_{k=0}^n |t - t_k|^{\beta_k}, \tag{3.3}$$

where

$$\beta_0 = p - 2 - \beta - \sum_{k=1}^n \beta_k. \tag{3.4}$$

Then

$$\|aI+bS_{\mathbb{R}}\|_{p,\rho_1} = \|aI+bS_0\|_{p,\rho_2},\ \ |aI+bS_{\mathbb{R}}|_{p,\rho_1} = |aI+bS_0|_{p,\rho_2}. \tag{3.5}$$

for any complex numbers a, b.

(See [K3, Theorem 4.6] and [VK2].)

3.2. Let $\rho(t)=t^\beta$, then

$$\|aI+bS_{\mathbb{R}}\|_{p,\beta} = |aI+bS_{\mathbb{R}}|_{p,\beta}. \tag{3.6}$$

This follows from Lemma 2.1 with $R_n f(t) = n^{\frac{1+\beta}{p}} f(nt)$ (see [K3, Section 4] for details).

3.3 Let $p\ge 2$ and $\rho(x)=|x+i|^{p-2}$ or $1<p\le 2$ and $\rho(x)=|x+i|^{2-p}$, then

$$\|S_{\mathbb{R}}\|_{p,\rho} = |S_{\mathbb{R}}|_{p,\rho} = \gamma_p,\ \ \|Q_{\mathbb{R}}\|_{p,\rho} = |Q_{\mathbb{R}}|_{p,\rho} = \|P_{\mathbb{R}}\|_{p,\rho} = |P_{\mathbb{R}}|_{p,\rho} = \frac{1}{\sin(\pi/p)}.$$

This follows from (3.5), (3.6), (2.7) and (2.9).

3.4. Equalities (3.5), (3.6), (2.9), (2.10) imply:

$$\|S_{\mathbb{R}}\|_p = |S_{\mathbb{R}}|_p = \gamma_p,\ \ \|Q_{\mathbb{R}}\|_p = |Q_{\mathbb{R}}|_p = \|P_{\mathbb{R}}\|_p = |P_{\mathbb{R}}|_p = \frac{1}{\sin(\pi/p)}. \tag{3.7}$$

Generalization of equalities (3.7) to the space $L_p(\mathbb{R}, w)$ with power weights

$$w(x) = \prod_{j=1}^{n} |x-x_j|^{\beta_j} \text{ satisfying } -1 < \sum_{k=1}^{n} \beta_k < p-1 \tag{3.8}$$

follows also from (3.5), (3.6), (2.9), (2.10) by taking in ρ_2 the exponent $\beta_0 = p-2-\sum_{k=1}^n \beta_k$.

For example,

$$\begin{aligned}\|S_{\mathbb{R}}\|_{p,|x-x_1|^\alpha|x-x_2|^\beta} &= \|S_0\|_{p,|t-t_1|^\alpha|t-t_2|^\beta|t-1|^{p-2-\alpha-\beta}} \\ &= \max\left(\gamma(p,\alpha), \gamma(p,\beta), \gamma(p,\alpha+\beta)\right) \end{aligned} \tag{3.9}$$

The last equality in (3.9) follows from Corollary 2.6.1, see [K3, Theorem 5.4] for details.

Observe that the norm in the left-hand side of (3.9) does not depend on the location of the points x_1, x_2. Theorem 2.6 and Corollary 2.6.1 can be extended (with the help of Theorem 3.1) to the spaces $L_p(\mathbb{R}, (x^2+1)^{\beta/2}\prod|x-x_k|^{\beta_k})$.

3.5. Segment $[a, b]$. Denote for $p \ge 2$

$$\begin{aligned}\eta(p,\beta) &:= \begin{cases} \cot\frac{\pi(1+\beta)}{p} & \text{if} \quad -1<\beta<-1/2 \\ \cot\frac{\pi}{2p} & \text{if} \quad -1/2\le\beta\le p-3/2 \\ \cot\frac{\pi(p-1-\beta)}{p} & \text{if} \quad p-3/2<\beta<p-1 \end{cases} \\ &= \max\left(\cot\frac{\pi}{2p}, |\cot\frac{\pi(1+\beta)}{p}|\right) \end{aligned} \tag{3.10}$$

and $\eta(p,\beta) := \eta(q, \beta(1-q))$ $(p^{-1}+q^{-1}=1$, if $1<p<2$.

Theorem 3.5. *Let* $\Gamma = [a, b]$ *or any simple smooth arc with end-points* a *and* b, *and let*

$$\rho(t) = |t - a|^{\alpha}|t - b|^{\beta} \quad (t \in [a, b], \alpha, \beta \in (-1, p - 1)). \tag{3.11}$$

Then

$$|S_\Gamma| = \max(\eta(p, \alpha), \eta(p, \beta)). \tag{3.12}$$

For details see [K3, Section 7].

Remark 3.5.1. Let $\Gamma = [a, b]$. If in (3.11) one of the numbers α, β equals zero or $p = 2$, then

$$\|S_\Gamma\|_{p,\rho} = \max\left(\eta(p, \alpha), \eta(p, \beta)\right). \tag{3.13}$$

In particular,

$$\|S_\Gamma\|_{2,\rho} = \max\left(\tan\frac{\pi|\alpha|}{2}, \tan\frac{\pi|\beta|}{2}, 1\right). \tag{3.14}$$

Remark 3.5.2. For arbitrary α, β and p the norm in the left-hand side of (3.13) has not been computed yet.

Question 3. See Remark 3.5.2.

3.6. $\Gamma = \mathbb{R}_+$. Below in this section we denote $S_{\mathbb{R}_+}$ by S, and treat $\mathbb{R}_+ := [0, \infty]$ as a compact subset of $\dot{\mathbb{C}}$. We start with the following equalities:

$$\|S\|_{L_p(\mathbb{R}_+, t^\beta)} = |S|_{L_p(\mathbb{R}_+, t^\beta)} = \|S\|_{L_p(\Gamma_1, |t+1|^\beta |t-1|^{p-2-\beta})} = \eta(p, \beta), \tag{3.15}$$

where $p > 1$, $-1 < \beta < p - 1$, and $\eta(p, \beta)$ is defined by (3.10). Here Γ_1 is the (unit) upper semi-circle.

The first equality in (3.15) follows from Lemma 2.1 where we set $R_n f(t) = n^{\frac{1+\beta}{p}} f(nt)$, the second one from Theorem 3.1 and the last one from Theorem 3.5 and equality $\eta(p, p - 2 - \beta) = \eta(p, \beta.)$

Below in this section we denote

$$\|A\|_{p,\beta} := \|A\|_{L_p(\mathbb{R}_+, t^\beta)} \tag{3.16}$$

Consider operators Π and Π^* defined by (1.6) and acting on $L_p(\mathbb{R}_+)$. As was mentioned in the Introduction, these operators play an important role in mathematical physics (see, for example, [Bi], [HKV]). In 1984 a problem was stated by M.S. Birman [Bi] to figure out the exact value of the norms of operators $I - \Pi$, $I - \Pi^*$ on $L_p(\mathbb{R}_+)$, $1 < p < \infty$. It is easy to check (see Example 3.7.3 below) that the norms of these two operators on L_2 equal $\sqrt{2}$. For $p \neq 2$ this problem turned out to be not so simple and only in 2003 it was figured out in [HKV] that

$$\|I - \Pi\|_{L_p(\mathbb{R}_+)} = \|I - \Pi^*\|_{L_p(\mathbb{R}_+)} = C_p, \tag{3.17}$$

where

$$C_p = \sqrt{2} \max_{\theta \in [0, 2\pi]} \left[\frac{|\cos(\theta - \pi/4)|^p + |\cos(\theta - \pi/4 + \pi/p)|^p}{|\cos(\theta)|^p + |\cos(\theta + \pi/p)|^p}\right]^{\frac{1}{p}} \tag{3.18}$$

for all $1 < p < \infty$. In particular, $C_4 = \sqrt{4 + 2\sqrt{5}}$.

See Subsection 7.4 for some comments regarding the method used in [HKV].

3.7. Denote by H the complex Hilbert space $H := L_2(\mathbb{R}_+, t^\beta)$ with the inner product

$$\langle f, g\rangle = \int_0^\infty f(x)\overline{g(x)}x^\beta\, dx \quad (|\beta| < 1)$$

and by $\mathcal{A}_\beta$ the unital Banach subalgebra of $L(H)$ generated by one element S.

Theorem 3.7 *$\mathcal{A}_\beta$ is a C^*-algebra. In particular,*

$$S^* = t^{-\beta}St^\beta = [\cos(\pi\beta)S - i\sin(\pi\beta)I]\,[\cos(\pi\beta)I - i\sin(\pi\beta)S]^{-1} \tag{3.19}$$

(see [K3,Theorem 13.7])

The spectrum of the operator S coincides with the circular arc (or line segment) $\ell(\beta)$ which connects the points ± 1 and passes through the point $i\tan(\pi\beta/2)$. The Gelfand transform of the element S can be represented as

$$g(S)(z) = z \ (z \in \ell(\beta)) \text{ or } h(S)(\xi) = \operatorname{cth}(\xi + \pi i\gamma), \quad \xi \in \mathbb{R}. \tag{3.20}$$

It is not difficult to check that

$$\max_{z\in\ell(\beta)} |z| = \max\left[1, \tan\left((\pi|\beta|)/2\right)\right]$$

and we arrive at equality (3.15) with $p = 2$.

Theorem 3.7.1. *For each $A \in \mathcal{A}_\beta$,*

$$\|A\|_{2,\beta} = |A|_{2,\beta}. \tag{3.21}$$

As in 3.6, this statement follows from Lemma 2.4.

The following statement is usually used for constructing the matrix symbols of sio along the contours with intersections (see, for example, Section 8 below).

Proposition 3.7.2. *Let $\theta \in (0, 2)$. Then the operator*

$$K_\theta f(x) := \frac{1}{\pi i}\int_0^\infty \frac{f(y)dy}{y - e^{i\pi\theta}x} \tag{3.22}$$

belongs to algebra $\mathcal{A}_\beta$ and its Gelfand transform is given by

$$h(K_\theta)(\xi) = \frac{\exp(-\alpha(\xi + \pi\gamma i))}{\operatorname{sh}(\xi + \pi\gamma i)}, \quad h(K_{-\theta})(\xi) = \frac{\exp(\alpha(\xi + \pi\gamma i))}{\operatorname{sh}(\xi + \pi\gamma i)}, \ (\alpha = 1 - \theta) \tag{3.23}$$

Let, in particular, $\beta = 0$. Then

$$g(K_\theta)(z) = (1 - z)^s(1 + z)^{1-s}e^{-\pi i\theta/2} = 2(1 - x)^s x^{1-s}e^{-\pi i\theta/2} \tag{3.24}$$

and

$$g(K_{-\theta})(z) = -(1 - z)^{1-s}(1 + z)^s e^{\pi i\theta/2} = -2(1 - x)^{1-s}x^s e^{\pi i\theta/2}, \tag{3.25}$$

where $s = \frac{\theta}{2}$, $x = (1 + z)/2 \in [0, 1]$.

Example 3.7.3. Operators $I - \Pi$ and $I - \Pi^*$ considered above (see (3.17)) can be represented as $A = I \pm iS - iK_1$. Hence

$$|h(A)|^2 = |1 \pm iz - \sqrt{(1 - z)(1 + z)}|^2 = 2(1 - \sqrt{1 - z^2}),$$

and $\|A\| = |A| = \max_{|z|\le 1} |h(z)| = \sqrt{2}$.

4. A local principle for best constants

4.1. Let X be a compact Hausdorff topological space, and let μ be a non-negative (possibly infinite) measure defined on a σ-algebra which contains all the Borel subsets of X. Denote $\mathcal{B} = L_p(X, \mu)$, $L(\mathcal{B})$ the algebra of all linear bounded operators on $\mathcal{B}$ and $\mathcal{K}$ the ideal of all compact operators $K \in L(\mathcal{B})$. For each measurable subset $M \subset X$ we let F_M denote the operator of multiplication by the characteristic function $\chi(M)$ of M.

An operator $A \in L(\mathcal{B})$ is called a *local type* operator (see [S]) if $F_{M_1} A F_{M_2}$ is compact for every pair of disjoint closed sets $M_1, M_2 \subset X$. Denote by $\mathcal{L}$ the set of all operators of local type.

Following [S] we set

$$q(x, A) = \inf_{U \ni x} |F_U A|, \tag{4.1}$$

where the infimum is taken over all neighborhoods U of the point $x \in X$. For a local type operator A acting on $\mathcal{B} = L_p(X, \mu)$ ($p \in [1, \infty)$), the following equality holds (see[K3, Theorem 6.3])

$$\inf_{U \ni x} \|F_U A\| = \inf_{U \ni x} |F_U A| := q(x, A); \tag{4.2}$$

the constant $q(x, A)$ is called the *local norm* of the operator A at the point x.

The following estimate was obtained in [S] for any operator A of local type:

$$|A| \leq (r + 1) \sup_{x \in X} q(x, A), \tag{4.3}$$

where r denotes the dimension of the topological space X. This estimate played an important role in the famous Simonenko local principle for Fredholmness of the operators of local type. An interesting problem was to figure out the best constant (say C) so that (4.3) holds with $r+1$ replaced by C. It follows from (4.1) that $|A| \geq q(x, A)$ for each $x \in X$. Therefore a local principle for computation of the essential norms of operators of local type could be constructed, only if this constant C happens to equal 1. The exact value of C was established in 1984 (see [K4]) and (we were lucky!) it indeed turned out that $C = 1$. Thus, the following equality holds:

$$|A| = \sup_{x \in X} q(x, A). \tag{4.4}$$

Let $A, B \in \mathcal{L}$. We say that A and B are equivalent an the point $x \in X$ (and write $A \overset{x}{\sim} B$) if $q(x, A - B) = 0$. Observe that if $A \overset{x}{\sim} B$, then $q(x, A) = q(x, B)$.

Let $\{A_x\}_{x \in X}$ be a family of operators of local type and $A \in \mathcal{L}$. If $A \overset{x}{\sim} A_x$ for each $x \in X$, then A is called *an envelope of the family* $\{A_x\}$ and A_x is the *local representative of operator A in the point* x. We say that A_x is a *nice local representative* if

$$\sup_{t \in X} q(t, A_x) = q(x, A_x). \tag{4.5}$$

The following theorem can be easily obtained using equality (4.4):

Theorem 4.1. *Let $A \in \mathcal{L}$ be an envelope of a family $\{A_x\}$ of local type operators. Then*

$$|A| \le \sup_{x\in X} |A_x|. \tag{4.6}$$

If, in addition, A_x are nice representatives, then

$$|A| = \sup_{x\in X} |A_x|. \tag{4.7}$$

For a general look at local principles with special emphasis on the norms computation aspect see the paper [BKS].

Below in this section we use notation $|A|_{p,\rho}$ for the essential norm of the operator $A \in L(L_p(\Gamma,\rho))$ and $|A|^x_{p,\rho}$ for the local norm of the operator $A \in \mathcal{L}$ at the point $x \in \Gamma$. It is not difficult to check that

$$|A|^x_{p,\rho}\ (= q(x,A)) = \inf\{\|a_x A\|_{p,\rho} : a_x \in M_x(\Gamma)\}, \tag{4.8}$$

where $M_x(\Gamma)$ denotes the set of all continuous functions on Γ which equal 1 in some (varying with the function) neighborhood of the point x.

A contour

$$\Gamma = \bigcup_{k=1}^{m} \Gamma_k \tag{4.9}$$

is called a *star* (or a ν-star) with a node z if Γ_k are smooth simple arcs having a common endpoint z and pairwise disjoint otherwise. Let $\nu = (\nu_1, \nu_2, \ldots, \nu_m)$, where ν_k are the angles between the arcs Γ_k.

We say that two weights ρ and w on Γ are *equivalent at the point* $x \in \Gamma$ ($\rho \overset{x}{\sim} w$) if there exists a function $h \in C(\Gamma),\ h(t) \ne 0$, such that $w(t) = h(t)\rho(t)$ in some neighborhood of the point x.

Also, by ρ_x we denote any weight equivalent to ρ at the point x.

Theorem 4.2. *Let Γ and γ be two composed contours, $z \in \Gamma$ and $w \in \gamma$. Suppose there exist neighborhoods $\Gamma_z \subset \Gamma$, $\gamma_w \subset \gamma$ of the points z, w respectively which happen to be ν-stars with the same $\nu = (\nu_1, \nu_2, \ldots, \nu_m)$, $\nu_j \in (0, 2\pi)$). Let further ρ be a Muckenhoupt weight on Γ and $A_\Gamma = aI + bS_\Gamma$, where a, b are some complex constants. Then*

$$|A_\Gamma|^z_{p,\rho} = |A_{\Gamma_z}|^z_{p,\rho_z} = |A_{\gamma_z}|^w_{p,\rho_z\circ\alpha} = |A_\gamma|^w_{p,\tilde\rho} = q(z, A_\Gamma), \tag{4.10}$$

where α is a diffeomorphism of γ_w into Γ_z and $\tilde\rho$ is an arbitrary weight on γ equivalent to $\rho_z \circ \alpha$ at the point w.

The proof uses the same arguments as the proof of Proposition 4.1 in [KS].

We mention also the following statements.

Let Γ and γ be two composed contours, $z \in \Gamma$ and $w \in \gamma$. We write $\Gamma \overset{z,w}{\sim} \gamma$ if $\{\Gamma, \gamma, z, w\}$ satisfy the conditions of Theorem 4.2. If $w = z$, we abbreviate this to $\Gamma \overset{z}{\sim} \gamma$.

Proposition 4.3. *Let A be an operator of local type on $L_p(\Gamma,\rho)$, where ρ is defined by* (1.7) *and $x\in\Gamma$. Then*

$$|A|^x_{p,\rho}=|A|^x_{p,\rho_x},\quad \textit{where}\quad \rho_x(t)=\begin{cases}|t-t_j|^{\beta_j} & \textit{if } x=t_j\\ 1 & \textit{if } x\neq t_1,\dots,t_n\end{cases},$$

and

$$|A|_{p,\rho}=\sup_{x\in\Gamma}|A|^x_{p,\rho}=\sup_{x\in\Gamma}|A|^x_{p,\rho_x}. \tag{4.11}$$

Proposition 4.4. *Let $a,b\in C(\Gamma)$. Then for singular integral operators $aI+bS_\Gamma$ we can localize simultaneously the coefficients a,b, the contour Γ and the weight ρ. Namely,*

$$|aI+bS_\Gamma|_{p,\rho}=\sup_{x\in\Gamma}|a(x)I+b(x)S_{\Gamma_x}|_{p,\rho_x}, \tag{4.12}$$

where Γ_x is an arbitrary contour smooth at each point $t\neq x$ and such that $\Gamma_x\overset{x}{\sim}\Gamma$.

If, in particular, Γ is a smooth closed (bounded) contour and ρ is a power weight (1.6) then

$$|S_\Gamma|_{p,\rho}=\max_k\gamma(p,\beta_k),\quad |P_\Gamma|_{p,\rho}=|Q_\Gamma|_{p,\rho}=\max_k\delta(p,\beta_k), \tag{4.13}$$

where $\gamma(p,\beta)$ and $\delta(p,\beta)$ are defined by (2.7), (2.8).

In the next theorem we give the formulas for essential norms of the operator S_Γ on $L_2(\Gamma,\rho)$ for a general simple contour and power weight defined by (1.7). We assume that all possible knots of the contour – corner-points, cusps and end-points, – are contained in the set $\{t_1,\dots,t_n\}$. Let ν_k ($k=1,\dots,n$; $\nu_k\in[0,2\pi]$) be the angles at the points t_j measured from the inner domain. When $\nu_j=0$ or $\nu_j=2\pi$ the corresponding knot is called a cusp. We suppose that the cusps are of finite order, i.e., for some parametrization $\omega_j^\pm:[0,1]\to\Gamma_j^\pm$ of smooth curves $\Gamma_j^\pm\subset\Gamma$ joining at a cusp $t_j\in\Gamma$ the inequality $|\omega_j^+-\omega_j^-|\geq Cx^\gamma$ holds for some $C>0,\gamma>1$ and all $x\in]0,1[$. By t_0 we denote an arbitrary (but fixed) point on $\Gamma\setminus\{t_1,t_2,\dots,t_m\}$.

Theorem 4.5. *Let $\mathcal{B}=L_2(\Gamma,\rho)$, where Γ is a simple contour and ρ is a power weight* (1.7)*. Then*

$$|S_\Gamma|=\max_{0\leq j\leq n}\mu_j, \tag{4.14}$$

where μ_k are defined by the following equalities:

(i) *If $t_k\in int(\Gamma)$ and Γ is smooth at the point t_k, then* [K1]

$$\mu_k=\tan(\pi/4)(1+|\beta_k|). \tag{4.15}$$

(ii) *If t_k is an end-point, then*

$$\mu_k=\max\left[1,\tan\frac{|\beta_k|}{2}\right], \tag{4.16}$$

see equality (3.12).

(iii) *If t_k is a cusp, then* (*see* [DK], [DKS])

$$\mu_k=\max\{1+\sqrt{2},\tan\frac{\pi}{4}(1+|\beta_k|)\}. \tag{4.17}$$

(iv) *In the remaining cases*

$$\mu_k = D(\nu_k, \beta_k) + \sqrt{D(\nu_k, \beta_k)^2 + 1}, \tag{4.18}$$

where

$$D(\nu, \beta)^2 := \sup_{s\in\mathbb{R}} \left|\frac{\operatorname{ch}(s\varphi) - \cos(\pi\beta)}{\operatorname{ch} s + \cos(\pi\beta)}\right| \quad (\varphi = |\pi - \nu|/\pi) \tag{4.19}$$

and ν_k *is the angle at the point* t_k (see [AK]).

If in particular $\beta = 0$, then

$$D(\nu) := D(\nu, 0) = \max_{\xi\in[0,\infty)} \frac{\operatorname{sh}(\varphi\xi)}{\operatorname{ch}(\xi)}, \tag{4.20}$$

where $\varphi = |\pi - \nu|/\pi$.

Corollary 4.6. *If* $\mathcal{B} = L_2(\Gamma)$, *then* $|S_\Gamma| = 1$ *for any smooth contour. If* Γ *has at least one cusp, then* $|S_\Gamma| = 1 + \sqrt{2}$. *In the remaining cases* $1 < |S_\Gamma| < 1 + \sqrt{2}$.

Example 4.7. Let Γ_j $(j = 1, 2, 3, 4, 5, 6, 7)$ be respectively a segment, an astroid, an equilateral triangle, a square, regular pentagon, hexagon, or n-sided polygon. Denote $S_j := S_{\Gamma_j}$. Then

$$|S_1| = 1; \quad |S_2| = 1 + \sqrt{2}; \quad |S_3| = \frac{1 + \sqrt{5}}{2}; \quad |S_4| = \sqrt{2};$$
$$|S_5| = \frac{\sqrt{63\sqrt{21} - 243} + \sqrt{63\sqrt{21} - 143}}{10};$$
$$|S_6| = \frac{1}{3}\sqrt{3 + 4\sqrt{3} + 2\sqrt{6(\sqrt{3} - 1)}}, \ |S_7| = \max_{\xi\geq 0} \frac{\operatorname{sh}(2\xi)}{\operatorname{ch}(n\xi)}. \tag{4.21}$$

Questions 4.9. It is clear that $\|S_1\| = |S_1| = 1$. It would be interesting to compute the norms $\|S_j\|$ for all (or at least some of) the operators S_j, $j = 2, 3, \dots, 7$.

The results of Theorem 4.5 for closed contours can be extended to operators $A = aI + bS_\Gamma$ $(a, b \in \mathbb{C})$ using the following proposition.

Theorem 4.8. *Let* S *be an operator on a Hilbert space, such that* $S^2 = I$ *and* $P = (I + S)/2$, $Q = I - P$. *Then*

$$\|P\| = \|Q\| = \left(\|S\| + \|S\|^{-1}\right)/2 \tag{4.22}$$

and

$$2\|cP + dQ\| = \left(\sqrt{|c - d|^2\delta + (|c| + |d|)^2} + \sqrt{|c - d|^2\delta + (|c| - |d|)^2}\right), \tag{4.23}$$

where $c, d \in \mathbb{C}$ *and* $\delta = \|P\|^2 - 1$.

Same equalities hold for essential norms $|S|, |P|, |Q|$.

Generalizations of equalities (4.22), (4.23) will be considered (and the corresponding references given) in Section 6.

5. Composed contours. Norms and essential norms

We start with

Theorem 5.1. *Suppose that Γ is a simple smooth contour and the operator S_Γ is bounded on $L_p(\Gamma,\rho)$ $(1<p<\infty)$. Then*

$$|aI+bS_\Gamma|_{p,\rho} \geq |aI+bS_\Gamma|_p \ (a,b\in\mathbb{C}). \tag{5.1}$$

Proof of this theorem under an additional condition that Γ is a Lyapunov contour is given in [VK2, Theorem 6] (see also [K3, Theorem 6.4]).

Remark 5.2. The additional (Lyapunov's) condition was used to establish boundedness of the operator S_Γ and compactness of the operator

$$Kf(t) = \frac{1}{2\pi i}\int_{\Gamma_0}\left[\frac{\alpha'(t)}{\alpha(t)-\alpha(\tau)} - \frac{1}{t-\tau}\right] f(\tau)d\tau, \tag{5.2}$$

where the function $z=\alpha(t)$ transforms the unit circle Γ_0 onto the contour Γ.

Somewhat later it was shown that the boundedness of operators S_Γ and compactness of operators K for non-weighted spaces remain valid for smooth contours Γ (see [Ca2], [Gru], [DG]). For Lebesgue spaces with Muckenhoupt weights the compactness of K can be proved by applying the weighted version of the Krasnosel'skii interpolation theorem (in fact, this was done in [KRA, Lemma 4.3]).

Theorem 5.3 (see [GK7, Ch. 7, Theorem 2.1]). *Let S_Γ be bounded on some $L_p(\Gamma,\rho)$ and $a,b,\in L_\infty(\Gamma)$. If $\Gamma=\bigcup_{j=1}^m \Gamma_j$ and Γ_j are disjoint then*

$$|aI+bS_\Gamma|_{p,\rho} = \max_j |a_jI+b_jS_{\Gamma_j}|_{p,\rho_j}. \tag{5.3}$$

Here h_j denotes the restriction of a function $h\in L_1(\Gamma)$ onto Γ_j.

The following statement shows that Theorem 5.3 fails if we replace the essential norms with the norms.

Theorem 5.4. *The norm of S_n acting on the family of concentric circles with radii $R_1<R_2<\cdots<R_n$ equals $r+\sqrt{1+r^2}$, where r is the spectral radius of the $n\times n$ real skew symmetric matrix $A(R_1,\dots,R_n)$ defined by the formula*

$$A_{ij}(R_1,\dots,R_n) = (-1)^{j+1}\left(\frac{R_i}{R_j}\right)^{1/2},\ (i<j). \tag{5.4}$$

In particular,

$$\begin{aligned} \|S_2\|_2 &= \sqrt{\frac{R_1}{R_2}} + \sqrt{\frac{R_1+R_2}{R_2}}; \\ \|S_3\|_2 &= \left(\frac{R_1}{R_3}+\frac{R_1}{R_2}+\frac{R_2}{R_3}\right)^{1/2} + \left(1+\frac{R_1}{R_3}+\frac{R_1}{R_2}+\frac{R_2}{R_3}\right)^{1/2}. \end{aligned} \tag{5.5}$$

The case of four concentric circles can also be tackled, but the final formula is too cumbersome.

Theorem 5.4 was obtained in [FKS, Section 4]. The particular case (n = 2) was earlier treated in [AK2]. In Theorem 5.4 $\Gamma = \bigcup \Gamma_j$, where Γ_j are disjoint circles, and it is clear that $|S_\Gamma| = |S_{\Gamma_j}| = \|S_{\Gamma_j}\| = 1 \neq \|S_\Gamma\|$.

Essential norms of sio along smooth contours do not depend on the contours. Namely, the following statement holds:

Theorem 5.5 (see [GK7, Ch. 7, Lemma 2.1]). *Let* Γ *be a simple smooth closed curve, and let* $\alpha : \Gamma \to S_0$ *be a smooth bijective mapping whose derivative* α' *is continuous and does not vanish on* Γ. *Further let* ρ *be the weight defined by* (1.7), *and*

$$\rho_0(z) := \prod_{k=1}^{n} |z - z_k|^{\beta_k} \quad (z, z_k \in S_0,\ t_k = \alpha(z_k))\,. \tag{5.6}$$

Then

$$|aI + bS_\Gamma|_{p,\rho} = |aI + bS_{\Gamma_0}|_{p,\rho_0} \quad (a, b \in \mathbb{C}). \tag{5.7}$$

A similar result (with the unit circle Γ_0 replaced by the segment $[0,1]$) holds for simple smooth non-closed arcs. See [GK7, Ch. 7, Lemma 2.2].

It follows from Theorem 5.5 that

$$|aI + bS_\Gamma|_p = |aI + bS_0|_p$$

for any smooth closed contour Γ, i.e., the essential norms do not depend on the (smooth) contour.

In contrast with essential norms, the norms of sio depend on the contour (even for smooth contours). Let, for example, $\Gamma := \{a\cos\theta + ib\sin\theta\}$ be an ellipse. Then

$$|S_\Gamma| = 1 \text{ while } 1 < \|S_\Gamma\| < 1 + \sqrt{2} \tag{5.8}$$

for each pair $a, b \in \mathbb{R},\ a \neq b$. For the second inequality in (5.8) see (5.21) below; the first one follows from the following

Theorem 5.6. *Let* Γ *be a smooth closed contour. Equality*

$$\|S\|_{L_2(\Gamma,\rho)} = 1 \tag{5.9}$$

holds if and only if Γ *is a circle and* $\rho(t)$ *is a constant.*

Indeed, the norm of any involution ($S^2 = I$) equals one if and only if S is self-adjoint. It was proved in [I] (see also [GK6, Theorem 7.2, Ch. I and [K5]), that S_Γ is selfadjoint if and only if Γ is a circle and $\rho(t)$ is a constant.

Corollary 5.7. *A smooth simple closed contour* Γ *is a circle if and only if* $\|S_\Gamma\|_2 = |S_\Gamma|_2$.

The adjective "smooth" in the Corollary 5.7 cannot be dropped. Indeed:

Theorem 5.8. *Let a closed contour* Γ *consist of two circular arcs or of an arc and a line segment. Then*

$$\|S_\Gamma\|_2 = |S_\Gamma|_2 = D(\nu) + \sqrt{D(\nu)^2 + 1}, \tag{5.10}$$

where ν *is the angle between these two arcs and* $D(\nu)$ *is defined by* (4.20).

In particular, for Γ consisting of a semicircle and a line segment, $\|S_\Gamma\|_2 = |S_\Gamma|_2 = \sqrt{2}$.

The second equality in (5.10) follows from Theorem 4.5, while the first equality is a consequence of the following

Theorem 5.9. *Let a contour Γ consist of several circular arcs (including possibly the line segment $[-1,1]$) with the endpoints ± 1, and let $\rho(z) = |z-1|^\beta |z+1|^{-\beta}$. Then $\|aI + bS_\Gamma\| = |aI + bS_\Gamma|$ for any complex constants a, b.*

This theorem follows from Lemma 2.1 with

$$R_n f(z) := \frac{2n^{\beta+1/2}}{n+1+z(n-1)} f\left(\frac{(n+1)z+n-1}{n+1+z(n-1)}\right). \tag{5.11}$$

A particular case of this statement was obtained in [AK2]. Now we are going to confirm the second inequality in (5.8).

In [FKS] we computed the norm of S_Γ on a weighted L_2 space, with Γ being an ellipse $t = \omega(\theta) = a\cos\theta + ib\sin\theta$ ($\theta \in [0, 2\pi]$, $a > b$). Namely:

Proposition 5.10 [FKS]. *Let Γ be the ellipse described above. Then*

$$\|S_\Gamma\|_{2,\rho} = \frac{a-b}{a+b} + \sqrt{1 + \left(\frac{a-b}{a+b}\right)^2}, \tag{5.20}$$

where $\rho(\omega(\theta)) = |\omega'(\theta)|^{-1/2}$.

It follows from (5.20) and Theorem 5.1 that

$$\|S_\Gamma\|_2 \le \frac{a-b}{a+b} + \sqrt{1 + \left(\frac{a-b}{a+b}\right)^2}. \tag{5.21}$$

Inequality (5.21) delivers an upper bound $1+\sqrt{2}$ for the norm $\|S_\Gamma\|$ on unweighted L_2-spaces for *all* ellipses Γ. Observe that the same upper bound $1+\sqrt{2}$ holds for the norms of sio on all contours consisting of two circular arcs or of a circular arc and a line segment (see (5.10) and Corollary 4.6).

It is interesting to compare our estimate (5.21) with the one obtained earlier in [Po]: $\|S\|_2 \le F + \sqrt{1+F^2}$, where

$$F = \frac{a^2-b^2}{8ab} \prod_{\ell=1}^{\infty} \left(1 + \frac{1}{4\ell^2(4\ell-3)}\right) \approx 0.159 \frac{a^2-b^2}{ab}. \tag{5.22}$$

For "rounded" ellipses (having $\varepsilon := (a^2-b^2)/(a^2+b^2)$ approximately between 0.25 and 1) the estimate (5.22) is sharper; for "prolonged" ellipses (5.21) is sharper. Moreover, the right-hand side of (5.22) increases unboundedly when $b = 1$, $a \to \infty$ while the right-hand side of (5.21) has a limit $1+\sqrt{2}$ under the same behavior of a, b.

Question 10. What is the norm of sio along the ellipse in the space $L_2(\Gamma)$?

Question 11. Same question for at least one smooth closed contour Γ such that $\|S_\Gamma\| \neq 1$.

Next we consider the contour consisting of n parallel lines or a family of circles (including maybe one line) having a common tangent point. It turns out that the norm of S_Γ depends only on the number of the parallel lines and not on the distances between them. Namely, the following result holds.

Theorem 5.11 ([FKS, Section 3]). *Let Γ consist of n parallel lines or of n circles (including possible a line) having a common tangent point. Then*

$$\|S_\Gamma\|_2 = \cot \frac{\pi}{4n}. \tag{5.23}$$

It was shown in Theorem 5.8 that the norm of sio on the contour consisting of two circular arcs depends on the angle at the point of intersection. Surprisingly, the norm of sio on the union of two circles does not depend on the angle they form at the point of intersection. This can be seen from the following theorem.

Theorem 5.12 [GaK]. *Let Γ consist of n straight lines having a common point or of n circles having two common points (one of the circles can degenerate into a straight line). Then*

$$\|S_\Gamma\|_2 = |S_\Gamma|_2 \geq \cot \frac{\pi}{4n}. \tag{5.24}$$

If in addition $n \leq 3$, then

$$\|S_\Gamma\|_2 = |S_\Gamma|_2 = \cot \frac{\pi}{4n}. \tag{5.25}$$

We conjectured in [GaK] that equality (5.25) holds for any $n > 1$. Recently [KS] this conjecture was confirmed in the case when the angles between the neighboring lines are all equal (see the next theorem).

Let Γ be the union of $m = 2n$ rays stemming from the same center and having alternating orientation. By shifting the curve, we may without loss of generality suppose that the center coincides with the origin. Then

$$\Gamma := \Gamma_\theta = \bigcup_{k=1}^{m} \Gamma_k, \text{ where } \Gamma_k = e^{\pi i \theta_k}\mathbb{R}_+ \tag{5.26}$$

and

$$\theta = (\theta_1, \dots, \theta_m), \quad 0 \leq \theta_1 < \theta_2 < \cdots < \theta_m < 2. \tag{5.27}$$

Rotating the curve, we may also without loss of generality suppose that $\theta_1 = 0$.

Theorem 5.13. *Let Γ be as in (5.26). Then*

$$\cot \frac{\pi}{2m} \leq \|S_\Gamma\|_2 = |S_\Gamma|_2 < \cot \frac{\pi}{4m} \tag{5.28}$$

and all the values in $[\cot\frac{\pi}{2m}, \cot\frac{\pi}{4m})$ *are attained by the norm of* S_Γ *for an appropriate choice of the* m*-tuple* (5.27). *In particular,*

$$\|S_\Gamma\|_2 = |S_\Gamma|_2 = \cot\frac{\pi}{2m} \tag{5.29}$$

for $\Gamma = \Gamma_\theta$ with $\theta_j = \frac{2j}{m}$; $j = 0, \dots, m-1$.

Consider the general case of a composed closed curve Γ without cusps. Each point $z \in \Gamma$ has a neighborhood the intersection of which with Γ consists of an even number of simple arcs, all having z as a common endpoint and otherwise disjoint. Denoting the number of these arcs by $2\nu(z)$, observe that it is different from 2 only for finitely many points (called the nodes of Γ) and let

$$\nu(\Gamma) = \max\{\nu(z) : z \in \Gamma\}.$$

Then the following statement holds ([KS, Theorem 1.3]):

Theorem 5.14. *Let* $\nu(\Gamma) = n$. *Then*

$$\cot\frac{\pi}{4n} \le |S_\Gamma|_2 < \cot\frac{\pi}{8n}. \tag{5.30}$$

Conversely, for any $d \in [\cot\frac{\pi}{4n}, \cot\frac{\pi}{8n})$ *there exists a composed closed curve* Γ *such that*

$$\nu(\Gamma) = n \quad \text{and} \quad |S_\Gamma|_2 = d. \tag{5.31}$$

If cusps are allowed, the value $\cot\frac{\pi}{8n}$ may be attained by the essential norm of S_Γ. In particular, for an astroid $n = 1$ while (see Example 4.7) $|S_\Gamma| = 1 + \sqrt{2}\,(= \cot\frac{\pi}{8})$. On the other hand, for Γ consisting of n circles (one of which may be a line) having a common tangent point, $|S_\Gamma| = \cot\frac{\pi}{4n}$ (see Theorem 5.11) while $\nu(\Gamma) = n$.

The method used in the proof of Theorems 5.13, 5.14 will be briefly explained in Section 8.

Question 12. It would be interesting to prove that for Γ defined by (5.26), $|S_\Gamma| = \cot\frac{\pi}{4n}$ if and only if Γ is the intersection of n straight lines having one common point.

Remark 5.15. We restricted our attention in this section (as well as in the whole survey) to piece-wise smooth contours because the best constant for sio on contours from wider classes are not known. We mention, however, some lower estimates for the essential norms of operators S_Γ, P_Γ, Q_Γ on $L_p(\Gamma, \rho)$ with Muckenhoupt weights ρ over Carleson curves Γ. See, for example, [Ka1, Ka2].

6. On the norms of polynomials in S and S^*

Let $F(X, Y)$ be an arbitrary polynomial in two non-commuting variables. We associate with this polynomial a function f defined by the following procedure. Let

$$P_z := \begin{pmatrix} 1 & z \\ 0 & 0 \end{pmatrix}, \text{ then of course } P_z^* = \begin{pmatrix} 1 & 0 \\ \bar{z} & 0 \end{pmatrix}, \quad z \in \mathbb{C}. \tag{6.1}$$

It is easy to show that $F(P_z, P_z^*)$ has the form

$$F(P_z, P_z^*) = \begin{pmatrix} A_{11}(|z|^2) & A_{12}(|z|^2)z \\ A_{21}(|z|^2)\overline{z} & A_{22}(|z|^2) \end{pmatrix}, \tag{6.2}$$

where A_{jk} are polynomials in one variable with complex coefficients. We set

$$r(x) := |A_{11}(x)|^2 + |A_{22}(x)|^2 + \left(|A_{11}(x)|^2 + |A_{11}(x)|^2\right) x, \tag{6.3}$$

$$s(x) := 2|A_{11}(x)A_{22}(x) - xA_{12}(x)A_{21}(x)| \tag{6.4}$$

and

$$f(x) := \frac{1}{2}\left(\sqrt{r(x)+s(x)} + \sqrt{r(x)-s(x)}\right). \tag{6.5}$$

Theorem 6.1 [FKM]. *Let H be a complex Hilbert space, S a non-trivial involutive operator* ($S^2 = I,\ S \neq \pm I$), and $P := (I+S)/2,\ Q := I - P$.

(i) *If* $\dim H = 2$, *then*

$$\|F(P,P^*)\| = f\left(\|P\|^2 - 1\right) \text{ and } |F(P,P^*)| = f\left(|P|^2 - 1\right). \tag{6.6}$$

(ii) *Equalities* (6.6) *hold for any Hilbert space H and any involution S if and only if the function f is non-decreasing on* $[0,\infty)$.

Consider an example which illustrates Theorem 6.1.

Example 6.2. The following equality holds: $2\|aI + bS + cS^*\| =$

$$\begin{aligned} &\sqrt{|a+b+c|^2 + |a-b-c|^2 + 4(|b|^2+|c|^2)\delta + 2|a^2 - (b+c)^2 - 4bc\delta|} + \\ &\sqrt{|a+b+c|^2 + |a-b-c|^2 + 4(|b|^2+|c|^2)\delta - 2|a^2 - (b+c)^2 - 4bc\delta|}, \end{aligned} \tag{6.7}$$

where $\delta := \|P\|^2 - 1$ and a, b, c are arbitrary complex numbers.

A similar equality holds for essential norms.

Let us mention several particular cases of equality (6.7):

$$\|Q\| = \|P\|;\ \|S\| = \|P\| + \sqrt{\|P\|^2 - 1}; \tag{6.8}$$

$$\|\operatorname{Re} S\| = \frac{1}{2}\left(\|S\| + \|S\|^{-1}\right) = \|P\|;\ \|\operatorname{Im} S\| = \frac{1}{2}\left(\|S\| - \|S\|^{-1}\right), \tag{6.9}$$

where $\operatorname{Re} A$ ($\operatorname{Im} A$) denotes the real (respectively, imaginary) part of the operator A.

The first equality in (6.8) was obtained in [L], the second one in [Sp1]. Some other examples are presented in [FKM]. Equalities (6.9) were used in [FKS], [GaK] and some other publications. Equality $\|SS^* + S^*S - 2I\| = \|P\|^2 - I$, which also follows from Theorem 6.1, was used in [DKS].

The following question was considered in [FKM]:

Question 6.3. For which Banach spaces $\mathcal{B}$

$$\|aI + bS\| = f_{\mathcal{B}}(a, b, \|P\|) \tag{6.10}$$

for all complex numbers a, b and involutive operator S?

The answer is given by the following

Theorem 6.4. *Let $\mathcal{B}$ be a Banach space with* $\dim \mathcal{B} > 2$. *If there exists a function g defined on $[1, \infty)$ such that $\|I - P\| = g(\|P\|)$ for any projection $P \neq 0, I$ on $\mathcal{B}$, then $\mathcal{B}$ is a Hilbert space and $g(x) = x$.*

Remark 6.5. In the case $\dim \mathcal{B} = 2$ there exists a wide class of non-Hilbert norms for which equality $\|I - P\| = \|P\|$ holds for any one-dimensional projection P. See [FKM], Remark 1.5 for details.

In the light of Theorem 6.4 we replace Question 6.3 by a more specific problem:

Problem 6.6. Let $\mathcal{B} = L_p(\Gamma_0, |t - t_1|^\beta), S := S_0,\ P = (I + S_0)/2,\ Q = I - P$. Find the function $g(a, b, p, \beta)$ $(a, b \in \mathbb{C}, p > 1, 1 < \beta < p - 1)$ such that

$$\|aI + bS_0\|_{p,\beta} = g(a, b, p, \beta). \tag{6.11}$$

In [AK] (see also [K3, page 24]) the following estimate was obtained:

$$\begin{aligned}\|aI + bS_0\|_{p,\rho} = |aI + bS_0|_{p,\rho} \geq &\left[|b|^2 \cot^2 \frac{\pi(1 + \beta)}{p} + \left(\frac{|a + b| - |a - b|}{2}\right)^2\right]^{1/2} \\ &+ \left[|b|^2 \cot^2 \frac{\pi(1 + \beta)}{p} + \left(\frac{|a + b| + |a - b|}{2}\right)^2\right]^{1/2}.\end{aligned} \tag{6.12}$$

For $p = 2$ this estimate is sharp (see 4.23). If $ab = 0$ or $a = \pm b$, then actually the equality holds for all $p \in (1, \infty)$ (see (2.9) and (2.10)). However, for arbitrary pair of complex numbers a, b Problem 6.6 is still open (even with $\beta = 0$). The following case was recently disposed of in [HKV].

Theorem 6.7. *Let $A := aI + biS_0$, where a, b are arbitrary real numbers, and $\mathcal{B} = L_p(\Gamma_0)$. Then*

$$\|A\|_p^p = \max_{x \in \mathbb{R}} \frac{|ax - b + (bx + a)\delta|^p + |ax - b - (bx + a)\delta|^p}{|x + \delta|^p + |x - \delta|^p} \tag{6.13}$$

$$= (a^2 + b^2)^{p/2} \max_{0 \leq \theta \leq 2\pi} \frac{|\cos(\theta + s)|^p + |\cos(\theta + s + \pi/p)|^p}{|\cos\theta|^p + |\cos(\theta + \pi/p)|^p} = \|\mathcal{A}\|_{\ell_p^2}^p, \tag{6.14}$$

where $\delta := \tan(\pi/2p),\ s := \arctan(b/a)$ and

$$\mathcal{A} := \begin{pmatrix} a + b\cot\frac{\pi}{p} & -b\csc\frac{\pi}{p} \\ b\csc\frac{\pi}{p} & a - b\cot\frac{\pi}{p} \end{pmatrix} \tag{6.15}$$

is the operator acting on the two-dimensional real space ℓ_p^2 with the norm

$$\|(x, y)\|^p = |x|^p + |y|^p.$$

See Subsection 7.4 for the explanation of the method used in [HKV].

It follows from (6.14), (6.15) that the estimate (6.12) in general is not sharp. Let, for example, $A = 3I + iS$. According to (6.15), in this case $\|\mathcal{A}\|_4 = 500^{1/4} \approx 4.7$ while (6.12) yields the value $1 + \sqrt{11} \approx 4.3$.

We conclude this section with the following

Theorem 6.8. (Nakazi–Yamomoto [NY]). Let $a, b \in L^\infty(\Gamma_0)$. Then

$$\|aP_0 + bQ_0\|_2 = \inf_{k\in H^\infty} \left\| \frac{|a|^2+|b|^2}{2} + \sqrt{|a\bar{b}+k|^2 + \left(\frac{|a|^2+|b|^2}{2}\right)^2} \right\|_\infty .$$

7. Some important inequalities and their applications

7.1. Let Γ be a closed contour such that S_Γ is bounded on $L_r(\Gamma)$ for some $r \in [2,\infty)$. Then

$$\|S_\Gamma\|_{2r} \le \|S_\Gamma\|_r + \sqrt{1+\|S_\Gamma\|_r^2}. \tag{7.1}$$

This inequality was obtained by Misha Cotlar ([Co], see also [GK6, Section 2, Ch. I]).

The next statement follows from (7.1).

Theorem 7.1. *Let S_Γ be bounded on $L_2(\Gamma)$. Denote $\phi = \operatorname{arccot}\|S_\Gamma\|_2$. Then*

$$\|S_\Gamma\|_p \le \cot\frac{2\phi}{p} \quad (p = 2^n, n = 1,2,3,\dots). \tag{7.2}$$

Example 7.1.1. Let Γ consist of k circles having one common point, or k parallel straight lines, or k straight lines having one common points and forming equal angles between the neighboring lines. Then (see Section 5) $\|S_\Gamma\|_2 = \cot\frac{\pi}{4k}$, and by Theorem 7.1

$$\|S_\Gamma\|_p \le \cot\frac{\pi}{2kp} \quad (p = 2^n, n = 1,2,\dots). \tag{7.3}$$

Two questions:

13. Is inequality (7.3) sharp?

14. Does the estimate (7.2) hold for any $p \ge 2$?

7.2. Calderón-Pichorides inequalities

$$|\sin\varphi|^p \le a(p)\cos^p\varphi - b(p)|\cos(p\varphi)| \quad \left(1 < p \le 2,\ 0 \le \varphi \le \frac{\pi}{2}\right) \tag{7.4}$$

([Ca], see also [Z, Chap. VII, Sec. 2, equality (2.18)]).

This equality was obtained and used by A. Calderón for a direct (not using interpolation theorems) proof of the boundedness of the operator S_0 on L_p. The best values for constants $a(p), b(p)$ in Calderón's inequality (7.4) were obtained by R. Pichorides [Pi]:

$$a(p) = \tan^p\frac{\pi}{2p}, \quad b(p) = \frac{\sin^{p-1}(\pi/2p)}{\cos(\pi/2p)} \tag{7.5}$$

and this allowed to confirm the equalities (2.4). An important role in Pichorides's proof was played by the subharmonicity of the function

$$g(z) = |z|^p \cos p(\alpha(z)) \ (z \neq 0) \text{ and } \ g(0) := 0, \tag{7.6}$$

where $\alpha(x+iy) = \arctan(y/|x|)$ and $1 < p \le 2$.

7.3. Hollenbeck-Verbitsky inequality B. Hollenbeck and I. Verbitsky confirmed the conjecture

$$\|P_0\|_p = \|Q_0\|_p \le \frac{1}{\sin \pi/p} \tag{7.7}$$

(see Subsection 2.2) along the following lines. They proved that the function

$$\Phi(w,z) = \frac{1}{\sin^p \frac{\pi}{p}} |w+\overline{z}|^p - \max(|w|^p, |z|^p), \quad (w,z) \in \mathbb{C}^2, \tag{7.8}$$

has a *plurisubharmonic* minorant $F(w,z) = b_p \operatorname{Re}[(wz)^{p/2}]$ on $\mathbb{C}^2$ such that $F(0,0) = 0$. More precisely, the following equality, which might be of independent interest, was established:

$$\max(|w|^p, |z|^p) \le a_p |w+\overline{z}|^p - b_p \operatorname{Re}[(wz)^{p/2}] \tag{7.9}$$

for all $(w,z) \in \mathbb{C}^2$ and $1 < p \le 2$, with the sharp constants a_p, b_p given by

$$a_p = \frac{1}{\sin^p \frac{\pi}{p}}, \quad b_p = \frac{2|\cos(\pi/p)|^{1-p/2}}{\sin(\pi/p)} \quad (b_2 := 2). \tag{7.10}$$

This allowed [HV] to confirm the conjecture (2.6) as well as (2.10) and also to obtain the inequality

$$\|\max(|Pf|, |Qf|)\|_p \le \frac{1}{\sin \frac{\pi}{p}} \|f\|_p \quad (f \in L_p(\Gamma_0)) \quad (1 < p \le 2) \tag{7.11}$$

which is stronger than (7.7).

7.4. Hollenbeck-Kalton-Verbitsky inequality B. Hollenbeck, N.J. Kalton and I.E. Verbitsky obtained the following inequalities.

Proposition 7.4.1. *Let*

$$C_p := \sqrt{2} \max_{\theta \in [0,2\pi]} \left[\frac{|\cos(\theta - \pi/4)|^p + |\cos(\theta - \pi/4 + \pi/p)|^p}{|\cos(\theta)|^p + |\cos(\theta + \pi/p)|^p} \right]^{\frac{1}{p}} \quad (1 < p < \infty). \tag{7.12}$$

Then

$$|x-y|^p \le C_p^p |x|^p - G(x,y), \quad (x,y) \in \mathbb{R}^2 \tag{7.13}$$

where $G(x,y)$ *is the subharmonic minorant of* $\Phi(x,y) = C_p^p |x|^p - |x-y|^p$ *in the plane satisfying* $G(0,0) = 0$.

In the proof (see [HKV, Lemma 4.2]), an explicit construction for the maximal subharmonic minorant of $\Phi(x,y)$ was given, and it was also shown that the constant C_p is sharp. Moreover, the following more general statement was proved.

Proposition 7.4.2. *Let*

$$B_p = (a^2+b^2)^{p/2} \max_{0\le\theta\le 2\pi} \frac{|\cos(\theta+s)|^p + |\cos(\theta+s+\pi/p)|^p}{|\cos\theta|^p + |\cos(\theta+\pi/p)|^p}, \tag{7.14}$$

where $s := \arctan(b/a)$. *Then there exists a subharmonic function* $G(z)$ *such that*

$$|a \operatorname{Re} z - b \operatorname{Im} z|^p \le B_p |\operatorname{Re} z|^p - G(z) \tag{7.15}$$

and the constant B_p *is sharp.*

These inequalities were used in the proof of (3.17) and Theorem 6.7.

8. Symmetric symbols and their applications

Recall that an algebra $\mathcal{A}$ of operators acting on a Banach space X admits a *matrix symbol* if there exists an algebraic homomorphism K of $\mathcal{A}$ into an algebra of matrix functions (possibly of variable size) depending on the parameter x with the domain determined by $\mathcal{A}$ and such that an element of $\mathcal{A}$ is Fredholm if and only if its image under K is point-wise invertible. Matrix symbols for *scalar* sio with piecewise continuous coefficients were first constructed and used to derive Fredholm conditions in [GK8]. In [K1], [AK1,2], [N] the matrix symbol was first used for computing the best constants for some sio. Of course, not every algebra of operators admits a matrix symbol. The description of Banach algebras which admit matrix symbols was obtained in [K7] (see also [K3, Theorems 21.1 and 22.2]). See [GK8], [K3,6], [BGKKRSS] for the symbol construction for various algebras of singular integral operators and for further bibliographical remarks.

For a C^*-algebra $\mathcal{A}$, the notion of a *symmetric* symbol K can be introduced. Namely, the symbol K is symmetric if it satisfies the additional requirement $K(A^*) = (K(A))^*$. This additional property implies that for any $A \in \mathcal{A}$,

$$|A| = \max_x \|K_x(A)\|, \tag{8.1}$$

where the norm in the right-hand side is by definition the maximal s-number of the matrix.

Consider, for example, the space $X = L_2(\Gamma)$ with Γ given by (5.26) and the C^*-algebra $\mathcal{A}$ generated by operator S_Γ and the orthogonal projections p_k – multiplications by the characteristic functions of the rays Γ_k, $k = 1, \dots, m = 2n$.

The symmetric symbol K for this algebra can be extracted from [K6] or [K3, pp. 125,98,100]. Namely, the parameter x in this case varies on $[0,1]$, all the values of K_x are of the same size $m \times m$, and K is an algebraic homomorphism of $\mathcal{A}$ into the algebra of $m \times m$ matrix functions depending on the parameter $x \in [0,1]$. For example, $K_x(p_k) = \operatorname{diag}(0, \dots 0, 1, 0, \dots 0)$ is a diagonal matrix with 1 on the kth place. Let $A_\Gamma := (S_\Gamma - S_\Gamma^*)/2$. The matrix $K_x(A_\Gamma) := C(x)$ is real skew-symmetric, with the entries

$$c_{jk}(x) = (-1)^j (1-x)^{1-s_{jk}} x^{s_{jk}} + (-1)^k (1-x)^{s_{jk}} x^{1-s_{jk}}, \quad j < k \tag{8.2}$$

above the main diagonal (here $s_{jk} = \frac{\theta_k - \theta_j}{2}$). Consequently, $\|A_\Gamma\| = |A_\Gamma| = \max_{x\in[0,1]} \|C(x)\|$.

Remark 8.2. As soon as $|A_\Gamma|$ is known, the norms of operators $S_\Gamma, aP_\Gamma + bQ_\Gamma$ and many others (see Theorem 6.1 and equalities (6.7)–(6.9)) can also be explicitly expressed.

Remark 8.3. Γ is a compact subset of $\dot{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and $|S_\Gamma|^\infty = |S_\Gamma|^0 \geq |S_\Gamma|^t = 1$, $t \in (0, \infty)$. It follows from (4.4), (4.8) that $|S_\Gamma| = |S_\Gamma|^0$.

The last equality along with Theorems 4.1, 4.2 allow us to obtain the results of Theorem 5.14 as a corollary of Theorem 5.13.

References

[AK1] R. Avendanio and N. Krupnik, *Local principle of computation of the factor-norms of singular integral operators*, Funkts. Analiz Priloz. **22** (1988), no. 2, 57–58 (Russian). English translation: Funct. Anal. Appl. **22** (1988), 130–131.

[AK2] R. Avendanio and N. Krupnik, *On the connection between the norms and essential norms of SIO*, Proc. of the I.N. Vekua seminar on applied math. **1** (1988), no. 2, 5–8 (Russian).

[Ba] A. Baernstein, *Some sharp inequalities for conjugate functions*, Indiana Univ. Math. J. **27** (1978), 833–852.

[Bi] M.S. Birman, *Re-expansion operators as objects of spectral analysis*, in: Linear and Complex Analysis Problem Book, Lecture Notes in Math., Springer **1043** 1984, 130–134.

[BKS] A. Böttcher, N. Krupnik and B. Silbermann, *A general look at local principles with special emphasis on the norm computation aspect*, Integr. Equat. Oper. Th. **11** (1988), 455–479.

[BGKKRSS] A. Böttcher, I. Gohberg, Yu. Karlovich, N. Krupnik, S. Roch, B. Silbermann, I. Spitkovsky, *Banach algebras generated by N idempotents and applications*, Singular integral operators and related topics (Tel-Aviv,1995), Oper. Theory Adv. Appl., Vol l90, Birkhäuser, Basel, 1996, 19–54.

[Ca] A.P. Calderón, *On the theorems of M. Riesz and Zygmund*, Proc. AMS **1** (1950), 533–534.

[Ca1] A.P. Calderón, *Cauchy integrals on Lipschitz curves and related operators*, Proc. Nat. Acad. Sci, USA **74** (1977), 1324.

[Co] M. Cotlar, *A unified theory of Hilbert transforms and ergodic theorems*, Revista Mat. Cuyana **1**, (1955), no. 2, 105–167.

[DG] V. Dybin, S. Grudsky, *The Riemann boundary value problem with discontinuities of almost periodic type in the coefficient*, Soviet Math. Dokl. **18** (1977), no. 6, 1383–1387.

[DK] R. Duduchava N. and Krupnik, *On the norms of singular integral operator on curves with cusps*, Integr. Equat. Oper. Th. **20** (1994), no. 4, 377–382

[DKS] R. Duduchava, N. Krupnik and E. Shargorodsky, *An algebra of integral operators with fixed singularities in kernels*, Integr. Equat Oper. Th. **33** (1999), no. 4, 406–425. See also: Preprint KCL-MTH-98-26 (King's College, London).

[FKM] I. Feldman, N. Krupnik and A. Markus, *On the norm of polynomials of two adjoint projections*, Integr. Equat. Oper. Th. **14** (1991), 70–90.

[FKS] I. Feldman, N. Krupnik and I.M. Spitkovsky, *Norms of the singular integral operator with Cauchy kernel along certain contours*, Integr. Equat. Oper. Th. **24** (1996), no. 1, 68–80.

[Gru] S. Grudsky, *On the compactness of the integral operator*, Dep. in VINITY, 18. **11** (1980), N 4856-80, 10 pp.

[GD] K.A. Georgiev and V.M. Deundyak, *A criterion for the weighted Cauchy singular operators to belong to the algebra of singular integral operators with coefficients in a Sarason algebra*, Func. Anal. Appl. **28** (1994), 287–288.

[GaK] J. Galperin and N. Krupnik, *On the norms of singular integral operators along certain curves with intersections*, Integr. Equat. Oper. Theory **29** (1997), 10–16.

[GoKr] I. Gohberg, M. Krein, *Theory and Applications of Volterra Operators in Hilbert Space*, American Mathematical Society, Providence, 1970, 430 pp.

[GK1] I. Gohberg and N. Krupnik,*On the norm of the Hilbert transform in the space* L_p, Funk. Anal. Prilozhen. **2**(1968), no. 2 91–92 (Russian). English translation: Functional Anal. Appl. **2** (1968), 180–181.

[GK2] I. Gohberg and N. Krupnik, *On the spectrum of SIO on* L_p, Studia Mathematica, T. XXXI (1968), 347–362.

[GK3] I. Gohberg and N. Krupnik, *On the quotient norm of singular integral operators*, Mat. Issled. Kishinev **4** (1969), no. 3, 136–139 (Russian). English translation: Amer. Math. Soc. Transl., (2), **III** (1978), 117–119.

[GK4] I. Gohberg and N. Krupnik, *On the spectrum of SIO on weighted spaces* L_p, DAN SSSR **185** (1969), no. 4, 745–748.

[GK5] I. Gohberg and N. Krupnik, *Introduction to the theory of one-dimensional SIO*, Shtiintsa, Kishinev, 1973 (Russian).

[GK6] I. Gohberg and N. Krupnik, *One-Dimensional Linear Singular Integral Equations, Vol. I, Introduction*, O.T. 53, Birkhäuser Verlag, Basel-Boston, 1992, 226 pp.

[GK7] I. Gohberg and N. Krupnik, *One-Dimensional Linear Singular Integral Equations, Vol. II, General Theory and Applications*, OT 54, Birkhäuser Verlag, Basel-Boston, 1993, 232 pp.

[GK8] I. Gohberg and N. Krupnik, *Singular integral operators with piecewise continuous coefficients and their symbols*, Izv. Akad. Nauk SSSR, Ser. Mat. **35** (1971), 940–964.

[HL] G.H. Hardy and J.E. Littlewood, *Some more theorems concerning Fourier series and Fourier power series*, Duke Math. J. **2** (1936), 354–382.

[HV] B. Hollenbeck, I.E. Verbitsky, *Best constants for the Riesz projection*, Journal of Functional Analysis **175** (2000), 370–392.

[HKV] B. Hollenbeck, N.J. Kalton and I.E. Verbitsky, *Best constants for some operators associated with the Fourier and Hilbert transforms*, Studia Mathematica **157** (2003), no. 3, 237–277.

[I] I. Itskovich, *Integral of Cauchy type considered as operator in Hilbert space*, Uchen. zap. Kishinevs. Univ. **5** (1952), 37–41.

[K1] N. Krupnik, *On the quotient norm of a singular integral operator*, Mat. Issled. Kishinev **10** (1975), no. 2, 255–263 (Russian).

[K2] N. Krupnik, *On singular integral operators with matrix coefficients*, Mat. Issled. Kishinev **45** (1977), 93–100 (Russian).

[K3] N. Krupnik, *Banach algebras with symbol and singular integral operators*, Stiintsa, Kishinev, 1984 (Russian). English translation: Birkhäuser Verlag, Basel-Boston, 1987, 205 pp.

[K4] N. Krupnik,*The exact constants in Simonenko's theorem on an envelope of a family of operators of local type*, Funkt. Anal. Prilozhen. **20** (1986), no. 2, 119–120. (Russian). English translation: Funct. Anal. Appl. **20**, (1986), no. 2, 144–145.

[K5] N. Krupnik, *The conditions of selfadjointness of the operator of singular integration*, Integr. Equat. Oper. Th. **14** (1991), 760–763.

[K6] N. Krupnik, *Symmetrization of the symbol in Banach algebras generated by idempotents*, Integr. Equat. Oper. Th. **31** (1998), 470–481.

[Ka1] A.Yu. Karlovich, *On the essential norm of the Cauchy singular operator in weighted rearrangement-invariant spaces*, Integr. Equat. Oper. Th. **38** (2000), 28–50.

[Ka2] A.Yu. Karlovich, *Fredholmness of singular integral operators with piecewise continuous coefficients on weighted Banach function spaces*, J. Integr. Equat. Appl. **15** (2003), 263–320.

[KN] N. Krupnik and V.I. Neagu, *On singular integral operators in weighted L_p spaces*, Mat. Issled. Kishinev **9** (1974), no. 3, 206–209 (Russian).

[KP] N. Krupnik and E.P. Polonskii, *The norm of an operator of singular integration*, Funkt. Anal. Prilozhen. **9** (1975), no. 3, 73-74 (Russian). English translation: Funct. Anal. App. **9**, (1975), no. 4, 337–339.

[KS] N. Krupnik, I.M. Spitkovsky, *On the norms of singular integral operators on contour with intersections*, Complex Analysis and Operator Theory **2** (2008), no. 4, 617–626.

[KRA] Yu.I. Karlovich and E. Ramírez de Arellano, *Singular integral operators with fixed singularities on weighted Lebesgue spaces*, Integr. Equat. Oper. Theory **48** (2004), 331–363.

[L] V.E. Ljance, *Some properties of idempotent operators*, Teor. i Prikl. Mat. **1** (1959), 16–22 (Russian).

[N] V.I. Neagu, *On the essential norms of SIO for some non-smooth contours*, Mat. Issled. **9** (1974), no. 2, 109–125.

[Ni] N.K. Nikolskii, *Treatise of the shift operators: Spectral function theory*, Springer Verlag, Berlin, New York, 1988.

[NY] T. Nakazi and T. Yamamoto, *Norms of some singular integral operators and their inverse operators*, J. Operator Theory **40** (1998), 185–207.

[Pa] S. Papadopoulos, *A note on the M. Riesz theorem for the conjugate function*, Bull. Polish Acad. Sci. Math. **47** (1999), 283–288.

[Pe] A. Pelczyński, *Norms of classical operators in function spaces*, Colloque en l'honneur of Laurent Schwartz, Vol. I, Astérisk **131** (1985), 137–162.

[Pi] S.K. Pichorides, *On the best values of the constants in the theorems of M. Riesz, Zygmund and Kolmogorov*, Studia mathematica **19** (1972), no. 2, 165–170.

[Po] I. Pokorny, *On the norm of singular operator acting on an ellipse*, Casopis pro Pestovani Matematiky **110** (1985), 158–171.

[R] M. Riesz, *Sur les functions conjuguées*, Math.Z. (1927), 213–244.

[S] I.B. Simonenko, *A new general method of investigating linear operator equations of singular integral type* I. Izv. Akad. Nauk SSSR, Ser. Mat. **29** (1965), no. 3, 567–586 (Russian).

[Sp] I.M. Spitkovsky, *The factorization of matrix-valued functions whose Hausdorff set lies inside an angle*, Soobsch. Akad. Nauk Gruzin. SSR **86** (1977), no. 3, 561–564 (Russian).

[Sp1] I.M. Spitkovsky, *The partial indices of continuous matrix-valued functions*, Dokl. Akad. Nauk SSSR **229** (1976), no. 5, 1059–1062; Soviet Math. Dokl. **17** (1976), no. 4, 1155–1159.

[T] E.C. Titchmarsh, *Reciprocal formulae for series and integrals*, M. Z. **25** (1926), 321–347.

[V] I. Verbitsky, *Estimate of the norm of a function in Hardy space in terms of the norms of its real and imaginary part*, Math. Issled. **54** (1980), 16–20 (Russian). English translation: Amer. Math. Soc. Transl. (2) **124** (1984), 11–15.

[VK1] I. Verbitsky and N. Krupnik, *The exact constants in the theorems of K.I. Babenko and B.V. Khvedelidze on boundedness of singular operators*, Soob. Akad. Nauk Gruz. SSR **85** (1977), no. 1, 21–24 (Russian).

[VK2] I. Verbitsky and N. Krupnik, *The exact constants in theorems on boundedness of singular operators in weighted spaces and their application*, Mat. Issled. **54** (1980), 21–35 (Russian).

[VK3] I. Verbitsky and N. Krupnik, *The norm of analytical projection*, Lect. Notes in Math., **1043** (1984), 325–327.

[Z] A. Zygmund, *Trigonometric series*, Vols. I, II, Cambridge Univ. Press, 1959.

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Received: February 13, 2009
Accepted: April 8, 2009

Operator Theory:
Advances and Applications, Vol. 202, 395–410

Gantmacher–Krein Theorem for 2-totally Nonnegative Operators in Ideal Spaces

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Abstract. The tensor and exterior squares of a completely continuous nonnegative linear operator A acting in the ideal space $X(\Omega)$ are studied. The theorem representing the point spectrum (except, probably, zero) of the tensor square $(A \otimes A)_M$ in the terms of the spectrum of the initial operator A is proved. The existence of the second (according to the module) positive eigenvalue λ_2, or a pair of complex conjugate eigenvalues of a completely continuous non-negative operator A is proved under the additional condition, that its exterior square $(A \wedge A)_M$ is also nonnegative.

Mathematics Subject Classification (2000). Primary 47B65. Secondary 47A80, 47B38, 46E30.

Keywords. Total positivity, ideal spaces, tensor products, exterior products, point spectrum.

1. Introduction

This paper presents the results of our study of the spectrum of 2-totally-nonnegative operators in ideal spaces. The theory of total positivity is mainly based on the results of F.R. Gantmacher and M.G. Krein (see [7]), concerning the properties of the spectrum of *k-totally primitive* matrices (i.e., matrices, which are nonnegative and primitive along with their jth compound matrices $(1 < j \leq k)$ up to the order k). In the most important case $k = n$ k-totally primitive matrices are called *oscillatory*. In monograph [7] the following statement was proved: *if the matrix* $\mathbf{A}$ *of a linear operator* A *in the space* $\mathbb{R}^n$ *is* k*-totally primitive, then the operator* A *has* k *positive simple eigenvalues* $0 < \lambda_k < \cdots < \lambda_2 < \lambda_1$, *with a positive eigenvector* e_1 *corresponding to the maximal eigenvalue* λ_1, *and an eigenvector* e_j, *which has exactly* $j-1$ *changes of sign, corresponding to* j*th eigenvalue* λ_j (see [7], p. 310, Theorem 9).

Communicated by L. Rodman.

The study of linear integral operators with analogous properties predates the study of oscillatory matrices. The main results concerning this problem were received by O.D. Kellog (see [16]). He proved the theorem about spectral properties of continuous symmetric totally nonnegative kernels. Later this theorem was generalized by F.R. Gantmacher for the non-symmetric case. This result one can find in monograph [7] in the following form: *let* $k(t,s) \in C[0,1]^2$ *satisfy the following conditions*:

(a) *for any* $0 < t_0 < t_1 < \cdots < t_n < 1$ *and* $0 < s_0 < s_1 < \cdots < s_n < 1$ $n = 0,\ 1,\ \ldots$ *the inequality*
$$k\begin{pmatrix} t_0 & t_1 & \ldots & t_n \\ s_0 & s_1 & \ldots & s_n \end{pmatrix} \geq 0$$
is true;

(b) *for any* $0 < t_0 < t_1 < \cdots < t_n < 1 \quad n = 0,\ 1,\ \ldots$ *the inequality*
$$k\begin{pmatrix} t_0 & t_1 & \ldots & t_n \\ t_0 & t_1 & \ldots & t_n \end{pmatrix} > 0$$
is true.

Then all the eigenvalues of the linear integral equation
$$\int_0^1 k(t,s)x(s)ds = \lambda x(t)$$
are positive and simple:
$$0 < \cdots < \lambda_n < \cdots < \lambda_2 < \lambda_1,$$
with a strictly positive on $(0,1)$ *eigenfunction* $e_1(t)$ *corresponding to the maximal eigenvalue* λ_1*, and an eigenfunction* $e_n(t)$*, which has exactly* $n-1$ *changes of sign and no other zeroes on* $(0,1)$*, corresponding to the nth eigenvalue* λ_n (see [7], p. 211).

The proof of this statement one can find also in [21], where the history of the theory of totally positive matrices and kernels is presented in detail. Unlike monograph [7], in which the basic statements of the theory are given in the form most suitable for the study of small oscillations of mechanical systems, in [21] definitions and theorems about the properties of totally positive kernels are given in the pure form.

In paper [15] by S.P. Eveson the result mentioned was spread onto a wider class of kernels. The existence of k positive eigenvalues was proved under some additional assumptions for the case of a compact linear integral operator, acting in $L_2[0,1]$, which kernel is totally positive of order k. A substantial contribution into the development of the theory of totally positive and sign-symmetric kernels was made by S. Karlin (see [6]).

Once a great number of papers are devoted to the theory of totally positive matrices and kernels, in the case of abstract (not necessarily integral) compact linear operators the situation is absolutely different. Here we can mention only a small number of papers. In paper [23] oscillatory operators in $C[a,b]$ were studied by the method of the passage to the limit from finite-dimensional approximants. In paper [24] another method of generalization was suggested. But this method was realized also only for the space $C[a,b]$. Many results, related to the applications of the theory of oscillation to differential operators, were included into monograph [22] by Yu.V. Pokornyi and his group.

In paper [17] we studied 2-totally indecomposable operators (i.e., indecomposable operators that are nonnegative with respect to some cone K, and such that their exterior squares are also nonnegative and indecomposable) in the spaces $L_p(\Omega)$ $(1 \leq p \leq \infty)$ and $C(\Omega)$. We proved the existence of the second (according to the module) eigenvalue λ_2 of a completely continuous non-negative operator A under the condition that its exterior square $A \wedge A$ is also non-negative. The simplicity of the first and second eigenvalues was proved and the interrelation between the indices of imprimitivity of A and $A \wedge A$ was examined for the case, when the operators A and $A \wedge A$ are indecomposable. The difference (according to the module) of λ_1 and λ_2 from each other and from other eigenvalues was proved for the case, when A and $A \wedge A$ are primitive.

In the present paper we are going to generalize the results, received in [17], for 2-totally nonnegative operators in some ideal spaces. As the authors believe, the natural method of the examination of such operators is a crossway from studying an operator A, acting in an ideal space X to the study of the operators $(A \otimes A)_M$ and $(A \wedge A)_M$, acting in spaces with mixed norms. Let us turn now to a more detailed outline of the paper. In Section 2 we briefly consider the basic properties of ideal spaces. Tensor and exterior squares of ideal spaces are described in Section 3. The connection between the topological exterior square of the ideal space X and the space with mixed norm is also clarified in Section 3. In Section 4 we work with the tensor and exterior square of a linear operator $A : X \to X$. These operators act in the tensor and exterior square of the initial space X respectively. Generally, in Sections 3 and 4 we develop the mathematical tools that will enable us to define the class of abstract totally nonnegative operators and to generalize the results of Gantmacher and Krein. Sections 5 and 6 present a number of results on the description of the spectrum and the parts of the spectrum of the tensor square and the exterior square of a completely continuous non-negative linear operator A acting in the ideal space $X(\Omega)$ in terms of the spectrum of the initial operator. The main mathematical results of this paper are concentrated in Section 6, where we prove the existence of the second according to the module positive eigenvalue λ_2, or a pair of complex adjoint eigenvalues of a completely continuous non-negative operator A under the additional condition, that its exterior square $(A \wedge A)_M$ is also nonnegative. For the case when A is a linear integral operator, the main theorem is formulated in terms of kernels.

2. Ideal spaces. Basic definitions and statements

Let $(\Omega, \mathfrak{A}, \mu)$ be a triple, consisting of a set Ω, σ-algebra $\mathfrak{A}$ of measurable subsets and σ-finite and σ-additive complete measure on $\mathfrak{A}$. Denote by $S(\Omega)$ the space of all measurable finite almost everywhere on Ω functions (further we consider the equivalent functions to be identical). Let $X(\Omega)$ be a Banach ideal space, i.e., a Banach space of all measurable on Ω functions having the following property: from $|x_1| \leq |x_2|$, $x_1 \in S(\Omega)$, $x_2 \in X$, it follows that $x_1 \in X$ $\|x_1\|_X \leq \|x_2\|_X$ (the definition and basic properties of ideal spaces are taken from paper [25], see also [13]). Consider the support supp$X$ of the space X to be the least measurable subset, outside which all the functions from X are equal to zero. Let supp$X = \Omega$, i.e., there exist functions from the space X, which are positive almost everywhere on Ω. The Banach ideal space X is called *regular*, if the norm in X is order-continuous, i.e., for every sequence $\{x_n\} \subset X$ from $0 \leq x_n \downarrow 0$ it follows that $\|x_n\| \to 0$. X is called *almost perfect*, if the norm in X is order-semicontinuous, i.e., for every sequence $\{x_n\} \subset X$ from $0 \leq x_n \uparrow x \in X$ it follows that $\|x_n\| \to \|x\|$. It's easy to see (see, for example, [25]), that every regular space is almost perfect.

Let us denote by $X_{12}(\Omega \times \Omega)$ the set of all measurable with respect to all the variables functions $x(t_1, t_2)$ on $\Omega \times \Omega$, which satisfy the following conditions:

(a) for almost every t_2 the function $x(\cdot, t_2)$ belongs to X;
(b) the function $\|x(\cdot, t_2)\|$ also belongs to X.

If the space X is almost perfect then the set X_{12} is linear (see, for example [13], [14]). The norm in X_{12} is introduced according to the following rule:

$$\|x(t_1, t_2)\|_{12} = \|\|x(t_1, t_2)\|_{(1)}\|_{(2)},$$

where indices (1) and (2) mean, that the norm of the space X is used firstly for the first variable, and then for the second variable.

The space $X_{21}(\Omega \times \Omega)$ with the norm

$$\|x(t_1, t_2)\|_{21} = \|\|x(t_1, t_2)\|_{(2)}\|_{(1)}$$

is defined similarly.

In the case of an almost perfect X both the space X_{12} and the space X_{21} are almost perfect Banach ideal spaces (see [25], [13], and also [3], p. 1234, theorem 3, where the completeness of the space with mixed norm is proved).

Further we will be interested in the space $\widetilde{X} = X_{12} \cap X_{21}$ of functions that are common for the spaces X_{12} and X_{21}. The norm in this space is introduced by the formula:

$$\|x(t_1, t_2)\|_M = \max\{\|x(t_1, t_2)\|_{12}, \|x(t_1, t_2)\|_{21}\}.$$

Note that the space $\widetilde{X}$ is regular if and only if the space X is regular.

In connection with the introduced intersection of the spaces X_{12} and X_{21} there arises a natural question of the possibility of their coincidence. For $X = L_p$ from the Fubini theorem it follows that X_{12} and X_{21} coincide according to the fund of elements and according to their norms. However, this is not true in the general case. Moreover, the coincidence of the spaces X_{12} and X_{21} is characteristic for the class of the spaces L_p. For the regular Banach ideal space X N.J. Nielsen proved, that from $X_{12} = X_{21}$ it follows that X is lattice-isomorphic to L_p-space (see [20]). The results concerning this problem one can also find in [4] and [1].

3. Tensor and exterior squares of ideal spaces

The algebraic tensor square $X \otimes X$ of the space X is defined as the set of all functions of the form

$$x(t_1, t_2) = \sum_{i=1}^{n} x_1^i(t_1) x_2^i(t_2),$$

where $x_1^i, x_2^i \in X$. Further call the elements of $X \otimes X$ *degenerate functions.* By the way, *the algebraic exterior square* $X \wedge X$ of the space X is defined as the set of all antisymmetric functions (i.e., functions $x(t_1, t_2)$, for which $x(t_1, t_2) = -x(t_2, t_1)$) from $X \otimes X$.

Generally, the norm on $X \otimes X$ can be defined by different ways. Let us go through definitions, that will be used further. The norm α on $X \otimes X$ is called a *crossnorm*, if for any $x_1, x_2 \in X$ the following equality holds:

$$\|x_1(t_1)x_2(t_2)\|_\alpha = \|x_1(t_1)\| \|x_2(t_2)\|.$$

There exists the greatest crossnorm π (see, for example [19]), which is defined by the equality:

$$\|x(t_1, t_2)\|_\pi = \inf \sum_{i=1}^{n} \|x_1^i\| \|x_2^i\|,$$

where inf is taken over all representations

$$x(t_1, t_2) = \sum_{i=1}^{n} x_1^i(t_1) x_2^i(t_2), \;\; x_1^i, x_2^i \in X.$$

The least crossnorm does not exist, however there exists the least *reasonable* crossnorm. Here the norm α on $X \otimes X$ is called reasonable, if α is a crossnorm, and the dual norm α' is also a crossnorm on $X' \otimes X'$, where the dual norm is defined on $X' \otimes X'$ by the equality

$$\|x'\|_{\alpha'} = \sup \{\langle x, x' \rangle, \|x\|_\alpha \leq 1\}, \quad x \in X \otimes X, \quad x' \in X' \otimes X'.$$

The least reasonable crossnorm is denoted by ϵ and is defined by the following rule:

$$\|x\|_\epsilon = \sup \left\{ \left| \sum_{i=1}^{n} \langle x_1^i, x_1' \rangle \langle x_2^i, x_2' \rangle \right|, \|x_1'\|_{X'}, \|x_2'\|_{X'} \leq 1 \right\}.$$

Note that the completion of the algebraic tensor square $X \otimes X$ of the ideal space X with respect to the norms π or ϵ will not be ideal. It is natural to define the norm on $X \otimes X$ in such a way that the completion with respect to this norm will be an ideal space or its part. With this purpose V.L. Levin introduced in [18] the following crossnorm on $X \otimes X$:

$$\|x\|_L = \inf\left\{ \|u\|_X : u \geq \left|\sum_i x_1^i \langle x_2^i, x^* \rangle\right|, \|x^*\|_{X^*} \leq 1 \right\}.$$

V.L. Levin proved (see [18], p. 55, proof of the theorem 1), that the topology of the space X_{21} induces on $X \otimes X$ the same topology, as the norm L.

Note that every norm α on $X \otimes X$ which satisfies inequalities $\epsilon \leq \alpha \leq \pi$ will be reasonable. In particular, the Levin's norm L is reasonable (see [18], p. 53, lemma 3).

Further let us call the completion of $X \otimes X$ with respect to the Levin's norm the *L-complete* tensor square of the space X and let us denote it by $(\widetilde{X \otimes X})_L$. As it was noticed above, $(\widetilde{X \otimes X})_L$ is a closed subspace of X_{21}. The space $(\widetilde{X \otimes X})_L$ was studied thoroughly by A.V. Bukhvalov in papers [2] and [3]. He also proved the criteria of the coincidence of this space and X_{21} (see [3], p. 7, theorem 0.1, and also [2], p. 1235, theorem 4).

Bukhvalov's theorem. *Let X be a Banach ideal space. Then the following statements are equivalent*:

(i) the set of all degenerate functions is dense in X_{21};
(ii) the equality $(\widetilde{X \otimes X})_L = X_{21}$ is true;
(iii) the space X is regular.

Bukhvalov's theorem implies that for a regular Banach ideal space the following equality holds:

$$(\widetilde{X \otimes X})_L^a = X_{21}^a(\Omega \times \Omega),$$

where $X_{21}^a(\Omega \times \Omega)$ is a subspace of the space $X_{21}(\Omega \times \Omega)$, which consists of antisymmetric functions.

Further note that $\wedge$-product of arbitrary functions $x_1, x_2 \in X$

$$(x_1 \wedge x_2)(t_1, t_2) = x_1(t_1)x_2(t_2) - x_1(t_2)x_2(t_1)$$

belongs to the space $X_{21}(\Omega \times \Omega)$, and the following equality holds:

$$x_1 \wedge x_2(t_1, t_2) = -x_1 \wedge x_2(t_2, t_1).$$

Therefore, for any $x_1, x_2 \in X$ the function $x_1 \wedge x_2$ belongs to the subspace $X_{21}^a(\Omega \times \Omega)$.

Moreover, *an arbitrary antisymmetric function $x(t_1, t_2)$, which belongs to the space X_{21}, at the same time belongs to X_{12}, and the following equality holds*

$$\|x(t_1, t_2)\|_{21} = \|x(t_1, t_2)\|_{12}.$$

Really, let $x(t_1, t_2)$ be equal to $-x(t_2, t_1)$. In this case

$$\begin{aligned}\|x(t_1, t_2)\|_{21} &= \|\|x(t_1, t_2)\|_{(2)}\|_{(1)} = \|\| - x(t_2, t_1)\|_{(2)}\|_{(1)} \\ &= \|\|x(t_1, t_2)\|_{(1)}\|_{(2)} = \|x(t_1, t_2)\|_{12}.\end{aligned}$$

That is why further we will assume that the completion of the algebraic exterior square $X \wedge X$ of the space X is taken with respect to the symmetric crossnorm

$$\|x(t_1,t_2)\|_M = \max\{\|x(t_1,t_2)\|_{12}, \|x(t_1,t_2)\|_{21}\}.$$

This completion coincides with the closed subspace of antisymmetric functions of the space $\widetilde{X}(\Omega \times \Omega)$.

The subspace $\widetilde{X}^a(\Omega\times\Omega)$ is isomorphic in the category of Banach spaces to the space $\widetilde{X}(W)$, where W is the measurable subset $\Omega \times \Omega$, for which the sets $W \cap \widetilde{W}$ and $(\Omega\times\Omega)\setminus(W\cup\widetilde{W})$ have zero measure; here $\widetilde{W} = \{(t_2,t_1) : (t_1,t_2) \in W\}$ (such sets do always exist). Really, extending the functions from $\widetilde{X}(W)$ as antisymmetric functions to $\Omega\times\Omega$, we obtain the set of all the functions from $\widetilde{X}^a(\Omega\times\Omega)$. Further, setting the norm of a function in $\widetilde{X}(W)$ to be equal to the norm of its extension, we get that the spaces $\widetilde{X}^a(\Omega \times \Omega)$ and $\widetilde{X}(W)$ are isomorphic in the category of normed spaces.

Therefore, for an almost perfect Banach ideal space the following equality holds:

$$(\widetilde{X \wedge X})_M = (\widetilde{X \otimes X})_M^a = \widetilde{X}_d(W),$$

where $\widetilde{X}_d(W)$ is the closure of the set of all degenerate functions from $\widetilde{X}(W)$ in the norm of $\widetilde{X}(W)$.

4. Tensor and exterior squares of linear operators in ideal spaces

Let A, B be continuous linear operators acting in the ideal space X. Define the *algebraic tensor product* of the operators A and B as the operator $A \otimes B$ in the space $X \otimes X$, defined on degenerate functions by the equality

$$(A \otimes B)x(t_1,t_2) = \sum_j Ax_1^j(t_1)\cdot Bx_2^j(t_2) \qquad \Big(x(t_1,t_2) = \sum_j x_1^j(t_1)\cdot x_2^j(t_2)\Big).$$

A crossnorm α on $X \otimes X$ is called *quasiuniform*, if for any continuous linear operators A, B one has:

$$\|A \otimes B\|_\alpha \le c\|A\|\|B\|,$$

where c is some constant. If $c = 1$, then such crossnorm is called *uniform.*

Define the *α-tensor product* of the operators A and B as the linear operator $(A \otimes B)_\alpha$ in the completion $(\widetilde{X \otimes X})_\alpha$ of the algebraic tensor square $X \otimes X$ with respect to the norms α, defined on degenerate functions by the equality

$$(A \otimes B)_\alpha x(t_1,t_2) = \sum_j Ax_1^j(t_1)\cdot Bx_2^j(t_2) \qquad \Big(x(t_1,t_2) = \sum_j x_1^j(t_1)\cdot x_2^j(t_2)\Big),$$

and on arbitrary functions by extension via continuity from the subspace of degenerate functions onto the whole of $(\widetilde{X \otimes X})_\alpha$ (if the norm α is quasiuniform, such an extension will be bounded).

The greatest and the least reasonable norms π and ϵ are uniform. However Levin's norm L in general case will not be quasiuniform. That is why we can define the *L-tensor product* of the operators A and B as the linear operator $(A \otimes B)_L$ in the space $(\widetilde{X \otimes X})_L$, only when the operator $(A \otimes B)_L$ exists, i.e., when the extension via continuity from the subspace of degenerate functions onto the whole of $(\widetilde{X \otimes X})_L$ is bounded.

Let us formulate the following statement concerning with the tensor product of two operators in ideal spaces (see [18], p. 62, proposition 6).

Levin's theorem. *Let X be a Banach ideal space, $A : X \to X$ be a regular operator, i.e., an operator, which can be represented in the form $A = A_1 - A_2$, where A_1 and A_2 are nonnegative linear operators* (*with respect to the cone of nonnegative functions*), *and let $B : X \to X$ be a continuous linear operator. Then the tensor product $(A \otimes B)_L$ does exist.*

Further, when studying the spectral properties of the tensor square $(A \otimes A)_L$ of the operator A, we will have to impose conditions on the operator A, that are stronger than nonnegativity or regularity. Let us give the following definition. A linear operator $A : X \to X$ is called *resolvent-regular*, if for any λ, which is not in $\sigma(A)$, the resolvent operator $R(\lambda, A) = (\lambda I - A)^{-1}$ is regular. The class of resolvent-regular operators includes, for example, Hilbert-Schmidt operators and operators, such that their certain power is a Hilbert-Schmidt operator.

Further let us examine the operator $(A \wedge A)_M$, defined as the restriction of the operator $(A \otimes A)_M$ onto the subspace $(\widetilde{X \overset{a}{\otimes} X})_M$. It is obvious that for degenerate antisymmetric functions the operator $(A \wedge A)_M$ can be defined by the equality

$$(A \wedge A)_M x(t_1, t_2) = \sum_j Ax_1^j(t_1) \wedge Ax_2^j(t_2) \qquad x(t_1, t_2) = \sum_j x_1^j(t_1) \wedge x_2^j(t_2).$$

5. Spectrum of the tensor square of linear operators in ideal spaces

As usual, we denote by $\sigma(A)$ the spectrum of an operator A, and we denote by $\sigma_p(A)$ the point spectrum, that is, the set of all eigenvalues of the operator A. Denote by $\sigma_{eb}(A)$ the Browder essential spectrum of the operator A. Thus $\sigma(A) \setminus \sigma_{eb}(A)$ will be the set of all isolated finite-dimensional eigenvalues of the operator A (for more detailed information see [11], [12]).

In papers [11]–[10] T. Ichinose obtained the results, representing the spectra and the parts of the spectra of the tensor products of linear bounded operators in terms of the spectra and parts of the spectra of the operators given under the natural assumptions, that the corresponding crossnorm α is reasonable and quasi-uniform. Among the mentioned results there are the explicit formulae, expressing the set of all isolated finite-dimensional eigenvalues and the Browder essential spectrum of the operator $(A \otimes A)_\alpha$ in terms of the parts of the spectrum of the operator given (see [11], p. 110, Theorem 4.2). In particular, Ichinose proved, that for the tensor square of a linear bounded operator A the following equalities hold:

$$\sigma(A \otimes A)_\alpha = \sigma(A)\,\sigma(A);$$
$$\sigma(A \otimes A)_\alpha \setminus \sigma_{eb}(A \otimes A)_\alpha = (\sigma(A) \setminus \sigma_{eb}(A))\,(\sigma(A) \setminus \sigma_{eb}(A)) \setminus (\sigma_{eb}(A)\sigma(A)); \quad (1)$$
$$\sigma_{eb}(A \otimes A)_\alpha = \sigma_{eb}(A)\,\sigma(A).$$

For a completely continuous operator the following equalities hold:

$$(\sigma(A) \setminus \sigma_{eb}(A)) \setminus \{0\} = \sigma_p(A) \setminus \{0\}; \ \sigma_{eb}(A) = \{0\} \text{ or } \emptyset.$$

So from (1) we can get the complete information about the nonzero eigenvalues of the tensor square of a completely continuous operator:

$$\sigma_p(A \otimes A)_\alpha \setminus \{0\} = (\sigma_p(A)\,\sigma_p(A)) \setminus \{0\}. \quad (2)$$

Here zero can be either a finite- or infinite-dimensional eigenvalue of $A \otimes A$ or a point of the essential spectrum.

In paper [17] there have been examined the case, when a linear operator acts in the space $L_p(\Omega)$ $(C(\Omega))$, and the corresponding crossnorm is reasonable and quasiuniform. That is why the formula for the spectrum of the tensor product of A directly follows from the results of T. Ichinose.

However in general case the crossnorm L is reasonable, but not quasiuniform (see [5]), and therefore we need a different proof for the statement about the spectrum of $(A \otimes A)_L$. The proof, given below, is based on the reasoning, made by A.S. Kalitvin (see [13], p. 83, theorem 3.10), for the case of the operator $A \otimes I + I \otimes A$ in a regular ideal space.

Theorem 1. *Let X be an almost perfect Banach ideal space, and let $A : X \to X$ be a completely continuous nonnegative with respect to the cone of nonnegative functions in X resolvent-regular operator. Then for the point spectrum of the operator $(A \otimes A)_L$, acting in the space $(\widetilde{X \otimes X})_L$, the following equality holds:*

$$\sigma_p(A \otimes A)_L \setminus \{0\} = (\sigma_p(A)\,\sigma_p(A)) \setminus \{0\}.$$

Proof. Let us examine the operators $(A \otimes I)_L$ and $(I \otimes A)_L$, acting in $(\widetilde{X \otimes X})_L$. Let us prove, that the following inclusions are true: $\sigma(A \otimes I)_L \subseteq \sigma(A)$ and $\sigma(I \otimes A)_L \subseteq \sigma(A)$. Let us prove the first inclusion (the second inclusion can be proved by analogy). Let λ does not belong to $\sigma(A)$. Then the operator $\lambda I - A$ is invertible. Let us define the operator $(\lambda I - A)^{-1} \otimes I$ on $X \otimes X$. Since the operator $(\lambda I - A)^{-1}$ is regular, we can apply the Levin's theorem. As it follows from Levin's theorem, the operator $(\lambda I - A)^{-1} \otimes I$ can be extended from $X \otimes X$ onto the whole of $(\widetilde{X \otimes X})_L$. It is easy to see, that the operator $(\lambda I - A)^{-1} \otimes I$ is inverse on $X \otimes X$ for the operator $\lambda I - (A \otimes I)$. So, its extension $((\lambda I - A)^{-1} \otimes I)_L$ will be inverse for the operator $\lambda I - (A \otimes I)_L$ on the whole of $(\widetilde{X \otimes X})_L$. That is why λ does not belong to $\sigma(A \otimes I)_L$, and the inclusion $\sigma(A \otimes I)_L \subseteq \sigma(A)$ is proved.

Thus as $(A\otimes A)_L = (A\otimes I)_L(I\otimes A)_L$ and the operators $(A\otimes I)_L$ and $(I\otimes A)_L$ are, obviously, commuting, the following relation is true

$$\sigma(A\otimes A)_L = \sigma((A\otimes I)_L(I\otimes A)_L) \subseteq \sigma(A\otimes I)_L\sigma(I\otimes A)_L.$$

Now, applying the inclusions $\sigma(A\otimes I)_L \subseteq \sigma(A)$ and $\sigma(I\otimes A)_L \subseteq \sigma(A)$, proved above, we see that the following inclusion is true as well

$$\sigma(A\otimes A)_L \subseteq \sigma(A)\sigma(A).$$

Due to the complete continuity of the operator A, its spectrum, except, probably, zero, consists of isolated finite-dimensional eigenvalues. That is why the following relations hold:

$$\begin{aligned}\sigma_p(A\otimes A)_L\setminus\{0\} \subseteq \sigma(A\otimes A)_L\setminus\{0\} &\subseteq (\sigma(A)\sigma(A))\setminus\{0\}\\ &= (\sigma_p(A)\sigma_p(A))\setminus\{0\},\end{aligned}$$

i.e., we proved:

$$\sigma_p(A\otimes A)_L\setminus\{0\} \subseteq (\sigma_p(A)\sigma_p(A))\setminus\{0\}.$$

Now, let us prove the reverse inclusion. For this we will examine the extension $(A\otimes A)_\epsilon$ of the operator $A\otimes A$ onto the whole $(\widetilde{X\otimes X})_\epsilon$, where $(\widetilde{X\otimes X})_\epsilon$ is a completion $X\otimes X$ with respect to the "weak" crossnorm ϵ. As it follows from the results of J.R. Holub (see [8], p. 401, theorem 2), the operator $(A\otimes A)_\epsilon$ is completely continuous in $(\widetilde{X\otimes X})_\epsilon$. Let us prove, that $\sigma_p(A\otimes A)_\epsilon\setminus\{0\} \subseteq \sigma_p(A\otimes A)_L\setminus\{0\}$. To check this it is enough to prove, that any eigenfunction of the operator $(A\otimes A)_\epsilon$, corresponding to a nonzero eigenvalue, belongs to the space $(\widetilde{X\otimes X})_L$. Let λ be an arbitrary nonzero eigenvalue of the operator $(A\otimes A)_\epsilon$. Since the crossnorm ϵ is reasonable and quasiuniform, then formula (2) follows from the results of T. Ichinose. Formula (2) implies that there exist indices i, j, for which $\lambda = \lambda_i\lambda_j$ (here $\{\lambda_k\}$ is the set of all nonzero eigenvalues of the operator A, enumerated without regard to multiplicity). Since λ is an isolated finite-dimensional eigenvalue, i, j can take only finite number of different values. Let us enumerate all the pairs of such values. Let $\lambda = \lambda_{i_k}\lambda_{j_k}$ $(k = 1,\dots,p)$. Decompose the space X into the direct sum of subspaces:

$$X = X_1\oplus\cdots\oplus X_p\oplus R,$$

where $X_k = \ker(A-\lambda_{i_k})^{m_k}$, m_k are the multiplicities of λ_{i_k}. Under this decomposition $(\widetilde{X\otimes X})_\epsilon$ also decomposes into the direct sum of subspaces:

$$\begin{aligned}&(\widetilde{X\otimes X})_\epsilon\\ &= (X_1\otimes X_1)\oplus\cdots\oplus(X_1\otimes X_p)\oplus\cdots\oplus(X_p\otimes X_1)\oplus\cdots\oplus(X_p\otimes X_p)\\ &\quad\oplus(X_1\otimes R)\oplus\cdots\oplus(X_p\otimes R)\oplus(R\otimes X_1)\oplus\cdots\oplus(R\otimes X_p)\oplus(\widetilde{R\otimes R})_\epsilon.\end{aligned}$$

Since $X_l \otimes X_m$, $X_l \otimes R$, $R \otimes X_l$, $\widetilde{(R \otimes R)}_\epsilon$ are invariant subspaces for the operator $(A \otimes A)_\epsilon$, the following equality holds:

$$\sigma(A\otimes A)_\epsilon =$$

$$\bigcup_{l,m}(\sigma(A\otimes A, X_l\otimes X_m)\cup\sigma(A\otimes A, R\otimes X_m)\cup\sigma(A\otimes A, X_l\otimes R))\cup\sigma(A\otimes A, \widetilde{R\otimes R}_\epsilon),$$

where the notation $\sigma(A \otimes A, X_l \otimes X_m)$ means the spectrum of the restriction of the operator $(A \otimes A)_\epsilon$ onto the corresponding subspace. Since $X_l \otimes X_m$, $R \otimes X_m$, $X_l \otimes R$, $\widetilde{(R \otimes R)}_\epsilon$ are the spaces with uniform crossnorms, we can apply the results of T. Ichinose, and therefore the following equalities hold:

$$\sigma(A \otimes A, X_l \otimes X_m) = \sigma(A, X_l)\sigma(A, X_m);\ \sigma(A \otimes A, X_l \otimes R) = \sigma(A, X_l)\sigma(A, R);$$

$$\sigma(A \otimes A, R \otimes X_m) = \sigma(A, R)\sigma(A, X_m);\ \sigma(A \otimes A, \widetilde{(R \otimes R)}_\epsilon) = \sigma(A, R)\sigma(A, R).$$

Since λ_{i_k} and λ_{j_k} do not belong $\sigma(A, R)$ for any values of indices k $(k = 1, \dots, p)$, λ does not belong to $\bigcup_{l,m}(\sigma(A\otimes A, R\otimes X_m)\cup\sigma(A\otimes A, X_l\otimes R))\cup\sigma(A\otimes A, \widetilde{(R \otimes R)}_\epsilon)$. As it follows, $\lambda \in \bigcup_{l,m}(\sigma(A \otimes A, X_l \otimes X_m))$. Further it is obvious, that for any l, m $(1 \leq l, m \leq p)$ $X_l \otimes X_m$ belongs to the algebraic tensor square $X \otimes X$ and therefore it belongs to the space $\widetilde{(X \otimes X)}_L$. So, for an arbitrary $\lambda \in \sigma_p(A\otimes A)_\epsilon \setminus \{0\}$ the inclusion $\lambda \in \sigma_p(A \otimes A)_L \setminus \{0\}$ is true. □

Let us notice, that under conditions of Theorem 1 the inclusion:

$$\sigma_{eb}(A \otimes A)_L \subseteq \sigma_{eb}(A)\sigma_{eb}(A) = \{0\}$$

follows from its proof. Moreover, for an arbitrary $\lambda \in (\sigma_p(A \otimes A)_L \setminus \{0\})$ the following equality holds:

$$\ker((A \otimes A)_L - \lambda(I \otimes I)_L) = \bigoplus_k \ker(A - \lambda_{i_k} I) \otimes \ker(A - \lambda_{j_k} I),$$

where the summation is taken over all the numbers k of the pairs $\lambda_{i_k}, \lambda_{j_k}$, for which $\lambda_{i_k}, \lambda_{j_k} \in (\sigma_p(A) \setminus \{0\})$ and $\lambda = \lambda_{i_k}\lambda_{j_k}$.

The statements of Theorem 1 remain true also for the case, when the tensor square $(A \otimes A)_M$ of the operator A acts in the space $\widetilde{(X \otimes X)}_M$. The proof can be developed by analogy.

6. Spectrum of the exterior square of linear operators in ideal spaces

For the exterior square, which is the restriction of the tensor square, the following inclusions hold:

$$\sigma(A \wedge A)_M \subset \sigma(A \otimes A)_M. \tag{3}$$

$$\sigma_p(A \wedge A)_M \subset \sigma_p(A \otimes A)_M. \tag{4}$$

In the finite-dimensional case it is known that the matrix $A \wedge A$ in a suitable basis appears to be the second-order compound matrix to the matrix A, and we conclude that all the possible products of the type $\{\lambda_i\lambda_j\}$, where $i < j$, form the set of all eigenvalues of the exterior square $A \wedge A$, repeated according to multiplicity (see [7], p. 80, Theorem 23).

In the infinite-dimensional case we can also obtain some information concerning eigenvalues of the exterior square of a linear bounded operator. Applying Theorem 1, we can prove the following statement:

Theorem 2. *Let X be an almost perfect ideal space and let $A : X \to X$ be a completely continuous nonnegative with respect to the cone of nonnegative functions in X resolvent-regular operator. Let $\{\lambda_i\}$ be the set of all nonzero eigenvalues of the operator A, repeated according to multiplicity. Then all the possible products of the type $\{\lambda_i\lambda_j\}$, where $i < j$, form the set of all the possible (except, probably, zero) eigenvalues of the exterior square of the operator $(A \wedge A)_M$, repeated according to multiplicity.*

Proof. The proof copies the proof of the corresponding statement from [17] (see [17], p. 12, Theorem 1).

Inclusion $\{\lambda_i\lambda_j\}_{i<j} \subset \sigma_p(A \wedge A)_M$, i.e., each product of the form $\lambda_i\lambda_j$, where $i < j$, is an eigenvalue of $(A \wedge A)_M$, comes out from the following reasoning:

Let $\lambda_i,\ \lambda_j \in \sigma_p(A)$. Then there exist such functions $x(t)$, $y(t)$ from X, that $(A - \lambda_i I)x(t) = 0$ and $(A - \lambda_j I)y(t) = 0$. Let us examine the value of the operator $A \wedge A - \lambda_i\lambda_j I \wedge I$ on the degenerate antisymmetric function $(x \wedge y)(t_1, t_2)$:

$$\begin{aligned}(A \wedge A - \lambda_i\lambda_j I \wedge I)(x \wedge y) &= Ax \wedge Ay - \lambda_i\lambda_j x \wedge y\\ &= Ax \wedge Ay - \lambda_i x \wedge Ay + \lambda_i x \wedge Ay - \lambda_i\lambda_j x \wedge y\\ &= [\text{because of the linearity of the exterior product}]\\ &= (Ax - \lambda_i x) \wedge Ay + \lambda_i x \wedge (Ay - \lambda_j y) = 0.\end{aligned}$$

From this we see that $\lambda_i\lambda_j \in \sigma_p(A \wedge A)_M$

Now we shall prove the reverse inclusion: $\sigma_p(A \wedge A)_M \subset \{\lambda_i\lambda_j\}_{i<j}$. As it was shown above in formulae (3) and (4):

$$\sigma_p(A \wedge A)_M \setminus \{0\} \subset \sigma_p(A \otimes A)_M \setminus \{0\} = \sigma_p(A)\sigma_p(A) \setminus \{0\},$$

i.e., the operator $(A \wedge A)_M$ has no other eigenvalues, except products of the form $\lambda_i\lambda_j$. Enumerate the set of pairs $\{(i, j)\}\ \ i, j = 1, 2, \ \ldots$. In this way we get a numeration of $\{\lambda_i\lambda_j\}$ – the set of all eigenvalues of $(A \otimes A)_M$, repeated according to multiplicity. Decompose the obtained finite or countable set of indices Λ in the following way:

$$\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3,$$

where the set Λ_1 includes the numbers of those pairs (i, j), for which $i < j$, Λ_2 includes those pairs, for which $i = j$, and Λ_3 includes those pairs, for which $i > j$. The set of all eigenvalues of $(A \otimes A)_M$, repeated according to multiplicity, will be then decomposed into three parts:

$$\{\lambda_\alpha\}_{\alpha\in\Lambda} = \{\lambda_\alpha\}_{\alpha\in\Lambda_1} \cup \{\lambda_\alpha\}_{\alpha\in\Lambda_2} \cup \{\lambda_\alpha\}_{\alpha\in\Lambda_3}.$$

Decompose $(\widetilde{X \otimes X})_M$ into the direct sum of subspaces invariant with respect to $(A \otimes A)_M$:

$$(\widetilde{X \otimes X})_M = (\widetilde{X \otimes X})^a_M \oplus (\widetilde{X \otimes X})^s_M,$$

where $(\widetilde{X \otimes X})^s_M$ is the subspace of all symmetric functions from $(\widetilde{X \otimes X})_M$. The operator $(A \otimes A)_M$ can be represented in the block form:

$$(A \otimes A)_M = \begin{pmatrix} (A \wedge A)_M & 0 \\ 0 & (A \otimes A)|_{(\widetilde{X \otimes X})^s_M} \end{pmatrix}$$

Since the operator $(A \otimes A)_M$ has a block structure, $\sigma_p(A \otimes A)_M$ can be decomposed into two subsets:

$$\sigma_p(A \otimes A)_M = \sigma_p(A \wedge A)_M \cup \sigma_p((A \otimes A)|_{(\widetilde{X \otimes X})^s_M}),$$

where $(\widetilde{X \otimes X})^s_M$ is the subset of all symmetric functions from $(\widetilde{X \otimes X})_M$. In order to prove that the eigenvalues of $(A \otimes A)_M$, belonging to the sets $\{\lambda_\alpha\}_{\alpha \in \Lambda_2}$ and $\{\lambda_\alpha\}_{\alpha \in \Lambda_3}$, will not be the eigenvalues of $(A \wedge A)_M$, it is enough to show, that they will be the eigenvalues of $(A \otimes A)|_{(\widetilde{X \otimes X})^s_M}$. Indeed, let $x_i(t) \in X$ be an eigenfunction of the operator A, corresponding to the eigenvalue λ_i. Let us examine the value of the operator $(A \otimes A - \lambda_i^2 I \wedge I)$ on the function $x_i(t_1)x_i(t_2) \in (\widetilde{X \otimes X})^s_M$:

$$\begin{aligned}
(A \otimes A - \lambda_i^2 I \otimes I)x_i(t_1)x_i(t_2) &= Ax_i(t_1)Ax_i(t_2) - \lambda_i^2 x_i(t_1)x_i(t_2) \\
&= Ax_i(t_1)Ax_i(t_2) - \lambda_i x_i(t_1)Ax_i(t_2) + \lambda_i x_i(t_1)Ax_i(t_2) - \lambda_i^2 x_i(t_1)x_i(t_2) \\
&= (Ax_i - \lambda_i x_i)(t_1)Ax_i(t_2) + \lambda_i x_i(t_1)(Ax_i - \lambda_i x_i)(t_2) = 0.
\end{aligned}$$

From this we see, that $\lambda_i^2 \in \sigma_p((A \otimes A)|_{(\widetilde{X \otimes X})^s_M})$. In analogous way we can prove that a product of the form $\lambda_i \lambda_j$ will also be an eigenvalue of $(A \otimes A)|_{(\widetilde{X \otimes X})^s_M}$ (with the corresponding symmetric function $x_i(t_1)x_j(t_2) + x_j(t_1)x_i(t_2)$). □

7. Generalization of the Gantmacher–Krein theorems in the case of 2-totally nonnegative operators in ideal spaces

Let us formulate some modification of the Krein–Rutman theorem (see [26], p. 81, theorem 1) about completely continuous operators, leaving invariant an almost reproducing cone K in a Banach space (for such operators the spectral radius $\rho(A)$ belongs to $\sigma_p(A)$).

Modified Krein–Rutman's theorem. *Let X be a real Banach space, $A : X \to X$ be a linear bounded operator. Let A leave invariant an almost reproducing cone K ($\overline{K - K} = X$). Let $\rho_c(A) < \rho(A)$ (here $\rho_c(A)$ is the Fredholm spectral radius of the operator A). Then $\lambda_1 = \rho(A) \in \sigma_p(A)$ and there exists a nonzero element $x_1 \in K$ and functional $x_1^* \in K^*$, for which $Ax_1 = \lambda_1 x_1$, $A^* x_1^* = \lambda_1 x_1^*$.*

It is easy to see, that the conditions of the modified Krein–Rutman theorem holds for all the operators, the Fredholm essential spectrum of which consists of only one point zero, in particular, for completely continuous operators.

Let us prove some generalizations of the Gantmacher–Krein theorems in the case of operators in ideal spaces, using the modified Krein–Rutman theorem:

Theorem 3. *Let $X(\Omega)$ be an almost perfect ideal space, and, respectively, let $\widetilde{X}_d(W)$ be the closure of the set of all degenerate functions in intersection of the spaces with mixed norms. Let $A:\ X(\Omega) \to X(\Omega)$ be a completely continuous nonnegative with respect to the cone of nonnegative functions in X resolvent-regular operator, such that $\rho(A) > 0$. Let the exterior square $(A \wedge A)_M : \widetilde{X}_d(W) \to \widetilde{X}_d(W)$ be nonnegative with respect to the cone of nonnegative functions in $\widetilde{X}_d(W)$, and $\rho(A \wedge A)_M > 0$. Then the operator A has a positive eigenvalue $\lambda_1 = \rho(A)$. Moreover, if there is only one eigenvalue on the spectral circle $|\lambda| = \rho(A)$, then the operator A has the second positive eigenvalue $\lambda_2 < \lambda_1$. If there is more than one eigenvalue on the spectral circle $|\lambda| = \rho(A)$, then either there is at least one pair of complex conjugates among them, or λ_1 is a multiple eigenvalue.*

Proof. Enumerate eigenvalues of a completely continuous operator A, repeated according to multiplicity, in order of decrease of their modules:

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots$$

Applying the modified Krein–Rutman theorem to A we get: $\lambda_1 = \rho(A) > 0$. Now applying the modified Krein–Rutman theorem to the operator $(A \wedge A)_M$ we get: $\rho(A \wedge A)_M \in \sigma_p(A \wedge A)_M$. (Note, that from $\sigma_{eb}(A \wedge A)_M \subseteq \sigma_{eb}(A \otimes A)_M \subseteq \sigma_{eb}(A)\sigma_{eb}(A) = \{0\}$, it follows, that the operator $(A \wedge A)_M$ also satisfy the conditions of the modified Krein–Rutman theorem).

As it follows from the statement of Theorem 2, the exterior square of the operator A has no other nonzero eigenvalues, except for all the possible products of the form $\lambda_i\lambda_j$, where $i < j$. So, we conclude that $\rho(A \wedge A)_M > 0$ can be represented in the form of the product $\lambda_i\lambda_j$ with some values of the indices i, j, $i < j$. Thus, if there is only one eigenvalue on the spectral circle $|\lambda| = \rho(A)$, from the fact that eigenvalues are numbered in decreasing order it follows that $\rho(A \wedge A)_M = \lambda_1\lambda_2$. Therefore $\lambda_2 = \frac{\rho(A \wedge A)}{\lambda_1} > 0$.

If there is m ($m \geq 2$) eigenvalues on the spectral circle $|\lambda| = \rho(A)$, then $\rho(A \wedge A)_M = \lambda_i\lambda_j$, where $1 \leq i < j \leq m$. So, both λ_i and λ_j are situated on the spectral circle $|\lambda| = \rho(A)$, and from the positivity of their product it follows, that λ_i λ_j are either a pair of complex conjugate eigenvalues, or both are positive and coincide with $\rho(A)$. □

It is well known (see, for example, [8], p. 55, corollary from the proposition 2.1), that a linear integral operator A, acting in the Banach ideal space $X(\Omega)$, is nonnegative if and only if its kernel $k(t, s)$ is nonnegative almost everywhere on Ω. It is also well known (see [17]), that the exterior power of a linear integral operator can be considered as a linear integral operator, acting in the space $\widetilde{X}_d(W)$ with the kernel equal to the second compound to the kernel of the operator given. That is why it is not difficult to reformulate Theorem 3 in terms of kernels of linear integral operators. In this case the conditions of Theorem 3 can be easily verified.

Theorem 4. *Let a completely continuous resolvent-regular linear integral operator A act in an almost perfect ideal space $X(\Omega)$. Let the kernel $k(t,s)$ of the operator A be nonnegative almost everywhere on the Cartesian square $\Omega\times\Omega$. Let the second compound kernel $k\wedge k(t_1,\ t_2,\ s_1,\ s_2)$ be nonnegative almost everywhere on the Cartesian square $W\times W$, where W is a measurable subset, possessing the following properties:*

1) $\mu(W\cap\widetilde{W})=0$;
2) $\mu((\Omega\times\Omega)\setminus(W\cup\widetilde{W}))=0.\qquad (\widetilde{W}=\{(t_2,t_1):\ (t_1,t_2)\in W\})$

Let, in addition, $\rho(A)>0$ and $\rho(A\wedge A)_M>0$. Then the operator A has a positive eigenvalue $\lambda_1=\rho(A)$. Moreover, if there is only one eigenvalue on the spectral circle $|\lambda|=\rho(A)$, then the operator A has the second positive eigenvalue $\lambda_2<\lambda_1$. If there is more than one eigenvalue on the spectral circle $|\lambda|=\rho(A)$, then either there is at least one pair of complex conjugates among them, or λ_1 is a multiple eigenvalue.

Note that in Theorem 4 the kernel is not presupposed to be continuous, we assume only, that the operator A acts in one of almost perfect ideal spaces.

Moreover, Theorem 3 can be generalized in the case, when the exterior square $(A\wedge A)_M$ of the operator A leaves invariant an arbitrary almost reproducing cone $\widetilde{K}$ in $\widetilde{X}_d(W)$. But in this case certain difficulties, related to the testing of the assumption of the generalized theorem, can arise.

References

[1] A. Boccuto, A.V. Bukhvalov, and A.R. Sambucini, *Some inequalities in classical spaces with mixed norms*, Positivity, **6** (2002), 393–411.

[2] A.V. Bukhvalov, *Vector-valued function spaces and tensor products*, Siberian Math. J., **13** (1972), 1229–1238 (Russian).

[3] ———, *On spaces with mixed norm*, Vestnik Leningrad. Univ., **19** (1973), 512 (Russian); English transl.: Vestnik Leningrad Univ. Math., **6** (1979), 303–311.

[4] ———, *Generalization of the Kolmogorov-Nagumo theorem on the tensor product*, Qualitative and Approximate Methods for the Investigation of Operator Equations, **4** (1979), 4865, Jaroslavl (Russian).

[5] ———, *Application of methods of the theory of order-bounded operators to the theory of operators in L^p-spaces*, Uspekhi Mat. Nauk., **38** (1983), 37–83.

[6] S.P. Eveson, *Eigenvalues of totally positive integral operators*, Bull. London Math. Soc., **29** (1997), 216–222.

[7] F.R. Gantmacher and M.G. Krein, *Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems*, Amer. Math. Soc., Providence, PI, 2002.

[8] J.R. Holub, *Compactness in topological tensor products and operator spaces*, Proc. Amer. Math. Soc., **36** (1972), 398–406.

[9] T. Ichinose, *Operators on tensor products of Banach spaces*, Trans. Amer. Math. Soc., **170** (1972), 197–219.

[10] ______, *Operational calculus for tensor products of linear operators in Banach spaces*, Hokkaido Math. J., **4** (1975), 306–334.

[11] ______, *Spectral properties of tensor products of linear operators. I*, Trans. Amer. Math. Soc., **235** (1978), 75–113.

[12] ______, *Spectral properties of tensor products of linear operators. II*, Trans. Amer. Math. Soc., **237** (1978), 223–254.

[13] A.S. Kalitvin, *Linear operators with partial integrals*, Voronezh, 2000.

[14] L.V. Kantorovich and G.P. Akilov, *Functional Analysis*, 2nd rev. Moscow, 1977 (Russian); English transl.: Pergamon Press, Oxford, 1982.

[15] S. Karlin, *Total positivity*, Stanford University Press, California, 1968.

[16] O.D. Kellog, *Orthogonal function sets arising from integral equations*, American Journal of Mathematics, **40** (1918), 145–154.

[17] O.Y. Kushel and P.P. Zabreiko, *Gantmacher–Krein theorem for* 2*-nonnegative operators in spaces of functions*, Abstract and Applied Analysis, Article ID 48132 (2006), 1–15.

[18] V.L. Levin, *Tensor products and functors in categories of Banach spaces defined by KB-lineals*, Trudy Moskov. Mat. Obshch., **20** (1969), 43–81 (Russian); English transl.: Trans. Moscow Math. Soc., **20** (1969), 41–78.

[19] T.-W. Ma, *Classical analysis on normed spaces*, World Scientific Publishing, 1995.

[20] N.J. Nielsen, *On Banach ideals determined by Banach lattices and their applications*, Diss. Math. **109** (1973).

[21] A. Pinkus A, *Spectral properties of totally positive kernels and matrices*, Total positivity and its applications, M. Gasca, C.A. Micchelli, eds., Dordrecht, Boston, London, Kluwer Acad. Publ., 1996, 1–35.

[22] Yu.V. Pokornyi Yu.V., O.M. Penkin, V.L. Pryadiev, A.V. Borovskikh, K.P. Lazarev, and S.A. Shabrov, *Differential equations on geometrical graphs*, FIZMATLIT, Moscow, 2004.

[23] A.V. Sobolev, *Abstract oscillatory operators*, Proceedings of the Seminar on differential equations, Kuibishev, 1977, no. 3, 72–78 (Russian).

[24] V.I. Yudovich, *Spectral properties of an evolution operator of a parabolic equation with one space variable and its finite-dimensional analogues*, Uspekhi Mat. Nauk, **32** (1977), 230–232 (Russian).

[25] P.P. Zabreiko, *Ideal spaces of functions*, Vestnik Yaroslav. Univ., **4** (1974), 12–52 (Russian).

[26] P.P. Zabreiko and S.V. Smitskikh, *A theorem of M.G. Krein and M.A. Rutman*, Functional Analysis and Its Applications, **13** (1980), 222–223.

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Received: November 30, 2008
Accepted: January 5, 2009

Operator Theory:
Advances and Applications, Vol. 202, 411–434

Conditions for Linear Dependence of Two Operators

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Dedicated to Israel Gohberg on the occasion of his 80th birthday

Abstract. The linear dependence property of two Hilbert space operators is expressed in terms of equality of size of values of certain sesquilinear and quadratic forms associated with the operators. The forms are based on q-numerical ranges.

Mathematics Subject Classification (2000). Primary 47A12; Secondary 47A99.

Keywords. Hilbert space, linear operators, linear dependence, numerical values, generalized numerical range.

1. Introduction

Let H be a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, and let $L(\mathsf{H})$ be the algebra of bounded linear operators on H. If H is of finite dimension n, we will identify H with C^n, the complex vector space of n-component column vectors with the standard inner product $\langle x, y \rangle = y^*x$, $x, y \in \mathsf{C}^n$, and will identify $L(\mathsf{H})$ with M_n, the algebra of $n \times n$ complex matrices. We assume throughout that H has dimension at least 2.

In this paper, we are interested in studying the following problem.

This work started during the visit of B. Kuzma, G. Lešnjak and T. Petek at College of William and Mary in Williamsburg. They are very grateful to the Department of Mathematics for the very warm hospitality they enjoyed during that visit. Research of C.-K. Li and L. Rodman was supported by the William and Mary Plumeri Awards and by NSF grants DMS 0600859 and DMS-0456625, respectively. B. Kuzma, G. Lešnjak, and T. Petek were supported by the grant BI-US/06-07-001.
Communicated by J.A. Ball.

Problem 1.1. *Suppose $A, B \in L(\mathsf{H})$. Find effective criteria to show that $\{A, B\}$ is linearly independent. In connection with this problem, when do $\{I, A\}$ and $\{I, B\}$ generate the same algebras in $L(\mathsf{H})$?*

Of course, if A and B are given, this is an easy question to answer. But in Problem 1.1 we assume that only partial information about the operators is given. For example, it is well known that if there is $\mu \in \mathsf{C}$ such that $\langle Ax, x\rangle = \mu\langle Bx, x\rangle$ for all normalized elements $x \in \mathsf{H}$, then $A = \mu B$. We will prove that for two nonzero operators $A, B \in L(\mathsf{H})$, at least one of the pairs A and B or A and B^* is linearly dependent if and only if there is $r > 0$ such that $|\langle Ax, x\rangle| = r|\langle Bx, x\rangle|$ for all normalized elements $x \in \mathsf{H}$; see Theorem 2.2 below. (Alternatively, we can say $|\langle Ax, x\rangle|/|\langle Bx, x\rangle|$ assumes only a single value whenever $|\langle Bx, x\rangle| \neq 0$.) The analysis turns out to be more involved, and inspires other equivalent conditions for $\{A, B\}$ to be linearly dependent.

Instead of comparing the absolute values of quadratic forms $\langle Ax, x\rangle$ and $\langle Bx, x\rangle$, we can also deduce our conclusion by considering the absolute values of general sesquilinear forms $\langle Ax, y\rangle$ and $\langle Bx, y\rangle$ for normalized elements $x, y \in \mathsf{H}$ with inner product $\langle x, y\rangle = q$ for a fixed value $q \in [0, 1]$. Clearly, if $q = 1$, we have $x = y$, and we are back to the quadratic form result. In general, we can show that two nonzero operators A and B are linearly dependent if and only if there is $r > 0$ and $q \in (0, 1)$ such that $|\langle Ax, y\rangle| = r|\langle Bx, y\rangle|$ for all normalized elements $x, y \in \mathsf{H}$ with $\langle x, y\rangle = q$; see Theorem 2.2. We also show that if $q = 0$, the above condition forces linear dependence of $A - aI$ and $B - bI$, for some $a, b \in \mathsf{C}$. Thus, $\{I, A\}$ and $\{I, B\}$ will generate the same algebra.

Note that $A \to \langle Ax, y\rangle$ can be viewed as the image of A under a (bounded) linear functional. The results we described above can be reformulated in the following form: two nonzero operators A and B are linearly dependent if and only if there is $r > 0$ such that $|f(A)| = r|f(B)|$ for all linear functionals in a certain specific class $\mathcal{S}$.

Clearly, to see whether $|f(A)| = r|f(B)|$ for all the linear functionals in a set $\mathcal{S}$, we can detect r by finding a linear functional $\hat{f} \in \mathcal{S}$ such that $\hat{f}(B) \neq 0$, and set $r = |\hat{f}(A)|/|\hat{f}(B)|$. Then we can replace B by B/r and test the condition $|f(A)| = |f(B)|$ for all $f \in \mathcal{S}$. We will do that in our subsequent discussion.

2. Main results

We consider Problem 1.1 in the context of numerical values of an operator defined by a constrained sesquilinear form, namely, q-numerical ranges. For a fixed q, $0 \leq q \leq 1$, consider the *q-numerical range*

$$W_q(A) = \{\langle Ax, y\rangle \ : \ x, y \in \mathsf{H}, \ \langle x, x\rangle = \langle y, y\rangle = 1, \ \langle x, y\rangle = q\}$$

of an operator $A \in L(\mathsf{H})$. The q-numerical range has been extensively studied during last twenty years or so; see, for example, [11, 10, 13, 2] among many works on the subject. If $q = 1$, then $W_1(A)$ coincides with the classical numerical range

$$W(A) = \{\langle Ax, x\rangle \ : \ x \in \mathsf{H}, \ \langle x, x\rangle = 1\}.$$

Let $q = \cos t$, $0 \le t \le \pi/2$. Then it is easy to see that we have

$$W_q(A) = \{\langle Ax, (\cos t)x + (\sin t)y\rangle \ : \ x, y \in \mathsf{H}, \ (x, y) \text{ orthonormal pair}\}.$$

We say that the numbers $\langle Ax, (\cos t)x + (\sin t)y\rangle$, where (x, y) varies through the set of orthonormal pairs, form the *q-numerical values* of the operator A. The characterization of operators having the same q-numerical values is easy to obtain (and the case $q = 1$ is well known):

Proposition 2.1. *Fix $q = \cos t$, $0 \le t \le \pi/2$. Two operators $A, B \in L(\mathsf{H})$ have the property that*

$$\langle Ax, (\cos t)x + (\sin t)y\rangle = \langle Bx, (\cos t)x + (\sin t)y\rangle \tag{2.1}$$

for every orthonormal pair (x, y), $x, y \in \mathsf{H}$, if and only if $A = B$ in case $t < \pi/2$, or $A - B$ is scalar (i.e., a scalar multiple of I) in case $t = \pi/2$.

Proof. The "if" part is obvious. For the "only if" part, if $t = \pi/2$, then for every nonzero $x \in \mathsf{H}$, the element $(A - B)x$ is orthogonal to span $\{x\}^{\perp}$, and therefore $(A - B)x$ is a scalar multiple of x: $(A - B)x = \lambda_x x$ for some $\lambda_x \in \mathsf{C}$; a priori λ_x may depend on x, but the additivity of $A - B$ easily implies that in fact λ_x is independent of x. Assume now $t < \pi/2$. The condition (2.1) implies that for a fixed orthonormal pair (x, y), the two circles in the complex plane

$$\{\langle Ax, x + (\tan t)ye^{\mathrm{i}\theta}\rangle \ : \ 0 \le \theta \le 2\pi\}, \ \{\langle Bx, x + (\tan t)ye^{\mathrm{i}\theta}\rangle \ : \ 0 \le \theta \le 2\pi\}$$

coincide, therefore their centers are the same: $\langle Ax, x\rangle = \langle Bx, x\rangle$. Since this equality holds for every normalized element x, we have $A = B$, as required. $\square$

In this paper we consider $A, B \in L(\mathsf{H})$ for which we require only equality in size in (2.1):

$$\begin{aligned} |\langle Ax, (\cos t)x + (\sin t)y\rangle| \quad &= \quad |\langle Bx, (\cos t)x + (\sin t)y\rangle|, \\ &\forall \quad \text{orthonormal pairs } (x, y), \quad x, y \in \mathsf{H}. \end{aligned} \tag{2.2}$$

Besides independent interest, the problem of characterization of operators A and B satisfying (2.2) came up (for $t = 0$) in the study of norm preservers of Jordan products [9].

A complete characterization of such A and B is given in our main result:

Theorem 2.2. *Fix $q = \cos t$, $0 \le t \le \pi/2$. Two operators $A, B \in L(\mathsf{H})$ have the property* (2.2) *if and only if*

(1) *$A = \mu B$ or $A = \mu B^*$ for some $\mu \in \mathsf{C}$, $|\mu| = 1$ in case $t = 0$;*
(2) *$A = \mu B$ for some $\mu \in \mathsf{C}$, $|\mu| = 1$ in case $0 < t < \pi/2$;*
(3) *$A = \mu B + \nu I$ for some $\mu, \nu \in \mathsf{C}$, $|\mu| = 1$ in case $t = \pi/2$.*

Remark 2.3. It is interesting to observe that the case $t = 0$ fails if one replaces the modulus by the real or imaginary part or by the argument of the complex number. To see this, pick any two positive definite $A, B \in L(\mathsf{H})$ and note that $\mathrm{Re}(\langle \mathrm{i}Ax, x\rangle) = 0 = \mathrm{Re}(\langle \mathrm{i}Bx, x\rangle)$, and $\mathrm{Arg}(\langle Ax, x\rangle) = 0 = \mathrm{Arg}(\langle Ax, x\rangle)$ for any normalized $x \in \mathsf{H}$.

Corollary 2.4. *Let $A, B \in L(\mathsf{H})$. Then*

$$|\langle Ax, y\rangle| = |\langle Bx, y\rangle|, \quad \forall \;\; x, y \in \mathsf{H}$$

if and only if $A = \mu B$ for some $\mu \in \mathsf{C}$, $|\mu| = 1$.

Proof. The part "if" is obvious, and the part "only if" is immediate from Theorem 2.2, the case $0 < t < \pi/2$.

However, Corollary 2.4 is actually used to prove Theorem 2.2, so we will deduce the corollary from the case $t = \pi/2$ of Theorem 2.2. Indeed, we have $A = \mu B + \nu I$ for some $\mu, \nu \in \mathsf{C}$, $|\mu| = 1$, and hence

$$|\langle \mu Bx, y\rangle + \nu\langle x, y\rangle| = |\langle Bx, y\rangle|, \quad \forall \;\; x, y \in \mathsf{H}.$$

Assuming $\nu \neq 0$, and taking y orthogonal to Bx we see that y is also orthogonal to x. Thus, $(\mathrm{span}\,(Bx))^{\perp} \subseteq (\mathrm{span}\,x)^{\perp}$, and so Bx is a scalar multiple of x: $Bx = \lambda_x x$, $\lambda_x \in \mathsf{C}$, for every $x \in \mathsf{H}$. Linearity of B easily implies that B is scalar, and now clearly $A = \mu' B$ for some $\mu' \in \mathsf{C}$, $|\mu'| = 1$. $\square$

Sections 3, 4, and 5 will be devoted to the proof of Theorem 2.2. In the last Section 6 we extend Proposition 2.1 to functionals given by trace class operators, and formulate an open problem and a conjecture concerning extension of Theorem 2.2 to such functionals.

We use notation $\mathbf{e}_j$ for the jth standard unit vector in C^n. $\mathrm{Re}\,(z)$ and $\mathrm{Im}\,(z)$ stand for the real and imaginary parts of the complex number $z = \mathrm{Re}\,(z) + \mathrm{i}\mathrm{Im}\,(z)$. We denote by X^{tr} the transpose of a matrix or vector X. The (block) diagonal matrix or operator with diagonal matrix or operator blocks $X_1, \ldots, X_p$ (in that order) will be denoted $\mathrm{diag}\,(X_1, \ldots, X_p)$.

3. Proof of Theorem 2.2: $t = 0$

For the proof of (1) we need preliminary results in matrix analysis which are of independent interest. We state and prove them first.

We start with the following known facts:

Proposition 3.1.

(a) *If $T \in M_n$ is not the zero matrix, then there exists a unitary U such that the diagonal entries of UTU^* are all nonzero.*
(b) *If $R, S \in M_n$ are such that U^*RU and U^*SU have the same diagonal for every unitary $U \in M_n$, then $R = S$.*

Proof. Part (b) is obvious because under the hypotheses of part (b) we have $\langle Rx, x\rangle = \langle Sx, x\rangle$ for every $x \in \mathsf{C}^n$.

Part (a). Note that every matrix is unitarily equivalent to a matrix with equal entries on the main diagonal, see [7, Problem 3, p. 77]. So we are done if $\operatorname{trace}(A) \neq 0$. Assume $\operatorname{trace}(A) = 0$. Due to $A \neq 0$ there exists a unit vector x_1 with $\mu_1 := \langle Ax_1, x_1\rangle \neq 0$. Choose any unitary U_1 with $U_1\mathbf{e}_1 = x_1$. Then, the first diagonal entry of $U_1^* A U_1$ is $\mu_1 \neq 0$. Due to $\operatorname{trace}(U_1^* A U_1) = \operatorname{trace}(A) = 0$, the main lower-right $(n-1) \times (n-1)$ submatrix $\widehat{A}$ of $U_1^* A U_1$, occupying the rows/columns $2, 3, \ldots, n$ has a nonzero trace. By an induction argument, there exists an $(n-1)\times(n-1)$ unitary V such that $V\widehat{A}V^*$ has all diagonal entries equal and nonzero. Then, the unitary $U := (1 \oplus V)U_1$ does the job. □

We denote by $\operatorname{diagv} A$ the diagonal vector of $A \in M_n$: If $A = [a_{ij}]_{i,j=1}^n$, then $\operatorname{diagv} A = [a_{11}\ a_{22}\ \ldots\ a_{nn}]^{\mathrm{tr}} \in \mathsf{C}^n$.

Theorem 3.2. *Let $A, B \in M_n$, where $n \geq 2$. Then the following three statements are equivalent:*

(i)
$$|\langle Ax, x\rangle| = |\langle Bx, x\rangle| \quad \text{for all} \quad x \in \mathsf{C}^n. \tag{3.1}$$

(ii) *For each unitary V there exists a unimodular number $\gamma(V)$, and a map $h_V : \mathsf{C} \to \mathsf{C}$ which is either identity or complex conjugation, such that*
$$\operatorname{diagv}(VBV^*) = \gamma(V)\operatorname{diagv}(VAV^*)^{h_V}.$$

(iii) *$B = \gamma A$ or $B = \gamma A^*$ for some unimodular number γ.*

Proof of Theorem 3.2. The proof consists of several steps.

Step 1. (iii)$\Longrightarrow$ (ii) Trivial. The implication (ii)$\Longrightarrow$ (i) is also immediate: By scaling, it suffices to prove (i) only for vectors x of norm one; then apply (ii) with unitary V whose first row is x^*.

Step 2. We prove (ii)$\Longrightarrow$ (iii), for $n \geq 3$. If $A = 0$, the result follows immediately from Proposition 3.1(b). We assume therefore that $A \neq 0$.

We first show that map h_V is independent of the unitary V. So assume, to reach a contradiction, that

$$\operatorname{diagv}(V_0BV_0^*) = \gamma_0 \operatorname{diagv}(V_0AV_0^*) \notin \{e^{\mathrm{i}\theta} \cdot \overline{\operatorname{diagv}(V_0AV_0^*)}\ :\ 0 \leq \theta \leq 2\pi\}, \tag{3.2}$$

for some unitary U_0 and unimodular γ_0, while

$$\operatorname{diagv}(V_1BV_1^*) = \gamma_1 \overline{\operatorname{diagv}(V_1AV_1^*)} \notin \{e^{\mathrm{i}\theta} \cdot \operatorname{diagv}(V_1AV_1^*)\ :\ 0 \leq \theta \leq 2\pi\} \tag{3.3}$$

for some other unitary U_1 and unimodular γ_1. Choose hermitian S_0, S_1 with $e^{\mathrm{i}S_0} = V_0$ and $e^{\mathrm{i}S_1} = V_1$. Then, $V_t := e^{\mathrm{i}(tS_1+(1-t)S_0)}$ is a path that connects V_0 and V_1 in the set of unitaries. Clearly, V_t and $V_t^* = e^{-\mathrm{i}(tS_1+(1-t)S_0)}$ are both analytic functions of the real variable $t \in [0,1]$. Moreover, $\mathbf{f}(t) := \operatorname{diagv}(V_tAV_t^*)$, as well as $\mathbf{g}(t) := \operatorname{diagv}(V_tBV_t^*)$ are also analytic vector-valued functions of real variable t. It is implicit in Eqs. (3.2)–(3.3) that $\mathbf{f}(0) \neq \mathbf{0}$ and $\mathbf{f}(1) \neq \mathbf{0}$. So at least one, say the first one $a_1(t)$, component of a vector-valued function $\mathbf{f}(t)$ is not identically zero. Now, being analytic, $a_1(t)$ has only finitely many zeros on $[0,1]$.

In view of hypothesis (ii), the zeros of $a_1(t)$ precisely match those of $b_1(t)$, the first component of $\mathbf{g}(t)$. Moreover, at each t off the set Λ of their common zeros, one of $\gamma(t) := b_1(t)/a_1(t)$ and $\gamma_1(t) := b_1(t)/\overline{a_1(t)}$ is unimodular. Clearly then, both are unimodular for all t off the common zeros. Then, however, they must have only removable singularities at common zeros, so both $\gamma(t)$ and $\gamma_1(t)$ are analytic functions of $t \in [0,1]$.

We next rewrite hypothesis (ii) into

$$\|\mathbf{g}(t) - \gamma(t)\mathbf{f}(t)\|^2 \cdot \|\mathbf{g}(t) - \gamma_1(t)\overline{\mathbf{f}(t)}\|^2 \equiv 0, \quad t \in [0,1] \setminus \Lambda. \tag{3.4}$$

Both factors in the left-hand side of (3.4) are analytic functions of a real variable t. We therefore conclude that at least one of them must vanish identically. Suppose the first one does, i.e., $\mathbf{g}(t) - \gamma(t)\mathbf{f}(t) \equiv 0$. Then, however, $\operatorname{diagv}(V_t B V_t^*) = \gamma(t)\operatorname{diagv}(V_t B V_t^*)$ for each t, contradicting Eq. (3.3). Likewise we get a contradiction if $\big(\mathbf{g}(t) - \gamma_1(t)\overline{\mathbf{f}(t)}\big) \equiv 0$.

If necessary, we replace B with B^*. In doing so, we can now guarantee that for each unitary V,

$$\operatorname{diagv}(VBV^*) = \gamma(V)\operatorname{diagv}(VAV^*), \quad |\gamma(V)| = 1. \tag{3.5}$$

We next show the unimodular factor $\gamma(V)$ is independent of V. If the trace of A is nonzero, this is obvious: $\gamma(V) = \operatorname{trace}(B)/\operatorname{trace}(A)$, by (3.5). Thus, assume $\operatorname{trace}(A) = 0$. By Proposition 3.1, there is a unitary $U \in M_n$ such that UAU^* has nonzero diagonal entries $\mu_1, \ldots, \mu_n$. We may assume that $U = I$; otherwise, replace (A, B) by (UAU^*, UBU^*). Let $A = (a_{ij})$ and $B = (b_{ij})$. The hypothesis (ii), and the above consideration, now imply $\operatorname{diagv} B = \gamma \operatorname{diagv} A$. We may assume that $\gamma = 1$; otherwise, replace B by $\overline{\gamma}B$. Thus,

$$\operatorname{diagv} B = \operatorname{diagv} A = [\mu_1, \ldots, \mu_n]^{\mathrm{tr}}, \quad \mu_1, \ldots, \mu_n \in \mathsf{C} \setminus \{0\}. \tag{3.6}$$

For $k = 1, \ldots, n$, let $A_k \in M_{n-1}$ be the submatrix of A obtained from it by removing its kth row and kth column. Similarly, we construct the matrices $B_1, \ldots, B_n$.

We claim that A_k and B_k are the same for all $k = 1, \ldots, n$. It will then follow that A and B are the same (the hypothesis that $n \geq 3$ is used here), and, in particular, $\gamma(V) = 1$ for all unitary V, which will complete the proof. To prove our claim, let $V \in M_n$ be a unitary matrix with (k,k) entry equal to 1. Since VAV^* and VBV^* have the same nonzero (k,k) entry μ_k (by (3.6)), we see from Eq. (3.5), that the two matrices actually have the same corresponding diagonal entries. As a result, UA_kU^* and UB_kU^* have the same diagonal entries for all unitary matrices $U \in M_{n-1}$. So $\operatorname{diagv}(U(A_k - B_k)U^*) = 0$ for all unitaries, which implies numerical range of $A_k - B_k$ consists only of zero. Thus, $A_k = B_k$ for $k = 1, 2, \ldots, n$, as required.

Step 3. (i) $\Longrightarrow$ (iii), for $n = 2$.

If $A = 0$ then $\langle Bx, x\rangle = 0$ for all $x \in \mathsf{C}^2$, so $B = 0$, and we are done. Otherwise, by (a) of Proposition 3.1, there exists a unitary U such that all diagonal entries of U^*AU are nonzero. Obviously,

$$|\langle U^*AUx, x\rangle| = |\langle A(Ux), Ux\rangle| = |\langle B(Ux), Ux\rangle| = |\langle U^*BUx, x\rangle|.$$

Consequently, we may replace (A, B) with (U^*AU, U^*BU) without changing the validity of assumptions (i) and conclusion (iii). This way, $a_{11} \neq 0$ (we denote by a_{11}, resp., b_{11}, the top left entry of A, resp., B). Choose a vector $x := \mathbf{e}_1$ to deduce $|a_{11}| = |b_{11}|$. We may, thus, further replace (A, B) with $(1/a_{11}U, \gamma B/a_{11})$ where $\gamma := a_{11}/b_{11}$ is unimodular. In doing so, we can assume $a_{11} = 1 = b_{11}$. Hence it remains to see that $B = A$ or $B = A^*$.

To see this, write

$$A := \begin{bmatrix} 1 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B := \begin{bmatrix} 1 & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

and choose a vector $x := \begin{bmatrix} 1 \\ r + \mathrm{i}s \end{bmatrix}$, where r, s are real. Then,

$$\begin{aligned} \langle Ax, x\rangle &= 1 + (r+\mathrm{i}s)a_{12} + \overline{(r+\mathrm{i}s)}a_{21} + |(r+\mathrm{i}s)|^2 a_{22} \\ &= 1 + r\mathrm{Re}\,(a_{12}+a_{21}) + s\mathrm{Im}\,(a_{21}-a_{12}) + (r^2+s^2)\mathrm{Re}\,(a_{22}) \\ &+ \mathrm{i}\left(r\mathrm{Im}\,(a_{12}+a_{21}) + s\mathrm{Re}\,(a_{12}-a_{21}) + (r^2+s^2)\mathrm{Re}\,(a_{22})\right). \end{aligned} \tag{3.7}$$

Tedious, but straightforward computation shows that

$$\begin{aligned} |\langle Ax, x\rangle|^2 &= 1 + (r^2+s^2)^2\,|a_{22}|^2 + 2\mathrm{Re}\,(\overline{a_{22}}(a_{12}+a_{21}))\,r^3 \\ &- 2\mathrm{Im}\,(\overline{a_{2,2}}(a_{12}-a_{21}))\,r^2 s + (|a_{12}+a_{21}|^2 + 2\mathrm{Re}\,a_{22})\,r^2 \\ &- 4\mathrm{Im}\,(\overline{a_{21}}a_{12})\,rs + 2(\mathrm{Re}\,(a_{12}+a_{21}))\,r \\ &+ 2\mathrm{Re}\,(\overline{a_{22}}(a_{12}+a_{21}))rs^2 - 2\mathrm{Im}\,(\overline{a_{22}}(a_{12}-a_{21}))\,s^3 \\ &- 2(\mathrm{Im}\,(a_{12}-a_{21}))\,s + (|a_{12}-a_{21}|^2 + 2\mathrm{Re}\,a_{22})\,s^2. \end{aligned}$$

Comparing the coefficients with the corresponding formula for $|\langle Bx, x\rangle|^2$ gives the following set of equations:

$$|b_{22}|^2 = |a_{22}|^2 \tag{3.8}$$

$$\mathrm{Re}\,(\overline{b_{22}}(b_{12}+b_{21})) = \mathrm{Re}\,(\overline{a_{22}}(a_{12}+a_{21})) \tag{3.9}$$

$$\mathrm{Im}\,(\overline{b_{22}}(b_{12}-b_{21})) = \mathrm{Im}\,(\overline{a_{22}}(a_{12}-a_{21})) \tag{3.10}$$

$$\mathrm{Re}\,(b_{12}+b_{21}) = \mathrm{Re}\,(a_{12}+a_{21}) \tag{3.11}$$

$$\mathrm{Im}\,(b_{12}-b_{21}) = \mathrm{Im}\,(a_{12}-a_{21}) \tag{3.12}$$

$$|b_{12}+b_{21}|^2 + 2\mathrm{Re}\,b_{22} = |a_{12}+a_{21}|^2 + 2\mathrm{Re}\,a_{22} \tag{3.13}$$

$$|b_{12}-b_{21}|^2 + 2\mathrm{Re}\,b_{22} = |a_{12}-a_{21}|^2 + 2\mathrm{Re}\,a_{22} \tag{3.14}$$

$$\mathrm{Im}\,(\overline{b_{21}}b_{12}) = \mathrm{Im}\,(\overline{a_{21}}a_{12}). \tag{3.15}$$

Subtracting (3.14) from (3.13) gives, after an easy simplification,

$$4\mathrm{Re}\,(\overline{a_{21}}a_{12}) = 4\mathrm{Re}\,(\overline{b_{21}}b_{12}). \tag{3.16}$$

Decompose now $a_{12} = z_1 + \mathrm{i}z_2$, and $a_{21} = y_1 + \mathrm{i}y_2$, with z_1, z_2, y_1, y_2 real, and $b_{12} = \widetilde{z}_1 + \mathrm{i}\widetilde{z}_2$, etc. Then, Eqs. (3.11)–(3.12), (3.15)–(3.16) give:

$$\begin{aligned} \widetilde{y}_1 + \widetilde{z}_1 &= z_1 + y_1 \\ \widetilde{z}_2 - \widetilde{y}_2 &= z_2 - y_2 \\ \widetilde{y}_1\widetilde{z}_1 + \widetilde{y}_2\widetilde{z}_2 &= z_1y_1 + z_2y_2 \qquad (3.17) \\ z_2y_1 - z_1y_2 &= \widetilde{y}_1\widetilde{z}_2 - \widetilde{y}_2\widetilde{z}_1. \qquad (3.18) \end{aligned}$$

From the first two equations we get

$$\widetilde{y}_1 = y_1 + z_1 - \widetilde{z}_1 \qquad \text{and} \qquad \widetilde{y}_2 = y_2 - z_2 + \widetilde{z}_2. \tag{3.19}$$

Substitute this into (3.17), (3.18), and simplify, to get

$$\begin{aligned} (y_1 - \widetilde{z}_1)(\widetilde{z}_1 - z_1) &= (z_2 - \widetilde{z}_2)(y_2 + \widetilde{z}_2) \qquad (3.20) \\ z_2(y_1 - \widetilde{z}_1) + y_2(\widetilde{z}_1 - z_1) &= (y_1 + z_1 - 2\widetilde{z}_1)\widetilde{z}_2. \qquad (3.21) \end{aligned}$$

We are now facing two possibilities:

Possibility 1. $\widetilde{z}_1 = z_1$. Then, the last two equations further simplify into $(z_2 - \widetilde{z}_2)(y_2 + \widetilde{z}_2) = 0$, respectively, $(y_1 - z_1)z_2 = (y_1 - z_1)\widetilde{z}_2$. So, either $\widetilde{z}_2 = z_2$ or else $(y_1, y_2) = (z_1, -\widetilde{z}_2)$. In the former case, Eq. (3.19) brings $(\widetilde{y}_1, \widetilde{y}_2) = (y_1, y_2)$, so

$$b_{12} = \widetilde{z}_1 + \mathrm{i}\widetilde{z}_2 = a_{12}, \quad b_{21} = \widetilde{y}_1 + \mathrm{i}\widetilde{y}_2 = a_{21}.$$

In the latter case, we similarly deduce $y_1 = z_1 = \widetilde{z}_1 = \widetilde{y}_1$ and $y_2 = -\widetilde{z}_2$, and $\widetilde{y}_2 = -z_2$. Therefore,

$$b_{12} = \overline{a_{21}}, \quad b_{21} = \overline{a_{12}}. \tag{3.22}$$

Possibility 2. $\widetilde{z}_1 \neq z_1$. Then, (3.20) gives

$$y_1 = \frac{(\widetilde{z}_1 - z_1)\widetilde{z}_1 + (z_2 - \widetilde{z}_2)(y_2 + \widetilde{z}_2)}{\widetilde{z}_1 - z_1}.$$

This simplifies the remaining (3.21) into

$$\frac{\left((z_1 - \widetilde{z}_1)^2 + (z_2 - \widetilde{z}_2)^2\right)(y_2 + \widetilde{z}_2)}{z_1 - \widetilde{z}_1} = 0.$$

Note that the sum of squares in the left factor is nonzero because $z_1 - \widetilde{z}_1 \neq 0$. Hence, $\widetilde{z}_2 = -y_2$. From the previous equation we now read $\widetilde{z}_1 = y_1$. Moreover, Eq. (3.19) brings $\widetilde{y}_1 = z_1$ and $\widetilde{y}_2 = -z_2$. Therefore, $b_{12} = \overline{a_{21}}$ and $b_{21} = \overline{a_{12}}$, as in Eq. (3.22).

It only remains to compare b_{22} with a_{22}. Now, since

$$|\langle Bx, x\rangle| = |(\langle Bx, x\rangle)^*| = |\langle B^*x, x\rangle|,$$

we are free to replace (A, B) with (A, B^*). This way, we can always assume $b_{12} = a_{12}$ and $b_{21} = a_{21}$. Hence, we are done if we also show $b_{22} = a_{22}$.

To this end, Eq. (3.14) immediately gives $\operatorname{Re} b_{22} = \operatorname{Re} a_{22}$, while from $|b_{22}| = |a_{22}|$ we deduce that either $b_{22} = a_{22}$ or else $b_{22} = \overline{a_{22}} \neq a_{22}$. In the former case we are done. In the latter case, consider Eqs. (3.9)–(3.10) with $b_{22} := \overline{a_{22}}$. Simplifying these equations yields

$$\mathrm{Im}\,(a_{12} + a_{21})\,\mathrm{Im}\,a_{22} = 0 = \mathrm{Re}\,(a_{12} - a_{21})\,\mathrm{Im}\,a_{22}.$$

We may divide by nonzero $\mathrm{Im}\,a_{22}$. What we end up with easily simplifies into $a_{12} = \overline{a_{21}}$. Then, however,

$$A^* = \begin{bmatrix} 1 & \overline{a_{21}} \\ a_{21} & a_{22} \end{bmatrix}^* = \begin{bmatrix} 1 & \overline{a_{21}} \\ a_{21} & \overline{a_{22}} \end{bmatrix} = B,$$

which completes the proof of Step 3.

Step 4. Assuming (i) holds, we will prove that there exists a unimodular complex number γ such that either $\mathrm{diagv}(A) = \gamma\mathrm{diagv}\,(B)$ or else $\mathrm{diagv}(A) = \overline{\gamma\mathrm{diagv}\,(B)}$.

Let $A = [a_{ij}]_{i,j=1}^n$, $B = [b_{ij}]_{i,j=1}^n$. Choose any pair of distinct indices (i, j), and let $x := \lambda\mathbf{e}_i + \mu\mathbf{e}_j$ be in the subspace spanned by $\mathbf{e}_i, \mathbf{e}_j$. Then, $\langle Ax, x\rangle = \langle A_{ij}z, z\rangle$, where $z := [\lambda, \mu]^{\mathrm{tr}}$, and A_{ij} is the 2×2 matrix formed by the ith and jth rows and columns of A. The identity (3.1) then reduces to

$$|\langle A_{ij}z, z\rangle| \equiv |\langle B_{ij}z, z\rangle|.$$

Here, B_{ij} is the 2×2 matrix formed by the ith and jth rows and columns of B. By Step 3, $B_{ij} \in \{\gamma A_{ij}, \overline{\gamma A_{ij}}\}$, where γ is a unimodular number. Considering diagonal entries yields

$$(b_{ii}, b_{jj}) = \gamma(a_{ii}, a_{jj}) \quad \text{or} \quad (b_{ii}, b_{jj}) = \overline{\gamma(a_{ii}, a_{jj})}. \tag{3.23}$$

Consequently, either $\mathrm{diagv}\,(A) = 0 = \mathrm{diagv}\,(B)$ or else both diagonals have at least one nonzero entry. In the former case we are done.

In the latter case, we assume for simplicity the $(1, 1)$ entries of A and B are nonzero. Since $|a_{11}| = |b_{11}|$ we may replace (A, B) with $(A/a_{11}, \gamma\, B/a_{11})$ where $\gamma := a_{11}/b_{11}$ is unimodular. The identity (3.1) as well as the end result will not change. This way we achieve $a_{11} = 1 = b_{11}$. Moreover, when $i = 1$ Eq. (3.23) yields

$$(1, b_{jj}) \in \{(1, a_{jj}), \overline{(1, a_{jj})}\}.$$

Hence, it remains to see that $\mathrm{diagv}(A) = \mathrm{diagv}\,(B)$ or $\mathrm{diagv}(A) = \overline{\mathrm{diagv}\,(B)}$.

Now, arguing by contradiction, suppose that

$$(1, b_{i_0i_0}) = (1, a_{i_0i_0}) \neq \overline{(1, a_{i_0i_0})}, \quad (1, b_{i_1i_1}) = \overline{(1, a_{i_1i_1})} \neq (1, a_{i_1i_1}),$$

for two different indices i_0 and i_1. This is possible only when $b_{i_0i_0} = a_{i_0i_0}$ and $b_{i_1i_1} = \overline{a_{i_1i_1}}$ are both nonreal (hence also nonzero). Now, by Eq.(3.23),

$$(b_{i_0i_0}, b_{i_1i_1}) \in \{\gamma(a_{i_0i_0}, a_{i_1i_1}), \overline{\gamma(a_{i_0i_0}, a_{i_1i_1})}\}$$

implies

$$\frac{b_{i_0i_0}}{b_{i_1i_1}} \in \left\{\frac{a_{i_0i_0}}{a_{i_1i_1}}, \overline{\left(\frac{a_{i_0i_0}}{a_{i_1i_1}}\right)}\right\}. \tag{3.24}$$

On the other hand, $\frac{b_{i_0i_0}}{b_{i_1i_1}} = \frac{a_{i_0i_0}}{a_{i_1i_1}}$, and in view of (3.24) we obtain either $a_{i_1i_1} = \overline{a_{i_1i_1}}$ or else $a_{i_0i_0} = \overline{a_{i_0i_0}}$. This is the desired contradiction.

Therefore, either $(1, b_{jj}) = (1, a_{jj})$ for all j or else $(1, b_{jj}) = \overline{(1, a_{jj})}$ for all j. In the first case, $\mathrm{diagv}(A) = \mathrm{diagv}\,(B)$ while in the second one, $\mathrm{diagv}(A) = \overline{\mathrm{diagv}\,(B)}$.

Step 5. (i) $\Longrightarrow$ (iii), for $n \geq 3$.

Fix any unitary U and consider $(A_U, B_U) := (U^*AU, U^*BU)$. Clearly,

$$|\langle A_U x, x\rangle| = |\langle A(Ux), Ux\rangle| = |\langle B(Ux), Ux\rangle| = |\langle B_U x, x\rangle|.$$

Then apply the result of Step 4 to (A_U, B_U). We see that

$$\mathrm{diagv}(U^*BU) = \gamma(U)\mathrm{diagv}(U^*AU) \quad \text{or} \quad \mathrm{diagv}(U^*BU) = \overline{\gamma(U)\mathrm{diagv}(U^*AU)}$$

for each unitary U. By Step 2, $B = \gamma A$ or else $B = \overline{\gamma} A^*$, as required.

This completes the proof of Theorem 3.2. $\square$

Proof of Theorem 2.2 *in case* $t = 0$. We prove the nontrivial "only if" part. We may assume $A \neq 0$, $B \neq 0$. Multiplying A and B by nonzero complex numbers of the same absolute value, we may further suppose that

$$\langle Ae, e\rangle = \langle Be, e\rangle = 1 \tag{3.25}$$

for a fixed normalized element $e \in \mathsf{H}$. If $A \neq B$ and also $A \neq B^*$, then we have

$$\langle Af_1, f_1\rangle \neq \langle Bf_1, f_1\rangle \quad \text{and} \quad \langle Af_2, f_2\rangle \neq \langle B^*f_2, f_2\rangle \tag{3.26}$$

for some elements $f_1, f_2 \in \mathsf{H}$.

On the other hand, let P be the selfadjoint projection on the finite-dimensional subspace $\mathsf{H}_1 \subset \mathsf{H}$, generated by e and f_1, f_2, and let $\widehat{A} := PAP$ and $\widehat{B} := PBP$ be the operators acting on H_1. Clearly, $\langle Ag, g\rangle = \langle \widehat{A}g, g\rangle$ for any element $g \in \mathsf{H}_1$; likewise for $\widehat{B}$. Hence, by the assumptions, $|\langle \widehat{A}g, g\rangle| = |\langle \widehat{B}g, g\rangle|$ for every $g \in \mathsf{H}_1$. Therefore, by Theorem 3.2, we must have $\widehat{B} = \gamma\widehat{A}$ or $\widehat{B} = \gamma\widehat{A}^*$. Actually, $\gamma = 1$, by Eq. (3.25). Then however,

$$\langle Af_1, f_1\rangle = \langle \widehat{A}f_1, f_1\rangle = \langle \widehat{B}f_1, f_1\rangle = \langle Bf_1, f_1\rangle$$

(respectively, $\langle Af_1, f_1\rangle = \langle B^*f_2, f_2\rangle$, if $\widehat{B} = \widehat{A}^*$), a contradiction with (3.26). $\square$

4. Proof of Theorem 2.2: $t = \pi/2$

Assume $A, B \in L(\mathsf{H})$ are such that

$$|\langle Ax, y\rangle| = |\langle Bx, y\rangle|, \quad \forall \text{ orthonormal pairs } (x, y),\ x, y \in \mathsf{H}. \tag{4.1}$$

We proceed in steps to derive (3) of Theorem 2.2.

Step 1. Suppose that the implication (4.1) $\Longrightarrow$ Theorem 2.2(3) has been proved for C^2 and C^3. We will prove the implication for general Hilbert space H.

We may assume B is not scalar (otherwise Ax is orthogonal to span $\{x\}^\perp$ so $Ax = \lambda_x x$, and we are done as in the proof of Proposition 2.1). Therefore, there exists a normalized element $x \in \mathsf{H}$ such that Bx is not a scalar multiple of x, and hence there is an orthonormal pair (x, y) such that $\langle Bx, y\rangle \neq 0$. Let $\Omega := \{x, y, z\}$ be an orthonormal triple, where x and y are fixed, and let P be the orthogonal projection on $\operatorname{span}\Omega$. By considering operators PAP and PBP on $\operatorname{span}\Omega$ and using the supposition, we see that

$$PAP = \mu_\Omega PBP + \nu_\Omega P, \quad \mu_\Omega, \nu_\Omega \in \mathsf{C}, \quad |\mu_\Omega| = 1. \tag{4.2}$$

In fact, μ_Ω and ν_Ω are independent of Ω. Indeed, for two orthonormal triples Ω and Ω' we have in view of (4.2):

$$\begin{bmatrix} \mu_\Omega\langle Bx, x\rangle + \nu_\Omega & \mu_\Omega\langle By, x\rangle \\ \mu_\Omega\langle Bx, y\rangle & \mu_\Omega\langle By, y\rangle + \nu_\Omega \end{bmatrix} = \begin{bmatrix} \mu_{\Omega'}\langle Bx, x\rangle + \nu_{\Omega'} & \mu_{\Omega'}\langle By, x\rangle \\ \mu_{\Omega'}\langle Bx, y\rangle & \mu_{\Omega'}\langle By, y\rangle + \nu_{\Omega'} \end{bmatrix}.$$

Since $\langle Bx, y\rangle \neq 0$, we obtain $\mu_\Omega = \mu_{\Omega'}$ and $\nu_\Omega = \nu_{\Omega'}$. Thus,

$$PAP = \mu PBP + \nu P, \quad \mu, \nu \in \mathsf{C}, \quad |\mu| = 1. \tag{4.3}$$

Since any element $z \in \mathsf{H}$ can be included in the range of P, for some orthonormal triple Ω, we obtain from (4.3):

$$\langle Az, z\rangle = \mu\langle Bz, z\rangle + \nu\langle z, z\rangle, \quad \forall \ z \in \mathsf{H},$$

and (3) of Theorem 2.2 follows.

Step 2. We prove the implication (4.1) $\Longrightarrow$ Theorem 2.2(3) for C^2 and C^3.

Applying simultaneous unitary similarity and addition of scalar matrices to A and to B we may assume that

$$A = [a_{ij}]_{i,j=1}^n, \qquad B = [b_{ij}]_{i,j=1}^n, \qquad a_{ij}, b_{ij} \in \mathsf{C},$$

where A is upper triangular, $a_{11} = 0$, $a_{12}, \ldots, a_{1n}$ are nonnegative and $b_{11} = 0$. (We need only the cases $n = 2, 3$, but this transformation can be applied for $L(\mathsf{C}^n)$ for any integer $n \geq 2$.) Applying (4.1) with $x = \mathbf{e}_i$, $y = \mathbf{e}_j$, $i < j$, we see that B is a also upper triangular. Applying (4.1) with $x = \mathbf{e}_i$, $y = \mathbf{e}_j$, $i > j$, we see that $|b_{ij}| = |a_{ij}|$ for all $i < j$.

We proceed with $n = 2$. If $a_{12} = 0$ then also $b_{12} = 0$ in which case $A = \operatorname{diag}(0, a_{22})$ and $B = \operatorname{diag}(0, b_{22})$. With orthonormal

$$x = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}, \quad y = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix},$$

(4.1) easily gives $|a_{22}| = |b_{22}|$, and we are done. If $a_{12} \neq 0$ we further assume (replacing B with $e^{\mathrm{i}s}B$ for some real s) that $b_{12} = a_{12}$. So under $n = 2$ we are left with

Case (a).

$$A = \begin{bmatrix} 0 & a_{12} \\ 0 & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & a_{12} \\ 0 & b_{22} \end{bmatrix},$$

where $a_{12} > 0$.

Multiplying (4.1) with suitable scalar we see it holds for any (possibly not normalized) orthogonal vectors x, y. Apply (4.1) with $x = \begin{bmatrix} z \\ 1 \end{bmatrix}$, $y = \begin{bmatrix} 1 \\ -\overline{z} \end{bmatrix}$, $z \in \mathsf{C}$. We obtain:

$$(a_{12} - a_{22}z)(a_{12} - \overline{a_{22}z}) = (a_{12} - b_{22}z)(a_{12} - \overline{b_{22}}\overline{z}), \quad \forall \; z \in \mathsf{C},$$

which yields $|a_{22}| = |b_{22}|$ and $a_{12}a_{22} = a_{12}b_{22}$. So, $a_{22} = b_{22}$ hence $A = B$, which proves case (a).

Next assume $n = 3$. If $a_{12} = 0 = a_{13}$ (and hence also $b_{12} = 0 = b_{13}$) then Corollary 2.4 is applicable to the already proven case of 2×2 matrices $[a_{ij}]_{i,j=2}^3$ and $[b_{ij}]_{i,j=2}^3$, and we are done using induction on n. Thus, we can assume that not both a_{12}, a_{13} are zeros, and letting a_{1r} be the first positive number among a_{12}, a_{13}, we further assume (replacing B with $e^{\mathrm{i}s}B$ for some real s) that $b_{1r} = a_{1r}$. So we are left with the following two cases to consider:

Case (b).

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & a_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix},$$

where $a_{12} > 0$, $a_{13} \geq 0$, $|b_{13}| = a_{13}$, $|b_{23}| = |a_{23}|$;

Case (c).

$$A = \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix},$$

where $a_{13} > 0$, $|b_{23}| = |a_{23}|$.

Consider Case (b). Applying the proof of Case (a) to the upper left 2×2 submatrices of A and B, we see that $a_{22} = b_{22}$. Now use (4.1) with

$$x = \begin{bmatrix} 0 \\ 1 \\ z \end{bmatrix}, \quad y = \mathbf{e}_1.$$

We obtain:

$$|a_{12} + a_{13}z|^2 = |a_{12} + b_{13}z|^2, \qquad \forall \; z \in \mathsf{C},$$

and hence $a_{13}a_{12} = b_{13}a_{12}$. Since $a_{12} \neq 0$, we have $a_{13} = b_{13}$. Analogous consideration of (4.1) with

$$x = \mathbf{e}_3, \qquad y = \begin{bmatrix} 1 \\ \overline{z} \\ 0 \end{bmatrix}$$

yields $a_{23}a_{13} = b_{23}a_{13}$. Thus, either $a_{13} = b_{13} = 0$, or $a_{13} = b_{13} \neq 0$ and $a_{23} = b_{23}$.

Therefore, one of the following three situations occurs:

(b1) $a_{13} = b_{13} \neq 0$, $a_{23} = b_{23}$;
(b2) $a_{13} = b_{13} = 0$, $a_{23} \neq 0$ (then also $b_{23} \neq 0$);
(b3) $a_{13} = b_{13} = 0$, $a_{23} = b_{23} = 0$.

If (b1) holds, then the proof of Case (a) applied to the 2×2 principal submatrices of A and B in the 1st and 3rd rows and columns yields $a_{33} = b_{33}$, i.e., $A = B$. If (b3) holds, then we apply (4.1) with

$$x = \begin{bmatrix} w \\ 1 \\ 1 \end{bmatrix}, \qquad y = \begin{bmatrix} 1 \\ 0 \\ -\overline{w} \end{bmatrix}, \qquad w \in \mathsf{C},$$

resulting in

$$|a_{12} - a_{33}w|^2 = |a_{12} - b_{33}w|^2, \qquad \forall \ \ w \in \mathsf{C}.$$

It follows that $|a_{33}| = |b_{33}|$ and

$$\mathrm{Re}\,(b_{33}a_{12}w - a_{33}a_{12}w) = 0, \qquad \forall \ \ w \in \mathsf{C}.$$

Since $a_{12} \neq 0$, we conclude $a_{33} = b_{33}$, thus $A = B$. Finally, if (b2) holds, then we apply (4.1) with

$$x = \begin{bmatrix} -w \\ 1 \\ 1 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ \overline{w} \\ 0 \end{bmatrix}, \quad w \in \mathsf{C}.$$

It follows that

$$|a_{12} + (a_{22} + a_{23})w|^2 = |a_{12} + (a_{22} + b_{23})w|^2.$$

In particular,

$$\mathrm{Re}\ ((a_{22} + a_{23})wa_{12} - (a_{22} + b_{23})wa_{12}) = 0,$$

and since this equality holds for all $w \in \mathsf{C}$, we obtain $a_{23} = b_{23}$. Now we apply the proof of Case (a) to the lower right 2×2 submatrices of $A - a_{22}I$ and $B - a_{22}I$, and the equality $A = B$ follows.

This concludes our consideration of Case (b).

Consider now Case (c). Applying the proof of Case (a) to the 2×2 submatrices of A and B generated by the 1st and 3rd rows and columns, we see that $b_{33} = a_{33}$. Next, apply (4.1) with

$$x = \begin{bmatrix} -zp \\ z \\ 1 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ \overline{p} \\ 0 \end{bmatrix}, \quad z, p \in \mathsf{C}.$$

It follows that

$$|a_{13} + (a_{22}z + a_{23})p|^2 = |a_{13} + (b_{22}z + b_{23})p|^2, \qquad \forall \ \ z, p \in \mathsf{C}.$$

Consider this as a polynomial of the real and imaginary parts of p, with z as a parameter. In particular,

$$\mathrm{Re}\,((a_{22}z + a_{23})pa_{13} - (b_{22}z + b_{23})pa_{13}) = 0.$$

Since $a_{13} \neq 0$ and the equality holds for all $p \in \mathsf{C}$, we have

$$a_{22}z + a_{23} = b_{22}z + b_{23}, \quad \forall \ \ z \in \mathsf{C}.$$

Clearly, $a_{22} = b_{22}$ and $a_{23} = b_{23}$. This completes the proof of Step 2. $\square$

5. Proof of Theorem 2.2: $0 < t < \pi/2$

Again, we prove only the nontrivial "only if" part.

Step 1. Assume the dimension of H is at least 3.

Let (x, y) be an orthonormal pair. Then

$$|\langle Ax, x + (\tan t)ye^{is}\rangle| = |\langle Bx, x + (\tan t)ye^{is}\rangle|, \quad \forall\ s \in [0, 2\pi). \tag{5.1}$$

Consider the two circles

$$C_A := \{\langle Ax, x + (\tan t)ye^{is}\rangle \ : \ 0 \le s < 2\pi\},$$
$$C_B := \{\langle Bx, x + (\tan t)ye^{is}\rangle \ : \ 0 \le s < 2\pi\}$$

with centers and radii $\langle Ax, x\rangle$, $(\tan t)|\langle Ax, y\rangle|$, and $\langle Bx, x\rangle$, $(\tan t)|\langle Bx, y\rangle|$, respectively. Condition (5.1) implies that

$$\min_{z\in C_A} |z| = \min_{z\in C_B} |z| \qquad \text{and} \qquad \max_{z\in C_A} |z| = \max_{z\in C_B} |z|,$$

and therefore

$$|\langle Ax, x\rangle| + (\tan t)|\langle Ax, y\rangle| = |\langle Bx, x\rangle| + (\tan t)|\langle Bx, y\rangle|$$

and

$$|\ |\langle Ax, x\rangle| - (\tan t)|\langle Ax, y\rangle|\ | = |\ |\langle Bx, x\rangle| - (\tan t)|\langle Bx, y\rangle|\ |.$$

We see that one of the two possibilities holds: either

(a) $$|\langle Ax, x\rangle| = |\langle Bx, x\rangle| \quad \text{and} \quad |\langle Ax, y\rangle| = |\langle Bx, y\rangle| \tag{5.2}$$

(this happens if the origin is not situated inside one of the circles C_A and C_B and outside of the other circle);

or

(b) there exist positive numbers $p \neq q$ such that $|\langle Ax, x\rangle| = p$, $|\langle Bx, x\rangle| = q$, $(\tan t)|\langle Ax, y\rangle| = q$, $(\tan t)|\langle Bx, y\rangle| = p$ (this happens if the origin is situated inside one circle and outside of the other).

Clearly, for every fixed normalized $x \in \mathsf{H}$, either (a) holds for all $y \in \mathsf{H}$ such that (x, y) form an orthonormal pair, or (b) holds for all such y. We claim that (b) is not possible (here we use the hypothesis that $\dim \mathsf{H} \ge 3$). Indeed, under (b) we have

$$|\langle Ax, y\rangle| = |\langle Bx, x\rangle|(\tan t)^{-1} \neq 0 \tag{5.3}$$

for every normalized y orthogonal to x. If y_1, y_2 are orthonormal elements both orthogonal to x, then there is a nonzero linear combination of y_1, y_2 which is orthogonal to Ax, a contradiction with (5.3). Thus, we have (a) for every orthonormal pair (x, y), $x, y \in \mathsf{H}$, and by the part of Theorem 2.2 for the cases $t = 0$ and $t = \pi/2$, we obtain $B = \mu A$ or $B = \mu A^*$ for some $\mu \in \mathsf{C}$, $|\mu| = 1$, as well as $B = \gamma A + \nu I$ for some $\gamma, \nu \in \mathsf{C}$ with $|\gamma| = 1$.

We claim that $B = \mu' A$, for some $\mu' \in \mathsf{C}$ with $|\mu'| = 1$, always holds. Indeed, suppose $B = \mu A^*$, $|\mu| = 1$. Without loss of generality we may take $\mu = 1$. Taking squares in (5.1), and using (5.2), we obtain for every orthonormal pair (x, y) and every s, $0 \leq s < 2\pi$:

$$\operatorname{Re}\left(\langle Ax, ye^{\mathrm{i}s}\rangle \cdot \overline{\langle Ax, x\rangle}\right) = \operatorname{Re}\left(\langle Bx, ye^{\mathrm{i}s}\rangle \cdot \overline{\langle Bx, x\rangle}\right).$$

Thus,

$$\langle Ax, y\rangle \cdot \overline{\langle Ax, x\rangle} = \langle Bx, y\rangle \cdot \overline{\langle Bx, x\rangle}.$$

Substituting in this equality $B = \gamma A + \nu I$, we have

$$\langle Ax, y\rangle \cdot \overline{\langle Ax, x\rangle} = \gamma\langle Ax, y\rangle(\overline{\gamma\langle Ax, x\rangle + \nu}). \tag{5.4}$$

If x is not an eigenvector of A, then we can take $y \not\perp Ax$, and (5.4) gives

$$\langle Ax, x\rangle = \langle Ax, x\rangle + \overline{\gamma}\nu,$$

thus $\nu = 0$ and we are done. If every normalized $x \in \mathsf{H}$ is an eigenvector of A, then $A = zI$, $z \in \mathsf{C}$, and

$$B = A^* = \overline{z}I = \frac{\overline{z}}{z}A,$$

and we are done again (the case $z = 0$ is trivial).

Step 2. Assume $\mathsf{H} = \mathsf{C}^2$.

We need to show that, for fixed $A, B \in M_2$, the equality

$$|x^*Ax + (\tan t)y^*Ax| = |x^*Bx + (\tan t)y^*Bx| \tag{5.5}$$

for every orthonormal pair (x, y), $x, y \in \mathsf{C}^2$, implies

$$A = \mu B \quad \text{for some unimodular} \quad \mu. \tag{5.6}$$

We consider a special case first.

Case 1. Suppose

$$A = \begin{bmatrix} 1 & a_1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \nu & b_1 \\ 0 & 0 \end{bmatrix}, \quad |\nu| = 1,$$

and (5.5) holds. We may assume $\nu = 1$. Write

$$a_1 = a_1' + \mathrm{i}a_1'', \quad b_1 = b_1' + \mathrm{i}b_1'',$$

where a_1', a_1'', b_1', b_1'' are real, and let $\tau = \tan t$. Applying (5.5) to the orthogonal pair

$$x = \begin{bmatrix} u + \mathrm{i}v \\ 1 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ -u + \mathrm{i}v \end{bmatrix}, \quad u, v \in \mathsf{R},$$

(note that x and y have equal lengths, and therefore (5.5) is applicable), we obtain

$$|u^2 + v^2 + (u - \mathrm{i}v)a_1 + \tau(u + \mathrm{i}v) + \tau a_1| = |u^2 + v^2 + (u - \mathrm{i}v)b_1 + \tau(u + \mathrm{i}v) + \tau b_1|.$$

Taking squares in this equality, and expressing the modulus squared of a complex number as the sum of squares of its real and imaginary parts, yields

$$\begin{aligned}&(u^2 + v^2 + ua_1' + va_1'' + \tau u + \tau a_1')^2 + (ua_1'' - va_1' + \tau v + \tau a_1'')^2 \\ &= (u^2 + v^2 + ub_1' + vb_1'' + \tau u + \tau b_1')^2 + (ub_1'' - vb_1' + \tau v + \tau b_1'')^2.\end{aligned} \tag{5.7}$$

This equality holds for all real u, v, and both sides are polynomials in u, v. Equating the coefficients of u^3 in both sides of (5.7) gives $2(a_1' + \tau) = 2(b_1' + \tau)$, and equating the coefficients of v^3 gives $2a_1'' = 2b_1''$. Thus, $a_1 = b_1$, as required.

To continue with the proof of Step 2, we bring a general fact. Given fixed $\alpha, \beta, \gamma, \delta \in \mathsf{C}$, assume the identity

$$|\alpha + e^{i\xi}\beta| = |\gamma + e^{i\xi}\delta|, \qquad \xi \in \mathsf{R}, \tag{5.8}$$

holds. Note that (5.8) is equivalent to

$$|\alpha|^2 + |\beta|^2 - |\gamma|^2 - |\delta|^2 + 2\mathrm{Re}\left(e^{i\xi}\left(\beta\overline{\alpha} - \delta\overline{\gamma}\right)\right) = 0. \tag{5.9}$$

Due to arbitrariness of $\xi \in \mathsf{R}$ (5.9) is further equivalent to

$$|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\delta|^2, \qquad \beta\overline{\alpha} = \delta\overline{\gamma}. \tag{5.10}$$

Adding two times the absolute values of the second equation in (5.10) to the first one, and subtracting the same from the first equation in (5.10), we easily find that at least one of the following two conditions holds:

(i) $(|\gamma|, |\delta|) = (|\alpha|, |\beta|)$;
(ii) $(|\gamma|, |\delta|) = (|\beta|, |\alpha|)$;

Multiply $\beta\overline{\alpha} = \delta\overline{\gamma}$ with α and use either $\alpha\overline{\alpha} = \gamma\overline{\gamma}$ or $\alpha\overline{\alpha} = \delta\overline{\delta}$ to obtain

$$\text{(i}'\text{)}\ (\gamma, \delta) = \mu(\alpha, \beta), \quad \text{or} \quad \text{(ii}'\text{)}\ (\gamma, \delta) = \mu(\overline{\beta}, \overline{\alpha}); \quad \text{for some } \mu \in \mathsf{C},\ |\mu| = 1. \tag{5.11}$$

Now, write $A = \sum a_{ij}E_{ij}$ and $B = \sum b_{ij}E_{ij}$, where E_{ij} are the standard matrix units in M_2: E_{ij} has 1 in the (i, j)th position and zeros elsewhere. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis of unit vectors for C^2.

Case 2. Suppose A or B is zero, say, $A = 0$. Applying (5.5) with $(x, y) = (\mathbf{e}_1, \mathbf{e}_2e^{i\xi})$ for every $\xi \in [0, 2\pi)$, we see that the first column of B is zero. Applying (5.5) with $(x, y) = (\mathbf{e}_2, \mathbf{e}_1e^{i\xi})$ for every $\xi \in [0, 2\pi)$, we see that the second column of B is also zero. So, (5.6) holds.

Case 3. Suppose both A and B are nonzero nilpotent. Replacing (A, B) by (U^*AU, U^*BU) for a suitable unitary U, we may assume that $a_{11} = a_{21} = a_{22} = 0$. Applying (5.5) with $(x, y) = (\mathbf{e}_1, \mathbf{e}_2e^{i\xi})$ for every $\xi \in [0, 2\pi)$, we see that the first column of B is zero. Since B is nilpotent, we see that $b_{22} = 0$. Applying (5.5) with $(x, y) = (\mathbf{e}_2, \mathbf{e}_1e^{i\xi})$ for every $\xi \in [0, 2\pi)$, we see that $|a_{12}| = |b_{12}|$. So, (5.6) holds.

Case 4. Suppose A and B are nonzero, and at least one of them, say, A is not nilpotent. Replacing (A, B) by $(U^*AU/\gamma, U^*BU/\gamma)$ for a suitable unitary U and a suitable $\gamma \in \mathsf{C}$, we may assume that $(a_{11}, a_{21}) = (1, 0)$ and $|a_{22}| \le 1$ (see the Schur unitary triangularization theorem [7, Theorem 2.3.1]).

Now, for $(x, y) = (c\mathbf{e}_1 + s\mathbf{e}_2, e^{-i\xi}(-s\mathbf{e}_1 + c\mathbf{e}_2))$ with $\xi, c, s \in \mathsf{R}$ such that $(c, s) = (\cos u, \sin u)$ for some $u \in \mathsf{R}$, equation (5.5) is valid. Hence,

$$\begin{aligned} &|c^2 + a_{12}cs + s^2a_{22} + e^{i\xi}\tan t\,(-cs(1 - a_{22}) - s^2a_{12})| \\ &\quad = |b_{11}c^2 + (b_{12} + b_{21})cs + s^2b_{22} + e^{i\xi}\tan t\,(b_{21}c^2 - cs(b_{11} - b_{22}) - s^2b_{12})|. \end{aligned} \tag{5.12}$$

It follows (see the implication (5.8) $\Rightarrow$ (5.11)) that for any pair $(c,s) = (\cos u, \sin u)$ with $c, s > 0$, at least one of the two pairs of equalities (i$''$) and (ii$''$) below holds:

$$\begin{aligned} c^2 + a_{12}cs + s^2 a_{22} &= \mu_s(b_{11}c^2 + (b_{12}+b_{21})cs + s^2 b_{22}) \\ \tan t\,(-(1-a_{22})cs - s^2 a_{12}) &= \mu_s \tan t\,(b_{21}c^2 - (b_{11}-b_{22})cs - s^2 b_{12}) \end{aligned} \tag{i$''$}$$

for some unimodular $\mu_s \in \mathsf{C}$;

$$\begin{aligned} c^2 + a_{12}cs + s^2 a_{22} &= \mu_s \tan t\,(\overline{b_{21}}c^2 - (\overline{b_{11}} - \overline{b_{22}})cs - s^2\overline{b_{12}}) \\ \tan t\,(-(1-a_{22})cs - s^2 a_{12}) &= \mu_s(\overline{b_{11}}c^2 + (\overline{b_{12}} + \overline{b_{21}})cs + s^2\overline{b_{22}}) \end{aligned} \tag{ii$''$}$$

for some unimodular $\mu_s \in \mathsf{C}$. Rewrite (i$''$) and (ii$''$) into equivalent forms

$$\begin{aligned} (1-\mu_s b_{11})\left(\tfrac{s}{c}\right)^{-1} + a_{12} - \mu_s(b_{12}+b_{21}) + \left(\tfrac{s}{c}\right)(a_{22} - \mu_s b_{22}) &= 0 \\ (-\mu_s b_{21})\left(\tfrac{s}{c}\right)^{-1} - ((1-a_{22}) - \mu_s(b_{11}-b_{22})) - \left(\tfrac{s}{c}\right)(a_{12} - \mu_s b_{12}) &= 0 \end{aligned} \tag{i$'''$}$$

and

$$\begin{aligned} (1-\mu_s\tau\,\overline{b_{21}})\left(\tfrac{s}{c}\right)^{-1} + a_{12} + \mu_s\tau\,(\overline{b_{11}} - \overline{b_{22}}) + \left(\tfrac{s}{c}\right)(a_{22} + \mu_s\tau\,\overline{b_{12}}) &= 0 \\ (-\mu_s\overline{b_{11}})\left(\tfrac{s}{c}\right)^{-1} - (\tau\,(1-a_{22}) + \mu_s(\overline{b_{12}} + \overline{b_{21}})) - \left(\tfrac{s}{c}\right)(\tau\,a_{12} + \mu_s\overline{b_{22}}) &= 0 \end{aligned} \tag{ii$'''$}$$

respectively, with $\tau := \tan t > 0$.

Fix a sequence of pairs of positive numbers (c_i, s_i), with $c_i^2 + s_i^2 = 1$, converging to $(1,0)$. Passing to a subsequence, we have that at least one of (i$'''$) and (ii$'''$) holds for all its members, and we may also assume that $\lim_{i\to\infty}\mu_{s_i} = \mu$ for some unimodular μ.

Suppose (i$'''$) holds for all (c_i, s_i). Clearly $(s_i/c_i)^{-1}$ converges to ∞, while $|\mu_{s_i}| = 1$ is bounded. It follows from the first equation of (i$'''$) that $\lim_{i\to\infty}(1 - \mu_{s_i}b_{11}) = 0$, so $1 - \mu b_{11} = 0$ and $b_{11} = \mu^{-1}$. The second equation in (i$'''$) yields that $\lim_{i\to\infty}(-\mu_{s_i}b_{21}) = 0$, hence $b_{21} = 0$. Now the second equation in (i$'''$) takes the form

$$a_{22} - 1 + \mu_{s_i}(\mu^{-1} - b_{22}) - \left(\tfrac{s_i}{c_i}\right)(a_{12} - \mu_{s_i}b_{12}) = 0, \tag{5.13}$$

and passing to the limit when $i \to \infty$ gives

$$a_{22} - 1 + \mu(\mu^{-1} - b_{22}) = 0, \tag{5.14}$$

i.e., $b_{22} = \mu^{-1}a_{22}$. Next, substitute zero for b_{21} and μ^{-1} for b_{11} in the first equation in (i$'''$), and pass to the limit. The result is

$$\lim_{i\to\infty}(1 - \mu_{s_i}\mu^{-1})\left(\tfrac{s_i}{c_i}\right)^{-1} = -a_{12} + \mu b_{12}. \tag{5.15}$$

On the other hand, substituting $b_{22} = \mu^{-1}a_{22}$ into (5.13) yields, after some rearrangements

$$(a_{22} - 1)\left(1 - \mu_{s_i}\mu^{-1}\right)\left(\tfrac{s_i}{c_i}\right)^{-1} - (a_{12} - b_{12}\mu_{s_i}) = 0.$$

Using (5.15) it follows after simplification that

$$\mu a_{22}b_{12} - a_{12}a_{22} = 0.$$

Thus, either $b_{12} = \mu^{-1}a_{12}$, and then (5.6) holds: $B = \mu^{-1}A$, or $a_{22} = 0$, and then (5.6) holds by virtue of Case 1.

Thus, the proof of Step 2 is complete if there is a sequence of positive numbers (c_i, s_i) with $c_i^2 + s_i^2 = 1$ converging to $(1, 0)$ such that (i‴) holds for all (c_i, s_i).

We now assume that (ii‴) holds for all positive (c, s) with $c^2 + s^2 = 1$ and s sufficiently close to zero. It follows from the first equation of (ii‴) that $\lim_{s\to 0}(1 - \mu_s \tau\, \overline{b_{21}}) = 0$. Denoting by μ any partial limit of μ_s as $s \to 0$, we have $1 - \mu\tau\, \overline{b_{21}} = 0$, or

$$\overline{b_{21}} = \mu\tau^{-1}. \tag{5.16}$$

(By the way this shows that μ is unique, i.e., $\mu = \lim_{s\to 0} \mu_s$.) The second equation in (ii‴) yields $\lim_{s\to 0}(-\mu_s \overline{b_{11}}) = 0$, hence $b_{11} = 0$. Letting $s \to 0$, the second equation in (ii‴) gives

$$\tau(1 - a_{22}) + \mu(\overline{b_{12}} + \overline{b_{21}}) = 0. \tag{5.17}$$

Thus,

$$\overline{b_{12}} = \mu^{-1}(-\mu\overline{b_{21}} - \tau(1 - a_{22})),$$

and using $\overline{b_{21}} = \mu^{-1}\tau^{-1}$ we obtain

$$\overline{b_{12}} = \mu^{-1}(-\tau - \tau^{-1} + \tau a_{22}). \tag{5.18}$$

It follows from (5.16) and (5.18) that

$$\overline{b_{12}} + \overline{b_{21}} = \mu^{-1}\tau(-1 + a_{22}). \tag{5.19}$$

Substituting μ^{-1} for $\tau\overline{b_{21}}$ and zero for b_{11} in the first equation in (ii‴), we find

$$(1 - \mu_s\mu^{-1})\left(\tfrac{s}{c}\right)^{-1} + a_{12} - \mu_s\tau\overline{b_{22}} + \left(\tfrac{s}{c}\right)(a_{22} + \mu_s\tau\, \overline{b_{12}}) = 0,$$

and passing to the limit when $s \to 0$, it follows that

$$\lim_{s\to 0}(\mu - \mu_s)\left(\tfrac{s}{c}\right)^{-1} = \mu^2\tau\overline{b_{22}} - \mu a_{12}. \tag{5.20}$$

On the other hand, using $b_{11} = 0$ and (5.17), the second equation in (ii‴) reads

$$(\mu - \mu_s)(\overline{b_{12}} + \overline{b_{21}}) - \left(\tfrac{s}{c}\right)(\tau a_{12} + \mu_s\overline{b_{22}}) = 0.$$

In view of (5.20) we have

$$(\mu^2\tau\overline{b_{22}} - \mu a_{12})(\overline{b_{12}} + \overline{b_{21}}) = \tau a_{12} + \mu\overline{b_{22}}. \tag{5.21}$$

Using (5.19), we solve (5.21) for $\overline{b_{22}}$:

$$\overline{b_{22}} = \frac{a_{12}\tau a_{22}}{\mu\tau^2 a_{22} - \mu\tau^2 - \mu}. \tag{5.22}$$

Note that the condition $|a_{22}| \le 1$ guarantees that the denominator in (5.22) is nonzero.

Next, we show that $b_{12} + b_{21} = b_{22} = 0$. Arguing by contradiction, let us suppose that

$$b_{12} + b_{21} \ne 0 \quad \text{or} \quad b_{22} \ne 0. \tag{5.23}$$

Then the second equation in (ii$'''$) can be solved for μ_s (assuming s is close enough to zero):

$$\mu_s = \frac{\tau(1-a_{22}) + w\tau a_{12}}{-\overline{b_{12}} - \overline{b_{21}} - w\overline{b_{22}}}, \quad w := \tfrac{s}{c}. \tag{5.24}$$

Substituting the right-hand side of (5.24) into the first equation in (ii$'''$), after some simple algebra, we obtain:

$$\begin{aligned}
&(\overline{b_{22}}a_{22} - \tau^2\overline{b_{12}}a_{12})w^3 \\
&+ \left(\left(\tau^2+1\right)\overline{b_{22}}a_{12} + \overline{b_{21}}a_{22} + \overline{b_{12}}\left(\left(\tau^2+1\right)a_{22} - \tau^2\right)\right)w^2 \\
&+ ((\overline{b_{12}} + \left(\tau^2+1\right)\overline{b_{21}})a_{12} + \overline{b_{22}}(-a_{22}\tau^2 + \tau^2 + 1))w \\
&+ \overline{b_{12}} + \overline{b_{21}}\left(-a_{22}\tau^2 + \tau^2 + 1\right) = 0
\end{aligned} \tag{5.25}$$

The equation holds for all w close to zero; equating coefficients of powers of w on the right-hand and on the left-hand sides of (5.25), the following equalities result ((5.26), ((5.27), ((5.28), ((5.29) correspond to the coefficients of w^3, w^0, w^2, w^1, respectively):

$$-a_{22}\overline{b_{22}} + \tau^2 a_{12}\overline{b_{12}} = 0, \tag{5.26}$$

$$\overline{b_{12}} + \overline{b_{21}} + \tau^2(1-a_{22})\overline{b_{21}} = 0, \tag{5.27}$$

$$a_{12}\overline{b_{22}} + \tau^2 a_{12}\overline{b_{22}} + a_{22}(\overline{b_{12}} + \overline{b_{21}}) - \tau^2(1-a_{22})\overline{b_{12}} = 0, \tag{5.28}$$

$$-\overline{b_{22}} - \tau^2 a_{12}\overline{b_{21}} + a_{12}(-\overline{b_{12}} - \overline{b_{21}}) - \tau^2(1-a_{22})\overline{b_{22}} = 0. \tag{5.29}$$

Substituting the right-hand sides of (5.22) and (5.18) for $\overline{b_{22}}$ and $\overline{b_{12}}$, respectively, in (5.26) yields after simplification:

$$a_{12}\left(a_{22}-1\right)\left(-\tau^2 + \left(\tau^2-1\right)a_{22} - 1\right) = 0$$

Thus, at least one of the three equalities holds:

$$a_{12} = 0, \tag{5.30}$$

$$a_{22} = 1, \tag{5.31}$$

$$a_{22}(\tau^2-1) = \tau^2 + 1. \tag{5.32}$$

However, (5.32) is impossible because it contradicts $\tau > 0$ and $|a_{22}| \le 1$. In the case (5.30) holds we have $b_{22} = 0$, by (5.26). Substitute $b_{22} = 0$ and the right-hand sides of (5.18) and (5.19) for $\overline{b_{12}}$ and $\overline{b_{12}} + \overline{b_{21}}$, respectively, in (5.28), to obtain:

$$\frac{\tau\left(\tau^2+1\right)\left(a_{22}-1\right)^2}{\mu} = 0,$$

and since $\tau > 0$ we have $a_{22} = 1$. But then $b_{12} + b_{21} = 0$ by (5.19), a contradiction with (5.23). So (5.30) cannot be true and hence we must have $a_{22} = 1$. Then $b_{12} + b_{21} = 0$. Now (5.28) gives

$$a_{12}\overline{b_{22}}(1+\tau^2) = 0,$$

so either $a_{12} = 0$ or $b_{22} = 0$, and in either case a contradiction with (5.23) results.

Thus, (5.23) cannot hold, and we have $b_{12}+b_{21}=0$ and $b_{22}=0$. By (5.19) $a_{22}=1$ and then by (5.22) $a_{12}=0$. Keeping in mind (5.16), the result is that

$$A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad B=\mu\begin{bmatrix}0 & -\cot t\\ \cot t & 0\end{bmatrix}.$$

We now can finish the proof of Step 2 as follows. If already $A=\mu B$ for some $|\mu|=1$, then we are done. Assume lastly $A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and $B=\mu\begin{bmatrix}0 & -\cot t\\ \cot t & 0\end{bmatrix}$. Here we replace the pair (A,B) with

$$(A',B') := (UAU^*, UBU^*), \quad A'=\sum a'_{ij}E_{ij}, \quad B'=\sum b'_{ij}E_{ij},$$

for the unitary $U := \operatorname{diag}(\mathfrak{i},1)$. Clearly, the new pair still satisfies the defining identity (5.5), and still $(a'_{11},a'_{21})=(1,0)$ and $|a'_{22}|\le 1$. This allows us to use the same arguments as above in Case 4. In particular, Eq. (5.12) with (a_{ij},b_{ij}) replaced by (a'_{ij},b'_{ij}) gives either $A'=\mu' B'$ ($|\mu'|=1$) wherefrom $A=\mu' B$, or else $A'=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and $B'=\mu'\begin{bmatrix}0 & -\cot t\\ \cot t & 0\end{bmatrix}$. But the last case is contradictory, namely recall that

$$\mu'\begin{bmatrix}0 & -\cot t\\ \cot t & 0\end{bmatrix}=B'=UBU^*=\mu\begin{bmatrix}0 & -\mathfrak{i}\cot t\\ -\mathfrak{i}\cot t & 0\end{bmatrix},$$

giving $\mu'=0=\mu$, a contradiction.

This concludes the proof of Step 2. □

6. Linear dependence in terms of trace functionals

If $C\in L(\mathsf{H})$ is a trace class operator, then the formula

$$W_C(A)=\{\operatorname{trace}(CU^*AU) \;:\; U\in L(\mathsf{H}), \quad U \text{ unitary}\}$$

defines the *C-numerical range* of an operator $A\in L(\mathsf{H})$. The C-numerical ranges also have been extensively studied, see [5, 8, 1, 12, 3], a representative sample of relevant works. In particular, C-numerical ranges of matrices have been applied recently in quantum computing and control [4, 14, 6]. It is easy to see that the q-numerical range is actually the C-numerical range with C given by

$$Cx=q\langle x,y\rangle y+\sqrt{1-q^2}\langle x,z\rangle y, \quad x\in\mathsf{H}, \tag{6.1}$$

where (y,z), $y,z\in\mathsf{H}$, is a fixed orthonormal pair. Note that every rank one operator is unitarily similar (after appropriate scaling) to an operator of the form (6.1); thus, the q-numerical ranges represent the C-numerical ranges with rank one operators C.

The result of Proposition 2.1 extends to C-numerical ranges, as follows.

Theorem 6.1. *Let f be the bounded linear functional on $L(\mathsf{H})$, given by a trace class operator C:*

$$f(X)=\operatorname{trace}(CX), \qquad X\in L(\mathsf{H}). \tag{6.2}$$

Assume that C is not scalar. Suppose $A, B \in L(\mathsf{H})$. Then

$$f(U^*AU) = f(U^*BU)$$

holds for every unitary U if and only if either (1) $\operatorname{trace} C \neq 0$ *and* $A = B$, *or* (2) $\operatorname{trace} C = 0$ *and* $A - B$ *is scalar.*

For the proof of Theorem 6.1 a few lemmas will be needed. We start with a simple observation.

Lemma 6.2. *An operator $A \in L(\mathsf{H})$ has the property that*

$$\langle Ax, x\rangle = \langle Ay, y\rangle \quad \forall \text{ orthonormal pairs } (x, y), \quad x, y \in \mathsf{H} \tag{6.3}$$

if and only if A is scalar.

Proof. The "if" part is trivial, and for the "only if" part note that if $z, w \in \mathsf{H}$ are normalized elements such that (z, y) and (w, y) are orthonormal pairs for some $y \in \mathsf{H}$, then

$$\langle Az, z\rangle = \langle Aw, w\rangle. \tag{6.4}$$

Thus, if the dimension of H is at least 3, then (6.4) holds for any normalized z and w. Hence the numerical range of A is a singleton, and A is scalar. If the dimension of A is 2, then the statement of Lemma 6.2 can be easily verified by a straightforward computation: Subtracting from A a suitable scalar, we can assume that

$$\langle A\mathbf{e}_1, \mathbf{e}_1\rangle = \langle A\mathbf{e}_2, \mathbf{e}_2\rangle = 0.$$

So $A = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$ for some $a, b \in \mathsf{C}$, and further consideration using property (6.3) shows that we must have $a = b = 0$. □

We denote by $L_1(\mathsf{H})$ the ideal of trace operators in $L(\mathsf{H})$, and by $L_{10}(\mathsf{H})$ the (closed in the trace-class norm) subspace of trace operators with zero trace.

Lemma 6.3. *Let $C \in L_1(\mathsf{H})$ be a nonzero operator with zero trace. Then $X \in L(H)$ satisfies the property that* $\operatorname{trace}(UCU^*X) = 0$ *for every unitary U if and only if X is scalar.*

The statement and proof of this and the following lemma is inspired by [15] (these lemmas are essentially proved in [15] in the case H is finite dimensional).

Proof. The "if" part being trivial, we prove the "only if" part. Suppose the operator UCU^*X has zero trace for every unitary U but X is not scalar. We may replace C by any (finite) nonzero linear combination of operators in the unitary orbit of C. By doing so, we may (and do) assume without loss of generality that, for some orthonormal pair (x, y), $x, y \in \mathsf{H}$, and with respect to the orthogonal decomposition

$$\mathsf{H} = (\operatorname{span} x) \oplus (\operatorname{span} y) \oplus (\operatorname{span}\{x, y\})^{\perp}, \tag{6.5}$$

the operator C has the following matrix form:

$$C = \operatorname{diag}(c_1, c_2, C_0), \tag{6.6}$$

where $c_1, c_2 \in \mathsf{C}$ and $c_1 \neq c_2$. Indeed, let $x \in \mathsf{H}$ be a normalized element such that $\langle Cx, x\rangle \neq 0$; the condition that C has zero trace guarantees that there exists a normalized y orthogonal to x such that $\langle Cy, y\rangle \neq \langle Cx, x\rangle$. Now let U_1, U_2, U_3 be self-adjoint unitary operators given by

$$U_1 = \operatorname{diag}[1, -1, I],\ U_2 = \operatorname{diag}[-1, 1, I],\ U_3 = \operatorname{diag}[-1, -1, I],\ I = I_{(\operatorname{span}\{x,y\})^\perp},$$

with respect to the decomposition (6.5). It is easy to see that the operator

$$C + U_1CU_1 + U_2CU_2 + U_3CU_3$$

has the desired form (6.6). Independently, X can be also replaced by V^*XV, for any unitary V. Since X is not scalar, $\langle Xx', x'\rangle \neq \langle Xy', y'\rangle$ for some orthonormal pair (x', y') by Lemma 6.2. Applying a transformation $X \to V^*XV$, we may assume $(x', y') = (x, y)$. So

$$X = \begin{bmatrix} x_1 & * & * \\ * & x_2 & * \\ * & * & X_0 \end{bmatrix}, \quad x_1, x_2 \in \mathsf{C},\ x_1 \neq x_2, \quad X_0 \in L((\operatorname{span}\{x,y\})^\perp),$$

with respect to (6.5). Now

$$0 = \operatorname{trace}(CX) = c_1x_1 + c_2x_2 + \operatorname{trace}(C_0X_0), \tag{6.7}$$

and letting

$$U = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I_{(\operatorname{span}\{x,y\})^\perp} \end{bmatrix},$$

we also have

$$0 = \operatorname{trace}(UCUX) = c_2x_1 + c_1x_2 + \operatorname{trace}(C_0X_0). \tag{6.8}$$

Comparing (6.7) and (6.8) we see that

$$(c_1 - c_2)(x_1 - x_2) = 0,$$

a contradiction with $c_1 \neq c_2$, $x_1 \neq x_2$. ☐

The result of the next lemma was proved in [15, 5] in case H is finite dimensional.

Lemma 6.4. *Let $C \in L_1(\mathsf{H})$ be a nonzero operator. Then the closure (in the trace norm) of the linear span of operators of the form UCU^*, U unitary, coincides with $L_1(\mathsf{H})$ if $\operatorname{trace} C \neq 0$, and coincides with $L_{10}(\mathsf{H})$ if $\operatorname{trace} C = 0$.*

Proof. Denote by $\mathcal{U}(C)$ the closure of the linear span of operators of the form UCU^*, U unitary. Suppose trace $C = 0$, and arguing by contradiction, assume $\mathcal{U}(C) \neq L_{10}(\mathsf{H})$. Then (because $L(\mathsf{H})$ is the dual of $L_1(\mathsf{H})$) there exists $X \in L(\mathsf{H})$ such that $\operatorname{trace}(TX) = 0$ for every $T \in \mathcal{U}(C)$ but $\operatorname{trace}(T_0X) \neq 0$ for some $T_0 \in L_{10}(\mathsf{H})$. Being nonscalar, $C \neq 0$, so by Lemma 6.3, the first condition implies that X is scalar, which contradicts the second condition.

Next, suppose trace $C \neq 0$. Since C is not scalar, we have $\langle Cx, x\rangle \neq \langle Cy, y\rangle$ for some orthonormal pair (x, y) by Lemma 6.2; hence $\widehat{C} := C - VCV^* \neq 0$ for some unitary V. Clearly trace $\widehat{C} = 0$ and $\mathcal{U}(C) \supseteq \mathcal{U}(\widehat{C})$. By the first part of the lemma we have $\mathcal{U}(\widehat{C}) = L_{10}(\mathsf{H})$, hence $\mathcal{U}(C) \supseteq L_{10}(\mathsf{H})$. On the other hand, since $C \in \mathcal{U}(C)$ and trace $C \neq 0$, we have $\mathcal{U}(C) \neq L_{10}(\mathsf{H})$, hence $\mathcal{U}(C) = L_1(\mathsf{H})$. $\square$

Proof of Theorem 6.1. The "if" part is trivial. We prove the "only if" part. The condition implies that trace $(AUCU^*) =$ trace $(BUCU^*)$, i.e., trace $((A-B)UCU^*) = 0$, for every unitary U. Since the closure of the linear span of $\{UCU^* : U \text{ unitary }\}$ is either $L_1(\mathsf{H})$ or $L_{10}(\mathsf{H})$ by Lemma 6.4, we see that (1) or (2) holds. $\square$

We were not able to prove a generalization of the result of Theorem 2.2 to the framework of trace functionals. Therefore the following open problem is suggested:

Open Problem 6.5. *Suppose f is a bounded linear functional on $L(\mathsf{H})$ given by* (6.2)*, where the trace class operator C is not scalar. Characterize pairs $A, B \in L(\mathsf{H})$ such that*

$$|f(U^*AU)| = |f(U^*BU)| \qquad \forall \ \textit{unitary} \ U \in L(\mathsf{H}). \tag{6.9}$$

By analogy with Theorem 2.2, we conjecture:

Conjecture 6.6. *Under the hypotheses of the open problem,* (6.9) *holds if and only if:*

(1) trace $C = 0$, $C = C^*$, *and either* $A = \mu B + \nu I$ *or* $A = \mu B^* + \nu I$ *for some* $\mu, \nu \in \mathsf{C}$, $|\mu| = 1$;
(2) trace $C = 0$, $C \neq C^*$, *and* $A = \mu B + \nu I$ *for some* $\mu, \nu \in \mathsf{C}$, $|\mu| = 1$;
(3) trace $C \neq 0$, $C = C^*$, *and either* $A = \mu B$ *or* $A = \mu B^*$ *for some* $\mu \in \mathsf{C}$, $|\mu| = 1$;
(4) trace $C \neq 0$, $C \neq C^*$, *and* $A = \mu B$ *for some* $\mu \in \mathsf{C}$, $|\mu| = 1$.

Theorem 2.2 proves the conjecture in the case when C is any rank one operator.

References

[1] W.-S. Cheung and N.-K. Tsing, The C-numerical range of matrices is star-shaped, *Linear and Multilinear Algebra* **41** (1996), 245–250.

[2] M.-T. Chien and N. Nakazato, Davis-Wielandt shell and q-numerical range, *Linear Algebra Appl.* **340** (2002), 15–31.

[3] M.-D. Choi, C.-K. Li, and T.-Y. Poon, Some convexity features associated with unitary orbits, *Canadian J. Mathematics* **55** (2003), 91–111.

[4] G. Dirr, U. Helmke, H. Kleinsteuber, and Th. Schulte-Herbrüggen, Relative C-numerical ranges for applications in quantum control and quantum information, *Linear Multilinear Algebra* **56** (2008), 27–51.

[5] M. Goldberg and E.G. Straus, Norm properties of C-numerical radii, *Linear Algebra Appl.* **24** (1979), 113–131.

[6] U. Helmke, K. Hüper, J.B. Moore, and Th. Schulte-Herbrüggen, Gradient flows computing the C-numerical range with applications in NMR spectroscopy, *J. Global Optim.* **23** (2002), 283–308.

[7] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[8] M.S. Jones, A note on the shape of the generalized C-numerical range, *Linear and Multilinear Algebra* **31** (1992), 81–84.
[9] B. Kuzma, C.-K. Li, G. Lešnjak, T. Petek, and L. Rodman, Norm preservers of Jordan products, paper in preparation.
[10] C.-K. Li, A survey on linear preservers of numerical ranges and radii. *Taiwanese J. Math.* **5** (2001), 477–496.
[11] C.-K. Li, P.P. Mehta, and L. Rodman, A generalized numerical range: the range of a constrained sesquilinear form, *Linear and Multilinear Algebra* **37** (1994), 25–49.
[12] C.-K. Li and L. Rodman, Multiplicative preservers of C-numerical ranges and radii *Linear and Multilinear Algebra* **52** (2004), 265–279.
[13] C.-K. Li, B.-S. Tam, and N.-K. Tsing, Linear operators preserving the (p,q)-numerical range, *Linear Algebra Appl.* **110** (1988), 75–89.
[14] Th. Schulte-Herbrüggen, G. Dirr, U. Helmke, and S.J. Glaser, The significance of the C-numerical range and the local C-numerical range in quantum control and quantum information, *Linear and Multilinear Algebra* **56** (2008), 3–26.
[15] B.-S. Tam, A simple proof of the Goldberg–Straus theorem on numerical radii, *Glasgow Math. J.* **28** (1986), 139–141.

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Received: March 25, 2009
Accepted: July 9. 2009

Operator Theory:
Advances and Applications, Vol. 202, 435–450

Matrix Inequalities and Twisted Inner Products

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Abstract. We will demonstrate that several known inequalities involving generalized Schur functions, also known as generalized matrix functions, follow from either the Cauchy-Schwartz inequality, or from certain monotonicity relations that exist between inner products on spaces of multilinear functions. Connections between our inner products and permanent inequalities are presented, and a connection to some unresolved problems in partial differential equations is indicated.

Mathematics Subject Classification (2000). 15A15, 15A45, 15A63, 15A69.

Keywords. Matrix inequalities, inner products, tensors, multilinear functions, tensor inequalities.

1. Introduction

Suppose the $n \times n$ matrix $A = [a_{ij}]$ is a member of $\mathcal{H}_n$, the set of all $n \times n$ positive semi-definite Hermitian matrices, and let k be an integer such that $1 \leq k \leq n-1$. It is known [13] that

$$\operatorname{per}(A) \geq \frac{1}{k} \sum_{i=1}^{k} \sum_{j=1}^{k} a_{ij} \operatorname{per}(A(i|j)) \tag{1}$$

where $\operatorname{per}(\cdot)$ denotes the permanent function, and $A(i|j)$ denotes the matrix obtained from A by deleting A's ith row and jth column. Distinct from (1), except when $k = 1$, is the Lieb permanent inequality [3] which states that for each $A \in \mathcal{H}_n$, and each k such that $1 \leq k \leq n-1$, we have

$$\operatorname{per}(A) \geq \operatorname{per}(A_{11}) \operatorname{per}(A_{22}) \tag{2}$$

Communicated by L. Rodman.

where A_{11} and A_{22} are, respectively, the upper left $k \times k$, and lower right $(n-k) \times (n-k)$ matrices in the partition

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{3}$$

In this case A_{12} is $k\times(n-k)$ and $A_{21} = A_{12}^*$. Actually, the right side of (1) refines the Lieb inequality (2), so we also have

$$\operatorname{per}(A) \;\geq\; \frac{1}{k}\sum_{i=1}^{k}\sum_{j=1}^{k} a_{ij}\operatorname{per}(\,A(i|j)\,) \;\geq\; \operatorname{per}(A_{11})\operatorname{per}(A_{22}) \tag{4}$$

for all $A \in \mathcal{H}_n$ partitioned as in (3).

A conjectured inequality, related to the permanent dominance conjecture and the conjecture of Soules, see [12], is that if $x = (x_1, x_2, \ldots, x_n)^t \in \mathbb{C}^n$, then

$$\|x\|^2 \operatorname{per}(A) \;\geq\; \sum_{i=1}^{n}\sum_{j=1}^{n} x_i\overline{x_j}\, a_{ij}\operatorname{per}(\,A(i|j)\,) \tag{5}$$

for all $A \in \mathcal{H}_n$, where, in this case, $\|x\|$ denotes the 2-norm of x. Of course, (1) implies that (5) holds in case x is a $(0,1)$-vector. From this there are a number of other special cases of (5) that can be proved true. See [12].

Another known, though not well-known, inequality involving the permanent function restricted to $\mathcal{H}_n$, like (2), involves matrix partitions. Suppose $A \in \mathcal{H}_n$ and let k be a positive integer such that $2k \leq n$. We partition A as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \tag{6}$$

where A_{11}, A_{12}, A_{21}, and A_{22} are $k \times k$, A_{13} and A_{23} are $k\times(n-2k)$, and A_{33} is $(n-2k) \times (n-2k)$. In this case it has been shown [11, Theorem 4', page 34] that

$$\left(\operatorname{per}\left(\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}\right)\right)^2$$

$$\leq \;\operatorname{per}\left(\begin{bmatrix} A_{11} & A_{11} & A_{13} \\ A_{11} & A_{11} & A_{13} \\ A_{31} & A_{31} & A_{33} \end{bmatrix}\right)\operatorname{per}\left(\begin{bmatrix} A_{22} & A_{22} & A_{23} \\ A_{22} & A_{22} & A_{23} \\ A_{32} & A_{32} & A_{33} \end{bmatrix}\right). \tag{7}$$

It is a remarkable fact that inequalities (1), (2), (4), and (7), despite there apparent dissimilarities, follow from a single theorem that describes the monotonicity properties of a certain array of inner products. We will describe these inner products. It is our hope that similar efforts will lead to a complete resolution of (5), as well as a number of other conjectures involving permanents. Several authors have used techniques in Multilinear Algebra to obtain matrix inequalities, notably M. Marcus who wrote many papers on this subject. See [4], for example.

2. Multilinear functions, contractions, main theorem

Given a non-trivial complex vector space V, and positive integers u and v, we let $\mathbf{T}_{u,v}(V)$ denote the set of all functions from $V\times V\times\cdots\times V$-($u+v$ copies) to $\mathbb{C}$ that are linear in the first u places, and conjugate linear in the last v places. We denote $\mathbf{T}_{u,0}(V)$ by $\mathbf{T}_u(V)$, and $\mathbf{T}_{0,0}(V)$ is $\mathbb{C}$. By $\mathbf{S}_{u,v}(V)$ we mean the subspace of $\mathbf{T}_{u,v}(V)$ consisting of the elements of $\mathbf{T}_{u,v}(V)$ that are symmetric in the first u positions and in the last v positions. If $v=0$, then $\mathbf{S}_{u,v}(V)$ is denoted by $\mathbf{S}_u(V)$, and is the set of all fully symmetric $F\in\mathbf{T}_u(V)$.

We assume that V has dimension m, and is equipped with an inner product, $\langle\cdot,\cdot\rangle$. We will derive from $\langle\cdot,\cdot\rangle$ an inner for each of the spaces $\mathbf{T}_{u,v}(V)$. For positive integers k and p we let $\Gamma_{k,p}$ denote the set of all sequences of length k each of whose terms is a member of $\mathbf{I}_p$, where $\mathbf{I}_p=\{1,2,\dots,p\}$. Generally $p=m$, so we abbreviate $\mathbf{I}_{k,m}$ to $\mathbf{I}_k$. If $x_1,x_2,\dots,x_k$ are in V, and $f\in\mathbf{I}_k$, then we let x_f denote the vector sequence $x_{f(1)},x_{f(2)},\dots,x_{f(k)}$. To extend the inner product to the spaces $\mathbf{T}_{u,v}(V)$ we choose an arbitrary orthonormal basis $\{e_i\}_{i=1}^m$ for V and define

$$\langle A,B\rangle = \sum_{f\in\Gamma_u}\sum_{g\in\Gamma_v} A(e_f;e_g)\overline{B(e_f;e_g)}, \quad \forall\, A,B\in\mathbf{T}_{u,v}(V). \tag{8}$$

The extended inner product $\langle\cdot,\cdot\rangle$ is independent of $\{e_i\}_{i=1}^m$.

We require operations on the spaces $\mathbf{T}_{u,v}(V)$ which we call insertions, and other operations called contractions. Both are actually just special linear maps amongst the various spaces $\mathbf{T}_{u,v}(V)$. If $F\in\mathbf{T}_{u,v}(V)$, and both $0\le s\le u$, and $0\le t\le v$ are satisfied, then the insertion $F(x_1,x_2,\dots,x_s;y_1,y_2,\dots,y_t)$, where $x_1,x_2,\dots,x_s,y_1,y_2,\dots,y_t\in V$, is in $\mathbf{T}_{u-s,v-t}(V)$, and is defined by

$$\begin{aligned}&F(x_1,x_2,\dots,x_s;y_1,y_2,\dots,y_t)(z_1,z_2,\dots,z_{u+v-s-t}) \\ &= F(x_1,x_2,\dots,x_s,z_1,z_2,\dots,z_{u-s};y_1,y_2,\dots,y_t,z_{u-s+1},z_{u-s+1},\dots,z_{u+v-s-t})\end{aligned} \tag{9}$$

for all $z_1,z_2,\dots,z_{u+v-s-t}\in V$. Of course definition (9) includes the case of insertions into members of $\mathbf{T}_u(V)$. If $F\in\mathbf{S}_{u,v}(V)$ then we need not be concerned about the placement of the vectors $x_1,x_2,\dots,x_s$ and $y_1,y_2,\dots,y_t$ as long as each of $x_1,x_2,\dots,x_s$ is placed somewhere in the first u positions, and each of $y_1,y_2,\dots,y_t$ is placed somewhere in the last v positions.

If $1\le t\le\min\{u,v\}$, then we define the linear contraction map $\mathbf{C}_t:\mathbf{S}_{u,v}(V)\to\mathbf{S}_{u-t,v-t}(V)$ by

$$\mathbf{C}_t(F) = \sum_{\phi\in\Gamma_t} F(e_\phi;e_\phi) = \sum_{\phi\in\Gamma_t} F_{\phi,\phi}, \quad \forall\, F\in\mathbf{S}_{u,v}(V). \tag{10}$$

In the above $F_{\phi,\phi}$ is an abbreviation of $F(e_\phi,e_\phi)$. We identify $\mathbf{C}_0$ with the identity map.

Explicitly, if $x_1, x_2, \dots, x_{u-t}, y_1, y_2, \dots, y_{v-t} \in V$, then

$$\begin{aligned}
&\mathbf{C}_t(F)(x_1, x_2, \dots, x_{u-t}; y_1, y_2, \dots, y_{v-t}) \\
&\quad = \sum_{\phi \in \Gamma_t} F(e_\phi, x_1, x_2, \dots, x_{u-t}; e_\phi, y_1, y_2, \dots, y_{v-t}) \\
&\quad = \sum_{\phi \in \Gamma_t} F\big(e_{\phi(1)}, e_{\phi(2)}, \dots, e_{\phi(t)}, x_1, x_2, \dots, x_{u-t}; \\
&\qquad\qquad\qquad e_{\phi(1)}, e_{\phi(2)}, \dots, e_{\phi(t)}, y_1, y_2, \dots, y_{v-t}\big). \qquad (11)
\end{aligned}$$

As in the case of $\langle \cdot, \cdot \rangle$, the $\mathbf{C}_t$ do not depend upon the orthonormal basis $\{e_i\}_{i=1}^m$.

Given t such that $1 \le t \le \min\{u, v\}$, we define a positive semi-definite sesquilinear form $[\cdot, \cdot]_t$ on $\mathbf{S}_{u,v}(V)$ so that

$$\big[A, B\big]_t = \big\langle \mathbf{C}_t(A), \mathbf{C}_t(B) \big\rangle, \quad \forall\, A, B \in \mathbf{S}_{u,v}(V). \qquad (12)$$

If $t = 0$, then $[\cdot, \cdot]_t = \langle \cdot, \cdot \rangle$. That $[\cdot, \cdot]_t$ is linear in its first position and conjugate linear in its second position follows because $\mathbf{C}_t$ is linear and $\langle \cdot, \cdot \rangle$ is itself an inner product. The semi-definiteness of $[\cdot, \cdot]_t$ follows from the fact that $\langle \cdot, \cdot \rangle$ is positive definite. If $a = (a_0, a_1, a_2, \dots, a_w)$ is a sequence of non-negative numbers, where $w = \min\{u, v\}$, then we construct sesquilinear form $[\cdot, \cdot]_a$ by defining

$$[A, B]_a = \sum_{t=0}^{w} a_t\, [A, B]_t, \quad \forall A, B \in \mathbf{S}_{u,v}(V). \qquad (13)$$

It is clear that $[\cdot, \cdot]_a$ is positive semi-definite and sesquilinear. Moreover, $[\cdot, \cdot]_a$ is positive definite, and therefore an inner product on $\mathbf{S}_{u,v}(V)$, whenever $a_0 > 0$. Of course, one can always construct new inner products from old ones by adding together non-negative scalar multiples. This is nothing new. The interesting fact is that for certain special sequences $(a_0, a_1, a_2, \dots, a_w)$, the resulting inner products, $[\cdot, \cdot]_a$, are meaningful within the context of multilinear algebra, and provide some known matrix inequalities. This leads one to the idea of trying to extend to an even more general setting, one that would imply (5) as well as perhaps the conjecture of Soules.

For $[\cdot, \cdot]_a$ to be an inner product it is not really necessary that all of the coefficients a_0, $a_1, a_2, \dots, a_k$ be non-negative. This follows from Lemma 1 below wherein we show if $1 \le t \le w$, then the set of all quotients of the form $\|\mathbf{C}_{t-1}(F)\|^2 / \|\mathbf{C}_t(F)\|^2$, where $F \in \mathbf{S}_{u,v}(V)$ and $F \ne 0$, is bounded below by $1/m$. In other words, $\inf\big\{\|\mathbf{C}_{t-1}(F)\|^2 / \|\mathbf{C}_t(F)\|^2 : F \ne 0\big\} \ge (1/m)$ when $1 \le t \le w$. In its simplest form this inequality implies that if $X = [x_{ij}]$ is an $m \times m$ complex matrix, then the square modulus of the trace of X does not exceed m times the Frobenius norm of X, that is,

$$|\mathrm{Tr}(X)|^2 = \Big|\sum_{i=1}^{m} x_{ii}\Big|^2 \le m \sum_{i,j=1}^{m} \big|x_{ij}\big|^2 = m\|X\|_F^2, \qquad (14)$$

where $\mathrm{Tr}(\cdot)$ is the trace function, and $\|\cdot\|_F$ is the Frobenius matrix norm. Inequality (14), though seemingly very crude, cannot be improved in general, because it reduces to equality if and only if X is a scalar multiple of the identity matrix. However, if we are willing to restrict the set of matrices, X, somewhat, then some improvement is possible. In particular, if r denotes the rank of X, then we have

$$|\mathrm{Tr}(X)|^2 \leq \Big|\sum_{i=1}^{m} x_{ii}\Big|^2 \ \leq \ r \sum_{i,j=1}^{m} \big|x_{ij}\big|^2 = r\|X\|_F^2. \tag{15}$$

Inequality (15) is listed in the reference work [2]. We will use it to prove Lemma 1. An obvious question at this point is what is a useful necessary and sufficient condition on the sequence $a_0,\ a_1, a_2, \ldots, a_k$ in order that the sesquilinear form $[\cdot,\cdot]_a$ be positive definite? An answer to this questions could have far-reaching effects.

Lemma 1. *Suppose u and v are positive integers and let $w = \min\{u, v\}$. If $1 \leq t \leq w$, then*

$$\|\mathbf{C}_t(F)\|^2 \ \leq \ m\,\|\mathbf{C}_{t-1}(F)\|^2, \quad \forall F \in \mathbf{S}_{u,v}(V). \tag{16}$$

Proof. Suppose $F \in \mathbf{S}_{u,v}(V)$ and $F \neq 0$. Let $\{e_i\}_{i=1}^m$ be an orthonormal basis for V. Then,

$$\|\mathbf{C}_t(F)\|^2 = \sum_{\alpha\in\Gamma_{u-t}} \sum_{\beta\in\Gamma_{v-t}} \left| \sum_{i=1}^{m} \left\{ \sum_{\phi\in\Gamma_{t-1}} F(e_\alpha, e_i, e_\phi; e_\phi, e_i, e_\beta) \right\} \right|^2 . \tag{17}$$

Now, define $k_{\alpha,\beta}(i,j)$ according to

$$k_{\alpha,\beta}(i,j) = \sum_{\phi\in\Gamma_{t-1}} F(e_\alpha, e_i, e_\phi; e_\phi, e_j, e_\beta) \tag{18}$$

for each $\alpha \in \Gamma_{u-t}$, $\beta \in \Gamma_{v-t}$, and i and j such that $1 \leq i, j \leq m$. Application of (14) to the matrix $[k_{\alpha,\beta}(i,j)]$ yields the inequality

$$\left|\sum_{i=1}^{m} k_{\alpha,\beta}(i,i)\right|^2 \ \leq m \sum_{i,j=1}^{m} \big|k_{\alpha,\beta}(i,j)\big|^2, \tag{19}$$

which holds for all α and β. Substitution of (18) into (17) followed by an application of (19) produces

$$\begin{aligned}
\|\mathbf{C}_t(F)\|^2 &= \sum_{\alpha,\beta} \left|\sum_{i=1}^{m} k_{\alpha,\beta}(i,i)\right|^2 \ \leq m \sum_{\alpha,\beta} \sum_{i,j=1}^{m} \big|k_{\alpha,\beta}(i,j)\big|^2 \\
&= m \sum_{\alpha,\beta} \sum_{i,j=1}^{m} \left| \sum_{\phi\in\Gamma_{t-1}} F(e_\alpha, e_i, e_\phi; e_\phi, e_j, e_\beta) \right|^2 \\
&= m \sum_{\alpha\in\Gamma_{u-t+1}} \sum_{\beta\in\Gamma_{v-t+1}} \left| \sum_{\phi\in\Gamma_{t-1}} F(e_\alpha, e_\phi; e_\phi, e_\beta) \right|^2 \\
&= m\,\|\mathbf{C}_{t-1}(F)\|^2.
\end{aligned} \tag{20}$$

□

Letting $\mathcal{N}(T)$ denote the nullspace of linear map T it is obvious from Lemma 1 that $\mathcal{N}(\mathbf{C}_{t-1}) \subset \mathcal{N}(\mathbf{C}_t)$ for each t such that $1 \le t \le w$. Thus,

$$\mathcal{N}(\mathbf{C}_1) \subset \mathcal{N}(\mathbf{C}_2) \subset \mathcal{N}(\mathbf{C}_3) \subset \cdots \subset \mathcal{N}(\mathbf{C}_w).$$

This fact also follows because of

Lemma 2. *If u and v are positive integers and $w = \min\{u, v\}$, then the maps $\mathbf{C}_t$, $0 \le t \le w$, defined on $\mathbf{S}_{u,v}(V)$, satisfy $\mathbf{C}_{s+t} = \mathbf{C}_s \circ \mathbf{C}_t = \mathbf{C}_t \circ \mathbf{C}_s$ for all non-negative integers s and t such that $s + t \le w$.*

Proof. That $\mathbf{C}_{s+t} = \mathbf{C}_s \circ \mathbf{C}_t = \mathbf{C}_t \circ \mathbf{C}_s$ follows from (11) by inspection. □

The following provides a simple condition on $a = (a_0, a_1, a_2, \ldots, a_w)$ that is necessary if $[\cdot,\cdot]_a$ is to be an inner product on $\mathbf{S}_{u,v}(V)$.

Lemma 3. *Suppose u and v are positive integers, and let $w = \min\{u, v\}$. If $a = (a_0, a_1, a_2, \ldots, a_w)$ is a sequence of real numbers such that $[\cdot,\cdot]_a$ is an inner product on $\mathbf{S}_{u,v}(V)$, and $\dim(V) \ge 2$, then $a_0 > 0$.*

Proof. We know that $\mathcal{N}(\mathbf{C}_{t-1}) \subset \mathcal{N}(\mathbf{C}_t)$ for each t such that $1 \le t \le w$, and $\mathbf{C}_0 = \mathrm{Id}$, the identity map. To prove the lemma it is therefore sufficient to produce a non-zero $F \in \mathbf{S}_{u,v}(V)$ such that $\mathbf{C}_1(F) = 0$. For such an F we would have $[F, F]_a = a_0\|F\|^2$, where $\|\cdot\|$ is the norm associated with the basic inner product $\langle\cdot,\cdot\rangle$. If a_0 were less than or equal to 0, then the contradiction $F \ne 0$ and $[F, F]_a \le 0$ is manifest.

To produce such an F we choose an orthonormal basis $\{e_i\}_{i=1}^m$ for V, and define f_1 and f_2 to be the linear functionals generated by e_1 and e_2; that is, $f_i(x) = \langle x, e_i\rangle$ for $i \in \{1, 2\}$ and $x \in V$. Define F by

$$F(x_1, x_2, \ldots, x_u; y_1, y_2, \ldots, y_v) = \prod_{i=1}^{u} f_1(x_i) \prod_{j=1}^{v} \overline{f_2(y_j)}$$

for all $x_1, x_2, \ldots, x_u, y_1, y_2, \ldots, y_v \in V$. Clearly, F is symmetric in its first u places and last v places, and $F \ne 0$. Since f_1 and f_2 are orthogonal with respect to the inner product on V^*, the dual of V, it is also easy to see that $\mathbf{C}_1(F) = 0$. □

If $F \in \mathbf{T}_n(V)$ and $G \in \mathbf{T}_p(V)$, then the standard tensor product of $F \otimes G$ is the member of $\mathbf{T}_{n+p}(V)$ defined by

$$(F \otimes G)(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_p) = F(x_1, x_2, \ldots, x_n)G(y_1, y_2, \ldots, y_p) \quad (21)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_p \in V$. We require a variant, $F\overline{\otimes}G$, on $F \otimes G$ defined by

$$(F\overline{\otimes}G)(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_p) = F(x_1, x_2, \ldots, x_n)\overline{G(y_1, y_2, \ldots, y_p)} \quad (22)$$

for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_p \in V$. The map $\overline{\otimes}$ maps $\mathbf{T}_n(V) \times \mathbf{T}_p(V)$ into $\mathbf{T}_{n,p}(V)$. Moreover, we have $\|F \otimes G\|^2 = \|F\overline{\otimes}G\|^2 = \|F\|^2\|G\|^2$ for all $F \in \mathbf{T}_n(V)$ and $G \in \mathbf{T}_p(V)$.

For each integer $k > 0$ let $\mathbb{S}_k$ denote the symmetric group on $\mathbf{I}_k$, and define the action of $\mathbb{S}_k$ on $\mathbf{T}_k(V)$ by

$$(\sigma F)(x_1, x_2, \ldots, x_k) = F(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}),$$

for all $\sigma \in \mathbb{S}_k$, $F \in \mathbf{T}_k(V)$, and $x_1, x_2, \ldots, x_k \in V$. The symmetrizer $\mathcal{P}_k$ is the linear map from $\mathbf{T}_k(V)$ to $\mathbf{S}_k(V)$ defined by

$$\mathcal{P}_k(F) = (k!)^{-1} \sum_{\sigma \in \mathbb{S}_k} \sigma F, \quad \forall F \in \mathbf{T}_k(V). \tag{23}$$

Note that $\sigma \mathcal{P}_k = \mathcal{P}_k \sigma = \mathcal{P}_k$ for all $\sigma \in \mathbb{S}_k$, so $\mathcal{P}_k^2 = \mathcal{P}_k$. Therefore, $\mathcal{P}_k$ is a Hermitian idempotent in the appropriate algebra of functions. Since it is also true that $\mathcal{P}_k^* = \mathcal{P}_k$, we have $\langle \mathcal{P}_k A, B \rangle = \langle A, \mathcal{P}_k^* B \rangle = \langle A, \mathcal{P}_k B \rangle$. The symmetrizer $\mathcal{P}_k$ is therefore the orthogonal projection of $\mathbf{T}_k(V)$ onto $\mathbf{S}_k(V)$. At first impression it might be surprising that there is any relationship between symmetric products of multilinear functions and the contraction maps $\mathbf{C}_t$. Nevertheless, we have the following identity [18].

Theorem 1. *Suppose n, p, q, and r are non-negative integers at least one of which is positive, and assume that $n+p = q+r$. If $A \in \mathbf{S}_n(V)$, $B \in \mathbf{S}_p(V)$, $C \in \mathbf{S}_q(V)$, and $D \in \mathbf{S}_r(V)$, then*

$$\binom{n+p}{q} \langle A \cdot B, C \cdot D \rangle = \sum_{s=\kappa_1}^{\kappa_2} \binom{n}{n-s} \binom{p}{q-s} \langle \mathbf{C}_{n-s}(A \bar{\otimes} D), \mathbf{C}_{q-s}(C \bar{\otimes} B) \rangle \tag{24}$$

where $\kappa_1 = \max\{0, n-r\} = \max\{0, q-p\}$, *and* $\kappa_2 = \min\{n, q\}$.

We will relate Theorem 1 to the sesquilinear forms $[\cdot, \cdot]_a$. If $q = n$ and $r = p$, then $\kappa_1 = \max\{0, n-p\}$ and $\kappa_2 = n$; thus, (24) reduces to

$$\binom{n+p}{n} \langle A \cdot B,\, C \cdot D \rangle = \sum_{s=\kappa_1}^{n} \binom{n}{n-s} \binom{p}{n-s} \langle \mathbf{C}_{n-s}(A \bar{\otimes} D),\, \mathbf{C}_{n-s}(C \bar{\otimes} B) \rangle. \tag{25}$$

Substituting $t = n-s$ in (25), and noting that the upper limit of summation is now $\min\{n, p\}$, which we denote by κ, we obtain

$$\binom{n+p}{n} \langle A \cdot B,\, C \cdot D \rangle = \sum_{t=0}^{\kappa} \binom{n}{t} \binom{p}{t} \langle \mathbf{C}_t(A \bar{\otimes} D),\, \mathbf{C}_t(C \bar{\otimes} B) \rangle. \tag{26}$$

The identity above is essentially Neuberger's identity [5] extended to the complex case. If we set

$$a_t = \frac{\binom{n}{t}\binom{p}{t}}{\binom{n+p}{n}}, \quad \forall t \text{ such that } 0 \leq t \leq \kappa, \tag{27}$$

and $a = (a_0, a_1, a_2, \ldots, a_\kappa)$, then we obtain that

$$\big\langle A \cdot B, C \cdot D \big\rangle = \sum_{t=0}^{\kappa} a_t \left\langle \mathbf{C}_t(A \overline{\otimes} D), \mathbf{C}_t(C \overline{\otimes} B) \right\rangle = \big[A \overline{\otimes} D, C \overline{\otimes} B\big]_a, \tag{28}$$

for all $A, C \in \mathbf{S}_n(V)$ and $B, D \in \mathbf{S}_p(V)$. Thus,we have

Theorem 2. *Suppose n and p are positive integers, and let $\kappa = \min\{n, p\}$. For each t such that $0 \le t \le \kappa$, let $a_t = \binom{n}{t}\binom{p}{t}/\binom{n+p}{n}$, and let $a = (a_0, a_1, a_2, \ldots, a_\kappa)$. Then, $[\cdot, \cdot]_a$ is an inner product on $\mathbf{S}_{n,p}(V)$, and*

$$\big\langle A \cdot B,\, C \cdot D \big\rangle = \big[A \overline{\otimes} D, C \overline{\otimes} B\big]_a \tag{29}$$

for all $A, B \in \mathbf{S}_n(V)$ and $B, D \in \mathbf{S}_p(V)$.

Letting $\| \cdot \|$ denote the norm associated with the standard inner product $\langle \cdot, \cdot \rangle$ defined by (8), and letting $\| \cdot \|_a$ denote the norm associated with $[\cdot, \cdot]_a$, we obtain from Theorem 2 that

$$\big\| A \cdot B \big\|^2 = \big\| A \overline{\otimes} B \big\|_a^2 \tag{30}$$

for all $A \in \mathbf{S}_n(V)$ and $B \in \mathbf{S}_p(V)$.

The above leads to some very interesting results. If $p = n$, $C = B$ and $D = A$, then $A \cdot B = B \cdot A$, hence

$$\begin{aligned} \|A \cdot B\|^4 &= \big(\langle A \cdot B, B \cdot A \rangle\big)^2 = \big(\big[A \overline{\otimes} A, B \overline{\otimes} B\big]_a\big)^2 \\ &\le \big[A \overline{\otimes} A, A \overline{\otimes} A\big]_a \big[B \overline{\otimes} B, B \overline{\otimes} B\big]_a \end{aligned} \tag{31}$$

by the Cauchy-Schwartz inequality. But, $\big[A \overline{\otimes} A, A \overline{\otimes} A\big]_a = \|A \cdot A\|^2$ and

$$\big[B \overline{\otimes} B, B \overline{\otimes} B\big]_a = \|B \cdot B\|^2$$

by (29). Therefore,

$$\|A \cdot B\|^4 \le \|A \cdot A\|^2 \, \|B \cdot B\|^2 \tag{32}$$

for all $A, B \in \mathbf{S}_n(V)$.

More general results arise from (29) if we eliminate the requirement that $p = n$. Suppose $E, F \in \mathbf{S}_n(V)$, and $G, H \in \mathbf{S}_p(V)$. Then, by applying (29) in the special case $A = G \cdot E$, $C = H \cdot F$, $B = F$, and $D = E$ we obtain that

$$\big\langle E \cdot F \cdot G, E \cdot F \cdot H \big\rangle = \big\langle (G \cdot E) \cdot F, (H \cdot F) \cdot E \big\rangle = \big[(G \cdot E) \overline{\otimes} E, (H \cdot F) \overline{\otimes} F\big]_a. \tag{33}$$

Therefore, we may apply the Cauchy-Schwartz inequality to (33) to obtain

$$\begin{aligned} \big|\big\langle E \cdot F \cdot G, E \cdot F \cdot H \big\rangle\big|^2 &= \big|\big[(G \cdot E) \overline{\otimes} E, (H \cdot F) \overline{\otimes} F\big]_a\big|^2 \\ &\le \big\| (G \cdot E) \overline{\otimes} E \big\|_a^2 \, \big\| (H \cdot F) \overline{\otimes} F \big\|_a^2 \\ &= \big\| G \cdot E \cdot E \big\|^2 \, \big\| H \cdot F \cdot F \big\|^2. \end{aligned} \tag{34}$$

We have proven the following.

Theorem 3. *If n and p are positive integers, then for all $A, B \in \mathbf{S}_n(V)$, and $C, D \in \mathbf{S}_p(V)$, we have*

$$\left|\left\langle A \cdot B \cdot C, A \cdot B \cdot D\right\rangle\right|^2 \le \left\|A \cdot A \cdot C\right\|^2 \left\| B \cdot B \cdot D\right\|^2. \tag{35}$$

Theorem 3 is an extension of the result presented in [11]. If we set $D = C$, then we obtain the inequality

$$\left(\left\|A \cdot B \cdot C\right\|^2\right)^2 \le \left\|A \cdot A \cdot C\right\|^2 \left\| B \cdot B \cdot C\right\|^2, \tag{36}$$

which holds for all $A, B \in \mathbf{S}_n(V)$, and $C \in \mathbf{S}_p(V)$. The result (36) is Theorem 3 of [11], which implies the permanental inequality (7). See [11] for details on how to transform (36) into the permanental inequality (7).

There are other similar results derivable from Theorem 2. Again we suppose $A, B \in \mathbf{S}_n(V)$, and $B, D \in \mathbf{S}_p(V)$. Then,

$$\left\langle A \cdot A \cdot C, B \cdot B \cdot D\right\rangle = \left\langle (A \cdot C) \cdot A, (B \cdot D) \cdot B\right\rangle = \left[(A \cdot C)\overline{\otimes} B, (B \cdot D)\overline{\otimes} A\right]_a; \tag{37}$$

therefore,

$$\left|\left\langle A{\cdot}A{\cdot}C, B{\cdot}B{\cdot}D\right\rangle\right|^2 \le \left\|(A{\cdot}C)\overline{\otimes} B\right\|_a^2 \left\|(B{\cdot}D)\overline{\otimes} A\right\|_a^2 = \left\|A{\cdot}B{\cdot}C\right\|^2 \left\|A{\cdot}B{\cdot}D\right\|^2. \tag{38}$$

Setting $D = C$ in (38) we obtain the inequality

$$\left|\left\langle A \cdot A \cdot C, B \cdot B \cdot C\right\rangle\right|^2 \le \left(\|A \cdot B \cdot C\|^2\right)^2, \tag{39}$$

which, in conjunction with (36), implies

Theorem 4. *If $A, C \in \mathbf{S}_n(V)$, and $B, D \in \mathbf{S}_p(V)$, then*

$$\left|\left\langle A \cdot A \cdot C, B \cdot B \cdot C\right\rangle\right|^2 \le \|A \cdot B \cdot C\|^4 \le \left\|A \cdot A \cdot C\right\|^2 \left\| B \cdot B \cdot C\right\|^2. \tag{40}$$

Of course the Cauchy-Schwartz inequality applied to the basic inner product $\langle \cdot, \cdot \rangle$ gives the inequality $\left|\left\langle A \cdot A \cdot C, B \cdot B \cdot C\right\rangle\right|^2 \le \left\|A \cdot A \cdot C\right\|^2 \left\| B \cdot B \cdot C\right\|^2$. The significance of Theorem 4 is that $\left\|A{\cdot}A{\cdot}C\right\|^2 \left\| B{\cdot}B{\cdot}C\right\|^2$ and $\left|\left\langle A{\cdot}A{\cdot}C, B{\cdot}B{\cdot}C\right\rangle\right|^2$ are, respectively, upper and lower bounds for $\|A{\cdot}B{\cdot}C\|^4$. Theorem 4 is Corollary 6 of [19].

3. Additional inner products, more inequalities

We have considered the inner product $[\cdot, \cdot]_a$ where $a = (a_0, a_1, a_2, \ldots, a_\kappa)$ is the coefficient sequence such that $a_t = \binom{n}{t}\binom{p}{t}/\binom{n+p}{n}$ for each t such that $0 \le t \le \kappa$. Henceforth, we let $[\cdot, \cdot]_{0,0}$ denote $\binom{n+p}{n}[\cdot, \cdot]_a$; thus, $[\cdot, \cdot]_{0,0}$ is $[\cdot, \cdot]_b$, where b is the sequence $(b_0, b_1, b_2, \ldots, b_\kappa)$ such that $b_i = \binom{n+p}{n} a_i$ for each i. We will consider inner products generated by similar sequences. Suppose n and p are positive integers, and let κ denote $\min\{n, p\}$. For all integers s and t such that $0 \le s, t \le \kappa$, we let $\mu_{s,t}$ denote $\min\{n{-}s, p{-}t\}$, and

$$a_{s,t}(w) = \binom{n-s}{w}\binom{p-t}{w}$$

for all integers w such that $0 \le w \le \mu_{s,t}$. We note that when $s = t = 0$ we have $\mu_{s,t} = \kappa$, and $a_{s,t}(w)$ is, except for the term $\binom{n+p}{n}$, the same as (27). Therefore, the sequence $a_{0,0} = (a_{0,0}(0), a_{0,0}(1), \ldots, a_{0,0}(\kappa))$ is simply a positive multiple of the sequence $(a_0, a_1, a_2, \ldots, a_\kappa)$ considered previously. Since $a_{s,t}(0) > 0$ in each case, and all of the numbers $a_{s,t}(w)$ are non-negative, we know that $[\cdot,\cdot]_{a_{s,t}}$, which we shall abbreviate to $[\cdot,\cdot]_{s,t}$, is an inner product for each s and t such that $0 \le s, t \le \kappa$. In term of the contractions $\mathbf{C}_w$ we have

$$[F, G]_{s,t} = \sum_{w=0}^{\mu_{s,t}} a_{s,t}(w)\big\langle \mathbf{C}_w(F), \mathbf{C}_w(G)\big\rangle, \qquad \forall\ F, G \in \mathbf{S}_{n,p}(V).$$

Suppose $F \in \mathbf{S}_{n,p}(V)$, and consider the difference $\big\|F\big\|_{s,t} - \big\|F\big\|_{s+1,t}$, where $\|\cdot\|_{u,v}$ is the norm associated with $[\cdot,\cdot]_{u,v}$ for each u and v such that $0 \le u, v \le \kappa$. Since $\mu_{s,t} \ge \mu_{s+1,t}$, we have

$$\big\|F\big\|_{s,t} - \big\|F\big\|_{s+1,t} \ge \sum_{w=0}^{\mu_{s+1,t}} \big\{a_{s,t}(w) - a_{s+1,t}(w)\big\}\|\mathbf{C}_w(F)\|^2. \tag{41}$$

But,

$$a_{s,t}(w) - a_{s+1,t}(w) = \binom{n-s}{w}\binom{p-t}{w} - \binom{n-s-1}{w}\binom{p-t}{w} \ge 0 \tag{42}$$

for all w such that $0 \le w \le \mu_{s+1,t}$. Therefore, (41) implies that $\big\|F\big\|_{s,t} - \big\|F\big\|_{s+1,t} \ge 0$ for all $F \in \mathbf{S}_{n,p}(V)$. Using the similar calculation we can show that $\big\|F\big\|_{s,t} - \big\|F\big\|_{s,t+1} \ge 0$ for all $F \in \mathbf{S}_{n,p}(V)$. This proves the following.

Theorem 5. *Suppose n and p are positive integers, and let κ denote* $\min\{n,p\}$. *If $u, v, s,$ and t be integers such that $0 \le s \le u \le \kappa$ and $0 \le t \le v \le \kappa$, then $\big\|F\big\|_{s,t} \ge \big\|F\big\|_{u,v}$ for all $F \in \mathbf{S}_{n,p}(V)$.*

To understand Theorem 5, we think in terms of an array of inner products $\big[[\cdot,\cdot]_{s,t}\big]_{s,t=0}^{\kappa}$. If $F \in \mathbf{S}_{n,p}(V)$, and we insert F into each position in each inner product in the array, then we obtain a $(\kappa+1)\times(\kappa+1)$ non-negative matrix that descends down its rows and columns. Of course Theorem 5 is easy to prove. The remarkable fact is that it actually has non-trivial applications to multilinear algebra and matrix theory. We will describe some of these applications.

We recall some notation. Let $\{e_i\}_{i=1}^m$ be an orthonormal basis for V. If $F \in \mathbf{S}_{n,p}(V)$, $g \in \Gamma_s$, and $h \in \Gamma_t$, where $0 \le s \le n$ and $0 \le t \le p$, then $F_{g,h}$ denotes

$$F\big(e_{g(1)}, e_{g(2)}, \ldots, e_{g(s)}; e_{h(1)}, e_{h(2)}, \ldots, e_{h(t)}\big),$$

which is then a member of $\mathbf{S}_{n-s,p-t}(V)$. If $A \in \mathbf{S}_n(V)$, then A_i denotes the insertion $A(e_i)$, and A_{ij} denotes $A(e_i, e_j)$. To make the connection between the inner products $[\cdot,\cdot]_{s,t}$, and certain permanental inequalities, and inequalities for norms of symmetrized tensors, we require the following. It is, in essence, Theorem 2 of [19].

Theorem 6. *Suppose* $\{A^i\}_{i=1}^u \subset \mathbf{S}_n(V)$, $\{C^j\}_{j=1}^v \subset \mathbf{S}_n(V)$, $\{B^i\}_{i=1}^u \subset \mathbf{S}_p(V)$, *and* $\{D^j\}_{j=1}^v \subset \mathbf{S}_p(V)$. *If* $F = \sum_{i=1}^u A^i \bar{\otimes} B^i$ *and* $G = \sum_{j=1}^v C^j \bar{\otimes} D^j$, *then*

$$\big[F, G\big]_{s,t} = \binom{n+p-s-t}{n-s} \sum_{i=1}^u \sum_{j=1}^v \sum_{f\in\Gamma_s} \sum_{g\in\Gamma_t} \langle A^i_f \cdot D^j_g, C^j_f \cdot B^i_g \rangle \tag{43}$$

for all integers s *and* t *such that* $0 \le s \le \kappa$ *and* $0 \le t \le \kappa$.

Theorems 5 and 6 have many corollaries, some of which are listed below.

Corollary 1. *If* $A \in \mathbf{S}_n(V)$ *and* $B \in \mathbf{S}_p(V)$, *then*

$$\|A \cdot B\|^2 \ge \frac{n}{n+p} \sum_{i=1}^m \|A_i \cdot B\|^2 \quad \text{and} \quad \|A \cdot B\|^2 \ge \frac{p}{n+p} \sum_{i=1}^m \|A \cdot B_i\|^2. \tag{44}$$

Proof. By (30), and the definition of $[\cdot,\cdot]_{0,0}$, we have

$$\big[A\overline{\otimes}B, A\overline{\otimes}B\big]_{0,0} = \binom{n+p}{n} \big\|A \cdot B\big\|^2. \tag{45}$$

Moreover, by applying Theorem 6 in the special case $C = A$, $D = B$, $s = 1$ and $t = 0$, we obtain that

$$\big[A\overline{\otimes}B, A\overline{\otimes}B\big]_{1,0} = \binom{n+p-1}{n-1} \sum_{i=1}^m \|A_i \cdot B\|^2. \tag{46}$$

But Theorem 5 says that $\big[A\overline{\otimes}B, A\overline{\otimes}B\big]_{0,0} \ge \big[A\overline{\otimes}B, A\overline{\otimes}B\big]_{1,0}$. Combining this with (45) and (46) we obtain that

$$\binom{n+p}{n} \big\|A \cdot B\big\|^2 \ge \binom{n+p-1}{n-1} \sum_{i=1}^m \|A_i \cdot B\|^2, \tag{47}$$

which is immediately seen to be equivalent to the inequality

$$\big\|A \cdot B\big\|^2 \ge \frac{n}{n+p} \sum_{i=1}^m \|A_i \cdot B\|^2 \tag{48}$$

that we wished to prove. The second inequality in (44) is obtained by setting $s = 0$ and $t = 1$ in (43), or by reversing the roles of A and B in (48). □

Proceeding as in the proof of Corollary 1 we can obtain many other inequalities. If $A \in \mathbf{S}_n(V)$ and $B \in \mathbf{S}_p(V)$, then according to Theorem 5 we have

$$[A\overline{\otimes}B, A\overline{\otimes}B]_{1,0} \ge [A\overline{\otimes}B, A\overline{\otimes}B]_{1,1} \tag{49}$$

while Theorem 6 implies that

$$\big[A\overline{\otimes}B, A\overline{\otimes}B\big]_{1,1} = \binom{n+p-2}{n-1} \sum_{i=1}^m \sum_{j=1}^m \|A_i \cdot B_j\|^2. \tag{50}$$

Combining (46), (47), (49), and (50) we obtain the inequality

$$\binom{n+p}{n}\|A\cdot B\|^2 \geq \binom{n+p-1}{n-1}\sum_{i=1}^{m}\|A_i\cdot B\|^2 \geq \binom{n+p-2}{n-1}\sum_{i=1}^{m}\sum_{j=1}^{m}\|A_i\cdot B_j\|^2 \quad (51)$$

which implies that

$$\|A\cdot B\|^2 \geq \frac{np}{(n+p)(n+p-1)}\sum_{i=1}^{m}\sum_{j=1}^{m}\|A_i\cdot B_j\|^2. \quad (52)$$

Given a positive integer q, and a number or indeterminate x we let x_q denote $x(x-1)(x-2)\cdots(x-q+1)$. This is standard factorial notation. By $x_{(0)}$ we mean 1. Collecting results like (51) into a single theorem we obtain

Theorem 7. *Suppose n and p are positive integers, and $\kappa = \min\{n,p\}$. If $0\leq s\leq u\leq\kappa$, and $0\leq t\leq v\leq\kappa$, then,*

$$\left[\frac{n_{(s)}p_{(t)}}{(n+p)_{(s+t)}}\right]\sum_{f\in\Gamma_s}\sum_{g\in\Gamma_t}\|A_f\cdot B_g\|^2 \geq \left[\frac{n_{(u)}p_{(v)}}{(n+p)_{(u+v)}}\right]\sum_{f\in\Gamma_u}\sum_{g\in\Gamma_v}\|A_f\cdot B_g\|^2, \quad (53)$$

for all $A\in\mathbf{S}_n(V)$ and $B\in\mathbf{S}_p(V)$.

If we set $s=t=0$, $u=1$, and $v=0$ in Theorem 7, then we obtain the inequality of Corollary 1, which in turn implies the permanental inequality (1), listed as Corollary 2. To see how to transform (44) into Corollary 2 see [16, Lemma 5]. Moreover, if we set $s=t=0$ and $u=v=\kappa$ in (53) then we obtain Lieb's inequality (2). Theorem 7 implies many other curious inequalities.

Corollary 2. *If $A=[a_{ij}]$ is an $(n+p)\times(n+p)$ positive semidefinite Hermitian matrix, then*

$$\operatorname{per}(A) \geq \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\operatorname{per}\big(A(i|j)\big).$$

As noted above, Corollary 2 establishes (5) in case x is a $(0,1)$-vector. In fact, Theorem 6 implies an entire array of permanental inequalities. For each u and v such that $u\leq v$ we let $Q_{u,v}$ denote the set of all strictly increasing functions from $\mathbf{I}_u$ to $\mathbf{I}_v$, where, for positive integer q, we let $\mathbf{I}_q$ denote $\{1,2,\ldots,q\}$. By $\mathbf{I}_0$ we shall mean the empty set. For non-negative integers u, v, and q, we let $Q^q_{u,v}$ denote the set of all strictly increasing functions from $\mathbf{I}_u$ to $\{q+1,q+2,\ldots,q+v\}$. If f and g are finite sequences, then $f\cup g$ denotes the sequence obtained by appending g to the end of f; thus, if $f\in Q_{s,n}$ and $g\in Q^n_{t,p}$, then $f\cup g\in Q_{s+t,n+p}$. Given an $(n+p)\times(n+p)$ matrix M, and increasing functions $f:\mathbf{I}_s\to\mathbf{I}_{n+p}$ and $g:\mathbf{I}_t\to\mathbf{I}_{n+p}$, we let $M(f|g)$ denote the matrix obtained from M by deleting the rows corresponding to the elements in the range of f, and the columns corresponding to the elements in the range of g; similarly $M[f|g]$ is obtained from M by deleting all rows except those corresponding to the elements in the range of f and deleting all columns except those corresponding to the elements in the range of g. We are now ready to define special matrix functions.

For all s and t such that $0 \le s, t \le \kappa$ define the matrix function $\mathcal{L}^{n,p}_{s,t}(\cdot)$ on the set of $(n+p) \times (n+p)$ matrices by

$$\mathcal{L}^{n,p}_{s,t}(M) = \left[1/\tbinom{n}{s}\tbinom{p}{t}\right] \times \sum_{\alpha,\delta\in Q_{s,n}} \sum_{\beta,\gamma\in Q^n_{t,p}} \operatorname{per}\big(M[\alpha|\delta]\big)\operatorname{per}\big(M[\beta\,|\,\gamma]\big)\operatorname{per}\big(M(\alpha\cup\beta\,|\,\delta\cup\gamma)\big). \tag{54}$$

To understand the definition of $\mathcal{L}^{n,p}_{s,t}(M)$ imagine that the matrix M is partitioned in the form

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

where M_{11} is $n\times n$ and M_{22} is $p\times p$. Assume that $\alpha, \delta \in Q_{s,n}$, and $\beta, \gamma \in Q^n_{t,p}$. Then what is the term $\operatorname{per}\big(M[\alpha|\delta]\big)\operatorname{per}\big(M[\beta\,|\,\gamma]\big)\operatorname{per}\big(M(\alpha\cup\beta\,|\,\delta\cup\gamma)\big)$? We see that $M[\beta\,|\,\gamma]$ is an $s \times s$ submatrix of M_{11} while $M[\beta\,|\,\gamma]$ is a $t \times t$ submatrix of M_{22}. The matrix $M(\alpha\cup\beta\,|\,\delta\cup\gamma)$ is the $(n+p-s-t)\times(n+p-s-t)$ submatrix of M obtained by deleting all rows indicated by either α or β, and all columns indicated by either δ or γ. The matrices $M[\alpha\,|\,\delta]$, $M[\beta\,|\,\gamma]$, and $M(\alpha\cup\beta\,|\,\delta\cup\gamma)$ are therefore non-overlapping submatrices of M such that $M[\alpha\,|\,\delta]$ is an $s\times s$ submatrix of M_{11}, $M[\beta\,|\,\gamma]$, is a $t\times t$ submatrix of M_{22}, and $M(\alpha\cup\beta\,|\,\delta\cup\gamma)$ is the rest of M in that it consists of the submatrix of M consisting of the rows not indicated by α or δ, and the columns not indicated by β or γ. The function $\mathcal{L}^{n,p}_{s,t}(M)$ is then the sum of all possible products of the form

$$\operatorname{per}\big(M[\alpha\,|\,\delta]\big)\operatorname{per}\big(M[\beta\,|\,\gamma]\big)\operatorname{per}\big(M(\alpha\cup\beta\,|\,\delta\cup\gamma)\big)$$

divided by the integer $\binom{n}{s}\binom{p}{t}$. We note that if M is the $(n+p)\times(n+p)$ identity matrix, then $\mathcal{L}^{n,p}_{s,t}(M) = 1$. This means that the coefficient in the definition is chosen especially to make each of the $\mathcal{L}^{n,p}_{s,t}(\cdot)$ assume the value 1 at the identity matrix. To understand the following theorem, visualize for each $M \in \mathcal{H}_{n+p}$ the $(\kappa+1)\times(\kappa+1)$ array $\big[\mathcal{L}^{n,p}_{s,t}(M)\big]^{\kappa}_{s,t=0}$. The theorem below says that if $M \in \mathcal{H}_{n+p}$, then this array is non-increasing from left to right and from top to bottom.

Theorem 8. *Suppose q is a positive integer and both n and p are positive integers such that $n+p = q$. If $s, u \in \{0, 1, 2, \ldots, n\}$, $t, v \in \{0, 1, 2, \ldots, p\}$, $s \le u$, and $t \le v$, then*

$$\mathcal{L}^{n,p}_{s,t}(M) \ge \mathcal{L}^{n,p}_{u,v}(M)$$

for all $q\times q$ positive semi-definite Hermitian matrices M, with equality if M is the $q\times q$ identity matrix.

Theorem 8 follows from Theorems 5 and 6. See [19] for details. We note that $\mathcal{L}^{n,p}_{0,0}(M) = \operatorname{per}(M)$, $\mathcal{L}^{n,p}_{1,0}(M) = [1/n]\sum_{i,j=1}^{n} m_{ij}\operatorname{per}(M(i|j))$, and $\mathcal{L}^{n,p}_{\kappa,\kappa}(M) = \operatorname{per}(M_{11})\operatorname{per}(M_{22})$; thus, the inequalities

$$\mathcal{L}^{n,p}_{0,0}(M) \ \ge\ \mathcal{L}^{n,p}_{1,0}(M) \ \ge\ \mathcal{L}^{n,p}_{\kappa,\kappa}(M),$$

implied by Theorem 8, translate into the inequalities (4). Theorem 8 therefore provides an extensive refinement of the Lieb inequality (2).

4. Indication of additional applications, concluding remarks

The inequality (1), and the others listed in Theorem 8, originally grew out of an effort to resolve the permanent dominance conjecture for immanants, and figured prominently in results presented in [17], which is currently the last word on this subject. Could it actually be that most conjectured permanent inequalities, restricted to Hermitian positive semidefinite matrices, are simply specializations of monotonicity relationships amongst appropriately defined norms? At this point it seems likely that this is the case. For example, instead of the spaces $\mathbf{S}_{n,p}(V)$, one could study, spaces of mixed type such as $\mathbf{SA}_{n,p}(V)$, which denotes the set of all members of $\mathbf{T}_{n,p}(V)$ that are symmetric in the first n places, and anti-symmetric in the last p places. It is then possible to define contractions $\mathbf{C}_w$, though one must be mindful of the anti-symmetry in the final p places. All such contractions, $\mathbf{C}_w$, are zero except $\mathbf{C}_1$, and $\mathbf{C}_0$, which is the identity map. The questions then become what inner products like (13) exist, which yield interesting tensor inequalities, and which are useful within the context of matrix theory. It seems likely that the monotonicity of the single-hook immanants [1] is simply a manifestation of this type of monotonicity involving inner products appropriately defined on the space $\mathbf{SA}_{n,p}(V)$.

The original inequality from among the many present in Theorem 7 is Neuberger's inequality which asserts that if $A \in \mathbf{S}_n(V)$ and $B \in \mathbf{S}_p(V)$, then

$$\|A \cdot B\|^2 \geq \frac{n!p!}{(n+p)!}\|A\|^2\|B\|^2. \tag{55}$$

This inequality arose from a desire to obtain minimum eigenvalue estimates for certain operators that figured into an iterative scheme for producing solutions to partial differential equations. See [6], [7], [8], [9], [14], and [15]. Suppose $A \in \mathbf{S}_n(V)$ is fixed and non-zero. For each positive integer p we define a linear map $\mathbf{M}_p : \mathbf{S}_p(V) \to \mathbf{S}_p(V)$ so that for all $B \in \mathbf{S}_p(V)$, we have

$$\mathbf{M}_p(B) = (A \cdot B)(A),$$

where $(A \cdot B)(A)$ denotes the insertion of A into $A \cdot B$. In general, if $q \geq n$, then the insertion of A into an element $C \in \mathbf{S}_q(V)$, is the element of $\mathbf{S}_{q-n}(V)$ defined by

$$C(A)(x_1, x_2, \ldots, x_{q-n}) = \sum_{f \in \Gamma_n} C(e_f, x_1, x_2, \ldots, x_{q-n})\overline{A(e_f)}.$$

In general, if $E, G \in \mathbf{S}_n(V)$ and $F, H \in \mathbf{S}_p(V)$, then we have

$$\langle E \cdot F,\, G \cdot H \rangle = \langle F,\, (G \cdot H)(E) \rangle = \langle E,\, (G \cdot H)(F) \rangle$$

This means that the operations of dotting and inserting are adjoint to one another. Therefore, we have

$$\langle \mathbf{M}_p(B), C \rangle = \langle (A \cdot B)(A), C \rangle = \langle A \cdot B, A \cdot C \rangle = \langle B, (A \cdot C)(A) \rangle = \langle B, \mathbf{M}_p(C) \rangle,$$

for all $B, C \in \mathbf{S}_p(V)$. The operators $\mathbf{M}_p$ are therefore all self-adjoint (or Hermitian). Moreover, since

$$\big\langle \mathbf{M}_p(B), B\big\rangle = \big\langle A \cdot B, A \cdot B\big\rangle = \big\|A \cdot B\big\|^2 > 0$$

for all non-zero $B \in \mathbf{S}_p(B)$, we deduce that each of the operators $\mathbf{M}_p$ is positive definite. A difficult problem has been further spectral analysis of the operators $\mathbf{M}_p$ which were originally introduced in Neuberger's paper [6]. On account of (28) with $C = A$ and $C = B$, we have

$$\big\langle A \cdot B, A \cdot B\big\rangle = \sum_{t=0}^{\kappa} a_t \big\langle \mathbf{C}_t(A \overline{\otimes} B), C_t(A \overline{\otimes} B)\big\rangle,$$

where a_t is defined as in (27). This suggests that further progress with analysis of the operators $\mathbf{M}_p$ depends upon an understanding of the contraction maps $\mathbf{C}_t$. Lemmas 1, 2, and 3 point in that direction. Other useful information about the operators $\mathbf{M}_p$ is contained in [10].

References

[1] P. Heyfron, Immanant dominance orderings for hook partitions, Linear and Multilinear Algebra, (1) **24** (1988), 65–78.

[2] R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[3] E.H. Lieb, Proofs of some conjectures on permanents, J. Math. and Mech., **16** (1966), 127–134.

[4] M. Marcus, Multilinear methods in linear algebra, Linear Algebra and Its Applications, **150** (1991), 41–59.

[5] J.W. Neuberger, Norm of symmetric product compared with norm of tensor product, Linear and Multilinear Algebra, **2** (1974), 115–122.

[6] J.W. Neuberger, Tensor products and successive approximations for partial differential equations, Israel Journal of Mathematics, **6** (1968), 121–132.

[7] J.W. Neuberger, An iterative method for solving non-linear partial differential equations, Advances in Mathematics, (2) **19** (1976), 245–265.

[8] J.W. Neuberger, A resolvent for an iteration method for nonlinear partial differential equations, Transactions of the American Mathematical Society, **226** (1977), 321–343.

[9] J.W. Neuberger, *Sobolev Gradients and Differential Equations*, Springer Lecture Notes in Mathematics, **1670**, Springer, 1997.

[10] T.H. Pate, Lower bounds for the norm of the symmetric product, Linear Algebra and Its Applications, (3) **14** (1976), 285–291.

[11] T.H. Pate, Tensor products, symmetric products, and permanents of positive semidefinite Hermitian matrices, Linear and Multilinear Algebra, (1-4) **31** (1992), 27–36.

[12] T.H. Pate, On Permanental compounds, Linear Algebra and Its Applications, (5-6) **429** (2008), 1093–1101.

[13] T.H. Pate, Group algebras, monotonicity, and the Lieb Permanent Inequality, Linear and Multilinear Algebra, **40** (1996), 207–220.

[14] T.H. Pate, Some improvements in Neuberger's iteration procedure for solving partial differential equations, Journal of Differential Equations, (1) **34** (1979), 261–272.

[15] T.H. Pate, An application of the Neuberger iteration procedure to the constant coefficient linear partial differential equation, Journal of Mathematical Analysis and Applications, (2) **72** (1979), 771–782.

[16] T.H. Pate, A new lower bound inequality for the norm of the symmetric product, Linear and Multilinear Algebra (1) **57** (2009), 87–102.

[17] T.H. Pate, Row appending maps, Ψ-functions, and immanant inequalities for Hermitian positive semi-definite matrices, Proceedings of the London Mathematical Society, (2) **76** (1998), 307–358.

[18] T.H. Pate, A generalized Neuberger identity for the inner product of symmetric products of tensors, Linear and Multilinear Algebra, (5) **56** (2008), 555–563.

[19] T.H. Pate, Twisted inner products and contraction inequalities on spaces of contravariant and covariant tensors, Linear Algebra and Its Applications, (7) **429** (2008), 1489–1503.

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Received: February 20, 2009
Accepted: June 16, 2009

Operator Theory:
Advances and Applications, Vol. 202, 451–467

The Spectrum of a Composition Operator and Calderón's Complex Interpolation

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In honor of Israel Gohberg on the occasion of his 80*th birthday.*

Abstract. Using the method of complex interpolation due to A.P. Calderón, we give a general theorem for identifying the spectrum of an operator acting on a family of interpolation spaces. We then use this to determine the spectrum of certain composition operators acting on the weighted Dirichlet and analytic Besov spaces of the unit disk.

Mathematics Subject Classification (2000). Primary 46B70, 47B33.

Keywords. Composition operators, complex interpolation, spectra.

1. Introduction

If φ is an analytic map of the unit disk into itself, then we may define the composition operator C_φ with symbol φ by the rule $C_\varphi f = f \circ \varphi$ which maps $H(\mathbb{D})$ into $H(\mathbb{D})$. The first results concerning the spectrum of a composition operator date back to the late 1960s and a paper by Eric Nordgren [18] where results were given for an invertible composition operator, i.e., a composition with automorphic symbol, acting on the Hardy space of the unit disk $H^2(\mathbb{D})$. To obtain his results, Nordgren characterized the invertible composition operators by symbol, elliptic, parabolic or hyperbolic automorphism, and then constructed eigenfunctions for each type of symbol. Specifically:

- if φ is elliptic, the $H^2(\mathbb{D})$ spectrum of C_φ is the unit circle or a finite subgroup of the unit circle;
- if φ is parabolic, the $H^2(\mathbb{D})$ spectrum of C_φ is the unit circle;

This work forms part of the author's dissertation written at the University of Virginia under the direction of Professor Barbara D. MacCluer.
Communicated by I.M. Spitkovsky.

- if φ is hyperbolic, the $H^2(\mathbb{D})$ spectrum of C_φ is an annulus
$$\left\{\lambda : \varphi'(a)^{1/2} \leq |\lambda| \leq \varphi'(a)^{-1/2}\right\}$$
where a is the Denjoy-Wolff point of φ.

These results were then extended to the weighted Bergman spaces of the unit disk $A^2_\alpha(\mathbb{D})$ where $\alpha > -1$ (Theorem 7.1, 7.4 and 7.5 of [7]) and recently to the Dirichlet space of the unit disk $\mathcal{D}$ (Theorem 5.1 in [10] or Theorems 3.1 and 3.2 of [13]). The spaces mentioned thus far are part of the one parameter family of spaces known as the weighted Dirichlet spaces of the unit disk, denoted $\mathcal{D}_\alpha$ where $\alpha > -1$, and we seek to extend these results to the entire range of weighted Dirichlet spaces. In addition, the Dirichlet space is also a member of the family of analytic Besov spaces, and while little work has been done regarding the spectrum of a composition operator on these spaces, some general results involving composition operators can be found in [9] and [22].

In the next section we define the spaces mentioned above, characterize the automorphisms of the disk, and discuss various means for extending Nordgren's results. In Section 3, we develop a method for treating these spaces in a unified manner using the method of complex interpolation due to A.P. Calderón. Section 4 then focuses on determining the spectrum of a composition operator with automorphic symbol acting on the weighted Dirichlet and analytic Besov spaces of the unit disk. In Section 5, we consider the spectrum of a certain class of non-invertible composition operators.

2. Preliminaries

2.1. The weighted Dirichlet spaces

Let $\mathbb{D}$ denote the open unit disk in the complex plane, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Nordgren's results concerned the Hardy space of the unit disk
$$H^2(\mathbb{D}) = \left\{f \text{ analytic in } \mathbb{D} : \|f\|^2_{H^2} \equiv \lim_{r\to 1^-} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty\right\}$$
where $d\theta$ is Lebesgue arc-length measure on the unit circle. For $\alpha > -1$, the weighted Bergman space is defined by
$$A^2_\alpha(\mathbb{D}) = \left\{f \text{ analytic in } \mathbb{D} : \|f\|^2_{A^2_\alpha} \equiv \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^\alpha \, dA < \infty\right\}$$
where dA is Lebesgue area measure normalized so that $A(\mathbb{D}) = 1$. The Dirichlet space is given by
$$\mathcal{D}(\mathbb{D}) = \left\{f \text{ analytic in } \mathbb{D} : \|f\|^2_{\mathcal{D}} \equiv |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA < \infty\right\}.$$

The weighted Dirichlet spaces encompass all the spaces listed above. For $\alpha > -1$,

$$\mathcal{D}_\alpha(\mathbb{D}) = \left\{ f \text{ analytic in } \mathbb{D} : \|f\|_\alpha^2 \equiv \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^\alpha \, dA < \infty \right\};$$

we then equip $\mathcal{D}_\alpha$ with the norm

$$\|f\|_{\mathcal{D}_\alpha}^2 \equiv |f(0)|^2 + \|f\|_\alpha^2.$$

Basic facts about these spaces can be found in [16] and Section 2.1 of [7] and we summarize a few here. We have

- $\mathcal{D}_1 = H^2$ with an equivalent norm;
- for $\alpha > 1$, $\mathcal{D}_\alpha = A^2_{\alpha-2}$ with an equivalent norm;
- $\mathcal{D} = \mathcal{D}_0$ with equal norm.

Also, if $-1 < \alpha < \beta < \infty$, $\mathcal{D}_\alpha \subset \mathcal{D}_\beta$ with continuous inclusion; moreover, the analytic polynomials are dense in $\mathcal{D}_\alpha$ for each $\alpha > -1$. If f is in $\mathcal{D}_\alpha$ with power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then the quantity

$$\sum_{n=0}^{\infty} (n+1)^{1-\alpha} |a_n|^2$$

is equivalent to the norm given above. Thus $\mathcal{D}_\alpha$ can also be recognized as

$$\mathcal{D}_\alpha(\mathbb{D}) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} (n+1)^{1-\alpha} |a_n|^2 < \infty \right\}. \tag{1}$$

2.2. The analytic Besov spaces

For $1 < p < \infty$, the analytic Besov space is given by

$$B_p(\mathbb{D}) = \left\{ f \text{ analytic in } \mathbb{D} : \|f\|_p^p = \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} \, dA(z) < \infty \right\}.$$

The quantity $\|\cdot\|_p$ defines a semi-norm on B_p and we equip it with the norm

$$\|f\|_{B_p} = |f(0)| + \|f\|_p.$$

As with the weighted Dirichlet spaces, the analytic Besov are nested; if $1 < p < q < \infty$, then $B_p \subset B_q$ with continuous inclusion ([22]) and, as above, the analytic polynomials are dense in each Besov space. We note that B_2 defines the same set of functions as the Dirichlet space $\mathcal{D} = \mathcal{D}_0$ with an equivalent norm. One of the most beneficial properties of these spaces is that they are Möbius invariant, i.e., the quantity $\|\cdot\|_p$ is invariant under composition with any automorphism of the disk. For more properties of these spaces, we point the reader to [22] and Section 5.3 of [23]. As stated before, there have been few, if any, spectral results for composition operators acting on these spaces.

2.3. Automorphisms of the disk and their composition operators

The automorphisms of the disk are characterized in terms of their fixed points. An automorphism has the following classification:

- elliptic if it has one fixed point in $\mathbb{D}$ and the other fixed point is outside $\overline{\mathbb{D}}$;
- parabolic if it has no fixed point in $\mathbb{D}$ and a boundary fixed point of multiplicity two;
- hyperbolic if it has no fixed point in $\mathbb{D}$ and two distinct boundary fixed points.

Concerning the spectrum of a composition operator with automorphic symbol acting on $\mathcal{D}_\alpha$, there are gaps in the information when α is in the range $(-1,0)$ or $(0,1)$. Theorem 4 below will fill in these gaps for the elliptic and parabolic cases. As an aside, we also mention that this theorem will resolve the issue raised in Remark 3.4 of [11]; there the authors consider the relationship between hypercyclicity and spectra for composition operators on $\mathcal{D}_\alpha$ with $1/2 < \alpha$ ($\nu < 1/2$ in their notation). For the Besov spaces, the only known results seem to be for B_2. Theorem 6 will give a complete classification for the entire range of Besov spaces. We now consider each type of automorphism briefly and discuss some known spectral results.

An elliptic automorphism is conformally equivalent to a rotation of the disk; moreover, two conformally equivalent maps give rise to composition operators which are similar and therefore have the same spectrum. On all of the spaces described earlier the rotations of the disk induce composition operators which are invertible isometries and thus have spectrum contained in the unit circle; the elliptic automorphism result stated for the Hardy space is in fact true on all of these spaces. Theorem 7.1 of [7] provides a proof for the weighted Dirichlet spaces and it is a simple matter to modify it for the analytic Besov spaces. We remark however that the method we develop in Section 3 will recover these results.

Lemma 7.2 of [7] implies that the spectral radius of a composition operator whose symbol is a parabolic automorphism acting on the Hardy or weighted Bergman spaces is 1. Our Lemma 2 will extend this to the entire range of weighted Dirichlet spaces. Since a composition operator of this type is invertible, the spectral mapping theorem will then imply that the spectrum is contained in the unit circle for C_φ acting on any weighted Dirichlet space. Section 4 will complete this circle of ideas showing that the spectrum is in fact the unit circle. For α in restricted ranges, the following provides two constructive methods for verifying this fact.

Consider the parabolic automorphism of the disk given by

$$\varphi(z) = \frac{(1+i)z-1}{z+i-1},$$

which has 1 as its only fixed point in $\overline{\mathbb{D}}$. In the case of H^2 and $s \geq 0$, Nordgren showed that the function

$$f(z) = \exp\left(\frac{s(z+1)}{(z-1)}\right), \tag{2}$$

which is bounded in the disk, is an eigenfunction for C_φ with eigenvalue e^{-2is}. Since f is in $H^\infty(\mathbb{D}) \subset H^2(\mathbb{D})$, this gives each point of the unit circle as an eigenvalue (of infinite multiplicity) for C_φ on H^2. These eigenfunctions also suffice for the weighted Bergman spaces since $H^\infty(\mathbb{D}) \subset A^2_\alpha(\mathbb{D})$ for all $\alpha > -1$. Now H^∞ is not a proper subset of the Dirichlet space and, more importantly, it can be shown that the functions given in Equation (2) are not in $\mathcal{D}$. In fact, H^∞ is not a proper subset of $\mathcal{D}_\alpha$ when $\alpha < 1$ (Equation (1) and Exercise 2.1.11 in [7]), however a norm calculation shows that the functions given in Equation (2) are in $\mathcal{D}_\alpha$ for $\alpha > 1/2$. To see this, let Φ be the Cayley Transform,

$$\Phi(z) = \frac{i(1+z)}{1-z}$$

which is a biholomorphic map of the unit disk onto the (open) upper half-plane Π^+. Letting $g(w) = e^{isw}$, the function f in Equation (2) may be written as $f = g \circ \Phi$. To calculate the norm of f, we use a change of variables and consider an integral over the upper half-plane as follows:

$$\begin{aligned}\|f\|_\alpha^2 &= \int_{\mathbb{D}} |(g\circ\Phi)'(z)|^2(1-|z|^2)^\alpha \, dA(z)\\ &= \int_{\Pi^+} |g'(w)|^2(1-|\Phi^{-1}(w)|^2)^\alpha \, dA(w).\end{aligned}$$

Writing $w = x + iy$ and calculating $1 - |\Phi^{-1}(w)|^2$ we obtain

$$\|f\|_\alpha^2 = s^2 \int_0^\infty \int_{-\infty}^\infty e^{-2sy}\left(\frac{4y}{x^2+(y^2+1)}\right)^\alpha \, dxdy;$$

note that the inner integral is finite if and only if $\alpha > 1/2$, after which the second integral can be bounded above by a convergent integral which is independent of α. Thus we have every point of the unit circle is an eigenvalue of infinite multiplicity for C_φ on $\mathcal{D}_\alpha$ provided $\alpha > 1/2$.

For the Dirichlet space, Higdon shows that the unit circle is contained in the approximate point spectrum of C_φ. Recall that a complex number λ is in the approximate point spectrum of an operator T acting on a Banach space $\mathcal{Y}$ if there is a sequence of unit vectors x_n in $\mathcal{Y}$ with $\|(T-\lambda)x_n\| \to 0$ as $n \to \infty$. Higdon's approximate eigenfunctions are necessarily more complicated than the functions given in Equation (2) and we omit the details of an explicit construction. However, the fact that the Dirichlet space is contained in $\mathcal{D}_\alpha$ when $\alpha > 0$ allows one to use a modified version of these approximate eigenfunctions to show that every point of the unit circle is an approximate eigenvalue for C_φ acting on $\mathcal{D}_\alpha$ with $\alpha > 0$. See [19] for this construction.

For the Besov spaces, Donaway gives the following result which we restate for our intended purposes.

Lemma 1 ([9] **Corollary 3.11**). *If φ is an automorphism of the unit disk, then $r(C_\varphi) = 1$ for C_φ acting on B_p with $1 < p < \infty$.*

As before the spectral mapping theorem now implies that the spectrum of any such composition operator is contained in the unit circle. In Section 4 we will verify that the spectrum is in fact the unit circle.

The hyperbolic automorphism case for the weighted Dirichlet spaces seems to be more difficult than the parabolic case and our Lemma 2 will provide some information, however a clear determination of the spectrum in this case remains unanswered. On the Besov spaces, Donaway's spectral radius result applies and the results of Section 4 will identify the spectrum as the unit circle.

3. Interpolating spectra

Let $(X_0, \|\cdot\|_0)$ and $(X_1, \|\cdot\|_1)$ be Banach spaces which form a compatible pair in the sense of Calderón (see [4]). Both X_0 and X_1 may be continuously embedded in the complex topological vector space $X_0 + X_1$ when equipped with the norm

$$\|x\|_{X_0+X_1} = \inf\{\|y\|_0 + \|z\|_1 : x = y + z, y \in X_0, z \in X_1\}.$$

Moreover, the space $X_0 \cap X_1$ with norm

$$\|x\|_{X_0\cap X_1} = \max(\|x\|_0,\ \|x\|_1)$$

is continuously embedded in each of X_0 and X_1. Further assume that the space $X_0 \cap X_1$ is dense in both X_0 and X_1 and define the interpolation algebra $\mathcal{I}[X_0, X_1]$ to be the set of all linear operators $T : X_0 \cap X_1 \to X_0 \cap X_1$ that are both 0-continuous and 1-continuous. For properties and applications of this algebra, see [2], [3], [12], and [20]. For a Banach space $\mathcal{Y}$, let $\mathcal{B}(\mathcal{Y})$ denote the set of all bounded operators on $\mathcal{Y}$. Then any operator $T \in \mathcal{I}[X_0, X_1]$ induces a unique operator $T_i \in \mathcal{B}(X_i)$, $i = 0, 1$. Letting $X_t = [X_0, X_1]_t$ be the interpolation space obtained via Calderón's method of complex interpolation, it follows that T also induces a unique operator $T_t \in \mathcal{B}(X_t)$ satisfying

$$\|T_t\|_{\mathcal{B}(X_t)} \leq \|T_0\|^{1-t}_{\mathcal{B}(X_0)} \|T_1\|^{t}_{\mathcal{B}(X_1)}, \quad t \in (0, 1). \tag{3}$$

The interpolation algebra defined above first appeared in the L^p-space setting in [2]; it has since been used in the study of the map $t \mapsto \sigma_{\mathcal{B}(X_t)}(T_t)$ and has provided very useful information regarding the spectrum of these operators. The result which is most relevant to our work provides a bound for the spectra of the interpolation operators T_t in terms of the endpoint operators. Note $\sigma(T) \equiv \sigma_{\mathcal{I}[X_0,X_1]}(T)$ for $T \in \mathcal{I}[X_0, X_1]$ and $\mathcal{B}(X_0 \cap X_1) \equiv \mathcal{B}((X_0 \cap X_1,\ \|\cdot\|_{X_0\cap X_1}))$.

Theorem 1 (part of [20] **Theorem 2).** *For $T \in \mathcal{I}[X_0, X_1]$,*

$$\sigma_{\mathcal{B}(X_t)}(T_t) \subseteq \sigma(T) = \sigma_{\mathcal{B}(X_0)}(T_0) \cup \sigma_{\mathcal{B}(X_1)}(T_1) \cup \sigma_{\mathcal{B}(X_0\cap X_1)}(T)$$

for all t in $(0, 1)$.

While this result is very useful, it can be difficult to explicitly determine the set on the far right, $\sigma_{\mathcal{B}(X_0\cap X_1)}(T)$. For specific examples see [12] and [21]. However, the situation often arises when X_0 is continuously contained in X_1, i.e., there is a positive constant K such that $\|x\|_1 \leq K\|x\|_0$ for all x in X_0. Then it must be the case that

$$\sigma_{\mathcal{B}(X_0 \cap X_1)}(T) \subseteq \sigma_{\mathcal{B}(X_0)}(T_0)$$

and Theorem 1 reduces to

$$\sigma_{\mathcal{B}(X_t)}(T_t) \subseteq \sigma_{\mathcal{B}(X_0)}(T_0) \cup \sigma_{\mathcal{B}(X_1)}(T_1) \tag{4}$$

for all t in $(0,1)$. This reduction was used in [15] to determine the spectrum of certain composition operators acting on the Bloch and weighted Bergman spaces.

Before stating our first result we recall the basics of reiteration of complex interpolation (see [8]). If $0 \leq x \leq t \leq y \leq 1$ and $\alpha \in [0,1]$ with $t = (1-\alpha)x + \alpha y$, the reiteration theorem states that

$$X_t = [X_x, X_y]_\alpha$$

with equality of norms. The idea contained in Equation (3) becomes

$$\|T_t\|_{\mathcal{B}(X_t)} \leq \|T_x\|_{\mathcal{B}(X_x)}^{1-\alpha} \|T_y\|_{\mathcal{B}(X_y)}^{\alpha}. \tag{5}$$

This immediately yields the following relationship concerning the spectral radius of the operators involved,

$$r_{\mathcal{B}(X_t)}(T_t) \leq (r_{\mathcal{B}(X_x)}(T_x))^{1-\alpha} (r_{\mathcal{B}(X_y)}(T_y))^{\alpha}. \tag{6}$$

We now state the main result of this section which involves a case where the map $t \mapsto \sigma_{\mathcal{B}(X_t)}(T_t)$ is constant on the interval $(0,1)$.

Theorem 2. *Let X_0 and X_1 be Banach spaces such that X_0 is continuously contained in X_1 and let $T \in \mathcal{I}[X_0, X_1]$. Then if $\sigma_{\mathcal{B}(X_0)}(T_0)$ and $\sigma_{\mathcal{B}(X_1)}(T_1)$ are contained in the unit circle, $\sigma_{\mathcal{B}(X_t)}(T_t)$ is the same set for all $t \in (0,1)$.*

Proof. Fix t in $(0,1)$ and fix any $x, y \in [0,1]$ with $0 \leq x < t < y \leq 1$. Equation (4) and the conditions on $\sigma_{\mathcal{B}(X_0)}(T_0)$ and $\sigma_{\mathcal{B}(X_1)}(T_1)$ guarantee that each of the sets $\sigma(T)$, $\sigma_{\mathcal{B}(X_t)}(T_t)$, $\sigma_{\mathcal{B}(X_x)}(T_x)$ and $\sigma_{\mathcal{B}(X_y)}(T_y)$ are contained in $\partial\mathbb{D}$. Now choose $\alpha \in (0,1)$ with $t = (1-\alpha)x + \alpha y$ so that

$$X_t = [X_x, X_y]_\alpha,$$

by the reiteration theorem, with equality of norms. Also, fix $\lambda \in \sigma_{\mathcal{B}(X_t)}(T_t)$ and set $\mu = r\lambda$ for some $r \in (1, \infty)$. Since $\mu \notin \partial\mathbb{D}$, the operator $T - \mu$ must be invertible in $\mathcal{I}[X_0, X_1]$ and it follows that

$$\left[(T-\mu)^{-1}\right]_t = (T_t - \mu)^{-1}$$

for all $t \in [0,1]$.

For any Banach algebra $\mathcal{A}$ it is true that

$$d(\mu,\, \sigma_{\mathcal{A}}(a)) = \frac{1}{r[(a-\mu)^{-1}]}$$

for $a \in \mathcal{A}$ and $\mu \notin \sigma_{\mathcal{A}}(a)$ (Theorem 3.3.5 of [1]). Setting $S \equiv (T-\mu)^{-1}$, we have

$$r_{\mathcal{B}(X_t)}(S_t) = r_{\mathcal{B}(X_t)}\left((T_t - \mu)^{-1}\right) = \frac{1}{d(\mu,\ \sigma_{\mathcal{B}(X_t)}(T_t))} = \frac{1}{|1-r|}.$$

Similarly,

$$r_{\mathcal{B}(X_x)}(S_x) = r_{\mathcal{B}(X_x)}\left((T_x - \mu)^{-1}\right) = \frac{1}{d(\mu,\ \sigma_{\mathcal{B}(X_x)}(T_x))} \leq \frac{1}{|1-r|}$$

and likewise for y. Applying Equation (6),

$$\begin{aligned}
\frac{1}{|1-r|} = r_{\mathcal{B}(X_t)}(S_t) &\leq \left(r_{\mathcal{B}(X_x)}(S_x)\right)^{1-\alpha}\left(r_{\mathcal{B}(X_y)}(S_y)\right)^{\alpha} \\
&\leq \left(\frac{1}{|1-r|}\right)^{1-\alpha}\left(\frac{1}{|1-r|}\right)^{\alpha} \\
&= \frac{1}{|1-r|}.
\end{aligned}$$

The positivity of the quantities involved yields

$$r_{\mathcal{B}(X_x)}(S_x) = r_{\mathcal{B}(X_y)}(S_y) = \frac{1}{|1-r|}$$

which in turn implies

$$d(\mu,\ \sigma_{\mathcal{B}(X_x)}(T_x)) = d(\mu,\ \sigma_{\mathcal{B}(X_y)}(T_y)) = |1-r|.$$

Thus λ must be in $\sigma_{\mathcal{B}(X_x)}(T_x) \cap \sigma_{\mathcal{B}(X_y)}(T_y)$, and it is clear that $\sigma_{\mathcal{B}(X_t)}(T_t) \subseteq \sigma_{\mathcal{B}(X_x)}(T_x)$ for all $x \in [0,1]$ since x and y were arbitrary in $[0,1]$. Moreover, since t was also arbitrary in $(0,1)$ we have that $\sigma_{\mathcal{B}(X_t)}(T_t)$ must be the same set for all t in $(0,1)$. $\square$

As an immediate consequence of the proof, we have the following corollary.

Corollary 1. *Let X_0 and X_1 be Banach spaces such that X_0 is continuously contained in X_1 and let $T \in \mathcal{I}[X_0, X_1]$. Then if $\sigma_{\mathcal{B}(X_0)}(T_0)$ and $\sigma_{\mathcal{B}(X_1)}(T_1)$ are contained in the unit circle and $\sigma_{\mathcal{B}(X_t)}(T_t) = \partial\mathbb{D}$ for some t in $(0,1)$, $\sigma_{\mathcal{B}(X_x)}(T_x) = \partial\mathbb{D}$ for all x in $[0,1]$.*

We note that the conclusion of the corollary is of particular importance since it also provides information about the spectrum for the endpoint spaces.

The hypothesis that the sets $\sigma_{\mathcal{B}(X_1)}(T_1)$ and $\sigma_{\mathcal{B}(X_0)}(T_0)$ be contained in the unit circle can be relaxed in the sense that the unit circle can be replaced by many other types of sets: finite sets, intervals, and other circles are the most obvious. This is due to the fact that in the proof the only property of the unit circle employed is that given a $\lambda \in \partial\mathbb{D}$, we could find a μ not in $\partial\mathbb{D}$ such that λ is the unique point of the circle satisfying $d(\mu, \partial\mathbb{D}) = |\lambda - \mu|$. Rephrasing, the unit circle could be replaced by any subset of the complex plane E with the property that for each $\lambda \in E$ there is a point $\mu \notin E$ such that $|\zeta - \mu| > |\lambda - \mu|$ for all $\zeta \in E \setminus \{\lambda\}$. With this point of view, it is apparent that the theorem only applies to sets with no interior which will have particular importance to composition operators; Section 5 will discuss this and provide a more exotic example of a set with this property.

To apply any of the results of this section, we must verify that the families of spaces mentioned above are in fact interpolation spaces. The result for the weighted Dirichlet spaces is often referenced, however we supply a proof.

Proposition 1. *Suppose* $-1 < \alpha < \gamma < \beta < \infty$. *If* $t \in (0,1)$ *with* $\gamma = (1-t)\alpha + t\beta$, *then* $[\mathcal{D}_\alpha, \mathcal{D}_\beta]_t = \mathcal{D}_\gamma$ *with an equivalent norm.*

Proof. For the proof we will use the series norm given in Equation (1), which is equivalent to the original norm given for a weighted Dirichlet space, and Theorem 1.1 of [17] which illustrates a canonical method for constructing interpolating Hilbert spaces. Choose α and β in the interval $(-1, \infty)$ with $\alpha < \beta$ and let $\langle \cdot, \cdot \rangle_\alpha$ denote the inner product in $\mathcal{D}_\alpha$. Also, recall that $\mathcal{D}_\alpha = \mathcal{D}_\alpha \cap \mathcal{D}_\beta$ is dense in $\mathcal{D}_\beta$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are in $\mathcal{D}_\alpha$, then

$$\langle f, g \rangle_\alpha = \sum_{n=0}^{\infty} a_n \overline{b_n} (n+1)^{1-\alpha}$$

and we define a positive operator A on $\mathcal{D}_\alpha$ by the rule

$$A\left(\sum_{n=0}^{\infty} b_n z^n\right) = \sum_{n=0}^{\infty} b_n (n+1)^{\alpha-\beta} z^n.$$

It is then easy to check that

$$\langle f, g \rangle_\beta = \langle f, Ag \rangle_\alpha$$

for all f and g in $\mathcal{D}_\alpha$.

For $t \in (0,1)$, let $\mathcal{H}_t$ be the closure of $\mathcal{D}_\alpha$ with respect to the norm induced by the inner product $\langle \cdot, A^t(\cdot) \rangle_\alpha$; the action of A^t is given by

$$A^t\left(\sum_{n=0}^{\infty} b_n z^n\right) = \sum_{n=0}^{\infty} b_n (n+1)^{(\alpha-\beta)t} z^n.$$

Theorem 1.1 of [17] asserts that $\mathcal{H}_t$ is an interpolation space for $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$. Moreover, the theorem guarantees that Calderón's method of complex interpolation produces the same space.

For the last part of the theorem, we identify $\mathcal{H}_t$ as a weighted Dirichlet space. Choose $\gamma \in (\alpha, \beta)$ and find $t \in (0,1)$ with $\gamma = (1-t)\alpha + t\beta$. A calculation gives

$$\begin{aligned} \left\langle f, A^t g \right\rangle_\alpha &= \sum_{n=0}^{\infty} a_n \overline{b_n} (n+1)^{(\alpha-\beta)t} (n+1)^{1-\alpha} \\ &= \sum_{n=0}^{\infty} a_n \overline{b_n} (n+1)^{1-\gamma} \\ &= \langle f, g \rangle_\gamma \end{aligned}$$

for all f and g in $\mathcal{D}_\alpha$. Thus $\mathcal{H}_t$ defines the same set of functions as $\mathcal{D}_\gamma$ with the series norm of Equation (1) completing the proof. $\square$

To identify the Besov spaces as interpolating spaces, we have the following theorem.

Theorem 3 **([23] Theorem 5.25).** *Suppose* $1 < p_0 < p_1 < \infty$ *and* $t \in [0,1]$, *then* $[B_{p_0}, B_{p_1}]_t = B_p$ *with equivalent norms, where*

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}.$$

4. Spectra of composition operators with automorphic symbol

4.1. The weighted Dirichlet spaces

We will use the symbols $\sigma_\alpha(C_\varphi)$ and $r_\alpha(C_\varphi)$ to denote the spectrum and spectral radius of C_φ when acting on the space $\mathcal{D}_\alpha$ and our main result is as follows.

Theorem 4. *Let* φ *be an elliptic or parabolic automorphism of the disk. Then* $\sigma_\alpha(C_\varphi) = \sigma_\beta(C_\varphi)$ *for all* $-1 < \alpha < \beta < \infty$. *Moreover, the following hold:*

(1) *if* φ *is a parabolic automorphism, then* $\sigma_\alpha(C_\varphi)$ *is the unit circle;*
(2) *if* φ *is an elliptic automorphism, then*

$$\sigma_\alpha(C_\varphi) = \overline{\{\varphi'(a)^k : k = 0, 1, 2, \ldots\}}$$

where a *is the interior fixed point of* φ. *This closure is either the unit circle or a finite subgroup of the unit circle if* $\varphi'(a)^n = 1$ *for some* n.

Our first task is to find a suitable estimate on the spectral radius of C_φ. The elliptic automorphism case is a well known result.

Theorem 5. *Suppose* φ *is an elliptic automorphism of the unit disk. Then for* $\alpha > -1$ *and* C_φ *acting on* $\mathcal{D}_\alpha$, C_φ *is similar to a unitary operator. In particular, the spectral radius of* C_φ *is 1.*

For the parabolic and hyperbolic automorphism cases we have the following extension of Lemma 7.2 in [7]

Lemma 2. *If* φ *is a parabolic or hyperbolic automorphism of the unit disk and* $\alpha > -1$, *then the spectral radius of* C_φ *on* $\mathcal{D}_\alpha$ *satisfies* $r_\alpha(C_\varphi) \le \varphi'(a)^{-|\alpha|/2}$ *where* a *is the Denjoy-Wolff point of* φ.

Proof. Let φ be a parabolic or hyperbolic automorphism of the disk which will have the form

$$\varphi(z) = \lambda \frac{u - z}{1 - \overline{u}z}$$

for some $|\lambda| = 1$ and $|u| < 1$; it follows that φ^{-1} is given by

$$\varphi^{-1}(z) = \overline{\lambda} \frac{\lambda u - z}{1 - \overline{\lambda u}z}.$$

Assuming $f \in \mathcal{D}_\alpha$, we first estimate $\|C_\varphi f\|_\alpha^2$; the change of variables $w = \varphi(z)$ and the identity

$$1 - |\varphi^{-1}(w)|^2 = \frac{(1-|w|^2)(1-|u|^2)}{|1-\overline{\lambda u} w|^2}$$

yields

$$\begin{aligned}\|C_\varphi f\|_\alpha^2 &= \int_{\mathbb{D}} |(f\circ\varphi)'(z)|^2 (1-|z|^2)^\alpha \, dA(z) \\ &= \int_{\mathbb{D}} |f'(\varphi(z))|^2 |\varphi'(z)|^2 (1-|z|^2)^\alpha \, dA(z) \\ &= \int_{\mathbb{D}} |f'(w)|^2 (1-|\varphi^{-1}(w)|^2)^\alpha \, dA(w) \\ &= \int_{\mathbb{D}} |f'(w)|^2 \frac{(1-|w|^2)^\alpha (1-|u|^2)^\alpha}{|1-\overline{\lambda u} w|^{2\alpha}} \, dA(w).\end{aligned}$$

At this point we consider cases for α positive, negative and zero. If α is zero, we have $\|C_\varphi f\|_\alpha = \|f\|_\alpha$. When α is positive or negative, we use the triangle inequality

$$1 - |u| \leq |1 - \overline{\lambda u} w| \leq 1 + |u|.$$

In particular, using the lower estimate when $\alpha > 0$ and the upper estimate when $\alpha < 0$, we can bring the cases together with the common estimate

$$\begin{aligned}\|C_\varphi f\|_\alpha^2 &= \int_{\mathbb{D}} |f'(w)|^2 \frac{(1-|w|^2)^\alpha (1-|u|^2)^\alpha}{|1-\overline{\lambda u} w|^{2\alpha}} \, dA(w) \\ &\leq \left(\frac{1+|u|}{1-|u|}\right)^{|\alpha|} \int_{\mathbb{D}} |f'(w)|^2 (1-|w|^2)^\alpha \, dA(w).\end{aligned}$$

Since $|\varphi(0)| = |u|$, we conclude that

$$\|C_\varphi f\|_\alpha \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{|\alpha|/2} \|f\|_\alpha.$$

Letting $\mathcal{D}_\alpha^0 = \{f \in \mathcal{D}_\alpha : f(0) = 0\}$, we can write $\mathcal{D}_\alpha = \mathcal{D}_\alpha^0 \oplus \mathbb{C}$; furthermore, consider the compression of C_φ to $\mathcal{D}_\alpha^0$ which is given by

$$\widetilde{C_\varphi} f = f \circ \varphi - f(\varphi(0)).$$

The quantity $\|\cdot\|_\alpha$ defines a norm on $\mathcal{D}_\alpha^0$ and combining this with the fact that $\|\widetilde{C_\varphi} f\|_\alpha = \|C_\varphi f\|_\alpha$ for all f in $\mathcal{D}_\alpha$ yields the estimate

$$\|\widetilde{C_\varphi} : \mathcal{D}_\alpha^0 \to \mathcal{D}_\alpha^0\| \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{|\alpha|/2}. \tag{7}$$

Since the constant functions are in $\mathcal{D}_\alpha$ and are invariant under composition, we can represent C_φ as the matrix

$$C_\varphi = \begin{pmatrix} \widetilde{C_\varphi} & 0 \\ K_{\varphi(0)}^\alpha & I \end{pmatrix}$$

where $K^\alpha_{\varphi(0)}$ is the point evaluation functional of $\mathcal{D}_\alpha$ at $\varphi(0)$, i.e., $K^\alpha_{\varphi(0)}(f) = f(\varphi(0))$. From Lemma 7.17 in [7], we have $\sigma_\alpha(C_\varphi) = \{1\} \cup \sigma_{\mathcal{B}(\mathcal{D}^0_\alpha)}(\widetilde{C_\varphi})$ and it is clear that $r_\alpha(C_\varphi) = \max\left(1, r_{\mathcal{B}(\mathcal{D}^0_\alpha)}(\widetilde{C_\varphi})\right)$. If we now let φ_n denote the nth iterate of φ, the definition of $\widetilde{C_\varphi}$ immediately implies that $\widetilde{C_\varphi}^n = \widetilde{C_{\varphi_n}}$. Using the familiar spectral radius formula and the estimate from Equation (7) gives

$$r_{\mathcal{B}(\mathcal{D}^0_\alpha)}(\widetilde{C_\varphi}) = \lim_{n\to\infty} \|\widetilde{C_\varphi}^n\|^{1/n} \leq \lim_{n\to\infty} \left(\frac{1+|\varphi_n(0)|}{1-|\varphi_n(0)|}\right)^{|\alpha|/(2n)}.$$

In the proof of Lemma 7.2 of [7], it was shown that this last limit is equal to $\varphi'(a)^{-|\alpha|/2}$, guaranteeing that

$$r_{\mathcal{B}(\mathcal{D}^0_\alpha)}(\widetilde{C_\varphi}) \leq \varphi'(a)^{-|\alpha|/2}$$

where a is the Denjoy-Wolff point of φ. If φ is parabolic, then $\varphi'(a) = 1$ and if φ is hyperbolic, $\varphi'(a) < 1$. Thus, in either case, $\varphi'(a)^{-|\alpha|/2} \geq 1$ and we have

$$r_\alpha(C_\varphi) = \max\left(1, r_{\mathcal{B}(\mathcal{D}^0_\alpha)}(\widetilde{C_\varphi})\right) \leq \varphi'(a)^{-|\alpha|/2}$$

as desired. □

Since a composition operator with automorphic symbol is invertible (with symbol of the same type) on $\mathcal{D}_\alpha$, an application of the spectral mapping theorem now provides the following characterization:

- if φ is elliptic or parabolic, then $\sigma_\alpha(C_\varphi)$ is contained in the unit circle;
- if φ is hyperbolic, then $\sigma_\alpha(C_\varphi)$ is contained in the annulus

$$\left\{\lambda : \varphi'(a)^{|\alpha|/2} \leq |\lambda| \leq \varphi'(a)^{-|\alpha|/2}\right\}$$

  where a is the Denjoy-Wolff point of φ.

With this information, it is clear that Theorem 2 will only apply to the elliptic and parabolic cases. For the hyperbolic case and $\alpha > 1$, the spectrum is the annulus given above ([7] Theorem 7.4). Though Theorem 2 will not apply to this case when α is in $(-1, 0)$ or $(0, 1)$, it may be possible to use interpolation to gain more information in this case.

Proof of Theorem 4. Let φ be an elliptic or parabolic automorphism of the unit disk and choose α and β with $-1 < \alpha < \beta < \infty$. For the first part of the theorem, we need to verify that C_φ satisfies the conditions of Theorem 2. Since C_φ is continuous on $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$, it is clear that $C_\varphi \in \mathcal{I}[\mathcal{D}_\alpha, \mathcal{D}_\beta]$. Then for $t \in (0, 1)$, the interpolated operator $(C_\varphi)_t$ is simply C_φ since C_φ is continuous on each weighted Dirichlet space. Furthermore, $\sigma_\alpha(C_\varphi)$ and $\sigma_\beta(C_\varphi)$ are both contained in the unit circle via Theorem 5 and Lemma 2. Thus we may apply Theorem 2 and we have that $\sigma_\gamma(C_\varphi)$ is the same set for all $\gamma \in (\alpha, \beta)$. Moreover, since α and β were arbitrary, we have that $\sigma_\gamma(C_\varphi)$ is the same set for all $\gamma \in (-1, \infty)$. For (1) and (2), recall that $\mathcal{D}_1 = H^2$. □

4.2. The analytic Besov spaces

For this section, $\sigma_p(C_\varphi)$ will denote the spectrum of C_φ when acting on the space B_p.

Theorem 6. *Let φ be an automorphism of the disk. Then $\sigma_p(C_\varphi) = \sigma_q(C_\varphi)$ for all $1 < p < q < \infty$. Moreover, the following hold:*

(1) *if φ is a parabolic or hyperbolic automorphism, then $\sigma_p(C_\varphi)$ is the unit circle;*
(2) *if φ is an elliptic automorphism, then*

$$\sigma_p(C_\varphi) = \overline{\{\varphi'(a)^k : k = 0, 1, 2, \ldots\}}$$

where a is the interior fixed point of φ. This closure is either the unit circle or a finite subgroup of the unit circle if $\varphi'(a)^n = 1$ for some n.

As stated in Section 2.3, Lemma 1 and the spectral mapping theorem assure us that, on the Besov spaces, the spectrum of a composition operator with automorphic symbol is contained in the unit circle.

Proof of Theorem 6. The first part of the proof is nearly identical to the proof given for Theorem 4 since C_φ is continuous on B_p for $1 < p < \infty$. For (1) and (2), recall that $B_2 = \mathcal{D}$ with an equivalent norm. The elliptic and parabolic cases then follow from Theorem 4. The spectrum of a composition operator whose symbol is a hyperbolic automorphism was shown to be the unit circle independently in Theorem 3.2 of [13] and Theorem 5.1 of [10], completing the proof. $\square$

5. A non-automorphic example

For a composition operator acting on the Hardy space, it is usually the case that the spectrum contains some non-trivial disk or annulus ([7]), which indicates that the techniques developed in Section 3 are not applicable to the weighted Dirichlet spaces. However, as we have already seen with two of the automorphism cases, there are certain instances where this is not the case.

Let φ be a parabolic non-automorphic linear fractional self-map of the unit disk, i.e., φ has the form

$$\varphi(z) = \frac{az + b}{cz + d}$$

where a, b, c, d are complex numbers and φ fixes a single point – with multiplicity two – in $\partial\mathbb{D}$. Any such φ is conformally equivalent to a translation of the (open) upper half-plane $\tau_\omega(z) = z + \omega$ where $\omega \in \Pi^+$. Moving between $\mathbb{D}$ and Π^+ via the Cayley Transform, we see that any parabolic non-automorphic linear fractional self-map of the disk is conformally equivalent to a self-map of the disk of the form

$$\varphi_\omega(z) = \frac{(2i - \omega)z + \omega}{-\omega z + 2i + \omega}$$

for some $\omega \in \Pi^+$. For the remainder of this section we will use the notation C_ω for the composition operator C_{φ_ω}. In [6] Theorem 6.1 (also [7] Theorem 7.41), the theory of holomorphic semigroups is used to show that the H^2 spectrum of C_ω is the logarithmic spiral

$$\{e^{i\omega t} : t \in [0, \infty)\} \cup \{0\}.$$

For C_ω acting on a weighted Bergman space, the proof for the H^2 setting is easily modified to show that the spectrum is again this particular spiral. Furthermore, the result also holds on the Dirichlet space ([13] and [10]). We will show that the result holds on the entire range of weighted Dirichlet spaces.

To apply our interpolation methods to this setting, one must verify that for each λ in the set $E = \{e^{i\omega t} : t \in [0, \infty)\}$ there is a $\mu \notin E$ such that $|\zeta - \mu| > |\lambda - \mu|$ for all $\zeta \in E \setminus \{\lambda\}$. When $\Re\omega = 0$, E is the line segment $[0, 1]$, and when $\Im\omega = 0$, E is the unit circle (this is exactly the parabolic automorphism case). It is easy to see that these types of sets have the desired property. If $\Im\omega \neq 0$ and $\Re\omega \neq 0$, the argument is a simple geometric construction using tangent and normal vectors and we omit the details.

Using Equation (4) and Corollary 1 (appropriately generalized) as well as the spectral information discussed above for C_ω acting on the Dirichlet and weighted Bergman spaces, we see that the spectrum of $C_\omega : \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ is $\{e^{i\omega t} : t \in [0, \infty)\} \cup \{0\}$ when $0 \leq \alpha < \infty$. A little more work is required when $-1 < \alpha < 0$ and we begin with a lemma.

Lemma 3. *If $\omega \in \Pi^+$ and $-1 < \alpha < \infty$, then $r_\alpha(C_\omega) \leq 1$.*

Proof. When $\alpha \geq 1$, the result is given as [7] Theorem 3.9 for H^2 and a similar proof will suffice for the weighted Bergman spaces. Theorem 3.11 of [9] considers the $\alpha = 0$ case; with this information, Equation (6) easily shows that the result holds for $0 < \alpha < 1$. Consider the case when $-1 < \alpha < 0$. As in the proof of Lemma 2, we first estimate $\|\widetilde{C_\omega} : \mathcal{D}^0_\alpha \to \mathcal{D}^0_\alpha\|$. Using a triangle inequality estimate on the denominator of $|\varphi'_\omega(z)|$ shows that

$$\begin{aligned}
\|\widetilde{C_\omega} f\|^2_\alpha &= \int_{\mathbb{D}} |f'(\varphi_\omega(z))|^2 |\varphi'_\omega(z)|^2 (1 - |z|^2)^\alpha \, dA \\
&\leq \frac{16}{(|2i + \omega| - |\omega|)^2} \int_{\mathbb{D}} |f'(\varphi_\omega(z))|^2 (1 - |z|^2)^\alpha \, dA.
\end{aligned}$$

Since each φ_ω induces a bounded composition operator on A^2_α together with the fact that $f \in \mathcal{D}_\alpha$ if and only if $f' \in A^2_\alpha$ leads to the bound

$$\|\widetilde{C_\omega} f\|^2_\alpha \leq \frac{16}{(|2i + \omega| - |\omega|)^2} \left(\frac{1 + |\varphi_\omega(0)|}{1 - |\varphi_\omega(0)|} \right)^{\alpha+2} \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^\alpha \, dA.$$

Letting φ^n_ω denote the nth iterate of φ_ω, it follows that $\varphi^n_\omega = \varphi_{n\omega}$; the above estimate now yields

$$\begin{aligned}
r_{\mathcal{B}(\mathcal{D}^0_\alpha)}(\widetilde{C_\omega}) &= \lim_{n\to\infty} \|\widetilde{C_\omega}^{\,n}\|^{1/n} = \lim_{n\to\infty} \|\widetilde{C_{n\omega}}\|^{1/n} \\
&\leq \lim_{n\to\infty} \frac{4^{1/n}}{(|2i + n\omega| - |n\omega|)^{1/n}} \left(\frac{1 + |\varphi^n_\omega(0)|}{1 - |\varphi^n_\omega(0)|} \right)^{(\alpha+2)/2n}.
\end{aligned}$$

Writing $\omega = x + iy$,

$$\lim_{n\to\infty} |2i + n\omega| - |n\omega| = \frac{2y}{|\omega|} > 0$$

and the proof of Theorem 3.9 in [7] guarantees us that

$$\lim_{n\to\infty} \left(\frac{1 + |\varphi_\omega^n(0)|}{1 - |\varphi_\omega^n(0)|}\right)^{1/n} = 1$$

since φ is a parabolic self-map of $\mathbb{D}$. Combining these last two facts guarantees $r_{\mathcal{B}(\mathcal{D}_\alpha^0)}(\widetilde{C_\omega}) \leq 1$ and as in the proof of Lemma 2 we have $r_\alpha(C_\omega) \leq 1$ as desired. □

Theorem 7. *Let φ be a parabolic non-automorphic linear fractional self-map of $\mathbb{D}$ and suppose $\alpha > -1$. Then*

$$\sigma_\alpha(C_\varphi) = \{e^{i\omega t} : t \in [0, \infty)\} \cup \{0\}$$

for some $\omega \in \Pi^+$.

Proof. From the comments preceding the statement of Lemma 3, it is clear that we only need to consider the case when $-1 < \alpha < 0$. Since such a φ is conformally equivalent to φ_ω for some $\omega \in \Pi^+$, it suffices to show the result holds for each C_ω. Moreover, if we can show that $\sigma_\alpha(C_\omega) \subset \{e^{i\omega t} : t \in [0, \infty)\} \cup \{0\}$ then the desired result will follow immediately from Corollary 1. For one final reduction, we apply Lemma 7.17 from [7] (as in the proof of Lemma 2) and see that it suffices to show that $\sigma_{\mathcal{B}(\mathcal{D}_\alpha^0)}(\widetilde{C_\omega}) \subset \{e^{i\omega t} : t \in [0, \infty)\} \cup \{0\}$. As in the proof of [7] Theorem 7.41, we will use the theory of holomorphic semi-groups to show this inclusion. The fact that for each $f \in \mathcal{D}_\alpha^0$ and each $z \in \mathbb{D}$ the map $\omega \mapsto \langle \widetilde{C_\omega} f, K_z^\alpha \rangle = f(\Phi^{-1}(\Phi(z) + \omega))$, with kernel function in $\mathcal{D}_\alpha^0$, is analytic in ω ensures us that for $f, g \in \mathcal{D}_\alpha^0$, the map $\omega \mapsto \langle \widetilde{C_\omega} f, g \rangle$ is analytic in ω. Theorem 3.10.1 of [14] now guarantees that $\{\widetilde{C_\omega} : \omega \in \Pi^+\}$ is a holomorphic semigroup of operators on $\mathcal{D}_\alpha^0$.

Letting $\mathcal{A}$ be the norm closed subalgebra of $\mathcal{B}(\mathcal{D}_\alpha^0)$ generated by the identity and $\{\widetilde{C_\omega} : w \in \Pi^+\}$, the Gelfand Theory asserts that $\mathcal{A}$ a unital commutative Banach algebra (Section VII.8 of [5]) and identifies the spectrum of elements of $\mathcal{A}$ as

$$\sigma_{\mathcal{A}}(T) = \{\Lambda(T) : \Lambda \text{ is a multiplicative linear functional on } \mathcal{A}\}.$$

For Λ a multiplicative linear functional on $\mathcal{A}$, define a function $\lambda(\omega) = \Lambda(\widetilde{C_\omega})$ which is analytic in Π^+ since $\{\widetilde{C_\omega} : \omega \in \Pi^+\}$ is a holomorphic semigroup. By the multiplicative property of Λ,

$$\lambda(w_1 + w_2) = \Lambda(\widetilde{C}_{w_1+w_2}) = \Lambda(\widetilde{C}_{w_1}\widetilde{C}_{w_2}) = \Lambda(\widetilde{C}_{w_1})\Lambda(\widetilde{C}_{w_2}) = \lambda(w_1)\lambda(w_2)$$

and thus

$$\lambda \equiv 0 \quad \text{or} \quad \lambda(\omega) = e^{\beta\omega}$$

for some $\beta \in \mathbb{C}$. If λ is not identically zero, the multiplicative property of Λ gives $\|\Lambda\| = 1$ and

$$\begin{aligned} |e^{\beta\omega}| &= \lim_{n\to\infty} |e^{\beta n\omega}|^{1/n} = \lim_{n\to\infty} |\lambda(\omega)^n|^{1/n} = \lim_{n\to\infty} |\Lambda(\widetilde{C_\omega})^n|^{1/n} \\ &= \lim_{n\to\infty} |\Lambda(\widetilde{C_\omega^n})|^{1/n} \leq \lim_{n\to\infty} \|\widetilde{C_\omega^n}\|^{1/n} \leq 1 \end{aligned}$$

for all $\omega \in \Pi^+$, where we have used Lemma 3 in the last inequality. Hence $\beta \in \{it : t \in [0, \infty)\}$ and it follows that

$$\begin{aligned} \sigma_{\mathcal{A}}(\widetilde{C_\omega}) &= \{\Lambda(\widetilde{C_\omega}) : \Lambda \text{ is a multiplicative linear functional on } \mathcal{A}\} \\ &\subseteq \{e^{i\omega t} : t \in [0, \infty)\} \cup \{0\}. \end{aligned}$$

Applying Theorem VII.5.4 in [5] yields

$$\sigma_{\mathcal{B}(\mathcal{D}_\alpha^0)}(\widetilde{C_\omega}) \subseteq \sigma_{\mathcal{A}}(\widetilde{C_\omega}) \subseteq \{e^{i\omega t} : t \in [0, \infty)\} \cup \{0\}$$

completing the proof. □

Acknowledgment

Many thanks to the referee for their encouraging remarks and insightful comments.

References

[1] B. Aupetit, *A Primer on Spectral Theory*, Springer-Verlag, Berlin-Heidelberg, 1991.

[2] B. Barnes, Interpolation of spectrum of bounded operators on Lebesgue spaces, *Rocky Mountain J. Math.* **20** (1987), 359–378.

[3] B. Barnes, Continuity properties of the spectrum of operators on Lebesgue spaces, *Proc. Amer. Math. Soc.* **106** (1989), 415–421.

[4] A. Calderón, Intermediate spaces and interpolation, the complex method, *Stud. Math.* **24** (1964), 113–190.

[5] J. Conway, *A Course in Functional Analysis*, Springer-Verlag, New York, 1990.

[6] C. Cowen, Composition operators on H^2, *J. Operator Theory* **9** (1983), 77–106.

[7] C. Cowen and B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, 1995.

[8] M. Cwikel, Complex interpolation spaces, a discrete definition and reiteration, *Indiana Univ. Math. J.* **27** (1978), 1005–1009.

[9] R. Donaway, Norm and essential norm estimates of composition operators on Besov type spaces, Ph.D. thesis, The University of Virginia, 1999.

[10] E. Gallardo-Gutiérrez and A. Montes-Rodríguez, Adjoints of linear fractional composition operators on the Dirichlet space, *Math. Ann.* **327** (2003), 117–134.

[11] E. Gallardo-Gutiérrez and A. Montes-Rodríguez, The role of the spectrum in the cyclic behavior of composition operators, *Mem. Amer. Math. Soc.* **167** (2004), no. 791, x + 81 pp.

[12] D. Herrero and K. Saxe Webb, Spectral continuity in complex interpolation, *Math. Balkanica* **3** (1989), 325–336.

[13] W. Higdon, The spectra of composition operators from linear fractional maps acting upon the Dirichlet space, *J. Func. Anal.* **220** (2005), 55–75.

[14] E. Hille and R. Phillips, *Functional Analysis and Semi-groups*, revised ed., American Math. Society, Providence, 1957.

[15] B. MacCluer and K. Saxe, Spectra of composition operators on the Bloch and Bergman spaces, *Isr. J. Math.* **128** (2002), 325–354.

[16] B. MacCluer and J. Shapiro, Angular derivatives and compact composition operators on the Hardy and Bergman spaces, *Canadian J. Math.* **38** (1986), 878–906.

[17] J. McCarthy, Geometric interpolation between Hilbert spaces, *Ark. Mat.* **30** (1992), no. 2, 321–330.

[18] E. Nordgren, Composition operators, *Canadian J. Math.* **20** (1968), 442–449.

[19] M. Pons, Composition operators on Besov and Dirichlet type spaces, Ph.D. thesis, The University of Virginia, 2007.

[20] K. Saxe, Compactness-like operator properties preserved by complex interpolation, *Ark. Mat.* **35** (1997), 353–362.

[21] K. Saxe, On complex interpolation and spectral continuity, *Stud. Math.* **130** (1998), no. 3, 223–229.

[22] M. Tjani, Compact composition operators on some Möbius invariant Banach spaces, Ph.D. thesis, Michigan State University, 1996.

[23] K. Zhu, *Operator Theory in Function Spaces*, 2nd ed., American Mathematical Society, Providence, 2007.

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Received: December 18, 2008
Accepted: January 17, 2009

Operator Theory:
Advances and Applications, Vol. 202, 469–487

Almost Periodic Factorization of 2 × 2 Triangular Matrix Functions: New Cases of Off Diagonal Spectrum

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Dedicated to Professor Israel Gohberg on occasion of his 80th birthday

Abstract. Many known results on almost periodic factorization of almost periodic 2×2 triangular matrix functions of the form $\begin{bmatrix} e^{i\lambda x} & 0 \\ * & e^{-i\lambda x} \end{bmatrix}$ are reviewed from a unified point of view, with particular attention to the case when the off diagonal entry is at most a quadrinomial almost periodic function. New results are obtained on almost periodic factorization for off diagonal entry having its Bohr-Fourier spectrum in a union of two shifted grids, i.e., arithmetic progressions, with the same difference, and perhaps an additional point. When specializing these results to the case of off diagonal almost periodic trinomials, new cases of factorability are obtained. The main technical tool is the Portuguese transformation, a known algorithm.

Mathematics Subject Classification (2000). Primary 47A68; Secondary 42A75.

Keywords. Almost periodic functions, factorization, Portuguese transformation.

1. Introduction

Let APP be the algebra of *almost periodic polynomials*, that is, finite linear combinations of elementary exponential functions

$$e_\lambda(x) := e^{i\lambda x}, \quad x \in \mathbb{R},$$

The research leading to this paper was done while the first author was an undergraduate at the College of William and Mary. All authors were partially supported by NSF grant DMS-0456625.
Communicated by J.A. Ball.

with real λ. The uniform closure of APP is the Bohr algebra AP of almost periodic functions. For each $f \in AP$, the limit

$$M(f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\,dx$$

exists and is called the *Bohr mean value* of f. Since $e_{-\lambda}f \in AP$ along with f for any $\lambda \in \mathbb{R}$, there also exist the *Bohr-Fourier coefficients* $\widehat{f}(\lambda) := M(e_{-\lambda}f)$. The set

$$\Omega(f) = \{\lambda \in \mathbb{R} \colon \widehat{f}(\lambda) \neq 0\},$$

is at most countable, and is called the *Bohr-Fourier spectrum* of $f \in AP$. We say that $f \in APW$ if its *Bohr-Fourier series*

$$\sum_{\lambda\in\Omega(f)} \widehat{f}(\lambda)e_\lambda$$

converges absolutely. Of course, APW is just the closure of APP with respect to the norm

$$\|\sum_j c_j e_{\lambda_j}\|_W := \sum_j |c_j|, \qquad c_j \in \mathbb{C},$$

and as such also is an algebra.

Denote by $AP_\pm$ ($APW_\pm$) the subalgebra of AP (respectively, APW) consisting of all functions f with $\Omega(f) \subset \pm[0,\infty)$. If X is an algebra of scalar valued functions, we denote by $X^{m\times m}$ the algebra of $m \times m$ matrices with entries in X.

A (left) *AP factorization* of an $n \times n$ matrix function G is a representation

$$G(x) = G_+(x)\Lambda(x)G_-(x) \tag{1.1}$$

such that Λ is a diagonal matrix $\mathrm{diag}[\mathrm{e}_{\lambda_1}, \dots, \mathrm{e}_{\lambda_n}]$ for some real numbers λ_j, and $G_+^{\pm1} \in AP_+^{n\times n}$, $G_-^{\pm1} \in AP_-^{n\times n}$. The numbers $\lambda_1, \dots, \lambda_n$ are called the (left) AP indices, and are uniquely determined up to a permutation. Representation (1.1) is a (left) *APW factorization* of G if in fact $G_+^{\pm1} \in APW_+^{n\times n}$, $G_-^{\pm1} \in APW_-^{n\times n}$, and a (left) *APP factorization* if in addition the entries of $G_\pm^{\pm1}$ are in APP.

A canonical (left) AP or APW factorization is one such that $\Lambda(x) = I$ in (1.1), i.e., all AP indices are zero. We say that G is AP (APW) *factorable* if it admits an AP (APW) factorization; the notion of AP (APW, APP) canonically factorable G is introduced in a similar way.

Of course, G must be an invertible element of $AP^{n\times n}$ ($APW^{n\times n}$, $APP^{n\times n}$) in order to be AP (respectively, APW, APP) factorable. It is a rather deep result in AP factorization theory that a canonical AP factorization of $G \in APW^{n\times n}$ is automatically an APW factorization, see [3, Corollary 10.7].

Of particular interest is AP factorization of 2×2 matrices of the form

$$G_f^{(\lambda)} := \begin{bmatrix} e_\lambda & 0 \\ f & e_{-\lambda} \end{bmatrix}, \quad f \in APW. \tag{1.2}$$

Factorizations of matrices of the form G_f play a key role in studies of corona theorems and of convolution type (in particular, linear difference) equations on finite intervals, for example (see, e.g., [3]).

The literature on AP factorization, and in particular on AP factorization of triangular functions of the form (1.2), is voluminous. We mention here the book [3], where many references (up to 2002) may be found, and more recent [8, 5, 7, 4].

By [3, Proposition 13.4], AP (or APW) factorability of $G_f^{(\lambda)}$ with $f \in APW$ is equivalent to that of $G^{(\lambda)}_{P_{(-\lambda,\lambda)}f}$, where

$$P_{(-\lambda,\lambda)}\left(\sum_{\mu\in S} c_\mu e_\mu\right) = \sum_{\mu\in S\cap(-\lambda,\lambda)} c_\mu e_\mu .$$

Therefore, in our discussions we will often replace f with $P_{(-\lambda,\lambda)}f$. In particular, f can be replaced with 0 whenever $\lambda \le 0$, which implies APW factorability of $G_f^{(\lambda)}$ with AP indices equal $\pm\lambda$. To avoid trivialities, we will therefore assume $\lambda > 0$.

Another simple but useful fact regarding factorization properties of matrices (1.2) is given in [3, Section 13.2] and states that they are exactly the same as those of $G_{f^*}^{(\lambda)}$, where

$$f^*(x) = \overline{f(x)}.$$

In this paper, we will consider matrix functions of the form $G_f^{(\lambda)}$ with $f \in APP$. We obtain several new classes of matrix functions whose APW factorability can be determined combining a recent result from [1] with the so-called Portuguese transformation. The latter provides an algorithm for constructing $G_{f_1}^{(\lambda_1)}$ from $G_f^{(\lambda)}$, where $\lambda_1 = -\min\Omega(f) < \lambda$, such that the two matrix functions are AP (APW) factorable only simultaneously and, in case when they are, have the same AP indices. The Portuguese transformation was introduced and used for the first time in [2]. It is covered in detail in [3, Chapter 13].

The paper is organized as follows. In Section 2, we describe the algorithm known as the Portuguese transformation, which we use to obtain the new results that follow. Section 3 contains a unified description of essentially known results concerning factorization of $G_f^{(\lambda)}$ with a quadrinomial f having its Bohr-Fourier spectrum in a set of the form $\{\beta-\lambda, \alpha-\lambda, \beta, \alpha\}$. Here, a result from [1] plays a key role. In Section 4, we introduce the notion of λ-admissibility of sets, and summarize known results in terms of this notion. In Section 5, we present new results concerning factorization of $G_f^{(\lambda)}$ which can be obtained from Section 3 by using one or two applications of the Portuguese transformation. In a short Section 6, we present formulas for computation of the geometric mean in cases of canonical factorization. Finally, in Section 7, we consider applications to a new class of trinomials whose factorability is determined by the results in Section 5.

The following notation will be used throughout:

$\mathbb{R}$ the real numbers
$\mathbb{R}_+$ the nonnegative real numbers
$\mathbb{R}_-$ the nonpositive real numbers
$\mathbb{R}_+^k$ the k-tuples of nonnegative real numbers
$\mathbb{C}$ the complex numbers
$\mathbb{N}$ the natural numbers
$\mathbb{Z}_+$ the nonnegative integers
$\mathbb{Z}_+^k$ the k-tuples of nonnegative integers
$|\mathbf{n}| := n_1 + \cdots + n_k$ for $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{Z}_+^k$
$(\mathbf{n}, \boldsymbol{\gamma}) := n_1\gamma_1 + \cdots n_k\gamma_k$, where

$$\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{Z}_+^k, \quad \boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_k) \in \mathbb{R}_+^k.$$

$\mathbb{Q}$ the rational numbers
$\lfloor x \rfloor$ the largest integer not exceeding $x \in \mathbb{R}$
$\lceil x \rceil = -\lfloor -x \rfloor$ the smallest integer not exceeded by $x \in \mathbb{R}_+$.

2. Portuguese transformation

In the Portuguese transformation, we are interested in reducing the factorization of $G_f^{(\lambda)}$ to the factorization of some other matrix function $G_{f_1}^{(\nu)}(x)$, where $\nu < \lambda$. Towards this end, we seek to construct an invertible in $APP_+^{2\times 2}$ matrix function

$$\begin{bmatrix} u & v \\ g_1 & g_2 \end{bmatrix}$$

such that

$$\begin{bmatrix} u & v \\ g_1 & g_2 \end{bmatrix} G_f^{(\lambda)} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = G_{f_1}^{(\nu)}. \tag{2.1}$$

Then, factorability of $G_f^{(\lambda)}$ would be equivalent to that of $G_{f_1}^{(\nu)}$. Condition (2.1) is equivalent to the following system

$$ve_{-\lambda} = e_\nu, \quad g_2 e_{-\lambda} = f_1, \quad ue_\lambda + vf = 0, \quad g_1 e_{\lambda+\nu} + g_2 e_\nu f = 1.$$

As described in [3, Section 13.3], we can deduce the following conditions

$$u = -e_\nu f, \quad v = e_{\lambda+\nu}, \quad \Omega(e_\nu f) \subset [0, \infty),$$

with $\nu \in (-\lambda, \lambda)$ and g_1, $g_2 \in APW_+$.

A complete description of the Portuguese transformation in the almost periodic polynomial case is given by the following theorem; see again [3] for details.

Theorem 2.1. *Let $f \in APP$ with $\nu = -\min\{\Omega(f) \in (-\lambda, \lambda)\}$, that is*

$$f(x) = ae_{-\nu}\left(1 - \sum_k b_k e_{\gamma_k}\right), \tag{2.2}$$

where $a \neq 0$ and $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_m < \lambda + \nu$. Then

(a) *the Portuguese transformation applied to* $G_f^{(\lambda)}$ *yields* $G_{f_1}^{(\nu)}$, *where* $f_1 = 0$ *if* $\nu \le 0$ *and*

$$f_1(x) = a^{-1} \cdot \sum_{\boldsymbol{n}} c_{\boldsymbol{n}} e_{(\boldsymbol{n},\boldsymbol{\gamma})-\lambda} \in APP, \quad \boldsymbol{\gamma} := (\gamma_1, \gamma_2, \ldots, \gamma_m) \tag{2.3}$$

if $\nu > 0$. *Here*

$$c_{\boldsymbol{n}} = \frac{|\boldsymbol{n}|!}{n_1! \ldots n_m!} b_1^{n_1} \ldots b_m^{n_m}, \tag{2.4}$$

with the sum in (2.3) *taken over all* $\boldsymbol{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m$ *such that* $-\nu < (\boldsymbol{n}, \boldsymbol{\gamma}) - \lambda < \nu$.

Consequently,

(b) *If* $\nu \le 0$, *then the matrix function* $G_f^{(\lambda)}$ *is APP factorable with AP indices equal* $\pm\nu$. *If* $\nu > 0$, *then* $G_f^{(\lambda)}$ *admits an AP* (*APW*, *APP*) *factorization only simultaneously with – and has the same AP indices as –* $G_{f_1}^{(\nu)}$.

It will be convenient to use the notation

$$\mathcal{PT}(G_f^{(\lambda)}) = G_{f_1}^{(\nu)}$$

for the Portuguese transformation as in Theorem 2.1.

Note that the part of Theorem 2.1 corresponding to the case $\nu \le 0$, combined with the transition from f to f^* if necessary, implies *APW* factorablity of $G_f^{(\lambda)}$ in the so-called *one-sided case*. This is the case when $\Omega(f)$ lies to the one side of the origin, and the *AP* indices in this case equal $\pm\,\mathrm{dist}\{\Omega(f), 0\}$. The one-sided case was first disposed of in [11] by a different method, and then (with the use of the Portuguese transformation) in [15]. See also [3, Theorem 14.1].

3. Quadrinomial off diagonal entry

In this section we collect some essentially known results, stated in the form and detail that will be needed later, concerning *APW* factorization of matrices $G_f^{(\lambda)}$, where f belongs to a certain class of almost periodic quadrinomials. Namely, we assume throughout this section that

$$\Omega(f) \subset \{\beta - \lambda, \alpha - \lambda, \beta, \alpha\}, \tag{3.1}$$

and $0 < \beta \le \lambda$, $0 < \alpha \le \lambda$, $\alpha \ne \beta$. We write such a function f as

$$f = c_{-2}e_{\beta-\lambda} + c_{-1}e_{\alpha-\lambda} + c_2 e_\beta + c_1 e_\alpha. \tag{3.2}$$

We say that f is a *proper quadrinomial*, resp. *proper trinomial* or *proper binomial*, if $\Omega(f)$ consists of exactly four, resp., three or two, elements.

Note that the parameter α (respectively, β) in (3.2) is not defined uniquely if $c_1 = c_{-1} = 0$ (resp., $c_2 = c_{-2} = 0$). In this case we agree to choose α, resp. β, in such a way that $(\alpha - \beta)/\lambda$ is rational. Under this convention, and also agreeing that $0^0 = 1$, the following result holds.

Theorem 3.1. *Suppose f is of the form* (3.2). *Then:*

(a) *If $\frac{\alpha-\beta}{\lambda}$ is rational, then $G_f^{(\lambda)}$ is APP factorable. Its factorization is canonical if and only if*

$$(-1)^n c_1^{n-m-k} c_{-1}^{m+k} \neq c_2^{n-k} c_{-2}^{k} \quad \text{for} \quad k = k_1, k_2, \tag{3.3}$$

where $\frac{m}{n} = \frac{\alpha-\beta}{\lambda}$ in lowest terms (with $n \in \mathbb{N}$), and $k_1 = \lfloor n\beta/\lambda \rfloor$, $k_2 = \lceil n\beta/\lambda \rceil$.

(b) *If $\frac{\alpha-\beta}{\lambda}$ is irrational, then the condition*

$$|c_1|^{\lambda-\alpha}|c_{-1}|^{\alpha} \neq |c_2|^{\lambda-\beta}|c_{-2}|^{\beta} \tag{3.4}$$

is necessary and sufficient for $G_f^{(\lambda)}$ to admit a canonical APW factorization. If (3.4) *fails with $c_{-1} = c_{-2} = 0$, or $c_2 = 0$, $\beta \neq \lambda$, or $c_1 = 0$, $\alpha \neq \lambda$, then $G_f^{(\lambda)}$ admits a non-canonical APP factorization. Finally, $G_f^{(\lambda)}$ is not AP factorable in all other cases when* (3.4) *does not hold.*

Proof. Since $m + k_1 = \lfloor n\alpha/\lambda \rfloor$, conditions of the theorem are invariant under the relabeling $\alpha \longleftrightarrow \beta$, $c_{\pm 1} \longleftrightarrow c_{\pm 2}$. Thus, we may without loss of generality suppose that $0 < \beta < \alpha \leq \lambda$.

(a) For $\frac{\alpha-\beta}{\lambda} = \frac{m}{n}$, the Bohr-Fourier spectrum of f lies on the grid $-\nu + h\mathbb{Z}_+$, where $\nu = \lambda - \beta$ and $h = \lambda/n$. This is the so-called *commensurable case*, and *APW* factorability of $G_f^{(\lambda)}$ is guaranteed by [12, Theorem 3.1], see also [3, Theorem 14.13]. The *APP* factorability was not stated in [12, 3] explicitly (since the notion itself was not introduced then) but it can be derived easily from the explicit factorization procedures used there.

Moving on to the canonical factorability criterion, note that it is actually available for any f satisfying $\Omega(f) \subset -\nu + h\mathbb{Z}_+$, not just for a quadrinomial (3.2). According to [14, Theorem 3.1] (see also [3, Theorem 14.14]), in somewhat different (but more convenient for our purposes) notation it can be stated as follows.

Write f as

$$f = \sum_j t_j e_{\tau + jh}, \tag{3.5}$$

where $\tau := \beta - k_1 h$ is the smallest non-negative point of $-\nu + h\mathbb{Z}_+$, and introduce $n \times n$ Toeplitz matrices

$$T_1 = [t_{j-i}]_{i,j=0}^{n-1}, \quad T_2 = [t_{j-i-1}]_{i,j=0}^{n-1}.$$

The *AP* factorization of the matrix $G_f^{(\lambda)}$ with f given by (3.5) is canonical if and only if $\det T_1 \neq 0$ (for $\tau = 0$) and $\det T_1 \det T_2 \neq 0$ (for $\tau > 0$).

Observe now that

$$\beta = \tau + k_1 h,\ \alpha = \tau + (k_1 + m)h,\ \alpha - \lambda = \tau + (k_1 + m - n)h, \text{ and } \beta - \lambda = \tau + (k_1 - n)h.$$

Consequently, for f of the form (3.2), the matrices T_j have at most four non-zero diagonals. Namely, T_1 contains the entry c_1 everywhere in its $(k_1 + m)$th diagonal, c_2 in the k_1th, c_{-2} in the $(k_1 - n)$th, and c_{-1} in the $(k_1 + m - n)$th diagonal. (The diagonals are parallel to the main diagonal; they are counted from the lower left corner to the upper right corner, with the main diagonal being the 0th diagonal.) Elementary row and column operations lead to the conclusion that in this case

$$(-1)^{k_1(n-k_1)} \det T_1 = c_2^{n-k_1} c_{-2}^{k_1} - (-1)^n c_1^{n-m-k_1} c_{-1}^{m+k_1},$$

with a convention that zero number of multiples yields the product equal to 1. In more detail, the matrix T_1 is cut vertically and the left portion is moved to the right, so that the resulting matrix has three nonzero diagonals, one of them being the main diagonal.

If $\tau = 0$, then $k_1 = k_2$, which makes condition $\det T_1 \neq 0$ equivalent to (3.3). For $\tau > 0$, on the other hand, $k_2 = k_1 + 1$ and the matrix T_2 contains the entry c_1 everywhere in its $(k_2 + m)$th diagonal, c_2 in the k_2th, c_{-2} in the $(k_2 - n)$th, and c_{-1} in the $(k_2 + m - n)$th diagonal. Then, similarly to what we had for T_1,

$$(-1)^{k_2(n-k_2)} \det T_2 = c_2^{n-k_2} c_{-2}^{k_2} - (-1)^n c_1^{n-m-k_2} c_{-1}^{m+k_2}.$$

This makes condition $\det T_1 \det T_2 \neq 0$ equivalent to (3.3) when $\tau > 0$.

(b) According to our convention, at least one coefficient in each pair $\{c_{\pm 1}\}$ and $\{c_{\pm 2}\}$ is different from zero, and f is at least a binomial. The remaining possibilities are as follows.

(b-i) f is a proper binomial, which can happen in four possible ways:

$$f = c_{-2}e_{\beta-\lambda} + c_1 e_\alpha,\ c_{-2}c_1 \neq 0, \text{ or } f = c_{-1}e_{\alpha-\lambda} + c_2 e_\beta,\ c_{-1}c_2 \neq 0, \tag{3.6}$$

and

$$f = c_{-2}e_{\beta-\lambda} + c_{-1}e_{\alpha-\lambda},\ c_{-2}c_{-1} \neq 0, \text{ or } f = c_1 e_\alpha + c_2 e_\beta,\ c_1 c_2 \neq 0. \tag{3.7}$$

For any binomial f, the distances between the points in $\Omega(f)$ are commensurable, so that the matrix function $G_f^{(\lambda)}$ is APP factorable.

If (3.6) holds, then the distance between the exponents of f equals $\lambda \pm (\alpha - \beta)$ which is not commensurable with λ. According to [3, Theorem 14.10], the AP factorization of $G_f^{(\lambda)}$ in this situation is canonical if and only if one of the exponents of f is zero, which in our setting means that f is given by the second formula in (3.6) and $\alpha = \lambda$. This is exactly the situation when (3.4) holds. All other cases covered by (3.6) are listed in the statement of the theorem as those for which a non canonical APP factorization exists, which is indeed the case.

If (3.7) holds, then we are in the one-sided setting, when the AP factorization is canonical if and only if one of the exponents is equal to zero. This does not occur in the second case of (3.7), since $\alpha, \beta \neq 0$. In agreement with this observation, condition (3.4) fails. The first case of (3.7) corresponds to a canonical AP factorization if and only if $\alpha = \lambda$ (recall that without loss of generality we have imposed the condition $\beta < \alpha$). Once again, this is exactly when (3.4) holds, and the cases when it does not are listed in the statement of the theorem as admitting a non canonical APP factorization.

The binomial case is settled.

(b-ii) $\alpha = \lambda$ and $c_{-2}c_{-1}c_2 \neq 0$. Condition (3.4) takes the form

$$|c_2|^{\lambda-\beta}\,|c_{-2}|^{\beta} \neq |c_{-1}|^{\lambda} \tag{3.8}$$

(not surprisingly, the coefficient c_1 corresponding to the exponent outside of the interval $(-\lambda, \lambda)$ becomes irrelevant). According to [3, Theorem 15.5], the matrix function $G_f^{(\lambda)}$ admits a canonical APW factorization when (3.8) holds and is not AP factorable when this condition fails.

(b-iii) $\alpha \neq \lambda$ and f is a proper trinomial, that is, exactly one of the coefficients $c_{\pm 1}, c_{\pm 2}$ is equal to zero. Then condition (3.4) holds, so that we need to show that $G_f^{(\lambda)}$ admits a canonical AP factorization.

If one of the "inner" coefficients c_{-1} and c_2 equals zero, then one element of $\Omega(f)$ is at a distance exactly λ from the rest of the set. This is the "big gap" case (see [15, Theorem 2.3] or [3, Section 14.2]), in which the canonical factorization exists.

If, on the other hand, an "outer" coefficient c_{-2} or c_1 is equal to zero, the distance between the endpoints of $\Omega(f)$ equals λ and the remaining term is non-zero. According to [14, Theorem 4.6] [1], $G_f^{(\lambda)}$ is canonically factorable.

(b-iv) $\alpha \neq \lambda$ and f is a proper quadrinomial, that is, $c_{-2}c_{-1}c_1c_2 \neq 0$.

In this case the equivalence of (3.4) to the canonical AP factorability of $G_f^{(\lambda)}$ follows from [1, Section 3.2]. This result was obtained in somewhat different terms, namely, as the invertibility criterion for the operator $A = I - qW$ acting on the Lebesgue space $L^2(\mathbb{T})$, where $\mathbb{T}$ is the unit circle. Here W is the rotation operator through the angle $\omega = 2\pi(\alpha - \beta)/\lambda$, and the coefficient q is a piecewise constant function with three points of discontinuity, assuming the values c_{-1}/c_{-2} and $c_{\pm 1}/c_2$. The relation between $G_f^{(\lambda)}$ and the operator A actually allows one to prove that, when A is not invertible, an AP factorization of $G_f^{(\lambda)}$ also does not exist.

To this end, consider any $g \in \operatorname{Ker} A$. Since $g = qWg$, the zero set of g is invariant under the rotation by ω. Due to irrationality of $(\alpha - \beta)/\lambda$, this rotation is ergodic, and therefore either $g = 0$ or g is different from 0 a.e.

For two non-zero functions $g_1, g_2 \in \operatorname{Ker} A$ we may therefore introduce a measurable function $h = g_1/g_2$. Clearly, h is invariant under rotations by the angle ω, and because of that (once again, due to the ergodicity) for any measurable $\Delta \subset \mathbb{C}$ the measure of the inverse image $h^{-1}(\Delta)$ is either 0 or 1. Consequently, h is constant a.e., that is, $\dim \operatorname{Ker} A \leq 1$. But AP factorability of $G_f^{(\lambda)}$ with non-zero AP indices would imply infinite dimensionality of $\operatorname{Ker} A$. Thus, $G_f^{(\lambda)}$ is not AP factorable when condition (3.4) fails. □

Several comments are in order.

[1] See also [3, Theorem 15.9] and its more recent discussion in [9].

Remark 1. The relation between canonical factorability of $G_f^{(\lambda)}$ and invertibility of the operator A was actually used in [1] to derive conditions (3.3) as well. The proof presented here was discussed with the first author of [1] while that paper was in preparation, and is referred to there as an "unpublished manuscript [15]".

On the other hand, for f being at most a binomial, more explicit conditions (though of course equivalent to (3.3)) follow from the already mentioned [3, Theorem 14.10]: the AP factorization of $G_f^{(\lambda)}$ is canonical if and only if $0 \in \Omega(f)$ or the distance d between the endpoints of $\Omega(f) \cap (-\lambda, \lambda)$ is such that $\lambda/d \in \mathbb{N}$.

Furthermore, for a trinomial f with vanishing c_{-1} or c_2 conditions (3.3) hold. This is in accordance with the "big gap" situation. For other two types of trinomials part (a) of Theorem 3.1 implies the following:

Corollary 3.2. *Let*

$$f = d_{-1}e_{-\nu} + d_0 e_\mu + d_1 e_\alpha, \quad d_{-1}d_0d_1 \neq 0, \ \alpha + \nu = \lambda, \ -\nu < \mu < \alpha \tag{3.9}$$

with rational $\frac{\alpha-\mu}{\lambda} = \frac{m}{n}$ *in lowest terms. Then* $G_f^{(\lambda)}$ *admits a canonical factorization, unless*

$$n\,|\mu| < \lambda \text{ and } (-1)^n d_1^{n-m} d_{-1}^m = d_0^n. \tag{3.10}$$

Proof. Due to the invariance of (3.10) under the transition from f to f^*, it suffices to consider the case $\mu \geq 0$. Then, in the notation of Theorem 3.1, $\beta = \mu$, $\lambda - \alpha = \nu$, $d_{\pm 1} = c_{\pm 1}$, $d_0 = c_2$ and $c_{-2} = 0$. If $n\mu \geq \lambda$, then $k_1, k_2 \neq 0$, and condition (3.3) holds automatically. If, on the other hand, $n\mu < \lambda$, then (3.3) holds for $k = k_2 (= 1)$ and is equivalent to negation of the equality in (3.10) for $k = k_1 (= 0)$. □

The case $\mu = 0$ of Corollary 3.2 was covered by [3, Theorem 23.9].

Remark 2. Theorem 23.9 of [3] also contains explicit factorization formulas for matrix functions $G_f^{(\lambda)}$ in the setting of case (b-ii). According to these formulas, the factorization (when exists) is "true" APW (that is, not APP). On the other hand, careful analysis of the explicit construction of the factorization in case (b-iii) shows that it is in fact an APP factorization. The situation in case (b-iv) is to be figured out, though we believe it is similar to that in case (b-ii).

4. λ-admissibility of sets

In this section, we introduce notions of λ-admissibility.

Definition 4.1. A set $\Omega \subset \mathbb{R}$ is said to be λ-*admissible* if for every $f \in APW$ with

$$\sup\{\Omega(f) \cap (-\lambda, 0]\}, \inf\{\Omega(f) \cap [0, \lambda)\} \in \Omega(f) \subseteq \Omega, \tag{4.1}$$

the matrix function $G_f^{(\lambda)}$ is AP factorable.

A set $\Omega \subset \mathbb{R}$ is said to be λ-*conditionally admissible* if a criterion is known, in terms of the Bohr-Fourier spectrum and the coefficients f_μ of any function

$$f = \sum_{\mu \in \Omega} f_\mu e_\mu \in APW, \tag{4.2}$$

with $\Omega(f)$ satisfying (4.1), for $G_f^{(\lambda)}$ to be AP factorable.

By default, every λ-admissible set is automatically λ-conditionally admissible.

More precisely, we require that the criterion for factorability of $G_f^{(\lambda)}$ in Definition 4.1 be given in terms of a finite number of equations and inequalities that involve expressions in f_j formed by combinations of polynomial functions and the function $\log|\cdot|$, where the f_μ's are taken from (4.2). For example, (3.4) can be recast in the form

$$(\lambda - \alpha)\log|c_1| + \alpha \log|c_{-1}| \neq (\lambda - \beta)\log|c_2| + \beta \log|c_{-2}|,$$

(assuming $c_j \neq 0$, $j = \pm 1, \pm 2$). For all known λ-admissible and λ-conditionally admissible sets, these polynomial expressions have been explicitly written down, although often they are unwieldy. Also, for all known λ-admissible and λ-conditionally admissible sets, explicit formulas for the indices are available (in case of factorability).

In the following theorem, we summarize many known results on admissible sets.

Theorem 4.2.

(a) *A set $\Omega \subset \mathbb{R}$ is λ-admissible if one of the following conditions is satisfied:*
   (1) $\Omega \cap (-\lambda, \lambda)$ *is a subset of an arithmetic progression, i.e., the distances between the points in* $\Omega \cap (-\lambda, \lambda)$ *are commensurable;*
   (2) $\Omega \cap (-\lambda, 0) = \emptyset$ *or* $\Omega \cap (0, \lambda) = \emptyset$ *(one-sided case);*
   (3) *there are points* $\mu_1, \mu_2 \in \Omega \cap (-\lambda, \lambda)$ *such that* $\mu_2 - \mu_1 \geq \lambda$ *and* $(\mu_1, \mu_2) \cap \Omega = \emptyset$ *(the big gap case);*
   (4) $\Omega = \{-\nu, \mu, \delta\}$, *where* $-\nu < \mu < \delta$, $\mu \neq 0$, *and* $\nu + \delta + |\mu| \geq \lambda$*;*
   (5) $\Omega = \{-\nu, 0, \delta\}$, *where* $\nu + \delta > \lambda$.

(b) Ω *is λ-conditionally admissible but not λ-admissible if* $\Omega = \{\beta - \lambda, \alpha - \lambda, \beta, \alpha\}$, *where* $\frac{\alpha - \beta}{\lambda} \notin \mathbb{Q}$ *and at least three terms of* Ω *lie strictly between* $-\lambda$ *and* λ.

Proof. Part (a), (1) and (2) follow from the results stated in [3, Chapter 14]; the same Chapter also covers part (3) in the particular case when either $(-\lambda, \mu_1)$ or (μ_2, λ) is disjoint with Ω. Part (4) (under a stronger condition $\nu + \delta \geq \lambda$ and (5) follow from [3, Chapter 15]. For the full strength versions of (3) and (4), see [6] and [15, Theorem 4.6], respectively.

Part (b) is a restatement of a part of Theorem 3.1. Note that if $\Omega \cap (-\lambda, \lambda)$ is in fact a triplet, then it is of the form $(-\nu, 0, \delta)$ with $\nu + \delta = \lambda$, and the result also follows from [3, Section 15.1]. $\square$

Remark 3. Lifting the "$\in$" part of condition (4.1) makes questionable the *AP* factorability of matrix functions $G_f^{(\lambda)}$ with $\Omega(f) \subset \Omega$ and Ω as in parts (2), (3) of Theorem 4.2. It is not known, in particular, whether every matrix function $G_f^{(\lambda)}$ with $f \in APW \setminus APP$ and $\Omega(f) \subset (0, \lambda)$ is *AP* factorable (simple stability argument shows that its *AP* factorization, if exists, is not canonical). This is our reason for including the "$\in$" part of condition (4.1) in the definition of λ-admissibility.

The usefulness of the notions of λ-factorability and λ-conditional factorability lies in their persistence under the Portuguese transformation. We now formalize this feature. Let

$$\Omega \cap (-\lambda, \lambda) = \{\omega_1 < \omega_2 < \cdots < \omega_t\}$$

be a finite set, and let $\omega_1, \omega_2, \ldots, \omega_s$ be all the negative numbers in Ω. For $j = 1, 2, \ldots, s$, let

$$\boldsymbol{\gamma}^{(j)} = \{\omega_{j+1} - \omega_j, \omega_{j+2} - \omega_j, \ldots, \omega_t - \omega_j\},$$

and

$$\Omega^{(j)} = \{(\boldsymbol{n}, \boldsymbol{\gamma}^{(j)}) - \lambda \, : \, \boldsymbol{n} \in \mathbb{Z}_+^{t-j}\} \cap (\omega_j, -\omega_j).$$

With this notation, we have the following result:

Theorem 4.3. *Assume that $\Omega^{(j)}$ is $-\omega_j$-admissible for all $j = 1, 2, \ldots, s$, resp., $-\omega_j$-conditionally admissible for all $j = 1, 2, \ldots, s$. Then Ω is λ-admissible, resp., λ-conditionally admissible.*

Proof. For $s = 0$ the sets are void. This is in agreement with the fact that then Ω is one-sided. For $s > 0$, consider $f \in APW$ with $\Omega(f) \subseteq \Omega$. If $\Omega(f) \cap (-\lambda, 0) = \emptyset$, the matrix function $G_f^{(\lambda)}$ is *APW* factorable, this being the one-sided case. Otherwise, let ω_j be the smallest point in $\Omega(f) \cap (-\lambda, 0)$. According to formula (2.3), the Portuguese transformation applied to $G_f^{(\lambda)}$ yields the matrix function $G_{f_1}^{(-\omega_j)}$ with $\Omega(f_1) \subset \Omega^{(j)}$. The rest is clear. $\square$

In the next section, we will combine Theorem 4.2 (especially part (b) of it) and Theorem 4.3 to obtain new information about factorability of matrix functions of the form $G_f^{(\lambda)}$.

5. Main results

In this section we consider a class of matrix functions that can be reduced to the case in Theorem 3.1 by means of one or two applications of the Portuguese transformation. Throughout this section we make a blanket assumption that all parameters denoted by lowercase Greek letters are positive.

Theorem 5.1. *The set*

$$\Omega_1 := (-\nu + \nu\mathbb{Z}_+) \cup (\alpha + \nu\mathbb{Z}_+) \tag{5.1}$$

is λ-admissible if ν/α is rational or $\max\{\alpha,\nu\} \geq \lambda$*, and λ-conditionally admissible if*

$$\lambda \leq \nu + 2\alpha. \tag{5.2}$$

Proof. Rationality of ν/α implies that the distances between the points of Ω_1 are commensurable. For $\alpha \geq \lambda$,

$$\Omega_1 \cap (-\lambda,\lambda) = (-\nu + \nu\mathbb{Z}_+) \cap (-\lambda,\lambda),$$

and for $\nu \geq \lambda$ simply $\Omega_1 \cap (-\lambda,\lambda) = \{0,\alpha\}$. Either way, case (1) of Theorem 4.2 applies.

Suppose now that (5.2) holds and consider a function $f \in APW$ with $\Omega(f)$ contained in Ω_1. An additional condition $\widehat{f}(-\nu) = 0$ makes the set $\Omega(f)$ nonnegative, and the matrix function $G_f^{(\lambda)}$ APP factorable. It remains therefore to consider the case $-\nu \in \Omega(f)$.

Under this condition and in accordance with (2.3), the Portuguese transformation of $G_f^{(\lambda)}$ is the matrix function $G_{f_1}^{(\nu)}$ with

$$\Omega(f_1) \subset \{n_1\nu + n_2\alpha - \lambda \colon n_1, n_2 \in \mathbb{Z}_+,\ n_2 \leq n_1\} \cap (-\nu,\nu).$$

Due to (5.2), $n_1\nu + n_2\alpha - \lambda \geq \nu$ whenever $(n_1 \geq)\ n_2 \geq 2$. So, all the terms of $\Omega(f_1)$ are of the form $n_1\nu + n_2\alpha - \lambda$ with $n_2 = 0, 1$. Moreover, for a fixed n_2 there are at most two values of n_1 for which $n_1\nu + n_2\alpha - \lambda$ falls between $-\nu$ and ν:

$$n_1 = \ell, \ell+1 \text{ for } n_2 = 0 \text{ and } n_1 = p, p+1 \text{ for } n_2 = 1,$$

where

$$\ell = \lfloor \lambda/\nu \rfloor, \quad p = \lfloor (\lambda-\alpha)/\nu \rfloor. \tag{5.3}$$

Consequently,

$$\Omega(f_1) \subset \{\ell\nu - \lambda, p\nu + \alpha - \lambda, (\ell+1)\nu - \lambda, (p+1)\nu + \alpha - \lambda\}. \tag{5.4}$$

Note that $\ell\nu = p\nu + \alpha$ and f_1 is therefore at most a binomial, provided that $\alpha/\nu \in \mathbb{N}$. This of course implies AP factorability of $G_{f_1}^{(\nu)}$ (and therefore of $G_f^{(\lambda)}$) but this case is already covered by the first alternative of the theorem.

For a non-integer α/ν (rational or not), the righthand side of (5.4) is ν-conditionally admissible due to Theorem 4.2, part (1), (4), or (b). In view of Theorem 4.3, the result follows. □

Of course, it is possible to work out the explicit factorability conditions for matrix functions $G_f^{(\lambda)}$ with $\Omega(f) \subset \Omega_1$. This requires computing the coefficients of f_1 resulting from the Portuguese transformation.

To this end, define $\mathbf{a}_m = (1,2,3,\ldots,m) \in \mathbb{Z}_+m$. With this notation, the set

$$\{\mathbf{n} \in \mathbb{Z}_+^m \colon (\mathbf{n},\mathbf{a}_m) = m\}$$

represents all positive integer partitions of the number $m \in \mathbb{N}$.

For the function $f \in APW$ given by

$$f = \sum_{k=-1}^{\infty} d_k e_{k\nu} + \sum_{k=0}^{\infty} \tilde{d}_k e_{k\nu+\alpha}, \quad d_k, \tilde{d}_k \in \mathbb{C},\ d_{-1} \neq 0 \tag{5.5}$$

after one application of the Portuguese transformation we have

$$f_1 = c_{-2} e_{\ell\nu-\lambda} + c_{-1} e_{p\nu+\alpha-\lambda} + c_2 e_{(\ell+1)\nu-\lambda} + c_1 e_{(p+1)\nu+\alpha-\lambda}, \tag{5.6}$$

where ℓ, p are given by (5.3) and the c_i's, according to (2.3) in our setting, are as follows:

$$c_{-2} = \sum_{(\mathbf{n},\mathbf{a}_\ell)=\ell} (-1)^{|\mathbf{n}|} \frac{|\mathbf{n}|!}{n_1! n_2! \dots n_\ell!} \frac{d_0^{n_1} d_1^{n_2} \dots d_{\ell-1}^{n_\ell}}{d_{-1}^{1+|\mathbf{n}|}} \tag{5.7}$$

$$c_{-1} = \frac{-\tilde{d}_{p-1}}{d_{-1}^2} + \sum_{k=1}^{p-1} \sum_{(\mathbf{n},\mathbf{a}_k)=k} (-1)^{1+|\mathbf{n}|} \frac{(1+|\mathbf{n}|)!}{n_1! n_2! \dots n_k!} \frac{\tilde{d}_{p-k-1} d_0^{n_1} d_1^{n_2} \dots d_{k-1}^{n_k}}{d_{-1}^{2+|\mathbf{n}|}} \tag{5.8}$$

$$c_1 = \frac{-\tilde{d}_p}{d_{-1}^2} + \sum_{k=1}^{p} \sum_{(\mathbf{n},\mathbf{a}_k)=k} (-1)^{1+|\mathbf{n}|} \frac{(1+|\mathbf{n}|)!}{n_1! n_2! \dots n_k!} \frac{\tilde{d}_{p-k} d_0^{n_1} d_1^{n_2} \dots d_{k-1}^{n_k}}{d_{-1}^{2+|\mathbf{n}|}} \tag{5.9}$$

$$c_2 = \sum_{(\mathbf{n},\mathbf{a}_{\ell+1})=\ell+1} (-1)^{|\mathbf{n}|} \frac{|\mathbf{n}|!}{n_1! n_2! \dots n_{\ell+1}!} \frac{d_0^{n_1} d_1^{n_2} \dots d_\ell^{n_{\ell+1}}}{d_{-1}^{1+|\mathbf{n}|}}. \tag{5.10}$$

Of course, the size of the q-tuple $\mathbf{n} = (n_1, \dots, n_q) \in \mathbb{Z}_+^q$, $q = k, \ell, \ell+1$, in these formulas is such that the scalar products involved make sense.

For simplicity, we state the respective result only in the case of irrational α/ν. Note that this is the most interesting case, because otherwise the results for the commensurable situation are applicable.

Theorem 5.2. *Let f be given by* (5.5). *Assuming that $\alpha/\nu \notin \mathbb{Q}$ and* (5.2) *holds, the matrix function $G_f^{(\lambda)}$ admits a canonical AP factorization if and only if*

$$|c_1|^{\lambda-\alpha-p\nu} |c_{-1}|^{(p+1)\nu+\alpha-\lambda} \neq |c_2|^{\lambda-\ell\nu} |c_{-2}|^{(\ell+1)\nu-\lambda}, \tag{5.11}$$

where ℓ, p are given by (5.3) *and c_j by* (5.7)–(5.10), *respectively. If* (5.11) *fails with $c_{-2} = 0$, $\lambda/\nu \notin \mathbb{N}$ or $c_{-1} = 0$, $(\lambda-\alpha)/\nu \notin \mathbb{N}$, then $G_f^{(\lambda)}$ admits a non-canonical APP factorization, and $G_f^{(\lambda)}$ is not AP factorable in all other cases when* (5.11) *does not hold.*

Proof. One needs only to apply Theorem 3.1 to $G_{f_1}^{(\nu)} = \mathcal{PT}(G_f^{(\lambda)})$, taking into consideration the changes $\lambda \to \nu$, $\alpha \to (p+1)\nu + \alpha - \lambda$, $\beta \to (l+1)\nu - \lambda$. □

Theorem 5.2 under the additional condition $(\lambda - \alpha)/\nu \in \mathbb{N}$ (respectively, $\lambda/\nu \in \mathbb{N}$) goes into Theorem 4.1 (respectively, 4.2) of [2]. Of course, irrationality of α/ν then implies that the inequality (5.2) must be strict, as it indeed was supposed in [2, Section 4].

Though formulas (5.7)–(5.10) look cumbersome, for low values of p and ℓ they yield rather explicit factorability criteria.

Example. Suppose that f is given by the formula (5.5) in which

$$\lambda < 2\nu, \qquad \alpha + \nu < \lambda < \alpha + 2\nu.$$

Then $\ell = p = 1$, while formulas (5.7)–(5.10) take the form

$$c_{-2} = -\frac{d_0}{d_{-1}^2}, \ c_{-1} = -\frac{\tilde{d}_0}{d_{-1}^2}, \ c_1 = \frac{2d_0\tilde{d}_0}{d_{-1}^3} - \frac{\tilde{d}_1}{d_{-1}^2}, \ c_2 = \frac{d_0^2}{d_{-1}^3} - \frac{d_1}{d_{-1}^2}.$$

According to Theorem 5.2 the matrix function $G_f^{(\lambda)}$ admits a canonical AP factorization if and only if

$$|c_1|^{\lambda-\alpha-\nu}|c_{-1}|^{2\nu+\alpha-\lambda} \neq |c_2|^{\lambda-\nu}|c_{-2}|^{2\nu-\lambda},$$

a non-canonical APP factorization if $c_{-2} = c_1c_{-1} = 0$ or $c_{-1} = c_2c_{-2} = 0$, and is not AP factorable in all other cases.

We now move to the case when

$$\Omega(f) \subset \{-\nu\} \cup \{k\alpha + j\beta + \lambda - \nu \ : \ k+1 \geq j \geq 0, \ k, j \in \mathbb{Z}\}$$

with $\nu \leq \alpha + 2\beta$. Since under these conditions only the terms corresponding to $j = 0, 1$ can possibly lie to the left of λ, we may without loss of generality suppose that in fact $\Omega(f) \subset \Omega_2$, where

$$\Omega_2 = \{-\nu\} \cup (\lambda - \alpha - \nu + \alpha\mathbb{Z}_+) \cup (\lambda + \beta - \nu + \alpha\mathbb{Z}_+), \quad \nu \leq \alpha + 2\beta. \tag{5.12}$$

Lemma 5.3. *The set Ω_2 given by* (5.12) *is λ-admissible if*

(i) $\alpha \geq \lambda \geq \nu$, *or*
(ii) $\lambda > \nu$, $(\alpha + \beta)/\lambda \in \mathbb{Q}$,

and it is conditionally λ-admissible if

(iii) $\alpha = \lambda < \nu$.

Proof. (i) Condition $\alpha \geq \lambda$ guarantees that the interval $(-\nu, \lambda - \nu)$ is disjoint with Ω_2. This makes part (2) or (3) of Theorem 4.2 applicable, if respectively $\nu = \lambda$ or $\nu < \lambda$.
(ii) Follows from part (1) of Theorem 4.2.
(iii) Due to (ii), it suffices to consider the case when β/λ is irrational. But then statement (b) of Theorem 4.2 applies. $\square$

The (conditional) λ-admissibility of the set (5.12) in general is not known. However, the following result is helpful in determining the factorability of matrices $G_f^{(\lambda)}$ with

$$-\nu \in \Omega(f) \subset \Omega_2 \tag{5.13}$$

under some additional requirements.

Theorem 5.4. *Let in* (5.12) $\alpha < \lambda$, $\nu < \lambda$, *and let* $G_{f_1}^{(\nu)} = \mathcal{PT}(G_f^{(\lambda)})$ *be the Portuguese transformation of the matrix function* $G_f^{(\lambda)}$ *for some* $f \in APW$ *satisfying* (5.13)*. Then* $\Omega(f_1)$ *is* ν*-conditionally admissible* (*and therefore the factorability properties of* $G_f^{(\lambda)}$ *can be determined*) *if* $\lambda \geq \nu + 2\alpha$, *or* $\lambda/\alpha \in \mathbb{N}$, *or* $\lambda = r\alpha + \beta$ *for some integer* $r \geq 2$.

Proof. Due to (5.12) and (5.13), the distances from the leftmost point of $\Omega(f)$ to the remaining terms of this set are all of the form $\lambda - \alpha + k\alpha$ and $\lambda + \beta + k\alpha$, $k \in \mathbb{Z}_+$. Theorem 2.1 then implies that

$$\Omega(f_1) \subset \{\{-\alpha, \beta\} + k_1(\lambda - \alpha) + k_2\alpha \colon k_1, k_2 \in \mathbb{Z}_+\} \cap (-\nu, \nu). \tag{5.14}$$

Case 1. $\lambda \geq \nu + 2\alpha$. Then terms with $k_1 \neq 0$ in (5.14) fall to the right of ν. Therefore, in fact

$$\Omega(f_1) \subset \{\{-\alpha, \beta\} + k\alpha \colon k \in \mathbb{Z}_+\}. \tag{5.15}$$

With an obvious change of notation, this is the setting of Theorem 5.1.

Case 2. $\lambda/\alpha \in \mathbb{N}$. Due to Lemma 5.3, we need to consider only the case of $\lambda = r\alpha$ with $r \geq 2$. Then

$$k_1(\lambda - \alpha) + k_2\alpha = (k_1(r-1) + k_2)\alpha,$$

so that (5.14) again implies (5.15).

Case 3. $\lambda = r\alpha + \beta$, $r \geq 2$. Then

$$k_1(\lambda - \alpha) + k_2\alpha = (k_1(r-1) + k_2)\alpha + k_1\beta = m_1\alpha + m_2(\alpha + \beta),$$

where $m_1, m_2 \in \mathbb{Z}_+$. Since

$$\left.\begin{array}{r} -\alpha + m_1\alpha + m_2(\alpha+\beta) \\ \beta + m_1\alpha + m_2(\alpha+\beta) \end{array}\right\} \geq \alpha + 2\beta \ (\geq \nu) \text{ when } \left\{\begin{array}{c} m_2 \geq 2 \\ m_2 \geq 1, \end{array}\right.$$

the only terms possibly in $\Omega(f_1)$ are $-\alpha + m_1\alpha$ and $\beta + m_1\alpha$. Thus, once again, (5.15) holds. $\square$

6. On the geometric mean computations

The (left) *geometric mean* $\tilde{\mathbf{d}}(G)$ is defined for any matrix function G admitting a canonical AP factorization. Namely, for $G = G_+G_-$,

$$\tilde{\mathbf{d}}(G) = M(G_+)M(G_-),$$

where the Bohr mean value in the matrix case is understood element-wise.

The geometric means of AP matrix functions play the crucial role in Fredholm theory of Toeplitz and Wiener-Hopf operators with semi almost periodic symbols, see, e.g., [3, Chapter 10]. In particular, the geometric means of matrices arise naturally in consideration of convolution type equations on finite intervals.

Presently, $\tilde{\mathbf{d}}(G_f^{(\lambda)})$ in the setting of Theorem 3.1 (b) with a truly quadrinomial f has not been computed, and therefore the situation of Theorems 5.2, 5.4 also remains out of reach. We will, however, state the result which relates the geometric mean of matrix functions $G_f^{(\lambda)}$ satisfying (5.2), (5.5) with that of their Portuguese transformation.

To this end, we need the relation between the geometric means of matrix functions $G_f^{(\lambda)}$ and their Portuguese transformations. It can be easily extracted from formulas (13.33), (13.42), (13.43) in [3], and reads as follows.

Proposition 6.1. *Let f be given by* (2.2), *with $\nu > 0$. Suppose that $G_f^{(\lambda)}$ admits a canonical factorization, and let $G_{f_1}^{(\nu)} = \mathcal{PT}(G_f^{(\lambda)})$. Then*

$$\tilde{\mathbf{d}}(G_f^{(\lambda)}) = \begin{bmatrix} -a^{-1} & 0 \\ x & a \end{bmatrix}^{-1} \tilde{\mathbf{d}}(G_{f_1}^{(\nu)}) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where

$$x = \sum_{k=1}^{m} b_k \sum_{(\boldsymbol{n},\boldsymbol{\gamma})=\lambda-\gamma_k+\nu} c_{\boldsymbol{n}},$$

with $c_{\boldsymbol{n}}$ given by (2.4).

To state the specific result following from Proposition 6.1 for f given by (5.5), define

$$b_k = -d_{k-1}/d_{-1} \quad \text{for } 1 \leq k \leq \ell+1,$$

and

$$\tilde{b}_k = -\tilde{d}_{k-1}/d_{-1} \quad \text{for } 1 \leq k \leq p+1.$$

Further, let

$$c_{\boldsymbol{n},\boldsymbol{m}} = \frac{(n_1 + \cdots + n_{\ell+1} + m_1 + \cdots + m_{p+1})!}{n_1! \cdots n_{\ell+1}! m_1! \cdots m_{p+1}!} b_1^{n_1} \cdots b_{\ell+1}^{n_{\ell+1}} \tilde{b}_1^{m_1} \cdots \tilde{b}_{p+1}^{m_{p+1}},$$

for

$$\boldsymbol{n} = (n_1, n_2, \dots, n_{\ell+1}) \in \mathbb{Z}^{\ell+1}, \quad \boldsymbol{m} = (m_1, m_2, \dots, m_{p+1}) \in \mathbb{Z}^{p+1}.$$

After rather straightforward computations, we obtain

Theorem 6.2. *Let f be such that* (5.5) *holds, and assume $\lambda \leq \nu + 2\alpha$, $\alpha/\nu \notin \mathbb{Q}$. Let $G_{f_1}^{(\nu)} = \mathcal{PT}(G_f^{(\lambda)})$. If $G_f^{(\lambda)}$, and hence also $G_{f_1}^{(\nu)}$, admit canonical factorization, then*

$$\tilde{\mathbf{d}}(G_f^{(\lambda)}) = \begin{bmatrix} -d_{-1}^{-1} & 0 \\ \Delta & d_{-1} \end{bmatrix} \tilde{\mathbf{d}}(G_{f_1}^{(\nu)}) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{6.1}$$

where

$$\Delta = \begin{cases} \displaystyle\sum_{k=1}^{\ell+1} b_k \sum_{(\boldsymbol{n},\mathbf{a}_{\ell+1})=\ell-k+1} c_{\boldsymbol{n},\mathbf{0}} & \text{if } \lambda = \ell\nu, \\ \displaystyle\sum_{k=1}^{p+1} b_k \sum_{j=1}^{p+1-k} \sum_{(\boldsymbol{n},\mathbf{a}_{p+1})=p-k-j+1} c_{\boldsymbol{n},\boldsymbol{e}^j} + \sum_{k=1}^{p+1} \tilde{b}_k \sum_{(\boldsymbol{n},\mathbf{a}_{p+1})=p-k+1} c_{\boldsymbol{n},\mathbf{0}} & \\ & \text{if } \lambda = p\nu + \alpha, \\ \tilde{b}_1^2 & \text{if } \lambda = \nu + 2\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

and where we denote by $\boldsymbol{e}^j$ the jth unit coordinate vector in $\mathbb{Z}^{p+1}$: the ith component of $\boldsymbol{e}^j$ is the Kronecker delta δ_{ij}.

7. Off diagonal trinomials

In this section we discuss new cases of trinomial functions f such that the factorability of $G_f^{(\lambda)}$ can be established using results of Section 5. Factorability of $G_f^{(\lambda)}$ with trinomial almost periodic function f has been extensively studied in the literature, see [10, 13, 2, 15], but so far there is no complete resolution of this problem.

We write a general trinomial as

$$f = d_{-1}e_{-\nu} + d_0 e_\mu + d_1 e_\delta,$$

with $-\lambda < -\nu < \mu < \delta < \lambda$; we assume that $d_{-1}d_0d_1 \neq 0$, or else f is actually a binomial or monomial, which is covered by Theorem 4.2 part (1). It also covers the case of rational δ/ν. We may also require $\delta, \nu > 0$, or else this is simply the one-sided case.

The case $\nu + \delta + |\mu| \geq \lambda$ is covered by parts (a) (4),(5) and (b) of the same Theorem 4.2. Therefore, the case of interest is

$$\nu + \delta + |\mu| < \lambda, \quad \delta/\nu \notin \mathbb{Q}.$$

No new results for the trinomial case arise from Theorem 5.4: under condition (5.13), the distance from the leftmost point $-\nu$ of $\Omega(f)$ to the second closest is at least ν. Theorem 5.1, however, immediately yields the following:

Theorem 7.1. *Let $\nu > 0$, $\delta > 0$. Then the set $\{-\nu, k\nu, \delta\}$ is λ-conditionally admissible whenever $k \in \mathbb{Z}_+$ and $\lambda \leq \nu + 2\delta$.*

Proof. Observe that for $\alpha = \delta$ the set Ω_1 given by (5.1) contains $\{-\nu, k\nu, \delta\}$, and that (5.1) holds. □

Corollary 7.2. *Let $\nu > 0$, $\delta > 0$. Then the set $\{-\nu, 0, \delta\}$ is λ-conditionally admissible if $\lambda \leq \nu + \delta + \max\{\nu, \delta\}$.*

Proof. For $\lambda \leq \nu + 2\delta$ this is simply a particular case of Theorem 7.1 corresponding to $k = 0$. The possibility $\lambda \leq 2\nu + \delta$ is reduced to the previous one by passing from f to f^*. □

Note that AP factorability of matrix functions $G_f^{(\lambda)}$ with

$$f = d_{-1}e_{-\nu} + d_0 + d_1 e_\delta, \quad \nu/\delta \notin \mathbb{Q}$$

and $\lambda > \nu + 2\delta$, $\lambda > 2\nu + \delta$ remains an open problem.

Formulas (5.7)–(5.10) in the setting of Theorem 7.1 for irrational δ/ν take the following form; here $m|n$, resp, $m \not| n$ denotes the property that m divides n, resp., m does not divide n, for positive integers m and n:

$$c_{-2} = \begin{cases} 0 & \text{if} \quad k+1 \not| \ell \\ \frac{(-d_0)^{\frac{\ell}{k+1}}}{d_{-1}^{\frac{\ell}{k+1}+1}} & \text{if} \quad k+1 | \ell \end{cases}$$

$$c_{-1} = \begin{cases} 0 & \text{if} \quad k+1 \not| p-1 \\ (-1)^{\frac{p-1}{k+1}+1}(\frac{p-1}{k+1}+1)\frac{d_0^{\frac{p-1}{k+1}} d_1}{d_{-1}^{\frac{p-1}{k+1}+2}} & \text{if} \quad k+1 | p-1 \end{cases}$$

$$c_{1} = \begin{cases} 0 & \text{if} \quad k+1 \not| p \\ (-1)^{\frac{p}{k+1}+1}(\frac{p}{k+1}+1)\frac{d_0^{\frac{p}{k+1}} d_1}{d_{-1}^{\frac{p}{k+1}+2}} & \text{if} \quad k+1 | p \end{cases}$$

$$c_{2} = \begin{cases} 0 & \text{if} \quad k+1 \not| \ell+1 \\ \frac{(-d_0)^{\frac{\ell+1}{k+1}}}{d_{-1}^{\frac{\ell+1}{k+1}+1}} & \text{if} \quad k+1 | \ell+1. \end{cases}$$

Here, ℓ and p are defined as in (5.3).

These formulas of course can be used to derive explicit factorability conditions for matrix functions $G_f^{(\lambda)}$ with $\Omega(f) \subset \{-\nu, k\nu, \delta\}$. We will provide only one result in this direction, corresponding to the (simplest) case $k = 0$. Then

$$\begin{aligned} c_{-2} &= \frac{(-d_0)^{\ell}}{(d_{-1})^{\ell+1}}, \qquad & c_{-1} &= (-1)^p p \frac{d_0^{p-1} d_1}{d_{-1}^{p+1}}, \\ c_1 &= (-1)^{p+1}(p+1)\frac{d_0^p d_1}{d_{-1}^{p+2}}, \qquad & c_2 &= \frac{(-d_0)^{\ell+1}}{d_{-1}^{\ell+2}}, \end{aligned} \tag{7.1}$$

and Theorem 3.1 applied to $G_{f_1}^{(\nu)}$ with f_1 given by (5.6) reveals:

Theorem 7.3. *Let $f = d_{-1}e_{-\nu} + d_0 + d_1 e_\delta$, where $0 < \nu, \delta$, and ν/δ is irrational. Then for any $\lambda \leq \nu + 2\delta$ the matrix function $G_f^{\lambda)}$ admits a canonical AP factorization if*

$$p^{(p+1)\nu+\delta-\lambda}(p+1)^{\lambda-\delta-p\nu}\,|d_1|^{\nu}\,|d_{-1}|^{\delta} \neq |d_0|^{\nu+\delta},$$

where $p = \lfloor (\lambda - \delta)/\nu \rfloor$, and is not AP factorable otherwise.

References

[1] S. Avdonin, A. Bulanova, and W. Moran, *Construction of sampling and interpolating sequences for multi-band signals. The two-band case*, Int. J. Appl. Math. Comput. Sci. **17** (2007), no. 2, 143–156.

[2] M.A. Bastos, Yu.I. Karlovich, I.M. Spitkovsky, and P.M. Tishin, *On a new algorithm for almost periodic factorization*, Recent progress in operator theory (Regensburg, 1995) (I. Gohberg, R. Mennicken, and C. Tretter, eds.), Operator Theory: Advances and Applications, vol. 103, Birkhäuser Verlag, Basel and Boston, 1998, pp. 53–74.

[3] A. Böttcher, Yu.I. Karlovich, and I.M. Spitkovsky, *Convolution operators and factorization of almost periodic matrix functions*, Operator Theory: Advances and Applications, vol. 131, Birkhäuser Verlag, Basel and Boston, 2002.

[4] M.C. Câmara and C. Diogo, *Invertibility of Toeplitz operators and corona conditions in a strip*, J. Math. Anal. Appl. **342** (2008), no. 2, 1297–1317.

[5] M.C. Câmara, A.F. dos Santos, and M.C. Martins, *A new approach to factorization of a class of almost-periodic triangular symbols and related Riemann-Hilbert problems*, J. Funct. Anal. **235** (2006), no. 2, 559–592.

[6] M.C. Câmara, Yu.I. Karlovich, and I.M. Spitkovsky, *Almost periodic factorization of some triangular matrix functions*, Operator Theory: Advances and Applications, vol. 190, Birkhäuser Verlag, Basel and Boston, 2009, pp. 171–190.

[7] M.C. Câmara and M.C. Martins, *Explicit almost-periodic factorization for a class of triangular matrix functions*, J. Anal. Math. **103** (2007), 221–260.

[8] Yu.I. Karlovich, *Approximation approach to canonical APW factorability*, Izv. Vuzov., Sev.-Kavk. Region, 2005, pp. 143–151.

[9] Yu.I. Karlovich and I.M. Spitkovsky, *Almost periodic polynomial factorization of some triangular matrix functions*, Operator Theory: Advances and Applications, vol. 202, Birkhäuser Verlag, Basel and Boston, 2010, pp. 337–354 (this volume).

[10] ——, *On the theory of systems of equations of convolution type with semi-almost-periodic symbols in spaces of Bessel potentials*, Soviet Math. Dokl. **33** (1986), 145–149.

[11] ——, *Factorization of almost periodic matrix-valued functions and the Noether theory for certain classes of equations of convolution type*, Mathematics of the USSR, Izvestiya **34** (1990), 281–316.

[12] ——, *Almost periodic factorization: An analogue of Chebotarev's algorithm*, Contemporary Math. **189** (1995), 327–352.

[13] ——, *Factorization of almost periodic matrix functions*, J. Math. Anal. Appl. **193** (1995), 209–232.

[14] Yu.I. Karlovich, I.M. Spitkovsky, and R.A. Walker, *Almost periodic factorization of block triangular matrix functions revisited*, Linear Algebra Appl. **293** (1999), 199–232.

[15] D. Quint, L. Rodman, and I.M. Spitkovsky, *New cases of almost periodic factorization of triangular matrix functions*, Michigan Math. J. **45** (1998), 73–102.

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Received: December 3, 2008
Accepted: April 7, 2009

Operator Theory:
Advances and Applications, Vol. 202, 489–495

Infinite Hankel Block Matrices, Extremal Problems

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Abstract. In this paper we use the matrix analogue of eigenvalue $\rho^2_{\min}$ to formulate and to solve the extremal Nehari problem. Our approach coincides with Arov, Adamyan, Krein approach when $\rho^2_{\min}$ is a scalar matrix.

Mathematics Subject Classification (2000). Primary 15A57; Secondary 47B10.

Keywords. Matrix Nehari problem, minimal solution, matrix analogue of eigenvalue.

1. Introduction

In the paper we consider a matrix version of the extremal Nehari problem [1],[4]. Our approach is based on the notion of a matrix analogue of the eigenvalue $\rho^2_{\min}$. The notion of $\rho^2_{\min}$ was used in a number of the extremal interpolation problems [2],[3],[7]. We note that $\rho^2_{\min}$ is a solution of a non-linear matrix inequality of the Riccati type [2],[6], [7]. Our approach coincides with the Arov, Adamyan, Krein approach [1], when $\rho^2_{\min}$ is a scalar matrix.

Now we introduce the main definitions. Let H be a fixed separable Hilbert space. By $\ell_2(H)$ we denote the Hilbert space of the sequences $\xi = \{\xi_k\}_1^\infty$, where $\xi_k \in H$ and

$$\|\xi\|^2 = \sum_{k=1}^{\infty} ||\xi_k||^2 < \infty.$$

The space of the bounded linear operators acting from $\ell_2(H_1)$ into $\ell_2(H_2)$ is denoted by $[\ell_2(H_1), \ell_2(H_2)]$. The Hankel operator $\Gamma \in [\ell_2(H_1), \ell_2(H_2)]$ has the form

$$\Gamma = \{\gamma_{j+k-1}\}, \quad 1 \le j, k \le \infty, \quad \gamma_k \in [H_1, H_2].$$

Communicated by V. Bolotnikov.

Let $L_\infty[H_1, H_2]$ be the space of the measurable operator-valued functions $F(\xi)\in[H_1, H_2]$, $|\xi| = 1$ with the norm

$$||F||_\infty = \mathrm{esssup}||F(\xi)|| < \infty, \quad |\xi| = 1.$$

We shall say that an operator $\rho\in[H, H]$ is *strongly positive* and will write $\rho\gg 0$ if there exists such a number $\delta > 0$ that $\rho > \delta I_H$, where I_H is the identity operator in the space H. Further we use the following version of the well-known theorem (see [1]) and references there).

Theorem 1.1. *Suppose given a sequence* $\gamma_k\in[H_1, H_2]$, $1\leq k < \infty$ *and a strongly positive operator* $\rho\in[H_2, H_2]$. *In order for there to exist an operator function* $F(\xi)\in L_\infty[H_1, H_2]$ *such that*

$$c_k(F) = \frac{1}{2\pi}\int_{|\xi|=1} \xi^k F(\xi)|d\xi| = \gamma_k, \quad k = 1, 2, \ldots \tag{1.1}$$

and

$$F^\star(\xi)F(\xi)\leq\rho^2 \tag{1.2}$$

it is necessary and sufficient that

$$\Gamma^\star\Gamma\leq R^2, \quad \textit{where} \quad R = \mathrm{diag}\{\rho, \rho, \ldots\}. \tag{1.3}$$

(*The integral in the right-hand side of* (1.1) *converges in the weak sense.*)

Proof. Let us introduce the denotations

$$F_\rho(\xi) = F(\xi)\rho^{-1}, \quad \gamma_{k,\rho} = \gamma_k\rho^{-1}. \tag{1.4}$$

Relations (1.1) and (1.2) take the forms

$$\frac{1}{2\pi}\int_{|\xi|=1} \xi^k F_\rho(\xi)|d\xi| = \gamma_{k,\rho}, \quad k = 1, 2, \ldots \tag{1.5}$$

and

$$F_\rho^\star(\xi)F_\rho(\xi)\leq I_{H_2}. \tag{1.6}$$

In case (1.5) and (1.6) the theorem is true (see [1]). Hence in case (1.1) and (1.2) the theorem is true as well. □

The aim of this work is to solve the following extremal problem.

Problem 1.2. *In the class of functions* $F(\xi)\in[H_1, H_2]$, $|\xi| = 1$ *satisfying condition* (1.1) *to find the function with the least deviation from the zero.*

As a deviation measure we do not choose a number but a strictly positive operator $\rho_{\min}$ such that

$$F^\star(\xi)F(\xi)\leq\rho_{\min}^2. \tag{1.7}$$

The case of the scalar matrix $\rho_{\min}$ was considered in the article [1]. The transition from the scalar matrix $\rho_{\min}$ to the general case considerably widens the class of the problems having one and only one solution. This is important both from the theoretical and the applied view points. We note that the $\rho_{\min}^2$ is an analogue of the eigenvalue of the operator $\Gamma^\star\Gamma$.

2. Extremal problem

In this section we consider a particular extremal problem. Namely, we try to find $\rho_{\min}$ which satisfies the condition

$$\Gamma^{\star}\Gamma \leq R^2_{\min}, \quad R_{\min} = \operatorname{diag}\{\rho_{\min}, \rho_{\min}, \dots\}. \tag{2.1}$$

In order to explain the notion of $\rho_{\min}$ we introduce the notations $B_r = [\gamma_2, \gamma_3, \dots]$, $B_c = \operatorname{col}[\gamma_2, \gamma_3, \dots]$. Then the matrix Γ has the following structure

$$\Gamma = \begin{bmatrix} \gamma_1 & B_r \\ B_c & \Gamma_1 \end{bmatrix}$$

where

$$\Gamma_1 = \{\gamma_{j+k}\}, \quad 1 \leq j, k < \infty, \quad \gamma_k \in [H_1, H_2].$$

It means that

$$\Gamma^{\star}\Gamma = \begin{bmatrix} A_{11} & A_{12} \\ A^{\star}_{12} & A_{22} \end{bmatrix},$$

where

$$A_{11} = \gamma_1^{\star}\gamma_1 + B_c^{\star}B_c, \quad A_{12} = \gamma_1^{\star}B_r + B_c^{\star}\Gamma, \quad A_{22} = \Gamma_1^{\star}\Gamma_1 + B_r^{\star}B_r. \tag{2.2}$$

Further we suppose that

$$R^2 - A_{22} \gg 0. \tag{2.3}$$

Then relation (1.3) is equivalent to the relation

$$\rho^2 \geq A_{11} + A_{12}(R^2 - A_{22})^{-1}A^{\star}_{12}. \tag{2.4}$$

Definition 2.1. *We shall call the strongly positive operator* $\rho \in [H_2, H_2]$ *a minimal solution of inequality* (2.1) *if inequality* (2.3) *is valid and*

$$\rho^2_{\min} = A_{11} + A_{12}(R^2_{\min} - A_{22})^{-1}A^{\star}_{12}. \tag{2.5}$$

It follows from (2.5) that $\rho^2_{\min}$ coincides with the solution of the non-linear equation

$$q^2 = A_{11} + A_{12}(Q^2 - A_{22})^{-1}A^{\star}_{12}, \tag{2.6}$$

where $q \in [H_2, H_2]$, $Q = \operatorname{diag}\{q, q, \dots\}$. Let us note that a solution q^2 of equation (2.8) is *an analogue of the eigenvalue* of the operator $\Gamma^{\star}\Gamma$.

Now we will present the method of constructing $\rho_{\min}$. We apply the method of successive approximation. We let

$$q_0^2 = A_{11}, \quad q^2_{n+1} = A_{11} + A_{12}(Q_n^2 - A_{22})^{-1}A^{\star}_{12}, \tag{2.7}$$

where

$$Q_n = \operatorname{diag}\{q_n, q_n, \dots\}, \quad n \geq 0. \tag{2.8}$$

Further we suppose that

$$Q_0^2 - A_{22} \gg 0. \tag{2.9}$$

It follows from relations (2.7)–(2.9) that

$$q_n^2 \geq q_0^2, \quad Q_n^2 \geq Q_0^2 \gg 0, \quad n \geq 0.$$

As the right-hand side of (2.7) decreases with the growth of q_n^2. the following assertions are true (see[7]).

Lemma 2.2.

1. *The sequence* $q_0^2, q_2^2, \dots$ *monotonically increases and has the strong limit* $\underline{q}^2$.
2. *The sequence* $q_1^2, q_3^2, \dots$ *monotonically decreases and has the strong limit* $\overline{q}^2$.
3. *The inequality*
$$\underline{q}^2 \leq \overline{q}^2 \tag{2.10}$$
is true.

Corollary 2.3. *If condition* (2.9) *is fulfilled and* $\underline{q}^2 = \overline{q}^2$, *then*

$$\rho_{\min}^2 = \underline{q}^2 = \overline{q}^2$$

A. Ran and M. Reurings [6] investigated equation (2.6) when A_{ij} are finite-order matrices. Slightly changing their argumentation we shall prove that the corresponding results are true in our case as well.

Theorem 2.4. *Let* A_{ij} *be defined by relations* (2.2) *and let condition* (2.9) *be fulfilled. If the inequalities*

$$A_{11} \geq 0, \quad A_{22} \geq 0, \quad A_{12}A_{12}^{\star} \gg 0 \tag{2.11}$$

are valid, then equation (2.6) *has one and only one strongly positive solution* q^2 *and*

$$\rho_{\min}^2 = q^2 = \underline{q}^2 = \overline{q}^2. \tag{2.12}$$

Proof. In view of Lemma 2.2, we have the relations

$$\underline{q}^2 = A_{11} + A_{12}(\overline{Q}^2 - A_{22})^{-1}A_{12}^{\star}, \tag{2.13}$$
$$\overline{q}^2 = A_{11} + A_{12}(\underline{Q}^2 - A_{22})^{-1}A_{12}^{\star},$$

where $\underline{Q} = \operatorname{diag}\{\underline{q}, \underline{q}, \dots\}$, $\quad \overline{Q} = \operatorname{diag}\{\overline{q}, \overline{q}, \dots\}$. According to (2.10) the inequality

$$y = \overline{q}^2 - \underline{q}^2 \geq 0 \tag{2.14}$$

holds. The direct calculation gives

$$y = B^{\star}YB, \quad Y = \operatorname{diag}\{y, y, \dots\}, \tag{2.15}$$

with

$$B = T(I + TYT)^{-1/2}TA_{12}^{\star} \tag{2.16}$$

Here $T = (\underline{Q} - A_{22})^{-1/2}$. Let us introduce the operator

$$P = \operatorname{diag}\{p, p, \dots\}, \quad p = \underline{q}^2 - A_{11}.$$

From assumption (2.9) and relations (2.11) we deduce that

$$B^{\star}PB \ll B^{\star}(\underline{Q}^2 - A_{22})B = p. \tag{2.17}$$

Let us explain the last equality in formula (2.17). Using (2.16) we have

$$(\underline{Q}^2 - A_{22})B = A_{12}T(I + TYT)^{-1}TA_{12}^{\star} = A_{12}(\underline{Q}^2 - A_{22} + Y)A_{12}^{\star}.$$

Taking into account relations (2.13) and (2.14) we prove that

$$(\underline{Q}^2 - A_{22})B = A_{12}(\overline{Q}^2 - A_{22})A_{12}^\star = p.$$

Relation (2.17) can be written in the form

$$B_1^\star B_1 \ll I, \quad where \quad B_1 = P^{1/2}Bp^{-1/2}. \tag{2.18}$$

Formula (2.15) takes the form

$$y_1 = B_1^\star Y_1 B_1, \quad y_1 = p^{-1/2}yp^{-1/2}, \quad Y_1 = P^{-1/2}YP^{-1/2}. \tag{2.19}$$

Inequality (2.18) implies that equation (2.19) has only the trivial solution $y_1 = 0$. The theorem is proved. □

We can omit the condition $A_{12}A_{12}^\star \gg 0$, when

$$\dim H_k = m < \infty, \quad k = 1, 2. \tag{2.20}$$

In this case the following assertion is true.

Theorem 2.5. *Let A_{ij} be defined by relations* (2.2) *and let conditions* (2.9) *and* (2.20) *be fulfilled. If the inequalities $A_{11} \geq 0$ and $A_{22} \geq 0$ are valid, then equation* (2.6) *has one and only one strongly positive solution q^2 and*

$$\rho_{\min}^2 = q^2 = \underline{q}^2 = \overline{q}^2.$$

Proof. Let us consider the maps

$$F(q^2) = A_{11} + A_{12}(Q^2 - A_{22})^{-1}A_{12}^\star,$$
$$G(q^2) = I_m + U(Q^2 - D)^{-1}U^\star,$$

where

$$U = q_0^{-1}A_{12}Q_0^{-1}, \quad D = Q_0^{-1}A_{22}Q_0^{-1}. \tag{2.21}$$

The fixed points q_F^2 and q_G^2 of the maps G and F respectively are related by

$$q_G = q_0^{-1}q_F q_0^{-1}.$$

In view of (2.2) and (2.21) the matrix U has the form $U = [u_1, u_2 \ldots]$, where u_k are $m \times m$ matrices. Then a vector $x \in \mathbb{C}^m$ belongs to $\ker U^\star$ if and only if $u_k^\star x = 0$ for all $k \geq 1$. Let $d = \dim\ker U^\star$. We shall use the decomposition

$$((\ker U^\star)^\perp) \bigoplus (\ker U^\star),$$

with respect to which the matrices $u_k^\star$ and q^2 are of the form

$$u_k^\star = \begin{bmatrix} u_{1,k}^\star & 0 \\ u_{2,k}^\star & 0 \end{bmatrix}, \quad q^2 = \begin{bmatrix} q_{11}^2 & 0 \\ 0 & I_d \end{bmatrix},$$

where $u_{1,k}^\star$, $u_{2,k}^\star$ and q_{11}^2 are matrices of order $(m-d)\times(m-d)$, $d\times(m-d)$ and $(m-d)\times(m-d)$ respectively. We note that

$$q_{11}^2 \geq I_{m-d}. \tag{2.22}$$

Changing the decomposition of the space $\ell_2(H_2)$ we can represent $U^\star$, D and Q^2 in the form

$$U^\star = \begin{bmatrix} U_1^\star & 0 \\ U_2^\star & 0 \end{bmatrix}, \quad D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}, \quad Q^2 = \begin{bmatrix} Q_{11}^2 & 0 \\ 0 & I \end{bmatrix}, \tag{2.23}$$

where $U_p^\star = \mathrm{col}[u_{p,1}^\star u_{p,2}^\star, \dots]$, $\quad (p = 1, 2)$ and $Q_{11}^2 = \mathrm{diag}\{q_{11}^2, q_{11}^2, \dots\}$. By direct calculation we deduce that

$$(Q^2 - D)^{-1} = T\mathrm{diag}\{Q_{11}^2 - d_{11} - d_{11}(I - d_{22})^{-1}d_{12}^\star, I - d_{22}\}^{-1}T^\star, \tag{2.24}$$

where

$$T = \begin{bmatrix} I & 0 \\ (I - d_{22})^{-1}d_{12}^\star & I \end{bmatrix}. \tag{2.25}$$

Using formulas (2.23)–(2.25) we reduce the map $G(q^2)$ to the form

$$G_1(q_{11}^2) = \hat{A}_{11} + \hat{A}_{12}(Q_{11}^2 - \hat{A}_{22})^{-1}\hat{A}_{12}^\star,$$

where

$$\hat{A}_{11} = I_{m-d} + u_2((I - d_{22})^{-1}u_2^\star, \quad \hat{A}_{12} = u_1 + u_2(I - d_{22})^{-1}d_{12}^\star,$$
$$\hat{A}_{22} = d_{11} + d_{12}(I - d_{22})^{-1}d_{12}^\star.$$

Relations (2.12), (2.21) and (2.22) imply that

$$D \ll I, \quad Q_{11}^2 \geq I$$

and hence the map $G_1(q_{11}^2)$ satisfies condition (2.9). By repeating the described reduction method we obtain the following result: either $\hat{A}_{12}^\star = 0$ or $\ker\hat{A}_{12}^\star = 0$. It is obvious that the theorem is true if $\hat{A}_{12}^\star = 0$. If $\ker\hat{A}_{12}^\star = 0$, then the $(m-d)\times(m-d)$ matrix $\hat{A}_{12}\hat{A}_{12}^\star$ is positive, i.e., this matrix is strongly positive. Now the assertion of the theorem follows directly from Theorem 2.4. □

Proposition 2.6. *Let conditions of either Theorem* 2.4 *or of Theorem* 2.5 *be fulfilled. Then there exists one and only one operator function* $F(\xi)$ *which satisfies conditions* (1.1) *and* (1.7).

Proof. The formulated assertion is true when

$$\rho_{\min} = \alpha I_{H_2}, \quad \alpha = \|\Gamma^\star\Gamma\|. \tag{2.26}$$

Using formulas (1.4) we reduce the general case to (2.26). The proposition is proved. □

Remark 2.7. The method of constructing the corresponding operator function is given in paper [1] for case (2.26). Using this method we can construct the operator function $F_{\rho_{\min}}(\xi)$ and then $F(\xi)$.

Remark 2.8. Condition (2.26) is valid in a few cases. By our approach (minimal ρ) we obtain the uniqueness of the solution for a broad class of problems.

References

[1] V.M. Adamyan, D.Z. Arov and M.G. Krein, *Infinite Hankel Block Matrices and Related Extension Problems*, Amer. Math. Soc. Transl. **111** (1978), 133–156.

[2] B. Fritzsche, B. Kirstein and L.A. Sakhnovich, *Extremal Classical Interpolation Problems* (*matrix case*), Lin. Alg. Appl. **430** (2009), 762–781.

[3] J.W. Helton and L.A. Sakhnovich, *Extremal Problems of Interpolation Theory*, Rocky Mount. J. Math., **35** (2005), 819–841.

[4] Z. Nehari, *On Bounded Linear Forms*, Ann. of Math. **65** (1957), 153–162.

[5] V.V. Peller and N.J. Joung, *Superoptimal Analytic Approximation of Matrix Functions*, J. Funct. Anal. **120** (1994), 300–343.

[6] A.C.M. Ran and M.C.B. Reurings *A Nonlinear Matrix Equation Connected to Interpolation Theory*, Lin. Alg. Appl. **379** (2004), 289–302.

[7] L.A. Sakhnovich, *Interpolation Theory and its Applications*, Kluwer, Dordrecht, 1997.

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Received: January 9, 2009
Accepted: June 19, 2009

Operator Theory:
Advances and Applications, Vol. 202, 497–508

On Compactness of Operators in Variable Exponent Lebesgue Spaces

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Abstract. We give a short discussion of known statements on compactness of operators in variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega, \varrho)$ and show that the existence of a radial integrable decreasing dominant of the kernel of a convolution operator guarantees its compactness in the space $L^{p(\cdot)}(\Omega, \varrho)$ whenever the maximal operator is bounded in this space, where $|\Omega| < \infty$ and ϱ is an arbitrary weight such that $L^{p(\cdot)}(\Omega, \varrho)$ is a Banach function space. In the non-weighted case $\varrho = 1$ we also give a modification of this statement for $\Omega = \mathbb{R}^n$.

Mathematics Subject Classification (2000). Primary 46E30; Secondary 47B38, 47G10.

Keywords. Variable exponent spaces, compact operator, integral operators, convolution operators, radial decreasing dominants.

1. Introduction

The area called now variable exponent analysis, is mainly concerned with the so-called function spaces with non-standard growth or variable exponent Lebesgue spaces. Last decade it became a rather branched field with many results related to Harmonic Analysis, Approximation Theory, Operator Theory, Pseudo-differential Operators. This topic continues to attract a strong interest of researchers, influenced in particular by possible applications revealed in the book [27]. We refer in particular to the survey articles [7, 11, 17, 28]. In particular, there are about a hundred of papers devoted to the study of the boundedness of various operators, including the classical operators of Harmonic Analysis, in variable exponent Lebesgue spaces. Although the importance of the compactness theorems for the operator theory is well known, the compactness of operators in such spaces was less touched. Some episodes related to compactness may be found in [15, 16, 24, 26].

Communicated by I.M. Spitkovsky.

Probably, only the recent paper [25] is specially devoted to the topic of compactness in variable exponent Lebesgue spaces. The present paper is aimed to partially fill the gap.

We recall and give slight improvements of known general results on compactness in variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ and give conditions of compactness of some classes of operators. Although the compactness of operators is well studied in more general setting, including the general case of Banach function spaces, in applications – in particular to various topics of the operator theory related to weighted variable exponent Lebesgue spaces – it is important to have easy to check conditions of compactness. In this paper we study some conditions of this kind.

As is well known, the Young theorem for convolutions in general is not valid in case of variable exponents. A convolution operator $k * f$ may be bounded in $L^{p(\cdot)}$ only in the case where the kernel has a singularity at the origin (unless we do not impose some special restrictions on the variability of $p(x)$). Thus singular and potential type operators are among the candidates to be studied in variable exponent spaces. Due to Stein's pointwise estimate via the maximal operator, convolutions with radial decreasing dominant of the kernel are bounded in $L^{p(\cdot)}(\Omega)$ whenever the maximal operator is bounded in this spaces. For such a class of integral operators we show, in particular, that the same holds with respect to their compactness in $L^{p(\cdot)}(\Omega)$, when $|\Omega| < \infty$, with some modification in the case $\Omega = \mathbb{R}^n$. In case $|\Omega| < \infty$ we also admit weighted spaces $L^{p(\cdot)}(\Omega, \varrho)$ with an arbitrary weight such that $L^{p(\cdot)}(\Omega, \varrho)$ is a Banach function space.

2. Preliminaries

Let $\Omega \subseteq \mathbb{R}^n$ be an open set in $\mathbb{R}^n$ and $p(x)$ a measurable function on Ω with values in $[1, \infty)$. By $L^{p(\cdot)}(\Omega, \varrho)$ we denote the space of functions $f(x)$ on Ω such that

$$I_p(\varrho f) = \int_\Omega \left(\frac{\varrho(x)|f(x)|}{\lambda} \right)^{p(x)} dx \ < \infty$$

for some $\lambda > 0$. This is a Banach space with respect to the norm

$$\|f\|_{L^{p(\cdot)}(\Omega,\varrho)} = \inf \left\{ \lambda > 0 : I_p \left(\frac{\varrho f}{\lambda} \right) \leq 1 \right\}. \tag{2.1}$$

We write $L^{p(\cdot)}(\Omega, \varrho) = L^{p(\cdot)}(\Omega)$ and $\|f\|_{L^{p(\cdot)}(\Omega,\varrho)} = \|f\|_{p(\cdot)}$ in the case $\varrho \equiv 1$. Let

$$p_- = \operatorname*{ess\,inf}_{x\in\Omega} p(x), \quad p_+ = \operatorname*{ess\,sup}_{x\in\Omega} p(x), .$$

In the sequel we always assume that

$$1 \leq p_- \leq p(x) \leq p_+ < \infty, \quad x \in \Omega, \tag{2.2}$$

admitting a possibility for $p(x)$ to attain value 1, this possibility being of a special interest in the variable exponent analysis, but some statements will be given under the stronger condition

$$1 < p_- \leq p(x) \leq p_+ < \infty, \quad x \in \Omega. \tag{2.3}$$

The space $L^{p(\cdot)}(\Omega)$ is a BFS (Banach function space) in the well-known sense [1], as verified in [8]. Recall that under the condition

$$|p(x) - p(y)| \leq \frac{C}{-\ln|x-y|}, \quad |x-y| \leq \frac{1}{2}, \tag{2.4}$$

and the condition that there exists $p(\infty) = \lim\limits_{x\to\infty} p(x)$ and

$$|p(x) - p(\infty)| \leq \frac{C}{\ln(2+|x|)}, \tag{2.5}$$

the maximal operator

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)\cap\Omega} |f(y)|\,dy,$$

is bounded in the space $L^{p(\cdot)}(\Omega)$, condition (2.5) appearing in the case of unbounded Ω, see [6, 4].

Definition 2.1. By w-Lip(Ω) we denote the class of exponents $p \in L^\infty(\Omega)$ satisfying the (local) logarithmic condition (2.4).

Definition 2.2. By $\mathcal{P}_\infty(\Omega)$ we denote the class of exponents $p \in L^\infty(\Omega)$ which satisfy assumptions (2.2), (2.5).

3. Two general results on compactness of operators

In this section we dwell on two general results on compactness which are known to be well suited for applications to variable exponent spaces. One is the so-called dominated compactness theorem for integral operators, another is a consequence of the general unilateral compactness interpolation theorems.

3.1. Dominated compactness theorem

For classical Lebesgue spaces $L^p(\Omega)$, $|\Omega| < \infty$ with a constant $p \in (1,\infty)$ there is known the result on compactness, which goes back to Krasnoselskii [19] and states that the compactness in L^p of an integral operator with a positive kernel yields that of the operator with a smaller kernel. To the case of variable exponent spaces this was extended in [25], where it was in general proved within the frameworks of BFS. In Theorem 3.12 of [25] a slightly more general version of the following Theorem 3.1 was proved. Let

$$\mathbb{K}f(x) = \int_\Omega \mathcal{K}(x,y)f(y)\,dy \quad \text{and} \quad \mathbb{K}_0 f(x) = \int_\Omega \mathcal{K}_0(x,y)f(y)\,dy \tag{3.1}$$

be two integral operators and $\mathcal{K}_0(x,y) \geq 0$.

Theorem 3.1. *Let* X *and* Y *be BFS with absolutely continuous norms. Let*

$$|\mathcal{K}(x,y)| \le \mathcal{K}_0(x,y)$$

and suppose that the operator $\mathbb{K}_0 : \mathsf{X} \to \mathsf{Y}$ *is compact. Then* $\mathbb{K}$ *is also a compact operator from* X *to* Y.

It is known that a BFS has an absolutely continuous norm if and only if its dual and associate spaces are isomorphic.

Corollary 3.2. *The statement of Theorem* 3.1 *is valid for the space* $L^{p(\cdot)}(\Omega,\varrho)$, *if condition* (2.2) *and the conditions*

$$\|\varrho\|_{p(\cdot)} < \infty, \quad \|\varrho^{-1}\|_{p'(\cdot)} < \infty, \quad \frac{1}{p'(x)} + \frac{1}{p(x)} \equiv 1, \tag{3.2}$$

are satisfied.

Proof. It suffices to note that conditions (3.2) are equivalent to the embeddings

$$L^\infty(\Omega) \subset L^{p(\cdot)}(\Omega,\varrho) \subset L^1(\Omega)$$

under which $L^{p(\cdot)}(\Omega,\varrho)$ is a BFS, while condition (2.2) yields the coincidence of the dual and associate spaces, see Theorem 2.5 in [1], and thereby this space has an absolutely continuous norm. □

3.2. Compactness interpolation theorem

In 1960 it was proved by M.A. Krasnoselskii [18] that it is possible to "one-sidedly" interpolate the compactness property in L^p-spaces with a constant p. After that an extension to the case of general Banach space setting was a matter of a study in a series of papers, we refer for instance to [2, 9, 20, 23], where such an extension was made under some hypotheses on the space, which were finally removed in [5].

For the spaces $L^{p(\cdot)}(\Omega)$ with the interpolation spaces realized directly as $L^{p_\theta(\cdot)}(\Omega)$, $\frac{1}{p_\theta(x)} = \frac{\theta}{p_1(x)} + \frac{1-\theta}{p_2(x)}$, $\theta \in (0,1)$, such a one-sided compactness interpolation theorem was derived in [24] from results of [23] and runs as follows.

Theorem 3.3. *Let* $\Omega \subseteq \mathbb{R}^n$ *be an open set and let the variable exponents* $p_j : \Omega \to [1,\infty)$, $j = 1,2$, *satisfy condition* (2.3). *Let a linear operator* A *defined on* $L^{p_1(\cdot)}(\Omega) \cup L^{p_2(\cdot)}(\Omega)$ *be bounded in the spaces* $L^{p_j(\cdot)}(\Omega), j = 1,2$. *If it is compact in the space* $L^{p_1(\cdot)}(\Omega)$, *then it is also compact in every space* $L^{p_\theta(\cdot)}(\Omega)$, *where*

$$\frac{1}{p_\theta(x)} = \frac{\theta}{p_1(x)} + \frac{1-\theta}{p_2(x)}, \qquad \theta \in (0,1].$$

In applications it is convenient to make use of the following statement.

Theorem 3.4. *Let* $\Omega \subseteq \mathbb{R}^n$ *and there are given a function* $p : \Omega \to [1,\infty)$, $p(x)$ *such that* $1 \le p_- \le p(x) \le p_+ < \infty$, *and a number*

$$\begin{cases} p_0 \in (1,\infty), & \text{if } p_- > 1, \\ p_0 = 1, & \text{if } p_- = 1. \end{cases}$$

There exists a function $q : \Omega \to [1, \infty)$ with the similar property $1 \le q_- \le q(x) \le q_+ < \infty$ and a number $\theta \in [0, 1)$ such that $L^{p(\cdot)}(\Omega)$ is an intermediate space between $L^{p_0}(\Omega)$ and $L^{q(\cdot)}(\Omega)$ corresponding to the interpolation parameter θ. Moreover, $q(x)$ may be also chosen so that $q_- > 1$ when $p_- > 1$.

Proof. The interpolating equality $\frac{1}{p(x)} = \frac{\theta}{p_0} + \frac{1-\theta}{q(x)}$ gives the expression for q:

$$q(x) = \frac{p_0(1-\theta)p(x)}{p_0 - \theta p(x)}$$

so that we have only to take care about the choice of $\theta \in (0, 1)$ such that the conditions $q_- > 1$ and $q_+ < \infty$ are fulfilled. This gives the restriction

$$\theta \in (0, \theta_0), \quad \theta_0 = \min\left\{1, \frac{p_0}{p_+}, \frac{p_0'}{p_-'}\right\},$$

(with $\frac{p_0'}{p_-'}$ interpreted as 1 in the case $p_0 = p_- = 1$), which is always possible. □

The importance for applications of the above statement, combined with the compactness interpolation theorem, is obvious: it allows us just to know that an operator is compact in L^{p_0}, then, if it is bounded in variable exponent spaces, it is also compact in such spaces. This approach has already been used in [24] in the study of the normal solvability of pseudodifferential operators. We illustrate this approach by another application in Section 5.

4. Compactness of an integral operator with integrable almost decreasing radial dominant of the kernel in the case $|\Omega| < \infty$

In this section we study the compactness of integral operators

$$Kf(x) = \int_\Omega \mathcal{K}(x, y) f(y)\, dy, \tag{4.1}$$

over an open set Ω of a bounded measure, $|\Omega| < \infty$, whose kernel $K(x, y)$ is dominated by difference kernel, that is,

$$|K(x, y)| \le \mathcal{A}(|x - y|). \tag{4.2}$$

It is well known that in the case $p(x) \equiv p = \text{const}$, operators

$$Kf(x) = \int_\Omega k(x - y) f(y)\, dy$$

over a set Ω with $|\Omega| < \infty$ are compact in $L^p(\Omega), 1 \le p < \infty$, for any integrable kernel $k(x)$ (which follows from a simple fact that a kernel $k \in L^1(\Omega)$ may be approximated in L^1-norm by bounded kernels).

In case of variable $p(x)$ this no more is valid for arbitrary integrable kernels, convolutions with such kernels even are unbounded in general: the Young theorem is not valid for an arbitrary integrable kernel.

There is known a class of convolutions which may be bounded operators in the case of variable $p(x)$. This is the class of convolutions which have radial decreasing integrable majorants, see [6]. Such convolutions are bounded operators in $L^{p(\cdot)}(\Omega), \Omega \subseteq \mathbb{R}^n$, whenever the maximal operator is bounded in this space, which is the consequence of Stein's pointwise estimate

$$|Kf(x)| \leq \|\mathcal{A}\|_1 Mf(x),$$

known under the assumption that $\mathcal{A}(r), r = |x|$, is decreasing. In [3] it was shown that the integrability of the decreasing dominant $\mathcal{A}$ is sufficient for the boundedness of the convolution operator also in the case $p_- = 1$.

The requirement for $\mathcal{A}$ to be decreasing may be slightly weakened to almost decreasing. Recall that a non-negative function $f(t), t \in \mathbb{R}^1_+$, is called almost decreasing if there exists a constant $C = C_f \geq 1$ such that $f(t_2) \leq Cf(t_1)$ for all $t_2 \geq t_1$. This is equivalent to saying that there exists a decreasing function $g(t)$ such that $c_1 g(t) \leq f(t) \leq c_2 g(t)$ where $c_1 > 0, c_2 > 0$. The constant

$$C_f = \sup_{t_2 \geq t_1} \frac{f(t_1)}{f(t_2)}$$

sometimes is called *the coefficient of almost decrease* of f.

In the sequel, when saying that the kernel $k(x)$ has a radial integrable almost decreasing dominant $\mathcal{A}$, we mean that

$$|k(x)| \leq \mathcal{A}(|x|),$$

where $\mathcal{A}(|x|) \in L^1(\mathbb{R}^n)$ and $\mathcal{A}(r)$ is an almost decreasing function

The results on compactness in weighted variable exponent spaces we prove in Subsection 4.2 are based on obtaining a version of Stein's estimate, see Lemma 4.2.

4.1. Non-weighted case

In the non-weighted case, the following compactness theorem for integral operators (4.1) is an immediate consequence of the interpolation Theorems 3.3 and 3.4.

Theorem 4.1. *Let $|\Omega| < \infty, 1 \leq p_- \leq p_+ < \infty$. An integral operator of form* (4.1) *with radial decreasing integrable dominant $\mathcal{A}(|x|)$ of its kernel is compact in the space $L^{p(\cdot)}(\Omega)$, if the maximal operator is bounded in this space.*

In the next subsection we provide another approach which does not use interpolation theorem and allows to cover the weighted case, at the least for a certain class of integral operators. Note that the validity of an interpolation theorem of type of Theorem 3.3 for weighted variable exponent spaces is an open question.

4.2. Weighted case

We assume that the dominant $\mathcal{A}$ in (4.2) is integrable:

$$\int_{B(0,R)} \mathcal{A}(|x|)\, dx < \infty, \quad R = 2 \operatorname{diam} \Omega \tag{4.3}$$

and almost decreasing.

We split the operator K in the standard way:

$$\begin{aligned} Kf(x) &= \int\limits_{|x-y|<\varepsilon} K(x,y)f(y)\,dy + \int\limits_{|x-y|>\varepsilon} K(x,y)f(y)\,dy \\ &=: K_\varepsilon f(x) + T_\varepsilon f(x). \end{aligned} \tag{4.4}$$

The following lemma is crucial for our purposes. In this lemma, in particular, we give a new proof of pointwise Stein's estimation

$$|Kf(x)| \le \|\mathcal{A}\|_1 Mf(x) \tag{4.5}$$

known in form (4.5) for radially decreasing dominants $\mathcal{A}$.

Lemma 4.2. *Let* (4.2) *be satisfied and let* $\mathcal{A}$ *be almost decreasing. Then the following pointwise estimate*

$$|K_\varepsilon f(x)| \le a(\varepsilon) Mf(x), \quad x \in \Omega \tag{4.6}$$

holds, where

$$a(\varepsilon) = (C_{\mathcal{A}})^2 \int\limits_{B(0,\varepsilon)} \mathcal{A}(|x|)\,dx \to 0 \ \text{as} \ \varepsilon \to 0 \tag{4.7}$$

and $C_{\mathcal{A}}$ *is the coefficient of the almost decrease of the function* $\mathcal{A}$*. In the case* $\Omega = \mathbb{R}^n$ *we also have*

$$|Kf(x)| \le (C_{\mathcal{A}})^2 \|\mathcal{A}\|_1 Mf(x), \quad x \in \mathbb{R}^n. \tag{4.8}$$

Proof. To prove (4.6), we use the decomposition

$$|K_\varepsilon f(x)| \le \sum_{k=0}^{\infty} \int\limits_{\lambda^{-k-1}\varepsilon<|y-x|<\lambda^{-k}\varepsilon} \mathcal{A}(|x-y|)|f(y)|\,dy$$

with an arbitrary $\lambda > 1$. Then after standard estimations we obtain (4.6) with

$$a(\varepsilon) = C_{\mathcal{A}} \frac{|\mathbb{S}^{n-1}|}{n} \sum_{k=0}^{\infty} \mathcal{A}(\lambda^{-k}\varepsilon)(\lambda^{-k}\varepsilon)^n. \tag{4.9}$$

To arrive at (4.7), we estimate the integral $\int\limits_{B(0,\varepsilon)} \mathcal{A}(|y|)\,dy$ as follows

$$\begin{aligned} \int\limits_{B(0,\varepsilon)} \mathcal{A}(|y|)\,dy &= \sum_{k=0}^{\infty} \int\limits_{\lambda^{-k-1}\varepsilon<|y|<\lambda^{-k}\varepsilon} \mathcal{A}(|y|)\,dy \\ &\geqq \frac{1}{C_{\mathcal{A}}} \sum_{k=0}^{\infty} \mathcal{A}(\lambda^{-k-1}\varepsilon) \int\limits_{\lambda^{-k-1}\varepsilon<|y|<\lambda^{-k}\varepsilon} dy \end{aligned}$$

which after easy calculations yields

$$\int\limits_{B(0,\varepsilon)} \mathcal{A}(|y|)\,dy \geq \frac{\lambda^n - 1}{C_{\mathcal{A}}} \frac{|\mathbb{S}^{n-1}|}{n} \left[\sum_{k=0}^{\infty} \mathcal{A}(\lambda^{-k}\varepsilon)(\lambda^{-k}\varepsilon)^n - \mathcal{A}(\varepsilon)\varepsilon^n \right].$$

Then by (4.9)

$$a(\varepsilon) \leq (C_{\mathcal{A}})^2 \left(\frac{1}{\lambda^n - 1} + 1 \right) \int\limits_{B(0,\varepsilon)} \mathcal{A}(|y|)\,dy.$$

Since the left-hand side of (4.6) does not depend on $\lambda > 1$, we may pass to the limit as $\lambda \to \infty$, which yields the validity of (4.6)–(4.8). □

Observe that the kernel of the operator T_ε in the representation (4.4) is a bounded function for each $\varepsilon > 0$. Therefore, from Lemma 4.2 we immediately arrive at the following statement.

Theorem 4.3. *An integral operator with radial almost decreasing integrable dominant $\mathcal{A}(|x|)$ of its kernel is compact in a Banach function space $X = X(\Omega)$ with $|\Omega| < \infty$, if*

1. *the maximal operator is bounded in X;*
2. *integral operators with bounded kernel are compact in X.*

In the case where X is a Banach function space with absolutely continuous norm, assumption 2. *may be omitted.*

Proof. The compactness of the operator K under both the assumptions (1)–(2) is obvious in view of representation (4.4) and estimate (4.6). A possibility to omit assumption (2) follows from Theorem 3.1, since the integral operator with constant kernel is one dimensional and consequently compact in every Banach function space. □

Corollary 4.4. *Let $|\Omega| < \infty, 1 \leq p_- \leq p_+ < \infty$ and the weight ϱ satisfy condition* (3.2)*. An integral operator with radial almost decreasing integrable dominant $\mathcal{A}(|x|)$ of its kernel is compact in the space $L^{p(\cdot)}(\Omega, \varrho)$, if the maximal operator is bounded in this space.*

Proof. It suffices to note that $L^{p(\cdot)}(\Omega, \varrho)$ is a Banach function space with absolute norm, under conditions (2.2) and (3.2). □

Remark 4.5. The boundedness of the maximal operator for the case $X = L^{p(\cdot)}(\Omega)$ is known [6] at least under conditions (2.3) and (2.4). Results on the weighted boundedness of the maximal operator in the space $L^{p(\cdot)}(\Omega, \varrho)$ for some classes of weights were given in [16, 12, 13, 14].

5. The case $\Omega = \mathbb{R}^n$: compactness of convolution type operators with coefficients vanishing at infinity

Definition 5.1. *A function $a(x) \in L^\infty(\mathbb{R}^n)$ is said to belong to the class $B_0^{\sup}(\mathbb{R}^n)$, if*

$$\lim_{N\to\infty} \operatorname*{ess\,sup}_{|x|>N} |a(x)| \;=\; 0. \tag{5.1}$$

The following statement is known (see [10, p. 39] and references therein) which is of importance in application to the Fredholmness theory of convolution type equations, see [10, Section 3].

Theorem 5.2. *The operator*

$$(Tf)(x) = a(x) \int_{R^n} k(x-y)b(y)f(y)\,dy\ , \quad x \in \mathbb{R}^n, \tag{5.2}$$

where $k \in L^1(\mathbb{R}^n)$ and $a, b \in L^\infty(\mathbb{R}^n)$, is compact in $L_p(\mathbb{R}^n)$, $1 \le p \le \infty$, when either $a \in B_0^{\sup}$ or $b \in B_0^{\sup}$.

Then from Theorems 5.2 and 3.3–3.4 we arrive at the following statement.

Theorem 5.3. *Let the kernel $k(x)$ have a radial integrable almost decreasing dominant, and $a, b \in L^\infty(R^n)$. Under condition* (2.2) *on $p(x)$, operators T of form* (5.2) *are compact in the space $L^{p(\cdot)}(R^n)$, if*

i) *the maximal operator M is bounded in the space $L^{q(\cdot)}(R^n)$ with any $q(\cdot)$ such that $\frac{1}{q(x)} = \frac{\lambda}{p(x)} - c$, where $\lambda \in (1,\infty), c \in (0,\infty)$ and $\frac{\lambda}{c} \ge p_-$,*
ii) *either $a \in B_0^{\sup}$ or $b \in B_0^{\sup}$.*

To complete Theorem 5.3, it remains to refer to known conditions on $p(x)$ sufficient for the boundedness of the maximal operator over the whole space R^n. As is well known, conditions (2.3), (2.4), (2.5) guarantee such a boundedness. A condition weaker than decay condition (2.5), is the known Nekvinda's condition from [21]. Recently, in [22] that condition was weakened as follows. Let $\ln_k x$ be the k-iterated logarithm, and $s(t)$ a monotone function on $\mathbb{R}^1_+$ such that

$$1 < \inf s(t),\ \sup s(t) < \infty \tag{5.3}$$

and

$$\left|\frac{d}{dt}s(t)\right| \le -K\frac{d}{dt}\left(\frac{1}{\ln_k^\alpha(t)}\right) \quad \text{for large } t, \tag{5.4}$$

where $\alpha > 0, K > 0$. In [22] the following statement was proved.

Proposition 5.4. *Let p satisfy assumptions* (2.3), (2.4). *If $p(x)$ satisfies the condition*

$$\int\limits_{p(x)\neq s(|x|)} c^{\frac{1}{|p(x)-s(|x|)|}}\,dx < \infty \tag{5.5}$$

with $c > 0$ and a monotone function $s(t)$ on $\mathbb{R}^1_+$ fulfilling (5.3)–(5.4), *then the maximal operator is bounded in the space $L^{p(\cdot)}(\mathbb{R}^n)$.*

Then from Theorem 5.3 we obtain the following result.

Theorem 5.5. *Let the kernel $k(x)$ have a radial integrable almost decreasing dominant, and $a, b \in L^\infty(R^n)$ and either $a \in B_0^{\sup}$ or $b \in B_0^{\sup}$. Operators T of form* (5.2) *are compact in the space $L^{p(\cdot)}(R^n)$, if $p(x)$ satisfies conditions* (2.3), (2.4) *and decay-type condition* (5.5) *with some monotone function $s(t)$ satisfying* (5.3)–(5.4).

Proof. Apply Theorem 5.3. Given $p(x)$, Theorem 5.3 requires the existence of the constants $\lambda > 1, c > 0$ such that the maximal operator is bounded in the space $L^{q(\cdot)}(\mathbb{R}^n)$, where $\frac{1}{q(x)} = \frac{\lambda}{p(x)} - c$. It suffices to observe that the validity of condition (5.5) for $p(x)$ with some $s(|x|)$ implies its validity for such $q(x)$ with another monotone function $s_1(t)$ defined by

$$\frac{1}{s_1(t)} = \frac{1}{s(t)} + \frac{c}{\lambda},$$

where $c > 0$ may be chosen small enough to get $\inf s_1(t) > 1$. This completes the proof. □

Acknowledgement

The author is thankful to the anonymous referee for the useful remark which led to the appearance of the final statement of Section 5 in the form of Theorem 5.5.

References

[1] C. Bennett and R. Sharpley. *Interpolation of operators.*, volume 129 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1988.

[2] F. Cobos, T. Kühn and T. Schonbek. One-sided compactness results for Aronsjain-Gagliardo functors. *J. Funct. Anal.*, 106:274–313, 1992.

[3] D. Cruz-Uribe and A. Fiorenza. Approximate identities in variable L^p spaces. *Math. Nachr.*, 280(3):256–270, 2007.

[4] D. Cruz-Uribe, A. Fiorenza and C.J. Neugebauer. The maximal function on variable L^p-spaces. *Ann. Acad. Scient. Fennicae, Math.*, 28:223–238, 2003.

[5] M. Cwikel. Real and complex interpolation and extrapolation of compact operators. *Duke Math. J.*, 65(2): 333–343, 1992.

[6] L. Diening. Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$. *Math. Inequal. Appl.*, 7(2):245–253, 2004.

[7] L. Diening, P. Hästö and A. Nekvinda. Open problems in variable exponent Lebesgue and Sobolev spaces. In *"Function Spaces, Differential Operators and Nonlinear Analysis", Proceedings of the Conference held in Milovy, Bohemian-Moravian Uplands, May* 28*–June* 2*,* 2004. Math. Inst. Acad. Sci. Czech Republick, Praha, 38–58, 2005.

[8] D.E. Edmunds, J. Lang and A. Nekvinda. On $l^{p(x)}$ norms. *Proc. R. Soc. Lond.* A 455: 219–225, 1999.

[9] K. Hayakawa. Interpolation by the real method preserves compactness of operators. *J. Math. Soc. Japan*, 21:189–199, 1969.

[10] N.K. Karapetiants and S.G. Samko. *Equations with Involutive Operators.* Birkhäuser, Boston, 2001.

[11] V. Kokilashvili. On a progress in the theory of integral operators in weighted Banach function spaces. In *"Function Spaces, Differential Operators and Nonlinear Analysis", Proceedings of the Conference held in Milovy, Bohemian-Moravian Uplands, May* 28*–June* 2*,* 2004. Math. Inst. Acad. Sci. Czech Republick, Praha, 152–175, 2005.

[12] V. Kokilashvili, N. Samko and S. Samko. The maximal operator in variable spaces $L^{p(\cdot)}(\Omega, \rho)$. *Georgian Math. J.*, 13(1):109–125, 2006.

[13] V. Kokilashvili, N. Samko and S. Samko. The Maximal Operator in Weighted Variable Spaces $L^{p(\cdot)}$. *J. Function spaces and Appl.*, 5(3):299–317, 2007.

[14] V. Kokilashvili and S. Samko. The maximal operator in weighted variable Lebesgue spaces on metric spaces. *Georgian Math. J.*, 15, no 4, 683–712, 2008.

[15] V. Kokilashvili and S. Samko. Singular Integral Equations in the Lebesgue Spaces with Variable Exponent. *Proc. A. Razmadze Math. Inst.*, 131:61–78, 2003.

[16] V. Kokilashvili and S. Samko. Maximal and fractional operators in weighted $L^{p(x)}$ spaces. *Revista Matematica Iberoamericana*, 20(2):493–515, 2004.

[17] V. Kokilashvili and S. Samko. Weighted boundedness of the maximal, singular and potential operators in variable exponent spaces. In A.A.Kilbas and S.V.Rogosin, editors, *Analytic Methods of Analysis and Differential Equations*, pages 139–164. Cambridge Scientific Publishers, 2008.

[18] M.A. Krasnosel′skii. On a theorem of M. Riesz. *Soviet Math. Dokl.*, 1:229–231, 1960.

[19] M.A. Krasnosel′skii, P.P. Zabreiko, E.I. Pustyl'nik and P.E. Sobolevskii. *Integral Operators in Spaces of Summable Functions.* (Russian). Moscow: Nauka, 1968. 499 pages.

[20] J.L. Lions and J. Peetre. Sur une classe d'espaces d'interpolation. *Inst. Hautes Etudes Sci. Publ. Math.*, 19:5–66, 1964.

[21] A. Nekvinda. Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R}^n)$. *Math. Inequal. and Appl.*, 7(2):255–265, 2004.

[22] A. Nekvinda. Maximal operator on variable Lebesgue spaces for almost monotone radial exponent. *J. Math. Anal. and Appl.*, 337(2):1345–1365, 2008.

[23] A. Persson. Compact linear mappings between interpolation spaces. *Ark. Mat.*, 5:215–219 (1964), 1964.

[24] V.S. Rabinovich and S.G Samko. Boundedness and Fredholmness of pseudodifferential operators in variable exponent spaces. *Integr. Eq. Oper. Theory*, 60(4):507–537, 2008.

[25] H. Rafeiro and S. Samko. Dominated compactness theorem in Banach function spaces and its applications. *Compl. Anal. Oper. Theory.* 2(4): 669–681, 2008.

[26] H. Rafeiro and S. Samko. On a class of fractional type integral equations in variable exponent spaces. *Fract. Calc. and Appl. Anal.*, (4):399–421, 2007.

[27] M. Ružička. *Electroreological Fluids: Modeling and Mathematical Theory.* Springer, Lecture Notes in Math., 2000. vol. 1748, 176 pages.

[28] S.G. Samko. On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators. *Integr. Transf. and Spec. Funct*, 16(5-6):461–482, 2005.

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Received: March 11, 2009
Accepted: June 9, 2009

Operator Theory:
Advances and Applications, Vol. 202, 509–518

Extension to an Invertible Matrix in Convolution Algebras of Measures Supported in $[0, +\infty)$

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Abstract. Let $\mathcal{M}_+$ denote the Banach algebra of all complex Borel measures with support contained in $[0, +\infty)$, with the usual addition and scalar multiplication, and with convolution $*$, and the norm being the total variation of μ. We show that the maximal ideal space $X(\mathcal{M}_+)$ of $\mathcal{M}_+$, equipped with the Gelfand topology, is contractible as a topological space. In particular, it follows that every left invertible matrix with entries from $\mathcal{M}_+$ can be completed to an invertible matrix, that is, the following statements are equivalent for $f \in (\mathcal{M}_+)^{n\times k}$, $k < n$:

1. There exists a matrix $g \in \mathcal{M}_+^{k\times n}$ such that $g * f = I_k$.
2. There exist matrices $F, G \in \mathcal{M}_+^{n\times n}$ such that $G * F = I_n$ and $F_{ij} = f_{ij}$, $1 \le i \le n$, $1 \le j \le k$.

We also show a similar result for all subalgebras of $\mathcal{M}_+$ satisfying a mild condition.

Mathematics Subject Classification (2000). Primary 54C40; Secondary 46J10, 32A38, 93D15.

Keywords. Contractibility of the maximal ideal space, convolution algebra of measures, Hermite ring, Tolokonnikov's lemma, coprime factorization.

1. Introduction

The aim of this paper is to show that the maximal ideal space $X(\mathcal{M}_+)$ of the Banach algebra $\mathcal{M}_+$ of all complex Borel measures with support in $[0, +\infty)$ (defined below), is contractible. We also apply this result to the problem of completing a left invertible matrix with entries in $\mathcal{M}_+$ to an invertible matrix over $\mathcal{M}_+$.

Definition 1.1. Let $\mathcal{M}_+$ denote the set of all complex Borel measures with support contained in $[0, +\infty)$. Then $\mathcal{M}_+$ is a complex vector space with addition and

Communicated by J.A. Ball.

scalar multiplication defined as usual, and it becomes a complex algebra if we take convolution of measures as the operation of multiplication. With the norm of μ taken as the total variation of μ, $\mathcal{M}_+$ is a Banach algebra. Recall that the *total variation* $\|\mu\|$ *of* μ is defined by

$$\|\mu\| = \sup \sum_{n=1}^{\infty} |\mu(E_n)|,$$

the supremum being taken over all *partitions of* $[0, +\infty)$, that is over all countable collections $(E_n)_{n\in\mathbb{N}}$ of Borel subsets of $[0, +\infty)$ such that $E_n \bigcap E_m = \emptyset$ whenever $m \neq n$ and $[0, +\infty) = \bigcup_{n\in\mathbb{N}} E_n$. The identity with respect to convolution in $\mathcal{M}_+$ is the *Dirac measure* δ, given by

$$\delta(E) = \begin{cases} 1 & \text{if } 0 \in E, \\ 0 & \text{if } 0 \notin E. \end{cases}$$

The above Banach algebra is classical, and we refer the reader to the book [1, §4, p. 141–150] for a detailed exposition.

Notation 1.2. Let $X(\mathcal{M}_+)$ denote the maximal ideal space of the Banach algebra $\mathcal{M}_+$, that is the set of all nonzero complex homomorphisms from $\mathcal{M}_+$ to $\mathbb{C}$. We equip $X(\mathcal{M}_+)$ with the Gelfand topology, that is, the weak-$*$ topology induced from the dual space $\mathcal{L}(\mathcal{M}_+; \mathbb{C})$ of the Banach space $\mathcal{M}_+$.

We will show that $X(\mathcal{M}_+)$ is contractible. We recall the notion of contractibility below:

Definition 1.3. A topological space X is said to be *contractible* if there exists a continuous map $H : X \times [0, 1] \to X$ and an $x_0 \in X$ such that for all $x \in X$, $H(x, 0) = x$ and $H(x, 1) = x_0$.

Our main result is the following:

Theorem 1.4. $X(\mathcal{M}_+)$ *is contractible.*

In particular, by a result proved by V.Ya. Lin, the above implies that the ring $\mathcal{M}_+$ is Hermite. Before stating this result, we recall the definition of a Hermite ring:

Definition 1.5. Let R be a ring with an identity element denoted by 1. Let us denote by $I_k \in R^{k\times k}$ the diagonal matrix of size $k \times k$ with all the diagonal entries equal to the identity element 1. A matrix $f \in R^{n\times k}$ is called *left invertible* if there exists a matrix $g \in R^{k\times n}$ such that $gf = I_k$.

The ring R is called a *Hermite ring* if for all $k, n \in \mathbb{N}$ with $k < n$ and all left invertible matrices $f \in R^{n\times k}$, there exist matrices $F, G \in R^{n\times n}$ such that $GF = I_n$ and $F_{ij} = f_{ij}$ for all $1 \leq i \leq n$ and $1 \leq j \leq k$.

We now recall Lin's result; [2, Theorem 3, p. 127]:

Proposition 1.6. *Let R be a commutative Banach algebra with identity. If the maximal ideal space $X(R)$ (equipped with the Gelfand topology) of the Banach algebra R is contractible, then R is a Hermite ring.*

Using the above result, our main result given in Theorem 1.4 then implies the following.

Corollary 1.7. *$\mathcal{M}_+$ is a Hermite ring, that is, the following statements are equivalent for $f \in (\mathcal{M}_+)^{n\times k}$, $k < n$:*

1. *There exists a matrix $g \in \mathcal{M}_+^{k\times n}$ such that $g * f = I_k$.*
2. *There exist matrices $F, G \in \mathcal{M}_+^{n\times n}$ such that $G * F = I_n$ and $F_{ij} = f_{ij}$, $1 \le i \le n$, $1 \le j \le k$.*

(In the above, $*$ denotes convolution, and F_{ij}, f_{ij} denote the entries in the ith row and jth column, of the matrices F and f, respectively.)

1.1. Relevance of the Hermiteness of $\mathcal{M}_+$ in control theory

The motivation for proving that $\mathcal{M}_+$ is a Hermite ring arises from control theory, where it plays an important role in the problem of stabilization of linear systems. Let $\widehat{\mathcal{M}_+}$ denote the integral domain of Laplace transforms of elements of $\mathcal{M}_+$. Then $\widehat{\mathcal{M}_+}$ is a class of "stable" transfer functions, in the sense that if the plant transfer function $g = \widehat{\mu}$ belongs to $\widehat{\mathcal{M}_+}$, then nice inputs are mapped to nice outputs in a continuous manner: if the initial state of the system is 0, and the input $u \in L^p(0,+\infty)$, where $1 \le p \le +\infty$, then the corresponding output[1] $y = \mu * u$ is in $L^p(0,+\infty)$ (here μ is the inverse Laplace transform of g). Moreover,

$$\sup_{0\neq u\in L^p(0,+\infty)} \frac{\|y\|_p}{\|u\|_p} \le \|g\|.$$

In fact one has equality above if $p = 1$ or $p = +\infty$.

The result that $\mathcal{M}_+$ is Hermite implies that if a system with a transfer function G in the field of fractions of $\widehat{\mathcal{M}_+}$ has a right (or left) coprime factorization, then G has a doubly coprime factorization, and the standard Youla parameterization yields all stabilizing controllers for G. For further details on the relevance of the Hermite property in control theory, see [5, Theorem 66, p. 347].

Unfortunately, a nice analytic test for checking right invertibility is not available; see [1, Theorem 4.18.5, p. 149]. This has been the reason that in control theory, one uses the subalgebra $\mathcal{A}$ of $\mathcal{M}_+$ consisting of those measures from $\mathcal{M}_+$ for which the non-atomic singular part is 0, for which an analytic condition for left invertibility is indeed available [1, Theorem 4.18.6]. The Hermite property of $\mathcal{A}$, which was mentioned as an open problem in Vidyasagar's book [5, p. 360], was proved in [4]. The proof of the Hermite property of $\mathcal{M}_+$ we give here is inspired from the calculation done in [4].

In Section 3, we will give the proof of Theorem 1.4, but before doing that, in Section 2, we first prove a few technical results which will be used in the sequel.

[1]equivalently $\widehat{y}(s) = g(s)\widehat{u}(s)$, for all s in some right half-plane in $\mathbb{C}$

2. Preliminaries

In this section, we prove a few auxiliary facts, which will be needed in order to prove our main result.

Definition 2.1. If $\mu \in \mathcal{M}_+$ and $\theta \in [0,1)$, then we define the complex Borel measure μ_θ as follows:

$$\mu_\theta(E) := \int_E (1-\theta)^t d\mu(t),$$

where E is a Borel subset of $[0,+\infty)$. If $\theta = 1$, then we define

$$\mu_\theta = \mu_1 := \mu(\{0\})\delta.$$

It can be seen that $\mu_\theta \in \mathcal{M}_+$ and that $\|\mu_\theta\| \leq \|\mu\|$. Also $\delta_\theta = \delta$ for all $\theta \in [0,1]$.

Proposition 2.2. *If $\mu, \nu \in \mathcal{M}_+$, then for all $\theta \in [0,1]$,*

$$(\mu * \nu)_\theta = \mu_\theta * \nu_\theta.$$

Proof. If E is a Borel subset of $[0,+\infty)$, then

$$(\mu * \nu)_\theta(E) = \int_E (1-\theta)^t d(\mu*\nu)(t) = \iint_{\substack{\sigma+\tau\in E\\ \sigma,\tau\in[0,+\infty)}} (1-\theta)^{\sigma+\tau} d\mu(\sigma)d\nu(\tau).$$

On the other hand,

$$\begin{aligned}(\mu_\theta * \nu_\theta)(E) &= \int_{\tau\in[0,+\infty)} \mu_\theta(E-\tau)d\nu_\theta(\tau)\\ &= \int_{\tau\in[0,+\infty)} \left(\int_{\substack{\sigma\in E-\tau\\ \sigma\in[0,+\infty)}} (1-\theta)^\sigma d\mu(\sigma)\right) d\nu_\theta(\tau)\\ &= \iint_{\substack{\sigma+\tau\in E\\ \sigma,\tau\in[0,+\infty)}} (1-\theta)^{\sigma+\tau} d\mu(\sigma)d\nu(\tau).\end{aligned}$$

This completes the proof. □

The following result says that for a fixed μ, the map $\theta \mapsto \mu_\theta : [0,1] \to \mathcal{M}_+$ is continuous.

Proposition 2.3. *If $\mu \in \mathcal{M}_+$ and $\theta_0 \in [0,1]$, then*

$$\lim_{\theta\to\theta_0} \mu_\theta = \mu_{\theta_0}$$

in $\mathcal{M}_+$.

Proof. Consider first the case when $\theta_0 \in [0,1)$. Given an $\epsilon > 0$, first choose an $R > 0$ large enough so that $|\mu|((R,+\infty)) < \epsilon$. Let $\theta \in [0,1)$. There exists a Borel measurable function w such that $d(\mu_\theta - \mu_{\theta_0})(t) = e^{-iw(t)} d|\mu_\theta - \mu_{\theta_0}|(t)$. Thus

$$\begin{aligned}
\|\mu_\theta - \mu_{\theta_0}\| &= |\mu_\theta - \mu_{\theta_0}|([0,+\infty)) \\
&= \int_{[0,+\infty)} e^{iw(t)} d(\mu_\theta - \mu_{\theta_0})(t) \\
&= \left| \int_{[0,+\infty)} e^{iw(t)} d(\mu_\theta - \mu_{\theta_0})(t) \right| \\
&= \left| \int_{[0,+\infty)} e^{iw(t)} \Big((1-\theta)^t - (1-\theta_0)^t \Big) d\mu(t) \right| .
\end{aligned}$$

Hence

$$\begin{aligned}
\|\mu_\theta - \mu_{\theta_0}\| &\leq \left| \int_{[0,R]} e^{iw(t)} \Big((1-\theta)^t - (1-\theta_0)^t \Big) d\mu(t) \right| \\
&\quad + \left| \int_{(R,+\infty)} e^{iw(t)} \Big((1-\theta)^t - (1-\theta_0)^t \Big) d\mu(t) \right| \\
&\leq \max_{t\in[0,R]} \left| (1-\theta)^t - (1-\theta_0)^t \right| |\mu|([0,R]) + 2|\mu|((R,+\infty)) \\
&\leq \max_{t\in[0,R]} \left| (1-\theta)^t - (1-\theta_0)^t \right| |\mu|([0,+\infty)) + 2\epsilon.
\end{aligned}$$

But by the mean value theorem applied to the function $\theta \mapsto (1-\theta)^t$,

$$(1-\theta)^t - (1-\theta_0)^t = (\theta-\theta_0)t(1-c)^{t-1} = (\theta-\theta_0)t\frac{(1-c)^t}{1-c},$$

for some c (depending on t, θ and θ_0) in between θ and θ_0. Since c lies between θ and θ_0, and since both θ and θ_0 lie in $[0,1)$, and $t \in [0,R]$, it follows that $(1-c)^t \leq 1$ and

$$\frac{1}{1-c} \leq \max\left\{ \frac{1}{1-\theta}, \frac{1}{1-\theta_0} \right\}.$$

Thus using the above and the fact that $|t| \leq R$,

$$\begin{aligned}
\max_{t\in[0,R]} \left| (1-\theta)^t - (1-\theta_0)^t \right| &= \max_{t\in[0,R]} |\theta-\theta_0||t||(1-c)^t| \frac{1}{|1-c|} \\
&\leq |\theta-\theta_0| \cdot R \cdot 1 \cdot \max\left\{ \frac{1}{1-\theta}, \frac{1}{1-\theta_0} \right\}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\limsup_{\theta\to\theta_0}\left(\max_{t\in[0,R]}\left|(1-\theta)^t-(1-\theta_0)^t\right||\mu|([0,+\infty))\right)\\
\leq\;& \limsup_{\theta\to\theta_0}\left(|\theta-\theta_0|\cdot R\cdot\max\left\{\frac{1}{1-\theta},\frac{1}{1-\theta_0}\right\}\cdot|\mu|([0,+\infty))\right)\\
=\;& 0\cdot R\cdot\frac{1}{1-\theta_0}|\mu|([0,+\infty))\\
=\;& 0.
\end{aligned}$$

Consequently,

$$\limsup_{\theta\to\theta_0}\|\mu_\theta-\mu_{\theta_0}\|\leq 2\epsilon.$$

But the choice of $\epsilon>0$ was arbitrary, and so

$$\limsup_{\theta\to\theta_0}\|\mu_\theta-\mu_{\theta_0}\|=0.$$

Since $\|\mu_\theta-\mu_{\theta_0}\|\geq 0$, we can conclude that

$$\lim_{\theta\to\theta_0}\|\mu_\theta-\mu_{\theta_0}\|=0.$$

Now let us consider the case when $\theta_0=1$. Let us assume for the moment that $\mu(\{0\})=0$. We will show that

$$\lim_{\theta\to 1}\mu_\theta=0$$

in $\mathcal{M}_+$. Given an $\epsilon>0$, first choose a $r>0$ small enough so that $|\mu|([0,r]))<\epsilon$. (This is possible, since $\mu(\{0\})=0$.) There exists a Borel measurable function w such that $d\mu_\theta(t)=e^{-iw(t)}d|\mu_\theta|(t)$. Thus

$$\begin{aligned}
\|\mu_\theta\| &= |\mu_\theta|([0,+\infty))=\int_{[0,+\infty)}e^{iw(t)}d\mu_\theta(t)\\
&= \int_{[0,+\infty)}e^{iw(t)}(1-\theta)^t d\mu(t)=\left|\int_{[0,+\infty)}e^{iw(t)}(1-\theta)^t d\mu(t)\right|\\
&\leq \left|\int_{[0,r]}e^{iw(t)}(1-\theta)^t d\mu(t)\right|+\left|\int_{(r,+\infty)}e^{iw(t)}(1-\theta)^t d\mu(t)\right|\\
&\leq |\mu|([0,r])+(1-\theta)^r|\mu|((r,+\infty))\\
&\leq \epsilon+(1-\theta)^r|\mu|([0,+\infty)).
\end{aligned}$$

Consequently,

$$\limsup_{\theta\to 1}\|\mu_\theta-\mu_{\theta_0}\|\leq\epsilon.$$

But the choice of $\epsilon>0$ was arbitrary, and so

$$\limsup_{\theta\to 1}\|\mu_\theta\|=0.$$

Since $\|\mu_\theta\| \geq 0$, we can conclude that

$$\lim_{\theta \to 1} \|\mu_\theta\| = 0.$$

Finally, if $\mu(\{0\}) \neq 0$, then define

$$\nu := \mu - \mu(\{0\})\delta \in \mathcal{M}_+.$$

It is clear that $\nu(\{0\}) = 0$ and $\nu_\theta = \mu_\theta - \mu(\{0\})\delta$. Since

$$\lim_{\theta \to 1} \nu_\theta = 0,$$

we obtain

$$\lim_{\theta \to 1} \mu_\theta = \mu(\{0\})\delta$$

in $\mathcal{M}_+$. □

3. Contractibility of $X(\mathcal{M}_+)$

In this section we will prove our main result.

Proof of Theorem 1.4. Define $\varphi_{+\infty} : \mathcal{M}_+ \to \mathbb{C}$ by $\varphi_{+\infty}(\mu) = \mu(\{0\})$, $\mu \in X(\mathcal{M}_+)$. It can be checked that $\varphi_{+\infty} \in X(\mathcal{M}_+)$; see [1, Theorem 4.18.1, p. 147]. We will construct a continuous map $H : X(\mathcal{M}_+) \times [0,1] \to X(\mathcal{M}_+)$ such that

$$\begin{aligned}
&\text{for all } \varphi \in X(\mathcal{M}_+),\ H(\varphi, 0) = \varphi, \text{ and}\\
&\text{for all } \varphi \in X(\mathcal{M}_+),\ H(\varphi, 1) = \varphi_{+\infty}.
\end{aligned}$$

The map H is defined as follows:

$$(H(\varphi,\theta))(\mu) = \varphi(\mu_\theta), \quad \mu \in \mathcal{M}_+, \quad \theta \in [0,1]. \tag{1}$$

We show that H is well defined. From the definition, $H(\varphi, 1) = \varphi_{+\infty} \in X(\mathcal{M}_+)$ for all $\varphi \in X(\mathcal{M}_+)$. If $\theta \in [0, 1)$, then the linearity of $H(\varphi, \theta) : \mathcal{M}_+ \to \mathbb{C}$ is obvious. Continuity of $H(\varphi,\theta)$ follows from the fact that φ is continuous and $\|\mu_\theta\| \leq \|\mu\|$. That $H(\varphi,\theta)$ is multiplicative is a consequence of Proposition 2.2, and the fact that φ respects multiplication. Finally $(H(\varphi,\theta))(\delta) = \varphi(\delta_\theta) = \varphi(\delta) = 1$.

That $H(\cdot, 0)$ is the identity map and $H(\cdot, 1)$ is a constant map is obvious.

Finally, we show below that H is continuous. Since $X(\mathcal{M}_+)$ is equipped with the Gelfand topology, we just have to prove that for every convergent net $(\varphi_i, \theta_i)_{i\in I}$ with limit (φ, θ) in $X(\mathcal{M}_+)\times[0,1]$, there holds that $(H(\varphi_i,\theta_i))(\mu) \to (H(\varphi,\theta))(\mu)$. We have

$$\begin{aligned}
|(H(\varphi_i,\theta_i))(\mu) - (H(\varphi,\theta))(\mu)| &= |\varphi_i(\mu_{\theta_i}) - \varphi_i(\mu_\theta) + \varphi_i(\mu_\theta) - \varphi(\mu_\theta)|\\
&\leq |\varphi_i(\mu_{\theta_i}) - \varphi_i(\mu_\theta)| + |\varphi_i(\mu_\theta) - \varphi(\mu_\theta)|\\
&= |\varphi_i(\mu_{\theta_i} - \mu_\theta)| + |(\varphi_i - \varphi)(\mu_\theta)|\\
&\leq \|\varphi_i\| \cdot \|\mu_{\theta_i} - \mu_\theta\| + |(\varphi_i - \varphi)(\mu_\theta)|\\
&\leq 1 \cdot \|\mu_{\theta_i} - \mu_\theta\| + |(\varphi_i - \varphi)(\mu_\theta)| \to 0.
\end{aligned}$$

This completes the proof. □

In [4], we had used the explicit description of the maximal ideal space $X(\mathcal{A})$ of the algebra $\mathcal{A}$ (of those complex Borel measures that do not have a singular non-atomic part) in order to prove that $X(\mathcal{A})$ is contractible. Such an explicit description of the maximal ideal space $X(\mathcal{M}_+)$ of $\mathcal{M}_+$ does not seem to be available explicitly in the literature on the subject.

Our definition of the map H is based on the following consideration, which can be thought of as a generalization of the Riemann-Lebesgue Lemma for functions $f_a \in L^1(0,+\infty)$ (which says that the limit as $s \to +\infty$ of the Laplace transform of f_a is 0):

Theorem 3.1. *If $\mu \in \mathcal{M}_+$, then*

$$\lim_{s\to+\infty} \int_0^{+\infty} e^{-st} d\mu(t) = \mu(\{0\}).$$

The set $X(\mathcal{M}_+)$ contains the half-plane

$$\mathbb{C}_{\geq 0} := \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\}$$

in the sense that each $s \in \mathbb{C}_{\geq 0}$, gives rise to the corresponding complex homomorphism $\varphi_s : \mathcal{M}_+ \to \mathbb{C}$, given simply by point evaluation of the Laplace transform of μ at s:

$$\mu \mapsto \varphi_s(\mu) = \int_0^{+\infty} e^{-st} d\mu(t), \quad \mu \in \mathcal{M}_+.$$

If we imagine s tending to $+\infty$ along the real axis we see, in light of the Theorem 3.1 stated above, that φ_s starts looking more and more like $\varphi_{+\infty}$. So we may define

$$H(\varphi_s, \theta) = \varphi_{s-\log(1-\theta)},$$

which would suggest that at least the part $\mathbb{C}_{\geq 0}$ of $X(\mathcal{M}_+)$ is contractible to $\varphi_{+\infty}$. But we see that we can view the action of $H(\varphi_s,\theta)$ defined above as follows:

$$\begin{aligned}
(H(\varphi_s,\theta))(\mu) &= \varphi_{s-\log(1-\theta)}(\mu) \\
&= \int_0^{+\infty} e^{-(s-\log(1-\theta))t} d\mu(t) \\
&= \int_0^{+\infty} e^{-st}(1-\theta)^t d\mu(t) \\
&= \varphi_s(\nu),
\end{aligned}$$

where ν is the measure such that $d\nu(t) = (1-\theta)^t d\mu(t)$. This motivates the definition of H given in (1).

4. Hermite-ness of some subalgebras of $\mathcal{M}_+$

The proof of Theorem 1.4 shows that in fact it works for all subalgebras R of $\mathcal{M}_+$ which are closed under the operation $\mu \mapsto \mu_\theta$, $\theta \in [0,1]$.

Theorem 4.1. *Suppose that R is a Banach subalgebra of $\mathcal{M}_+$, such that it has the property:*

$$\text{(P) For all } \mu \in R \text{ and for all } \theta \in [0,1], \ \mu_\theta \in R.$$

Then the maximal ideal space $X(R)$ equipped with the Gelfand topology is contractible. In particular, the ring R is Hermite, that is, the following statements are equivalent for $f \in R^{n\times k}$, $k < n$:

1. *There exists a matrix $g \in R^{k\times n}$ such that $g * f = I_k$.*
2. *There exist matrices $F, G \in R^{n\times n}$ such that $G * F = I_n$ and $F_{ij} = f_{ij}$, $1 \le i \le n$, $1 \le j \le k$.*

As specific examples of R, we consider the following:

(a) Consider the Wiener-Laplace algebra $\mathcal{W}^+$ of the half-plane, of all functions defined in the half-plane $\mathbb{C}_{\geq 0}$ that differ from the Laplace transform of an $L^1(0,+\infty)$ function by a constant. The Wiener-Laplace algebra $\mathcal{W}^+$ is a Banach algebra with pointwise operations and the norm

$$\|\widehat{f} + \alpha\|_{W^+} = \|\widehat{f}\|_{L^1} + |\alpha|, \quad f \in L^1(0,+\infty),\ \alpha \in \mathbb{C}.$$

Then $\mathcal{W}^+$ is precisely the set of Laplace transforms of elements of the subalgebra of $\mathcal{M}_+$ consisting of all complex Borel measures of the type $\mu_a + \alpha\delta$, where μ_a is absolutely continuous (with respect to the Lebesgue measure) and $\alpha \in \mathbb{C}$. This subalgebra of $\mathcal{M}_+$ has the property (P) demanded in the statement of Theorem 4.1, and so the maximal ideal space $X(\mathcal{W}^+)$ is contractible.

(b) Also we recover the main result in [4], but this time without recourse to the explicit description of the maximal ideal space of $\mathcal{A}$. Indeed, the subalgebra $\mathcal{A}$ of $\mathcal{M}_+$, consisting of all complex Borel measures that do not have a singular non-atomic part, also possesses the property (P).

(c) Finally, we consider the algebra almost-periodic Wiener algebra APW^+, of sums

$$f(s) = \sum_{k=1}^{\infty} f_k e^{-st_k}, \quad s \in \mathbb{C}_{\geq 0}$$

where $t_0 = 0 < t_1, t_2, t_3, \dots$ and $\displaystyle\sum_{k=0}^{\infty} |f_k| < +\infty$.

This algebra is isometrically isomorphic to the subalgebra of $\mathcal{M}_+$ of atomic measures μ. Since this subalgebra has the property (P), it follows that APW^+ is a Hermite ring.

In each of the above algebras $\mathcal{W}^+$, $\mathcal{A}$ or APW^+, the corona theorem holds, that is, there is an analytic condition which is equivalent to left-invertibility. (The proofs/references of the corona theorems for $\mathcal{W}^+$, $\mathcal{A}$ and APW^+ can be found for example in [3, Theorem 4.3].) Combining the Hermite-ness with the corona theorem, we obtain the following:

Corollary 4.2. *Let R be any one of the algebras $\mathcal{W}^+$, $\mathcal{A}$ or APW^+. Then the following statements are equivalent for $f \in R^{n\times k}$, $k < n$:*

1. *There exists a matrix $g \in R^{k\times n}$ such that $gf = I_k$.*
2. *There exist matrices $F, G \in R^{n\times n}$ such that $GF = I_n$ and $F_{ij} = f_{ij}$ for all $1 \le i \le n$, $1 \le j \le k$.*
3. *There exists a $\delta > 0$ such that for all $s \in \mathbb{C}_{\geq 0}$, $f(s)^* f(s) \geq \delta^2 I$.*

Acknowledgement

Thanks are due to Serguei Shimorin who raised the question of whether $\mathcal{M}_+$ is Hermite or not from the audience when I gave a talk on the result in [4] at the Royal Institute of Technology (KTH), Stockholm in August, 2008. I am grateful to Adam Ostaszewski from the London School of Economics for showing me a proof of the generalization of the Riemann-Lebesgue theorem (Theorem 3.1) for measures in $\mathcal{M}_+$.

References

[1] E. Hille and R.S. Phillips. *Functional analysis and semi-groups.* Third printing of the revised edition of 1957. American Mathematical Society Colloquium Publications, Vol. XXXI. American Mathematical Society, Providence, R.I., 1974.

[2] V.Ya. Lin. Holomorphic fiberings and multivalued functions of elements of a Banach algebra. *Functional Analysis and its Applications*, no. 2, 7:122–128, 1973, English translation.

[3] K.M. Mikkola and A.J. Sasane. Bass and Topological Stable Ranks of Complex and Real Algebras of Measures, Functions and Sequences. To appear in *Complex Analysis and Operator Theory.*

[4] A.J. Sasane. The Hermite property of a causal Wiener algebra used in control theory. To appear in *Complex Analysis and Operator Theory.*

[5] M. Vidyasagar. *Control System Synthesis: A Factorization Approach.* MIT Press Series in Signal Processing, Optimization, and Control, 7, MIT Press, Cambridge, MA, 1985.

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Received: November 1, 2008
Accepted: March 24, 2009

Operator Theory:
Advances and Applications, Vol. 202, 519–534

The Invariant Subspace Problem via Composition Operators-redux

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Abstract. The Invariant Subspace Problem ("ISP") for Hilbert space operators is known to be equivalent to a question that, on its surface, seems surprisingly concrete: *For composition operators induced on the Hardy space H^2 by hyperbolic automorphisms of the unit disc, is every nontrivial minimal invariant subspace one dimensional (i.e., spanned by an eigenvector)?* In the hope of reviving interest in the contribution this remarkable result might offer to the studies of both composition operators and the ISP, I revisit some known results, weaken their hypotheses and simplify their proofs. *Sample results:* If φ is a hyperbolic disc automorphism with fixed points at α and β (both necessarily on the unit circle), and C_φ the composition operator it induces on H^2, then for every $f \in \sqrt{(z-\alpha)(z-\beta)}\, H^2$, the doubly C_φ-cyclic subspace generated by f contains many independent eigenvectors; more precisely, the point spectrum of C_φ's restriction to that subspace intersects the unit circle in a set of positive measure. Moreover, this restriction of C_φ is hypercyclic (some forward orbit is dense). Under the stronger restriction $f \in \sqrt{(z-\alpha)(z-\beta)}\, H^p$ for some $p > 2$, the point spectrum of the restricted operator contains an open annulus centered at the origin.

Mathematics Subject Classification (2000). Primary 47B33; Secondary 47A15.

Keywords. Composition operator, hyperbolic automorphism, Invariant Subspace Problem.

1. Introduction

More than twenty years ago Nordgren, Rosenthal, and Wintrobe [8] made a surprising connection between composition operators on the Hardy space H^2 and the Invariant Subspace Problem – henceforth, the "ISP". The ISP asks if every operator on a separable Hilbert space has a nontrivial invariant subspace (following tradition: "operator" means "bounded linear operator," "subspace" means "closed

Communicated by I.M. Spitkovsky.

linear manifold," and for a subspace, "nontrivial" means "neither the whole space nor the zero-subspace"). Nordgren, Rosenthal, and Wintrobe proved the following [8, Corollary 6.3, page 343]:

> *Suppose φ is a hyperbolic automorphism of the open unit disc $\mathbb{U}$. Let C_φ denote the composition operator induced by φ on the Hardy space H^2. Then the ISP has a positive solution if and only if every nontrivial minimal C_φ-invariant subspace of H^2 has dimension one.*

It is easy to see that, for each nontrivial minimal invariant subspace V of a Hilbert space operator T, every non-zero vector $x \in V$ is cyclic; i.e., $\text{span}\,\{T^n x : n = 0, 1, 2, \ldots\}$ is dense in V. If, in addition, T is invertible, then so is its restriction to V (otherwise the range of this restriction would be a nontrivial invariant subspace strictly contained in V, contradicting minimality). Thus for T invertible, V a nontrivial minimal invariant subspace of T, and $0 \neq x \in V$,

$$V = \overline{\text{span}}\,\{T^n x : n = 0, 1, 2, \ldots\} = \overline{\text{span}}\,\{T^n x : n \in \mathbb{Z}\},$$

where now "$\overline{\text{span}}$" means "closure of the linear span."

The result of Nordgren, Rosenthal, and Wintrobe therefore suggests that for φ a hyperbolic disc automorphism we might profitably study how the properties of a function f in $H^2\backslash\{0\}$ influence the operator-theoretic properties of $C_\varphi|_{D_f}$, the restriction of C_φ to the "doubly cyclic" subspace subspace

$$D_f := \overline{\text{span}}\,\{C_\varphi^n f : n \in \mathbb{Z}\} = \overline{\text{span}}\,\{f \circ \varphi_n : n \in \mathbb{Z}\}, \tag{1.1}$$

with particular emphasis on the question of when the point spectrum of the restricted operator is nonempty. (Here, for n is a positive integer, φ_n denotes the nth compositional iterate of φ, while φ_{-n} is the nth iterate of φ^{-1}; φ_0 is the identity map.)

Along these lines, Valentin Matache [4, 1993] obtained a number of interesting results on minimal invariant subspaces for hyperbolic-automorphically induced composition operators. He observed, for example, that if a minimal invariant subspace for such an operator were to have dimension larger than 1, then, at either of the fixed points of φ, none of the non-zero elements of that subspace could be both continuous and non-vanishing (since φ is a hyperbolic automorphism of the unit disc, its fixed points must necessarily lie on the unit circle; see §2.1 below). Matache also obtained interesting results on the possibility of minimality for invariant subspaces generated by inner functions.

Several years later, Vitaly Chkliar [3, 1996] proved the following result for hyperbolic-automorphic composition operators C_φ:

> *If $f \in H^2\backslash\{0\}$ is bounded in a neighborhood of one fixed point of φ, and at the other fixed point vanishes to some order $\varepsilon > 0$, then the point spectrum of $C_\varphi|_{D_f}$ contains an open annulus centered at the origin.*

Later Matache [5] obtained similar conclusions under less restrictive hypotheses.

In the work below, after providing some background (in §2), I revisit in §3 and §4 the work of Chkliar and Matache, providing simpler proofs of stronger results. Here is a sample: *for* φ *a hyperbolic automorphism of* $\mathbb{U}$ *with fixed points* α *and* β *(necessarily on* $\partial\mathbb{U}$*):*

(a) *If* $f \in [(z-\alpha)(z-\beta)]^{1/2}H^2\backslash\{0\}$, *then* $\sigma_p(C_\varphi|_{D_f})$ *intersects the unit circle in a set of positive measure.*

(b) *If* $f \in [(z-\alpha)(z-\beta)]^{1/2}H^p\backslash\{0\}$ *for some* $p > 2$, *then* $\sigma_p(C_\varphi|_{D_f})$ *contains an open annulus centered at the origin.*

Note that the function $[(z-\alpha)(z-\beta)]^{1/2}$ is an outer function, so the set of functions f satisfying the hypotheses in each of parts (a) and (b) is dense in H^2.

2. Background material

2.1. Disc automorphisms

An *automorphism* of a domain in the complex plane is a univalent holomorphic mapping of that domain onto itself. Every automorphism of the open unit disc $\mathbb{U}$ is a linear fractional map [9, Theorem 12.6, page 255].

Linear fractional maps can be regarded as homeomorphisms of the Riemann Sphere; as such, each one that is not the identity map has one or two fixed points. The maps with just one fixed point are the *parabolic* ones; each such map is conjugate, via an appropriate linear fractional map, to one that fixes only the point at infinity, i.e., to a translation. A linear fractional map that fixes two distinct points is conjugate, again via a linear fractional map, to one that fixes both the origin and the point at infinity, i.e., to a dilation $w \to \mu w$ of the complex plane, where $\mu \neq 1$ is a complex number called the *multiplier* of the original map (actually $1/\mu$ can just as well occur as the multiplier – depending on which fixed point of the original map is taken to infinity by the conjugating transformation). The original map is called *elliptic* if $|\mu| = 1$, hyperbolic if μ is positive, and *loxodromic* in all other cases (see, for example, [10, Chapter 0] for more details).

Suppose φ is a hyperbolic automorphism of $\mathbb{U}$. Then the same is true of its inverse. The fixed points of φ must necessarily lie on $\partial\mathbb{U}$, the unit circle To see this note that if the attractive fixed point of φ lies outside the closed unit disc, then the compositional iterates of φ pull $\mathbb{U}$ toward that fixed point, and hence outside of $\mathbb{U}$, which contradicts the fact that $\varphi(\mathbb{U}) = \mathbb{U}$. If, on the other hand, the attractive fixed point lies in $\mathbb{U}$, then its reflection in the unit circle is the repulsive fixed point, which is the attractive one for φ^{-1}. Thus φ^{-1} can't map $\mathbb{U}$ into itself, another contradiction. Conclusion: both fixed points lie on $\partial\mathbb{U}$.

Let's call a hyperbolic automorphism φ of $\mathbb{U}$ *canonical* if it fixes the points ± 1, with $+1$ being the attractive fixed point. We'll find it convenient to move between the open unit disc $\mathbb{U}$ and the open right half-plane Π^+ by means of the *Cayley transform* $\kappa : \Pi^+ \to \mathbb{U}$ and its inverse $\kappa^{-1} : \mathbb{U} \to \Pi^+$, where

$$\kappa(w) = \frac{w-1}{w+1} \quad \text{and} \quad \kappa^{-1}(z) = \frac{1+z}{1-z} \qquad (z \in \mathbb{U}, w \in \Pi^+).$$

In particular, if φ is a canonical hyperbolic automorphism of $\mathbb{U}$, then $\Phi := \kappa^{-1}\circ\varphi\circ\kappa$ is an automorphism of Π^+ that fixes 0 and ∞, with ∞ being the attractive fixed point. Thus $\Phi(w) = \mu w$ for some $\mu > 1$, and $\varphi = \kappa \circ \Phi \circ \kappa^{-1}$, which yields, after a little calculation,

$$\varphi(z) = \frac{r+z}{1+rz} \qquad \text{where} \qquad \varphi(0) = r = \frac{\mu-1}{\mu+1} \in (0,1). \tag{2.1}$$

If φ is a hyperbolic automorphism of $\mathbb{U}$ that is not canonical, then it can be conjugated, via an appropriate automorphism of $\mathbb{U}$, to one that is. This is perhaps best seen by transferring attention to the right half-plane Π^+, and observing that if $\alpha < \beta$ are two real numbers, then the linear fractional map Ψ of Π^+ defined by

$$\Psi(w) \;=\; i\,\frac{w-i\beta}{w-i\alpha}$$

preserves the imaginary axis, and takes the point 1 into Π^+. Thus it is an automorphism of Π^+ that takes the boundary points $i\beta$ to zero and $i\alpha$ to infinity. Consequently if Φ is any hyperbolic automorphism of Π^+ with fixed points $i\alpha$ (attractive) and $i\beta$ (repulsive), then $\Psi \circ \Phi \circ \Psi^{-1}$ is also hyperbolic automorphism with attractive fixed point ∞ and repulsive fixed point 0. If, instead, $\alpha > \beta$ then $-\Psi$ does the job.

Since any hyperbolic automorphism φ of $\mathbb{U}$ is conjugate, via an automorphism, to a canonical one, C_φ is similar, via the composition operator induced by the conjugating map, to a composition operator induced by a canonical hyperbolic automorphism. For this reason the work that follows will focus on the canonical case.

2.2. Spectra of hyperbolic-automorphic composition operators

Suppose φ is a hyperbolic automorphism of $\mathbb{U}$ with multiplier $\mu > 1$. Then it is easy to find lots of eigenfunctions for C_φ in H^2. We may without loss of generality assume that φ is canonical, and then move, via the Cayley map, to the right half-plane where φ morphs into the dilation $\Phi(w) = \mu w$. Let's start by viewing the composition operator C_Φ as just a linear map on $\mathrm{Hol}\,(\Pi^+)$, the space of all holomorphic functions on Π^+. For any complex number a define $E_a(w) = w^a$, where $w^a = \exp(a \log w)$, and "log" denotes the principal branch of the logarithm. Then $E_a \in \mathrm{Hol}\,(\Pi^+))$ and $C_\Phi(E_a) = \mu^a E_a$; i.e., E_a is an eigenvector of C_Φ (acting on $\mathrm{Hol}\,(\Pi^+))$ and the corresponding eigenvalue is μ^a (again taking the principal value of the "ath power"). Upon returning via the Cayley map to the unit disc, we see that, when viewed as a linear transformation of $\mathrm{Hol}\,(\mathbb{U})$, the operator C_φ has, for each $a \in \mathbb{C}$, the eigenvector/eigenvalue combination (f_a, μ^a), where the function

$$f_a(z) = \left(\frac{1+z}{1-z}\right)^a \qquad\qquad (z \in \mathbb{U}) \tag{2.2}$$

belongs to H^2 if and only if $|\mathrm{Re}\,(a)| < 1/2$. Thus the corresponding H^2-eigenvalues μ^a cover the entire open annulus

$$A := \{\lambda \in \mathbb{C} : \frac{1}{\sqrt{\mu}} < |\lambda| < \sqrt{\mu}\}. \tag{2.3}$$

In particular $\sigma(C_\varphi)$, the H^2-spectrum of C_φ, contains this annulus, and since the map $a \to \mu^a$ takes the strip $|\mathrm{Re}\,(a)| < 1/2$ infinitely-to-one onto A, each point of A is an eigenvalue of C_φ having infinite multiplicity.

As for the rest of the spectrum, an elementary norm calculation shows that $\sigma(C_\varphi)$ is just the closure of A. To see this, note first that the change-of-variable formula from calculus shows that for each $f \in H^2$ and each automorphism φ of $\mathbb{U}$ (not necessarily hyperbolic):

$$\|C_\varphi f\|^2 = \int_{\partial\mathbb{U}} |f|^2 P_a \, dm \tag{2.4}$$

where m is normalized arc-length measure on the unit circle $\partial\mathbb{U}$, and P_a is the Poisson kernel for $a = \varphi(0)$; more generally, for any $a \in \mathbb{U}$:

$$P_a(\zeta) \;=\; \frac{1-|a|^2}{|\zeta - a|^2} \qquad (\zeta \in \partial\mathbb{U}) \tag{2.5}$$

(see also Nordgren's neat argument [7, Lemma 1, page 442], which shows via Fourier analysis that (2.4) holds for any inner function).

Now suppose φ is the canonical hyperbolic automorphism of $\mathbb{U}$ with multiplier $\mu > 1$. Then φ is given by (2.1), so by (2.5)

$$P_r(\zeta) \;=\; \frac{1-r^2}{|\zeta - r|^2} \;\le\; \frac{1+r}{1-r} \;=\; \mu$$

which, along with (2.4) shows that

$$\|C_\varphi\| \;\le\; \sqrt{\mu}\,. \tag{2.6}$$

Since also

$$P_r(\zeta) \;\ge\; \frac{1-r}{1+r} \;=\; \mu^{-1}$$

we have, for each $f \in H^2$

$$\|C_\varphi f\| \;\ge\; \frac{1}{\sqrt{\mu}}\,\|f\|,$$

which shows that (2.6) holds with C_φ replaced by C_φ^{-1}. Thus the spectra of both C_φ and its inverse lie in the closed disc of radius $\sqrt{\mu}$ centered at the origin, so by the spectral mapping theorem, $\sigma(C_\varphi)$ is contained in the closure of the annulus (2.3). Since we have already seen that this closed annulus contains the spectrum of C_φ we've established the following result, first proved by Nordgren [7, Theorem 6, page 448] using precisely the argument given above:

Theorem 2.1. *If φ is a hyperbolic automorphism of $\mathbb{U}$ with multiplier μ (> 1), then $\sigma(C_\varphi)$ is the closed annulus $\{\lambda \in \mathbb{C} : \mu^{-1/2} \le |\lambda| \le \mu^{1/2}\}$. The interior of this annulus consists entirely of eigenvalues of C_φ, each having infinite multiplicity.*

In fact the interior of $\sigma(C_\varphi)$ is precisely the point spectrum of C_φ; see [6] for the details.

2.3. Poisson kernel estimates

Formula (2.5), giving the Poisson kernel for the point $a = \rho e^{i\theta_0} \in \mathbb{U}$, can be rewritten

$$P_a(e^{i\theta}) \;=\; \frac{1-\rho^2}{1-2\rho\cos(\theta-\theta_0)+\rho^2} \qquad (0 \le \rho < 1, \theta \in \mathbb{R}).$$

We will need the following well-known estimate, which provides a convenient replacement for the Poisson kernel (cf. for example [1, page 313]).

Lemma 2.2. *For $0 \le \rho < 1$ and $|\theta| \le \pi$:*

$$P_\rho(e^{i\theta}) \;\le\; 4\,\frac{(1-\rho)}{(1-\rho)^2 + (\theta/\pi)^2} \tag{2.7}$$

Proof.

$$P_\rho(e^{i\theta}) := \frac{1-\rho^2}{1-2\rho\cos\theta+\rho^2} = \frac{1-\rho^2}{(1-\rho)^2+\rho(2\sin\frac{\theta}{2})^2} \le \frac{2(1-\rho)}{(1-\rho)^2+4\rho(\theta/\pi)^2}$$

so, at least when $\rho \ge \frac{1}{4}$, inequality (2.7) holds with constant "2" in place of "4". For the other values of ρ one can get inequality (2.7) by checking that, over the interval $[0,\pi]$, the minimum of the right-hand side exceeds the maximum of the left-hand side. □

Remark 2.3. The only property of the constant "4" on the right-hand side of (2.7) that matters for our purposes is its independence of ρ and θ.

For the sequel (especially Theorem 3.5 below) we will require the following upper estimate of certain infinite sums of Poisson kernels.

Lemma 2.4. *For φ the canonical hyperbolic automorphism of $\mathbb{U}$ with multiplier μ:*

$$\sum_{n=0}^{\infty} P_{\varphi_n(0)}(e^{i\theta}) \;\le\; \frac{16\mu}{\mu-1}\,\frac{\pi}{|\theta|} \qquad (|\theta| \le \pi). \tag{2.8}$$

In the spirit of Remark 2.3 above, the precise form of the positive constant that multiplies $\pi/|\theta|$ on the right-hand side of (2.8) is unimportant (as long as it does not depend on θ).

Proof. The automorphism φ is given by equations (2.1). For each integer $n \ge 0$ the nth iterate φ_n of φ is just the canonical hyperbolic automorphism with multiplier μ^n, so upon substituting μ^n for μ in (2.1) we obtain

$$\varphi_n(z) = \frac{r_n + z}{1 + r_n z} \qquad \text{where} \qquad \varphi_n(0) = r_n = \frac{\mu^n - 1}{\mu^n + 1} \in (0,1). \tag{2.9}$$

Thus $1 - r_n = 2/(\mu^n + 1)$, and so

$$\mu^{-n} \;<\; 1 - r_n \;<\; 2\,\mu^{-n} \quad (n = 0, 1, 2, \dots), \tag{2.10}$$

(in particular, r_n approaches the attractive fixed point $+1$ with exponential speed as $n \to \infty$; this is true of the φ-orbit of any point of the unit disc).

Fix $\theta \in [-\pi, \pi]$. We know from (2.7) and (2.10) that for each integer $n \geq 0$,

$$P_{r_n}(e^{i\theta}) \;\leq\; \frac{4(1-r_n)}{(1-r_n)^2 + (\theta/\pi)^2} \;\leq\; \frac{8\mu^{-n}}{\mu^{-2n} + (\theta/\pi)^2}\,,$$

whereupon, for each non-negative integer N:

$$\begin{aligned}
\frac{1}{8}\sum_{n=0}^{\infty} P_{r_n}(e^{i\theta}) \;&\leq\; \sum_{n=0}^{\infty} \frac{\mu^{-n}}{\mu^{-2n} + (\theta/\pi)^2} \\
&\leq\; \sum_{n=0}^{N-1} \frac{\mu^{-n}}{\mu^{-2n}} \;+\; \left(\frac{\pi}{\theta}\right)^2 \sum_{n=N}^{\infty} \mu^{-n} \\
&=\; \sum_{n=0}^{N-1} \mu^{n} \;+\; \left(\frac{\pi}{\theta}\right)^2 \sum_{n=N}^{\infty} \mu^{-n} \\
&=\; \frac{\mu^N - 1}{\mu - 1} \;+\; \left(\frac{\pi}{\theta}\right)^2 \mu^{-N}(1-\mu^{-1})^{-1}
\end{aligned}$$

where the geometric sum in the next-to-last line converges because $\mu > 1$.

We need a choice of N that gives a favorable value for the quantity in the last line of the display above. Let $\nu = \log_\mu(\pi/|\theta|)$, so that $\mu^\nu = \pi/|\theta|$. Since $|\theta| \leq \pi$ we are assured that $\nu \geq 0$. Let N be the least integer $\geq \nu$, i.e., the unique integer in the interval $[\nu, \nu+1)$. The above estimate yields for any integer $N \geq 0$, upon setting $C := 8\mu/(\mu-1)$ (which is > 0 since $\mu > 1$),

$$\sum_{n=0}^{\infty} P_{r_n}(e^{i\theta}) \leq C\left[\mu^{N-1} + \left(\frac{\pi}{\theta}\right)^2 \mu^{-N}\right] \leq C\frac{\pi}{|\theta|}\left[\frac{|\theta|}{\pi}\mu^\nu + \left(\frac{|\theta|}{\pi}\mu^\nu\right)^{-1}\right]. \qquad (2.11)$$

By our choice of ν, both summands in the square-bracketed term at the end of (2.11) have the value 1 and this implies (2.8). □

3. Main results

Here I extend work of Chkliar [3] and Matache [5] that provides, for a hyperbolic-automorphically induced composition operator C_φ, sufficient conditions on $f \in H^2$ for the doubly-cyclic subspace D_f, as defined by (1.1), to contain a rich supply of linearly independent eigenfunctions. I'll focus mostly on canonical hyperbolic automorphisms, leaving the general case for the next section. Thus, until further notice, φ will denote a canonical hyperbolic automorphism of $\mathbb{U}$ with multiplier $\mu > 1$, attractive fixed point at $+1$ and repulsive one at -1; i.e., φ will be given by equations (2.1).

Following both Chkliar and Matache, I will use an H^2-valued Laurent series to produce the desired eigenvectors. The idea is this: for $f \in H^2$, and λ a non-zero complex number, if the series

$$\sum_{n\in\mathbb{Z}} \lambda^{-n}(f\circ\varphi_n) \tag{3.1}$$

converges strongly enough (for example, in H^2) then the sum F_λ, whenever it is not the zero-function, will be a λ-eigenfunction of C_φ that lies in D_f. Clearly the convergence of the series (3.1) will depend crucially on the behavior of φ at its fixed points, as the next result indicates. For convenience let's agree to denote by $A(R_1, R_2)$ the open annulus, centered at the origin, of inner radius R_1 and outer radius R_2 (where, of course, $0 < R_1 < R_2 < \infty$).

Theorem 3.1. (cf. [3]) *Suppose* $0 < \varepsilon, \delta \le 1/2$, *and that*

$$f \in (z-1)^{\frac{1}{2}+\varepsilon}(z+1)^{\frac{1}{2}+\delta}H^2\backslash\{0\}.$$

Then $\sigma_p(C_\varphi|_{D_f})$ *contains, except possibly for a discrete subset,* $A(\mu^{-\varepsilon},\mu^{\delta})$.

Proof. Our hypothesis on the behavior of f at the point $+1$ (the attractive fixed point of φ) is that $f = (z-1)^{\frac{1}{2}+\varepsilon}g$ for some $g \in H^2$, i.e., that

$$\infty \;>\; \int_{\partial\mathbb{U}} |g|^2 dm \;=\; \int_{\partial\mathbb{U}} \frac{|f(\zeta)|^2}{|\zeta-1|^{2\varepsilon+1}}\, dm(\zeta) \;\ge\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|f(e^{i\theta})|^2}{|\theta|^{2\varepsilon+1}}\, d\theta\,. \tag{3.2}$$

Upon setting $a = \varphi_n(0) := r_n$ in (2.4) we obtain

$$\|f\circ\varphi_n\|^2 \;=\; \int |f|^2 P_{r_n}\, dm\,, \qquad (n\in\mathbb{Z}) \tag{3.3}$$

which combines with estimates (2.7) and (2.10) to show that if n is a non-negative integer (thus insuring that $r_n > 0$):

$$\begin{aligned}
\|f\circ\varphi_n\|^2 \;&\le\; 2\pi\int_{-\pi}^{\pi} |f(e^{i\theta})|^2 \frac{1-r_n}{(1-r_n)^2+\theta^2}\, d\theta \\
&\le\; 4\pi\int_{-\pi}^{\pi} |f(e^{i\theta})|^2 \frac{\mu^{-n}}{\mu^{-2n}+\theta^2}\, d\theta \\
&=\; 4\pi\mu^{-2n\varepsilon}\int_{-\pi}^{\pi} \frac{|f(e^{i\theta})|^2}{|\theta|^{1+2\varepsilon}} \left\{\frac{(\mu^n|\theta|)^{1+2\varepsilon}}{1+(\mu^n|\theta|)^2}\right\} d\theta \\
&\le\; 4\pi\mu^{-2n\varepsilon}\sup_{x\in\mathbb{R}}\left\{\frac{|x|^{1+2\varepsilon}}{1+x^2}\right\}\int_{-\pi}^{\pi} \frac{|f(e^{i\theta})|^2}{|\theta|^{1+2\varepsilon}}\, d\theta\;.
\end{aligned}$$

By (3.2) the integral in the last line is finite, and since $0 < \varepsilon \le 1/2$, the supremum in that line is also finite. Thus

$$\|f\circ\varphi_n\| \;=\; \mathrm{O}(\mu^{-n\varepsilon}) \quad \text{as} \quad n\to\infty,$$

which guarantees that the subseries of (3.1) with positively indexed terms converges in H^2 for all $\lambda\in\mathbb{C}$ with $|\lambda| > \mu^{-\varepsilon}$.

As for the negatively indexed subseries of (3.1), note from (2.1) that $\varphi^{-1}(z) = -\varphi(-z)$, so $\varphi_{-n}(z) = -\varphi_n(-z)$ for each integer n. Let $g(z) = f(-z)$, so our hypothesis on f implies that $g \in (z-1)^{\frac{1}{2}+\delta} H^2\backslash\{0\}$. Let $\psi_n(z) = \varphi_n(-z)$ (the subscript on ψ does not now indicate iteration). Then for each positive integer n we have $\psi_n(0) = \varphi_n(0) = r_n$, hence:

$$\|f \circ \varphi_{-n}\|^2 \;=\; \|g \circ \psi_n\|^2 \;=\; \int_{\partial\mathbb{U}} |g|^2 P_{r_n}\, dm$$

so by the result just obtained, with g in place of f and ε replaced by δ,

$$\|f \circ \varphi_{-n}\| \;=\; \mathrm{O}(\mu^{-n\delta}) \quad \text{as} \quad n \to \infty.$$

Thus the negatively indexed subseries of (3.1) converges in H^2 for all complex numbers λ with $|\lambda| < \mu^{\delta}$.

Conclusion: For each λ in the open annulus $A(\mu^{-\varepsilon}, \mu^{\delta})$ the H^2-valued Laurent series (3.1) converges in the norm topology of H^2 to a function $F_\lambda \in H^2$. Now F_λ, for such a λ, will be a C_φ-eigenfunction unless it is the zero-function, and – just as for scalar Laurent series – this inconvenience can occur for at most a discrete subset of points λ in the annulus of convergence (the relevant uniqueness theorem for H^2-valued holomorphic functions follows easily from the scalar case upon applying bounded linear functionals). □

Remark 3.2. Chkliar [3] has a similar result, where there are uniform conditions on the function f at the fixed points of φ (see also Remark 3.10 below); as he suggests, it would be of interest to know whether or not the "possible discrete subset" that clutters the conclusions of results like Theorem 3.1 can actually be nonempty.

Remark 3.3. The limiting case $\delta = 0$ of Theorem 3.1 is also true (see Theorem 3.6 below); it is a slight improvement on Chkliar's result (see also the discussion following Theorem 3.6).

Remark 3.4. Note that the restriction $\varepsilon, \delta \leq 1/2$ in the hypothesis of Theorem 3.1 cannot be weakened since, as mentioned at the end of §2.2, the point spectrum of C_φ is the open annulus $A(\mu^{-\frac{1}{2}}, \mu^{\frac{1}{2}})$.

Here is a companion to Theorem 3.1, which shows that even in the limiting case $\delta = \varepsilon = 0$ (in some sense the "weakest" hypothesis on f) the operator $C_\varphi|_{D_f}$ still has a significant supply of eigenvalues.

Theorem 3.5. *If $f \in \sqrt{(z+1)(z-1)}\, H^2$ then $\sigma_p(C_\varphi|_{D_f})$ intersects $\partial\mathbb{U}$ in a set of positive measure.*

Proof. We will work in the Hilbert space $L^2(H^2, dm)$ consisting of H^2-valued (m-equivalence classes of) measurable functions F on $\partial\mathbb{U}$ with

$$|||F|||^2 \;:=\; \int_{\partial\mathbb{U}} \|F(\omega)\|^2 dm(\omega) < \infty.$$

I will show in a moment that the hypothesis on f implies

$$\sum_{n\in\mathbb{Z}} \|f \circ \varphi_n\|^2 < \infty\,. \tag{3.4}$$

Granting this, it is easy to check that the H^2-valued Fourier series

$$\sum_{n\in\mathbb{Z}}(f\circ\varphi_n)\,\omega^{-n} \qquad (\omega\in\partial\mathbb{U}) \tag{3.5}$$

converges unconditionally in $L^2(H^2,dm)$, so at least formally, we expect that for a.e. $\omega\in\partial\mathbb{U}$ we'll have $C_\varphi(F(\omega))=\omega F(\omega)$. This is true, but a little care is needed to prove it. The "unconditional convergence" mentioned above means this: If, for each finite subset E of $\mathbb{Z}$,

$$S_E(\omega):=\sum_{n\in E}(f\circ\varphi_n)\omega^{-n} \qquad (\omega\in\partial\mathbb{U}),$$

then the net ($S_E : E$ a finite subset of $\mathbb{Z}$) converges in $L^2(H^2,dm)$ to F. In particular, if for each non-negative integer n we define $F_n=S_{[-n,n]}$, then $F_n\to F$ in $L^2(H^2,dm)$, hence some subsequence $(F_{n_k}(\omega))_{k=1}^\infty$ converges in H^2 to $F(\omega)$ for a.e. $\omega\in\partial\mathbb{U}$. Now for any n and any $\omega\in\partial\mathbb{U}$:

$$C_\varphi F_n(\omega)=\omega F_n(\omega)-\omega^{n+1}f\circ\varphi_{-n}+\omega^{-n}f\circ\varphi_{n+1}$$

which implies, since (3.4) guarantees that $\|f\circ\varphi_n\|\to 0$ as $n\to\infty$, that

$$C_\varphi F_n(\omega)-\omega F_n(\omega)\to 0 \text{ in } H^2 \qquad (n\to\infty).$$

This, along with the a.e. convergence of the subsequence (F_{n_k}) to F, shows that $C_\varphi F(\omega)=\omega F(\omega)$ for a.e. $\omega\in\partial\mathbb{U}$. Now the H^2-valued Fourier coefficients $f\circ\varphi_n$ are not all zero (in fact, none of them are zero) so at least for a subset of points $\omega\in\partial\mathbb{U}$ having positive measure we have $F(\omega)\neq 0$. The corresponding H^2-functions $F(\omega)$ are therefore eigenfunctions of C_φ that belong to D_f, thus $\sigma_p(C_\varphi|_{D_f})\cap\partial\mathbb{U}$ has positive measure.

It remains to prove (3.4). As usual, we treat the positively and negatively indexed terms separately. Since $f\in\sqrt{z-1}\,H^2$ we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|f(e^{i\theta})|^2}{|\theta|}d\theta \;\le\; \int_{\partial\mathbb{U}}\frac{|f(\zeta)|^2}{|\zeta-1|}dm(\zeta)\;<\;\infty$$

so successive application of (2.4) and (2.8) yields

$$\sum_{n=0}^{\infty}\|f\circ\varphi_n\|^2=\int_{\partial\mathbb{U}}|f|^2\left(\sum_{n=0}^{\infty}P_{r_n}\right)dm\;\le\;\text{const.}\int_{-\pi}^{\pi}\frac{|f(e^{i\theta})|^2}{|\theta|}\,d\theta\;<\;\infty\,.$$

For the negatively indexed terms in (3.4), note that our hypothesis on f guarantees that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|f(e^{i(\theta-\pi)})|^2}{|\theta|}d\theta\;\le\;\int_{\partial\mathbb{U}}\frac{|f(\zeta)|^2}{|\zeta+1|}dm(\zeta)\;<\;\infty\,. \tag{3.6}$$

Recall from the proof of Theorem 3.1 that $\varphi_{-n}(z)=-\varphi_n(-z)$ for $z\in\mathbb{U}$ and $n>0$, and so

$$\|f\circ\varphi_{-n}\|^2=\int|f|^2P_{-r_n}dm=\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(e^{i\theta})|^2P_{r_n}(\theta-\pi)\,d\theta.$$

Thus

$$\begin{aligned}
\sum_{n=1}^{\infty} \|f \circ \varphi_{-n}\|^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 \left(\sum_{n=1}^{\infty} P_{r_n}(\theta - \pi) \right) d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i(\theta-\pi)})|^2 \left(\sum_{n=1}^{\infty} P_{r_n}(\theta) \right) d\theta \\
&\le \text{const.} \int_{-\pi}^{\pi} \frac{|f(e^{i(\theta-\pi)})|^2}{|\theta|} d\theta \\
&< \infty
\end{aligned}$$

where the last two lines follow, respectively, from inequalities (2.8) and (3.6). This completes the proof of (3.4), and with it, the proof of the Theorem. □

It would be of interest to know just how large the set $\sigma_p(C_\varphi|_{D_f})$ has to be in Theorem 3.5. Might it always be the whole unit circle? Might it be even larger? What I do know is that if the hypothesis of the Theorem is strengthened by replacing the hypothesis "$f \in \sqrt{(z+1)(z-1)}\, H^2$" with the stronger "$f \in \sqrt{(z+1)(z-1)}\, H^p$ for some $p > 2$", then the conclusion improves dramatically, as shown below by the result below, whose proof reprises the latter part of the proof of Theorem 3.1.

Theorem 3.6. (cf. [5, Theorem 5.5]) *If $f \in \sqrt{(z+1)(z-1)}\, H^p \backslash \{0\}$ for some $p > 2$, then $\sigma_p(C_\varphi|_{D_f})$ contains, except possibly for a discrete subset, the open annulus $A(\mu^{-\varepsilon}, \mu^{\varepsilon})$ where $\varepsilon = \frac{1}{2} - \frac{1}{p}$.*

Proof. I will show that the hypothesis implies that $f \in [(z-1)(z+1)]^{\frac{1}{2}+\delta} H^2$ for each positive $\delta < \varepsilon$. This will guarantee, by the proof of Theorem 3.1, that the series (3.1) converges in the open annulus $A(\mu^{-\delta}, \mu^{\delta})$ for each such δ, and hence it converges in $A(\mu^{-\varepsilon}, \mu^{\varepsilon})$, which will, just as in the proof of Theorem 3.1 finish the matter. The argument below, suggested by Paul Bourdon, greatly simplifies my original one. Our hypotheses on f imply that for some $g \in H^p$,

$$f = [(z-1)(z+1)]^{\frac{1}{2}+\delta} h \quad \text{where} \quad h = [(z-1)(z+1)]^{-\left(\frac{1}{2}+\delta\right)} g.$$

To show: $h \in H^2$. The hypothesis on δ can be rewritten: $2p\delta/(p-2) < 1$, so the function $[(z-1)(z+1)]^{-\delta}$ belongs to $H^{\frac{2p}{p-2}}$, hence an application of Hölder's inequality shows that h is in H^2 with norm bounded by the product of the H^p-norm of g and the $H^{\frac{2p}{p-2}}$-norm of $[(z-1)(z+1)]^{-\delta}$. □

In both [3] and [5, Theorem 5.3] there are results where the hypotheses on f involve uniform boundedness for f at one or both of the fixed points of φ. In [5, Theorem 5.4] Matache shows that these uniform conditions can be replaced by boundedness of a certain family of Poisson integrals, and from this he derives the following result.

> [5, Theorem 5.5] *If* $f \in (z-1)^{\frac{2}{p}} H^p$ *for some* $p > 2$*, and* f *is bounded in a neighborhood of* -1*, then* $\sigma_p(C_\varphi|_{D_f})$ *contains an open annulus centered at the origin.*

I'll close this section by presenting some results of this type, where uniform boundedness at one of the fixed points is replaced by boundedness of the Hardy-Littlewood maximal function. This is the function, defined for g non-negative and integrable on $\partial\mathbb{U}$, and $\zeta \in \partial\mathbb{U}$, by:

$$M[g](\zeta) := \sup\left\{\frac{1}{m(I)}\int_I g\,dm : I \text{ an arc of } \partial\mathbb{U} \text{ centered at } \zeta\right\}.$$

The *radial maximal function* $R[g]$ of g at $\zeta \in \partial\mathbb{U}$ is the supremum of the values of the Poisson integral of g on the radius $[0,\zeta)$. It is easy to check that $M[g]$ is dominated pointwise on $\partial\mathbb{U}$ by a constant multiple of $R[g]$. What is perhaps surprising, but still elementary, is the fact that there is a similar inequality in the other direction: the radial maximal function of the non-negative integrable function g is dominated pointwise on $\partial\mathbb{U}$ by a constant multiple of its Hardy-Littlewood maximal function (see [9, Theorem 11.20, page 242]). This and (2.4) yield

Lemma 3.7. *For* $f \in H^2$,

$$M[|f|^2](-1) < \infty \quad \Longrightarrow \quad \sup\{\|f \circ \varphi_n\| : n < 0\} < \infty.$$

To see that the hypotheses of Lemma 3.7 can be satisfied by functions in H^2 that are unbounded as $z \to -1$, one need only observe that

$$M[|f|^2](-1) \;\le\; \text{const.} \int \frac{|f(\zeta)|^2}{|1+\zeta|}\,dm(\zeta),$$

hence, along with (2.4), the Lemma implies:

Corollary 3.8. *If* $f \in \sqrt{z+1}\,H^2$ *then* $\sup\{\|f \circ \varphi_n\| : n < 0\} < \infty$.

Thus if $f \in \sqrt{z+1}\,H^2$, or more generally if $M[|f|^2](-1) < \infty$, the negatively indexed subseries of (3.1) will converge in H^2 for all $\lambda \in \mathbb{U}$. We have seen in the proof of Theorem 3.1 that if $f \in (z-1)^{\frac{1}{2}+\varepsilon} H^2$ for some $\varepsilon \in (0,1/2]$ then the positively indexed subseries of (3.1) converges for $|\lambda| > \mu^{-\varepsilon}$. Putting it all together we obtain the promised "$\delta = 0$" case of Theorem 3.1:

Theorem 3.9. *Suppose* $f \in (z+1)^{\frac{1}{2}}(z-1)^{\frac{1}{2}+\varepsilon} H^2\backslash\{0\}$ *for some* $0 < \varepsilon < 1/2$*. Then* $\sigma_p(C_\varphi|_{D_f})$ *contains, with the possible exception of a discrete subset, the open annulus* $A(\mu^{-\varepsilon}, 1)$.

Remark 3.10. By the discussion preceding this theorem, the hypothesis on f could be replaced by the weaker: "$f \in (z-1)^{\frac{1}{2}+\varepsilon} H^2\backslash\{0\}$ and $M[|f|^2](-1) < \infty$, " (cf. [3]). If, in either version, the hypotheses on the attractive and repulsive fixed points are reversed, then the conclusion will assert that $\sigma_p(C_\varphi|_{D_f})$ contains, except for perhaps a discrete subset, the annulus $A(1, \mu^{\varepsilon})$ (see §4.1, especially the discussion preceding Corollary 4.2).

Remark 3.11. Note how the previously mentioned Theorem 5.5 of [5] follows from the work above. Indeed, if $f \in (z-1)^{2/p} H^p$ for some $p > 2$ then by Hölder's inequality $f \in (z-1)^{\frac{1}{2}+\varepsilon} H^2$, for each $\varepsilon < 1/p$. Thus, as in the proof of Theorem 3.1, the positively indexed subseries of (3.1) converges for $|\lambda| > \mu^{-1/p}$, and by Lemma 3.7 the boundedness of f in a neighborhood of -1 insures that the negatively indexed subseries of (3.1) converges in the open unit disc. Thus as in the proof of Theorem 3.1, $\sigma_p(C_\varphi|_{D_f})$ contains, with the possible exception of a discrete subset, the open annulus $A(\mu^{-1/p}, 1)$.

4. Complements and comments

In this section I collect some further results and say a few more words about the theorem of Nordgren, Rosenthal, and Wintrobe.

4.1. Non-canonical hyperbolic automorphisms

The results of §3, which refer only to canonical hyperbolic automorphisms φ, can be easily "denormalized". Here is a sample:

Theorem 4.1. *Suppose φ is a hyperbolic automorphism of $\mathbb{U}$ with attractive fixed point α, repulsive one β, and multiplier $\mu > 1$. Then*

(a) (cf. Theorem 3.1) *Suppose, for $0 < \varepsilon, \delta < 1/2$ we have*
$$f \in (z-\alpha)^{\frac{1}{2}+\varepsilon}(z-\beta)^{\frac{1}{2}+\delta} H^2\backslash\{0\}.$$
Then $\sigma_p(C_\varphi|_{D_f})$ contains, except possibly for a discrete subset, the open annulus $A(\mu^{-\varepsilon}, \mu^{\delta})$.

(b) (cf. Theorem 3.5) *If $f \in \sqrt{(z-\alpha)(z-\beta)}\, H^2$ then $\sigma_p(C_\varphi|_{D_f})$ intersects $\partial\mathbb{U}$ in a set of positive measure.*

(c) (cf. Theorem 3.6) *If $f \in \sqrt{(z-\alpha)(z-\beta)}\, H^p\backslash\{0\}$ for some $p > 2$, then $\sigma_p(C_\varphi|_{D_f})$ contains, except possibly for a discrete subset, the open annulus $A(\mu^{-\varepsilon}, \mu^{\varepsilon})$ where $\varepsilon = \frac{1}{2} - \frac{1}{p}$.*

Proof. I'll just outline the idea, which contains no surprises. Suppose α and β (both on $\partial\mathbb{U}$) are the fixed points of φ, and – for the moment – that $\tilde{\alpha}$ and $\tilde{\beta}$ are any two distinct points of $\partial\mathbb{U}$. Then, as we noted toward the end of §2.1, there is an automorphism ψ of $\mathbb{U}$ that takes $\tilde{\alpha}$ to α and $\tilde{\beta}$ to β. Thus $\tilde{\varphi} := \psi^{-1} \circ \varphi \circ \psi$ is a hyperbolic automorphism of $\mathbb{U}$ that is easily seen to have attractive fixed point $\tilde{\alpha}$ and repulsive one $\tilde{\beta}$. Furthermore:

- $C_{\tilde{\varphi}} = C_\psi C_\varphi C_\psi^{-1}$, so $C_{\tilde{\varphi}}$ is similar to C_φ.
- For $f \in H^2$: $C_\psi D_f = D_{f\circ\psi}$.
- $F \in H^2$ is a λ-eigenvector for C_φ if and only if $C_\psi = F \circ \psi$ is one for $C_{\tilde{\varphi}}$.
- For $f \in H^2$, $M[|f|^2](\beta) < \infty \iff M[|f \circ \psi|^2](\beta) < \infty$.
- For any $\gamma > 0$, $f \in (z-\alpha)^\gamma H^2 \iff C_\psi f \in (z-\tilde{\alpha})^\gamma H^2$.

Only the last of these needs any comment. If $f \in (z-\alpha)^\gamma H^2$ then

$$\begin{aligned} C_\psi f &\in (\psi(z)-\alpha)^\gamma C_\psi(H^2) \\ &= \left(\frac{\psi(z)-\psi(\tilde\alpha)}{z-\tilde\alpha}\right)^\gamma (z-\tilde\alpha)^\gamma H^2 \\ &= (z-\tilde\alpha)^\gamma H^2 \end{aligned}$$

where the last line follows from the fact that the quotient in the previous one is, in a neighborhood of the closed unit disc, analytic and non-vanishing (because ψ is univalent there), hence both bounded and bounded away from zero on the closed unit disc. Thus $C_\psi((z-\alpha)^\gamma H^2) \subset (z-\tilde\alpha) H^2$, and the opposite inclusion follows from this by replacing ψ by ψ^{-1} and applying C_ψ to both sides of the result.

Theorem 4.1 now follows, upon setting $(\tilde\alpha, \tilde\beta) = (+1,-1)$, from Theorems 3.1, 3.5, and 3.6. □

What happens if we interchange attractive and repulsive fixed points of φ in the hypotheses of Theorem 4.1(a)? Then the hypotheses apply to φ^{-1}, hence so does the conclusion. Since $C_{\varphi^{-1}} = C_\varphi^{-1}$, Theorem 4.1(a) and the spectral mapping theorem yield, for example, the following complement to Theorem 3.9:

Corollary 4.2. *Suppose φ is a hyperbolic automorphism of $\mathbb{U}$ with attractive fixed point α, repulsive one β, and multiplier $\mu > 1$. Suppose further that $f \in (z-\alpha)^{\frac{1}{2}}(z-\beta)^{\frac{1}{2}+\varepsilon}$ for some $\varepsilon \in (0, \frac{1}{2})$. Then $\sigma_p(C_\varphi|_{D_f})$ contains, except possibly for a discrete subset, the open annulus $A(1, \mu^\varepsilon)$*

The reader can easily supply similar "reversed" versions of the other results on the point spectrum of $C_\varphi|_{D_f}$.

4.2. The Nordgren-Rosenthal-Wintrobe Theorem

Recall that this result equates a positive solution to the Invariant Subspace Problem for Hilbert space with a positive answer to the question: "For φ a hyperbolic automorphism of $\mathbb{U}$, does does every nontrivial minimal C_φ-invariant subspace of H^2 contain an eigenfunction?" The theorem comes about in this way: About forty years ago Caradus [2] proved the following elementary, but still remarkable, result:

> *If an operator T maps a separable, infinite-dimensional Hilbert space onto itself and has infinite-dimensional null space, then* every *operator on a separable Hilbert space is similar to a scalar multiple of the restriction of T to one of its invariant subspaces.*

Consequently the invariant subspace lattice of T contains an order isomorphic copy of the invariant subspace lattice of every operator on a separable Hilbert space. Thus, if the invariant subspace problem has a negative solution; i.e., if some operator on a separable Hilbert space has no nontrivial invariant subspace, the same will be true of the restriction of T to one of its invariant subspaces.

Now all composition operators (except the ones induced by constant functions) are one-to-one, so none of them obeys the Caradus theorem's hypotheses.

However Nordgren, Rosenthal, and Wintrobe were able to show that if φ is a hyperbolic automorphism, then for every eigenvalue λ of C_φ the operator $C_\varphi - \lambda I$, which has infinite-dimensional kernel (recall Theorem 2.1), maps H^2 onto itself. Their restatement of the Invariant Subspace Problem follows from this via the Caradus theorem and the fact that C_φ and $C_\varphi - \lambda I$ have the same invariant subspaces.

4.3. Cyclicity

Minimal invariant subspaces for invertible operators are both cyclic and doubly invariant – this was the original motivation for studying the subspaces D_f. Thus it makes sense, for a given doubly invariant subspace, and especially for a doubly cyclic one D_f, to ask whether or not it is cyclic. Here is a result in that direction in which the cyclicity is the strongest possible: *hypercyclicity* – some orbit (with no help from the linear span) is dense. I state it for canonical hyperbolic automorphisms; the generalization to non-canonical ones follows from the discussion of §4.1 and the similarity invariance of the property of hypercyclicity.

Proposition 4.3. *Suppose φ is a canonical hyperbolic automorphism of $\mathbb{U}$ and $f \in \sqrt{(z+1)(z-1)}\, H^2$. Then $C_\varphi|_{D_f}$ is hypercyclic.*

Proof. A sufficient condition for an invertible operator on a Banach space X to be hypercyclic is that for some dense subset of the space, the positive powers of both the operator and its inverse tend to zero pointwise in the norm of X (see [10, Chapter 7, page 109], for example; much weaker conditions suffice). In our case the dense subspace is just the linear span of $S := \{f \circ \varphi_n : n \in \mathbb{Z}\}$. As we saw in the proof of Theorem 3.5, our hypothesis on f insures that $\sum_{n\in\mathbb{Z}} \|f \circ \varphi_n\|^2 < \infty$ so both $(C_\varphi^n)_0^\infty$ and $(C_\varphi^{-n})_0^\infty$ converge pointwise to zero on S, and therefore pointwise on its linear span. □

Remark 4.4. One can obtain the conclusion of Proposition 4.3 under different hypotheses. For example if f is continuous with value zero at both of the fixed points of φ, then the same is true of the restriction of $|f|^2$ to $\partial\mathbb{U}$. Thus the Poisson integral of $|f|^2$ has radial limit zero at each fixed point of φ (see [9, Theorem 11.3, page 244], for example), so by (3.3), just as in the proof of Proposition 4.3, $C_\varphi|_{D_f}$ satisfies the sufficient condition for hypercyclicity. In fact, all that is really needed for this argument is that the measure

$$E \to \int_E |f|^2\, dm \qquad (E \text{ measurable} \subset \partial\mathbb{U})$$

have symmetric derivative zero at both fixed points of φ (see the reference above to [9]).

Acknowledgement

I wish to thank the referees and Professor Paul Bourdon of Washington and Lee University for many suggestions that corrected and improved preliminary versions of this paper.

References

[1] Patrick Ahern, *The mean modulus and the derivative of an inner function*, Indiana Univ. Math. J. 28 (1979), 311–347.

[2] S.R. Caradus, *Universal operators and invariant subspaces,* Proc. Amer. Math. Soc. 23 (1969), 526–527.

[3] Vitali Chkliar, *Eigenfunctions of the hyperbolic composition operator,* Integr. Equ. Oper. Theory 29 (1997), 264–367.

[4] Valentin Matache, *On the minimal invariant subspaces of the hyperbolic composition operator,* Proc. Amer. Math. Soc. 119 (1993), 837–841.

[5] Valentin Matache, *The eigenfunctions of a certain composition operator,* Contemp. Math. 213 (1998), 121–136.

[6] Valentin Matache, *Notes on hypercyclic operators,* Acta Sci. Math. (Szeged) 58 (1993), 397–406.

[7] Eric Nordgren, *Composition operators,* Canadian J. Math. 20 (1968), 442–449.

[8] Eric Nordgren, Peter Rosenthal, and F.S. Wintrobe, *Invertible composition operators on H^p,* J. Functional Anal. 73 (1987), 324–344.

[9] Walter Rudin, *Real and Complex Analysis,* third ed., McGraw-Hill, New York, 1987.

[10] Joel H. Shapiro, *Composition Operators and Classical Function Theory,* Springer-Verlag 1993.

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Received: February 19, 2009
Accepted: May 7, 2009

Operator Theory:
Advances and Applications, Vol. 202, 535–538

On Norms of Completely Positive Maps

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Abstract. King and Ruskai asked whether the norm of a completely positive map acting between Schatten classes of operators is equal to that of its restriction to the real subspace of self-adjoint operators. Proofs have been promptly supplied by Watrous and Audenaert. Here we provide one more proof, in fact of a slightly more general fact, under the (slightly weaker) assumption of 2-positivity. The argument is elementary and self-contained.

Mathematics Subject Classification (2000). 46L07, 46N50, 47L07, 47L20, 81P68.

Keywords. Norms of completely positive maps, Schatten norms, 2-positivity.

Let $\mathcal{M}_n$ denote the space of $n \times n$ (real or complex) matrices and, for $A \in \mathcal{M}_n$ and $p \geq 1$, let $\|A\|_p := (\mathrm{tr}(A^\dagger A)^{p/2})^{1/p}$ be the Schatten p-norm of A, with the limit case $p = \infty$ corresponding to the usual operator norm. Further, if $\Phi : \mathcal{M}_m \to \mathcal{M}_n$ is a linear map and $p, q \in [1, \infty]$, we consider

$$\|\Phi\|_{p\to q} := \max\{\|\Phi(\sigma)\|_q \ : \ \sigma \in \mathcal{M}_m, \|\sigma\|_p \leq 1\}, \tag{1}$$

i.e., the norm of Φ as an operator between the normed spaces $(\mathcal{M}_m, \|\cdot\|_p)$ and $(\mathcal{M}_n, \|\cdot\|_q)$. Such quantities were studied (in the context of quantum information theory) in [1], where the question was raised under what conditions (1) coincides with the *a priori* smaller norm

$$\|\Phi\|^H_{p\to q} := \max\{\|\Phi(\sigma)\|_q \ : \ \sigma \in \mathcal{M}_m, \sigma = \sigma^\dagger, \|\sigma\|_p \leq 1\} \tag{2}$$

of the restriction of Φ to the (real linear) subspace of *Hermitian* matrices and, in particular, whether this holds when Φ is *completely positive.* The latter was subsequently confirmed in [2, 3], the first of which also contains an assortment of examples showing when such equalities may or may not hold (see also the Appendix in [4] and [5, 6]). Here we provide one more proof. More precisely, we will show

Proposition. *If Φ is 2-positive, then $\|\Phi\|_{p\to q} = \|\Phi\|^H_{p\to q}$ and both norms are attained on positive semi-definite (p.s.d.) matrices. Moreover, the statement also holds if the domain and the range of Φ are endowed with any unitarily invariant norms.*

Communicated by L. Rodman.

Recall that $\Phi : \mathcal{M}_m \to \mathcal{M}_n$ is called *k-positive* if $\Phi \otimes Id_{\mathcal{M}_k}$ is *positivity preserving* (i.e., p.s.d.-preserving); Φ is completely positive if it is k-positive for all $k \in \mathbb{N}$. We note that if Φ is just positivity preserving ($\Leftrightarrow$ 1-positive), then it maps *all* Hermitian matrices to Hermitian matrices. (In the complex case, this property is equivalent to $\Phi(\sigma^\dagger) = \Phi(\sigma)^\dagger$ for $\sigma \in \mathcal{M}_m$; in the real case the latter property is stronger, but is implied by 2-positivity, as we shall see below.) A norm $\|\cdot\|$ on $\mathcal{M}_n$ is called *unitarily invariant* if $\|UAV\| = \|A\|$ for any $A \in \mathcal{M}_n$ and any $U, V \in U(n)$ (resp., $O(n)$ in the real case); see [7, 8]. This is equivalent to requiring that the norm of a matrix depends only on its *singular values* (called in some circles "Schmidt coefficients").

Besides using a slightly weaker hypothesis and yielding a slightly more general assertion, the argument we present is self-contained and uses only definitions and elementary facts and concepts from linear algebra, of which the most sophisticated is the singular value decomposition. It may thus be argued that it is the "right" proof. (Note that an analysis of [3] and its references shows that in fact only 2-positivity is needed there, too; this was reflected in the published version of [3].)

Proof. For clarity, we will consider first the case when $p = 1$, i.e., when the domain of Φ is endowed with the trace class norm. In this case the extreme points of the respective unit balls (on which the maxima in (1) and (2) are necessarily achieved) are particularly simple: they are rank one operators. Accordingly, the question reduces to showing that

$$\max_{|u|=|v|=1} \|\Phi(|v\rangle\langle u|)\|_q \;\leq\; \max_{|u|=1} \|\Phi(|u\rangle\langle u|)\|_q, \tag{3}$$

where $u, v \in \mathbb{C}^m$ (or $\mathbb{R}^m$, depending on the context) and $|\cdot|$ is the Euclidean norm. Given such u, v, consider the block matrix $M_{u,v} = \begin{bmatrix} |u\rangle\langle u| & |u\rangle\langle v| \\ |v\rangle\langle u| & |v\rangle\langle v| \end{bmatrix} \in \mathcal{M}_{2m}$ and note that $M_{u,v} = |\xi\rangle\langle\xi|$ where $|\xi\rangle = (|u\rangle, |v\rangle) \in \mathbb{C}^r \oplus \mathbb{C}^r$ (in particular $M_{u,v}$ is p.s.d.). Considering $M_{u,v}$ as an element of $\mathcal{M}_m \otimes \mathcal{M}_2$ and appealing to 2-positivity of Φ we deduce that $(\Phi \otimes Id_{\mathcal{M}_2})(M_{u,v}) = \begin{bmatrix} \Phi(|u\rangle\langle u|) & \Phi(|u\rangle\langle v|) \\ \Phi(|v\rangle\langle u|) & \Phi(|v\rangle\langle v|) \end{bmatrix}$ is p.s.d. In particular, $\Phi(|v\rangle\langle u|) = \Phi(|u\rangle\langle v|)^\dagger$ and the conclusion now follows from the following lemma (see, e.g., [8], Theorem 3.5.15; for completeness we include a proof at the end of this note).

Lemma. *Let $A, B, C \in \mathcal{M}_r$ be such that the $2r \times 2r$ block matrix $M = \begin{bmatrix} A & B \\ B^\dagger & C \end{bmatrix}$ is positive semi-definite, and let $\|\cdot\|$ be a unitarily invariant norm on $\mathcal{M}_r$. Then $\|B\|^2 \leq \|A\|\,\|C\|$.*

The case of arbitrary $p \in [1, \infty]$ is almost as simple. First, for $\sigma \in \mathcal{M}_m$ with $\|\sigma\|_p \leq 1$ we consider the positive semi-definite matrix

$$M_\sigma = \begin{bmatrix} (\sigma\sigma^\dagger)^{1/2} & \sigma \\ \sigma^\dagger & (\sigma^\dagger\sigma)^{1/2} \end{bmatrix}.$$

(Positive semi-definiteness is seen, e.g., by writing down the singular value decompositions of the entries and expressing M_σ as a positive linear combination of matrices of the type $M_{u,v}$ considered above, or by looking at the polar decomposition $\sigma = UP$ and noting that $M_\sigma = S^\dagger S$, where $S = [P^{1/2}U^\dagger, P^{1/2}]$.) Since unitarily invariant norms depend only on singular values of a matrix, we have $\|(\sigma\sigma^\dagger)^{1/2}\|_p = \|(\sigma^\dagger\sigma)^{1/2}\|_p = \|\sigma\|_p \le 1$. On the other hand, arguing as in the special case $p = 1$, we deduce from the Lemma that $\|\Phi(\sigma)\|_q^2 \le \|\Phi((\sigma\sigma^\dagger)^{1/2})\|_q \, \|\Phi((\sigma^\dagger\sigma)^{1/2})\|_q \le \big(\|\Phi\|^H_{p\to q}\big)^2$, and the conclusion follows by taking the maximum over σ. The proof for general unitarily invariant norms is the same (once the appropriate notation is introduced).

Note that since $(\sigma\sigma^\dagger)^{1/2}$ and $(\sigma^\dagger\sigma)^{1/2}$ are positive semi-definite, the argument also yields the second assertion of the Proposition. However, the fact that the norm $\|\Phi\|^H$ of the Hermitian restriction of Φ is attained on positive semi-definite matrices is even more elementary and requires only that Φ be a positivity preserving map. $\square$

Proof of the lemma. (Written for $\|\cdot\| = \|\cdot\|_q$, but the general case works in the same way.) Let $B = \sum_{j=1}^r \lambda_j |\varphi_j\rangle\langle\psi_j|$ be the singular value decomposition. Consider the orthonormal basis of $\mathbb{C}^{2r}$ which is a concatenation of $(|\varphi_j\rangle)$ and $(|\psi_j\rangle)$. The representation of M in that basis is

$$M' := \begin{bmatrix} (\langle\varphi_j|A|\varphi_k\rangle)_{j,k=1}^r & \mathrm{Diag}(\lambda) \\ \mathrm{Diag}(\lambda) & (\langle\psi_j|C|\psi_k\rangle)_{j,k=1}^r \end{bmatrix},$$

where $\mathrm{Diag}(\mu)$ is the diagonal matrix with the sequence $\mu = (\mu_j)$ on the diagonal. Given $j \in \{1,\ldots,r\}$, the 2×2 matrix $\begin{bmatrix} \langle\varphi_j|A|\varphi_j\rangle & \lambda_j \\ \lambda_j & \langle\psi_j|C|\psi_j\rangle \end{bmatrix}$ is a minor of M' and hence positive semi-definite, and so $\lambda_j \le \sqrt{\langle\varphi_j|A|\varphi_j\rangle\langle\psi_j|C|\psi_j\rangle} \le (\langle\varphi_j|A|\varphi_j\rangle + \langle\psi_j|C|\psi_j\rangle)/2$. Consequently

$$\begin{aligned} \|B\|_q = \big(\sum_j \lambda_j^q\big)^{1/q} &\le \Big(\big(\sum_j \langle\varphi_j|A|\varphi_j\rangle^q\big)^{1/q} + \big(\sum_j \langle\psi_j|C|\psi_j\rangle^q\big)^{1/q}\Big)/2 \\ &\le (\|A\|_q + \|C\|_q)/2. \end{aligned} \tag{4}$$

The last inequality in (4) follows from the well-known fact that, for *any* square matrix $S = (S_{jk})$, $\|S\|_q \ge \big(\sum_j |S_{jj}|^q\big)^{1/q}$ (which in turn is a consequence of $(S_{jk}\delta_{jk})$, the diagonal part of S, being the average of $\mathrm{Diag}(\varepsilon)\, S\, \mathrm{Diag}(\varepsilon)$, where $\varepsilon = (\varepsilon_j)$ varies over all choices of $\varepsilon_j = \pm 1$). The bound from (4) is already sufficient to prove (3) (and the Proposition). To obtain the stronger statement from the Lemma we use the inequality $ab \le \frac{1}{2}(ta + b/t)$ (for $t > 0$, instead of $ab \le \frac{1}{2}(a+b)$) to obtain $\|B\|_q \le \frac{1}{2}(t\|A\|_q + \|C\|_q/t)$, and then specify the optimal value $t = (\|C\|_q/\|A\|_q)^{1/2}$. Passing to a general unitarily invariant norm requires just replacing everywhere $\big(\sum_j \mu_j^q\big)^{1/q}$ by $\|\mathrm{Diag}(\mu)\|$; equalities such as $\|B\| =$

$\|\mathrm{Diag}(\lambda)\|$ or $\|A\| = \|\left(\langle\varphi_j|A|\varphi_k\rangle\right)_{j,k=1}^{r}\|$ just express the unitary invariance of the norm. □

Acknowledgement

Supported in part by grants from the National Science Foundation (USA). The author thanks K. Audenaert and M.B. Ruskai for comments on the first version of this note, which appeared as an arxiv.org e-print `quant-ph/0603110`.

References

[1] C. King and M.B. Ruskai, Comments on multiplicativity of maximal p-norms when $p = 2$, Quantum Inf. Comput. 4 (2004), no. 6-7, 500–512.

[2] J. Watrous, Notes on super-operator norms induced by Schatten norms, Quantum Inf. Comput. 5 (2005), 58–68.

[3] K. Audenaert, A Note on the $p \to q$ norms of Completely Positive Maps, arxiv.org e-print math-ph/0505085. [A note on the $p \to q$ norms of 2-positive maps. Linear Algebra Appl. 430 (2009), no. 4, 1436–1440].

[4] A.S. Kholevo, Multiplicativity of p-norms of completely positive maps and the additivity problem in quantum information theory, Russian Math. Surveys 61 (2006), no. 2, 301–339.

[5] I. Devetak, M. Junge, C. King and M.B. Ruskai, Multiplicativity of completely bounded p-norms implies a new additivity result, Commun. Math. Phys. 266 (2006), 37–63.

[6] P. Hayden and A. Winter, Counterexamples to the maximal p-norm multiplicativity conjecture for all $p > 1$, Comm. Math. Phys. 284 (2008), no. 1, 263–280.

[7] R. Schatten, Norm Ideals of Completely Continuous Operators. Springer-Verlag, Berlin, 1970.

[8] R.A. Horn and C.R. Johnson, Topics in matrix analysis. Cambridge University Press, Cambridge, 1994.

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Received: February 27, 2009
Accepted: March 25, 2009

Operator Theory:
Advances and Applications, Vol. 202, 539–552

Some Exponential Inequalities for Semisimple Lie Groups

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Abstract. Let $|||\cdot|||$ be any give unitarily invariant norm. We obtain some exponential relations in the context of semisimple Lie group. On one hand they extend the inequalities (1) $|||e^A||| \le |||e^{\mathrm{Re}\, A}|||$ for all $A \in \mathbb{C}_{n\times n}$, where $\mathrm{Re}\, A$ denotes the Hermitian part of A, and (2) $|||e^{A+B}||| \le |||e^A e^B|||$, where A and B are $n\times n$ Hermitian matrices. On the other hand, the inequalities of Weyl, Ky Fan, Golden-Thompson, Lenard-Thompson, Cohen, and So-Thompson are recovered. Araki's relation on $(e^{A/2}e^Be^{A/2})^r$ and $e^{rA/2}e^{rB}e^{rA/2}$, where A, B are Hermitian and $r \in \mathbb{R}$, is extended.

Mathematics Subject Classification (2000). Primary 15A45, 22E46; Secondary 15A42.

Keywords. Singular values, eigenvalue moduli, spectral radius, pre-order.

1. Introduction

A norm $|||\cdot||| : \mathbb{C}_{n\times n} \to \mathbb{R}$ is said to be unitary invariant if $|||A||| = |||UAV|||$ for all $U, V \in U(n)$. It is known [3, Theorem IX.3.1, Theorem IX.3.7] that for any unitarily invariant norm $|||\cdot||| : \mathbb{C}_{n\times n} \to \mathbb{R}$,

$$|||e^A||| \le |||e^{\mathrm{Re}\, A}|||, \quad A \in \mathbb{C}_{n\times n}, \tag{1.1}$$

$$|||e^{A+B}||| \le |||e^A e^B|||, \quad A, B \in \mathbb{C}_{n\times n} \text{ are Hermitian}, \tag{1.2}$$

where $\mathrm{Re}\, A$ denotes the Hermitian part of $A \in \mathbb{C}_{n\times n}$. Inequality (1.2) is a generalization of the famous Golden-Thompson inequality [6, 21]

$$\mathrm{tr}\, e^{A+B} \le \mathrm{tr}\, (e^A e^B), \qquad A, B \text{ Hermitian}. \tag{1.3}$$

It is because that the Ky Fan n-norm, denoted by $\|\cdot\|_n$, is unitarily invariant, where $\|A\|_n$ is the sum of the singular values of $A \in \mathbb{C}_{n\times n}$. See [16, 22, 1, 2] for some generalizations of Golden-Thompson's inequality.

Communicated by L. Rodman.

A result in [3, Theorem IX.3.5] implies that for any irreducible representation π of the general linear group $\mathrm{GL}_n(\mathbb{C})$,

$$|\pi(e^{A+B})| \le |\pi(e^{\mathrm{Re}\,A}e^{\mathrm{Re}\,B})|, \quad A, B \in \mathbb{C}_{n\times n}, \tag{1.4}$$

where $|X|$ denotes the spectral radius of the linear map X.

The inequalities (1.1) and (1.2) compare two matrix exponentials using unitarily invariant norm. Apparently unitarily invariant norm plays no role in the inequality (1.4). But we will obtain Theorem 3.1 as unified extension of (1.1), (1.2) and (1.4).

After the preliminary materials are introduced in Section 2, Theorem 3.1 is obtained in the context of semisimple Lie group. It contains two sets of inequalities concerning a pre-order of Kostant [14]. To further demonstrated the importance of Theorem 3.1, in a sequence of remarks, we derive from Theorem 3.1 the inequalities of

1. Weyl [3]: the moduli of the eigenvalues of A are log majorized by the singular values of $A \in \mathbb{C}_{n\times n}$.
2. Ky Fan [3]: the real parts of the eigenvalues of A are majorized by the real singular values of $A \in \mathbb{C}_{n\times n}$.
3. Lenard-Thompson [16, 22]: $|||e^{A+B}||| \le |||e^{A/2}e^{B}e^{A/2}|||$, where $A, B \in \mathbb{C}_{n\times n}$ are Hermitian.
4. Cohen [4]: the eigenvalues of the positive definite part of e^A (with respect to the usual polar decomposition) are log majorized by the eigenvalues of $e^{\mathrm{Re}\,A}$, where $A \in \mathbb{C}_{n\times n}$.
5. So-Thompson [18]: the singular values of e^A are weakly log majorized by the exponentials of the singular values of $A \in \mathbb{C}_{n\times n}$.

In Section 4 we extend, in the context of semisimple Lie group, Araki's result [1] on the relation of $(e^{A/2}e^{B}e^{A/2})^r$ and $e^{rA/2}e^{rB}e^{rA/2}$, where $A, B \in \mathbb{C}_{n\times n}$ are Hermitian, $r \ge 0$.

2. Preliminaries

We recall some basic notions, especially a pre-order of Kostant and some results in [14].

A matrix in $\mathrm{GL}_n(\mathbb{C})$ is called *elliptic* (respectively *hyperbolic*) if it is diagonalizable with norm 1 (respectively real positive) eigenvalues. It is called *unipotent* if all its eigenvalues are 1. The *complete multiplicative Jordan decomposition* of $g \in \mathrm{GL}_n(\mathbb{C})$ asserts that $g = ehu$ for $e, h, u \in \mathrm{GL}_n(\mathbb{C})$, where e is elliptic, h is hyperbolic, u is unipotent, and these three elements commute. The decomposition is obvious when g is in a Jordan canonical form with diagonal entries (i.e., eigenvalues) $z_1, \ldots, z_n$, in which

$$e = \mathrm{diag}\left(\frac{z_1}{|z_1|}, \ldots, \frac{z_n}{|z_n|}\right), \qquad h = \mathrm{diag}\left(|z_1|, \ldots, |z_n|\right),$$

and $u = h^{-1}e^{-1}g$ is a unit upper triangular matrix. The above decomposition can be extended to semisimple Lie groups.

Let $\mathfrak{g}$ be a real semisimple Lie algebra. Let G be any connected Lie group having $\mathfrak{g}$ as its Lie algebra. An element $X \in \mathfrak{g}$ is called *real semisimple* if $\operatorname{ad} X \in \operatorname{End} \mathfrak{g}$ is diagonalizable over $\mathbb{R}$ and is called *nilpotent* if $\operatorname{ad} X \in \operatorname{End} \mathfrak{g}$ is a nilpotent endomorphism. An element $g \in G$ is called *hyperbolic* if $g = \exp X$, where $X \in \mathfrak{g}$ is real semisimple and is called *unipotent* if $g = \exp X$, where $X \in \mathfrak{g}$ is nilpotent. An element $g \in G$ is *elliptic* if $\operatorname{Ad} g \in \operatorname{Aut} \mathfrak{g}$ is diagonalizable over $\mathbb{C}$ with eigenvalues of modulus 1. The complete multiplicative Jordan decomposition (CMJD) [14, Proposition 2.1] for G asserts that each $g \in G$ can be uniquely written as

$$g = ehu,$$

where e is elliptic, h is hyperbolic and u is unipotent and the three elements e, h, u commute. We write $g = e(g)h(g)u(g)$.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a fixed Cartan decomposition of $\mathfrak{g}$. Let $K \subset G$ be the analytic group of $\mathfrak{k}$ so that $\operatorname{Ad} K$ is a maximal compact subgroup of $\operatorname{Ad} G$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace in $\mathfrak{p}$. Then $A := \exp \mathfrak{a}$ is the analytic subgroup of $\mathfrak{a}$. Let W be the Weyl group of $(\mathfrak{a}, \mathfrak{g})$ which may be defined as the quotient of the normalizer of A in K modulo the centralizer of A in K. The Weyl group operates naturally in $\mathfrak{a}$ and A and the isomorphism $\exp : \mathfrak{a} \to A$ is a W-isomorphism.

For each real semisimple $X \in \mathfrak{g}$, let

$$c(X) := \operatorname{Ad} G(X) \cap \mathfrak{a}$$

denote the set of all elements in $\mathfrak{a}$ which are conjugate to X (via the adjoint representation of G). For each hyperbolic $h \in G$, let

$$C(h) := \{ghg^{-1} : g \in G\} \cap A$$

denote the set of all elements in A which are conjugate to h. It turns out that $X \in \mathfrak{g}$ ($h \in G$, $e \in G$) is real semisimple (hyperbolic, elliptic) if and only if it is conjugate to an element in $\mathfrak{a}$ (A, K, respectively) [14, Proposition 2.3 and 2.4]. Thus $c(X)$ and $C(h)$ are single W-orbits in $\mathfrak{a}$ and A respectively. Moreover

$$C(\exp(X)) = \exp c(X).$$

Denote by $\operatorname{conv} W(X)$ the convex hull of the orbit $Wc(X) \subset \mathfrak{a}$ under the action of the Weyl group W. For arbitrary $g \in G$, define

$$C(g) := C(h(g)),$$

where $h(g)$ is the hyperbolic component of g and

$$\mathcal{A}(g) := \exp(\operatorname{conv} W(\log h(g)))$$

(For a hyperbolic $h \in G$, we write $\log h = X$ if $e^X = h$ and X is real semisimple. The element X is unique since $\operatorname{Ad}(e^X) = e^{\operatorname{ad} X}$ and the restriction of the usual matrix exponential map $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$ on the set of diagonalizable matrices over $\mathbb{R}$ is one-to-one.) Clearly $\mathcal{A}(g) \subset A$ and is invariant under the Weyl group. It is the "convex hull" of $C(g)$ in the multiplicative sense. Given $f, g \in G$, we say that $f \prec g$ if

$$\mathcal{A}(f) \subset \mathcal{A}(g),$$

or equivalently

$$C(f) \subset \mathcal{A}(g).$$

Notice that $\prec$ is a pre-order on G and $\mathcal{A}(\ell g \ell^{-1}) = \mathcal{A}(g)$ since $h(\ell g \ell^{-1}) = \ell h(g) \ell^{-1}$ for all $\ell \in G$. It induces a partial order on the equivalence classes of hyperbolic elements under the conjugation of G. The order $\prec$ is different from Thompson's pre-order [22] on $\mathrm{SL}_n(\mathbb{C})$ which simplifies the one made by Lenard [16]. Indeed the orders of Lenard and Thompson agree on the space of positive definite matrices.

We denote by $\hat{G}$ the index set of the irreducible representations of G, $\pi_\lambda : G \to \mathrm{Aut}\,(V_\lambda)$ a fixed representation in the class corresponding to $\lambda \in \hat{G}$, $|\pi_\lambda(g)|$ the spectral radius of the automorphism $\pi_\lambda(g) : V_\lambda \to V_\lambda$, where $g \in G$, that is, the maximum modulus of the eigenvalues of $\pi_\lambda(g)$, and χ_λ the character of π_λ. The following nice result of Kostant describes the pre-order $\prec$ via the irreducible representations of G and plays an important role in the coming sections.

Theorem 2.1. (Kostant [14, Theorem 3.1]) *Let $f, g \in G$. Then $f \prec g$ if and only if $|\pi_\lambda(f)| \le |\pi_\lambda(g)|$ for all $\lambda \in \hat{G}$, where $|\cdot|$ denotes the spectral radius.*

The following proposition describes $\prec$ in terms of inequalities when $G = \mathrm{SL}_n(\mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Proposition 2.2. *Let $G = \mathrm{SL}_n(\mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and let $f, g \in G$. Denote by $\alpha_1, \dots, \alpha_n$ the eigenvalues of f and $\beta_1, \dots, \beta_n$ the eigenvalues of g arranged in the way that $|\alpha_1| \ge |\alpha_2| \ge \cdots \ge |\alpha_n|$ and $|\beta_1| \ge |\beta_2| \ge \cdots \ge |\beta_n|$. Then $f \prec g$ if and only if $|\alpha|$ is multiplicatively majorized by $|\beta|$, that is,*

$$\begin{aligned} \prod_{i=1}^k |\alpha_i| &\le \prod_{i=1}^k |\beta_i|, \quad k = 1, \dots, n-1, \\ \prod_{i=1}^n |\alpha_i| &= \prod_{i=1}^n |\beta_i|. \end{aligned}$$

Proof. We just deal with the real case (the complex case is similar) and we first describe the CMJD. Let $G = \mathrm{SL}_n(\mathbb{R})$ with $K = \mathrm{SO}(n)$, $A \subset \mathrm{SL}_n(\mathbb{R})$ consists of positive diagonal matrices of determinant 1, and $\mathfrak{a}$ is the space of diagonal matrices of zero trace. Now $\mathrm{Ad}\, g = g(\cdot)g^{-1}$, $g \in \mathrm{SL}_n(\mathbb{R})$, that is, $\mathrm{Ad}\, g$ is the conjugation via g. It is known that $s \in \mathfrak{sl}_n(\mathbb{R})$ real semisimple means that s is diagonalizable over $\mathbb{R}$ (see [12, Theorem 6.4] and [15, 558]); $n \in \mathfrak{sl}_n(\mathbb{R})$ nilpotent means $n^k = 0$ for some integer $k > 0$. So $h \in \mathrm{SL}_n(\mathbb{R})$ hyperbolic means that h is diagonalizable over $\mathbb{R}$ and the eigenvalues of h are positive; $e \in \mathrm{SL}_n(\mathbb{R})$ elliptic means that e is diagonalizable over $\mathbb{R}$ and the eigenvalues of e have modulus 1; $u \in \mathrm{SL}_n(\mathbb{R})$ is unipotent if $u - 1 \in \mathfrak{sl}_n(\mathbb{R})$ is nilpotent. Then follow [9, Lemma 7.1]: viewing

$g \in \mathrm{SL}_n(\mathbb{R})$ as an element in $\mathfrak{gl}_n(\mathbb{R})$, the additive Jordan decomposition [11, p. 153] for $\mathfrak{gl}_n(\mathbb{R})$ yields

$$g = s + n_1$$

($s \in \mathrm{SL}_n(\mathbb{R})$ semisimple, that is, diagonalizable over $\mathbb{C}$, $n_1 \in \mathfrak{sl}_n(\mathbb{R})$ nilpotent and $sn_1 = n_1 s$). Moreover these conditions determine s and n_1 completely [12, Proposition 4.2]. Put $u := 1 + s^{-1}n_1 \in \mathrm{SL}_n(\mathbb{R})$ and we have the multiplicative Jordan decomposition

$$g = su,$$

where s is semisimple, u is unipotent, and $su = us$. By the uniqueness of the additive Jordan decomposition, s and u are also completely determined. Since s is diagonalizable,

$$s = eh,$$

where e is elliptic, h is hyperbolic, $eh = he$, and these conditions completely determine e and h. The decomposition can be obtained by observing that there is $k \in \mathrm{SL}_n(\mathbb{C})$ such that

$$k^{-1}sk = s_1 I_{r_1} \oplus \cdots \oplus s_m I_{r_m},$$

where $s_1 = e^{i\xi_1}|s_1|, \ldots, s_m = e^{i\xi_m}|s_m|$ are the distinct eigenvalues of s with multiplicities $r_1, \ldots, r_m$ respectively. Set

$$e := k(e^{i\xi_1} I_{r_1} \oplus \cdots \oplus e^{i\xi_m} I_{r_m})k^{-1}, \quad h := k(|s_1| I_{r_1} \oplus \cdots \oplus |s_m| I_{r_m})k^{-1}.$$

Since

$$ehu = g = ugu^{-1} = ueu^{-1}uhu^{-1}u,$$

the uniqueness of s, u, e and h implies e, u and h commute. Since g is fixed under complex conjugation, the uniqueness of e, h and u imply $e, h, u \in \mathrm{SL}_n(\mathbb{R})$ [9, p. 431]. Thus $g = ehu$ is the CMJD for $\mathrm{SL}_n(\mathbb{R})$. The eigenvalues of h are simply the eigenvalue moduli of s and thus of g.

We now are to describe $\prec$. Let $\mathfrak{sl}_n(\mathbb{R}) = \mathfrak{so}(n) + \mathfrak{p}$ be the fixed Cartan decomposition of $\mathfrak{sl}_n(\mathbb{R})$, that is, $\mathfrak{k} = \mathfrak{so}(n)$ and $\mathfrak{p}$ is the space of real symmetric matrices of zero trace. So $K = \mathrm{SO}(n)$. Let $\mathfrak{a} \subset \mathfrak{p}$ be the maximal abelian subspace of $\mathfrak{sl}_n(\mathbb{R})$ in $\mathfrak{p}$ containing the diagonal matrices. So the analytic group A of $\mathfrak{a}$ is the group of positive diagonal matrices of determinant 1. The Weyl group W of $(\mathfrak{a}, \mathfrak{g})$ is the full symmetric group S_n [13] which acts on A and $\mathfrak{a}$ by permuting the diagonal entries of the matrices in A and $\mathfrak{a}$. Now

$$C(f) := C(h(f)) = \{\mathrm{diag}\,(|\alpha_{\sigma(1)}|, \ldots, |\alpha_{\sigma(n)}|) : \sigma \in S_n\},$$

where $\alpha_1, \ldots, \alpha_n$ denote the eigenvalues of $f \in \mathrm{SL}_n(\mathbb{C})$ with the order $|\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_n|$. So

$$c(\log h(f)) = \{\mathrm{diag}\,(\log|\alpha_{\sigma(1)}|, \ldots, \log|\alpha_{\sigma(n)}|) : \sigma \in S_n\}$$

and

$$\mathcal{A}(f) = \exp\mathrm{conv}\,\{\mathrm{diag}\,(\log|\alpha_{\sigma(1)}|, \ldots, \log|\alpha_{\sigma(n)}|) : \sigma \in S_n\}.$$

So $f \prec g$, $f, g \in \mathrm{SL}_n(\mathbb{R})$ means that $\log|\alpha|$ is majorized by $\log|\beta|$ [3, p. 33], usually denoted by $|\alpha| \prec_{\log} |\beta|$ and is called log majorization [2], where β's are the eigenvalues of g. □

Remark 2.3. In the above example, the pre-order $\prec$ in $\mathrm{SL}_n(\mathbb{R}) \subset \mathrm{SL}_n(\mathbb{C})$ coincides with that in $\mathrm{SL}_n(\mathbb{C})$ since the Weyl groups are identical. But it is pointed out in [14, Remark 3.1.1] that the pre-order $\prec$ is not necessarily the same as the pre-order on the semisimple G that would be induced by a possible embedding of G in $\mathrm{SL}_n(\mathbb{C})$ for some n.

3. A pre-order of Kostant and some order relations

Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the real semisimple Lie algebra $\mathfrak{g}$. For each $X \in \mathfrak{g}$, write $X = X_{\mathfrak{k}} + X_{\mathfrak{p}}$, where $X_{\mathfrak{k}} \in \mathfrak{k}$ and $X_{\mathfrak{p}} \in \mathfrak{p}$. Let

$$G = KP$$

be the Cartan decomposition of analytic group G of $\mathfrak{g}$ [9], where $P := \exp \mathfrak{p}$.

Define $g^* := pk^{-1}$ if $g = kp$ with respect to the Cartan decomposition $G = KP$. When $G = \mathrm{SL}_n(\mathbb{C})$ with $K = \mathrm{SU}(n)$, g^* is simply the complex conjugate transpose of g.

Theorem 3.1. *Let $\mathfrak{g}$ be a real semisimple Lie algebra. Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Then for any $g \in G$,*

$$g^{2n} \prec (g^*)^n g^n \prec (g^* g)^n, \qquad n = 1, 2, \ldots. \tag{3.1}$$

Moreover for any $X, Y \in \mathfrak{g}$,

$$e^{X+Y} \prec e^{-\theta(X+Y)/2} e^{(X+Y)/2} \prec e^{X_{\mathfrak{p}}} e^{Y_{\mathfrak{p}}}, \tag{3.2}$$

where θ is the Cartan involution of $\mathfrak{g}$ with respect to the given Cartan decomposition.

Remark 3.2. When $G = \mathrm{SL}_n(\mathbb{C})$ or $\mathrm{GL}_n(\mathbb{C})$, the relation $g^{*n} g^n \prec (g^* g)^n$ was established in [4] and $g^{2n} \prec (g^* g)^n$ was obtained in [22]. The inequality $g^{2n} \prec (g^*)^n g^n$ is reduced to Weyl's inequality by Proposition 2.2. See Remark 3.8. Kostant [14, proof of Theorem 6.3] also proved $g^{2n} \prec (g^* g)^n$ and $e^{A+B} \prec e^A e^B$, $A, B \in \mathfrak{p}$, for general G. The generalization as a whole is new.

Proof. Let $\theta \in \operatorname{Aut} \mathfrak{g}$ be the Cartan involution of $\mathfrak{g}$, that is, θ is 1 on $\mathfrak{k}$ and -1 on $\mathfrak{p}$. Set $P = e^{\mathfrak{p}}$. We have the (global) Cartan decomposition

$$G = KP.$$

The involution θ induces an automorphism Θ of G such that the differential of Θ at the identity is θ [13, p. 387]. Explicitly

$$\Theta(kp) = kp^{-1}, \quad k \in K,\ p \in P.$$

For any $g \in G$ let

$$g^* := \Theta(g^{-1}).$$

If $g = kp$, then

$$g^* = \Theta(p^{-1}k^{-1}) = \Theta(p^{-1})k^{-1} = pk^{-1},$$

and hence $g^*g = p^2 \in P$, since the centralizer $G^\Theta = \{g \in G : \Theta(g) = g\}$ coincides with K [13, p. 305]. So

$$g^* := \Theta(g^{-1}) = (\Theta(g))^{-1}, \quad (g^*)^* = g, \quad (fg)^* = g^*f^*, \quad (g^*)^n = (g^n)^*,$$

for all $f, g \in G$, $n \geq 1$. Since θ is the differential of Θ at the identity, we have [9, 110]

$$\Theta(e^X) = e^{\theta X},$$

for all $X \in \mathfrak{g}$. So

$$(e^X)^* = \Theta(e^{-X}) = e^{-\theta X}. \tag{3.3}$$

The relation $g^{2n} \prec (g^*g)^n$ in (3.1) is known in [14, p. 448] and we use similar idea to establish (3.1). Actually the original idea can be found in [22] when $G = \mathrm{SL}_n(\mathbb{C})$.

We denote by $\Pi_\lambda : \mathfrak{g} \to \operatorname{End} V_\lambda$ the differential at the identity of the representation $\pi_\lambda : G \to \operatorname{Aut} V_\lambda$. So [9, p. 110]

$$\exp \circ \Pi_\lambda = \pi_\lambda \circ \exp, \tag{3.4}$$

where the exponential function on the left is $\exp : \operatorname{End} V_\lambda \to \operatorname{Aut} V_\lambda$ and the one on the right is $\exp : \mathfrak{g} \to G$. Now $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$ (direct sum) is a compact real form of $\mathfrak{g}_\mathbb{C}$ (the complexification of $\mathfrak{g}$). The representation $\Pi_\lambda : \mathfrak{g} \to \operatorname{End} V_\lambda$ naturally defines a representation $\mathfrak{u} \to \operatorname{End} V_\lambda$ of $\mathfrak{u}$, also denoted by Π_λ and vice versa. Let U be a simply connected Lie group of $\mathfrak{u}$ [24, p. 101] so that it is compact [5, Corollary 3.6.3]. There is a unique homomorphism $\hat\pi_\lambda : U \to \operatorname{Aut} V_\lambda$ such that the differential of $\hat\pi_\lambda$ at the identity is Π_λ [24, Theorem 3.27]. Thus there exists an inner product $\langle \cdot, \cdot \rangle$ on V_λ such that $\hat\pi_\lambda(u)$ is orthogonal for all $u \in U$. We will assume that V_λ is endowed with this structure from now on. Differentiate the identity

$$\langle \hat\pi_\lambda(e^{tZ})X, \hat\pi_\lambda(e^{tZ})Y \rangle = \langle X, Y \rangle,$$

for all $X, Y \in V_\lambda$ at $t = 0$ we have

$$\langle \Pi_\lambda(Z)X, Y \rangle = -\langle X, \Pi_\lambda(Z)Y \rangle$$

by (3.4). Thus, with respect to $\langle \cdot, \cdot \rangle$, $\Pi_\lambda(Z)$ is skew Hermitian for all $Z \in \mathfrak{u}$ [13, Proposition 4.6], [14, p. 435]. Then $\Pi_\lambda(Z)$ is skew Hermitian if $Z \in \mathfrak{k}$ and is Hermitian if $Z \in \mathfrak{p}$. So $\pi_\lambda(z)$ is unitary if $z \in K$ and is positive definite if $z \in P$ by (3.4).

Since each $g \in G$ can be written as $g = kp$, $k \in K$ and $p \in P$,

$$\begin{aligned} \langle u, \pi_\lambda(g^*)v \rangle &= \langle u, \pi_\lambda(pk^{-1})v \rangle \\ &= \langle u, \pi_\lambda(p)\pi_\lambda(k^{-1})v \rangle \\ &= \langle \pi_\lambda(k)\pi_\lambda(p)u, v \rangle \\ &= \langle \pi_\lambda(g)u, v \rangle, \end{aligned}$$

for all $u, v \in V_\lambda$. Thus

$$\pi_\lambda(g)^* = \pi_\lambda(g^*), \tag{3.5}$$

where $\pi_\lambda(g)^*$ denotes the (Hermitian) adjoint of $\pi_\lambda(g)$. Thus

$$\pi_\lambda(g^*g) = \pi_\lambda(g)^*\pi_\lambda(g) \in \operatorname{Aut} V_\lambda$$

is a positive definite operator for all $g \in G$. Denote by

$$\|\pi_\lambda(g)\| := \max_{0\neq v\in V_\lambda} \frac{\|\pi_\lambda(g)v\|}{\|v\|},$$

the operator norm of $\pi_\lambda(g) \in \operatorname{Aut} V_\lambda$, where $\|v\| := \langle v, v\rangle^{1/2}$ is the norm induced by $\langle\cdot,\cdot\rangle$. Thus the spectral theorem for self-adjoint operators implies

$$|\pi_\lambda(p)| = \|\pi_\lambda(p)\|, \quad \text{for all } p \in P.$$

Because of Theorem 2.1, to arrive at the claim (3.1) it suffices to show

$$|\pi_\lambda(g^{2n})| \le |\pi_\lambda((g^*)^n g^n)| \le |\pi_\lambda((g^*g)^n)|, \quad \text{for all } \lambda \in \hat{G}.$$

Now

$$\begin{aligned}
|\pi_\lambda((g^*)^n g^n)| &= |\pi_\lambda((g^n)^* g^n)| \\
&= \|\pi_\lambda((g^n)^* g^n)\| \qquad \text{since } \pi_\lambda((g^n)^* g^n) \in \operatorname{End} V_\lambda \text{ is p.d.} \\
&= \|\pi_\lambda(g^n)^*\pi_\lambda(g^n)\| \qquad \text{by (3.5)} \\
&= \|\pi_\lambda(g^n)\|^2 \qquad \text{since } \|T\|^2 = \|T^*T\|.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|\pi_\lambda((g^*g)^n)| &= |\pi_\lambda(g^*g)|^n \\
&= \|\pi_\lambda(g^*g)\|^n \qquad \text{since } \pi_\lambda((g^*g) \in \operatorname{End} V_\lambda \text{ is p.d.} \\
&= \|\pi_\lambda(g)^*\pi_\lambda(g)\|^n \\
&= \|\pi_\lambda(g)\|^{2n} \qquad \text{since } \|T\|^2 = \|T^*T\| \\
&\ge \|\pi_\lambda(g^n)\|^2 \qquad \text{since } \|T^n\| \le \|T\|^n,
\end{aligned}$$

where the inequality is due to the well-known fact that the spectral radius is no greater than the operator norm. So we have $(g^*)^n g^n \prec (g^*g)^n$. Now

$$|\pi_\lambda((g^*)^n)g^n)| = |\pi_\lambda((g^n)^*)\pi_\lambda(g^n)| = \|\pi_\lambda(g^n)\|^2 \ge |\pi_\lambda(g^n)|^2 = |\pi_\lambda(g^{2n})|.$$

Hence $g^{2n} \prec (g^*)^n g^n$ and we just proved the claim.

By the first relation in (3.1), if $g = xy$, where $x, y \in G$, then for any $m \in \mathbb{N}$,

$$(xy)^{2^{m+1}} \prec (y^*x^*)^{2^m}(xy)^{2^m}.$$

Set $x = e^{X/2^m}$, $y = e^{Y/2^m}$, where $X, Y \in \mathfrak{g}$. From (3.3)

$$\begin{aligned}
((e^{X/2^m}e^{Y/2^m})^{2^m})^2 &\prec ((e^{Y/2^m})^*(e^{X/2^m})^*)^{2^m}(e^{X/2^m}e^{Y/2^m})^{2^m} \\
&= (e^{-\theta Y/2^m}e^{-\theta X/2^m})^{2^m}(e^{X/2^m}e^{Y/2^m})^{2^m}.
\end{aligned}$$

Since $\lim_{t\to\infty}(e^{X/t}e^{Y/t})^t = e^{X+Y}$ [9, p. 115] (Lie-Trotter formula; as pointed out in [7, p. 35], Trotter's formula is for suitable unbounded operators on an infinite-dimensional Hilbert space [17, VIII.8]), and the relation $\prec$ remains valid as we take limits on both sides because the spectral radius is a continuous function on $\operatorname{Aut} V_\lambda$, by Theorem 2.1 we have $e^{2(X+Y)} \prec e^{-\theta(X+Y)}e^{(X+Y)}$. As a result

$$e^{X+Y} \prec e^{-\frac{1}{2}\theta(X+Y)}e^{\frac{1}{2}(X+Y)}, \quad \text{for all } X, Y \in \mathfrak{g}$$

and we just established the first part of (3.2).

Let $g = e^{(X+Y)/n}$, $X, Y \in \mathfrak{g}$. By the second relation of (3.1) and (3.3)

$$(e^{-\theta(X+Y)/n})^n(e^{(X+Y)/n})^n \prec ((e^{-\theta(X+Y)/n}e^{(X+Y)/n}))^n.$$

As before

$$e^{-\theta(X+Y)}e^{X+Y} \prec e^{2(X+Y)_{\mathfrak{p}}} = e^{2X_{\mathfrak{p}}+2Y_{\mathfrak{p}}} \prec e^{2X_{\mathfrak{p}}}e^{2Y_{\mathfrak{p}}},$$

where the last relation is established in [14, Theorem 6.3]. □

Similar technique of the proof is also used in [19, 20]. By setting $Y = 0$ or $Y = X$ in the second set of inequalities of Theorem 3.1, we have

Corollary 3.3. *Let $X \in \mathfrak{g}$. Then $e^X \prec e^{-\theta X/2}e^{X/2} \prec e^{X_{\mathfrak{p}}}$.*

Remark 3.4. The statement $e^{X+Y} \prec e^{X_{\mathfrak{k}}}e^{Y_{\mathfrak{k}}}$ is not true by simply considering $G = \mathrm{SL}_n(\mathbb{C})$ in which $K = \mathrm{SU}(n)$ and $\mathfrak{k} = \mathfrak{su}(n)$. Clearly $e^{X_{\mathfrak{k}}}e^{Y_{\mathfrak{k}}} \in \mathrm{SU}(n)$ and we may pick $X, Y \in \mathfrak{sl}_n(\mathbb{C})$ such that $X + Y$ is nonzero Hermitian matrix with a positive eigenvalue. Viewing each $g \in \mathrm{SL}_n(\mathbb{C})$ as a linear operator on $V_\lambda = \mathbb{C}^n$ (the natural representation of $\mathrm{SL}_n(\mathbb{C})$), the spectral radius $|e^{X_{\mathfrak{k}}}e^{Y_{\mathfrak{k}}}| = 1$ but $|e^{X+Y}| > 1$.

Remark 3.5. (Cohen's inequalities.) When $G = \mathrm{GL}_n(\mathbb{C})$ the second relation in (3.1)

$$g^{*n}g^n \prec (g^*g)^n, \qquad n = 1, 2, \ldots$$

is equivalent to

$$p(g^n) \prec (p(g))^n, \qquad n = 1, 2, \ldots$$

where $g = k(g)p(g)$ is the polar decomposition of $g \in G$. If we set $g = e^{X/n}$, then

$$p(e^X) \prec [p(e^{X/n})]^n, \qquad n = 1, 2, \ldots$$

Now $p(e^{X/n}) = ((e^{X/n})^*e^{X/n})^{1/2} = (e^{-\theta X/n}e^{X/n})^{1/2}$. By $\lim_{t\to\infty}(e^{X/t}e^{Y/t})^t = e^{X+Y}$, we have

$$\lim_{n\to\infty}[p(e^{X/n})]^n = \lim_{n\to\infty}[(e^{-\theta X/n}e^{X/n})^n]^{1/2} = [\lim_{n\to\infty}(e^{-\theta X/n}e^{X/n})^n]^{1/2} = e^{X_{\mathfrak{p}}},$$

and thus

$$p(e^X) \prec e^{X_{\mathfrak{p}}}.$$

In particular the singular values of e^A is log majorized by the eigenvalue moduli of $e^{\operatorname{Re} A}$, i.e., Cohen's result [4] when $G = \mathrm{GL}_n(\mathbb{C})$ (with appropriate scaling on $\mathrm{SL}_n(\mathbb{C})$).

Remark 3.6. (Ky Fan's inequality and inequality (1.1).) Continuing with Proposition 2.2, for $A \in \mathfrak{sl}_n(\mathbb{C})$, the moduli of the eigenvalues of e^A are the exponentials of the real parts of the eigenvalues of A, counting multiplicities. The matrix $e^{\mathrm{Re}\, A}$ is positive definite. So the eigenvalues of $e^{\mathrm{Re}\, A}$ are indeed the singular values, and are the exponentials of the eigenvalues of $\mathrm{Re}\, A$. The eigenvalues of $\mathrm{Re}\, A$ are known as the real singular values of A, denoted by $\beta_1 \geq \cdots \geq \beta_n$. Denote the real parts of the eigenvalues of A by $\alpha_1 \geq \cdots \geq \alpha_n$. By Corollary 3.3 $e^A \prec e^{\mathrm{Re}\, A}$ which amounts to

$$\begin{aligned} \prod_{i=1}^{k} e^{\alpha_i} &\leq \prod_{i=1}^{k} e^{\beta_i}, \quad i = 1, \ldots, n-1, \\ \prod_{i=1}^{n} e^{\alpha_i} &= \prod_{i=1}^{n} e^{\beta_i}, \end{aligned}$$

that is $e^\alpha \prec_{\log} e^\beta$. Thus, by taking log on the above relation, the relation $e^A \prec e^{\mathrm{Re}\, A}$ amounts to the usual majorization relation $\alpha \in \operatorname{conv} S_n \beta$, a well-known result of Ky Fan [3, Proposition III.5.3] for $\mathfrak{gl}_n(\mathbb{C})$ with appropriate scaling on $\mathfrak{sl}_n(\mathbb{C})$).

From the second relation of Corollary 3.3, $e^A e^{A^*} \prec e^{A+A^*}$ which amounts to the fact that the singular values of e^A (that is, the square roots of the eigenvalues of $e^A e^{A^*}$) are multipicatively majorized, and hence weakly majorized [3, p. 42], [2], by the singular values (also the eigenvalues) of the positive definite $e^{\mathrm{Re}\, A}$. Thus

$$|||e^A||| \leq |||e^{\mathrm{Re}\, A}|||,$$

for all unitarily invariant norms $|||\cdot|||$ [3, Theorem IX.3.1] by Ky Fan Dominance Theorem [3, Theorem IV.2.2]. Thus we have (1.1).

Remark 3.7. (So-Thompson's inequality.) For $A \in \mathbb{C}_{n\times n}$, So-Thompson inequalities [18, Theorem 2.1] asserts that

$$\prod_{i=1}^{k} s_i(e^A) \leq \prod_{i=1}^{k} e^{s_i(A)}, \quad k = 1, \ldots, n.$$

From $e^A e^{A^*} \prec e^{A+A^*}$, $A \in \mathbb{C}_{n\times n}$, So-Thompson inequalities can be derived via Fan-Hoffman inequalities [3, proposition III.5.1]

$$\lambda_i(\mathrm{Re}\, A) \leq s_i(A), \quad i = 1, \ldots, n,$$

where $s_1(A) \geq \cdots \geq s_n(A)$ denote the singular values of $A \in \mathbb{C}_{n\times n}$.

Remark 3.8. (Weyl's inequality and inequalities (1.2) and (1.4).) Let $A \in \mathrm{SL}_n(\mathbb{C})$. By (3.5) $A^2 \prec A^*A$. By Proposition 2.2, $|\lambda^2(A)| \prec_{\log} |\lambda(A^*A)| = |s(A^*A)|$, that is,

$$|\lambda(A)| \prec_{\log} s(A).$$

By scaling and continuity argument, the log majorization remains valid for $A \in \mathbb{C}_{n\times n}$, that is, Weyl's inequality [3, p. 43]. In the literature, Weyl's inequality is often proved via the kth exterior power once $|\lambda_1(A)| \le s_1(A)$ is established, for example [3, p. 42–43]. Such an approach shares some favor of Theorem 2.1.

If $A, B \in \mathbb{C}_{n\times n}$ are Hermitian, then e^A, e^B and e^{A+B} are positive definite. Though e^Ae^B is not positive definite in general, its eigenvalues, denoted by $\delta_1 \ge \cdots \ge \delta_n$, are positive since e^Ae^B and the positive definite $e^{A/2}e^Be^{A/2}$ share the same eigenvalues, counting multiplicities. Denote the eigenvalues of e^{A+B} by $\gamma_1 \ge \cdots \ge \gamma_n$. Thus γ is multiplicatively majorized by δ because of $e^{A+B} \prec e^Ae^B$ (Theorem 3.1). Notice that δ is also multiplicatively majorized by the singular values $s_1 \ge \cdots \ge s_n$ of e^Ae^B, by Weyl's inequality. Hence we have the weak majorization relation $\gamma \prec_w s$ [3, p. 42] so that (1.2) follows. Finally (1.4) follows from Theorem 3.1 and Theorem 2.1.

Remark 3.9. (Lenard-Thompson's inequality.) Lenard's result [16] together with [22, Theorem 2] imply that

$$|||e^{A+B}||| \le |||e^{A/2}e^Be^{A/2}|||, \quad A, B \in \mathbb{C}_{n\times n} \text{ Hermitian}, \tag{3.6}$$

from which Golden-Thompson's result follows. It is because e^{A+B} and $e^{A/2}e^Be^{A/2}$ are positive definite and their traces are indeed the Ky Fan n-norm, that is, sum of singular values which is unitarily invariant. Indeed Lenard's original result asserts that any arbitrary neighborhood of e^{A+B} contains X such that $X \prec e^{A/2}e^Be^{A/2}$ [16, p. 458]. By a limit argument and Thompson's argument, (3.6) follows. The inequality (3.6) follows from the stronger relation:

$$e^{A+B} \prec e^{A/2}e^Be^{A/2}, \quad A, B \text{ Hermitian}. \tag{3.7}$$

Let us establish (3.7). From Theorem 3.1

$$e^{A+B} \prec e^Ae^B, \quad A, B \text{ Hermitian}$$

is a generalization of Golden-Thompson's inequality (1.3). Now (3.7) is true because $\pi_\lambda(e^Ae^B)$ and $\pi_\lambda(e^{A/2}e^Be^{A/2})$ have the same spectrum (by the fact that XY and YX have the same spectrum and π_λ is a representation) and thus have the same spectral radius. Then apply Theorem 2.1.

4. Extension of Araki's result

Araki's result [1] asserts that if $A, B \in \mathbb{C}_{n\times n}$ are Hermitian, then

$$(e^{A/2}e^Be^{A/2})^r \prec e^{rA/2}e^{rB}e^{rA/2}, \qquad r > 1. \tag{4.1}$$

It appears in the proof of the main result in [1, p. 168–169]. Also see [10] for a short proof. Notice that $e^{A/2}e^Be^{A/2}$ and $e^{rA/2}e^{rB}e^{rA/2}$ in (4.1) are positive definite so that their eigenvalues and singular values coincide. So (4.1) amounts to

$$s((e^{A/2}e^Be^{A/2})^r) \prec_{\log} s(e^{rA/2}e^{rB}e^{rA/2}), \qquad r > 1,$$

or equivalently

$$s((e^{qA/2}e^{qB}e^{qA/2})^{1/q}) \prec_{\log} s((e^{pA/2}e^{pB}e^{pA/2})^{1/p}), \qquad 0 < q \le p.$$

Using (4.1) and Lie's product formula [9, Lemma 1.8, p. 106]

$$e^{A+B} = \lim_{r\to 0}(e^{rA/2}e^{rB}e^{rA/2})^{1/r},$$

Golden-Thompson's result is strengthened [2]:

$$|||e^{pA/2}e^{pB}e^{pA/2}|||$$

decreases down to $|||e^{A+B}|||$ as $p \downarrow 0$ for any unitarily invariant norm $|||\cdot|||$ on $\mathbb{C}_{n\times n}$ and in particular

$$\operatorname{tr} e^{A+B} \le \operatorname{tr}\,[e^{pA/2}e^{pB}e^{pA/2}]^{1/p}, \qquad p > 0.$$

Araki's result also implies a result of Wang and Gong [23] (also see [3, Theorem IX.2.9]).

In order to extend (4.1) for general G, we need a result of Heinz [8] concerning two positive semidefinite operators. Indeed the original proof of Araki's result [1] also makes use of Heinz's result. Given two positive semidefinite operators A, B, the spectrum (counting multiplicities) $\lambda(AB) = \lambda(A^{1/2}BA^{1/2})$ and thus all eigenvalues of AB are positive. So the largest eigenvalue of AB, $\lambda_1(AB)$, is the spectral radius of AB. The first part of the following theorem is due to Heinz [8] (see [p. 255–256] for two nice proofs of Heinz's result). The second part is proved via the Heinz's result in [3, Theorem IX.2.6] in a somewhat lengthy way. See [19] for some generalization of Heniz's theorem.

Theorem 4.1. *The following two statements are equivalent and valid.*

1. (*Heinz*) *For any two positive semidefinite operators A, B,*
$$\|A^sB^s\| \le \|AB\|^s, \quad 0 \le s \le 1.$$
2. *For any two positive semidefinite operators A, B,*
$$\lambda_1(A^sB^s) \le \lambda_1^s(AB), \quad 0 \le s \le 1.$$

Proof. We just establish the equivalence of the two statements. Since $\|T\| = \|T^*T\|^2$,

$$\begin{aligned}\|A^sB^s\| &= \|(A^sB^s)A^sB^s\|^{1/2} = \|B^sA^{2s}B^s\|^{1/2}\\ &= \lambda_1^{1/2}(B^sA^{2s}B^s) = \lambda_1^{1/2}(A^{2s}B^{2s}),\end{aligned}$$

and

$$\|AB\|^s = \|ABBA\|^{s/2} = \lambda_1^{s/2}(AB^2A) = \lambda_1^{s/2}(A^2B^2). \qquad \square$$

Remark 4.2. An equivalent statement to Heniz's result is that for any positive operators A, B, $\|A^tB^t\| \ge \|AB\|^t$ if $t \ge 1$, or equivalently $\lambda_1(A^tB^t) \ge \lambda_1^t(AB)$ [3, p. 256–257].

Since $P := e^{\mathfrak{p}}$, each element of P is of the form e^A, $A \in \mathfrak{p}$ so that $(e^A)^r := e^{rA} \in P$, where $r \in \mathbb{R}$. So $f^r, g^r \in P$, f^rg^r (hyperbolic, since f^rg^r is conjugate to $f^{r/2}g^rf^{r/2}$), $r \in \mathbb{R}$, are well defined for $f, g \in P$.

When $A, B \in \mathfrak{p}$, $e^{A/2}e^Be^{A/2} \in P$ since it is of the form g^*g, where $g = e^{B/2}e^{A/2}$. Thus $(e^{A/2}e^Be^{A/2})^r \in P$ ($r \in \mathbb{R}$) is well defined.

Theorem 4.3. *Let $A, B \in \mathfrak{p}$. Then*

$$\begin{array}{rcll} (e^{A/2}e^Be^{A/2})^r & \prec & e^{rA/2}e^{rB}e^{rA/2}, & r > 1, \\ e^{rA/2}e^{rB}e^{rA/2} & \prec & (e^{A/2}e^Be^{A/2})^r, & 0 \le r \le 1. \end{array}$$

Moreover, for all $\lambda \in \hat{G}$

$$\begin{array}{rcll} \chi_\lambda((e^{A/2}e^Be^{A/2})^r) & \le & \chi_\lambda(e^{rA/2}e^{rB}e^{rA/2}), & r > 1, \\ \chi_\lambda(e^{rA/2}e^{rB}e^{rA/2}) & \le & \chi_\lambda((e^{A/2}e^Be^{A/2})^r), & 0 \le r \le 1. \end{array}$$

Proof. Notice that $\pi_\lambda(e^A)$ is positive definite and

$$\pi_\lambda((e^A)^r) = (\pi_\lambda(e^A))^r, \qquad r \in \mathbb{R},$$

where $(\pi_\lambda(e^A))^r$ is the usual rth power of the positive definite operator $\pi_\lambda(e^A) \in \operatorname{Aut} V_\lambda$. In particular $|\pi_\lambda((e^A)^r)| = |\pi_\lambda(e^A)|^r$. So for $r \in \mathbb{R}$,

$$\begin{array}{rcll} |\pi_\lambda(e^{A/2}e^Be^{A/2})^r| & = & |\pi_\lambda(e^{A/2}e^Be^{A/2})|^r & (e^{A/2}e^Be^{A/2} \in P) \\ & = & |\pi_\lambda(e^Ae^B)|^r & \\ & = & |\pi_\lambda(e^A)\pi_\lambda(e^B)|^r, & \end{array}$$

and

$$|\pi_\lambda(e^{rA/2}e^{rB}e^{rA/2})| = |\pi_\lambda(e^{rA}e^{rB})| = |(\pi_\lambda(e^A))^r(\pi_\lambda(e^B))^r|.$$

Since the operators $\pi_\lambda(e^A)$ and $\pi_\lambda(e^B)$ are positive definite, by Theorem 4.1 (2) and Remark 4.2,

$$\begin{array}{rcll} |\pi_\lambda(e^{A/2}e^Be^{A/2})^r| & \le & |\pi_\lambda(e^{rA/2}e^{rB}e^{rA/2})|, & r \ge 1, \\ |\pi_\lambda(e^{A/2}e^Be^{A/2})^r| & \ge & |\pi_\lambda(e^{rA/2}e^{rB}e^{rA/2})|, & 0 \le r \le 1. \end{array}$$

By Theorem 2.1, the desired relations then follow.

Now $(e^{A/2}e^Be^{A/2})^r \in P$ since $e^{A/2}e^Be^{A/2} \in P$. Clearly $e^{rA/2}e^{rB}e^{rA/2} \in P$. Thus $(e^{A/2}e^Be^{A/2})^r$ and $e^{rA/2}e^{rB}e^{rA/2}$ in P and thus are hyperbolic [14, Proposition 6.2] and by [14, Theorem 6.1], the desired inequalities follow. □

References

[1] H. Araki, On an inequality of Lieb and Thirring, Lett. Math. Phys., **19** (1990) 167–170.

[2] T. Ando and F. Hiai, Log majorization and complementary Golden-Thompson type inequalities, Linear Algebra Appl., **197/198** (1994) 113–131.

[3] R. Bhatia, *Matrix Analysis*, Springer, New York, 1997.

[4] J.E. Cohen, Spectral inequalities for matrix exponentials, Linear Algebra Appl., **111** (1988) 25–28.

[5] J.J. Duistermaat and J.A.C. Kolk, *Lie Groups*, Springer, Berlin, 2000.

[6] S. Golden, Lower bounds for the Helmholtz function, Phys. Rev., **137** (1965) B1127–B1128.

[7] B.C. Hall, *Lie Groups, Lie Algebras, and Representations*, Springer, New York, 2003.

[8] E. Heinz, Beitrağe zur Störungstheorie der Spektralzerlegung, Math. Ann., **123** (1951), 415–438.

[9] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.

[10] F. Hiai, Trace norm convergence of exponential product formula, Lett. Math. Phys., **33** (1995), 147–158.

[11] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge Univ. Press, 1991.

[12] J.E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, New York, 1972.

[13] A.W. Knapp, *Lie Groups Beyond an Introduction*, Birkhäuser, Boston, 1996.

[14] B. Kostant, On convexity, the Weyl group and Iwasawa decomposition, Ann. Sci. Ecole Norm. Sup. (4), **6** (1973) 413–460.

[15] S. Lang, *Algebra*, Addison-Wesley, Reading, Mass., 1965.

[16] A. Lenard, Generalization of the Golden-Thompson inequality $\mathrm{Tr}(e^A e^B) \geq \mathrm{Tr}\, e^{A+B}$, Indiana Univ. Math. J. **21** (1971/1972) 457–467.

[17] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, I, Functional Analysis*, second edition, Academic Press, New York, 1980.

[18] W. So and R.C. Thompson, Singular values of matrix exponentials, Linear and Multilinear Algebra, **47** (2000) 249–258.

[19] T.Y. Tam, Heinz-Kato's inequalities for semisimple Lie groups, Journal of Lie Theory, **18** (2008) 919–931.

[20] T.Y. Tam and H. Huang, An extension of Yamamoto's theorem on the eigenvalues and singular values of a matrix, Journal of Math. Soc. Japan, **58** (2006) 1197–1202.

[21] C.J. Thompson, Inequality with applications in statistical mechanics, J. Mathematical Phys., **6** (1965) 1812–1813.

[22] C.J. Thompson, Inequalities and partial orders on matrix spaces, Indiana Univ. Math. J., **21** (1971/72) 469–480.

[23] B. Wang and M. Gong, Some eigenvalue inequalities for positive semidefinite matrix power products, Linear Algebra Appl. **184** (1993) 249–260.

[24] F. Warmer, *Foundation of Differentiable Manifolds and Lie Groups*, Scott Foresman and Company, 1971.

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Received: February 14, 2009
Accepted: June 18, 2009

Operator Theory:
Advances and Applications, Vol. 202, 553–568

Parabolic Quasi-radial Quasi-homogeneous Symbols and Commutative Algebras of Toeplitz Operators

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Abstract. We describe new Banach (not C^* !) algebras generated by Toeplitz operators which are commutative on each weighted Bergman space over the unit ball $\mathbb{B}^n$, where $n > 2$. For $n = 2$ all these algebras collapse to the single C^*-algebra generated by Toeplitz operators with quasi-parabolic symbols. As a by-product, we describe the situations when the product of mutually commuting Toeplitz operators is a Toeplitz operator itself.

Mathematics Subject Classification (2000). Primary 47B35; Secondary 47L80, 32A36.

Keywords. Toeplitz operator, weighted Bergman space, commutative Banach algebra, parabolic quasi-radial quasi-homogeneous symbol.

1. Introduction

The commutative C^*-algebras generated by Toeplitz operators acting on the weighted Bergman spaces on the unit ball $\mathbb{B}^n$ were studied in [4]. The main result of the paper states that, given any maximal commutative subgroup of biholomorphisms of the unit ball, the C^*-algebra generated by Toeplitz operators, whose symbols are invariant under the action of this group, is commutative on each (commonly considered) weighted Bergman space on $\mathbb{B}^n$.

Under some technical assumption on "richness" of symbol classes, this result is exact for the case of the unit disk ($n = 1$). The results of [1] state that a C^*-algebra generated by Toeplitz operators is commutative on each weighted Bergman space if and only if the symbols of generating Toeplitz operators are invariant under the action of a maximal commutative subgroup of the Möbius transformations of the unit disk.

This work was partially supported by CONACYT Project 80503, México.
Communicated by I.M. Spitkovsky.

The natural and very strong guess was that the situation for $n > 1$ is the same, i.e., that the algebras described in [4] exhaust all possible algebras of Toeplitz operators that are commutative on each weighted Bergman space. But the reality turns out to be more interesting and unpredictable. It has been shown in [5] that for $n > 1$ there are many other Banach (not C^* !) algebras generated by Toeplitz operators which are commutative on each weighted Bergman space on $\mathbb{B}^n$. The symbols of generating Toeplitz operators of such algebras, in a certain sense, are originated from, or subordinated to separately radial symbols, one of the $(n+2)$ model classes of symbols for the n-dimensional unit ball (see [4] for details).

In the present paper we show that the another model class of symbols from [4], the class of quasi-parabolic symbols, also originates, for $n > 2$, Banach algebras of Toeplitz operators which are commutative on each weighted Bergman space. For $n = 2$ all these algebras collapse to the single C^*-algebra generated by Toeplitz operators with quasi-parabolic symbols.

As a by-product, we describe the situations when the product of mutually commuting Toeplitz operators is a Toeplitz operator itself.

2. Preliminaries

Let $\mathbb{B}^n$ be the unit ball in $\mathbb{C}^n$, that is,

$$\mathbb{B}^n = \{w = (w_1, \dots, w_n) \in \mathbb{C}^n : \ |w|^2 = |w_1|^2 + \dots + |w_n|^2 < 1\}.$$

Denote by $dV = dx_1dy_1 \dots dx_ndy_n$, where $w_l = x_l + iy_l$, $l = 1, 2, \dots, n$, the standard Lebesgue measure in $\mathbb{C}^n$, and let $d\mu_\lambda$, with $\lambda > -1$, be the standard (see, for example, [7, Section 1.3]) weighted measure:

$$d\mu_\lambda(w) = \frac{\Gamma(n+\lambda+1)}{\pi^n\,\Gamma(\lambda+1)}\,(1-|w|^2)^\lambda\,dV(w).$$

We introduce the weighted space $L_2(\mathbb{B}^n, d\mu_\lambda)$ and its subspace, the weighted Bergman space $\mathcal{A}^2_\lambda = \mathcal{A}^2_\lambda(\mathbb{B}^n)$, which consists of all functions analytic in $\mathbb{B}^n$. We denote by $B_{\mathbb{B}^n,\lambda}$ the (orthogonal) Bergman projection of $L_2(\mathbb{B}^n, dv_\lambda)$ onto the Bergman space $\mathcal{A}^2_\lambda(\mathbb{B}^n)$.

Given a function $a(z) \in L_\infty(\mathbb{B}^n)$, the Toeplitz operator T_a with symbol a acts on $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ by the rule

$$T_a : \ \varphi \in \mathcal{A}^2_\lambda(\mathbb{B}^n) \longmapsto B_{\mathbb{B}^n,\lambda}(a\varphi) \in \mathcal{A}^2_\lambda(\mathbb{B}^n).$$

In what follows it is convenient to pass to the unbounded realization of the unit ball, known as the Siegel domain

$$D_n = \{w = (z, w_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \ \operatorname{Im} w_n - |z|^2 > 0\}.$$

Recall in this connection that the Cayley transform $w = \omega(\zeta)$, where

$$\begin{aligned} z_k &= i\,\frac{\zeta_k}{1+\zeta_n}, \quad k = 1,\dots,n-1, \\ w_n &= i\,\frac{1-\zeta_n}{1+\zeta_n}, \end{aligned}$$

maps biholomorphically the unit ball $\mathbb{B}^n$ onto the Siegel domain D_n, and that the unitary operator

$$U_\lambda : L_2(\mathbb{B}^n, \mu_\lambda) \longrightarrow L_2(D_n, \widetilde{\mu}_\lambda),$$

defined by

$$(U_\lambda f)(w) = \left(\frac{2}{1-iw_n}\right)^{n+\lambda+1} f(\omega^{-1}(w)),$$

maps $L_2(\mathbb{B}^n, d\mu_\lambda)$ onto $L_2(D_n, \widetilde{\mu}_\lambda)$, where

$$\widetilde{\mu}_\lambda(w) = \frac{c_\lambda}{4}\,(\operatorname{Im} w_n - |z|^2)^\lambda, \quad \text{with} \quad c_\lambda = \frac{\Gamma(n+\lambda+1)}{\pi^n\Gamma(\lambda+1)},$$

and maps $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ onto the (weighted) Bergman space $\mathcal{A}^2_\lambda(D_n)$ on the Siegel domain D_n.

We recall now necessary facts from [4]. Let $\mathcal{D} = \mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}_+$; the mapping

$$\kappa \,:\, w = (z, u, v) \in \mathcal{D} \longmapsto (z, u + iv + i|z|^2) \in D_n,$$

is obviously a diffeomorphism between $\mathcal{D}$ and D_n. The operator

$$(U_0 f)(w) = f(\kappa(w)),$$

is unitary from $L_2(D_n, \widetilde{\mu}_\lambda)$ onto $L_2(\mathcal{D}, \eta_\lambda)$, where

$$\eta_\lambda = \eta_\lambda(v) = \frac{c_\lambda}{4}\,v^\lambda, \quad \lambda > -1.$$

We represent the space $L_2(\mathcal{D}, \eta_\lambda)$ as the following tensor product

$$L_2(\mathcal{D}, \eta_\lambda) = L_2(\mathbb{C}^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda),$$

and consider the unitary operator $U_1 = I \otimes F \otimes I$ acting on it. Here

$$(Ff)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u) e^{-i\xi u} du$$

is the standard Fourier transform on $L_2(\mathbb{R})$.

Passing to polar coordinates in $\mathbb{C}^{n-1}$ we represent

$$\begin{aligned} L_2(\mathcal{D}, \eta_\lambda) &= L_2(\mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}_+, \eta_\lambda) \\ &= L_2(\mathbb{R}^{n-1}_+, rdr) \otimes L_2(\mathbb{T}^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda), \end{aligned}$$

where

$$rdr = \prod_{k=1}^{n-1} r_k dr_k, \quad L_2(\mathbb{T}^{n-1}) = \bigotimes_{k=1}^{n-1} L_2\left(\mathbb{T}, \frac{dt_k}{it_k}\right).$$

Introduce the unitary operator $U_2 = I \otimes \mathcal{F}_{(n-1)} \otimes I \otimes I$ which acts from

$$L_2(\mathbb{R}^{n-1}_+, rdr) \otimes L_2(\mathbb{T}^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda)$$

onto

$$\begin{gathered} L_2(\mathbb{R}_+^{n-1}, rdr) \otimes l_2(\mathbb{Z}^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda) = \\ l_2(\mathbb{Z}^{n-1}, L_2(\mathbb{R}_+^{n-1}, rdr) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda)), \end{gathered}$$

where $\mathcal{F}_{(n-1)} = \mathcal{F} \otimes \cdots \otimes \mathcal{F}$, and each $\mathcal{F}$ is the one-dimensional discrete Fourier transform:

$$\mathcal{F} : f \longmapsto c_l = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} f(t)\, t^{-l}\, \frac{dt}{it}, \quad l \in \mathbb{Z}.$$

In what follows we will use the standard multi-index notation. That is, for a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \in \mathbb{Z}_+^{n-1}$:

$$\begin{aligned} |\alpha| &= \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}, \\ \alpha! &= \alpha_1!\, \alpha_2! \cdots \alpha_{n-1}!, \\ z^\alpha &= z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_{n-1}^{\alpha_{n-1}}. \end{aligned}$$

Two multi-indices α and β are called orthogonal, $\alpha \perp \beta$, if

$$\alpha \cdot \beta = \alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_{n-1}\beta_{n-1} = 0.$$

We denote by $\mathcal{A}_2(\mathcal{D}) = U_2U_1U_0(\mathcal{A}_\lambda^2(D_n))$ the image of the Bergman space $\mathcal{A}_\lambda^2(D_n)$. By [4] the space $\mathcal{A}_2(\mathcal{D})$ consists of all sequences $\{d_\alpha(r,\xi,v)\}_{\alpha\in\mathbb{Z}_+^{n-1}}$, where

$$d_\alpha(r,\xi,v) = \left(\frac{2^{n+1}}{c_\lambda} \frac{(2\xi)^{|\alpha|+\lambda+n}}{\alpha!\,\Gamma(\lambda+1)} \right)^{\frac{1}{2}} r^\alpha\, e^{-\xi(|r|^2+v)}\, c_\alpha(\xi), \qquad \xi \in \mathbb{R}_+,$$

with $c_\alpha = c_\alpha(\xi) \in L_2(\mathbb{R}_+)$.

Thus the space $\mathcal{A}_1(\mathcal{D}) = U_2^{-1}(\mathcal{A}_2(\mathcal{D})) = U_1U_0(\mathcal{A}_\lambda^2(D_n))$ is the subspace of $L_2(\mathcal{D}, \nu_\lambda) = L_2(\mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}_+, \eta_\lambda)$, which consists of all functions of the form

$$\begin{aligned} \psi(z,\xi,v) &= (2\pi)^{-\frac{n-1}{2}} \sum_{\alpha\in\mathbb{Z}_+^{n-1}} d_\alpha(r,\xi,v)\, t^\alpha \\ &= \sum_{\alpha\in\mathbb{Z}_+^{n-1}} \left(\frac{4}{c_\lambda \pi^{n-1}} \frac{(2\xi)^{|\alpha|+\lambda+n}}{\alpha!\,\Gamma(\lambda+1)} \right)^{\frac{1}{2}} c_\alpha(\xi) z^\alpha\, e^{-\xi(v+|z|^2)} \\ &= \sum_{\alpha\in\mathbb{Z}_+^{n-1}} \psi_\alpha(z,\xi,v), \end{aligned}$$

where $c_\alpha(\xi) \in L_2(\mathbb{R}_+)$ for all $\alpha \in \mathbb{Z}_+^{n-1}$.

Introduce now the operator

$$V : L_2(\mathbb{C}^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda) \longrightarrow L_2(\mathbb{C}^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda)$$

by the rule

$$(V\psi)(z,\xi,v) = (2\xi)^{-\frac{\lambda+n}{2}}\, \psi\left(\frac{z}{\sqrt{2\xi}},\ \xi,\ \frac{v}{2\xi} \right).$$

The operator V is obviously unitary, and the space $\widetilde{\mathcal{A}}_1(\mathcal{D}) = V(\mathcal{A}_1(\mathcal{D}))$ consists of all functions of the form

$$\widetilde{\psi}(z,\xi,v) = V\psi = \sum_{\alpha\in\mathbb{Z}_+^{n-1}} \widetilde{\psi}_\alpha(z,\xi,v),$$

where

$$\widetilde{\psi}_\alpha(z,\xi,v) = (V\psi_\alpha)(z,\xi,v) = c_\alpha(\xi)\left(\frac{4}{c_\lambda \pi^{n-1}\,\alpha!\,\Gamma(\lambda+1)}\right)^{\frac{1}{2}} z^\alpha\, e^{-\frac{|z|^2}{2}}\, e^{-\frac{v}{2}}.$$

We note that the function

$$\ell_0(v) = \left(\frac{4}{c_\lambda\,\Gamma(\lambda+1)}\right)^{\frac{1}{2}} e^{-\frac{v}{2}}$$

belongs to $L_2(\mathbb{R}_+,\eta_\lambda)$ and $\|\ell_0(v)\| = 1$. We denote by L_0 the one-dimensional subspace of $L_2(\mathbb{R}_+,\eta_\lambda)$ generated by $\ell_0(v)$.

For each $\alpha\in\mathbb{Z}_+^{n-1}$, the function

$$e_\alpha(z) = (\pi^{n-1}\alpha!)^{-\frac{1}{2}} z^\alpha e^{-\frac{|z|^2}{2}}$$

belongs to $L_2(\mathbb{C}^{n-1})$ and

$$\begin{aligned}
\|e_\alpha(z)\|^2 &= \frac{1}{\pi^{n-1}\alpha!}\int_{\mathbb{C}^{n-1}} \prod_{l=1}^{n-1} |z_l|^{2\alpha_l}\, e^{-\frac{|z|^2}{2}}\, dV(z) \\
&= \frac{1}{\pi^{n-1}\alpha!}\int_{\mathbb{R}_+^{n-1}} \prod_{l=1}^{n-1} r_l^{2\alpha_l}\, e^{-\frac{r^2}{2}}\, rdr\,(2\pi)^{n-1} \\
&= \frac{1}{\alpha!}\int_{\mathbb{R}_+^{n-1}} r^\alpha\, e^{-(r_1+\cdots+r_{n-1})} dr = 1.
\end{aligned}$$

That is,

$$\widetilde{\psi}(z,\xi,v) = e_\alpha(z)\, c_\alpha(\xi)\, \ell_0(v).$$

Moreover it is easy to check that the system of functions $\{e_\alpha(z)\}_{\alpha\in\mathbb{Z}_+^{n-1}}$ is orthonormal. Denoting by $\widetilde{\mathcal{A}}(\mathbb{C}^{n-1})$ the Hilbert space with the basis $\{e_\alpha(z)\}_{\alpha\in\mathbb{Z}_+^{n-1}}$ we have finally that

$$\widetilde{\mathcal{A}}_1(\mathcal{D}) = \widetilde{\mathcal{A}}(\mathbb{C}^{n-1}) \otimes L_2(\mathbb{R}_+) \otimes L_0 \subset L_2(\mathbb{C}^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+,\eta_\lambda).$$

We define now the isometry

$$R_0:\ l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+)) \longrightarrow L_2(\mathcal{D},\eta_\lambda) = L_2(\mathbb{C}^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+,\eta_\lambda)$$

by the rule

$$R_0:\ \{c_\alpha(\xi)\}_{\alpha\in\mathbb{Z}_+^{n-1}} \longmapsto \chi_+(\xi) \sum_{\alpha\in\mathbb{Z}_+^{n-1}} c_\alpha(\xi) e_\alpha(z) \ell_0(v),$$

where $\chi_+(\xi)$ is the characteristic function of $\mathbb{R}_+$ and the functions $c_\alpha(\xi)$ are extended by zero for $\xi\in\mathbb{R}\setminus\mathbb{R}_+$.

The adjoint operator

$$R_0^* : \; L_2(\mathcal{D}, \eta_\lambda) \longrightarrow l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+))$$

has obviously the form

$$R_0^* : \varphi(z, \xi, v) \longmapsto \left\{ \chi_+(\xi) \int_{\mathbb{C}^{n-1}\times\mathbb{R}_+} \varphi(z, \xi, v) \overline{e_\alpha(z)}\, \overline{\ell_0(v)}\, \eta_\lambda(v) dV(z) dv \right\}_{\alpha \in \mathbb{Z}_+^{n-1}} .$$

The operators R_0 and R_0^* obey the following properties

(i) $R_0^* R_0 = I : \; l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+)) \longrightarrow l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+))$;

(ii) $R_0 R_0^* = \widetilde{P} : L_2(\mathcal{D}, \eta_\lambda) \longrightarrow \widetilde{\mathcal{A}}_1(\mathcal{D})$,
where $\widetilde{P}$ is the orthogonal projection of $L_2(\mathcal{D}, \eta_\lambda)$ onto $\widetilde{\mathcal{A}}_1(\mathcal{D})$,

(iii) the image of the operator R_0 coincides with $\widetilde{\mathcal{A}}_1(\mathcal{D})$, and being considered as

$$R_0 : \; l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+)) \longrightarrow \widetilde{\mathcal{A}}_1(\mathcal{D})$$

this operator is unitary with

$$R_0^{-1} = R_0^*|_{\widetilde{\mathcal{A}}_1(\mathcal{D})} : \; \widetilde{\mathcal{A}}_1(\mathcal{D}) \longrightarrow l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+)).$$

Now the operator $R = R_0^* V U_1 U_0$ maps $L_2(D_n, \widetilde{\mu}_\lambda)$ onto $l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+))$, and its restriction onto the Bergman space $\mathcal{A}_\lambda^2(D_n)$:

$$R|_{\mathcal{A}_\lambda^2(D_n)} : \; \mathcal{A}_\lambda^2(D_n) \longrightarrow l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+))$$

is an isometric isomorphism. The adjoint operator

$$R^* = U_0^* U_1^* V^* R_0 : \; l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+)) \longrightarrow \mathcal{A}_\lambda^2(D_n) \subset L_2(D_n, \widetilde{\mu}_\lambda)$$

is an isometric isomorphism of $l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+))$ onto the subspace $\mathcal{A}_\lambda^2(D_n)$ of $L_2(D_n, \widetilde{\mu}_\lambda)$.

Furthermore

$$\begin{aligned} RR^* = I \quad &: \quad l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+)) \longrightarrow l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+)), \\ R^*R = B_{D_n,\lambda} \quad &: \quad L_2(D_n, \widetilde{\mu}_\lambda) \longrightarrow \mathcal{A}_\lambda^2(D_n), \end{aligned}$$

where $B_{D_n,\lambda}$ is the Bergman projection on the Siegel domain D_n.

Lemma 2.1. *Let $a(w) \in L_\infty(D_n)$ is of the form $a = a(z, y_n)$, where $z \in \mathbb{C}^{n-1}$, $y_n = \operatorname{Im} w_n$. Then, for the Toeplitz operator T_a acting on $\mathcal{A}_\lambda^2(D_n)$, we have*

$$RT_aR^* = R_0^*\, a\left(\frac{z}{\sqrt{2\xi}}, \frac{v+|z|^2}{2\xi}\right) R_0.$$

Proof. We have

$$\begin{aligned} RT_aR^* &= RB_{D_n,\lambda} a B_{D_n,\lambda} R^* = RR^*RaR^*RR^* = RAR^* \\ &= R_0^* V U_1 U_0 a(z, y_n) U_0^* U_1^* V^* R_0 = R_0^* V U_1 a(z, v+|z|^2) U_1^* V^* R_0 \\ &= R_0^* V a(z, v+|z|^2) V^* R_0 = R_0^*\, a\left(\frac{z}{\sqrt{2\xi}}, \frac{v+|z|^2}{2\xi}\right) R_0. \end{aligned}$$

□

Corollary 2.2 ([4],Theorem 10.2). *Let $a = a(r, y_n) \in L_\infty(D_n)$. Then*

$$RT_aR^* = \{\gamma_a(\alpha,\xi)\}_{\alpha\in\mathbb{Z}_+^{n-1}} I,$$

where

$$\begin{aligned}\gamma_a(\alpha,\xi) &= \frac{1}{\alpha!\,\Gamma(\lambda+1)}\int_{\mathbb{R}_+^n} a\left(\sqrt{\frac{r}{2\xi}}, \frac{v+r_1+\cdots+r_{n-1}}{2\xi}\right) r^\alpha v^\lambda \\ &\quad \times e^{-(v+r_1+\cdots+r_{n-1})}drdv, \quad \xi\in\mathbb{R}_+.\end{aligned}$$

Proof. Just a matter of a direct calculation. □

We finish the section with recalling of a known equality (see, for example, [7, Section 1.3]), which will be used frequently in what follows. Let $\mathbb{S}^k$ be the unit sphere and let dS be the corresponding (not normalized) surface measure. Then, for each α, $\beta \in \mathbb{Z}_+^k$, we have

$$\int_{\mathbb{S}^k} \zeta^\alpha\overline{\zeta}^\beta dS(\zeta) = \delta_{\alpha,\beta}\,\frac{2\pi^k\,\alpha!}{(k-1+|\alpha|)!}. \tag{2.1}$$

3. Parabolic quasi-radial symbols

Let $k = (k_1,\ldots,k_m)$ be a tuple of positive integers whose sum is equal to $n-1$: $k_1+\cdots+k_m = n-1$. The length of such a tuple may obviously vary from 1, for $k = (n-1)$, to $n-1$, for $k = (1,\ldots,1)$.

Given a tuple $k = (k_1,\ldots,k_m)$, we rearrange the $n-1$ coordinates of $z\in\mathbb{C}^{n-1}$ in m groups, each one of which has k_j, $j = 1,\ldots,m$, entries and introduce the notation

$$z_{(1)} = (z_{1,1},\ldots,z_{1,k_1}),\ z_{(2)} = (z_{2,1},\ldots,z_{2,k_2}),\ \ldots,\ z_{(m)} = (z_{m,1},\ldots,z_{m,k_m}).$$

We represent then each $z_{(j)} = (z_{j,1},\ldots,z_{j,k_j}) \in \mathbb{C}^{k_j}$ in the form

$$z_{(j)} = r_j\zeta_{(j)}, \quad \text{where} \quad r_j = \sqrt{|z_{j,1}|^2+\cdots+|z_{j,k_j}|^2} \text{ and } \zeta_{(j)}\in\mathbb{S}^{k_j}. \tag{3.1}$$

Given a tuple $k = (k_1,\ldots,k_m)$, a bounded measurable function $a = a(w)$, $w \in D_n$, will be called *parabolic k-quasi-radial* if it depends only on r_1, …, r_m and $y_n = \operatorname{Im} w_n$. We note that for $k = (1,\ldots,1)$ this is exactly a quasi-parabolic function of [4].

Varying k we have a collection of the partially ordered by inclusion sets $\mathcal{R}_k$ of k-quasi-radial functions. The minimal among these sets is the set $\mathcal{R}_{(n)}$ of *radial* (with respect to z) functions and the maximal one is the set $\mathcal{R}_{(1,\ldots,1)}$ of *separately radial* (with respect to z) functions ≡ quasi-parabolic functions of [4].

As in [5], to avoid repetitions of unitary equivalent algebras and ambiguities in what follows we will always assume first, that $k_1 \le k_2 \le \cdots \le k_m$, and second, that

$$\begin{aligned}&z_{1,1} = z_1,\ z_{1,2} = z_2,\ \ldots,\ z_{1,k_1} = z_{k_1},\quad z_{2,1} = z_{k_1+1},\ \ldots,\\ &z_{2,k_2} = z_{k_1+k_2},\ \ldots,\ z_{m,k_m} = z_{n-1}.\end{aligned} \tag{3.2}$$

Given $k = (k_1, \dots, k_m)$ and any $(n-1)$-tuple $\alpha = (\alpha_1, \dots, \alpha_{n-1})$, we denote

$$\alpha_{(1)} = (\alpha_1, \dots, \alpha_{k_1}), \qquad \alpha_{(2)} = (\alpha_{k_1+1}, \dots, \alpha_{k_1+k_2}), \quad \dots, \\ \dots, \quad \alpha_{(m)} = (\alpha_{n-k_m}, \dots, \alpha_{n-1}).$$

As each set $\mathcal{R}_k$ is a subset of the set $\mathcal{R}_{(1,\dots,1)}$ of quasi-parabolic functions, the operator RT_aR^* with $a \in \mathcal{R}_k$, by Corollary 2.2, is diagonal on $l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+))$. The exact form of the corresponding diagonal representation gives the next lemma.

Lemma 3.1. *Given a parabolic k-quasi-radial function $a = a(r_1, \dots, r_m, y_n)$, we have*

$$RT_aR^* : \ \{c_\alpha(\xi)\}_{\alpha\in\mathbb{Z}_+^{n-1}} \longmapsto \{\gamma_{a,k}(\alpha, \xi)\, c_\alpha(\xi)\}_{\alpha\in\mathbb{Z}_+^{n-1}}$$

where

$$\begin{aligned} \gamma_{a,k}(\alpha,\xi) \quad &= \quad \frac{1}{\Gamma(\lambda+1)\prod_{j=1}^m (k_j - 1 + |\alpha_{(j)}|)!} \\ &\quad \times \int_{\mathbb{R}_+^{m+1}} a\left(\sqrt{\frac{r_1}{2\xi}}, \dots, \sqrt{\frac{r_m}{2\xi}}, \frac{v + r_1 + \dots + r_m}{2\xi}\right) \\ &\quad \times v^\lambda e^{-(v+r_1+\dots+r_m)} dv \prod_{j=1}^m r_j^{|\alpha_{(j)}|+k_j-1} dr_j, \qquad \xi \in \mathbb{R}_+. \end{aligned} \tag{3.3}$$

Proof. By Lemma 2.1 we have

$$RT_aR^* = R_0^*\, a\left(\frac{r_1}{\sqrt{2\xi}}, \dots, \frac{r_m}{\sqrt{2\xi}}, \frac{v + r_1^2 + \dots + r_m^2}{2\xi}\right) R_0.$$

Thus

$$\begin{aligned} &RT_aR^*\, \{c_\alpha(\xi)\}_{\alpha\in\mathbb{Z}_+^{n-1}} \\ &= \quad R_0^* \left[a\left(\frac{r_1}{\sqrt{2\xi}}, \dots, \frac{r_m}{\sqrt{2\xi}}, \frac{v+r^2}{2\xi}\right) \chi_+(\xi) \sum_{\alpha\in\mathbb{Z}_+^{n-1}} c_\alpha(\xi) e_\alpha(z) \ell_0(v) \right] \\ &= \quad \left\{ \chi_+(\xi) \int_{\mathbb{C}^{n-1}\times\mathbb{R}_+} a\left(\frac{r_1}{\sqrt{2\xi}}, \dots, \frac{r_m}{\sqrt{2\xi}}, \frac{v+r^2}{2\xi}\right) \right. \\ &\qquad \left. \times \sum_{\alpha\in\mathbb{Z}_+^{n-1}} c_\alpha(\xi) e_\alpha(z) \ell_0(v) \overline{e_\beta(z)} \ell_0(v) \eta_\lambda(v) dV(z) dv \right\}_{\beta\in\mathbb{Z}_+^{n-1}} \\ &= \quad \left\{ \chi_+(\xi) \sum_{\alpha\in\mathbb{Z}_+^{n-1}} c_\alpha(\xi)\, I_{\alpha,\beta} \right\}_{\beta\in\mathbb{Z}_+^{n-1}}. \end{aligned}$$

Using representation (3.1) and formula (2.1) we have

$$\begin{aligned}
I_{\alpha,\beta} &= \int_{\mathbb{C}^{n-1}\times\mathbb{R}_+} a\left(\frac{r_1}{\sqrt{2\xi}},\ldots,\frac{r_m}{\sqrt{2\xi}},\frac{v+r^2}{2\xi}\right) e_\alpha(z)\overline{e_\beta(z)}\ell_0^2(v)\eta_\lambda(v)dV(z)dv \\
&= \int_{\mathbb{R}_+^m\times\mathbb{R}_+} a\left(\frac{r_1}{\sqrt{2\xi}},\ldots,\frac{r_m}{\sqrt{2\xi}},\frac{v+r^2}{2\xi}\right) \frac{e^{-r^2}}{\pi^{n-1}\sqrt{\alpha!\,\beta!}} \prod_{j=1}^m r_j^{|\alpha_{(j)}|+|\beta_{(j)}|+2k_j-1} dr_j \\
&\quad\times \frac{1}{\Gamma(\lambda+1)} e^{-v}v^\lambda dv \prod_{j=1}^m \int_{\mathbb{S}^{k_j}} \zeta_{(j)}^{\alpha_{(j)}}\overline{\zeta}_{(j)}^{\beta_{(j)}} dS \\
&= \begin{cases} 0, & \text{if} \quad \alpha\neq\beta \\ I_{\alpha,\alpha}, & \text{if} \quad \alpha=\beta \end{cases},
\end{aligned}$$

where

$$\begin{aligned}
\gamma_{a,k}(\alpha,\xi) &= I_{\alpha,\alpha} \\
&= 2^m \prod_{j=1}^m \frac{\alpha_{(j)}!\,\pi^{k_j}}{(k_j-1+|\alpha_{(j)}|)!} \int_{\mathbb{R}_+^{m+1}} a\left(\frac{r_1}{\sqrt{2\xi}},\ldots,\frac{r_m}{\sqrt{2\xi}},\frac{v+r^2}{2\xi}\right) \frac{e^{-r^2}}{\pi^{n-1}\alpha!} \\
&\quad\times \prod_{j=1}^m r_j^{2(|\alpha_{(j)}|+k_j)-1} dr_j \, \frac{e^{-v}v^\lambda}{\Gamma(\lambda+1)} dv \\
&= \frac{2^m}{\Gamma(\lambda+1)} \prod_{j=1}^m \frac{1}{(k_j-1+|\alpha_{(j)}|)!} \int_{\mathbb{R}_+^{m+1}} a\left(\frac{r_1}{\sqrt{2\xi}},\ldots,\frac{r_m}{\sqrt{2\xi}},\frac{v+r^2}{2\xi}\right) \\
&\quad\times v^\lambda e^{-(v+r^2)}dv \prod_{j=1}^m r_j^{2(|\alpha_{(j)}|+k_j)-1} dr_j \\
&= \frac{1}{\Gamma(\lambda+1)\prod_{j=1}^m (k_j-1+|\alpha_{(j)}|)!} \\
&\quad\times \int_{\mathbb{R}_+^{m+1}} a\left(\sqrt{\frac{r_1}{2\xi}},\ldots,\sqrt{\frac{r_m}{2\xi}},\frac{v+r_1+\cdots+r_m}{2\xi}\right) \\
&\quad\times v^\lambda e^{-(v+r_1+\cdots+r_m)}dv \prod_{j=1}^m r_j^{|\alpha_{(j)}|+k_j-1} dr_j, \qquad \xi\in\mathbb{R}_+.
\end{aligned}$$

This finishes the proof. □

Given $k=(k_1,\ldots,k_m)$, we use the representations $z_{(j)}=r_j\zeta_{(j)}$, $j=1,\ldots,m$, to define the vector

$$\zeta = (\zeta_{(1)},\zeta_{(2)},\ldots,\zeta_{(m)}) \in \mathbb{S}^{k_1}\times\mathbb{S}^{k_2}\times\cdots\times\mathbb{S}^{k_m}.$$

We introduce now the *quasi-homogeneous* extension of the parabolic k-quasi-radial functions (see [2, 3, 6, 5]). Let p, $q\in\mathbb{Z}_+^{n-1}$ be a pair of orthogonal ($p\perp q$) multi-indices.

A function $\varphi \in L_\infty(D_n)$ is called parabolic quasi-homogeneous (or *parabolic k-quasi-homogeneous*) function if it has the form

$$\varphi(z) = \varphi(z_{(1)}, z_{(2)}, \dots, z_{(m)}, y_n) = a(r_1, r_2, \dots, r_m, y_n)\, \zeta^p \overline{\zeta}^q, \tag{3.4}$$

where $a(r_1, r_2, \dots, r_m, y_n) \in \mathcal{R}_k$. We will call the pair (p, q) the *quasi-homogeneous degree* of the parabolic k-quasi-homogeneous function $a(r_1, r_2, \dots, r_m, y_n)\, \zeta^p\, \overline{\zeta}^q$.

For each $\alpha \in \mathbb{Z}_+^{n-1}$, we denote by $\widehat{e}_\alpha = \{\delta_{\alpha,\beta}\}_{\beta\in\mathbb{Z}_+^{n-1}}$ the α's element of the standard orthonormal basis in $l_2(\mathbb{Z}_+^{n-1})$. Given $c(\xi) \in L_2(\mathbb{R}_+)$, let

$$\widehat{e}_\alpha(c(\xi)) = \widehat{e}_\alpha \otimes c(\xi) = \{\delta_{\alpha,\beta}\, c(\xi)\}_{\beta\in\mathbb{Z}_+^{n-1}}$$

be the corresponding one-component element of $l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+))$.

Lemma 3.2. *Given a parabolic k-quasi-radial quasi-homogeneous symbol* (3.4), *we have*

$$RT_\varphi R^* : \widehat{e}_\alpha(c(\xi)) \longmapsto \begin{cases} 0, & \text{if } \exists\, l \text{ such that} \\ & \quad \alpha_l + p_l - q_l < 0, \\ \widetilde{\gamma}_{a,k,p,q}(\alpha, \xi)\, \widehat{e}_{\alpha+p-q}(c(\xi)), & \text{if } \forall\, l \quad \alpha_l + p_l - q_l \geq 0, \end{cases}$$

where

$$\begin{aligned} \widetilde{\gamma}_{a,k,p,q}(\alpha, \xi) &= \frac{2^m (\alpha+p)!}{\sqrt{\alpha!\,(\alpha+p-q)!}\,\Gamma(\lambda+1)} \prod_{j=1}^m \frac{1}{(k_j - 1 + |\alpha_{(j)} + p_{(j)}|)!} \\ &\quad \times \int_{\mathbb{R}_+^{m+1}} a\left(\frac{r_1}{\sqrt{2\xi}}, \dots, \frac{r_m}{\sqrt{2\xi}}, \frac{v + r^2}{2\xi}\right) \\ &\quad \times v^\lambda e^{-(v+r^2)} dv \prod_{j=1}^m r_j^{2|\alpha_{(j)}| + |p_{(j)}| - |q_{(j)}| + 2k_j - 1} dr_j, \quad \xi \in \mathbb{R}_+. \end{aligned} \tag{3.5}$$

Proof. Using Lemma 2.1 we have

$$\begin{aligned} RT_\varphi R^*\, \widehat{e}_\alpha(c(\xi)) &= R_0^*\, a\left(\frac{r_1}{\sqrt{2\xi}}, \dots, \frac{r_m}{\sqrt{2\xi}}, \frac{v + r_1^2 + \dots + r_m^2}{2\xi}\right) \zeta^p \overline{\zeta}^q R_0\, \widehat{e}_\alpha(c(\xi)) \\ &= R_0^* \left[a\left(\frac{r_1}{\sqrt{2\xi}}, \dots, \frac{r_m}{\sqrt{2\xi}}, \frac{v + r^2}{2\xi}\right) \zeta^p \overline{\zeta}^q \chi_+(\xi) c(\xi) e_\alpha(z) \ell_0(v) \right] \\ &= \left\{ \chi_+(\xi) \int_{\mathbb{C}^{n-1}\times\mathbb{R}_+} a\left(\frac{r_1}{\sqrt{2\xi}}, \dots, \frac{r_m}{\sqrt{2\xi}}, \frac{v + r^2}{2\xi}\right) \zeta^p \overline{\zeta}^q \right. \\ &\quad \left. \times\, c(\xi) e_\alpha(z) \ell_0(v) \overline{e_\beta(z)} \ell_0(v) \eta_\lambda(v) dV(z) dv \right\}_{\beta\in\mathbb{Z}_+^{n-1}} \\ &= \{\chi_+(\xi) c(\xi)\, I_\beta\}_{\beta\in\mathbb{Z}_+^{n-1}}\,. \end{aligned}$$

Using representation (3.1) and formula (2.1) we calculate

$$\begin{aligned}
I_\beta &= \int_{\mathbb{R}_+^m\times\mathbb{R}_+} a\left(\frac{r_1}{\sqrt{2\xi}},\dots,\frac{r_m}{\sqrt{2\xi}},\frac{v+r^2}{2\xi}\right)\frac{e^{-r^2}}{\pi^{n-1}\sqrt{\alpha!\,\beta!}}\prod_{j=1}^m r_j^{|\alpha_{(j)}|+|\beta_{(j)}|+2k_j-1}dr_j\\
&\quad\times\frac{1}{\Gamma(\lambda+1)}e^{-v}v^\lambda dv\prod_{j=1}^m\int_{\mathbb{S}^{k_j}}\zeta_{(j)}^{\alpha_{(j)}+p_{(j)}}\overline{\zeta}_{(j)}^{\beta_{(j)}+q_{(j)}}dS\\
&= \begin{cases} 0, & \text{if}\quad \alpha+p\neq\beta+q,\\ I_{\alpha+p-q}, & \text{if}\quad \alpha+p=\beta+q\ \text{ and }\ \alpha_l+p_l-q_l\geq 0,\\ & \qquad\text{for each}\ \ l=1,2,\dots,n-1,\end{cases}
\end{aligned}$$

where

$$\begin{aligned}
I_{\alpha+p-q} &= 2^m\prod_{j=1}^m\frac{(\alpha_{(j)}+p_{(j)})!\,\pi^{k_j}}{(k_j-1+|\alpha_{(j)}+p_{(j)}|)!}\int_{\mathbb{R}_+^{m+1}} a\left(\frac{r_1}{\sqrt{2\xi}},\dots,\frac{r_m}{\sqrt{2\xi}},\frac{v+r^2}{2\xi}\right)\\
&\quad\times\frac{e^{-(v+r^2)}v^\lambda\,dv}{\pi^{n-1}\sqrt{\alpha!\,(\alpha+p-q)!}\,\Gamma(\lambda+1)}\prod_{j=1}^m r_j^{|\alpha_{(j)}|+|\alpha_{(j)}+p_{(j)}-q_{(j)}|+2k_j-1}dr_j\\
&= \frac{2^m(\alpha+p)!}{\sqrt{\alpha!\,(\alpha+p-q)!}\,\Gamma(\lambda+1)}\prod_{j=1}^m\frac{1}{(k_j-1+|\alpha_{(j)}+p_{(j)}|)!}\\
&\quad\times\int_{\mathbb{R}_+^{m+1}} a\left(\frac{r_1}{\sqrt{2\xi}},\dots,\frac{r_m}{\sqrt{2\xi}},\frac{v+r^2}{2\xi}\right)\\
&\quad\times v^\lambda e^{-(v+r^2)}dv\prod_{j=1}^m r_j^{2|\alpha_{(j)}|+|p_{(j)}|-|q_{(j)}|+2k_j-1}dr_j,\qquad \xi\in\mathbb{R}_+,
\end{aligned}$$

and the result follows. □

4. Commutativity results

The next theorem describes the condition under which the Toeplitz operators with parabolic quasi-radial and parabolic quasi-radial quasi-homogeneous symbols commute. It is important to mention that this condition is on a quasi-homogeneous degree and that under it the commutativity property remains valid for arbitrary chose of parabolic k-quasi-radial functions.

Theorem 4.1. *Let $k=(k_1,k_2,\dots,k_m)$ and p, q be a pair of orthogonal multi-indices. Then for each pair of non identically zero parabolic k-quasi-radial functions a_1 and a_2, the Toeplitz operators T_{a_1} and $T_{a_2\xi^p\overline{\xi}^q}$ commute on each weighted Bergman space $\mathcal{A}^2_\lambda(D_n)$ if and only if $|p_{(j)}|=|q_{(j)}|$ for each $j=1,2,\dots,m$.*

Proof. We have

$$\begin{aligned}
T_1 &= RT_{a_2\zeta^p\overline{\zeta}^q}T_{a_1}R^* = RT_{a_2\zeta^p\overline{\zeta}^q}R^*\cdot RT_{a_1}R^*,\\
T_2 &= RT_{a_1}T_{a_2\zeta^p\overline{\zeta}^q}R^* = RT_{a_2\zeta^p\overline{\zeta}^q}R^*\cdot RT_{a_1}R^*.
\end{aligned}$$

Then for those multi-indices α with $\alpha_l + p_l - q_l \geq 0$, for each $l = 1, 2, \ldots, n-1$, Lemmas 3.1 and 3.2 yield

$$
\begin{aligned}
T_1 \widehat{e}_\alpha(c(\xi)) \;=\; & \frac{2^m (\alpha+p)!}{\sqrt{\alpha!\,(\alpha+p-q)!}\,\Gamma(\lambda+1)} \prod_{j=1}^m \frac{1}{(k_j - 1 + |\alpha_{(j)} + p_{(j)}|)!} \\
& \times \int_{\mathbb{R}_+^{m+1}} a_2 \left(\frac{r_1}{\sqrt{2\xi}}, \ldots, \frac{r_m}{\sqrt{2\xi}}, \frac{v + r^2}{2\xi} \right) \\
& \times v^\lambda e^{-(v+r^2)} dv \prod_{j=1}^m r_j^{2|\alpha_{(j)}| + |p_{(j)}| - |q_{(j)}| + 2k_j - 1} dr_j \\
& \times \frac{1}{\Gamma(\lambda+1)} \prod_{j=1}^m \frac{1}{(k_j - 1 + |\alpha_{(j)}|)!} \\
& \times \int_{\mathbb{R}_+^{m+1}} a_1 \left(\sqrt{\frac{r_1}{2\xi}}, \ldots, \sqrt{\frac{r_m}{2\xi}}, \frac{v + r_1 + \cdots + r_m}{2\xi} \right) \\
& \times v^\lambda e^{-(v + r_1 + \cdots + r_m)} dv \prod_{j=1}^m r_j^{|\alpha_{(j)}| + k_j - 1} dr_j \; \widehat{e}_{\alpha+p-q}(c(\xi))
\end{aligned}
$$

and

$$
\begin{aligned}
T_2 \widehat{e}_\alpha(c(\xi)) \;=\; & \frac{1}{\Gamma(\lambda+1)} \prod_{j=1}^m \frac{1}{(k_j - 1 + |\alpha_{(j)} + p_{(j)} - q_{(j)}|)!} \\
& \times \int_{\mathbb{R}_+^{m+1}} a_1 \left(\sqrt{\frac{r_1}{2\xi}}, \ldots, \sqrt{\frac{r_m}{2\xi}}, \frac{v + r_1 + \cdots + r_m}{2\xi} \right) \\
& \times v^\lambda e^{-(v + r_1 + \cdots + r_m)} dv \prod_{j=1}^m r_j^{|\alpha_{(j)} + p_{(j)} - q_{(j)}| + k_j - 1} dr_j \\
& \times \frac{2^m (\alpha+p)!}{\sqrt{\alpha!\,(\alpha+p-q)!}\,\Gamma(\lambda+1)} \prod_{j=1}^m \frac{1}{(k_j - 1 + |\alpha_{(j)} + p_{(j)}|)!} \\
& \times \int_{\mathbb{R}_+^{m+1}} a_2 \left(\frac{r_1}{\sqrt{2\xi}}, \ldots, \frac{r_m}{\sqrt{2\xi}}, \frac{v + r^2}{2\xi} \right) \\
& \times v^\lambda e^{-(v+r^2)} dv \prod_{j=1}^m r_j^{2|\alpha_{(j)}| + |p_{(j)}| - |q_{(j)}| + 2k_j - 1} dr_j \; \widehat{e}_{\alpha+p-q}(c(\xi))
\end{aligned}
$$

That is $T_{a_2 \zeta^p \overline{\zeta}^q} T_{a_1} = T_{a_1} T_{a_2 \zeta^p \overline{\zeta}^q}$ if and only if $|p_{(j)}| = |q_{(j)}|$ for each $j = 1, 2, \ldots, m$ □

Remark 4.2. For those j for which $k_j = 1$ both $p_{(j)}$ and $q_{(j)}$ are of length one, and the condition $|p_{(j)}| = |q_{(j)}|$ is equivalent to $p_{(j)} = q_{(j)} = 0$.

We note that under the condition $|p_{(j)}| = |q_{(j)}|$, for each $j = 1, 2, \ldots, m$, formula (3.5) becomes of the form

$$
\begin{aligned}
&\widetilde{\gamma}_{a,k,p,q}(\alpha,\xi) \qquad (4.1)\\
&= \frac{2^m(\alpha+p)!}{\sqrt{\alpha!\,(\alpha+p-q)!}\,\Gamma(\lambda+1)} \prod_{j=1}^{m} \frac{1}{(k_j-1+|\alpha_{(j)}+p_{(j)}|)!}\\
&\quad \times \int_{\mathbb{R}_+^{m+1}} a\left(\frac{r_1}{\sqrt{2\xi}},\ldots,\frac{r_m}{\sqrt{2\xi}},\frac{v+r^2}{2\xi}\right) v^{\lambda} e^{-(v+r^2)} dv \prod_{j=1}^{m} r_j^{2(|\alpha_{(j)}|+k_j)-1} dr_j\\
&= \frac{(\alpha+p)!}{\sqrt{\alpha!\,(\alpha+p-q)!}\,\Gamma(\lambda+1)} \prod_{j=1}^{m} \frac{1}{(k_j-1+|\alpha_{(j)}+p_{(j)}|)!}\\
&\quad \times \int_{\mathbb{R}_+^{m+1}} a\left(\sqrt{\frac{r_1}{2\xi}},\ldots,\sqrt{\frac{r_m}{2\xi}},\frac{v+r_1+\cdots+r_m}{2\xi}\right)\\
&\quad \times v^{\lambda} e^{-(v+r_1+\cdots+r_m)} dv \prod_{j=1}^{m} r_j^{|\alpha_{(j)}|+k_j-1} dr_j\\
&= \prod_{j=1}^{m}\left[\frac{(k_j-1+|\alpha_{(j)}|)!}{(k_j-1+|\alpha_{(j)}+p_{(j)}|)!}\,\frac{(\alpha_{(j)}+p_{(j)})!}{\sqrt{\alpha_{(j)}!\,(\alpha_{(j)}+p_{(j)}-q_{(j)})!}}\right]\gamma_{a,k}(\alpha,\xi).
\end{aligned}
$$

As in [5] we have rather surprising corollaries in which the product of mutually commuting Toeplitz operators turns out to be a Toeplitz operator.

Corollary 4.3. *Given $k = (k_1, k_2, \ldots, k_m)$, for each pair of orthogonal multi-indices p and q with $|p_{(j)}| = |q_{(j)}|$, for all $j = 1, 2, \ldots, m$, and each $a(r_1, r_2, \ldots, r_m, y_n) \in \mathcal{R}_k$, we have*

$$T_a T_{\zeta^p\overline{\zeta}^q} = T_{\zeta^p\overline{\zeta}^q} T_a = T_{a\zeta^p\overline{\zeta}^q}.$$

Given $k = (k_1, k_2, \ldots, k_m)$, and a pair of orthogonal multi-indices p and q with $|p_{(j)}| = |q_{(j)}|$, for all $j = 1, 2, \ldots, m$, let

$$\widetilde{p}_{(j)} = (0, \ldots, 0, p_{(j)}, 0, \ldots, 0) \quad \text{and} \quad \widetilde{q}_{(j)} = (0, \ldots, 0, q_{(j)}, 0, \ldots, 0).$$

Then, of course, $p = \widetilde{p}_{(1)} + \widetilde{p}_{(2)} + \cdots + \widetilde{p}_{(m)}$ and $q = \widetilde{q}_{(1)} + \widetilde{q}_{(2)} + \cdots + \widetilde{q}_{(m)}$.

For each $j = 1, 2, \ldots, m$, we introduce the Toeplitz operator $T_j = T_{\zeta^{\widetilde{p}_{(j)}}\overline{\zeta}^{\widetilde{q}_{(j)}}}$.

Corollary 4.4. *The operators T_j, $j = 1, 2, \ldots, m$, mutually commute.*

Given an h-tuple of indices $(j_1, j_2, \ldots, j_h)$, where $2 \le h \le m$, let

$$\widetilde{p}_h = \widetilde{p}_{(j_1)} + \widetilde{p}_{(j_2)} + \cdots + \widetilde{p}_{(j_h)} \quad \text{and} \quad \widetilde{q}_h = \widetilde{q}_{(j_1)} + \widetilde{q}_{(j_2)} + \cdots + \widetilde{q}_{(j_h)}.$$

Then

$$\prod_{g=1}^{h} T_{j_g} = T_{\zeta^{\widetilde{p}_h}\overline{\zeta}^{\widetilde{q}_h}}.$$

In particular,

$$\prod_{j=1}^{m} T_j = T_{\zeta^p\overline{\zeta}^q}.$$

Given $k = (k_1, k_2, \ldots, k_m)$, we consider any two bounded measurable parabolic k-quasi-radial quasi-homogeneous symbols $a(r_1, r_2, \ldots, r_m, y_n)\zeta^p\overline{\zeta}^q$ and $b(r_1, r_2, \ldots, r_m, y_n)\zeta^u\overline{\zeta}^v$, which satisfy the conditions of Theorem 4.1, i.e., $a(r_1, r_2, \ldots, r_m)$ and $b(r_1, r_2, \ldots, r_m)$ are arbitrary parabolic k-quasi-radial functions, $p \perp q$, $u \perp v$, and

$$|p_{(j)}| = |q_{(j)}| \quad \text{and} \quad |u_{(j)}| = |v_{(j)}|, \quad \text{for all} \quad j = 1, 2, \ldots, m.$$

Theorem 4.5. *Let*

$$a(r_1, r_2, \ldots, r_m, y_n)\zeta^p\overline{\zeta}^q \quad \textit{and} \quad b(r_1, r_2, \ldots, r_m, y_n)\zeta^u\overline{\zeta}^v$$

be as above. Then the Toeplitz operators $T_{a\zeta^p\overline{\zeta}^q}$ and $T_{b\zeta^u\overline{\zeta}^v}$ commute on each weighted Bergman space $\mathcal{A}^2_\lambda(D_n)$ if and only if for each $l = 1, 2, \ldots, n-1$ one of the next conditions is fulfilled

1. $p_l = q_l = 0$;
2. $u_l = v_l = 0$;
3. $p_l = u_l = 0$;
4. $q_l = v_l = 0$.

Proof. We calculate and compare first

$$T_1\,\widehat{e}_\alpha(c(\xi)) = RT_{a\zeta^p\overline{\zeta}^q}T_{b\zeta^u\overline{\zeta}^v}R^*\,\widehat{e}_\alpha(c(\xi)) = RT_{a\zeta^p\overline{\zeta}^q}R^*\cdot RT_{b\zeta^u\overline{\zeta}^v}R^*\,\widehat{e}_\alpha(c(\xi))$$

and

$$T_2\,\widehat{e}_\alpha(c(\xi)) = RT_{b\zeta^u\overline{\zeta}^v}T_{a\zeta^p\overline{\zeta}^q}R^*\,\widehat{e}_\alpha(c(\xi)) = RT_{b\zeta^u\overline{\zeta}^v}R^*\cdot RT_{a\zeta^p\overline{\zeta}^q}R^*\,\widehat{e}_\alpha(c(\xi))$$

for those multi-indices α when both above expressions are non zero.

By (4.1) we have

$$\begin{aligned} T_1\,\widehat{e}_\alpha(c(\xi)) &= \frac{(\alpha+u-v+p)!}{\sqrt{(\alpha+u-v)!\,(\alpha+u-v+p-q)!}} \\ &\quad\times\prod_{j=1}^{m}\frac{(k_j-1+|\alpha_{(j)}+u_{(j)}-v_{(j)}|)!}{(k_j-1+|\alpha_{(j)}+u_{(j)}-v_{(j)}+p_{(j)}|)!}\;\gamma_{a,k}(\alpha+u-v,\xi) \\ &\quad\times\frac{(\alpha+u)!}{\sqrt{\alpha!\,(\alpha+u-v)!}}\prod_{j=1}^{m}\frac{(k_j-1+|\alpha_{(j)}|)!}{(k_j-1+|\alpha_{(j)}+u_{(j)}|)!} \\ &\quad\times\gamma_{b,k}(\alpha,\xi)\;\widehat{e}_{\alpha+p+u-q-v}(c(\xi)) \end{aligned}$$

and

$$
\begin{aligned}
T_2\,\widehat{e}_\alpha(c(\xi)) \;=\;& \frac{(\alpha+p-q+u)!}{\sqrt{(\alpha+p-q)!\,(\alpha+p-q+u-v)!}} \\
&\times \prod_{j=1}^{m} \frac{(k_j-1+|\alpha_{(j)}+p_{(j)}-q_{(j)}|)!}{(k_j-1+|\alpha_{(j)}+p_{(j)}-q_{(j)}+u_{(j)}|)!}\,\gamma_{b,k}(\alpha+p-q,\xi) \\
&\times \frac{(\alpha+p)!}{\sqrt{\alpha!\,(\alpha+p-q)!}} \prod_{j=1}^{m} \frac{(k_j-1+|\alpha_{(j)}|)!}{(k_j-1+|\alpha_{(j)}+p_{(j)}|)!} \\
&\times \gamma_{a,k}(\alpha,\xi)\,\widehat{e}_{\alpha+p+u-q-v}(c(\xi)).
\end{aligned}
$$

As $|p_{(j)}| = |q_{(j)}|$ and $|u_{(j)}| = |v_{(j)}|$, for all $j = 1, 2, \ldots, m$, and , by (3.3),

$$
\begin{aligned}
\gamma_{a,k}(\alpha+u-v,\xi) &= \gamma_{a,k}(|\alpha_{(1)}+u_{(1)}-v_{(1)}|,\ldots,|\alpha_{(m)}+u_{(m)}-v_{(m)}|,\xi) \\
&= \gamma_{a,k}(|\alpha_{(1)}|,\ldots,|\alpha_{(m)}|,\xi) = \gamma_{a,k}(\alpha,\xi), \\
\gamma_{b,k}(\alpha+p-q,\xi) &= \gamma_{b,k}(|\alpha_{(1)}+p_{(1)}-q_{(1)}|,\ldots,|\alpha_{(m)}+p_{(m)}-q_{(m)}|,\xi) \\
&= \gamma_{b,k}(|\alpha_{(1)}|,\ldots,|\alpha_{(m)}|,\xi) = \gamma_{b,k}(\alpha,\xi),
\end{aligned}
$$

we have that $T_1\,\widehat{e}_\alpha(c(\xi)) = T_2\,\widehat{e}_\alpha(c(\xi))$ if and only if

$$
\frac{(\alpha+u-v+p)!\,(\alpha+u)!}{(\alpha+u-v)!} = \frac{(\alpha+p-q+u)!\,(\alpha+p)!}{(\alpha+p-q)!}.
$$

Varying α it is easy to see that the last equality holds if and only if for each $l = 1, 2, \ldots, n-1$ one of the next conditions is fulfilled

1. $p_l = q_l = 0$; 2. $u_l = v_l = 0$; 3. $p_l = u_l = 0$; 4. $q_l = v_l = 0$.

To finish the proof we mention that under either of the above conditions both quantities $T_1\,\widehat{e}_\alpha(c(\xi))$ and $T_2\,\widehat{e}_\alpha(c(\xi))$ are zero or non zero simultaneously only. □

To systematize the commutative algebras generated by Toeplitz operators with parabolic quasi-radial quasi-homogeneous symbols we proceed as follows (see [5]). To avoid the repetition of the unitary equivalent algebras and to simplify the classification of the (non unitary equivalent) algebras, in addition to (3.2), we can rearrange the variables z_l and correspondingly the components of multi-indices in p and q so that

(i) for each j with $k_j > 1$, we have

$$
p_{(j)} = (p_{j,1},\ldots,p_{j,h_j},0,\ldots,0) \quad \text{and} \quad q_{(j)} = (0,\ldots,0,q_{j,h_j+1},\ldots,q_{j,k_j}); \tag{4.2}
$$

(ii) if $k_{j'} = k_{j''}$ with $j' < j''$, then $h_{j'} \leq h_{j''}$.

Now, given $k = (k_1,\ldots,k_m)$, we start with m-tuple $h = (h_1,\ldots,h_m)$, where $h_j = 0$ if $k_j = 1$ and $1 \leq h_j \leq k_j - 1$ if $k_j \geq 1$; in the last case, if $k_{j'} = k_{j''}$ with $j' < j''$, then $h_{j'} \leq h_{j''}$.

We denote by $\mathcal{R}_k(h)$ the linear space generated by all parabolic k-quasi-radial quasi-homogeneous functions

$$
a(r_1, r_2, \ldots, r_m, y_n)\,\zeta^p\overline{\zeta}^q,
$$

where $a(r_1, r_2, \ldots, r_m, y_n) \in \mathcal{R}_k$, and the components $p_{(j)}$ and $q_{(j)}$, $j = 1, 2, \ldots, m$, of multi-indices p and q are of the form (4.2) with

$$p_{j,1} + \cdots + p_{j,h_j} = q_{j,h_j+1} + \cdots + q_{j,k_j}, \quad p_{j,1}, \ldots, p_{j,h_j}, q_{j,h_j+1}, \ldots, q_{j,k_j} \in \mathbb{Z}_+.$$

We note that $\mathcal{R}_k \subset \mathcal{R}_k(h)$ and that the identity function $e(z) \equiv 1$ belongs to $\mathcal{R}_k(h)$. The main result of the paper gives the next corollary.

Corollary 4.6. *The Banach algebra generated by Toeplitz operators with symbols from $\mathcal{R}_k(h)$ is commutative.*

We note that,

- for $n > 2$ and $k \neq (1, 1, \ldots, 1)$, these algebras are just Banach; extending them to C^*-algebras they become non commutative;
- these algebras are commutative for each weighted Bergman space $\mathcal{A}^2_\lambda$, with $\lambda > -1$,
- for $n = 2$ all these algebras collapse to the single C^*-algebra generated by Toeplitz operators with quasi-parabolic symbols (see [4, Section 10.1]).

References

[1] S. Grudsky, R. Quiroga-Barranco, and N. Vasilevski. Commutative C^*-algebras of Toeplitz operators and quantization on the unit disk. *J. Funct. Anal.*, 234(1):1–44, 2006.

[2] I. Louhichi and N.V. Rao. On Toeplitz operators with quasihomogeneous symbols. *Atch. Math.*, 851:248–257, 2005.

[3] Issam Louhichi, Elizabeth Strouse, and Lova Zakariasy. Products of Toeplitz operators on the Bergman space. *Integral Equations Operator Theory*, 54(4):525–539, 2006.

[4] Raul Quiroga-Barranco and Nikolai Vasilevski. Commutative C^*-algebras of Toeplitz operators on the unit ball, I. Bargmann-type transforms and spectral representations of Toeplitz operators. *Integral Equations Operator Theory*, 59(3):379–419, 2007.

[5] Nikolai Vasilevski. Quasi-radial quasi-homogeneous symbols and commutative Banach algebras of Toeplitz operators. Reporte Interno #386, Departamento de Matemáticas, CINVESTAV del I.P.N., Mexico City, 13 p., 2009.

[6] Ze-Hua Zhou and Xing-Tang Dong. Algebraic properties of Toeplitz operators with radial symbols on the Bergman space of the unit ball. *Integral Equations and Operator Theory*, 64(1):137–154, 2009.

[7] Kehe Zhu. *Spaces of Holomorphic Functions in the Unit Ball.* Springer Verlag, 2005.

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Received: February 20, 2009
Accepted: April 16, 2009

Operator Theory:
Advances and Applications, Vol. 202, 569–578

Algebraic Aspects of the Paving and Feichtinger Conjectures

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Abstract. The Paving Conjecture in operator theory and the Feichtinger Conjecture in frame theory are both problems that are equivalent to the Kadison-Singer problem concerning extensions of pure states. In all three problems, one of the difficulties is that the natural multiplicative structure appears to be incompatible – the unique extension problem of Kadison-Singer is compatible with a linear subspace, but not a subalgebra; likewise, the pavable operators is known to be a linear subspace but not a subalgebra; the Feichtinger Conjecture does not even have a linear structure. The Paving Conjecture and the Feichtinger Conjecture both have special cases in terms of exponentials in $L^2[0,1]$. We introduce convolution as a multiplication to demonstrate a possible attack for these special cases.

Mathematics Subject Classification (2000). Primary: 46L99; Secondary 46B99, 42B35.

Keywords. Kadison-Singer Problem, Paving, Laurent operator, frame.

1. Introduction

The Paving Conjecture of Anderson [1] states that every bounded operator on $\ell^2(\mathbb{Z})$ can be paved, that is, given $T \in \mathcal{B}(\ell^2(\mathbb{Z}))$ and given $\epsilon > 0$, there exists a finite partition $\{A_j\}_{j=1}^J$ of $\mathbb{Z}$ such that for every $j = 1, \dots, J$

$$\|P_{A_j}\,(T - \mathcal{D}(T))\,P_{A_j}\| < \epsilon,$$

where $\mathcal{D}(T)$ is the diagonal of T, and P_{A_j} denotes the canonical projection onto $\ell^2(A_j) \subset \ell^2(\mathbb{Z})$.

The Paving Conjecture is a reformulation of the Kadison-Singer Problem [11]. The Kadison-Singer problem is whether every pure state on $\mathcal{D}(\ell^2(\mathbb{Z})) \subset \mathcal{B}(\ell^2(\mathbb{Z}))$ has a unique extension to a pure state on all of $\mathcal{B}(\ell^2(\mathbb{Z}))$. Here $\mathcal{D}(\ell^2(\mathbb{Z}))$ denotes the masa of diagonal operators on $\ell^2(\mathbb{Z})$. We note that Kadison and Singer "incline to the view" that the extensions are not in general unique.

Communicated by J.A. Ball.

A special case of the Paving Conjecture is whether all Laurent operators are pavable. A Laurent operator is an element of $\mathcal{B}(\ell^2(\mathbb{Z}))$ which commutes with the bilateral shift. Equivalently, via the (inverse) Fourier transform, a Laurent operator T is such that $\mathcal{F}^{-1}T\mathcal{F}$ is a multiplication operator on $L^2[0,1]$. Here we define the Fourier transform $\mathcal{F} : L^2[0,1] \to \ell^2(\mathbb{Z})$ in the usual way. Since $\mathcal{F}^{-1}T\mathcal{F}$ is a multiplication operator, we will denote it by $M_\phi : L^2[0,1] \to L^2[0,1] : f \mapsto \phi f$; ϕ is the symbol of T. We will denote the Laurent operator in matrix form by T_ϕ.

Currently it is unknown if every Laurent operator is pavable. The best known result is that a Laurent operator is pavable if its symbol is Riemann integrable [9].

Recently, Casazza and Tremain have shown a direct connection between the Paving Conjecture and several problems in frame theory [5, 4].

Let H be a separable Hilbert space. A Bessel sequence $\mathbb{X} := \{x_n\} \subset H$ is such that the synthesis operator

$$\Theta^*_{\mathbb{X}} : \ell^2(\mathbb{Z}) \to H : (c_n) \mapsto \sum_n c_n x_n$$

is bounded. The square of the norm of $\Theta^*_{\mathbb{X}}$ is called the Bessel bound.

The sequence is a frame if the dual Grammian satisfies

$$C_1 I \leq \Theta^*_{\mathbb{X}}\Theta_{\mathbb{X}} \leq C_2 I.$$

The optimal constants C_1 and C_2 which satisfy these inequalities are called the lower and upper frame bounds, respectively.

The sequence is a Riesz basic sequence if the Grammian satisfies

$$(D_1)^2 I \leq \Theta_{\mathbb{X}}\Theta^*_{\mathbb{X}} \leq (D_2)^2 I.$$

The optimal constants D_1 and D_2 which satisfy these inequalities are called the lower and upper Riesz basis bounds, respectively.

We say $\mathbb{X}$ is a $\|x\| + \epsilon$ RBS if $\|x_n\| = \|x\|$ and

$$\|x\| - \epsilon \leq D_1 \leq D_2 \leq \|x\| + \epsilon.$$

The Feichtinger Conjecture says that given a frame $\{x_n\}_{n\in\mathbb{Z}} \subset H$, with the property that $\inf\{\|x_n\|\} > 0$, then there exists a finite partition $\{A_j\}_{j=1}^J$ of $\mathbb{Z}$ such that for each j, $\{x_n\}_{n\in A_j}$ is a Riesz basic sequence.

The $\mathcal{R}_\epsilon$-Conjecture says that given a frame $\{x_n\}_{n\in\mathbb{Z}} \subset H$, with the property that $\|x_n\| = \|x\|$ for every n, and given an $\epsilon > 0$, then there exists a finite partition $\{A_j\}_{j=1}^J$ of $\mathbb{Z}$ such that for each j, $\{x_n\}_{n\in A_j}$ is a $\|x\| + \epsilon$ Riesz basic sequence.

The connections between the Paving, $\mathcal{R}_\epsilon$-, and Feichtinger Conjectures can be summarized as thus:

Theorem. *The following are equivalent:*

1. *the Kadison-Singer problem has a positive solution;*
2. *the Paving Conjecture has a positive solution;*
3. *the $\mathcal{R}_\epsilon$-Conjecture has a positive solution;*
4. *the Feichtinger conjecture has a positive solution.*

One remark regarding matrices: If B is a $p \times q$ submatrix of an $m \times n$ matrix A, we will use B to denote both a $p \times q$ matrix as well as an $m \times n$ matrix (with 0's in appropriate coordinates). Note that the norm of B is the same in either case. Likewise if B and A are infinite matrices. As such, this should cause no confusion.

2. Paving Laurent operators and frame theory

We shall define three classes in $L^\infty[0,1]$ in relation to the Paving, $\mathcal{R}_\epsilon$-, and Feichtinger Conjectures.

Definition 1. If $f \in L^\infty[0,1]$ and the Laurent operator T_f, with symbol f, is pavable, we will say $f \in \mathcal{P}_\mathcal{L}$.

Proposition 1. *The set $\mathcal{P}_\mathcal{L} \subset L^\infty[0,1]$ has the following properties:*

1. *Subspace of $L^\infty[0,1]$;*
2. *Closed in norm;*
3. *Closed under conjugation;*
4. *Closed under convolution.*

Proof. Items 1 and 2 follow from the fact that the set of pavable operators in $\mathcal{B}(\ell^2(\mathbb{Z}))$ is a closed subspace.

Item 3 follows from the fact that $T_{\overline{f}} = T_f^*$, and the submatrices satisfy

$$\|P_{A_j}\left(T_f^* - \mathcal{D}(T_f^*)\right) P_{A_j}\| = \|P_{A_j}\left(T_f - \mathcal{D}(T_f)\right) P_{A_j}\|.$$

Item 4 follows from the fact that if $f, g \in \mathcal{P}_\mathcal{L}$, $f * g \in L^\infty[0,1]$ is continuous, hence Riemann integrable, and hence by [9], $f * g \in \mathcal{P}_\mathcal{L}$. □

An important question concerning $\mathcal{P}_\mathcal{L}$ is whether it is closed under (pointwise) multiplication.

Definition 2. If $f \in L^\infty[0,1]$, we say f satisfies the $\mathcal{R}_\epsilon$-Conjecture if given $\epsilon > 0$ there exists a finite partition $A_1, \ldots, A_J$ of $\mathbb{Z}$ such that each

$$\{e^{2\pi i n x} f(x)\}_{n \in A_j}$$

is a $\|f\| + \epsilon$ Riesz basic sequence. We will denote this by $f \in \mathcal{R}_\epsilon$. We make the convention that $0 \in \mathcal{R}_\epsilon$.

Proposition 2. *The set $\mathcal{R}_\epsilon \subset L^\infty[0,1]$ has the following properties:*

1. *Closed in norm;*
2. *Closed under scalar multiplication;*
3. *Closed under conjugation;*
4. *Closed under convolution;*

Proof. Item 1 follows from a variation of the Paley-Wiener theorem [12, Theorem XXXVII]. Suppose that $\{f_k\} \subset \mathcal{R}_\epsilon$ with $f_k \to f$. Without loss of generality, assume $\|f_k\| = \|f\| = 1$. Given $\epsilon > 0$, find f_k such that $\|f_k - f\| < \frac{\epsilon}{2}$; then find a partition $\{A_j\}_{j=1}^J$ such that each $\{e^{2\pi inx} f_k(x) : n \in A_j\}$ is a $1 + \frac{\epsilon}{2}$ RBS. Then by the proof of the Paley-Wiener theorem, or the proof of Theorem 1 in [7],

$$\{e^{2\pi inx} f(x) : n \in A_j\} = \{e^{2\pi inx} f_k(x) + e^{2\pi inx}(f(x) - f_k(x)) : n \in A_j\}$$

is a $1 + \epsilon$ RBS.

Item 2 follows from the fact that if $\{e^{2\pi inx} f(x) : n \in A_j\}$ is a $\|f\| + \epsilon$ RBS, and $\lambda \in \mathbb{C}$ is nonzero, then $\{e^{2\pi inx}\lambda f(x) : n \in A_j\}$ is a $\|\lambda f\| + |\lambda|\epsilon$ RBS.

Item 3 follows from the fact that the Grammian matrices for $\{e^{2\pi inx} f(x) : n \in A_j\}$ and $\{e^{2\pi inx}\overline{f}(x) : n \in A_j\}$ are equal, whence if one is a $\|f\| + \epsilon$ RBS, then the other is also.

Item 4 follows from Theorem 1 below: if $f, g \in \mathcal{R}_\epsilon$, $f * g$ is continuous, whence Riemann integrable. Therefore, $|f * g|^2$ is also Riemann integrable and thus is in $\mathcal{P}_\mathcal{L}$. Finally, $f * g \in \mathcal{R}_\epsilon$. □

Remark 1. It is unknown if $\mathcal{R}_\epsilon$ is closed under addition. If $f, g \in \mathcal{R}_\epsilon$ and have disjoint support, then $f + g \in \mathcal{R}_\epsilon$. See Corollary 1 below.

Definition 3. If $f \in L^\infty[0,1]$, we say f satisfies the Feichtinger Conjecture if there exists a finite partition $A_1, \ldots, A_J$ of $\mathbb{Z}$ such that each

$$\{e^{2\pi inx} f(x)\}_{n \in A_j}$$

is a Riesz basic sequence. We will denote this by $f \in \mathcal{F}$. We make the convention that $0 \in \mathcal{F}$.

Proposition 3. *The set $\mathcal{F} \subset L^\infty[0,1]$ has the following properties:*

1. *Closed under scalar multiplication;*
2. *Closed under conjugation;*
3. *Closed under convolution.*

Proof. Item 1 is obvious. Item 2 follows as above from the fact that the Grammian matrices for the two sequences are identical. Item 3 follows again from Theorem 1. □

Fundamental Question. Which, if any, of $\mathcal{F}$, $\mathcal{R}_\epsilon$, $\mathcal{P}_\mathcal{L}$ are all of $L^\infty[0,1]$?

Theorem 1. *The three classes have the following relations:*

1. $f \in \mathcal{R}_\epsilon \Leftrightarrow |f|^2 \in \mathcal{P}_\mathcal{L}$
2. $\mathcal{P}_\mathcal{L} \subset \mathcal{F}$
3. $\mathcal{P}_\mathcal{L} \cap \mathcal{R}_\epsilon \cap \mathcal{F}$ *contains Riemann integrable functions.*

Proof. Item 1: Consider the Gram matrix G_f and the Laurent matrix $T_{|f|^2}$. The entries of the matrix G_f are as follows:

$$\begin{aligned} G_f[m,n] &= \langle e^{2\pi inx} f(x), e^{2\pi imx} f(x)\rangle \\ &= \int_0^1 |f(x)|^2 e^{2\pi i(n-m)x} dx. \end{aligned}$$

Likewise, the entries of the matrix $T_{|f|^2}$ are as follows:

$$\begin{aligned} T_{|f|^2}[m,n] &= \langle T_{|f|^2}\delta_n, \delta_m\rangle \\ &= \langle M_{|f|^2}e^{2\pi inx}, e^{2\pi imx}\rangle \\ &= \int_0^1 |f(x)|^2 e^{2\pi i(n-m)x}dx. \end{aligned}$$

Therefore the matrices are identical, $G_f = T_{|f|^2}$.

Furthermore, if $A \subset \mathbb{Z}$, note that the Grammian matrix for $\{e^{2\pi inx}f(x) : n \in A\}$, denoted by G_f^A, is such that

$$G_f^A = P_A T_{|f|^2} P_A.$$

Suppose that $|f|^2 \in \mathcal{P}_{\mathcal{L}}$, and let $\epsilon > 0$ be given. Choose $0 < \delta < \|f\|^2$ such that

$$\|f\| - \epsilon < \sqrt{\|f\|^2 - \delta} < \sqrt{\|f\|^2 + \delta} < \|f\| + \epsilon.$$

Since $T_{|f|^2}$ is pavable, let $\{A_j\}_{j=1}^J$ be a partition of $\mathbb{Z}$ such that

$$\|P_{A_j}\left(T_{|f|^2} - \mathcal{D}(T_{|f|^2})\right)P_{A_j}\| < \delta$$

for $j = 1, \ldots, J$. Note that $\mathcal{D}(T_{|f|^2}) = \|f\|^2 I$. For a fixed j, the Grammian matrix of $\{e^{2\pi inx}f(x) : n \in A_j\}$ can be written as

$$G_f^{A_j} = \|f\|^2 I_{A_j} + M^{A_j} \tag{1}$$

(where I_{A_j} is the identity on $\ell^2(A_j)$) as well as

$$G_f^{A_j} = P_{A_j} T_{|f|^2} P_{A_j}. \tag{2}$$

Thus we have

$$M^{A_j} = P_{A_j}\left(T_{|f|^2} - \mathcal{D}(T_{|f|^2})\right)P_{A_j}, \tag{3}$$

and so by the spectral mapping theorem,

$$\left(\|f\|^2 - \delta\right) I_{A_j} \le G_f^{A_j} \le \left(\|f\|^2 + \delta\right) I_{A_j}.$$

It follows that $\{e^{2\pi inx}f(x) : n \in A_j\}$ is a Riesz basic sequence, with basis bounds that satisfy

$$\|f\| - \epsilon < \sqrt{\|f\|^2 - \delta} \le D_1 \le D_2 < \sqrt{\|f\|^2 + \delta} < \|f\| + \epsilon$$

and hence $\{e^{2\pi inx}f(x) : n \in A_j\}$ is a $\|f\| + \epsilon$ RBS. Therefore, $f \in \mathcal{R}_\epsilon$.

Conversely, suppose that $f \in \mathcal{R}_\epsilon$ and let $\epsilon > 0$ be given. Choose $0 < \delta$ such that

$$-\epsilon < -2\|f\|\delta + \delta^2 < 2\|f\|\delta + \delta^2 < \epsilon.$$

Since $f \in \mathcal{R}_\epsilon$, there exists a partition $\{A_j\}_{j=1}^J$ of $\mathbb{Z}$ such that for each j,

$$\{e^{2\pi inx}f(x) : n \in A_j\}$$

is a $\|f\| + \delta$ RBS. Therefore, for a fixed j,

$$(\|f\| - \delta)^2 I_{A_j} \le G_f^{A_j} \le (\|f\| + \delta)^2 I_{A_j}.$$

From Equation 1 we have

$$\left(-2\|f\|\delta+\delta^2\right) I_{A_j} \le M^{A_j} \le \left(2\|f\|\delta+\delta^2\right) I_{A_j}$$

and thus by Equations 2 and 3,

$$\begin{aligned} -\epsilon I_{A_j} \;<\; \left(-2\|f\|\delta+\delta^2\right) I_{A_j} &\le P_{A_j}\left(T_{|f|^2}-\mathcal{D}(T_{|f|^2})\right) P_{A_j} \\ &\le \left(2\|f\|\delta+\delta^2\right) I_{A_j} \;<\; \epsilon I_{A_j}. \end{aligned}$$

It follows that $\{A_j\}_{j=1}^J$ is a paving of $T_{|f|^2}$, and $|f|^2 \in \mathcal{P}_{\mathcal{L}}$.

Item 2: the statement follows from Casazza-Weber [6] as well as Bownik-Speegle [2].

Item 3: Halpern-Kaftal-Weiss [9] prove that $\mathcal{P}_{\mathcal{L}}$ contains the Riemann integrable functions; $\mathcal{F}$ contains the Riemann integrable functions by Item 2; and $\mathcal{R}_\epsilon$ contains the Riemann integrable functions by Item 1. Indeed, if f is Riemann integrable, then $|f|^2$ is also, and thus $|f|^2 \in \mathcal{P}_{\mathcal{L}}$, whence by Item 1, $f \in \mathcal{R}_\epsilon$. □

Corollary 1. *If $f, g \in \mathcal{R}_\epsilon$ and the supports of f and g are disjoint, then $f+g \in \mathcal{R}_\epsilon$.*

Proof. By Theorem 1, since $f, g \in \mathcal{R}_\epsilon$, then $|f|^2, |g|^2 \in \mathcal{P}_{\mathcal{L}}$. Since the supports of f, g are disjoint, $|f+g|^2 = |f|^2 + |g|^2$. By Proposition 1, $|f|^2 + |g|^2 \in \mathcal{P}_{\mathcal{L}}$, hence $|f+g|^2 \in \mathcal{P}_{\mathcal{L}}$. Therefore, $f+g \in \mathcal{R}_\epsilon$. □

Corollary 2. *If $\mathcal{R}_\epsilon$ is a subspace of $L^\infty[0,1]$, then $\mathcal{R}_\epsilon = \mathcal{P}_{\mathcal{L}} = L^\infty[0,1]$. Likewise, if $\mathcal{R}_\epsilon$ is convex, then $\mathcal{R}_\epsilon = \mathcal{P}_{\mathcal{L}} = L^\infty[0,1]$.*

Proof. Let $E \subset [0,1]$ be measurable, and consider $f = \chi_E + i\chi_{\widetilde{E}}$, where $\widetilde{E}$ is the complement of E. Note that $f \in \mathcal{R}_\epsilon$. Since $\mathcal{R}_\epsilon$ is closed under conjugation and is by hypothesis a subspace, we have that

$$\chi_E = \frac{1}{2}(f + \overline{f})$$

is in $\mathcal{R}_\epsilon$. By Theorem 1, $|\chi_E|^2 = \chi_E \in \mathcal{P}_{\mathcal{L}}$, hence every projection is in $\mathcal{P}_{\mathcal{L}}$, and so $\mathcal{P}_{\mathcal{L}} = L^\infty[0,1]$. Finally, since every $|g|^2 \in \mathcal{P}_{\mathcal{L}}$, every $g \in \mathcal{R}_\epsilon$.

Similarly, if $\mathcal{R}_\epsilon$ is convex, then $\chi_E \in \mathcal{P}_{\mathcal{L}}$ for every measurable E. □

Corollary 3. *If $\mathcal{R}_\epsilon = \mathcal{P}_{\mathcal{L}}$, then $\mathcal{R}_\epsilon = \mathcal{P}_{\mathcal{L}} = L^\infty[0,1]$.*

Proof. By Corollary 2, if $\mathcal{R}_\epsilon = \mathcal{P}_{\mathcal{L}}$, then $\mathcal{R}_\epsilon$ is a subspace, whence the conclusion follows. □

3. Convolution and Segal algebras

We have seen that $\mathcal{P}_{\mathcal{L}}, \mathcal{R}_\epsilon$, and $\mathcal{F}$ are closed under convolution. They are also invariant under convolution by elements of $L^1[0,1]$, and so we can consider these classes in terms of ideals in the Banach algebra $L^1[0,1]$.

Theorem 2. *If $f \in \mathcal{P}_{\mathcal{L}}, (\mathcal{R}_\epsilon, \mathcal{F})$ and $g \in L^1[0,1]$, then $f * g$ is again in $\mathcal{P}_{\mathcal{L}}, (\mathcal{R}_\epsilon, \mathcal{F}$ resp.).*

Proof. If $f \in L^\infty[0,1]$ and $g \in L^1[0,1]$, then $f * g$ is continuous and hence Riemann integrable. Therefore $f * g \in \mathcal{P}_{\mathcal{L}}$, $(\mathcal{R}_\epsilon, \mathcal{F}$ resp.) □

Corollary 4. *$\mathcal{P}_{\mathcal{L}} \subset L^1[0,1]$ is an ideal in the Banach algebra $L^1[0,1]$. $\mathcal{P}_{\mathcal{L}}$ is also an ideal in the Banach algebra $(L^\infty[0,1], *)$.*

Likewise, $\mathcal{R}_\epsilon, \mathcal{F}$ are "almost" ideals in these Banach algebras.

Question 1. Is $\mathcal{P}_{\mathcal{L}}$ a maximal ideal in $L^\infty[0,1]$?

We will see that it is not a proper maximal ideal.

Of course, while $\mathcal{P}_{\mathcal{L}}, \mathcal{R}_\epsilon$, and $\mathcal{F}$ are subsets of $L^1[0,1]$, and are ("almost") ideals therein, they are not closed in norm in $L^1[0,1]$ (they are, after all, dense). However, they are closed in norm in $L^\infty[0,1]$, and as such we can regard them in terms of (abstract) Segal algebras.

Definition 4. An ideal $\mathcal{B} \subset L^1[0,1]$ is a Segal algebra if

1. $\mathcal{B}$ is dense in $L^1[0,1]$;
2. $\mathcal{B}$ is a Banach algebra with respect to a norm $\|\cdot\|'$;
3. $\mathcal{B}$ is translation invariant, and translations are strongly continuous in $\|\cdot\|'$.

We do not know if $\mathcal{R}_\epsilon$ or $\mathcal{F}$ are ideals in $L^1[0,1]$. We do know that $\mathcal{P}_{\mathcal{L}}$ is an ideal in $L^1[0,1]$, is dense, and is a Banach algebra with respect to the norm in $L^\infty[0,1]$. However, translations are not strongly continuous in $L^\infty[0,1]$, and hence $\mathcal{P}_{\mathcal{L}}$ is not a Segal algebra in $L^1[0,1]$. However, it is an abstract Segal algebra:

Definition 5. If $\mathcal{A}$ is any Banach algebra, an ideal $\mathcal{B} \subset \mathcal{A}$ is an abstract Segal algebra (ASA) if

1. $\mathcal{B}$ is dense in $\mathcal{A}$;
2. $\mathcal{B}$ is a Banach algebra with respect to a norm $\|\cdot\|'$;
3. $\exists M > 0$ so that $\|x\| \le M\|x\|', \forall x \in \mathcal{B}$;
4. $\exists C > 0$ so that $\|xy\|' \le C\|x\|\|y\|', \forall x, y \in \mathcal{B}$.

Proposition 4. *Both $\mathcal{P}_{\mathcal{L}}$ and $(L^\infty[0,1], *)$ are ASA's in $L^1[0,1]$ with respect to the norm $\|\cdot\|_\infty$.*

Proof. Clearly, $L^\infty[0,1]$ is dense in $L^1[0,1]$; $\mathcal{P}_{\mathcal{L}}$ is also dense since it contains all continuous functions. A simple computation shows that $\mathcal{P}_{\mathcal{L}}$ and $L^\infty[0,1]$ are Banach algebras with respect to $\|\cdot\|_\infty$. Additionally, $\|\cdot\|_1 \le \|\cdot\|_\infty$. Finally, we have

$$\begin{aligned}|f * g(y)| &= \left|\int_0^1 f(y-x)g(x)dx\right| \\ &\le \|g\|_\infty \int_0^1 |f(y-x)|dx = \|f\|_1\|g\|_\infty.\end{aligned}$$

Hence, $\|f * g\|_\infty \le \|f\|_1\|g\|_\infty$. □

The following theorem is called the Fundamental Theorem of Abstract Segal Algebras by Burnham [3]:

Theorem (Burnham, 1.1). *If $\mathcal{B}$ is an ASA in $\mathcal{A}$ and every right approximate unit of $\mathcal{B}$ is also a left approximate unit, then the following are true:*

1. *If J is a closed ideal in $\mathcal{A}$, then $J \cap \mathcal{B}$ is a closed ideal in $\mathcal{B}$;*
2. *If I is a closed ideal in $\mathcal{B}$, then $cl(I)$ (in $\mathcal{A}$-norm) is a closed ideal in $\mathcal{A}$ and $I = cl(I) \cap \mathcal{B}$.*

Conceivably, since $L^1[0,1]$ is commutative, every right approximate unit of $L^\infty[0,1]$ is also a left approximate unit. Therefore Burnham's theorem, applied with $\mathcal{A} = L^1[0,1]$, $\mathcal{B} = L^\infty[0,1]$, and $I = \mathcal{P}_\mathcal{L}$, would yield $\mathcal{P}_\mathcal{L} = L^\infty[0,1]$. However, $L^\infty[0,1]$ does not have any right approximate units, and the proof of Burnham's theorem requires the existence of one.

Theorem (Burnham, 2.1). *If $\mathcal{B}$ is an ASA in a commutative Banach algebra $\mathcal{A}$, then the regular maximal ideal spaces of $\mathcal{B}$ and $\mathcal{A}$ are homeomorphic.*

The proof actually shows that the complex homomorphisms if $\mathcal{A}$ and $\mathcal{B}$ are identical, and possess the same topology.

Corollary 5. *$\mathcal{P}_\mathcal{L}$ is not a proper maximal ideal in $L^\infty[0,1]$.*

Proof. Applying the theorem to $\mathcal{A} = L^1[0,1]$ and $\mathcal{B} = L^\infty[0,1]$, we have that the complex homomorphisms for the two Banach algebras are identical. We see that $\mathcal{P}_\mathcal{L}$ is not the kernel of a complex homomorphism, since for any $n \in \mathbb{Z}$, there exists an $f \in \mathcal{P}_\mathcal{L}$ with

$$\int_0^1 f(x) e^{2\pi i n x} dx \neq 0. \qquad \square$$

4. Gabor systems

We conclude the paper with one positive result. It is based on the observation that not only are $\mathcal{P}_\mathcal{L}$, $\mathcal{R}_\epsilon$, and $\mathcal{F}$ invariant under convolution, but if $f, g \in L^\infty[0,1]$, then $f * g \in \mathcal{P}_\mathcal{L}$, $\mathcal{R}_\epsilon$, or $\mathcal{F}$, respectively, even if we don't know whether either f or g are in $\mathcal{P}_\mathcal{L}$, $\mathcal{R}_\epsilon$, or $\mathcal{F}$.

A Gabor system in $L^2(\mathbb{R})$ has the form

$$\{e^{2\pi i b n x} f(x - am) : n, m \in \mathbb{Z}\}$$

and is denoted by (f, a, b). If $a = b = 1$, then the Zak transform is available:

$$\mathcal{Z} : L^2(\mathbb{R}) \to L^2\left([0,1] \times [0,1]\right)$$

where

$$[\mathcal{Z} f](\omega, \xi) = \sum_{n \in \mathbb{Z}} e^{2\pi i n \xi} f(\omega - n).$$

The Zak transform is a unitary operator, and if $h(x) = e^{2\pi ikx}g(x-l)$, then

$$[\mathcal{Z}h](\omega,\xi) = e^{2\pi ik\omega}e^{-2\pi il\xi}[\mathcal{Z}g](\omega,\xi).$$

The Gabor system $(f,1,1)$ is Bessel if and only if $\mathcal{Z}f \in L^\infty([0,1]\times[0,1])$.

Theorem 3. *Suppose* $(g,1,1)$ *is a Gabor Bessel system such that the Zak transform* $[\mathcal{Z}g](\omega,\xi)$ *can be factored as* $[\mathcal{Z}g](\omega,\xi) = h_1(\omega)h_2(\xi)$. *Then the system*

$$\{e^{2\pi inx}g(x-n) : n \in \mathbb{Z}\}$$

satisfies the Feichtinger conjecture.

Proof. We compute the Grammian matrix of the set $\{e^{2\pi inx}g(x-n) : n \in \mathbb{Z}\}$, utilizing the Zak transform:

$$\begin{aligned}
G[p,q] &= \langle e^{2\pi iqx}g(x-q), e^{2\pi ipx}g(x-p)\rangle \\
&= \langle \mathcal{Z}\left(e^{2\pi iqx}g(x-q)\right), \mathcal{Z}\left(e^{2\pi ipx}g(x-p)\right)\rangle \\
&= \langle e^{-2\pi iq\xi}e^{2\pi iq\omega}[\mathcal{Z}g](\omega,\xi), e^{-2\pi ip\xi}e^{2\pi ip\omega}[\mathcal{Z}g](\omega,\xi)\rangle \\
&= \int_0^1\int_0^1 e^{-2\pi i(q-p)\xi}e^{2\pi i(q-p)\omega}|h_1(\omega)|^2|h_2(\xi)|^2 d\omega d\xi \\
&= \left(\int_0^1 e^{-2\pi i(q-p)\xi}|h_2(\xi)|^2 d\xi\right)\left(\int_0^1 e^{2\pi i(q-p)\omega}|h_1(\omega)|^2 d\omega\right)
\end{aligned}$$

Thus, G is a Laurent operator. We can define the following two matrices

$$\begin{aligned}
H_1[p,q] &= \int_0^1 e^{2\pi i(q-p)\omega}|h_1(\omega)|^2 d\omega; \\
H_2[p,q] &= \int_0^1 e^{-2\pi i(q-p)\xi}|h_2(\xi)|^2 d\xi = \int_0^1 e^{2\pi i(q-p)\xi}|h_2(-\xi)|^2 d\xi
\end{aligned}$$

which both are (bounded) Laurent operators.

We see that G is the Schur (entrywise) product of the matrices H_1 and H_2; via the Fourier transform, the Laurent operator G has symbol J which is the convolution of $|h_2(-x)|^2$ and $|h_1(x)|^2$. Therefore, the matrix G is pavable by the observation at the beginning of this section.

Since $J \in \mathcal{P}_{\mathcal{L}}$, by Item 2 of Theorem 1, $J \in \mathcal{F}$. This computation shows that through the Zak transform, the set $\{e^{2\pi imx}J(x) : m \in \mathbb{Z}\}$ is unitarily equivalent to $\{e^{2\pi inx}g(x-n) : n \in \mathbb{Z}\}$, therefore the latter also satisfies the Feichtinger Conjecture. □

References

[1] J. Anderson, *Extreme points in sets of positive linear maps on* $\mathcal{B}(\mathcal{H})$, J. Funct. Anal. **31** (1979), no. 2, 195–217.

[2] M. Bownik and D. Speegle, *The Feichtinger conjecture for wavelet frames, Gabor frames and frames of translates*, Canad. J. Math. **58** (2006), no. 6, 1121–1143.

[3] J.T. Burnham, *Closed ideals in subalgebras of Banach algebras. I*, Proc. Amer. Math. Soc. **32** (1972), 551–555.

[4] P. Casazza, M. Fickus, J. Tremain, and E. Weber, *The Kadison-Singer problem in mathematics and engineering: a detailed account*, Operator theory, operator algebras, and applications, Contemp. Math., vol. 414, Amer. Math. Soc., Providence, RI, 2006, pp. 299–355.

[5] P. Casazza and J. Tremain, *The Kadison-Singer problem in mathematics and engineering*, Proc. Natl. Acad. Sci. USA **103** (2006), no. 7, 2032–2039 (electronic).

[6] P.G. Casazza and E. Weber, *The Kadison-Singer problem and the uncertainty principle*, Proc. Amer. Math. Soc. **136** (2008), no. 12, 4235–4243.

[7] O. Christensen, *A Paley-Wiener theorem for frames*, Proc. Amer. Math. Soc. **123** (1995), no. 7, 2199–2201.

[8] ______, *An introduction to frames and Riesz bases*, Applied and Numerical Harmonic Analysis, Birkhäuser Boston Inc., Boston, MA, 2003.

[9] H. Halpern, V. Kaftal, and G. Weiss, *Matrix pavings and Laurent operators*, J. Operator Theory **16** (1986), no. 2, 355–374.

[10] D. Han, K. Kornelson, D. Larson, and E. Weber, *Frames for undergraduates*, Student Mathematical Library, vol. 40, American Mathematical Society, Providence, RI, 2007.

[11] R. Kadison and I. Singer, *Extensions of pure states*, American J. Math. **81** (1959), 547–564.

[12] R. Paley and N. Wiener, *Fourier transforms in the complex domain*, American Mathematical Society Colloquium Publications, vol. 19, American Mathematical Society, Providence, RI, 1987, Reprint of the 1934 original.

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Received: February 19, 2009
Accepted: March 18, 2009

Operator Theory:
Advances and Applications, Vol. 202, 579–600

Dominating the Commutator

David Wenzel

Abstract. The values of the smallest possible constant C in the inequality $\|XY - YX\| \leq C\|X\|\|Y\|$ on the space of real or complex $n \times n$-matrices are investigated for different norms.

Mathematics Subject Classification (2000). Primary: 15A45; Secondary: 15A69.

Keywords. Commutator, operator norm, Schatten norm, inequality.

1. Introduction

"We wrote several papers about it ... Well, that's what mathematicians do."

Hans Schneider

In [1] we started a topic and asked how big the quotient

$$\frac{\|XY - YX\|_{\mathsf{F}}}{\|X\|_{\mathsf{F}}\|Y\|_{\mathsf{F}}}$$

can be. Here, X and Y are finite matrices with real or complex entries and $\|\cdot\|_{\mathsf{F}}$ is the Hilbert-Schmidt norm, which is also called the Frobenius or Euclidean norm. We showed that the quotient is typically very small in higher dimensions. One of the consequences of this observation is that it's of no use to design a computer test for commutativity of large matrices.

Based on results for restrictions of the quotient to special classes of matrices we also conjectured that it does not exceed $\sqrt{2}$, a bound that is better than the trivial bound 2. Later on, the validity of the bound $\sqrt{2}$ was extended to other classes by Lajos László [4], Zhiqin Lu [5], Seak-Weng Vong and Xiao-Qing Jin [6] and finally proven for all complex $n \times n$-matrices in [2]. In the last paper we also investigated the problem of determining

$$C := \sup_{X,Y \neq 0} \frac{\|XY - YX\|}{\|X\|\|Y\|} \tag{1}$$

for other unitarily invariant norms $\|\cdot\|$.

Communicated by I.M. Spitkovsky.

This constant may be interpreted as a generalization of operator norms to the bilinear function that maps matrices X, Y to their commutator. Obviously, the problem is equivalent to the determination of

$$C = \inf\{c > 0 \ : \ \|XY - YX\| \leq c\|X\|\|Y\| \quad \forall X, Y\}.$$

We also studied pairs of matrices attaining the supremum.

The present paper aims at extending the investigation to other norms and proving some results that were presented at the 2008 IWOTA but not yet published or were subject to speculation at that time. To emphasize the connection to that event and our talk about this paper's topic given there, each section will be accompanied by a quote catching memorable impressions.

We will bound the general constant C and determine it exactly for the operator, vector and Schatten p norms of matrices, except for one situation, in which we obtain only very tight estimates. Furthermore, we give criteria for the case of equality between the quotient of a particular pair of matrices and the supremum.

As usual, $\mathbb{K}$ stands for either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$, $\mathbb{K}^n$ denotes the space of vectors with n entries in $\mathbb{K}$ and $M_n(\mathbb{K})$ represents the set of all $n \times n$-matrices over $\mathbb{K}$. We will use the Lie bracket notation $[X, Y] := XY - YX$ as an abbreviation and write C instead of the supremum (1) for simplicity, often with a subscript indicating the utilized norm.

2. Appearances and other deceits

"To every problem in mathematics there is a solution that is simple ... elegant ... and wrong."

Michael Jury adopting a saying of Henry Louis Mencken

Obviously, many people already noticed the above fact and undoubtly it should be of particular relevance to any mathematician. Don't panic, we do not intend to revoke anything of the previous publications to that topic. However, we will use the opportunity to demonstrate the complexity of the original problem and to show briefly some intuitional ideas that met our way but have been unwilling to work. Nevertheless, they carry some interesting aspects, of which some will reappear throughout this paper.

School's method.
The pretty first thought is, of course, to look for the extremal points of the real-valued multivariate function $\|XY - YX\|_{\mathsf{F}}$ under the restrictions $\|X\|_{\mathsf{F}} = 1$ and $\|Y\|_{\mathsf{F}} = 1$. Assume X and Y to be real matrices. Then this can be done with help of Lagrange's method by determining the stationary points of

$$\|XY - YX\|_{\mathsf{F}}^2 + \lambda(1 - \|X\|_{\mathsf{F}}^2) + \mu(1 - \|Y\|_{\mathsf{F}}^2).$$

Clearly, taking the square of the functions does not alter the maximal points. Forming all partial derivatives and putting them zero yields a system of equations. Choosing an adequate depiction, we obtain

$$\lambda = \mu = \|XY - YX\|_{\mathsf{F}}^2,$$

$$\lambda X = ZY^* - Y^*Z, \quad \mu Y = X^*Z - ZX^*$$

as a characterization of local extrema, where $Z := XY - YX$ and X^* is the adjoint of X. Sadly, we do not see a chance to solve that system. But we get an interesting property that links extremal points X and Y with their commutator Z in a kind of circle of Lie bracket operations.

A computational attempt.
Forming the commutator

$$(X, Y) \mapsto XY - YX$$

is a bilinear operation

$$M_n(\mathbb{K}) \times M_n(\mathbb{K}) \to M_n(\mathbb{K})$$

with respect to the variables X and Y. But, since every entry of $XY - YX$ is a linear combination of the numbers $x_{ij}y_{kl}$, i.e.,

$$(XY - YX)_{pq} = \sum_{i,j,k,l} \alpha_{ijkl}^{(pq)} x_{ij} y_{kl}$$

with $\alpha_{ijkl}^{(pq)} \in \{-1, 0, 1\}$, passage to the commutator can also be regarded as a linear map

$$X \otimes Y \mapsto XY - YX$$

acting on the tensor product (or Kronecker product) of X and Y. By writing

$$Z = \left(z_{(i-1)n+k,(j-1)n+l}\right)_{ij,kl} \mapsto \left(\sum_{i,j,k,l} \alpha_{ijkl}^{(pq)} z_{(i-1)n+k,(j-1)n+l}\right)_{pq}$$

we may simply regard this as a linear operator

$$M_{n^2}(\mathbb{K}) \to M_n(\mathbb{K})$$

on the whole space of $n^2 \times n^2$-matrices. In this manner the constant C_{F} (in the case of the Hilbert-Schmidt norm) can be estimated by calculating the spectral norm of a matrix within the precision of the computer and the algorithm.

To do so, we transform matrices from $M_n(\mathbb{K})$ to vectors in $\mathbb{K}^{n^2}$ by row stacking, which in the case $n = 2$ is

$$\operatorname{vec}\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} = \begin{pmatrix} z_{11} & z_{12} & z_{21} & z_{22} \end{pmatrix}^\top.$$

Then, the linear map can be represented by the matrix K in the equality

$$K \operatorname{vec}(X \otimes Y) = \operatorname{vec}(XY - YX).$$

In the two-dimensional case K is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & +1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Sadly, this matrix already possesses norm 2. For larger dimensions n the norm turns out to be even bigger with a value of $\sqrt{2n}$. This is a sign that the tensor product structure of the argument vectors is essential for restricting the quotient (1).

Since actually every entry of $XY - YX$ is a linear combination of the terms $x_{ij}y_{kl} - y_{ij}x_{kl}$, we may even regard the commutator as a linear map defined on $X \otimes Y - Y \otimes X$ instead of $X \otimes Y$. This attempt is not really more effective. The induced matrix admits spectral norm $\sqrt{n}$. This is at least a proof for $n = 2$, with an idea that is correlated to the proof of $C_{\mathsf{F}} = \sqrt{2}$ for 2×2-matrices given in [1] as Theorem 4.2.

Nice try – Bad luck.
An excellent idea is using the unitary invariance of the Hilbert-Schmidt norm. As in the first step of the proof given in [2], let $X = USV$ be the singular value decomposition of X with $S = \operatorname{diag}(s_1, \ldots, s_n)$. Then, putting $B = VYV^*$ and $D = U^*YU$ yields

$$\begin{aligned} &\|XY - YX\|_{\mathsf{F}}^2 = \|SB - DS\|_{\mathsf{F}}^2 \\ &\leq \sum_{j \neq k} \left(s_j^2 |b_{jk}|^2 + s_k^2 |b_{jk}|^2 + s_j^2 |d_{jk}|^2 + s_k^2 |d_{jk}|^2 \right) + \sum_{j=1}^{n} s_j^2 |b_{jj} - d_{jj}|^2. \end{aligned}$$

If we could prove

$$|b_{jj} - d_{jj}|^2 \leq \sum_{k=1}^{n} \left(|b_{kk}|^2 + |d_{kk}|^2 \right) \tag{2}$$

for every j, then we would obtain

$$\|SB - DS\|_{\mathsf{F}}^2 \leq \|S\|_{\mathsf{F}}^2 (\|B\|_{\mathsf{F}}^2 + \|D\|_{\mathsf{F}}^2) = 2\|X\|_{\mathsf{F}}^2 \|Y\|_{\mathsf{F}}^2$$

due to unitary equivalencies and thus the desired bound. Clearly,

$$\sum_{j=1}^{n} |b_{jj} - d_{jj}|^2 \leq n \sum_{k=1}^{n} \left(|b_{kk}|^2 + |d_{kk}|^2 \right)$$

for $n \geq 2$. Hence, (2) is true for at least one j, which we assume to be 1. Similarly,

$$\sum_{j=2}^{n} |b_{jj} - d_{jj}|^2 \leq (n-1) \sum_{k=2}^{n} \left(|b_{kk}|^2 + |d_{kk}|^2 \right)$$

without restriction results in

$$|b_{22} - d_{22}|^2 \leq \sum_{k=2}^{n} \left(|b_{kk}|^2 + |d_{kk}|^2 \right).$$

Repeating this procedure inductively, we see (2) to be valid for all but one j. Note that this statement is true for arbitrary matrices B and D. However, the hope to force validity on the last index whenever B and D are unitarily equivalent dashes.

False friends.
It is known that the Hilbert-Schmidt norm and the matrix and tensor products are compatible in the following sense:

$$\|XY\|_{\mathsf{F}} \leq \|X\|_{\mathsf{F}}\|Y\|_{\mathsf{F}} = \|X \otimes Y\|_{\mathsf{F}}. \tag{3}$$

This can be read as a monotonicity between the two products. More precisely, replacing the matrix product with the tensor product does not reduce the norm. Now, one can hope that this property extends to

$$\|XY - YX\|_{\mathsf{F}} \leq \|X \otimes Y - Y \otimes X\|_{\mathsf{F}} \tag{4}$$

as well. The last inequality is of special interest, since then the estimate

$$\|X \otimes Y - Y \otimes X\|_{\mathsf{F}}^2 = 2\|X\|_{\mathsf{F}}^2\|Y\|_{\mathsf{F}}^2 - 2|\mathrm{tr}\,(Y^*X)|^2 \leq 2\|X\|_{\mathsf{F}}^2\|Y\|_{\mathsf{F}}^2$$

would lead to the desired $C_{\mathsf{F}} \leq \sqrt{2}$. Here, the trace $\mathrm{tr}\,(Y^*X)$ denotes the Hilbert-Schmidt inner product of X and Y. Actually, as shown by Theorem 3.1 in [2], $C_{\mathsf{F}} = \sqrt{2}$ implies the stronger inequality (4). However, there can be no way to shift the property (3) to (4) by a direct argumentation based on the respective spaces of matrices. This can be seen by the fact that inequality (4) with $-$ replaced by $+$ is not true in general (see Remark 3.3 of [2]).

3. Bounding the problem

"This is our definition ... And it is a good definition."

Jürgen Leiterer

Since in finite dimensions all norms are equivalent, the quality in the behaviour of the quotient $\frac{\|XY-YX\|}{\|X\|\|Y\|}$ is always similar. In particular, the overwhelming majority of matrix pairs concentrates near commutativity with growing dimension. Nevertheless, the quantity and especially the supremum of the quotient may be different. We've already seen in [2] that $C = \sqrt{2}$ for the Hilbert-Schmidt norm and $C = 2$ for all Ky Fan norms. This section is devoted to the question on what values the constant C may attain in general.

On the one hand, the answer is quite simple. The following arguments show that in principle every value $C \in (0, \infty)$ may be achieved. If $\|\cdot\|$ is an arbitrary norm and $\alpha > 0$ some positive number, then $\|\cdot\|_\alpha := \alpha\|\cdot\|$ is a norm, too. Inserting the last definition in the quotient (1), one obtains

$$C_\alpha = \frac{C}{\alpha}.$$

So, just by scaling of, e.g., the Hilbert-Schmidt norm, C can take an arbitrary value.

On the other hand, a more refined look into this topic is necessary. In Proposition 5.1 of [2] we already observed that for all unitarily invariant norms C cannot be lower than $\sqrt{2}$. The following result extends this lower bound by weakening the assumptions to a very reasonable scaling condition on the elementary matrices $E_{jk} = e_j \otimes e_k^* \in M_n(\mathbb{K})$.

Proposition 1. *Suppose $\|E_{jk}\| = 1$ and $\|E_{kj}\| \le 1$ for some $j \neq k$. Then $C \ge \sqrt{2}$.*

Proof. Think of the entries of the following 2×2-matrices placed in the positions a_{jj}, a_{jk}, a_{kj} and a_{kk} of $n \times n$-matrices that are zero elsewhere. We consider the two examples

$$\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5}$$

and

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}. \tag{6}$$

The supremum has to exceed both of the corresponding quotients. Hence, defining $\mu := \left\| \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\|$ we get

$$C \ge \max\left\{ \frac{\mu}{\|E_{kj}\|}, \frac{2}{\mu} \right\} \ge \max\left\{ \mu, \frac{2}{\mu} \right\}$$

and clearly, both numbers μ and $\frac{2}{\mu}$ cannot be strictly smaller than $\sqrt{2}$ simultaneously. □

For any norm an upper bound to C can be obtained by plain usage of the triangle inequality:

$$\|XY - YX\| \le \|XY\| + \|YX\|.$$

This inequality can be weakened to

$$\|XY - YX\| \le 2\|X\|\|Y\|,$$

giving $C \le 2$ whenever the norm $\|\cdot\|$ is submultiplicative, that is

$$\|XY\| \le \|X\|\|Y\| \quad \forall X, Y \in M_n(\mathbb{K}).$$

Since not every norm on $M_n(\mathbb{K})$ is submultiplicative, the general result reads as follows.

Lemma 2. *If $\beta > 0$ and $\|XY\| \le \beta\|X\|\|Y\|$ for all $X, Y \in M_n(\mathbb{K})$ then $C \le 2\beta$.*

Keep in mind that with Proposition 1 the special property of minimality of the Hilbert-Schmidt norm amongst all unitarily invariant norms shown in [2] is extended to a special property amongst basically all norms. We also want to remark that the scaling condition of Proposition 1 turns the question asked at the beginning of this section into a well-posed question.

4. Fundamental examples

"I'm very fascinated by Matlab ... It's much simpler than thinking."

Mr. Linear Algebra (real name known to the intended audience)

We want to study the problem in greater depth and will determine the constant C for three special classes of norms. As seen in the proof of Proposition 1, specific examples may unveil lots of information. Be aware that Matlab made the hunt for appropriate matrices a lot easier. Also note that not all members of the classes may be handled by the means presented here. The more delicate cases will be discussed in the next section.

Example 1. (*Operator p norms*)
In this example let $\|\cdot\|_p$ be one of the matrix norms

$$\sup_{v\neq 0}\frac{\sqrt[p]{\sum_j |(Av)_j|^p}}{\sqrt[p]{\sum_j |v_j|^p}} \quad \text{for } p\in[1,\infty) \quad \text{or} \quad \sup_{v\neq 0}\frac{\max_j |(Av)_j|}{\max_j |v_j|} \quad \text{for } p=\infty.$$

Since all of these norms are submultiplicative, Lemma 2 yields $C_p \leq 2$. Now consider matrices X and Y with 2×2-blocks as in (6) in the upper left corners and 0 elsewhere. By generating the associated quotient we obtain

$$C_p \geq \frac{\|XY-YX\|_p}{\|X\|_p\|Y\|_p} = 2.$$

Together both inequalities result in $C_p = 2$ for all operator p norms. We remark that this result easily extends to operator norms based on other vector norms with a few restrictions such as symmetry in the entries and permutational invariance.

Example 2. (*Vector p norms*)
Another type of norms on matrices is defined entry-wise by one of the rules

$$\sqrt[p]{\sum_{j,k}|a_{jk}|^p} \quad \text{for } p\in[1,\infty) \quad \text{or} \quad \max_{j,k}|a_{jk}| \quad \text{for } p=\infty.$$

For $p=1$ we again have submultiplicativity and hence $C_1 \leq 2$. Moreover, the pair of matrices in (5) gives equality. Again we think of these 2×2-matrices as being extended to $n\times n$-matrices with zeros.

The case $p=\infty$ is only a little more trickier than $p=1$. We do not have submultiplicativity here, but instead

$$\|XY\|_\infty \leq n\|X\|_\infty\|Y\|_\infty$$

is valid. So, Lemma 2 yields $C_\infty \leq 2n$. The example of $n \times n$-matrices

$$\left[\begin{pmatrix} -1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \\ 1 & -1 & \cdots & -1 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 0 & \cdots & 0 \\ 2n & 0 & \cdots & 0 \end{pmatrix}$$

then gives $C_\infty = 2n$.

With $p = 2$ we get the Hilbert-Schmidt norm, for which it is now known that $C_2 = \sqrt{2}$. The constants for other values of p cannot simply be tackled in such a way and will be deduced in the next section. Although for $p > 2$ similar attempts as for $p = \infty$ could perhaps work by determining β for use with Lemma 2, there is no chance to do so for $p \in (1, 2)$, since $\beta = 1$ gives 2 as the best upper bound for C_p. Estimates of C_p obtained with the help of Matlab by examples – and hence also lower bounds to C_p – are seen in Figure 1.

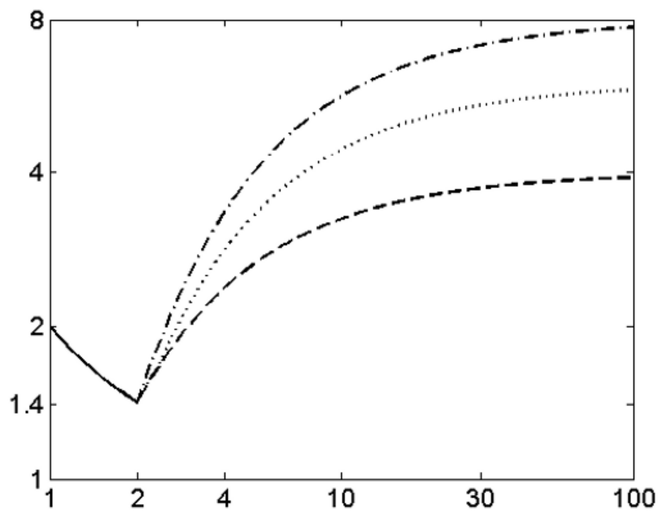

Figure 1. Value estimates for C_p of vector p norms.
The y-axis marks C_p in dependence of p (x-axis) in a doubly logarithmic scale for $1 \leq p \leq 2$ (solid line), $p \geq 2, n = 2$ (dashed line), $p \geq 2, n = 3$ (dotted line) and $p \geq 2, n = 4$ (dash-dotted line).

Example 3. (*Schatten p norms*)
A special class of unitarily invariant norms is given by the Schatten norms, defined via the matrix' singular values $s_1, \ldots, s_n$ by

$$\sqrt[p]{\sum_j s_j^p} \quad \text{for } p \in [1, \infty) \quad \text{or} \quad \max_j s_j \quad \text{for } p = \infty.$$

We can also write $\|X\|_p := \|\sigma(X)\|_p$ with the vector $\sigma(X) = (s_1, \ldots, s_n)$.

All of these norms are submultiplicative and therefore 2 is an upper bound to all the constants C_p. Again the cases $p = 1$ and $p = \infty$ can easily be handled with the examples (5) and (6), respectively, resulting in $C_1 = C_\infty = 2$.

The Schatten 2 norm is just the Hilbert-Schmidt norm and hence one has $C_2 = \sqrt{2}$. All the other cases are more complicated and will be subject to the following section. Estimates are shown in Figure 2.

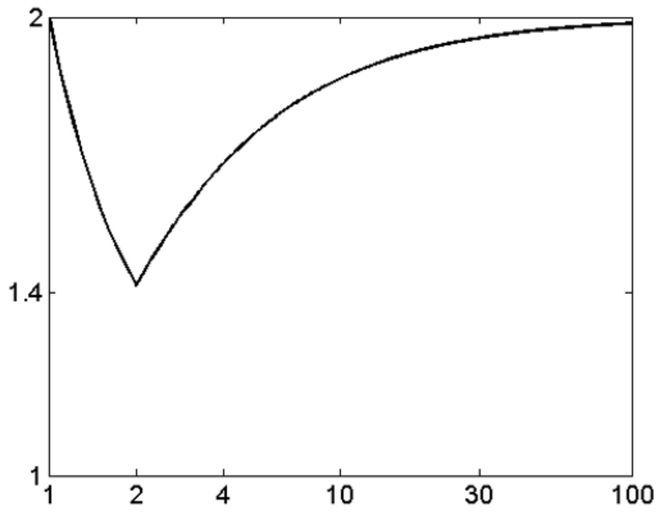

FIGURE 2. Value estimates for C_p of Schatten p norms.
The y-axis marks C_p in dependence of p (x-axis) in a logarithmic scale.

5. The constants for general p

"All I will do is computation ... And I like computation."

Paul Fuhrmann

In this section we will see that the determination of the constants C_p for the Schatten and vector norms turns out to be nothing but the result of a simple calculation. Of course, that process will be based on a deep result – the Riesz-Thorin interpolation theorem.

In a convenient formulation the statement is that if for a linear operator T and $1 \leq p_0 < p_1 \leq \infty$ there are $M_{p_0}, M_{p_1} > 0$ such that

$$\|Tf\|_{p_0} \leq M_{p_0}\|f\|_{p_0} \quad \text{and} \quad \|Tf\|_{p_1} \leq M_{p_1}\|f\|_{p_1} \tag{7}$$

for all arguments f, then for any $p \in [p_0, p_1]$ and every vector f the inequality

$$\|Tf\|_p \leq M_{p_0}^{1-\theta} M_{p_1}^{\theta}\|f\|_p, \quad \text{with } \theta \in [0,1] \text{ defined by } \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

holds. The theorem can be shown to be valid even in a very general infinite-dimensional setting on measurable spaces as in [3]. In particular, f may be taken from a subspace only, the proof can be ported to the Schatten class or the norm indices of initial and target space may be different. The result has to be read uniformly, i.e., bounds over all f can be interpolated. By this, the hardly accessible operator p norms of the matrix T can be estimated by the values of the easy computable 1,2 and ∞ operator norms.

Furthermore, we need a multiplicativity property with respect to the tensor product that we already hinted in Section 2 for the Hilbert-Schmidt norm. For arbitrary matrices $X, Y \in M_n(\mathbb{K})$ one has

$$\|X \otimes Y\|_p = \|X\|_p \|Y\|_p \tag{8}$$

for any Schatten or vector p norm. This can be shown to be true for the vector norms by direct calculation. As for the Schatten norms, it suffices to check that the singular values of $X \otimes Y$ are given by all of the possible products of a singular value of X with a singular value of Y, i.e., $\sigma(X \otimes Y) = \sigma(X) \otimes \sigma(Y)$ if we ignore the order of the entries.

We are now in a position to get the bounds we encountered in Figures 1 and 2. As already noted in Section 2, a linear map K is induced by the commutator on all tensor products,

$$X \otimes Y \mapsto XY - YX,$$

and subsequently extended to an operator on all $n^2 \times n^2$-matrices. So it should be pretty clear that we want to utilize the Riesz-Thorin theorem. But we are in need to explain why we may write $X \otimes Y$ instead of f. This is indeed necessary, since the set of all tensor products is no subspace of $M_{n^2}(\mathbb{K})$.

Theorem 3. *Let $1 \leq p_0 < p_1 \leq \infty$ and assume $X, Y \in M_n(\mathbb{C})$. Let T be a linear operator $M_{n^2}(\mathbb{C}) \to M_n(\mathbb{C})$. Suppose there are $M_{p_0}, M_{p_1} > 0$ such that*

$$\|T(X \otimes Y)\|_\rho \leq M_\rho \|X \otimes Y\|_\rho \quad \forall X, Y \quad (\rho \in \{p_0, p_1\}).$$

Then for any $p \in [p_0, p_1]$ the inequality

$$\|T(X \otimes Y)\|_p \leq M_{p_0}^{1-\theta} M_{p_1}^{\theta} \|X \otimes Y\|_p$$

holds for every pair X and Y, where $\theta \in [0, 1]$ is given by

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Proof. We refer to the proof of the Riesz-Thorin theorem given in [3] as Theorem 1.3.4 and show that it can be modified in the required manner. The statement can be founded on complex analysis. The conditions (7) are assumed for all simple functions and the interpolated inequality is proven for any fixed simple f in the vector space of the arguments. However, in the finite-dimensional case, actually all vectors are simple functions. So, the proof covers indeed all possible arguments without the necessity of subsequent extension by a density argumentation. In our situation the counting measure is used.

An analytic function $F(z)$ is defined on the strip $0 \leq \operatorname{Re} z \leq 1$ with help of other simple functions f_z on which the linear operator T is applied – see (1.3.17) of [3]. The assertions, more precisely the inequalities (7) for the interpolation bases p_0 and p_1, or (1.3.12) in [3], are used within the proof only in (1.3.18) and (1.3.19) with the new simple function f_z. There,

$$\|T(f_z)\|_{p_0} \leq M_{p_0} \|f_z\|_{p_0} \quad (\operatorname{Re} z = 0) \quad \text{and} \quad \|T(f_z)\|_{p_1} \leq M_{p_1} \|f_z\|_{p_1} \quad (\operatorname{Re} z = 1)$$

are inferred. The proof then concludes with the application of Hadamard's three lines theorem to interpolate the obtained bounds for $|F(z)|$ that are independent of $\operatorname{Im} z$, since

$$\|f_z\|_{p_0}^{p_0} = \|f\|_p^p \quad (\operatorname{Re} z = 0) \quad \text{and} \quad \|f_z\|_{p_1}^{p_1} = \|f\|_p^p \quad (\operatorname{Re} z = 1)$$

are true by the way f is transformed into f_z in (1.3.17).

But, we are only interested into interpolating bounds for very special arguments $f = X \otimes Y$. In that case, f_z is for all z of the same type as f, i.e., a tensor product, too. Indeed, we have

$$(X \otimes Y)_z = X_z \otimes Y_z \tag{9}$$

by virtue of the exponential representation of complex numbers,

$$x_{ij} y_{kl} = r_{ij} e^{\alpha_{ij}} \cdot s_{kl} e^{\beta_{kl}} = (r_{ij} s_{kl}) e^{\alpha_{ij} + \beta_{kl}},$$

and the rules in $\mathbb{C}$, which imply for the definition of f_z:

$$(r_{ij} s_{kl})^{P(z)} e^{\alpha_{ij} + \beta_{kl}} = r_{ij}^{P(z)} e^{\alpha_{ij}} \cdot s_{kl}^{P(z)} e^{\beta_{kl}}.$$

So, the sharper bounds that are valid on all tensor products may be applied in (1.3.18) and (1.3.19) and are hence available thereafter. $\square$

Theorem 4. *Let $p \in (1, \infty)$ and q be the corresponding dual value, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Then for the Schatten p norms on $M_n(\mathbb{K})$ we have*

$$C_p = 2^{1/\min(p,q)}.$$

Proof. First assume $p \in (1, 2)$. Example 3 and (8) then ensure

$$\|K(X \otimes Y)\|_\rho = \|XY - YX\|_\rho \leq C_\rho \|X\|_\rho \|Y\|_\rho = C_\rho \|X \otimes Y\|_\rho$$

for $\rho = 1$ and $\rho = 2$ with $C_1 = 2$ and $C_2 = \sqrt{2}$. Now, by interpolation on tensor products with help of Theorem 3 we obtain for all p in between

$$\|XY - YX\|_p = \|K(X \otimes Y)\|_p \leq C_p \|X \otimes Y\|_p = C_p \|X\|_p \|Y\|_p$$

with $\theta = 2 - \frac{2}{p}$ and

$$C_p \leq 2^{2/p-1} \sqrt{2}^{2-2/p} = 2^{1/p}.$$

Similarly, in the case $p \in (2, \infty)$ we have $C_2 = \sqrt{2}$, $C_\infty = 2$ and with $\theta = 1 - \frac{2}{p}$ finally

$$C_p \leq \sqrt{2}^{2/p} 2^{1-2/p} = 2^{1/q}.$$

So far, we only obtained an upper bound to the actual values C_p, but examples for realizing these bounds were already given in [2]. See also (5) and (6) again. These are even the specific matrix pairs that resulted into the picture of Figure 2. □

We already explained in [2] for the Hilbert-Schmidt norm that such a result can be extended to the infinite-dimensional setting of the Schatten norm, since a limiting process $n \to \infty$ reveals Theorem 4 to be true also in the countable case.

Theorem 5. *Let $p \in (1, \infty)$ and q its dual. Then for the vector p norm on $M_n(\mathbb{K})$ we have*

$$C_p = \begin{cases} 2^{1/p} & \text{for} \quad p \in (1, 2] \\ 2^{1/q} n^{1-2/p} & \text{for} \quad p \in (2, \infty),\ n \text{ even.} \end{cases} \tag{10}$$

For odd dimensions n and $p \in (2, \infty)$ we have

$$C_p \leq 2^{1/q} n^{1-2/p} \tag{11}$$

and

$$C_p \geq \begin{cases} 2^{1/q} (n-1)^{1-2/p} & \text{for} \quad p \in (2, P] \quad (12) \\ 2^{1/q} n^{1-4/p} (n^2 - 1)^{1/p} & \text{for} \quad p \in [P, \infty). \quad (13) \end{cases}$$

with

$$P := \frac{\ln\left((n+1)(n-1)^3 n^{-4}\right)}{\ln\left((n-1)n^{-1}\right)}.$$

Proof. The proof is similar to the one of Theorem 4, but based on the values computed in Example 2. For $p \in (1,2)$ we get $C_p \leq 2^{1/p}$ again. With an eye on (5) it gets clear that this upper bound can be attained.

Now, for $p \in (2,\infty)$ the different value $C_\infty = 2n$ results in

$$C_p \leq \sqrt{2}^{2/p}(2n)^{1-2/p} = 2^{1/q} n^{1-2/p},$$

which is (11). If we take a look at

$$X = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ -1 & -1 & \cdots & -1 \\ 1 & 1 & \cdots & 1 \\ -1 & -1 & \cdots & -1 \\ \vdots & \vdots & & \vdots \end{pmatrix} \in M_n(\mathbb{K}),$$

the commutator $[X, -X^*]$ becomes the chessboard matrix

$$[X, -X^*] = \begin{pmatrix} 0 & 2n & 0 & \cdots \\ 2n & 0 & 2n & \cdots \\ 0 & 2n & 0 & \cdots \\ 2n & 0 & 2n & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix},$$

yielding (10) in even dimensions. But whenever n is odd the quotient $\frac{\|[X,-X^*]\|_p}{\|X\|_p^2}$ takes the value in (13) which is only a lower bound. That key example was investigated in Example 2.4 of [1].

For p close to 2, extending such a chessboard example of dimension $n-1$ with zeros gives a better lower bound. By this (12) follows from (10) immediately. The point P of equality between (12) and (13) is the result of a simple calculation. We have

$$2^{1/q}(n-1)^{1-2/p} = 2^{1/q} n^{1-4/p}(n^2-1)^{1/p}$$

if and only if

$$\frac{n-1}{n} = \frac{(n+1)^{1/p}(n-1)^{3/p}}{n^{4/p}}.$$

Taking the logarithm on both sides yields the stated formula for P. □

Remarks. We emphasize that the proof of the Riesz-Thorin theorem, which we modified to show Theorem 3, is the complex version. Thus, a priori, the proven constants apply to $\mathbb{K} = \mathbb{C}$ only. However, we always have $C_\mathbb{R} \leq C_\mathbb{C}$ and all the examples used to demonstrate that the bounds in Theorems 4 and 5 can be attained are real.

Theorem 3 is also true when the norm indices of the initial and the target spaces do not coincide. In a similar fashion, the tensor product structure is preserved even if X and Y have different sizes or are not square. Note that in the infinite-dimensional setting a similar result is not true in general and can only be given for simple functions.

In contrast to even dimensions the upper bound (11) in Theorem 4 seems unreachable for odd n. In fact, it is heavily indicated that the lower bounds (12) and (13) also represent C_p, the maximal value of the quotient. Figure 3 gives an overview of that really odd thing for $n = 3$ and $n = 5$. Remember that (12) is just (10) with n replaced by $n-1$ and so the values C_p in even dimensions $n = 2$ and $n = 4$ are also pictured. Although the actual value of C_p is not known, the pictures suggest that the estimates are very tight. Indeed, we have $P \to 2$ as $n \to \infty$ and the maximal distance between the lower and upper bounds tends to zero quite fast as $n \to \infty$. Moreover, for any fixed n, the distance of (11) and (13) annihilates rapidly as $p \to \infty$. Figure 4 is intended to give an impression of that behaviour by numerical examples.

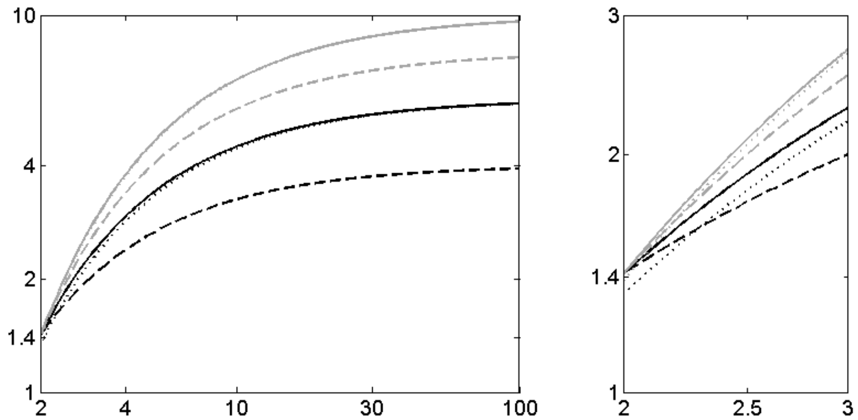

FIGURE 3. Bounds for the vector p norm ($p \geq 2$) with odd n.

Both pictures show the bounds for dimensions $n = 3$ (dark color) and $n = 5$ (light color): upper bound (11) (solid line) and lower bounds (12) (dashed line) and (13) (dotted line).

n	3	5	9	49	99
P	2.2905	2.1829	2.1055	2.0202	2.01
d	0.0851	0.0309	0.0095	0.0003	0.0001

FIGURE 4. Swapping point P of lower bounds (12) and (13) for C_p of vector p norms and the maximal distance d between upper and lower bounds in different dimensions n.

6. Pinpoints

"Conclusion: Toeplitz matrices are the center of the universe."

Albrecht Böttcher

Well, here we are in a parallel universe since in our world the focus is on the Hilbert-Schmidt norm. We already investigated pairs of maximal matrices for this norm, that is, non-zero matrices realizing the supremum (1). We have shown in [2] that at the cross of the Schatten and vector class – the case $p = 2$ – a pair can only be maximal if several orthogonality conditions hold, starting with

$$\operatorname{tr} X = 0, \quad \operatorname{tr} Y = 0 \quad \text{and} \quad \operatorname{tr}(Y^* X) = 0.$$

Sadly these restrictions are not yet sufficient. Now we want to study the problem of finding conditions to X and Y ensuring $\|XY - YX\|_p = C_p \|X\|_p \|Y\|_p$ for the Schatten and vector p norms as well.

First we take a closer look at conditions for 1-maximality.

Proposition 6. *Suppose X and Y to be non-zero matrices in $M_n(\mathbb{K})$. Then (X, Y) is a Schatten 1-maximal pair if and only if*

$$\operatorname{rank} X = 1, \quad \operatorname{tr} X = 0 \quad \textit{and} \quad Y = \alpha X^*$$

for some $\alpha \in \mathbb{K} \backslash \{0\}$.

Proof. Known properties of the Schatten 1 norm yield

$$\|XY - YX\|_1 \le \|XY\|_1 + \|YX\|_1 \le \begin{cases} 2\|X\|_\infty \|Y\|_1 \\ 2\|X\|_1 \|Y\|_\infty \end{cases} \le 2\|X\|_1 \|Y\|_1.$$

Hence, for a 1-maximal pair we need to have $\|X\|_\infty = \|X\|_1$ and $\|Y\|_\infty = \|Y\|_1$ which means that X and Y must be matrices of rank one.

Assume the pair (X, Y) to be 2-maximal, additionally. Then Proposition 4.5 of [2] implies that

$$\operatorname{tr} X = 0 \quad \text{and} \quad Y = \alpha X^*.$$

Conversely, matrices with the claimed three properties fulfil

$$X = \|X\|_{\mathsf{F}} e_1 e_2^*, \quad Y = \alpha \|X\|_{\mathsf{F}} e_1 e_2^*,$$

where e_1 and e_2 are orthogonal unit vectors. We obtain

$$\begin{aligned} XY - YX &= \alpha \|X\|_{\mathsf{F}}^2 (e_1 e_1^* - e_2 e_2^*) \\ &= \begin{pmatrix} e_1 & e_2 & \cdots \end{pmatrix} \begin{pmatrix} \alpha\|X\|_{\mathsf{F}}^2 & 0 & 0 & \cdots \\ 0 & \alpha\|X\|_{\mathsf{F}}^2 & 0 & \\ 0 & 0 & 0 & \\ \vdots & & & \ddots \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \end{pmatrix}, \end{aligned}$$

yielding

$$\sigma(X) = (\|X\|_{\mathsf{F}}, 0, \ldots), \quad \sigma(Y) = (|\alpha|\|X\|_{\mathsf{F}}, 0, \ldots),$$
$$\sigma(XY - YX) = (|\alpha|\|X\|_{\mathsf{F}}^2, |\alpha|\|X\|_{\mathsf{F}}^2, 0, \ldots)$$

and thus a 1-maximal quotient.

To complete the proof we show that 1-maximal pairs of matrices are always 2-maximal. Since $\operatorname{rank} X = \operatorname{rank} Y = 1$, clearly $\operatorname{rank}(XY - YX) \le 2$. We may assume $\|X\|_1 = \|X\|_{\mathsf{F}} = 1 = \|Y\|_{\mathsf{F}} = \|Y\|_1$ and therefore the singular values t_i of $XY - YX$ fulfil without restriction

$$t_1 + t_2 = 2 \quad \text{and} \quad \sqrt{t_1^2 + t_2^2} \le \sqrt{2}.$$

Due to the norm inequality

$$\|t\|_1 \le \sqrt{2}\|t\|_2$$

on $\mathbb{R}^2$ the inequality above then automatically sharpens to

$$\sqrt{t_1^2 + t_2^2} = \sqrt{2},$$

which is the 2-maximality. □

For the vector norm we can formulate a similar, but even more restrictive result.

Proposition 7. *Suppose X and Y to be non-zero matrices in $M_n(\mathbb{K})$. Then (X, Y) is a vector 1-maximal pair if and only if*

$$X = \alpha E_{jk} \quad \textit{and} \quad Y = \beta E_{kj}$$

for some $j \ne k$ and numbers $\alpha, \beta \in \mathbb{K}\backslash\{0\}$.

Proof. The chain of inequalities

$$2 = \frac{\sum_{j,l} |\sum_k x_{jk}y_{kl} - y_{jk}x_{kl}|}{\sum_{j,k} |x_{jk}| \sum_{i,l} |y_{il}|} \le \frac{\sum_{j,k,l} |x_{jk}y_{kl}| + \sum_{j,k,l} |y_{jk}x_{kl}|}{\sum_{i,j,k,l} |x_{jk}y_{il}|} \le 2$$

implies that a pair can only be 1-maximal if

$$\sum_{j,k,l} |x_{jk}y_{kl}| = \sum_{j,k,l} |y_{jk}x_{kl}| = \sum_{i,j,k,l} |x_{jk}y_{il}|.$$

So, we need to have

$$x_{jk} \ne 0 \quad \Longrightarrow \quad y_{il} = 0 \quad \forall i \ne k, \forall l,$$
$$y_{jk} \ne 0 \quad \Longrightarrow \quad x_{il} = 0 \quad \forall i \ne k, \forall l.$$

Now let x_{jk} be a non-zero entry of X. The conditions above annihilate all rows in Y except for the kth. Suppose $y_{kl} \ne 0$, then all rows of X except for the lth would be eliminated. This restricts y_{kj} to be the only non-zero entry of Y. Swapping the roles of X and Y yields the same for x_{jk}. Obviously, matrix pairs of the given type are 1-maximal whenever $j \ne k$. □

Remark. Propositions 6 and 7 can be read in quite the same way:
A non-zero pair of matrices (X, Y) *is* 1*-maximal if and only if*

1) $Y = \alpha X^*$ *for some* $\alpha \in \mathbb{K}\backslash\{0\}$,
2) $\operatorname{tr} X = 0$ *and*
3) X *has only one non-zero* $\begin{cases} \textit{singular value (Schatten norm)} \\ \textit{entry (vector norm).} \end{cases}$

It is easy to check that pairs of matrices which meet the criteria stated in Propositions 6 and 7 are p-maximal even for all $p \in [1, 2]$ with respect to the Schatten or vector p norm, respectively. This observation gives hope to expand the necessity of these three conditions to all p between 1 and 2 by interpolation arguments.

Proposition 8. *Let* $p \in (1, 2)$ *and suppose* $X, Y \in M_n(\mathbb{K})$ *to be non-zero matrices.*

a) *Then* (X, Y) *is a Schatten* p*-maximal pair if and only if*

$$\operatorname{rank} X = 1, \quad \operatorname{tr} X = 0 \quad \textit{and} \quad Y = \alpha X^*$$

for some $\alpha \in \mathbb{K}\backslash\{0\}$.

b) *Then* (X, Y) *is a vector* p*-maximal pair if and only if*

$$X = \alpha E_{jk} \quad \textit{and} \quad Y = \beta E_{kj}$$

for some $j \neq k$ *and numbers* $\alpha, \beta \in \mathbb{K}\backslash\{0\}$.

Proof. It suffices to show that there can be no other p-maximal pairs than those determined by the stated structures. For both norm classes, the claim follows from an analysis based on the Riesz-Thorin theorem. So, we return to the end of the proof of Theorem 3. In the last step, Hadamard's three lines theorem went into action. This is a generalization of the well-known maximum principle for analytic functions to the infinite strip. Actually, the general result can be reduced to the original principle. Back to our situation, first observe that in the finite-dimensional case the supremum in (1) is actually a maximum. So, the task is well posed.

The function $F(z)$ (introduced to calculate the p norm with help of a functional in the proof [3]) is analytic. The same is true for the function

$$G(z) := \frac{F(z)}{B_0^{1-z} B_1^z}$$

with any fixed positive numbers B_0 and B_1. Here, B_0 and B_1 are the bounds for the interpolation bases p_0 and p_1 seen in (1.3.18) and (1.3.19). We have $|G(z)| \leq 1$ (as remarked in the proof of Lemma 1.3.5 of [3]) and also $|G(\theta)| = 1$ whenever f realizes the interpolated bound. Of course, the maximality of f is linked with the existence of an appropriate simple function g with $\|g\|_{p'} = 1$ (p' being the dual of p) that enables us to calculate the norm $\|T(f)\|_p = \int T(f) g d\nu$ as a scalar product.

The maximum principle now ensures that G is constant on any finite rectangle (as a truncation of the infinite strip), since θ is an interior point. In particular, we have $G(0) = G(\theta)$, yielding $|F(0)| = B_0$ and $\|T(f_0)\|_{p_0} = M_{p_0} \|f_0\|_{p_0}$ – see again (1.3.18) and keep in mind that $\|g_0\|_{p_0'} = 1$. Hence, we obtain that f_0 realizes the

p_0-bound. More precisely, if

$$f = \left(r_{ij} e^{\alpha_{ij}}\right)_{i,j}$$

is a p-maximal element $f = X \otimes Y$, then

$$f_0 = \left(r_{ij}^{P(0)} e^{\alpha_{ij}}\right)_{i,j} = \left(r_{ij}^{p} e^{\alpha_{ij}}\right)_{i,j}$$

is 1-maximal. By virtue of (9), (X_0, Y_0) is necessarily a 1-maximal pair. Then by Proposition 7 for the vector norm, X_0 and Y_0 have only one non-trivial entry and by Proposition 6 for the Schatten norm, both matrices have rank one. Clearly, these conditions carry over to the original matrices X and Y. □

Remarks. As the criteria for ∞-maximality are less restrictive, attempts for $p > 2$ are not very successful. A pair (X, Y) of non-zero matrices is Schatten ∞-maximal if and only if

$$\begin{aligned} \|XY\|_\infty &= \|X\|_\infty \|Y\|_\infty = \|YX\|_\infty, \\ \|XY - YX\|_\infty &= 2\|X\|_\infty \|Y\|_\infty. \end{aligned}$$

This result just reflects that for ∞-maximal pairs necessarily equality is given in the triangle inequality as well as in both usages of the submultiplicativity property.

Similarly, by usage of Cauchy's inequality one can verify that (X, Y) is a vector ∞-maximal pair if and only if after an appropriate scaling of X and Y there are $j \neq k$ and $|\alpha| = 1$ such that

$$\begin{aligned} &|x_{il}| \leq 1, |y_{il}| \leq 1 \quad \forall i, l \quad \text{and} \quad |x_{jl}| = 1 = |x_{lk}| \quad \forall l, \\ &y_{jl} = \alpha \overline{x_{lk}} \quad \forall l \quad \text{and} \quad y_{lk} = -\alpha \overline{x_{jl}} \quad \forall l. \end{aligned}$$

Notice that this requires that $x_{jj} = -x_{kk}$ and that this is basically the same statement as for the Schatten norm.

Note also that pairs involving matrices of rank 2 can only be p-maximal if $p \geq 2$. In fact, there are pairs of rank greater than 1 for that case: For every $s_2 \in [0, 1]$ the example

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ s_2 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 2 \\ -2s_2 & 0 \end{pmatrix}$$

is Schatten p-maximal.

The case $p = 1$ fits well into the scheme of $p < 2$. In contrast to this, $p = \infty$ breaks out of the possible patterns for $p > 2$. This gets clear with Example 2 as we have given an ∞-maximal pair that is not maximal for any other p. Also 2-maximality is more comprehensive. Indeed, a 2-maximal pair can also consist of two rank 1 matrices, since Schatten 1-maximal pairs are also 2-maximal. Furthermore, there are examples such as

$$\left[\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}\right] = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix}$$

that are p-maximal only for $p = 2$.

Finally note that the proofs given here are again complex. But, since $C_{\mathbb{R}}$ and $C_{\mathbb{C}}$ coincide, we encounter no problems in restating the real versions.

7. The border $\sqrt{2}$

"One way to get the inverse is to guess the inverse and then to prove it is right."

Harold Widom

In Theorem 5.4 of [2] we've proven that for 2×2-matrices the Hilbert-Schmidt norm is the only unitarily invariant norm realizing the lower bound $C = \sqrt{2}$. Naturally, the question arises whether there is a non-unitarily invariant norm with $C = \sqrt{2}$ in dimension $n = 2$. Inspired by the quote, we guess a norm and prove that it does the job.

Proposition 9. *Let* $\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| := \sqrt{a^2 + d^2} + \sqrt{b^2 + c^2}$. *Then* $C = \sqrt{2}$.

Proof. Writing

$$X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$$

and

$$XY - YX = \begin{pmatrix} x_2y_3 - x_3y_2 & x_1y_2 + x_2y_4 - x_2y_1 - x_4y_2 \\ x_3y_1 + x_4y_3 - x_1y_3 - x_3y_4 & x_3y_2 - x_2y_3 \end{pmatrix}$$

the inequality

$$\|XY - YX\| \leq \sqrt{2}\|X\|\|Y\|$$

is equivalent to

$$\begin{aligned} &\sqrt{2(x_2y_3 - x_3y_2)^2} + \\ &\sqrt{((x_1 - x_4)y_2 + x_2(y_4 - y_1))^2 + (x_3(y_1 - y_4) + (x_4 - x_1)y_3)^2} \\ &\leq \sqrt{2}\left(\sqrt{x_1^2 + x_4^2} + \sqrt{x_2^2 + x_3^2}\right)\left(\sqrt{y_1^2 + y_4^2} + \sqrt{y_2^2 + y_3^2}\right). \end{aligned}$$

The latter is true whenever

$$\begin{aligned} &\sqrt{((x_1 - x_4)y_2 + x_2(y_4 - y_1))^2 + (x_3(y_1 - y_4) + (x_4 - x_1)y_3)^2} \\ &\leq \sqrt{2}\left(\sqrt{x_1^2 + x_4^2}\sqrt{y_1^2 + y_4^2} + \sqrt{x_1^2 + x_4^2}\sqrt{y_2^2 + y_3^2} + \sqrt{x_2^2 + x_3^2}\sqrt{y_1^2 + y_4^2}\right) \end{aligned}$$

holds, which is a consequence of

$$\begin{aligned}&((x_1-x_4)y_2+x_2(y_4-y_1))^2+(x_3(y_1-y_4)+(x_4-x_1)y_3)^2\\&\le 2\Bigg((x_1^2+x_4^2)(y_1^2+y_4^2)+(x_1^2+x_4^2)(y_2^2+y_3^2)+(x_2^2+x_3^2)(y_1^2+y_4^2)\\&+2(y_1^2+y_4^2)\sqrt{x_1^2+x_4^2}\sqrt{x_2^2+x_3^2}+2(x_1^2+x_4^2)\sqrt{y_1^2+y_4^2}\sqrt{y_2^2+y_3^2}\\&+2\sqrt{x_1^2+x_4^2}\sqrt{x_2^2+x_3^2}\sqrt{y_1^2+y_4^2}\sqrt{y_2^2+y_3^2}\Bigg).\end{aligned}$$

We further strengthen this inequality to

$$\begin{aligned}&2(x_2y_2+x_3y_3)(x_1-x_4)(y_4-y_1)\le 2\Bigg((x_1^2+x_4^2)(y_1^2+y_4^2)\\&+2(y_1^2+y_4^2)\sqrt{x_1^2+x_4^2}\sqrt{x_2^2+x_3^2}+2(x_1^2+x_4^2)\sqrt{y_1^2+y_4^2}\sqrt{y_2^2+y_3^2}\\&+2\sqrt{x_1^2+x_4^2}\sqrt{x_2^2+x_3^2}\sqrt{y_1^2+y_4^2}\sqrt{y_2^2+y_3^2}\Bigg)\end{aligned}$$

and finally to

$$2(x_2y_2+x_3y_3)(x_1-x_4)(y_4-y_1)\le 4\sqrt{x_1^2+x_4^2}\sqrt{x_2^2+x_3^2}\sqrt{y_1^2+y_4^2}\sqrt{y_2^2+y_3^2},$$

which is obviously true by Cauchy's inequality. □

There may be many other norms for 2×2-matrices having $C=\sqrt{2}$. That problem seems to be a topic of its own.

8. Open questions

"There are several different methods to attack a problem.
Another point of view could make it very easy."

Vadim Olshevsky

Right before the end we will summarize some questions that are still unanswered.

Problem 1. In Theorem 3.1 of [2] one can see that the Hilbert-Schmidt norm ($p=2$) allows to deduce the inequality

$$\|XY-YX\|_p\le\|X\otimes Y-Y\otimes X\|_p \tag{14}$$

we already mentioned in Section 2. We also observed that this inequality is even sharper than the inequality

$$\|XY-YX\|_2\le\sqrt{2}\|X\|_2\|Y\|_2$$

since

$$\|X\otimes Y-Y\otimes X\|_2\le\sqrt{2}\|X\|_2\|Y\|_2$$

is true. Naturally, the question arises, whether a similar result can be given for the Schatten or vector p norms.

The answer is definitely no for $p \in (2, \infty]$ – for both types of norms. For this, again remember (6). We have

$$X \otimes Y - Y \otimes X = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and as a consequence (14) is true if

$$2 \leq \sqrt[p]{4}$$

holds for the vector norm, and

$$2 \leq \sqrt[p]{\sqrt{2}^p + \sqrt{2}^p}$$

for the Schatten norm. However, these inequalities fail to be valid whenever $p > 2$. Note that in case $p > 2$ the inequality can be true for certain pairs of matrices. For this you may consider (5). But the inequality is very likely true whenever $p \leq 2$.

Be aware that $p = 2$ is in general not the critical point for swapping the validity of (14) for a fixed pair (X, Y). By modifying the previous example to

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -d \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

with $d \in [0, 1]$ for the vector norms and $d \in [\sqrt{2} - 1, 1]$ for the Schatten norms one obtains that for any fixed $p_0 \in [2, \infty]$ there are matrices such that inequality (14) is true for all $p \leq p_0$ and wrong for all $p > p_0$. In the case $p_0 = \infty$ (14) is true for all p.

Problem 2. In regard of Section 6 we ask for characterizations of p-maximal pairs of matrices in the cases $p > 2$. Since Propositions 6 and 7 already found their application in the proof of Proposition 8, it would be especially interesting to have criteria for the ∞-maximality. Knowledge of these matrix pairs could also help to close the gap between odd and even dimensions for the vector norms. In this context P-maximality could give another exception beside the 2 and ∞-maximality and should be of particular interest, too.

Problem 3. Although Section 7 gives a partial answer to the problem raised in [2], still open is the question whether in dimensions $n \geq 3$ there are unitarily invariant norms with constant $C = \sqrt{2}$. Possible candidates for such norms are given by

$$\|X\| = \|U\mathrm{diag}(s_1, \ldots, s_n)V\| := \sqrt{s_1^2 + s_2^2},$$

a mixture of the second Ky Fan and the Hilbert-Schmidt norms.

References

[1] A. Böttcher and D. Wenzel, How big can the commutator of two matrices be and how big is it typically? *Linear Algebra Appl.* **403** (2005), pp. 216–228.

[2] A. Böttcher and D. Wenzel, The Frobenius norm and the commutator, *Linear Algebra Appl.* **429** (2008), pp. 1864–1885.

[3] L. Grafakos, *Classical Fourier Analysis*, Graduate Texts in Mathematics (2008), Springer, New York.

[4] L. László, Proof of Böttcher and Wenzel's conjecture on commutator norms for 3-by-3 matrices, *Linear Algebra Appl.* **422** (2007), pp. 659–663.

[5] Zhiqin Lu, Proof of the normal scalar curvature conjecture,
Available from: arxiv:0711.3510v1 [math.DG] 22 November 2007.

[6] Seak-Weng Vong and Xiao-Qing Jin, Proof of Böttcher and Wenzel's conjecture, *Oper. Matrices* **2** (2008), pp. 435–442.

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Received: February 26, 2009
Accepted: April 22, 2009