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The Functional Calculus for Sectorial Operators

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Preface

In 1928 the austrian author Egon Friedell wrote in the introduction to his opus magnum *Kulturgeschichte der Neuzeit* [92]:

> *Alle Dinge haben ihre Philosophie, ja noch mehr: alle Dinge* sind *Philosophie. Alle Menschen, Gegenstände und Ereignisse sind Verkörperungen eines bestimmten Naturgedankens, einer eigentümlichen Weltabsicht. Der menschliche Geist hat nach der Idee zu forschen, die in jedem Faktum verborgen liegt, nach dem Gedanken, dessen bloße Form es ist. Die Dinge pflegen oft erst spät, ihren wahren Sinn zu offenbaren.*[1]

And, a few lines after:

> *Daß die Dinge geschehen, ist nichts. Daß sie gewusst werden, ist alles.*[2]

Although spoken in the context of cultural history these words may also be applied towards the interpretation of mathematical thought. Friedell seems to say that nothing is just a 'brute fact' but the form of an idea which is hidden and has to be discovered in order to be shared by human beings. What really matters is not the mere fact (which in mathematics would be: the truth of a theorem) but is the form of our knowledge of it, the way (*how*) we know things. This means that in order to obtain substantial understanding ('revealing its true meaning') it is not enough to just state and prove theorems.

This conviction is at the heart of my efforts in writing this book. It came out of my attempt to deepen (or to establish in the first place) my *own* understanding of its subject. But I hope of course that it will also prove useful to others, and eventually will have its share in the advance of our understanding in general of the mathematical world.

[1] All things have their philosophy, even more: all things actually *are* philosophy. All men, objects, and events embody a certain thought of nature, a proper intention of the world. The human mind has to inquire the idea which is hidden in each fact, the thought its mere form it is. Things tend to reveal their true meaning only after a long time. (Translation by the author)

[2] That things happen, is nothing. That they are known, is everything. (Translation by the author)

Topic of the book

The main theme, as the title indicates, is functional calculus. Shortly phrased it is about 'inserting operators into functions', in order to render meaningful such expressions as

$$A^\alpha, \quad e^{-tA}, \quad \log A,$$

where A is an (in general unbounded) operator on a Banach space. The basic objective is quite old, and in fact the Fourier transform provides an early example of a method to define $f(A)$, where $A = \Delta$ is the Laplacian, $X = \mathbf{L^2}(\mathbb{R})$ and f is an arbitrary measurable function on $\mathbb{R}$. A straightforward generalisation involving self-adjoint (or normal) operators on Hilbert spaces is provided by the Spectral Theorem, but to leave the Hilbert space setting requires a different approach.

Suppose that a class of functions on some set Ω has a reproducing kernel, i.e.,

$$f(z) = \int_\Omega f(w)K(z,w)\,\mu(dw) \qquad (z \in \Omega)$$

for some measure μ, and — for whatever reason — one already 'knows' what operator the expression $K(A,w)$ should yield; then one may try to define

$$f(A) := \int_\Omega f(w)K(A,w)\,\mu(dw).$$

The simplest reproducing kernel is given by the Cauchy integral formula, so that $K(z,w) = (w-z)^{-1}$, and $K(A,w) = R(w,A)$ is just the resolvent of A. This leads to the 'ansatz'

$$f(A) = \frac{1}{2\pi i}\int_{\partial\Omega} f(w)R(w,A)\,dw,$$

an idea which goes back already to Riesz and Dunford, with a more recent extension towards functions which are singular at some points of the boundary of the spectrum. The latter extension is indeed needed, e.g. to treat fractional powers A^α, and is one of the reasons why functional calculus methods nowadays can be found in very different contexts, from abstract operator theory to evolution equations and numerical analysis of partial differential equations. We invite the reader to have a look at Chapter 9 in order to obtain some impressions of the possible applications of functional calculus.

Overview

The Cauchy formula encompasses a great flexibility in that its application requires only a spectral condition on the operator A. Although we mainly treat sectorial operators the approach itself is generic, and since we shall need to use it also for so-called strip-type operators, it seemed reasonable to ask for a more *axiomatic treatment*. This is provided in Chapter 1. *Sectorial operators* are introduced in

Chapter 2 and we give a full account of the basic functional calculus theory of these operators. As an application of this theory and as evidence for its elegance, in Chapter 3 we treat *fractional powers* and *holomorphic semigroups.* Chapter 4 is devoted to the interplay between a sectorial operator A and its *logarithm* $\log A$. One of the main aspects in the theory, subject to extensive research during the last two decades, is the *boundedness* of the H^∞-calculus. Chapter 5 provides the necessary background knowledge including *perturbation theory*, Chapter 6 investigates the relation to real *interpolation spaces.* Here we encounter the suprising fact that an operator improves its functional calculus properties in certain interpolation spaces; this is due to the 'flexible' descriptions of these spaces in terms of the functional calculus.

Hilbert spaces play a special role in analysis in general and in functional calculus in particular. On the one hand, boundedness of the functional calculus can be deduced directly from *numerical range* conditions. On the other hand, there is an intimate connection with *similarity* problems. Both aspects are extensively studied in Chapter 7.

Chapters 8 and 9 account for *applications* of the theory. We study elliptic operators with constant coefficients and the relation of the functional calculus to Fourier multiplier theory. Then we apply functional calculus methods to a problem from numerical analysis regarding time-discretisation schemes of parabolic equations. Finally, we discuss the so-called maximal regularity problem and the functional calculus approach to its solution.

To make the book as self-contained as possible, we have provided an ample appendix, often also listing the more elementary results, since we thought the reader might be grateful for a comprehensive and nevertheless surveyable account. The appendix consists of six parts. Appendix A deals with operators, in particular their basic *spectral theory.* Our opinion is that a slight increase of generality, namely towards *multi-valued* operators, renders the whole account much easier. (Multi-valued operators will appear in the main text occasionally, but not indispensably.) Appendix B provides basics on *interpolation spaces.* Two more appendices (Appendix C and D) deal with forms and operators on *Hilbert spaces* as well as the *Spectral Theorem.* Finally, Appendix F quotes two results from complex *approximation theory*, but giving proofs here would have gone far beyond the scope of this book.

Instead of giving numbers to definitions I decided to incorporate the definitions into the usual text body, with the defined terms printed in boldface letters. All these definitions and some other key-words are collected in the index at the end of the book. There one will find also a list of symbols.

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I dedicate this book to my father, *Hans Albert Haase* (1929–1997).

February 2006, Markus Haase

Chapter 1

Axiomatics for Functional Calculi

We convey the fundamental *intuition* behind the concept "functional calculus" (Section 1.1). Then we present a *formalisation* of certain ideas common to many functional calculus constructions. In particular, we introduce a method of extending an elementary functional calculus to a larger algebra (Section 1.2). In Section 1.3 we introduce the notion of a *meromophic functional calculus* for a closed operator A on a Banach space and specialise the abstract results from Section 1.2. As an important example we treat *multiplication operators* (Section 1.4). In Section 1.3.2 we prove an abstract composition rule for a pair of meromorphic functional calculi.

1.1 The Concept of Functional Calculus

Consider the Banach space $X := \mathbf{C}[0,1]$ of continuous functions on the unit interval with values in the complex numbers $\mathbb{C}$. Each function $a \in X$ determines a bounded linear operator

$$M_a = (g \longmapsto ag) : \mathbf{C}[0,1] \longrightarrow \mathbf{C}[0,1]$$

on X called the *multiplication* operator associated with a. Its spectrum $\sigma(M_a)$ is simply the range $a[0,1]$ of a. Given any other continuous function $f : \sigma(M_a) \longrightarrow \mathbb{C}$ one can consider the multiplication operator $M_{f \circ a}$ associated with $f \circ a$. This yields an *algebra homomorphism*

$$\Phi = (f \longmapsto M_{f \circ a}) : \mathbf{C}(\sigma(M_a)) \longrightarrow \mathcal{L}(X)$$

into the algebra of bounded linear maps on X. Since one has $\Phi(z) = M_a$[†] and $\Phi((\lambda - z)^{-1}) = R(\lambda, M_a)$ for $\lambda \in \varrho(M_a)$, and since the operator $\Phi(f)$ is simply multiplication by $f(a(z))$, one says that $\Phi(f)$ is obtained by 'inserting' the operator M_a into the function f and writes $f(M_a) := \Phi(f)$. Generalising this example to the Banach space $X = \mathbf{C_0}(\mathbb{R})$ and continuous functions $a \in \mathbf{C}(\mathbb{R})$, one realises that boundedness of the operators is not an essential requirement.

[†]We simply write $z := (z \longmapsto z)$ for the coordinate function on $\mathbb{C}$. Hence the symbols $f(z)$ and f are used interchangeably.

The intuition of functional calculus now consists, roughly, in the idea that to *every* closed operator A on a Banach space X there corresponds an algebra of complex-valued functions on its spectrum in which the operator A can somehow be 'inserted' in a reasonable way. Here 'reasonable' means *at least* that $f(A)$ should have the expected meaning *if* one expects something, e.g., if $\lambda \in \varrho(A)$ then one expects $(\lambda - z)^{-1}(A) = R(\lambda, A)$ or if A generates a semigroup T then $e^{tz}(A) = T(t)$. (This is just a minimal requirement. There may be other reasonable criteria.) In summary we think of a mapping $f \longmapsto f(A)$ which we (informally) call a *functional calculus* for A. Unfortunately, up to now there is no overall formalisation of this idea. The best thing achieved so far is a case by case construction.

We return to some examples. If one knows that the operator A is 'essentially' (i.e., is similar to) a multiplication operator, then it is straightforward to construct a functional calculus. By one version of the spectral theorem, this is the case if A is a normal operator on a Hilbert space (cf. Appendix D). As is well known, the Fourier transform on $\mathbf{L}^2(\mathbb{R})$ is an instance of this, and hence it provides one of the earliest examples of a non-trivial functional calculus. (The operator A in that case is just id/dt.) In general, there is no canonical 'diagonalisation' of the normal operator A, and so the resulting functional calculus depends — at least a priori — on the chosen unitary equivalence. Hence an additional argument is needed to ensure independence of the construction (cf. Theorem D.6.1).

Having this in mind as well as for the sake of generality, one looks for *intrinsic* definitions. It was the idea of Riesz and Dunford to base the construction of $f(A)$ on a Cauchy-type integral

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(z) R(z, A)\, dz. \tag{1.1}$$

The idea behind this is readily sketched. Let us assume that we are given a *bounded* operator A and an open superset U of the spectrum $K := \sigma(A)$. The so-called 'homology version' of the Cauchy Integral Formula says the following. One can choose a generalised contour (a 'cycle') Γ with the following properties:

1) $\Gamma^* \subset U \setminus K$;
2) each point of $\mathbb{C} \setminus \Gamma^*$ has index either 0 or 1 with respect to Γ;
3) each point of K has index 1 with respect to Γ.

(The set Γ^* is the trace of the cycle.) With respect to such a contour Γ, one has

$$f(a) = \frac{1}{2\pi i} \int_\Gamma f(z) \frac{1}{z - a}\, dz \tag{1.2}$$

for each $a \in K$ and for every holomorphic function defined on U. (See [48, Proposition VIII.1.1] for a proof.) Replacing a by A on both sides in the formula (1.2) and insisting that '$1/(z - A)$' should be the same as $R(z, A)$, we obtain (1.1).

The so-defined functional calculus is called the *Dunford–Riesz* calculus and it turns out that it is only a special case of a general construction in Banach algebras, cf. [79, VII.3 and VII.11], [49, Chapter VII, §4]. Actually it yields an *algebra homomorphism*

$$\Phi := (f \longmapsto f(A)) : \mathcal{A} \longrightarrow \mathcal{L}(X),$$

where $\mathcal{A}$ is the algebra of *germs* of holomorphic functions on $\sigma(A)$.

Let us continue with a second example. Consider the Banach space $X := \mathbf{C}[0,1]$ again, and thereon the **Volterra operator** $V \in \mathcal{L}(X)$ defined by

$$(Vf)(t) := \int_0^t f(s)\,ds \qquad (t \in [0,1]).$$

As is well known, V is a compact, positive contraction with $\sigma(A) = \{0\}$. Given a holomorphic function f defined in a neighbourhood of 0, simply choose $\varepsilon > 0$ small enough and define

$$f(V) := \frac{1}{2\pi i} \int_{\partial B_\varepsilon(0)} f(z) R(z,V)\,dz.$$

Since $\|V^n\| = 1/n!$ for all $n \in \mathbb{N}$, one can equally replace z by V in the power-series expansion of f.

Now note that V is injective but not invertible, and that z^{-1} is not holomorphic at 0. Nevertheless it seems reasonable to write $V^{-1} = (z^{-1})(V)$. This suggests an extension of the functional calculus to a larger algebra (of germs of holomorphic functions) which contains the function z^{-1}. In the concrete example of the Volterra operator this extension is easy. (Consider functions holomorphic in a *pointed* neighbourhood of 0 and without essential singularity at 0. This means that the principal part of the Laurent series is a polynomial in z^{-1}. Now insert V as in Appendix A.7 into the principal part and as above into the remainder. Finally add the results.)

In more general situations one cannot go back to power or Laurent series, but has to make use of an abstract extension procedure. This is the topic of the next section.

1.2 An Abstract Framework

In this section we describe abstractly how to extend a certain basic functional calculus to a wider class of functions. It is a little 'Bourbakistic' in spirit, and the reader who is not so fond of axiomatic treatment may skip it on first reading and come back to it when needed.

To have a model in mind, assume that we are given an operator A on a Banach space X and a basic class of functions $\mathcal{E}$ on the spectrum of A. Assume further that we have a 'method'

$$\Phi = (f \longmapsto f(A)) : \mathcal{E} \longrightarrow \mathcal{L}(X)$$

of 'inserting' the operator into functions from $\mathcal{E}$, i.e., $\mathcal{E}$ is an algebra and Φ is a homomorphism of algebras. Moreover, the mapping Φ should have something to do with the operator A and so we assume for the moment that for some $\lambda \in \varrho(A)$ the function $(\lambda - z)^{-1}$ is contained in $\mathcal{E}$ and $\Phi((\lambda - z)^{-1}) = R(\lambda, A)$. Then one can think of defining

$$f(A) := (\lambda - A)^n \Phi[f(z)(\lambda - z)^{-n}] = \Phi[(\lambda - z)^{-n}]^{-1} \Phi[f(z)(\lambda - z)^{-n}]$$

for each function f such that $f(z)(\lambda - z)^{-n}$ is contained in $\mathcal{E}$ for some $n \in \mathbb{N}$. This would clearly yield a closed operator and one only has to make sure that the definition is independent of the chosen $n \in \mathbb{N}$. Hence one has defined '$f(A)$' for f taken from a larger algebra.

To proceed even further assume now that e is *any* function from $\mathcal{E}$ such that $e(A)$ is injective, and f is a function with $ef \in \mathcal{E}$. Then, as above where we had $e = (\lambda - z)^{-n}$, one can define

$$f(A) = e(A)^{-1}(ef)(A) = \Phi(e)^{-1}\Phi(ef).$$

Of course one has to make sure that the use of different e's does not lead to different $f(A)$'s. This will shortly be proved, after we have cast all this imprecise reasoning into an abstract, axiomatic framework.

1.2.1 The Extension Procedure

The starting point of what we call the **extension procedure** is a Banach space X, a commutative algebra $\mathcal{M}$ with $\mathbf{1}$ together with a subalgebra $\mathcal{E} \subset \mathcal{M}$ (with $\mathbf{1} \notin \mathcal{E}$ in general) and a homomorphism $\Phi : (e \longrightarrow \Phi(e)) : \mathcal{E} \longrightarrow \mathcal{L}(X)$. Let us call the triple $(\mathcal{E}, \mathcal{M}, \Phi)$ an **abstract functional calculus** (in short: **afc**) over X. Sometimes we denote this object just by the pair $(\mathcal{E}, \mathcal{M})$ and suppress explicit reference to the homomorphism $\Phi : \mathcal{E} \longrightarrow \mathcal{L}(X)$. We say that the afc is **non-degenerate** or **proper** if the set

$$\operatorname{Reg}(\mathcal{E}) := \{e \in \mathcal{E} \mid \Phi(e) \text{ is injective}\}$$

is not empty. Each member of $\operatorname{Reg}(\mathcal{E})$ is called a **regulariser**. Take $f \in \mathcal{M}$. If there is $e \in \operatorname{Reg}(\mathcal{E})$ such that also $ef \in \mathcal{E}$, we call f **regularisable** by $\mathcal{E}$ and e a **regulariser for** f. Note that $\mathbf{1}$ is regularisable if and only if the afc is proper. In this case

$$\mathcal{M}_r := \{f \in \mathcal{M} \mid f \text{ is regularisable}\}$$

clearly is a subalgebra of $\mathcal{M}$ which contains $\mathcal{E}$.

Let $(\mathcal{E}, \mathcal{M}, \Phi)$ be a proper afc. For $f \in \mathcal{M}_r$ we define

$$\Phi(f) := \Phi(e)^{-1}\Phi(ef), \tag{1.3}$$

where $e \in \operatorname{Reg}(\mathcal{E})$ is a regulariser for f. We often write $f_\bullet$ instead of $\Phi(f)$. (So (1.3) reads $f_\bullet := e_\bullet^{-1}(ef)_\bullet$ under this convention.) The next lemma shows that the definition (1.3) is independent of the chosen regulariser e.

Lemma 1.2.1. *Let $(\mathcal{E}, \mathcal{M}, \Phi)$ be a proper afc. Then by* (1.3) *a closed operator on X is well defined and the so-defined mapping*

$$\Phi = (f \longmapsto \Phi(f)) : \mathcal{M}_r \longrightarrow \{\textit{closed operators on } X\}$$

extends the original mapping $\Phi : \mathcal{E} \longrightarrow \mathcal{L}(X)$.

Proof. Let $h \in \mathrm{Reg}(\mathcal{E})$ be a second regulariser for f, and define $A := (e_\bullet)^{-1}(ef)_\bullet$ and $B := (h_\bullet)^{-1}(hf)_\bullet$. Because $e_\bullet h_\bullet = (eh)_\bullet = (he)_\bullet = h_\bullet e_\bullet$, inverting yields $(e_\bullet)^{-1}(h_\bullet)^{-1} = (h_\bullet)^{-1}(e_\bullet)^{-1}$. Now it follows that

$$\begin{aligned} A &= (e_\bullet)^{-1}(ef)_\bullet = (e_\bullet)^{-1}(h_\bullet)^{-1}h_\bullet(ef)_\bullet = (h_\bullet)^{-1}(e_\bullet)^{-1}(hef)_\bullet \\ &= (h_\bullet)^{-1}(e_\bullet)^{-1}e_\bullet(hf)_\bullet = (h_\bullet)^{-1}(hf)_\bullet = B. \end{aligned}$$

This shows that $f_\bullet$ is the same whichever regulariser one chooses.

We are left to show that the new Φ extends the old one. Since the afc is proper, $\mathcal{E} \subset \mathcal{M}_r$. If $e, f \in \mathcal{E}$ with $e_\bullet$ injective, one has $(e_\bullet)^{-1}(ef)_\bullet = (e_\bullet)^{-1}e_\bullet f_\bullet = f_\bullet$, whence it is shown that the map on $\mathcal{M}_r$ is indeed an extension of the original. □

Sometimes we call the original mapping $\Phi : \mathcal{E} \longrightarrow \mathcal{L}(X)$ the **primary** (functional) calculus (in short: **pfc**) and the extension defined above the **extended** (functional) calculus. The algebra $\mathcal{M}_r$ is called the **domain** of the afc $(\mathcal{E}, \mathcal{M}, \Phi)$.

1.2.2 Properties of the Extended Calculus

We collect some basic properties.

Proposition 1.2.2. *Let $(\mathcal{E}, \mathcal{M}, \Phi)$ be a proper abstract functional calculus over the Banach space X. Then the following assertions hold.*

a) *If $T \in \mathcal{L}(X)$ commutes with each $e_\bullet$, $e \in \mathcal{E}$, then it commutes with each $f_\bullet$, $f \in \mathcal{M}_r$.*

b) *One has $1 \in \mathcal{M}_r$ and $1_\bullet = I$.*

c) *Given $f, g \in \mathcal{M}_r$, one has*

$$\begin{aligned} f_\bullet + g_\bullet &\subset (f+g)_\bullet \\ f_\bullet\, g_\bullet &\subset (fg)_\bullet \end{aligned}$$

with $\mathcal{D}(f_\bullet g_\bullet) = \mathcal{D}((fg)_\bullet) \cap \mathcal{D}(g_\bullet)$.

d) *If $f, g \in \mathcal{M}_r$ such that $fg = 1$, then $f_\bullet$ is injective with $(f_\bullet)^{-1} = g_\bullet$.*

e) *Let $f \in \mathcal{M}_r$, and let F be a subspace of $\mathcal{D}(f_\bullet)$. Suppose that there is a sequence $(e_n)_n \subset \mathcal{E}$ such that $e_{n\bullet} \to I$ strongly as $n \to \infty$ and $\mathcal{R}(e_{n\bullet}) \subset F$ for all $n \in \mathbb{N}$. Then F is a core for $f_\bullet$.*

Proof. a) Let $T \in \mathcal{L}(X)$ commute with every $e_\bullet$, $e \in \mathcal{E}$. Take $f \in \mathcal{M}_r$ and a regulariser $e \in \mathcal{E}$ for f. Then we have $Tf_\bullet = T(e_\bullet)^{-1}(ef)_\bullet \subset (e_\bullet)^{-1}T(ef)_\bullet = (e_\bullet)^{-1}(ef)_\bullet T = f_\bullet T$.

b) Since the afc is proper, there exists $e \in \mathcal{E}$ such that $e_\bullet$ is injective. Clearly, e regularises 1. Moreover, we have $1_\bullet = e_\bullet^{-1}(e1)_\bullet = e_\bullet^{-1}e_\bullet = I$.

c) Take $f, g \in \mathcal{M}_r$ and let $e_1, e_2 \in \mathcal{E}$ be regularisers for f, g, respectively. Then $e := e_1e_2$ is a regulariser for both f and g, hence for $f + g$. Also $efg = (e_1f)(e_2g)$ is in $\mathcal{E}$, whence e is also a regulariser for fg. We have

$$
\begin{aligned}
f_\bullet + g_\bullet &= e_\bullet^{-1}(ef)_\bullet + e_\bullet^{-1}(eg)_\bullet \subset e_\bullet^{-1}[(ef)_\bullet + (eg)_\bullet] \\
&= e_\bullet^{-1}(e(f+g))_\bullet = (f+g)_\bullet \qquad\qquad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
f_\bullet g_\bullet &= e_{1\bullet}^{-1}(e_1f)_\bullet e_{2\bullet}^{-1}(e_2g)_\bullet \subset e_{1\bullet}^{-1}e_{2\bullet}^{-1}(e_1f)_\bullet(e_2g)_\bullet = [e_{2\bullet}e_{1\bullet}]^{-1}(efg)_\bullet \\
&= e_\bullet^{-1}(efg)_\bullet = (fg)_\bullet.
\end{aligned}
$$

To prove the assertions concerning the domains, let $x \in \mathcal{D}((fg)_\bullet) \cap \mathcal{D}(g_\bullet)$. Since $(e_1f)_\bullet$ commutes with $e_{2\bullet}$ it also commutes with $e_{2\bullet}^{-1}$. By assumption, $y := (e_2g)_\bullet x \in \mathcal{D}(e_{2\bullet}^{-1})$, whence also $(e_1f)_\bullet y \in \mathcal{D}(e_{2\bullet}^{-1})$ and

$$
(e_1f)_\bullet g_\bullet x = (e_1f)_\bullet e_{2\bullet}^{-1}y = e_{2\bullet}^{-1}(e_1f)_\bullet y = e_{2\bullet}^{-1}(efg)_\bullet x \in \mathcal{D}(e_{1\bullet}^{-1})
$$

by assumption and the identity $e_\bullet^{-1} = e_{1\bullet}^{-1}e_{2\bullet}^{-1}$. Consequently, $g_\bullet x \in \mathcal{D}(f_\bullet)$, and hence $x \in \mathcal{D}(f_\bullet g_\bullet)$.

d) Suppose $f, g \in \mathcal{M}_r$ with $fg = 1$. By b) and c) we have $g_\bullet f_\bullet \subset (fg)_\bullet = 1_\bullet = I$ and $\mathcal{D}(g_\bullet f_\bullet) = \mathcal{D}(I) \cap \mathcal{D}(f_\bullet) = \mathcal{D}(f_\bullet)$. Interchanging f and g proves the statement.

e) Let $x \in \mathcal{D}(f_\bullet)$ and define $y := f_\bullet x$. With $x_n := e_{n\bullet}x$ and $y_n := e_{n\bullet}y$ we have $x_n \to x$, $x_n \in F$ and $f_\bullet x_n = f_\bullet e_{n\bullet}x = e_{n\bullet}f_\bullet x = y_n \to y$. $\square$

In general one cannot expect equality in c) of Proposition 1.2.2. However, if we define

$$
\mathcal{M}_b := \{f \in \mathcal{M}_r \mid f_\bullet \in \mathcal{L}(X)\},
$$

we obtain the following.

Corollary 1.2.3. *Let $\mathcal{E}, \mathcal{M}, \Phi, X$ be as above.*

a) *For $f \in \mathcal{M}_r, g \in \mathcal{M}_b$ one has $f_\bullet + g_\bullet = (f+g)_\bullet$ and $f_\bullet\, g_\bullet = (fg)_\bullet$.*

b) *The set $\mathcal{M}_b$ is a subalgebra with* **1** *of $\mathcal{M}$, and the map*

$$
(f \longmapsto f_\bullet) : \mathcal{M}_b \longrightarrow \mathcal{L}(X)
$$

is a homomorphism of algebras with **1**.

c) *If $f \in \mathcal{M}_b$ is such that $f_\bullet$ is injective, then*

$$
(f_\bullet)^{-1}g_\bullet f_\bullet = g_\bullet
$$

holds for all $g \in \mathcal{M}_r$.

Proof. a) By c) of Proposition 1.2.2 we have $f_\bullet + g_\bullet \subset (f+g)_\bullet$ and $(f+g)_\bullet - g_\bullet \subset (f+g-g)_\bullet = f_\bullet$. Since $g_\bullet$ is bounded, $\mathcal{D}((f+g)_\bullet) = \mathcal{D}(f_\bullet)$. This readily implies $f_\bullet + g_\bullet = (f+g)_\bullet$. The second assertion is immediate from c) of Proposition 1.2.2 since we have $\mathcal{D}(g_\bullet) = X$.

b) This follows from a).

c) We can find $e \in \mathcal{E}$ which regularises both f and g. Now we compute

$$f_\bullet^{-1} g_\bullet f_\bullet = f_\bullet^{-1} e_\bullet^{-1} (eg)_\bullet f_\bullet = f_\bullet^{-1} e_\bullet^{-1} f_\bullet (eg)_\bullet = e_\bullet^{-1} f_\bullet^{-1} f_\bullet (eg)_\bullet = e_\bullet^{-1} (eg)_\bullet = g_\bullet.$$

Here we used b) and the identity $e_\bullet^{-1} f_\bullet^{-1} = f_\bullet^{-1} e_\bullet^{-1}$, which is true since $e_\bullet$ and $f_\bullet$ are both bounded and injective. $\square$

Using this new information we can improve d) of Proposition 1.2.2.

Corollary 1.2.4. *Let $\mathcal{E}, \mathcal{M}, \Phi, X$ be as above. Suppose that $f \in \mathcal{M}_r, g \in \mathcal{M}$ such that $fg = 1$. Then*

$$g \in \mathcal{M}_r \quad \Longleftrightarrow \quad f_\bullet \text{ is injective.}$$

In this case, we have $g_\bullet = f_\bullet^{-1}$.

Proof. One direction of the equivalence is simply d) of Proposition 1.2.2. Suppose that $f_\bullet$ is injective and let $e \in \mathcal{E}$ be a regulariser for f. Then $fe \in \mathcal{E}, (fe)g = e \in \mathcal{E}$ and $(fe)_\bullet = f_\bullet e_\bullet$ is injective. This means that fe is a regulariser for g. $\square$

Although the reader will encounter several examples of the extension procedure in the following chapters, we illustrate it here in the case of the Volterra operator. We let $\mathcal{M}$ be the algebra of germs of functions holomorphic in a pointed neighbourhood of 0, and let $\mathcal{E}$ be the subalgebra of germs of functions holomorphic in a whole neighbourhood of 0. The primary calculus is given by the Cauchy integral (1.1) (or, alternatively, by insertion into the power series). Since $(z)(V) = V$, this afc is proper and there is a natural extension to $\mathcal{M}_r$. It is easy to see that $\mathcal{M}_r$ is exactly the algebra of germs of meromorphic functions at 0. If f is such a germ with $z^n f$ holomorphic at 0, we have $f(V) = V^{-n}(z^n f)(V)$. In particular, $(z^{-n})(V) = V^{-n}$ for each $n \in \mathbb{N}$.

1.2.3 Generators and Morphisms

Let us return to the abstract treatment. We start with a proper afc $(\mathcal{E}, \mathcal{M}, \Phi)$ over the Banach space X. A subalgebra $\mathcal{D} \subset \mathcal{M}_b$ is called **admissible** if the set $\{f \in \mathcal{D} \mid f_\bullet \text{ is injective}\}$ is not empty. In this case $(\mathcal{D}, \mathcal{M}, \Phi)$ is another proper afc over X. Let us denote by

$$\langle \mathcal{D} \rangle := \{f \in \mathcal{M} \mid \text{there is } d \in \mathcal{D} \text{ such that } df \in \mathcal{D} \text{ and } d_\bullet \text{ is injective}\}$$

the regularisable elements of this 'sub-afc'.

Proposition 1.2.5. *Let $(\mathcal{E}, \mathcal{M}, \Phi)$ be a proper afc over the Banach space X, and let $\mathcal{D}$ be an admissible subalgebra of $\mathcal{M}_b$. If $f \in \mathcal{M}$ and $g \in \langle \mathcal{D} \rangle$ are such that $g_\bullet$ is injective and $gf \in \langle \mathcal{D} \rangle$, then already $f \in \langle \mathcal{D} \rangle$.*

Proof. By assumption there are regularising elements $d_1, d_2 \in \mathcal{D}$ for g, fg, respectively. Letting $d := d_1 d_2$ we see that d regularises both g and fg. Since $(d_\bullet)^{-1}(dg)_\bullet = g_\bullet$ is injective, the operator $(dg)_\bullet$ also is. Hence dg regularises f, whence $f \in \langle \mathcal{D} \rangle$. □

The proposition shows in particular that $\mathcal{M}_r = \langle \mathcal{E} \rangle = \langle \mathcal{M}_b \rangle$. A **generator** of the afc is an admissible subalgebra $\mathcal{D}$ such that $\langle \mathcal{D} \rangle = \mathcal{M}_r$. One is interested in small generators. To check that a given admissible subalgebra is a generator, it suffices to cover any generator, as the following corollary shows.

Corollary 1.2.6. *Let $(\mathcal{E}, \mathcal{M}, \Phi)$ be a proper afc over the Banach space X. Let $\mathcal{D}, \mathcal{D}'$ be admissible subalgebras of $\mathcal{M}_b$ such that $\mathcal{D}' \subset \langle \mathcal{D} \rangle$. Then $\langle \mathcal{D}' \rangle \subset \langle \mathcal{D} \rangle$. In particular, an admissible subalgebra $\mathcal{D}$ of $\mathcal{M}_b$ is a generator of the afc if and only if $\mathcal{E} \subset \langle \mathcal{D} \rangle$.*

Proof. This follows immediately from Proposition 1.2.5. □

Abstract functional calculi over the Banach space X are the objects of a category. We describe the morphisms of this category. Let $(\mathcal{E}, \mathcal{M}, \Phi)$ and $(\mathcal{E}', \mathcal{M}', \Phi')$ be proper afc[1] over the Banach space X. A **morphism**

$$\theta : (\mathcal{E}, \mathcal{M}, \Phi) \longrightarrow (\mathcal{E}', \mathcal{M}', \Phi')$$

between these afc consists of a homomorphism of algebras $\theta : \mathcal{M} \longrightarrow \mathcal{M}'$ with $\theta(\mathcal{E}) \subset \mathcal{E}'$ and $\theta(e)_\bullet = e_\bullet$ for all $e \in \mathcal{E}$. Not surprisingly, the extension procedure is functorial, as the next proposition shows.

Proposition 1.2.7. *Let $\theta : (\mathcal{E}, \mathcal{M}, \Phi) \longrightarrow (\mathcal{E}', \mathcal{M}', \Phi')$ be a morphism of proper afc on the Banach space X. Then $\theta(\mathcal{M}_r) \subset (\mathcal{M}')_r$ with $\theta(f)_\bullet = f_\bullet$ for every $f \in \mathcal{M}_r$.*

Proof. Let $f \in \mathcal{M}_r$ and let $e \in \operatorname{Reg}(\mathcal{E})$ be a regulariser for f. Since $\theta(e)_\bullet = e_\bullet$ is injective and $\theta(e)\theta(f) = \theta(ef) \in \theta(\mathcal{E}) \subset \mathcal{E}'$, the element $\theta(e)$ is a regulariser for $\theta(f)$. Moreover,

$$\theta(f)_\bullet = [\theta(e)_\bullet]^{-1}[\theta(e)\theta(f)]_\bullet = (e_\bullet)^{-1}\theta(ef)_\bullet = (e_\bullet)^{-1}(ef)_\bullet = f_\bullet. \qquad \square$$

The above lemma may seem to be another bit of abstract nonsense at a first glance. However, it will be applied several times in the sequel, where $\mathcal{M}'$ is in fact a superalgebra of $\mathcal{M}$ and θ is just the inclusion mapping. In such a situation Proposition 1.2.7 implies that for a consistent extension of an afc one has to take care only of the *primary* calculus, whence consistency of the extended calculus is then automatic.

[1] The correct plural of the word *calculus* is *calculi*, whence the plural of *afc* should be again *afc*

Remark 1.2.8. The extension procedure actually makes use only of the *multiplicative* structure of $\mathcal{M}$. Except from those statements which explicitly involve addition — the first part of Proposition 1.2.2 c) and Corollary 1.2.3 a) — everything remains true when we merely assume that $\mathcal{M}$ is a multiplicative *monoid* extending the multiplicative structure of $\mathcal{E} \subset \mathcal{M}$. In such a setting the above-mentioned statements involving addition can be appropriately modified. In fact, call $h \in \mathcal{M}$ a *sum* of $f, g \in \mathcal{M}$ if $ef + eg = eh$ for all $e \in \mathcal{E}$ such that $ef, eg \in \mathcal{E}$, and if such elements e exist. Then all statements remain true, replacing '$f + g$' by 'h'. As a matter of fact, one has to adapt the notion of morphism which should be an algebra homomorphism on the level of $\mathcal{E}$, but only a homomorphism of monoids on the level of $\mathcal{M}$.

1.3 Meromorphic Functional Calculi

We want to apply the results of the previous section to functional calculi for operators. Let $\Omega \subset \mathbb{C}$ be an open subset of the complex plane. We denote by

$$\mathcal{O}(\Omega) \quad \text{and} \quad \mathcal{M}(\Omega)$$

the algebras of **holomorphic** and **meromorphic** functions on the set Ω. (Note that if Ω is connected, $\mathcal{M}(\Omega)$ is in fact a field with respect to the pointwise operations.)

Let $\mathcal{E}(\Omega)$ be a subalgebra of $\mathcal{M}(\Omega)$, and let $\Phi : \mathcal{E}(\Omega) \longrightarrow \mathcal{L}(X)$ be an algebra homomorphism, where X is a Banach space. Hence we are given an abstract functional calculus $(\mathcal{E}(\Omega), \mathcal{M}(\Omega), \Phi)$ in the terminology of the previous section. Assume that the following hold:

1. The function $z := (w \longmapsto w) \in \mathcal{M}(\Omega)$ is regularisable by $\mathcal{E}(\Omega)$ whence the operator $A := \Phi(z)$ is well defined by the extension procedure.
2. An operator $T \in \mathcal{L}(X)$ which commutes with A also commutes with each $\Phi(e)$, $e \in \mathcal{E}(\Omega)$.

Then we call the afc $(\mathcal{E}(\Omega), \mathcal{M}(\Omega), \Phi)$ a **(meromorphic) functional calculus** for A. We write $\mathcal{M}(\Omega)_A := \mathcal{M}(\Omega)_r$ and

$$f(A) := \Phi(f) \qquad (f \in \mathcal{M}(\Omega)_A)$$

in order to use a more common notation. Also we introduce the alternative notation

$$H(A) := \{f \in \mathcal{M}(\Omega)_A \mid f(A) \in \mathcal{L}(X)\}$$

for the set which was called $\mathcal{M}(\Omega)_b$ in our abstract setting of Section 1.2.

Remark 1.3.1. Note that the terminology $\mathcal{M}(\Omega)_A$ is inaccurate in that this set heavily depends on the primary functional calculus $(\mathcal{E}(\Omega), \Phi)$. This is even more true for the notation $H(A)$, where also the reference to the set Ω has to be understood from the context.

We can now rephrase the properties of afc in terms of meromorphic functional calculi.

Theorem 1.3.2 (The Fundamental Theorem of the Functional Calculus). *Let A be a closed operator on the Banach space X, and let $(\mathcal{E}(\Omega), \mathcal{M}(\Omega), \Phi)$ be a meromorphic functional calculus for A. Let $f \in \mathcal{M}(\Omega)_A$. Then the following assertions hold.*

a) *If $T \in \mathcal{L}(X)$ commutes with A, then it also commutes with $f(A)$. If $f(A) \in \mathcal{L}(X)$, i.e., $f \in H(A)$, then $f(A)$ commutes with A.*

b) *$\mathbf{1}(A) = I$ and $(z)(A) = A$.*

c) *Let also $g \in \mathcal{M}(\Omega)_A$. Then*

$$f(A) + g(A) \subset (f+g)(A) \quad \text{and} \quad f(A)\,g(A) \subset (fg)(A).$$

Furthermore, $\mathcal{D}((fg)(A)) \cap \mathcal{D}(g(A)) = \mathcal{D}(f(A)g(A))$ and one has equality in the above relations if $g \in H(A)$.

d) *The mapping $(f \mapsto f(A)) : H(A) \longrightarrow \mathcal{L}(X)$ is a homomorphism of algebras.*

e) *One has $f(A) = g(A)^{-1} f(A) g(A)$ if $g \in H(A)$ and $g(A)$ is injective.*

f) *Let $\lambda \in \mathbb{C}$ such that $1/(\lambda - z) \in \mathcal{M}(\Omega)$. Then*

$$\frac{1}{\lambda - f(z)} \in \mathcal{M}(\Omega)_A \qquad \Longleftrightarrow \qquad \lambda - f(A) \text{ is injective.}$$

In this case $(\lambda - f(z))^{-1}(A) = (\lambda - f(A))^{-1}$. In particular, $\lambda \in \varrho(f(A))$ if and only if $(\lambda - f(z))^{-1} \in H(A)$.

Proof. This is more or less a restatement of Proposition 1.2.2 and Corollaries 1.2.3 and 1.2.4. Note that in b) the statement $(z)(A) = A$ is just a reformulation of the hypothesis that the given meromorphic functional calculus is a calculus *for A*. □

Our definition of meromorphic functional calculus for an operator A immediately raises the question of uniqueness. Unfortunately we are unable to prove a positive result at this level of abstraction, knowing too little e.g. about the space $\mathcal{E}(\Omega)$. We present a uniqueness result for the functional calculus for sectorial operators in Section 5.3.5, where additional topological assumptions are needed. The next results show that for rational functions there is no problem.

1.3.1 Rational Functions

Recall that for *any* operator A with non-empty resolvent set there is a definition of '$r(A)$' where r is a rational function on $\mathbb{C}$ with all its poles outside of $\sigma(A)$ (see Appendix A.6). Also, for any *injective* operator A there is a definition of $p(A)$ where $p \in \mathbb{C}[z, z^{-1}]$ is a polynomial in the variables z and z^{-1} (see Appendix A.7). The following results show that we meet these general definitions with *any* meromorphic functional calculus for A.

Proposition 1.3.3. *Let A be a closed operator on the Banach space X such that $\varrho(A) \neq \emptyset$, and let $(\mathcal{E}(\Omega), \mathcal{M}(\Omega), \Phi)$ be a meromorphic functional calculus for A. Let r be a rational function with all its poles being contained in $\varrho(A)$. Then $r \in \mathcal{M}(\Omega)_A$ and '$r(A)$' has its standard meaning as defined in Appendix* A.6.

Proof. Let us first deal with polynomials $p \in \mathbb{C}[z]$. We prove the statement by induction on $\deg p$. If $\deg p = 0$ or $\deg p = 1$, the assertion is trivial. So suppose that $\deg p = n + 1$ and write $p = zq + \mu$ where μ is a scalar and $\deg q = n$. The induction hypothesis implies that $q \in \mathcal{M}(\Omega)_A$ and $\mathcal{D}(q(A)) = \mathcal{D}(A^n)$. Since $\mathcal{M}(\Omega)_A$ is an algebra, $p \in \mathcal{M}(\Omega)_A$ and we have $B := p(A) \supset q(A)A + \mu$. By Theorem 1.3.2 c) we also have $\mathcal{D}(B) \cap \mathcal{D}(A) = \mathcal{D}(q(A)A) = \mathcal{D}(A^{n+1})$. Hence to prove the assertion we only have to show $\mathcal{D}(B) \subset \mathcal{D}(A)$. So let $x \in \mathcal{D}(B)$ and choose $\lambda \in \varrho(A)$. Employing a) and f) of Theorem 1.3.2 we have $R(\lambda, A)B \subset BR(\lambda, A)$, whence $R(\lambda, A)x \in \mathcal{D}(A) \cap \mathcal{D}(B)$. This implies $R(\lambda, A)x \in \mathcal{D}(A^{n+1})$, from which we readily infer that $x \in \mathcal{D}(A^n)$.

Turning to rational functions we observe that by Theorem 1.3.2, for $\lambda \in \varrho(A)$ one has $(\lambda - z)^{-1} \in H(A)$ and $(\lambda - z)^{-1}(A) = R(\lambda, A)$. Hence if $q \in \mathbb{C}[z]$ is such that all zeros are contained in $\varrho(A)$, we obtain $q^{-1} \in H(A)$ and $q^{-1}(A) = q(A)^{-1}$. Then $r(A) = (p/q)(A) = p(A)q^{-1}(A) = p(A)q(A)^{-1}$, again by Theorem 1.3.2. The last expression is exactly how $r(A)$ is defined in Appendix A.6. □

Next we turn to injective operators.

Proposition 1.3.4. *Let A be an injective, closed operator on the Banach space X such that $\varrho(A) \neq \emptyset$, and let $(\mathcal{E}(\Omega), \mathcal{M}(\Omega), \Phi)$ be a meromorphic functional calculus for A. Let $p = \sum_{k\in\mathbb{Z}} a_k z^k \in \mathbb{C}[z, z^{-1}]$ be a polynomial in z, z^{-1}. Then $p \in \mathcal{M}(\Omega)_A$ and $p(A) = \sum_{k\in\mathbb{Z}} a_k A^k$.*

Proof. It follows from Theorem 1.3.2 and the assumptions that $z, z^{-1} \in \mathcal{M}(\Omega)_A$. Hence $\mathbb{C}[z, z^{-1}] \subset \mathcal{M}(\Omega)_A$. Now take $p \in \mathbb{C}[z, z^{-1}]$ as in the hypothesis of the statement. We may suppose that $p \notin \mathbb{C}[z]$. Hence we can write $p(z) = \sum_{k=-n}^{m} a_k z^k$ where $a_{-n} \neq 0$ and $n \geq 1$. It follows from Theorem 1.3.2 that $p(A) = \tau(A)^{-n} p(A) \tau(A)^n = \Lambda^n (\tau^n p)(A)$, where $\tau(z) := z/(\lambda - z)^2$ and $\Lambda := \tau(A)^{-1}$. By Theorem 1.3.2 c) we obtain

$$
\begin{aligned}
p(A) = \Lambda^n (\tau^n p)(A) &= \Lambda^n \left((\sum_{k=0}^{m+n} a_{k-n} z^k)(\lambda - z)^{-2n} \right)(A) \\
&= \Lambda^n \left(\sum_{k=0}^{m+n} a_{k-n} z^k \right)(A)(\lambda - A)^{-2n} \\
&= (\lambda - A)^{2n} A^{-n} \left(\sum_{k=0}^{m+n} a_{k-n} A^k \right)(\lambda - A)^{-2n}
\end{aligned}
$$

and this is contained in $(\lambda - A)^{2n}\left(\sum_{k=0}^{m+n} a_{k-n}A^k\right)A^{-n}(\lambda - A)^{-2n}$, by Lemma A.7.2. Applying Lemma A.7.1 and Lemma A.6.1 together with the previous Proposition 1.3.3, we see that this is equal to

$$\begin{aligned}&\left(\sum_{k=0}^{m+n} a_{k-n}A^k\right)(\lambda - A)^{2n}(\lambda - A)^{-2n}A^{-n} = \left(\sum_{k=0}^{m+n} a_{k-n}A^k\right)A^{-n}\\ &= \sum_{k=-n}^{m} a_k A^k = \sum_{k=-n}^{m}(a_k z^k)(A) \subset \left(\sum_{k=-n}^{m} a_k z^k\right)(A) = p(A).\end{aligned}$$ □

Taking into account also other points $\lambda \in \mathbb{C}$ such that $\lambda - A$ is injective leads us to the following.

Theorem 1.3.5. *Let A be a closed operator on the Banach space X with $\varrho(A) \neq \emptyset$. Let r be any rational function on $\mathbb{C}$ with no pole of r being an eigenvalue of A. Then $r(A)$ is uniquely defined by any meromorphic functional calculus for A.*

Proof. Let $(\mathcal{E}(\Omega), \mathcal{M}(\Omega), \Phi)$ be a meromorphic functional calculus for A. Then $r \in \mathcal{M}(\Omega)_A$ since $\mathcal{M}(\Omega)_A$ is an algebra and $z, \mathbf{1}, (\lambda - z)^{-1} \in \mathcal{M}(A)_A$ for all $\lambda \notin P\sigma(A)$ (cf. Theorem 1.3.2 b) and f)). Let $r = p/q$, and fix $\lambda \in \varrho(A)$. Take $n > \deg p, \deg q$. Then

$$\begin{aligned} r(A) &= \frac{p}{q}(A) = \left(\frac{(\lambda - z)^n}{q}\frac{p}{(\lambda - z)^n}\right)(A) = \left(\frac{(\lambda - z)^n}{q}\right)(A)\left(\frac{p}{(\lambda - z)^n}\right)(A)\\ &= \left(\frac{q}{(\lambda - z)^n}\right)(A)^{-1}\left(\frac{p}{(\lambda - z)^n}\right)(A),\end{aligned}$$

where we have again used Theorem 1.3.2. Now Proposition 1.3.3 shows that $r(A)$ is independent of the chosen functional calculus. □

The previous result has to be complemented by the following remark. The discussion of Appendix A.6 shows that any operator A with $\varrho(A) \neq \emptyset$ has a meromorphic functional calculus on the whole of $\mathbb{C}$, with

$$\mathcal{E}(\mathbb{C}) = \left\{\frac{p}{q} \;\middle|\; p, q \in \mathbb{C}[z],\ \deg(p) \le \deg(q),\ \{q = 0\} \subset \varrho(A)\right\}$$

as the domain of the primary calculus. The domain of the extended calculus consists exactly of the rational functions r on $\mathbb{C}$ with no pole of r being an eigenvalue of A. Theorem 1.3.5 shows that this calculus is *minimal* in a sense.

1.3.2 An Abstract Composition Rule

The intention behind using a functional calculus consists not only in generating new operators from old ones but also to be able to perform *computations* with them. Beside the basic rules given in Theorem 1.3.2 it is also very important to have at one's disposal the so-called **composition rule** by which we mean an identity of the form

$$f(g(A)) = (f \circ g)(A). \tag{1.4}$$

Examples are formulae like

$$e^{\log A} = A, \quad (A^\alpha)^\beta = A^{\alpha\beta}, \quad \log(A^\alpha) = \alpha \log(A) \quad \text{etc.}$$

To make sense of (1.4), one does need not only a functional calculus for A but also one for $g(A)$, and $(f \circ g)$ should be meaningful. As with the uniqueness question above, we are unable to prove (1.4) in full generality without knowing more about the involved functional calculi. In concrete cases, as for sectorial and strip-type operators, we are in a much better situation (see Theorem 2.4.2 and Theorem 4.2.4). All we can say in general is that the problem — if there is one — occurs already at the level of the primary functional calculus.

Proposition 1.3.6. *Let $\Omega, \Omega' \subset \mathbb{C}$ be two open subsets of the plane. Suppose that A is a closed operator on the Banach space X and that $(\mathcal{E}(\Omega), \mathcal{M}(\Omega), \Phi)$ is a meromorphic functional calculus for A. Suppose further that $g : \Omega \longrightarrow \Omega'$ is meromorphic, $g \in \mathcal{M}(\Omega)_A$, and $(\mathcal{E}(\Omega'), \mathcal{M}(\Omega'), \Phi')$ is a meromorphic functional calculus for $g(A)$. Then the statement*

$$f \circ g \in \mathcal{M}(\Omega)_A \qquad \text{and} \qquad (f \circ g)(A) = f(g(A))$$

holds for all $f \in \mathcal{M}(\Omega')_{g(A)}$ provided it holds for all $f \in \mathcal{E}(\Omega')$.

Note that since $g(\Omega) \subset \Omega'$, the function g is actually holomorphic on Ω.

Proof. Define $B := g(A)$. Let $f \in \mathcal{M}(\Omega')_{g(A)}$ and $e \in \mathcal{E}(\Omega')$ be a regulariser for f. Then $ef \in \mathcal{E}(\Omega')$ and $e(B)$ is injective. By assumption $e \circ g, (ef) \circ g \in \mathcal{M}(\Omega)_A$ and $(e \circ g)(A) = e(B)$ as well as

$$[(e \circ g)(f \circ g)](A) = [(ef) \circ g](A) = (ef)(B).$$

This shows that $e \circ g \in H(A)$ is a regulariser for $f \circ g$. Employing Proposition 1.2.5 we conclude that $f \circ g \in \mathcal{M}(\Omega)_A$ and $(f \circ g)(A) = f(B)$. $\square$

The previous proposition is important in that it shows that for nice composition rules one only has to look at the primary functional calculi. Nevertheless, this still may be a tedious task, cf. Theorem 2.4.2.

1.4 Multiplication Operators

In this section we make sure that the axiomatics developed in Section 1.2 meet our intuition in the case of multiplication operators.

Consider a σ-finite measure space (Ω, Σ, μ) and thereon the scale of spaces $\mathbf{L}^{\boldsymbol{p}}(\Omega, \mu)$, where $1 \leq p \leq \infty$. We usually omit reference to the underlying σ-algebra Σ. Let $\mathfrak{M}(\Omega, \mu; \mathbb{C})$ be the space of (μ-equivalence classes of) measurable complex-valued functions. Each $a \in \mathfrak{M}(\Omega, \mu; \mathbb{C})$ determines a **multiplication operator**

$$M_a := (f \mapsto af)$$

on each space $\mathbf{L}^{\boldsymbol{p}}(\Omega, \mu)$, with the maximal domain

$$\mathcal{D}(M_a) = \{f \in \mathbf{L}^{\boldsymbol{p}} \mid af \in \mathbf{L}^{\boldsymbol{p}}(\Omega, \mu)\}.$$

It is easy to see that M_a is closed.

Lemma 1.4.1. *The operator M_a is bounded if and only if $a \in \mathbf{L}^{\boldsymbol{\infty}}(\Omega, \mu)$. The operator M_a is injective if and only if $\mu(a = 0) = 0$. If $1 \le p < \infty$, the operator M_a is densely defined.*

Proof. We restrict to the case that $p < \infty$. If $a \in \mathbf{L}^{\boldsymbol{\infty}}$ then clearly M_a is bounded with $\|M_a\| \le \|a\|_{\mathbf{L}^{\boldsymbol{\infty}}}$. Conversely, suppose that M_a is bounded, and let $c > 0$ such that $A_c := \{|a| > c\}$ is not a null set. Then we can find $A \subset A_c$ such that $0 < \mu(A) < \infty$. Let $f := \mu(A)^{-1/p}\mathbf{1}_A$. (Here, $\mathbf{1}_A$ denotes the characteristic function of the set A, i.e. $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$ and $\mathbf{1}_A(\omega) = 0$ if $\omega \notin A$.) Then $\|f\|_p = 1$ and

$$\|M_a\|^p \ge \|af\|_p^p = \mu(A)^{-1} \int_A |a|^p \, d\mu \ge c^p.$$

Hence $c \le \|M_a\|$. Taking the supremum over c yields $\|a\|_{\mathbf{L}^{\boldsymbol{\infty}}} \le \|M_a\|$.

If $\mu(a = 0) > 0$ one can find a set A such that $0 < \mu(A) < \infty$ and $a = 0$ on A. Clearly $M_a\mathbf{1}_A = 0$ and $\mathbf{1}_A \ne 0$ in $\mathbf{L}^{\boldsymbol{p}}$. On the other hand, if M_a is not injective, one finds that $0 \ne f \in \mathbf{L}^{\boldsymbol{p}}$ such that $af = 0$. Then $\{f \ne 0\} \subset \{a = 0\}$. Now let $1 \le p < \infty$ and $f \in \mathbf{L}^{\boldsymbol{p}}(\Omega, \mu)$. Then $f\mathbf{1}_{\{|a| \le n\}} \to f$ pointwise as $n \to \infty$, whence in $\mathbf{L}^{\boldsymbol{p}}$ by Lebesgue's theorem. But it is clear that $f\mathbf{1}_{\{|a| \le n\}} \in \mathcal{D}(M_a)$ for all $n \in \mathbb{N}$. $\square$

We set up a proper abstract functional calculus for M_a on $X := \mathbf{L}^{\boldsymbol{p}}(\Omega, \mu)$. Let

$$K := \operatorname{essran} a := \{\lambda \in \mathbb{C} \mid \mu(a \in U) > 0 \text{ for each neighbourhood } U \text{ of } \lambda \text{ in } \mathbb{C}\}$$

be the **essential range** of the function a. Then K is a closed subset of $\mathbb{C}$ with $a(\omega) \in K$ for μ-almost all $\omega \in \Omega$. On K we consider the measure $a[\mu]$ defined by $a[\mu](B) = \mu(a \in B)$ for all Borel subsets $B \subset K$. For $e \in \mathbf{L}^{\boldsymbol{\infty}}(K, a[\mu])$ we define $\Phi(e) := M_{e \circ a}$. This determines an algebra homomorphism

$$\Phi : \mathbf{L}^{\boldsymbol{\infty}}(K, a[\mu]) \longrightarrow \mathcal{L}(X)$$

with $\Phi(\mathbf{1}) = I$. Let $\mathbb{C}_\infty$ denote the **Riemann sphere**. We embed $\mathbf{L}^{\boldsymbol{\infty}}(K, a[\mu])$ into the set $\mathfrak{M}(K, a[\mu]; \mathbb{C}_\infty) := \{f : K \longrightarrow \mathbb{C}_\infty \mid f \text{ is } a[\mu]\text{-measurable}\}$. This set is not an algebra, but certainly a multiplicative monoid. Taking into account Remark 1.2.8 we obtain a proper abstract functional calculus

$$(\mathbf{L}^{\boldsymbol{\infty}}(K, a[\mu]), \, \mathfrak{M}(K, a[\mu]; \mathbb{C}_\infty), \, \Phi)\,. \tag{1.5}$$

Theorem 1.4.2. *Let* (Ω, μ), $a \in \mathfrak{M}(\Omega, \mu; \mathbb{C})$, *and* K *be as above. Then the set* K *equals the spectrum* $\sigma(M_a)$ *of the operator* M_a *(cf. Appendix* A.3*). For* $\lambda \in \mathbb{C} \setminus K$ *one has*

$$\|R(\lambda, M_a)\| = \frac{1}{\operatorname{dist}(\lambda, K)}.$$

A function $f \in \mathfrak{M}(K, a[\mu]; \mathbb{C}_\infty)$ *is regularisable within the abstract functional calculus* (1.5) *if and only if* $\mu(a \in \{f = \infty\}) = 0$. *In this case we have*

$$f(M_a) = M_{f \circ a}.$$

Proof. The first assertion follows from the equivalence

$$\lambda \notin K \quad \Longleftrightarrow \quad (\lambda - a)^{-1} \in \mathbf{L}^\infty$$

and Lemma 1.4.1. The second assertion follows from the identity

$$\|f\|_{\mathbf{L}^\infty} = \sup\{|\lambda| \mid \lambda \in \operatorname{essran} f\},$$

which is straightforward to prove. Now take $f \in \mathfrak{M}(K, a[\mu]; \mathbb{C}_\infty)$ such that f is regularisable. By definition, there is $e : K \longrightarrow \mathbb{C}$ measurable and bounded such that also ef is bounded and $e(A) = M_{e\circ a}$ is injective. Hence $\{f = \infty\} \subset \{e = 0\}$ and, by Lemma 1.4.1, $\mu(a \in \{e = 0\}) = \mu\{e \circ a = 0\} = 0$. This implies that $\mu(a \in \{f = \infty\}) = 0$. If conversely this condition holds, define $e := \min(1, 1/|f|)$. Clearly, e is measurable and bounded and one has $|ef| \leq 1$. Now, it is clear that $\{e = 0\} = \{f = \infty\}$, whence from the hypothesis and Lemma 1.4.1 it follows that $e(A) = M_{e\circ a}$ is injective. Hence e is a regulariser for f. □

Note that if $\lambda \in K \setminus P\sigma(A)$ then, by Lemma 1.4.1, the operator $f(M_a)$ is independent of the value of f at the point λ.

1.5 Concluding Remarks

The considerations of the previous sections lead to a somewhat surprising result: Once one has the primary calculus, the extension to a larger class of functions is easy or at least straightforward. The crucial point in constructing a functional calculus for an operator A on a Banach space X lies in setting up the *primary* calculus. This sometimes requires a good knowledge of the operator in question and often is not at all obvious. If one favours the approach via Cauchy integrals (as we do in this treatment), this knowledge consists mostly of insight into the growth behaviour of the resolvent near the spectrum.

To illustrate this, let us turn back to the Volterra operator V introduced above. On first sight one may think that the functional calculus for V obtained so far is the end of the story. However, this is false. The reason is that the resolvent

$R(\lambda, V)$ behaves nicely for $\lambda \to 0$ when λ is restricted to the left half-plane. In fact, the inverse operator V^{-1} is just d/dt with domain

$$\mathcal{D}(V^{-1}) = \{f \in \mathbf{C^1}[0,1] \mid f(0) = 0\},$$

and one can show that both operators $A = V, V^{-1}$ satisfy a resolvent estimate of the form

$$\|R(\lambda, A)\| \leq \frac{C}{|\operatorname{Re}\lambda|} \qquad (\operatorname{Re}\lambda < 0)$$

(cf. Section 8.5). Consequently, V^{-1} and V are both *sectorial* operators of angle $\pi/2$ (cf. Section 2.1). This means that in approaching the spectral point 0 on a straight line with an angle φ such that $\pi \geq |\arg\varphi| > \pi/2$ the growth behaviour of the resolvent of V is optimal, i.e. $\|\lambda R(\lambda, V)\|$ remains bounded. (One can show that the growth is like $1/|\lambda|^2$ when approaching 0 on the imaginary axis, and is more or less exponential when coming from the right.)

Now suppose that one is given a function f, holomorphic on some open sector $S_\varphi := \{z \in \mathbb{C} \setminus \{0\} \mid 0 < |\arg z| < \varphi\}$, where $\varphi \in (\pi/2, \pi)$ and such that $|f(z)| = O(|z|^\alpha)$ as $z \to 0$, for some $\alpha > 0$. Then the Cauchy integral (1.1) still makes sense, Γ being a contour as in Figure 1 below.

As we shall see in Chapter 2, in this way a primary calculus is set up which contains (germs of) functions holomorphic at 0 as well as (germs of) functions holomorphic on some sector S_φ and with a good behaviour at 0. In particular, the *fractional powers* V^α ($\operatorname{Re}\alpha > 0$) and the *logarithm* $\log V$ can be defined (cf. Chapter 3) via this functional calculus.

It seems hard to judge if *this* is now the end of the story for the Volterra operator. Although there is a basic intuition of a kind of a 'maximal functional calculus' for each individual operator, this has not been made precise yet. The best achieved so far is a bundle of constructions of primary calculi for certain classes of operators, like bounded operators, operators described by certain resolvent growth conditions, or semigroup generators.

1.6 Comments

The contents of this chapter mainly rest on [107] where the notion of abstract functional calculus was introduced. As already said, this approach axiomatises the constructions prevalent in the literature. The Fundamental Theorem 1.3.2 thus contains the essence of what an unbounded functional calculus should satisfy.

We remark that our treatment here is 'purely commutative' since we think of algebras of scalar functions. However, with some more-or-less obvious modifications one can also give an abstract setting for 'operator-valued' functional calculi, where non-commutativity must (and can) be allowed to a certain extent. Let us sketch the necessary abstract framework. Suppose again that $\Phi : \mathcal{E} \longrightarrow \mathcal{L}(X)$ is

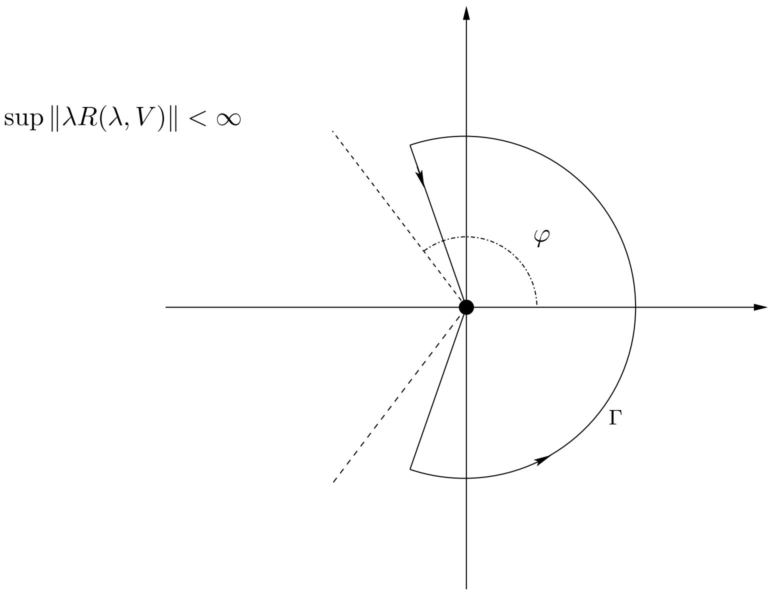


Figure 1: The contour Γ surrounds the region of rapid growth of $R(., V)$.

a homomorphism of algebras and that $\mathcal{E} \subset \mathcal{M}$ as before, but $\mathcal{E}$ (and hence $\mathcal{M}$) need not be commutative. Define the **centre** of $\mathcal{E}$ to be

$$Z(\mathcal{E}) := \{s \in \mathcal{E} \mid es = se \text{ for all } e \in \mathcal{E}\}.$$

An element $f \in \mathcal{M}$ is now called regularisable if there is $e \in Z(\mathcal{E})$ such that $e_\bullet$ is injective and $ef \in \mathcal{E}$. Hence the only change to the commutative situation is that the regularisers have to be taken from $Z(\mathcal{E})$. Then practically all results remain true, apart from the very obvious ones where it is then necessary to incorporate the commutativity into the assumptions (e.g. in Corollary 1.2.4).

The reader will find two more paragraphs on abstract functional calculi in later chapters, namely on dual functional calculi in Section 2.6.1 and on approximations of functional calculi in Section 2.6.3.

A more detailed historical account of roots and development of the idea of functional calculus is deferred to the end of the next chapter (see Section 2.8) when the construction of the functional calculus for sectorial operators has been carried out.

Chapter 2

The Functional Calculus for Sectorial Operators

In Section 2.1 the basic theory of sectorial operators is developed, including examples and the concept of sectorial approximation. In Section 2.2 we introduce some notation for certain spaces of holomorphic functions on sectors. A functional calculus for sectorial operators is constructed in Section 2.3 along the lines of the abstract framework of Chapter 1. Fundamental properties like the composition rule are proved. In Section 2.5 we give natural extensions of the functional calculus to larger function spaces in the case where the given operator is bounded and/or invertible. In this way a panorama of functional calculi is developed. In Section 2.6 some mixed topics are discussed, e.g., adjoints and restrictions of sectorial operators and some fundamental boundedness and some first approximation results. Section 2.7 contains a spectral mapping theorem.

2.1 Sectorial Operators

In the following, X always denotes a (non-trivial) Banach space and A a (single-valued, linear) operator on X. (Note the 'agreement' on page 279 in Appendix A.) For $0 \le \omega \le \pi$ let

$$S_\omega := \begin{cases} \{z \in \mathbb{C} \mid z \neq 0 \,\text{and}\, |\arg z| < \omega\} & \text{if} \quad \omega \in (0, \pi], \\ (0, \infty) & \text{if} \quad \omega = 0. \end{cases}$$

Hence if $\omega > 0$, S_ω denotes the open sector symmetric about the positive real axis with opening angle 2ω.

Let $\omega \in [0, \pi)$. An operator A on X is called **sectorial** of angle ω — in short: $A \in \mathrm{Sect}(\omega)$ — if

1) $\sigma(A) \subset \overline{S_\omega}$ and
2) $M(A, \omega') := \sup\left\{\|\lambda R(\lambda, A)\| \;\middle|\; \lambda \in \mathbb{C} \setminus \overline{S_{\omega'}}\right\} < \infty$ for all $\omega' \in (\omega, \pi)$.

Figure 2 illustrates this notion. An operator A is called **quasi-sectorial** (of angle ω) if there exists $\lambda \in \mathbb{R}$ such that $\lambda + A$ is sectorial (of angle ω). Given a sectorial

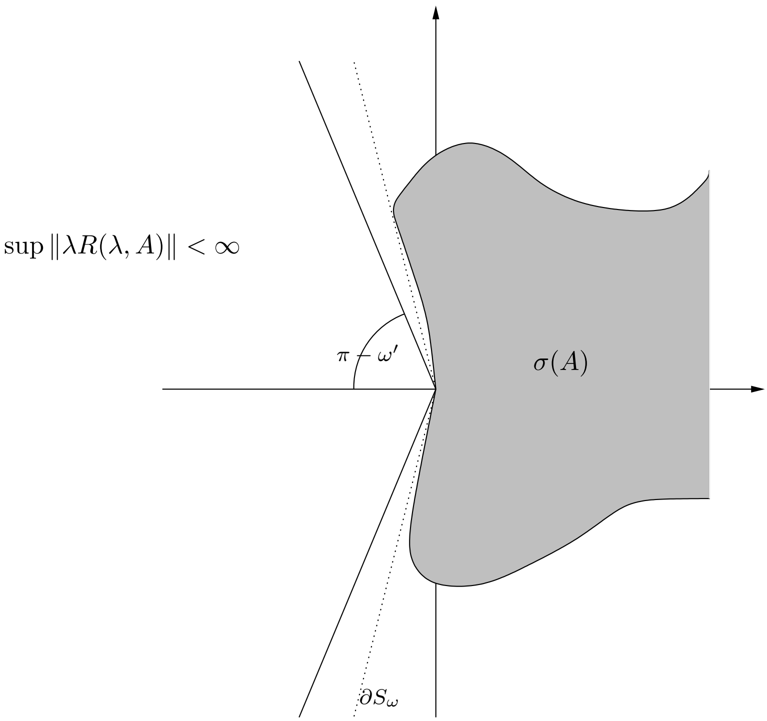


Figure 2: Spectrum of a sectorial operator.

operator A, we call

$$\omega_A := \min\{0 \leq \omega < \pi \mid A \in \mathrm{Sect}(\omega)\}$$

the **spectral angle** (or **sectoriality angle**) of A.

A family of operators $(A_\iota)_\iota$ is called **uniformly sectorial** of angle $\omega \in [0, \pi)$ if $A_\iota \in \mathrm{Sect}(\omega)$ for each ι, and $\sup_\iota M(A_\iota, \omega') < \infty$ for all $\omega' \in (\omega, \pi)$.

Proposition 2.1.1. *Let A be a closed operator on a Banach space X.*

a) *If $(-\infty, 0) \subset \varrho(A)$ and*

$$M(A) := M(A, \pi) := \sup_{t>0} \left\| t(t + A)^{-1} \right\| < \infty,$$

then $M(A) \geq 1$ and $A \in \mathrm{Sect}\left(\pi - \arcsin(M(A)^{-1})\right)$.

Let $A \in \mathrm{Sect}(\omega)$ for some $\omega \in [0, \pi)$.

b) *If A is injective, then $A^{-1} \in \mathrm{Sect}(\omega)$ and the* **fundamental identity**

$$\lambda \left(\lambda + A^{-1}\right)^{-1} = I - \frac{1}{\lambda} \left(\frac{1}{\lambda} + A\right)^{-1} \tag{2.1}$$

holds for all $0 \neq \lambda \in \mathbb{C}$. *In particular,* $M(A^{-1}, \omega') \leq 1 + M(A, \omega')$ *for all* $\omega' \in (\omega, \pi)$.

c) *Let* $n \in \mathbb{N}$ *and* $x \in X$. *Then one has*

$$\begin{array}{lcl} x \in \overline{\mathcal{D}(A)} & \iff & \lim_{t\to\infty} t^n (t+A)^{-n} x = x \\ & \iff & \lim_{t\to\infty} A^n (t+A)^{-n} x = 0, \text{ and} \\ x \in \overline{\mathcal{R}(A)} & \iff & \lim_{t\to 0} A^n (t+A)^{-n} x = x \\ & \iff & \lim_{t\to 0} t^n (t+A)^{-n} x = 0. \end{array}$$

d) *We have* $\mathcal{N}(A) \cap \overline{\mathcal{R}(A)} = 0$. *If* $\overline{\mathcal{R}(A)} = X$, *then* A *is injective.*

e) *The identity* $\mathcal{N}(A^n) = \mathcal{N}(A)$ *holds for all* $n \in \mathbb{N}$.

f) *The family of operators* $(A+\delta)_{\delta \geq 0}$ *is uniformly sectorial of angle* ω. *Indeed,*

$$M(A+\delta, \omega') \leq c(\omega') M(A, \omega') \qquad (\delta > 0, \omega' \in (\omega, \pi))$$

where $c(\omega') = (\sin \omega')^{-1}$ *if* $\omega' \in (0, \pi/2]$ *and* $c(\omega') = 1$ *if* $\omega' \in [\pi/2, \pi)$.

The family of operators $(\varepsilon A)_{\varepsilon \geq 0}$ *is uniformly sectorial of angle* ω. *Indeed,* $M(\varepsilon A, \omega') = M(A, \omega')$ *for all* $\omega' \in (\omega, \pi)$ *and all* $\varepsilon > 0$.

The family of operators $\{(A+\delta)(A+\varepsilon+\delta)^{-1} \mid \varepsilon > 0,\ \delta \geq 0\}$ *is uniformly sectorial of angle* ω.

g) *Let* $\varepsilon > 0$, $n, m \in \mathbb{N}$, *and* $x \in X$. *Then we have*

$$\left(A(A+\varepsilon)^{-1}\right)^n x \in \mathcal{D}(A^m) \quad \iff \quad x \in \mathcal{D}(A^m).$$

h) *If the Banach space* X *is reflexive, one has* $\overline{\mathcal{D}(A)} = X$ *and*

$$X = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}.$$

i) *If* $0 \neq \lambda \in \mathbb{C}$ *with* $|\arg \lambda| + \omega < \pi$, *then* $\lambda A \in \operatorname{Sect}(\omega + |\arg \lambda|)$ *and* $M(\lambda A, \omega') \leq M(A, \omega' - |\arg \lambda|)$ *for each* $\omega' \in (|\arg \lambda| + \omega, \pi]$.

j) *If* $\overline{\mathcal{D}(A)} = X$ *then* $A' \in \operatorname{Sect}(\omega, X')$ *with* $M(\omega', A') = M(\omega, A)$ *for all* $\omega' \in (\omega, \pi]$.

Proof. a) Set $M := M(A)$. For each $x \in D(A)$ and $\lambda > 0$ we have

$$\|x\| \leq \frac{1}{\lambda} M \|Ax\| + M \|x\|.$$

This implies that $\|x\| \leq M \|x\|$ for all $x \in \mathcal{D}(A)$. If $M < 1$, then $\mathcal{D}(A) = 0$, but this is impossible, since $X \neq 0$ and $\varrho(A) \neq \emptyset$.

Let $\lambda_0 < 0$ and $M' > M$. For each $\mu \in \mathbb{C}$ with $|\mu - \lambda_0| \leq |\lambda_0| / M'$ we have that $\mu \in \varrho(A)$ and $R(\mu, A) = \sum_n (\mu - \lambda_0)^n R(\lambda_0, A)^{n+1}$ (Neumann series, see

Proposition A.2.3). Hence it follows that

$$|\mu| \, \|R(\mu, A)\| \le \frac{|\mu|}{|\lambda_0|} \sum_n \frac{|\mu - \lambda_0|^n}{|\lambda_0|^n} M^{n+1} \le M \frac{|\mu|}{|\lambda_0|} \sum_n \left(\frac{M}{M'}\right)^n$$
$$\le \left(1 + \frac{|\mu - \lambda_0|}{|\lambda_0|}\right) \frac{M}{1 - \frac{M}{M'}} \le \left(1 + \frac{1}{M'}\right) \frac{MM'}{M' - M} = \frac{(M'+1)M}{M' - M}.$$

Now, if we choose $\omega' > \pi - \arcsin(1/M)$ and define $M' := 1/\sin(\pi - \omega')$, then clearly $M' > M$. Choose $\mu \in \mathbb{C}$ such that $\pi \ge |\arg \mu| \ge \omega'$ and define $\lambda_0 := \operatorname{Re} \mu$. Then $\lambda_0 < 0$ and $|\mu - \lambda_0| \le |\lambda_0| / M'$, whence $\|\mu R(\mu, A)\| \le (M'+1)M/(M'-M)$.

b) The identity (2.1) is true for all operators and all $\lambda \neq 0$, cf. Lemma A.2.1. The second statement follows readily.

c) Consider the stated equivalence $x \in \overline{\mathcal{D}(A)} \iff \lim_{t\to\infty} t^n (t+A)^{-n} x = x$. One implication is clear. For the other implication, let first $x \in \mathcal{D}(A)$ and consider the identity $x = t(t+A)^{-1}x + (1/t)[t(t+A)^{-1}]Ax$. After repeatedly inserting this identity into itself one arrives at

$$x = [t(t+A)^{-1}]^n x + \frac{1}{t} \sum_{k=1}^{n} [t(t+A)^{-1}]^k Ax.$$

This shows $\lim_{t\to\infty} [t(t+A)^{-1}]^n x = x$ for $x \in \mathcal{D}(A)$. By the uniform boundedness of the operators $([t(t+A)^{-1}]^n)_{t>0}$ this is then true for all $x \in \overline{\mathcal{D}(A)}$. The other equivalences are treated similarly.

d) is an immediate consequence of c).

e) Evidently $\mathcal{N}(A) \subset \mathcal{N}(A^n)$. But if $x \in \mathcal{N}(A^n)$ and $n \ge 2$, then in particular $x \in \mathcal{D}(A^{n-1})$ and one has $0 = (t+A)^{-1}A^n x = A(t+A)^{-1}A^{n-1}x$ for all $t > 0$. Because $A^{n-1}x \in \mathcal{R}(A)$, one may apply c) to obtain $A^{n-1}x = 0$. By repeating this argument one finally arrives at $Ax = 0$.

f) Define (for the moment) $A_\delta := A + \delta$, and let $\omega' \in (\omega, \pi)$. From $\lambda \in S_{\pi-\omega'}$ it follows that $\lambda + \delta \in S_{\pi-\omega'}$, and because

$$\lambda(\lambda + A_\delta)^{-1} = \frac{\lambda}{\delta + \lambda} (\delta + \lambda)(A + \delta + \lambda)^{-1},$$

we obtain $M(A_\delta, \omega') \le M(A, \omega') \sup_{\lambda \in S_{\pi-\omega'}} |\lambda/\lambda + \delta|$. Hence the family $(A_\delta)_{\delta \ge 0}$ is uniformly sectorial of angle ω and $M(A_\delta, \omega') \le c(\omega')M(A, \omega')$ with $c(\omega')$ as described.

The uniform sectoriality of the family $(\varepsilon A)_{\varepsilon \ge 0}$ is clear from the fact that sectors are invariant under dilations with positive factors.

Finally, let $\varepsilon > 0$. Because $\lambda \in S_{\pi-\omega'}$, we have $\lambda(1+\lambda)^{-1}\varepsilon + \delta \in S_{\pi-\omega'}$, and so

$$\lambda + A_\delta(A_\delta + \varepsilon)^{-1} = (\lambda(A_\delta + \varepsilon) + A_\delta)(A_\delta + \varepsilon)^{-1} = (\lambda + 1)\left(A_\delta + \varepsilon\lambda/(1+\lambda)\right)(A_\delta + \varepsilon)^{-1}$$

for these λ. Hence the operator $\lambda + A_\delta(A_\delta+\varepsilon)^{-1}$ is invertible, and

$$\begin{aligned}\lambda[\lambda + A_\delta(A_\delta+\varepsilon)^{-1}]^{-1} &= \frac{\lambda}{1+\lambda}(A_\delta+\varepsilon)[A_\delta + \frac{\lambda}{1+\lambda}\varepsilon]^{-1}\\ &= \frac{\lambda}{1+\lambda}(A_\delta + \frac{\lambda}{1+\lambda}\varepsilon + \frac{\varepsilon}{1+\lambda})[A_\delta+\frac{\lambda}{1+\lambda}\varepsilon]^{-1}\\ &= \frac{\lambda}{1+\lambda}I + \frac{1}{1+\lambda}\left[\frac{\lambda\varepsilon}{1+\lambda}\left(A_\delta + \frac{\lambda\varepsilon}{1+\lambda}\right)^{-1}\right].\end{aligned}$$

The uniform sectoriality of the family $(A_\delta)_\delta$ now gives the first statement of f).

g) We consider first the reverse implication. To prove it we select $x \in \mathcal{D}(A^m)$. Then $A(A+\varepsilon)^{-1}x = x - \varepsilon(A+\varepsilon)^{-1}x \in \mathcal{D}(A^m)$, and an iteration of this argument yields $(A(A+\varepsilon)^{-1})^n x \in \mathcal{D}(A^m)$. The proof of the direction '$\Rightarrow$' can also be reduced to the case $n = 1$, which is proved by induction on m.

h) Let $x \in X$, and suppose that X is reflexive. The sequence $(n(n+A)^{-1}x)_{n\in\mathbb{N}}$ is bounded and hence it has a weakly convergent subsequence $n_k(n_k+A)^{-1}x \rightharpoonup y$. This means that $A(n_k+A)^{-1}x \rightharpoonup x-y$. Now, $(n_k+A)^{-1}x \to 0$ even strongly. Because the graph of A is closed and a linear subspace of $X \oplus X$, it is weakly closed, whence $x - y = 0$. This means that x lies in the weak closure of $\mathcal{D}(A)$, but the weak and the strong closure of $\mathcal{D}(A)$ coincide, by the Hahn–Banach theorem. Hence it follows that $x \in \overline{\mathcal{D}(A)}$.

If $x \in \mathcal{N}(A) \cap \overline{\mathcal{R}(A)}$, then $0 = Ax = \lim_{t\to 0}(t+A)^{-1}Ax = \lim_{t\to 0} A(t+A)^{-1}x = x$ by c). Therefore the sum is direct. For arbitrary $x \in X$, by the reflexivity of X one can find numbers $t_n \searrow 0$ and a $y \in X$ such that $t_n(t_n+A)^{-1}x \rightharpoonup y$. But we have $t_nA(t_n+A)^{-1}x \to 0$. The graph of A is weakly closed, hence $y \in \mathcal{N}(A)$. This implies that $A(t_n+A)^{-1}x \rightharpoonup x-y$. Therefore $x-y$ is in the weak closure of $\mathcal{R}(A)$, but the weak and the strong closure of $\mathcal{R}(A)$ coincide. It follows that $x \in \mathcal{N}(A) + \overline{\mathcal{R}(A)}$.

i) If $\mu \notin \overline{S_{\omega'}}$, then $\mu\lambda^{-1} \notin \overline{S_{\omega'-|\arg\lambda|}}$. Hence $\mu R(\mu,\lambda A) = (\mu\lambda^{-1})R(\mu\lambda^{-1},A)$ is uniformly bounded by $M(A,\omega'-|\arg\lambda|)$ for such μ.

k) This follows since $\varrho(A) = \varrho(A')$ and $\|R(\lambda,A)\| = \|R(\lambda,A)'\| = \|R(\lambda,A')\|$ for all $\lambda \in \varrho(A)$, cf. Proposition A.4.2 and Corollary A.4.3. □

Remark 2.1.2. Let $A \in \mathrm{Sect}(\omega)$ be a sectorial operator. Then

$$M(A,\varphi) = \inf_{\omega' \in (\omega,\varphi)} M(A,\omega')$$

for each $\varphi \in (\omega,\pi]$. This follows from the proof of Proposition 2.1.1 a). Actually, the function

$$(\varphi \longmapsto M(A,\varphi)) : (\omega,\pi] \longrightarrow [1,\infty)$$

is continuous, see [161, Proposition 1.2.2].

Let $A \in \mathrm{Sect}(\omega)$ on the Banach space X. We define $Y := \overline{\mathcal{R}(A)}$ and denote by B the part of A in Y, i.e.,

$$\mathcal{D}(B) := \mathcal{D}(A) \cap \overline{\mathcal{R}(A)} \quad \text{with} \quad By := Ay \qquad (y \in \mathcal{D}(B)).$$

It is easy to see that $B \in \mathrm{Sect}(\omega)$ on Y with $M(\omega', B) \le M(\omega', A)$ for all $\omega' \in (\omega, \pi]$. Moreover, B is injective, so we call B the **injective part** of A. The identity

$$\mathcal{D}(B^n) = \mathcal{D}(A^n) \cap Y \qquad (n \in \mathbb{N})$$

is easily proved by induction. Proposition 2.1.1 yields the fact that $\mathcal{N}(A) \cap \overline{\mathcal{R}(A)} = \mathcal{N}(A) \cap Y = 0$, and by a short argument one obtains

$$R(\lambda, A)(x \oplus y) = \frac{1}{\lambda} x \oplus R(\lambda, B)y$$

for all $x \in \mathcal{N}(A)$ and $y \in Y$.

2.1.1 Examples

Let us browse through a list of examples.

Multiplication Operators

Let (Ω, Σ, μ) be a σ-finite measure space, and let $a \in \mathrm{Mes}(\Omega, \mu; \mathbb{C})$ be as in Section 1.4. Define $A := M_a$ to be the multiplication operator on $\mathbf{L}^p(\Omega, \mu)$. Theorem 1.4.2 shows that $K = \operatorname{essran} a$ equals the spectrum $\sigma(A)$. We claim that A is sectorial of angle $\omega \in [0, \pi)$ if and only if $K \subset \overline{S_\omega}$. This clearly is a necessary condition, but we also have

$$\|R(\lambda, A)\| = \frac{1}{\operatorname{dist}(\lambda, K)} \le \frac{1}{\operatorname{dist}(\lambda, \overline{S_\omega})}$$

for all $\lambda \in \mathbb{C} \setminus \overline{S_\omega}$ by Theorem 1.4.2. So the pure location of the spectrum already implies the resolvent growth condition. (This is not true for general operators.)

Generators of Semigroups

Let $-A$ generate a bounded semigroup $(T(t))_{t>0}$ in the sense of Appendix A.8. Then with $M := \sup_{t>0} \|T(t)\|$ one has

$$\left\|(\lambda + A)^{-1}\right\| = \|R(\lambda, -A)\| \le \frac{M}{\operatorname{Re} \lambda} \qquad (\operatorname{Re} \lambda > 0) \tag{2.2}$$

by (the easy part of) the Hille–Yosida theorem (Theorem A.8.6). This shows that A is sectorial of angle $\pi/2$.

An estimate of the form (2.2) does not suffice to ensure that $-A$ generates a semigroup, neither if $M = 1$ (e.g. the Volterra operator on $\mathbf{C}[0, 1]$, cf. Section 8.5), nor if A is densely defined (see [233]). However, there is a perfect correspondence between generators of bounded *analytic* semigroups and sectorial operators of angle strictly *less* than $\pi/2$. See Section 3.4 for more information.

Normal Operators and Numerical Range Conditions

Let $X = H$ be a Hilbert space, and let A be a normal operator on H. By the Spectral Theorem D.5.1 the operator A is similar to a multiplication operator on some $\mathbf{L^2}$-space. Hence by our first example, A is sectorial of angle ω if and only if $\sigma(A) \subset \overline{S_\omega}$.

If A is not normal but has its numerical range $W(A)$ contained in $\overline{S_\omega}$ for some $\omega \in [0, \pi/2]$ then A is sectorial of angle ω if and only if A has some resolvent point outside the sector. This follows from Appendix C.3. The Hilbert space theory is dealt with more deeply in Chapter 7.

For more concrete examples see Chapter 8.

2.1.2 Sectorial Approximation

The present section formalises an important approximation tool in the theory of sectorial operators.

A uniformly sectorial sequence $(A_n)_{n\in\mathbb{N}}$ of angle ω is called a **sectorial approximation** on S_ω for the operator A if

$$\lambda \in \varrho(A) \quad \text{and} \quad R(\lambda, A_n) \to R(\lambda, A) \text{ in } \mathcal{L}(X) \tag{2.3}$$

for some $\lambda \notin \overline{S_\omega}$.[1] From Proposition A.5.3 it follows that in this case (2.3) is true for *all* $\lambda \notin \overline{S_\omega}$. Moreover, A itself is sectorial of angle ω.

If $(A_n)_n$ is a sectorial approximation for A on S_ω, we write $A_n \to A$ (S_ω) and speak of **sectorial convergence**.

Proposition 2.1.3. a) *If $A_n \to A$ (S_ω) and all A_n as well as A are injective, then $A^{-1} \to A^{-1}$ (S_ω).*

b) *If $A_n \to A$ (S_ω) and $A \in \mathcal{L}(X)$, then $A_n \in \mathcal{L}(X)$ for large $n \in \mathbb{N}$, and $A_n \to A$ in norm.*

c) *If $A_n \to A$ (S_ω) and $0 \in \varrho(A)$, then $0 \in \varrho(A_n)$ for large n.*

d) *If $(A_n)_n \subset \mathcal{L}(X)$ is uniformly sectorial of angle ω, and if $A_n \to A$ in norm, then $A_n \to A$ (S_ω).*

e) *If $A \in \mathrm{Sect}(S_\omega)$, then $(A + \varepsilon)_{\varepsilon>0}$ is a sectorial approximation for A on S_ω.*

f) *If $A \in \mathrm{Sect}(S_\omega)$, then $(A_\varepsilon)_{0<\varepsilon\leq 1}$, where*

$$A_\varepsilon := (A + \varepsilon)\,(1 + \varepsilon A)^{-1},$$

is a sectorial approximation for A on S_ω.

[1] Obviously this concept extends to nets. However, in all relevant situations in this book everything can be reduced to sequences.

Proof. a) This follows from b) in Proposition 2.1.1.

b) If $A_n \to A\,(S_\omega)$ and $A \in \mathcal{L}(X)$, then $(1+A_n)^{-1} \to (1+A)^{-1}$ in norm. But the set of bounded invertible operators on X is open with the inversion mapping being continuous, hence eventually $(1+A_n)^{-1}$ is invertible, and $(1+A_n) \to (1+A)$ in norm.

c) Define $B_n := A_n^{-1}$ and $B := A^{-1}$. Then B_n and B are possibly multi-valued operators. However, by the Spectral Mapping Theorem for the resolvent (Proposition A.3.1) and Lemma A.2.1 we have $(1+B)^{-1}, (1+B_n)^{-1} \in \mathcal{L}(X)$ and $(1+B_n)^{-1} \to (1+B)^{-1}$ in norm. With the same argument as in the proof of b) we conclude that $B_n \in \mathcal{L}(X)$ for large $n \in N$.

d) Suppose that $(A_n)_n$ is uniformly sectorial with $A_n \to A$ in norm. Then $(1+A_n) \to (1+A)$ in norm and $\sup_n \left\|(1+A_n)^{-1}\right\| < \infty$. This implies that $(1+A)^{-1} \in \mathcal{L}(X)$ and that $(1+A_n)^{-1} \to (1+A)^{-1}$ in norm.

e) Let A be sectorial. The uniform sectoriality of $(A+\varepsilon)_{\varepsilon>0}$ has been shown in the proof of Proposition 2.1.1 f). But it is clear that $(1+\varepsilon+A)^{-1} \to (1+A)^{-1}$ in norm.

f) This is a consequence of Proposition 2.1.1 f) and the identity

$$(A+\varepsilon)(1+\varepsilon A)^{-1} = \varepsilon^{-1}(A+\varepsilon)\left(A+\varepsilon+\frac{1-\varepsilon^2}{\varepsilon}\right)^{-1}.$$

□

Remark 2.1.4. Although we have assumed throughout the present section that A is single-valued, the *definition* of sectoriality makes perfect sense even if A is multi-valued. Admittedly, we deal mostly with single-valued operators, but sometimes it is quite illuminating to have the multi-valued case in mind. Therefore, we speak of a 'multi-valued, sectorial operator' whenever it is convenient. Note that the fundamental identity (2.1) still holds in the multi-valued case and readily implies that the inverse of a sectorial operator is sectorial, with the same angle. One has $x \in \overline{\mathcal{D}(A)} \Leftrightarrow \lim_{t\to\infty} t(t+A)^{-1}x = x$ in the multi-valued case as well (cf. c) of Proposition 2.1.1). This shows $A0 \cap \overline{\mathcal{D}(A)} = 0$. Most statements of this section remain true in the multi-valued case, at least after adapting notation a little. As a rule, one has to replace expressions of the form $B(B+\lambda)^{-1}$ by $I - \lambda(B+\lambda)^{-1}$. For example we obtain that $A_\varepsilon = 1/\varepsilon - \left((1-\varepsilon^2)/\varepsilon\right)(1+\varepsilon A)^{-1}$ is a sectorial approximation of A by bounded and invertible operators (cf. Proposition 2.1.3 f)). Note also that part k) of Proposition 2.1.1 holds even without the assumption that A is densely defined. In this case the adjoint A' is a multi-valued sectorial operator.

2.2 Spaces of Holomorphic Functions

In the next section we construct a functional calculus for sectorial operators. This is done by proceeding along the lines of Chapter 1 with the primary calculus constructed by means of a Cauchy integral. Since the spectrum of a sectorial operator is in general unbounded, one has to integrate along infinite lines (here: the boundary of a sector). As a matter of fact, this is only possible for a restricted collection of functions. Dealing with these functions requires some notation, which we now introduce.

As in Chapter 1 we write $\mathcal{O}(\Omega)$ for the space of all **holomorphic functions** and $\mathcal{M}(\Omega)$ for the space of all **meromorphic functions** on the open set $\Omega \subset \mathbb{C}$. Suppose that A denotes a sectorial operator of angle ω on a Banach space X. We wish to define operators of the form

$$f(A) := \frac{1}{2\pi i} \int_\Gamma f(z) R(z, A)\, dz, \tag{2.4}$$

where $f \in \mathcal{O}(S_\varphi)$, $\varphi \in (\omega, \pi]$, and the path Γ 'surrounds' the sector S_ω in the positive sense. This means in particular that — considered as a curve on the Riemann sphere — Γ passes through the point ∞. To give meaning to the above integral, the function f should have a rapid decay at ∞. We therefore introduce the notion of a *polynomial limit*.

Polynomial Limits

Let $\varphi \in (0, \pi]$, and let $f \in \mathcal{M}(S_\varphi)$. We say that f has **polynomial limit** $c \in \mathbb{C}$ at 0 if there is $\alpha > 0$ such that $f(z) - c = O(|z|^\alpha)$ as $z \to 0$. We say that f has polynomial limit ∞ at 0 if $1/f$ has polynomial limit 0 at 0. Similarly, we say that f has polynomial limit $d \in \mathbb{C}_\infty$ at ∞ if $f(z^{-1})$ has polynomial limit d at 0.

We say that f has a **finite polynomial limit** at 0 (at ∞) if there is $c \in \mathbb{C}$ such that f has polynomial limit c at 0 (at ∞). If f has polynomial limit 0 at 0 (at ∞), we call f **regularly decaying** at 0 (at ∞).

Clearly, if f, g both have finite polynomial limits at 0 (at ∞) also fg and $f + g$ do so.

Remarks 2.2.1. 1) For a function f to have a polynomial limit at ∞ means that f has a limit (within $\mathbb{C}_\infty$) and this limit is approached at least polynomially fast. Note that if this limit is ∞ this does not imply that f is polynomially bounded at ∞. Indeed, considered as a function on S_φ with $\varphi \in [0, \pi/2)$, the function e^z has polynomial limit ∞ at ∞ but is far from being polynomially bounded.

2) If f is meromorphic at 0, then f has a polynomial limit at 0 and this limit is finite if and only if f is holomorphic at 0. The same remark applies to the point ∞.

Now look again at the Cauchy integral (2.4). By the sectoriality of A, the function f being regularly decaying at ∞ guarantees integrability at infinity, at least if Γ is eventually straight. The same holds if f is regularly decaying at ∞. It is therefore natural to consider the so-called **Dunford–Riesz class** on S_φ, defined by

$$H_0^\infty(S_\varphi) := \{f \in H^\infty(S_\varphi) \mid f \text{ is regularly decaying at } 0 \text{ and at } \infty\},$$

where

$$H^\infty(S_\varphi) := \{f \in \mathcal{O}(S_\varphi) \mid f \text{ is bounded}\}$$

is the Banach algebra of all **bounded, holomorphic functions** on S_φ, endowed with the norm

$$\|f\|_\infty = \|f\|_{\infty, S_\varphi} = \sup\{|f(z)| \mid z \in S_\varphi\}.$$

Obviously $H_0^\infty(S_\varphi)$ is an algebra ideal in the algebra $H^\infty(S_\varphi)$. With each $f(z)$ also the function $f(1/z)$ is contained in $H_0^\infty(S_\varphi)$. The following description is often helpful.

Lemma 2.2.2. *Let $\varphi \in (0, \pi]$, and let $f : S_\varphi \longrightarrow \mathbb{C}$ be holomorphic. The following assertions are equivalent:*

(i) *The function f belongs to $H_0^\infty(S_\varphi)$.*

(ii) *There is $C \geq 0$ and $s > 0$ such that $|f(z)| \leq C \min\left\{|z|^s, |z|^{-s}\right\}$ for all $z \in S_\varphi$.*

(iii) *There is $C \geq 0$ and $s > 0$ such that $|f(z)| \leq C \frac{|z|^s}{1+|z|^{2s}}$ for all $z \in S_\varphi$.*

(iv) *There is $C \geq 0$ and $s > 0$ such that $|f(z)| \leq C \left(\frac{|z|}{1+|z|^2}\right)^s$ for all $z \in S_\varphi$.*

Proof. The proof is easy and we omit it. □

It is obvious that neither the rational function $(1 + z)^{-1}$ nor the constant function $\mathbf{1}$ is contained in $H_0^\infty(S_\varphi)$. We therefore define

$$\mathcal{E}(S_\varphi) := H_0^\infty(S_\varphi) \oplus \left\langle (1 + z)^{-1} \right\rangle \oplus \langle \mathbf{1} \rangle$$

called the **extended Dunford–Riesz class**. This set is in fact a subalgebra of $H^\infty(S_\varphi)$, due to the identity

$$\frac{1}{(1 + z)^2} = \frac{1}{1 + z} - \frac{z}{(1 + z)^2}.$$

Lemma 2.2.3. *Let $\varphi \in (0, \pi]$, and let $f : S_\varphi \longrightarrow \mathbb{C}$ be holomorphic. The following assertions are equivalent:*

(i) *The function f belongs to $\mathcal{E}(S_\varphi)$.*

(ii) *The function f is bounded and has finite polynomial limits at 0 and ∞.*

Proof. The implication (i)⇒(ii) is obvious. To prove the converse, suppose that $f \in H^\infty(S_\varphi)$ has finite polynomial limits at 0 and ∞. Then the function $g(z) := f(z) - f(\infty) - [f(0) - f(\infty)]/(1 + z)$ is contained in $H_0^\infty(S_\varphi)$. □

We remark that both algebras $H_0^\infty(S_\varphi)$ and $\mathcal{E}(S_\varphi)$ are invariant under inversion, i.e., with $f(z)$ also $f(z^{-1})$ belongs to the set. This is clear for $H_0^\infty(S_\varphi)$. For the larger algebra it follows from

$$\frac{1}{1+z^{-1}} = \frac{z}{1+z} = \mathbf{1} - \frac{1}{1+z}.$$

If the precise sector S_φ is understood from the context or unimportant, we simply write H_0^∞ instead of $H_0^\infty(S_\varphi)$ in the sequel. The same applies to the other function spaces.

Example 2.2.4. Let $f \in H^\infty(S_\varphi)$ such that f is regularly decaying at ∞ and has a holomorphic extension to a neighbourhood of 0. Then $f \in \mathcal{E}(S_\varphi)$. Indeed, by holomorphy there is a constant C such that $|f(z) - f(0)| \leq C\,|z|$ for z near 0. Hence f has a finite polynomial limit at 0.

An important special case is the function e^{-z}, provided one takes $\varphi \in (0, \pi/2)$.

Example 2.2.5. Let $0 < \operatorname{Re}\beta < \operatorname{Re}\alpha$. Then for all $\varphi \in (0, \pi)$

$$\frac{z^\beta}{(1+z)^\alpha} \in H_0^\infty(S_\varphi), \quad \frac{1}{(1+z)^\alpha} \in \mathcal{E}(S_\varphi), \quad \text{and} \quad \frac{z^\alpha}{(1+z)^\alpha} \in \mathcal{E}(S_\varphi).$$

Proof. The function $(1+z)^{-\alpha}$ clearly is regularly decaying at ∞ and has a holomorphic extension to a neighbourhood of 0. Hence by Example 2.2.4 it lies in $\mathcal{E}$. This implies that $z^\beta(1+z)^{-\alpha} \in H_0^\infty$. Since $(zw)^\alpha = z^\alpha w^\alpha$ whenever $|\arg z|$, $|\arg w|$, $|\arg z + \arg w| < \pi$, we have

$$\frac{1}{(1+z^{-1})^\alpha} = \left(\frac{1}{1+z^{-1}}\right)^\alpha = \left(\frac{z}{1+z}\right)^\alpha = \frac{z^\alpha}{(1+z)^\alpha}. \qquad \square$$

Next, we consider an important construction.

Example 2.2.6. Let $\psi \in H_0^\infty(S_\varphi)$, and define

$$h(z) := \int_0^1 \psi(sz)\,\frac{ds}{s} \quad \text{and} \quad g(z) := \int_1^\infty \psi(sz)\,\frac{ds}{s}$$

for $z \in S_\varphi$. Then $h, g \in \mathcal{E}(S_\varphi)$, $h(0) = g(\infty) = 0$, and

$$h(z) + g(z) = \int_0^\infty \psi(s)\,\frac{ds}{s} \qquad (z \in S_\varphi)$$

is constant.

Proof. Choose $C, \alpha > 0$ such that $|\psi(z)| \leq C \min\left\{|z|^\alpha, |z|^{-\alpha}\right\}$. Then

$$|h(z)| \leq C \int_0^1 |sz|^\alpha\,\frac{ds}{s} = (C/\alpha)\,|z|^\alpha.$$

Analogously, $|g(z)| \leq (C/\alpha)\,|z|^{-\alpha}$. This shows not only that g, h are well defined, but also that h behaves well at 0 and g behaves well at ∞. By Morera's theorem, g and h are holomorphic. The function

$$c(z) := \int_0^\infty \psi(sz)\,\frac{ds}{s}$$

is constant on $(0, \infty)$ (by change of variables) hence on the whole sector S_φ (by holomorphy). Thus we can write $h - c = -g$ thereby showing that $g, h \in \mathcal{E}(S_\varphi)$ and $g(0) = h(\infty) = c$. $\square$

2.3 The Natural Functional Calculus

In this section A denotes always a sectorial operator of angle ω on a Banach space X. We pursue our idea of defining a functional calculus by means of a Cauchy integral. Following the abstract setting of Section 1.2 and Section 1.3 we start with the primary calculus.

2.3.1 Primary Functional Calculus via Cauchy Integrals

Fix $\varphi \in (0, \pi)$ and $\delta > 0$. We denote by $\Gamma_\varphi := \partial S_\varphi$ the boundary of the sector S_φ, oriented in the positive sense, i.e.,

$$\Gamma_\varphi := -\mathbb{R}_+ e^{i\varphi} \oplus \mathbb{R}_+ e^{-i\varphi}.$$

For $\varphi \in (\omega, \pi)$ and $f \in H_0^\infty(S_\varphi)$ we define

$$f(A) := \frac{1}{2\pi i} \int_{\Gamma_{\omega'}} f(z)\,R(z, A)\,dz, \tag{2.5}$$

where $\omega' \in (\omega, \varphi)$ is arbitrary. A standard argument using Cauchy's integral theorem shows that this definition is actually independent of ω'. (Figure 3 below illustrates the definition of $f(A)$.)

Lemma 2.3.1. *Let $A \in \mathrm{Sect}(\omega)$, and let $\varphi \in (\omega, \pi)$. Then the following assertions hold.*

a) *The mapping $(h \mapsto h(A)) : H_0^\infty(S_\varphi) \longrightarrow \mathcal{L}(X)$ is a homomorphism of algebras.*

b) *If B is a closed operator commuting with the resolvents of A, then B commutes with $f(A)$. In particular, $f(A)$ commutes with A and with $R(\lambda, A)$ for all $\lambda \in \varrho(A)$.*

c) *We have $R(\lambda, A)f(A) = ((\lambda - z)^{-1} f)(A)$ for each $\lambda \notin \overline{S_\varphi}$.*

Proof. a) is a simple application of Fubini's theorem and the resolvent identity.
b) is trivial.
c) Define $g := (\lambda - z)^{-1} f$ and $\Gamma := \Gamma_{\omega'}$. Then

$$\begin{aligned}(\lambda - A)^{-1} g(A) &= \frac{1}{2\pi i} \int_\Gamma g(z)(\lambda - A) R(z, A)\, dz \\ &= \frac{1}{2\pi i} \int_\Gamma g(z) \big[(\lambda - z) R(z, A) + I\big]\, dz \\ &= \frac{1}{2\pi i} \int_\Gamma f(z) R(z, A)\, dz + \frac{1}{2\pi i} \int_\Gamma g(z)\, dz = f(A)\end{aligned}$$

since the latter summand equals 0 by Cauchy's theorem. □

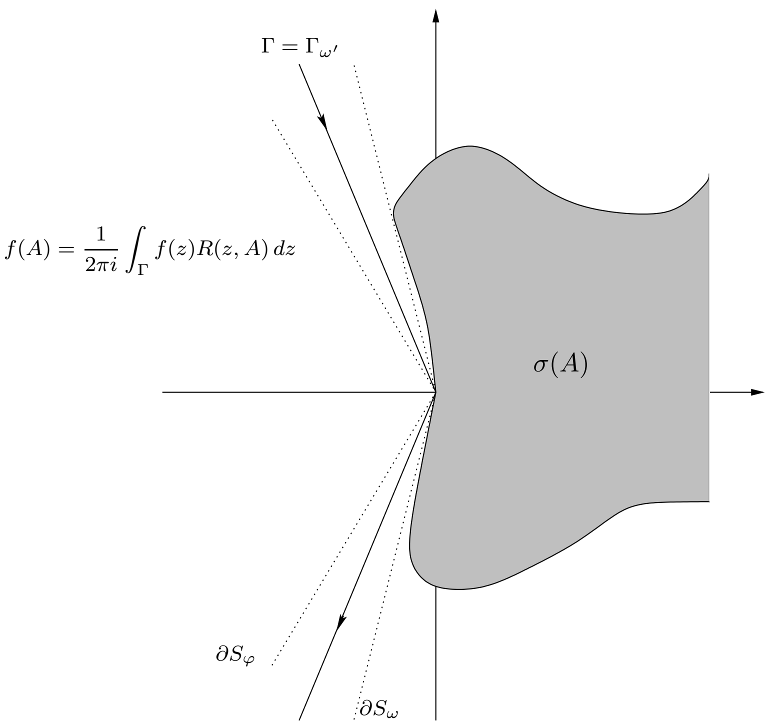


Figure 3: Integration into 0 is possible due to rapid decay of f.

We extend the definition of $f(A)$ from $f \in H_0^\infty(S_\varphi)$ to all $f \in \mathcal{E}(S_\varphi)$. This can be done by defining

$$g(A) := f(A) + c(1 + A)^{-1} + d \tag{2.6}$$

whenever $g = f + c(1+z)^{-1} + d$ with $f \in H_0^\infty(S_\varphi)$ and $c, d \in \mathbb{C}$. We will show that this yields an algebra homomorphism

$$\Phi_A := (g \longmapsto g(A)) : \mathcal{E}(S_\varphi) \longrightarrow \mathcal{L}(X)$$

which gives rise to a meromorphic functional calculus $(\mathcal{E}(S_\varphi), \mathcal{M}(S_\varphi), \Phi)$ in the sense of Section 1.3. To establish this fact, there is still some work to do.

Let us (for the moment) introduce the notation

$$H_{(0)}^\infty(S_\varphi) := \{f \in H^\infty(S_\varphi) \mid f \text{ is regularly decaying at } \infty \\ \text{and holomorphic at } 0\}.$$

We saw in Example 2.2.4 that $H_{(0)}^\infty(S_\varphi) \subset \mathcal{E}(S_\varphi)$. The next lemma shows that for $f \in H_{(0)}^\infty(S_\varphi)$ the value of $f(A)$ can also be computed by a Cauchy integral.

Lemma 2.3.2. *Let $f \in H_{(0)}^\infty(S_\varphi)$, $\omega' \in (\omega, \varphi)$, and let $\delta > 0$ be small enough so that f is holomorphic in a neighbourhood of $\overline{B_\delta(0)}$. Then*

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(z) R(z, A)\, dz$$

where $\Gamma = \Gamma_{\omega',\delta}$ is the positively oriented boundary of $S_{\omega'} \cup B_\delta(0)$.

Figure 4 below gives an impression of what is going on.

Proof. If $f \in H_{(0)}^\infty(S_\varphi) \cap H_0^\infty(S_\varphi)$ then the statement is true since one can shrink the path around the point 0 to 0 without changing the value of the integral. As a general $f \in H_{(0)}^\infty$ may be written as $f = g + c(1+z)^{-1}$ with $g \in H_{(0)}^\infty \cap H_0^\infty$, we are left to show the claim for $f(z) = (1+z)^{-1}$. The idea is to introduce the contour $\Gamma' := -\Gamma_{\omega',R}$ with $R > 1$. Cauchy's theorem shows that $\int_{\Gamma'} (1+z)^{-1} R(z, A)\, dz = 0$. Indeed, without changing the value of this integral the path may be shifted to the left, whereupon the value of the integral tends to 0. However, if we add the integral over Γ' and the initial integral over Γ, some parts cancel and there remains only a simple closed curve around the singularity -1. So by Cauchy's theorem one obtains $-R(-1, A) = (1 + A)^{-1}$ as its final value. $\square$

Now we can put the pieces together.

Theorem 2.3.3. *Let $A \in \mathrm{Sect}(\omega)$ on X, and let $\varphi \in (\omega, \pi)$. The mapping*

$$\Phi_A := (g \longmapsto g(A)) : \mathcal{E}(S_\varphi) \longrightarrow \mathcal{L}(X)$$

defined by (2.6) *is a homomorphism of algebras. Moreover, it has the following properties:*

a) $\left(z(1+z)^{-1}\right)(A) = A(1+A)^{-1}$.

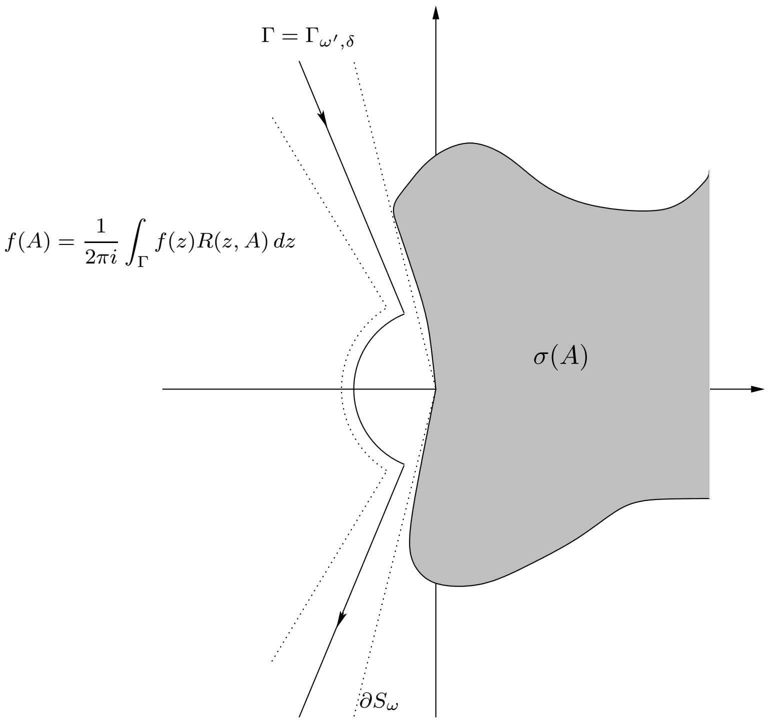


Figure 4: The contour of integration avoids 0 with f being analytic there.

b) *If B is a closed operator commuting with the resolvents of A, then B also commutes with $f(A)$ for each $f \in \mathcal{E}(S_\varphi)$. In particular, each $f(A)$ commutes with A.*

c) *If $x \in \mathcal{N}(A)$ and $f \in \mathcal{E}(S_\varphi)$, then $f(A)x = f(0)x$.*

d) *Let B denote the injective part of A. Then $Y := \overline{\mathcal{R}(A)}$ is invariant under the action of each $f(A)$, and one has $f(B) = f(A)|_Y$.*

Proof. Take two functions $g_i = f_i + c_i(1+z)^{-1} + d_i$ $(i = 1, 2)$ in $\mathcal{E}$ with $f_i \in H_0^\infty$ and $c_i, d_i \in \mathbb{C}$. Since Φ is linear one has to take care of all mixed products. When a constant d_i is involved there is nothing to show; the products $f_1 f_2$ and $f_i(1+z)^{-1}$ are dealt with using Lemma 2.3.1. The assertion for the remaining product $(1+z)^{-1}(1+z)^{-1} = (1+z)^{-2}$ follow from Lemma 2.3.2, Fubini's theorem and the resolvent identity.

a) By what we have already proved, $(z(1+z)^{-1})(A) = \mathbf{1}(A) - (1+z)^{-1}(A) = I - (1+A)^{-1}A(1+A)^{-1}$.

b) is trivial.

c) For $x \in \mathcal{N}(A)$ and $z \in \varrho(A)$ one has $R(z, A)x = (1/z)x$. Hence one has

$$f(A)x = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z} \, dz \, x = 0 \cdot x = 0$$

for $f \in H_0^\infty$, by Cauchy's theorem. The rest follows.
d) follows from the fact that Y is $R(\lambda, A)$-invariant, with $R(\lambda, A)|_Y = R(\lambda, B)$ for all $\lambda \in \varrho(A)$. ☐

We call the algebra homomorphism

$$\Phi_A = (f \longmapsto f(A)) : \mathcal{E}(S_\varphi) \longrightarrow \mathcal{L}(X) \tag{2.7}$$

the **primary functional calculus** (in short: **pfc**) on S_φ for A as a sectorial operator.

Remark 2.3.4. Note that the definition of this primary calculus is perfectly meaningful even if A is multi-valued. Theorem 2.3.3 holds true (after reformulating part a) appropriately) even in this more general case. We shall not use this fact except for Section 3.4 when we treat holomorphic semigroups.

We close this section with a result similar to Lemma 2.3.2.

Corollary 2.3.5. *Let $A \in \operatorname{Sect}(\omega)$, $\varphi \in (\omega, \pi)$, and let $f \in \mathcal{O}(S_\varphi)$ be holomorphic at both points 0 and ∞. Then $f \in \mathcal{E}(S_\varphi)$ and $f(A)$ is given by*

$$f(A) = f(\infty) + \frac{1}{2\pi i} \int_\Gamma f(z) \, R(z, A) \, dz$$

where the contour $\Gamma = \Gamma_{\omega', \delta, R}$ is as is shown in Figure 5 (with $\omega' \in (\omega, \varphi)$, $\delta > 0$ small and $R > 0$ large enough).

Proof. Define $g(z) = f(z) - f(\infty) \in H_{(0)}^\infty$. Then $f(A) = f(\infty) + g(A)$ and we use Lemma 2.3.2 to compute $g(A) = \frac{1}{2\pi i} \int_{\Gamma_{\omega',\delta}} g(z) R(z, A) \, dz$. Applying Cauchy's theorem we may replace the (infinite) path of integration $\Gamma_{\omega',\delta}$ by the finite path $\Gamma_{\omega',\delta,R}$, with R large enough. After another application of Cauchy's theorem the constant $f(\infty)$ cancels and we are done. ☐

2.3.2 The Natural Functional Calculus

As before, let $A \in \operatorname{Sect}(\omega)$ and $\varphi \in (\omega, \pi)$. By Theorem 2.3.3 we have established an abstract functional calculus

$$(\mathcal{E}(S_\varphi), \mathcal{M}(S_\varphi), \Phi_A),$$

which is proper since $(1+z)^{-1}(A) = (1 + A)^{-1}$ is injective. In fact, by Theorem 2.3.3 a) the function $(1 + z)^{-1}$ regularises z and

$$z(A) = (1 + A)(z(1 + z)^{-1})(A) = (1 + A)A(1 + A)^{-1} = A.$$

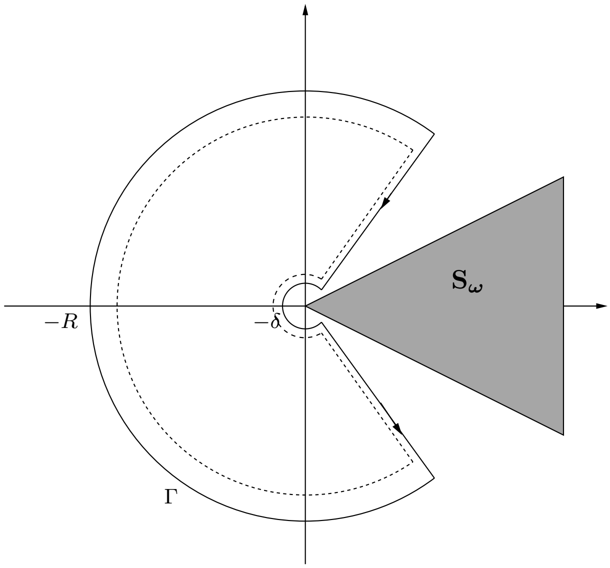


Figure 5: The function f is holomorphic outside the dashed line, including the point ∞.

Hence in the terminology of Section 1.3 our abstract functional calculus is even a *meromorphic* functional calculus for A on S_φ. We write

$$\mathcal{M}(S_\varphi)_A := \left\{ f \in \mathcal{M}(S_\varphi) \;\middle|\; \exists\, e \in \mathcal{E}(S_\varphi) : e(A) \text{ is injective and } ef \in \mathcal{E}(S_\varphi) \right\}$$

to denote the class of functions which are regularisable within this abstract functional calculus. For $f \in \mathcal{M}(S_\varphi)$, the operator $f(A)$ is defined by

$$f(A) := e(A)^{-1}(ef)(A)$$

with $e \in \mathcal{E}(S_\varphi)$ being an arbitrary regulariser of f. The next lemma is quite useful.

Lemma 2.3.6. *Let $A \in \mathrm{Sect}(\omega)$, $\varphi \in (\omega, \pi)$, and $f \in \mathcal{M}(S_\varphi)_A$. Then one can find a regulariser $e \in \mathcal{E}(S_\varphi)$ for f with $e(\infty) = 0$. If A is injective, one can even find a regulariser $e \in H_0^\infty(S_\varphi)$.*

Proof. If d is any regulariser for f then $e := (1+z)^{-1}d$ is a regulariser as well which moreover satisfies $e(\infty) = 0$. If A is injective, then $e := z(1+z)^{-2}d$ is a regulariser for f with $e \in H_0^\infty$. □

Since we have a meromorphic functional calculus for A, it follows from Theorem 1.3.2 that

$$(\mu - z)(A) = \mu - A, \quad (\lambda - z)^{-1}(A) = R(\lambda, A) \tag{2.8}$$

for all $\mu \in \mathbb{C}$ and $\lambda \in \varrho(A)$. More generally, $r(A)$ has its usual meaning (as explained in Appendix A.6) for every rational function r with poles in $\varrho(A)$ (Proposition 1.3.3).

Now note that we can choose φ arbitrarily close to ω. For $\varphi_1 > \varphi_2 > \omega$ we may regard $\mathcal{M}(S_{\varphi_1})$ in a natural way as a subalgebra of $\mathcal{M}(S_{\varphi_2})$ and we have likewise $\mathcal{E}(S_{\varphi_1}) \subset \mathcal{E}(S_{\varphi_2})$. The corresponding functional calculi are *consistent*, which means that these inclusions give rise to a morphism (cf. Proposition 1.2.7)

$$(\mathcal{E}(S_{\varphi_1}), \mathcal{M}(S_{\varphi_1}), \Phi_A) \hookrightarrow (\mathcal{E}(S_{\varphi_2}), \mathcal{M}(S_{\varphi_2}), \Phi_A).$$

Writing $\mathcal{E}[S_\omega] := \bigcup_{\varphi\in(\omega,\pi)} \mathcal{E}(S_\varphi)$ and $\mathcal{M}[S_\omega] := \bigcup_{\varphi\in(\omega,\pi)} \mathcal{M}(S_\varphi)$ we thus (as an 'inductive limit') obtain a meromorphic functional calculus

$$(\mathcal{E}[S_\omega], \mathcal{M}[S_\omega], \Phi)$$

which we call the **natural functional calculus** for A as a sectorial operator on S_ω. Note that again we may view $\mathcal{M}[S_\omega]$ as a subalgebra of $\mathcal{M}(S_\omega)$ so that this calculus is even a meromorphic functional calculus for A on S_ω (in the terminology of Section 1.3). The set

$$\mathcal{M}[S_\omega]_A := \bigcup_{\varphi\in(\omega,\pi)} \mathcal{M}(S_\varphi)_A$$

is called the **domain** of this calculus. As in Chapter 1 we write

$$H(A) := \{f \in \mathcal{M}[S_\omega]_A \mid f(A) \in \mathcal{L}(X)\}$$

as an abreviation.

For simplicity, we frequently omit explicit reference to the sector and simply write '$\mathcal{M}_A$' instead of '$\mathcal{M}[S_\omega]_A$', provided that no ambiguities occur. Also we use the abbreviation $\mathcal{O}[S_\omega]_A := \mathcal{O}[S_\omega] \cap \mathcal{M}_A$ and similar ones for other function classes.

Remarks 2.3.7. 1) One should keep in mind that a function f belonging to $\mathcal{E}[S_\omega]$ or $\mathcal{M}[S_\omega]_A$, is actually defined on a larger sector S_φ for some $\varphi \in (\omega, \pi]$.

2) As in Section 1.3 we warn the reader not to forget that our notations $\mathcal{M}(S_\varphi)_A$, $\mathcal{M}[S_\omega]_A$ and $H(A)$ are somewhat inaccurate. In fact, these sets do not only depend on A but also on the domain of the primary calculus. Since we are going to encounter other primary calculi in later sections, this remark should always be recalled.

3) To distinguish different functional calculi which may be defined for the same operator A (even on the same set Ω) we may add the words 'as a sectorial operator' to the notion of 'natural functional calculus'. If for example the operator A is also bounded, then one has in addition the usual Dunford calculus at hand (see the Preface). This calculus could be named the 'natural functional calculus for A as a bounded operator'. If A is not invertible then $z^{1/2}$ is in the domain of the natural funcional calculus for A as a sectorial, but not as a bounded operator. Similarly, the function $e^{1/(r-z)}$ for $r > r(A)$ is in the domain of the natural functional calculus for A as a bounded, but not as a sectorial operator.

The general properties of the so-defined 'natural functional calculus' for A as a sectorial operator are summarised in Theorem 1.3.2 with $\Omega := S_\omega$ (and, in fact, $\mathcal{M}[S_\omega]_A$ instead of $\mathcal{M}(S_\omega)_A$). We often shall use them without further mention.

Which functions f are contained in $\mathcal{M}_A$? The next lemma shows that a limit behaviour at 0 is necessary if A is not injective.

Lemma 2.3.8. *Let $A \in \mathrm{Sect}(\omega)$, $f \in \mathcal{M}_A$, and $\lambda \in \overline{S_\omega}$. Then at least one of the following assertions holds.*

1) *There is $c \in \mathbb{C}$ and $\alpha > 0$ such that $f(z) - c = O(|z|^\alpha)$ as $z \to \lambda$.*
2) *The operator $\lambda - A$ is injective.*

If A is not injective, then $f(A)x = f(0)x$ for all $x \in \mathcal{N}(A)$.

Proof. Let $e \in \mathcal{E}$ be a regulariser for f. Then also $ef \in \mathcal{E}$. Suppose that f fails 1), and that $\lambda \neq 0$. Then λ is a pole of f, whence $e(\lambda) = 0$. Hence also $h(z) := e(z)/(z - \lambda) \in \mathcal{E}$ and $e(A) \supset h(A)(A - \lambda)$. Since $e(A)$ is injective and $h(A) \in \mathcal{L}(X)$, $A - \lambda$ must be injective.

To complete the proof, suppose $\lambda = 0$ and A is not injective. Since $e(A)x = e(0)x$ for every $x \in \mathcal{N}(A)$ (Theorem 2.3.3) and $e(A)$ is injective, $e(0) \neq 0$. Hence

$$f(z) - f(0) = \frac{[e(z)f(z) - e(0)f(0)] - f(0)[e(z) - e(0)]}{e(z)}$$

for z near 0, thereby showing the assertion. □

Corollary 2.3.9. *Let X be reflexive, and let $A \in \mathrm{Sect}(\omega)$ with $\mathcal{N}(A) \neq 0$. Denote by B the injective part of A. If $f \in \mathcal{M}_A$, then also $f \in \mathcal{M}_B$ with*

$$\mathcal{D}(f(A)) = \mathcal{N}(A) \oplus \mathcal{D}(f(B)) \quad \textit{and} \quad f(A)(x \oplus y) = f(0)x \oplus f(B)y$$

for all $x \oplus y \in \mathcal{N}(A) \oplus \mathcal{D}(f(B))$.

Proof. Note first that $f(0)$ makes sense since A is not injective. Let e be a regulariser for f in the functional calculus for A. Then $e, ef \in \mathcal{E}$ with $e(0) \neq 0$ (since A is not injective) and $e(A)$ is injective. By Theorem 2.3.3, $e(A)(x \oplus y) = e(0)x \oplus e(B)y$ for all $x \oplus y \in \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$, whence $e(B)$ is also injective. This shows that $f \in \mathcal{M}_B$ and

$$f(A) = e(A)^{-1}(ef)(A) = e(0)^{-1} \oplus e(B)^{-1}[(ef)(0) \oplus (ef)(B)] = f(0) \oplus f(B)$$

in suggestive notation. □

2.3.3 Functions of Polynomial Growth

To determine subclasses of $\mathcal{M}_A$, one looks for natural regularisers. Taking elementary rational functions leads to the definition

$$\mathcal{A}(S_\varphi) := \{f : S_\varphi \longrightarrow \mathbb{C} \mid \exists\, n \in \mathbb{N} : f(z)(1 + z)^{-n} \in \mathcal{E}(S_\varphi)\}.$$

and $\mathcal{A}[S_\omega] := \bigcup_{\varphi\in(\omega,\pi)} \mathcal{A}(S_\varphi)$. If φ is understood or not important, we write simply $\mathcal{A}$ instead of $\mathcal{A}(S_\varphi)$.

Lemma 2.3.10. *For $f \in \mathcal{O}[S_\omega]$ the following assertions are equivalent:*

(i) *The function f belongs to $\mathcal{A}$.*

(ii) *The function f has the following two properties:*

1) *$f(z) = O(|z|^\alpha)$ $(z \to \infty)$ for some $\alpha \in \mathbb{R}$, and*
2) *f has a finite polynomial limit at* 0.

(iii) *There is $c \in \mathbb{C}$, $n \in \mathbb{N}$, and $F \in H_0^\infty$ such that $f(z) = c + (1+z)^n F(z)$.*

If f is bounded, one can take $n = 1$ in (iii).

In particular, $\mathcal{A}[S_\omega]$ is an algebra of functions containing every rational function with poles outside of $\overline{S_\omega}$.

Proof. (i)⇒(ii). If $g(z) := f(z)(1+z)^{-n} \in \mathcal{E}$, then clearly condition 1) is satisfied. Because g has a finite polynomial limit at 0, also $f = (1+z)^n g$ has a finite polynomial limit at 0.
(ii)⇒(iii). Choose α as in (ii), and let $n > \alpha$. Then $(f(z) - f(0))(1+z)^{-n} \in H_0^\infty$.

(iii)⇒(i). This is trivial. □

Proposition 2.3.11. *Let $A \in \mathrm{Sect}(\omega)$. Then $\mathcal{A}[S_\omega] \subset \mathcal{M}_A$. For $f \in \mathcal{A}[S_\omega]$ the following assertions hold.*

a) *If A is bounded, so is $f(A)$.*

b) *One has $[(z-\mu)f(z)](A) = (A-\mu)f(A)$ for all $\mu \in \mathbb{C}$.*

c) *If $\overline{\mathcal{D}(A)} = X$ and $f(z)(1+z)^{-n} \in \mathcal{E}$, then $D(A^n)$ is a core for $f(A)$.*

Proof. a) It is immediate that $\mathcal{A}[S_\omega] \subset \mathcal{M}_A$. Let $f \in \mathcal{A}$, and choose n such that $F := f(1+z)^{-n} \in \mathcal{E}$. If A is bounded, then $f(A) = (1+A)^n F(A) \in \mathcal{L}(X)$.
b) By (2.8) and Lemma A.6.1 we have

$$\begin{aligned}
((z-\mu)f(z))(A) &= (1+A)^{n+1}\left(\frac{f(z)(z-\mu)}{(1+z)^{n+1}}\right)(A)\\
&= (1+A)^{n+1}[(A-\mu)(1+A)^{-1}]F(A) = (A-\mu)(1+A)^{n+1}(1+A)^{-1}F(A)\\
&= (A-\mu)(1+A)^n F(A) = (A-\mu)f(A).
\end{aligned}$$

c) Note that the operator $F(A)$ commutes with $(1+A)^{-n}$, whence $\mathcal{D}(A^n)$ is $F(A)$-invariant. This gives $\mathcal{D}(A^n) \subset \mathcal{D}(f(A))$. For arbitrary $x \in \mathcal{D}(f(A))$ we have $T_t(x) := (t(t+A)^{-1})^n x \to x$ as $t \to \infty$. As a matter of fact, $T_t(x) \in \mathcal{D}(A^n)$ and — by Proposition 2.1.1 c) — also $f(A)T_t x = T_t f(A)x \to f(A)x$. □

Remark 2.3.12. Interestingly enough, polynomial growth at ∞ is not the best one can achieve for an arbitrary sectorial operator. In fact, we shall see in Section 3.4 that if $\omega_A < \pi/2$ then $e^{-A} := e^{-z}(A)$ is injective (cf. Proposition 3.4.4.) Hence e^{-z} is a regulariser which compensates even exponential growth at ∞. If ω_A happens to be larger than $\pi/2$, one can use the functions e^{-z^α} with suitable $\alpha \in (0,1)$. Therefore we see that $\mathcal{O}[S_\omega]_A$ is much larger than $\mathcal{A}[S_\omega]$. However, a property like b) of the last proposition cannot be expected for more general functions since its proof rests on the fact that the regulariser is the inverse of a polynomial.

2.3.4 Injective Operators

Throughout this section we consider an *injective*, sectorial operator $A \in \mathrm{Sect}(\omega)$ on the Banach space X. (If X is reflexive, then automatically $\overline{\mathcal{D}(A)} = \overline{\mathcal{R}(A)} = X$.) With A being injective also the function $\tau(z) := z(1+z)^{-2} \in H_0^\infty$ (and each of its powers) is a regulariser since $\tau(A) = A(1+A)^{-2}$ is injective. Let us denote its inverse by

$$\Lambda_A := \tau(A)^{-1} = (1+A)^2 A^{-1} = (1+A)A^{-1}(1+A) = (2 + A + A^{-1}),$$

with $\mathcal{D}(\Lambda_A) = \mathcal{D}(A) \cap \mathcal{R}(A)$. The class of functions regularised by powers of τ is

$$\mathcal{B}(S_\varphi) := \{f : S_\varphi \longrightarrow \mathbb{C} \mid \text{there is } n \in \mathbb{N} : \tau(z)^n f(z) \in H_0^\infty(S_\varphi)\},$$

where $\varphi \in (0,\pi]$. As usual, we write $\mathcal{B} := \mathcal{B}[S_\omega] := \bigcup_{\varphi \in (\omega,\pi)} \mathcal{B}(S_\varphi)$. Obviously, $\mathcal{B}(S_\varphi)$ is an algebra of functions and $\mathcal{A}(S_\varphi) \subset \mathcal{B}(S_\varphi)$. A holomorphic function f on S_φ belongs to $\mathcal{B}$ if and only if f has at most polynomial growth at 0 and at ∞ (and is bounded in between). In particular, $H^\infty(S_\varphi)$ is a subalgebra of $\mathcal{B}(S_\varphi)$.

Proposition 2.3.13. *Let $A \in \mathrm{Sect}(\omega)$ be injective. Then $\mathcal{B}[S_\omega] \subset \mathcal{M}_A$. Moreover, the following assertions hold.*

a) *If X is reflexive, then $\overline{\mathcal{D}(A)} = \overline{\mathcal{R}(A)} = \overline{\mathcal{D}(A^n) \cap \mathcal{R}(A^n)}$ for every $n \in \mathbb{N}$.*

b) *If $\overline{\mathcal{D}(A)} = X = \overline{\mathcal{R}(A)}$ and $f(z)\tau^n \in H_0^\infty$, then $D(A^n) \cap \mathcal{R}(A^n)$ is a core for $f(A)$.*

Proof. a) Let $x \in X$. Then $A^n(t+A)^{-n}(1+tA)^{-n}x \in \mathcal{D}(A^n) \cap \mathcal{R}(A^n)$. By Proposition 2.1.1 we have $\lim_{t\to 0} A^n(t+A)^{-n}(1+tA)^{-n}x = x$ if $x \in \overline{\mathcal{D}(A)} \cap \overline{\mathcal{R}(A)}$.

b) is proved in the same way as c) of Proposition 2.3.11. □

Remark 2.3.14. A law of the form $(zf(z))(A) = Af(A)$ cannot hold in general for all $f \in \mathcal{B}$ (cf. Proposition 2.3.11 b)). Indeed, by Theorem 1.3.2 f) we have $(z^{-1})(A) = A^{-1}$, but $(zz^{-1})(A) = (\mathbf{1})(A) = I \neq AA^{-1} = A[(z^{-1})(A)]$ in general.

We conclude this section with an important example.

Example 2.3.15. Let (Ω, Σ, μ) be a σ-finite measure space, and let $a \in \mathfrak{M}(\Omega, \mu; \mathbb{C})$ be as in Sections 1.4 and 2.1.1. Let $A := M_a$ be the multiplication operator on $\mathbf{L}^{\boldsymbol{p}}(\Omega, \mu)$ and $K := \operatorname{essran} a = \sigma(A)$ its spectrum. We suppose that $A \in \operatorname{Sect}(\psi)$, i.e., $K \subset \overline{S_\psi}$. For simplicity we also suppose that A is injective, i.e., $\mu(a = 0) = 0$ (Lemma 1.4.1). Take $\varphi \in (\psi, \pi)$. Then we have defined two abstract functional calculi

$$(\mathbf{L}^{\boldsymbol{\infty}}(K, a[\mu]), \mathfrak{M}(K, a[\mu]; \mathbb{C}_\infty), \Phi_1) \quad \text{and} \quad (\mathcal{E}(S_\varphi), \mathcal{M}(S_\varphi), \Phi_2)$$

where $\Phi_1(f) = M_{f \circ a}$ and $\Phi_2(f) = f(A)$. Since the set $\{a = 0\}$ is a μ-null set by assumption, $\mathcal{M}(S_\varphi)$ 'embeds' naturally into $\mathfrak{M}(K, a[\mu]; \mathbb{C}_\infty)$. More precisely, one considers the mapping

$$\theta := \big(f \longmapsto [f\big|_K]\big) : \mathcal{M}(S_\varphi) \longrightarrow \mathfrak{M}(K, a[\mu]; \mathbb{C}_\infty).$$

We claim that *this mapping is a morphism of abstract functional calculi* in the sense of Section 1.2.3.

Proof. The only thing to show is that $f(A) = M_{f \circ a}$ for $f \in H_0^\infty(S_\varphi)$. Take $x \in \mathbf{L}^{\boldsymbol{p}}(\Omega, \mu)$ and $x' \in \mathbf{L}^{\boldsymbol{p}'}(\Omega, \mu)$, where p' is the dual exponent. Using Fubini's and Cauchy's theorem we compute

$$\begin{aligned}
\langle x', f(A)x \rangle &= \frac{1}{2\pi i} \int_\Gamma f(z) \, \langle x', R(z, A)x \rangle \; dz \\
&= \frac{1}{2\pi i} \int_\Gamma f(z) \int_\Omega x'(\omega) \frac{x(\omega)}{z - a(\omega)} \, \mu(d\omega) \, dz \\
&= \int_\Omega x'(\omega) x(\omega) \, \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - a(\omega)} \, dz \, \mu(d\omega) \\
&= \int_\Omega x'(\omega) x(\omega) f(a(\omega)) \, \mu(d\omega) = \langle x', M_{f \circ a} x \rangle \, .
\end{aligned}$$

Since x' was arbitrary, we obtain $f(A) = M_{f \circ a}$ as desired. $\square$

The consequence of this fact is that also for the extended calculi one has compatibility, i.e., $f(A) = M_{f \circ a}$ holds for *all* $f \in \mathcal{M}(S_\varphi)_A$.

If A is not injective, all the above statements remain true, for the formal argument one however has to replace $\mathcal{M}(S_\varphi)$ by its subalgebra of functions with limits at 0.

2.4 The Composition Rule

Let $A \in \mathrm{Sect}(\omega)$ be injective. Then also A^{-1} is sectorial, of same angle (Proposition 2.1.1). A straightforward conjecture now would be the validity of an identity

$$f(z^{-1})(A) = f(A^{-1})$$

for $f \in \mathcal{M}[S_\omega]_{A^{-1}}$, at least if also the left-hand side is defined. In fact, even more is true.

Proposition 2.4.1. *Let $A \in \mathrm{Sect}(\omega)$ be injective, and let $f \in \mathcal{M}(S_\varphi)$ for some $\varphi \in (\omega, \pi)$. Then*

$$f \in \mathcal{M}(S_\varphi)_{A^{-1}} \qquad \Longleftrightarrow \qquad f(z^{-1}) \in \mathcal{M}(S_\varphi)_A$$

and in this case $f(A^{-1}) = f(z^{-1})(A)$.

Proof. We employ Proposition 1.3.6. Namely, we have meromorphic functional calculi $(H_0^\infty(S_\varphi), \mathcal{M}(S_\varphi), \Phi_A)$ and $(H_0^\infty(S_\varphi), \mathcal{M}(S_\varphi), \Phi_{A^{-1}})$ and $g(z) := z^{-1}$ maps S_φ to itself. By Proposition 1.3.6 one only has to prove the assertion for all $f \in H_0^\infty(S_\varphi)$. But this follows from an easy change of variables in the defining Cauchy integral, using the fundamental identity (2.1). $\square$

The foregoing was an instance of the so-called **composition rule**

$$(f \circ g)(A) = f(g(A)).$$

(see Section 1.3). As a matter of fact, the rule as it stands does not make sense unless we require some additional hypotheses. Basically, we need that A is sectorial, $g(A)$ is defined and also sectorial, and g maps a sector into another sector. More precisely, we require the following:

1) $A \in \mathrm{Sect}(\omega)$.
2) $g \in \mathcal{M}[S_\omega]_A$ and $g(A) \in \mathrm{Sect}(\omega')$.
3) For every $\varphi' \in (\omega', \pi)$ there is $\varphi \in (\omega, \pi)$ with $g \in \mathcal{M}(S_\varphi)$ and $g(S_\varphi) \subset \overline{S_{\varphi'}}$.

Under these requirements obviously $g(S_\omega) \subset \overline{S_{\omega'}}$.

Theorem 2.4.2. (Composition Rule) *Let the operator A and the function g satisfy the conditions* 1), 2), *and* 3) *above. Then $f \circ g \in \mathcal{M}[S_\omega]_A$ and*

$$(f \circ g)(A) = f(g(A)) \tag{2.9}$$

for every $f \in \mathcal{M}[S_{\omega'}]_{g(A)}$.

Let us first discuss the case that $g = c$ is a constant. Then $g(A) = c$, and if $c \neq 0$, everything is easy by Cauchy's theorem. If $c = 0$, i.e., $g(A) = 0$, then $f \in \mathcal{M}[S_{\omega'}]_{g(A)}$ need to have a 'nice' behaviour at 0 (Lemma 2.3.8). So $f \circ g$ is in fact defined (it is the constant $f(0)$) and again by Lemma 2.3.8 the composition rule holds.

Hence in the following we may suppose without loss of generality that g is not a constant. Given this, by the Open Mapping Theorem one has $g(S_\omega) \subset S_{\omega'}$ and the stronger property

3)′ For every $\varphi' \in (\omega', \pi)$ there is $\varphi \in (\omega, \pi)$ with $g \in \mathcal{M}(S_\varphi)$ and $g(S_\varphi) \subset S_{\varphi'}$.

We now appeal to the *abstract* composition rule Proposition 1.3.6 from Chapter 1 with the data

$$\begin{aligned}
&(\mathcal{E}[S_\omega], \mathcal{M}(S_\omega), \Phi_A) && \text{(meromorphic f.c. for } A \text{ on } S_\omega\text{)},\\
&(\mathcal{E}[S_{\omega'}], \mathcal{M}(S_{\omega'}), \Phi_{g(A)}) && \text{(meromorphic f.c. for } g(A) \text{ on } S_{\omega'}\text{)},\\
&g : S_\omega \longrightarrow S_{\omega'}.
\end{aligned}$$

Hence it suffices to prove the assertions of Theorem 2.4.2 only for $f \in \mathcal{E}[S_\omega]_A$.

Lemma 2.4.3. *Let A and g be as in Theorem* 2.4.2*, and let $f \in \mathcal{E}[S_{\omega'}]$. Then $f \circ g \in H^\infty[S_\omega]_A$.*

Proof. We may suppose that g is not a constant. Clearly $f \circ g \in H^\infty[S_\omega]$. If A is injective, nothing is to prove. If A is not injective, g has a finite polynomial limit $g(0)$ at 0 (Lemma 2.3.8). But f has a finite polynomial limit at $f(0)$. (If $g(0) \neq 0$ this follows from the holomorphy of f at c). So also $f \circ g$ has a finite polynomial limit at 0, and therefore $(f \circ g)(A)$ is defined. □

Lemma 2.4.4. *Let A and g be as above, and let $f \in \mathcal{E}[S_{\omega'}]$. Then the composition rule $f(g(A)) = (f \circ g)(A)$ holds.*

Proof. Again we suppose that g is not a constant. Let $f \in \mathcal{E}[S_{\omega'}]$. Then there are constants $c, d \in \mathbb{C}$ and $f_1 \in H_0^\infty[S_{\omega'}]$ such that $f = c\mathbf{1} + d/(1+z) + f_1$. Now for $f = c\mathbf{1}$ there is nothing to prove. If $f = d/(1+z)$ then the assertion is contained in Theorem 1.3.2 f). Hence without loss of generality we may suppose that $f \in H_0^\infty$. For $\lambda \notin \overline{S_{\omega'}}$ the function $(\lambda - g(z))^{-1}$ certainly is bounded and holomorphic on S_φ. Now

$$f(g(A)) = \frac{1}{2\pi i}\int_{\Gamma'} f(\lambda) R(\lambda, g(A))\, d\lambda = \frac{1}{2\pi i}\int_{\Gamma'} f(\lambda)\left(\frac{1}{\lambda - g(z)}\right)(A)\, d\lambda$$

where $\Gamma' = \partial S_{\omega_1'}$ for suitable $\omega_1' \in (\omega', \pi)$. We choose $\varphi' \in (\omega', \omega_1')$ and according to 3)′ we find $\varphi \in (\omega, \pi)$ such that $g(S_\varphi) \subset S_{\varphi'}$. We consider two cases:

1) A is injective.

2) A is not injective.

The first case is easier to handle. If A is injective we can use $\tau(z) := z(1+z)^{-2}$

as a regulariser, let Γ surround S_ω within S_φ, and compute

$$\begin{aligned}
f(g(A)) &= \Lambda_A \tau(A) f(g(A)) = \Lambda_A \frac{1}{2\pi i}\int_{\Gamma'} f(\lambda)\tau(A)R(\lambda, g(A))\, d\lambda \\
&= \Lambda_A \frac{1}{2\pi i}\int_{\Gamma'} f(\lambda)\left(\frac{z}{(1+z)^2(\lambda - g(z))}\right)(A)\, d\lambda \\
&= \Lambda_A \frac{1}{(2\pi i)^2}\int_{\Gamma'}\int_{\Gamma} f(\lambda)\frac{z}{(1+z)^2(\lambda - g(z))}\, R(z, A)\, dz\, d\lambda \\
&\overset{(1)}{=} \Lambda_A \frac{1}{2\pi i}\int_{\Gamma} \frac{f(g(z))\; z}{(1+z)^2}\, R(z, A)\, dz \\
&= \Lambda_A \left(f(g(z))\, \tau(z)\right)(A) = (f \circ g)(A).
\end{aligned}$$

(Recall the definition $\Lambda_A := \tau(A)^{-1}$.) Equality (1) is an application of Cauchy's integral theorem. Before, one has to interchange the order of integration. To justify this, note that the function

$$\frac{f(\lambda)}{(\lambda - g(z))(1+z)^2} = \frac{\lambda}{\lambda - g(z)}\frac{f(\lambda)}{\lambda}\frac{1}{(1+z)^2}$$

is product integrable on $\Gamma \times \Gamma'$ since the first factor is uniformly bounded.

To cover the second case, suppose that A is not injective. Then g has a finite polynomial limit $c := g(0)$ at 0 (Lemma 2.3.8). Define $g_1(z) := g(z) - g(0)$ and choose a regulariser $e \in \mathcal{E}$ for g_1 with $e(\infty) = 0$. Then $eg_1 \in H_0^\infty$. Furthermore, for $\lambda \notin \overline{S_{\varphi'}}$ the function $[(\lambda - g(z))^{-1} - (\lambda - c)^{-1}]$ is also regularised by e. Thus we compute

$$\begin{aligned}
f(g(A)) &= e(A)^{-1}e(A)f(g(A)) = e(A)^{-1}\frac{1}{2\pi i}\int_{\Gamma'} f(\lambda)e(A)R(\lambda, g(A))\, d\lambda \\
&= e(A)^{-1}\frac{1}{2\pi i}\int_{\Gamma'} f(\lambda)\left(\frac{e(z)}{\lambda - g(z)}\right)(A)\, d\lambda \\
&= e(A)^{-1}\frac{1}{2\pi i}\int_{\Gamma'} f(\lambda)\left(\frac{g_1(z)e(z)}{(\lambda - g(z))(\lambda - g(0))} + \frac{e(z)}{\lambda - c}\right)(A)\, d\lambda \\
&= e(A)^{-1}\frac{1}{2\pi i}\int_{\Gamma'} f(\lambda)\left(\frac{(g_1 e)(z)}{(\lambda - g(z))(\lambda - c)}\right)(A) + \frac{f(\lambda)}{\lambda - c}e(A)\, d\lambda \\
&= e(A)^{-1}\frac{1}{2\pi i}\int_{\Gamma'} f(\lambda)\left(\frac{(g_1 e)(z)}{(\lambda - g(z))(\lambda - c)}\right)(A)\, d\lambda \\
&\qquad + e(A)^{-1}\frac{1}{2\pi i}\int_{\Gamma'} \frac{f(\lambda)}{\lambda - c}\, d\lambda\; e(A).
\end{aligned}$$

The second summand equals $f(c)$ by Cauchy's theorem. The first satisfies

$$\begin{aligned}
&e(A)^{-1}\frac{1}{2\pi i}\int_{\Gamma'} f(\lambda)\left(\frac{(g_1e)(z)}{(\lambda-c)(\lambda-g(z))}\right)(A)\,d\lambda\\
&\quad= e(A)^{-1}\frac{1}{2\pi i}\int_{\Gamma'} f(\lambda)\frac{1}{2\pi i}\int_{\Gamma}\frac{(g_1e)(z)}{(\lambda-c)(\lambda-g(z))}R(z,A)\,dz\,d\lambda\\
&\quad\overset{(1)}{=} e(A)^{-1}\frac{1}{2\pi i}\int_{\Gamma}\left(\frac{1}{2\pi i}\int_{\Gamma'} f(\lambda)\left[\frac{1}{\lambda-g(z)}-\frac{1}{\lambda-c}\right]d\lambda\right)e(z)R(z,A)\,dz\\
&\quad\overset{(2)}{=} e(A)^{-1}\frac{1}{2\pi i}\int_{\Gamma}\left(f(g(z))-f(c)\right)e(z)R(z,A)\,dz\\
&\quad\overset{(3)}{=} e(A)^{-1}\left[(f(g(z))-f(c))e(z)\right](A)\\
&\quad= e(A)^{-1}\left((f\circ g)(A)-f(c)\right)e(A)=(f\circ g)(A)-f(c)
\end{aligned}$$

where we used Fubini's theorem in (1) and Cauchy's theorem in (2). To justify the application of Fubini's theorem in (1) one has, after estimating the resolvent, to consider the function

$$F(\lambda,z):=\frac{f(\lambda)(g_1e)(z)}{(\lambda-c)(\lambda-g(z))z}$$

and prove its product integrability. The representation

$$F(\lambda,z)=\left(\frac{f(\lambda)}{\lambda}\right)\left(\frac{\lambda}{(\lambda-c)(\lambda-g(z))}\right)\left(\frac{(g_1e)(z)}{z}\right)$$

shows that $c\neq 0$ is harmless since $\lambda/(\lambda-g(z))$ is uniformly bounded because of the conditions $g(z)\in\overline{S_{\varphi'}}$ and $\lambda\in\Gamma'$. (Recall that $(eg_1)\in H_0^\infty$.) If $c=0$ (hence $g_1=g$) we write

$$\begin{aligned}
F(\lambda,z)&=\left(\frac{f(\lambda)}{\lambda}\right)\left(\frac{1}{\lambda-g(z)}\right)\left(\frac{(ge)(z)}{z}\right)\\
&=\left(\frac{f(\lambda)}{\lambda^{1+\alpha}}\right)\left(\frac{\lambda^\alpha g(z)^{1-\alpha}}{\lambda-g(z)}\right)\left(\frac{g(z)^\alpha e(z)}{z}\right).
\end{aligned}$$

(Recall that g has no poles within S_φ.) Here $\alpha\in(0,1)$ is chosen in such a way that the first factor remains integrable. Then the middle term is still uniformly bounded. It is easily seen that also $eg^\alpha\in H_0^\infty$, whence F is integrable. □

This concludes the proof of Theorem 2.4.2. We shall encounter several applications of the composition rule throughout the remaining parts of this book, in particular in Chapter 3.

2.5 Extensions According to Spectral Conditions

In this section we treat sectorial operators which are invertible and/or bounded. The additional hypothesis makes it possible to set up a new primary functional calculus extending the $\mathcal{E}$-calculus. Strictly speaking this leads to completely new abstract functional calculi extending the one from the previous section.

2.5.1 Invertible Operators

Let $A \in \mathrm{Sect}(\omega)$ be invertible. Then $B_r(0) \subset \varrho(A)$, where $1/r = r_{A^{-1}}$ is the spectral radius of A^{-1}. It should be clear that in setting up a functional calculus for A on a sector S_φ with $\varphi \in (\omega, \pi]$, the behaviour of the functions at 0 does not matter. Define

$$\mathcal{E}_\infty(S_\varphi) := \big\{ f \in \mathcal{O}(S_\varphi) \;\big|\; f = O(|z|^{-\alpha}) \; (z \to \infty) \text{ for some } \alpha > 0 \big\}.$$

For $A \in \mathrm{Sect}(\omega)$ satisfying $0 \in \varrho(A)$ and $f \in \mathcal{E}_\infty(S_\varphi)$ ($\varphi \in (\omega, \pi]$) we define as usual

$$\Phi(f) := f(A) := \frac{1}{2\pi i} \int_\Gamma f(z) R(z, A)\, dz,$$

where the path Γ bounds a sector $S_{\omega'}$ (with $\omega' \in (\omega, \varphi)$), except for the region near 0 where it avoids 0 at small distance, cf. Figure 6 below.

As a matter of fact, one can prove in a similar way a theorem analogous to Theorem 2.3.3. Hence a meromorphic functional calculus $(\mathcal{E}_\infty(S_\varphi), \mathcal{M}(S_\varphi), \Phi)$ for A on S_φ is defined. As in Section 2.3 one can form the inductive limit as $\varphi \to \omega$ and obtains a meromorphic functional calculus

$$(\mathcal{E}_\infty[S_\omega], \mathcal{M}[S_\omega], \Phi)$$

on $\Omega := S_\omega$. We call it the **natural functional calculus** on S_ω for A as an invertible, sectorial operator. It clearly forms a consistent extension of the natural functional calculus for A as a sectorial operator (defined in Section 2.3).

Remark 2.5.1. The construction of the calculus for invertible sectorial operators can of course be refined. Indeed, one can pass to functions f which are not even defined near the origin. Since this would make necessary a lot more notation we omit it.

In order to prove a *composition rule* as in Theorem 2.4.2, we have to face several cases, namely the combinations

1) A sectorial, $g(A)$ invertible and sectorial;
2) A invertible and sectorial, $g(A)$ sectorial;
3) Both $A, g(A)$ invertible and sectorial.

The proofs are similar to the proof of Theorem 2.4.2. However, in cases 1) and 3) one needs the additional assumption $0 \notin \overline{g(S_\varphi)}$ on the function g.

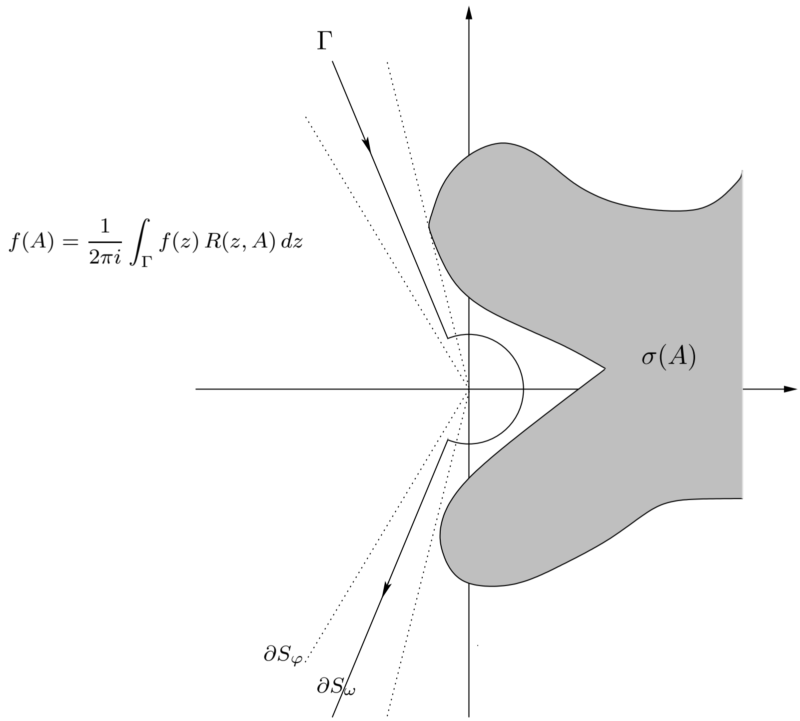


Figure 6: Cauchy integral in the case of invertibility of A.

2.5.2 Bounded Operators

Now we are concerned with the 'dual' situation, namely with a *bounded* operator $A \in \mathrm{Sect}(\omega)$. It is then clear that for the expression $f(A)$ to make sense, the limit behaviour of f at ∞ is irrelevant. Define

$$\mathcal{E}_0(S_\varphi) := \{f \in \mathcal{O}(S_\varphi) \mid f = O(|z|^\alpha)\ (z \to 0) \text{ for some } \alpha > 0\}.$$

For $A \in \mathrm{Sect}(\omega) \cap \mathcal{L}(X)$ and $f \in \mathcal{E}_0(S_\varphi)$ ($\varphi \in (\omega, \pi]$) we define as usual

$$\Phi(f) := f(A) := \frac{1}{2\pi i} \int_\Gamma f(z) R(z, A)\, dz,$$

where the contour Γ bounds a sector $S_{\omega'}$ (with $\omega' \in (\omega, \varphi)$), except for the region near ∞ where it avoids ∞ and stays away from $\sigma(A)$. If A is injective we may leave it at this and obtain a meromorphic functional calculus $(\mathcal{E}_0(S_\varphi), \mathcal{M}(S_\varphi), \Phi)$ for A on S_ω. If A is not injective — as in the case of a general sectorial operator — we have to include also functions which are holomorphic at 0. The easiest way to do this is to pass to the algebra $\widetilde{\mathcal{E}}_0(S_\varphi) := \mathcal{E}_0(S_\varphi) \oplus \mathbb{C}(1+z)^{-1}$ and define the primary calculus by

$$\Phi\left(f + \frac{c}{1+z}\right) := f(A) + c(1+A)^{-1},$$

where $f \in \mathcal{E}_0(S_\varphi)$. In any case, one ends up with a meromorphic functional calculus for A on S_φ and by forming the inductive limit as $\varphi \to \omega$ one obtains a meromorphic functional calculus

$$(\widetilde{\mathcal{E}}_0[S_\omega], \mathcal{M}[S_\omega], \Phi)$$

for A on S_ω; this we call the **natural functional calculus** on S_ω for A as a bounded, sectorial operator. It clearly forms a consistent extension of the natural functional calculus for A as a sectorial operator (defined in Section 2.3).

Remark 2.5.2. Also in this case the so-obtained calculus is not the most general. In fact, one can extend it to functions f which are not even defined near ∞.

As above, to prove a *composition rule* as in Theorem 2.4.2, we have to face the following cases:

1) A sectorial, $g(A)$ bounded and sectorial.
2) A invertible and sectorial, $g(A)$ bounded and sectorial.
3) Both $A, g(A)$ bounded and sectorial.
4) A bounded and sectorial, $g(A)$ sectorial.
5) A bounded and sectorial, $g(A)$ invertible and sectorial.

Again, the proofs are similar to the proof of Theorem 2.4.2, but sometimes one needs an additional assumption on the mapping behaviour of g: In cases 1)–3) one needs $\infty \notin \overline{g(S_\omega)}$ and in case 5) one needs $0 \notin \overline{g(S_\omega)}$.

2.5.3 Bounded and Invertible Operators

If $A \in \mathrm{Sect}(\omega)$ is bounded and invertible, then for a primary calculus no assumptions on the behaviour of f at 0 or ∞ are needed. In fact, one may use the usual Dunford calculus for A. Easy arguments show that this yields a consistent extension of either the natural functional calculus for A as an invertible, sectorial operator or the natural functional calculus for A as a bounded, sectorial operator. Again, composition rules (under suitable assumptions on the function g) could be proved linking all calculi introduced up to now.

Final Remarks

Our systematic introduction of functional calculi for sectorial operators has now come to an end. Depending on how 'good' the operator A is in terms of certain spectral conditions, we have given meaning to the symbol '$f(A)$' where 'f'

denotes a meromorphic function on a sector. As introduced above, the resulting mapping $(f \longmapsto f(A))$ is called the *natural functional calculus* or **nfc** for A as a (bounded, invertible, bounded and invertible) sectorial operator A. The set of functions f such that '$f(A)$' is defined depends heavily on the used *primary* calculus. Hence the assertions '$f(A)$ is defined by the nfc for sectorial operators' and '$f(A)$ is defined by the nfc for bounded, sectorial operators' have different meanings. However, in the following chapters we use mostly the nfc for sectorial operators.

2.6 Miscellanies

In this section we examine the relation between the functional calculi for A and its adjoint A'. Then we look at the part A_Y of A in a continuously embedded Banach space $Y \subset X$. After this we examine the behaviour of the functional calculus with respect to sectorial approximation and prove a fundamental boundedness result.

2.6.1 Adjoints

Let A be a multi-valued, sectorial operator on the Banach space X. Then its adjoint is again a multi-valued, sectorial operator, see Proposition 2.1.1 k) and Remark 2.1.4. How do the functional calculi of A and A' relate? We begin with an abstract consideration.

Let $(\mathcal{E}, \mathcal{M}, \Phi)$ be a proper abstract functional calculus over the Banach space X. Then by $e_* := \Phi(e)' = (e_\bullet)'$ a homomorphism

$$\Phi^* : (e \longmapsto e_*) : \mathcal{E} \longrightarrow \mathcal{L}(X')$$

is defined. Hence $(\mathcal{E}, \mathcal{F}, \Phi^*)$ is an abstract functional calculus over X', the so-called **dual (functional) calculus**. This afc is proper if and only if there is $e \in \mathcal{E}$ such that $\overline{\mathcal{R}(e_\bullet)} = X$. It is now natural to ask, for which $f \in \mathcal{M}$ the identity $f_* = (f_\bullet)'$ holds. We obtain the following result.

Proposition 2.6.1. *Let $(\mathcal{E}, \mathcal{M}, \Phi)$ be an afc with dual calculus $(\mathcal{E}, \mathcal{M}, \Phi^*)$, and let $f \in \mathcal{M}$. Suppose that there is a Φ-regulariser $e \in \mathcal{E}$ for f with the following property: There exists $(f_n)_n \subset \mathcal{M}_b$ such that*

1) *$f_{n\bullet} \to I$ strongly on X;*
2) *$\mathcal{R}(f_{n\bullet}) \subset \mathcal{R}(e_\bullet)$.*

Then the space $\mathcal{R}(e_\bullet)$ is dense in X, it is a core for $f_\bullet$, and $f_* = (f_\bullet)'$.

Proof. It is immediately clear from 1) and 2) that $\mathcal{R}(e_\bullet)$ is dense in X. Hence e is a regulariser for f in the dual calculus. Let $x \in \mathcal{D}(f_\bullet)$ and $y := f_\bullet x$. Then $x_n := f_{n\bullet} x \to x$ and $x_n \in \mathcal{R}(e_\bullet) \subset \mathcal{D}(f_\bullet)$. Moreover $f_\bullet x_n = f_\bullet f_{n\bullet} x = f_{n\bullet} y \to y$.

Since $x \in \mathcal{D}(f_\bullet)$ was arbitrary, we see that $\mathcal{R}(e_\bullet)$ is a core for $f_\bullet$. This means that $\overline{(ef)_\bullet e_\bullet^{-1}} = f_\bullet$. Using this we compute

$$\begin{aligned} f_* &= (e_*)^{-1}(ef)_* = [(e_\bullet)']^{-1}[(ef)_\bullet]' \overset{(1)}{=} [e_\bullet^{-1}]'[(ef)_\bullet]' \\ &\overset{(2)}{=} \left[(ef)_\bullet e_\bullet^{-1}\right]' \overset{(3)}{=} \left[\overline{(ef)_\bullet e_\bullet^{-1}}\right]' = (f_\bullet)'. \end{aligned}$$

We used b), k) and a) of Proposition A.4.2 for (1), (2), and (3), respectively. □

Let us return to sectorial operators. Suppose $A \in \mathrm{Sect}(\omega)$ and $\overline{\mathcal{D}(A)} = X$. This extra condition ensures that A' is again single-valued, whence also $A' \in \mathrm{Sect}(\omega)$. Note that if A is in addition bounded or invertible, the same is true for A'. Hence the primary functional calculi for A and A' have the same domain.

Lemma 2.6.2. *Let $A \in \mathrm{Sect}(\omega)$ with $\overline{\mathcal{D}(A)} = X$. Then the pfc for A' is the dual of the pfc for A, i.e., $f(A') = f(A)'$, whenever $f(A)$ is defined by the primary functional calculus for A.*

Recall that the hypothesis on f means: $f \in \mathcal{E}[S_\omega]$ if A is neither bounded nor invertible, $f \in \mathcal{E}_\infty[S_\omega]$ if $A \in \mathrm{Sect}(\omega)$ is invertible and $f \in \widetilde{\mathcal{E}}_0[S_\omega]$ if $A \in \mathrm{Sect}(\omega)$ is bounded.

Proof. Let f be as required. Then we may write $f(A) = \frac{1}{2\pi i}\int_\Gamma f(z)R(z,A)\,dz$ for some contour Γ. Hence

$$\begin{aligned} f(A)' &= \left(\frac{1}{2\pi i}\int_\Gamma f(z)R(z,A)\,dz\right)' = \frac{1}{2\pi i}\int_\Gamma f(z)R(z,A)'\,dz \\ &= \frac{1}{2\pi i}\int_\Gamma f(z)R(z,A')\,dz = f(A'). \end{aligned}$$
□

Combining this result with Proposition 2.6.1 we can prove the identity $f(A)' = f(A')$ for certain classes of functions f.

Proposition 2.6.3. *Let $A \in \mathrm{Sect}(\omega)$ with $\overline{\mathcal{D}(A)} = X$, and let $f \in \mathcal{M}[S_\omega]$. Then the identity*

$$f(A)' = f(A')$$

holds in the following cases:

a) $f \in \mathcal{A}$;

b) $\overline{\mathcal{R}(A)} = X$ *and* $f \in \mathcal{B}$.

Proof. In the case a) f is regularisable by $e := (1+z)^{-m}$ for some m. Since $\mathcal{D}(A)$ is dense, the functions $f_n := [n(n+z)^{-1}]^m$ satisfy conditions 1) and 2) of Proposition 2.6.1, whence the statement follows. The case b) is treated similarly, with $e = z^m/(1+z)^{2m}$ for some $m \in \mathbb{N}$. Note that the additional condition $\overline{\mathcal{R}(A)} = X$ ensures that both A and A' are injective. □

Remarks 2.6.4. 1) As a matter of course similar results hold for bounded sectorial and for invertible sectorial operators A.

2) Even if nothing more than the sectoriality of A is required, the theorem holds for a much larger class than $\mathcal{A}$. Indeed, suppose $\omega_A < \pi/2$ for simplicity. Then, since $\mathcal{D}(A)$ is dense, $e^{-(1/n)A} \to I$ strongly (cf. Proposition 3.4.1). Therefore even subexponential growth is admissible.

2.6.2 Restrictions

Let A be a sectorial operator on the Banach space X, and let Y be another Banach space, continuously embedded into X. Let $A_Y := A\big|_Y$ be the *part* of A in Y, i.e.,

$$\mathcal{D}(A_Y) := \{x \in Y \cap \mathcal{D}(A) \mid Ax \in Y\}, \quad A_Y y = Ay \quad (y \in \mathcal{D}(A_Y)).$$

(cf. Proposition A.2.8). In general, A_Y is not necessarily sectorial. However, if it is, one may ask how $f(A_Y)$ is computed from $f(A)$. The answer, given in the next proposition, is not surprising.

Proposition 2.6.5. *Let $A \in \mathrm{Sect}(\omega)$ on the Banach space X, and let $Y \subset X$ be another Banach space, continuously included in X. If $A_Y \in \mathrm{Sect}(\omega)$ then the following assertions hold:*

a) *If $e \in \mathcal{E}[S_\omega]$, then Y is invariant under $e(A)$ and $e(A_Y) = e(A)\big|_Y$.*

b) *If $f \in \mathcal{M}[S_\omega]_A$, then $f \in \mathcal{M}[S_\omega]_{A_Y}$ and $f(A_Y) = f(A)_Y$.*

Proof. a) Since A_Y is sectorial of angle ω on Y, the space Y must be invariant under all resolvents $R(\lambda, A)$, $\lambda \notin \overline{S_\omega}$, with $R(\lambda, A_Y) = R(\lambda, A)\big|_Y$ for these λ. If $e \in \mathcal{E}[S_\omega]$, then by Proposition A.2.8 we have $e(A_Y) = e(A)\big|_Y$. In fact, e can be decomposed as $f(z) = d + c(1+z)^{-1} + h(z)$ where $h(A)$ is defined by a Cauchy integral, i.e., as an integral over resolvents.

b) Take a general f and find a regulariser e, i.e., $e, (ef)$ are elementary and $e(A)$ is injective. But then also $e(A_Y) = e(A)_Y = e(A)\big|_Y$ is injective. Furthermore, $(x, y) \in f(A) \cap Y \oplus Y \Leftrightarrow x, y \in Y,\ (ef)(A)x = e(A)y \Leftrightarrow x, y \in Y,\ (ef)(A_Y)x = e(A_Y)y \Leftrightarrow (x, y) \in f(A_Y)$. $\square$

Remark 2.6.6. Proposition 2.6.5 has an obvious analogue for the class of invertible (bounded, bounded and invertible) sectorial operators.

2.6.3 Sectorial Approximation

Suppose that $A \in \mathrm{Sect}(\omega)$ on X and that $(A_n)_n$ is a sectorial approximation of A on S_ω. By Proposition 2.1.3, if A is bounded and/or invertible, the same is true for eventually all A_n. Hence A and the A_n eventually have the same primary functional calculus.

Lemma 2.6.7. *Let $A \in \mathrm{Sect}(\omega)$, and let $(A_n)_n$ be a sectorial approximation of A on S_ω. Then $f(A_n) \to f(A)$ in norm whenever $f(A)$ is defined by the primary functional calculus for A.*

Proof. Let f be as required. Then we can write $f(A) = \frac{1}{2\pi i}\int_\Gamma f(z)R(z,A)\,dz$ for a certain contour Γ. Hence

$$\begin{aligned}\lim_n f(A_n) &= \lim_n \left(\frac{1}{2\pi i}\int_\Gamma f(z)R(z,A_n)\,dz\right) = \frac{1}{2\pi i}\int_\Gamma f(z)\lim_n R(z,A_n)\,dz \\ &= \frac{1}{2\pi i}\int_\Gamma f(z)R(z,A)\,dz = f(A)\end{aligned}$$

by an easy application of Lebesgue's Dominated Convergence theorem. □

It is certainly interesting whether $f(A_n) \to f(A)$ (in some sense) for more general functions f. However, this is a delicate matter since it is not even clear that $f(A_n)$ is defined for large n whenever $f(A)$ is. And even if one assumes this, the regulariser for f might depend on n. Let us quickly skip to the abstract setting.

Let $(\mathcal{E},\mathcal{M},\Phi), (\mathcal{E},\mathcal{M},\Phi_n)$, $n \in \mathbb{N}$, be a family of proper afc over the Banach space X. We call the sequence $[(\mathcal{E},\mathcal{M},\Phi_n)]_n$ **convergent** to the afc $(\mathcal{E},\mathcal{M},\Phi)$ if $\Phi_n(e) \to \Phi(e)$ in norm for every $e \in \mathcal{E}$.

Proposition 2.6.8. *Let $[(\mathcal{E},\mathcal{M},\Phi_n)]_n$ be a sequence of proper afc converging to the proper afc $(\mathcal{E},\mathcal{M},\Phi)$. Let $f \in \mathcal{M}$, and let e be a* **uniform regulariser** *for f, i.e., $e, ef \in \mathcal{E}$ and all $e_\bullet, e_{\bullet_n}$ are injective. If $x_n \in \mathcal{D}(f_{\bullet_n})$ such that $x_n \to x$ and $f_{\bullet_n}x_n \to y$, then $(x,y) \in f_\bullet$.*
In particular, if all $f_{\bullet_n} \in \mathcal{L}(X)$ and $f_{\bullet_n} \to T \in \mathcal{L}(X)$ strongly, then $f_\bullet = T$.

Proof. Let f, e, x_n, x, y be as in the hypothesis. We have $(ef)_\bullet x = \lim_n (ef)_{\bullet_n} x_n = \lim_n e_{\bullet_n} f_{\bullet_n} x_n = e_\bullet y$. This yields $(x,y) \in f_\bullet$. □

Coming back to sectorial operators, Proposition 2.6.8 shows that for a general convergence result one needs uniform regularisers. The next proposition provides an example.

Proposition 2.6.9. *Let $A \in \mathrm{Sect}(\omega)$, and let $(A_n)_n$ be a sectorial approximation of A on S_ω. Take $f \in \mathcal{M}[S_\omega]$ and suppose either*

1) *$f \in \mathcal{A}$, or*
2) *$f \in \mathcal{B}$ and all operators A, A_n are injective.*

If $x_n \in \mathcal{D}(f(A_n))$ with $x_n \to x$ and $f(A_n)x_n \to y$, then also $x \in \mathcal{D}(f(A))$ and $y = f(A)x$. In particular, if $f(A_n) \in \mathcal{L}(X)$ with $f(A_n) \to T \in \mathcal{L}(X)$ strongly, then $f(A) = T$.

Proof. In case 1) a uniform regulariser is $(1+z)^{-m}$ for some m, in case 2) a power of $z/(1+z)^2$ can be used. The rest is only the application of Proposition 2.6.8. □

2.6.4 Boundedness

In the theory of functional calculus it is of ultimate importance to know which families of functions lead to uniformly bounded families of operators. The whole Chapter 5 is dedicated to that question. At the present point we provide only some simple although useful results.

Lemma 2.6.10. *Let $A \in \mathrm{Sect}(\omega)$, $\varphi \in (\omega, \pi)$, and $(f_\alpha)_\alpha \subset H_0^\infty(S_\varphi)$. Let there be numbers $C, s, c_\alpha > 0$ such that*

$$|f_\alpha(z)| \le C \min\left\{ |c_\alpha z|^s, |c_\alpha z|^{-s} \right\} \tag{2.10}$$

for all $z \in S_\varphi$ and all indices α. Then

$$\sup_\alpha \|f_\alpha(A)\| \le (2C/s\pi)\, M(A, \varphi).$$

Proof. Choose $\omega' \in (\omega, \varphi)$. Then, by a simple change of variables in the integral, one obtains the estimate

$$\|f_\alpha(A)\| \le M(A, \omega')\, \frac{C}{2\pi} \int_{\Gamma_{\omega'}} \min\left\{ |z|^s, |z|^{-s} \right\} \frac{|dz|}{|z|} = (2C/s\pi)\, M(A, \omega').$$

Now let ω' tend to φ, cf. Remark 2.1.2. □

Proposition 2.6.11. *Let $\varphi \in (0, \pi]$, and let $f \in \mathcal{E}(S_\varphi)$. Then there is a constant $C_f > 0$ such that*

$$\sup_{t>0} \|f(tA)\| \le C_f M(A, \varphi)$$

for each sectorial operator $A \in \mathrm{Sect}(\omega)$, $\omega \in (0, \varphi)$, on a Banach space X.

Moreover, given $\theta \in [0, \varphi - \omega)$ one has

$$\|f(\lambda A)\| \le C_f M(A, \varphi - \theta)$$

for all $\lambda \in \mathbb{C}, |\arg \lambda| \le \theta$.

Proof. Write $f = \psi + a/(1+z) + b$ with $\psi \in H_0^\infty(S_\varphi), a, b \in \mathbb{C}$. We can choose $C, s > 0$ such that

$$|\psi(z)| \le C \min\left\{ |z|^s, |z|^{-s} \right\} \qquad (z \in S_\varphi).$$

By a change of variable in the defining Cauchy integral (or, if you wish, an application of the composition rule), we obtain $\psi(tA) = \psi(tz)(A)$ for all $t > 0$. Hence by Lemma 2.6.10 we have

$$\|\psi(tA)\| \le \frac{2C}{s\pi} M(A, \varphi).$$

This implies that

$$\|f(tA)\| \le \frac{2C}{s\pi} M(A,\varphi) + |a|\; M(A) + |b| \le \left[\frac{2C}{s\pi} + |a| + |b|\right] M(A,\varphi)$$

since $1 \le M(A) \le M(A,\varphi)$, cf. Proposition 2.1.1 a).

To prove the second assertion, write $\lambda = te^{-i\mu}$ with $|\mu| \le \theta$. Note that $e^{i\mu}A \in \mathrm{Sect}(\omega+\theta)$. Since $\omega+\theta<\varphi$, we may apply the (already proved) first part of the Proposition and obtain

$$\|f(\lambda A)\| = \left\|f(te^{i\mu}A)\right\| \le C_f M(e^{i\mu}A,\varphi) \le C_f M(A,\varphi-\theta). \qquad \square$$

Remark 2.6.12. Proposition 2.6.11 remains meaningful and true if A is a multi-valued operator (cf. Remark 2.3.4). This will play a role in Section 3.4. For more results on boundedness and on approximation we refer to Section 5.2 and Theorem 9.2.4.

2.7 The Spectral Mapping Theorem

In this section we prove a spectral mapping theorem for the natural functional calculus for sectorial operators. The first step is a spectral inclusion theorem.

2.7.1 The Spectral Inclusion Theorem

We begin with two auxiliary results.

Lemma 2.7.1. *Let $A \in \mathrm{Sect}(\omega)$, and let $\lambda \in \mathbb{C}$ be such that $\lambda - A$ is injective. Let $e \in H(A)$, and let $0 \ne c \in \mathbb{C}$ be such that*

$$f(z) := \frac{e(z)-c}{\lambda - z} \in H(A).$$

Then $e(A)(\lambda-A)^{-1} = (\lambda-A)^{-1}e(A)$.

Proof. By Theorem 1.3.2, the inclusion $e(A)(\lambda-A)^{-1} \subset (\lambda-A)^{-1}e(A)$ is always true. To prove the converse, take $x \in X$ such that $e(A)x \in \mathcal{D}((\lambda-A)^{-1})$. Then there is $z \in \mathcal{D}(A)$ with $e(A)x = (\lambda-A)z$. Since $e = (\lambda-z)f + c$ we have $(\lambda-A)z = cx + (\lambda-A)f(A)x$, whence $cx = (\lambda-A)(z-f(A)x)$. Now, $c \ne 0$ by assumption, hence $x \in \mathcal{R}(\lambda-A) = \mathcal{D}((\lambda-A)^{-1})$. $\square$

Lemma 2.7.2. *Let $A \in \mathrm{Sect}(\omega)$, and let $f \in \mathcal{M}[S_\omega]_A$. Given $0 \ne \lambda \in \overline{S_\omega}$ there is a regulariser e for f satisfying $e(\lambda) \ne 0$.*

Proof. Let g be any regulariser for f, i.e., $g, fg \in \mathcal{E}$, and $g(A)$ is injective. Define $e := g/(\lambda-z)^n$ where $n \in \mathbb{N}$ is the order of the zero λ of g. (This order n may be zero.) Then $e(\lambda) \ne 0$. Furthermore, $e \in \mathcal{E}$ and $g(A) = (\lambda-A)^n e(A)$, whence $e(A)$ is injective. Clearly, also $ef = fg/(\lambda-z)^n \in \mathcal{E}$. Hence e is a regulariser for f with $e(\lambda) \ne 0$. $\square$

We now come to the main part.

Proposition 2.7.3. *Let $A \in \mathrm{Sect}(\omega)$, $f \in \mathcal{M}[S_\omega]_A$, and $0 \neq \lambda \in \overline{S_\omega}$. If $f(\lambda) = 0$ and $f(A)$ is invertible, then $\lambda \in \varrho(A)$.*

Proof. Choose a regulariser $e \in \mathcal{E}$ for f with $c := e(\lambda) \neq 0$. Then $(ef) \in \mathcal{E}$ and $(ef)(\lambda) = 0$. This implies that also $h := ef/(\lambda - z) \in \mathcal{E}$, whence

$$h(A)(\lambda - A) \subset (ef)(A) = f(A)e(A).$$

Because $e(A)$ and $f(A)$ are both injective, this shows that $\lambda - A$ must be injective. Also, e is a regulariser for $f/(z - \lambda)$.

It is obvious that $g := (e - c)/(\lambda - z) \in \mathcal{E}$. By Lemma 2.7.1, $e(A)(\lambda - A)^{-1} = (\lambda - A)^{-1}e(A)$. Inverting both sides of this equation yields $(\lambda - A)e(A)^{-1} = e(A)^{-1}(\lambda - A)$. Hence

$$\begin{aligned} f(A) &= \left((\lambda - z)\frac{f}{\lambda - z}\right)(A) = e(A)^{-1}\left((\lambda - z)\frac{ef}{\lambda - z}\right)(A) \\ &= e(A)^{-1}(\lambda - A)\left(\frac{ef}{\lambda - z}\right)(A) = (\lambda - A)e(A)^{-1}\left(\frac{ef}{\lambda - z}\right)(A) \\ &= (\lambda - A)\left(\frac{f}{\lambda - z}\right)(A). \end{aligned}$$

Since $f(A)$ is surjective, $\lambda - A$ must also be surjective, hence $\lambda \in \varrho(A)$. □

Theorem 2.7.4 (Spectral Inclusion Theorem). *Let $A \in \mathrm{Sect}(\omega)$, and let $f \in \mathcal{M}[S_\omega]_A$. Then $f(\sigma(A) \setminus \{0\}) \subset \tilde{\sigma}(f(A))$.*

Proof. Let $0 \neq \lambda \in \sigma(A)$, and define $\mu := f(\lambda)$. If $\mu \neq \infty$, Proposition 2.7.3 applied to the function $\mu - f$ shows that $\mu - f(A)$ cannot be invertible. Hence $\mu \in \sigma(f(A))$.

Suppose that $\mu = \infty \notin \tilde{\sigma}(f(A))$. Then $f(A) \in \mathcal{L}(X)$ and there is $\lambda_0 \in \mathbb{C}$ such that $f(A) - \lambda_0$ is invertible. Hence $g := 1/(f - \lambda_0) \in \mathcal{M}[S_\omega]_A$ with $g(A)$ being invertible and $g(\lambda) = 0$. Another application of Proposition 2.7.3 yields $\lambda \in \varrho(A)$, contradicting the assumption made on λ. □

It is natural to ask what happens at the 'critical' points 0 and ∞. Since $\tilde{\sigma}(f(A))$ is a compact subset of the Riemann sphere, we immediately obtain

$$\overline{f(\sigma(A) \setminus \{0\})}^{\mathbb{C}_\infty} \subset \tilde{\sigma}(f(A)). \tag{2.11}$$

However, consider the Volterra operator V on $\mathbf{C}[0,1]$ (cf. Chapter 1). We know that $\sigma(V) = \{0\}$, whence the above inclusion is trivial. The next result gives a sufficient condition, involving the notion of *polynomial limit* (see page 27).

Theorem 2.7.5. *Let $A \in \mathrm{Sect}(\omega)$, and let $\lambda_0 \in \{0, \infty\}$. If $f \in \mathcal{M}[S_\omega]_A$ has polynomial limit μ at λ_0 and $\lambda_0 \in \tilde{\sigma}(A)$, then $\mu \in \tilde{\sigma}(f(A))$.*

Proof. For $\mu \in \mathbb{C}$ we can consider $f - \mu$ instead of f and in this way reduce the problem to the cases $\mu = 0$ and $\mu = \infty$.

Let us start with the case $\mu = 0, \lambda_0 = \infty$. That is, we suppose that $f(A)$ is invertible and f has polynomial limit 0 at $\lambda_0 = \infty$. If $\infty \in \overline{\sigma(A) \setminus \{0\}}^{\mathbb{C}_\infty}$ then the assertion follows from (2.11). Otherwise there is $R > 0$ such that $\sigma(A) \subset \overline{B_R(0)}$. Define

$$Q := \frac{1}{2\pi i} \int_\Gamma R(z, A)\, dz,$$

where $\Gamma = \partial B_\delta(0)$ for (any) $R < \delta$. It is easy to see that Q is a bounded projection. Let $P := I - Q$ be the complementary projection, and let $Y := \mathcal{R}(P)$ be its range space. Since P commutes with A, $Ay \in Y$ for every $y \in Y \cap \mathcal{D}(A)$. It is also easy to see that $B := A|_Y$ is a sectorial operator of angle ω on Y with $\varrho(A) \subset \varrho(B)$ and $R(\lambda, B) = R(\lambda, A)|_Y$ for all $\lambda \in \varrho(A)$.

Moreover, $\tilde{\sigma}(B) \subset \{\infty\}$. Indeed, it is easily verified that for each $\mu \in \sigma(A)$ one has

$$R(\mu, B) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{\mu - z} R(z, A)\, dz$$

on Y. Clearly $e(B) = e(A)|_Y$ for each $e \in \mathcal{E}[S_\omega]$. This implies that $\mathcal{D}(f(B)) = \mathcal{D}(f(A)) \cap Y$ and $f(B)x = f(A)x \in Y$ for all $x \in \mathcal{D}(f(A)) \cap Y$. Since $f(A)$ is invertible by assumption, also $f(B)$ is invertible with $f(B)^{-1} = f(A)^{-1}|_Y$.

Since f has polynomial limit 0 at ∞, clearly f is in the domain of the primary functional calculus for B! Moreover, for large $n \in \mathbb{N}$ the function $g := zf^n$ still has this property. This gives $g(B) = Bf(B)^n \in \mathcal{L}(X)$. But $f(B)$ is invertible, whence B must be bounded. Since $\tilde{\sigma}(B) \subset \{\infty\}$, we conclude that $\tilde{\sigma}(B) = \emptyset$, hence $Y = 0$. This shows that A is bounded, i.e., $\infty \notin \tilde{\sigma}(A)$.

We now consider the case $\mu = 0, \lambda_0 = 0$. Suppose again that $f(A)$ is invertible and f has polynomial limit 0 at 0. Let $e \in \mathcal{E}$ be a regulariser for f. Then $ef \in \mathcal{E}$, $(ef)(A)$ is injective and $(ef)(0) = 0$. Lemma 2.3.8 now shows that A must be injective. Let $B := A^{-1}$, and define $g(z) := f(z^{-1})$. By Proposition 2.4.1, $g \in \mathcal{M}[S_\omega]_B$, $g(B) = f(A)$ is invertible and g has polynomial limit 0 at ∞. By what we have shown above, B is bounded, whence A is invertible.

Finally, we deal with the case $\mu = \infty$. Suppose that $\infty = \mu \notin \tilde{\sigma}(f(A))$. By definition, $f(A)$ must be bounded. Hence we can find $\lambda \in \mathbb{C}$ such that $\lambda - f(A)$ is invertible. By Theorem 1.3.2, $g := 1/(\lambda - f) \in \mathcal{M}[S_\omega]_A$, $g(A)$ is invertible and g has polynomial limit 0 at λ_0. By the results achieved so far we can conclude that $\lambda_0 \notin \tilde{\sigma}(A)$. □

Cf. [108] for a slightly stronger result.

2.7.2 The Spectral Mapping Theorem

We are heading towards the final goal. As usual, we start with an auxiliary result.

Lemma 2.7.6. *Let $A \in \mathrm{Sect}(\omega)$ and $f \in \mathcal{M}[S_\omega]$. Suppose that f has finite polynomial limits at $\tilde{\sigma}(A) \cap \{0, \infty\}$ and all poles of f are contained in $\varrho(A)$. Then the following assertions are true.*

a) *If $\{0, \infty\} \subset \tilde{\sigma}(A)$, $f(A)$ is defined by the nfc for sectorial operators.*

b) *If A is invertible, $f(A)$ is defined by the nfc for invertible sectorial operators.*

c) *If A is bounded, $f(A)$ is defined by the nfc for bounded sectorial operators.*

In either case, $f(A) \in \mathcal{L}(X)$.

Proof. Let f be as required, and let λ_0 be a pole of f. Then, for suitably large $n_0 \in \mathbb{N}$, the function

$$f_1 := \frac{(\lambda_0 - z)^{n_0}}{(1+z)^{n_0}} f(z)$$

also has finite polynomial limits at $\tilde{\sigma}(A) \cap \{0, \infty\}$, but one pole less than f. Moreover, letting $r_0 := (\lambda_0 - z)^{n_0}/(1+z)^{n_0}$, we see that $r_0(A)$ is bounded and invertible.

Suppose that we are in the situation of a). Then f can have only finitely many poles. By induction we find a bounded rational function r such that $r(A)$ is bounded and invertible and $rf \in \mathcal{E}$. Hence r regularises f to $\mathcal{E}$ and $f(A) = r(A)^{-1}(rf)(A) \in \mathcal{L}(X)$. (Cf. also Proposition 1.2.5.)

A similar reasoning applies in the other cases. Namely, only finitely many poles of f are situated in the 'relevant' part of the domain of f. For example, if A is invertible, the poles of f may accumulate at 0, but for the nfc for A, behaviour of f near 0 is irrelevant (cf. Remark 2.5.1). $\square$

Proposition 2.7.7. *Let $A \in \mathrm{Sect}(\omega)$, and let $f \in \mathcal{M}[S_\omega]_A$ have polynomial limits at the points $\tilde{\sigma}(A) \cap \{0, \infty\}$. Then $\tilde{\sigma}(f(A)) \subset f(\tilde{\sigma}(A))$.*

Proof. Note that by assumption $f(\lambda)$ is defined for each $\lambda \in \tilde{\sigma}(A)$. Take $\mu \in \mathbb{C}$ such that $\mu \notin f(\tilde{\sigma}(A))$. Then the function $(\mu - f)^{-1} \in \mathcal{M}[S_\omega]$ has finite polynomial limits at $\{0, \infty\} \cap \tilde{\sigma}(A)$ and all of its poles (namely, the points $\lambda \in \overline{S_\omega} \setminus \{0\}$ where $f(\lambda) = \mu$) are contained in the resolvent set of A. Hence one may apply Lemma 2.7.6 to conclude that $(\mu - f)^{-1}(A)$ is defined and bounded. But this implies that $\mu - f(A)$ is invertible, whence $\mu \notin \tilde{\sigma}(f(A))$.

Now suppose $\mu = \infty \notin f(\tilde{\sigma}(A))$. This implies that the poles of f are contained in the resolvent of A. An application of Lemma 2.7.6 yields that $f(A)$ is a bounded operator, whence $\infty \notin \tilde{\sigma}(f(A))$. $\square$

Theorem 2.7.8 (Spectral Mapping Theorem). *Let $A \in \mathrm{Sect}(\omega)$, and let $f \in \mathcal{M}[S_\omega]_A$ have polynomial limits at $\{0, \infty\} \cap \tilde{\sigma}(A)$. Then*

$$f(\tilde{\sigma}(A)) = \tilde{\sigma}(f(A)).$$

Proof. Combine Proposition 2.7.7 and Theorem 2.7.5. $\square$

2.8 Comments

2.1 Sectorial Operators. Sectorial operators in our sense were introduced by KATO in [126] who however used the name 'sectorial' for something different (see below). At the same time BALAKRISHNAN in [24] considered operators A satisfying a resolvent estimate $\sup_{t>0} \left\| t(t+A)^{-1} \right\| < \infty$. These operators were later called *non-negative operators* by KOMATSU. On Banach spaces the concepts of non-negative and sectorial operators coincide (Theorem 2.1.1), but this is no longer true when one passes to more general locally convex spaces, cf. [161].

Many articles follow KATO's definition of the name 'sectorial' to denote operators on Hilbert spaces associated with sectorial forms (cf. [130]). We have decided to call them *Kato-sectorial* instead, see Chapter 7. Other texts reserve the name 'sectorial operator' for generators of holomorphic semigroups, like [85] or [157]. Up to a minus sign these operators conform to our sectorial operators of angle strictly less than $\pi/2$. Finally, a sectorial operator is often required to have dense domain and range. However, we felt it more convenient to drop this additional density assumption.

Most of the material of this section is adapted from [161, Section 1.2], where also the standard examples are presented. The notions 'uniform sectoriality' and 'sectorial approximation' unify methods well known in the literature. E.g., the family of operators $A_\varepsilon = (A+\varepsilon)(1+\varepsilon A)^{-1}$ is used in [192, Section 8.1] and [145, Section 2] and is called 'Nollau approximation' in [179]. Basics on multi-valued sectorial operators can be found in [160, Section 2].

2.2 Spaces of Holomorphic Functions. The name 'Dunford–Riesz class' is taken from [216, Section 1.3.3.1] and [218, Section 2] (where the symbol $\mathcal{DR}$ is used). In [51] this class is denoted by $\Psi(S_\varphi)$ but meanwhile the notation H_0^∞ is prevalent. Example 2.2.6 is from MCINTOSH's seminal paper [167].

2.3 The Natural Functional Calculus. As for the historical roots of the functional calculus developed here, we have already given a short account in Chapter 1. The 'mother of all functional calculi', so to speak, is provided by the Fourier transform or, more generally, by the spectral theorem for bounded normal operators on a Hilbert space (see Appendix D). The aim to obtain a similar tool for general bounded operators on a Banach space lead to the so called Dunford–Riesz calculus, see [79, Section VII.3] for the mathematics and [79, Section VII.11] for some historical remarks. This turned out to be only a special case of a general construction in Banach algebras; an account of it can be found in [49, Chapter VII, §4], cf. also Chapter 1. However, in the Banach space situation only bounded operators were considered so far.

The first approach to a (Banach space) calculus for *unbounded* operators was to reduce it to the bounded case by an application of a resolvent/elementary rational function, cf. [79, Section VII.9] or [3, Lecture 2]. This functional calculus is sometimes called **Taylor** calculus. Here as in the bounded case, only functions are used which are holomorphic in a neighbourhood of the spectrum, where ∞ has to be considered a member of the spectrum if A is unbounded (see Section A.2). Transforming back to the original operator one obtains a formula of the form

$$f(A) = f(\infty) + \frac{1}{2\pi i} \int_\Gamma f(z)\, R(z,A)\, dz$$

where Γ is a suitable finite cycle that avoids $\sigma(A)$, cf. Corollary 2.3.5.

With the Hilbert space (i.e. multiplication operator) examples in mind it became clear that boundedness of operators should not play an essential role in a general framework, neither as a requirement for the original operator A nor as a restriction for the result $f(A)$. That $f(A)$ is always bounded in the Taylor setting can be regarded as a mere coincidence due to the particular functions used. The decisive step to abandon boundedness was to realise how to involve functions which are singular in certain boundary points of the spectrum. Despite later development this was first done for strip-type operators (see Section 4.1) by Bade [21]. He introduced the 'regularisation trick' which we have formalised in the extension procedure of Section 1.2. However, since ∞ is somehow a 'hidden' spectral point, it was not obvious what was going on and in fact became fully clear only after McIntosh [167] constructed the setup for sectorial operators. de Laubenfels [65] constructs primary functional calculi for several classes of operators and uses regularisation by resolvents.

More or less *systematic* accounts of the natural functional calculus for sectorial operators can be found, e.g., in [3], [216] or [222]. Our presentation differs from these approaches in two decisive points. First, they mostly consider only injective operators, with the obvious disadvantage that the usual fractional powers of a non-injective sectorial operator (which may even be bounded) is not covered by their methods. The second main difference between our and other treatments lies in that we do not make any density assumptions on either the domain or the range of the operator. This is due to the fact that there are important examples of operators which are not densely defined, like the Dirichlet Laplacian on $\mathbf{C}(\overline{\Omega})$ where Ω is some open and bounded subset of $\mathbb{R}^n$, cf. [10, Section 6.1]. Moreover, it seems to be advantageous to discard density assumptions in a systematic treatment because then one is not tempted to prove things by approximation and closure arguments (which usually are a tedious matter).

The focus on injective operators (with dense range and dense domain) in the literature is of course not due to unawareness but a matter of convenience. As one can see, e.g., in the proof of Lemma 2.4.4, working only with injective operators often makes life easier. And McIntosh [167] outlines how to treat the matter without the injectivity assumption.

2.4 The Composition Rule. Most of the main properties of the natural functional calculus as enumerated, e.g., in Theorem 1.3.2 are folklore (even if e) and f) appeared quite late in printed form (in [107]). The composition rule (Theorem 2.4.2) however has been underestimated for a long time. Our account is based on [107], cf. [142]. We believe that only the composition rule opens the door for a fruitful use of the natural functional calculus beyond the mere definition.

2.5 & 2.6 Extensions According to Spectral Conditions and Miscellanies. These sections supplement the two previous ones. To sum up, we have defined four compatible 'natural' functional calculi for certain classes of (sectorial) operators. This is by far not the end of the story. For example, suppose that A is sectorial but there is $\varepsilon > 0$ such that the vertical line $\{\operatorname{Re} z = \varepsilon\}$ separates the spectrum of A. Then by using appropriate contours a new primary calculus can be defined, leading to a natural functional calculus for such operators. Obviously one can think of infinitely many modifications which all show the same pattern.

The results on adjoints may also be found in [107]. Lemma 2.6.10 and as Proposition 2.6.11 are folklore results. The very Proposition 2.6.11 plays a decisive role in practically all non-trivial boundedness results in connection with functional calculus, cf. Theorem 9.2.4.

2.7 The Spectral Mapping Theorem. Spectral mapping theorems (SMTs) have a long tradition and are ultimately important. In semigroup theory for example a spectral mapping theorem $\sigma(e^{tA}) = e^{t\sigma(A)}$ has a wealth of consequences regarding the asymptotics of the semigroup. (One can find a thorough discussion in [85, Chapter IV, Section 3].) It is well known that the SMT holds for the Dunford–Riesz calculus, see [79, Section VII.3] or [49, Chapter VII, §4]. Also, there is a SMT for the Hirsch functional calculus, see below and [161, Chapter 4]. Dore [74] has obtained partial results for our functional calculus, with the key notion of polynomial limits, although implicit. Our presentation is based on [108] where even more general results are presented. The proofs are generic, whence similar results hold for other types of holomorphic functional calculi, cf. Remark 4.2.7.

Other Functional Calculi. Although not particularly important for our purposes, we mention that there is a wealth of other functional calculi in the literature. The common pattern is this: Suppose that you are given an operator A and a function f for which you would like to define $f(A)$. Take some (usually: integral) representation of f in terms of other functions g for which you already 'know' $g(A)$. Then plug in A into the known parts and hope that the formulas still make sense. Obviously, our Cauchy integral-based natural calculus is of this type (the known parts are resolvents). Other examples are:

1) The **Hirsch** functional calculus, based on a representation

$$f(z) = a + \int_{\mathbb{R}_+} \frac{z}{1+tz}\,\mu(dt)$$

where μ is a suitable complex measure on $\mathbb{R}_+$. Details can be found in [161, Chapter 4]. In [107] it is shown that if A fails to be injective, there exist functions f which allow a representation as above but are not in the domain of the natural functional calculus. This can be remedied by extending the natural functional calculus by topological means, see [107, Section 5].

2) The **Phillips** calculus, based on the Laplace transform

$$f(z) = \int_{\mathbb{R}_+} e^{-zt}\,\mu(dt)$$

where μ is a finite complex measure on $\mathbb{R}_+$. (Here, the 'known' part is $e^{-zt}(A) = e^{-tA}$, i.e., $-A$ is assumed to generate a bounded C_0-semigroup.) We shall encounter this calculus in Section 3.3.

3) The **Mellin transform** calculus as developed in [193] and [218].

4) A functional calculus based on the **Poisson integral formula**, see [36] and [64].

Of course, each of these calculi can be extended by the 'regularisation trick'. For the Phillips calculus, this is done in [216, Section 1.3.4].

Let us mention that there are first attempts to define a functional calculus for *multi-valued* sectorial operators, see [1] and [160].

Chapter 3

Fractional Powers and Semigroups

In this chapter we present the basic theory of fractional powers A^α of a sectorial operator A, making efficient use of the functional calculus developed in Chapter 2. In Section 3.1 we introduce fractional powers with positive real part and give proofs for the scaling property, the laws of exponents, the spectral mapping theorem, and the Balakrishnan representation. Furthermore, we examine for variable $\varepsilon > 0$ the behaviour of $(A+\varepsilon)^\alpha$, and the behaviour of $A^\alpha x$ for variable α. In Section 3.2 we generalise the results from Section 3.1 to fractional powers with arbitrary real part. (Here, the operator A has to be injective.) In Section 3.3 we introduce the Phillips calculus for generators of bounded semigroups. Then the definition and the fundamental properties of holomorphic semigroups are presented in Section 3.4. The usual generator/semigroup correspondence is extended to the case of multi-valued operators. In Section 3.5 the logarithm of an injective sectorial operator A is defined and Nollau's theorem is proved. Finally, the connection of $\log A$ with the family of imaginary powers $(A^{is})_{s\in\mathbb{R}}$ of A.

3.1 Fractional Powers with Positive Real Part

In this section X always denotes a Banach space and A a sectorial operator of angle ω on X. (Recall the general agreement on terminology on page 279.)

Fix $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$ and consider the function $f(z) = z^\alpha$. It obviously has polynomial limit 0 at 0 and polynomial limit ∞ at ∞ (cf. Section 2.7). Hence it belongs to the class $\mathcal{A}(S_\varphi)$, for every $\varphi \in (0,\pi]$. Therefore $f(A)$ is defined by the natural functional calculus for sectorial operators. Being more specific, choose $n \in \mathbb{N}$ such that $n > \operatorname{Re}\alpha$. Then $z^\alpha(1+z)^{-n} \in H_0^\infty(S_\varphi)$ and

$$A^\alpha := (z^\alpha)(A) = (1+A)^n \left(\frac{z^\alpha}{(1+z)^n}\right)(A) \qquad (0 < \operatorname{Re}\alpha < n).$$

We call A^α the **fractional power** with exponent α of A.

Proposition 3.1.1. *Let A be a sectorial operator on the Banach space X. Then the following assertions hold.*

a) *If A is bounded, then also A^α is bounded, and the mapping*

$$(\alpha \longmapsto A^\alpha) : \{\alpha \in \mathbb{C} \mid \operatorname{Re}\alpha > 0\} \longrightarrow \mathcal{L}(X)$$

is holomorphic.

b) *Let* $n \in \mathbb{N}$ *and* $\operatorname{Re}\alpha \in (0, n)$. *Then* $\mathcal{D}(A^n) \subset \mathcal{D}(A^\alpha)$, *and the mapping*

$$(\alpha \longmapsto A^\alpha x) : \{\alpha \in \mathbb{C} \mid 0 < \operatorname{Re}\alpha < n\} \longrightarrow X$$

is holomorphic for each $x \in \mathcal{D}(A^n)$.

c) **(First Law of Exponents)** *For all* $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$ *the identity*

$$A^{\alpha+\beta} = A^\alpha\, A^\beta$$

holds. In particular, $\mathcal{D}(A^\gamma) \subset \mathcal{D}(A^\alpha)$ *for* $0 < \operatorname{Re}\alpha < \operatorname{Re}\gamma$.

d) *One has* $\mathcal{N}(A^\alpha) = \mathcal{N}(A)$ *for all* $\operatorname{Re}\alpha > 0$.

e) *If is* A *injective, then* $(A^{-1})^\alpha = (A^\alpha)^{-1}$. *If* $0 \in \varrho(A)$, *then also* $0 \in \varrho(A^\alpha)$.

f) *If* $T \in \mathcal{L}(X)$ *commutes with* A, *then it also commutes with* A^α.

g) *Let* $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$, *let* $x \in X$ *and* $\varepsilon > 0$. *Then*

$$\left(A(A+\varepsilon)^{-1}\right)^\alpha x \in \mathcal{D}(A^\beta) \quad \Longleftrightarrow \quad x \in \mathcal{D}(A^\beta).$$

h) *If* A *is densely defined,* $n \in \mathbb{N}$, *and* $\operatorname{Re}\alpha \in (0, n)$, *then the space* $\mathcal{D}(A^n)$ *is a core for* A^α.

i) *If the Banach space* X *is reflexive, then*

$$A^\alpha(x \oplus y) = 0 \oplus B^\alpha y$$

for all $x \in \mathcal{N}(A), y \in \overline{\mathcal{R}(A)}$, *where* $B := A|_{\overline{\mathcal{R}(A)}}$ *is the injective part of* A.

j) **(Spectral Mapping Theorem)**

$$\sigma(A^\alpha) = \{\mu^\alpha \mid \mu \in \sigma(A)\}$$

for all $\operatorname{Re}\alpha > 0$.

Proof. a) This follows easily from the definition and the Dominated Convergence Theorem. The same applies to b) since $A^\alpha x = (z^\alpha/(1+z)^n)(A)(1+A)^n x$ for $x \in \mathcal{D}(A^n)$ and $0 < \operatorname{Re}\alpha < n$.

c) From the general statements on the functional calculus (Theorem 1.3.2) we know that $A^\alpha A^\beta \subset A^{\alpha+\beta}$, with $\mathcal{D}(A^\alpha A^\beta) = \mathcal{D}(A^{\alpha+\beta}) \cap \mathcal{D}(A^\beta)$. Let $n > \operatorname{Re}\alpha, \operatorname{Re}\beta$ be fixed. We define $\Phi_\alpha := (z^\alpha/(1+z)^n)(A) \in \mathcal{L}(X)$ and Φ_β analogously. Let $x \in \mathcal{D}(A^{\alpha+\beta})$. Then $\Phi_\alpha\Phi_\beta x = [z^{\alpha+\beta}/(1+z)^{2n}](A)x \in \mathcal{D}(A^{2n})$. From this it follows that

$$A^n(1+A)^{-2n}\Phi_\beta x = \frac{z^{n+\beta}}{(1+z)^{3n}}(A)x = \frac{z^{n-\alpha}}{(1+z)^n}(A)\Phi_\alpha\Phi_\beta x \in \mathcal{D}(A^{2n}).$$

Applying g) of Proposition 2.1.1 we obtain $(1+A)^{-n}\Phi_\beta x \in \mathcal{D}(A^{2n})$. But this gives $\Phi_\beta x \in \mathcal{D}(A^n)$, hence $x \in \mathcal{D}(A^\beta)$.

d) follows from $A^n x = A^{n-\alpha}A^\alpha x$ and Proposition 2.1.1 e).

e) Note that $f(z) := z^{-\alpha} \in \mathcal{B}$. Hence by Proposition 2.4.1 we have

$$(A^{-1})^\alpha = f(z^{-1})(A^{-1}) = f(A) = (f^{-1}(A))^{-1} = (A^\alpha)^{-1}.$$

The second assertion follows from the first by using a).

f) is a special case of Theorem 1.3.2 a).

g) Recall that $A(A+\varepsilon)^{-1}$ is bounded and sectorial and commutes with A. Hence by f), $(A(A+\varepsilon)^{-1})^\alpha$ commutes with A, whence it commutes with A^β by another application of f). This yields one implication. Let $(A(A+\varepsilon)^{-1})^\alpha x \in \mathcal{D}(A^\beta)$, and fix a natural number $n > \operatorname{Re}\alpha, \operatorname{Re}\beta$. Applying the implication just proved with α replaced by $n-\alpha$, we obtain $(A(A+\varepsilon)^{-1})^n x \in \mathcal{D}(A^\beta)$. From this we conclude that $(A(A+\varepsilon)^{-1})^n \Phi_\beta x \in \mathcal{D}(A^n)$, where $\Phi_\beta := (z^\beta/(1+z)^n)(A)$. By part g) of Proposition 2.1.1 this implies that $\Phi_\beta x \in \mathcal{D}(A^n)$, hence $x \in \mathcal{D}(A^\beta)$.

h) follows from Proposition 2.3.11, and i) is obvious.

j) Consider the function $f(z) = z^\alpha$. Since f has polynomial limits both at 0 and at ∞ one can apply Theorem 2.7.8 to obtain $f(\tilde{\sigma}(A)) = \tilde{\sigma}(f(A)) = \tilde{\sigma}(A^\alpha)$. But $\mu^\alpha = \infty$ if and only if $\mu = \infty$. ☐

Consider an operator $A \in \operatorname{Sect}(\omega)$ and $\alpha \in (0, \pi/\omega)$. Then by part j) of the previous proposition $\sigma(A^\alpha) = \sigma(A)^\alpha \subset \overline{S_{\alpha\omega}}$. But there is more to say in this case.

Proposition 3.1.2 (Scaling Property). *Let $A \in \operatorname{Sect}(\omega)$ for some $\omega \in (0,\pi)$, and let $[\varepsilon, \delta] \subset (0, \pi/\omega)$ be a compact interval. Then the family $(A^\alpha)_{\varepsilon \le \alpha \le \delta}$ is uniformly sectorial of angle $\delta\omega$.*

In particular, for every $\alpha \in (0, \pi/\omega_A)$ the operator A^α is sectorial with $\omega_{A^\alpha} = \alpha\omega_A$.

Proof. Clearly, the second statement follows from the first (by taking $\varepsilon = \delta = 1$). So let $J := [\varepsilon, \delta] \subset (0, \pi/\omega)$ be a compact interval, and fix $\varphi \in (\delta\omega, \pi)$. For $\alpha \in J$, $|\arg\lambda| \in [\varphi, \pi]$ we define

$$\psi_{\lambda,\alpha}(z) := \frac{\lambda}{\lambda - z^\alpha} - \frac{|\lambda|^{\frac{1}{\alpha}}}{z + |\lambda|^{\frac{1}{\alpha}}} = \frac{\lambda z + |\lambda|^{\frac{1}{\alpha}} z^\alpha}{(\lambda - z^\alpha)(z + |\lambda|^{\frac{1}{\alpha}})}.$$

Since obviously $\psi_{\lambda,\alpha} \in H_0^\infty(S_{\delta\omega})$ (see 3.1 below), we have

$$\frac{1}{\lambda - z^\alpha} = \frac{1}{\lambda - z^\alpha} = \frac{1}{\lambda}\left(\frac{|\lambda|^{\frac{1}{\alpha}}}{z + |\lambda|^{\frac{1}{\alpha}}} + \psi_{\lambda,\alpha}(z)\right) \in H(A).$$

So we have shown that $\lambda \in \varrho(A^\alpha)$ and

$$\lambda R(\lambda, A^\alpha) = |\lambda|^{\frac{1}{\alpha}} \left(|\lambda|^{\frac{1}{\alpha}} + A\right)^{-1} + \psi_{\lambda,\alpha}(A).$$

Using the scaling property $\psi_{t^\alpha\lambda,\alpha}(tz) = \psi_\lambda(z)$ for $t > 0$ we see that

$$\begin{aligned}\|\lambda R(\lambda, A^\alpha)\| &\le M(A) + \left\|\psi_{\lambda/|\lambda|,\alpha}(|\lambda|^{\frac{-1}{\alpha}} A)\right\| \\ &\le M(A) + \frac{M(A,\omega')}{2\pi}\int_{\partial S_{\omega'}} \left|\psi_{\lambda/|\lambda|,\alpha}(z)\right| \frac{|dz|}{|z|}\end{aligned}$$

for $\omega' \in (\omega, \varphi)$. So what we actually have to prove is that

$$\sup\left\{\int_{\partial S_{\omega'}} |\psi_{\lambda,\alpha}(z)| \frac{|dz|}{|z|} \;\middle|\; |\lambda| = 1, |\arg\lambda| \in [\varphi,\pi], \alpha\in J\right\} \;<\; \infty.$$

Now for $|\lambda| = 1$, $|\arg\lambda| \in [\varphi, \pi]$, $\alpha \in J$, and $z \in \overline{S_{\omega'}}$ we have

$$|\psi_{\lambda,\alpha}(z)| = \left|\frac{\lambda z + z^\alpha}{(\lambda - z^\alpha)(1+z)}\right| \le \left|\frac{z^\varepsilon}{\lambda - z^\alpha}\right| \frac{|z|^{1-\varepsilon}}{|1+z|} + \left|\frac{z^{\alpha-\varepsilon}}{\lambda - z^\alpha}\right| \frac{|z|^\varepsilon}{|1+z|}, \tag{3.1}$$

and the factors $|z^\varepsilon/(\lambda - z^\alpha)|, |z^{\alpha-\varepsilon}/(\lambda - z^\alpha)|$ are uniformly bounded in our parameters λ, α, and z. This concludes the proof. ☐

Corollary 3.1.3. *Let $(A_\iota)_{\iota\in I} \subset \mathrm{Sect}(\omega)$ be uniformly sectorial for some $\omega \in (0,\pi)$, and let $J := [\varepsilon,\delta] \subset (0,\pi/\omega)$ be a compact interval. Then the family $(A_\iota^\alpha)_{\iota\in I,\alpha\in J}$ is uniformly sectorial of angle $\delta\omega$. Furthermore, if $A_\iota \to A$ (S_ω), then one has $A_\iota^\alpha \to A^\alpha$ $(S_{\delta\omega})$ for each $\alpha \in J$.*

Proof. This follows from the proof of Proposition 3.1.2. ☐

The next result is an immediate consequence of Proposition 3.1.2 and the general composition rule (Theorem 2.4.2).

Proposition 3.1.4. *Let $A \in \mathrm{Sect}(\omega)$ for some ω, and let $\alpha \in (0,\pi/\omega)$ and $\varphi \in (\omega,\pi/\alpha)$. If $f \in H_0^\infty(S_{\alpha\varphi})$ ($f \in \mathcal{A}(S_{\alpha\varphi})$, $f \in \mathcal{M}[S_{\alpha\omega}]_{A^\alpha}$), then the function $f(z^\alpha)$ is in $H_0^\infty(S_\varphi)$ ($\mathcal{A}(S_\varphi)$, $\mathcal{M}[S_\omega]_A$), and the identity*

$$f(A^\alpha) = \big(f(z^\alpha)\big)(A).$$

holds.

Corollary 3.1.5. (Second Law of Exponents) *Let $A \in \mathrm{Sect}(\omega)$ with $\omega \in (0,\pi)$, and let $\alpha \in (0,\pi/\omega)$. Then*

$$(A^\alpha)^\beta = A^{\alpha\beta}$$

for all $\mathrm{Re}\,\beta > 0$.

Corollary 3.1.6. *Let $A \in \mathrm{Sect}(\omega)$, $\mathrm{Re}\,\gamma > 0$, and let $x \in \mathcal{D}(A^\gamma)$. Then the mapping*

$$(\alpha \longmapsto A^\alpha x) : \{\alpha \in \mathbb{C} \mid 0 < \mathrm{Re}\,\alpha < \mathrm{Re}\,\gamma\} \longrightarrow X$$

is holomorphic.

Proof. Without loss of generality we may suppose $\gamma > 0$. Choose $n \in \mathbb{N}, n > \gamma$. Then $A^\alpha x = B^{\alpha n/\gamma} x$ and $x \in \mathcal{D}(B^n)$ where $B := A^{\gamma/n}$. Now the claim follows from Proposition 3.1.1, b). □

We now compare the operators A^α and $(A+\varepsilon)^\alpha$ for $\varepsilon > 0$.

Proposition 3.1.7. *Let* $\operatorname{Re}\alpha \in (0,1)$ *and* $\varepsilon > 0$. *Then*

$$T_\varepsilon := ((z+\varepsilon)^\alpha - z^\alpha)(A) \in \mathcal{L}(X) \quad \textit{and} \quad A^\alpha + T_\varepsilon = (A+\varepsilon)^\alpha.$$

Moreover, we have $\|T_\varepsilon\| \le C\varepsilon^{\operatorname{Re}\alpha}$, *where* $C = C(\alpha, \varphi, M(A,\varphi))$ *for* $\varphi \in (\omega, \pi)$.

Proof. Let $\psi_1(z) := (z+1)^\alpha - z^\alpha - (z+1)^{-1}$. A short computation shows that $\psi_1 \in H_0^\infty$. Hence $\psi_\varepsilon(z) := \varepsilon^\alpha \psi_1(z/\varepsilon) = (z+\varepsilon)^\alpha - z^\alpha - \varepsilon^\alpha(\varepsilon^{-1}z+1)^{-1}$ is also contained in H_0^∞. This shows that $(z+\varepsilon)^\alpha - z^\alpha \in \mathcal{E}(S_\varphi)$. By part c) of Theorem 1.3.2 and the composition rule we obtain $A^\alpha + T_\varepsilon = (A+\varepsilon)^\alpha$. Furthermore, we have

$$\varepsilon^{-\alpha} T_\varepsilon = \varepsilon^{-\alpha}\psi_\varepsilon(A) - (1+\varepsilon^{-1}A)^{-1} = \psi_1(\varepsilon^{-1}A) - (1+\varepsilon^{-1}A)^{-1}.$$

The claim now follows from

$$\sup_{\varepsilon>0} \left\|(1+\varepsilon^{-1}A)^{-1}\right\| = M(A) \quad \text{and} \quad \left\|\psi_1(\varepsilon^{-1}A)\right\| \le M(A,\varphi) C(\psi_1, \varphi)$$

for $\varphi \in (\omega, \pi)$ (apply Proposition 2.6.11 with $\psi = 0$). □

Remark 3.1.8. With the help of the Balakrishnan representation (see below) the last proposition can be improved with respect to the constant C. Namely, one can explicitly determine a constant which depends only on $M(A), \operatorname{Re}\alpha$, and $|\sin \alpha\pi|$, see [161, Proposition 5.1.14].

The last proposition implies in particular that $\mathcal{D}((A+\varepsilon)^\alpha) = \mathcal{D}(A^\alpha)$ for $\varepsilon > 0$ and $0 < \operatorname{Re}\alpha < 1$. However, this is true for *all* $\operatorname{Re}\alpha > 0$ as it is shown by the next result.

Proposition 3.1.9. *Let* $\operatorname{Re}\alpha > 0$ *and* $\varepsilon > 0$. *Then the following assertions hold.*

a) $\mathcal{D}(A^\alpha) = \mathcal{D}((A+\varepsilon)^\alpha)$.

b) $A^\alpha\,((A+\varepsilon)^{-1})^\alpha = \left(A(A+\varepsilon)^{-1}\right)^\alpha$.

c) $\lim_{\varepsilon\to 0}(A+\varepsilon)^\alpha x = A^\alpha x$ *for each* $x \in \mathcal{D}(A^\alpha)$.

Proof. Apply the composition rule to the functions $f(z) := (z+\varepsilon)^{-1}$ and $g(z) := z^\alpha$ to obtain $(z+\varepsilon)^{-\alpha}(A) = \left((A+\varepsilon)^{-1}\right)^\alpha \in \mathcal{L}(X)$. Hence $(z+\varepsilon)^{-\alpha} \in H(A)$ and

$$(A(A+\varepsilon)^{-1})^\alpha = \left(\frac{z^\alpha}{(z+\varepsilon)^\alpha}\right)(A) = z^\alpha(A)(z+\varepsilon)^{-\alpha}(A) = A^\alpha((A+\varepsilon)^{-1})^\alpha$$

by the composition rule again, whence b) is proved. Furthermore, we have

$$\left((A+\varepsilon)^{-1}\right)^\alpha = (z+\varepsilon)^{-\alpha}(A) = ((z+\varepsilon)^\alpha(A))^{-1} = ((A+\varepsilon)^\alpha)^{-1}$$

by the composition rule and Theorem 1.3.2 f). Together with b) this shows that

$$\mathcal{D}((A+\varepsilon)^\alpha) = \mathcal{R}(((A+\varepsilon)^{-1})^\alpha) \subset \mathcal{D}(A^\alpha).$$

To prove the other inclusion of a), choose $x \in \mathcal{D}(A^\alpha)$ and $n > \operatorname{Re}\alpha$. Applying the first law of exponents and b) we obtain

$$\begin{aligned}\mathcal{D}(A^n) \ni (A+\varepsilon)^{-n}A^\alpha x = A^\alpha(A+\varepsilon)^{-n}x &= A^\alpha((A+\varepsilon)^{-1})^\alpha((A+\varepsilon)^{-1})^{n-\alpha}x \\ &= (A(A+\varepsilon)^{-1})^\alpha((A+\varepsilon)^{-1})^{n-\alpha}x.\end{aligned}$$

This yields $((A+\varepsilon)^{-1})^{n-\alpha}x \in \mathcal{D}(A^n)$ by Proposition 3.1.1 g). Hence it follows that $x = (A+\varepsilon)^{n-\alpha}((A+\varepsilon)^{-1})^{n-\alpha}x \in \mathcal{D}((A+\varepsilon)^\alpha)$.

We are left to show c). The family of operators $(A+\varepsilon)(A+1)^{-1}$ is a sectorial approximation of $A(A+1)^{-1}$ (apply Proposition 2.1.3 c) together with Proposition 2.1.1 f)). Lemma 2.6.7 together with b) implies that

$$(A+\varepsilon)^\alpha((A+1)^{-1})^\alpha = ((A+\varepsilon)(A+1)^{-1})^\alpha \to (A(A+1)^{-1})^\alpha = A^\alpha((A+1)^\alpha)^{-1}$$

in norm. In particular we have $\lim_{\varepsilon\searrow 0}(A+\varepsilon)^\alpha x = A^\alpha x$ for all $x \in \mathcal{D}((A+1)^\alpha)$. However, $\mathcal{D}((A+1)^\alpha) = \mathcal{D}(A^\alpha)$. $\square$

Remark 3.1.10. The last result together with the rule $(A^\alpha)^{-1} = (A^{-1})^\alpha$ (in the case where A is injective) reduces the definition of A^α for (general) sectorial operators A to the one for operators $A \in \mathcal{L}(X)$ with $0 \in \varrho(A)$ where the usual Dunford calculus is at hand. Therefore, the last result may be viewed as an 'interface' to the literature where often the fractional powers are defined in a different way.

Instead of proving a statement for fractional powers directly with recourse to the definition one can proceed in three steps:

1) The validity of the statement is proved for $A \in \mathcal{L}(X)$ with $0 \in \varrho(A)$.
2) One shows that in the case where A is injective the statement for A follows from the statement for A^{-1}.
3) One shows that the statement is true for A if it is true for all $A+\varepsilon$ (with small $\varepsilon > 0$).

In fact, many proofs follow this scheme.

Corollary 3.1.11. *One has $\mathcal{D}(A^\alpha) \subset \overline{\mathcal{D}(A)}$ and $\mathcal{R}(A^\alpha) \subset \overline{\mathcal{R}(A)}$ for each $\operatorname{Re}\alpha > 0$ and each sectorial operator A.*

Proof. Suppose first that A is bounded, i.e., $A \in \mathcal{L}(X)$. Then with Γ being an appropriate *finite* path, we have

$$A^\alpha = \frac{1}{2\pi i}\int_\Gamma z^\alpha R(z,A)\,dz = \frac{1}{2\pi i}\int_\Gamma z^{\alpha-1}AR(z,A)\,dz$$

by Cauchy's theorem. This gives $\mathcal{R}(A^\alpha) \subset \overline{\mathcal{R}(A)}$ if $A \in \mathcal{L}(X)$. For arbitrary A we apply this to the operator $A(A+1)^{-1}$ and obtain

$$\mathcal{R}(A^\alpha) = \mathcal{R}[A^\alpha((A+1)^\alpha)^{-1}] = \mathcal{R}[(A(A+1)^{-1})^\alpha] \subset \overline{\mathcal{R}(A(A+1)^{-1})} = \overline{\mathcal{R}(A)}.$$

(Here we used b) of the last proposition.) Finally we conclude from this and Proposition 3.1.9 a) that

$$\mathcal{D}(A^\alpha) = \mathcal{D}[(A+1)^\alpha] = \mathcal{R}[((A+1)^{-1})^\alpha] \subset \overline{\mathcal{R}((A+1)^{-1})} = \overline{\mathcal{D}(A)}. \qquad \square$$

We now turn to an integral representation which historically was one of the first approaches to fractional powers.

Proposition 3.1.12 (Balakrishnan Representation). *Let $A \in \operatorname{Sect}(\omega)$ on the Banach space X, and let $0 < \operatorname{Re}\alpha < 1$. Then*

$$A^\alpha x = \frac{\sin \alpha\pi}{\pi} \int_0^\infty t^{\alpha-1}\,(t+A)^{-1}Ax\,dt \tag{3.2}$$

for all $x \in \mathcal{D}(A)$. More generally, we have

$$A^\alpha x = \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^\infty t^{\alpha-1}[A(t+A)^{-1}]^m x\,dt \tag{3.3}$$

where $0 < \operatorname{Re}\alpha < n \le m$ and $x \in \mathcal{D}(A^n)$.

Proof. Suppose first that $0 < \operatorname{Re}\alpha < 1$. For $x \in \mathcal{D}(A)$ we have

$$A^\alpha x = \frac{1}{2\pi i} \int_{\Gamma_\varphi} z^{\alpha-1} R(z,A)Ax\,dz, \tag{3.4}$$

where $\varphi \in (\omega, \pi)$. In fact, we can compute

$$\begin{aligned} A^\alpha x &= \left(\frac{z^\alpha}{z+\varepsilon}\right)(A)(\varepsilon + A)x \\ &= \left(\frac{z^\alpha}{z+\varepsilon}\right)(A)Ax + \varepsilon\left(\frac{z^\alpha}{(z+\varepsilon)(z+1)}\right)(A)(1+A)x \\ &= \frac{1}{2\pi i}\int_{\Gamma_\varphi} z^{\alpha-1}\left(\frac{z}{z+\varepsilon}\right)R(z,A)Ax\,dz \\ &\qquad + \frac{\varepsilon}{2\pi i}\int_{\Gamma_\varphi}\left(\frac{z^\alpha}{(z+\varepsilon)(z+1)}\right)R(z,A)(1+A)x\,dz, \end{aligned}$$

where $\varepsilon > 0$. As $\varepsilon \searrow 0$ the second summand vanishes and we obtain (3.4). Note that the function $z \longmapsto z^{\alpha-1}R(z,A)Ax$ is integrable on the boundary $\Gamma_\varphi = \partial S_\varphi$. Indeed, $R(z,A)Ax$ is bounded at 0 and $O(|z|^{-1})$ as $z \to \infty$. The functions $(z/z+\varepsilon)$ are bounded on Γ_φ uniformly in ε, hence Lebesgue's theorem is applicable.

Starting from (3.4) we 'deform' the path Γ_φ onto the negative real axis. This means that the opening angle φ of Γ_φ is enlarged until the angle π is reached. Cauchy's theorem ensures that the integral does not change its value during this deforming procedure. Lebesgue's theorem shows that the limit is exactly the right-hand side of (3.2).

So we have proved the first part of the Proposition. For the second we suppose $m = n$ and $n-1 < \operatorname{Re}\alpha < n$. Then for $x \in \mathcal{D}(A^n)$ we write

$$\begin{aligned} A^\alpha x = A^{\alpha-(n-1)}A^{n-1}x &= \frac{\sin(\alpha-n+1)\pi}{\pi}\int_0^\infty t^{\alpha-n}\,(t+A)^{-1}A^n x\,dt \\ &\overset{\text{Int.b.p.}}{=} \frac{\sin((\alpha-n+1)\pi)(n-1)!}{\pi(\alpha-n+1)(\alpha-n+2)\dots(\alpha-1)}\int_0^\infty t^{\alpha-1}(t+A)^{-n}A^n x\,dt \\ &= \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n-\alpha)}\int_0^\infty t^{\alpha-1}(t+A)^{-n}A^n x\,dt. \end{aligned}$$

Here we used the standard formulae $\sin(\pi z)/\pi = 1/(\Gamma(z)\Gamma(1-z))$ and $z\Gamma(z) = \Gamma(z+1)$ for the Gamma function. An holomorphy argument allows us to replace the assumption $n-1 < \operatorname{Re}\alpha$ by $0 < \operatorname{Re}\alpha < n = m$. Thus we have proved that (3.3) holds for $n = m$.

To prove the general statement we use induction on m. The assertion is already known for $n = m$. Define

$$c_m := \frac{\Gamma(m)}{\Gamma(m-\alpha)\Gamma(\alpha)} \quad \text{and} \quad I_m := \int_0^\infty t^{\alpha-1}[A(t+A)^{-1}]^m x\,dt.$$

Then we have

$$\begin{aligned} I_m &\overset{\text{I.b.p.}}{=} \frac{t^\alpha}{\alpha}[A(t+A)^{-1}]^m x|_0^\infty + \frac{m}{\alpha}\int_0^\infty t^\alpha[A(t+A)^{-1}]^m(t+A)^{-1}x\,dt \\ &= \frac{m}{\alpha}\int_0^\infty t^\alpha[A(t+A)^{-1}]^m(t+A)^{-1}x\,dt \\ &= \frac{m}{\alpha}\int_0^\infty t^{\alpha-1}([A(t+A)^{-1}]^m x - [A(t+A)^{-1}]^{m+1})x\,dt \\ &= \frac{m}{\alpha}(I_m - I_{m+1}). \end{aligned}$$

This means that $m/(m-\alpha)I_{m+1} = I_m$. Since $c_m(m/(m-\alpha)) = c_{m+1}$, the induction is complete. $\square$

Corollary 3.1.13. *Let $n \in \mathbb{N}$ and $\alpha \in (0,n)$. Then*

$$\sup_{t>0}\left\|(t(t+A)^{-1})^\alpha\right\| \le M(A)^n \quad \textit{and} \quad \sup_{t>0}\left\|(A(t+A)^{-1})^\alpha\right\| \le (M(A)+1)^n.$$

Proof. Let $\alpha \in (0,1)$. Then we have

$$\begin{aligned}\big((t+A)^{-1}\big)^{\alpha} &= \frac{\sin \pi\alpha}{\pi}\int_0^\infty s^{\alpha-1}(t+A)^{-1}[s+(t+A)^{-1}]^{-1}\,ds\\ &= \frac{\sin \pi\alpha}{\pi}\int_0^\infty s^{\alpha-1}(st+1+A)^{-1}\,ds, \quad \text{whence}\\ \Big\|\big((t+A)^{-1}\big)^{\alpha}\Big\| &\le \frac{\sin \pi\alpha}{\pi}\int_0^\infty s^{\alpha-1}(st+1)^{-1}\,dtM(A) = t^{-\alpha}M(A).\end{aligned}$$

The general statement follows with the help of an easy induction argument. The proof of the second assertion is similar, see [161, Remark 5.1.2]. □

Corollary 3.1.14. *Let $A \in \mathrm{Sect}(\omega)$ and $\mathrm{Re}\,\alpha \in (0,1)$. Then*

$$A^\alpha x = \frac{\sin \pi\alpha}{\pi}\int_0^\infty t^{-\alpha}(t+A^{-1})^{-1}x\,dt$$

for $x \in \mathcal{D}(A)$. (The operator A^{-1} may be multi-valued, see Remark 2.1.4.*)*

Proof. Starting from the Balakrishnan representation (3.2) we obtain

$$\begin{aligned}A^\alpha x &= \frac{\sin \pi\alpha}{\pi}\int_0^\infty s^\alpha (s+A)^{-1}Ax\,\frac{ds}{s} \overset{t=1/s}{=} \frac{\sin \pi\alpha}{\pi}\int_0^\infty t^{-\alpha}(\frac{1}{t}+A)^{-1}Ax\,\frac{dt}{t}\\ &= \frac{\sin \pi\alpha}{\pi}\int_0^\infty t^{-\alpha}(I-\frac{1}{t}(\frac{1}{t}+A)^{-1})x\,\frac{dt}{t} = \frac{\sin \pi\alpha}{\pi}\int_0^\infty t^{-\alpha}t(t+A^{-1})^{-1}x\,\frac{dt}{t}\\ &= \frac{\sin \pi\alpha}{\pi}\int_0^\infty t^{-\alpha}(t+A^{-1})^{-1}x\,dt\end{aligned}$$

by the fundamental identity (2.1). □

Proposition 3.1.15. *Let $A \in \mathrm{Sect}(\omega)$, $x \in \mathcal{D}(A)$, and $\varphi \in [0,\pi/2)$. Then the following assertions hold.*

a) $x \in \overline{\mathcal{R}(A)} \iff \lim_{\alpha\to 0,\,\alpha\in S_\varphi} A^\alpha x = x.$

b) $Ax \in \overline{\mathcal{D}(A)} \iff \lim_{\alpha\to 1,\,\alpha\in 1-S_\varphi} A^\alpha x = Ax.$

Proof. a) The implication '⇐' is immediate from Corollary 3.1.11. To prove the reverse direction we use the Balakrishnan representation (3.2) and write

$$\begin{aligned}
\|A^\alpha x - Ax\| &= \left\| \frac{\sin\alpha\pi}{\pi} \int_0^\infty t^{\alpha-1}((t+A)^{-1}Ax - \frac{1}{t+1}x)dt \right\| \\
&\le \left| \frac{\sin\alpha\pi}{\pi} \right| \left| \int_0^L t^{\operatorname{Re}\alpha - 1} \left\|(t+A)^{-1}Ax - x\right\| + t^{\operatorname{Re}\alpha} \|x\| \, dt \right. \\
&\quad + \left| \frac{\sin\alpha\pi}{\pi} \right| \left| \int_L^\infty t^{\operatorname{Re}\alpha - 2} \left\| t(t+A)^{-1}Ax - \frac{t}{1+t}x \right\| dt \right. \\
&\le \left| \frac{\sin\alpha\pi}{\alpha\pi} \right| \frac{|\alpha|}{\operatorname{Re}\alpha} \Bigg(L^{\operatorname{Re}\alpha} \sup_{t\le L} \left\|(t+A)^{-1}Ax - x\right\| + \frac{\operatorname{Re}\alpha}{\operatorname{Re}\alpha + 1} L^{\operatorname{Re}\alpha+1} \|x\| \\
&\quad + \frac{\operatorname{Re}\alpha}{1-\operatorname{Re}\alpha} L^{\operatorname{Re}\alpha - 1}(M\|Ax\| + \|x\|) \Bigg).
\end{aligned}$$

Note that $\sin\alpha\pi/\alpha\pi$ is continuous at 0 and that $|\alpha| / \operatorname{Re}\alpha$ is bounded by $(\cos\varphi)^{-1}$. Since $x \in \overline{\mathcal{R}(A)}$ we may choose the number L such that $\left\|(t+A)^{-1}Ax - x\right\|$ is small for $t \le L$. For a fixed L the other summands tend to zero as $\operatorname{Re}\alpha \to 0$.

The proof of b) requires similar arguments, see [161, p.62]. □

Remark 3.1.16. Let $A \in \operatorname{Sect}(\omega)$ and $\alpha \in (0,1)$. Then $A^\alpha \in \operatorname{Sect}(\alpha\omega)$ as we know from Proposition 3.1.2. By applying the same technique as in the proof of Proposition 3.1.12 one obtains a Balakrishnan-type representation for the resolvent of A^α, i.e.,

$$R(\lambda, A^\alpha) = \frac{-\sin\alpha\pi}{\pi} \int_0^\infty \frac{t^\alpha}{(\lambda - t^\alpha e^{i\pi\alpha})(\lambda - t^\alpha e^{-i\pi\alpha})} (t+A)^{-1}\, dt$$

for $|\arg\lambda| > \alpha\pi$. One can deduce $M(A^\alpha) \le M(A)$ from this, see [161, (5.24) and (5.25)] and [210, (2.23)].

3.2 Fractional Powers with Arbitrary Real Part

To introduce fractional powers with arbitrary real part, i.e., in order to render the definition

$$A^\alpha := (z^\alpha)(A) \qquad (\alpha \in \mathbb{C})$$

meaningful we have to suppose that the sectorial operator A is *injective.* (Note that $z^\alpha \in \mathcal{B}(S_\varphi)$ for all $\alpha \in \mathbb{C}$ and all $\varphi \in (0,\pi)$.)

Proposition 3.2.1. *Let* $A \in \operatorname{Sect}(\omega)$ *be injective, and let* $\alpha, \beta \in \mathbb{C}$. *Then the following assertions hold.*

a) *The operator* A^α *is injective with* $(A^\alpha)^{-1} = A^{-\alpha} = (A^{-1})^\alpha$.

b) *We have* $A^\alpha A^\beta \subset A^{\alpha+\beta}$ *with* $\mathcal{D}(A^\beta) \cap \mathcal{D}(A^{\alpha+\beta}) = \mathcal{D}(A^\alpha A^\beta)$.

c) *If* $\overline{\mathcal{D}(A)} = X = \overline{\mathcal{R}(A)}$, *then* $A^{\alpha+\beta} = \overline{A^\alpha A^\beta}$.

d) *If* $0 < \operatorname{Re}\alpha < 1$, *then*

$$A^{-\alpha}x = \frac{\sin \pi\alpha}{\pi} \int_0^\infty t^{-\alpha}(t+A)^{-1}x\,dt \qquad (x \in \mathcal{R}(A)).$$

e) *If* $\alpha \in \mathbb{R}$ *satisfies* $|\alpha| < \pi/\omega$, *then* $A^\alpha \in \operatorname{Sect}(|\alpha|\,\omega)$ *and one has*

$$(A^\alpha)^\beta = A^{\alpha\beta} \qquad (\beta \in \mathbb{C}).$$

f) *Let* $\operatorname{Re}\alpha_0, \operatorname{Re}\alpha_1 > 0$. *Then* $\mathcal{D}(A^{\alpha_1}) \cap \mathcal{R}(A^{\alpha_0}) \subset \mathcal{D}(A^\alpha)$ *for each* α *with* $-\operatorname{Re}\alpha_0 < \operatorname{Re}\alpha < \operatorname{Re}\alpha_1$. *The mapping*

$$(\alpha \longmapsto A^\alpha x) : \{\alpha \in \mathbb{C} \mid -\operatorname{Re}\alpha_0 < \operatorname{Re}\alpha < \operatorname{Re}\alpha_1\} \longrightarrow X$$

is holomorphic for each $x \in \mathcal{D}(A^{\alpha_1}) \cap \mathcal{R}(A^{\alpha_0})$.

Proof. a) follows from Theorem 1.3.2 f), cf. the proof of Proposition 3.1.1 e).

b) is immediate from c) of Theorem 1.3.2.

c) For $1 < n \in \mathbb{N}$ we define

$$\tau_n(A) := n(n+A)^{-1} - \frac{1}{n}\left(\frac{1}{n} + A\right)^{-1} = (n+A)^{-1}\left(\frac{n-1}{n}\right) A \left(\frac{1}{n} + A\right)^{-1}.$$

Then it is easy to see that for $k \in \mathbb{N}$ the following assertions hold.

a) $\sup_n \left\|\tau_n(A)^k\right\| < \infty$.

b) $\lim_{n\to\infty} \tau_n(A)^k x = x$ for $x \in \overline{\mathcal{D}(A)} \cap \overline{\mathcal{R}(A)}$.

c) $\mathcal{R}(\tau_n(A)^k) \subset \mathcal{D}(A^k) \cap \mathcal{R}(A^k)$, and $\overline{\mathcal{D}(A^k) \cap \mathcal{R}(A^k)} = \overline{\mathcal{D}(A)} \cap \overline{\mathcal{R}(A)}$.

We now choose $k \in \mathbb{N}$ such that $\mathcal{D}(A^k) \cap \mathcal{R}(A^k) \subset \mathcal{D}(A^\beta)$. Let $x \in \mathcal{D}(A^{\alpha+\beta})$, and define $x_n := \tau_n(A)^k x$. Then $x_n \in \mathcal{D}(A^{\alpha+\beta}) \cap \mathcal{D}(A^\beta) \subset \mathcal{D}(A^\alpha A^\beta)$ and $x_n \to x$. Furthermore, $A^\alpha A^\beta x_n = \tau_n(A)^k A^{\alpha+\beta} x \to A^{\alpha+\beta} x$. This shows the claim.

d) follows from Corollary 3.1.14 by replacing A by A^{-1}.

e) follows from a) and Proposition 3.1.2 together with the composition rule (Theorem 2.4.2).

To prove f), let $\operatorname{Re}\alpha_0, \operatorname{Re}\alpha_1 > 0$, $x \in \mathcal{D}(A^{\alpha_1}) \cap \mathcal{R}(A^{\alpha_0})$, and $-\operatorname{Re}\alpha_0 < \alpha < \operatorname{Re}\alpha_1$. If $\alpha \notin i\mathbb{R}$, then it is clear from Proposition 3.1.1 that $x \in D(A^\alpha)$. Let $\alpha = is \in i\mathbb{R}$. Choose $n \in \mathbb{N}$ such that $\operatorname{Re}\alpha_0, \operatorname{Re}\alpha_1 < n$ and define $B := A^{1/n}$. Then $x \in \mathcal{D}(B) \cap \mathcal{R}(B)$, hence $x \in \mathcal{D}(B^{ins}) = \mathcal{D}(A^{is}) = \mathcal{D}(A^\alpha)$. Since $A^\alpha x = A^{\alpha+\alpha_0} A^{-\alpha_0} x$ the second statement follows from Corollary 3.1.6. $\square$

For actual computations the following integral representations are of great importance.

Proposition 3.2.2 (Komatsu Representation)). *Let $A \in \mathrm{Sect}(\omega)$ be injective. The identities*

$$\begin{aligned} A^\alpha x &= \frac{\sin \pi\alpha}{\pi}\left[\frac{1}{\alpha}x - \frac{1}{1+\alpha}A^{-1}x + \int_0^1 t^{\alpha+1}(t+A)^{-1}A^{-1}x\,dt \right. \\ &\qquad \left. + \int_1^\infty t^{\alpha-1}(t+A)^{-1}Ax\,dt\right] \end{aligned} \tag{3.5}$$

$$\begin{aligned} &= \frac{\sin \pi\alpha}{\pi}\left[\frac{1}{\alpha}x + \int_0^1 t^{-\alpha}(1+tA)^{-1}Ax\,dt \right. \\ &\qquad \left. - \int_0^1 t^\alpha(1+tA^{-1})^{-1}A^{-1}x\,dt\right] \end{aligned} \tag{3.6}$$

hold for $|\mathrm{Re}\,\alpha| < 1$ and $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$.

Note that the second formula is symmetric with respect to A and A^{-1}, whence it can again be seen that $A^{-\alpha}x = (A^{-1})^\alpha x$ for $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$.

Proof. We suppose first that $0 < \mathrm{Re}\,\alpha < 1$. Starting from the Balakrishnan representation (3.2) we obtain

$$\begin{aligned} \frac{\pi}{\sin \alpha\pi}A^\alpha x &= \int_0^\infty t^{\alpha-1}(t+A)^{-1}Ax\,dt \\ &= \int_0^1 t^{\alpha-1}(t+A)^{-1}Ax\,dt + \int_1^\infty t^{\alpha-1}(t+A)^{-1}Ax\,dt \\ &= \int_0^1 t^{\alpha-1}(1+tA^{-1})^{-1}x\,dt + \int_1^\infty t^{\alpha-1}(t+A)^{-1}Ax\,dt \\ &= \int_0^1 t^{\alpha-1}(1-(1+tA^{-1})^{-1}tA^{-1})x\,dt + \int_1^\infty t^{\alpha-1}(t+A)^{-1}Ax\,dt \\ &= \frac{1}{\alpha}x - \int_0^1 t^\alpha(1+tA^{-1})^{-1}A^{-1}x\,dt + \int_1^\infty t^{\alpha-1}(t+A)^{-1}Ax\,dt \qquad (*) \\ &= \frac{1}{\alpha}x - \int_0^1 t^\alpha(1+tA^{-1})^{-1}A^{-1}x\,dt + \int_0^1 t^{-\alpha}(1+tA)^{-1}Ax\,dt \end{aligned}$$

for $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$, where in the last step we have replaced t by t^{-1} in the second integral. This last formula makes sense even for $-1 < \mathrm{Re}\,\alpha < 1$. Hence by holomorphy we obtain (3.6) for $|\mathrm{Re}\,\alpha| < 1$. The representation (3.5) now follows from $(*)$ with the help of the identity $t^{-1}(t+A)^{-1} = t^{-1}A^{-1} - (t+A)^{-1}A^{-1}$. $\square$

A particularly important subclass of the injective sectorial operators are the invertible ones.

Proposition 3.2.3. *Let $A \in \mathrm{Sect}(\omega)$ such that $0 \in \varrho(A)$. Then the mapping*

$$(\alpha \longmapsto A^{-\alpha}) : \{\alpha \in \mathbb{C} \mid \mathrm{Re}\,\alpha > 0\} \longrightarrow \mathcal{L}(X)$$

is holomorphic. For every $\varphi \in (0, \pi/2)$ *one has*

$$\sup\left\{\left\|A^{-\alpha}\right\| \;\middle|\; \operatorname{Re}\alpha \in (0,1),\ |\arg\alpha| \le \varphi\right\} \;<\; \infty \tag{3.7}$$

and the equivalence $\quad x \in \overline{\mathcal{D}(A)} \quad \Longleftrightarrow \quad \lim_{\alpha\to 0, |\arg\alpha|\le\varphi} A^{-\alpha}x = x.$

Proof. The first assertion follows from $A^{-\alpha} = (A^{-1})^{\alpha}$ and Proposition 3.1.1 a). To prove (3.7) we employ Proposition 3.2.1 d) and obtain

$$\left\|A^{-\alpha}x\right\| \le K\frac{|\sin\pi\alpha|}{\pi}\int_0^\infty t^{-\operatorname{Re}\alpha}(t+1)^{-1}\,dt\ \|x\| = K\frac{|\sin\pi\alpha|}{\sin(\pi\operatorname{Re}\alpha)}\,\|x\|\,,$$

where $K := \sup_{t>0}\|(t+1)(t+A)^{-1}\| < \infty$. Hence

$$\left\|A^{-\alpha}\right\| \le K\left|\frac{\sin\pi\alpha}{\pi\alpha}\right|\frac{\pi\operatorname{Re}\alpha}{\sin(\pi\operatorname{Re}\alpha)}\frac{|\alpha|}{\operatorname{Re}\alpha}.$$

If $A^{-\alpha}x \to x$, then $x \in \overline{\mathcal{D}(A)}$ by Corollary 3.1.11. The converse implication follows from Proposition 3.1.15 a). $\square$

3.3 The Phillips Calculus for Semigroup Generators

In this section we wish to give a formula for the fractional powers in the case that $-A$ generates a *bounded* semigroup $(T(t))_{t\ge 0}$. This is actually part of a more general construction.

For $\mu \in \mathbf{M}[0,\infty)$ we define its **Laplace transform** $\mathcal{L}(\mu)$ by

$$\mathcal{L}(\mu)(z) := \int_{[0,\infty)} e^{-zt}\,\mu(dt) \qquad (\operatorname{Re} z \ge 0)$$

It is well known (or easy to see) that $\mathcal{L}(\mu)$ is a bounded holomorphic function on $\mathbb{C}_+$ and continuous on $\overline{\mathbb{C}_+}$. Moreover,

$$\mathcal{L}(\mu)(is) = (\mathcal{F}\mu)(s) \qquad (s \in \mathbb{R})$$

is the Fourier transform of μ (see Appendix E.1). From this follows that μ is uniquely determined by $\mathcal{L}(\mu)$. A simple computation involving Fubini's theorem yields the product law

$$\mathcal{L}(\mu)\cdot\mathcal{L}(\nu) = \mathcal{L}(\mu * \nu) \qquad (\mu, \nu \in \mathbf{M}[0,\infty)).$$

Now take $\varphi \in (\pi/2, \pi)$ and $f \in \mathcal{E}(S_\varphi)$. Obviously $f \in H^\infty(\mathbb{C}_+) \cap \mathbf{C}(\overline{\mathbb{C}_+})$, so we may ask whether $f = \mathcal{L}(\mu)$ for some $\mu \in \mathbf{M}[0,\infty)$. Clearly $\mathcal{L}(\delta_0) = \mathbf{1}$, where δ_0 denotes the **Dirac measure** at 0, and

$$\mathcal{L}(e^{\lambda t}\,dt) = \frac{1}{z-\lambda} \qquad (\operatorname{Re}\lambda < 0),$$

hence $\mathcal{L}(e^{-t}\,dt) = (1+z)^{-1}$. The following lemma deals with the case $f \in H_0^\infty$.

Lemma 3.3.1. *Let $\varphi \in (\pi/2, \pi)$ and $\psi \in H_0^\infty(S_\varphi)$. Then there is (a unique) $g \in \mathbf{L^1}(0,\infty)$ such that $\psi = \mathcal{L}[g(t)\,dt]$. More precisely, g is given by*

$$g(t) := \frac{-1}{2\pi i} \int_\Gamma \psi(z) e^{zt}\, dz \tag{3.8}$$

where $\Gamma = \partial S_{\omega'}$, and $\omega' \in (\pi/2, \varphi)$ is arbitrary.

From the representation (3.8) one can see that g has a holomorphic extension to a sector S_η for some $\eta > 0$ and that $|\lambda g(\lambda)|$ is bounded on S_η. We will not use these facts in the sequel.

Proof. Take $z_0 \in \mathbb{C}_+$. Then $\psi(z_0)$ is given by the Cauchy integral

$$\begin{aligned}\psi(z_0) &= \frac{1}{2\pi i} \int_\Gamma \frac{\psi(z)}{z - z_0}\, dz = \frac{-1}{2\pi i} \int_\Gamma \psi(z) \int_0^\infty e^{(z-z_0)t}\, dt\, dz \\ &= \int_0^\infty e^{z_0 t} \frac{-1}{2\pi i} \int_\Gamma \psi(z) e^{zt}\, dz\, dt.\end{aligned}$$

Note that for $z \in \Gamma$ we have $\operatorname{Re}(z - z_0) < 0$. The application of Fubini's theorem is justified since with $c := \sup_{0 \neq z \in \Gamma} |z| \,/\, |\operatorname{Re} z| < \infty$ we have

$$\int_\Gamma \int_0^\infty |\psi(z)|\, e^{\operatorname{Re}(z-z_0)t}\, dt\, |dz| \le \int_\Gamma \left| \frac{\psi(z)}{z} \right| \frac{|z|}{|\operatorname{Re} z|}\, |dz| \le c \int_\Gamma \left| \frac{\psi(z)}{z} \right| dz < \infty.$$

Hence with g defined by (3.8), $\psi(z_0) = \mathcal{L}(g)(z_0)$. $\square$

To sum up, given $f \in \mathcal{E}(S_\varphi)$ we find a uniquely determined $g \in \mathbf{L^1}(0,\infty)$ such that

$$f = \mathcal{L}(\mu), \quad \text{where} \quad \mu := g\, dt \,\oplus\, ce^{-t}\, dt \,\oplus\, d\delta_0$$

and $d = f(\infty), c = f(0) - f(\infty)$. Now, if $-A$ generates a bounded semigroup on X, we expect the identity

$$f(A)x = \int_{[0,\infty)} T(t)x\, \mu(dt)$$

to hold for all $x \in X$. This is true, and we have an even more general result.

Proposition 3.3.2. *Let $-A$ be the generator of a bounded semigroup $(T(t))_{t>0}$ on the Banach space X. If $\mu \in \mathbf{M}[0,\infty)$ is such that $f := \mathcal{L}(\mu)$ is contained in $\mathcal{M}[S_{\pi/2}]_A$, then $f(A) \in \mathcal{L}(X)$ and*

$$f(A)x = \mathcal{L}(\mu)(A)x = \int_0^\infty T(t)x\, \mu(dt) \qquad (x \in X).$$

Proof. Take $\varphi \in (\pi/2, \pi)$, and let $f \in \mathcal{O}(S_\varphi)$. First one proves the assertion for $f \in \mathcal{E}(S_\varphi)$. For $f = \mathbf{1}$ the assertion is trivially true. The case $f = (1+z)^{-1}$ is immediate from Proposition A.8.1. The case $f = \psi \in H_0^\infty$ is dealt with by means of Lemma 3.3.1:

$$\begin{aligned}\psi(A)x &= \frac{1}{2\pi i}\int_\Gamma \psi(z)R(z,A)x\,dz = \frac{-1}{2\pi i}\int_\Gamma \psi(z)R(-z,-A)x\,dz\\ &= \frac{-1}{2\pi i}\int_\Gamma \psi(z)\int_0^\infty e^{zt}T(t)x\,dt\,dz = \int_0^\infty g(t)T(t)x\,dt.\end{aligned}$$

(We have used again Proposition A.8.1 and Fubini's theorem.)

In the general case we take $e \in \mathcal{E}(S_\varphi)$ a regulariser of f. Let $\nu \in \mathbf{M}[0,\infty)$ be such that $\mathcal{L}(\nu) = e$. Then $\mathcal{L}(\mu * \nu) = ef$, and since $ef \in \mathcal{E}$, we may write

$$\begin{aligned}(ef)(A)x &= \int_0^\infty T(t)x\,(\mu*\nu)(dt) = \int_0^\infty\int_0^\infty T(t+s)x\,\nu(ds)\,\mu(dt)\\ &= \int_0^\infty T(t)\int_0^\infty T(s)x\,\nu(ds)\,\mu(dt) = \int_0^\infty T(t)e(A)x\,\mu(dt)\\ &= e(A)\int_0^\infty T(t)x\,\mu(dt),\end{aligned}$$

and this proves the claim. $\square$

Remark 3.3.3. Actually, one can *define*

$$f(A)x := \int_{[0,\infty)} T(t)x\,\mu(dt) \qquad (x \in X)$$

if $f = \mathcal{L}(\mu)$ and $\mu \in \mathbf{M}[0,\infty)$. The class $E := \{\mathcal{L}(\mu) \mid \mu \in \mathbf{M}[0,\infty)\}$ is a subalgebra of $H^\infty(\mathbb{C}_+) \cap \mathbf{C}(\overline{\mathbb{C}_+})$ and the mapping $(f \longmapsto f(A)) : E \longrightarrow \mathcal{L}(X)$ is an algebra homomorphism. This mapping is called the **Phillips calculus** for A. Proposition 3.3.2 shows that this calculus is in fact an extension of the natural functional calculus for A as defined in Chapter 2.

Let us apply these results to fractional powers.

Lemma 3.3.4. *Let* $\operatorname{Re}\alpha > 0$. *Then*

$$z^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-tz}\,dt \qquad (\operatorname{Re} z > 0).$$

Proof. Define $\psi(z) := z^\alpha e^{-z}$, which is in $H_0^\infty(S_\varphi)$ for each $\varphi \in (0, \pi/2)$. As in Example 2.2.6 we see that $c := \int_0^\infty \psi(tz)\,dt/t$ is constant, and by the very definition of the Gamma function, letting $z = 1$ we find $c = \Gamma(\alpha)$. The rest is some trivial algebra. $\square$

Proposition 3.3.5. *Let $-A$ be the generator of a bounded semigroup $(T(t))_{t\geq 0}$. Then*

$$(\varepsilon + A)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\varepsilon t} T(t)\, dt \tag{3.9}$$

in the strong sense, for all $\varepsilon > 0, \operatorname{Re}\alpha > 0$.

Proof. Replacing z by $z + \varepsilon$, Lemma 3.3.4 shows that the function $(z+\varepsilon)^{-\alpha}$ is the Laplace transform of $\Gamma(\alpha)^{-1} t^{\alpha-1} e^{-\varepsilon t}\, dt$. Hence the assertion follows from Proposition 3.3.2. (We use the composition rule here.) □

Corollary 3.3.6. *Let $-A$ be the generator of an exponentially stable semigroup $T = (T(t))_{t\geq 0}$. Then*

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t)\, dt \tag{3.10}$$

in the strong sense, for all $\operatorname{Re}\alpha > 0$.

Proof. We find $\varepsilon > 0$ such that the semigroup $e^{\varepsilon t} T(t)$ is still bounded. If we apply Proposition 3.3.5 to this semigroup (which has generator $-(A - \varepsilon)$) we obtain the assertion. □

Note that in no result of this section did we assume that the semigroup is strongly continuous at 0.

3.4 Holomorphic Semigroups

In this section we survey the basic properties of (bounded) holomorphic semigroups. In contrast to general semigroups, the holomorphic ones are accessible via the natural functional calculus. This is one reason why the natural functional calculus is so important in the framework of parabolic problems (where the semigroups are holomorphic) but is of minor relevance in a hyperbolic setting.

Let A be a sectorial operator of angle $\omega \in [0, \pi/2)$. In contrast to other sections we allow the operator A *to be multi-valued.* For $0 \neq \lambda \in \mathbb{C}$ with $|\arg\lambda| < \pi/2 - \omega$ the function $e^{-\lambda z}$ is in $\mathcal{E}(S_\varphi)$ for each $\varphi \in (\omega, \pi/2 - |\arg\lambda|)$ (it decays fast at ∞ and is holomorphic at 0, cf. Example 2.2.4 and Lemma 2.3.2).

By Remark 2.3.4, this allows us to define

$$e^{-\lambda A} := (e^{-\lambda z})(A) \;\in\; \mathcal{L}(X). \tag{3.11}$$

The family $(e^{-\lambda A})_{|\arg\lambda| < \pi/2-\omega}$ is called the **holomorphic semigroup** generated by $-A$ (see below for a general introduction of this terminology). We summarise some basic properties.

Proposition 3.4.1. *Let A be a multi-valued sectorial operator of angle $\omega \in [0, \pi/2)$. Then the following assertions hold.*

a) $e^{-\lambda A} e^{-\mu A} = e^{-(\lambda+\mu)A}$ *for all* $\lambda, \mu \in S_{\pi/2-\omega}$.

b) *The mapping*

$$(\lambda \longmapsto e^{-\lambda A}) : S_{\frac{\pi}{2}-\omega} \longrightarrow \mathcal{L}(X)$$

is holomorphic.

c) *Let* $\theta \in (0, \pi/2 - \omega)$. *Then*

$$\sup \left\{ \|e^{-\lambda A}\| \;\middle|\; |\arg \lambda| \le \theta \right\} \;<\; \infty.$$

More precisely, for each choice of $\omega' \in (\omega, \pi/2 - \theta)$ *there exists a constant* $C = C(e^{-z}, \omega' + \theta)$ *with* $\|e^{-\lambda A}\| \le CM(A, \omega')$ *for all* $|\arg \lambda| \le \theta$.

d) *The identity*

$$(\mu + A)^{-1} = \int_0^\infty e^{-\mu t} e^{-tA}\, dt$$

holds true for all $\operatorname{Re} \mu > 0$.

e) *For all* $\lambda \in S_{\pi/2-\omega}$ *we have* $\mathcal{R}(e^{-\lambda A}) \subset \bigcap_{n\in\mathbb{N}} \mathcal{D}(A^n)$.

f) *If* $x \in \overline{\mathcal{D}(A)}$, *then*

$$\lim_{\lambda \to 0, |\arg \lambda| \le \varphi} e^{-\lambda A} x = x$$

for each $\varphi \in (0, \pi/2 - \omega)$.

g) *Suppose that* A *is single-valued. Then*

$$\sigma(e^{-tA}) = e^{-t\tilde{\sigma}(A)} \qquad (t \ge 0).$$

Proof. a) is immediate from Theorem 2.3.3.
b) is easily proved in writing $e^{-\lambda A}$ as a Cauchy integral (Lemma 2.3.2) and applying the Dominated Convergence Theorem. We obtain

$$\frac{d^n}{d\lambda^n}(e^{-\lambda A}) = ((-z)^n e^{-\lambda z})(A).$$

c) is a special case of Proposition 2.6.11 (writing $\varphi := \omega' + \theta$) in noting that we have $e^{-\lambda A} = (e^{-z})(\lambda A)$ (an instance of the composition rule).
d) Given $\operatorname{Re} \mu > 0$ we compute

$$\begin{aligned}\int_0^\infty e^{-\mu t} e^{-tA}\, dt &= \int_0^\infty e^{-\mu t} \frac{1}{2\pi i} \int_\Gamma e^{-tz} R(z, A)\, dz \\ &= \frac{1}{2\pi i} \int_\Gamma \int_0^\infty e^{-\mu t} e^{-tz}\, dt\, R(z, A)\, dz = \frac{1}{2\pi i} \int_\Gamma \frac{1}{\mu + z} R(z, A)\, dz = (\mu + A)^{-1}\end{aligned}$$

by Lemma 2.3.2. (Here, Γ is an appropriate contour avoiding 0, see Section 2.3.)
e) Take $n \in \mathbb{N}$ and $\lambda \in S_{\pi/2-\omega}$. Since both functions $(1+z)^{-n}$ and $(1+z)^n e^{-\lambda z}$ are contained in $\mathcal{E}$ we obtain $e^{-\lambda A} = (1 + A)^{-n}((1 + z)^n e^{-\lambda z})(A)$ by multiplicativity.
f) Choose $\varphi \in (0, \pi/2 - \omega)$. Then $(e^{-\lambda z}(1 + z)^{-1})(A) \to (1 + A)^{-1}$ in norm as

$S_\varphi \ni \lambda \to 0$, by Lebesgue's theorem. Hence $e^{-\lambda A}x = (e^{-\lambda z}(1+z)^{-1})(A)z \to x$ for $x \in \mathcal{D}(A)$ and $z \in (1+A)x$. By the uniform boundedness proved in c), we obtain $e^{-\lambda A}x \to x$ even for all $x \in \overline{\mathcal{D}(A)}$.

g) follows from the Spectral Mapping Theorem (Theorem 2.7.8). □

Remark 3.4.2. We remark that by a)–d) of the previous proposition the operator family $T := (e^{-tA})_{t\geq 0}$ is a bounded semigroup with generator $-A$ (cf. Appendix A.8). Part g) implies in particular that $r(e^{-tA}) = e^{ts(-A)}$ for each $t \geq 0$, where $s(-A) := \sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(-A)\}$ is the **spectral bound** of the generator. Since always $e^{t\omega_0(T)} = r(e^{-tA})$ (see [85, Proposition IV.2.2]), this in turn yields $\omega_0(T) = s(-A)$. Such an identity fails for general semigroups, see [85, IV.2.7].

Proposition 3.4.3. *Let $\omega \in [0, \pi/2)$, and let $A \in \operatorname{Sect}(\omega)$ be single-valued. Then $f(A)e^{-\lambda A} \in \mathcal{L}(X)$ for all $\lambda \in S_{\pi/2-\omega}$ and all $f \in \mathcal{A}[S_\omega]$. Let $\varphi \in (0, \pi/2-\omega)$ and $\operatorname{Re}\alpha > 0$. Then*

$$\sup\left\{ |\lambda|^{\operatorname{Re}\alpha} \left\| A^\alpha e^{-\lambda A} \right\| \;\middle|\; |\arg\lambda| \leq \varphi \right\} < \infty.$$

More precisely, for each choice of $\omega' \in (\omega, \pi/2-\theta)$ there is $C = C(z^\alpha e^{-z}, \omega'+\theta)$ such that

$$\left\| A^\alpha e^{-\lambda A} \right\| \leq C\, M(A, \omega') e^{(\operatorname{Im}\alpha \arg\lambda)} \, |\lambda|^{-\operatorname{Re}\alpha} \qquad (|\arg\lambda| \leq \theta).$$

Proof. Take $f \in \mathcal{A}[S_\omega]$. Then $f(z)e^{-\lambda z}$ is (super)polynomially decaying at ∞ and $f(z)e^{-\lambda z} - f(0)$ is polynomially decaying at 0, hence $f(z)e^{-\lambda z} \in \mathcal{E}$. By a standard functional calculus argument we obtain $f(A)e^{-\lambda A} = (f(z)e^{-\lambda z})(A) \in \mathcal{L}(X)$. The last statement follows from Proposition 2.6.11 in noting that $\lambda^\alpha A^\alpha e^{-\lambda A} = (z^\alpha e^{-z})(\lambda A)$ (composition rule). □

Let $\theta \in (0, \pi/2]$. A mapping $T : S_\theta \longrightarrow \mathcal{L}(X)$ is called a **bounded holomorphic (degenerate) semigroup** (of angle θ) if it has the following properties:

1) The **semigroup law** $T(\lambda)T(\mu) = T(\lambda+\mu)$ holds for all $\lambda, \mu \in S_\theta$.
2) The mapping $T : S_\theta \longrightarrow \mathcal{L}(X)$ is holomorphic.
3) The mapping T satisfies $\sup_{\lambda \in S_\varphi} \|T(\lambda)\| < \infty$ for each $\varphi \in (0, \theta)$.

(By a), b) and c) of Proposition 3.4.1, $(e^{-\lambda A})_{\lambda \in S_\theta}$ is a bounded holomorphic semigroup of angle $\theta = \pi/2 - \omega$, whenever $A \in \operatorname{Sect}(\omega)$ with $\omega \in [0, \pi/2)$.) By holomorphy, T is uniquely determined by its values on $(0, \infty)$. Moreover, if the semigroup law holds for real values, then it holds for all λ (see the proof of [10, Proposition 3.7.2]). Also by holomorphy and the semigroup law, the space $\mathcal{N}_T := \mathcal{N}(T(\lambda))$ is independent of $\lambda \in S_\theta$. Hence either each or none of the operators $T(\lambda)$ is injective.

If we are given a bounded holomorphic semigroup T and restrict it to the positive real axis $(0, \infty)$, we obtain a bounded semigroup as defined in Section A.8.

This semigroup has a **generator** B, defined via its resolvent by

$$R(\lambda, B) := \int_0^\infty e^{-\lambda t} T(t)\, dt \qquad (\operatorname{Re} \lambda > 0).$$

Let $A := -B$. Then we have $A0 = \mathcal{N}_T$ by (A.1) on page 298. Hence A is single-valued if and only if $T(w)$ is injective for one/all $w \in S_\theta$.

The multi-valued operator A is sectorial (at least of angle $\pi/2$), since

$$\left\| \lambda(\lambda + A)^{-1} \right\| = \| \lambda R(\lambda, B) \| \;\le\; (\sup_{t>0} \|T(t)\|) \, \frac{|\lambda|}{\operatorname{Re} \lambda}$$

for all $\operatorname{Re} \lambda > 0$. But even more is true: *The multi-valued operator A is sectorial of angle $\pi/2 - \theta$, and we have $T(\lambda) = e^{-\lambda A}$ for all $\lambda \in S_\theta$.*

Proof. Let $\varphi \in (-\theta, \theta)$, and consider the bounded holomorphic semigroup $T(e^{i\varphi} \cdot)$ on $S_{\theta - |\varphi|}$. We know already that there is a multi-valued operator $A_\varphi \in \operatorname{Sect}(\pi/2)$ such that $(\lambda + A_\varphi)^{-1} = \int_0^\infty e^{-\lambda t} T(te^{i\varphi})\, dt$ for all $\operatorname{Re} \lambda > 0$. Let $\Gamma = (0, \infty)e^{i\varphi}$. By Cauchy's theorem,

$$\begin{aligned} (s + A)^{-1} &= \int_0^\infty e^{-st} T(t)\, dt = \int_\Gamma e^{-sz} T(z)\, dz = e^{i\varphi} \int_0^\infty e^{-se^{i\varphi} t} T(te^{i\varphi})\, dt \\ &= e^{i\varphi}(se^{i\varphi} + A_\varphi)^{-1} = (s + e^{-i\varphi} A_\varphi)^{-1} \end{aligned}$$

for all $s > 0$. This yields $e^{i\varphi} A = A_\varphi$. Since φ ranges between $-\theta$ and θ and each A_φ is sectorial of angle $\pi/2$ we obtain that A is sectorial of angle $\pi/2 - \theta$. Employing d) of Proposition 3.4.1 and the injectivity of the Laplace transform we conclude that $T(t) = e^{-tA}$ for all $t > 0$. By holomorphy, this implies that $T(\lambda) = e^{-\lambda A}$, $\lambda \in S_\theta$. $\square$

Combining the above considerations with Proposition 3.4.1 we obtain the following.

Proposition 3.4.4. *There is a one-one correspondence between multi-valued sectorial operators A of angle $\omega \in [0, \pi/2)$ and bounded holomorphic semigroups T on $S_{\pi/2 - \omega}$, given by the relations*

$$T(z) = e^{-zA} \quad (z \in S_{\frac{\pi}{2} - \omega}), \qquad (\lambda + A)^{-1} = \int_0^\infty e^{-\lambda t} T(t)\, dt \qquad (\operatorname{Re} \lambda > 0).$$

The operator A is single-valued if and only if $T(z)$ is injective for all $z \in S_{\pi/2 - \omega}$.

The last result establishes a fact pointed at in Remark 2.3.12. Namely, if A is sectorial and single-valued and $\omega_A < \pi/2$, then e^{-tz} for $t > 0$ is a regulariser which compensates exponential growth at ∞.

Remark 3.4.5. Fix $\theta \in (0, \pi/2]$. An **exponentially bounded holomorphic semigroup** of angle θ is a holomorphic mapping $T : S_\theta \longrightarrow \mathcal{L}(X)$ satisfying the semigroup law and such that $\{T(\lambda) \mid \lambda \in S_\varphi,\, |\lambda| \le 1\}$ is bounded for each $\varphi \in (0, \theta)$. Given such a semigroup, for each $\varphi \in (0, \theta)$ one can find numbers $M_\varphi \ge 1, w_\varphi \ge 0$ such that

$$\|T(\lambda)\| \le M_\varphi e^{w_\varphi \operatorname{Re} \lambda} \qquad (\lambda \in S_\varphi).$$

In particular, $T|_{(0,\infty)}$ is an exponentially bounded semigroup in the sense of Section A.8. Thus it has a generator $-A$, which is characterised by

$$(\lambda + A)^{-1} = \int_0^\infty e^{-\lambda t} T(t)\, dt$$

for sufficiently large $\operatorname{Re} \lambda$. Given $\varphi \in (0, \theta)$ and w_φ as above, we have

$$e^{-w_\varphi \lambda}\, T(\lambda) = e^{-\lambda(A + w_\varphi)} \qquad (\lambda \in S_\varphi).$$

Hence all statements on exponentially bounded holomorphic semigroups can be reduced to statements on bounded holomorphic semigroups. For example, given $\varphi \in (-\theta, \theta)$, the mapping $(t \longmapsto T(e^{i\varphi} t))$ is an exponentially bounded semigroup with generator $-e^{i\varphi} A$. (See the proof before Proposition 3.4.4.)

From Corollary A.8.3 we know that the **space of strong continuity** of $T|_{(0,\infty)}$ is exactly $\overline{\mathcal{D}(A)}$. Employing Proposition 3.4.4 and part f) of Proposition 3.4.1 we see that even

$$\lim_{S_\varphi \ni \lambda \to 0} T(\lambda) x = x$$

for $x \in \overline{\mathcal{D}(A)}$ and each $\varphi \in [0, \theta)$.

Let us describe two situations where holomorphic semigroups appear in a natural way.

Example 3.4.6 (Subordinated Semigroups). Let $-A$ generate a bounded semigroup, and let $\alpha \in (0,1)$. By scaling (Proposition 3.1.2) $A^\alpha \in \operatorname{Sect}(\alpha\pi/2)$ and since $\alpha < 1$ the operator $-A^\alpha$ generates a bounded holomorphic semigroup of angle at least $(1 - \alpha)\pi/2$. For $t > 0$, the function e^{-tz^α} belongs to $\mathcal{E}[S_{\pi/2}]$ and vanishes at ∞, so there must be a function $f_{t,\alpha} \in \mathbf{L^1}(0, \infty)$ such that

$$e^{-tz^\alpha} = \mathcal{L}(f_{t,\alpha}) = \int_0^\infty e^{-sz} f_{t,\alpha}(s)\, ds.$$

One then has $e^{-tA^\alpha} x = \int_0^\infty f_{t,\alpha}(s) T(s) x\, ds$ for all $x \in X$, by the Phillips calculus (Proposition 3.3.2). Since $f_{t,\alpha}$ is the inverse Fourier transform of the function $r \mapsto e^{-t(ir)^\alpha}$, one has

$$f_{t,\alpha}(s) = \frac{1}{2\pi i} \int_{i\mathbb{R}} e^{sz - tz^\alpha}\, dz \qquad (s, t > 0,\ \alpha \in (0,1)).$$

Deforming the path yields the explicit formula

$$f_{t,\alpha}(s) = \frac{1}{\pi} \int_0^\infty e^{sr\cos\theta - tr^\alpha \cos\alpha\theta} \sin(sr\sin\theta - tr^\alpha \sin\alpha\theta + \theta)\, dr,$$

where $\theta \in [\pi/2, \pi]$ is arbitrary, see [230, p.263]. In the case $\alpha = 1/2$ this reduces to the particularly nice form

$$f_{t,\frac{1}{2}}(s) = \frac{t}{\sqrt{4\pi s^3}} e^{-\frac{t^2}{4s}},$$

see [230, p.268] or [10, Thm.3.8.3]. One can show that for all $\alpha \in (0,1), t > 0$ the function $f_{t,\alpha}$ is positive [230, p.261]. The semigroups $(e^{tA^\alpha})_{t>0}$ for $\alpha \in (0,1)$ play an important role in the theory of Markov diffusion processes, and are called **subordinated semigroups**.

Example 3.4.7. Let A be a sectorial operator on a Banach space X, and let $\alpha \in (0, 1/2]$. Then $A^\alpha \in \operatorname{Sect}(\alpha\omega_A)$ (Proposition 3.1.2), hence $(-A^\alpha)$ generates a holomorphic semigroup. One can derive Balakrishnan-type representation formulae for these semigroups, namely

$$e^{-tA^\alpha} = \frac{1}{\pi} \int_0^\infty e^{-tr^\alpha \cos\pi\alpha} \sin(tr^\alpha \sin\pi\alpha)\, (A + r)^{-1}\, dr \qquad (t > 0)$$

where the integral is absolutely convergent in the case where $\alpha \in (0, 1/2)$ and is convergent in the improper sense when $\alpha = 1/2$. See [161, Section 5.5] for details.

Let $A \in \operatorname{Sect}(\omega)$ on the Banach space X, and suppose that $0 \in \varrho(A)$. We know from Proposition 3.2.3 that $(A^{-z})_{\operatorname{Re} z > 0}$ is an exponentially bounded holomorphic semigroup of angle $\pi/2$, with $\overline{\mathcal{D}(A)}$ as its space of strong continuity. In the next section we identify its generator.

3.5 The Logarithm and the Imaginary Powers

We return to our terminological agreement that 'operator' always is to be read as 'single-valued operator' (see p. 279). In fact, we work with an *injective*, single-valued, sectorial operator A. The function $\log z$ has subpolynomial growth at 0 and at ∞, whence it belongs to the class $\mathcal{B}(S_\varphi)$ for each $\varphi \in (0, \pi]$. Since A is assumed to be injective, the operator

$$\log A := (\log z)(A)$$

is defined and is called the **logarithm** of the operator A. Because of the identity $\log(z^{-1}) = -\log z$ we have $\log(A^{-1}) = -\log A$. The next result is fundamental.

Lemma 3.5.1 (Nollau). *Let* $A \in \mathrm{Sect}(\omega)$ *be injective. If* $|\mathrm{Im}\,\lambda| > \pi$*, then* $\lambda \in \varrho(\log A)$ *and*

$$R(\lambda, \log A) = \int_0^\infty \frac{-1}{(\lambda - \log t)^2 + \pi^2}(t + A)^{-1}\,dt. \tag{3.12}$$

Hence $\|R(\lambda, \log A)\| \leq \pi\, M(A)\,(|\mathrm{Im}\,\lambda| - \pi)^{-1}$.

We call the formula (3.12) the **Nollau representation** of the resolvent of $\log A$.

Proof. Suppose first that $A, A^{-1} \in \mathcal{L}(X)$. We choose $\varphi \in (\omega, \pi)$, $a > 0$ small and $b > 0$ large enough and denote by Γ the positively oriented boundary of the bounded sector $S_\varphi(a, b)$. Then

$$\begin{aligned}
&(\lambda - \log z)^{-1}(A) \\
&\quad = \frac{1}{2\pi i}\int_\Gamma \frac{1}{\lambda - \log z} R(z, A)\,dz \\
&\quad = \frac{1}{2\pi i}\int_a^b \frac{e^{-i\varphi}}{\lambda - \log t + i\varphi} R(te^{-i\varphi}, A)\,dt + \frac{1}{2\pi}\int_{-\varphi}^{\varphi} \frac{be^{is}}{\lambda + \log b - is} R(be^{is}, A)\,ds \\
&\qquad - \frac{1}{2\pi i}\int_a^b \frac{e^{i\varphi}}{\lambda - \log t - i\varphi} R(te^{i\varphi}, A)\,dt - \frac{1}{2\pi}\int_{-\varphi}^{\varphi} \frac{ae^{is}}{\lambda + \log a - is} R(ae^{is}, A)\,ds \\
&\quad \overset{(1)}{=} \int_a^b \frac{-1}{(\lambda - \log t)^2 + \pi^2}(t + A)^{-1}\,dt + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{be^{is}}{\lambda + \log b - is} R(be^{is}, A)\,ds \\
&\qquad - \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{ae^{is}}{\lambda + \log a - is} R(ae^{is}, A)\,ds \\
&\quad \overset{(2)}{=} \int_0^\infty \frac{-1}{(\lambda - \log t)^2 + \pi^2}(t + A)^{-1}\,dt =: J(A),
\end{aligned}$$

where we have let $\varphi \to \pi$ in (1) and $a \to 0, b \to \infty$ in (2). Hence we have proved the claim in the special case where A and A^{-1} are both bounded. In the general case define $A_\varepsilon := (A + \varepsilon)(1 + \varepsilon A)^{-1}$. Then $(A_\varepsilon)_{\varepsilon > 0}$ is a sectorial approximation of A (see Proposition 2.1.3). Let $f(z) := (\lambda - \log z)^{-1}$. Obviously $f \in H^\infty(S_\varphi)$ for $\varphi > \omega$. We have already shown that $f(A_\varepsilon) = J(A_\varepsilon) \in \mathcal{L}(X)$. It follows from the Dominated Convergence Theorem that

$$f(A_\varepsilon) \to J(A) = \int_0^\infty \frac{-1}{(\lambda - \log t)^2 + \pi^2}(t + A)^{-1}\,dt.$$

in norm. Applying Proposition 2.6.9 we obtain $f(A) = J(A)$. From Theorem 1.3.2 f) we conclude that in fact $f(A) = (\lambda - \log A)^{-1}$. Having shown this we compute

$$
\begin{aligned}
\|R(\lambda, \log A)\| &\leq \int_0^\infty \frac{M(A)}{|(\lambda - \log t)^2 + \pi^2|} \frac{dt}{t} = \int_{\mathbb{R}} \frac{M(A)}{|(i \operatorname{Im} \lambda - s)^2 + \pi^2|} \, ds \\
&= \int_{\mathbb{R}} \frac{M(A)}{\sqrt{(s^2 - ((\operatorname{Im} \lambda)^2 - \pi^2)^2 + 4s^2 (\operatorname{Im} \lambda)^2}} \, ds \\
&\leq \int_{\mathbb{R}} \frac{M(A)}{(s^2 + ((\operatorname{Im} \lambda)^2 - \pi^2)} \, ds = \frac{M(A)\pi}{\sqrt{(\operatorname{Im} \lambda)^2 - \pi^2}} \leq \frac{M(A)\pi}{|\operatorname{Im} \lambda| - \pi},
\end{aligned}
$$

where we have used the (easily verified) inequalities

$$
\begin{aligned}
(s^2 - ((\operatorname{Im} \lambda)^2 - \pi^2))^2 + 4s^2 (\operatorname{Im} \lambda)^2 &\geq (s^2 - ((\operatorname{Im} \lambda)^2 - \pi^2))^2 + 4s^2 ((\operatorname{Im} \lambda)^2 - \pi^2) \\
&= (s^2 + ((\operatorname{Im} \lambda)^2 - \pi^2))^2, \qquad \text{and} \\
\sqrt{(\operatorname{Im} \lambda)^2 - \pi^2} &\geq |\operatorname{Im} \lambda| - \pi.
\end{aligned}
$$

□

Proposition 3.5.2. *Let $A \in \operatorname{Sect}(\omega)$ be injective. If $|\operatorname{Im} \lambda| > \omega$, then $\lambda \in \varrho(\log A)$ and for each $\varphi > \omega$ there is a constant M_φ such that*

$$
\|R(\lambda, \log A)\| \leq \frac{M_\varphi}{|\operatorname{Im} \lambda| - \varphi} \qquad (|\operatorname{Im} \lambda| > \varphi).
$$

In fact $M_\varphi = \pi M(A^{\pi/\varphi})$. Furthermore, $\mathcal{D}(\log A) \subset \overline{\mathcal{D}(A)} \cap \overline{\mathcal{R}(A)}$.

Proof. Let $\alpha := \pi/\varphi$. Then $A^\alpha \in \operatorname{Sect}(\alpha\omega)$ is injective. The composition rule yields $\log(A^\alpha) = \alpha \log(A)$. Applying Nollau's Lemma 3.5.1 we obtain

$$
\|R(\mu, \alpha \log A)\| \leq M(A^\alpha)\pi/(|\operatorname{Im} \mu| - \pi) \qquad (|\operatorname{Im} \mu| > \pi).
$$

Letting $\lambda = \mu/\alpha$ and $\varphi = \pi/\alpha$ we arrive at

$$
\|R(\lambda, \log A)\| \leq M(A^\alpha)\pi/(|\operatorname{Im} \lambda| - \varphi) \qquad (|\operatorname{Im} \lambda| > \varphi).
$$

From the Nollau representation (3.12) for $R(\cdot, \log A)$ it follows that $\mathcal{D}(\log A) \subset \overline{\mathcal{D}(A)}$. But $-\log A = \log A^{-1}$, hence also $\mathcal{D}(\log A) \subset \overline{\mathcal{D}(A^{-1})} = \overline{\mathcal{R}(A)}$. □

As promised, we can now identify the generator of the holomorphic semigroup $(A^{-z})_{\operatorname{Re} z > 0}$, where A is a sectorial and invertible operator.

Proposition 3.5.3. *Let $A \in \operatorname{Sect}(\omega)$ with $0 \in \varrho(A)$. Then $-\log A$ is the generator of the holomorphic semigroup $(A^{-z})_{\operatorname{Re} z > 0}$. In particular, $\overline{\mathcal{D}(\log A)} = \overline{\mathcal{D}(A)}$.*

Proof. By semigroup theory, there is $c > 0$ such that $(e^{-ct} A^{-t})_{t > 0}$ is a bounded semigroup. Clearly it suffices to show that

$$
(\lambda + \log A)^{-1} = \int_0^\infty e^{-\lambda t} A^{-t} \, dt
$$

for some $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > c$ and $|\operatorname{Im}\lambda > \pi|$. So choose such a λ and let $0 < a < b < \infty$. Then

$$\begin{aligned}\int_a^b e^{-\lambda t} A^{-t}\, dt &= \int_a^b e^{-\lambda t} \frac{1}{2\pi i} \int_\Gamma z^{-t} R(z,A)\, dz\, dt \\ &= \frac{1}{2\pi i} \int_\Gamma \int_a^b e^{-\lambda t} z^{-t}\, dt R(z,A)\, dz = \frac{1}{2\pi i} \int_\Gamma \frac{e^{-\lambda a} z^{-a} - e^{-\lambda b} z^{-b}}{\lambda + \log z} R(z,A)\, dz \\ &= e^{-\lambda a} A^{-a} (\lambda + \log A)^{-1} - e^{-\lambda b} (\lambda + \log A)^{-1},\end{aligned}$$

where Γ is an appropriate path avoiding 0 (see Section 2.5). The last equality is due to the fact that $(\lambda + \log z)^{-1}(A) = (\lambda + \log A)^{-1} \in \mathcal{L}(X)$ by Nollau's Lemma 3.5.1. Since $\operatorname{Re}\lambda > c$ we have $\left\| e^{-\lambda b} A^{-b} \right\| \to 0$ as $b \to \infty$. From $\mathcal{D}(\log A) \subset \overline{\mathcal{D}(A)}$ and Proposition 3.2.3 we see that $e^{-\lambda a} A^{-a} (\lambda + \log A)^{-1} \to (\lambda + \log A)^{-1}$ strongly. This concludes the proof. □

Remark 3.5.4. Suppose that A is a *bounded* sectorial operator. As a matter of fact, the fractional powers $(A^z)_{\operatorname{Re} z > 0}$ of A form a quasi-bounded holomorphic semigroup as well. What is its generator? If A is injective, the answer is of course $\log A$. But if A is not injective, the generator is not single-valued and cannot be obtained by the natural functional calculus. Since also in this case the generator should be $\log A$ in a sense, we feel the necessity of a sophisticated functional calculus for multi-valued operators. However, we do not further pursue this matter here.

We conclude our investigation of the logarithm with the following nice observation. Consider the function $f(z) := \log z - \log(z + \varepsilon) = \log(z(z + \varepsilon)^{-1})$. A short computation reveals that f is holomorphic at ∞ with $f(\infty) = 0$. Hence if A is sectorial and invertible, then $f(A) \in \mathcal{L}(X)$, i.e., $f \in H(A)$. From the usual rules of functional calculus we see that $\log(A + \varepsilon)$ is a bounded perturbation of $\log A$. In particular, $\mathcal{D}(\log A) = \mathcal{D}(\log(A + \varepsilon))$.

Let us turn to the **imaginary powers** A^{is} of an injective operator $A \in \operatorname{Sect}(\omega)$.

Proposition 3.5.5. *Let $A \in \operatorname{Sect}(\omega)$ be injective, and let $0 \neq s \in \mathbb{R}$. Then the following assertions hold.*

a) *If $A^{is} \in \mathcal{L}(X)$, then $\mathcal{D}(A^\alpha) \subset \mathcal{D}(A^{\alpha + is})$ for all $\alpha \in \mathbb{C}$. Conversely, if $\mathcal{D}(A^\alpha) \subset \mathcal{D}(A^{\alpha + is})$ for $\alpha \in \{-1, 1\}$, then $A^{is} \in \mathcal{L}(X)$.*

b) *If $0 \in \varrho(A)$ and $\varepsilon > 0$, then*

$$(A + \varepsilon)^{is} = (1 + \varepsilon A^{-1})^{is} A^{is}.$$

In particular, $\mathcal{D}((A+\varepsilon)^{is}) = \mathcal{D}(A^{is})$. Moreover, $(A+\varepsilon)^{is} x \to A^{is} x$ as $\varepsilon \to 0$, whenever $x \in \mathcal{D}(A^{is})$.

c) *The space $D := \mathcal{D}((A + \varepsilon)^{is})$ is independent of $\varepsilon > 0$. If $x \in D$ and $\lim_{\varepsilon \to 0} (A + \varepsilon)^{is} x =: y$ exists, then $x \in \mathcal{D}(A^{is})$ with $y = A^{is} x$.*

d) *If* $A^{is} \in \mathcal{L}(X)$, *then also* $(A+\varepsilon)^{is} \in \mathcal{L}(X)$ *for all* $\varepsilon > 0$. *Moreover,* $\sup_{0\le\varepsilon\le 1} \|(A+\varepsilon)^{is}\| < \infty$ *and* $\lim_{\varepsilon\to 0}(A+\varepsilon)^{is}x = A^{is}x$ *for all* $x \in \overline{\mathcal{R}(A)}$.

e) *Let* $0 \in \varrho(A)$. *If* $A^{is} \in \mathcal{L}(X)$, *then* $\sup_{0<\alpha<1} \|A^{-\alpha+is}\| < \infty$ *and*

$$\lim_{\alpha\searrow 0} A^{-\alpha+is}x = A^{is}x \qquad (x \in \overline{\mathcal{D}(A)}).$$

Conversely, if $\sup_{0<\alpha<1} \|A^{-\alpha+is}\| < \infty$ *and* $\overline{\mathcal{D}(A)} = X$, *then* $A^{is} \in \mathcal{L}(X)$.

Proof. a) Suppose $A^{is} \in \mathcal{L}(X)$, and let $\alpha \in \mathbb{C}$. Then $A^{is}A^{\alpha} \subset A^{\alpha+is}$, hence $\mathcal{D}(A^{\alpha}) = \mathcal{D}(A^{is}A^{\alpha}) \subset \mathcal{D}(A^{\alpha+is})$. Conversely, suppose $\mathcal{D}(A) \subset \mathcal{D}(A^{1+is})$ and $\mathcal{R}(A) \subset \mathcal{D}(A^{-1+is})$. Since always $\mathcal{D}(A^{1+is}) \cap \mathcal{D}(A) = \mathcal{D}(A^{is}A)$, we have $\mathcal{D}(A) = \mathcal{D}(A^{is}A)$. Hence $\mathcal{R}(A) \subset \mathcal{D}(A^{is})$. Similarly we obtain $\mathcal{D}(A) = \mathcal{R}(A^{-1}) \subset \mathcal{D}(A^{is})$. But $\mathcal{D}(A) + \mathcal{R}(A) = X$, whence $A^{is} \in \mathcal{L}(X)$.

b) Since $A(A+\varepsilon)^{-1}$ is bounded and invertible, we have $(A(A+\varepsilon)^{-1})^{is} \in \mathcal{L}(X)$. But

$$(1+\varepsilon A^{-1})^{-is} = (A(A+\varepsilon)^{-1})^{is} = \left(\frac{z}{z+\varepsilon}\right)^{is}(A)$$

by the composition rule. Hence $(z(z+\varepsilon)^{-1})^{is} \in H(A)$. Therefore,

$$(A+\varepsilon)^{is}(1+\varepsilon A^{-1})^{-is} = \left[(z+\varepsilon)^{is}\left(\frac{z}{z+\varepsilon}\right)^{is}\right](A) = (z^{is})(A) = A^{is}$$

(see Theorem 1.3.2). From this we conclude that

$$(A+\varepsilon)^{is} = (1+\varepsilon A^{-1})^{is}(A+\varepsilon)^{is}(1+\varepsilon A^{-1})^{-is} = (1+\varepsilon A^{-1})^{is}A^{is}$$

by Theorem 1.3.2 e). The last statement follows from the fact that $(1+\varepsilon A^{-1})^{is} \to I$ in norm as $\varepsilon \to 0$ (apply Lemma 2.6.7).

c) The first statement is immediate from b). Now take $x \in D$ and suppose that the limit $\lim_{\varepsilon\searrow 0}(A+\varepsilon)^{is}x =: y$ exists. Then

$$(\tau(z)z^{is})(A) = \left(\left(\frac{z}{z+\varepsilon}\right)^{is}\tau(z)\right)(A+\varepsilon)^{is}x \to \tau(A)y \qquad (\varepsilon \searrow 0)$$

where we use Lemma 5.1.2 below. (The reader will surely realise that although we have postponed that lemma, there is no hidden logical circle in this reasoning.) But this is just to say that $A^{is}x = y$.

d) Suppose that $A^{is} \in \mathcal{L}(X)$. Then

$$\mathcal{D}(A+\varepsilon) = \mathcal{D}(A) \subset \mathcal{D}(A^{1+is}) = \mathcal{D}((A+\varepsilon)^{1+is})$$

by a) and Proposition 3.1.9. Hence

$$(A+\varepsilon)^{is} = (z^{is})(A+\varepsilon) = (z^{1+is}z^{-1})(A+\varepsilon) = (A+\varepsilon)^{1+is}(A+\varepsilon)^{-1} \in \mathcal{L}(X).$$

From Proposition 3.1.7 and Corollary 3.1.13 and

$$(A+\varepsilon)^{is}x = \left[(A+\varepsilon)^{\frac{1}{2}+is} - A^{\frac{1}{2}+is}\right](A+\varepsilon)^{-\frac{1}{2}}x + A^{is}\left[A(A+\varepsilon)^{-1}\right]^{\frac{1}{2}}x$$

we conclude that $\sup_{0\le\varepsilon\le 1}\|(A+\varepsilon)^{is}\| < \infty$. Now, let $x = A^{1/2}y \in \mathcal{R}(A^{1/2})$. Then the first summand is $O(\varepsilon^{1/2})$. The second summand may be written as $(A(A+\varepsilon)^{-1})A^{is}(A+\varepsilon)^{1/2}y$ and this tends to $A^{is}A^{1/2}y = A^{is}x$ as $\varepsilon \searrow 0$ since $A^{is} \in \mathcal{L}(X)$, $(A+\varepsilon)^{1/2}y \to A^{1/2}y$ (by Proposition 3.1.9), and $A(A+\varepsilon)^{-1} \to I$ strongly on $\overline{\mathcal{R}(A)}$ (by Proposition 2.1.1).

e) The statements follow almost immediately from Proposition 3.2.3. $\square$

In [161, Example 7.3.3] (which goes back to Komatsu) one can find a *bounded* sectorial operator A on a Banach space X such that $A^{is} \notin \mathcal{L}(X)$ for all $0 \neq s \in \mathbb{R}$. This shows in particular that in general $\mathcal{D}(A^{is}) \neq \mathcal{D}((A+\varepsilon)^{is})$.

The next result shows that — in a way — the operator $i\log A$ may be considered the 'generator' of the operator family $(A^{is})_{s\in\mathbb{R}}$. Recall the notation $\Lambda_A^{-1} = A(1+A)^{-2}$ for an injective sectorial operator A and note that $(A^{is}\Lambda_A^{-1})_{s\in\mathbb{R}}$ is a strongly continuous family of bounded operators on X.

Proposition 3.5.6. *Let $A \in \operatorname{Sect}(\omega)$ be injective. For each $\theta \in (\omega,\pi]$ there is a constant $C_\theta \ge 0$ such that*

$$\left\|A^{is}\Lambda_A^{-1}\right\| \le M(A,\theta)C_\theta e^{|s|\theta}$$

for all $s \in \mathbb{R}$. Moreover, for all $\operatorname{Re}\lambda > \omega$,

$$(\lambda - i\log A)^{-1} = \Lambda_A \int_0^\infty e^{-\lambda s}A^{is}\Lambda_A^{-1}\,ds. \tag{3.13}$$

Proof. Choose $\theta > \omega$. Writing $\tau(z) = z(1+z)^{-2}$ we have $A^{is}\Lambda_A^{-1} = (z^{-is}\tau)(A)$, whence

$$\left\|A^{is}\Lambda_A^{-1}(A)\right\| \le \frac{M(A,\theta)}{2\pi}\int_{\Gamma_\theta}\left|z^{is}\right|\frac{1}{|1+z|^2}\,|dz| \le e^{|s|\theta}\frac{M(A,\theta)}{2\pi}\int_{\Gamma_\theta}\frac{1}{|1+z|^2}\,|dz|\,.$$

Let $0<a<b$. Then

$$\begin{aligned}\int_a^b e^{-\lambda s}A^{is}\Lambda_A^{-1}\,ds &= \int_a^b e^{-\lambda s}\frac{1}{2\pi i}\int_\Gamma \frac{z^{is}z}{(1+z)^2}R(z,A)\,dz\\ &= \frac{1}{2\pi i}\int_\Gamma \frac{z}{(1+z)^2}\left(\int_a^b z^{is}e^{-\lambda s}\,ds\right)R(z,A)\,dz\end{aligned}$$

by Fubini's theorem. Evaluating the inner integral yields

$$\begin{aligned}
&\frac{1}{2\pi i}\int_\Gamma \frac{z}{(1+z)^2}\frac{e^{-(\lambda - i\log z)a} - e^{-(\lambda - i\log z)b}}{\lambda - i\log z} R(z,A)\,dz \\
&= \frac{e^{-\lambda a}}{2\pi i}\int_\Gamma \frac{z^{ia}z}{(1+z)^2(\lambda - i\log z)} R(z,A)\,dz - \\
&\qquad \frac{e^{-\lambda b}}{2\pi i}\int_\Gamma \frac{z^{ib}z}{(1+z)^2(\lambda - i\log z)} R(z,A)\,dz \\
&= e^{-\lambda a}A^{ia}\Lambda_A^{-1}R(\lambda, i\log A) - e^{-\lambda b}A^{ib}\Lambda_A^{-1}R(\lambda, i\log A).
\end{aligned}$$

The second summand tends to 0 in norm as $b \to \infty$ since $\operatorname{Re}\lambda > \omega$. The first summand tends to $\Lambda_A^{-1}R(\lambda, i\log A)$ strongly as $a \to 0$, by Proposition 3.2.1 f). Hence we obtain

$$\int_0^\infty e^{-\lambda s}A^{is}\Lambda_A^{-1}\,ds = \Lambda_A^{-1}R(\lambda, i\log A),$$

which immediately implies (3.13). $\square$

Corollary 3.5.7. *Let $A \in \operatorname{Sect}(\omega)$ be injective. The following assertions are equivalent.*

(i) $\overline{\mathcal{D}(A) \cap \mathcal{R}(A)} = X$ *and* $A^{is} \in \mathcal{L}(X)$ *for all* $s \in \mathbb{R}$.

(ii) *The operators* $(A^{is})_{s\in\mathbb{R}}$ *form a* C_0*-group of bounded operators on* X.

(iii) *The operator* $i\log A$ *generates a* C_0*-group* $(T(s))_{s\in\mathbb{R}}$ *of bounded operators on* X.

In this case we have $T(s) = A^{is}$ *for all* $s \in \mathbb{R}$.

Proof. (i)⇒(ii). Define $T(s) := A^{is}$ for $s \in \mathbb{R}$. Obviously, $(T(s))_{s\in\mathbb{R}}$ is a group. For every $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$, $T(\cdot)x$ is continuous. This follows from Proposition 3.2.1. Since $\overline{\mathcal{D}(A) \cap \mathcal{R}(A)} = X$, we have that for *each* $x \in X$ the trajectory $T(\cdot)x$ is at least strongly measurable. From [119, Theorem 10.2.3] we infer that T is strongly continuous on $(0,\infty)$, but this implies readily that T is strongly continuous on the whole real line.

(ii)⇒(iii). This follows from Proposition 3.5.6 since the hypothesis allows us to put the operator Λ_A^{-1} in front of the integral.

(iii)⇒(i). Suppose that $i\log A$ generates the C_0-group T. In particular, $\mathcal{D}(\log A)$ must be dense in X. Now, $\mathcal{D}(\log A) \subset \overline{\mathcal{D}(A)} \cap \overline{\mathcal{R}(A)} = \overline{\mathcal{D}(A) \cap \mathcal{R}(A)}$ by Proposition 3.5.2. Thus, we are left to show that $T(s) = A^{is}$ for all $s \in \mathbb{R}$. Employing Proposition 3.5.6 we obtain

$$\int_0^\infty e^{-\lambda s}A^{is}\Lambda_A^{-1}\,ds = \Lambda_A^{-1}R(\lambda, i\log A) = \int_0^\infty e^{-\lambda s}\Lambda_A^{-1}T(s)\,ds$$

for $\operatorname{Re}\lambda$ large. Since the Laplace transform is injective, $A^{is}\Lambda_A^{-1} = \Lambda_A^{-1}T(s)$ for all $s > 0$. Multipying both sides of this equation with Λ_A yields $A^{is} = T(s)$ for $s > 0$. From this we infer that $A^{-is} = (A^{is})^{-1} = T(s)^{-1} = T(-s)$ for all $s > 0$. $\square$

If A satisfies the equivalent conditions in Corollary 3.5.7 we say that A has **bounded imaginary powers** and write $A \in \mathrm{BIP}(X)$. Note that $\overline{\mathcal{D}(A) \cap \mathcal{R}(A)} = X$ is included in this definition; there exist operators A such that $A^{is} \in \mathcal{L}(X)$ for all s and even $\sup_{s\in\mathbb{R}} \|A^{is}\| < \infty$, but $\overline{\mathcal{D}(A) \cap \mathcal{R}(A)} \neq X$ (see Example 7.3.1 in [161]). If $A \in \mathrm{BIP}(X)$, by semigroup theory, we know that

$$\theta_A := \inf\Big\{\theta \geq 0 \mid \exists C : \big\|A^{is}\big\| \leq C\, e^{|s|\theta}\ (s \in \mathbb{R})\Big\} \ < \ \infty. \tag{3.14}$$

We write $A \in \mathrm{BIP}(X, \theta)$ if $A \in \mathrm{BIP}(X)$ and $\theta \geq \theta_A$. This notation slightly differs from the terminology used in [161, Definition 8.1.1]. The first part of the celebrated Prüss–Sohr theorem states that $\theta_A \geq \omega_A$ if $A \in \mathrm{BIP}(X)$. We shall reprove and generalise this result in Section 4.3, see Corollary 4.3.4.

Remark 3.5.8. Suppose that $(A^{is})_{s>0} \subset \mathcal{L}(X)$ is strongly continuous and norm bounded as $s \to 0$. This implies that $T(s) := A^{is}$ is an exponentially bounded semigroup in the terminology of Section A.8. It follows from Proposition 3.5.6 that the operator $i\log A$ is the generator of T even in this situation. The same conclusion holds (with an appropriate notion of 'generator') in the case where $(s \longmapsto A^{is})$ is continuous only with respect to a 'very weak' topology, like those considered in [140] and [88]. In any case, Proposition 3.5.6 implies that $(A^{is}\Lambda_A^{-1})_{s\in\mathbb{R}}$ is an exponentially bounded Λ_A^{-1}-regularised group with generator $i\log A$, see [66] and cf. also [179, Addendum] and [178].

3.6 Comments

Literature on Fractional Powers, past and present. Beginning with the seminal papers by Krasnoselskii and Sobolevski [137], Balakrishnan [24], Yosida [229], Kato [126], around 1960, fractional powers have been the subject of extensive research. A first attempt to exhaustively present the theory was undertaken by Komatsu in a series of papers [131, 132, 134, 133, 135] in the years from 1965 to 1970. Since then they appeared — at least in the special case of invertible operators — in practically every book on semigroup theory or evolution equations, beginning with Krein's book [139] up to [186], [85] and [10], just to mention a few. Besides, several articles underlined the importance of fractional — in particular purely imaginary — powers for the regularity theory of evolution equations, see [200, 75, 76, 227, 228].

The first monograph to appear on the theory of fractional powers was [161] by Martinez and Sanz. There the construction of the fractional powers is based on the Balakrishnan representation, which is a real integral. As a consequence, the authors are forced to give a definition of fractional powers 'by cases' as is sketched in Remark 3.1.10.

3.1 Fractional Powers with Positive Real Part. The results presented in this section are standard and in fact all included in [161, Chapter V]. The proof of Proposition 3.1.2 is from [19, Proposition 3.5]. Proposition 3.1.7 and its proof are also in [150, Lemma 3.3]. The proof of the Balakrishnan representation and its corollaries is from [161]. The Spectral Mapping Theorem for fractional powers (part j) of Proposition 3.1.1) was originally proved by Balakrishnan in [24], reprinted in [161, p. 121]. Martinez and Sanz [161, Thm. 5.2.1] give also their own proof, based on the Spectral Mapping Theorem for the Hirsch functional calculus.

3.2 Fractional Powers with Arbitrary Real Part. As in Section 3.1, the results are well known. Proposition 3.2.3 and Proposition 3.3.5 are from [5, Section III.4.6].

3.4 Holomorphic Semigroups. This section is inspired by [157]. The generalisation to the multi-valued case seems to be new as it stands. However, there are similar results in [89, Chapter III]. Holomorphic semigroups abound within the field of evolution equations of parabolic type, see e.g. [157] and the references therein.

3.5 The Logarithm and the Imaginary Powers. The logarithm of an injective sectorial operator was introduced by Nollau [176], where Lemma 3.5.1 is proved. Then for a long time the operator logarithm was out of focus but reappeared finally during the 1990's in works of Boyadzhiev [35] and Okazawa [177, 179]. The monograph [161] mentions them only in the Hilbert space case. From [103] it finally became clear that logarithms play a fundamental rule in the *theory* of sectorial operators. (This is the topic of Chapter 4 below.)

Without question, the imaginary powers of a sectorial operator are the most misterious objects in the field. Venni [219] has constructed examples of sectorial operators A where A^{is} is bounded for some values of $s \in \mathbb{R}$ and unbounded for others, see also [161, Example 7.3.4]. The boundedness of all imaginary powers has surprising consequences in regularity theory (the celebrated *Dore–Venni theorem*, see Theorem 9.3.11) and in interpolation theory (Theorem 6.6.9) as well as for functional calculus on Hilbert spaces (Theorem 7.3.1).

Lemma 3.5.1 was first proved by Nollau [176]. Our proof is an adaptation of [161, Lemma 10.1.5] and [179, Lemma 5.1] (for the norm estimate). Proposition 3.5.2 is stated and proved in the Hilbert space case in [161, Theorem 10.1.6] (though without using facts particular to Hilbert spaces). Our proof is slightly different. The basic facts on the imaginary powers collected in Proposition 3.5.5 are extracted from [161, Chapter 7]. Part d) is a perturbation result, see also Section 5.5.2. Corollary 3.5.7 is essentially in [161, Theorem 10.1.3 and Theorem 10.1.4]. An earlier account can be found in [179, Addendum] and [178]. From Uiterdijk [216, Proposition 2.2.31] we learned the argument in the proof of the implication (i)⇒(ii).

Chapter 4

Strip-type Operators and the Logarithm

As a straightforward abstraction of the logarithm of an injective sectorial operator we introduce the notion of a strip-type operator (Section 4.1). Since the resolvent of a strip-type operator by definition is bounded outside a horizontal strip, a functional calculus based on Cauchy integrals can be set up (Section 4.2). Section 4.3 is devoted to prove the main result, which states equality between the spectral angle of an injective sectorial operator A and the spectral height of the strip-type operator $\log A$. As a corollary one obtains an important theorem of Prüss and Sohr, saying that in the case where $A \in \mathrm{BIP}$, the group type of $(A^{is})_{s\in\mathbb{R}}$ is always larger than the spectral angle of A. In Section 4.4 the problem of 'inversion' is discussed, namely the question, which strip-type operators are actually logarithms of sectorial operators. Here we present a theorem of Monniaux, slightly generalised. In Section 4.5 we construct the example of an injective sectorial operator $A \in \mathrm{BIP}$ on a UMD space with the property that the group type of $(A^{is})_{s\in\mathbb{R}}$ is larger than π.

4.1 Strip-type Operators

We are going to introduce a new class of operators, in order to have an abstract concept at hand when dealing with logarithms of sectorial operators. For $\omega > 0$ we denote by

$$H_\omega := \{z \in \mathbb{C} \mid |\operatorname{Im} z| < \omega\}$$

the **horizontal strip** of height 2ω which is symmetric with respect to the real axis. In the case where $\omega = 0$ we define $H_0 := \mathbb{R}$. An operator B on a Banach space X is said to be a **strip-type operator** of height ω — in short: $B \in \mathrm{Strip}(\omega)$) — if

1) $\sigma(B) \subset \overline{H_\omega}$ and
2) $L(B,\omega') := \sup\left\{\|R(\lambda,B)\| \;\middle|\; |\operatorname{Im}\lambda| \geq \omega'\right\} < \infty$ for all $\omega' > \omega'$.

It is clear that $B \in \mathrm{Strip}(\omega)$ if and only if $-B \in \mathrm{Strip}(\omega)$, and in this case one has $L(B,\omega') = L(-B,\omega')$ for each $\omega' > \omega$. We call

$$\omega_{st}(B) := \min\{\omega \geq 0 \mid B \in \mathrm{Strip}(\omega)\}$$

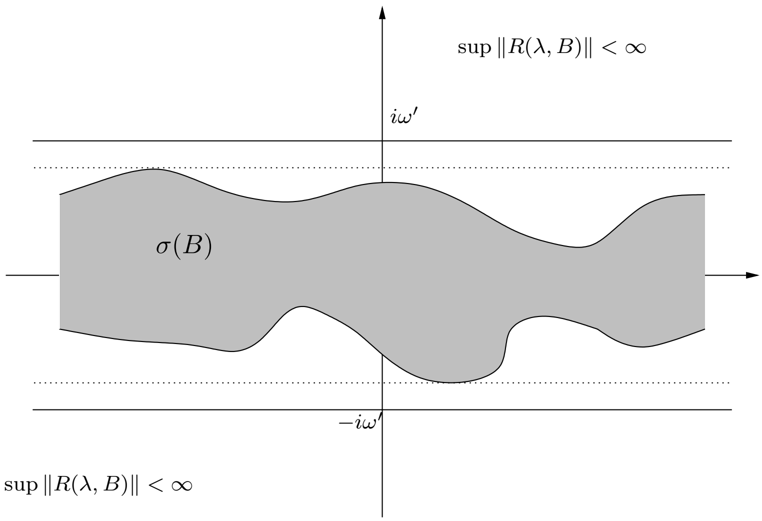


Figure 7: Spectrum of a strip-type operator.

the **spectral height** of B. Figure 7 illustrates the notion of a strip-type operator.

Let us say that an operator B is a **strong strip-type operator** of height ω if for each $\omega' > \omega$ there is $L_{\omega'}$ such that

$$\|R(\lambda, B)\| \leq \frac{L_{\omega'}}{|\mathrm{Im}\,\lambda| - \omega'} \qquad (|\mathrm{Im}\,\lambda| > \omega').$$

Such operators obviously have the property that for each $\omega' > \omega$ both operators $\omega' - iB$ and $\omega' + iB$ are sectorial of angle $\pi/2$.

Examples 4.1.1. We describe three classes of strip-type operators that arise in a natural manner.

1) Let $\omega \in [0, \pi)$, and let $A \in \mathrm{Sect}(\omega)$ be injective. Then $B := \log A$ is a strong strip-type operator of height ω, as we learn from Nollau's theorem (Proposition 3.5.2). We shall prove in Section 4.3 that $\omega_A = \omega_{st}(\log A)$.

2) Let iB generate a C_0-group T. Then B is a strong strip-type operator of height $\theta(T)$, where $\theta(T)$ is the group type of T. In general, it may occur that $\omega_{st}(B) < \theta(T)$. However, Gearhart's Theorem C.8.2 implies that we have $\omega_{st}(B) = \theta(T)$ in the case where X is a Hilbert space.

3) Let H be a Hilbert space, and let B be a self-adjoint operator on H. Then $\sigma(B) \subset \mathbb{R}$ and

$$\|R(\lambda, B)\| \leq \frac{1}{|\mathrm{Im}\,\lambda|} \qquad (\lambda \notin \mathbb{R})$$

(see Proposition C.4.2). In particular, B is a strong strip-type operator of height 0. Of course this is a special case of 2) since iB generates a unitary group on H, by Stone's Theorem C.7.4.

Remark 4.1.2. As a matter of fact, instead of dealing with *horizontal* strips we could have defined all notions for *vertical* strips. (The strip-type operators in the horizontal and vertical case correspond to each other via the mapping $(B \longmapsto iB)$.) This in fact seems more natural for generators of groups and therefore was done in [101] and [105]. Since the logarithm of a sectorial operator is our guiding example, we have chosen horizontal strips for this exposition.

4.2 The Natural Functional Calculus

We wish to define a functional calculus for a strip-type operator $B \in \mathrm{Strip}(\omega)$ analogously to the sectorial case. Given $\varphi > 0$, we let

$$\mathcal{F}(H_\varphi) := \left\{ f \in \mathcal{O}(H_\varphi) \;\middle|\; f(z) = O(|\mathrm{Re}\, z|^{-\alpha}) \ (|z| \to \infty) \text{ for some } \alpha > 1 \right\}$$

and $\gamma_\varphi := \partial H_\varphi$ (oriented in the positive sense). Then for $B \in \mathrm{Strip}(\omega)$, $\omega' \in (\omega, \varphi)$, and $f \in \mathcal{F}(H_\varphi)$, the Cauchy integral

$$f(B) := \frac{1}{2\pi i} \int_{\gamma_{\omega'}} f(z) R(z, B)\, dz$$

is absolutely convergent in $\mathcal{L}(X)$ since $R(\cdot, B)$ is bounded on the path $\gamma_{\omega'}$, see Figure 8 below. By Cauchy's theorem the definition of $f(B)$ is independent of the actual choice of ω'. The following result is not surprising.

Proposition 4.2.1. a) *The mapping $(f \longmapsto f(B)) : \mathcal{F}(H_\varphi) \longrightarrow \mathcal{L}(X)$ is a homomorphism of algebras.*

b) *For $\lambda, \mu \notin \overline{H_\varphi}$ the identity $((\lambda - z)(\mu - z))^{-1}(B) = R(\lambda, B)R(\mu, B)$ holds.*

c) *We have $(f(z)(\lambda - z)^{-1})(B) = R(\lambda, B)f(B) = f(B)R(\lambda, B)$ for $\lambda \notin \overline{H_\varphi}$.*

d) *If C is a closed operator commuting with the resolvents of B, then C also commutes with $f(B)$. In particular, $f(B)$ commutes with B and with $R(\lambda, B)$ for all $\lambda \in \varrho(B)$.*

Proof. a), c) and d) are proved analogously to the corresponding statements in Lemma 2.3.1. The only difficulty is in b). We give a sketch of the proof but leave the details to the reader. Let $f(z) := (\lambda - z)^{-1}(\mu - z)^{-1}$ with $\lambda, \mu \notin \overline{H_\varphi}$. Let us suppose that λ, μ lie on the same side of H_φ, say above. $f(B)$ is given by a Cauchy integral on the contour $(\mathbb{R} - i\delta_1) \ominus (\mathbb{R} + i\delta_2)$ where $\delta_1, \delta_2 \in (\varphi, \infty)$ are close to φ. Since the integrand has no singularity within the lower half-plane, we shift the lower path (as $\delta_1 \to \infty$) towards $\mathrm{Im}\, z = -\infty$ without changing its value. This value must therefore be equal to 0. As δ_2 increases, nothing happens in the beginning, but when crossing λ we obtain a residue, which is $(\mu - \lambda)^{-1} R(\lambda, B)$,

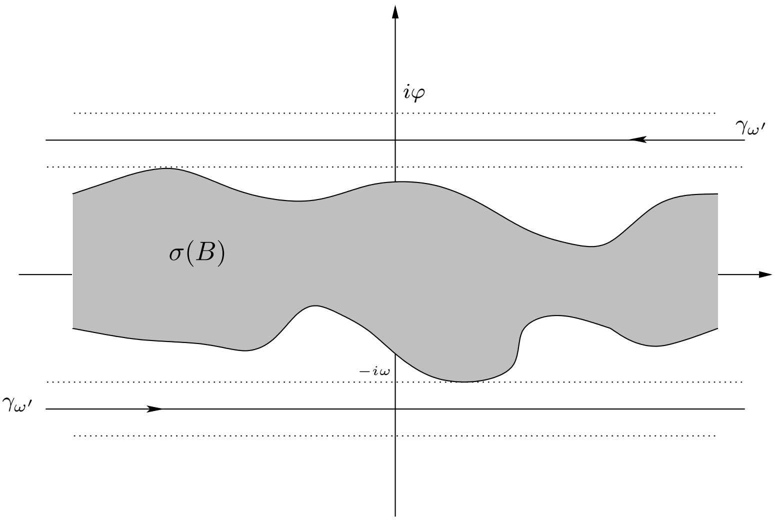


Figure 8: Cauchy integral for strip-type operators.

and when crossing μ we obtain in addition $(\lambda - \mu)^{-1}R(\mu, B)$. After this the path wanders off to $\operatorname{Im} z = +\infty$ without changing the value any more. So this value is equal to 0. Summing up, we obtain

$$f(B) = \frac{R(\lambda, B)}{\mu - \lambda} + \frac{R(\mu, B)}{\lambda - \mu} = R(\lambda, B)R(\mu, B)$$

by the resolvent identity. (When $\lambda = \mu$ a similar reasoning applies, also in the case that λ, μ are situated on different sides of the strip.) $\square$

We define $\mathcal{F}[H_\omega] := \bigcup_{\varphi > \omega} \mathcal{F}(H_\varphi)$ and $\mathcal{M}[H_\omega] := \bigcup_{\varphi > \omega} \mathcal{M}(H_\varphi)$, and call the mapping

$$\Phi_B := (f \longmapsto f(B)) : \mathcal{F}[H_\omega] \longrightarrow \mathcal{L}(X)$$

the **primary functional calculus** (in short: **pfc**) for the strip-type operator B. Obviously, $\mathcal{F}[H_\omega]$ is a subalgebra of $\mathcal{M}[H_\omega] \subset \mathcal{M}(H_\omega)$. Hence an abstract functional calculus

$$(\mathcal{F}[H_\omega], \mathcal{M}(H_\omega), \Phi_B)$$

in the sense of Chapter 1 is established; this is proper, by Proposition 4.2.1 b). Employing the abstract results of Section 1.2, the primary functional calculus has a natural extension to a map $\mathcal{M}[H_\omega]_B \longrightarrow \{\text{closed operators}\}$, where

$$\mathcal{M}[H_\omega]_B := \left\{ f \in \mathcal{M}[H_\omega] \;\middle|\; \exists e \in \mathcal{F}(H_\omega) : e(B) \text{ is injective and } ef \in \mathcal{F}[H_\omega] \right\}$$

is the set of *regularisable* elements of $\mathcal{M}[H_\omega]$. The operator $f(B)$ is then defined as

$$f(B) = e(B)^{-1}(ef)(B)$$

whenenver $f \in \mathcal{M}[H_\omega]_B$ and $e \in \mathcal{F}[H_\omega]$ is such that $e(B)$ is injective and $ef \in \mathcal{F}[H_\omega]$. The so-obtained abstract functional calculus $(\mathcal{F}[H_\omega], \mathcal{M}(H_\omega), \Phi_B)$ is called the **natural functional calculus** for the strip-type operator B and it is even a *meromorphic* functional calculus *for* B (see Section 1.3), as the following Lemma shows.

Lemma 4.2.2. *Let* $B \in \mathrm{Strip}(\omega)$. *Then* $(z)(B) = B$.

Proof. Let $\lambda \notin \overline{H_\omega}$. Then $(\lambda - z)^{-3}(B) = R(\lambda, B)^3$, by Proposition 4.2.1 b) and c). Hence $(\lambda - z)^{-3}$ regularises z. Now, $z(\lambda - z)^{-3} = \lambda(\lambda - z)^{-3} - (\lambda - z)^{-2}$, whence $z(\lambda - z)^{-3}(B) = \lambda R(\lambda, B)^3 - R(\lambda, B)^2 = BR(\lambda, B)^3$. This yields $z(B) = (\lambda - B)^3(z(\lambda - z)^{-3})(B) = (\lambda - B)^3 BR(\lambda, B)^3 = B(\lambda - B)^3 R(\lambda, B)^3 = B$. $\square$

As a consequence, we have at hand all the results on meromorphic functional calculi from Section 1.3, in particular Theorem 1.3.2. It is now natural to ask which meromorphic functions can be guaranteed to be contained in $\mathcal{M}[H_\omega]_B$.

Lemma 4.2.3. *Let* $B \in \mathrm{Strip}(\omega)$.

a) *If* $f \in \mathcal{O}[H_\omega]$ *is polynomially bounded at* ∞, *then* $f \in \mathcal{M}[H_\omega]_B$. *In particular,* $f(B)$ *is defined for every* $f \in H^\infty[S_\omega]$.

b) *Let* $\omega \in [0, \pi)$, *and let* $\lambda \in \mathbb{C} \setminus \{0\}$ *such that* $|\arg \lambda| \in (\omega, \pi]$. *Then one has* $(\lambda - e^z)^{-1} \in H^\infty[H_\omega]$, *and the operator* $(\lambda - e^z)^{-1}(B)$ *is injective.*

c) *Let* $\omega \in [0, \pi)$, *and let* $\lambda, \mu \in \mathbb{C} \setminus \{0\}$ *such that* $|\arg \lambda|, |\arg \mu| \in (\omega, \pi]$. *Then the operator*
$$T(\lambda, \mu) := \frac{e^z}{(\lambda - e^z)(\mu - e^z)}(B)$$
is injective.

Note that the function $f(z) := e^z/(\lambda - e^z)(\mu - e^z)$ is indeed a member of $\mathcal{F}[H_\omega]$.

Proof. a) For $\mu > \omega$ one may use powers of $e_\mu(z) := (\mu^2 + z^2)^{-1} \in \mathcal{F}[H_\omega]$ as regularisers for polynomially bounded holomorphic functions. (Note that $e_\mu(B) = -R(i\mu, B)R(-i\mu, B)$ by Proposition 4.2.1 b), and this is injective.)

b) and c) Let $f \in \mathcal{F}[H_\omega]$ be such that $f(B)$ is injective. Then, for every H^∞-function g, f is a regulariser for g and $\ker g(B) = \ker(fg)(B)$. Let $\omega < |\arg \lambda|$, $|\arg \mu| \leq \pi$. Since
$$\frac{\lambda f(z)}{\lambda - e^z} = \frac{\mu f(z)}{\mu - e^z} + \frac{(\mu - \lambda)e^z}{\mu(\lambda - e^z)} \frac{\mu f(z)}{\mu - e^z}$$
it follows that $\ker(\mu - e^z)^{-1}(B) \subset \ker(\lambda - e^z)^{-1}(B)$, whence equality by symmetry. Now, as $\lambda \to -\infty$ we have $[\lambda f(z)/(\lambda - e^z)](B) \to f(B)$ in norm (use the definition by Cauchy integrals). Hence $\ker(\lambda - e^z)^{-1}(B) = \ker f(B) = 0$, and this proves b).

In a totally analogous way one can prove that the operator $(1/\mu - e^{-z})^{-1}(B)$ is injective. Then c) follows from b) and

$$-f(B)^2 T(\lambda,\mu) = \frac{f(z)}{\lambda - e^z}(B)\,\frac{e^z f(z)}{e^z - \mu}(B) = \frac{f(z)}{\lambda - e^z}(B)\,\frac{(1/\mu) f(z)}{(1/\mu) - e^{-z}}(B). \qquad \square$$

From the last result it is clear that $f(B)$ is defined whenever $B \in \mathrm{Strip}(\omega)$ and $f \in \mathcal{O}[H_\omega]$ is such that $f = O(e^{\alpha|\mathrm{Re}\, z|})$ as $|\mathrm{Re}\, z| \to \infty$ for some $\alpha \in (0, \pi/\omega)$. In particular

$$e^B := (e^z)(B)$$

is defined for any strip-type operator B. This operator is injective and satisfies $\mathbb{C} \setminus \overline{S_\omega} \cap P\sigma(e^B) = \emptyset$.

Proof. Let $\lambda \in \mathbb{C} \setminus \overline{S_\omega}$. Then $1/(\lambda - e^z) \in H^\infty[H_\omega] \subset \mathcal{M}[H_\omega]_B$, whence $\lambda - e^B$ is injective by Theorem 1.3.2 f). Since $(\lambda - e^B)^{-1}(\mu - e^B)^{-1} e^B \subset T(\lambda, \mu)$, the operator e^B is injective. $\square$

By Proposition 3.5.2 we know that $\log A \in \mathrm{Strip}(\omega)$ if $A \in \mathrm{Sect}(\omega)$ is injective. So we expect $e^{\log A} = A$ in this case. In fact, this is an instance of the following composition rule.

Theorem 4.2.4 (Composition Rule). *Let $A \in \mathrm{Sect}(\omega)$, and let $g \in \mathcal{M}[S_\omega]_A$ such that $g(A) \in \mathrm{Strip}(\omega')$. Suppose in addition that for every $\varphi' \in (\omega', \infty)$ there is $\varphi \in (\omega, \pi)$ such that $g(S_\varphi) \subset \overline{H_{\varphi'}}$. Then $f \circ g \in \mathcal{M}[S_\omega]_A$ and*

$$(f \circ g)(A) = f(g(A))$$

for every $f \in \mathcal{M}[H_{\omega'}]_{g(A)}$.
Analogous statements hold if A is strip-type and $g(A)$ is sectorial or also strip-type.

Proof. We prove only the first statement of the theorem. Its proof is totally analogous to the sectorial situation (Section 2.4). Obviously we may suppose that g is not a constant. By the Open Mapping Theorem one then has $g(S_\omega) \subset H_{\omega'}$. Next, an appeal to the abstract composition rule (Proposition 1.3.6) reduces everything to $f \in \mathcal{F}[H_{\omega'}]$. Since we are mainly interested in the case $g(z) = \log(z)$ we suppose for simplicity that A is injective. Choose $\omega' < \varphi' < \omega_1'$ such that f is defined on a strip larger than $H_{\omega_1'}$, and take $\Gamma' = \partial H_{\omega_1'}$. Then choose $\varphi \in (\omega, \pi)$ such that $g(S_\varphi) \subset \overline{H_{\varphi'}}$. Set $\Gamma := \partial S_{\omega_1}$ for some $\omega_1 \in (\omega, \varphi)$. Using the convention $\tau(z) := z(1+z)^{-2}$ and $\Lambda_A := \tau(A)^{-1}$ we have

$$\begin{aligned} f(g(A)) &= \frac{1}{2\pi i}\int_{\Gamma'} f(\lambda) R(\lambda, g(A))\, d\lambda \\ &= \Lambda_A \frac{1}{2\pi i}\int_{\Gamma'} f(\lambda) \frac{1}{2\pi i}\int_{\Gamma} \frac{\tau(z)}{\lambda - g(z)} R(z, A)\, dz\, d\lambda. \end{aligned}$$

Since the function $f(\lambda)(\lambda - g(z))^{-1}\tau(z)/z$ is clearly product integrable on $\Gamma' \times \Gamma$, we can apply Fubini's theorem to obtain

$$\begin{aligned}\ldots &= \Lambda_A \frac{1}{2\pi i}\int_\Gamma \left[\frac{1}{2\pi i}\int_{\Gamma'} \frac{f(\lambda)}{\lambda - g(z)}\, d\lambda\right] \tau(z)R(z,A)\, dz \\ &= \Lambda_A \frac{1}{2\pi i}\int_\Gamma \tau(z)(f\circ g)(z)\, R(z,A)\, dz = \Lambda_A\left(\tau(f\circ g)\right)(A) = (f\circ g)(A).\end{aligned}$$

If A is not injective, modifications as in the proof of Lemma 2.4.4 are needed. □

Here is an immediate application.

Corollary 4.2.5. *Let $\omega \in [0,\pi)$, and let $A \in \mathrm{Sect}(\omega)$ be injective. Then*

$$e^{\log A} = A \qquad \textit{and} \qquad f(\log A) = (f(\log z))(A)$$

for all $f \in H^\infty(S_\varphi)$ and all $\varphi \in (\omega,\pi)$. In particular we have

$$\left(\frac{1}{\lambda - e^z}\right)(\log A) = R(\lambda, A) \qquad \textit{and}$$

$$\left(\frac{e^z}{(\lambda - e^z)(\mu - e^z)}\right)(\log A) = AR(\lambda,A)R(\mu,A)$$

for all $\lambda,\mu \in \mathbb{C}\setminus \overline{S_\omega}$.

Corollary 4.2.6. *An injective, sectorial operator on a Banach space X is uniquely determined by its logarithm, i.e., if A and B are injective, sectorial operators on X such that $\log A = \log B$, then $A = B$.*

Remark 4.2.7. 1) Suppose $B \in \mathrm{Strip}(\omega)$ is bounded. Then one can set up the usual Dunford calculus for B. It is easy to see that if $f \in \mathcal{O}[H_\omega]_B$, then $f(B)$ (defined by the natural functional calculus for strip-type operators) is bounded and coincides with $f(B)$ defined by the Dunford calculus. Hence the Dunford calculus is an extension of the natural functional calculus in this case.

2) By methods analogous to those developed in Section 2.7 one can prove a spectral inclusion theorem and a spectral mapping theorem for the natural functional calculus for strip-type operators.

Theorem 4.2.8 (Spectral Mapping Theorem). *Let $A \in \mathrm{Sect}(\omega)$ be injective. Then the identity*

$$\tilde{\sigma}(\log A) = \log\left(\tilde{\sigma}(A)\right) \tag{4.1}$$

holds.

Proof. Theorem 2.7.4 (spectral inclusion) yields that $\log\left(\sigma(A)\setminus\{0\}\right) \subset \sigma(\log A)$. Now, by Remark 4.2.7 one has also a spectral inclusion theorem for strip-type operators. Hence we may infer that $e^{\sigma(\log A)} \subset \sigma(e^{\log A})\setminus\{0\} = \sigma(A)\setminus\{0\}$, whence $\sigma(\log A) \subset \log(\sigma(A))\setminus\{0\}$. On the other hand, it is obvious that $\log(A)$ is bounded if and only if A is bounded and invertible. This concludes the proof. □

4.3 The Spectral Height of the Logarithm

Let A be an injective, sectorial operator on the Banach space X. Recall that Nollau's theorem (Proposition 3.5.2) says that $\log A$ is a strip-type operator with $\omega_{st}(\log A) \le \omega_A$. The purpose of this section is to prove the following.

Theorem 4.3.1. *Let X be a Banach space, and let $A \in \mathrm{Sect}(\omega)$ be injective. If $\log A \in \mathrm{Strip}(\omega')$ for some $\omega' \ge 0$, then $A \in \mathrm{Sect}(\omega')$. In particular, we have*

$$\omega_{st}(\log A) = \omega_A.$$

We let $B := \log A$. Without loss of generality we may suppose that $\omega' < \omega$. Let us consider the operator family

$$T(\lambda, \mu) := \left(\frac{e^z}{(\lambda - e^z)(\mu - e^z)} \right)(B) \;\in\; \mathcal{L}(X)$$

defined for $\lambda, \mu \notin \overline{S_{\omega'}}$, cf. Lemma 4.2.3. From Corollary 4.2.5 we obtain

$$T(\lambda, \mu) = AR(\lambda, A)R(\mu, A) \tag{4.2}$$

for $|\arg \lambda|, |\arg \mu| \in (\omega, \pi]$.

Lemma 4.3.2. *Fix μ with $|\arg \mu| > \omega'$. Then the mapping*

$$(\lambda \longmapsto T(\lambda, \mu)) : \mathbb{C} \setminus \overline{S_{\omega'}} \longrightarrow \mathcal{L}(X)$$

is holomorphic.

Proof. It suffices to show that the function

$$z \longmapsto \frac{e^z}{(\lambda - e^z)(\mu - e^z)} R(z, B)$$

is integrable on horizontal lines uniformly in $\lambda \in K$, for each compact $K \subset \mathbb{C} \setminus \overline{S_{\omega'}}$.

Choose $\varphi \in (\omega', |\arg \mu|)$ and define $C_\nu := \mathrm{dist}(\nu, \overline{S_\varphi})$ for $|\arg \nu| > \varphi$. Let $|\arg \lambda| \in (\varphi, \pi]$, and define $f_\lambda(z) := e^z(\lambda - e^z)^{-1}(\mu - e^z)^{-1}$. Then

$$|f_\lambda(z)| \le \frac{e^{\mathrm{Re}\, z}}{C_\mu C_\lambda} \quad \text{and} \quad |f_\lambda(z)| \le \frac{e^{-\mathrm{Re}\, z}}{|\lambda|\,|\mu|\, C_{\frac{1}{\lambda}} C_{\frac{1}{\mu}}}$$

for $z \in H_\varphi$. Hence there is $C(\lambda)$, locally bounded in λ, such that $|f_\lambda(z)| \le C(\lambda) e^{-|\mathrm{Re}\, z|}$. This proves the claim. $\square$

We need another lemma.

Lemma 4.3.3. *Let $\varphi \in (\omega', \pi)$. Then the set*

$$\left\{ \|(\mu - \lambda)T(\lambda, \mu)\| \;\middle|\; |\arg \lambda| \geq \varphi,\ \mu = -|\lambda| \right\}$$

is bounded.

Proof. Choose $\omega_1 \in (\omega', \varphi)$. We write $(\mu - \lambda)T(\lambda, \mu)$ as a Cauchy integral along the path $\gamma := \partial H_{\omega_1}$ and obtain

$$\begin{aligned}
\|(\mu - \lambda)T(\lambda, \mu)\| &= \left\| \frac{1}{2\pi i} \int_\gamma \frac{(\mu - \lambda)e^z}{(\lambda - e^z)(\mu - e^z)} R(z, B)\, dz \right\| \\
&\leq \frac{L(B, \omega_1)}{2\pi} \int_\gamma \frac{|(\mu - \lambda)e^z|\; |dz|}{|\lambda - e^z)|\, |(\mu - e^z)|}.
\end{aligned}$$

The integral over the path $\mathbb{R} + i\omega_1$ is estimated by

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{|\mu - \lambda|\, \left|e^{r + i\omega_1}\right|}{|\lambda - e^{r+i\omega_1}|\, |\mu - e^{r+i\omega_1}|}\, dr \overset{t = e^r}{=} \int_0^\infty \frac{|\mu - \lambda|}{|\lambda - te^{i\omega_1}|\, |\mu - te^{i\omega_1}|}\, dt \\
&= \int_0^\infty \frac{|\mu - \lambda|\, |\lambda|}{|\lambda - |\lambda|\, te^{i\omega_1}|\, |\mu - |\lambda|\, te^{i\omega_1}|}\, dt \overset{\mu = -|\lambda|}{=} \int_0^\infty \frac{\left|1 + \frac{\lambda}{|\lambda|}\right|}{\left|\frac{\lambda}{|\lambda|} - te^{i\omega_1}\right| |1 + te^{i\omega_1}|}\, dt \\
&\leq \int_0^\infty \frac{2}{\left|\frac{\lambda}{|\lambda|} - te^{i\omega_1}\right| |1 + te^{i\omega_1}|}\, dt.
\end{aligned}$$

The last term is uniformly bounded since $\varphi \leq |\arg \lambda| \leq \pi$. Needless to say that the second integral can be treated analogously. $\square$

We are now able to *complete the proof* of Theorem 4.3.1. By elementary calculations we obtain from (4.2) the identity

$$\lambda R(\lambda, A) = (\mu - \lambda)T(\lambda, \mu) + \mu R(\mu, A), \tag{4.3}$$

which holds for all λ, μ with $|\arg \lambda|, |\arg \mu| \in (\omega, \pi]$. Keeping μ fixed we see that the right-hand side of this equation is defined even for $|\arg \lambda| \in (\omega', \pi]$ and is in fact holomorphic as a function of λ (Lemma 4.3.2). Since the norm of the resolvent blows up if one approaches a spectral value (Proposition A.2.3), no λ with $|\arg \lambda| \in (\omega', \pi]$ can belong to $\sigma(A)$. Furthermore, if we choose $\varphi \in (\omega', \pi)$ and let $\mu = \mu_\lambda := -|\lambda|$ in (4.3), we arrive at

$$\|\lambda R(\lambda, A)\| \leq \|(\mu_\lambda - \lambda)T(\lambda, \mu_\lambda)\| + \left\| |\lambda|\, (|\lambda| + A)^{-1} \right\|.$$

This is bounded uniformly for $|\arg \lambda| \geq \varphi$, by Lemma 4.3.3 and sectoriality. $\square$

Let us state two important corollaries. Recall that, if iB generates a group T on a Banach space X, one always has $\theta(T) \geq \omega_{st}(B)$ by the Hille–Yosida

Theorem A.8.6. If $X = H$ is a Hilbert space, even $\theta(T) = \omega_{st}(B)$ holds, by Gearhart's Theorem C.8.2. Now, if A is sectorial and $A \in \mathrm{BIP}(X)$, we know that $i \log A$ generates the group $(A^{is})_{s\in\mathbb{R}}$ (Proposition 3.5.6). Employing the identity $\omega_A = \omega_{st}(\log A)$ we obtain the following two results.

Corollary 4.3.4 (Prüss–Sohr). *Let X be a Banach space, and let A be an injective, sectorial operator on X such that $A \in \mathrm{BIP}(X)$. Then $\omega_A \leq \theta_A$, i.e., the group type of the group $(A^{is})_{s\in\mathbb{R}}$ is always larger than the spectral angle of A.*

Corollary 4.3.5 (McIntosh). *Let H be a Hilbert space, and let A be an injective, sectorial operator on H such that $A \in \mathrm{BIP}$. Then $\theta_A = \omega_A$, i.e., the group type of $(A^{is})_{s\in\mathbb{R}}$ equals the spectral angle of A.*

We shall see in Section 7.3.3 that even more is true in the Hilbert space situation.

4.4 Monniaux's Theorem and the Inversion Problem

The logarithm of an injective, sectorial operator A is a strip-type operator, and in fact we have $e^{\log A} = A$ by the composition rule (Corollary 4.2.5). So it is natural to ask whether e^B is sectorial whenever $B \in \mathrm{Strip}(\omega)$, $\omega \in [0, \pi)$. Let us give a name to that question and call it the **inversion problem** for strip-type operators. Unfortunately, the answer is 'not always', even if we allow B to be a strong strip-type operator.

Example 4.4.1. Let $X := \mathbf{L^1}(\mathbb{R})$, and let $B := -id/dt$, where d/dt denotes the usual derivative with domain $\mathcal{D}(d/dt) = \mathbf{W^{1,1}}(\mathbb{R})$. Then iB generates a strongly continuous isometric group. In particular, B is a strong strip-type operator of angle 0. However, e^B has empty resolvent set, whence it is not sectorial, see Corollary 8.4.6.

So one may ask for additional sufficient conditions to ensure that $A := e^B$ is sectorial. The following lemma is obviously only a small step.

Lemma 4.4.2. *Let $B \in \mathrm{Strip}(\omega)$ such that $\omega < \pi$, and let $A := e^B$. If $\lambda \in \varrho(A)$ for some λ with $|\arg \lambda| > \omega$, then this is true for all such λ.*

Proof. This follows from $\lambda/(\lambda - e^z) = (\mu-\lambda)e^z/(\lambda - e^z)(\mu - e^z) + \mu/(\mu - e^z)$. □

Unfortunately this lemma does not give estimates for $\|t(t + A)^{-1}\|$ as $\lambda \to 0$ and $\lambda \to \infty$. In fact, it is an open problem whether the existence of the resolvent already implies the sectoriality of e^B. However, there is a remarkable result by Monniaux from [172]. See Appendix E.6 for the definition of UMD spaces.

Theorem 4.4.3 (Monniaux). *Let X be a Banach space with the UMD property, and let iB be the generator of a strongly continuous group on X. If $\omega_{st}(B) < \pi$, then e^B is sectorial.*

Proof. In [172, Theorem 4.3] the theorem is proved under the additional assumption that $\theta(U) < \pi$, where U denotes the group generated by iB and $\theta(U)$ is its group type, cf. page 302. Using this result, we deal with the general case.

If $\theta := \theta(U) \geq \pi$, then one can choose $\alpha \in (0,1)$ such that $\theta\alpha < \pi$. Then $i\alpha B$ generates the group $(U(\alpha s))_{s\in\mathbb{R}}$, which has group type $\alpha\theta$. Furthermore, $\alpha B \in \mathrm{Strip}(\alpha\omega)$. Applying [172, Theorem 4.3] we can find an injective, sectorial operator C on X such that $C^{is} = U(\alpha s)$ for all $s \in \mathbb{R}$. Theorem 4.3.1 yields that $\omega_C \leq \alpha\omega$. If we define $A := C^{1/\alpha}$ we know from Proposition 3.1.2 and Proposition 3.2.1 that also A is an injective, sectorial operator and $A^{is} = U(s)$ for all $s \in \mathbb{R}$. Hence $i \log A$ generates U, whence $\log A = B$. □

It would lead us too far astray to give a complete proof of Monniaux's original result from [172, Theorem 4.3].

4.5 A Counterexample

Theorem 4.3.1 allows us to answer in the positive the following

Question: Is there a Banach space X and a sectorial operator $A \in \mathrm{BIP}(X)$ such that $\theta_A \geq \pi$?

On UMD spaces, the question stated above is intimately connected with the failing of Gearhart's theorem (Theorem C.8.2) for C_0-groups. We state this as a proposition.

Proposition 4.5.1. *Let X be a Banach space with the* UMD *property. Then the following assertions are equivalent.*

(i) *There is a sectorial operator $A \in \mathrm{BIP}(X)$ such that $\theta_A \geq \pi$.*

(ii) *There is a sectorial operator $A \in \mathrm{BIP}(X)$ such that $\omega_A < \theta_A$.*

(iii) *There is an operator B which generates a C_0-semigroup T on X such that $s_0(B) < \omega_0(T)$ and T is a group.*

Recall the definition (C.7) of $s_0(B)$ on page 329.

Proof. The implication (i)⇒(ii) is trivial.

(ii)⇒(i). We may suppose that $\theta_A < \pi$. Then consider the scaled operator A^α with α being defined as $\alpha := \pi/\theta_A$.

(ii)⇒(iii). Let $A \in \mathrm{BIP}(X)$ with $\omega_A < \theta_A$. Then $i \log A$ generates $T_1 := (A^{is})_{s\geq 0}$ and $-i \log A$ generates $T_2 := (A^{-is})_{s\geq 0}$. But

$$\max\{s_0(i \log A), s_0(-i \log A)\} = \omega_{st}(\log A) = \omega_A < \theta_A = \max\{\omega_0(T_1), \omega_0(T_2)\}.$$

Hence we can take $B := i \log A$ or $B := -i \log A$.

(iii)⇒(ii). Let $(U(s))_{s\in\mathbb{R}}$ be a C_0-group on X, let B be the generator of the semigroup $T := (U(s))_{s\geq 0}$, and suppose that $s_0(B) < \omega_0(T)$. Changing B to $\alpha(B+\lambda)$ with suitable $\lambda, \alpha > 0$ we may suppose that $\sup_{s\leq 0} \|U(s)\| < \infty$ and $0 \leq s_0(B) < \omega_0(T) < \pi$. By Monniaux's Theorem 4.4.3 there is a sectorial operator $A \in \mathrm{BIP}(X)$ such that $A^{is} = U(s)$ for all $s \in \mathbb{R}$. Hence $i \log A = B$ by the uniqueness of generators. Obviously,

$$\omega_{st}(\log A) = s_0(B) < \omega_0(T) = \theta(U) = \theta_A.$$

Since $\omega_A = \omega_{st}(\log A)$ by Theorem 4.3.1, assertion (ii) follows readily. $\square$

We are now going to give an example of a Banach space X with the UMD property such that (iii) of Proposition 4.5.1 holds. Let $1 < p < 2 < q < \infty$, and let $a > q$ such that $2/p < a/q$. We define the weight $w : \mathbb{R} \to [0, \infty)$ by

$$w(x) := \begin{cases} e^{ax} & x \leq 0, \\ 1 & x \geq 0. \end{cases}$$

Now we let $X := \mathbf{L}^{\boldsymbol{p}}(\mathbb{R}, e^{2x}dx) \cap \mathbf{L}^{\boldsymbol{q}}(\mathbb{R}, w(x)dx)$ with its natural (sum) norm. It can be shown that X has in fact the UMD property. *The space $\mathbf{C_c}(\mathbb{R})$ of compactly supported continuous functions is dense in X.*

Proof. Let $f \in X$. Then $\mathbf{1}_{[-n,n]}f \to f$ in X as $n \to \infty$. Hence we may suppose that f is compactly supported. Now we approximate f in $\mathbf{L}^{\boldsymbol{q}}(\mathbb{R})$ by functions $f_n \in \mathbf{C_c}(\mathbb{R})$ such that $\mathrm{supp}(f_n) \subset [a, b]$, where $a < b$ and a, b do not depend on n. Since

$$\mathbf{L}^{\boldsymbol{q}}((a,b), w(x)dx) \cong \mathbf{L}^{\boldsymbol{q}}(a,b) \hookrightarrow \mathbf{L}^{\boldsymbol{p}}(a,b) \cong \mathbf{L}^{\boldsymbol{p}}((a,b), e^{2x}dx)$$

the sequence f_n tends to f in the norm of X. $\square$

On X we consider the **left shift group** $(T(t))_{t\in\mathbb{R}}$ defined by

$$[T(t)f](x) := f(x+t) \qquad (x \in \mathbb{R},\, t \in \mathbb{R}).$$

Then we have the norm inequalities

$$\begin{aligned} \|T(t)f\|_X &\leq e^{-\frac{2}{p}t} \|f\|_p + \|f\|_q \leq \|f\|_X\,, \\ \|T(-t)f\|_X &\leq e^{\frac{2}{p}t} \|f\|_p + e^{\frac{a}{q}t} \|f\|_q \qquad\qquad (f \in X,\, t \geq 0). \end{aligned}$$

Proof. Obviously, $\|T(t)f\|_p = e^{-(2/p)t} \|f\|_p$ for $t \in \mathbb{R}$. If $t \geq 0$, we have

$$\begin{aligned} \|T(t)f\|_q^q &= \int_{-\infty}^{\infty} |f(x)|^q \, w(x-t)\, dt \\ &= e^{-at} \int_{-\infty} |f(x)|^q \, e^{ax}\, dx + e^{-at} \int_0^t |f(x)|^q \, e^{ax}\, dx + \int_t^{\infty} |f(x)|^q \, dx \\ &\leq e^{-at} \int_{-\infty} |f(x)|^q \, e^{ax}\, dx + \int_0^{\infty} |f(x)|^q \, dx \leq \|f\|_q^q\,. \end{aligned}$$

The computation for $\|T(-t)f\|_q^q$ is similar. $\square$

In particular, it follows that $\|T(t)\| \le 1$ for all $t \ge 0$ and that T is a group. Since $\mathbf{C_c}(\mathbb{R})$ is dense in X and $(t \longmapsto T(t)f) : \mathbb{R} \longrightarrow X$ is continuous for each $f \in \mathbf{C_c}(\mathbb{R})$, we conclude that T is in fact a C_0-group.

Claim. *We have* $\|T(t)\| = 1$ *for all* $t \ge 1$.

Proof. Let $t_0 \ge 0$. Choose $t_1 > t_0$ arbitrary. Since $\mathbf{L^p}((t_0, t_1), e^{2x}dx) \cong \mathbf{L^p}(t_0, t_1)$ and this is not embedded into $\mathbf{L^q}(t_0, t_1)$, there is no inequality of the form $\|f\|_q \le C\|f\|_p$ with $f \in \mathbf{C_c}(t_0, t_1)$. Hence there is a sequence $g_n \in \mathbf{C_c}(t_0, t_1)$ such that $\|g_n\|_q \ge n\|g_n\|_p$ for all n. Letting $f_n := g_n / \|g_n\|_q$ we have $\|f_n\|_q = 1$, $\|f_n\| \le 1/n$, and $f_n \equiv 0$ on $(-\infty, t_0]$. Thus, $\|T(t_0)f_n\|_X = e^{-(2/p)t_0}\|f_n\|_p + \|f_n\|_q \ge 1$ and $\|f_n\|_X \le 1/n + 1$. Hence $1 \ge \|T(t_0)\| \ge 1/(1 + 1/n)$ for all $n \in \mathbb{N}$. ☐

Claim. *The operator* A *given by*

$$\mathcal{D}(A) = \{f \in X \mid f' \in X\} \quad \textit{and} \quad Af = f'$$

(where f' *denotes the distributional derivative of* f*) is the generator* A *of* T.

Proof. Let us denote the derivative operator on distributions by ∂. If $f \in \mathcal{D}(A)$ there is a $g \in X$ such that $t^{-1}(T(t)f - f) \to g$ in X as $t \searrow 0$. Since $X \hookrightarrow \mathcal{D}(\mathbb{R})'$ we see that $g = f'$. Hence, $\mathcal{D}(A) \subset \mathcal{D} := \{f \in X \mid f' \in X\}$ and $Af = f'$ for $f \in \mathcal{D}(A)$. Now, $1 - A$ is bijective (since T is a contraction semigroup) and $1 - \partial : \mathcal{D} \longrightarrow X$ is injective (since $f' = f \in X$ implies that f is more or less the exponential function). This implies the claim. ☐

Claim. *We have* $s(A) := \sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A)\} \le -2/p$.

Proof. Let $X_p := \mathbf{L^p}(\mathbb{R}, e^{2x}dx)$ with the norm $\|f\|_p$ defined above, and denote by A_p the generator of the left shift group on X_p. Since $\|T(t)f\|_p = e^{-(2/p)t}\|f\|_p$ for all $t \in \mathbb{R}$ and all $f \in X_p$, we have $s_0(A_p) \le -2/p$. By the same reasoning as above one can show that $\mathcal{D}(A_p) = \{f \in X_p \mid f' \in X_p\}$ with $A_p f = f'$ for $f \in A_p$. Next, we claim that $\mathcal{D}(A_p) \hookrightarrow X$. Actually, by the Closed Graph Theorem, only inclusion has to be shown. If $f \in \mathcal{D}(A_p)$ we let $g := e^{(2/p)\cdot}f$ and note that $g \in \mathbf{L^p}(\mathbb{R}, dx)$ and $g' = (2/p)g + e^{(2/p)\cdot}f' \in \mathbf{L^p}(\mathbb{R}, dx)$. Hence $g \in \mathbf{W^{1,p}}(\mathbb{R})$, and since $\mathbf{W^{1,p}}(\mathbb{R}) \hookrightarrow \mathbf{C_0}(\mathbb{R})$, there is a constant $C > 0$ such that $|f(x)| \le Ce^{-(2/p)x}$ for $x \in \mathbb{R}$. This immediately implies that $f \in \mathbf{L^q}((0, \infty), dx)$. Moreover, we have $|f(x)|\, e^{(a/q)x} \le Ce^{(a/q - 2/p)x}$ for all $x \in \mathbb{R}$, and since $a/q > 2/p$ we conclude that $f \in \mathbf{L^q}((-\infty, 0), e^{ax}dx)$. Altogether we obtain $f \in X$.

Now take $\lambda \in \mathbb{C}$ with $-2/p < \operatorname{Re}\lambda$. Since $(\lambda - A)f = (\lambda - A_p)f$ for $f \in \mathcal{D}(A) \subset \mathcal{D}(A_p)$ we see that $\lambda - A$ is injective. If $f \in X$, then $f \in X_p$ and $g := R(\lambda, A_p)f \in \mathcal{D}(A_p) \subset X$. But this implies that $g \in \mathcal{D}(A)$ since $g' = \lambda g - f \in X$. Hence $\lambda - A$ is also surjective. This proves that $\lambda \in \varrho(A)$. ☐

We take the last step. Obviously, the space X is not only a Banach space but even a *Banach lattice* and the semigroup T is *positive*, i.e., $T(t)f \ge 0$ for all $0 \le f \in X$ and all $t \ge 0$. By [10, Theorem 5.3.1] we conclude that $s_0(A) = s(A) \le -2/p < 0 = \omega_0(T)$, whence we are done.

Corollary 4.5.2. *There is a Banach space X with the* UMD *property and an injective, sectorial operator A on X such that $A \in \mathrm{BIP}(X)$ and $\theta_A > \pi$.*

4.6 Comments

4.1 and 4.2. Strip-type Operators and their Natural Functional Calculus. The natural functional calculus for strip-type operators appears first in [21]. It is discussed in [65] in a general setting and used in [34] and companion papers. Our presentation here follows the lines of [103].

Assume that one is given an operator B with spectrum in a horizontal strip H_ω and such that the resolvent is bounded on some horizontal lines $\mathbb{R} \pm i\varphi$ with $\varphi > \omega$. Then one can construct the 'functional calculus' as we did in Section 4.2, since the basic Cauchy integrals converge. However, it may happen that this 'functional calculus' is the zero mapping. If one requires that the calculus behaves well for rational functions, i.e., is really a functional calculus for B, then B is forced to be strip-type. This was shown in [101].

4.3 The Spectral Height of the Logarithm. Theorem 4.3.1 is due to the author [101]. The Prüss–Sohr result (Corollary 4.3.4) is part of a celebrated theorem of Prüss and Sohr [193, Theorem 3.3]. (See Corollary 5.5.12 for the second part.) Its original proof rests on the Mellin transform calculus for C_0-groups, cf. also [216, Proposition 3.19], [218], and [161, Chapter 9]. Corollary 4.3.5 is originally due to McIntosh [167].

4.4 Monniaux's Theorem and the Inversion Problem. For the proof of Theorem 4.4.3 Monniaux [172] utilises the theory of analytic generators of C_0-groups. In the same paper she constructs Example 4.4.1 to show that the conclusion may fail when one discards the UMD assumption. This example also shows that (strong) strip-type operators are more general than sectorial operators. Up to now, Monniaux's theorem seems to be the only positive result on the inversion problem.

4.5 A Counterexample. The example of a group with differing growth bound and abszissa of uniform boundedness of the resolvent is due to the author [103] and is an adaptation of an example given by Wolff [225]. We are indebted to Batty for bringing this result to our attention. Although we do not know it for sure, we expect that in our example the operator does not have a bounded natural H^∞-calculus on some strip (cf. Section 5.3.3 for terminology). The derivative on $\mathbf{L}^{p}(\mathbb{R})$, $p \neq 2$ does not have bounded H^∞-calculus on strips, see Section 8.4. Kalton [123] has given an example of a sectorial operator with bounded H^∞-calculus whose sectoriality angle differs from the H^∞-angle, cf. Section 5.4.

Chapter 5

The Boundedness of the H^∞-Calculus

This chapter mainly provides technical background information. We start with the so-called *Convergence Lemma* (Section 5.1) and some fundamental boundedness and approximation results (Section 5.2). Then we prove equivalence of boundedness of $\mathcal{F}$-functional calculi for different subalgebras $\mathcal{F}$ of H^∞ (Section 5.3). We also introduce and study the H^∞-angle (Section 5.4) and present permanence results with respect to additive perturbations (Section 5.5). Finally, we prove a technical lemma which indicates the connections with Harmonic Analysis (Section 5.6).

One of the main issues in the theory of functional calculus of sectorial operators is the question for which operators A and functions $f \in H^\infty$ the resulting operator $f(A)$ is bounded. Without exaggerating one may say that Chapters 6–8 are devoted to this question or at least have it as a *leitmotif.* The present chapter collects the general and more technical aspects of the matter, serving as a main source for later reference.

5.1 Convergence Lemma

In this section we prove a basic approximation result, called the Convergence Lemma. Whereas in Section 2.6.3 we treated only approximation of the operators, we now consider the approximation of functions. More precisely, our hypothesis is *pointwise* convergence of holomorphic functions, thereby bringing us in need of the following fact.

Proposition 5.1.1 (Vitali). *Let $\Omega \subset \mathbb{C}$ be open and connected, and let $(f_\alpha)_\alpha$ be a locally bounded net of holomorphic functions on Ω. If the set $\{z \in \Omega \mid (f_\alpha(z))_\alpha$ converges$\}$ has a limit point in Ω, then f_α converges to a holomorphic function uniformly on compact subsets of Ω.*

Proof. See [14, Theorem 2.1] for an elegant proof. (For sequences instead of nets this proof can also be found in [10, Theorem A.5].) $\square$

5.1.1 Convergence Lemma for Sectorial Operators.

The next lemma, although very elementary, is fundamental.

Lemma 5.1.2. *Let $A \in \mathrm{Sect}(\omega)$, and let $\varphi \in (\omega, \pi)$. Let $(f_\alpha)_\alpha \subset H_0^\infty(S_\varphi)$ be a net of functions converging pointwise to a function f. Suppose that there are $C, s > 0$ such that*

$$|f_\alpha(z)| \le C \min\left\{|z|^s, |z|^{-s}\right\} \qquad (z \in S_\varphi) \tag{5.1}$$

independent of α. Then $f \in H_0^\infty(S_\varphi)$ and $\|f_\alpha(A) - f(A)\| \to 0$.

Proof. Because of (5.1) the family $(f_\alpha)_\alpha$ is locally bounded. Vitali's theorem implies that f is holomorphic, hence $f \in H_0^\infty(S_\varphi)$. Moreover, $f_\alpha \to f$ uniformly on compact sets. Now the claim follows from a version of the Dominated Convergence Theorem. ☐

Example 5.1.3. Let $A \in \mathrm{Sect}(\omega)$. Then $A^\alpha \to A$ (S_ω) as $\alpha \nearrow 1$ (sectorial approximation). Indeed, by Proposition 3.1.2 the family $(A^\alpha)_{\varepsilon \le \alpha \le 1}$ is uniformly sectorial, for fixed $\varepsilon \in (0, 1)$. The proof of Proposition 3.1.2 yields that

$$(1 + A^\alpha)^{-1} - (1 + A)^{-1} = \psi_{-1,\alpha}(A)$$

with $\psi_{-1,\alpha} \to 0$ as $\alpha \nearrow 1$. Moreover, the estimate (3.1) given in that proof shows that the family $(\psi_{-1,\alpha})_{\varepsilon \le \alpha \le 1}$ meets the conditions of Lemma 5.1.2.

Proposition 5.1.4 (Convergence Lemma). *Let $A \in \mathrm{Sect}(\omega)$, $\varphi \in (\omega, \pi)$, and $(f_\alpha)_\alpha \subset H^\infty(S_\varphi)$. Suppose that $\sup_\alpha \|f_\alpha\|_\infty < \infty$ and that the limit $f(z) := \lim_\alpha f_\alpha(z)$ exists pointwise on S_φ. Suppose furthermore that $f_\alpha(A)$ and $f(A)$ are defined by the natural functional calculus for sectorial operators. Then*

$$f_\alpha(A)x \to f(A)x$$

for all $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$. Moreover the following assertions hold:

a) *If A is injective, $f_\alpha(A) \in \mathcal{L}(X)$ for all α, and $f_\alpha(A) \to T \in \mathcal{L}(X)$ strongly, then $f(A) = T$.*

b) *If A is densely defined with dense range and $\sup_\alpha \|f_\alpha(A)\| < \infty$, then $f(A) \in \mathcal{L}(X)$ and $f_\alpha(A) \to f(A)$ strongly.*

Proof. Vitali's theorem implies that $f \in H^\infty(S_\varphi)$. Let $\tau(z) := z(1+z)^{-2}$, $g := f\tau$, and $g_\alpha := f_\alpha \tau$. The net $(g_\alpha)_\alpha \subset H_0^\infty(S_\varphi)$ obviously satisfies the hypotheses of Lemma 5.1.2. Hence

$$f_\alpha(A)\tau(A) = g_\alpha(A) \to g(A) = f(A)\tau(A)$$

in norm. Therefore $f_\alpha(A)x \to f(A)x$ for all $x \in \mathcal{R}(\tau(A)) = \mathcal{D}(A) \cap \mathcal{R}(A)$.
a) Suppose that A is injective, $f_\alpha(A) \in \mathcal{L}(X)$ for all α, and $f_\alpha(A) \to T \in \mathcal{L}(X)$ strongly. We write (as usual) $\Lambda_A := \tau(A)^{-1}$. Let $x \in X$. Then $g_\alpha(A)x =: y_\alpha \in \mathcal{D}(\Lambda_A)$ for each α and $y_\alpha \to y := g(A)x$. However, $\Lambda_A y_\alpha = f_\alpha(A)x \to Tx$ by assumption. Since the operator Λ_A is closed, we have $y \in \mathcal{D}(\Lambda_A)$ and $\Lambda_A y = Tx$. But this means $x \in \mathcal{D}(f(A))$ with $f(A)x = Tx$.
b) follows from a). ☐

Remark 5.1.5. One may wonder whether part a) of Proposition 5.1.4 remains true if we drop the hypothesis that A is injective. Of course in this case we cannot allow arbitrary H^∞-functions f_α, but have to require a finite polynomial limit at 0, see Lemma 2.3.8. So we suppose that $f, f_\alpha \in H^\infty \cap \mathcal{A}$, cf. Lemma 2.3.10. Then we can still conclude that $f(A) \in \mathcal{L}(X)$, and we have

$$T = f(A) \quad \text{on } \overline{\mathcal{R}(A)} \qquad \text{and} \qquad \mathcal{R}(T - f(A)) \subset \mathcal{N}(A).$$

This is the best one can expect, cf. Example 5.1.6 below.

Proof. Let $h \in H^\infty \cap \mathcal{A}$ be arbitrary. Then by definition of $\mathcal{A}$ and Lemma 2.2.3 we have $h(1+z)^{-1} \in \mathcal{E}$, whence $\mathcal{D}(A) \subset \mathcal{D}(h(A))$. Therefore, as $X = \mathcal{D}(A) + \mathcal{R}(A)$, to prove that $f(A) \in \mathcal{L}(X)$ it suffices to show that $\mathcal{R}(A) \subset \mathcal{D}(f(A))$.

Take $x \in \mathcal{R}(A)$. Then $(1+A)^{-1}x \in \mathcal{D}(A) \cap \mathcal{R}(A)$ and

$$y_\alpha := \left(\frac{f_\alpha(z)}{1+z}\right)(A)x = f_\alpha(A)(1+A)^{-1}x \to f(A)(1+A)^{-1}x = \left(\frac{f(z)}{1+z}\right)(A)x =: y$$

by Proposition 5.1.4. Since $f_\alpha(A) \in \mathcal{L}(X)$, $y_\alpha \in \mathcal{D}(A)$ and $(1+A)y_\alpha = f_\alpha(A)x \to Tx$. Because $(1+A)$ is closed, we have $y \in \mathcal{D}(A)$ and $(1+A)y = Tx$. This means that $x \in \mathcal{D}(f(A))$ and $f(A)x = Tx$.

So indeed $f(A) \in \mathcal{L}(X)$ and $f(A) = T$ on $\mathcal{R}(A)$, whence on $\overline{\mathcal{R}(A)}$, by boundedness. Let $x \in \mathcal{D}(A)$. Then $f(A)x, Tx \in \mathcal{D}(A)$ as well and

$$A(f(A)x - Tx) = f(A)Ax - TAx = 0.$$

□

Example 5.1.6. Let A be a sectorial operator on a reflexive space X such that $\mathcal{N}(A) \neq 0$, and let $P : X \longrightarrow \overline{\mathcal{R}(A)}$ be the projection along $\mathcal{N}(A)$, cf. Proposition 2.1.1 h). With $f := \mathbf{1}$ and $f_n(z) := z(1/n + z)^{-1}$ we have $f_n(A) \to P$ strongly, but $f(A) = I \neq P$.

5.1.2 Convergence Lemma for Strip-type Operators.

It should be clear that in the case of a strip-type operator $B \in \mathrm{Strip}(\omega)$ a result similar to Proposition 5.1.4 holds.

Proposition 5.1.7 (Convergence Lemma on the Strip). *Let X be a Banach space, and let $B \in \mathrm{Strip}(\omega)$ on X. Let $\varphi > \omega$, and let $(f_\alpha)_\alpha$ be a net of holomorphic functions on H_φ that converges pointwise to a function f on H_φ.*

a) *If $\beta > 1$ and*

$$\sup_\alpha \sup_{z \in H_\varphi} |f_\alpha(z)| \left(1 + |\mathrm{Re}\, z|^\beta\right) < \infty,$$

then $f \in \mathcal{F}(H_\varphi)$ and $f_\alpha(B) \to f(B)$ in norm.

b) *If $f_\alpha \in H^\infty(H_\varphi)$, $f_\alpha(B) \in \mathcal{L}(X)$ for all α,*

$$\sup_\alpha \sup_{z\in H_\varphi} |f(z)| < \infty, \quad \textit{and} \quad \sup_\alpha \|f_\alpha(B)\| < \infty,$$

then $f_\alpha(B)x \to f(B)x$ for all $x \in \overline{\mathcal{D}(B^2)}$. If in addition $f_\alpha(B) \to T \in \mathcal{L}(X)$ strongly, then $T = f(B)$.

Note that if $B \in \mathrm{Strip}(\omega)$ is densely defined, then $\overline{\mathcal{D}(B^2)} = \overline{\mathcal{D}(B)} = X$.

Proof. The proof is analogous to the proof of Proposition 5.1.4. $\square$

5.2 A Fundamental Approximation Technique

In this section we provide auxiliary results which will not be used until the next chapter, so the reader may skip this section on first reading. The main part is an approximation method that has been used extensively by McIntosh and his collaborators.

Lemma 5.2.1. *Let $[a,b] \subset \mathbb{R}$, and let $F : [a,b] \times S_\varphi \longrightarrow \mathbb{C}$ be continuous and such that $F(t,.) : S_\varphi \longrightarrow \mathbb{C}$ is holomorphic for each $t \in [a,b]$. Suppose that there are $C, s > 0$ such that*

$$|F(t,z)| \le C \min\left\{ |z|^s, |z|^{-s} \right\}$$

for all $t \in [a,b]$, $z \in S_\varphi$. Define $f(z) := \int_a^b F(t,z)\,dt$. Then the following assertions hold.

a) $f \in H_0^\infty(S_\varphi)$;

b) $(t \longmapsto F(t,A)) : [a,b] \longrightarrow \mathcal{L}(X)$ *is continuous.*

c) $f(A) = \int_a^b F(t,A)\,dt$.

Proof. Assertion a) is clear and b) is an immediate consequence of Lemma 5.1.2. To prove c) we compute

$$\begin{aligned}
(2\pi i)\int_a^b F(t,A) &= \int_a^b \int_\Gamma F(t,z)\,R(z,A)\,dz\,dt \\
&\overset{\text{Fub.}}{=} \int_\Gamma \left(\int_a^b F(t,z)\,dt \right) R(z,A)\,dz = \int_\Gamma f(z)R(z,A)\,dz \\
&= (2\pi i) f(A).
\end{aligned}$$

$\square$

Let us introduce the following notation. For $\psi \in \mathcal{E}(S_\varphi)$ and $t > 0$ we write

$$\psi_t := (z \longmapsto \psi(tz)) \qquad (z \in S_\varphi). \tag{5.2}$$

It is then clear that $\psi_t \in \mathcal{E}(S_\varphi)$ again, and $\psi(tA) = \psi_t(A)$ for all $t > 0$.

Theorem 5.2.2. *Let $A \in \mathrm{Sect}(\omega)$, $\varphi \in (\omega, \pi)$, and $\psi, \theta \in H_0^\infty(S_\varphi)$. Then the following statements hold.*

a) *For each $f \in H^\infty(S_\varphi)$ the mapping $(t \longmapsto (f\psi_t)(A)) : (0,\infty) \longrightarrow \mathcal{L}(X)$ is continuous. Moreover, there is a constant $C = C(\psi)$ such that*

$$\sup_{t>0} \|(f\psi_t)(A)\| \le C\, M(A,\varphi)\, \|f\|_{\infty,\varphi} \qquad (f \in H^\infty(S_\varphi)).$$

b) *One has*

$$\sup_{t>0} \int_0^\infty \|\theta(tA)\psi(rA)\| \,\frac{dr}{r} < \infty.$$

Proof. a) Choose $C, s > 0$ with $|\psi(z)| \le C \min\{|z|^s, |z|^{-s}\}$. Let $[a,b] \subset (0,\infty)$, and define $F(t,z) := f(z)\psi(tz)$ on $[a,b] \times S_\varphi$. Then we have

$$|F(t,z)| \le \|f\|_\infty \, C \min\left\{ |z|^s, |z|^{-s} \right\} \max\left\{t^s, t^{-s}\right\}.$$

This shows that F satisfies the hypothesis of Lemma 5.2.1, whence continuity is established. To prove the boundedness simply apply Lemma 2.6.10. This is possible since we have

$$|(f\psi_t)(z)| \le C\, \|f\|_{\infty,\varphi} \min\left\{ |tz|^s, |tz|^{-s} \right\}$$

for all $t > 0$, $z \in S_\varphi$.
b) Choose $\omega' \in (\omega, \varphi)$ and define $\Gamma := \partial S_{\omega'}$. Then

$$\begin{aligned}
\int_0^\infty \|\theta(tA)\psi(rA)\| \,\frac{dr}{r} &\lesssim \int_0^\infty \int_\Gamma |\theta(tz)\psi(rz)| \,\frac{|dz|}{|z|}\frac{dr}{r} \\
&= \int_0^\infty \int_\Gamma \left|\theta(z)\psi(rt^{-1}z)\right| \frac{|dz|}{|z|}\frac{dr}{r} \quad \le \int_\Gamma \int_0^\infty \left|\psi(re^{i\arg z})\right| \frac{dr}{r}\, |\theta(z)| \,\frac{|dz|}{|z|} \\
&\le \int_\Gamma \frac{|\theta(z)|}{|z|}\, |dz| \, \max_{\varepsilon=\pm1} \left(\int_0^\infty \left|\psi(re^{\varepsilon i\omega'})\right| \frac{dr}{r} \right).
\end{aligned}$$

The constant hidden in the symbol '$\lesssim$' is of course $M(A,\omega')/2\pi$. ☐

Let $A \in \mathrm{Sect}(\omega)$, $\varphi \in (\omega, \pi]$, $\psi \in H_0^\infty(S_\varphi)$, and define

$$\psi_{a,b}(z) := \int_a^b \psi(tz)\,\frac{dt}{t} \qquad (0 < a < b < \infty,\ z \in S_\varphi). \tag{5.3}$$

One knows from Example 2.2.6 that $\psi_{a,b}(z) \to c$ pointwise as $(a,b) \to (0,\infty)$, where c is the constant

$$c := \int_0^\infty \psi(t)\,\frac{dt}{t}. \tag{5.4}$$

The interesting question now is, for which $x \in X$ one has

$$\psi_{a,b}(A)x \to cx \qquad \text{as} \qquad (a,b) \to (0,\infty).$$

The first step towards a general result is the following.

Lemma 5.2.3. *Let $A \in \mathrm{Sect}(\omega)$, $\varphi \in (\omega, \pi]$, and $\psi \in H_0^\infty(S_\varphi)$. Define $\psi_{a,b}$ and c as in* (5.3) *and* (5.4). *Then the following assertions hold.*

a) $\psi_{a,b} \in H_0^\infty(S_\varphi)$.

b) $\sup_{a,b} \|\psi_{a,b}\|_\infty < \infty$.

c) $\psi_{a,b}(z) \to c$ *as* $(a,b) \to (0,\infty)$ *uniformly on compact sets of* S_φ.

d) $\psi_{a,b}(A) = \displaystyle\int_a^b \psi(tA)\,\frac{dt}{t}$.

e) $\sup_{a,b} \|\psi_{a,b}(A)\| < \infty$.

Proof. We write

$$\begin{aligned}\psi_{a,b}(z) &= \int_0^b \psi(tz)\,\frac{dt}{t} \;-\; \int_0^a \psi(tz)\,\frac{dt}{t} \\ &= \int_0^1 \psi(tbz)\,\frac{dt}{t} \;-\; \int_0^1 \psi(taz)\,\frac{dt}{t} = h(bz) - h(az),\end{aligned}$$

where $h(z) = \int_0^1 \psi(tz)\,dt/t$ as in Example 2.2.6. Using the results shown there, a)–c) follow immediately. To prove assertion d) we apply Lemma 5.2.1 to the function $F(t,z) := \psi(tz)$. Assertion e) is immediate from $\psi_{a,b}(A) = h(bA) - h(aA)$ and Proposition 2.6.11. ☐

Proposition 5.2.4. *Let $A \in \mathrm{Sect}(\omega)$, $\varphi \in (\omega, \pi]$, and $\psi \in H_0^\infty(S_\varphi)$. Define*

$$h(z) := \int_0^1 \psi(tz)\,\frac{dt}{t}, \quad g(z) := \int_1^\infty \psi(tz)\,\frac{dt}{t}, \quad c := \int_0^\infty \psi(t)\,\frac{dt}{t}$$

as in Example 2.2.6. *Let also* $\psi_{a,b}(z) := \displaystyle\int_a^b \psi(tz)\,\frac{dt}{t}$ *as above. Then for* $x \in X$ *the following assertions hold.*

a) *If* $\displaystyle\int_0^1 \|\psi(tA)x\|\,\frac{dt}{t} < \infty$, *then* $h(A)x = \lim_{a\searrow 0} \psi_{a,1}(A)x = \displaystyle\int_0^1 \psi(tA)x\,\frac{dt}{t}$.

b) *If* $\displaystyle\int_1^\infty \|\psi(tA)x\|\,\frac{dt}{t} < \infty$ *and A is injective, then*

$$g(A)x = \lim_{b\nearrow\infty} \psi_{1,b}(A)x = \int_1^\infty \psi(tA)x\,\frac{dt}{t}.$$

c) *If* $x \in \mathcal{R}(\theta(A))$ *for some* $\theta \in H_0^\infty(S_\varphi)$, *then* $\displaystyle\int_0^\infty \|\psi(tA)x\|\,\frac{dt}{t} < \infty$ *and*

$$\lim_{a\to 0, b\to\infty} \psi_{a,b}(A)x \;=\; \lim_{a\to 0, b\to\infty} \int_a^b \psi(tA)x\,\frac{dt}{t} \;=\; \int_0^\infty \psi(tA)x\,\frac{dt}{t} \;=\; c\,x.$$

Proof. Find $C, s > 0$ such that $|\psi(z)| \le C \min\{|z|^s, |z|^{-s}\}$.
a) It is clear that $\psi_{a,1}(z) \to h(z)$ pointwise, with

$$|\psi_{a,1}(z)| \le \int_0^\infty |\psi(tz)| \, \frac{dt}{t} \le C \int_0^\infty \min\{t^s, t^{-s}\} \, \frac{dt}{t} = \frac{2C}{s}.$$

Also,

$$|\psi_{a,1}(z)| \le \int_0^1 |\psi(tz)| \, \frac{dt}{t} \le \frac{C}{s} |z|^s .$$

Hence one can apply Lemma 5.1.2 to the functions $\psi_{a,1}/(1+z)$ and obtains the convergence $(1+A)^{-1}\psi_{a,1}(A) \to (1+A)^{-1}h(A)$ in norm as $a \to 0$. On the other hand, by assumption and Lemma 5.2.3 d) one has

$$(1+A)^{-1}\psi_{a,1}(A)x = (1+A)^{-1} \int_a^1 \psi(tA)x \, \frac{dt}{t} \to (1+A)^{-1} \int_0^1 \psi(tA) \, \frac{dt}{t}$$

as $a \searrow 0$. Since $(1+A)^{-1}$ is injective, it follows that $h(A) = \int_0^1 \psi(tA) \, dt/t$.
b) is proved in the same way as a), where one has to regularise at 0 with the function $z(1+z)^{-1}$. Note that one needs $A(1+A)^{-1}$ to be injective in the end of the argument.
c) The first assertion is straightforward from Theorem 5.2.2 b). For the second, let $x = \theta(A)y$ and consider the functions $\psi_{a,b}(z)\theta(z)$, which converge to the function $c\theta(z)$. Since $\sup_{a,b} \|\psi_{a,b}\|_\infty < \infty$ by Lemma 5.2.3, one can apply Lemma 5.1.2 and obtains $\psi_{a,b}(A)x = \psi_{a,b}(A)\theta(A)y \to c\theta(A)y = cx$. $\square$

Remark 5.2.5. Part b) of Proposition 5.2.4 is false in general if A is not injective. More precisely, if $c = \int_0^\infty \psi(t) \, dt/t \ne 0$ and $0 \ne x \in \mathcal{N}(A)$, then $\psi(tA)x = 0$ for *all* $t > 0$ but $g(A)x = cx \ne 0$.

The final result of this section is just a corollary of the above considerations.

Theorem 5.2.6 (McIntosh Approximation). *Let $A \in \operatorname{Sect}(\omega)$, and let $\psi \in H_0^\infty[S_\omega]$ such that $\int_0^\infty \psi(t) \, dt/t = 1$. Then*

$$\int_a^b \psi(tA)x \frac{dt}{t} = \left(\int_a^b \psi(tz) \frac{dt}{t} \right)(A)x \quad \overset{a \to 0, b \to \infty}{\longrightarrow} \quad \int_0^\infty \psi(tA)x \, \frac{dt}{t} = x$$

for all $x \in \overline{\mathcal{D}(A) \cap \mathcal{R}(A)}$.

5.3 Equivalent Descriptions and Uniqueness

Let $A \in \operatorname{Sect}(\omega)$ be a sectorial operator on the Banach space X and let $\varphi \in (\omega, \pi)$. Suppose we are given a subalgebra $\mathcal{F} \subset H^\infty(S_\varphi)$ such that $f(A)$ is defined by the natural functional calculus for each $f \in \mathcal{F}$. (This is a restriction only if A is not injective.) We say that **the natural $\mathcal{F}$-calculus for A is bounded** if $f(A) \in \mathcal{L}(X)$ for all $f \in \mathcal{F}$ and

$$\|f(A)\| \le C \, \|f\|_\varphi \qquad (f \in \mathcal{F}) \tag{5.5}$$

for some constant $C \ge 0$. Here, $\|f\|_\varphi$ is shorthand for

$$\|f\|_\varphi := \|f\|_{\infty, S_\varphi} = \sup\{|f(z)| \mid z \in S_\varphi\} \tag{5.6}$$

We call

$$\inf\{C \ge 0 \mid (5.5) \text{ holds}\}$$

the **bound** of the natural $\mathcal{F}$-calculus.

If $\mathcal{F}$ is a closed subalgebra of $H^\infty(S_\varphi)$ and A is injective, the Closed Graph Theorem together with part a) of the Convergence Lemma (Proposition 5.1.4) yields the existence of a constant C satisfying (5.5) if only $f(A) \in \mathcal{L}(X)$ for all $f \in \mathcal{F}$.

5.3.1 Subspaces

We first consider restrictions to subspaces.

Proposition 5.3.1. *Let $A \in \mathrm{Sect}(\omega)$ be an injective sectorial operator on the Banach space X. Let $Y := \overline{\mathcal{D}(A) \cap \mathcal{R}(A)}$ and A_Y be the part of A in Y (see Section 2.1). Then A_Y is a densely defined sectorial operator of angle ω with dense range. Moreover, if $f \in H^\infty(S_\varphi), \varphi \in (\omega, \pi)$ and $f(A) \in \mathcal{L}(X)$, then $f(A_Y) \in \mathcal{L}(Y)$ and $\|f(A_Y)\|_{\mathcal{L}(Y)} \le \|f(A)\|_{\mathcal{L}(X)}$.*

In particular, if the natural $H^\infty(S_\varphi)$-calculus for A is bounded with bound C, then also the natural $H^\infty(S_\varphi)$-calculus for A_Y is bounded with bound C.

Proof. The space Y is obviously invariant under the resolvent of A. Hence by Proposition A.2.8, $\varrho(A) \subset \varrho(A_Y)$, and since the norm of X on Y agrees with the norm of Y, the sectoriality of A_Y follows. We have to show dense domain and dense range. Obviously, $\mathcal{D}(A^2) \cap \mathcal{R}(A) \subset \mathcal{D}(A_Y)$, so to have $\overline{\mathcal{D}(A_Y)} = Y$ it suffices to show that

$$\mathcal{D}(A) \cap \mathcal{R}(A) \subset \overline{\mathcal{D}(A^2) \cap \mathcal{R}(A)}.$$

Now, for $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$ we have $t(t+A)^{-1}x \to x$ as $t \to \infty$ and $t(t+A)^{-1}x \in \mathcal{D}(A^2) \cap \mathcal{R}(A)$. If we apply this to the operator A^{-1} we obtain that $[A^{-1}]_Y$ has dense domain. But $\mathcal{D}([A^{-1}]_Y) = \mathcal{D}([A_Y]^{-1}) = \mathcal{R}(A_Y)$ and so A_Y has dense range.

Suppose now that $f(A) \in \mathcal{L}(X)$ for some $f \in H^\infty$. Then one has $\|f(A)x\| \le \|f(A)\| \, \|x\|$ for all $x \in X$, in particular for all $x \in \mathcal{D}(A_Y) \cap \mathcal{R}(A_Y)$. But on this space the operators $f(A)$ and $f(A_Y)$ agree (see also Section 2.6.2). Hence $\|f(A_Y)x\| \le \|f(A)\| \, \|x\|$ for all x from a dense subspace of Y. Since $f(A_Y)$ is closed, this proves the claim. $\square$

Clearly Proposition 5.3.1 is only relevant in the case where X is not reflexive. Although we do not know of an example, there may exist an injective sectorial operator (on a non-reflexive space, of course) such that A_Y has a bounded H^∞-calculus but A does not.

5.3.2 Adjoints

The next, in no way surprising, result shows that boundedness is preserved when taking adjoints.

Proposition 5.3.2. *Let $A \in \mathrm{Sect}(\omega)$ have dense domain and dense range, and let $\varphi \in (\omega, \pi)$. Then*

$$f(A) \in \mathcal{L}(X) \iff f(A') \in \mathcal{L}(X')$$

for every $f \in H^\infty(S_\varphi)$. Moreover, the natural $H^\infty(S_\varphi)$-calculus for A is bounded with bound C if, and only if, the natural $H^\infty(S_\varphi)$-calculus for A' is bounded with bound C.

Proof. By Proposition 2.6.3 we have $f(A)' = f(A')$. If $f(A)$ is bounded, then also $f(A)'$ is, by Proposition A.4.2. Moreover, the norms are the same. If $f(A') = f(A)'$ is bounded, then the same reasoning shows that $f(A)'' \in \mathcal{L}(X'')$. However, the hypotheses imply that $f(A)$ is densely defined in X. By Lemma A.1.2 and $f(A) = f(A)'' \cap (X \oplus X)$, it follows that $f(A) \in \mathcal{L}(X)$. The rest is clear. $\square$

5.3.3 Logarithms

It is clear that the definition of boundedness of the natural H^∞-calculus applies as well in the case of strip-type operators. The next result shows in a way that via the logarithm one may switch back and forth from the sectorial to the strip case.

Proposition 5.3.3. *Let $A \in \mathrm{Sect}(\omega)$ be injective, let $B := \log A$, and let $\varphi \in (\omega, \pi]$. If the natural $H^\infty(S_\varphi)$-calculus for A is bounded with bound C, i.e., one has*

$$f(A) \in \mathcal{L}(X) \quad \textit{and} \quad \|f(A)\| \leq C \, \|f\|_{S_\varphi} \quad \textit{for all } f \in H^\infty(S_\varphi), \tag{5.7}$$

then also

$$g(B) \in \mathcal{L}(X) \quad \textit{and} \quad \|g(B)\| \leq C \, \|g\|_{H_\varphi} \quad \textit{for all } g \in H^\infty(H_\varphi). \tag{5.8}$$

Conversely, if $B \in \mathrm{Strip}(\omega)$ such that (5.8) holds for some $\varphi \in (\omega, \pi]$, then $A := e^B$ is sectorial and (5.7) holds.

Proof. The first statement is clear from the composition rule (Theorem 4.2.4) since $z \longmapsto \log z : S_\varphi \longrightarrow H_\varphi$ is biholomorphic with inverse $z \longmapsto e^z$. Suppose that $B \in \mathrm{Strip}(\omega)$ and that (5.8) holds with $\omega < \varphi \leq \pi$. Then $(t + e^z)^{-1}(B)$ is bounded by M/t for some constant M and all $t > 0$. Hence $A := e^B$ is sectorial. By Theorem 4.3.1, $\omega_A = \omega_{st}(B) \leq \omega$. The rest follows again from the composition rule. $\square$

5.3.4 Boundedness on Subalgebras of H^∞

In the remaining part of this section we examine how — for different subalgebras $\mathcal{F} \subset H^\infty$ — the boundedness of the natural $\mathcal{F}$-calculus and the corresponding bounds interrelate. Here is the 'omnibus theorem' in the case where the operator has dense domain and range.

Proposition 5.3.4. *Let $A \in \mathrm{Sect}(\omega)$ have dense domain and dense range. Let $\varphi \in (\omega, \pi)$ and $C \geq 0$. The following assertions are equivalent.*

(i) *The natural $H_0^\infty(S_\varphi)$-calculus for A is bounded with bound C.*

(ii) *The natural $H^\infty(S_\varphi)$-calculus for A is bounded with bound C.*

(iii) *The natural $H^\infty(S_\varphi) \cap \mathbf{C_0}(\overline{S_\varphi})$-calculus for A is bounded with bound C.*

(iv) *The natural $\mathcal{R}_0^\infty(S_\varphi)$-calculus for A is bounded with bound C.*

(v) *The natural $\mathcal{R}^\infty(S_\varphi)$-calculus for A is bounded with bound C.*

Note that in a reflexive space an injective sectorial operator *automatically* has dense domain and dense range. Cf. Appendix F for the definition of the spaces $\mathcal{R}^\infty(S_\varphi)$ and $\mathcal{R}_0^\infty(S_\varphi)$.

Proof. The implications (ii)⇒(i) and (ii)⇒(v)⇒(iv) are trivial.

(iv)⇒(iii). By Proposition F.3 the space $\mathcal{R}_0^\infty(S_\varphi)$ is uniformly dense in the space $H^\infty(S_\varphi) \cap \mathbf{C_0}(\overline{S_\varphi})$. Now apply part a) of the Convergence Lemma (Proposition 5.1.4). (Note that here only injectivity of A is needed.)

(iii)⇒(i). Let $f \in H_0^\infty(S_\varphi)$, and define $f_n(z) := f(z + 1/n)$. One has the estimate $\|f_n\|_\varphi \leq \|f\|_\varphi$, $f_n \in H^\infty(S_\varphi) \cap \mathbf{C_0}(\overline{S_\varphi})$, and $f_n \to f$ pointwise on S_φ. Now (i) follows from another application of the Convergence Lemma.

(i)⇒(ii). Let $f \in H^\infty(S_\varphi)$, and define $\psi_s(z) := z^s/(1+z)^{2s}$ for $s > 0$. Then $\psi_s, f\psi_s \in H_0^\infty(S_\varphi)$. Moreover,

$$\|f\psi_s\|_\varphi \leq \|f\|_\varphi K^s$$

where $K := \|\psi_1\|_\varphi$. Employing (i) yields $\|(f\psi_s)(A)\| \leq C \|f\|_\infty K^s$. Now we apply the Convergence Lemma (Proposition 5.1.4) and infer $f(A) \in \mathcal{L}(X)$ with $(f\psi_s)(A) \to f(A)$ strongly as $s \searrow 0$. Since $\lim_{s\to 0} K^s = 1$, $\|f(A)\| \leq C \|f\|_\infty$. $\square$

The questions become more difficult if we drop the density assumptions. Recall that for a sectorial operator A we can always construct the uniformly sectorial family

$$A_\varepsilon := (\varepsilon + A)(1 + \varepsilon A)^{-1};$$

this forms a sectorial approximation of A consisting of bounded and invertible operators (see Proposition 2.1.3).

Lemma 5.3.5. *Let $A \in \mathrm{Sect}(\omega)$, $\varphi \in [\omega, \pi)$, and $C \geq 0$. The following assertions are equivalent.*

(i) *The natural* $\mathcal{R}^\infty(S_\varphi)$*-calculus for* A *is bounded with bound* C.

(ii) *The natural* $\mathcal{R}^\infty(S_\varphi)$*-calculus for* A_ε *is bounded with bound* C*, for each* $\varepsilon > 0$.

Proof. Let $r_\varepsilon(z) = (\varepsilon+z)(1+\varepsilon z)^{-1} \in \mathcal{R}^\infty(S_\varphi)$. Then $A_\varepsilon = r_\varepsilon(A)$. If $r \in \mathcal{R}^\infty(S_\varphi)$, then $r \circ r_\varepsilon \in \mathcal{R}^\infty(S_\varphi)$ with $\|r \circ r_\varepsilon\|_\varphi \le \|r\|_\varphi$ since r_ε maps S_φ into S_φ. The composition rule yields $r(A_\varepsilon) = (r\circ r_\varepsilon)(A)$, whence we have proved the implication (i)$\Rightarrow$(ii).

The reverse implication follows from the fact that $r(A_\varepsilon) \to r(A)$ in norm, by Lemma 2.6.7. (Note that $\mathcal{R}^\infty(S_\varphi) \subset \mathcal{E}(S_\varphi)$.) □

One should note that $\varphi = \omega$ is allowed in Lemma 5.3.5.

Proposition 5.3.6. *Let* $A \in \operatorname{Sect}(\omega)$, $\varphi \in (\omega, \pi)$ *and* $C \ge 0$. *We define* K *to be the closure of* S_φ *in* $\mathbb{C}_\infty$. *The following assertions are equivalent.*

(i) *The natural* $\mathcal{A}(S_\varphi) \cap \mathbf{C}(K)$*-calculus for* A *is bounded with bound* C.

(ii) *The natural* $\mathcal{R}^\infty(S_\varphi)$*-calculus for* A *is bounded with bound* C.

If A *is injective, one may replace* $\mathcal{A}(S_\varphi)$ *by* $H^\infty(S_\varphi)$ *in* a).

Proof. (i)$\Rightarrow$(ii). This is obviously true.

(ii)$\Rightarrow$(i). Take $f \in \mathcal{A}(S_\varphi)\cap\mathbf{C}(K)$. By Proposition F.3 there is a sequence $(r_n)_n \subset \mathcal{R}^\infty(S_\varphi)$ such that $\|r_n - f\|_\varphi \to 0$. By Lemma 5.3.5, the hypothesis (ii) implies that the natural $\mathcal{R}^\infty(S_\varphi)$-calculus for A_ε is bounded with bound C, for each $\varepsilon > 0$. Since each A_ε is bounded and invertible, we can apply Proposition 5.3.4 to conclude that the natural $H^\infty(S_\varphi)$-calculus for A_ε is bounded with bound C for each $\varepsilon > 0$. This implies that $r_n(A_\varepsilon) \to f(A_\varepsilon)$ *uniformly* in $\varepsilon > 0$. From Lemma 2.6.7 we see that $r_n(A_\varepsilon) \to r_n(A)$ as $\varepsilon \searrow 0$ for each n. A standard argument from functional analysis now implies that there is $T \in \mathcal{L}(X)$ with $f(A_\varepsilon) \to T$. An application of Proposition 2.6.9 yields $T = f(A)$.

The same proof applies in the case where A is injective and $f \in H^\infty(S_\varphi) \cap \mathbf{C}(K)$. □

Proposition 5.3.7. *Let* $A \in \operatorname{Sect}(\omega)$, $\varphi \in [\omega, \pi)$*, and* $C \ge 0$. *Suppose that the natural* $\mathcal{R}_0^\infty(S_\varphi)$*-calculus for* A *is bounded with bound* C. *If* A *is densely defined and* $\varphi \le \pi/2$*, then the natural* $\mathcal{R}^\infty(S_\varphi)$*-calculus for* A *is bounded with bound* C.

Proof. Let $r_n := n/(n+z)$. Then $\|r_n\|_\varphi = 1$ since $\varphi \le \pi/2$. Moreover, $r_n(A)x \to x$ as $n \to \infty$ (by Proposition 2.1.1) since $\mathcal{D}(A)$ is dense in X. Thus for a function $r \in \mathcal{R}^\infty(S_\varphi)$ we have for all $x \in X$

$$\|r(A)x\| = \lim_n \|r(A)r_n(A)x\| = \lim_n \|(rr_n)(A)x\| \le C\,\|rr_n\|_\varphi\,\|x\| \le C\,\|r\|_\varphi\,\|x\|.$$

□

We do not know whether in the second part of Proposition 5.3.7 one can omit either the assumption $\overline{\mathcal{D}(A)} = X$ or $\varphi \le \pi/2$.

5.3.5 Uniqueness

Let us turn (for the moment) to a more general situation. Let A be a closed operator on a Banach space X, let $\Omega \subset \mathbb{C}$ be open, and let $\mathcal{F} \subset H^\infty(\Omega)$ be a subalgebra containing the rationals $r_\lambda(z) = (\lambda - z)^{-1}$ for $\lambda \notin \overline{\Omega}$. We say that a mapping $\Phi : \mathcal{F} \to \mathcal{L}(X)$ is a **bounded $\mathcal{F}$-calculus** for A if the following conditions are satisfied:

1) The mapping Φ is a homomorphism of algebras.
2) We have $\Phi(r_\lambda) = R(\lambda, A)$ for all $\lambda \notin \overline{\Omega}$.
3) If $\mathbf{1} \in \mathcal{F}$, then $\Phi(\mathbf{1}) = I$.
4) There is $C \geq 0$ such that $\|\Phi(f)\| \leq C \|f\|_\Omega$ for all $f \in \mathcal{F}$.

Here, $\|f\|_\Omega$ denotes the supremum norm of f on Ω. We say that a sequence $(f_n)_n \subset \mathcal{F}$ **converges boundedly and pointwise** on Ω to a function f if $f_n \to f$ pointwise on Ω and $\sup_n \|f_n\|_\Omega < \infty$. We say that Φ is continuous with respect to boundedly and pointwise convergence (in short: **b.p.-continuous**) if it has the following property.

5) If $f_n, f \in \mathcal{F}$ such that $f_n \to f$ boundedly and pointwise on Ω as $n \to \infty$, then $\Phi(f_n) \to \Phi(f)$ strongly on X.

Note that if $\mathbf{1} \notin \mathcal{F}$ we may always extend Φ to $\mathcal{F}' = \mathcal{F} \oplus \mathbb{C}\mathbf{1}$ satisfying 3) and keeping all other properties.

Lemma 5.3.8. *Let A be a closed operator on the Banach space X. Let $\omega \in (0, \pi)$, and let Φ be a bounded $H^\infty(S_\omega)$-calculus for A with bound $C \geq 0$. Then A is sectorial with $\omega_A \leq \omega$.*

Take a sector S_φ with $\omega_A < \varphi$ and $\omega \leq \varphi$, and denote by K the closure of S_φ in $\mathbb{C}_\infty$. Then the following assertions hold.

a) *$\Phi(f) = f(A)$ for all $f \in \mathcal{A}(S_\varphi) \cap \mathbf{C}(K)$.*
b) *If A is injective, then $\Phi(f) = f(A)$ for $f \in H^\infty(S_\varphi) \cap \mathbf{C}(K)$.*
c) *If A has dense domain and dense range, then the natural $H^\infty(S_\varphi)$-calculus for A is bounded with bound C.*

Proof. The sectoriality of A follows by applying Φ to the rationals $\lambda/(\lambda - z)$. By Proposition F.3, we have $\Phi(f) = f(A)$ for $f \in \mathcal{R}^\infty(S_\omega)$. The assertions a) and b) now follow from Proposition 5.3.6, and c) from Proposition 5.3.4. □

Proposition 5.3.9. *Let $\omega \in (0, \pi)$, and let A be a closed operator on a Banach space X. Then there is at most one bounded and b.p.-continuous $H^\infty(S_\omega)$-calculus Φ for A. If such a Φ exists, the operator A is sectorial with $\omega_A \leq \omega$ and has dense domain and dense range. Moreover, Φ coincides with the natural functional calculus on $H^\infty(S_\varphi)$, where $\varphi > \omega_A$ with $\varphi \geq \omega$.*

Proof. Uniqueness follows from Proposition F.4. Define $f_n(z) := n(n + z)^{-1}$. Then $f_n \to \mathbf{1}$ pointwise on S_ω as $n \to \infty$ and $\sup_n \|f_n\|_\omega < \infty$. Since Φ is b.p.-continuous, $n(n+A)^{-1} = \Phi(f_n) \to \Phi(\mathbf{1}) = I$ strongly, whence A is densely defined. The same argument with $f_n(z) = z(1/n+z)^{-1}$ yields that A has dense range. Take $\varphi > \omega_A$ with $\varphi \geq \omega$. From Lemma 5.3.8 it follows that the natural $H^\infty(S_\varphi)$-calculus for A is bounded. By uniqueness, the natural calculus must coincide with Φ. $\square$

Remark 5.3.10. Let $A \in \mathrm{Sect}(\omega)$ have dense domain and range, and let $\varphi \in (\omega, \pi]$. Proposition 5.3.4 implies that if A has *some* bounded $H^\infty(S_\varphi)$-calculus with bound C, then the *natural* $H^\infty(S_\varphi)$-calculus for A is bounded with bound C. However, the question remains if there really can exist two different bounded H^∞-calculi for the same operator.

5.4 The Minimal Angle

Suppose we are given an injective sectorial operator A such that the natural $H^\infty(S_\varphi)$-calculus for A is bounded for some $\varphi \in (\omega_A, \pi]$. Then we may ask for the *minimal angle* φ such that this is true. In fact, such a minimal angle does not necessarily exist but we can define the H^∞**-angle**

$$\omega_{H^\infty}(A) := \inf\{\varphi > \omega_A \mid \text{the natural } H^\infty(S_\varphi)\text{-calculus for } A \text{ is bounded}\},$$

in which it is understood that $\omega_{H^\infty} = \infty$ if the set on the right-hand side is empty.

It is clear that if $\omega_{H^\infty}(A) < \infty$, then the imaginary powers A^{is} of A are bounded operators and for each $\mu \in (\omega_{H^\infty}(A), \pi)$ there is a constant C_μ such that

$$\left\|A^{is}\right\| \leq C_\mu e^{\mu|s|} \qquad (s \in \mathbb{R}).$$

Although the imaginary powers do not need to form a C_0-group (since the space $\mathcal{D}(A) \cap \mathcal{R}(A)$ need not be dense in X), the definition of the type θ_A from (3.14) is still meaningful:

$$\theta_A = \inf\left\{\theta > 0 \,\middle|\, \exists\, M \geq 1 : \left\|A^{is}\right\| \leq M e^{\theta|s|} \ (s \in \mathbb{R})\right\}.$$

So with this notation we obtain $\theta_A \leq \omega_{H^\infty}(A)$. We shall see below (Corollary 5.4.3) that one actually has $\theta_A = \omega_{H^\infty}(A)$ if $\omega_{H^\infty}(A) < \infty$. Since the proof anyway works with strips instead of sectors, we first formulate the result in the strip version.

Theorem 5.4.1 (Cowling, Doust, McIntosh, Yagi). *Let $B \in \mathrm{Strip}(\omega)$, and let $\alpha > \omega$ such that the natural $H^\infty(H_\alpha)$-calculus for B is bounded. Let $0 < \mu < \theta < \alpha$ and suppose that there is a constant C with*

$$\left\|e^{isB}\right\| \leq C e^{\mu|s|} \qquad (s \in \mathbb{R}).$$

Then the natural $H^\infty(H_\theta)$-calculus for B is bounded.

For the proof we utilise the following auxiliary result.

Lemma 5.4.2. *Let $\alpha > 0$. Then there is a constant $c_\alpha \geq 0$ such that the following holds. Given $\theta \in (0,\alpha)$ and $f \in H^\infty(H_\theta)$ there exists $(f_n)_{n\in\mathbb{Z}} \subset H^\infty(H_\alpha)$ with*

1) $\sum_{n\in\mathbb{Z}} e^{\mu|n|} \|f_n\|_{H_\alpha} \leq \dfrac{c_\alpha}{1-e^{\mu-\theta}} \|f\|_{H_\theta}$.
2) $f(z) = \sum_{n\in\mathbb{Z}} e^{inz} f_n(z)$ *uniformly for* $z \in H_\mu$

for every $\mu \in (0,\theta)$.

Proof. Choose $\varphi \in \boldsymbol{\mathcal{D}}(\mathbb{R})$ with the following properties:

$$\operatorname{supp}\varphi \subset [-1,1], \quad 0 \leq \varphi \leq 1, \quad \text{and} \quad \sum_{n\in\mathbb{Z}} \varphi(x-n) = 1 \qquad (x \in \mathbb{R}).$$

By the Paley–Wiener theorem [197, Theorem 7.22] one knows that the inverse Fourier transform $\check{\varphi}$ of φ is an entire function which decreases rapidly along horizontal lines, uniformly within bounded horizontal strips. Hence for $n \in \mathbb{Z}$ the function

$$f_n(z) := \int_{\mathbb{R}} \check{\varphi}(z-t) f(t) e^{-int}\, dt$$

is entire. Choose $\varepsilon \in (-1,1)$ such that $\varepsilon n \leq 0$. We can shift the contour of integration to obtain

$$f_n(z) = \int_{\mathbb{R}} \check{\varphi}(z-t-i\varepsilon\theta) f(t+i\varepsilon\theta) e^{-int} e^{\varepsilon n\theta}\, dt.$$

This (with $|\varepsilon| \to 1$) yields the estimate $|\check{\varphi}_n(z)| \leq C_\alpha e^{-|n|\theta} \|f\|_{H_\theta}$ for $z \in H_\alpha$, where $C_\alpha = \sup_{|y|\leq 2\alpha} \int_{\mathbb{R}} |\check{\varphi}(t+iy)|\, dt$. Hence for $\mu \in (0,\theta)$ we obtain

$$\sum_{n\in\mathbb{Z}} e^{\mu|n|} \|f_n\|_{H_\alpha} \leq C_\alpha \left(\sum_{n\in\mathbb{Z}} e^{(\mu-\theta)|n|} \right) \|f\|_{H_\theta} \leq \frac{2C_\alpha}{1-e^{\mu-\theta}} \|f\|_{H_\theta}.$$

So we know that the sum $\sum_{n\in\mathbb{Z}} e^{inz} f_n(z)$ converges uniformly within the strip H_μ to a bounded function g. However, in the distributional sense, we have

$$f = \delta_0 * f = \sum_{n\in\mathbb{Z}} (\varphi(\cdot - n))^\vee * f = \sum_{n\in\mathbb{Z}} \left(e^{in\cdot}\check{\varphi}\right) * f = \sum_{n\in\mathbb{Z}} e^{in\cdot} f_n(\cdot)$$

on $\mathbb{R}$. Hence we have $g = f$ on $\mathbb{R}$ and therefore on the whole of H_μ. □

Let us now turn to the proof of Theorem 5.4.1.

Proof of Theorem 5.4.1. Choose $f \in H^\infty(H_\theta)$ and employ Lemma 5.4.2 to find $(f_n)_{n\in\mathbb{Z}}$ in $H^\infty(H_\alpha)$ with properties 1) and 2). Denote by C' the bound of the $H^\infty(H_\alpha)$-calculus for B. Then we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \left\| \left(e^{inz} f_n(z)\right)(B) \right\| &= \sum_{n\in Z} \left\| e^{inB} f_n(B) \right\| \leq C\, C' \sum_{n\in\mathbb{Z}} e^{\mu|n|} \|f_n\|_{H_\alpha} \\ &\leq C\, C'\, c_{\mu,\theta,\alpha} \|f\|_{H_\theta}.\end{aligned}$$

Hence the sum $\sum_{n\in\mathbb{Z}} \left(e^{inz} f_n(z)\right)(B)$ converges uniformly in $\mathcal{L}(X)$ towards an operator T, say. By the Convergence Lemma for strip-type operators (Proposition 5.1.7) we see that $T = f(B)$ and, in fact, $\|f(B)\| \le CC' c_{\mu,\theta,\alpha} \|f\|_{H_\theta}$. □

Corollary 5.4.3. *Let $A \in \mathrm{Sect}(\omega)$ be injective with $\omega_{H^\infty}(A) < \infty$. Then*

$$\omega_{H^\infty}(A) = \theta_A.$$

Note that we do not suppose A to be densely defined or to have dense range.

Proof. We already know that the inequality $\theta_A \le \omega_{H^\infty}(A)$ holds. Choose parameters μ, θ, α with $\theta_A < \mu < \theta < \alpha \le \pi$ and such that the natural $H^\infty(S_\alpha)$-calculus for A is bounded. Let $B := \log(A)$. By Proposition 5.3.3, the operator B satisfies the hypotheses of Theorem 5.4.1. Applying again Proposition 5.3.3 we obtain that the natural $H^\infty(S_\theta)$-calculus for A is bounded. Since θ can be arbitrarily close to θ_A, we are done. □

Remark 5.4.4. In Section 9.1.3 we present an example of an injective sectorial operator A on $X := \mathbf{L}^{\boldsymbol{p}}(\mathbb{T})$ $(1 < p < \infty,\ p \ne 2)$ such that $A \in \mathrm{BIP}(X, 0)$ but $\omega_{H^\infty}(A) = \infty$.

5.5 Perturbation Results

In this section we examine permanence properties of the functional calculus for sectorial operators under *additive perturbations*, i.e., we start with a sectorial operator A and a linear operator $B : \mathcal{D}(A) \longrightarrow X$ and consider the operator $A + B$ with domain $\mathcal{D}(A + B) = \mathcal{D}(A)$. We first remark that in order that $A + B$ is quasi-sectorial *it is necessary that* $B \in \mathcal{L}(\mathcal{D}(A), X)$.

Proof. If $\lambda_0 + A + B$ is sectorial, then $A + B$ is closed. Hence also $A(1 + A)^{-1} + B(1 + A)^{-1} = (A + B)(1 + A)^{-1}$ is closed. From this it follows that $B(1 + A)^{-1}$ is closed and fully defined, whence bounded. □

5.5.1 Resolvent Growth Conditions

We consider conditions under which an additive perturbation $A \to A + B$ preserves (quasi)-sectoriality and boundedness of the H^∞-calculus.

A Domain Condition

The next lemma describes a class of 'well behaving' perturbation operators.

Lemma 5.5.1. *Let $A \in \mathrm{Sect}(\omega)$, $B \in \mathcal{L}(\mathcal{D}(A), X)$, and $\theta > 0$. Then the following assertions are equivalent.*

(i) $\sup_{t\ge 1} \left\| t^\theta B(t + A)^{-1} \right\| < \infty$.

(ii)
$$\sup\left\{|\lambda|^\theta \|BR(\lambda, A)\| \;\Big|\; |\lambda| \geq 1, |\arg\lambda| \in [\omega', \pi]\right\} < \infty$$

for each $\omega' \in (\omega, \pi)$.

In this case there is $\lambda_0 \in \mathbb{R}$ such that $\lambda_0 + A + B \in \mathrm{Sect}(\omega)$.

Proof. The implication (ii)⇒(i) is trivial.
Suppose that (i) holds, and set $d := \sup_{t\geq 1} \left\|t^\theta B(t + A)^{-1}\right\|$. Let $\omega' \in (\omega, \pi)$, and take $\lambda \in \mathbb{C}$ such that $\arg\lambda \in [\omega', \pi]$ and $|\lambda| \geq 1$. Then

$$\begin{aligned}
|\lambda|^\theta \|BR(\lambda, A)\| &\leq |\lambda|^\theta \left\|BR(\lambda, A) + B(|\lambda| + A)^{-1}\right\| + |\lambda|^\theta \left\|B(|\lambda| + A)^{-1}\right\| \\
&\leq |\lambda|^\theta \|B\left(R(\lambda, A) - R(-|\lambda|, A)\right)\| + d \\
&= |\lambda|^\theta \|{-B}(|\lambda| + \lambda)R(-|\lambda|, A)R(\lambda, A)\| + d \\
&\leq 2dM(A, \omega') + d.
\end{aligned}$$

To prove the additional assertion, we write

$$(\lambda - (A + B)) = (I - BR(\lambda, A))(\lambda - A)$$

for $\lambda \notin \overline{S_{\omega'}}$. Since $\lim_{|\lambda|\to\infty} \|BR(\lambda, A)\| = 0$ by (ii) we see that $\lambda - (A + B)$ is invertible for large $|\lambda|$ and $R(\lambda, A + B) = R(\lambda, A)(I - BR(\lambda, A))^{-1}$. The rest follows. □

Remark 5.5.2. We shall prove in Section 6.7 that $B : \mathcal{D}(A) \longrightarrow X$ satisfies the equivalent conditions (i), (ii) from Lemma 5.5.1 with $\theta \in (0, 1]$ if, and only if B extends to a bounded operator $B : (X, \mathcal{D}(A))_{1-\theta,1} \longrightarrow X$ (see Proposition 6.7.3). So the condition (i) is more or less a condition on the *domain* of the perturbing operator B. This accounts for the headline of the present section.

Let us formulate the main result.

Proposition 5.5.3. *Let $A \in \mathrm{Sect}(\omega)$ be injective, and let $\varphi \in (\omega, \pi)$. Let B be an operator satisfying the equivalent conditions* (i), (ii) *from Lemma* 5.5.1*, for some $\theta > 0$. Suppose in addition that $A + B \in \mathrm{Sect}(\omega)$ and $A + B$ is invertible. Then there are constants K_1, K_2 such that*

$$\|f(A + B)\| \leq K_1 \|f(A)\| + K_2 \|f\|_\varphi$$

for all $f \in H^\infty(S_\varphi)$ such that $f(A) \in \mathcal{L}(X)$.

Proof. Take $f \in H^\infty(S_\varphi)$ such that $f(A) \in \mathcal{L}(X)$ and a contour $\Gamma = \Gamma_{\omega'}$ for some $\omega' \in (\omega', \varphi)$. By definition,

$$f(A + B) = (2 + (A + B) + (A + B)^{-1})\frac{1}{2\pi i}\int_\Gamma \frac{f(z)z}{(1 + z)^2} R(z, A + B)\, dz.$$

Since $2 + (A+B)^{-1}$ is a bounded operator, we can estimate

$$\begin{aligned}&\left\|(2+(A+B)^{-1})\frac{1}{2\pi i}\int_\Gamma \frac{f(z)z}{(1+z)^2}R(z,A+B)\,dz\right\|\\&\qquad\le \|f\|_\varphi\, M(A+B,\omega')\left\|2+(A+B)^{-1}\right\|\frac{1}{2\pi}\int_\Gamma \frac{|dz|}{|1+z|^2}.\end{aligned}$$

As to the remaining part, writing $R(z,A+B) = R(z,A+B)BR(z,A) + R(z,A)$ within the integral we have to estimate the two summands

$$(1)\qquad (A+B)\frac{1}{2\pi i}\int_\Gamma \frac{f(z)z}{(1+z)^2}R(z,A)\,dz,$$

$$(2)\qquad (A+B)\frac{1}{2\pi i}\int_\Gamma \frac{f(z)z}{(1+z)^2}R(z,A+B)BR(z,A)\,dz.$$

Since $f(A)$ is a bounded operator we can write $(A+B)A(1+A)^{-2}f(A)$ for the first term and estimate

$$\left\|(A+B)A(1+A)^{-2}\right\| \le (M+1)^2 + (M+1)\left\|B(1+A)^{-1}\right\|,$$

where $M := M(A)$. The second summand is split into two parts by decomposing the contour Γ into $[\Gamma\cap\{|z|\le 1\}]\oplus[\Gamma\cap\{|z|\ge 1\}]$; thus we obtain

$$\begin{aligned}&\left\|(A+B)\frac{1}{2\pi i}\int_\Gamma \frac{f(z)z}{(1+z)^2}R(z,A+B)BR(z,A)\,dz\right\|\\ &\le \int_{\Gamma,|z|\le 1}\frac{|f(z)|}{|1+z|^2}\left\|(A+B)R(z,A+B)\right\|\left\|B(1+A)^{-1}\right\|\left\|z(A+1)R(z,A)\right\|\frac{|dz|}{2\pi}\\ &\qquad + \int_{\Gamma,|z|\ge 1}\frac{|f(z)|\,|z|^{1-\theta}}{|1+z|^2}\left\|(A+B)R(z,A+B)\right\||z|^\theta\left\|BR(z,A)\right\|\frac{|dz|}{2\pi}\\ &\le \|f\|_\varphi\,(M(A+B,\omega')+1)(2M(A,\omega')+1)\left\|B(1+A)^{-1}\right\|\frac{1}{2\pi}\int_{\Gamma,|z|\le 1}\frac{|dz|}{|1+z|^2}\\ &\qquad + \|f\|_\varphi\,(M(A+B,\omega')+1)\,C\,\frac{1}{2\pi}\int_{\Gamma,|z|\ge 1}\frac{|z|^{1-\theta}\,|dz|}{|1+z|^2}\end{aligned}$$

with $C := \left(\sup_{|\arg z|=\omega',|z|\ge 1}|z|^\theta\left\|BR(z,A\right\|\right)$. $\square$

Remark 5.5.4. The statement is false without the assumption that $A+B$ is invertible. This follows from the fact that there is a bounded, injective operator A (on a Hilbert space) with unbounded natural H^∞-calculus (see Corollary 9.1.8) However, A may be written as $A = (A+1)+(-1)$, and $A+1$ does have a bounded H^∞-calculus since it is bounded and invertible.

In passing by we note the following corollary.

Corollary 5.5.5. *Let A be an injective operator on a Banach space X, and let $B \in \mathcal{L}(X)$. Suppose that $A, A+B \in \mathrm{Sect}(\omega)$ and $0 \in \varrho(A+B)$. If for some $\varphi \in (\omega, \pi)$ the natural $H^\infty(S_\varphi)$-calculus for A is bounded, then this also holds for $A+B$.*

A Range Condition

The next lemma introduces another class of perturbations. Essentially, the required property is a condition on the *range* of the perturbing operator (see Remark 5.5.7 below).

Lemma 5.5.6. *Let $A \in \mathrm{Sect}(\omega)$, $B \in \mathcal{L}(\mathcal{D}(A), X)$ and $\theta > 0$. Then the following assertions are equivalent.*

(i) $\sup_{t\geq 1} \left\| t^\theta A(t+A)^{-1}B \right\|_{\mathcal{L}(\mathcal{D}(A),X)} < \infty$.

(ii) $\sup \left\{ |\lambda|^\theta \left\| AR(\lambda, A)B \right\|_{\mathcal{L}(\mathcal{D}(A),X)} \;\middle|\; |\lambda| \geq 1, |\arg \lambda| \in [\omega', \pi] \right\} < \infty$ *for each* $\omega' \in (\omega, \pi)$.

In this case there is $\lambda_0 \in \mathbb{R}$ such that $\lambda_0 + A + B \in \mathrm{Sect}(\omega)$.

Proof. The implication (ii)⇒(i) is trivial.

Suppose that (i) holds, and set $d := \sup_{t\geq 1} \left\| t^\theta A(t+A)^{-1}B \right\|$. Let $\omega' \in (\omega, \pi)$, and take $\lambda \in \mathbb{C}$ such that $\arg\lambda \in [\omega', \pi]$, $|\lambda| \geq 1$. Then

$$\begin{aligned}
|\lambda|^\theta \left\| AR(\lambda, A)B \right\| &\leq |\lambda|^\theta \left\| AR(\lambda, A)B + A(|\lambda| + A)^{-1}B \right\| + |\lambda|^\theta \left\| A(|\lambda| + A)^{-1}B \right\| \\
&\leq |\lambda|^\theta \left\| A\left(R(\lambda, A) - R(-|\lambda|, A)\right)B \right\| + d \\
&= |\lambda|^\theta \left\| -(|\lambda| + \lambda)R(\lambda, A)AR(-|\lambda|, A)B \right\| + d \\
&\leq 2dM(A, \omega') + d
\end{aligned}$$

(all the norms in $\mathcal{L}(\mathcal{D}(A), X)$). To prove the additional assertion, we write

$$(\lambda - (A+B)) = (\lambda - A)(I - R(\lambda, A)B)$$

for $\lambda \notin \overline{S_{\omega'}}$. Choose $\sigma \in (0, \theta]$ with $\sigma < 1$. Then our assumptions and (ii) imply actually that $\sup_{|\lambda|\geq 1} |\lambda|^\sigma \left\| (\lambda + A)^{-1}B \right\|_{\mathcal{D}(A)\longrightarrow\mathcal{D}(A)} < \infty$. (Note that $B \in \mathcal{L}(\mathcal{D}(A), X)$). Hence $\lim_{|\lambda|\to\infty} \left\| (\lambda + A)^{-1}B \right\|_{\mathcal{L}(\mathcal{D}(A))} = 0$, and so $(I - (\lambda + A)^{-1}B)$ is an isomorphism on $\mathcal{D}(A)$ for large $|\lambda|$. Therefore, for $|\lambda|$ large enough, we see that $\lambda - (A+B)$ is invertible and $R(\lambda, A+B) = (I - BR(\lambda, A))^{-1}R(\lambda, A)$. The rest follows. □

Remark 5.5.7. We shall prove in Section 6.7 that $B : \mathcal{D}(A) \longrightarrow X$ satisfies the equivalent conditions (i), (ii) from Lemma 5.5.1 with $\theta \in (0, 1]$ if, and only if $B : \mathcal{D}(A) \longrightarrow X$ is bounded and $\mathcal{R}(B) \subset (X, \mathcal{D}(A))_{\theta,\infty}$ (see Proposition 6.7.1). This accounts for the headline of the present section.

Proposition 5.5.8. *Let $A \in \operatorname{Sect}(\omega)$ be injective, and let $\varphi \in (\omega, \pi)$. Let B satisfy the equivalent properties* (i), (ii) *from Lemma* 5.5.6*, for some $\theta > 0$. Suppose in addition that $A + B \in \operatorname{Sect}(\omega)$ and $A + B$ is invertible. Then there are constants K_1, K_2 such that*

$$\|f(A+B)\| \le K_1 \|f(A)\| + K_2 \|f\|_\varphi$$

for all $f \in H^\infty(S_\varphi)$ such that $f(A) \in \mathcal{L}(X)$.

Proof. The proof follows the same lines as the proof of Proposition 5.5.3. The main difference is in writing

$$R(z, A+B) = R(z, A) - R(z, A)BR(z, A+B)$$

instead of $R(z, A+B) = R(z, A) + R(z, A+B)BR(z, A)$. $\square$

A Degenerate Case

We turn to a different class of operators, somehow 'dual' to the first one. Suppose that $A \in \operatorname{Sect}(\omega)$ has dense domain. Let $B : \mathcal{D}(A) \longrightarrow X$ be an operator such that

$$\sup_{t \ge 1} \left\| t^\theta (t + A)^{-1} B \right\| < \infty \tag{5.9}$$

for some $\theta \in (0, 1)$. If we replace A by $\varepsilon + A$ for $\varepsilon > 0$, condition (5.9) still holds. So in the following we take A to be invertible.

The problem is now that in general the operator $A+B$ with $\mathcal{D}(A+B) := \mathcal{D}(A)$ is not quasi-sectorial, and not even a closed operator. However, it always has a quasi-sectorial extension. To prove this we introduce a suitable *extrapolation space* of X. Namely, let

$$Y := (X, \left\|A^{-1}\cdot\right\|)^\sim,$$

i.e., Y is the completion of X with respect to the norm $\|x\|_Y := \left\|A^{-1}x\right\|_X$. Then, by density of $\mathcal{D}(A)$ in X, the operator A extends in a unique way to an isometric isomorphism $A_Y : X \longrightarrow Y$. Moreover, by (5.9) one has

$$\begin{aligned} \|Bx\|_Y = \left\|A^{-1}Bx\right\|_X &\le \left\|(A+1)A^{-1}\right\|_{\mathcal{L}(X)} \left\|(1+A)^{-1}Bx\right\|_X \\ &\le c \left\|(A+1)A^{-1}\right\|_{\mathcal{L}(X)} \|x\|_X \qquad (x \in \mathcal{D}(A)) \end{aligned}$$

for some $c > 0$. This implies that B extends uniquely to a bounded operator $B_Y : X \longrightarrow Y$. Using these extensions condition (5.9) becomes

$$\sup_{t \ge 1} \left\| t^\theta A_Y (t + A_Y)^{-1} B_Y \right\|_{\mathcal{L}(X,Y)} < \infty. \tag{5.10}$$

Considering the operator A_Y as an operator on Y with domain $\mathcal{D}(A_Y) = X$, we realise that A_Y is sectorial of angle ω, isometrically similar to the operator A on X. Moreover, $B_Y : \mathcal{D}(A_Y) \longrightarrow Y$ is a perturbation of A_Y of the type which we considered in Lemma 5.5.6. Hence we conclude that for some $\lambda_0 > 0$ the operator $\lambda_0 + A_Y + B_Y$ is also sectorial (of angle ω) and invertible (on the Banach space Y). If we take the part in X of this operator we have our sectorial extension.

Proposition 5.5.9. *Let A, B, λ_0 be as above, and let $C := (A_Y + B_Y)\big|_X$ be the part of $A_Y + B_Y$ in X. Then the following assertions hold.*

a) *The operator $\lambda_0 + C$ is an invertible sectorial operator of angle ω in X, and C extends $A + B$.*

b) *The following two assertions are equivalent:*

  (i) *$C = \overline{A + B}$;*

  (ii) *$\lambda_0 + A + B$ has dense range.*

c) *For each $\varphi \in (\omega, \pi)$ there exist constants K_1, K_2 such that for every function $f \in H^\infty(S_\varphi)$ such that $f(A) \in \mathcal{L}(X)$ one has $f(\lambda_0 + C) \in \mathcal{L}(X)$ and*

$$\|f(\lambda_0 + C)\| \le K_1 \|f(A)\| + K_2 \|f\|_\varphi .$$

Proof. a) Since $\mathcal{D}(A_Y + B_Y) = X$, the space X is invariant under the resolvent of $A_Y + B_Y$. Also it is clear that $R(\lambda, C) = R(\lambda, A_Y + B_Y)\big|_X$ for all $\lambda \in \varrho(A_Y + B_Y)$. This proves that $\sigma(\lambda_0 + C) \subset \overline{S_\omega}$. To prove the sectoriality estimate we simply write

$$(\lambda + A_Y + B_Y)^{-1} = (I + (\lambda + A_Y)^{-1} B_Y)^{-1} (\lambda + A_Y)^{-1}.$$

By restriction this gives $(\lambda + C)^{-1} = (I + (\lambda + A_Y)^{-1} B_Y)^{-1} (\lambda + A)^{-1}$. Now one combines the sectoriality of A with condition (5.10) (and the resolvent identity) to obtain the sectoriality of $\lambda_0 + C$.

b) If $C = \overline{A + B}$ we have

$$X = \mathcal{R}(\lambda_0 + C) = \mathcal{R}\left(\overline{\lambda_0 + A + B}\right) \subset \overline{\mathcal{R}(\lambda_0 + A + B)},$$

whence $\lambda_0 + A + B$ has dense range. On the other hand, if this is so, then from $\overline{A + B} \subset C$ it follows that $(\lambda_0 + C)' \subset (\lambda_0 + \overline{A + B})' = (\lambda_0 + A + B)'$. The latter is injective, by hypothesis. However, $(\lambda_0 + C)'$ is even invertible (since $\lambda_0 + C$ is), and this is only possible if one has equality $(\lambda_0 + C)' = (\lambda_0 + \overline{A + B})'$. Taking adjoints and intersecting with $X \oplus X$ yields $\lambda_0 + C = \lambda_0 + \overline{A + B}$.

c) Take $\varphi \in (\omega, \pi)$ and $f \in H^\infty(S_\varphi)$ such that $f(A) \in \mathcal{L}(X)$. Then clearly $f(A_Y) \in \mathcal{L}(Y)$ with $\|f(A_Y)\|_{\mathcal{L}(Y)} = \|f(A)\|_{\mathcal{L}(X)}$. By Proposition 5.5.8 we have $f(\lambda_0 + A_Y + B_Y) \in \mathcal{L}(Y)$ with

$$\|f(\lambda_0 + A_Y + B_Y)\|_{\mathcal{L}(Y)} \le K_1 \|f(A_Y)\|_{\mathcal{L}(Y)} + K_2 \|f\|_\varphi$$

for some constants K_1, K_2. Now note that, by Proposition 2.6.5, we have

$$f(\lambda_0 + C) = f((\lambda_0 + A_Y + B_Y)_X) = f(\lambda_0 + A_Y + B_Y)_X.$$

Moreover, $X = \mathcal{D}(\lambda_0 + A_Y + B_Y)$, which means that $\|(\lambda_0 + A_Y + B_Y)x\|_Y$ provides an equivalent norm on X. On the other hand, the space X is invariant under $f(\lambda_0 + A_Y + B_Y)$ with

$$\|f(\lambda_0 + C)\|_{\mathcal{L}(X)} = \|f(\lambda_0 + A_Y + B_Y)_X\|_{\mathcal{L}(X)} \lesssim \|f(\lambda_0 + A_Y + B_Y)\|_{\mathcal{L}(Y)} .$$

This concludes the proof. $\square$

One can show that each of the two equivalent conditions appearing in Proposition 5.5.9 is also equivalent to the identity $(A+B)' = A' + B'$. More on extrapolation spaces can be found in Section 6.3.

5.5.2 A Theorem of Prüss and Sohr

Recall that, by Proposition 3.5.5, one has

$$A^{is} \in \mathcal{L}(X) \quad \Rightarrow \quad (A+\varepsilon)^{is} \in \mathcal{L}(X)$$

for each injective, sectorial operator A and all $s \in \mathbb{R}, \varepsilon > 0$. The reverse implication is valid if A is invertible. Unfortunately, this perturbation result does not say anything about the group types θ_A and $\theta_{A+\varepsilon}$. In this section we wish to prove the following theorem.

Theorem 5.5.10 (Prüss–Sohr). *Let A be an injective, sectorial operator on the Banach space X. Suppose that there is a constant K such that $\left\|A^{is}\right\| \le Ke^{\theta|s|}$ $(s \in \mathbb{R})$ for some $\theta > \omega_A$. Then there is K' such that*

$$\left\|(A+\varepsilon)^{is}\right\| \le K' e^{\theta|s|} \qquad (s \in \mathbb{R})$$

uniformly in $\varepsilon > 0$.

To achieve this we have to study in more detail a special case of Corollary 5.5.5.

Proposition 5.5.11. *Let $A \in \mathrm{Sect}(\omega)$ be injective. Let $\varphi \in (\omega, \pi)$ and $f \in H^\infty(S_\varphi)$. If $f(A) \in \mathcal{L}(X)$, then $f(A+1) \in \mathcal{L}(X)$ and*

$$\|f(A+1)\| \le K_1 \|f(A)\| + K_2 \|f\|_\varphi ,$$

where $K_1 = 1 + M(A)$ and

$$K_2 = (2 + M(A) + M(A,\varphi))M(A,\varphi)C(\varphi),$$

where $C(\varphi)$ only depends on φ.

Proof. Take $\omega' \in (\omega, \varphi)$ and let $\Gamma = \partial S_{\omega'}$. One has to estimate

$$f(A+1) = (2 + (A+1) + (A+1)^{-1}) \frac{1}{2\pi} \int_\Gamma \frac{f(z)z}{(1+z)^2} R(z, A+1)\, dz.$$

As in the proof of Proposition 5.5.3 we split this expression into two summands, namely

$$\begin{aligned} f(A+1) = {} & (2 + (A+1)^{-1}) \frac{1}{2\pi i} \int_\Gamma \frac{f(z)z}{(1+z)^2} R(z, A+1)\, dz \\ & + (A+1) \frac{1}{2\pi i} \int_\Gamma \frac{f(z)z}{(1+z)^2} R(z, A+1)\, dz. \end{aligned}$$

The first summand is estimated by

$$\|f\|_\varphi\,(2+M(A))\,M(A+1,\omega')\,\frac{1}{2\pi}\int_\Gamma \frac{|dz|}{|1+z|^2}.$$

The second is written as the difference

$$\begin{aligned}
&(A+1)\int_\Gamma \frac{f(z)z}{(1+z)^2}R(z,A+1)\,\frac{dz}{2\pi i}\\
&= (A+1)\int_\Gamma \frac{f(z)z}{(1+z)^2}R(z,A)\,\frac{dz}{2\pi i}\;-\;(A+1)\int_\Gamma \frac{f(z)z}{(1+z)^2}R(z,A)R(z,A+1)\,\frac{dz}{2\pi i}\\
&= A(1+A)^{-1}f(A)\;-\;(A+1)\int_\Gamma \frac{f(z)z}{(1+z)^2}R(z,A)R(z,A+1)\,\frac{dz}{2\pi i}.
\end{aligned}$$

Here we obtain the bounds $(1+M(A))\,\|f(A)\|$ and

$$\|f\|_\varphi\;M(A,\omega')\,M(A+1,\omega')\,\frac{1}{2\pi}\int_\Gamma \frac{|dz|}{|1+z|^2}.$$

We now let $\omega'\to\varphi$ and apply Proposition 2.1.1 f) to obtain a constant $C(\varphi)$ depending only on φ such that

$$\begin{aligned}
\|f(A+1)\| &\le (1+M(A))\;\|f(A)\|\\
&\quad+[(2+M(A)+M(A,\varphi))M(A,\varphi)C(\varphi)]\;\|f\|_\varphi. \qquad \square
\end{aligned}$$

We now *prove Theorem* 5.5.10. Let A be sectorial and injective, and let $\theta>\omega_A$. Choose $\varphi\in(\omega,\min(\theta,\pi))$, and suppose that $\|A^{is}\|\le Ke^{\theta|s|}$ $(s\in\mathbb{R})$. Given $\varepsilon>0$ and $s\in\mathbb{R}$ we apply Proposition 5.5.11 to the operator $\varepsilon^{-1}A$ and the function $f(z)=z^{is}$ and obtain

$$\left\|(\varepsilon^{-1}A+1)^{is}\right\|\le K_1\left\|(\varepsilon^{-1}A)^{is}\right\|+K_2\left\|z^{is}\right\|_\varphi$$

for all $s\in\mathbb{R}$. A closer look at the shape of the constants K_1,K_2 in Proposition 5.5.11 reveals that they do *not* depend on ε (due to i) of Proposition 2.1.1). Now $|\varepsilon^{-is}|=1$, $\|z^{is}\|_\varphi=e^{\varphi|s|}$, and $(\varepsilon^{-1}A+1)^{is}=(\varepsilon^{-1}(A+\varepsilon))^{is}=\varepsilon^{-is}(A+\varepsilon)^{is}$ by a (trivial) application of the composition rule. Similarly, $(\varepsilon^{-1}A)^{is}=\varepsilon^{-is}A^{is}$. Altogether this yields

$$\left\|(A+\varepsilon)^{is}\right\|\le K_1\left\|A^{is}\right\|+K_2e^{\varphi|s|}\le(K_1K+K_2)e^{\theta|s|}\qquad(s\in\mathbb{R})$$

uniformly in $\varepsilon\ge 0$, whence Theorem 5.5.10 is proved. $\square$

We state a corollary which summarises our considerations.

Corollary 5.5.12. *Let A be sectorial and have dense domain and dense range, and let $\theta\ge 0$. If $A\in\mathrm{BIP}(X,\theta)$, then $A+\varepsilon\in\mathrm{BIP}(X,\theta)$ for all $\varepsilon>0$. In fact, if $\theta>\omega_A$, there is K' such that $\left\|(A+\varepsilon)^{is}\right\|\le K'e^{\theta|s|}$ $(s\in\mathbb{R})$ uniformly in $\varepsilon\ge 0$.*

5.6 A Characterisation

The next result, well known in Harmonic Analysis, is of great importance and shows that a bounded H_0^∞-calculus implies a large 'amount of unconditionality'.

Lemma 5.6.1 (Unconditionality Lemma). *Let $0 \le \omega < \varphi < \pi$ and $f \in H_0^\infty(S_\varphi)$. Then there is a constant $D > 0$ such that the following holds. If $A \in \mathrm{Sect}(\omega)$ on some Banach space X such that the natural $H_0^\infty(S_\varphi)$-calculus for A is bounded with bound C, then*

$$\Big\| \sum\nolimits_{k\in\mathbb{Z}} a_k\, f(t2^k A) \Big\| \le CD\, \|a\|_\infty \tag{5.11}$$

for all $t > 0$ and all finite sequences $a = (a_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$.

Proof. Choose $C' > 0$ and $s > 0$ such that $|f(z)| \le C' \min\Big(|z|^s, |z|^{-s}\Big)$ for all $z \in S_\varphi$. Let $a = (a_k)_{k\in\mathbb{Z}}$ be a finite sequence of complex numbers such that $\|a\|_\infty \le 1$, and let $t > 0$. We estimate

$$\begin{aligned}
\Big\| \sum\nolimits_k a_k f(t2^k A) \Big\| &= \Big\| \Big(\sum\nolimits_k a_k f(t2^k z) \Big)(A) \Big\| \le C \sup_{z\in S_\varphi} \Big| \sum\nolimits_k a_k f(t2^k z) \Big| \\
&\le C \sup_{z\in S_\varphi} \sum\nolimits_k \big| f(2^k z) \big| \le CC' \sup_{z\in S_\varphi} \sum\nolimits_k \min\big(|2^k z|^s, |2^k z|^{-s} \big) \\
&= CC' \sup_{1\le t\le 2} \sum\nolimits_k \min\big(2^{ks} t^s, 2^{-ks} t^{-s} \big) \le CC' \sum\nolimits_k \min\Big(2^{(k+1)s}, 2^{-ks} \Big) \\
&\le CC' 2 \sum\nolimits_{k\ge 0} 2^{-ks} = CC' \frac{2^{s+1}}{2^s - 1}.
\end{aligned}$$

□

The Unconditionality Lemma is in fact part of a general characterisation of the boundedness of the H^∞-calculus. We cite the following theorem from [141, Theorem 12.2] without proof.

Theorem 5.6.2. *Let $A \in \mathrm{Sect}(\omega)$ be a sectorial operator on a Banach space X such that A has dense domain and dense range. For $\varphi, \theta \in (\omega_A, \pi)$ let $\psi_{\pm\theta}(z) := z^{1/2}(e^{\pm i\theta} - z)^{-1}$ and consider the following statements:*

(i) *The natural $H^\infty(S_\varphi)$-calculus is bounded;*

(ii) $\displaystyle \sup_{N\in\mathbb{N}, t>0, \varepsilon\in\{-1,1\}^N} \Big\| \sum\nolimits_{k=-N}^{N} \varepsilon_k \psi_{\pm\theta}(2^k t A) \Big\| < \infty.$

Then (i)⇒(ii) *for $\theta < \varphi$ and* (ii)⇒(i) *for $\varphi < \theta$.*

5.7 Comments

5.1 Convergence Lemma. It is an open controversy if continuity properties should be incorporated into a general notion of functional calculus. As the reader might have already guessed we champion a purely algebraic concept and regard continuity properties as being accidental. Nevertheless, they may have important consequences. This is the case with the Convergence Lemma, which is crucial if one is interested in the boundedness of the H^∞-calculus. Lemma 5.1.2 is a refinement of [167, Section 4, Theorem, a)]. Part b) of the Convergence Lemma (Proposition 5.1.4) is [167, Section 5, Theorem] or [51, Lemma 2.1] or [3, Theorem D] and practically contained in all other papers on H^∞-calculus. Part a) is our contribution to the matter, see also Remark 5.1.5.

5.2 A Fundamental Approximation Technique. Lemma 5.2.1 was invented to pave the ground for McIntosh's approximation technique (Theorem 5.2.6), which was introduced by McIntosh [167] and often employed in the subsequent works [169, 3, 19]. Theorem 5.2.2, Lemma 5.2.3 and Proposition 5.2.4 give, more or less, a systematised account of results of [3].

5.3 Equivalent Descriptions and Uniqueness. Proposition 5.3.4 covers results from the literature. The proof of the implication (i)⇒(ii) is from [51], the one of (iv)⇒(iii) is inspired by [145] and [146], where Runge's theorem is invoked.

Discussing *uniqueness* of the functional calculus seems to be a delicate matter (see also the 'Concluding Remarks' in Section 2.8). Since usually the only thing linking the functional calculus to the operator is its behaviour on rational functions, a continuity assumption with respect to *bounded and pointwise* convergence of functions seems necessary to obtain uniqueness, at least if the function algebra is large. This is reflected in Proposition 5.3.9. Lemma 5.3.8 is the best we could achieve if such a continuity property is missing.

The reason why we included a section on uniqueness is that in the literature reference is often made to 'operators which have a bounded H^∞-functional calculus' where it is obviously meant that the *natural* (as we call it) H^∞-calculus is bounded. The cited phrase automatically raises the uniqueness question since it presupposes a general definition of *an* H^∞-calculus. It is this context which accounts for our Proposition 5.3.9. However, Proposition 5.3.4 shows that with a reasonable definition of 'bounded H^∞-calculus' the following assertions are equivalent:

(i) A has *a* (= *some*) bounded $H^\infty(S_\varphi)$-calculus.

(ii) The *natural* H^∞-calculus for A is bounded.

This justifies in retrospect the quoted manner of speaking.

5.4 The Minimal Angle. Theorem 5.4.1 and its proof via Lemma 5.4.2 is taken from [51, Theorem 5.4].

5.5 Perturbation Results. One of the first perturbation results in connection with functional calculus is Corollary 5.5.12 by Prüss and Sohr [193]. (Actually, they proved a slighly different result, allowing $\theta = \omega_A$ but requiring $\theta \in (0, \pi)$. In all relevant applications this difference is immaterial.)

A different proof being more or less in the same vein as ours was given by Monniaux [171] and subsequently by Uiterdijk [217]. The need to consider a perturbation $A - \varepsilon$ instead of $A + \varepsilon$ leads to a result for bounded perturbations in [12]. There it was remarked that weaker conditions than boundedness suffice, and Propositions 5.5.3 and 5.5.8 are elaborations of this remark, due to the author. In the same spirit are the perturbation results from [6] and [141, Proposition 13.1]. More intricate perturbation theorems were proved by Hieber, Kalton, Kunstmann, Prüss and Weis, see [141] and the references therein.

5.6 A Characterisation. A first characterisation of the boundednes of the H^∞-calculus was given by Boyadzhiev and deLaubenfels [33]. Cowling et al. [51] investigated more deeply the connection with so-called *quadratic estimates* that had appeared naturally in McIntosh's work on Hilbert spaces. Finally Kalton and Weis in [125] and subsequent work stressed the importance and fruitfulness of randomisation techniques, see e.g. [141].

Chapter 6

Interpolation Spaces

In the present chapter we examine the connections between functional calculus and interpolation spaces. As an 'appetiser', in Section 6.1 we present two central ideas: a model describing the real interpolation spaces $(X, \mathcal{D}(A))_{\theta,p}$ using the functional calculus and a theorem of DORE. In Section 6.2 we examine the first of these ideas, proving several representation results for the spaces $(X, \mathcal{D}(A^\alpha))_{\theta,p}$. Then we introduce extrapolation spaces for injective operators (Section 6.3). With the help of these spaces in Section 6.4 we derive two fundamental results (Theorem 6.4.2 and Theorem 6.4.5) that lead to more characterisations of interpolation spaces by functional calculus (Section 6.5.1) and a generalisation of Dore's theorem (Section 6.5.3). In Section 6.6 we establish all the common properties of fractional domain spaces as intermediate spaces: density, the moment inequality, reiteration. Moreover, we prove the intriguing fact that for an operator $A \in \mathrm{BIP}(X)$ the fractional domain spaces equal the *complex* interpolation spaces (Theorem 6.6.9). Finally we characterise growth conditions like $\sup_{t\geq 1} \left\|t^\theta B(t+A)^{-1}\right\| < \infty$ in terms of interpolation spaces (Section 6.7).

6.1 Real Interpolation Spaces

We assume that the reader is familiar with the basic theory of real interpolation spaces. A short account and references can be found in Appendix B. For convenience we recall our notation $\mathbf{L}^{p}_{*}((a,b);X)$ for the Bochner space of (equivalence classes of) measurable functions $f : (a,b) \longrightarrow X$ which are p-integrable with respect to the measure dt/t. In the case where $(a,b) = (0,\infty)$ we write simply $\mathbf{L}^{p}_{*}(X)$. Also, we define $\Lambda \subset [0,1] \times [1,\infty]$ by

$$(\theta, p) \in \Lambda \quad :\Longleftrightarrow \quad p \in [1,\infty),\, \theta \in (0,1) \quad \text{or} \quad p = \infty,\, \theta \in [0,1].$$

The present section should be viewed as an 'appetiser' for a deeper study of the connections of functional calculus with real interpolation spaces.

Let $A \in \mathrm{Sect}(\omega)$ on the Banach space X. We use the common notation

$$D_A(\theta,p) := (X, \mathcal{D}(A))_{\theta,p}$$

for $(\theta,p) \in \Lambda$; the respective norm is denoted by $\|\cdot\|_{\theta,p}$. Employing only elementary arguments (see [158, Proposition 3.1.1]) one can derive the following description.

Lemma 6.1.1. *Let $A \in \mathrm{Sect}(\omega)$ on the Banach space X. Then*

$$D_A(\theta,p) := \big\{x \in X \;\big|\; t^\theta A(t+A)^{-1}x \in \mathbf{L}^p_*((0,\infty);X)\big\} \tag{6.1}$$

with equivalence of norms

$$\|x\|_{\theta,p} \sim \|x\|_X + \big\|t^\theta A(t+A)^{-1}x\big\|_{\mathbf{L}^p_*((0,\infty);X)}.$$

If A is invertible, one can even drop the $\|\cdot\|_X$-part and then has the norm equivalence

$$\|x\|_{\theta,p} \sim \big\|t^\theta A(t+A)^{-1}x\big\|_{\mathbf{L}^p_*((0,\infty);X)}.$$

We do not prove the lemma here since it follows from more general descriptions we give in the next section. However, we point out that the result is in fact a characterisation of $D_A(\theta,p)$ in terms of functional calculus. Indeed, letting $\psi(z) := z(1+z)^{-1}$ after a change of parameter $t \mapsto t^{-1}$, the description (6.1) becomes

$$(X,\mathcal{D}(A))_{\theta,p} = \{x \in X \mid t^{-\theta}\psi(tA)x \in \mathbf{L}^p_*((0,\infty);X)\}.$$

This provides us with a basic intuition that will be made precise in Theorems 6.2.1, 6.2.9 and 6.5.3 below.

One of the central features of real interpolation spaces is that the operator A when restricted to $D_A(\theta,p)$ improves some of its functional calculus properties. This heuristic statement is made precise by a result of Dore from [72]. In order to prove it, we need some preliminaries.

Let again $A \in \mathrm{Sect}(\omega)$ on X. Clearly, the spaces $D_A(\theta,p)$ are invariant under application of the resolvent of A, i.e., $R(\lambda,A)D_A(\theta,p) \subset D_A(\theta,p)$ with

$$\|R(\lambda,A)x\|_{\theta,p} \le \|R(\lambda,A)\|_{\mathcal{L}(X)} \|x\|_{\theta,p} \tag{6.2}$$

for all $\lambda \in \varrho(A)$, $(\theta,p) \in \Lambda$, and $x \in D_A(\theta,p)$. Restricting the operator A to the space $D_A(\theta,p)$ therefore yields again a sectorial operator. The following proposition is stated for the sake of convenience. Its proof follows from Proposition 2.6.5.

Proposition 6.1.2. *Let $A \in \mathrm{Sect}(\omega)$, $\theta \in (0,1)$ and $p \in [1,\infty]$. Denote by $A_{\theta,p}$ the part of A in $D_A(\theta,p)$, i.e.,*

$$(x,y) \in A_{\theta,p} \quad\Longleftrightarrow\quad x \in \mathcal{D}(A) \quad \text{and} \quad y = Ax \in D_A(\theta,p)$$

for all $x,y \in X$. Then the following assertions hold.

a) $\varrho(A) \subset \varrho(A_{\theta,p})$ *with*

$$R(\lambda,A_{\theta,p}) = R(\lambda,A)\big|_{D_A(\theta,p)} \qquad (\lambda \in \varrho(A)).$$

b) $A_{\theta,p} \in \mathrm{Sect}(\omega)$ *in the Banach space $D_A(\theta,p)$.*

c) *If A is injective or invertible, so is $A_{\theta,p}$.*

d) *If $f \in \mathcal{M}[S_\omega]_A$, then $f \in \mathcal{M}[S_\omega]_{A_{\theta,p}}$ with $f(A_{\theta,p})$ being the part of $f(A)$ in $D_A(\theta,p)$, i.e.,*

$$(x,y) \in f(A_{\theta,p}) \quad \iff \quad \begin{cases} x \in \mathcal{D}(f(A)) \cap D_A(\theta,p), \\ y = f(A)x \in D_A(\theta,p) \end{cases}$$

for all $x, y \in X$.

Now we can state Dore's theorem. Note that that there are sectorial operators without a bounded H^∞-calculus (Remark 5.4.4).

Theorem 6.1.3 (Dore). *Let A be an invertible sectorial operator on the Banach space X, and let $\theta \in (0,1)$ and $p \in [1,\infty]$. Then for each $\varphi \in (\omega_A, \pi)$ the natural $H^\infty(S_\varphi)$-calculus for $A_{\theta,p}$ is bounded. In particular, $D_A(\theta,p) \subset \mathcal{D}(f(A))$ for all $f \in H^\infty[S_\omega]$.*

Proof. We first treat the case $p = \infty$. Employing the resolvent identity and the second part of Lemma 6.1.1, one sees that for each $\omega' \in (\omega_A, \pi)$ there is a constant $c(\omega')$ such that

$$\sup_{z \in \partial S_{\omega'}} \left\| z^\theta AR(z,A)x \right\| \le c(\omega') \left\| x \right\|_{\theta,\infty}$$

for all $x \in D_A(\theta,\infty)$. Take $\varphi \in (\omega_A, \pi)$, $f \in H^\infty(S_\varphi)$ and $x \in D_A(\theta,p)$. Then

$$\begin{aligned} \left(\frac{f(z)}{1+z}\right)(A)x &= \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{1+z} R(z,A)x\,dz \\ &= A^{-1}\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(1+z)z^\theta}\left[z^\theta AR(z,A)x\right]dz, \end{aligned}$$

where $\Gamma = \partial S_{\omega'}$ for some $\omega' \in (\omega, \varphi)$. This shows that $(f/(1+z))(A)x \in \mathcal{D}(A)$, whence $x \in \mathcal{D}(f(A))$. Moreover, for each $t > 0$ one has

$$\begin{aligned} \left\| t^\theta A(t+A)^{-1} f(A)x \right\|_X &= \frac{1}{2\pi}\left\| \int_\Gamma \frac{f(z)t^\theta}{(t+z)z^\theta} z^\theta AR(z,A)x\,dz \right\|_X \\ &\le \frac{1}{2\pi}\int_\Gamma \frac{|f(z)|\,t^\theta\,|dz|}{|t+z|\,|z|^\theta}\, c(\omega')\left\| x \right\|_{\theta,\infty} = \frac{1}{2\pi}\int_\Gamma \frac{|f(tz)|\,|dz|}{|1+z|\,|z|^\theta}\, c(\omega')\left\| x \right\|_{\theta,\infty} \\ &\le \left\| f \right\|_\varphi \frac{1}{2\pi}\int_\Gamma \frac{|dz|}{|1+z|\,|z|^\theta}\, c(\omega')\left\| x \right\|_{\theta,\infty}. \end{aligned}$$

Hence $f(A)x \in D_A(\theta,\infty)$ with $\left\| f(A)x \right\|_{\theta,\infty} \le C \left\| f \right\|_\varphi \left\| x \right\|_{\theta,\infty}$ for some constant C independent of x.

Let us turn to the case $1 \le p < \infty$. By the Reiteration Theorem B.2.9 one can view $D_A(\theta,p)$ as a real interpolation space between $D_A(\theta,\infty)$ and $D_A(\beta,\infty)$, where $\theta < \beta < 1$. Combining this with Proposition 6.1.2 concludes the proof. $\square$

Example 6.1.4. The hypothesis of invertibility cannot be dropped from Theorem 6.1.3. Let A be an unbounded, invertible, sectorial operator on a Hilbert space H such that the natural H^∞-calculus for A is not bounded (see Corollary 9.1.8). From the composition rule it is clear that A^{-1} cannot have a bounded H^∞-calculus. Form the diagonal operator

$$\mathcal{A} := \operatorname{diag}(A, A^{-1}) := \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$$

on the space $H \oplus H$. Then it is clear that $D_{\mathcal{A}}(\theta, p) = D_A(\theta, A) \oplus H$ with the induced operator $\mathcal{A}_{\theta,p}$ being the diagonal operator $\operatorname{diag}(A_{\theta,p}, A^{-1})$. Since obviously $f(\mathcal{A}) = \operatorname{diag}(f(A), f(A^{-1}))$, $\mathcal{A}$ cannot have a bounded H^∞-calculus on $D_{\mathcal{A}}(\theta, p)$, for any pair (θ, p).

Corollary 6.1.5. *Let* $A \in \operatorname{Sect}(\omega)$ *such that* $\mathcal{D}(A^\alpha) = D_A(\operatorname{Re}\alpha, p)$ *(with equivalent norms) for some* $\operatorname{Re}\alpha \in (0,1)$ *and some* $p \in [1, \infty]$. *Then for each* $\varepsilon > 0$ *and each* $\varphi \in (\omega, \pi]$ *the natural* $H^\infty(S_\varphi)$*-calculus for* $A + \varepsilon$ *is bounded.*

Proof. Since $(A + \varepsilon)^{-\alpha} : X \longrightarrow \mathcal{D}(A^\alpha)$ is an isomorphism, the assertion follows from Theorem 6.1.3. □

We shall prove below that the identity $\mathcal{D}(A^\alpha) = D_A(\alpha, p)$ for *one* α implies already that it holds for *all* α. Also we shall prove a much more general form of Dore's theorem without employing the Reiteration Theorem.

6.2 Characterisations

In the present section we use the functional calculus to give several descriptions of the real interpolation spaces associated with a sectorial operator A on a Banach space X. Given $\operatorname{Re}\alpha > 0$ the space $\mathcal{D}(A^\alpha)$ will be considered endowed with the norm $\|x\|_X + \|A^\alpha x\|_X$, which turns it into a Banach space continuously embedded in X. Hence the pair $(\mathcal{D}(A^\alpha), \mathcal{D}(A^\beta))$ is an interpolation couple whenever $\alpha, \beta \in \{z \mid \operatorname{Re} z > 0\} \cup \{0\}$. Note that, since $A^\alpha (A+1)^{-\alpha} = (A(A+1)^{-1})^\alpha$ is bounded, one has $\mathcal{D}(A^\alpha) = \mathcal{D}((A+1)^\alpha)$ not only as sets but also with equivalent norms. (Apply the Closed Graph Theorem.) Therefore, in considerations involving only the *space* $\mathcal{D}(A^\alpha)$ one may always suppose that A is invertible. In that case the homogeneous norm $\|x\|_{A^\alpha} := \|A^\alpha x\|_X$ is an equivalent norm on $\mathcal{D}(A^\alpha)$.

6.2.1 A First Characterisation

In this section we consider a description of the real interpolation spaces as follows.

Theorem 6.2.1. *Let* $A \in \operatorname{Sect}(\omega)$, $\varphi \in (\omega, \pi)$, *and* $\operatorname{Re}\alpha > 0$. *Furthermore, let* $\psi \in \mathcal{O}(S_\varphi)$ *be a function with the following properties:*

a) $\psi, \psi z^{-\alpha} \in \mathcal{E}(S_\varphi)$;

b) $\lim_{z\to 0} \psi(z)z^{-\alpha} \neq 0$;

c) $\psi(z) \neq 0$ *for all* $z \in S_\varphi$;

d)

$$\sup_{z\in S_\varphi, s\geq 1} \left| \frac{\psi(sz)}{s^\alpha \psi(z)} \right| < \infty. \tag{6.3}$$

Then for all $(\theta, p) \in \Lambda$ *one has the identity*

$$(X, \mathcal{D}(A^\alpha))_{\theta,p} = \left\{ x \in X \;\middle|\; t^{-\theta \operatorname{Re}\alpha} \psi(tA)x \in \mathbf{L}^{\boldsymbol{p}}_*((0,\infty); X) \right\} \tag{6.4}$$

with the equivalence of norms

$$\|x\|_{(X,\mathcal{D}(A^\alpha))_{\theta,p}} \quad \sim \quad \|x\|_X + \left\| t^{-\theta \operatorname{Re}\alpha} \psi(tA)x \right\|_{\mathbf{L}^{\boldsymbol{p}}_*((0,\infty);X)}.$$

If one is willing to do without the extremal case $\theta = 1, p = \infty$, the assumptions on ψ can be weakened, cf. Theorem 6.2.9.

Remarks 6.2.2. 1) By Proposition 2.6.11, the function $t \longmapsto \psi(tA)$ is bounded, whence $t^{-\theta \operatorname{Re}\alpha} \psi(tA) \in \mathbf{L}^{\boldsymbol{p}}_*((1,\infty); \mathcal{L}(X))$ for all $(\theta, p) \in \Lambda$. Hence one may replace (6.4) by

$$(X, \mathcal{D}(A^\alpha))_{\theta,p} = \left\{ x \in X \mid t^{-\theta \operatorname{Re}\alpha} \psi(tA)x \in \mathbf{L}^{\boldsymbol{p}}_*((0,1); X) \right\} \tag{6.5}$$

with the equivalence of norms

$$\|x\|_{(X,\mathcal{D}(A^\alpha))_{\theta,p}} \quad \sim \quad \|x\|_X + \left\| t^{-\theta \operatorname{Re}\alpha} \psi(tA)x \right\|_{\mathbf{L}^{\boldsymbol{p}}_*((0,1);X)}$$

for each $(\theta, p) \in \Lambda$.

2) If A is invertible, one may discard the $\|\cdot\|_X$-part and has the simpler norm equivalence

$$\|x\|_{(X,\mathcal{D}(A^\alpha))_{\theta,p}} \quad \sim \quad \left\| t^{-\theta \operatorname{Re}\alpha} \psi(tA)x \right\|_{\mathbf{L}^{\boldsymbol{p}}_*((0,\infty);X)}.$$

This is proved in Section 6.5.2, see Corollary 6.5.5.

Before we give the proof of Theorem 6.2.1, we formulate an important special case.

Corollary 6.2.3. *Let* $A \in \operatorname{Sect}(\omega)$, $\varphi \in (\omega, \pi)$, *and* $\operatorname{Re}\alpha > 0$. *Let* $\psi \in \mathcal{O}(S_\varphi)$ *be a function satisfying the following conditions:*

a) $\psi, \psi z^{-\alpha} \in \mathcal{E}(S_\varphi)$;

b) $\lim_{z\to 0} \psi(z)z^{-\alpha} \neq 0$;

c) $\psi(z) \neq 0$ *for all* $z \in S_\varphi$;

d) $\psi(\infty) \neq 0$.

Then the conclusion of Theorem 6.2.1 *holds.*

Proof. One only has to show that also (6.3) is satisfied, at least on a slightly smaller sector. Choose $\varphi' \in (\omega, \varphi)$. Then for $s \geq 1$,

$$\left|\frac{\psi(sz)}{s^\alpha \psi(z)}\right| = \left|\frac{\psi(sz)(sz)^{-\alpha}}{\psi(z)z^{-\alpha}}\right| \leq \left\|\psi z^{-\alpha}\right\|_\varphi \left|\frac{1}{\psi(z)z^{-\alpha}}\right|$$

and

$$\left|\frac{\psi(sz)}{s^\alpha \psi(z)}\right| \leq \|\psi\|_\varphi \left|\frac{1}{\psi(z)}\right|.$$

The first is bounded for $z \in S_{\varphi'}, |z| \leq 1$, the second for $z \in S_{\varphi'}, |z| \geq 1$. □

Let us now turn to the proof of Theorem 6.2.1. We begin by noticing that one of the desired inclusions is easy and in fact requires only condition a).

Lemma 6.2.4. *Let $A \in \operatorname{Sect}(\omega)$, $\varphi \in (\omega, \pi)$, and $\operatorname{Re}\alpha > 0$. Let $\psi \in \mathcal{E}(S_\varphi)$ such that $\gamma := \psi z^{-\alpha} \in \mathcal{E}(S_\varphi)$. Then*

$$\|\psi(tA)x\| \leq C\, K(t^{\operatorname{Re}\alpha}, x, X, \mathcal{D}(A^\alpha)) \qquad (x \in X, t > 0),$$

where $C = \max(C_\psi, C_\gamma)$, cf. Proposition 2.6.11. Consequently, there is a continuous inclusion

$$(X, \mathcal{D}(A^\alpha))_{\theta,p} \subset \left\{x \in X \;\middle|\; t^{-\theta \operatorname{Re}\alpha}\psi(tA)x \in \mathbf{L}^{p}_{*}((0,\infty); X)\right\}$$

for each $(\theta, p) \in \Lambda$.

Proof. Let $x = a + b$, where $a \in X$ and $b \in \mathcal{D}(A^\alpha)$. Then one has $\psi(tA)x = \psi(tA)a + t^\alpha \gamma(tA)A^\alpha b$, and this yields

$$\|\psi(tA)x\| \leq C_\psi \|a\|_X + t^{\operatorname{Re}\alpha} C_\gamma \|A^\alpha b\|_X \leq C\big(\|a\|_X + t^{\operatorname{Re}\alpha}\|b\|_{\mathcal{D}(A^\alpha)}\big).$$

Taking the infimum yields the first statement. The second follows readily. □

The other inclusion is not as easy to establish. In fact, we need several auxiliary results.

Lemma 6.2.5. *Let $0 \neq \psi \in \mathcal{E}(S_\varphi)$ and $\operatorname{Re}\alpha > 0$. Then there is $f \in H_0^\infty(S_\varphi)$ such that*

$$\int_0^\infty (f\psi)(sz)\,\frac{ds}{s} = 1 \qquad (z \in S_\varphi)$$

and $z^\alpha f \in H_0^\infty(S_\varphi)$.

Proof. Let $\overline{\psi}(z) := \overline{\psi(\overline{z})}$ and $\tau(z) := z/(1+z)^2$. Then $\psi(t)\overline{\psi}(t) = |\psi(t)|^2$ for all $t > 0$. Choose $m > \operatorname{Re}\alpha$. Since $0 \neq \psi$,

$$c := \int_0^\infty \tau(s)^m \psi(s)\overline{\psi}(s)\,\frac{ds}{s} > 0.$$

Then $f := c^{-1}\tau^m\overline{\psi}$ is a possible choice. □

The next lemma states the classical Hardy–Young inequality.

Lemma 6.2.6 (Hardy–Young Inequality). *Let* $\sigma > 0$, $p \in [1, \infty]$, *and* $f : (0, \infty) \longrightarrow [0, \infty)$.

a) *If* $t^{-\sigma} f \in \mathbf{L}^{\boldsymbol{p}}_{*}(0, \infty)$ *then* $f \in \mathbf{L}^{\mathbf{1}}_{*}(0, T)$ *for every* $T \in (0, \infty)$ *and with*

$$g(t) := \int_0^t f(s) \,\frac{ds}{s}$$

one has $t^{-\sigma} g \in \mathbf{L}^{\boldsymbol{p}}_{*}(0, \infty)$ *and* $\|t^{-\sigma} g\|_{\mathbf{L}^{\boldsymbol{p}}_{*}(0,\infty)} \le (1/\sigma) \, \|t^{-\sigma} f\|_{\mathbf{L}^{\boldsymbol{p}}_{*}(0,\infty)}$.

b) *If* $t^{\sigma} f \in \mathbf{L}^{\boldsymbol{p}}_{*}(0, \infty)$, *then* $f \in \mathbf{L}^{\mathbf{1}}_{*}(T, \infty)$ *for every* $T \in (0, \infty)$ *and with*

$$g(t) := \int_t^\infty f(s) \,\frac{ds}{s}$$

one has $t^{\sigma} g \in \mathbf{L}^{\boldsymbol{p}}_{*}(0, \infty)$ *and* $\|t^{\sigma} g\|_{\mathbf{L}^{\boldsymbol{p}}_{*}(0,\infty)} \le (1/\sigma) \, \|t^{\sigma} f\|_{\mathbf{L}^{\boldsymbol{p}}_{*}(0,\infty)}$.

Note that the constant appearing in the norm inequality is independent of the length of the interval $(0, T)$ and of the parameter p.

Proof. The second assertion follows from the first by a change of parameter $t \mapsto t^{-1}$. The first assertion can be proved easily by Riesz–Thorin interpolation since the case $p = \infty$ is trivial and the case $p = 1$ is just the Fubini theorem. (Only positive operators are involved here, so interpolation is elementary, cf. [104].) Of course one may also look into a book, e.g., in [115, p.245-246]. $\square$

Returning to the proof of Theorem 6.2.1 we look for a possibility to write the constant 1 as a sum of appropriate functions in $\mathcal{E}$. This will be accomplished by the next lemma.

Lemma 6.2.7. *Let* $A \in \mathrm{Sect}(\omega)$, $\varphi \in (\omega, \pi)$, *and* $\operatorname{Re} \alpha > 0$, *and let* $\psi \in \mathcal{E}(S_\varphi)$ *satisfy all the hypotheses of Theorem* 6.2.1. *Then there are functions* $f \in H_0^\infty(S_\varphi)$ *and* $g \in \mathcal{E}(S_\varphi)$ *such that*

$$\int_0^1 (f\psi)(sz) \,\frac{ds}{s} \;+\; g(z)\psi(z)z^{-\alpha} \;=\; 1 \qquad (z \in S_\varphi). \tag{6.6}$$

Proof. We apply Lemma 6.2.5 to find a function $f \in H_0^\infty$ such that $\tilde{f} := z^\alpha f \in H_0^\infty$ and $\int_0^\infty (f\psi)(sz) \, ds/s = 1$ for all $z \in S_\varphi$. Let $h(z) := \int_1^\infty (f\psi)(sz) \, ds/s$. From Example 2.2.6 it follows that $h \in \mathcal{E}$ and $h(0) = 1$. Define $g(z) := h(z)/\psi(z)z^{-\alpha}$. Since by assumption $\psi(z)z^{-\alpha}\big|_{z=0} \neq 0$, g is 'good' at 0. To see that g is also 'good' at ∞, we use condition (6.3) and write

$$|g(z)| = \left| \frac{h(z)}{\psi(z)z^{-\alpha}} \right| = \left| \int_1^\infty (sz)^\alpha f(sz) \frac{\psi(sz)}{s^\alpha \psi(z)} \,\frac{ds}{s} \right| \le C \int_1^\infty \left| \tilde{f}(sz) \right| \,\frac{ds}{s}$$

the latter being 'good' at ∞, cf. Example 2.2.6. $\square$

Now we take the last step.

Proposition 6.2.8. *Let $A \in \mathrm{Sect}(\omega)$, $\varphi \in (\omega, \pi)$, and $\operatorname{Re}\alpha > 0$, and let $\psi \in \mathcal{O}(S_\varphi)$ such that $\psi, \psi z^{-\alpha} \in \mathcal{E}(S_\varphi)$. Suppose that there exist functions $f, g \in \mathcal{E}(S_\varphi)$ with $(f\psi) \in H_0^\infty(S_\varphi)$ and*

$$\int_0^1 (f\psi)(sz)\,\frac{ds}{s} + g(z)\psi(z)z^{-\alpha} = 1 \qquad (z \in S_\varphi).$$

Then the conclusion of Theorem 6.2.1 holds.

Proof. As already noted, one inclusion is clear from Lemma 6.2.4. Let

$$h_1(z) := \int_0^1 (f\psi)(sz)\,\frac{ds}{s} \quad \text{and} \quad h_2(z) := g(z)\psi(z)z^{-\alpha}.$$

Clearly, $h_1, h_2 \in \mathcal{E}(S_\varphi)$, cf. Example 2.2.6. Moreover, for every $x \in X, t > 0$ we have $h_2(tA)x \in \mathcal{D}(A^\alpha)$ with

$$A^\alpha h_2(tA)x = t^{-\alpha} g(tA)\psi(tA)x,$$

hence $\|h_2(tA)x\|_{\mathcal{D}(A^\alpha)} \le t^{-\operatorname{Re}\alpha} C_g \|\psi(tA)x\| + C_{h_2}\|x\|$. (See Proposition 2.6.11 for the meaning of C_g, C_{h_2}.) This yields

$$K(t^{\operatorname{Re}\alpha}, x, X, \mathcal{D}(A^\alpha)) \le \|h_1(tA)x\| + C_g\|\psi(tA)x\| + t^{\operatorname{Re}\alpha} C_{h_2}\|x\|.$$

Since also $K(t^{\operatorname{Re}\alpha}, x, X, \mathcal{D}(A^\alpha)) \le \|x\|$, we can enlarge the constants to obtain

$$K(t^{\operatorname{Re}\alpha}, x, X, \mathcal{D}(A^\alpha)) \le C\left(\|h_1(tA)x\| + \|\psi(tA)x\| + \min\left(t^{\operatorname{Re}\alpha}, 1\right)\|x\|\right). \quad (6.7)$$

Now fix $(\theta, p) \in \Lambda, \theta \neq 0$ and $x \in X$. Suppose that $t^{-\theta \operatorname{Re}\alpha}\|\psi(tA)x\| \in \mathbf{L}^{\boldsymbol{p}}_*(0,\infty)$. Then also $t^{-\theta\operatorname{Re}\alpha}\|(f\psi)(tA)x\| \in \mathbf{L}^{\boldsymbol{p}}_*(0,\infty)$ since $\|(f\psi)(tA)x\| \le C_f\|\psi(tA)x\|$. By Hölder's inequality and $\theta \operatorname{Re}\alpha > 0$, one has

$$[s \mapsto (f\psi)(stA)x] \in \mathbf{L}^{\mathbf{1}}_*((0,1);X)$$

for each $t > 0$. An application of Proposition 5.2.4 a) yields

$$\|h_1(tA)x\| = \left\|\int_0^1 (f\psi)(stA)x\,\frac{ds}{s}\right\| \le \int_0^t \|(f\psi)(sA)x\|\,\frac{ds}{s} =: \sigma(t).$$

Employing the Hardy–Young inequality (Lemma 6.2.6), we obtain $t^{-\theta\operatorname{Re}\alpha}\sigma(t) \in \mathbf{L}^{\boldsymbol{p}}_*(0,\infty)$ with

$$\left\|t^{-\theta\operatorname{Re}\alpha}\sigma(t)\right\|_{\mathbf{L}^{\boldsymbol{p}}_*} \le \frac{1}{\theta\operatorname{Re}\alpha} C_f \left\|t^{-\theta\operatorname{Re}\alpha}\psi(tA)x\right\|_{\mathbf{L}^{\boldsymbol{p}}_*(0,\infty)}.$$

From (6.7) we can now infer that

$$t^{-\theta\operatorname{Re}\alpha} K(t^{\operatorname{Re}\alpha}, x, X, \mathcal{D}(A^\alpha)) \in \mathbf{L}^{\boldsymbol{p}}_*(0,\infty)$$

and estimate the $\mathbf{L}^{\boldsymbol{p}}_*$-norm of this function in terms of $\|x\|$ and $\left\|t^{-\theta\operatorname{Re}\alpha}\psi(tA)x\right\|_{\mathbf{L}^{\boldsymbol{p}}_*}$. □

Combining Lemma 6.2.7 and Proposition 6.2.8 completes the proof of Theorem 6.2.1.

6.2.2 A Second Characterisation

In this section we formulate and prove a result which is more general than Theorem 6.2.1, at least when the interpolation parameter θ is restricted to the *open* interval $(0,1)$.

Theorem 6.2.9. *Let $A \in \operatorname{Sect}(\omega)$, $\varphi \in (\omega, \pi)$ and $\operatorname{Re}\alpha > 0$. Take any function $0 \neq \psi \in \mathcal{O}(S_\varphi)$ such that $\psi, \psi z^{-\alpha} \in \mathcal{E}(S_\varphi)$. Then one has*

$$(X, \mathcal{D}(A^\alpha))_{\theta,p} = \left\{ x \in X \mid t^{-\theta \operatorname{Re}\alpha} \psi(tA)x \in \mathbf{L}^{\boldsymbol{p}}_*((0,\infty);X) \right\}$$

with the equivalence of norms

$$\|x\|_{(X,\mathcal{D}(A^\alpha))_{\theta,p}} \quad \sim \quad \|x\|_X + \left\| t^{-\theta \operatorname{Re}\alpha} \psi(tA)x \right\|_{\mathbf{L}^{\boldsymbol{p}}_*((0,\infty);X)}$$

for all $\theta \in (0,1), p \in [1,\infty]$.

(We shall give an even more general result for the case of an injective operator A in Section 6.5.1 below.) As before, it follows from Lemma 6.2.4 that we only have to take care of the inclusion

$$\left\{ x \;\middle|\; t^{-\theta \operatorname{Re}\alpha} \|\psi(tA)x\| \in \mathbf{L}^{\boldsymbol{p}}_*(0,\infty) \right\} \subset (X, \mathcal{D}(A^\alpha))_{\theta,p}.$$

The main ingredients of the proof are already known. There are just some further technicalities.

Lemma 6.2.10. *Let $A \in \operatorname{Sect}(\omega), \varphi \in (\omega, \pi)$, $f \in H^\infty_0(S_\varphi)$ and $\operatorname{Re}\alpha > 0$. Define*

$$h(z) := \int_1^\infty s^{-\alpha} f(sz) \frac{ds}{s}.$$

Then $h \in H^\infty_0(S_\varphi)$ and $h(A) = \displaystyle\int_1^\infty s^{-\alpha} f(sA) \frac{ds}{s}$.

Proof. If one chooses $0 < \varepsilon < \operatorname{Re}\alpha$ small enough such that $z^{\pm\varepsilon} f \in H^\infty$, one also has $z^{\pm\varepsilon} h \in H^\infty$. This shows that $h \in H^\infty_0$. The rest is simply Fubini's theorem and the definition the H^∞_0-functional calculus by the Cauchy integral. Note that the function $s \longmapsto s^{-\alpha} f(sA)$ is in $\mathbf{L}^{\boldsymbol{1}}_*((1,\infty);\mathcal{L}(X))$. □

We return to our main objective.

Proof of Theorem 6.2.9. As in the proof of Theorem 6.2.1 we try to write the constant function $\mathbf{1}$ as a sum of appropriate functions. Hence we start as before and choose a function $f \in H^\infty_0(S_\varphi)$ such that $\tilde{f} := z^\alpha f \in H^\infty_0(S_\varphi)$ and

$$\int_0^\infty (f\psi)(s) \frac{ds}{s} = 1$$

(Lemma 6.2.5). Then we define

$$h_1(z) := \int_0^1 (f\psi)(sz)\,\frac{ds}{s} \quad \text{and} \quad h_2(z) := \int_1^\infty (f\psi)(sz)\,\frac{ds}{s}.$$

By Example 2.2.6, $h_1, h_2 \in \mathcal{E}(S_\varphi)$ and $h_1 + h_2 = 1$. Moreover,

$$z^\alpha h_2(z) = \int_1^\infty z^\alpha f(sz)\psi(sz)\,\frac{ds}{s} = \int_1^\infty s^{-\alpha}(\tilde{f}\psi)(sz)\,\frac{ds}{s}$$

is — by Lemma 6.2.10 — a function in $H^\infty(S_\varphi)$. Moreover, Lemma 6.2.10 yields

$$\begin{aligned} A^\alpha h_2(tA) &= t^{-\alpha}(z^\alpha h_2)(tA) = t^{-\alpha}\int_1^\infty s^{-\alpha}(\tilde{f}\psi)(sz)\,\frac{ds}{s}\,(tA) \\ &= \int_1^\infty (st)^{-\alpha}(\tilde{f}\psi)(stA)\,\frac{ds}{s} = \int_t^\infty s^{-\alpha}(\tilde{f}\psi)(sA)\,\frac{ds}{s} \end{aligned}$$

for all $t > 0$. In particular, $h_2(tA)$ maps X into $\mathcal{D}(A^\alpha)$.

Now fix $\theta \in (0,1), p \in [1,\infty]$, and $x \in X$ such that $t^{-\theta \operatorname{Re}\alpha}\|\psi(tA)x\| \in \mathbf{L}^{\boldsymbol{p}}_*(0,\infty)$. By the above, $x = h_1(tA)x + h_2(tA)x$ and $h_2(tA)x \in \mathcal{D}(A^\alpha)$, hence

$$K(t^{\operatorname{Re}\alpha}, x, X, \mathcal{D}(A^\alpha)) \le \|h_1(tA)x\| + t^{\operatorname{Re}\alpha}\|h_2(tA)x\| + t^{\operatorname{Re}\alpha}\|A^\alpha h_2(tA)x\|$$

for $t > 0$. In the middle term we estimate $\|h_2(tA)x\| \le C_{h_2}\|x\|$ and — enlarging the constant — arrive at the inequality

$$\begin{aligned} &K(t^{\operatorname{Re}\alpha}, x, X, \mathcal{D}(A^\alpha)) \le \\ &\qquad C\left[\ \|h_1(tA)x\| + \min\left(t^{\operatorname{Re}\alpha}, 1\right)\ \|x\|\ +\ t^{\operatorname{Re}\alpha}\int_t^\infty s^{-\operatorname{Re}\alpha}\,\|\psi(sA)x\|\ \frac{ds}{s}\right]. \end{aligned}$$

Since we would like to have $t^{-\theta\operatorname{Re}\alpha}K(t^{\operatorname{Re}\alpha}, x, X, \mathcal{D}(A^\alpha)) \in \mathbf{L}^{\boldsymbol{p}}_*(0,\infty)$, the middle term is obviously good. The first term is dealt with *exactly* as in the proof of Proposition 6.2.8. For the third we define $g(s) := s^{-\operatorname{Re}\alpha}\|\psi(sA)x\|$ and observe that

$$s^{(1-\theta)\operatorname{Re}\alpha}g(s) = s^{\theta\operatorname{Re}\alpha}\|\psi(sA)x\| \in \mathbf{L}^{\boldsymbol{p}}_*(0,\infty)$$

by assumption. Hence we can apply part b) of the Hardy–Young inequality (Lemma 6.2.6), and we are done. □

6.2.3 Examples

Theorem 6.2.9 allows us to obtain various concrete descriptions of the real interpolation spaces.

The 'Komatsu Spaces'

Let A be a sectorial operator on the Banach space X and let $\operatorname{Re}\alpha > 0$. We consider the function

$$\psi(z) := \frac{z^\alpha}{(1+z)^\alpha},$$

which, by Example 2.2.5, satisfies the hypotheses of Corollary 6.2.3. Applying that result yields, after a change of variable $t \mapsto t^{-1}$, the description

$$(X, \mathcal{D}(A^\alpha))_{\theta,p} = \Big\{ x \in X \;\Big|\; t^{\theta \operatorname{Re}\alpha} [A(t+A)^{-1}]^\alpha x \in \mathbf{L}^{\boldsymbol{p}}_*(0,\infty) \Big\} \tag{6.8}$$

for all $(\theta, p) \in \Lambda$. If $\theta \in (0,1)$, this description is connected with the name of Komatsu, see [132], [136] and cf. also [161, Chapter 11]. The case where $\alpha = 1$ is the description

$$(X, \mathcal{D}(A))_{\theta,p} = \Big\{ x \in X \;\Big|\; t^{\theta} [A(t+A)^{-1}] x \in \mathbf{L}^{\boldsymbol{p}}_*(0,\infty) \Big\}$$

that we encountered in the beginning, cf. Lemma 6.1.1.

Description by Holomorphic Semigroups

Let A be a sectorial operator on the Banach space X with $\omega_A < \pi/2$. Choose $\varphi \in (\omega_A, \pi/2)$. The function $f(z) := e^{-z}$ belongs to $\mathcal{E}(S_\varphi)$ and $f(tA) = e^{-tA}$ is the bounded holomorphic semigroup generated by $-A$, see Section 3.4. Let $\operatorname{Re}\alpha > 0$. The function

$$\psi(z) := z^\alpha e^{-z}$$

does not satisfy the conditions of Corollary 6.2.3. However, it satisfies the more general condition (6.3) since

$$\frac{\psi(sz)}{s^\alpha \psi(z)} = \frac{e^{-sz}}{e^{-z}} = e^{-(s-1)z},$$

and this is uniformly bounded for $z \in S_\varphi$, $s \geq 1$. Hence Theorem 6.2.1 applies and yields the description

$$(X, \mathcal{D}(A^\alpha))_{\theta,p} = \Big\{ x \in X \;\Big|\; t^{(1-\theta)\operatorname{Re}\alpha} A^\alpha e^{-tA} x \in \mathbf{L}^{\boldsymbol{p}}_*(0,\infty) \Big\} \tag{6.9}$$

for all $(\theta, p) \in \Lambda$.

Another Characterisation

Let again A be a sectorial operator on the Banach space X and consider the function

$$\psi(z) := \frac{z^\alpha}{(1+z)^\beta}$$

where $0 < \operatorname{Re}\alpha < \operatorname{Re}\beta$. Then ψ satisfies the conditions of Theorem 6.2.9 (but not those of Theorem 6.2.1). Hence after a change of variables we obtain

$$(X, \mathcal{D}(A^\alpha))_{\theta,p} = \left\{ x \in X \;\middle|\; t^{\operatorname{Re}\beta-(1-\theta)\operatorname{Re}\alpha}[A^\alpha(t+A)^{-\beta}]x \in \mathbf{L}^{\boldsymbol{p}}_*(0,\infty) \right\}$$

for all $(\theta, p) \in \Lambda$. Specialising, e.g., $\alpha = 1, \beta = 2$ yields

$$(X, \mathcal{D}(A))_{\theta,p} = \left\{ x \in X \;\middle|\; t^{1+\theta}A(t+A)^{-2}x \in \mathbf{L}^{\boldsymbol{p}}_*(0,\infty) \right\}.$$

6.3 Extrapolation Spaces

From now on we only consider *injective* sectorial operators. We shall see in Theorem 6.5.3 that they allow a much more flexible description of interpolation spaces and we shall shed new light on Dore's theorem (Section 6.5.3). In order to do this we need what is called *extrapolation spaces*.

6.3.1 An Abstract Method

Assume that one is given a Banach space X and a bounded, linear, and injective operator $T \in \mathcal{L}(X)$. The space $\mathcal{R}(T)$ is then a Banach space endowed with the norm $\left\|T^{-1}x\right\|_X$, and $T : X \longrightarrow \mathcal{R}(T)$ is an isometric isomorphism. Now we consider the commuting diagram

$$\begin{tikzcd} X \arrow[r, "id"] & X \arrow[d, no head] \\ X \arrow[u, "T"] \arrow[r, "T"] & \mathcal{R}(T) \end{tikzcd}$$

and rename some of its components:

$$\begin{tikzcd} X_{-1} \arrow[r, "T_{-1}"] & X \arrow[d, no head] \\ X \arrow[u, "\iota"] \arrow[r, "T"] & \mathcal{R}(T) \end{tikzcd}$$

The map ι is bounded and injective, i.e., it is an *embedding*, and after some set-theoretical work we may view X_{-1} as a proper superspace of X. We arrive at the commuting diagram

$$\begin{tikzcd} X_{-1} \arrow[r, "T_{-1}"] \arrow[d, dash] & X \arrow[d, dash] \\ X \arrow[r, "T"] \arrow[d, dash] & \mathcal{R}(T) \arrow[d, dash] \\ \mathcal{R}(T) \arrow[r, "T"] & \mathcal{R}(T^2) \end{tikzcd}$$

which shows that $T_{-1}\big|_X = T$. Hence no confusion can arise in renaming T_{-1} by T. When we define $X_1 := \mathcal{R}(T), X_2 := \mathcal{R}(T^2) \ldots$ we obtain

$$\begin{tikzcd} X_{-1} \arrow[r, "T"] \arrow[d, dash] & X \arrow[d, dash] \\ X \arrow[r, "T"] \arrow[d, dash] & X_1 \arrow[d, dash] \\ X_1 \arrow[r, "T"] & X_2 \end{tikzcd}$$

with T being an isometric isomorphism at every stage. Iterating this procedure yields an upwards directed series of new spaces $(X_{-n})_{n\in\mathbb{N}}$, i.e.,

$$X = X_0 \hookrightarrow X_{-1} \hookrightarrow X_{-2} \hookrightarrow \cdots \qquad \hookrightarrow X_{-n} \hookrightarrow \cdots$$

where as before we may always view the embeddings as proper inclusions. Moreover, we have (compatible) isometric isomorphisms $T : X_{-n} \longrightarrow X_{-n-1}$.

As a last extension we finally construct the *algebraic inductive limit* of the directed family $(X_{-n})_{n\in\mathbb{N}}$,

$$U := X_{-\infty} := \varinjlim{}_{n\in\mathbb{N}}\, X_{-n} = \bigcup_{n\in\mathbb{N}} X_{-n}.$$

The space U may be called the **universal extrapolation space** corresponding to T. On U we define the following notion of convergence. Let $(x_\alpha)_{\alpha\in\mathsf{P}} \subset U$ be a net and $x \in U$. Then

$$x_\alpha \to x \text{ in } U \quad :\Longleftrightarrow$$
$$\exists n \in \mathbb{N}, \alpha_0 \in \mathsf{P} : \quad x, x_\alpha \in X_{-n}\ (\alpha \geq \alpha_0) \quad \text{and} \quad \|x_\alpha - x\|_{X_{-n}} \to 0.$$

(This does not give a proper topology on U but is well adapted to our purposes, cf. the comments in Section 6.8 below.) One easily sees that the limit of a net in U is unique, and that sum and scalar multiplication are 'continuous' with respect to the so-defined notion of convergence. Finally, since the operator T is defined on each space X_{-n}, $n \in \mathbb{N}$, it is a fortiori defined on the whole of U. The so-defined mapping $T : U \longrightarrow U$ is obviously surjective, whence it is an algebraic isomorphism, continuous with respect to the notion of convergence defined above.

6.3.2 Extrapolation for Injective Sectorial Operators

We now apply this abstract procedure to the case of an *injective* sectorial operator A on X. The operator T — which is constitutive for the extrapolation method — is defined by

$$T := A(1+A)^{-2} = (1+A)^{-1}(1+A^{-1})^{-1}.$$

Then T is injective since A is. Moreover, $\mathcal{R}(T) = \mathcal{D}(A) \cap \mathcal{R}(A)$. By the abstract method described above, we obtain a sequence of nested spaces

$$X = X_0 \subset X_{-1} \subset X_{-2} \subset \cdots \subset X_{-n} \subset \ldots U$$

with U being the 'universal' space. Recall that T extends to a 'topological' isomorphism on U.

Let us now extend the operator A to the whole of U. Using the isometric isomorphism $T : X_{-1} \longrightarrow X$ one can just transfer the operator A to X_{-1} by defining

$$A_{-1} := T^{-1}AT \quad \text{with} \quad \mathcal{D}(A_{-1}) := T^{-1}\mathcal{D}(A).$$

By construction, A_{-1} is an injective sectorial operator on X_{-1}, isometrically similar to A. Clearly, $X \subset \mathcal{D}(A_{-1})$. Moreover, A is the part of A_{-1} in X, i.e.,

$$\mathcal{D}(A) = \{x \in X \mid A_{-1}x \in X\} \quad \text{and} \quad A_{-1}x = Ax \quad (x \in \mathcal{D}(A)).$$

(This is due to the fact that the operator T — considered as an operator on X — commutes with A.) The inverse of A_{-1} is given by $[A_{-1}]^{-1} = T^{-1}A^{-1}T$ with the appropriate domain $\mathcal{D}(A_{-1}^{-1}) = \mathcal{R}(A_{-1}) = T^{-1}\mathcal{R}(A)$.

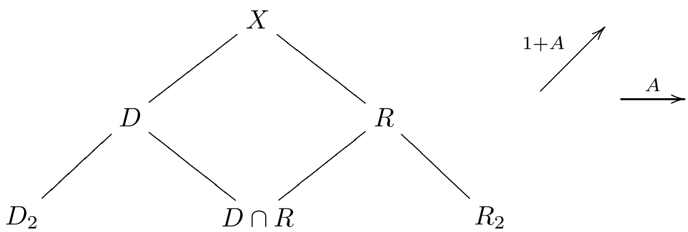
Iterating this procedure we obtain a sequence of isometrically similar sectorial operators A_{-n} on X_{-n}, with each A_{-n} being the part of $A_{-(n+1)}$ in X_{-n}. Since $X_{-n} \subset \mathcal{D}(A_{-(n+1)})$, we obtain an extension of A to the whole of U. This extension, by abuse of notation, is again denoted by A. It has the pleasant feature that it is invertible, i.e., $A : U \longrightarrow U$ is an isomorphism, continuous with respect to the notion of convergence we introduced above.

Within the space U we can define an array of spaces as follows. Already within X we have the following natural spaces:

- $D := D_1 := \mathcal{D}(A)$ with norm $\|x\|_D := \|(1+A)x\|$.
- $R := R_1 := \mathcal{R}(A)$ with norm $\|x\|_R := \left\|(1+A^{-1})x\right\|$.
- $D \cap R := \mathcal{D}(A) \cap \mathcal{R}(A)$ with the norm
$$\|x\|_{D\cap R} = \left\|(2+A+A^{-1})x\right\| = \left\|T^{-1}x\right\|.$$

- $D_2 := \mathcal{D}(A^2)$ with the norm $\|x\|_{D_2} = \left\|(1+A)^2 x\right\|$.
- $R_2 := \mathcal{R}(A^2) = \mathcal{D}(A^{-2})$ with the norm $\|x\|_{R_2} = \left\|(1+A^{-1})^2 x\right\|$.

These can be arranged into a diagram.



Here a downward meeting of two lines means intersection and an upward meeting of two lines means sum of the spaces. E.g., $X = D + R$ and $D = D_2 + (D \cap R)$. The operator $A + 1$ acts as an isometric isomorphism in the $\nearrow$-direction, i.e. $D_2 \xrightarrow{A+1} D \xrightarrow{A+1} X$ or $D \cap R \xrightarrow{A+1} R$. (One may replace $A + 1$ by $A + \lambda$ for each $\lambda > 0$, but then the isomorphisms cease to be isometric.) Furthermore, the operator A acts as an isometric isomorphism in the $\rightarrow$-direction, e.g., $D \xrightarrow{A} R$ or $D_2 \xrightarrow{A} D \cap R$.

Via the isometric ismorphism $T : X_{-1} \longrightarrow X$ this diagram can be transported to X_{-1}. That is, we form the spaces

- $D_{-1} := T^{-1}(D)$ with the norm $\|u\|_{D_{-1}} = \|Tu\|_D$ and
- $R_{-1} := T^{-1}(R)$ with the norm $\|u\|_{R_{-1}} = \|Tu\|_R$

and obtain a situation as shown in Figure 9.

Note that we already know $D_{-1} = \mathcal{D}(A_{-1}), R_{-1} = \mathcal{R}(A_{-1})$. By applying T^{-1} again and again, we generate the spaces $D_{-2}, D_{-3}, \dots$ and $R_{-2}, R_{-3}, \dots$. Since A is an isomorphism on U, we define

$$X^{(n)} := A^{-n}(X) \quad \text{with norm} \quad \|x\|_{X^{(n)}} := \left\|A^{-n}x\right\|_X .$$

We frequently write

$$D^{(n)} := X^{(n)} \quad \text{and} \quad R^{(n)} := X^{(-n)}$$

for $n \in \mathbb{N}$, and $\dot{D} := D^{(1)}$ and $\dot{R} := R^{(1)}$ in the case $n = 1$. These spaces are called the **homogeneous spaces** associated with the injective sectorial operator A. Figure 10 illustrates the situation. As before, T, A, $(A+1)$ and $(A^{-1}+1)$ act as isometric isomorphisms in the directions $\downarrow, \rightarrow, \nearrow, \nwarrow$, respectively. If $\mathcal{D}(A)$ is dense in X, then all inclusions in the $\nearrow$-direction are dense. If $\mathcal{R}(A)$ is dense then all inclusions in the $\nwarrow$-direction are dense.

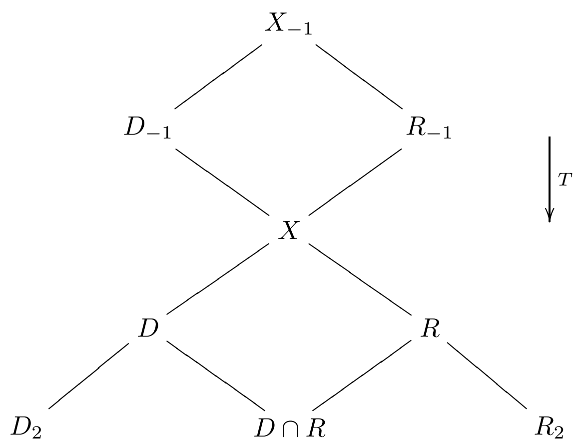


Figure 9: The first extrapolation space.

6.3.3 The Homogeneous Fractional Domain Spaces

Having constructed the universal space U we look at the behaviour of the functional calculus. Let $A \in \operatorname{Sect}(\omega)$ and $\varphi \in (\omega, \pi]$. Recall the definition of the function algebra

$$\mathcal{B}(S_\varphi) := \left\{ f \in \mathcal{O}(S_\varphi) \;\middle|\; \exists C, s > 0 : \; |f(z)| \leq C \max(|z|^s, |z|^{-s}) \right\}.$$

Since A is injective, $\mathcal{B}(S_\varphi) \subset \mathcal{M}(S_\varphi)_A$ in the notation of Chapter 2, and in fact $f(A)$ is defined for $f \in \mathcal{B}(S_\varphi)$ in each of the spaces X_{-n}. These definitions are consistent, i.e.,

$$f(A_{-(n+1)})\big|_{X_{-n}} = f(A_{-n}) \qquad (n \in \mathbb{N}).$$

By definition of the class $\mathcal{B}$, for each $f \in \mathcal{B}(S_\varphi)$ there is $m \in \mathbb{N}$ such that $\tau^m f \in \mathcal{E}(S_\varphi)$, where $\tau(z) = z/(1+z)^2$. This implies that actually

$$f(A) : X_{-n} \to X_{-(n+m)}$$

is bounded for each $n \in \mathbb{N}$. To sum up, one may say that $f(A)$ is actually a 'continuous' fully-defined operator on the whole of U. Thus, with a slight abuse of notation, we have turned our unbounded functional calculus into a proper algebra homomorphism

$$(f \longmapsto f(A)) : \mathcal{B}(S_\varphi) \longrightarrow \mathcal{L}(U).$$

The following lemma is important and straightforward to prove.

Lemma 6.3.1. *Let $f \in \mathcal{B}(S_\varphi)$. Then*

$$\mathcal{D}(f(A)) = \{x \in X \mid f(A)x \in X\},$$

i.e., the operator $f(A)$ considered as an operator in X is the part in X of $f(A)$ considered as an operator on U.

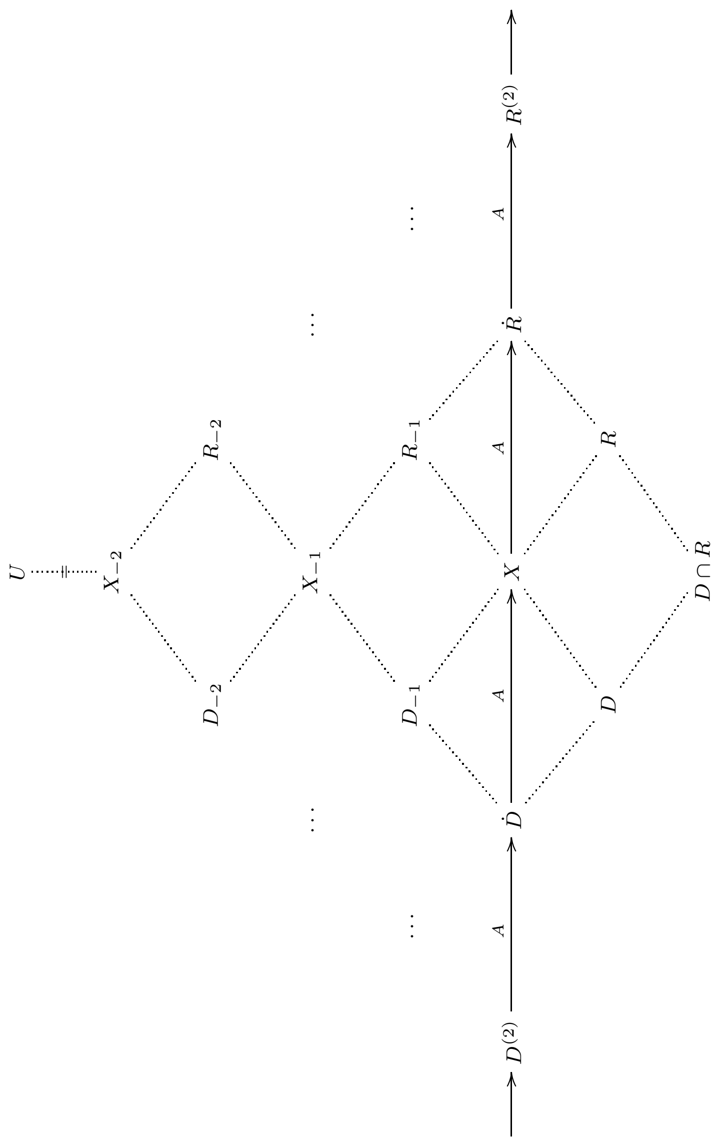


Figure 10: An array of extrapolation spaces.

Now note that the function z^α is contained in $\mathcal{B}$ for each $\alpha \in \mathbb{C}$. The resulting operators A^α are isomorphisms on U, and in fact

$$(\alpha \longmapsto A^\alpha) : \mathbb{C} \longrightarrow \mathcal{L}(U)$$

is a representation of the abelian group $(\mathbb{C}, +)$ on U. Consequently, we can generalise the definition of the homogeneous spaces:

$$X^{(\alpha)} := A^{-\alpha}(X) \quad \text{with norm} \quad \|x\|_{X^{(\alpha)}} = \|A^\alpha x\|_X \qquad (\alpha \in \mathbb{C}).$$

These are the **homogeneous fractional domain spaces**. Similar to the last section we write $D^{(\alpha)} := X^{(\alpha)}$ and $R^{(\alpha)} := X^{(-\alpha)}$ for $\operatorname{Re}\alpha > 0$. Obviously we have $X^{(\alpha)} \subset X_{-1}$ whenever $|\operatorname{Re}\alpha| < 1$.

Lemma 6.3.2. *Let A be an injective sectorial operator on the Banach space X. Then for $\operatorname{Re}\alpha > 0$ the following assertions hold:*

a) $D^{(\alpha)} \cap X = \mathcal{D}(A^\alpha)$ *and* $\|A^\alpha x\| + \|x\| \sim \|(1+A)^\alpha x\|$.

b) $R^{(\alpha)} \cap X = \mathcal{R}(A^\alpha)$ *and* $\|A^{-\alpha} x\| + \|x\| \sim \|(1+A)^\alpha A^{-\alpha} x\|$.

c) $D^{(\alpha)} \cap R^{(\alpha)} = \mathcal{D}(A^\alpha) \cap \mathcal{R}(A^\alpha)$ *with* $\|A^\alpha x\| + \|A^{-\alpha} x\| \sim \left\|(1+A)^{2\alpha} A^{-\alpha} x\right\|$.

d) $\mathcal{D}(A^\alpha) + \mathcal{R}(A^\alpha) = X$.

Proof. a) The identity $D^{(\alpha)} \cap X = \mathcal{D}(A^\alpha)$ follows from Lemma 6.3.1, and the equivalence of norms is just the well-known fact $\mathcal{D}(A^\alpha) = \mathcal{D}((A+1)^\alpha)$ (Proposition 3.1.9).

b) is the same as a) with A replaced by A^{-1}, in using the identity $(A+1)^\alpha A^{-\alpha} = (1 + A^{-1})^\alpha$, which follows from the composition rule.

c) Let $x \in D^{(\alpha)} \cap R^{(\alpha)}$. Then $y := A^{-\alpha}x, A^\alpha x = A^{2\alpha}x \in X$, whence $y \in \mathcal{D}(A^{2\alpha})$. But $A^{2\alpha} = A^\alpha A^\alpha$ even as operators in X, and so $x = A^\alpha y \in X$. This shows $D^{(\alpha)} \cap R^{(\alpha)} \subset X$ and together with a) and b) this implies the stated set equality. From a) we obtain the norm equivalence

$$\left\|A^{2\alpha}x\right\| + \|x\| \sim \left\|(1+A)^{2\alpha}x\right\|$$

(which holds for $x \in \mathcal{D}(A^{2\alpha})$). Now we replace x by $A^{-\alpha}x$, $x \in \mathcal{D}(A^\alpha) \cap \mathcal{R}(A^\alpha)$ and we are done.

d) Take $n > \operatorname{Re}\alpha$ and expand $I = [A(1+A)^{-1} + (1+A)^{-1}]^{2n}$. $\square$

In many concrete situations extrapolation spaces can be explicitly identified with spaces of well-known objects. In Section 8.3 we shall exemplify this for the negative Laplace operator $-\Delta$ on $\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d), p \in [1, \infty)$, identifying the universal extrapolation space with a space of distributions.

6.4 Homogeneous Interpolation

We make the overall assumption that $A \in \mathrm{Sect}(\omega)$ is an *injective* sectorial operator on the Banach space X. In this section we characterise the real interpolation spaces $(D^{(\alpha)}, R^{(\beta)})_{\theta,p}$ between the homogeneous fractional domain spaces by means of functional calculus. To achieve this we freely use the extrapolation space U constructed in Section 6.3.

6.4.1 Some Intermediate Spaces

Fix $p \in [1, \infty]$, $\theta \in \mathbb{R}$, and $0 \neq \psi \in \mathcal{O}[S_\omega]$ with $z^{-\theta}\psi \in H_0^\infty[S_\omega]$. Then we define

$$X_{\theta,\psi,p} := \left\{ x \in U \;\middle|\; t^{-\theta}\psi(tA)x \in \mathbf{L}_*^{\boldsymbol{p}}((0,\infty);X) \right\}.$$

For $x \in X_{\theta,\psi,p}$ we let

$$\|x\|_{\theta,\psi,p} := \left\| t^{-\theta}\psi(tA)x \right\|_{\mathbf{L}_*^{\boldsymbol{p}}}.$$

(Note that it may well be that $x \in X_{\theta,\psi,p}$ but $x \notin X$.)

Proposition 6.4.1. *Let $p \in [1,\infty]$, $\theta \in \mathbb{R}$, and ψ as above. Then the following statements hold.*

a) *The operator A^θ (defined on U) induces an isometric isomorphism*

$$A^\theta : X_{\theta,\psi,p} \longrightarrow X_{0,z^{-\theta}\psi,p}.$$

b) *The space $X_{\theta,\psi,p}$ is continuously included in $A^{-\theta}X_{-1}$.*

c) *The space $X_{\theta,\psi,p}$ is a Banach space.*

Proof. a) is clear from the identity

$$t^{-\theta}\psi(tA)x = (tA)^{-\theta}\psi(tA)A^\theta x = (z^{-\theta}\psi)(tA)A^\theta x$$

for each $x \in U$. To prove b) it suffices to consider the case that $\theta = 0$ (by a)). Therefore $\psi \in H_0^\infty$, by assumption. Take $x \in X_{0,\psi,p}$, i.e., $x \in U$ such that $\psi(tA)x \in \mathbf{L}_*^{\boldsymbol{p}}(X)$. Apply Lemma 6.2.5 with $\alpha = 0$ to find a function $f \in H_0^\infty$ with $\int_0^\infty (f\psi)(t)\,dt/t = 1$. By Theorem 5.2.2 the function $t \mapsto \tau(A)f(tA)$ is bounded and absolutely integrable in $\mathcal{L}(X)$. (We use the abbreviation $\tau(z) := z(1+z)^{-2}$ as usual.) In particular, by Hölder's inequality it is contained in $\mathbf{L}_*^{\boldsymbol{p'}}(\mathcal{L}(X))$, where p' is the conjugated exponent to p. Hence we obtain $\tau(A)(\tilde{\psi}\psi)(tA)x \in \mathbf{L}_*^{\boldsymbol{1}}(X)$ with

$$\int_0^\infty \left\| \tau(A)(\tilde{\psi}\psi)(tA)x \right\|_X \frac{dt}{t} \leq \|x\|_{X_{0,\psi,p}} \cdot \left\| \tau(A)\tilde{\psi}(tA) \right\|_{\mathbf{L}_*^{\boldsymbol{p'}}(\mathcal{L}(X))}.$$

Applying Proposition 5.2.4 we obtain $\int_0^\infty (f\psi)(tA)x\,dt/t = x$ in U (or, if you wish, in some space X_{-m}). Consequently $\tau(A)x \in X$, i.e., $x \in X_{-1}$, with

$$\|x\|_{X_{-1}} = \|\tau(A)x\|_X \leq \|x\|_{X_{0,\psi,p}} \cdot \left\| \tau(A)\tilde{\psi}(tA) \right\|_{\mathbf{L}_*^{\boldsymbol{p'}}(\mathcal{L}(X))}.$$

The proof of c) is now easy. Again, it suffices to consider the case that $\theta = 0$. Let $(x_n)_n \subset X_{0,\psi,p}$ be a Cauchy sequence. By b) there is $x \in X_{-1}$ with $x_n \to x$ in X_{-1}. Then $\psi(tA)x_n \to \psi(tA)x$ in X_{-1} uniformly in t. On the other hand there is $f \in \mathbf{L}^{\boldsymbol{p}}_{*}(X)$ with $\psi(tA)x_n \to f(t)$ in the $\mathbf{L}^{\boldsymbol{p}}_{*}(X)$-norm, hence a fortiori in the $\mathbf{L}^{\boldsymbol{p}}_{*}(X_{-1})$-norm. This shows that $f(t) = \psi(tA)x$ almost everywhere, whence we are done. □

We can now prove the main result.

Theorem 6.4.2. *The spaces $(X_{\theta,\psi,p}, \|.\|_{\theta,\psi,p})$ are independent of the chosen ψ.*

Proof. By a) of Proposition 6.4.1 it suffices to prove the theorem in the case that $\theta = 0$. We choose $\varphi \in (\omega, \pi)$, $0 \neq \psi, \gamma \in H_0^\infty(S_\varphi)$, and $x \in X_{0,\gamma,p}$. Then we apply Lemma 6.2.5 (with $\alpha = 0$) to find $f \in H_0^\infty(S_\varphi)$ such that $\int_0^\infty (\gamma f)(s)\, ds/s = 1$. By Theorem 5.2.2, the numbers

$$E := \sup_{s>0} \int_0^\infty \|\psi(sA)f(tA)\|_{\mathcal{L}(X)} \frac{dt}{t} \qquad \text{and}$$
$$F := \sup_{t>0} \int_0^\infty \|\psi(sA)f(tA)\|_{\mathcal{L}(X)} \frac{ds}{s}$$

are both finite. Also, Proposition 5.2.4 shows that for each $s > 0$,

$$\psi(sA)x = \int_0^\infty (\gamma f)(tA)\psi(sA)x \frac{dt}{t} \tag{6.10}$$

as a convergent integral in some space X_{-m}. Now, for $s, t > 0$ we have

$$(\gamma f)(tA)\psi(sA)x = [\psi(sA)f(tA)]\,[\gamma(tA)x],$$

and considered as a product of functions in t this is integrable within X since

$$\begin{aligned} \psi(sA)f(tA) \;&\in\; \mathbf{L}^{\boldsymbol{\infty}}_{*}(\mathcal{L}(X)) \cap \mathbf{L}^{\boldsymbol{1}}_{*}(\mathcal{L}(X)) \;\subset\; \mathbf{L}^{\boldsymbol{p'}}_{*}(\mathcal{L}(X)) \\ \text{and} \qquad \gamma(tA)x \;&\in\; \mathbf{L}^{\boldsymbol{p}}_{*}(X), \end{aligned}$$

by Proposition 5.2.4 and the choice of x. Hence we actually have

$$\psi(sA)x = \int_0^\infty (\gamma f)(tA)\psi(sA)x \frac{dt}{t} \;\in\; X \qquad \text{with}$$

$$\|\psi(sA)x\|_X \le \int_0^\infty \|\psi(sA)f(tA)\,\gamma(tA)x\|_X \frac{dt}{t} \le \|\psi(sA)f(\cdot A)\|_{\mathbf{L}^{\boldsymbol{p'}}_{*}(\mathcal{L}(X))} \|x\|_{0,\gamma,p}\,.$$

By (6.10) and Hölder's inequality, $h(s) := (s \longmapsto \psi(sA)x) : (0,\infty) \longrightarrow X$ is bounded. Moreover, similar estimates show that the continuous(!) functions

$$h_{a,b}(s) := \int_a^b (f\gamma)(tA)\psi(sA)x \frac{dt}{t} = \psi(sA)\left[\int_a^b f(tA)\gamma(tA)x \frac{dt}{t}\right]$$

converge uniformly to h as $a \searrow 0, b \nearrow \infty$. Hence the function h is in fact continuous. In particular, it is measurable.

Now, in the case $p = \infty$ we are already done, obtaining $x \in X_{0,\psi,\infty}$ with

$$\|x\|_{0,\psi,\infty} \leq E \, \|x\|_{0,\gamma,\infty} .$$

In the case $p < \infty$ we compute

$$\begin{aligned}
\|\psi(sA)x\|_X &\leq \int_0^\infty \|\psi(sA)f(tA)\gamma(tA)x\| \, \frac{dt}{t} \\
&\leq \left(\int_0^\infty \|\psi(sA)f(tA)\| \, \frac{dt}{t} \right)^{\frac{1}{p'}} \left(\int_0^\infty \|\psi(sA)f(tA)\| \, \|\gamma(tA)x\|^p \, \frac{dt}{t} \right)^{\frac{1}{p}} \\
&\leq E^{\frac{1}{p'}} \left(\int_0^\infty \|\psi(sA)f(tA)\| \, \|\gamma(tA)x\|^p \, \frac{dt}{t} \right)^{\frac{1}{p}} .
\end{aligned}$$

The last factor considered as a function in s is contained in $\mathbf{L}^{\boldsymbol{p}}_{\boldsymbol{*}}(0, \infty)$ since

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \|\psi(sA)f(tA)\| \, \|\gamma(tA)x\|^p \, \frac{dt}{t} \, \frac{ds}{s} \\
&\quad = \int_0^\infty \int_0^\infty \|\psi(sA)f(tA)\| \, \frac{ds}{s} \, \|\gamma(tA)x\|^p \, \frac{dt}{t} \\
&\quad \leq \left(\sup_{t>0} \int_0^\infty \|\psi(sA)f(tA)\| \, \frac{ds}{s} \right) \int_0^\infty \|\gamma(tA)x\|^p \, \frac{dt}{t} = F \, \|x\|^p_{0,\gamma,p} .
\end{aligned}$$

Therefore we end up with

$$\|x\|_{0,\psi,p} \leq E^{\frac{1}{p'}} \, F^{\frac{1}{p}} \, \|x\|_{0,\gamma,p} ,$$

and the theorem is completely proved. $\square$

In the following we will simply write $X_{\theta,p}$ instead of $X_{\theta,\psi,p}$. (This is justified by Theorem 6.4.2.) Note that one always has the (continuous) inclusions

$$X_{\theta,1} \subset X^{(\theta)} \subset X_{\theta,\infty}.$$

Proof. By Proposition 5.2.4 the inclusion $X_{\theta,1} \subset X_{\theta,p}$ is immediate. The proof of Theorem 6.4.2 shows that $X_{\theta,p} \subset X_{\theta,\infty}$. In fact it was proved that for $x \in X_{\theta,\psi,p}$ the function $(t \mapsto t^{-\theta}\psi(tA)x) : (0, \infty) \longrightarrow X$ is actually *continuous* and uniformly bounded. $\square$

Corollary 6.4.3. *The operators $A^{is}, s \in \mathbb{R}$, act as topological isomorphisms on each of the spaces $X_{\theta,p}$, $\theta \in \mathbb{R}$, $p \in [1, \infty]$.*

Proof. It suffices to prove the statement for $\theta = 0$. But if $0 \neq \psi \in H_0^\infty(S_\varphi)$ also $0 \neq z^{is}\psi \in H_0^\infty(S_\varphi)$, and since $\psi(tA)A^{is}x = t^{-is}(z^{is}\psi)(tA)x$ and $|t^{-is}| = 1$, we have $x \in X_{0,(z^{is}\psi),p}$ if and only if $A^{is}x \in X_{0,\psi,p}$. $\square$

6.4.2 ... Are Actually Real Interpolation Spaces

We are now going to show that the spaces $X_{\theta,p}$ are nothing else than real interpolation spaces between homogeneous fractional domain spaces.

Proposition 6.4.4. *Let* $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$, *and let* $p \in [1, \infty]$. *Then*

$$(D^{(\alpha)}, R^{(\beta)})_{\theta,p} = X_{0,p}$$

with equivalent norms, where $\theta := \operatorname{Re}\alpha / (\operatorname{Re}\alpha + \operatorname{Re}\beta) \in (0, 1)$.

Proof. Choose $\varphi \in (\omega_A, \pi)$ and $\psi \in H_0^\infty(S_\varphi)$ with $z^{-\alpha}\psi, z^\beta\psi \in \mathcal{E}(S_\varphi)$. For $x \in U$, $x = a + b$ with $a \in D^{(\alpha)}, b \in R^{(\beta)}$ we have

$$\psi(tA)x = t^\alpha (tA)^{-\alpha} \psi(tA) A^\alpha a + t^\beta (tA)^{-\beta} \psi(tA) A^{-\beta} b,$$

which yields

$$\|\psi(tA)x\|_X \leq t^{\operatorname{Re}\alpha} M_1 \|a\|_{D^{(\alpha)}} + t^{-\operatorname{Re}\beta} M_2 \|b\|_{R^{(\beta)}}$$

for all $t > 0$ and some constants M_1, M_2. Taking the infimum with respect to the representations $x = a + b$ yields

$$\|\psi(tA)x\| \leq M\, t^{\operatorname{Re}\alpha} K(t^{-(\operatorname{Re}\alpha + \operatorname{Re}\beta)}, x, D^{(\alpha)}, R^{(\beta)}) \qquad (t > 0)$$

for some constant M. This proves the continuous inclusion $(D^{(\alpha)}, R^{(\beta)})_{\theta,p} \subset X_{0,p}$.

For the converse inclusion choose a function $\psi \in H_0^\infty(S_\varphi)$ with $\int_0^\infty \psi(t)\, dt/t = 1$ and define

$$h(z) := \int_0^1 \psi(tz) \frac{dt}{t}, \qquad g(z) := \int_1^\infty \psi(tz) \frac{dt}{t}$$

as in Example 2.2.6 and Proposition 5.2.4. Without loss of generality one may suppose that the decay of ψ at 0 and at ∞ is rapid enough to ensure that both functions $\psi_1 := z^\alpha g$ and $\psi_2 := z^{-\beta} h$ are contained in $H_0^\infty(S_\varphi)$. Take $x \in X_{0,p}$ and write

$$x = g(tA)x + h(tA)x \qquad (t > 0).$$

Now observe that $g(tA)x \in D^{(\alpha)}$, because

$$A^\alpha g(tA)x = t^{-\alpha}(tA)^\alpha g(tA)x = t^{-\alpha}\psi_1(tA)x \in X.$$

Analogously, $h(tA)x \in R^{(\beta)}$ with

$$A^{-\beta} h(tA)x = t^\beta (tA)^{-\beta} h(tA)x = t^\beta \psi_2(tA)x.$$

Therefore, for each $s > 0$ one obtains

$$K(s, x, D^{(\alpha)}, R^{(\beta)}) \leq t^{-\operatorname{Re}\alpha} \|\psi_1(tA)x\|_X + s t^{\operatorname{Re}\beta} \|\psi_2(tA)x\|_X .$$

Letting $s := t^{-(\operatorname{Re}\alpha + \operatorname{Re}\beta)}$ yields

$$t^{\operatorname{Re}\alpha} K(t^{-(\operatorname{Re}\alpha + \operatorname{Re}\beta)}, x, D^{(\alpha)}, R^{(\beta)}) \leq \|\psi_1(tA)x\| + \|\psi_2(tA)x\| \qquad (t > 0).$$

The right-hand side (as a function of t) is in $\mathbf{L}^{\boldsymbol{p}}_{\boldsymbol{*}}$. This concludes the proof. $\square$

We remark that in the proof we made essential use of Theorem 6.4.2. Indeed, it was important that one can choose *every* H_0^∞-function to describe the space $X_{0,p}$. With the help of Proposition 6.4.4 we are now able to show that each real interpolation space between homogeneous spaces is in fact a space $X_{\theta,p}$.

Theorem 6.4.5 (Homogeneous Interpolation). *Let A be an injective, sectorial operator on the Banach space X, and let $\alpha, \beta \in \mathbb{C}$ with $\operatorname{Re}\alpha \neq \operatorname{Re}\beta$. Then, for all $\theta \in (0,1), p \in [1,\infty]$,*

$$\left(X^{(\alpha)}, X^{(\beta)}\right)_{\theta,p} = X_{(1-\theta)\operatorname{Re}\alpha + \theta\operatorname{Re}\beta, p}$$

with equivalent norms.

Proof. Without loss of generality we may suppose that $\operatorname{Re}\alpha > \operatorname{Re}\beta$ (replace θ by $1-\theta$ if necessary). Define $\delta := (1-\theta)\alpha + \theta\beta$. Then $\alpha' := \alpha - \delta = \theta(\alpha - \beta)$ and $\beta' := \delta - \beta = (1-\theta)(\alpha - \beta)$. Hence $\operatorname{Re}\alpha', \operatorname{Re}\beta' > 0$ and $\theta = \operatorname{Re}\alpha' / (\operatorname{Re}\alpha' + \operatorname{Re}\beta')$. By Proposition 6.4.4 we have

$$\left(X^{(\alpha-\delta)}, X^{(\beta-\delta)}\right)_{\theta,p} = \left(X^{(\alpha')}, X^{(-\beta')}\right)_{\theta,p} = \left(D^{(\alpha')}, R^{(\beta')}\right)_{\theta,p} = X_{0,p}.$$

Applying $A^{-\delta} = A^{-\operatorname{Re}\delta} A^{-i\operatorname{Im}\delta}$ to this identity concludes the proof. (Note that by Corollary 6.4.3 the operator $A^{-i\operatorname{Im}\delta}$ is an isomorphism on $X_{0,p}$.) □

Corollary 6.4.6. *Let $\theta \in (0,1), p \in [1,\infty]$. Then $(\dot{D}, \dot{R})_{\theta,p} = X_{1-2\theta,p}$.*

6.5 More Characterisations and Dore's Theorem

In this section we revisit Dore's theorem and give evidence to the heuristic statement that the functional calculus properties of an injective sectorial operator A improve in its interpolation spaces. But first, we complement the characterisations of the interpolation spaces obtained in Section 6.2.

6.5.1 A Third Characterisation (Injective Operators)

With Theorem 6.4.5 we have established a powerful description of real interpolation spaces in terms of functional calculus. However, the spaces $(X^{(\alpha)}, X^{(\beta)})_{\theta,p}$ are quite uncommon and we look for spaces which are naturally included in the original space X.

Lemma 6.5.1. *Let A be an injective sectorial operator on the Banach space X. Then*

$$(X, \mathcal{D}(A^\alpha) \cap \mathcal{R}(A^\alpha))_{\theta,p} = (X, D^{(\alpha)})_{\theta,p} \cap (X, R^{(\alpha)})_{\theta,p},$$
$$(X, \mathcal{D}(A^\alpha))_{\theta,p} = (X, D^{(\alpha)})_{\theta,p} \cap X,$$
$$(\mathcal{D}(A^\alpha), \mathcal{R}(A^\alpha))_{\theta,p} = A^\alpha(1+A)^{-\alpha}(D^{(\alpha)}, X)_{\theta,p}$$

for all $\operatorname{Re}\alpha > 0, \theta \in [0,1], p \in [1,\infty]$.

Proof. The last assertion follows from the fact that the isomorphism $A^{-\alpha}(1+A)^{\alpha}$ sends $\mathcal{D}(A^{\alpha})$ to $D^{(\alpha)}$ and $\mathcal{R}(A^{\alpha})$ to X, see Lemma 6.3.2. For the first two assertions we use Proposition B.2.7. Indeed, the second assertion follows directly from (B.4) and $D^{(\alpha)} \cap X = \mathcal{D}(A^{\alpha})$ (see Lemma 6.3.2).

To prove the first, let us denote for the moment $Y := D^{(\alpha)}, Z := R^{(\alpha)}$. Then we have $Y \cap Z = \mathcal{D}^{(\alpha)} \cap \mathcal{R}^{(\alpha)} = \mathcal{D}(A^{\alpha}) \cap \mathcal{R}(A^{\alpha}) \subset X$ and

$$(X \cap Y) + (X \cap Z) = (\mathcal{D}^{(\alpha)} \cap X) + (\mathcal{R}^{(\alpha)} \cap X) = \mathcal{D}(A^{\alpha}) + \mathcal{R}(A^{\alpha}) = X,$$

by Lemma 6.3.2. Hence we obtain $(X,Y)_{\theta,p} \cap (X,Z)_{\theta,p} \subset (X+Y) \cap (X+Z) = X$. This yields

$$\begin{aligned}(X,Y)_{\theta,p} \cap (X,Z)_{\theta,p} &= [(X,Y)_{\theta,p} \cap X] \cap [(X,Z)_{\theta,p} \cap X] \\ &= (X, X \cap Y)_{\theta,p} \cap (X, X \cap Z)_{\theta,p} = (X, X \cap Y \cap X \cap Z)_{\theta,p} \\ &= (X, Y \cap Z)_{\theta,p}.\end{aligned}$$

□

Remark 6.5.2. In the first statement of the last lemma one writes 'vertical' interpolation spaces as intersections of 'horizontal' ones. As a matter of fact, one can also go in the reverse direction. E.g., one has the equality

$$(D,R)_{\theta,p} = (D_{-1}, D)_{1-\theta,p} \cap (R_{-1}, R)_{\theta,p},$$

which can be deduced from an interpolation identity that is in the spirit of Proposition B.2.7 (see [111, Theorem 1]).

Combining Lemma 6.5.1 with Theorem 6.4.5 we arrive at a third representation theorem.

Theorem 6.5.3. *Let A be an injective sectorial operator on a Banach space X, let $\varphi \in (\omega_A, \pi)$, $\operatorname{Re}\alpha > 0$, $\theta \in (0,1)$, and $p \in [1,\infty]$.*

a) *One has*

$$(X, \mathcal{D}(A^{\alpha}))_{\theta,p} = \left\{ x \in X \;\middle|\; t^{-\theta \operatorname{Re}\alpha} \psi(tA)x \in \mathbf{L}^{p}_{*}((0,\infty);X) \right\}$$

with equivalence of norms

$$\|x\|_{(X,\mathcal{D}(A^{\alpha}))_{\theta,p}} \sim \|x\|_X + \left\| t^{-\theta \operatorname{Re}\alpha} \psi(tA)x \right\|_{\mathbf{L}^{p}_{*}((0,\infty);X)}$$

whenever $\psi \in \mathcal{O}(S_\varphi) \setminus \{0\}$ such that $z^{-\theta\alpha}\psi \in H_0^{\infty}(S_\varphi)$.

b) *One has*

$$\begin{aligned}&(X, \mathcal{D}(A^{\alpha}) \cap \mathcal{R}(A^{\alpha}))_{\theta,p} \\ &\quad = \left\{ x \in X \;\middle|\; t^{-\theta \operatorname{Re}\alpha} \psi_1(tA)x,\; t^{\theta \operatorname{Re}\alpha} \psi_2(tA)x \in \mathbf{L}^{p}_{*}((0,\infty);X) \right\}\end{aligned}$$

with equivalence of norms

$$\|x\|_{(X,\mathcal{D}(A^\alpha)\cap\mathcal{R}(A^\alpha))_{\theta,p}} \sim \left\|t^{-\theta\operatorname{Re}\alpha}\psi_1(tA)x\right\|_{\mathbf{L}^p_*} + \left\|t^{\theta\operatorname{Re}\alpha}\psi_2(tA)x\right\|_{\mathbf{L}^p_*}$$

whenever $\psi_1, \psi_2 \in \mathcal{O}(S_\varphi) \setminus \{0\}$ *such that* $z^{-\theta\alpha}\psi_1, z^{\theta\alpha}\psi_2 \in H_0^\infty(S_\varphi)$.

Proof. a) Let us specialise $\beta = 0$ and $\operatorname{Re}\alpha > 0$ in Theorem 6.4.5. Thus we obtain $(X, X^{(\alpha)})_{\theta,p} = X_{\theta\operatorname{Re}\alpha,p}$. Now we intersect both sides of this identity with X and employing Lemma 6.5.1 we arrive at $(X, \mathcal{D}(A^\alpha))_{\theta,p} = X_{\theta\operatorname{Re}\alpha,p} \cap X$. The rest follows from Theorem 6.4.2.

The proof of b) is similar. □

When A is invertible, some simplifications apply. This is the content of the next section.

6.5.2 A Fourth Characterisation (Invertible Operators)

Let us specialise the given characterisations to the case of an *invertible* sectorial operator A. If A is invertible, the array of extrapolation spaces becomes 'half-trivial' since we obtain

$$D^{(\alpha)} = \mathcal{D}(A^\alpha), \quad \mathcal{R}(A^\alpha) = X \qquad (\operatorname{Re}\alpha > 0).$$

Moreover, $D_{-1} = X$ and $R_{-1} = X_{-1}$. Consequently, one has

$$(X, \mathcal{D}(A^\alpha))_{\theta,p} = X_{\theta\operatorname{Re}\alpha,p}$$

by Theorem 6.4.5. Comparing this with the Characterisation Theorem 6.5.3 a) this means that in the case where A is invertible one may leave out the $\|\cdot\|_X$-part and simply has a norm equivalence

$$\|x\|_{(X,\mathcal{D}(A^\alpha))_{\theta,p}} \sim \left\|t^{-\theta\operatorname{Re}\alpha}\psi(tA)x\right\|_{\mathbf{L}^p_*((0,\infty);X)}.$$

The next proposition reproves this fact and moreover shows that it holds even in the extremal case $\theta = 1, p = \infty$, with ψ as in Theorem 6.2.9.

Proposition 6.5.4. *Let A be an invertible sectorial operator on X, $\varphi \in (\omega_A, \pi)$ and $0 \neq \psi \in \mathcal{B}(S_\varphi)$. Given* $\operatorname{Re}\alpha > 0, \theta \in (0,1], p \in [1,\infty]$ *there is a constant c such that*

$$\|x\|_X \le c\,\left\|t^{-\theta\operatorname{Re}\alpha}\psi(tA)x\right\|_{\mathbf{L}^p_*((0,\infty);X)} \qquad (x \in X).$$

Note that $\psi(tA)x \in U$ is always defined but the right-hand side might be infinite, in which case the inequality holds trivially.

Proof. Without harm we can replace α by $\theta\alpha$ and suppose that $\theta = 1$. Choose $f \in H_0^\infty(S_\varphi)$ such that $f_1 := z^{\alpha+1}f, \psi f \in H_0^\infty(S_\varphi)$ and $\int_0^\infty (f\psi)(t)\,dt/t = 1$ (Lemma 6.2.5). Define as usual

$$h(z) := \int_0^1 (f\psi)(tz)\,\frac{dt}{t}, \quad \text{and} \quad g(z) := \int_1^\infty (f\psi)(tz)\,\frac{dt}{t}.$$

Then $h+g=1$ (Example 2.2.6). Take $x \in X$ with $t^{-\theta \operatorname{Re}\alpha}\psi(tA)x \in \mathbf{L}^{\boldsymbol{p}}_{*}((0,\infty);X)$. Then

$$(f\psi)(tA)x = A^{-(\alpha+1)}\,[t^{-1}f_1(tA)]\,[t^{-\alpha}\psi(tA)x] \;\in\; \mathbf{L}^{\mathbf{1}}_{*}((1,\infty);X)$$

since by assumption the last factor is in $\mathbf{L}^{\boldsymbol{p}}_{*}$, the first factor is bounded, and the second factor is in $\mathbf{L}^{\boldsymbol{p'}}_{*}$. Hence by Proposition 5.2.4 we obtain

$$\|g(A)x\| = \left\| \int_1^\infty (f\psi)(tA)x\,\frac{dt}{t} \right\| \le c_1 \left\| t^{-\operatorname{Re}\alpha}\psi(tA)x \right\|_{\mathbf{L}^{\boldsymbol{p}}_{*}((1,\infty);X)},$$

with $c_1 = \|t^{-1}\|_{\mathbf{L}^{\boldsymbol{p'}}_{*}(1,\infty)}\,\|A^{-(\alpha+1)}\|\,\sup_{t>1}\|f_1(tA)\|$. On the other hand we have

$$(f\psi)(tA)x = [t^{\alpha}f(tA)][t^{-\alpha}\psi(tA)x] \;\in\; \mathbf{L}^{\mathbf{1}}_{*}((0,1);X)$$

since the first factor is in $\mathbf{L}^{\boldsymbol{p'}}_{*}((0,1),X)$. Again by Proposition 5.2.4 we obtain

$$\|h(A)x\| = \left\| \int_0^1 (f\psi)(tA)\,\frac{dt}{t} \right\| \le c_2 \left\| t^{-\operatorname{Re}\alpha}\psi(tA)x \right\|_{\mathbf{L}^{\boldsymbol{p}}_{*}((0,1);X)},$$

with $c_2 = \|t^{\operatorname{Re}\alpha}\|_{\mathbf{L}^{\boldsymbol{p'}}_{*}(0,1)}\,\sup_{t<1}\|f(tA)\|$. Thus, the proof is complete. ☐

Corollary 6.5.5. *Let A be an invertible sectorial operator on a Banach space X. Then one has equivalence of norms*

$$\|x\|_{(X,\mathcal{D}(A^\alpha))_{\theta,p}} \quad\sim\quad \left\|t^{-\theta\operatorname{Re}\alpha}\psi(tA)x\right\|_{\mathbf{L}^{\boldsymbol{p}}_{*}((0,\infty);X)}$$

in Theorem 6.2.1 and Theorem 6.5.3.

6.5.3 Dore's Theorem Revisited

Dore's theorem (Theorem 6.1.3) says that an invertible operator has a bounded H^∞-calculus in each of its real interpolation spaces $(X,\mathcal{D}(A))_{\theta,p}$, $\theta \in (0,1)$. We have seen in Example 6.1.4 that the assumption of invertibility cannot be dropped. This phenomenon can now be fully understood, since by the results of the last section we have the following theorem.

Theorem 6.5.6. *Let A be an injective sectorial operator on the Banach space X, and let $\alpha,\beta \in \mathbb{C}$ with $\operatorname{Re}\alpha \neq \operatorname{Re}\beta$. Then for each $\varphi \in (\omega_A,\pi)$, the operator A has a bounded $H^\infty(S_\varphi)$-calculus in each of the spaces*

$$\left(X^{(\alpha)}, X^{(\beta)}\right)_{\theta,p} \qquad (\theta \in (0,1), p \in [1,\infty]).$$

Proof. Employing Theorem 6.4.5 and Proposition 6.4.1 we are reduced to the spaces $X_{0,p}$. Let $x \in X_{0,p}$ and choose two functions $0 \neq \psi,\gamma \in H_0^\infty(S_\varphi)$. By Theorem 6.4.2 we have $X_{0,p} = X_{0,\gamma,p} = X_{0,(\psi\gamma),p}$. Now

$$(\psi\gamma)(tA)f(A)x = (\psi_t f)(A)\gamma(tA)x \qquad (t>0).$$

But $\sup_{t>0}\|(f\psi_t)(A)\|_{\mathcal{L}(X)} \le C(\psi)M(A,\varphi)\,\|f\|_{S_\varphi}$ by Theorem 5.2.2. ☐

We obtain an immediate corollary.

Corollary 6.5.7. *Let A be an injective sectorial operator on the Banach space X, and let $\varphi \in (\omega_A, \pi)$. Then A has a bounded $H^\infty(S_\varphi)$-calculus in each of the spaces*

$$\big(\mathcal{D}(A^\alpha), \mathcal{R}(A^\alpha)\big)_{\theta,p} \qquad (\operatorname{Re}\alpha > 0, \theta \in (0,1), p \in [1,\infty]).$$

Proof. By Lemma 6.5.1,

$$A^{-\alpha}(1+A)^\alpha : (\mathcal{D}(A^\alpha), \mathcal{R}(A^\alpha))_{\theta,p} \longrightarrow (X^{(\alpha)}, X^{(0)})_{\theta,p}$$

is an isomorphism which commutes with A. Hence the assertion follows from Theorem 6.5.6. □

We may apply Lemma 6.5.1 to pass from the 'horizontal' interpolation spaces to the 'vertical' ones.

Corollary 6.5.8 (Dore). *Let A be an injective sectorial operator on the Banach space X, and let $\varphi \in (\omega_A, \pi)$. Then A has a bounded $H^\infty(S_\varphi)$-calculus in each of the spaces*

$$\big(X, \mathcal{D}(A^\alpha) \cap \mathcal{R}(A^\alpha)\big)_{\theta,p} \qquad (\operatorname{Re}\alpha > 0, \theta \in (0,1), p \in [1,\infty]).$$

(For the special case $\alpha = 1$ and under density assumptions, this corollary was proved by Dore [73].) Assuming A to be invertible and taking $\alpha = 1$ gives back Dore's Theorem 6.1.3. Note that this new proof does not use the Reiteration Theorem.

6.6 Fractional Powers as Intermediate Spaces

In this section we examine the domains $\mathcal{D}(A^\alpha)$ as intermediate spaces. Here A is any sectorial operator on a Banach space X, not necessarily injective.

6.6.1 Density of Fractional Domain Spaces

If A is a sectorial operator with non-dense domain, then for no pair α, β with $0 < \operatorname{Re}\alpha < \operatorname{Re}\beta$ is the space $\mathcal{D}(A^\beta)$ dense in $\mathcal{D}(A^\alpha)$. However, within the real interpolation spaces we have a quite different behaviour.

Theorem 6.6.1 (Density). *Let A be a sectorial operator on the Banach space X and let $0 < \operatorname{Re}\alpha < \operatorname{Re}\beta$, $\theta \in (0,1), p \in [1,\infty)$. Then the space $\mathcal{D}(A^\beta)$ is dense in $(X, \mathcal{D}(A^\alpha))_{\theta,p}$.*

Proof. Without loss of generality one may suppose that A is invertible. Choose $\varphi \in (\omega_A, \pi)$ and $0 \neq \psi \in \mathcal{O}(S_\varphi)$ such that $z^{-\theta\alpha}\psi \in H_0^\infty(S_\varphi)$. Choose $0 \neq \tau \in H_0^\infty(S_\varphi)$ such that $(\tau\psi) \in H_0^\infty(S_\varphi)$. Then also $z^{-\theta\alpha}(\tau\psi) \in H_0^\infty(S_\varphi)$ and either function $\psi, \tau\psi$ may be used to describe $(X, \mathcal{D}(A^\alpha))_{\theta,p}$ (Corollary 6.5.5). We choose

$f \in H_0^\infty(S_\varphi)$ in such a way that $f\psi, z^\beta f\psi \in H_0^\infty(S_\varphi)$ and $\int_0^\infty (f\psi)(s)\, ds/s = 1$ (Lemma 6.2.5). Define

$$g_n(z) := \int_{\frac{1}{n}}^\infty (f\psi)(sz)\, \frac{ds}{s}.$$

Then $g_n \in \mathcal{E}(S_\varphi)$ as was shown in Example 2.2.6. We claim that (i) $g_n(A)(X) \subset \mathcal{D}(A^\beta)$ and (ii) $g_n(A)x \to x$ within $(X, \mathcal{D}(A^\alpha))_{\theta,p}$ for $x \in (X, \mathcal{D}(A^\alpha))_{\theta,p}$.

Claim (i) follows from the identity

$$z^\beta g_n(z) = \int_{\frac{1}{n}}^\infty (z^\beta f\psi)(sz) s^{-\beta}\, \frac{ds}{s},$$

which is a function in $H_0^\infty(S_\varphi)$.

To prove Claim (ii) we take $x \in (X, \mathcal{D}(A^\alpha))_{\theta,p}$. We have to show

$$t^{-\theta \operatorname{Re}\alpha}(\tau\psi)(tA) g_n(A)x \to t^{-\theta \operatorname{Re}\alpha}(\tau\psi)(tA)x$$

in $\mathbf{L}^p_*((0,\infty); X)$. The Convergence Lemma (Proposition 5.1.4, see also c) of Proposition 5.2.4) shows that the convergence is at least pointwise. Employing the Dominated Convergence Theorem we have to prove domination. Now

$$t^{-\theta \operatorname{Re}\alpha}(\tau\psi)(tA) g_n(A)x = \int_{\frac{1}{n}}^\infty (f\psi)(sz)\tau(tz)\, \frac{ds}{s}(A)\, t^{-\theta \operatorname{Re}\alpha}\psi(tA)x$$

and, by Proposition 5.2.4 and Theorem 5.2.2,

$$\begin{aligned}\sup_{t>0} \left\| \int_{\frac{1}{n}}^\infty (f\psi)(sz)\tau(tz)\, \frac{ds}{s}(A) \right\| &= \sup_{t>0} \left\| \int_{\frac{1}{n}}^\infty (f\psi)(sA)\tau(tA)\, \frac{ds}{s} \right\| \\ &\le \sup_{t>0} \int_0^\infty \|(f\psi)(sA)\tau(tA)\|\, \frac{ds}{s} =: c < \infty.\end{aligned}$$

This yields $\left\|t^{-\theta \operatorname{Re}\alpha}(\tau\psi)(tA) g_n(A)x\right\|_X \le c \left\|t^{-\theta \operatorname{Re}\alpha}\psi(tA)x\right\|_X$, which is in $\mathbf{L}^p_*(0,\infty)$ by assumption. □

6.6.2 The Moment Inequality

We first establish that — in the language of interpolation theory — the space $\mathcal{D}(A^\beta)$ is always of class $J_\theta \cap K_\theta$ between X and $\mathcal{D}(A^\alpha)$, provided $0 < \operatorname{Re}\beta < \operatorname{Re}\alpha$ and θ is defined as $\theta := \operatorname{Re}\beta / \operatorname{Re}\alpha$. The following result is of auxiliary nature.

Proposition 6.6.2. *Let $A \in \operatorname{Sect}(\omega)$, $\varphi \in (\omega, \pi)$, and $\psi \in \mathcal{E}(S_\varphi)$. Then the following assertions hold.*

a) *Let $\operatorname{Re}\alpha > 0$, and suppose that also $z^{-\alpha}\psi \in \mathcal{E}(S_\varphi)$. Then*

$$x \in \mathcal{D}(A^\beta) \quad \Longrightarrow \quad \sup_{t>0} \left\|t^{-\operatorname{Re}\beta}\psi(tA)x\right\|_X < \infty$$

for every $\beta \in \mathbb{C}$ with $\operatorname{Re}\beta \in (0, \operatorname{Re}\alpha)$ or $\beta \in \{0, \alpha\}$.

b) *Let* $\operatorname{Re}\alpha > 0$. *Then* $\displaystyle\int_0^1 t^{-\operatorname{Re}\alpha} \left\|\psi(tA)x\right\|_X \frac{dt}{t} < \infty \quad \Longrightarrow \quad x \in \mathcal{D}(A^\alpha)$.

Proof. a) Suppose that $x \in \mathcal{D}(A^\beta)$ where β is in the given range. From the hypotheses it is clear that $\tilde{\psi} := \psi z^{-\beta} \in \mathcal{E}$. Now, $\psi(tA)x = t^\beta \tilde{\psi}(tA)A^\beta x$, whence the assertion follows from Proposition 2.6.11.

b) Choose $\delta > \operatorname{Re}\alpha$ and $f \in H_0^\infty$ such that $z^\delta f \in H_0^\infty$ and $\int_0^\infty (f\psi)(t)\,dt/t = 1$, cf. Lemma 6.2.5. Define

$$h(z) := \int_0^1 (f\psi)(tz)\,\frac{dt}{t} \quad \text{and} \quad g(z) := \int_1^\infty (f\psi)(tz)\,\frac{dt}{t}$$

as in Example 2.2.6. Then $1 = h + g$, and $z^\alpha g \in H_0^\infty$ since gz^δ is still bounded. Hence $g(A)X \subset \mathcal{D}(A^\alpha)$. Take $x \in X$ such that $\int_0^1 t^{-\operatorname{Re}\alpha}\left\|\psi(tA)x\right\|dt/t < \infty$. Let $\tilde{f} := z^\alpha f \in H_0^\infty$. Then clearly

$$\int_0^1 \left\|f(tA)\psi(tA)x\right\| \frac{dt}{t} < \infty \quad \text{and} \quad \int_0^1 \left\|\tilde{f}(tA)t^{-\alpha}\psi(tA)x\right\| \frac{dt}{t} < \infty.$$

Proposition 5.2.4 implies that in fact

$$h(A)x = \int_0^1 f(tA)\psi(tA)x\,\frac{dt}{t},$$

and since $f(tA)\psi(tA)x \in \mathcal{D}(A^\alpha)$ for all $t \in (0,1)$ and the operator A^α is closed, we have $h(A)x \in \mathcal{D}(A^\alpha)$ with

$$A^\alpha h(A)x = \int_0^1 t^{-\alpha}\tilde{f}(tA)\psi(tA)x\,\frac{dt}{t}.$$

Thus we arrive at $x = h(A)x + g(A)x \in \mathcal{D}(A^\alpha)$. □

Specialising ψ to a function which describes the real interpolation spaces as in Theorem 6.2.9 yields the following.

Corollary 6.6.3. *Let* $\operatorname{Re}\gamma < \operatorname{Re}\beta < \operatorname{Re}\alpha$ *with* $\operatorname{Re}\gamma > 0$ *or* $\gamma = 0$. *Then*

$$\big(\mathcal{D}(A^\gamma), \mathcal{D}(A^\alpha)\big)_{\theta,1} \subset \mathcal{D}(A^\beta) \subset \big(\mathcal{D}(A^\gamma), \mathcal{D}(A^\alpha)\big)_{\theta,\infty} \tag{6.11}$$

where $\theta \in (0,1)$ *is such that* $\operatorname{Re}\beta = (1-\theta)\operatorname{Re}\gamma + \theta\operatorname{Re}\alpha$.

Proof. Since $(1+A)^{-\gamma} : X \longrightarrow \mathcal{D}(A^\gamma)$ is an isomorphism which carries $\mathcal{D}(A^{\beta-\gamma})$ to $\mathcal{D}(A^\beta)$ and $\mathcal{D}(A^{\alpha-\gamma})$ to $\mathcal{D}(A^\alpha)$, we may suppose that $\gamma = 0$. Then the statement follows from the characterisation in Theorem 6.2.9 by taking an appropriate function ψ. □

It is well known that the left-hand inclusion in (6.11) implies (and is in fact equivalent to) an inequality of the form $\left\|A^\beta x\right\| \le C \left\|A^\gamma x\right\|^{1-\theta} \left\|A^\alpha x\right\|^\theta$. The usual interpolation-theoretic proof requires a description of the interpolation spaces different from the K-method, cf. [29, Theorem 3.5.2] or [157, Theorem 1.2.15]. However, our methods allow also a direct proof.

Proposition 6.6.4 (Moment Inequality). *Let A be a sectorial operator on the Banach space X. Let $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\operatorname{Re}\gamma < \operatorname{Re}\beta < \operatorname{Re}\alpha$ and $\operatorname{Re}\gamma > 0$ or $\gamma = 0$. Then there is a constant C such that*

$$\left\|A^\beta x\right\| \le \frac{C}{\theta(1-\theta)} \left\|A^\gamma x\right\|^{1-\theta} \left\|A^\alpha x\right\|^\theta \qquad (x \in \mathcal{D}(A^\alpha)),$$

where θ is defined as $\theta := (\operatorname{Re}\beta - \operatorname{Re}\gamma)/(\operatorname{Re}\alpha - \operatorname{Re}\gamma)$.

Proof. Choose any $\psi \in H_0^\infty$ such that $\psi z^\alpha, \psi z^{-\alpha}$ are still bounded functions. Define $h(z) := \int_0^1 \psi(sz)\, ds/s$, $g(z) := \int_1^\infty \psi(sz)\, ds/s$ as in Example 2.2.6. Then $z^{-\alpha} h$ and $z^\alpha g$ are bounded functions, whence $\hat{h} := z^{-(\alpha-\beta)} h$ and $\hat{g} := z^{\beta-\gamma} g$ are both in H_0^∞. For $x \in \mathcal{D}(A^\alpha)$ we have

$$A^\beta x = h(tA)A^\beta x + g(tA)A^\beta x = t^{\alpha-\beta}\hat{h}(tA)A^\alpha x + t^{-(\beta-\gamma)}\hat{g}(tA)x.$$

This yields

$$\left\|A^\beta x\right\| \le t^{\operatorname{Re}\alpha(1-\theta}C_{\hat{h}} \left\|A^\alpha x\right\| + t^{-\operatorname{Re}\alpha\theta} C_{\hat{g}} \left\|A^\gamma x\right\|$$

(see Proposition 2.6.11). Taking the infimum with respect to $t > 0$ we arrive at

$$\left\|A^\beta x\right\| \le C \left[\left(\frac{1-\theta}{\theta}\right)^\theta + \left(\frac{\theta}{1-\theta}\right)^{1-\theta} \right] \left\|A^\gamma x\right\|^{1-\theta} \left\|A^\alpha x\right\|^\theta,$$

where $C := \max\{C_{\hat{g}}, C_{\hat{h}}\}$. The term in brackets is less than $(\theta(1-\theta))^{-1}$. □

Remark 6.6.5. A more detailed analysis would reveal the kind of dependence of the constant C on A and on α, β, γ. The classical estimate is [161, Lemma 3.1.7].

Corollary 6.6.6. *Let $A \in \operatorname{Sect}(\omega)$ and $0 \le \operatorname{Re}\gamma < \operatorname{Re}\beta < \operatorname{Re}\alpha$, where either $\gamma = 0$ or $\operatorname{Re}\gamma > 0$. Then*

$$\mathcal{D}(A^\beta) \in J_\theta(\mathcal{D}(A^\gamma), \mathcal{D}(A^\alpha)) \quad \textit{and} \quad \mathcal{D}(A^\beta) \in K_\theta(\mathcal{D}(A^\gamma), \mathcal{D}(A^\alpha)),$$

where $\theta \in (0,1)$ is such that $\operatorname{Re}\beta = (1-\theta)\operatorname{Re}\gamma + \theta \operatorname{Re}\alpha$.

6.6.3 Reiteration and Komatsu's Theorem

The next result could be obtained from Corollary 6.6.6 by means of the Reiteration Theorem B.2.9. However we can give a direct proof.

Proposition 6.6.7 (Reiteration). *Let A be a sectorial operator on a Banach space X. Then the following assertions hold.*

a) *If $0 < \operatorname{Re}\beta \le \operatorname{Re}\alpha$, then*

$$(X, \mathcal{D}(A^\alpha))_{\theta \frac{\operatorname{Re}\beta}{\operatorname{Re}\alpha}, p} = (X, \mathcal{D}(A^\beta))_{\theta,p}$$

for all $\theta \in (0,1), p \in [1,\infty]$.

b) *If $0 < \operatorname{Re}\gamma < \operatorname{Re}\beta \le \operatorname{Re}\alpha$, $\sigma \in (0,1)$, $p \in [1,\infty]$, and $x \in X$, then*

$$x \in (X, \mathcal{D}(A^\alpha))_{\theta,p} \quad \Longleftrightarrow \quad x \in \mathcal{D}(A^\gamma) \quad \text{and} \quad A^\gamma x \in (X, \mathcal{D}(A^{\beta-\gamma}))_{\sigma,p},$$

where $\theta := (1-\sigma)(\operatorname{Re}\gamma / \operatorname{Re}\alpha) + \sigma(\operatorname{Re}\beta/\operatorname{Re}\alpha)$.

c) *If $\alpha, \beta, \gamma, p, \sigma, \theta$ are as in* b), *then*

$$(X, \mathcal{D}(A^\alpha))_{\theta,p} = (\mathcal{D}(A^\gamma), \mathcal{D}(A^\beta))_{\sigma,p}.$$

Proof. a) Choose a function $\psi \in \mathcal{E}$ such that $z^{-\alpha}\psi \in H_0^\infty$. Then also $\psi z^{-\beta} \in H_0^\infty$ and one can apply Theorem 6.2.9 twice.

b) Choose $\tilde{\psi} \in \mathcal{E}$ such that $z^{-\alpha}\tilde{\psi} \in \mathcal{E}$ and use it to describe $(X, \mathcal{D}(A^\alpha))_{\theta,p}$. Observe that also $\psi := z^{-\gamma}\tilde{\psi} \in \mathcal{E}$ and $z^{-(\beta-\gamma)}\psi = z^{-\beta}\tilde{\psi} \in \mathcal{E}$. Hence we may use ψ to describe $(X, \mathcal{D}(A^{\beta-\gamma}))_{\sigma,p}$. (Note that $\theta, \sigma \in (0,1)$, whence Theorem 6.2.9 is applicable.) Now, since $\theta \operatorname{Re}\alpha - \operatorname{Re}\gamma > 0$, we have

$$\int_0^1 s^{-\operatorname{Re}\gamma} \left\| \tilde{\psi}(sA)x \right\| \frac{ds}{s} = \int_0^1 s^{\theta \operatorname{Re}\alpha - \operatorname{Re}\gamma} \left\| s^{-\theta \operatorname{Re}\alpha} \tilde{\psi}(sA)x \right\| \frac{ds}{s} < \infty$$

if $x \in (X, \mathcal{D}(A^\alpha))_{\theta,p}$. Applying b) of Proposition 6.6.2 yields $(X, \mathcal{D}(A^\alpha))_{\theta,p} \subset \mathcal{D}(A^\gamma)$. Moreover,

$$t^{-\sigma \operatorname{Re}(\beta-\gamma)} \psi(tA) A^\gamma x = t^{-\theta \operatorname{Re}\alpha} \tilde{\psi}(tA) x$$

for all $t > 0$. This concludes the proof of the stated equivalence.

c) follows immediately from b) since here one may suppose without loss of generality that A is invertible. ☐

Theorem 6.6.8 (Komatsu). *Let A be a sectorial operator on a Banach space X, and let $\theta \in (0,1), p \in [1,\infty]$. Suppose that the identity*

$$(X, \mathcal{D}(A^\alpha))_{\theta,p} = \mathcal{D}(A^{\theta\alpha}) \tag{6.12}$$

holds for some $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$. Then it holds for all *such α. Moreover, if A is invertible, its natural $H^\infty(S_\varphi)$-calculus is bounded on X for each $\varphi \in (\omega_A, \pi)$.*

Proof. Without loss of generality we may suppose that A is invertible (replace A by $A+1$). Then by Dore's theorem, A has bounded H^∞-calculus on $\mathcal{D}(A^{\theta\alpha})$, hence on X since $A^{-\theta\alpha}$ maps X isomorphically onto $\mathcal{D}(A^{\theta\alpha})$. In particular, $A \in \mathrm{BIP}$ and the spaces $\mathcal{D}(A^{\beta})$ depend only on $\operatorname{Re}\beta$ (see Proposition 3.5.5).

We show first that for fixed θ we have $\mathcal{D}(A^{\beta\theta}) = (X, \mathcal{D}(A^{\beta}))_{\theta,p}$ for arbitrary $\operatorname{Re}\beta > 0$. To obtain this we may suppose that $\operatorname{Re}\alpha \neq \operatorname{Re}\beta$. Suppose that $\operatorname{Re}\beta > \operatorname{Re}\alpha$. Then we let $\gamma := \beta - \alpha$ and $\theta_1 := \theta(\operatorname{Re}\beta - \operatorname{Re}\gamma)/(\operatorname{Re}\beta - \theta\operatorname{Re}\gamma)$ and find by Proposition 6.6.7 the identities

$$(X, \mathcal{D}(A^{\beta}))_{\theta,p} = (\mathcal{D}(A^{\theta\gamma}), \mathcal{D}(A^{\beta}))_{\theta_1,p}, \qquad (X, \mathcal{D}(A^{\beta-\gamma}))_{\theta,p} = (X, \mathcal{D}(A^{\beta-\theta\gamma}))_{\theta_1,p}.$$

Now $A^{-\theta\gamma} : (X, \mathcal{D}(A^{\beta-\theta\gamma}))_{\theta_1,p} \longrightarrow (\mathcal{D}(A^{\theta\gamma}), \mathcal{D}(A^{\beta}))_{\theta_1,p}$ is an isomorphism. Employing the assumption we obtain

$$(X, \mathcal{D}(A^{\beta}))_{\theta,p} = A^{-\theta\gamma}(X, \mathcal{D}(A^{\alpha}))_{\theta,p} = A^{-\theta\gamma}\mathcal{D}(A^{\theta\alpha}) = \mathcal{D}(A^{\theta\beta}).$$

The case $\operatorname{Re}\beta < \operatorname{Re}\alpha$ is proved analogously.

To complete the proof, let $\operatorname{Re}\gamma > 0$ and $0 < \theta_1 < 1$ with $\theta \neq \theta_1$. Find $\operatorname{Re}\beta > 0$ such that $\theta/\theta_1 = \operatorname{Re}\gamma/\operatorname{Re}\beta$. Then by Proposition 6.6.7 we have

$$(X, \mathcal{D}(A^{\gamma}))_{\theta_1,p} = (X, \mathcal{D}(A^{\beta}))_{\theta,p} = \mathcal{D}(A^{\theta\beta}) = \mathcal{D}(A^{\theta_1\gamma}). \qquad \square$$

6.6.4 The Complex Interpolation Spaces and BIP

So far we have encountered real interpolation spaces only. We have seen that these spaces allow characterisations in terms of the functional calculus and that the operator improves when restricted to these spaces. However, Komatsu's Theorem 6.6.8 shows that an identity of the form $(X, \mathcal{D}(A^{\alpha}))_{\theta,p} = \mathcal{D}(A^{\theta\alpha})$ can hold only if the operator A (or a translate of it) has bounded H^{∞}-calculus in the orginal space X.

On the other hand, using complex interpolation spaces we have the following intriguing result.

Theorem 6.6.9. *Let A be sectorial and densely defined such that $A+1 \in \mathrm{BIP}(X)$. Then, for $\theta \in (0,1)$, we have*

$$[\mathcal{D}(A^{\alpha}), \mathcal{D}(A^{\beta})]_{\theta} = \mathcal{D}(A^{(1-\theta)\alpha+\theta\beta})$$

for all $0 \leq \operatorname{Re}\alpha < \operatorname{Re}\beta$ with either $\operatorname{Re}\alpha > 0$ or $\alpha = 0$.

(The reader may consult Appendix B.3 for background information on the complex interpolation spaces.)

Proof. Replacing A by $A+1$ we may suppose without loss of generality that A is invertible. Moreover, applying the isomorphism A^{α} on both sides of the identity, reduces the whole statement to the identity

$$[X, \mathcal{D}(A^{\alpha})]_{\theta} = \mathcal{D}(A^{\theta\alpha}) \qquad (\operatorname{Re}\alpha > 0, \theta \in (0,1)).$$

From $A \in \mathrm{BIP}$ also follows that $\mathcal{D}(A^\alpha)$ depends only on the real part $\operatorname{Re}\alpha$, whence we may suppose that $\alpha > 0$.

By assumption, $\mathcal{D}(A)$ is dense in X and $(A^{-is})_{s\in\mathbb{R}}$ is a C_0-group on X, whence one can find $M \geq 1, \eta \geq 0$ such that

$$\|A^{is}\| \leq Me^{\eta|s|} \qquad (s \in \mathbb{R}).$$

Now, two inclusions have to be shown. To prove the first, we take $x \in \mathcal{D}(A^{\alpha\theta})$ and consider the function

$$f(z) := e^{(z-\theta)^2} A^{-\alpha z} A^{\theta\alpha} x \qquad (z \in \overline{S}).$$

Clearly $f(\theta) = x$ and $f : S \longrightarrow X$ is holomorphic. Since the norms $\|A^{is}\|$ of the imaginary powers grow at most exponentially as $s \to \infty$, $\sup_{z\in S} \|f(z)\|_X < \infty$. From Proposition 3.5.5 it follows that $f : \overline{S} \longrightarrow X$ is continuous and

$$\|f(1+is)\|_{\mathcal{D}(A^\alpha)} = \|A^\alpha f(1+is)\|_X \leq M\, e^{(1-\theta)^2} e^{-s^2} e^{\eta\alpha|s|} \|A^{\theta\alpha} x\|_X.$$

So in fact $f \in \mathcal{F}(X, \mathcal{D}(A^\alpha))$, and this implies that $x \in [X, \mathcal{D}(A^\alpha)]_\theta$.

The converse inclusion $[X, \mathcal{D}(A^\alpha)]_\theta \subset \mathcal{D}(A^\alpha)$ is shown by using the fact that $\mathcal{D}(A^\alpha)$ is dense in $[X, \mathcal{D}(A^\alpha)]_\theta$ (see Theorem B.3.3). Take $x \in \mathcal{D}(A^\alpha)$, and let $f \in \mathcal{V}(X, \mathcal{D}(A^\alpha))$ be such that $f(\theta) = x$ (cf. Proposition B.3.4). Then f is of the form $f(z) = \sum_{j=1}^n \varphi_j(z) a_j$ for certain $\varphi_j \in \mathcal{F}_0(\mathbb{C}, \mathbb{C})$ and $a_j \in \mathcal{D}(A^\alpha)$. Define

$$g(z) := e^{(z-\theta)^2} A^{\alpha z} f(z) = e^{(z-\theta)^2} \sum_{j=1}^n \varphi_j(z) A^{\alpha z} a_j \qquad (z \in \overline{S}).$$

Then $g \in \mathcal{F}_0(X, X)$ with $g(\theta) = A^{\theta\alpha} x$. Now we apply the Three Lines Lemma B.3.1:

$$\|x\|_{\mathcal{D}(A^{\theta\alpha})} = \|A^{\theta\alpha} x\|_X = \|f(\theta)\|_X \leq \sup_{s\in\mathbb{R}} \{\|g(is)\|_X\,, \|g(1+is)\|_X\}\,,$$

but for each $s \in \mathbb{R}$ we have

$$\begin{aligned} \|g(is)\|_X &\leq e^{\theta^2} e^{-s^2} \|A^{i\alpha s} f(is)\|_X \leq M e^{\theta^2} e^{-s^2} e^{\eta\alpha|s|} \|f(is)\|_X\,, \\ \|g(1+is)\|_X &\leq e^{(1-\theta)^2} e^{-s^2} \|A^\alpha A^{i\alpha s} f(1+is)\|_X \\ &\leq M e^{(1-\theta)^2} e^{-s^2} e^{\eta\alpha|s|} \|f(1+is)\|_{\mathcal{D}(A^\alpha)}\,. \end{aligned}$$

Hence we can find a constant $c > 0$ (not depending on f and x of course) such that $\|x\|_{\mathcal{D}(A^{\theta\alpha})} \leq c\,\|f\|_{\mathcal{F}}$. Taking the infimum with respect to f (and applying Proposition B.3.4) we find $\|x\|_{\mathcal{D}(A^{\theta\alpha})} \leq c\,\|x\|_{[X,\mathcal{D}(A^\alpha)]_\theta}$. Now density accounts for the rest. □

Remark 6.6.10. It is unknown in general whether the equality $[X, \mathcal{D}(A)]_\theta = \mathcal{D}(A^\theta)$ for some $\theta \in (0,1)$ is sufficient to have $A+1 \in \mathrm{BIP}$. It is true on Hilbert spaces due to the fact that $(X, \mathcal{D}(A))_{\theta,2} = [X, \mathcal{D}(A)]_\theta$ for all $\theta \in (0,1)$ (see [158, Corollary 4.3.12]).

Example 6.6.11. Consider the negative Laplacian $-\Delta_p$ on $X := \mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d)$, where $p \in (1, \infty)$ and $p \neq 2$. It has domain $\mathcal{D}(-\Delta_p) = \mathbf{W}^{\mathbf{2},\boldsymbol{p}}(\mathbb{R}^d)$ and is an injective, sectorial operator with spectral angle 0 and bounded H^∞-calculus. However,

$$(X, \mathcal{D}(A))_{\theta,p} \neq \mathcal{D}(A^\theta)$$

for all $\theta \in (0,1), p \in [1,\infty]$. See Section 8.3, in particular Remark 8.3.6.

It should be clear from these considerations that an identity of the form

$$(X, \mathcal{D}(A))_{\theta,p} = \mathcal{D}(A^\theta)$$

is — apart from the Hilbert space case — *extremely rare.*

6.7 Characterising Growth Conditions

When we examined perturbations B of a sectorial operator A in Section 5.5, an important condition proved to be an estimate of the form

$$\sup_{t>0} \left\| t^\theta B(t+A)^{-1} \right\| < \infty$$

(cf. Lemma 5.5.1 and Proposition 5.5.3). We remarked (in Remark 5.5.2) that one can describe such a condition in terms of interpolation spaces. The present section is to give evidence to this assertion.

Proposition 6.7.1. *Let A be an arbitrary sectorial operator on the Banach space X, let $B : Y \longrightarrow X$ be a bounded operator, where Y is another Banach space, and let $\theta \in [0,1]$. Then the equivalences*

$$\begin{aligned} \mathcal{R}(B) \subset (X, \mathcal{D}(A))_{\theta,\infty} &\iff \sup_{t>1} \left\| t^\theta (t+A)^{-1} B \right\|_{Y \to \mathcal{D}(A)} < \infty \\ &\iff \sup_{t>1} \left\| t^\theta (t+A)^{-1} B \right\|_{Y \to \dot{D}} < \infty \end{aligned}$$

hold, with the latter applying only in the case where A is injective. Furthermore, if A is injective, then also the equivalences

$$\begin{aligned} \mathcal{R}(B) \subset (X, \mathcal{R}(A))_{1-\theta,\infty} &\iff \sup_{0<t<1} \left\| t^\theta (t+A)^{-1} B \right\|_{Y \to X} < \infty \\ &\iff \sup_{0<t<1} \left\| t^\theta (t+A)^{-1} B \right\|_{Y \to \mathcal{D}(A)} < \infty \end{aligned}$$

hold true.

Proof. Let us consider the first assertion with its two equivalences. The case $\theta = 0$ is trivial, so let $\theta \in (0,1]$. We observe that, by the sectoriality of A, the second equivalence holds trivially. The first is derived immediately from Lemma

6.1.1 (see also the Komatsu representation in Section 6.2.3) and the Closed Graph Theorem. The second statement is obtained from the first by substituting $s := 1/t$ and replacing A by A^{-1}:

$$t^\theta(t+A)^{-1} = s^{1-\theta}\frac{1}{s}\left(\frac{1}{s}+A\right)^{-1} = s^{1-\theta}A^{-1}(s+A^{-1})^{-1}. \qquad \square$$

Combining both statements yields the following.

Corollary 6.7.2. *Let A be an injective sectorial operator on the Banach space X, let $B : Y \longrightarrow X$ be a bounded operator, where Y is another Banach space, and let $\theta \in [0,1]$. Then the condition*

$$\sup_{0<t<\infty} \left\| t^\theta(t+A)^{-1}B \right\|_{Y\to\mathcal{D}(A)} < \infty$$

is equivalent to $\mathcal{R}(B) \subset (\mathcal{D}(A), \mathcal{R}(A))_{1-\theta,\infty}$.

Proof. Combining both conditions from Proposition 6.7.1 yields the inclusion $\mathcal{R}(B) \subset (X, \mathcal{D}(A))_{\theta,\infty} \cap (X, \mathcal{R}(A))_{1-\theta,\infty}$. Now observe that $X = \mathcal{D}(A) + \mathcal{R}(A)$ and apply (B.7) from Proposition B.2.7. $\square$

Let us turn to the 'dual' conditions.

Proposition 6.7.3. *Let A be an arbitrary sectorial operator on the Banach space X, let $C : \mathcal{D}(A) \longrightarrow Y$ be a bounded operator, where Y is a second Banach space, and let $\theta \in [0,1]$. Then the condition*

$$\sup_{t>1} \left\| t^{1-\theta}C(t+A)^{-1} \right\|_{X\to Y} < \infty \tag{6.13}$$

is trivially satisfied if $\theta = 1$. In the case where $\theta \in (0,1)$, (6.13) is equivalent to the boundedness of

$$C : (X, \mathcal{D}(A))_{\theta,1} \longrightarrow Y$$

on the space $\mathcal{D}(A)$. In the case where $\theta = 0$ it is equivalent to the fact that C (defined on $\mathcal{D}(A)$) is bounded for the norm $\|\cdot\|_X$.

Proof. Let us start with the case $\theta \in (0,1)$. We use again the representation of the space $(X, \mathcal{D}(A))_{\theta,1}$ from Lemma 6.1.1. If $C : (X, \mathcal{D}(A))_{\theta,1} \longrightarrow X$ is bounded, we can estimate

$$\begin{aligned}
\left\| t^{1-\theta}C(t+A)^{-1}x \right\| &\le ct^{1-\theta}\left\| (t+A)^{-1}x \right\|_{(X,\mathcal{D}(A))_{\theta,1}} \\
&\le ct^{1-\theta}\left\|(t+A)^{-1}x\right\| + ct^{1-\theta}\int_0^\infty \left\| s^\theta A(s+A)^{-1}(t+A)^{-1}x \right\| \frac{ds}{s} \\
&\le ct^{-\theta}M(A) + c\int_0^\infty \left\| s^{\theta-1}A^{1-\theta}(1+s^{-1}A)^{-1}t^{-\theta}A^\theta(1+t^{-1}A)^{-1}x \right\| \frac{ds}{s} \\
&= ct^{-\theta}M(A) + c\int_0^\infty \left\| \varphi_{1/s}(A)\psi_{1/t}(A)x \right\| \frac{ds}{s}
\end{aligned}$$

for $x \in X$, $t > 0$ and some constant c, where we have used the notation $\varphi(z) := z^{1-\theta}/(1+z)$ and $\psi(z) := z^\theta/(1+z)$. By Theorem 5.2.2 b) the second summand is uniformly bounded in $t > 0$.

Suppose now that (6.13) holds, again under the assumption $\theta \in (0,1)$. We want to show the boundedness of $C : (X, \mathcal{D}(A))_{\theta,1} \longrightarrow Y$, and since $\mathcal{D}(A) = \mathcal{D}(A+1)$, we may without loss of generality suppose that A is invertible and

$$c' := \sup_{t>0} \left\| t^{1-\theta} C(t+A)^{-1} \right\| < \infty.$$

Theorem 6.6.1 yields that $\mathcal{D}(A^2)$ is dense in $(X, \mathcal{D}(A))_{\theta,1}$. Let $\tau(z) := z(1+z)^{-2}$ and $c := \int_0^\infty \tau(s^{-1})\, ds/s > 0$. Then we have $x = c^{-1} \int_0^\infty \tau(s^{-1}A)x\, ds/s$ for all $x \in \mathcal{D}(A)$ (Proposition 5.2.4). So, for $x \in \mathcal{D}(A^2)$ this integral converges in $\mathcal{D}(A)$, whence

$$\begin{aligned}
\|Cx\| &\le c^{-1} \int_0^\infty \left\| C\tau(s^{-1}A)x \right\| \frac{ds}{s} \\
&= c^{-1} \int_0^\infty \left\| C(1+s^{-1}A)^{-1} s^{-1}A(1+s^{-1}A)^{-1}x \right\| \frac{ds}{s} \\
&= c^{-1} \int_0^\infty \left\| s^{1-\theta} C(s+A)^{-1} s^\theta A(s+A)^{-1}x \right\| \frac{ds}{s} \\
&\le c'c^{-1} \int_0^\infty \left\| s^\theta A(s+A)^{-1}x \right\| \frac{ds}{s} \le c'c^{-1} \|x\|_{(X,\mathcal{D}(A))_{\theta,1}}
\end{aligned}$$

for $x \in \mathcal{D}(A^2)$. Since $\mathcal{D}(A^2)$ is dense in $(X, \mathcal{D}(A))_{\theta,1}$, we obtain the desired result.

We are left to deal with the cases $\theta = 0, 1$. It is easily seen that condition (6.13) with $\theta = 1$ is equivalent to $\|C\|_{\mathcal{D}(A)\to Y} < \infty$. So let $\theta = 0$. If C is bounded for the norm $\|\cdot\|_X$, then clearly (6.13) holds, by sectoriality of A. For the converse suppose that $\left\| tC(t+A)^{-1}x \right\|_Y \le \gamma \|x\|$ for each $x \in X$ and $t > 1$. Let $x \in \mathcal{D}(A)$. Then

$$\|Cx\| = \left\| tC(t+A)^{-1}x \right\| + \left\| C(t+A)^{-1}Ax \right\| \le \gamma \|x\| + t^{-1} \|Ax\|$$

for all $t > 1$. As $t \to \infty$ we obtain $\|Cx\| \le \gamma \|x\|$ for all $x \in \mathcal{D}(A)$. □

Proposition 6.7.4. *Let A be an injective sectorial operator on the Banach space X, let $C : \mathcal{D}(A) \longrightarrow Y$ a bounded operator, where Y is a second Banach space, and let $\theta \in [0,1]$. Then the condition*

$$\sup_{0<t<1} \left\| t^{1-\theta} C(t+A)^{-1} \right\|_{X\to Y} \tag{6.14}$$

is trivially satisfied for $\theta = 0$. In the case where $\theta \in (0,1)$, (6.14) is equivalent to the boundedness of

$$CA^{-1} : (X, \mathcal{R}(A))_{1-\theta,1} \longrightarrow Y$$

on $\mathcal{R}(A)$. In the case where $\theta = 1$ it is equivalent to the boundedness on $\mathcal{D}(A)$ of $C : \dot{D} \longrightarrow Y$.

Proof. Suppose first that $\theta \in (0,1)$. Writing $s = 1/t$ yields

$$t^{1-\theta}C(t+A)^{-1} = t^{-\theta}Ct(t+A)^{-1} = s^{\theta}C[I-s(s+A^{-1})^{-1} = s^{\theta}[CA^{-1}](s+A^{-1})^{-1}.$$

Now one can apply Proposition 6.7.3 with A replaced by A^{-1} and θ replaced by $1-\theta$.

The case $\theta = 0$ being easy, suppose now that $\theta = 1$. If $C : \dot{D} \longrightarrow Y$ is bounded, one has

$$\left\|C(t+A)^{-1}x\right\|_Y \leq \|C\|_{\dot{D}\to Y} \left\|A(t+A)^{-1}x\right\| \leq \|C\|_{\dot{D}\to Y} (M(A)+1),$$

hence (6.14). Conversely, suppose that $\left\|C(t+A)^{-1}x\right\| \leq \gamma \|x\|$ for all $x \in X$ and all $0 < t < 1$. Then

$$\|Cx\| \leq t\left\|C(t+A)^{-1}x\right\| + \left\|C(t+A)^{-1}Ax\right\| \leq t\gamma\|x\| + \gamma\|Ax\|$$

for $x \in \mathcal{D}(A)$. Hence if we let $t \to 0$, we obtain $\|Cx\| \leq \gamma \|x\|_{\dot{D}}$. □

Again, we combine the last two propositions.

Corollary 6.7.5. *Let A be an injective sectorial operator on the Banach space X, let $C : \mathcal{D}(A) \longrightarrow Y$ be a bounded operator, where Y is another Banach space, and let $\theta \in [0,1]$. Then the condition*

$$\sup_{0<t<\infty} \left\|t^{1-\theta}C(t+A)^{-1}\right\|_{X\to Y} < \infty$$

is equivalent to the boundedness on $\mathcal{D}(A)$ of

$$\begin{cases} C : X \longrightarrow Y, & \text{if} \quad \theta = 0, \\ C : (X, \dot{D})_{\theta,1} \longrightarrow Y, & \text{if} \quad \theta \in (0,1), \\ C : \dot{D} \longrightarrow Y, & \text{if} \quad \theta = 1. \end{cases}$$

Proof. In the cases $\theta = 0, 1$ this is just a combination of Propositions 6.7.3 and 6.7.4. Suppose that $\theta \in (0,1)$. Since A is injective, it induces a topological isomorphism

$$A : (\dot{D}, \mathcal{D}(A)) \longrightarrow (X, \mathcal{R}(A))$$

of Banach couples. Hence the boundedness of $CA^{-1} : (X, \mathcal{R}(A))_{1-\theta,1} \longrightarrow Y$ is then equivalent to the boundedness of $C : (\dot{D}, \mathcal{D}(A))_{1-\theta,1} \longrightarrow Y$. This yields the boundedness of $C : (X, \mathcal{D}(A))_{\theta,1} + (\dot{D}, \mathcal{D}(A))_{1-\theta,1} \longrightarrow Y$ as a characterisation. However, (B.6) of Proposition B.2.7 applies and we obtain the identity

$$(X, \mathcal{D}(A))_{\theta,1} + (\dot{D}, \mathcal{D}(A))_{1-\theta,1} = (X, \dot{D})_{\theta,1}.$$

□

6.8 Comments

6.2 and 6.5. Characterisations of Real Interpolation Spaces. Komatsu was probably the first who studied thoroughly the possible descriptions of real (i.e., Lions–Peetre) interpolation spaces for sectorial operators. The description by holomorphic semigroups (6.9) and (for $\alpha \in \mathbb{N}$) the proper 'Komatsu description' (6.8) can be found already in [132]. At least for the case $\alpha = 1$ (Lemma 6.1.1) they are now folklore and reproduced in many texts on parabolic equations, e.g. in [157]. Theorem 6.5.3 and the general characterisations in Section 6.2 are due to the author [106, 110]. Dore [73] proves a special case of Theorem 6.5.3 b).

6.1, 6.4, and 6.5. Dore's Theorem and Homogeneous Interpolation. Dore's results Theorem 6.1.3 from [72] and Corollary 6.5.8 (in the special case $\alpha = 1$) from [73] are by far not the first accounts of the fact that a sectorial operator improves some properties when restricted to its real interpolation spaces. Indeed, in 1975 Da Prato and Grisvard [58] discovered that an invertible sectorial operator always has *maximal regularity* in its interpolation spaces (cf. Section 9.3 below). And Auscher, McIntosh and Nahmod [19] pointed out that McIntosh's theorem on Hilbert spaces (Theorem 7.3.1) may also be read in this way (cf. Remark 7.3.2). Around the same time it turned out that an operator-valued version of Dore's theorem yields back the original results of Da Prato and Grisvard (cf. Section 9.3).

The crucial point lies in the fact that one has at hand a functional calculus description for the interpolation space where one is allowed to vary the auxiliary function (cf. the proof of Theorem 6.5.6). This idea is from [167] and [19], and in fact the whole Section 6.4 is a generalisation of Hilbert space results from (parts of) [19] to general Banach spaces. The central Theorem 6.1.3 is due to the author [110] and is itself only a special case of a far more general result which allows the functional calculus to be operator-valued and asserts even R-boundedness of the functional calculus in the case where $p \in (1, \infty)$. See [110] for more details.

6.3 Extrapolation Spaces. The concept of extrapolation space has proved quite useful in semigroup theory, in particular in the context of perturbations (see [85, Section III.3]). In the 'classical' approach one considers the norm of the (desired) extrapolation space X_{-1} on the original space X and takes the (abstract) completion. This embeds X densely into the new space X_{-1}. For example the homogeneous spaces $\dot{X}^{(\alpha)}$ would be the completion of X with respect to the norm $\|A^\alpha x\|$. In the case where A is densely defined this works well, but unfortunately breaks down otherwise. (The completion is then simply too small.) A one-step extrapolation without density was first given by Haak, Kunstmann and the author [100] in the context of perturbation theory.

Section 6.3 with its 'universal' extrapolation space goes back to the author's article [110, Appendix]. The construction can be considerably generalised. Let

X be a Banach space and $\mathcal{T} \subset \mathcal{L}(X)$ a set of pairwise commuting, injective, bounded linear operators on X, closed under multiplication. To each $T \in \mathcal{T}$ one can trivially construct a one-step extrapolation space $X_T \supset X$ together with an extension of T to an isometric isomorphism $T : X_T \longrightarrow T$. For two elements $S, T \in \mathcal{T}$ the space X_{ST} may be regarded as a common superspace of X_T and X_S since $ST = TS$. Hence the set of spaces $(X_T)_{T\in\mathcal{T}}$ is upward directed, and the inductive limit $U := \varinjlim_T X_T$ is an extrapolation space on which all operators $T \in \mathcal{T}$ become isomorphisms.

To identify extrapolation spaces in concrete situations with known spaces it is highly desirable to have a categorical definition of extrapolation spaces by means of a universal property. A first attempt can be found in [109].

Our notion of convergence on the universal space U is adapted to our purposes but is very unlikely to be induced by a proper vector space topology on U. Of course one thinks of the inductive limit topology as an alternative, but apart from very special cases (e.g., where one has weakly compact embeddings $X_{-n} \hookrightarrow X_{-(n+1)}$) one cannot guarantee that this topology is Hausdorff. This is so unpleasant a feature that we wanted to avoid it at any cost.

6.6 Intermediate Spaces and 6.7 Growth Conditions. The moment inequality (Proposition 6.6.4) and the Reiteration Theorem (Proposition 6.6.7) are well known, see e.g. [210], [211], [158], or [161, Chapter 11] and the references therein. We treat the matter for the sake of completeness, but also since it shows how the functional calculus descriptions of the interpolation spaces can be used. Theorem 6.6.9 on BIP and the complex interpolation spaces seems to go back to the paper [200] of Seeley for differential operators. Formulated and proved in full generality it may be found in [215, Theorem 1.15.3] or [161, Theorem 11.6.1].

Growth conditions as treated in Section 6.7 are common in perturbation theory. The characterisations given here in their full generality go back to the paper [100] by Haak, Kunstmann and the author.

Chapter 7

The Functional Calculus on Hilbert Spaces

We start with providing necessary background information on the functional calculus on Hilbert spaces. In Section 7.1 we show how numerical range conditions account for the boundedness of the H^∞-calculus. In the sector case this is essentially von Neumann's inequality (Section 7.1.3), in the strip case it is a result by CROUZEIX and DELYON (Section 7.1.5). From von Neumann's inequality we obtain certain 'mapping theorems for the numerical range' (Section 7.1.4). Section 7.2 is devoted to C_0-groups on Hilbert spaces. We discuss Liapunov's direct method (Section 7.2.1) from Linear Systems Theory and apply it to obtain a remarkable decomposition and similarity result for group generators (Section 7.2.2). This allows us to prove a theorem of DE LAUBENFELS and BOYADZHIEV on the boundedness of the H^∞-calculus on strips for such operators. This result can be approached also in a different way, yielding in addition a characterisation of group generators (Section 7.2.3). Section 7.3 is devoted mainly to the connection of the functional calculus with similarity theorems. The main tool is MCINTOSH's fundamental result on the boundedness of the H^∞-calculus (Section 7.3.1). The similarity questions we are interested in deal with operators defined by sesquilinear forms. We introduce these operators in Section 7.3.2 and obtain in Section 7.3.3 a characterisation of such operators up to similarity. Afterwards, several theorems on similarity are proved, related also to the so-called *square root problem.* We give an example of a C_0-semigroup which is not similar even to a quasi-contractive semigroup (Section 7.3.4). Finally, we present applications to generators of cosine functions, showing in particular that after a similarity transformation those operators always have numerical range in a horizontal parabola (Section 7.4).

Preliminaries on Functional Calculus

Before we can come to more substantial things we have to do some preliminary work.

Let $U \subset \mathbb{C}$ be open and invariant under complex conjugation, e.g., $U = S_\omega$ or $U = H_\omega$ for some ω. Let $f : U \longrightarrow \mathbb{C}_\infty$ be meromorphic. Then the function f^*, defined by

$$f^* := \left(z \longmapsto \overline{f(\overline{z})}\right) : U \longrightarrow \mathbb{C}_\infty$$

is called the **conjugate** of the function f. Obviously, f^* is meromorphic again.

Moreover, the spaces $H_0^\infty(S_\omega), H^\infty(S_\omega), \ldots$ and $\mathcal{F}(H_\omega), H^\infty(H_\omega), \ldots$ are invariant under conjugation.

The following proposition gives some background information on the functional calculus for sectorial operators on Hilbert spaces.

Proposition 7.0.1. *Let $A \in \operatorname{Sect}(\omega)$ be a sectorial operator on a Hilbert space H. Then the following assertions hold.*

a) *The operator A is densely defined and $H = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$. In particular, A is injective if and only if $\mathcal{R}(A)$ is dense in H.*

b) *The operator A^* is also sectorial of angle ω with $M(A, \omega') = M(A^*, \omega')$ for all $\omega' \in (\omega, \pi]$. In particular, we have $\omega_A = \omega_{A^*}$.*

c) *The operator A is injective (invertible, bounded) if and only if A^* is injective (invertible, bounded).*

d) *The identity $f(A^*) = [f^*(A)]^*$ holds whenever $f \in \mathcal{A}[S_\omega]$. If A is injective, it holds for all $f \in \mathcal{B}[S_\omega]$.*

e) *The identity*
$$(A^\alpha)^* = (A^*)^{\overline{\alpha}} \tag{7.1}$$
is true for all $\operatorname{Re}\alpha > 0$, and even for all $\alpha \in \mathbb{C}$ in the case that A is injective.

f) *If $A \in \operatorname{BIP}(H)$ then also $A^* \in \operatorname{BIP}(H)$ with $\theta_A = \theta_{A^*}$.*

g) *Let $\varphi \in (\omega, \pi]$. If the natural $H_0^\infty(S_\varphi)$-calculus for A is bounded, the same is true for A^* and the bounds are the same.*

Proof. a) follows from Proposition 2.1.1 since a Hilbert space is reflexive.

b) follows from Corollary C.2.2.

c) If A is injective, then $\mathcal{R}(A)$ is dense. By Proposition C.2.1 e) this implies that $\mathcal{N}(A^*) = 0$. The converse implication follows from $A^{**} = A$.

d) Let $f \in H_0^\infty(S_\varphi)$. Then

$$\begin{aligned}
[f^*(A)]^* &= \left(\frac{1}{2\pi i}\int_\Gamma f^*(z)R(z,A)\,dz\right)^* = \frac{-1}{2\pi i}\int_{\overline{\Gamma}} f(z)R(\overline{z},A)^*\,dz \\
&= \frac{-1}{2\pi i}\int_{\overline{\Gamma}} f(z)R(z,A^*)\,dz \stackrel{!}{=} \frac{1}{2\pi i}\int_\Gamma f(z)R(z,A^*)\,dz = f(A^*),
\end{aligned}$$

since Γ is such that $(z \longmapsto \overline{z})$ just reverses the orientation of Γ. Since $g^* = g$ if $g(z) = (1+z)^{-1}$ we thus obtain $f^*(A)^* = f(A^*)$ for all $f \in \mathcal{E}(S_\varphi)$. Now, if $f \in \mathcal{A}$ such that $F(z) := (1+z)^{-n} f(z) \in \mathcal{E}$ we have

$$f(A^*) = (1+A^*)^n F(A^*) \stackrel{(1)}{=} [(1+A)^n]^* [F^*(A)]^* \stackrel{(2)}{=} [F^*(A)(1+A)^n]^* \stackrel{(3)}{=} [f^*(A)]^*.$$

Here we have used Proposition C.2.3 in (1) and Proposition C.2.1 in (2). Equation (3) is justified by Proposition C.2.1 and the fact that $\mathcal{D}(A^n)$ is a core for $f^*(A)$ by Proposition 2.3.11 c).

The proof of the statement in the case that A is injective and $f \in \mathcal{B}$ is similar. One has to use the identity $[\Lambda_A^n]^* = \Lambda_{A^*}^n$, which holds for each $n \in \mathbb{N}$.

e), f) and g) are consequences of d). □

Of course, there is an analogous result for strip-type operators, but we refrain from stating it explicitly.

7.1 Numerical Range Conditions

In this section we examine sectorial and strip-type operators A on a Hilbert space H, with the additional property that the numerical range $W(A)$ is contained in the relevant set (sector or strip). The ultimate goal is to obtain boundedness of the natural H^∞-calculus for such operators.

7.1.1 Accretive and ω-accretive Operators

Let us recall that an operator A on a Hilbert space is called **accretive** if its **numerical range**

$$W(A) := \{ (Ax \,|\, x) \;|\; x \in \mathcal{D}(A),\ \|x\| = 1\}$$

(see Appendix C.3) is contained in the closed right half-plane $\overline{S_{\pi/2}}$. Since any restriction of an accretive operator is likewise accretive, an accretive operator need not be closed. However, if we assume in addition that $-1 \in \varrho(A)$ we obtain a so-called **m-accretive** operator, cf. Appendix C.7. By Proposition C.7.2 and Theorem C.7.3 the following assertions are equivalent for a closed operator A on H.

(i) A is m-accretive, i.e., $W(A) \subset \overline{S_{\pi/2}}$ and $\mathcal{R}(A + 1)$ is dense in H.

(ii) $-A$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ with $\|T(t)\| \leq 1$ for all $t \geq 0$.

(iii) $\{\operatorname{Re} z < 0\} \subset \varrho(A)$ and $\left\|(\lambda + A)^{-1}\right\| \leq (\operatorname{Re}\lambda)^{-1}$ for all $\operatorname{Re}\lambda > 0$.

In particular, $A \in \operatorname{Sect}(\pi/2)$.

As a matter of fact, one can consider also operators whose numerical range is contained in a smaller sector. Let $\omega \in [0, \pi/2]$. An operator A on a Hilbert space H is called **ω-accretive** if $W(A) \subset \overline{S_\omega}$. Note that this means

$$|\operatorname{Im}(Au \,|\, u)| \leq (\tan\omega)\operatorname{Re}(Au \,|\, u) \qquad (u \in \mathcal{D}(A)).$$

The operator A is called **m-ω-accretive** if it is ω-accretive and $\mathcal{R}(A + 1)$ is dense in H. Hence an operator is (m-)accretive if and only if it is (m-)$\pi/2$-accretive. Furthermore, each m-ω-accretive operator is m-accretive. A 0-accretive operator is symmetric (since our Hilbert spaces are complex). An operator is positive if and only if it is m-0-accretive. The following proposition gives a useful characterisation.

Proposition 7.1.1. *Let A be an operator on the Hilbert space H, let $\omega \in [0, \pi/2)$, and define $\theta := \pi/2 - \omega$. The following assertions are equivalent.*

(i) *The operator A is m-ω-accretive.*

(ii) *The operators $e^{\pm i\theta} A$ are m-accretive.*

(iii) *The operator $-A$ generates a holomorphic C_0-semigroup $(T(z))_{z \in S_\theta}$ on S_θ such that $\|T(z)\| \le 1$ for all $z \in S_\theta$.*

If one of these equivalent conditions is satisfied then $A \in \mathrm{Sect}(\omega)$.

Note that (ii) does not imply (i) if $\omega = 0$. For the meaning of (iii) cf. Section 3.4.

Proof. (i)$\Leftrightarrow$(ii). This is clear from Corollary C.3.2.
(i)$\Rightarrow$(iii). From (i) and Proposition C.3.1 we infer that $A \in \mathrm{Sect}(\omega)$. Since $\omega < \pi/2$ we conclude that $-A$ generates a bounded holomorphic semigroup $T : S_\theta \longrightarrow \mathcal{L}(H)$. For each $\varphi \in (-\theta, \theta)$ the operator $-e^{i\varphi} A$ generates the semigroup $(T(e^{i\varphi} t))_{t>0}$ (see Remark 3.4.5). Since $e^{i\varphi} A$ is m-accretive, $\left\|T(e^{i\varphi} t)\right\| \le 1$ for all $t > 0$, and this proves (iii).
(iii)$\Rightarrow$(i). As above, the operator $-e^{i\varphi} A$ generates the semigroup $(T(e^{i\varphi} t))_{t>0}$, for each $\varphi \in (-\theta, \theta)$. Since this is a contraction semigroup we conclude that $e^{i\varphi} A$ is m-accretive. Letting φ tend to $\pm\theta$ yields that $e^{\pm\theta} A$ is m-accretive.
The final statement follows from Proposition C.3.1. □

Parallel to sectorial operators we wish to study strip-type operators. Here we can say the following.

Proposition 7.1.2. *Let A be an operator on a Hilbert space H and let $\omega > 0$. Then the following assertions are equivalent.*

(i) $W(A) \subset \overline{H_\omega}$ *and* $\mathcal{R}(A \pm i(\omega + 1))$ *are dense in* H.

(ii) *The operators $\omega \pm iA$ are both m-accretive.*

(iii) *iA generates a C_0-group $(T(s))_{s \in \mathbb{R}}$ such that*

$$\|T(s)\| \le e^{\omega|s|} \qquad (s \in \mathbb{R}).$$

(iv) $\sigma(A) \subset \overline{H_\omega}$ *and*

$$\|R(\lambda, A)\| \le \frac{1}{|\mathrm{Im}\,\lambda| - \omega} \qquad (\lambda \notin \overline{H_\omega}).$$

(v) *There are self-adjoint operators B and C such that $-\omega \le C \le \omega$ and $A = B + iC$.*

(Note that the operator C in (v) *is necessarily bounded.) If these equivalent conditions are satisfied, $A \in \mathrm{SST}(\omega)$, $\mathcal{D}(A) = \mathcal{D}(A^*)$ and $A^* = B - iC$.*

Proof. The equivalence (i)$\Leftrightarrow$(ii) readily follows from the definition of 'm-accretive', while (ii)$\Leftrightarrow$ (iii) is due to the Lumer–Philips theorem C.7.3 and Proposition A.8.9. Assertions (iv) and (ii) are equivalent by Proposition C.7.2.

(v)$\Rightarrow$(i). Clearly the hypothesis implies that $W(A) \subset \overline{H_\omega}$. Moreover, iA is a bounded perturbation of the generator of a C_0-group, whence itself generates a C_0-group (Proposition A.8.8). In particular, A has some resolvent point outside the strip $\overline{H_\omega}$, and so (ii) follows.

Finally, assume that (i)–(iv) are satisfied. Applying the generalised Cauchy–Schwarz inequality (Proposition C.1.2) to the form

$$c(u,v) := \mathrm{Im}[(A\cdot \,|\, \cdot)](u,v) = \frac{1}{2i}\left((Au\,|\,v) - (u\,|\,Av)\right) \qquad (u,v \in \mathcal{D}(A))$$

on $V := \mathcal{D}(A)$ we obtain $|(Au\,|\,v) - (u\,|\,Av)| \le 2\omega\,\|u\|\,\|v\|$ for $u,v \in \mathcal{D}(A)$. Since $\mathcal{D}(A)$ is dense in H, the form c extends to a continuous symmetric form on H. Hence there exists a bounded symmetric operator $C \in \mathcal{L}(H)$ such that $(Cx\,|\,y) = ((Ax\,|\,y) - (x\,|\,Ay))/2i$. Obviously we have $-\omega \le C \le \omega$. Now we define $B := A - iC$. Then $(Bu\,|\,u) = (Au\,|\,u) - i\,\mathrm{Im}\,(Au\,|\,u) = \mathrm{Re}\,(Au\,|\,u)$ for all $u \in \mathcal{D}(A) = \mathcal{D}(B)$, whence $W(B) \subset \mathbb{R}$. Since A generates a C_0-group by (iii), also B does so (Proposition A.8.8), and this implies that B is skew-adjoint by Stone's theorem C.7.4.

The remaining assertions are clear since $A^* = (B+iC)^* = B^* - iC^* = B - iC$, by Proposition C.2.1 j). $\square$

An operator A on H is called **m-H_ω-accretive** if it satisfies the equivalent conditions of Proposition 7.1.2. Note that in Proposition 7.1.2, $\omega = 0$ is allowed, in which case the operator A is self-adjoint.

7.1.2 Normal Operators

Let us start with a discussion of normal operators. By the Spectral Theorem (see Appendix D) normal operators are unitarily equivalent to multiplication operators on $\mathbf{L^2}$-spaces over standard measure spaces.

Proposition 7.1.3. *Let (Ω,μ) be a standard measure space, let $\omega \in [0,\pi)$, and let $a : \Omega \to \mathbb{C}$ be a continuous function such that $a(\Omega) \subset \overline{S_\omega}$. Denote by $A := M_a$ the multiplication operator on $H := \mathbf{L^2}(\Omega,\mu)$. Then A is m-ω-accretive and $f(A) = M_{f\circ a}$ whenever $f \in \mathcal{O}[S_\omega]_A$.*

Proof. Let $\psi \in \mathcal{D}(A)$. Then

$$(A\psi\,|\,\psi)_{\mathbf{L^2}} = \int_\Omega a\psi\overline{\psi}\,d\mu = \int_\Omega a\,|\psi|^2\,d\mu.$$

Now $a(s)\,|\psi(s)|^2 \in \overline{S_\omega}$ for every $s \in \Omega$ by hypothesis and $\overline{S_\omega}$ is closed and convex. Hence also $\int_\Omega a\,|\psi|^2 \in \overline{S_\omega}$. This shows that $W(A) \subset \overline{S_\omega}$.

To prove the second assertion, suppose first that $f \in H_0^\infty[S_\omega]$. Then $f(A)$ is given by the integral $f(A) = (2\pi i)^{-1} \int_\Gamma f(z)R(z,A)\,dz$, which converges in the operator norm topology. By Proposition D.1.1, $R(z,A) = M_{(z-a)^{-1}}$ for each $z \notin \overline{S_\omega}$. Moreover, the mapping $(g \longmapsto M_g) : \mathbf{C^b}(\Omega) \longrightarrow \mathcal{L}(\mathbf{L^2}(\Omega,\mu))$ is an isometric embedding. Therefore, $f(A) = M_g$ for some bounded and continuous function on Ω. Since evaluation at a point $s \in \Omega$ is a continuous functional on $\mathbf{C^b}(\Omega)$, we obtain

$$g(s) = \left[\frac{1}{2\pi i}\int_\Gamma f(z)(z-a(\cdot))^{-1}\,dz\right](s) = \frac{1}{2\pi i}\int_\Gamma f(z)(z-a(s))^{-1}\,dz = f(a(s))$$

by Cauchy's theorem. Hence $f(A) = M_{f\circ a}$. Since this is obviously true for $f(z) = (1+z)^{-1}$ and for $f = \mathbf{1}$, it is true for all $f \in \mathcal{E}[S_\omega]$.

Let $f \in \mathcal{O}[S_\omega]_A$ and let $e \in \mathcal{E}[S_\omega]$ be a regulariser for f, i.e., $ef \in \mathcal{E}$ and $e(A)$ is injective. Since $e(A) = M_{e\circ a}$, the set $\{e \circ a = 0\}$ is locally μ-null. Therefore,

$$\begin{aligned}(x,y) \in f(A) \iff (ef)(A)x = e(A)y \iff (e\circ a)(f\circ a)x = (e\circ a)y \ \mu - a.e.\\ \iff (f\circ a)x = y \ \mu - a.e.\end{aligned}$$

This yields $f(A) = M_{f\circ a}$. □

Remark 7.1.4. Proposition 7.1.3 might look like a repetition of Example 2.3.15. But note that there we supposed the measure space to be σ-finite and applied Fubini's theorem. One could in fact justify this reasoning also in the case of a standard measure space (which in general is *not* σ-finite) but it is also instructive to see how continuity helps to avoid Fubini's theorem.

In the strip case we obtain the analogous result (with almost the same proof).

Proposition 7.1.5. *Let (Ω,μ) be a standard measure space, let $\omega \geq 0$, and let $a : \Omega \to \mathbb{C}$ be a continuous function such that $a(\Omega) \subset \overline{H_\omega}$. Denote by $A := M_a$ the multiplication operator on $H := \mathbf{L^2}(\Omega,\mu)$. Then $W(A) \subset \overline{H_\omega}$, $A \in \mathrm{Strip}(\omega)$ and $f(A) = M_{f\circ a}$ whenever $f \in \mathcal{O}[H_\omega]_A$.*

Corollary 7.1.6. *Let A be a self-adjoint operator on a Hilbert space H.*

a) *If $A \geq 0$, then $A \in \mathrm{Sect}(0)$ and for each $\varphi > 0$ the natural $H_0^\infty(S_\varphi)$-calculus is bounded. In fact $\|f(A)\| \leq \|f\|_{(0,\infty)}$ for all $f \in \mathcal{E}[S_0]$. If in addition A is injective, then for each $\varphi \in (0,\pi]$ the natural $H^\infty(S_\varphi)$-calculus for A is bounded (with bound 1).*

b) *We have $iA \in \mathrm{Sect}(\pi/2)$ and $\|f(iA)\| \leq \|f\|_{i\mathbb{R}}$ for all $f \in \mathcal{E}[S_{\pi/2}]$.*

c) *$A \in \mathrm{Strip}(0)$ and for each $\varphi > 0$ the natural $H^\infty(H_\varphi)$-calculus for A is bounded (with bound 1).*

Proof. By the Spectral Theorem D.5.1 there is a standard measure space (Ω,μ) and a continuous, real-valued function $a \in \mathbf{C}(\Omega)$ such that (H,A) is unitarily equivalent to $(\mathbf{L^2}(\Omega,\mu), M_a)$. Now the assertions follow from Proposition 7.1.3, Proposition 7.1.5 and Proposition D.1.1. □

Corollary 7.1.6 shows that self-adjoint operators behave well with respect to the functional calculus. Moreover, our restriction to self-adjoint operators is only for the sake of simplicity. Clearly the same proof works for normal operators. However, these results were obtained using the Spectral Theorem, which actually gives a much stronger result, namely a bounded Borel functional calculus (cf. Theorem D.6.1). It is interesting that in the context of *holomorphic* functional calculi sole conditions on the numerical range of A suffice. This will be shown in the next subsection.

7.1.3 Functional Calculus for m-accretive Operators

Here is the main result of this section.

Theorem 7.1.7. *Let $A \in \mathrm{Sect}(\pi/2)$ on the Hilbert space H. Then A is m-accretive if and only if*

$$\|f(A)\| \leq \|f\|_{\frac{\pi}{2}} \tag{7.2}$$

for all $f \in \mathcal{E}[S_{\pi/2}]$.

Before we give a proof of Theorem 7.1.7, let us draw an instant corollary.

Corollary 7.1.8. *Let A be a m-accretive operator on a Hilbert space H, and let $f \in H^\infty[S_{\pi/2}]_A$. Then $f(A)$ is bounded and $\|f(A)\| \leq \|f\|_{\pi/2}$.*

In particular, if A is injective then $A \in \mathrm{BIP}(H)$ and $\|A^{-is}\| \leq e^{\pi|s|/2}$, $s \in \mathbb{R}$.

Proof. If A is injective, the claim follows from Theorem 7.1.7 and Proposition 5.3.4; if A is not injective, approximate f by $f_n(z) := nf(z)(z+n)^{-1}$. Note that A is densely defined and $f(1+z)^{-1} \in \mathcal{E}$. (Cf. Lemma 2.3.8 and the proof of the Convergence Lemma (Proposition 5.1.4.)) □

Remark 7.1.9. 1) Theorem 7.1.7 is essentially equivalent to **von Neumann's inequality** for contractions T on a Hilbert space. Indeed, if A is m-accretive then its Cayley transform $T := (A-1)(A+1)^{-1}$ is a contraction, and von Neumann's inequality states that $\|p(T)\| \leq \|p\|_{\mathbb{D}}$ for every polynomial $p \in \mathbb{C}[z]$, where $\mathbb{D} := \{z \mid |z| \leq 1\}$ is the unit disc. This readily yields $\|r(T)\| \leq \|r\|_{\mathbb{D}}$ for every rational function r with poles outside of $\mathbb{D}$. Hence we can conclude that the natural $\mathcal{R}^\infty(S_{\pi/2})$-calculus for A is bounded with bound 1. Finally we apply Proposition 5.3.6. (Conversely, one can derive von Neumann's inequality from Theorem 7.1.7.)

2) Again, let A be an m-accretive operator on a Hilbert space. Our formulation of Theorem 7.1.7 seems artificial in that we take into account only functions f defined on sectors slightly larger than the half-plane $S_{\pi/2}$. This is of course due to the fact that we consider A as a sectorial operator. Clearly, one may ask for a functional calculus for A that incorporates functions just defined on

the half-plane $\{\operatorname{Re} z \geq 0\}$. Employing Proposition F.3 we can — by uniform approximation — extend the elementary functional calculus to the algebra

$$A(\mathbb{C}_+) := \left\{ f \in H^\infty(S_{\frac{\pi}{2}}) \cap \mathbf{C}(\overline{S_{\frac{\pi}{2}}}) \;\middle|\; \lim_{z \to \infty} f(z) \text{ exists} \right\}$$

such that the estimate (7.2) still holds. Whether one can go further, e.g. up to the whole of $H^\infty(S_{\pi/2})$, depends on particular properties of the operator A. In the analogous situation on the unit disc $\mathbb{D}$, these questions lead to the notion of *absolutely continuous* contractions, cf. [42].

We now give two proofs of Theorem 7.1.7. The first relies on the following theorem, which we state without proof. Recall that A is m-accretive if and only if $-A$ generates a contraction semigroup on H (Theorem C.7.3).

Theorem 7.1.10 (Szökefalvi-Nagy). *Let $(T(t))_{t \geq 0}$ be a contraction semigroup on a Hilbert space H. Then there exists a Hilbert space K, an isometric embedding $\iota : H \longrightarrow K$ and a unitary C_0-group $(U(t))_{t \in \mathbb{R}}$ on K such that*

$$P \circ U(t) \circ \iota = \iota \circ T(t)$$

for all $t \geq 0$. Here, $P : K \longrightarrow \iota(H)$ denotes the orthogonal projection on the closed subspace $\iota(H)$ of K.

The triple (K, U, ι) is called a **dilation** of the contraction semigroup T. For a proof see [61, Chapter 6, Section 3]. We now give our first proof of Theorem 7.1.7.

First Proof of Theorem 7.1.7. If we take $f(z) = (z-1)(z+1)^{-1}$ in (7.2) we obtain $\left\| (A-1)(A+1)^{-1} \right\| \leq 1$. So A is m-accretive by Proposition C.7.2, (iv).

The converse is proved by means of the Sz.-Nagy theorem. Suppose that A is m-accretive. Then $-A$ generates a contraction semigroup T. By Theorem 7.1.10 there is a dilation (K, U, ι) of T. By Stone's theorem (Theorem C.7.4) the generator of U is of the form $-iB$ where B is a self-adjoint operator on K. We claim that $PR(\lambda, iB)\iota = \iota R(\lambda, A)$ for each $\operatorname{Re} \lambda < 0$. In fact,

$$\begin{aligned} PR(\lambda, iB)\iota &= -P(-\lambda, -iB)\iota = -\int_0^\infty e^{\lambda s} PU(s)\iota \, ds = -\int_0^\infty e^{\lambda s} \iota T(s) \, ds \\ &= -\iota R(-\lambda, -A) = \iota R(\lambda, A). \end{aligned}$$

Now choose $\varphi \in (\pi/2, \pi]$ and $f \in H_0^\infty(S_\varphi)$. Then

$$\begin{aligned} Pf(iB)\iota &= \frac{1}{2\pi i} \int_\Gamma f(z) PR(z, iB)\iota \, dz = \frac{1}{2\pi i} \int_\Gamma f(z)\iota \, R(z, A) \, dz \\ &= \iota \, \frac{1}{2\pi i} \int_\Gamma f(z) R(z, A) \, dz = \iota f(A). \end{aligned}$$

Hence we obtain $Pf(iB)\iota = \iota f(A)$ for all $f \in \mathcal{E}$. Since ι is isometric, by b) of Corollary 7.1.6 we have

$$\begin{aligned}\|f(A)x\|_H &= \|\iota f(A)x\|_K = \|Pf(iB)\iota x\|_K \le \|f(iB)\iota x\|_K \\ &\le \|f\|_{\frac{\pi}{2}} \|\iota x\|_K = \|f\|_{\frac{\pi}{2}} \|x\|_H\end{aligned}$$

for every $x \in H$. This finishes the proof. □

This proof appears to be nice but does not really illuminate how the numerical range condition comes into play. We therefore give a second proof.

Second Proof of Theorem 7.1.7. By Propositions 5.3.6 and 5.3.7 combined with Lemma 5.3.5 it suffices to show (7.2) for $f \in \mathcal{R}_0(S_\varphi)$ for $\varphi \in (\pi/2, \pi)$ and A replaced by its sectorial approximation $A_\varepsilon := (\varepsilon + A)(1 + \varepsilon A)^{-1}$. Note that if A is m-accretive, A_ε is bounded and strictly accretive, i.e.,

$$\operatorname{Re}\,(\,A_\varepsilon x \,|\, x\,) \ge \varepsilon \,\|x\|^2 \qquad (x \in H).$$

Hence in the following we may suppose that A is bounded and strictly accretive. Given $f \in \mathcal{R}_0(S_\varphi)$ one can compute $f(A)$ by the usual Cauchy integral, but since A is bounded and invertible, one can shift the path of integration onto the imaginary axis (from $+i\infty$ to $-i\infty$). Now, on $i\mathbb{R}$ we have $\overline{z} = -z$, whence $R(z, A)^* = R(-z, A^*)$. We therefore can compute

$$\begin{aligned}f(A) &= \frac{1}{2\pi i}\int_{-i\mathbb{R}} f(z)R(z, A)\,dz \\ &= \frac{1}{2\pi i}\int_{-i\mathbb{R}} f(z)\left(R(z, A) + R(z, A)^*\right)\,dz - \frac{1}{2\pi i}\int_{-i\mathbb{R}} f(z)R(z, A)^*\,dz \\ &= \frac{-1}{2\pi i}\int_{-i\mathbb{R}} f(z)R(z, A)^*(A + A^*)R(z, A)\,dz - \frac{1}{2\pi i}\int_{-i\mathbb{R}} f(z)R(-z, A^*)\,dz.\end{aligned}$$

The latter integral equals zero since $f(z)R(-z, A^*)$ is holomorphic in the right half-plane and we can shift the path of integration to $\operatorname{Re} z = +\infty$. Hence we obtain

$$f(A) = \frac{1}{2\pi i}\int_{i\mathbb{R}} f(z)S(z)\,dz$$

with $S(z) := R(z, A)^*(A + A^*)R(z, A)$. Note that each $S(z)$ is a strictly positive self-adjoint operator. By inserting $f(z) := n/(n + z)$ and letting n tend to ∞, we obtain

$$I = \frac{1}{2\pi i}\int_{i\mathbb{R}} S(z)\,dz.$$

Thus, employing Corollary C.6.4 yields

$$\|f(A)\| = \left\|\frac{1}{2\pi}\int_{\mathbb{R}} f(it)S(it)\,dt\right\| \le \|f\|_\infty \left\|\frac{1}{2\pi}\int_{\mathbb{R}} S(it)\,dt\right\| = \|f\|_\infty\,.$$ □

Remark 7.1.11. Specialising $H = \mathbb{C}$ and $A = a$ in the above proof yields the **Poisson formula**

$$f(a) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(is) \frac{\operatorname{Re} a}{(\operatorname{Re} a)^2 + (\operatorname{Im} a - s)^2} \, ds \qquad (\operatorname{Re} a > 0)$$

for rational functions f bounded on the half-plane $\{\operatorname{Re} z \geq 0\}$.

Our second proof reveals a basic scheme: The numerical range condition implies positivity of certain self-adjoint operators; the integral over this family has a norm independent of the individual operator A; then, Proposition C.6.3 furnishes the final estimate. We shall encounter the same scheme in the proof of Theorem 7.1.16.

7.1.4 Mapping Theorems for the Numerical Range

As an application of Theorem 7.1.7 we prove what is called 'mapping theorems for the numerical range', a name somewhat inaccurate but not totally inappropriate. Here is the main result.

Theorem 7.1.12. *Let A be an m-accretive operator on the Hilbert space H, and let $f \in \mathcal{M}[S_{\pi/2}]_A$. Then*

$$W(f(A)) \subset \bigcap_{\varphi > \frac{\pi}{2}} \overline{\operatorname{conv}} f(S_\varphi).$$

Proof. Fix $\varphi \in (\pi/2, \pi)$ such that $f \in \mathcal{O}(S_\varphi)_A$. Without loss of generality we may suppose that the set $\overline{\operatorname{conv}(f(S_\varphi)}$ is not the whole plane, so there must be a proper closed half-plane containing it, and it is in fact the intersection of all such half-planes. Let P be any closed half-plane which contains $f(S_\varphi)$. Then there are numbers $a, b \in \mathbb{C}$ such that $a + bP$ equals the right half-plane $\{\operatorname{Re} z \geq 0\}$. Therefore,

$$\left| \frac{a + bf - 1}{a + bf + 1} \right| \leq 1 \qquad (z \in S_\varphi).$$

Hence Theorem 7.1.7 (or rather the succeeding corollary) and some general functional calculus arguments yield that the operator $a + bf(A) + 1$ is invertible and

$$\left\| (a + bf(A) + 1)(a + f(A) + 1)^{-1} \right\| \leq 1.$$

So $a + bf(A)$ is m-accretive, and this means that $a + bW(f(A)) = W(a + bf(A)) \subset \{\operatorname{Re} z \geq 0\}$, whence $W(f(A)) \subset P$. $\square$

Corollary 7.1.13. *Let A be m-accretive, and let $\alpha \in (0,1)$. Then the operator A^α is m-$\alpha\pi/2$-accretive, i.e., one has*

$$|\operatorname{Im}(A^\alpha x \,|\, x)| \leq \tan \frac{\alpha\pi}{2} \operatorname{Re}(A^\alpha x \,|\, x) \qquad (x \in \mathcal{D}(A^\alpha)).$$

Proof. Let $f(z) := z^\alpha$. Then $f(S_\varphi) = S_{\alpha\varphi}$ for every $\varphi \in (\pi/2, \pi)$, and the assertion follows from Theorem 7.1.12. □

We remark that the converse of Corollary 7.1.13 is not true, as SIMARD has shown. In fact there is $\alpha \in (0, 1)$ and an operator A on the two-dimensional Hilbert space such that A is not accretive but A^α is $\alpha\pi/2$-accretive (see [203, Proposition 6]). (Surprisingly, the converse does hold *modulo similarity*; this was proved by LE MERDY [149].) Thus it becomes clear that Theorem 7.1.12 is of restricted value when the original operator is not m-accretive but satisfies other numerical range conditions. Sometimes however one is lucky, like in the following example.

Corollary 7.1.14. *Let $\omega \in [0, \pi/2]$, and let A be an injective m-ω-accretive operator. Then* $\log A$ *is m-H_ω-accretive.*

Proof. Theorem 7.1.12 shows that $\log A$ is m-$H_{\pi/2}$-accretive whenever A is m-accretive. Define $\theta := \pi/2 - \omega$ and apply the foregoing to $e^{\pm i\theta}$. Since $\log(e^{\pm i\theta} A) = \pm i\theta + \log(A)$ we obtain

$$\pm i\theta + W(\log(A)) = W(\log(e^{\pm i\theta} A)) \subset H_{\frac{\pi}{2}}. \qquad \square$$

For completeness, we state another mapping theorem for the numerical range.

Corollary 7.1.15. *Let $\delta > 0$ and $A - \delta$ be m-accretive. Then $A^\alpha - \delta^\alpha$ is m-accretive for each $\alpha \in (0, 1)$.*

Proof. We reproduce the proof from [210, Lemma 2.3.6]. The assumption implies that $t \left\|(A - \delta + t)^{-1}\right\| \le 1$ for $t > 0$. Replacing t by $t + \delta$ we obtain

$$\begin{aligned} \operatorname{Re}\left(A(t+A)^{-1}x \,\middle|\, x \right) &= \|x\|^2 - t \operatorname{Re}\left((t+A)^{-1}x \,\middle|\, x \right) \\ &\ge \|x\|^2 - \frac{t}{t+\delta}\|x\|^2 = \frac{\delta}{t+\delta}\|x\|^2 . \end{aligned}$$

Now the Balakrishnan representation (3.2) yields for $x \in \mathcal{D}(A)$

$$\begin{aligned} \operatorname{Re}\left(A^\alpha x \,|\, x \right) &= \frac{\sin \alpha\pi}{\alpha} \int_0^\infty t^{\alpha-1} \operatorname{Re}\left((t+A)^{-1}Ax \,\middle|\, x \right) dt \\ &\ge \frac{\sin \alpha\pi}{\alpha} \int_0^\infty t^{\alpha-1} \frac{\delta}{t+\delta}\, dt \;\; \|x\|^2 = \delta^\alpha \|x\|^2 . \end{aligned}$$

Since $\mathcal{D}(A)$ is dense in H, it is a core for A^α. Thus by approximation we obtain $\operatorname{Re}\left(A^\alpha x \,|\, x \right) \ge \delta^\alpha \|x\|^2$ for all $x \in \mathcal{D}(A^\alpha)$, i.e., $A^\alpha - \delta^\alpha$ is m-accretive. □

7.1.5 The Crouzeix–Delyon Theorem

We now turn to the strip case, i.e., to the case of an m-H_ω-accretive operator for $\omega > 0$. (The case $\omega = 0$ has been treated in Corollary 7.1.6 c).) The following result was originally published in [55].

Theorem 7.1.16 (Crouzeix–Delyon). *Let $\omega > 0$ and let A be a m-H_ω-accretive operator on the Hilbert space H. Then*

$$\|f(A)\| \leq \left(2 + 2/\sqrt{3}\right) \|f\|_{H_\omega} \qquad (f \in H^\infty[H_\omega]).$$

Proof. We begin with some reducing steps. First, we may suppose that $f \in \mathcal{F}[H_\omega]$. (Use Proposition 5.1.7 and approximate f by $f_n(z) := f(z)\,(n/(n+\omega+iz))^2$.) Next we reduce the theorem to bounded operators being $H_{\varepsilon\omega}$-accretive for some $0 < \varepsilon < 1$. To this aim, consider the operator $A_n := (1-1/n)[n(n+\omega+iA)^{-1}]^* A[n(n+\omega+iA)^{-1}]$. Clearly, A_n is bounded and $H_{(1-1/n)\omega}$-accretive and $A_n x \to Ax$ for all $x \in \mathcal{D}(A)$. Using Proposition 7.1.2 we see that $\|R(\lambda, A_n)\|$ is bounded uniformly in $n \in \mathbb{N}$ and λ on horizontal lines. This implies that

$$R(\lambda, A_n)x - R(\lambda, A)x = R(\lambda, A_n)[A_n - A]R(\lambda, A)x \to 0$$

for each $x \in H$. Hence $f(A_n) \to f(A)$ strongly for every $f \in \mathcal{F}[H_\omega]$.

We thus may suppose that A is bounded and $H_{\omega'}$-accretive for some $0 < \omega' < \omega$. Write $A = B + iC$ with B, C bounded and self-adjoint and $-\omega' \leq C \leq \omega'$. For $f \in \mathcal{F}[H_\omega]$ we can shift the path of integration to ∂H_ω to obtain

$$\begin{aligned} f(A) &= \frac{1}{2\pi i} \int_{\partial H_\omega} f(z) R(z, A)\, dz \\ &= \frac{1}{2\pi i} \int_{\partial H_\omega} f(z)\left(R(z,A) - R(z,A)^*\right)\, dz + \frac{1}{2\pi i} \int_{\partial H_\omega} f(z) R(\overline{z}, A^*)\, dz \\ &= \int_{\partial H_\omega} f(z) T(z)\, dz + \frac{1}{2\pi i} \int_{\partial H_\omega} f(z) R(\overline{z}, A^*)\, dz, \end{aligned} \tag{7.3}$$

with

$$T(z) := \frac{1}{2\pi i}\left(R(z,A) - R(z,A)^*\right) = \frac{1}{\pi} R(z,A)^*(C - \operatorname{Im} z) R(z,A).$$

Note that $T(z) \geq 0$ when $\operatorname{Im} z \leq -\omega$ and $T(z) \leq 0$ when $\operatorname{Im} z \geq \omega$. Let us deal with the first summand. We define $S(z) := \begin{cases} -T(z) & \operatorname{Im} z \geq \omega \\ T(z) & \operatorname{Im} z \leq -\omega \end{cases}$. Then $T(z)\, dz = S(z)\, |dz|$ on ∂H_ω. Hence

$$S := \int_{\partial H_\omega} T(z)\, dz = \int_{\partial H_\omega} S(z)\ |dz|$$

is a positive operator. Moreover, by Corollary C.6.4 we can estimate

$$\left\| \int_{\partial H_\omega} f(z) T(z)\, dz \right\| \leq \|f\|_{H_\omega} \|S\|, \tag{7.4}$$

so we have to specify $\|S\|$. To this end we write

$$\begin{aligned} T(z) &= \frac{1}{2\pi i}\left(R(z,A) - \frac{1}{z}\right) + \frac{1}{2\pi i}\left(\frac{1}{z} - \frac{1}{\overline{z}}\right) + \frac{1}{2\pi i}\left(\frac{1}{\overline{z}} - R(\overline{z}, A^*)\right) \\ &= \frac{1}{2\pi i}\frac{R(z,A)A}{z} + \frac{-\operatorname{Im} z}{\pi\,|z|^2} - \frac{1}{2\pi i}\frac{R(\overline{z},A^*)A^*}{\overline{z}}. \end{aligned}$$

When we form $S = \int_{\partial H_\omega} T(z)\,dz$ the first and the last summand vanish by Cauchy's theorem (deform the path according to $\omega \to \infty$). Hence we arrive at

$$S = \int_{\partial H_\omega} \frac{-\operatorname{Im} z}{\pi\,|z|^2}\,dz = \int_{\mathbb{R}} \frac{\omega}{\pi\,|r - i\omega|^2}\,dr - \int_{\mathbb{R}} \frac{-\omega}{\pi\,|r + i\omega|^2}\,dr = 2. \tag{7.5}$$

We now treat the second summand, namely

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\partial H_\omega} f(z)\,R(\overline{z}, A^*)\,dz \\ &\quad= \frac{1}{2\pi i}\int_{\mathbb{R}} f(r - i\omega)\,R(r + i\omega, A^*)\,dz - \frac{1}{2\pi i}\int_{\mathbb{R}} f(r + i\omega)\,R(r - i\omega, A^*)\,dz \\ &\quad= \frac{1}{2\pi i}\int_{\operatorname{Im} z = -\omega} f(z)R(z + 2i\omega, A^*)\,dz - \frac{1}{2\pi i}\int_{\operatorname{Im} z = \omega} f(z)R(z - 2i\omega, A^*)\,dz \\ &\quad\overset{(*)}{=} \frac{1}{2\pi i}\int_{\mathbb{R}} f(z)R(z + 2i\omega, A^*)\,dz - \frac{1}{2\pi i}\int_{\mathbb{R}} f(z)R(z - 2i\omega, A^*)\,dz \\ &\quad= \frac{1}{2\pi i}\int_{\mathbb{R}} f(z)\left[R(z + 2i\omega, A^*) - R(z - 2i\omega, A^*)\right]\,dz \\ &\quad= \frac{-2\omega}{\pi}\int_{\mathbb{R}} f(t)R(t + 2i\omega, A^*)\,R(t - 2i\omega, A^*)\,dt, \end{aligned}$$

where in $(*)$ we have used Cauchy's theorem. Write

$$\begin{aligned} R(t{+}2i\omega, A^*)R(t - 2i\omega, A^*) &= \left((t - A^*)^2 + 4\omega^2\right)^{-1} = \left((t - B + iC)^2 + 4\omega^2\right)^{-1} \\ &= \left((t-B)^2 + 4\omega^2 - C^2 + i[2tC - CB - BC]\right)^{-1} \\ &= (M_t + iN_t)^{-1} = M_t^{-\frac{1}{2}}(I + iR_t)^{-1}M_t^{-\frac{1}{2}} \end{aligned}$$

with $M_t := (t - B)^2 + 4\omega^2 - C^2$, $N_t := 2tC - BC - CB$ and $R_t := M_t^{-1/2}N_tM_t^{-1/2}$. Note that all operators M_t, N_t, R_t are self-adjoint and one has $M_t \geq 3\omega^2$. Now $\left\|(I + iR_t)^{-1}\right\| \leq 1$ for all t since R_t is self-adjoint. Employing Proposition C.6.3 we obtain

$$\begin{aligned} \left\|\frac{1}{2\pi i}\int_{\partial H_\omega} f(z)\,R(\overline{z}, A^*)\,dz\right\| &= \frac{2\omega}{\pi}\left\|\int_{\mathbb{R}} f(t)M_t^{-\frac{1}{2}}(I + iR_t)^{-1}M_t^{-\frac{1}{2}}\,dt\right\| \\ &\leq \|f\|_{H_\omega}\,\frac{2\omega}{\pi}\left\|\int_{\mathbb{R}} M_t^{-1}\,dt\right\|. \end{aligned} \tag{7.6}$$

Now, $M_t \geq (t-B)^2 + 3\omega^2$ whence $M_t^{-1} \leq \left((t-B)^2 + 3\omega^2\right)^{-1}$ by Proposition C.4.7. This gives

$$\int_{\mathbb{R}} M_t^{-1}\, dt \leq \int_{\mathbb{R}} \left((t-B)^2 + 3\omega^2\right)^{-1}\, dt. \tag{7.7}$$

Claim: $\displaystyle\int_{\mathbb{R}} \left((t-B)^2 + 3\omega^2\right)^{-1}\, dt = \frac{\pi}{\sqrt{3}\omega}.$

Proof of Claim. The assertion is readily seen to be true by using the Spectral Theorem. Without this powerful tool one can argue as follows (with $\eta := \sqrt{3}\omega$):

$$\begin{aligned}
&\int_{\mathbb{R}} \left((t-B)^2 + \eta^2\right)^{-1}\, dt = \frac{1}{2i\eta}\int_{\mathbb{R}} \left(R(t-i\eta,B) - R(t+i\eta,B)\right)\, dt \\
&= \frac{1}{2i\eta}\int_{\mathbb{R}} \left(R(t-i\eta,B) - \frac{1}{t-i\eta}\right) + \left(\frac{1}{t-i\eta} - \frac{1}{t+i\eta}\right) - \left(R(t+i\eta,B) - \frac{1}{t+i\eta}\right) dt \\
&= \frac{1}{2i\eta}\int_{\mathbb{R}} \frac{R(t-i\eta,B)B}{t-i\eta} - \frac{R(t+i\eta,B)B}{t+i\eta}\, dt + \int_{\mathbb{R}} \frac{dt}{t^2+\eta^2} \\
&= \frac{1}{2i\eta}\int_{\partial H_\eta} \frac{R(z,B)B}{z}\, dz + \frac{\pi}{\eta} = \frac{\pi}{\eta},
\end{aligned}$$

by Cauchy's theorem (make a change of variable $w := z^{-1}$ in the integral and note that $w^{-1}R(w^{-1},B)$ is holomorphic at 0).

The Claim together with (7.7) shows that $\left\|\int_{\mathbb{R}} M_t^{-1}\, dt\right\| \leq \pi/(\sqrt{3}\omega)$. Combining this with (7.3), (7.4), (7.5), and (7.6) yields

$$\|f(A)\| \leq \|f\|_{H_\omega} 2 + \|f\|_{H_\omega} \frac{2\omega}{\pi}\frac{\pi}{\sqrt{3}\omega} = \left(2 + 2/\sqrt{3}\right)\|f\|_{H_\omega}. \qquad \square$$

As a corollary we obtain a statement for ω-accretive operators.

Corollary 7.1.17. *Let A be m-ω-accretive for some $\omega \in [0, \pi/2]$. Then*

$$\|f(A)\| \leq \left(2 + \frac{2}{\sqrt{3}}\right)\|f\|_{S_\omega} \tag{7.8}$$

for all $f \in \mathcal{R}(S_\omega)$. If A is injective, then (7.8) *holds for all $f \in H^\infty[S_\omega]$.*

Proof. Let $B := \log A$ and $c := 2 + 2/\sqrt{3}$. Then B is H_ω-accretive by Corollary 7.1.14. Hence

$$\|f(A)\| = \|f(e^z)(B)\| \leq c\,\|f(e^z)\|_{H_\omega} = c\,\|f\|_{S_\omega}$$

by the composition rule. $\square$

Remark 7.1.18. A different (and easier) proof can be given for the following, considerably weaker, statement: If A is H_ω-accretive and $\varphi \in (\omega, \pi)$, then the natural $H^\infty(H_\varphi)$-calculus for A is bounded. Indeed, one can view $A = B + iC$ as a bounded perturbation of a self-adjoint operator and use an argument similar to the proof of Proposition 5.5.3. This has been done in [105].

7.2 Group Generators on Hilbert Spaces

We have seen in the previous section that an m-H_ω-accretive operator A on a Hilbert space H has bounded natural $H^\infty(H_\varphi)$-calculi for all $\varphi > \omega$. Due to the numerical range condition, the bounds are even independent of φ. In this section we weaken the assumptions made on A and nevertheless retain the conclusion (with bounds now being dependent on φ.) More precisely, we shall prove the following theorem result from [34].

Theorem 7.2.1 (Boyadzhiev–de Laubenfels). *Let iA generate a C_0-group T on the Hilbert space H. Then, for each $\alpha > \theta(T)$, the natural $H^\infty(H_\alpha)$-calculus for A is bounded.*

Remark 7.2.2. Let us remark that Monniaux's Theorem 4.4.3 in the Hilbert space case is an immediate corollary of Theorem 7.2.1. Indeed, if iB generates the group T, then $\omega_{st(B)} = \theta(T)$ by Gearhart's theorem. If this is smaller than π, e^B must be sectorial by Proposition 5.3.3 and Theorem 7.2.1.

We shall prove Theorem 7.2.1 in two different ways. The first employs Theorem 7.1.16 (or Remark 7.1.18) by an elegant similarity result. The second approach yields even a characterisation of group generators on Hilbert spaces.

The key idea for our first proof of Theorem 7.2.1 is that the conclusion is not dependent on any particular scalar product on H. So if we can find, for fixed $\alpha > \theta(T)$, an equivalent scalar product $(\,.\,|\,.\,)_\circ$ with respect to which the operator A is H_α-accretive, we are done by the Crouzeix–Delyon Theorem 7.1.16. This method indeed works, see Theorem 7.2.8 below. However, we have to make a little detour before.

7.2.1 Liapunov's Direct Method for Groups

Recall the classical Liapunov theorem for linear dynamical systems in $\mathbb{C}^n$.

Theorem 7.2.3 (Liapunov). *Let $A \in \mathrm{Mat}(n, \mathbb{C})$ with $\sigma(A) \subset \{z \mid \operatorname{Re} z > 0\}$. Then there is a Hilbert norm $\|\cdot\|_\circ$ on $\mathbb{C}^n$ and $\varepsilon > 0$ such that*

$$\left\|e^{-tA}\right\|_\circ \le e^{-\varepsilon t} \qquad (t \ge 0).$$

The theorem has two components. *First* it states that the spectral condition

$$s(-A) := \sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(-A)\} < 0$$

for the generator $-A$ of a semigroup $T(t) = e^{-tA}$ on the Hilbert space $\mathbb{C}^n$ implies the exponential stability of the semigroup. *Second* it states that an exponentially stable semigroup on $(\mathbb{C}^n, \|\cdot\|_2)$ is similar to a contraction semigroup.

One can wonder about infinite-dimensional analogues. Concerning the first part, it is well known that a sole condition on the position of the spectrum does not imply exponential stability of the semigroup on a general Hilbert space. (See [10, Example 5.3.2] or [57, Example 5.1.4].) Concerning the second part, we shall see in Section 7.3.4 below that there is a Hilbert space H and a C_0-semigroup T on H that is not similar to a quasi-contractive semigroup, i.e., for *no* $\omega \in \mathbb{R}$ one can find an equivalent Hilbert norm $\|\cdot\|_\circ$ on H such that $\|T(t)\|_\circ \le e^{\omega t}$ for all $t \ge 0$.

So if one is interested in infinite-dimensional versions of Liapunov's theorem, one has to impose further conditions. One guiding principle is to look at the proof in finite dimensions. The central point is to find a **Liapunov function** for the dynamical system given by

$$\dot{u} + (A - \varepsilon)u = 0,$$

i.e., a function which decreases along the orbits $(t \longmapsto e^{\varepsilon t}T(t)x)$. In the finite-dimensional situation such a Liapunov function is given by

$$\|x\|_\circ^2 := \int_0^\infty \left\|e^{\varepsilon t}T(t)x\right\|^2 \, dt \tag{7.9}$$

for $x \in H$. (Note that $(e^{\varepsilon t}T(t))_{t\ge 0}$ is still exponentially stable.) This is sometimes called **Liapunov's direct method**. It is easily seen that by (7.9) a continuous norm is defined. In general, this norm may not be equivalent to the orginal one (see the remark after Proposition 7.2.7 below), but in the special case where T is a *group*, everything works.

Proposition 7.2.4. *Let A be the generator of an exponentially stable C_0-semigroup T on H. Then the operator Q defined by*

$$Q := \int_0^\infty T(t)^*T(t)\, dt$$

is a bounded, positive, and injective operator on H. By

$$(\, x \,|\, y\,)_\circ := (\, Qx \,|\, y\,) = \int_0^\infty (\, T(t)x \,|\, T(t)y\,)\; dt$$

a continuous scalar product is defined on H. The semigroup T is contractive with respect to $\|\cdot\|_\circ$, and Q satisfies the **Liapunov inclusion**

$$QA \subset -I - A^*Q. \tag{7.10}$$

Equivalently, $Q\mathcal{D}(A) \subset \mathcal{D}(A^*)$ and $QAx + A^*Qx = -x$ for all $x \in \mathcal{D}(A)$. If T is a group, then Q is invertible and $(\,\cdot\,|\,\cdot\,)_\circ$ is an equivalent scalar product on H. Moreover, one has

$$QA = -I - A^*Q \quad \textit{and} \quad A = -Q^{-1} - A^\circ, \tag{7.11}$$

where A° denotes the adjoint of A with respect to the scalar product $(\,\cdot\,|\,\cdot\,)_\circ$.

Proof. Boundedness and positivity of Q are clear from the definition. Since we have $(Qx \mid x) = \int_0^\infty \|T(t)x\|^2\, dt$, the operator Q is injective and $(\cdot \mid \cdot)_\circ$ is in fact a scalar product (not just a semi-scalar product, see Proposition C.5.1). A simple change of variable yields the formula $\|T(t_0)x\|_\circ \le \|x\|_\circ$ for all $x \in H$ and $t_0 \ge 0$. If T is a group, we find $M \ge 1$ and $\omega > 0$ such that $\|T(t)\| \le Me^{\omega|t|}$ for all $t \in \mathbb{R}$. Thus,

$$\|x\|_\circ^2 = (Qx \mid x) = \int_0^\infty \|T(t)x\|^2\, dt \ge \int_0^\infty M^{-2}e^{-2\omega t}\, dt\, \|x\|^2 = \frac{1}{2\omega M^2}\|x\|^2$$

for all $x \in H$. Hence in this case the new scalar product is equivalent to the old one and Q is invertible (Proposition C.5.1). For the proof of the Liapunov inclusion note that since T is exponentially stable, the resolvent of its generator A is given by

$$-A^{-1} = R(0, A) = \int_0^\infty T(s)\, ds$$

(see Proposition A.8.1). Therefore,

$$\begin{aligned} Q(-A^{-1}) &= \int_0^\infty T(t)^*T(t)\int_0^\infty T(s)\, ds\, dt = \int_0^\infty T^*(t)\int_t^\infty T(s)\, ds\, dt \\ &= \int_0^\infty \left(\int_0^s T(t)^*\, dt\right) T(s)\, ds. \end{aligned}$$

Similarly,

$$-(A^*)^{-1}Q = \int_0^\infty T(t)^*\int_0^\infty T(s)^*T(s)\, ds\, dt = \int_0^\infty \left(\int_s^\infty T(t)^*\, dt\right) T(s)\, ds.$$

Adding the two identities we obtain

$$-(QA^{-1} + (A^*)^{-1}Q) = \int_0^\infty \int_0^\infty T(t)^*\, dt\, T(s)\, ds = (A^*)^{-1}A^{-1}.$$

This yields $QA^{-1} = -(A^*)^{-1}(A^{-1} + Q)$, whence (7.10) holds. Suppose T is a group. Then Q is invertible and $A^\circ = Q^{-1}A^*Q$ is the adjoint with respect to the new scalar product (see Lemma C.5.2). Multiplying the Liapunov inclusion from the left by Q^{-1} yields $A \subset -A^\circ - Q^{-1}$. But $-A^\circ - Q^{-1}$ is a bounded perturbation of a C_0-semigroup generator, whence it is also a generator of a C_0-semigroup. This implies readily that $A = -A^\circ - Q^{-1}$. Multiplying by Q from the left yields (7.11). □

Corollary 7.2.5 (Liapunov's theorem for groups). *Let T be an exponentially stable C_0-semigroup on the Hilbert space H. Suppose that T is a group. Then for each $\varepsilon \in (0, -\omega_0(T))$ there is an equivalent Hilbert norm $\|\cdot\|_\circ$ on H such that*

$$\|T(t)\|_\circ \le e^{-\varepsilon t} \qquad (t \ge 0).$$

Remarks 7.2.6. 1) Since the ‘Liapunov inclusion’ $QA \subset -I - A^*Q$ is not an equation in general, we do not want to call it a ‘Liapunov equation’ (which is the name in the finite-dimensional setting). However, one can reformulate the inclusion as a *system* of equations

$$(Ax \,|\, Qy) + (Qx \,|\, Ay) = - (x \,|\, y) \qquad (x, y \in \mathcal{D}(A)).$$

In [57, p.160 and p.217] this system is called ‘Liapunov equation’.

2) We shall prove an infinite-dimensional Liapunov theorem for holomorphic semigroups in Corollary 7.3.8.

The next proposition shows in which cases the ‘direct method’ works. Consult [105] for a proof.

Proposition 7.2.7. *Let A be the generator of an exponentially stable C_0-semigroup T on H, and define $Q := \int_0^\infty T(t)^*T(t)\,dt$. Then the following assertions are equivalent.*

(i) *The semigroup T is a group.*

(ii) *The operator Q is invertible and $T(t)$ has dense range for some $t > 0$.*

(iii) *The operator Q is invertible and $T^*(t)$ is injective for some $t > 0$.*

(iv) *Both operators Q and $\tilde{Q} := \int_0^\infty T(t)T^*(t)\,dt$ are invertible.*

(v) *The operator Q is invertible and no left half-plane is contained in the residual spectrum of A.*

As a consequence we obtain that the direct method works in the case of holomorphic semigroups if and only if the generator is bounded.

7.2.2 A Decomposition Theorem for Group Generators

We come back to our original goal. Let A be the generator of a C_0-group T on the Hilbert space H. Recall the definition of the *group type* $\theta(T)$ in Appendix A on page 302. We fix $\omega > \theta(T)$ and define

$$\begin{aligned} (x \,|\, y)_\circ &:= \int_{\mathbb{R}} (T(t)x \,|\, T(t)y) \, e^{-2\omega|t|}\, dt \\ &= \int_0^\infty (T(t)x \,|\, T(t)y) \, e^{-2\omega t}\, dt + \int_0^\infty (T(-t)x \,|\, T(-t)y) \, e^{-2\omega t}\, dt \end{aligned} \tag{7.12}$$

for $x, y \in H$, i.e., we apply the Liapunov method simultaneously to the rescaled ’forward’ and ’backward’ semigroups obtained from the group T. From Proposition 7.2.4 it is immediate that $(\cdot \,|\, \cdot)_\circ$ is an equivalent scalar product on H. The following theorem summarises its properties.

Theorem 7.2.8 (Decomposition Theorem). *Let iA be the generator of a C_0-group T on a Hilbert space H and let $\omega > \theta(T)$. With respect to the (equivalent) scalar product $(\cdot\,|\,\cdot)_\circ$ defined by (7.12) the operator A is m-H_ω-accretive.*

If one writes $A = B + iC$ with B, C self-adjoint and $-\omega \le C \le \omega$ with respect to $(\,.\,|\,.\,)_\circ$, then $\mathcal{D}(A) = \mathcal{D}(A^\circ)$ is C-invariant, and $[B, C] = BC - CB$ has an extension to a bounded operator which is skew-adjoint with respect to $(\,.\,|\,.\,)_\circ$.

Note that by Proposition 7.1.2 the decomposition $A = B + iC$ exists.

Proof. For $s \in \mathbb{R}, x \in H$ one has

$$\begin{aligned}\|T(s)x\|_\circ^2 &= \int_{\mathbb{R}} \|T(t)T(s)x\|^2 e^{-2\omega|t|}\,dt = \int_{\mathbb{R}} \|T(t+s)x\|^2 e^{-2\omega|t|}\,dt \\ &= \int_{\mathbb{R}} \|T(t)x\|^2 e^{-2\omega|t-s|}\,dt = \int_{\mathbb{R}} \|T(t)x\|^2 e^{-2\omega|t|} e^{2\omega(|t|-|t-s|)}\,dt \\ &\le e^{2\omega|s|}\,\|x\|_\circ^2\end{aligned}$$

since $|t| - |t-s| \le |s|$ for all $s, t \in \mathbb{R}$ by the triangle inequality. This gives $\|T(t)\|_\circ \le e^{\omega|t|}$ for all $t \in \mathbb{R}$, whence iA is m-H_ω-accretive by Proposition 7.1.2. Now, let

$$Q_\oplus := \int_0^\infty T(t)^* T(t) e^{-2\omega t}\,dt, \quad Q_\ominus := \int_0^\infty T(-t)^* T(-t) e^{-2\omega t}\,dt,$$

and

$$Q := Q_\oplus + Q_\ominus = \int_{\mathbb{R}} T(t)^* T(t) e^{-2\omega|t|}\,dt.$$

Then $(x\,|\,y)_\circ = (Qx\,|\,y)$ for all $x, y \in H$. The Liapunov equations for $Q_\oplus$ and $Q_\ominus$ read

$$Q_\oplus(iA - \omega) = -I - (-iA^* - \omega)Q_\oplus \tag{7.13}$$

$$Q_\ominus(-iA - \omega) = -I - (iA^* - \omega)Q_\ominus. \tag{7.14}$$

Subtracting the second from the first yields

$$QA \subset A^*Q + 2i\omega(Q_\oplus - Q_\ominus).$$

If we multiply by Q^{-1} we arrive at $A \subset A^\circ + 2i\omega Q^{-1}(Q_\oplus - Q_\ominus)$. Since $\mathcal{D}(A) = \mathcal{D}(A^\circ)$ by Proposition 7.1.2, we actually have equality and

$$C = \omega Q^{-1}(Q_\oplus - Q_\ominus).$$

The C-invariance of $\mathcal{D}(A)$ is clear from the formulae (7.13) and (7.14) and the identity $\mathcal{D}(A) = \mathcal{D}(A^\circ) = Q^{-1}\mathcal{D}(A^*)$. Furthermore, employing again (7.13) and

(7.14) as well as $A^\circ = A - 2iC$, we compute

$$\begin{aligned} iCA &= \omega Q^{-1}(Q_\oplus iA - Q_\ominus iA) \\ &= \omega Q^{-1}(-1 + 2\omega Q_\oplus + iA^* Q_\oplus - 1 + 2\omega Q_\ominus - iA^* Q_\ominus) \\ &= \omega Q^{-1}(-2 + 2\omega Q + iA^*(Q_\oplus - Q_\ominus)) \\ &= -2\omega Q^{-1} + 2\omega^2 + i\omega A^\circ Q^{-1}(Q_\oplus - Q_\ominus) \\ &= -2\omega Q^{-1} + 2\omega^2 + iA^\circ C \\ &= -2\omega Q^{-1} + 2\omega^2 I + i(A - 2iC)C \\ &= -2\omega Q^{-1} + 2\omega^2 + 2C^2 + iAC. \end{aligned}$$

This shows that $[B, C] = [A, C]$ has an extension to a bounded operator which is skew-adjoint with respect to $(\cdot\,|\,\cdot)_\circ$. $\square$

A simple renaming of terms and forgetting the new scalar product yields the following stunning corollary.

Corollary 7.2.9. *Let A generate a C_0-group T on a Hilbert space H. Then there exists a bounded operator C such that $B := A - C$ generates a bounded C_0-group. Moreover, the operator C can be chosen in such a way that $\mathcal{D}(A)$ is C-invariant and the commutator $[A, C] = AC - CA$ has an extension to a bounded operator on H.*

Remark 7.2.10. Let A be an H_ω-accretive operator on H. By Proposition 7.1.2, $A = B + iC$ with B, C self-adjoint and $-\omega \le C \le \omega$. In general, however, $\mathcal{D}(A)$ is *not* C-invariant. In fact, let $H := \mathbf{L^2}(\mathbb{R})$ and $B = d/dt$ be the generator of the shift group. Furthermore, let $C := (f \mapsto \omega m f)$ where $m(x) = \operatorname{sgn} x$ is the sign function. Then C is bounded and self-adjoint, and $A := B + C$ generates a C_0-group T with $\|T(t)\| \le e^{\omega|t|}$. Obviously, $\mathcal{D}(A) = \mathcal{D}(B) = \mathbf{W^{1,2}}(\mathbb{R})$ is not invariant with respect to multiplication by m. This example shows that the additional statement in Theorem 7.2.8 is not a matter of course and is due to the particular way of renorming.

Theorem 7.2.8 together with the Crouzeix–Delyon Theorem 7.1.16 immediately imply Theorem 7.2.1.

7.2.3 A Characterisation of Group Generators

We now turn to a second proof of Theorem 7.2.1. In fact, we prove a more general result, characterising generators of C_0-groups on Hilbert spaces.

In this section we consider a strip-type operator $A \in \operatorname{Strip}(\omega)$ on the Hilbert space H. Let $\varphi_1, \varphi_2 \in \mathbb{R} \setminus [-\omega, \omega]$. Then we have

$$(t \mapsto R(t + i\varphi_1, A)x) \in \mathbf{L^2}(\mathbb{R}, H) \quad \Longleftrightarrow \quad (t \mapsto R(t + i\varphi_2, A)x) \in \mathbf{L^2}(\mathbb{R}, H)$$

for each $x \in H$. (Use the resolvent identity and the fact, that the resolvent is uniformly bounded on the horizontal lines $\mathbb{R} + i\varphi_1$ and $\mathbb{R} + i\varphi_2$).

We say that A admits **quadratic estimates** if for every $\varphi \in \mathbb{R} \setminus [-\omega, \omega]$ there exists $c = c(A, \varphi)$ such that

$$\int_{\mathbb{R}} \|R(t + i\varphi, A)x\|^2 \, dt \le c(A, \varphi) \|x\|^2 \qquad (x \in H).$$

From the remarks above and the Closed Graph Theorem it follows that A admits quadratic estimates if and only if there exists $\varphi \in \mathbb{R} \setminus [-\omega, \omega]$ such that

$$(t \longmapsto R(t + i\varphi, A)x) \in \mathbf{L^2}(\mathbb{R}, H)$$

for all $x \in H$. If A admits quadratic estimates then also $-A$ and $A + \lambda$ do so, for each $\lambda \in \mathbb{C}$.

Example 7.2.11. Let $A \in \mathrm{Strip}(\omega)$, and suppose that iA is the generator of a C_0-semigroup T on H. We claim that A admits quadratic estimates. In fact, we can find constants M, ω_0 such that $\|T(t)\| \le M e^{\omega_0 t}$ for all $t \ge 0$. Then

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt$$

for all $\mathrm{Re}\,\lambda > \omega_0$. An application of the Plancherel Theorem C.8.1 now yields

$$\begin{aligned} \int_{\mathbb{R}} \|R(t - i\varphi, A)x\|^2 \, dt = \int_{\mathbb{R}} \|R(it, A - \varphi)x\|^2 \, dt &= 2\pi \int_0^\infty \left\|e^{-\varphi s} T(s)x\right\|^2 \, ds \\ &\le \frac{\pi M^2}{\varphi - \omega_0} \|x\|^2 \end{aligned}$$

for all $\varphi > \omega, \omega_0$ and all $x \in H$. Since also $iA^* = -(iA)^*$ generates a C_0-semigroup, the operator A^* admits quadratic estimates as well.

We can now state the main result.

Theorem 7.2.12 (Characterisation Theorem). *Let H be a Hilbert space, let $\omega \ge 0$, and let $A \in \mathrm{Strip}(\omega)$ on H such that A is densely defined. Then the following assertions are equivalent.*

(i) *The natural $H^\infty(H_\alpha)$-calculus for A is bounded, for one / all $\alpha > \omega$.*

(ii) *The operator iA generates a C_0-group.*

(iii) *The operator iA generates a C_0-semigroup.*

(iv) *The operators A and A^* both admit quadratic estimates.*

If iA generates the C_0-group T, then $\omega(T) \le \omega$.

Proof. The implication (ii)$\Rightarrow$(iii) is obvious, and (iii)$\Rightarrow$(iv) is in Example 7.2.11. To prove (i)$\Rightarrow$(ii) one only has to note that the boundedness of the H^∞-functional calculus for A on some horizontal strip H_α implies the Hille–Yosida conditions for iA and $-iA$ (see Theorem A.8.6). Hence iA generates a C_0-group T. Futhermore, we see that in this case the group type of T is at most as large as α.

To establish the implication (iv)$\Rightarrow$(i) needs a little more effort. We fix $\alpha \in (\omega, \mu)$ and define the auxiliary function ψ by

$$\psi(z) := \frac{c}{(\mu^2 + z^2)^2} \qquad (z \in H_\alpha),$$

where c is chosen such that

$$\int_{\mathbb{R}} \psi(t)\, dt = \int_{\mathbb{R}} \frac{c}{(\mu^2 + t^2)^2}\, dt = 1.$$

(One can easily compute $c = 4\mu^3/\pi$). For a given $f \in H^\infty(H_\alpha)$ we now define the approximants f_n by

$$f_n(z) := \int_{-n}^{n} (f\psi_t)(z)\, dt = f(z) \int_{-n}^{n} \psi(z+t)\, dt \qquad (z \in H_\alpha), \tag{7.15}$$

where here and in the following for a function g on the strip H_α we denote by g_t the function $g_t := (z \longmapsto g(t+z)) : H_\alpha \longrightarrow \mathbb{C}$. We collect the properties of these approximants in a lemma. (Recall the definition of $\mathcal{F}(H_\alpha)$ in Section 4.2.)

Lemma 7.2.13. *Let $f \in H^\infty(S_\alpha)$ and let the sequence $(f_n)_n$ be defined by* (7.15). *Then the following holds.*

a) *$f_n \in \mathcal{F}(H_\alpha)$ for all n.*

b) $\sup_n \|f_n\|_\infty < \infty$.

c) *$f_n \to f$ pointwise on S_α.*

d) *The function $(t \mapsto (f\psi_t)(A)) : \mathbb{R} \longrightarrow \mathcal{L}(H)$ is continuous and*

$$f_n(A) = \int_{-n}^{n} (f\psi_t)(A)\, dt \in \mathcal{L}(H).$$

e) $\sup_n \|f_n(A)\| < \infty$.

Proof. a) By elementary Complex Analysis it is clear that f_n is holomorphic on H_α for each n. We can choose $d > 0$ such that

$$|\psi(z)| = \frac{c}{|\mu^2 + z^2|^2} \le \frac{d}{(1 + |\operatorname{Re} z|^2)^2} \qquad (z \in H_\alpha).$$

For fixed $n \in \mathbb{N}$ one can find $d_n > 0$ such that

$$\frac{1}{1 + |\operatorname{Re} z + t|^2} \le \frac{d_n}{1 + |\operatorname{Re} z|^2} \qquad (z \in H_\alpha,\ |t| \le n).$$

With the help of this we can compute

$$\begin{aligned}|f_n(z)| \le \|f\|_\infty \int_{-n}^{n} |\psi(z+it)|\, dt &\le \|f\|_\infty \int_{-n}^{n} \frac{d}{(1+|\mathrm{Re}\, z+t|^2)^2}\, dt \\ &\le \|f\|_\infty \frac{d_n}{1+|\mathrm{Re}\, z|^2} \left(\int_{\mathbb{R}} \frac{d}{1+t^2}\, dt \right) = \|f\|_\infty \frac{d d_n \pi}{1+|\mathrm{Re}\, z|^2}\end{aligned}$$

for $z \in H_\alpha$. This proves a). To establish b), let $n \in \mathbb{N}$ and $z \in H_\alpha$. Then

$$\begin{aligned}|f_n(z)| &\le \|f\|_\infty \int_{\mathbb{R}} |\psi(z+t)|\, dt \\ &\le \|f\|_\infty \int_{\mathbb{R}} \frac{d}{(1+|\mathrm{Re}\, z+t|^2)^2}\, dt = \|f\|_\infty \int_{\mathbb{R}} \frac{d}{(1+t^2)^2}\, dt.\end{aligned}$$

Part c) follows easily, since by b) and Vitali's theorem it suffices to show that $f_n(z) \to f(z)$ for all $z \in \mathbb{R}$. But this is obvious from (7.15). The statement d) is immediate from the Convergence Lemma (Proposition 5.1.7). To prove e) we let $\eta(z) := 1/(\mu^2 + z^2)$. Hence $\psi = c\eta^2$. Choose $\omega_1 \in (\omega, \alpha)$ and let $\gamma := \gamma_{\omega_1} = \partial H_{\omega_1}$. Now we fix $t \in \mathbb{R}$ and compute

$$\begin{aligned}\int_\gamma \frac{1}{|\mu^2+(z+t)^2|}\, d|z| &= 2\int_{\mathbb{R}} \frac{ds}{|\mu^2+(s+i\omega_1)^2|} \\ &= 2\int_{\mathbb{R}} \frac{ds}{|(s+i\omega_1+i\mu)(s+i\omega_1-i\mu)|} \\ &\le 2\left(\int_{\mathbb{R}} \frac{ds}{|s+i\omega_1+i\mu|^2}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \frac{ds}{|s+i\omega_1-i\mu|^2}\right)^{\frac{1}{2}} \\ &= 2\left(\int_{\mathbb{R}} \frac{ds}{(\mu+\omega_1)^2+s^2}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \frac{ds}{(\mu-\omega_1)^2+s^2}\right)^{\frac{1}{2}} \\ &= \frac{2\pi}{\sqrt{\mu^2-\omega_1^2}}.\end{aligned}$$

Using this we can estimate $\|(f\eta_t)(A)\|$ for each $t \in \mathbb{R}$ by

$$\begin{aligned}\|(f\eta_t)(A)\| &= \left\| \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{\mu^2+(z+t)^2} R(z,A)\, dz \right\| \\ &\le \frac{L(A,\omega_1)}{2\pi} \|f\|_\infty \int_\gamma \frac{1}{|\mu^2+(z+t)^2|}\, d|z| \\ &\le \frac{L(A,\omega_1)}{2\pi} \|f\|_\infty \frac{2\pi}{\sqrt{\mu^2-\omega_1^2}} \\ &= \frac{L(A,\omega_1)}{\sqrt{\mu^2-\omega_1^2}} \|f\|_\infty .\end{aligned}$$

Thus, for arbitrary $x, y \in H$ one has

$$\begin{aligned}
|(f_n(A)x \,|\, y)| &= \left| \int_{-n}^{n} ((f\psi_t)(A)x \,|\, y)\; dt \right| \\
&= c \left| \int_{-n}^{n} (\eta_t(A)(f\eta_t)(A)x \,|\, y)\; dt \right| \\
&= c \left| \int_{-n}^{n} ((f\eta_t)(A)R(i\mu - t, A)x \,|\, R(i\mu - t, A^*)y)\; dt \right| \\
&\leq \frac{c\, L(A, \omega_1)}{\sqrt{\mu^2 - \omega_1^2}} \, \|f\|_\infty \int_{\mathbb{R}} \|R(t + i\mu, A)x\| \, \|R(t + i\mu, A^*)y\| \; dt \\
&\leq \frac{c}{\sqrt{\mu^2 - \omega_1^2}} L(A, \omega_1) \; \|f\|_\infty \\
&\qquad \left(\int_{\mathbb{R}} \|R(t + i\mu, A)x\|^2 \; dt \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \|R(t + i\mu, A^*)y\|^2 \; dt \right)^{\frac{1}{2}} \\
&\leq \frac{c}{\sqrt{\mu^2 - \omega_1^2}} L(A, \omega_1)\, c(A, \mu)\, c(A^*, \mu) \; \|f\|_\infty \; \|x\| \, \|y\| \, .
\end{aligned}$$

In particular, this shows that

$$\|f_n(A)\| \leq \|f\|_\infty \frac{c}{\sqrt{\mu^2 - \omega_1^2}} L(A, \omega_1)\, c(A, \mu)\, c(A^*, \mu)$$

for each n. Thus e) is completely proved. □

It is now easy to complete the proof of Theorem 7.2.12. We simply apply the Convergence Lemma (Proposition 5.1.7) to the sequence $(f_n(A))_{n \in \mathbb{N}}$. Hence we obtain the boundedness of the functional calculus and, more explicitly,

$$\|f(A)\| \leq \|f\|_\infty \frac{4\mu^3}{\pi \sqrt{\mu^2 - \omega_1^2}} L(A, \omega_1)\, c(A, \mu)\, c(A^*, \mu) \tag{7.16}$$

for all $f \in H^\infty(H_\alpha)$. □

Corollary 7.2.14 (Liu). *Let A be the generator of a C_0-semigroup T on the Hilbert space H. If the resolvent of A exists and is uniformly bounded on a left half-plane, then T is a group.*

7.3 Similarity Theorems for Sectorial Operators

In the last section we have seen how the boundedness of an H^∞-calculus could be proved by first changing the scalar product appropriately. In the current section we do it the other way round. Indeed, we shall see how the boundedness of the H^∞-calculus for a sectorial operator leads to an interesting similarity theorem.

7.3.1 The Theorem of McIntosh

The objective of the present section is among the most important results in the theory of functional calculus.

Theorem 7.3.1 (McIntosh). *Let A be an injective sectorial operator on the Hilbert space H. Then the following assertions are equivalent:*

(i) *For some/each $0 \neq f \in H_0^\infty[S_{\omega_A}]$ there are constants $C_1(f), C_2(f) > 0$ such that*

$$C_1(f)\,\|x\|^2 \leq \int_0^\infty \|f(tA)x\|^2 \,\frac{dt}{t} \leq C_2(f)\,\|x\|^2 \tag{7.17}$$

for all $x \in H$.

(ii) *For one/all $\varphi \in (\omega_A, \pi)$, the natural $H^\infty(S_\varphi)$-calculus for A is bounded.*

(iii) $A \in \mathrm{BIP}(H)$.

Proof. (ii)⇒(iii) is obvious. To prove the converse, we pass to $B := \log A$. If $A \in \mathrm{BIP}(H)$, iB generates the C_0-group $T := (A^{is})_{s\in\mathbb{R}}$. By the Boyadzhiev–de Laubenfels Theorem 7.2.1, B has bounded H^∞-calculus on every strip H_α, where $\alpha > \theta(T) = \theta_A$. But $\theta_A = \omega_A$ (Corollary 4.3.5), and therefore (ii) follows from Proposition 5.3.3.

(i)⇒(ii). Reformulating statement (i) we obtain

$$H = \left\{x \in H \;\middle|\; t^{-1/2} f(tA)x \in \mathbf{L}^2_*((0,\infty);H)\right\}$$

with equivalent norms $\|x\|_H \sim \|t^{-1/2}f(tA)x\|_{\mathbf{L}^2_*((0,\infty);H)}$ (we use the terminology of Chapter 6). Employing Theorem 6.4.5 and some density arguments, this is equivalent to the statement

$$H = (\dot{D}, \dot{R})_{\frac{1}{2},2} \tag{7.18}$$

where $\dot{D}$ is the homogeneous domain and $\dot{R}$ is the homogeneous range space (see Section 6.3.3). Hence (ii) follows from Theorem 6.5.6.

(ii)⇒(i). Let C be such that $\|g(A)\| \leq C\,\|g\|_{S_\varphi}$ for all $g \in H_0^\infty(S_\varphi)$ and let $f \in H_0^\infty(S_\varphi)$. Employing the Rademacher functions (see Appendix E.7)

$$\begin{aligned}
\int_{2^{-N}}^{2^N} \|f(tA)x\|^2 \,\frac{dt}{t} &= \sum_{k=-N}^{N} \int_{2^k}^{2^{k+1}} \|f(tA)x\|^2 \,\frac{dt}{t} = \sum_{k=-N}^{N} \int_1^2 \left\|f(t2^kA)x\right\|^2 \frac{dt}{t} \\
&= \int_1^2 \sum_{k=-N}^{N} \left\|f(t2^kA)x\right\|^2 \frac{dt}{t} = \int_1^2 \left\| \sum\nolimits_{-N}^{N} r_k \otimes f(t2^kA)x \right\|_{\mathrm{Rad}(H)}^2 \frac{dt}{t} \\
&\leq \int_1^2 \left\| \sum\nolimits_{-N}^{N} r_k \otimes f(t2^kA) \right\|_{\mathrm{Rad}(\mathcal{L}(H))}^2 \frac{dt}{t}\, \|x\|^2 .
\end{aligned}$$

At this point we use the Unconditionality Lemma 5.6.1 to conclude that

$$\left\| \sum\nolimits_{-N}^{N} r_k(g) f(t2^k A) \right\|_{\mathcal{L}(H)} \leq DC$$

for all $g \in G$, $t > 0$, and $N \in \mathbb{N}$, where D is a constant which does only depend on f and φ. This implies that

$$\int_0^\infty \|f(tA)x\|^2 \frac{dt}{t} \leq C^2 D^2 (\log 2) \|x\|^2 \qquad (x \in H).$$

So we are left to show the second inequality. For this we observe that by Proposition 7.0.1 the operator A^* also satisfies the hypothesis (ii), even with the same constant C. By what we have already shown,

$$\int_0^\infty \|f(tA^*)x\|^2 \frac{dt}{t} \leq (\log 2) C^2 D^2 \|x\|^2 \qquad (x \in H).$$

Consider the function $g := f^* f$. Obviously, $g \in H_0^\infty(S_\varphi)$ and $g(t) := |f(t)|^2$ for $t > 0$. Since $f \neq 0$ and f is holomorphic, we have

$$\alpha := \int_0^\infty |f(t)|^2 \frac{dt}{t} > 0.$$

By Theorem 5.2.6 we can compute, for all $x \in \overline{\mathcal{D}(A) \cap \mathcal{R}(A)} = H$,

$$\begin{aligned}
\alpha \|x\|^2 &= (\alpha x \,|\, x) = \left(\int_0^\infty g(tA)x \frac{dt}{t} \,\middle|\, x \right) \\
&= \int_0^\infty (f^*(tA) f(tA)x \,|\, x) \frac{dt}{t} = \int_0^\infty (f(tA)x \,|\, f(tA^*)x) \frac{dt}{t} \\
&\leq \left(\int_0^\infty \|f(tA)x\|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \left(\int_0^\infty \|f(tA^*)x\|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \\
&\leq CD(\log 2)^{\frac{1}{2}} \|x\| \left(\int_0^\infty \|f(tA)x\|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.
\end{aligned}$$

Dividing by $(\log 2)^{1/2} CD \|x\|$ concludes the proof. □

Remark 7.3.2. Let us sketch a second proof of the implication (iii)⇒(i) using interpolation spaces. Namely, since $A, A^{-1} \in \mathrm{BIP}$, one has $\mathcal{D}(A^{1/2}) = [H, \mathcal{D}(A)]_{1/2}$ and $\mathcal{R}(A^{1/2}) = [H, \mathcal{R}(A)]_{1/2}$ by Theorem 6.6.9. On the other hand, it is known that on Hilbert spaces real interpolation spaces (with $p = 2$) and complex interpolation spaces are equal (see [158, Corollary 4.3.12]). Hence we have

$$(H, \mathcal{D}(A))_{\frac{1}{2},2} = \mathcal{D}(A^{\frac{1}{2}}) \quad \text{and} \quad (H, \mathcal{D}(A^{-1}))_{\frac{1}{2},2} = \mathcal{R}(A^{\frac{1}{2}}).$$

Now we can compute

$$\begin{aligned}A^{\frac12}(1+A)^{-1}(H) &= \mathcal{D}(A^{\frac12}) \cap \mathcal{R}(A^{\frac12}) = (H,D)_{\frac12,2} \cap (H,R)_{\frac12,2}\\ &= (D,R)_{\frac12,2} = (1+A)^{-1}(X,\dot{R})_{\frac12,2} = A^{\frac12}(1+A)^{-1}\left(X^{(\frac12)}, X^{(-\frac12)}\right)_{\frac12,2}.\end{aligned}$$

This yields $H = \left(D^{(1/2)}, R^{(1/2)}\right)_{1/2,2}$, and by Theorem 6.4.5 this is exactly what we intended to prove.

7.3.2 Interlude: Operators Defined by Sesquilinear Forms

This section has a preparatory character. We briefly review the construction of operators by means of elliptic forms. This construction provides us with two interesting questions concerning similarity that will be answered in the subsequent section with the help of McIntosh's theorem.

Let $d \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^d$ an open set, and let $H := \mathbf{L^2}(\Omega)$. Suppose that we are given a set of coefficients $a_{ij} \in \mathbf{L^\infty}(\Omega)$ $(i,j = 1\dots,d)$ satisfying a uniform strict ellipticity condition, i.e.,

$$\operatorname{Re} \sum_{i,j=1}^{d} a_{i,j}(x)\xi_i\overline{\xi_j} \;\geq\; \delta\,|\xi|^2 \qquad (\xi \in \mathbb{C}^d,\ x \in \mathbb{R}^d)$$

for some $\delta > 0$. Given $b_i, c_i \in \mathbf{L^\infty}(\Omega)$, $i = 1,\dots,d$, and $m \in \mathbf{L^\infty}(\Omega)$ one considers the operator

$$\mathcal{A} := -\sum\nolimits_{i,j=1}^{d} D_j(a_{i,j}\,D_i) + \sum\nolimits_{i=1}^{d} \big(b_iD_i - D_ic_i\big) \;+\; m,$$

which acts (in the distributional sense) from $\mathbf{W^{1,2}}(\Omega)$ to $\mathbf{D}(\Omega)$. Taking the part of this operator in $\mathbf{L^2}(\Omega)$ yields the so called *maximal* $\mathbf{L^2}$-realisation of the differential operator.

In general, only a further restriction of this operator (which amounts to incorporating boundary conditions) leads to an operator with reasonable properties. Such restrictions are furnished by considering a closed subspace $V \subset \mathbf{W^{1,2}}(\Omega)$ containing $\boldsymbol{\mathcal{D}}(\Omega)$ and thereon the sesquilinear form

$$a(u,v) := \int_\Omega \Big\{ \sum_{i,j=1}^{d} a_{i,j}(D_iu)\,\overline{D_jv} \;+\; \sum_{i}^{d} b_i(D_iu)\overline{v} + c_iu\overline{D_iv} \;+\; mu\overline{v}\Big\} dx.$$

The operator A is then defined by

$$(u,y) \in A \quad :\Longleftrightarrow \qquad u \in V, y \in H,\ \text{and}\ a(u,v) = (\,y\,|\,v\,)_H \;\;\forall\, v \in V.$$

It is easily seen that $\boldsymbol{\mathcal{D}}(\Omega) \subset \mathcal{D}(A)$ and $Au = \mathcal{A}u$ for all $u \in V$, whence A is a restriction of $\mathcal{A}$. The choice $V = \mathbf{W_0^{1,2}}(\Omega)$ yields Dirichlet and the choice

$V = \mathbf{W^{1,2}}$ yields Neumann boundary conditions. Other choices of V lead to so-called Robin or mixed boundary conditions, cf. [16, 17]. By a standard argument (see [210, Sec.2.2, Ex.1] or [87, Chapter 6]) one establishes the so-called **Gårding inequality**, i.e.,

$$\operatorname{Re} a(u,u) + \lambda_0 \|u\|_{\mathbf{L^2}}^2 \geq \delta \|u\|_{\mathbf{W^{1,2}}}^2 \qquad (u \in V)$$

for some constants $\lambda_0 \in \mathbb{R}$ and $\delta > 0$. As we shall see below, this inequality implies that the operator $A + \lambda_0$ is m-ω-accretive for some $\omega \in [0, \pi/2)$.

We now turn to the abstract description. Let H be a Hilbert space, $V \subset H$ a dense subspace and $a \in \operatorname{Ses}(V)$ a sesquilinear form on V. For $\lambda \in \mathbb{C}$ we define the form $a_\lambda \in \operatorname{Ses}(V)$ by

$$a_\lambda(u,v) := a(u,v) + \lambda\,(u \,|\, v)_H$$

for $u, v \in V$. The form a is called **elliptic** if there is $\lambda_0 \geq 0$ such that the following two conditions hold.

1) The form $\operatorname{Re} a_{\lambda_0} = (\operatorname{Re} a) + \lambda_0\,(\,.\,|\,.\,)_H$ is a scalar product on V; this turns V into a Hilbert space such that the inclusion mapping $V \subset H$ is continuous.
2) The form a is continuous with respect to this scalar product on V.

(A moment's reflection shows that in our example above indeed the two conditions are satisfied, see Remark 7.3.3 below.)

With an elliptic form a on $V \subset H$ we associate an operator A in the following way. We define a norm on V by

$$\|u\|_V^2 := \operatorname{Re} a(u) + \lambda_0 \|u\|_H^2$$

for $u \in V$. Condition 1) then implies that V is a Hilbert space. Since V is continuously embedded in H (also by condition 1)), there is a continuous embedding of H^* into V^* (injectivity follows since V is dense in H). If we identify H with its antidual by means of the scalar product $(\cdot\,|\,\cdot)_H$ of H (Riesz–Fréchet theorem), we obtain a sequence of continuous embeddings

$$V \subset H(\cong H^*) \subset V^*.$$

In doing this, H becomes a *dense* subspace of V^*. Now we define the mapping $\mathcal{A} : V \longrightarrow V^*$ by

$$\mathcal{A} := (u \longmapsto a(u,.)) : V \longrightarrow V^*.$$

By means of the identifications above we obtain $(\mathcal{A} + \lambda_0)(u) = a(u,.) + \lambda_0\,(u\,|\,.)_H$. The operator $\mathcal{B} := \mathcal{A} + \lambda_0 : V \longrightarrow V^*$ is an isomorphism.

Proof. Consider the form a_{λ_0}. Then $\operatorname{Re} a_{\lambda_0}$ *is* the scalar product of V, hence a is coercive (see (C.5) on page 324). By condition 1), a is also continuous, hence satisfies the hypotheses of the Lax–Milgram theorem (Theorem C.5.3). $\square$

Employing the embedding $H \subset V^*$ we define

$$\begin{aligned}\mathcal{D}(A) &:= \{u \in V \mid \text{there is } y \in H \text{ s.t. } a(u,\cdot) = (y\,|\,\cdot)_H\} \\ &= \{u \in V \mid \mathcal{A}u \in H\}\end{aligned}$$

and $Au := \mathcal{A}u$ regarded as an element of H. Then the two fundamental identities

$$a(u,v) = \langle \mathcal{A}u, v\rangle \qquad (u \in V) \tag{7.19}$$

$$a(u,v) = (Au\,|\,v)_H \quad (u \in \mathcal{D}(A)) \tag{7.20}$$

hold for $v \in V$. The operator A is called **associated** with the form a (notation: $A \sim a$). Note that the definition of A is actually independent of the chosen λ_0. An operator A on H is called **Kato-sectorial** if there is a dense subspace $V \subset H$ and an elliptic form $a \in \mathrm{Ses}(V)$ on V such that $a \sim A$. Clearly, if A is Kato-sectorial, then $A+\lambda$ is Kato-sectorial for each $\lambda \in \mathbb{C}$. Furthermore, also A^* is Kato-sectorial (if $A \sim a$ then $A^* \sim \overline{a}$).

Remarks 7.3.3. 1) We use the name 'Kato-sectorial' because these operators were treated extensively by Kato. In [130] he called them just 'sectorial' but nowadays this very name is used as we have defined it in Chapter 1.

2) In our introductory example of an elliptic differential operator with boundary conditions the starting point is a little different in that the space V already carries a (natural) Hilbert space structure such that V is densely and continuously embedded in H. We denote this given scalar product on V by $(\cdot\,|\,\cdot)_V$. Instead of the conditions 1) and 2) one rather has

3) $\mathrm{Re}\, a_\lambda(u) \geq \delta \,\|u\|_V^2$ for some $\delta > 0$ and all $u \in V$.

4) $|a(u,v)| \leq M \,\|u\|_V \,\|v\|_V$ for some $M \geq 0$ and all $u, v \in V$.

Here, condition 4) says that a is continuous on V, and this together with condition 3) implies that $\mathrm{Re}\, a_\lambda$ is an equivalent scalar product on V. So indeed conditions 1) and 2) are satisfied.

The following characterisation of Kato-sectorial operators is well known.

Proposition 7.3.4. *Let $A \sim a$ where $a : V \times V \longrightarrow H$ is an elliptic form. Then there is $\omega \in [0, \pi/2)$ such that $A + \lambda_0$ is an invertible m-ω-accretive operator on H. (Here, λ_0 is such that the conditions* 1) *and* 2) *above hold.)*

Conversely, let A be an m-ω-accretive operator on H for some $\omega \in [0, \pi/2)$. Then A is Kato-sectorial. More precisely, there is a dense subspace $V \subset H$ and an elliptic form $a \in \mathrm{Ses}(V)$ such that $A \sim a$ and $\mathrm{Re}\, a$ *is positive.*

Proof. Suppose that $A \sim a$ and $a : V \times V \longrightarrow H$ is elliptic, and let λ_0 be as above. Then the operator $B := A + \lambda_0 : \mathcal{D}(A) \longrightarrow H$ is bijective and B^{-1} is a bounded operator on H since B is the restriction of $\mathcal{B} : V \longrightarrow V^*$ to the range H. Because H is dense in V^*, $\mathcal{D}(A) = \mathcal{D}(B)$ is dense in V, hence a fortiori in H. This means that B is a densely defined closed operator in H with $0 \in \varrho(B)$.

Claim: The form a_{λ_0} is continuous with respect to the scalar product $\operatorname{Re} a_{\lambda_0}$.

Proof of Claim. Since the embedding $V \subset H$ is continuous, there is a constant M_1 such that $\|u\|_H^2 \leq M_1 \|u\|_V^2$ for all $u \in V$. The continuity of the form a (condition 2) yields the existence of a constant M_2 such that $|a(u,v)| \leq M_2 \|u\|_V \|v\|_V$. Putting the two inequalities together we obtain $|a_{\lambda_0}(u,v)| \leq (M_2 + \lambda_0 M_1) \|u\|_V^2$.

A short glance at Proposition C.1.3 shows that the form a_{λ_0} is sectorial of some angle ω which depends on the continuity constants of a_{λ_0} with respect to $\operatorname{Re} a_{\lambda_0}$. Since $(Bu \,|\, u)_H = a_{\lambda_0}(u,u)$ for $u \in \mathcal{D}(B)$ the operator B is m-ω-accretive.

Conversely, let A be m-ω-accretive. On $\mathcal{D}(A)$ we define the sesquilinear form a by

$$a(u,v) := (Au \,|\, v)_H$$

and a scalar product (!) by

$$(u \,|\, v)_V := (\operatorname{Re} a)(u,v) + (u \,|\, v)_H .$$

The form a is continuous with respect to this scalar product. In fact, since a is sectorial of angle ω, by Proposition C.1.3 we have

$$|a(u,v)| \leq (1 + \tan\omega)\sqrt{\operatorname{Re} a(u)}\sqrt{\operatorname{Re} a(v)} \leq (1 + \tan\omega) \|u\|_V \|v\|_V$$

for all $u, v \in \mathcal{D}(A)$. Obviously, the embedding $(\mathcal{D}(A), \|.\|_V) \subset H$ is continuous. Hence it has a continuous extension $\iota : V \longrightarrow H$, where V is the (abstract) completion of $\mathcal{D}(A)$ with respect to $\|.\|_V$

Claim: The mapping ι is injective.

Proof of Claim. Let $x \in V$ and $\iota x = 0$. This means that there is $(u_n)_n \subset \mathcal{D}(A)$ such that $\|u_n - u_m\|_V \to 0$ and $\|u_n\|_H \to 0$. Hence for all $n, m \in \mathbb{N}$ we have

$$\begin{aligned}
\operatorname{Re} a(u_n) &= \operatorname{Re}(Au_n \,|\, u_n)_H = \operatorname{Re}(Au_n \,|\, u_n - u_m)_H + \operatorname{Re}(Au_n \,|\, u_m)_H \\
&= \operatorname{Re}[a(u_n, u_n - u_m)] + \operatorname{Re}(Au_n \,|\, u_m)_H \\
&\leq |a(u_n, u_n - u_m)| + |(Au_n \,|\, u_m)_H| \\
&\leq M \|u_n\|_V \|u_n - u_m\|_V + \|Au_n\|_H \|u_m\|_H .
\end{aligned}$$

Since (u_n) is V-Cauchy, $C := \sup_n \|u_n\|_V < \infty$. Hence

$$\operatorname{Re} a(u_n) \leq MC \limsup_m \|u_n - u_m\|_V \to 0 \quad \text{as } n \to \infty.$$

This shows that $\|u_n\|_V \to 0$, whence $x = 0$.

We therefore can regard V as being continuously and densely embedded into H. The form a has a unique extension to a continuous sesquilinear form on V (again denoted by a). Clearly, the form a is elliptic (with $\lambda_0 = 1$). Hence it remains to show that A is associated with a.

We denote by B the operator which is associated with a and choose $u \in \mathcal{D}(A)$. The equation $a(u,v) = (Au \mid v)_H$ holds for all $v \in V$ since it holds for all $v \in \mathcal{D}(A)$, $\mathcal{D}(A)$ is dense in V, a is continuous on V, and V is continuously embedded in H. This shows $u \in \mathcal{D}(B)$ and $Bu = Au$, whence $A \subset B$. On the other hand, we have $-1 \in \varrho(B)$ by construction and $-1 \in \varrho(A)$ since A is m-ω-accretive. This implies that $A = B$. □

Combining Proposition 7.3.4 with Proposition 7.1.1 yields the following corollary.

Corollary 7.3.5. *An operator A on H is Kato-sectorial if and only if there is $\lambda \in \mathbb{R}$ such that $A + \lambda$ is m-ω-accretive for some $\omega < \pi/2$ if and only if $-A$ generates a holomorphic semigroup T on some sector S_θ such that*

$$\|T(z)\| \leq e^{\lambda \operatorname{Re} z} \qquad (z \in S_\theta)$$

for some $\lambda \in \mathbb{R}$.

Note that the corollary says that A is Kato-sectorial if and only if A is 'quasi-m-ω-accretive' for some $\omega \in [0, \pi/2)$.

The construction of operators by means of sesquilinear forms, though quite powerful, raises two highly non-trivial questions. The first is due to the fact that, in contrast to many other concepts, the notion of 'Kato-sectoriality' relies heavily on the particular chosen scalar product. That is, an operator may be Kato-sectorial with respect to the original but *not* Kato-sectorial with respect to some new (although equivalent) scalar product on H. (A concrete example is given in [12].) In fact, a result of Matolcsi says that if A is not a bounded operator, one *always* can find a scalar product on H such that A is not Kato-sectorial (not even quasi-m-accretive, see [162]). It is therefore reasonable to ask for a characterisation of Kato-sectorial operators *modulo similarity.* Such a characterisation can indeed be given with the help of the functional calculus (see Corollary 7.3.10 below.)

To understand the second problem, we have to cite a stunning theorem from [127].

Theorem 7.3.6 (Kato). *Let A be m-accretive and $\alpha \in [0, 1/2)$. Then the following assertions hold:*

1) $\mathcal{D}(A^\alpha) = \mathcal{D}(A^{*\alpha}) =: \mathcal{D}_\alpha$.
2) $\|A^{*\alpha}u\| \leq \tan\big(\pi(1+2\alpha)/4\big)\, \|A^\alpha u\|$ *for all* $u \in \mathcal{D}_\alpha$.
3) $\operatorname{Re}(A^\alpha u \mid A^{*\alpha}u) \geq (\cos \pi\alpha)\, \|A^\alpha u\|\, \|A^{*\alpha}u\|$ *for all* $u \in \mathcal{D}_\alpha$.

Proof. See [127, Theorem 1.1]. The proof proceeds in two steps. First the statements are proved under the additional hypothesis of A being bounded with $\operatorname{Re}(Au \mid u) \geq \delta \|u\|^2$ for all $u \in H$ and some $\delta > 0$. (This is the difficult part of the proof.). The second (easy) step uses 'sectorial approximation' (in our language) and Proposition 3.1.9. □

Kato's theorem is remarkable also in that it fails for $\alpha = 1/2$. Kato was not aware of this when he wrote the article [127] but only shortly afterwards, Lions in [154] produced a counterexample. For some time it had been an open question whether at least for m-ω-accretive operators A it is true that

$$\mathcal{D}(A^{\frac{1}{2}}) = \mathcal{D}(A^{\frac{1}{2}*}). \tag{7.21}$$

If A is associated with the form $a \in \mathrm{Ses}(V)$ and, say, we have $\lambda_0 = 0$, then (7.21) is equivalent to $V = \mathcal{D}(A^{1/2})$, as Lions in [154] and Kato in [128] have shown. Finally, McIntosh gave a counterexample in [163]. However, this was not the end of the story. In fact, even if the statement is false *in general* it might be true for *particular operators* such as second order elliptic operators on $\mathbf{L}^2(\mathbb{R}^n)$ in divergence form. For these operators the problem became famous under the name **Kato's Square Root Problem** and has stimulated a considerable amount of research which led to the discovery of deep connections between Operator Theory and Harmonic Analysis (see also the comments in Section 7.5).

Let us call an operator A on the Hilbert space H **square root regular** if $A+\lambda$ is sectorial and $\mathcal{D}((A+\lambda)^{1/2}) = \mathcal{D}((A+\lambda)^{1/2\,*})$ for large $\lambda \in \mathbb{R}$.

Whether an operator is square root regular does depend on the particular chosen scalar product. (If the scalar product changes, A stays the same while the adjoint A^* changes.) Hence it is reasonable to ask the following: Suppose that A is Kato-sectorial with respect to some scalar product. Is there an equivalent scalar product on H such that A is Kato-sectorial *and* square root regular with respect to the new scalar product? This is indeed the case (see Corollary 7.3.10 below).

7.3.3 Similarity Theorems

After so much propedeutical work we can now bring in the harvest. The first similarity result has an interesting historical background (see the comments in Section 7.5 of this chapter).

Theorem 7.3.7 (Callier–Grabowski–Le Merdy). *Let $-A$ be the generator of a bounded holomorphic C_0-semigroup T on the Hilbert space H. Let B be the injective part of A on $K := \overline{\mathcal{R}(A)}$. Then T is similar to a contraction semigroup if and only if $B \in \mathrm{BIP}(K)$.*

Recall that the injective part of A is the part of A in $\overline{\mathcal{R}(A)}$. Clearly, A is m-ω-accretive if and only if B is.

Proof. Suppose that T is similar to a contraction semigroup. Changing the scalar product we may suppose that A is m-accretive. Therefore, $B \in \mathrm{BIP}(K)$ by Theorem 7.1.7.

To prove the converse, let $B \in \mathrm{BIP}(K)$. Since $-A$ generates a bounded holomorphic semigroup, A is sectorial with $\omega_A < \pi/2$. By Theorem 7.3.1 we can change the norm on K to $(\int_0^\infty \|f(tA)x\|^2 \, dt/t)^{1/2}$ where $f \in H_0^\infty[S_{\omega_A}]$ is arbitrary. If we choose $f(z) := z^{1/2}e^{-z}$ we obtain the new norm

$$\|x\|_{\text{new}}^2 = \int_0^\infty \left\|(tA)^{\frac{1}{2}} e^{-tA} x\right\|^2 \frac{dt}{t} = \int_0^\infty \left\|A^{\frac{1}{2}} T(t)x\right\|^2 dt$$

on K. Employing the semigroup property it is easy to see that each $T(t)$ is contractive on K with respect to this new norm.

However, on $\mathcal{N}(A)$ each operator $T(t)$ acts as the identity. Since $H = \mathcal{N}(A) \oplus K$ we can choose the new scalar product on H in such a way that it is the old one on $\mathcal{N}(A)$, the one just constructed on K and $\mathcal{N}(A) \perp K$. With respect to this new scalar product, the semigroup T is contractive. $\square$

For a different proof of Theorem 7.3.7 using interpolation theory, see the comments in Section 7.5. As an immediate corollary we obtain the announced Liapunov theorem for holomorphic semigroups.

Corollary 7.3.8 (Liapunov's theorem for holomorphic semigroups). *Let A be an operator on a Hilbert space H such that $-A$ generates a holomorphic C_0-semigroup T and $s(-A) < 0$. If $A + \lambda \in \mathrm{BIP}(H)$ for some $\lambda > 0$ then, for every $\varepsilon \in (0, -s(-A))$ there is an equivalent Hilbert norm $\|\cdot\|_\circ$ on H such that*

$$\|T(t)\|_\circ \le e^{-\varepsilon t} \qquad (t \ge 0).$$

Proof. Choose $0 < \varepsilon < \delta := -s(-A)$ and consider the operator $B := A - \varepsilon$. We claim that B is still sectorial with $\omega_B < \pi/2$. To establish this it suffices to show that $\{\lambda \mid \operatorname{Re}\lambda < 0\} \subset \varrho(B)$ with $\|\lambda R(\lambda, B)\|$ being uniformly bounded for $\operatorname{Re}\lambda < 0$. Since $0 \in \varrho(B)$ we only have to consider $|\lambda| \ge R$ for some radius $R > 0$. We choose $\omega' \in (\omega_A, \pi/2)$ and let $R := \sqrt{\varepsilon^2(1 + \tan^2\omega')}$. Now $\lambda R(\lambda, B) = \lambda(\lambda+\varepsilon)^{-1}[(\lambda+\varepsilon)R(\lambda+\varepsilon, A)]$ and the factor $\lambda/(\lambda+\varepsilon)$ is uniformly bounded for $|\lambda| \ge R$. The second factor is uniformly bounded for $\operatorname{Re}\lambda \le -\varepsilon$ hence we have to check that

$$\sup\{\|(\lambda+\varepsilon)R(\lambda+\varepsilon, A)\| \mid -\varepsilon \le \operatorname{Re}\lambda \le 0,\ |\lambda| \ge R\} < \infty.$$

But $|\lambda| \ge R$ together with $-\varepsilon \le \operatorname{Re}\lambda \le 0$ implies that $|\operatorname{Im}(\lambda+\varepsilon)| = |\operatorname{Im}\lambda| \ge \sqrt{R^2 - \varepsilon^2} = \varepsilon\tan\omega'$, whence $|\arg(\lambda+\varepsilon)| \ge \omega'$. Since $\omega' > \omega_A$, the claim is proved.

So $-B$ generates a bounded holomorphic C_0-semigroup. Moreover, $B + \lambda \in \mathrm{BIP}(H)$ for some $\lambda > 0$. By b) of Proposition 3.5.5, $B^{is} \in \mathcal{L}(H)$ for all $s \in \mathbb{R}$. Applying Corollary 3.5.7 we obtain $B \in \mathrm{BIP}(H)$. We can now apply the Callier–Grabowski–Le Merdy Theorem 7.3.7. This yields an equivalent scalar product such that $(e^{\varepsilon t}T(t))_{t\ge 0}$ becomes contractive. $\square$

Recall that one can explicitly write down the new scalar product in Proposition 7.3.8. In fact, it follows from the argument in the proof of Theorem 7.3.7

that

$$\|x\|_\circ^2 = \int_0^\infty \left\|(A-\varepsilon)^{\frac{1}{2}} e^{\varepsilon t} T(t)x\right\|^2 \, dt \qquad (x \in H).$$

The function $\|\cdot\|_\circ^2$ is a Liapunov function for the dynamical system given by

$$\dot{u} + (A-\varepsilon)u = 0.$$

Now we can state the main result of this section.

Theorem 7.3.9 (Similarity Theorem). *Let A be a sectorial operator on the Hilbert space H with $\omega_A < \pi/2$. Suppose that A satisfies the equivalent properties* (i)–(iv) *of Theorem* 7.3.1. *Then for each $\varphi \in (\omega_A, \pi/2)$ there is an equivalent scalar product $(\cdot\,|\,\cdot)_\circ$ on H with the following properties.*

1) $\mathcal{N}(A) \perp \overline{\mathcal{R}(A)}$ *with respect to* $(\cdot\,|\,\cdot)_\circ$.
2) *The operator A is m-φ-accretive with respect to* $(\cdot\,|\,\cdot)_\circ$.
3) *One has $\mathcal{D}(A^\alpha) = \mathcal{D}(A^{\circ\alpha})$ for $0 \le \alpha \le \pi/(4\varphi)$. Here, A° denotes the adjoint of A with respect to* $(\cdot\,|\,\cdot)_\circ$.
4) *One has $\|f(A)\|_\circ \le \|f\|_\varphi$ for all $f \in \mathcal{E}(S_\varphi)$.*

Note that $\pi/(4\varphi) > 1/2$. Hence in particular $\mathcal{D}(A^{1/2}) = \mathcal{D}(A^{\circ(1/2)})$.

Proof. We choose $\omega_A =: \omega < \varphi' < \varphi$, and define $\beta := \pi/(2\varphi')$ and $B := A^\beta$. Then $B \in \mathrm{Sect}(\omega')$, where $\omega' = \beta\omega = (\omega/\varphi')(\pi/2) < \pi/2$. Hence $-B$ generates a bounded holomorphic C_0-semigroup. By hypothesis, A satisfies condition (i) of Theorem 7.3.1. Applying Proposition 3.1.4 we see that B has the same property. By Theorem 7.3.7 there is an equivalent scalar product $(\cdot\,|\,\cdot)_\circ$ that makes B m-accretive and such that $\mathcal{N}(A) \perp \overline{\mathcal{R}(A)}$. Now, $A = B^{1/\beta}$, whence by Corollary 7.1.13, A is m-$\pi/2\beta$-accretive with respect to the new scalar product. Moreover, Kato's Theorem 7.3.6 says that $\mathcal{D}(A^{\alpha\beta}) = \mathcal{D}(B^\alpha) = \mathcal{D}(B^{\circ\alpha}) = \mathcal{D}(A^{\circ\alpha\beta})$ for all $0 \le \alpha < 1/2$. Thus, assertion 3) follows since

$$0 < \alpha\beta < \frac{\beta}{2} = \frac{\pi}{4\varphi'} \quad \text{and} \quad \frac{\pi}{4\varphi'} > \frac{\pi}{4\varphi}.$$

If we take $f \in \mathcal{E}(S_\varphi)$ then

$$\|f(A)\|_\circ = \|f(z^{1/\beta})(B)\|_\circ \le \|f(z^{1/\beta})\|_{\frac{\pi}{2}} = \|f\|_{\frac{\pi}{2\beta}} = \|f\|_{\varphi'} \le \|f\|_\varphi$$

by Proposition 3.1.4 and Theorem 7.1.7. $\square$

As a corollary, we obtain a simultaneous solution to the similarity problems posed in Section 7.3.2 (see page 201).

Corollary 7.3.10. *Let A be a closed operator on a Hilbert space. Then A is Kato-sectorial with respect to some equivalent scalar product if and only if $-A$ generates a holomorphic C_0-semigroup and $A+\lambda \in \mathrm{BIP}(H)$ for large $\lambda \in \mathbb{R}$. In this case the scalar product can be chosen such that A is Kato-sectorial <u>and</u> square root regular.*

Corollary 7.3.11 (Franks–Le Merdy). *Let A be an injective sectorial operator on the Hilbert space H such that $A \in \mathrm{BIP}(H)$. Then for each $\varphi \in (\omega_A, \pi)$ there is an equivalent scalar product on H with respect to which the natural $H^\infty(S_\varphi)$-calculus for A is contractive.*

Proof. If $\omega_A < \pi/2$ one can apply the Similarity Theorem 7.3.9 in combination with Proposition 5.3.4. In the case that $\omega_A \geq \pi/2$ we apply this to the operator $A^{1/2}$ and use Proposition 3.1.4. □

Finally we give an application of the Similarity Theorem to derive a dilation theorem for groups on Hilbert spaces.

Corollary 7.3.12 (Dilation theorem for groups). *Let T be a C_0-group on the Hilbert space H. For each $\omega > \theta(T)$ there is a Hilbert space K, a (not necessarily isometric) embedding $\iota : H \longrightarrow K$ onto a closed subspace of K, and a normal C_0-group U on K with $\|U(s)\| \leq e^{\omega|s|}$ for all $s \in \mathbb{R}$ such that*

$$P \circ U(s) \circ \iota = \iota \circ T(s) \qquad (s \in \mathbb{R}).$$

Here, $P : K \longrightarrow \iota(H)$ denotes the orthogonal projection of K onto $\iota(H)$.

Proof. Choose $\alpha > 0$ such that $\alpha\omega = \pi/2$. We consider the group $T(\alpha\cdot)$, which has group type $\alpha\theta(T)$. By Monniaux's Theorem 4.4.3 (cf. also Remark 7.2.2) we find an injective sectorial operator A on H such that $A^{is} = T(\alpha s)$ for all $s \in \mathbb{R}$. Then $\omega_A = \alpha\theta(T) < \pi/2$ by Gearhart's Theorem C.8.2 and Theorem 4.3.1. By the Callier–Grabowski–Le Merdy Theorem 7.3.7 we can change the scalar product on H in order to have $-A$ generating a contraction semigroup $(e^{-tA})_{t\geq 0}$. The Sz.-Nagy Theorem 7.1.10 yields a new Hilbert space K and an isomorphic embedding $\iota : H \longrightarrow K$ (which is isometric with respect to the new scalar product) and a unitary group $(W(t))_{t\in\mathbb{R}}$ on K such that $PW(t)\iota = \iota e^{-tA}$ for all $t \geq 0$. Let $-B$ be the generator of W. In the proof of Proposition 7.1.7 we showed that $Pf(B)\iota = \iota f(A)$ for all $f \in \mathcal{E}[S_{\pi/2}]$. By applying the Convergence Lemma (Proposition 5.1.4) to the sequence $f_n(z) := z^{is} zn(z + 1/n)^{-1}(n + z)^{-1}$ we obtain $PB^{is}\iota = \iota A^{is} = \iota T(\alpha s)$ for $s \in \mathbb{R}$. Thus we have arrived at the desired dilation when we define $U(s) := B^{is/\alpha}$. Since $\left\|B^{is}\right\| \leq e^{|s|\pi/2}$ and $\omega = \pi/2\alpha$ we clearly have $\|U(s)\| \leq e^{\omega|s|}$ for all $s \in \mathbb{R}$. □

7.3.4 A Counterexample

Let $-A$ generate a bounded C_0-semigroup T and suppose that A is injective. The Callier–Grabowski–Le Merdy Theorem 7.3.7 states that if $A \in \mathrm{BIP}(H)$ and T is even bounded holomorphic, T is similar to a contraction semigroup. What if we drop the assumption on holomorphy? This question is answered by the following theorem.

Theorem 7.3.13. *There exists a Hilbert space H and an operator A on H such that the following conditions hold.*

1) *The operator $-A$ generates a bounded C_0-semigroup T on H.*
2) *The operator A is invertible and $A \in \mathrm{BIP}(H)$.*
3) *The semigroup T is not similar to a quasi-contractive semigroup.*

Here, a C_0-semigroup T on H is called **quasi-contractive**, if there is $\omega \in \mathbb{R}$ such that

$$\|T(t)\| \leq e^{\omega t} \qquad (t \geq 0).$$

We combine a counterexample given by Le Merdy with a technique due to Chernoff. Based on an example given by Pisier in [188], Le Merdy showed in [146, Proposition 4.8] that there exists a Hilbert space H, a bounded C_0-semigroup T on H with injective generator $-A \in \mathrm{BIP}(H)$ such that T is not similar to a contraction semigroup. Let $\alpha = (\alpha_n)_n$ be a scalar sequence with $\alpha_n > 0$ for all n. We consider the space $\mathcal{H} := \ell^2(H)$ and the operator $\mathcal{A}$ defined on $\mathcal{H}$ by

$$\mathcal{D}(\mathcal{A}) := \{x = (x_n)_n \in \mathcal{H} \mid x_n \in \mathcal{D}(A)\ \forall n \in \mathbb{N}\}, \qquad \mathcal{A}x := (\alpha_n A x_n)_n.$$

Denote by $|\!|\!|\cdot|\!|\!|$ the (Hilbert)-norm on $\mathcal{H}$.

Lemma 7.3.14. *The following assertions hold.*

a) *The operator $-\mathcal{A}$ generates the bounded C_0-semigroup $\mathcal{T}$ defined by*

$$\mathcal{T}(t)x := (T(\alpha_n t))_n \qquad (x = (x_n)_n \in \mathcal{H},\ t \geq 0).$$

b) *We have*

$$\varrho(\mathcal{A}) = \{\lambda \mid \lambda\alpha_n^{-1} \in \varrho(A) \text{ and } \sup_n \left\|(\lambda - \alpha_n A)^{-1}\right\| < \infty\},$$

with $R(\lambda, \mathcal{A})x = ((\lambda - \alpha_n A)^{-1}x_n)_n$ for $x = (x_n)_n \in \mathcal{H}$.

c) *The operator $\mathcal{A}$ is injective.*

d) *Let $\varphi \in (\pi/2, \pi)$ and $f \in H_0^\infty(S_\varphi)$. Then $f(\mathcal{A})x = (f(\alpha_n A)x_n)_n$ for each $x = (x_n) \in \mathcal{H}$. Moreover $|\!|\!| f(\mathcal{A})|\!|\!| \leq \sup_n \|f(\alpha_n A)\|$*

Proof. Since T is a bounded semigroup, all operators $\mathcal{T}(t)$ are well defined bounded operators on $\mathcal{H}$, and even uniformly bounded in t. Obviously, the semigroup law holds. Since the space of finite H-sequences is dense in $\mathcal{H}$ and each $T(\alpha_n \cdot)$ is strongly continuous on H, $\mathcal{T}$ is a C_0-semigroup on $\mathcal{H}$. Denote its generator by $\mathcal{B}$ and let $x = (x_n)_n \in \mathcal{D}(\mathcal{B})$. Then $\lim_{t\searrow 0} t^{-1}(\mathcal{T}(t)x - x) \to \mathcal{B}x$. This implies that $\lim_{t\searrow 0} t^{-1}(T(\alpha_n t)x - x) \to [\mathcal{B}x]_n$ for each n, whence $x_n \in \mathcal{D}(A)$ and $[\mathcal{B}x]_n = -\alpha_n A x_n$ for each n. Hence we have $\mathcal{B} \subset -\mathcal{A}$. Thus, by a resolvent argument, a) will be proved as soon as we will have established b).

Obviously, the inclusion '$\supset$' holds in b). Let $\lambda \in \varrho(\mathcal{A})$. Then

$$((x_n) \longmapsto ((\lambda - \alpha_n A)x_n)) : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{H}$$

is bijective. By composing this mapping with suitable injections and projections we see that $(\lambda - \alpha_n A) : \mathcal{D}(A) \longrightarrow H$ is bijective for each n and that $\sup_n \left\|(\lambda - \alpha_n A)^{-1}\right\| < \infty$.

The assertion c) follows from the fact that A is injective and each $\alpha_n > 0$. Part d) is immediate from b) and the representation of $f(A)$ as a Cauchy integral. $\square$

Since $A \in \mathrm{BIP}(H)$ we know from McIntosh's Theorem 7.3.1 that the natural $H^\infty(S_\varphi)$-calculus is bounded for each $\varphi > \pi/2$. Fix $\varphi > \pi/2$ and $C \geq 0$ such that $\|f(A)\| \leq C \|f\|_\varphi$ for all $f \in H^\infty(S_\varphi)$. If $f \in H_0^\infty(S_\varphi)$, Lemma 7.3.14 d) yields $\|f(\mathcal{A})\|_{\mathcal{L}(\mathcal{H})} \leq C \sup_n \|f(\alpha_n \cdot)\|_\varphi = C \|f\|_\varphi$. Hence also $\mathcal{A}$ has a bounded $H^\infty(S_\varphi)$-calculus (see Proposition 5.3.4), whence $\mathcal{A} \in \mathrm{BIP}(\mathcal{H})$. We turn to Chernoff's observation from [43] and specialise $\alpha_n := 1/n$ in the above construction.

Lemma 7.3.15 (Chernoff). *If $\mathcal{T}$ is similar to a quasi-contractive semigroup, then T is similar to a contractive semigroup.*

Proof. We denote by $(\cdot\,|\,\cdot)$ and $\langle\cdot,\cdot\rangle$ the scalar products on H and $\mathcal{H}$, respectively. Suppose that there is an equivalent scalar product $\langle\cdot,\cdot\rangle_{\mathrm{new}}$ on $\mathcal{H}$ and a scalar $\omega \geq 0$ such that $\|\mathcal{T}(t)x\|_{\mathrm{new}} \leq e^{\omega t} \|x\|_{\mathrm{new}}$ for all $x \in \mathcal{H}$ and all $t \geq 0$. Define $(x\,|\,y)_n := \langle\iota_n x, \iota_n y\rangle_{\mathrm{new}}$ for $x, y \in H$ and $n \in \mathbb{N}$, where $\iota_n : H \longrightarrow \mathcal{H}$ is the natural inclusion mapping onto the n-th coordinate. Obviously, each $(\cdot\,|\,\cdot)_n$ is an equivalent scalar product on H. More precisely, if $c > 0$ such that $c^{-1} \|x\|^2 \leq \|x\|_{\mathrm{new}} \leq c \|x\|$ for all $x \in \mathcal{H}$ then $c^{-1} \|x\|^2 \leq (x\,|\,x)_n \leq c \|x\|^2$ for all $n \in \mathbb{N}$ and all $x \in H$. In particular, for each pair of vectors $x, y \in H$, the sequence $((x\,|\,y)_n)_n$ is bounded by $c\|x\|\,\|y\|$. Now, choose some free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and define

$$(x\,|\,y)_{\mathrm{new}} := \mathcal{U} - \lim\,(x\,|\,y)_n$$

for $x, y \in H$. Obviously, this yields a positive sesquilinear form on H. Moreover, we have

$$c^{-1} \|x\|^2 \leq (x\,|\,x)_{\mathrm{new}} \leq c \|x\|^2$$

for all $x \in H$, whence $(\cdot\,|\,\cdot)_{\mathrm{new}}$ is an equivalent scalar product on H. Since

$$\|T(t)x\|_n^2 = \|\iota_n T(t)x\|_{\mathrm{new}}^2 = \|\mathcal{T}(t/n)\iota_n x\|_{\mathrm{new}}^2 \leq e^{2\omega\frac{t}{n}} \|\iota_n\|_{\mathrm{new}}^2 = e^{2\omega\frac{t}{n}} \|x\|_n^2,$$

we obtain $\|T(t)x\|_{\mathrm{new}}^2 \leq \|x\|_{\mathrm{new}}^2$ for all $x \in H$, $t \geq 0$. Hence T is contractive with respect to $(\cdot\,|\,\cdot)_{\mathrm{new}}$. $\square$

Since we started with a bounded semigroup that is not similar to a contraction semigroup, $\mathcal{T}$ is not similar to a quasi-contraction semigroup. Obviously this remains true also for each rescaled semigroup $(e^{-\varepsilon t}\mathcal{T}(t))_{t\geq 0}$. Hence for each $\varepsilon > 0$ the operator $\mathcal{A} + \varepsilon$ on the Hilbert space $\mathcal{H}$ has the properties required in Theorem 7.3.13.

Remark 7.3.16. Actually a statement slightly stronger than Theorem 7.3.13 is valid. In Le Merdy's example the operator A even has a bounded $\mathcal{R}^\infty(S_{\pi/2})$-calculus. By construction, this immediately carries over to the operator $\mathcal{A}$, and this is strictly stronger than to say that $\mathcal{A} \in \mathrm{BIP}(\mathcal{H})$.

7.4 Cosine Function Generators

A **cosine function** on a Banach space X is a strongly continuous mapping

$$\mathrm{Cos} : \mathbb{R}_+ \longrightarrow \mathcal{L}(X)$$

such that

$$\begin{aligned} \mathrm{Cos}(0) &= I, \\ 2\ \mathrm{Cos}(t)\ \mathrm{Cos}(s) &= \mathrm{Cos}(t+s) + \mathrm{Cos}(t-s) \qquad (t, s \in \mathbb{R}). \end{aligned}$$

In the following we cite some basic results of the theory of cosine functions from [10, Sections 3.14-3.16]. Given a cosine function, one can take its Laplace transform and define its **generator** B by

$$\lambda\, R(\lambda^2, B)x = \int_0^\infty e^{-\lambda t}\, \mathrm{Cos}(t)x\, dt$$

for $x \in X$ and $\operatorname{Re} \lambda$ sufficiently large. Then, for each pair $(x, y) \in X^2$ the function

$$u(t) := \mathrm{Cos}(t)x + \int_0^t \mathrm{Cos}(s)y\, ds$$

is the unique mild solution of the second order abstract Cauchy problem

$$\begin{cases} u''(t) = B\,u(t) \quad (t \geq 0), \\ u(0) = x, \\ u'(0) = y \end{cases}$$

(cf. [10, Corollary 3.14.8]). If B generates a cosine function, then it also generates an exponentially bounded holomorphic semigroup of angle $\pi/2$ (cf. [10, Theorem 3.14.17]).

Proposition 7.4.1. *Let A generate a cosine function on the Banach space X. Let the operator $\mathcal{A}$ on $X \times X$ be defined by*

$$\mathcal{D}(\mathcal{A}) := \mathcal{D}(A) \times X, \quad \mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ Ax \end{pmatrix}.$$

Then there exists a unique Banach space V such that $\mathcal{D}(A) \hookrightarrow V \hookrightarrow X$ and the part $\mathcal{B}$ of $\mathcal{A}$ in $V \times X$ generates a C_0-semigroup.

A proof is in [10, Theorem 3.14.11]. The space $V \times H$ is called the **phase space** associated with A. If A generates a cosine function and $\lambda \in \mathbb{C}$, then $A + \lambda$ generates a cosine function with the same phase space (cf. [10, Corollary 3.14.13]).

The connection with the theory of C_0-groups is given by the following: If an operator A generates a C_0-group $(U(t))_{t\in\mathbb{R}}$ on the Banach space X, then A^2 generates a cosine function Cos with phase space $D(A) \times X$, where $\mathrm{Cos}(t) = (U(t) + U(-t))/2$ $(t \geq 0)$ (cf. [10], Example 3.14.15). Moreover, a remarkable theorem of Fattorini states the (partial) converse.

Theorem 7.4.2 (Fattorini). *Let A be the generator of a cosine function on an UMD-space X. If $-A$ is sectorial, then $C := i(-A)^{1/2}$ generates a strongly continuous group and $C^2 = A$.*

A proof can be found in [10, Theorem 3.16.7]. Altogether this suggests that we consider squares of group generators.

Theorem 7.4.3. *Let A be m-H_ω-accretive for some $\omega \geq 0$. Then for every angle $0 \leq |\varphi| < \pi/2$ the operator*

$$e^{i\varphi}\left(\left(\frac{\omega}{\cos\varphi}\right)^2 + A^2\right) \tag{7.22}$$

is m-accretive. The operator $-A^2$ generates a holomorphic semigroup $(S(z))_{\mathrm{Re}\, z>0}$ of angle $\pi/2$ such that

$$\|S(z)\| \leq e^{\left(\frac{\omega}{\cos\varphi}\right)^2 \mathrm{Re}\, z} \qquad (|\arg z| \leq \varphi < \pi/2).$$

Moreover, one has

$$\mathcal{D}\left((\omega^2 + A^2)^{\frac{1}{2}}\right) = \mathcal{D}(A).$$

Proof. The case where $\omega = 0$ is trivial since then the group is unitary and iA is skew-adjoint. This implies that A^2 is self-adjoint with $-A^2 \leq 0$, and the assertions of the theorem are immediate.

So suppose that $\omega > 0$, let $0 \leq |\varphi| < \pi/2$ and fix $\varepsilon > 0$. Define $\alpha = \omega \tan\varphi$, i.e., $z := \omega - i\alpha = (\omega/\cos\varphi)e^{-i\varphi}$. By assumption and Proposition C.7.2 (ii), the operators $z - iA$ and $z + iA$ are m-accretive. Applying (iv) of Proposition C.7.2 we obtain

$$\left\|\frac{iA - (z-\varepsilon)}{iA - (z+\varepsilon)}\right\|, \left\|\frac{iA + (z-\varepsilon)}{iA + (z+\varepsilon)}\right\| \leq 1.$$

(Here and in the following we write $(iA+\lambda)/(iA+\mu)$ instead of $(iA+\lambda)(iA+\mu)^{-1}$ to make the computations more perspicious.) Hence

$$\left\|\frac{A^2 + (z-\varepsilon)^2}{A^2 + (z+\varepsilon)^2}\right\| = \left\|\left(\frac{iA - (z-\varepsilon)}{iA - (z+\varepsilon)}\right)\left(\frac{iA + (z-\varepsilon)}{iA + (z+\varepsilon)}\right)\right\| \leq 1.$$

Now,

$$\frac{A^2 + (z-\varepsilon)^2}{A^2 + (z+\varepsilon)^2} = \frac{A^2 + (z^2+\varepsilon^2) - 2z\varepsilon}{A^2 + (z^2+\varepsilon^2) + 2z\varepsilon} = \frac{e^{i\varphi}[A^2 + (z^2+\varepsilon^2)] - 2\varepsilon\,|z|}{e^{i\varphi}[A^2 + (z^2+\varepsilon^2)] + 2\varepsilon\,|z|}.$$

We can apply Proposition C.7.2 again (note that $2\varepsilon\,|z| > 0$) to conclude that

$$e^{i\varphi}[(z^2+\varepsilon^2) + A^2] = \big(\frac{\omega}{\cos\varphi}\big)^2 e^{-i\varphi} + \varepsilon^2 e^{i\varphi} + e^{i\varphi}A^2$$

is m-accretive. Letting $\varepsilon \searrow 0$ we see that $(\omega/\cos\varphi)^2 e^{-i\varphi} + e^{i\varphi}A^2$ and finally that

$$e^{i\varphi}\left(\big(\frac{\omega}{\cos\varphi}\big)^2 + A^2\right) = \big(\frac{\omega}{\cos\varphi}\big)^2 e^{i\varphi} + e^{i\varphi}A^2$$

is m-accretive. (Note that $e^{i\varphi}$ differs from $e^{-i\varphi}$ only by a purely imaginary number and this does not affect m-accretivity by (ii) of Proposition C.7.2.) This finishes the proof of the first part of the theorem. The second part follows from Proposition 7.1.1, cf. also [10, Chapter 3.4 and Chapter 3.9].

To prove the last assertion, note first that the operator $\omega^2 + A^2$ is m-accretive, hence sectorial. Thus the square root is well defined. Since iA generates a group, $-A^2$ generates a cosine function with phase space $\mathcal{D}(A) \times H$. By general cosine function theory (see the remarks at the beginning of this section), $-(A^2+\omega^2)$ also generates a cosine function with the same phase space. Fattorini's Theorem 7.4.2 implies that $B := i(\omega^2 + A^2)^{1/2}$ generates a group and $B^2 = -(A^2+\omega^2)$. Then $\mathcal{D}(B) = \mathcal{D}(A)$ follows from the uniqueness of the phase space (see Proposition 7.4.1). □

Combining this result with the Similarity Theorem 7.2.8 for groups yields the following corollary:

Corollary 7.4.4. *Let $-B$ be the generator of a cosine function on a Hilbert space H. Then B is Kato-sectorial and square root regular with respect to some equivalent scalar product. In particular, $\lambda + B \in \mathrm{BIP}(H)$ for large $\lambda \in \mathbb{R}$.*

Proof. First, one can find β such that $B+\beta$ is sectorial. Since $-(B+\beta)$ generates a cosine function as well, we can apply Fattorini's Theorem 7.4.2. Thus, the operator $iA := i(\beta+B)^{1/2}$ generates a strongly continuous group T on H. Choose $\omega > \theta(T)$. By Theorem 7.2.8 we can find a new scalar product $(\cdot\,|\,\cdot)_\circ$ that renders A m-H_ω-accretive and is such that $\mathcal{D}(A) = \mathcal{D}(A^\circ)$. Apply now Theorem 7.4.3 together with Corollary 7.3.5 to conclude that $A^2 = B+\beta$ is Kato-sectorial. This implies that B is Kato-sectorial. Finally, we apply Theorem 7.4.3 to the operators A and A° and obtain

$$\begin{aligned}\mathcal{D}((\omega^2+\beta+B)^{\frac12}) &= \mathcal{D}((\omega^2+A^2)^{\frac12}) = \mathcal{D}(A) = \mathcal{D}(A^\circ)\\ &= \mathcal{D}((\omega^2+A^{\circ 2})^{\frac12}) = \mathcal{D}((\omega^2+\beta+B)^{\circ\frac12}).\end{aligned}$$

This completes the proof. □

Let us turn to a second corollary. Namely, Theorem 7.4.3 can be seen as a mapping theorem for the numerical range (cf. also Section 7.1.4). We define the horizontal parabola

$$\Pi_\omega = \{z \mid (\operatorname{Im} z)^2 < 4\omega^2 \operatorname{Re} z\}$$

for $\omega > 0$, and $\Pi_0 := [0, \infty)$.

Corollary 7.4.5. *Let A be an H_ω-accretive operator on the Hilbert space H for some $\omega \in [0, \pi/2]$. Then $W(A^2) \subset \{z^2 \mid z \in \overline{H_\omega}\}$, i.e., the numerical range of $B := \omega^2 + A^2$ is contained in the horizontal parabola*

$$\overline{\Pi_\omega} = \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0 \text{ and } |\operatorname{Im} z| \leq 2\omega\sqrt{\operatorname{Re} z}\}.$$

This is equivalent to saying that B is m-accretive and

$$|\operatorname{Im}(Bu \mid u)| \leq 2\omega \operatorname{Re}(Bu \mid u)^{\frac{1}{2}} \|u\| \tag{7.23}$$

for all $u \in \mathcal{D}(B) = \mathcal{D}(A^2)$.

Proof. By Theorem 7.4.3 we obtain that $e^{i\varphi}((\omega/\cos\varphi)^2 + A^2)$ is m-acccretive for every $0 \leq |\varphi| < \pi/2$. Specialising $\varphi = 0$ yields that $B = \omega^2 + A^2$ is m-accretive. Now we write

$$e^{i\varphi}\left(\frac{\omega^2}{\cos^2\varphi} + A^2\right) = e^{i\varphi}\left(B + \left(\frac{\omega^2}{\cos^2\varphi} - \omega^2\right)\right) = e^{i\varphi}\left(B + \omega^2\tan^2\varphi\right).$$

Since this operator is m-accretive and $\tan^2\varphi = \tan^2(-\varphi)$, by Proposition 7.1.1 we obtain $W(B + \omega^2\tan^2\varphi) \subset \overline{S_{\pi/2-\varphi}}$, i.e.,

$$|\operatorname{Im}(Bu \mid u)| \leq \tan\left(\frac{\pi}{2} - \varphi\right)\left(\operatorname{Re}(Bu \mid u) + \omega^2\tan^2\varphi\,\|u\|^2\right) \qquad (u \in \mathcal{D}(B))$$

for $0 < \varphi < \pi/2$. Now $\tan(\pi/2, -\varphi) = (\tan\varphi)^{-1}$ and if we parametrise $\tau := \tan\varphi$ with $0 < \tau < \infty$ we obtain $|\operatorname{Im}(Bu \mid u)| \leq \tau^{-1}\operatorname{Re}(Bu \mid u) + \omega^2\tau\,\|u\|^2$ for every $u \in \mathcal{D}(B)$. The right-hand side has $2\sqrt{\operatorname{Re}(Bu \mid u)\,\omega^2\,\|u\|^2}$ as its minimum value, whence we arrive at (7.23). $\square$

Corollary 7.4.6. *Let $-B$ be the generator of a cosine function on the Hilbert space H. Then, with respect to an equivalent scalar product, B has numerical range in a horizontal parabola $\lambda + \overline{\Pi_\omega}$ for some $\lambda \in \mathbb{R}, \omega > 0$.*

Proof. Apply Fattorini's Theorem 7.4.2, change the scalar product according to Theorem 7.2.8, then apply Corollary 7.4.5. $\square$

Let us call an operator **m-Π_ω-accretive** if its numerical range is contained in $\overline{\Pi_\omega}$ and $\mathbb{C} \setminus \overline{\Pi_\omega}$ is contained in its resolvent set. The last corollary immediately raises the following question. Suppose $-B$ is m-Π_ω-accretive for some $\omega > 0$. Does it follow that B already generates a cosine function? This is in fact true since Crouzeix has proved in [54] the following result, which we quote without proof.

Theorem 7.4.7 (Crouzeix). *There is an absolute constant $c > 0$ such that*

$$\|f(A)\| \leq c \, \|f\|_{\Pi_\omega}$$

for all $f \in \mathcal{R}^\infty(\Pi_\omega)$ and every m-Π_ω-accretive operator on a Hilbert space H.

The proof follows *in principle* the lines of the proof of Theorem 7.1.16. However, it is much harder in detail.

Corollary 7.4.8. *Let B be an operator on a Hilbert space H. Then $-B$ generates a cosine function if and only if there is $\lambda \in \mathbb{R}$ and an equivalent scalar product on H with respect to which $\lambda + B$ is m-Π_ω-accretive.*

Proof. One direction is Corollary 7.4.6. For the converse, suppose that $\lambda + B$ is m-Π_ω-accretive. Without loss of generality we may suppose that B is m-Π_ω-accretive and invertible. Then we can set up a holomorphic functional calculus for B on horizontal parabolas Π_α for $\alpha > \omega$. Using Crouzeix' Theorem 7.4.7 together with similar arguments as in Section 5.3 we conclude that the natural $H_\infty(\Pi_\alpha)$-calculus is bounded for every $\alpha > \omega$. Inserting into this calculus the functions $e^{tiz^{(1/2)}}$ for $t \in \mathbb{R}$ we obtain the C_0-group generated by $-B^{1/2}$. (The density of $\mathcal{D}(B)$ in H follows from sectoriality.) Hence $-B = (iB^{1/2})^2$ generates a cosine function. $\square$

At the end of this section we would like to illustrate Crouzeix's result with an example. In Section 7.3.2 we considered operators A defined by means of an elliptic form $a : V \times V \longrightarrow H$. Suppose that condition 1) of page 198 is satisfied, i.e., we find $\lambda_0 \in \mathbb{R}$ such that $\operatorname{Re} a + \lambda_0 \left(\cdot \,|\, \cdot \right)_H$ is a scalar product on V which turns V into a Hilbert space, continuously embedded into H. Instead of condition 2) (which, by some trivial equivalences, merely postulates continuity of $\operatorname{Im} a$ with respect to this scalar product) we require the *stronger* condition

$$|\operatorname{Im} a(u,u)| \leq M \, \|u\|_H \, \|u\|_V \qquad (u \in V). \tag{7.24}$$

Then the numerical range of $A + \lambda_0$ is contained in Π_ω for some $\omega \geq 0$. Using approximation by rational functions, one can conlcude that $-(A + \lambda_0)$ and hence $-A$ generates a cosine function.

In the concrete case of an elliptic form with $\mathbf{L}^\infty$-coefficients

$$a(u,v) = \int_\Omega \left\{ \sum_{i,j} a_{i,j} D_i u \overline{D_j v} + \sum_i b_i D_i u \overline{v} + c_i u \overline{D_i v} + d u \overline{v} \right\} dx \qquad (u \in V)$$

on $H = \mathbf{L}^2(\Omega)$ and with V being some closed subspace of $\mathbf{W}^{1,2}(\Omega)$ containing $\boldsymbol{\mathcal{D}}(\Omega)$, condition (7.24) is satisfied if $\overline{a_{i,j}} = a_{j,i}$ for all i, j.

7.5 Comments

7.1 Numerical Range Conditions. The material on normal or self-adjoint operators is standard. The proof is usually given via a spectral measure respresentation, cf. [158, Section 4.3.1].

7.1.3 Functional Calculus for m-accretive Operators. The fundamental Theorem 7.1.7 has been folklore for a long time, due to its connection with von Neumann's inequality (cf. Remark 7.1.9 and below). The classical reference for the dilation theorem is the book [209] by Sz.-Nagy and Foias. Therein one can find also the construction of a functional calculus for so-called completely non-unitary contractions. The proof of Theorem 7.1.7 via the dilation theorem can also be found in [145, Theorem 4.5]. Our second proof is taken from [3], where it is attributed to Franks. This proof is in the spirit of Bernard and François Delyon's proof of the von Neumann inequality in [68]. Other proofs of von Neumann's inequality can be found in [185, Corollary 2.7] and [190, Chapter 1].

7.1.4 Numerical Range Mapping Theorems. To establish 'mapping theorems for the numerical range' Theorem 7.1.7 seems to be at the core of the theory since it ties closely contractivity and m-accretivity to the functional calculus. Kato [129] uses this connection to prove more general results. Based on that, Crouzeix and Delyon [55] essentially prove Corollary 7.1.14. Kato [127] gives a proof of Corollary 7.1.13 different from ours, see also [210, Lemma 2.3.6]. In general, 'mapping theorems for the numerical range' are a delicate matter. For example, Simard in [203] constructed an example of an m-$H_{\pi/2}$-accretive operator on the space $H = \mathbb{C}^2$ such that e^A is *not* m-accretive.

7.1.5 The Crouzeix–Delyon Theorem. The Crouzeix–Delyon Theorem 7.1.16 from [55] can be considered a major breakthrough in deriving boundedness of functional calculi from numerical range conditions. For a long time, the case of m-accretive operators was more or less the only example of this connection. The new insights came when a new proof of von Neumann's inequality was found by directly manipulating the corresponding Cauchy integrals (see above). This has been developed further by Crouzeix [53, 54] to obtain similar results for operators with numerical range in a parabola (see Theorem 7.4.7) and more general convex domains.

7.2 Group Generators on Hilbert Spaces. Theorem 7.2.1 is due to Boyadzhiev and de Laubenfels [34, Theorem 3.2], although in a little different form. The original proof proceeds in two steps. First, assuming $\omega < \pi/2$ without loss of generality, the authors construct the operator e^B (see Chapter 4; their construction however relies on the theory of regularised semigroups). Then they show that this operator is sectorial and has a bounded H^∞-functional calculus on a sector. The statement then follows by means of some special case of the composition rule.

Our first proof for Theorem 7.2.1 is just a combination of the Decomposition Theorem 7.2.8 and the Crouzeix–Delyon Theorem 7.1.16. Since the proof of the latter is quite involved, this proof cannot be regarded as simple. However, there is no need of the full power of the Crouzeix–Delyon theorem. In fact, the weaker version mentioned in Remark 7.1.18 suffices. Regarding the Spectral Theorem as known, this can be proved by a simple perturbation argument, as was done by the author [105].

7.2.1 Liapunov's Direct Method. In the case where A is bounded the Liapunov method is used in Chapter I of the book [59]. There the operator equation $QA + A^*Q = -I$ is directly linked to the problem of finding a Liapunov function for the semigroup. (This is sometimes called 'Liapunov's direct method'.) For the unbounded case the relevant facts are included in [57, Theorem 5.1.3], where a characterisation of exponential stability of the semigroup is given in terms of the existence of an operator Q satisfying the Liapunov equation(s). (Extensions of this result can be found in [98], [15] and [2].) It is shown in [232] that this method in fact gives an *equivalent* scalar product if the semigroup is a group. In the paper [156] on exact controllability LIU and RUSSELL establish almost the same results (and others).

The Liapunov theorem for holomorphic semigroups (Proposition 7.3.8) is due to ARENDT, BU, and the author [12, Theorem 4.1]. In that paper, Liapunov type theorems are established also for hyperbolic holomorphic semigroups and quasi-compact holomorphic semigroups. Moreover, Proposition 7.3.8 is applied to semilinear equations.

7.2.2 The Decomposition Theorem. Theorem 7.2.8 is due to the author [105]. The following well-known theorem by SZ.-NAGY [207] can be regarded as the 'limit case': Every generator of a bounded group is similar to a skew-adjoint operator. This result cannot be deduced directly from Theorem 7.2.8. However, ZWART [232] gives a proof using the Liapunov renorming and some approximation argument.

DE LAUBENFELS [67, Theorem 2.4] proves that, given a C_0-group T on a Hilbert space, one has $\|T(t)\|_\circ \le e^{\omega|t|}$ for some equivalent scalar product $(\cdot\,|\,\cdot)_\circ$ and some ω strictly larger than the group type $\theta(T)$. (This is covered by Theorem 7.2.8 a).) However, this proof is based on the boundedness of the H^∞-calculus and on Paulsen's theorem (see below), so the new scalar product cannot easily be made explicit.

One can wonder whether, given a group T such that $\|T(s)\| \le Ke^{\theta|s|}$ $(s \in \mathbb{R})$, one may always take $\omega = \theta$ in Theorem 7.2.8. However, SIMARD [203] has shown that this is not possible in general.

7.2.3 A Characterisation of Group Generators. Theorem 7.2.12 is due to the author [101]. The rough idea behind the proof of the crucial implication (iv)⇒(i) consists in adapting MCINTOSH's methods from [167] (in particular his use of quadratic estimates) to strip-type operators. Corollary 7.2.14 is a result by LIU [155, Theorem 1]. One may consult [101] for other characterisations of groups on Hilbert spaces, including results of LIU [155, Theorem 2] and ZWART [232, Theorem 2.2]. Further results on quadratic estimates in connection with semigroups can be found in [96, 202, 122, 27].

7.3.1 McIntosh's Theorem. Theorem 7.3.1 is due to McIntosh [167], based on ideas of Yagi [226], see also [169]. Norm inequalities such as (i) of McIntosh's theorem are called *quadratic estimates* or *square function estimates* in the terminology of these authors. The importance of quadratic estimates pertains when studying boundedness of H^∞-functional calculus on $\mathbf{L}^p$-spaces, see [51], [151] and [141, Section 11]. Auscher, Nahmod and McIntosh [19] have made explicit the connection with interpolation theory (cf. also Chapter 6 and the comments there). The proof given in Remark 7.3.2 is taken from the author's paper [110].

The proof of the implication (ii)⇒(i) in Theorem 7.3.1 via the Unconditionality Lemma 5.6.1 is taken from [150, Theorem 4.2]. Let us point out that in Theorem 7.3.1 the equivalence of (iii) and the other statements is proved without making use of interpolation theory (cf. Remark 7.3.2).

One can add further equivalences to McIntosh's Theorem 7.3.1, e.g.

(iv) There is equivalence of norms

$$\|x\|_H^2 \quad \sim \quad \int_{S_\alpha} \|\psi(zA)x\|_H^2 \, \frac{dz}{z^2}$$

where $0 \neq \psi \in H_0^\infty(S_\varphi)$ *and* $\varphi \in (\omega_A + \alpha, \pi)$.

This is done by Kunstmann and Weis [141, Theorem 11.13]. One can easily deduce this from the fact that in any Banach space the homogeneous interpolation space $(D^{(\alpha)}, R^{(\alpha)})_{1/2,p}$ can be described as

$$(D^{(\alpha)}, R^{(\alpha)})_{\frac{1}{2},p} = \left\{x \in U \;\middle|\; \psi(zA)x \in \mathbf{L}^p(S_\alpha, dz/z^2; X)\right\},$$

which in turn follows from the results of Section 6.4. (One has to note that the embedding constants associated with the family of sectorial operators $(e^{i\theta}A)_{|\theta|\leq\alpha}$ are uniformly bounded. We do not go into details.)

7.3.2 Sesquilinear Forms. The classical reference for operators constructed via sesquilinear forms is Kato's early paper [127] as well as his book [130]. This is the reason why we called this operator 'Kato-sectorial'. Proposition 7.1.1 essentially is Kato's 'First Representation Theorem' [130, Chapter VI, Theorem 2.1]. It is also included as Theorem 1.2 in [12]. The applications to PDE in the literature are numerous, see the recent book [180] by Ouhabaz and the references therein. In [12, Example 3.2] an example of two similar operators on a Hilbert space is given, one variational but not the other. As mentioned already, by Matolcsi's result [162] this is not a pathological case.

It would be interesting if there is a proof for Kato's Theorem 7.3.6 that differs essentially from Kato's original one. Apart from [210, Lemma 2.3.8] and [158, Theorem 4.3.4], which more or less copy Kato's arguments, we do not know of any other account. The Square Root Problem (cf. Remark 7.3.2) has been solved by Auscher, Hoffman, Lacey, Lewis, McIntosh, and Tchamitchian [18]. Surveys and a deeper introduction to these matters as well as the connection with Calderón's conjecture for Cauchy integrals on Lipschitz curves are in [165, 166, 168] and [20]. The first counterexample to Kato's original question was given by McIntosh [163]. A different one can be obtained from [20, Section 0, Theorem 6] employing an additional direct sum argument.

7.3.3 Similarity Theorems and 7.3.4 A Counterexample. Similarity problems have a long tradition in operator theory. In 1947 Sz.-Nagy [207] had observed that a bounded and invertible operator T on a Hilbert space H is similar to a unitary operator if and only if the discrete group $(T^n)_{n\in\mathbb{Z}}$ is uniformly bounded. His question was whether the same is true if one discards the invertibility of T. In [208] he showed that the answer is yes if T is compact, but Foguel [90] disproved the general conjecture by giving a counterexample. By von Neumann's inequality, if T is a contraction, then T is not only power-bounded but even polynomially bounded, i.e.,

$$\sup\{\|p(T)\| \ \mid p \in \mathbb{C}[z],\, \|p\|_\infty \le 1\} \,<\, \infty,$$

where $\|p\|_\infty$ denotes the uniform norm on the unit disc. So Halmos [113] asked whether polynomial boundedness in fact characterises the bounded operators on H that are similar to a contraction. This question remained open only until recently when Pisier [188] found a counterexample, see also [60]. Meanwhile, Paulsen [184] had shown that T is similar to a contraction if and only if T is *completely* polynomially bounded. His characterisation is a special case of a general similarity result [185, Theorem 8.1] for completely bounded homomorphisms of operator algebras (see [185] for definitions and further results). We address this result as **Paulsen's theorem** in the following.

One can set up a semigroup analogue of Sz.-Nagy's question (see above) namely whether every bounded C_0-semigroup on a Hilbert space is similar to a contraction semigroup. The corresponding result for groups is true as was also proved by Sz.-Nagy in the very same article [207]. (The question is a little more special than the original one since not *every* power-bounded operator is the Cayley transform of a C_0-semigroup generator.) It was answered in the negative by Packel [181]. By using this result, Chernoff [43] provided an example of a *bounded* operator with the generated C_0-semigroup being bounded but not similar to a contraction one. Furthermore, via Lemma 7.3.15 he constructed a C_0-semigroup which is not even similar to a quasi-contractive semigroup. (Our setting up this historical panorama is mainly based on [146, Introduction]. More on similarity problems and their connection with the theory of operator algebras can be found in [189].)

Being probably unaware of the history of this problem, Callier and Grabowski in an unpublished research report [97] proved Theorem 7.3.7 in the case where the semigroup is exponentially stable. Their arguments are based on two facts from interpolation theory: first, both the (complex) interpolation space $[H, \mathcal{D}(A)]_{1/2}$ and the (real) interpolation space $(H, \mathcal{D}(A))_{1/2,2}$ are equal to $\mathcal{D}(A^{1/2})$ if $A \in \mathrm{BIP}(H)$ (see Theorem 6.6.9) and, second, this real interpolation space is given by

$$(H, \mathcal{D}(A))_{\frac{1}{2},2} = \left\{ x \in H \;\middle|\; \int_0^\infty \left\| t^{\frac{1}{2}} A^{\frac{1}{2}} e^{-tA} x \right\|^2 \frac{dt}{t} < \infty \right\}$$

(cf. (6.9) in Section 6.2.3). It was Le Merdy [150, Theorem 4.2] who observed that the result follows from McIntosh's theorem.

Theorem 7.3.7 as it stands is also a consequence of Corollary 7.3.11 and is stated as such in [146, Theorem 4.3]. Le Merdy proves Corollary 7.3.11 with the help of Paulsen's theorem (see above). The same is done by Franks in [91, Section 4]. Note that — by using the scaling technique and the results of Kato on accretive operators — the Franks-Le Merdy theorem and the Callier–Grabowski–Le Merdy theorem are actually *equivalent*, and that Paulsen's theorem is *not necessary* to prove the Similarity Theorem 7.3.9. However, Le Merdy [148] proved with the help of Paulsen's theorem that the exponential e^A of an m-$H_{\pi/2}$-accretive operator A is always similar to an m-accretive operator. (As mentioned above, Simard has shown that e^A need not be m-accretive itself.) Up to now it is unknown whether one can avoid Paulsen's theorem here.

It has been noted in [12, Theorem 3.3] that the Franks–Le Merdy result solved the first similarity problem posed (see page 201). However, from their proof it was not clear whether also the second problem concerning the square roots can be solved. That it actually can is an observation due to the author [102, Theorem 4.26] and sheds an interesting light on the orginal square root problem. It also complements a result of Yagi [226, Theorem B] which says that a sectorial invertible operator A on a Hilbert space has bounded imaginary powers if $\mathcal{D}(A^\alpha) = \mathcal{D}(A^{*\alpha})$ for all α contained in a small interval $[0, \varepsilon)$. It is a consequence of Theorem 7.3.9 combined with the scaling technique that the converse holds *modulo similarity*. The following problem seems to be open until now.

Problem: Is there an m-accretive operator A on a Hilbert space H such that

$$\mathcal{D}(A^{\frac{1}{2}}) \neq \mathcal{D}(A^{\circ \frac{1}{2}})$$

for *every* equivalent scalar product $(\cdot \,|\, \cdot)_\circ$? (Note that, by Theorem 7.3.9, such an operator necessarily has to satisfy the relation $\omega_A = \pi/2$.)

Returning to the semigroup version of Sz.-Nagy's question (see above), by Theorem 7.1.7 (which is essentially von Neumann's inequality) one obtains that a bounded H^∞-calculus (or equivalently BIP) is as necessary a condition for similarity to a contraction semigroup as it was polynomial boundedness in the discrete case. Moreover, the Callier–Grabowski–Le Merdy Theorem 7.3.7 implies that for bounded *holomorphic* semigroups in fact the condition BIP already suffices. Using

Pisier's counterexample to Halmos's problem Le Merdy [145, Proposition 4.8] succeeded in showing that in general BIP is not sufficient. Theorem 7.3.13, based on Chernoff's ideas from [43] and due to the author [102, Theorem 4.31], states that one can even exclude similarity to a quasi-contraction semigroup.

7.4 Cosine Function Generators. The mapping theorem for the numerical range established in Theorem 7.4.3 and Corollary 7.4.5 is from [102, Corollary 5.17]. Its proof is based on Theorem 7.4.3, also due to the author [105]. McIntosh [164] shows that an operator B satisfying the conclusion of Corollary 7.4.5 also has the square root property $\mathcal{D}(B^{1/2}) = \mathcal{D}(B^{*1/2})$. Employing this one can eliminate the cosine function theory from the proof of Corollary 7.4.4. Crouzeix' result Theorem 7.4.7 from [54] is a milestone. This author is about to prove a general theorem on bounded H^∞-calculus for operators with numerical range in a convex subset of the plane, cf. [53]. For more information on cosine functions in general consult [10] and the references therein.

Chapter 8

Differential Operators

We treat constant coefficient elliptic operators on the euclidean space $\mathbb{R}^d$. The main focus lies on the connection of functional calculus with Fourier multiplier theory. The $\mathbf{L}^1$-theory is the subject of Section 8.1 while the $\mathbf{L}^p$-case is presented in Section 8.2. Then we apply the obtained results to the negative Laplace operator (Section 8.3). The universal extrapolation space for the Laplace operator is identified with a space of certain (equivalence classes) of tempered distributions. In Sections 8.4 and 8.5 we treat the derivative operator on the line, the half-line and finite intervals. The UMD property of a Banach space X is characterised by functional calculus properties of the derivative operator on X-valued functions.

Preliminaries

In this chapter we discuss a fairly concrete class of operators, namely (elliptic) differential operators with constant coefficients. We shall work on the whole space $\mathbb{R}^d$ and only sketch generalisations to differential operators on domains, cf. Section 8.6. In our discussion we will make extensive use of Fourier theory, and we refer the reader to Appendix E for notation and background information.

For a scalar function of tempered growth $a \in \boldsymbol{\mathcal{P}}(\mathbb{R}^d)$ we define the associated operator

$$A := \big(u \longmapsto \mathcal{F}^{-1}a(\mathcal{F}u)\big) : \mathbf{TD}(\mathbb{R}^d; X) \longrightarrow \mathbf{TD}(\mathbb{R}^d; X).$$

Given any Banach space $\mathcal{X} \subset \mathbf{TD}(\mathbb{R}^d; X)$ the operator $A_{\mathcal{X}}$ is defined as the part of A in $\mathcal{X}$, i.e.,

$$\mathcal{D}(A_{\mathcal{X}}) := \{u \in \mathbf{TD}(\mathbb{R}^d; X) \mid Au \in \mathcal{X}\}.$$

As the embedding $\mathcal{X} \hookrightarrow \mathbf{TD}(\mathbb{R}^d; X)$ is continuous, $A_{\mathcal{X}}$ is a closed operator on $\mathcal{X}$. For the special choices $\mathcal{X}_p := \mathbf{L}^p(\mathbb{R}^d; X)$, $p \in [1, \infty]$ we use the abbreviation A_p. This setting covers in particular differential operators with constant coefficients since they are induced by proper polynomials. E.g., the polynomial

$$a(s) = \sum_{|\alpha| \le m} a_\alpha (is)^\alpha \qquad (a_\alpha \in \mathbb{C}) \tag{8.1}$$

yields the operator

$$Au := \sum_{|\alpha|\le m} a_\alpha D^\alpha u \tag{8.2}$$

on $\mathbf{TD}(\mathbb{R}^d; X)$.

Lemma 8.0.1. *Let $a \in \boldsymbol{\mathcal{P}}(\mathbb{R}^d)$, let $A = \mathcal{F}^{-1} a \mathcal{F}$ as in (8.1) and (8.2) above, and let $\lambda \in \mathbb{C}$. If $\lambda \notin \overline{a(\mathbb{R}^d)}$, then $(\lambda - a)^{-1} \in \boldsymbol{\mathcal{P}}(\mathbb{R}^d)$, $(\lambda - A)$ is injective, and*

$$(\lambda - A)^{-1} = \mathcal{F}^{-1}(\lambda - a)^{-1}\mathcal{F}.$$

Let $\mathcal{X}$ be a Banach space continuously embedded into $\mathbf{TD}(\mathbb{R}^d; X)$. Then $\lambda \in \varrho(A_{\mathcal{X}})$ if and only if $\mathcal{X}$ is invariant under $\lambda - A$. In this case $R(\lambda, A_{\mathcal{X}}) = [(\lambda - A)^{-1}]_{\mathcal{X}}$.

Proof. Suppose that $\lambda \notin \overline{a(\mathbb{R}^d)}$. Then $(\lambda - A)^{-1} \in \boldsymbol{\mathcal{P}}(\mathbb{R}^d)$ follows from a general result about derivatives of quotients. Indeed, one proves by induction that, given f, g functions of d variables, for each multiindex $\alpha \in \mathbb{N}^d$ one has

$$D^\alpha \left(\frac{f}{g}\right) = \frac{f_\alpha}{g^{|\alpha|+1}},$$

where f_α is a polynomial in the derivatives $D^\beta g, D^\beta f$, $(0 \le \beta \le \alpha)$ and with integer coefficients. Applying this fact to the case $f = 1$ and $g = \lambda - a$, one sees easily that each derivative $D^\alpha(\lambda - a)^{-1}$ must be polynomially bounded. The remaining statements are then straightforward to prove. □

In general, very little may be said about such differential operators. This changes drastically when we restrict to the class of **homogeneous elliptic** operators. These arise from homogeneous polynomials

$$a(s) = \sum_{|\alpha| = m} a_\alpha (is)^\alpha \tag{8.3}$$

satisfying the condition

$$s \neq 0 \quad \Longrightarrow \quad a(s) \neq 0 \tag{8.4}$$

and a non-triviality postulate

$$-1 \notin a(\mathbb{R}^d). \tag{8.5}$$

If one requires the even stronger condition

$$0 \neq s \quad \Longrightarrow \quad \operatorname{Re} a(s) > 0, \tag{8.6}$$

the polynomial a and the corresponding operator A are called **strongly elliptic**. For our further investigations we are in need of some auxiliary information.

Lemma 8.0.2. *Let $a : \mathbb{R}^d \longrightarrow \mathbb{C}$ satisfy (8.3)–(8.5). Then the following assertions hold.*

a) *There are constants* $c_1, c_2 > 0$ *such that*

$$c_1 |s|^m \le |a(s)| \le c_2 |s|^m \qquad (s \in \mathbb{R}^d).$$

b) *There is* $\omega \in [0, \pi)$ *such that* $a(\mathbb{R}^d) = \overline{a(\mathbb{R}^d)} \subset \overline{S_\omega}$*. Let* ω_a *be the least of these* ω*. If* a *is strongly elliptic, i.e., if* (8.6) *holds, then* $\omega_a < \pi/2$*.*

c) *With* $c_3 := c_2/2$ *and* $L := (c_3)^{-m}$ *one has*

$$|a(s) + \lambda| \ge c_3 |s|^m \qquad (|s| \ge L,\ |\lambda| = 1)\,.$$

d) *If* $\lambda \notin a(\mathbb{R}^d)$*, then* $(\lambda - a)^{-1} \in \boldsymbol{\mathcal{P}}(\mathbb{R}^d)$ *and*

$$D^\alpha (\lambda - a)^{-1} = \frac{p_\alpha}{(\lambda - a)^{|\alpha|+1}} \qquad (\alpha \in \mathbb{N}^d),$$

where p_α *is a polynomial of degree* $\deg p_\alpha \le (m-1)\,|\alpha|$*.*

Proof. a) The existence of c_2 follows from the inequality $|s^\alpha| \le |s|^{|\alpha|}$. By (8.4) and the compactness of the unit sphere $c_2 := \inf_{|s|=1} |a(s)| > 0$. The rest follows from the homogeneity.

b) Since by a) one has $\lim_{|s|\to\infty} |a(s)| = \infty$, $a(\mathbb{R}^d) = \overline{a(\mathbb{R}^d)}$ is obvious. Condition (8.5) then insures that a small neighbourhood U of -1 does not intersect the image of a, whence by the homogeneity the whole cone generated by U does not intersect the image of a. If a is strongly elliptic, then $\{z \mid |z| = 1, |\arg z| \in [\pi/2, \pi]\}$ does not intersect $a(\mathbb{R}^d)$. Hence this is true also for $\{z \mid |z| = 1, |\arg z| \in [\omega, \pi]\}$, with some $\omega \in (0, \pi/2)$.

c) is a simple computation and d) is proved by induction on $|\alpha|$. □

In the following we look at the part A_p in $\mathbf{L^p}(\mathbb{R}^d; X)$ of a constant-coefficient elliptic operator A. By Lemma 8.0.1, the attempt to determine the resolvent leads to the question whether a certain function — here: $(\lambda - a)^{-1}$ — is an $\mathbf{L^p}(\mathbb{R}^d; X)$-Fourier multiplier. Moreover, general functional calculus philosophy lets us expect the formula

$$f(A)u = \mathcal{F}^{-1}(f \circ a)\widehat{u},$$

at least for certain u. Boundedness of $f(A)$ is therefore somehow linked to $f \circ a$ being an $\mathbf{L^p}(\mathbb{R}^d; X)$-Fourier multiplier. The next two sections will render more precise this intuitive reasoning.

8.1 Elliptic Operators: $\mathbf{L^1}$-Theory

Let $a : \mathbb{R}^d \longrightarrow \mathbb{C}$ satisfy (8.3)–(8.5), let A be the corresponding operator with symbol a , and let $c_1, c_2, c_3, L, \omega_a$ as in Lemma 8.0.2.

Proposition 8.1.1. *For each $\lambda \notin a(\mathbb{R}^d)$ there is a unique $r_\lambda \in \mathbf{L^1}(\mathbb{R}^d)$ such that $\widehat{r_\lambda} = (\lambda - a)^{-1}$. The mapping*

$$(\lambda \longmapsto r_\lambda) : \mathbb{C} \setminus a(\mathbb{R}^d) \longrightarrow \mathbf{L^1}(\mathbb{R}^d)$$

is holomorphic, and $\|\lambda r_\lambda\|_1 = \left\|r_{\lambda/|\lambda|}\right\|_1$.

Proof. First of all, the homogeneity of a implies that $\lambda \notin a(\mathbb{R}^d)$ if and only if $\mu := \lambda/|\lambda| \notin a(\mathbb{R}^d)$. We claim that there is an $r_\mu \in \mathbf{L^1}(\mathbb{R}^d)$ such that $\widehat{r_\mu} = (\mu - a)^{-1}$. To establish this, let $v_\mu := (\mu - a)^{-1}$. Then $D^\alpha v_\mu = p_\alpha/(\mu - a)^{|\alpha|+1}$ for some polynomial p_α of degree $\deg p_\alpha \leq (m-1)|\alpha|$ (Lemma 8.0.2). Hence

$$|s|^{|\alpha|+m} |D^\alpha v_\mu(s)| \leq \frac{|s|^{|\alpha|+m} |p_\alpha(s)|}{|\mu - a(s)|^{|\alpha|+1}} \leq (c_3)^{-|\alpha|+1} \frac{|p_\alpha(s)|}{|s|^{(m-1)|\alpha|}} \leq (c_3)^{-|\alpha|+1} c_\alpha$$

for $|s| \geq L$. This gives $v_\mu \in \mathcal{M}^m$ (defined in Theorem E.4.2), hence $v_\mu = \widehat{r_\lambda}$ for some $r_\lambda \in \mathbf{L^1}(\mathbb{R}^d)$. Now let $\varepsilon := |\lambda|^{-1/m}$ and use the formulae from Appendix E.2 and the homogeneity of a to compute

$$\begin{aligned}(\lambda - a)^{-1} &= \varepsilon^m(\mu - \varepsilon^m a)^{-1} = \varepsilon^m U_\varepsilon(\mu - a)^{-1} = \varepsilon^m U_\varepsilon \mathcal{F} r_\mu \\ &= \varepsilon^{m-d} \mathcal{F} U_\varepsilon^{-1} r_\mu = \varepsilon^m \mathcal{F} U_\varepsilon' r_\mu.\end{aligned}$$

Hence $(\lambda - a)^{-1} = \mathcal{F} r_\lambda$ with $r_\lambda = |\lambda|^{-1} U'_{|\lambda|^{-1/m}} r_\mu$. Since U'_ε is isometric on $\mathbf{L^1}$, we have $\|\lambda r_\lambda\|_1 = \|r_\mu\|_1$. It is now clear that $R(\lambda, A_1)$ is convolution with r_λ, and so the holomorphy follows from Lemma E.3.1. □

With this result at hand we are in a position to prove the main theorem on homogeneous elliptic constant-coefficient operators.

Theorem 8.1.2. *Let $a : \mathbb{R}^d \longrightarrow \mathbb{C}$ satisfy (8.3)–(8.5), and let $A = \sum_{|\alpha|=m} a_\alpha D^\alpha$ be the corresponding operator on $\mathbf{TD}(\mathbb{R}; X)$, with A_1 being its part in $\mathbf{L^1}(\mathbb{R}^d; X)$. Then the following assertions hold.*

a) *$R(\lambda, A_1)u = r_\lambda * u$ for all $u \in \mathbf{L^1}(\mathbb{R}^d; X)$ and $\lambda \notin a(\mathbb{R}^d)$.*

b) *$\|R(\lambda, A_1)\| = \|r_\lambda\|_1$ for $\lambda \notin a(\mathbb{R}^d)$.*

c) *A_1 is an injective, densely defined, sectorial operator of angle $\omega_{A_1} = \omega_a$.*

d) *$\sigma(A_1) = a(\mathbb{R}^d)$.*

Proof. a) is clear from Lemma 8.0.1 and Proposition 8.1.1.

b) follows from a) and Lemma E.3.1.

c) Proposition 8.1.1 shows that $\|\lambda r_\lambda\|_1$ is constant on rays originating at 0. Since $\|\lambda R(\lambda, A_1)\| \leq \|\lambda r_\lambda\|_1$ and $(\lambda \longmapsto r_\lambda)$ is continuous, the sectoriality of A_1 is immediate. Since $\|(\lambda - a)^{-1}\|_\infty \leq \|r_\lambda\|_1 = \|R(\lambda, A_1)\|$, the sectoriality angle of A_1 must in fact be equal to ω_a. Clearly $\boldsymbol{\mathcal{S}}(\mathbb{R}^d; X) \subset \mathcal{D}(A_1)$, so A_1 is also densely defined. Let us prove that A_1 is injective. If $u \in \mathbf{L^1}(\mathbb{R}^d; X)$ such that $A_1 u = 0$, then $0 = \mathcal{F}A_1 u = a\widehat{u}$. But $\widehat{u} \in \mathbf{C_0}$ and $a(s) \neq 0$ whenever $s \neq 0$. So $\widehat{u}(s) = 0$ for all $s \neq 0$, whence also $\widehat{u}(0) = 0$ by continuity. This implies that $u = 0$ since the Fourier transform is injective.

d) Fix $\lambda \in \varrho(A_1)$. For each $u \in \mathbf{L^1}(\mathbb{R}^d; X)$ we have $u = (\lambda - A)R(\lambda, A_1)u$. Taking Fourier transforms yields

$$\widehat{u} = (\lambda - a)\mathcal{F}R(\lambda, A_1)u, \qquad (u \in \mathbf{L^1}(\mathbb{R}^d; X)).$$

Since for given $s \in \mathbb{R}^d$ we can find a function u such that $\widehat{u}(s) \neq 0$, it follows that $a(s) \neq \lambda$ for all $s \in \mathbb{R}^d$, i.e., $\lambda \notin a(\mathbb{R}^d)$. $\square$

We also can say something about the domain $\mathcal{D}(A_1)$.

Proposition 8.1.3. *Let a and A be as above, and let $1 \leq k \leq m-1$. Then*

$$\mathbf{W^{m,1}}(\mathbb{R}^d; X) \;\subset\; \mathcal{D}(A_1) \;\subset\; (\mathbf{L^1}(\mathbb{R}^d; X), \mathcal{D}(A_1))_{\frac{k}{m},1} \;\subset\; \mathbf{W^{k,1}}(\mathbb{R}^d; X).$$

Proof. It is clear from the definition that $\mathbf{W^{1,m}}(\mathbb{R}^d; X) \subset \mathcal{D}(A_1)$. We claim that for $\beta \in \mathbb{N}^d$, $|\beta| \leq m-1$ the function $s^\beta(1 + a(s))^{-1}$ is an $\mathbf{L^1}(\mathbb{R}^d; X)$-multiplier. This implies in particular that the operator $D^\beta(1 + A)^{-1}$ restricts to a bounded operator on the space $\mathbf{L^1}(\mathbb{R}^d; X)$. In order to prove the claim, we use Theorem E.4.2. By induction on $|\alpha|$ one proves that for any $\alpha \in \mathbb{N}^d$ there exists a polynomial $p_{\alpha,\beta}$ such that

$$D^\alpha\left(\frac{s^\beta}{1 + a(s)}\right) = \frac{p_{\alpha,\beta}}{(1 + a(s))^{|\alpha|+1}}$$

and $\deg p_{\alpha,\beta} \leq (m-1)\,|\alpha| + |\beta|$. Hence one can estimate

$$\left|D^\alpha\left(\frac{s^\beta}{1 + a(s)}\right)\right| \leq c_{\alpha,\beta}\,|s|^{|\beta|-m-|\alpha|} \qquad (|s| \geq L)$$

(see Lemma 8.0.2), with constants $c_{\alpha,\beta} > 0$ not depending on s. This implies that $s^\beta(1 + a)^{-1} \in \mathcal{M}^{m-|\beta|} \subset \mathcal{F}\mathbf{L^1}$ by Theorem E.4.2, as long as $|\beta| \leq m-1$.

Now, fix $\beta \in \mathbb{N}^d$ with $|\beta| \leq m-1$, and choose $v \in \mathbf{L^1}$ such that $\widehat{v} = (is)^\beta(1 + a)^{-1}$. Let $t > 0$, and define $\varepsilon := t^{-1/m}$. Then

$$\begin{aligned}(is)^\beta(t + a)^{-1} &= \varepsilon^m (is)^\beta U_\varepsilon (1 + a)^{-1} = \varepsilon^{m-|\beta|} U_\varepsilon \mathcal{F}v \\ &= \varepsilon^{m-|\beta|}\mathcal{F}U'_\varepsilon v = t^{\frac{|\beta|}{m}-1}\mathcal{F}U'_\varepsilon v.\end{aligned}$$

Hence $t^{1-|\beta|/m}D^\beta(t + A_1)^{-1}$ is induced by convolution with $U'_\varepsilon v$, whence

$$\left\|t^{1-\frac{|\beta|}{m}}D^\beta(t + A_1)^{-1}\right\|_{\mathbf{L^1}\to\mathbf{L^1}} = \|U'_\varepsilon v\|_1 = \|v\|_1$$

is constant. By Corollary 6.7.5 this means

$$D^\beta : \left(\mathbf{L^1}(\mathbb{R}^d; X), \mathcal{D}(A_1)^\bullet\right)_{\frac{|\beta|}{m},1} \longrightarrow \mathbf{L^1}(\mathbb{R}^d; X),$$

where $\mathcal{D}(A_1)^\bullet = \dot{D}$ is the homogeneous domain as defined in Chapter 6. Hence

$$(\mathbf{L^1}(\mathbb{R}^d; X), \mathcal{D}(A_1))_{k,1} \subset \mathbf{W^{k,1}}(\mathbb{R}^d; X)$$

for all $1 \le k \le m-1$. $\square$

Let us now turn to the functional calculus. For $f \in H_0^\infty(S_\varphi)$, $\varphi \in (\omega_a, \pi)$ we may define

$$g := \frac{1}{2\pi i} \int_\Gamma f(z) r_z \, dz \ \in \ \mathbf{L^1}(\mathbb{R}^d), \tag{8.7}$$

where $\Gamma = \partial S_{\omega_1}$ for some $\omega_1 \in (\omega_a, \varphi)$. (The integral converges in $\mathbf{L^1}(\mathbb{R}^d)$ by Proposition 8.1.1.) Then clearly

$$f(A_1)u = g * u \qquad (u \in \mathbf{L^1}(\mathbb{R}^d; X)).$$

Since $(1 + A_1)^{-1}$ is convolution with $-r_{-1}$ and I is convolution with the Dirac measure δ_0, for every $f \in \mathcal{E}(S_\varphi)$ there exists $\mu_f \in \mathbf{L^1}(\mathbb{R}^d) \oplus \langle \delta_0 \rangle$ such that

$$f(A_1)u = \mu_f * u \qquad (u \in \mathbf{L^1}(\mathbb{R}^d; X)). \tag{8.8}$$

The Dirac part of μ_f is exactly $f(\infty)\delta_0$, i.e., $\mu_f \in \mathbf{L^1}$ if and only if f vanishes at ∞. In particular, if $\omega_a < \pi/2$, the holomorphic semigroup generated by $-A_1$ is given by convolution with $\mathbf{L^1}$-functions.

Corollary 8.1.4. *Suppose that $\omega_a < \pi/2$. Then for each $|\arg \lambda| < \pi/2 - \omega_a$ there is $F_\lambda \in \mathbf{L^1}(\mathbb{R}^d)$ such that*

$$e^{-\lambda A} u = F_\lambda * u \qquad (u \in \mathbf{L^1}(\mathbb{R}^d; X)).$$

The mapping

$$(\lambda \longmapsto F_\lambda) : S_{\frac{\pi}{2} - \omega_a} \longrightarrow \mathbf{L^1}(\mathbb{R}^d)$$

is holomorphic.

The identity (8.8) shows that one has $f(A_1)u = \mathcal{F}^{-1}\big((f \circ a)\widehat{u}\big)$ for all elementary functions $f \in \mathcal{E}(S_\varphi)$. To proceed towards more general functions f we are in need of a lemma that is also of independent interest.

Lemma 8.1.5. *Let $q \in \mathbb{C}[x_1, \dots x_d]$ be a polynomial in d variables. Then the following two assertions hold.*

a) *If $q \neq 0$ then $\{q = 0\}$ is a $\mathbb{R}^d$-null set.*

b) *If $q \in \mathbf{L^p}(\mathbb{R}^d)$ for some $p \in [1, \infty)$, then $q = 0$.*

c) *If $q \in \mathbf{L^\infty}(\mathbb{R}^d)$, then q is a constant.*

Proof. We prove assertions a) and b) by induction on d, the case $d = 1$ being trivial. So suppose that $d \ge 1$ and $0 \neq q \in \mathbb{C}[x_1, \dots, x_d, t]$. Write

$$q(x,t) = q_0(x) + q_1(x)t + \cdots + q_m(x)t^m$$

for certain $q_j \in \mathbb{C}[x_1, \dots, x_d]$, $q_m \neq 0$. By induction hypothesis, $\{q_m = 0\}$ is a $\mathbb{R}^d$-null set and whenever $x \in \mathbb{R}^d \setminus \{q_m = 0\}$, the set $\{t \in \mathbb{R} \mid q(x,t) = 0\}$ is finite, hence an $\mathbb{R}$-null set. By general (product) measure theory, this implies that $\{q = 0\}$ is an $\mathbb{R}^{d+1}$-null set.

To prove b) we fix $p \in [1, \infty)$ and suppose that $\|q\|_p < \infty$. This means

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} |q(x,t)|^p \, dx \, dt < \infty.$$

Hence $q(\cdot, t) \in \mathbf{L^p}(\mathbb{R}^d)$ for almost all $t \in \mathbb{R}$. By induction hypothesis, $q(\cdot, t) = 0$ for these $t \in \mathbb{R}$. Hence $\{q = 0\}$ is a set of infinite $\mathbb{R}^{d+1}$-measure; this by a) implies that $q = 0$.

The proof of c) is trivial. □

Now, take again $\varphi \in (\omega_a, \pi)$ and $f \in \mathcal{O}(S_\varphi)$. The function $f \circ a$ is defined on $\mathbb{R}^d$ except for the point 0. In fact, we have $f \circ a \in \mathbf{C}^\infty(\mathbb{R}^d \setminus \{0\})$.

Proposition 8.1.6. *Let a A be as above, let $\varphi \in (\omega_a, \pi)$, and let $f \in \mathcal{O}(S_\varphi)_{A_1}$. Then*

$$u \in \mathcal{D}(f(A_1)) \quad \iff \quad \exists\, v \in \mathbf{L^1}(\mathbb{R}^d; X) : (f \circ a)\, \widehat{u} \;=\; \widehat{v} \quad \text{on} \;\; \mathbb{R}^d \setminus \{0\}.$$

In this case $f(A_1)u = v = \mathcal{F}^{-1}[(f \circ a)\widehat{u}]$. In particular, $W(X) \subset \mathcal{D}(f(A_1))$. Moreover, the following statements are equivalent.

(i) *There is $c > 0$ such that $\|f(A_1)u\|_1 \le c \|u\|_1$ for all $u \in W(X)$.*

(ii) *$f \circ a$ is an $\mathbf{L^1}(\mathbb{R}^d; X)$-Fourier multiplier.*

(iii) *$f \circ a$ is an $\mathbf{L^1}(\mathbb{R}^d)$-Fourier multiplier.*

(iv) *$f(A_1)$ is a bounded operator on $\mathbf{L^1}(\mathbb{R}^d; X)$.*

(The definition of $W(X)$ can be found in Appendix E.5.) The stated equivalence is surprising on first glance since the space $W(X)$ is not dense in $\mathbf{L^1}(\mathbb{R}^d; X)$.

Proof. Let e be any regulariser for f. Then $u \in \mathcal{D}(f(A_1))$ if and only if there is $v \in \mathbf{L^1}(\mathbb{R}^d; X)$ such that $(ef)(A_1)u = e(A_1)v$. By taking Fourier transforms, this identity is equivalent to $[(ef) \circ a]\widehat{u} = \widehat{v}$. This already proves the implication '$\Leftarrow$'. To prove the converse, suppose that there is such a v. Then $(f \circ a)\widehat{u} = \widehat{v}$ except possibly on the set $\{0\} \cup \{e \circ a = 0\}$. However, e has at most countably many zeroes, so by Lemma 8.1.5 a) and the fact that a is a polynomial, $\{e \circ a\}$ is an $\mathbb{R}^d$-null set. Since both $\widehat{u}, \widehat{v}$ are continuous functions and $f \circ a$ is continuous apart from 0, one has $\widehat{v} = (f \circ a)\widehat{u}$ on $\mathbb{R}^d \setminus \{0\}$.

Within the stated equivalence the implication (iv)$\Rightarrow$(i) is obvious, and so is the equivalence (ii)$\Leftrightarrow$(iii) since $f \circ a$ is a scalar function. The implication (i)$\Rightarrow$(iii) follows easily from Theorem E.5.4. Suppose that (iii) holds, i.e., that $f \circ a$ is an

$\mathbf{L^1}$-Fourier multiplier. This means that there is a bounded measure $\mu \in \mathbf{M}(\mathbb{R}^d)$ with $\widehat{\mu} = f \circ a$. Now, with $e \in H_0^\infty(S_\varphi)$ being a regulariser for f as above, we have

$$\mathcal{F}(ef)(A_1)u = [(ef) \circ a]\widehat{u} = (e \circ a)(f \circ a)\widehat{u} = \mathcal{F}e(A_1)(\mu * u),$$

whence $(ef)(A_1)u = e(A_1)(\mu * u)$, for *every* $u \in \mathbf{L^1}(\mathbb{R}^d; X)$. This proves the claim. $\square$

The homogeneity of the polynomial a has an interesting consequence. (See Appendix E.2 for the definition of the operator U_ε.)

Corollary 8.1.7. *Let a, A be as above, let $\varphi \in (\omega_a, \pi)$ and $f \in \mathcal{O}(S_\varphi)_{A_1}$, and let $t > 0$. Then $f \in \mathcal{O}(S_\varphi)_{tA_1}$ and*

$$f(tA_1) = U_\varepsilon^{-1} f(A_1) U_\varepsilon$$

where $\varepsilon := t^{1/m}$. In particular, $f(A_1)$ is bounded if and only if $f(tA_1)$ is bounded, and in this case $\|f(A_1)\| = \|f(tA_1)\|$.

Proof. The operator tA is induced by the polynomial $ta(s) = a(\varepsilon s)$. If $e \in \mathcal{E}(S_\varphi)$ then the proof of Lemma E.4.1 shows that $e(tA_1) = U_\varepsilon^{-1} e(A_1) U_\varepsilon$. Hence $e(A_1)$ is injective if and only if $e(tA_1)$ is injective. Thus if e regularises f in the functional calculus of A_1, it does so also in the functional calculus for tA_1, and

$$\begin{aligned} f(tA_1) &= e(tA_1)^{-1}(ef)(tA) = [U_\varepsilon^{-1} e(A_1) U_\varepsilon]^{-1} U_\varepsilon^{-1}(ef)(A_1) U_\varepsilon \\ &= U_\varepsilon^{-1} e(A_1)^{-1} U_\varepsilon U_\varepsilon^{-1} (ef)(A_1) U_\varepsilon = U_\varepsilon^{-1} f(A_1) U_\varepsilon^{-1}. \end{aligned}$$ $\square$

The next lemma is just transitory in the $\mathbf{L^1}$-context (but cf. Corollary 8.2.5).

Lemma 8.1.8. *Let a be a homogeneous elliptic polynomial, and let A be its associated operator. For all $0 \neq r \in \mathbb{R}$ and $\varepsilon > 0$ one has the equivalence*

$$(A_1)^{ir} \text{ is bounded} \quad \Longleftrightarrow \quad (\varepsilon + A_1)^{ir} \text{ is bounded.}$$

In this case one has $\left\|A_1^{ir}\right\| \le \left\|(\varepsilon + A_1)^{ir}\right\| = \left\|(\delta + A_1)^{ir}\right\|$ for all $\varepsilon, \delta > 0$.

Proof. If A_1^{ir} is bounded, also $(\varepsilon + A_1)^{ir}$ must be bounded, by Proposition 3.5.5. If on the other hand $(\varepsilon + A_1)^{ir}$ is bounded, by Corollary 8.1.7, also $t^{-ir}(\varepsilon + tA_1)^{ir} = (t^{-1}\varepsilon + A_1)^{ir}$ is bounded for each $t > 0$, and the norm is independent of t. Hence by Proposition 8.1.6, all the functions $m_n(s) := (1/n + a(s))^{ir}$ are $\mathbf{L^1}$-Fourier multipliers, with uniform bound on the multiplier norm. Lemma E.4.1 b) shows that also $a(s)^{ir}$ must be an $\mathbf{L^1}$-multiplier. Applying Proposition 8.1.6 again proves the claim. $\square$

Corollary 8.1.9. *Let a be a homogeneous elliptic polynomial of degreee $m \ge 1$, and let A be its associated operator. Then each operator $(\varepsilon + A_1)^{ir}$, $r \neq 0$, $\varepsilon \ge 0$, is unbounded.*

Proof. Lemma 8.1.8 shows that it suffices to prove the assertion for $\varepsilon = 0$. If A_1^{ir} is bounded, the function a^{ir} must be a Fourier multiplier on $\mathbf{L}^{\mathbf{1}}$, by Proposition 8.1.6. This implies in particular that this function is continuous at 0. Take $0 \neq s_0 \in \mathbb{R}^d$. Then $a(s_0) \neq 0$ by ellipticity and $a(ts_0)^{ir} = t^{irm} a(s_0)^{ir}$ must be continuous in $t \geq 0$. But this is impossible since $\lim_{t \searrow 0} t^{irm}$ does not exist. $\square$

8.2 Elliptic Operators: $\mathbf{L}^{\boldsymbol{p}}$-Theory

We continue our study of operators A, given by a homogeneous elliptic polynomial a of degree $m \geq 1$. In this section we focus on the part A_p of A in the spaces $\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X)$, $p \in (1, \infty)$. The case $p = \infty$ will be treated later.

Theorem 8.2.1. *Let $a : \mathbb{R}^d \longrightarrow \mathbb{C}$ satisfy* (8.3)–(8.5)*, and let $A = \sum_{|\alpha|=m} a_\alpha D^\alpha$ be the corresponding operator. Let $1 < p < \infty$. Then the part A_p of A in $\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X)$ is a densely defined, injective, sectorial operator of angle $\omega_{A_p} = \omega_a$ with spectrum $\sigma(A_p) \subset a(\mathbb{R}^d)$. Its resolvent is given by*

$$R(\lambda, A_p)u = r_\lambda * u \qquad (u \in \mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X), \lambda \notin a(\mathbb{R}^d)),$$

where $(r_\lambda)_\lambda$ is as in Proposition 8.1.1*. For $1 \leq k \leq m-1$ one has*

$$\mathbf{W}^{\boldsymbol{m},\boldsymbol{p}}(\mathbb{R}^d; X) \;\subset\; \mathcal{D}(A_p) \;\subset\; (\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X), \mathcal{D}(A_p))_{\frac{k}{m},1} \;\subset\; \mathbf{W}^{\boldsymbol{k},\boldsymbol{p}}(\mathbb{R}^d; X).$$

If X is a UMD space, then $\mathcal{D}(A_p) = \mathbf{W}^{\boldsymbol{m},\boldsymbol{p}}(\mathbb{R}^d; X)$.

Proof. As in the case $p = 1$ the assertions about the resolvent set, the representation of the resolvent as convolution with r_λ and hence the sectoriality of A_p follow from Proposition 8.1.1 and Young's inequality. It is also clear that A_p is densely defined since $\boldsymbol{\mathcal{D}}(\mathbb{R}^d; X)$ is contained in $\mathcal{D}(A_p)$.

Let us prove injectivity. If $A_p u = 0$, then $u \in \mathbf{L}^{\boldsymbol{p}}$ and $a\widehat{u} = 0$. Since $a \neq 0$ away from the origin, $\operatorname{supp} \widehat{u} \subset \{0\}$, whence by Lemma E.2.1 the distribution $\widehat{u}$ is a linear combination of derivatives of δ_0. Hence its inverse Fourier transform v is a polynomial which, by Lemma 8.1.5, must then be equal to 0. Thus u is weakly zero, hence 0.

The proof of the embedding $\mathcal{D}(A_p) \subset (\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X), \mathcal{D}(A_p))_{k/m,1} \subset \mathbf{W}^{\boldsymbol{k},\boldsymbol{p}}(\mathbb{R}^d; X)$ works exactly as in the $\mathbf{L}^{\mathbf{1}}$-case. In fact, in the proof of Proposition 8.1.3 we have shown that the operator $t^{1-|\beta|/m} D^\beta (t + A)^{-1}$ is induced by convolution with a function $U_\varepsilon' v$, for $\varepsilon = t^{-1/m}$ and a fixed function $v \in \mathbf{L}^{\mathbf{1}}$. So the rest is just an application of Young's inequality.

Now, let X be a UMD Banach space. Then the vector-valued Mikhlin theorem (Theorem E.6.2) is at hand to show that the functions $s^\beta (1 + a)^{-1}$ are in fact $\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X)$-Fourier multipliers for all $|\beta| \leq m$. Cf. the computation in the proof of Proposition 8.1.3. $\square$

Remark 8.2.2. One can show that actually $\sigma(A_p) = a(\mathbb{R}^d)$, see [10, Lemma 8.1.1] and its use in the proof of [10, Lemma 8.3.2].

As to the functional calculus, if $f \in \mathcal{E}(S_\varphi), \varphi \in (\omega_A, \pi)$, we find as before $\mu_f \in \mathbf{L^1}(\mathbb{R}^d) \oplus \langle \delta_0 \rangle$ such that $f(A_p)$ is convolution with μ_f. In particular, if $\omega_a < \pi/2$, the holomorphic semigroup $e^{-\lambda A_p}$ is given by convolution with the $\mathbf{L^1}$-kernels F_λ as in Corollary 8.1.4. For more general functions f we have a result similar to Proposition 8.1.6.

Proposition 8.2.3. *Let a and A be as above, $p \in (1, \infty)$, $\varphi \in (\omega_a, \pi)$, $f \in \mathcal{O}(S_\varphi)_{A_p}$, and $u \in \mathbf{L^p}(\mathbb{R}^d; X)$. Then*

$$u \in \mathcal{D}(f(A_p)) \quad \Longleftrightarrow \quad \exists\, v \in \mathbf{L^p}(\mathbb{R}^d; X) : (f \circ a)\,\widehat{u} \;=\; \widehat{v} \quad \text{on} \quad \mathbb{R}^d \setminus \{0\}.$$

In this case $f(A_p)u = v$. In particular, $W(X) \subset \mathcal{D}(f(A_p))$. Moreover, the following assertions are equivalent:

(i) *$f(A_p)$ is bounded.*

(ii) *$f \circ a$ is an $\mathbf{L^p}(\mathbb{R}^d; X)$-Fourier multiplier.*

Note that the product $(f \circ a)\widehat{u}$ is a well-defined distribution on $\mathbb{R}^d \setminus \{0\}$ since $f \circ a \in \mathbf{C}^\infty(\mathbb{R}^d \setminus \{0\})$.

Proof. The proof slightly differs from the $\mathbf{L^1}$-case since Fourier transforms of $\mathbf{L^p}$-functions need not be proper functions any more. We first make sure that the stated first equivalence holds for elementary functions $f \in \mathcal{E}$. Let $\mu_f := \mathcal{F}^{-1}(f \circ a)$ be the corresponding measure (see (8.8)). Take any test function $\psi \in \boldsymbol{\mathcal{D}}(\mathbb{R}^d \setminus \{0\})$ and compute

$$\begin{aligned}\langle \mathcal{F}(f(A_p)u), \psi \rangle &= \left\langle \mu_f * u, \widehat{\psi} \right\rangle = \left\langle u, (\mathcal{S}\mu_f) * \widehat{\psi} \right\rangle = \langle u, \mathcal{F}((f \circ a)\psi) \rangle \\ &= \langle \widehat{u}, (f \circ a)\psi \rangle .\end{aligned}$$

Hence indeed $\mathcal{F}(f(A_p)u) = (f \circ a)\widehat{u}$ as distributions on $\mathbb{R}^d \setminus \{0\}$.

Now, take a general f and let e be any regulariser for f. Then $u \in \mathcal{D}(f(A_p))$ if and only if there is $v \in \mathbf{L^p}(\mathbb{R}^d; X)$ such that $e(A_p)v = (ef)(A_p)u$. If this is the case, taking Fourier transforms yields $(e \circ a)\widehat{v} = (e \circ a)(f \circ a)\widehat{u}$ on $\mathbb{R}^d \setminus \{0\}$. Hence $\widehat{v} = (f \circ a)\widehat{u}$ on $\mathbb{R}^d \setminus A_e$, where $A_e = \{0\} \cup \{e \circ a = 0\}$. The function e can have at most countably many zeroes, with $0, \infty$ being the only possibilities for accumulation. For any given compact set $K \subset \mathbb{R}^d \setminus \{0\}$ we can choose a regulariser e such that e has no zeroes on $a(K)$. (Cf. the proof of Lemma 2.7.2.) Since K was arbitrary, we conclude that $\widehat{v} = (f \circ a)\widehat{u}$ on $\mathbb{R}^d \setminus \{0\}$.

To prove the converse implication, suppose that there is $v \in \mathbf{L^p}(\mathbb{R}^d; X)$ with $\widehat{v} = (f \circ a)\widehat{u}$ on $\mathbb{R}^d \setminus \{0\}$. Then clearly $\mathcal{F}(e(A_p)v) = (e \circ a)\widehat{v} = ((fe) \circ a)\widehat{u} = \mathcal{F}((ef)(A_p)u)$ on $\mathbb{R}^d \setminus \{0\}$. Hence $w := e(A_p)v - (ef)(A_p)u$ is an $\mathbf{L^p}$-function whose Fourier transform has support in $\{0\}$. As in the proof of Theorem 8.2.1

(injectivity of A_p), employing Lemma 8.1.5 we conclude that $w = 0$. But this means that $v = f(A_p)u$, by definition.

Let us turn to the second equivalence. Since $W(X)$ is dense in $\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X)$ and $f(A_p)$ is always a closed operator, the boundedness of $f(A_p)$ is equivalent to its boundedness on $W(X)$. This is equivalent to (ii) by Corollary E.5.5. $\square$

Analogous to Corollary 8.1.7 and Lemma 8.1.8 we can prove the following.

Corollary 8.2.4. *Let a and A be as above, and let $p \in (1,\infty)$, $\varphi \in (\omega_a, \pi)$, and $f \in \mathcal{O}(S_\varphi)_{A_p}$. Then for each $t > 0$, $f \in \mathcal{O}(S_\varphi)_{tA_p}$ and*

$$f(tA_p) = U_\varepsilon^{-1} f(A_p) U_\varepsilon$$

where $\varepsilon := t^{1/m}$. In particular, $f(A_p)$ is bounded if and only if $f(tA_p)$ is bounded, and if this is the case, then $\|f(A_p)\| = \|f(tA_p)\|$.

Corollary 8.2.5. *Let a be a homogeneous elliptic polynomial, let A be its associated operator, and let $p \in (1,\infty)$. For each $0 \neq r \in \mathbb{R}$, $\varepsilon > 0$ one has the equivalence*

$$(A_p)^{ir} \in \mathcal{L}(\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X)) \quad \Longleftrightarrow \quad (\varepsilon + A_p)^{ir} \in \mathcal{L}(\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X)).$$

In this case one has $\left\|A_p^{ir}\right\| \leq \left\|(\varepsilon + A_p)^{ir}\right\| = \left\|(\delta + A_p)^{ir}\right\|$ for all $\varepsilon, \delta > 0$.

Now we are going to show that much more can be said about the functional calculus in the case where the Banach space X is an UMD space. For this, we need information about the growth behaviour of the derivatives of an H^∞-function.

Lemma 8.2.6. *Let $f \in H^\infty(S_\varphi)$ for some $\varphi \in (0,\pi]$. Then*

$$\sup_{\lambda \in S_\omega} \left|\lambda^n f^{(n)}(\lambda)\right| < \infty \qquad (n \in \mathbb{N}, \omega \in (0,\varphi)).$$

Proof. Let $n \geq 1$. The Cauchy integral formula yields

$$\lambda^n f^{(n)}(\lambda) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)\lambda^n}{(z-\lambda)^{n+1}}\,dz = \frac{1}{2\pi i} \int_\Gamma \frac{f(|\lambda|\,z)\,\lambda^n\,|\lambda|}{(|\lambda|\,z-\lambda)^{n+1}}\,dz$$

with $\Gamma = \partial S_{\omega'}$, $\omega' \in (\omega, \varphi)$ and $\lambda \in S_{\omega'}$. Setting $\mu := \lambda / |\lambda|$ we obtain

$$\left|\lambda^n f^{(n)}(\lambda)\right| \leq \|f\|_\infty \frac{1}{2\pi} \int_\Gamma \frac{|dz|}{|z-\mu|^{n+1}}.$$

$\square$

Theorem 8.2.7. *Let $p \in (1,\infty)$, and let A_p be the part of A in $\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X)$ where A is given by a homogeneous elliptic polynomial a as before. If X is a UMD space, then the natural $H^\infty(S_\varphi)$-calculus for A_p is bounded, for each $\varphi \in (\omega_a, \pi)$.*

Proof. We apply the multi-dimensional vector-valued Mikhlin theorem (Theorem E.6.2). By induction on $|\alpha|$ one can prove easily an identity of the form

$$D^\alpha(f \circ a) = \sum_{j=0}^{|\alpha|} (f^{(j)} \circ a)\, p_{\alpha,j}$$

where the $p_{\alpha,j}$ are polynomials with $|\alpha| + \deg p_j \le mj$. This yields the estimate

$$|s|^{|\alpha|} |D^\alpha(f \circ a)(s)| \le \sum_{j=0}^{|\alpha|} |s|^{|\alpha|} \left| f^{(j)}(a(s)) \right| |p_{\alpha,j}(s)| \le \sum_{j=0}^{|\alpha|} c_{\alpha,j} \frac{|s|^{jm}}{|a(s)|^j} \le \sum_{j=0}^{|\alpha|} c_{\alpha,j} c_1^{-j}$$

where $\alpha \in \mathbb{N}^d$ is arbitrary and c_1 is as in Lemma 8.0.2. (We used Lemma 8.2.6 in the estimation.) So the conditions of the Mikhlin theorem are satisfied and we conclude that $f \circ a$ is an $\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X)$-Fourier multiplier. A look at Proposition 8.2.3 concludes the proof. □

$\mathbf{L}^\infty$-Theory and Closed Subspaces

We conclude this section with a brief summary of the $\mathbf{L}^{\boldsymbol{\infty}}$- or $\mathbf{C_0}$-theory. Let as before A be an elliptic operator induced by an elliptic polynomial a. Since convolutions with $\mathbf{L^1}$-functions are bounded operators on $\mathbf{L}^{\boldsymbol{\infty}}$, the part A_∞ of A in $\mathbf{L}^{\boldsymbol{\infty}}(\mathbb{R}^d; X)$ is a sectorial operator of angle ω_a. Its kernel consists of the constant functions and its domain satisfies

$$\mathcal{D}(A_\infty) \subset \mathbf{BUC}(\mathbb{R}^d; X)$$

in the case where $m = 1$ and

$$\mathbf{W}^{\boldsymbol{m,\infty}}(\mathbb{R}^d; X) \;\subset\; \mathcal{D}(A_\infty) \;\subset\; (\mathbf{L}^{\boldsymbol{\infty}}(\mathbb{R}^d; X), \mathcal{D}(A_\infty))_{\frac{k}{m},1} \;\subset\; \mathbf{W}^{\boldsymbol{k,\infty}}(\mathbb{R}^d; X)$$

for all $1 \le k \le m-1$ in the case where $m \ge 2$.

The space $\mathbf{L}^{\boldsymbol{\infty}}$ is somehow inconvenient with regard to the functional calculus. This is due to the constant functions, whose Fourier images are multiples of the Dirac measure. Thus Fourier multiplier representations shall fail in general. However, one can restrict to subspaces, and in fact the next theorem works for arbitrary $p \in [1, \infty]$.

Theorem 8.2.8. *Let A be an elliptic operator, induced by an elliptic polynomial a. Fix $p \in [1, \infty]$ and a closed subspace $\mathcal{X} \subset \mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X)$ not containing the constant functions. Suppose that $\mathcal{X}$ is invariant with respect to the operators $R(\lambda, A_p)$, $\lambda \notin \overline{S_{\omega_a}}$. Then A restricts to an injective sectorial operator $A_{\mathcal{X}}$ on $\mathcal{X}$ of angle ω_a. Let $\varphi \in (\omega_a, \pi)$, and let $f \in \mathcal{O}(S_\varphi)_{A_{\mathcal{X}}}$. Then for $u \in \mathcal{X}$ one has*

$$u \in \mathcal{D}(f(A_{\mathcal{X}})) \qquad \Longleftrightarrow \qquad \exists\, v \in \mathcal{X} : \widehat{v} = (f \circ a)\, \widehat{u} \quad \text{on } \mathbb{R}^d \setminus \{0\}.$$

In this case $v = f(A_{\mathcal{X}})u$.

Proof. That $A_{\mathcal{X}} \in \operatorname{Sect}(\omega_a)$ is clear by general theory. Injectivity follows as in the proof of Theorem 8.2.1. To prove the statement about the functional calculus, one repeats word by word the arguments of the proof of Proposition 8.2.3, replacing the space $\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X)$ by the space $\mathcal{X}$. In the case where $p = \infty$ one uses Lemma 8.1.5 c) and the fact that, by assumption, the constants are not contained in $\mathcal{X}$. □

In the case where $p = \infty$, one particular example is the space $\mathcal{X} = \mathbf{C_0}(\mathbb{R}^d; X)$. It is indeed invariant under the resolvent of A_∞ since one has $\mathbf{L^1} * \mathbf{C_0} \subset \mathbf{C_0}$. The domain of the induced operator — let us call it A_0 — satisfies

$$\mathbf{C_0^m}(\mathbb{R}^d; X) \ \subset \ \mathcal{D}(A_0) \ \subset \ (\mathbf{C_0}(\mathbb{R}^d; X), \mathcal{D}(A_0))_{\frac{k}{m},1} \ \subset \ \mathbf{C_0^k}(\mathbb{R}^d; X)$$

for all $1 \le k \le m-1$.

Corollary 8.2.9. *Let a, A_0, φ, f be as above. Then $W(X) \subset \mathcal{D}(f(A_0))$, and the following assertions are equivalent:*

(i) *There is a $c > 0$ such that $\|f(A_0)u\|_\infty \le c\,\|u\|_\infty$ for all $u \in W(X)$.*

(ii) *$f \circ a$ is an $\mathbf{L}^{\boldsymbol{\infty}}(\mathbb{R}^d; X)$-Fourier multiplier.*

(iii) *$f \circ a$ is an $\mathbf{L}^{\boldsymbol{\infty}}(\mathbb{R}^d)$-Fourier multiplier.*

(iv) *$f(A_0)$ is a bounded operator on $\mathbf{C_0}(\mathbb{R}^d; X)$.*

In particular, for each $0 \neq r \in \mathbb{R}$ and $\varepsilon \ge 0$ the operator $(\varepsilon + A_0)^{ir}$ is unbounded.

Proof. That $W(X) \subset \mathcal{D}(f(A_0))$ follows directly from Theorem 8.2.8. The proof of the stated equivalence is practically the same as in Proposition 8.1.6. Since scalar $\mathbf{L}^{\boldsymbol{\infty}}$-multipliers and $\mathbf{L^1}$-multipliers coincide, the remaining statement follows from Corollary 8.1.9. □

8.3 The Laplace Operator

In this section we apply the general results to a very particular and prominent example. The negative **Laplace operator** $-\Delta$ is induced by the polynomial

$$a(s) := |s|^2 = \sum_{j=1}^d s_j^2 \qquad (s \in \mathbb{R}^d),$$

so that

$$Au = -\Delta u := -\sum_{j=1}^d D_j D_j u.$$

As above we denote by $-\Delta_p$ its part in $\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X)$.

We note some obvious facts. The polynomial a is homogeneous of degree $m = 2$, its range $a(\mathbb{R}^d) = [0, \infty)$ is the positive real line, and in fact, if $s \neq 0$, then $a(s) > 0$. Hence $-\Delta$ is strongly elliptic. So for each Banach space X and each $p \in [1, \infty]$ the operator $-\Delta_p$ is a sectorial operator on $\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X)$ of angle 0. It is injective and densely defined in the case where $p \neq \infty$. If $p = \infty$, its kernel consists of the constant functions. Its domain satisfies

$$\mathbf{W}^{\mathbf{2},\boldsymbol{p}}(\mathbb{R}^d; X) \subset \mathcal{D}(\Delta_p) \subset \big(\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X), \mathcal{D}(\Delta_p)\big)_{\frac{1}{2},1} \subset \mathbf{W}^{\mathbf{1},\boldsymbol{p}}(\mathbb{R}^d; X).$$

If $p \in (1, \infty)$ and X is a UMD space, $-\Delta_p$ has a bounded $H^\infty(S_\varphi)$-calculus, for any $\varphi \in (0, \pi)$, and one has the identity

$$\mathcal{D}(\Delta_p) = \mathbf{W}^{\mathbf{2},\boldsymbol{p}}(\mathbb{R}^d; X).$$

All imaginary powers $(\varepsilon - \Delta_1)^{ir}$ with $\varepsilon \geq 0, r \neq 0$, are unbounded, as well as all operators $(\varepsilon - \Delta_\infty)^{ir}$ or their parts in $\mathbf{C_0}(\mathbb{R}^d; X)$.

Since the sectorial operator $-\Delta_p$ has angle 0, its negative Δ_p generates a holomorphic semigroup of angle $\pi/2$, given by convolution with $\mathbf{L^1}$-kernels. This semigroup is the well-known **Gauss–Weierstrass semigroup**.

Proposition 8.3.1. *For any $p \in [1, \infty]$ and $u \in \mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X)$ one has*

$$e^{\lambda\Delta_p} u = G_\lambda * u \qquad (\operatorname{Re}\lambda > 0),$$

where

$$G_\lambda(t) = (4\pi\lambda)^{-\frac{d}{2}} e^{-\frac{|t|^2}{4\lambda}} \qquad (\operatorname{Re}\lambda > 0).$$

Moreover, $(\lambda \longmapsto G_\lambda) : \{\operatorname{Re}\lambda > 0\} \longrightarrow \mathbf{C_0}(\mathbb{R}^d) \cap \mathbf{L^1}(\mathbb{R}^d)$ is holomorphic.

Proof. Essentially one has to show that

$$G_\lambda := \mathcal{F}^{-1} e^{-\lambda|s|^2} = (4\pi\lambda)^{-\frac{d}{2}} e^{-\frac{|t|^2}{4\lambda}}$$

for all $\operatorname{Re}\lambda > 0$. This is done in two steps. If $\lambda > 0$, one uses the formula for the inverse Fourier transform and the well-known identity

$$\int_{\mathbb{R}^d} e^{-\pi|t|^2} e^{-2\pi i s\cdot t}\, dt = e^{-\pi|s|^2} \qquad (s \in \mathbb{R}^d)$$

employing some simple change of variables. Next, one shows that the function

$$\Big(\lambda \longmapsto e^{-\lambda|\cdot|^2}\Big) : \{\operatorname{Re}\lambda > 0\} \longrightarrow \mathbf{L^1}(\mathbb{R}^d)$$

is holomorphic. For this it suffices to show that the derivative $-|\cdot|^2 e^{-\lambda|\cdot|^2}$ is majorised by an integrable function, locally uniformly in λ. (This is obviously the case.) The inverse Fourier transform is a bounded linear operator from $\mathbf{L^1}$ to $\mathbf{C_0}$, whence it follows that in fact $G_\lambda(t)$ is holomorphic in λ for each $t \in \mathbb{R}^d$. So we obtain the final result by uniqueness of analytic continuation. $\square$

Corollary 8.3.2. *One has*

$$\left\|e^{\lambda\Delta_p}\right\| \le \|G_\lambda\|_1 = \left(\frac{|\lambda|}{\operatorname{Re}\lambda}\right)^{\frac{d}{2}} \qquad (\operatorname{Re}\lambda > 0).$$

From the concrete representation of the semigroup we can derive a formula for the operators $(\varepsilon - \Delta)^{-\alpha}$, $\operatorname{Re}\alpha > 0$.

Proposition 8.3.3. *Let $\varepsilon > 0$, $\operatorname{Re}\alpha > 0$, and $p \in [1,\infty]$. Then*

$$(\varepsilon - \Delta_p)^{-\alpha}u = B_{\varepsilon,\alpha} * u \qquad (u \in \mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X)),$$

where $B_{\varepsilon,\alpha} \in \mathbf{L}^{\mathbf{1}}(\mathbb{R}^d)$ is given by

$$B_{\varepsilon,\alpha}(x) = \frac{1}{(4\pi)^{\frac{d}{2}}\Gamma(\alpha)} \int_0^\infty t^{\alpha-\frac{d}{2}-1} e^{-t\varepsilon} e^{-\frac{|x|^2}{4t}}\, dt \qquad (x \neq 0).$$

Proof. We apply Proposition 3.3.5 to the operator $-\Delta_1$. Here, kernel representations and operators are two sides of the same coin (isometrically). Hence the integral

$$B_{\varepsilon,\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\varepsilon t} G_t\, dt$$

converges in $\mathbf{L}^{\mathbf{1}}(\mathbb{R}^d)$ yielding the convolution kernel for $(\varepsilon - \Delta_1)^{-\alpha}$. It is clear that this carries over to the $\mathbf{L}^{\boldsymbol{p}}$-setting. The only remaining thing to check is the given pointwise respresentation of $B_{\varepsilon,\alpha}$ on $\mathbb{R}^d \setminus \{0\}$. But if $K \subset \mathbb{R}^d \setminus \{0\}$ is compact, the integral above is absolutely convergent within $\mathbf{L}^{\mathbf{1}}(K) \cap \mathbf{C}(K)$, whence the pointwise representation follows. □

The convolution kernel $B_{\varepsilon,\alpha}$ is called a **Bessel potential** of degree α.

Proposition 8.3.4. *Let X be a UMD space, and let $p \in (1,\infty)$ and $\varphi \in (0,\pi)$. Then the operator $-\Delta_p$ has a bounded $H^\infty(S_\varphi)$-calculus on $\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X)$.*

On $\mathbf{L}^{\mathbf{1}}(\mathbb{R}^d)$, each of the operators $(\varepsilon - \Delta_1)^{ir}$, $r \neq 0, \varepsilon \ge 0$, is unbounded.

Proof. Just apply Theorem 8.2.7 and Corollary 8.1.9. □

We now describe briefly a concrete realisation of the extrapolation spaces (as constructed abstractly in Section 6.3.2) associated with the Laplace operator. For simplicity let us restrict to the case $p < \infty$, so the operator $A := -\Delta$ on $X = \mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d)$ is injective. Loosely speaking, on the universal extrapolation space U, which is a superspace of X, both operators A and $(1 + A)$ are bijective. To find a natural example of such a superspace we have to leave the setting of tempered distributions and pass to the factor space

$$[\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)] := \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)\big/\,\mathcal{P}$$

where $\mathcal{P} := \mathbb{C}[t_1, \dots, t_d]$ is the space of all polynomials. By Lemma 8.1.5, the natural embedding $\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d) \longrightarrow [\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)]$ is injective. By the Hahn–Banach theorem, one may identify $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d) \big/ \mathcal{P} \cong \Phi'(\mathbb{R}^d)$ with

$$\Phi(\mathbb{R}^d) := \{f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d) \mid \langle p, f\rangle = 0 \;\forall\, p \in \mathcal{P}\}.$$

The Fourier transform restricts to an isomorphism

$$\mathcal{F}\Phi(\mathbb{R}^d) \longrightarrow \Psi(\mathbb{R}^d) := \{f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d) \mid D^\alpha f(0) = 0 \;\forall\, \alpha \in \mathbb{N}^d\}.$$

By adjoint action, one defines the Fourier transform as an isomorphism

$$\mathcal{F} : \Phi'(\mathbb{R}^d) \longrightarrow \Psi'(\mathbb{R}^d)$$

and it is clear that this corresponds to the induced operator

$$\boldsymbol{\mathcal{S}}'(\mathbb{R}^d) \big/ \mathcal{P} \xrightarrow{[\mathcal{F}]} \boldsymbol{\mathcal{S}}'(\mathbb{R}^d) \big/ \mathcal{F}\,\mathcal{P}\,.$$

The space $\Phi'(\mathbb{R}^d)$ allows for a much greater variety of Fourier multiplier operators as $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$, namely with singularities at 0. In particular, for each $\alpha \in \mathbb{C}$ not only the operator

$$(1 - \Delta)^\alpha = \big(u \longmapsto \mathcal{F}^{-1}((1 + |s|)^{2\alpha}\mathcal{F}u)\big) : \Psi' \longrightarrow \Psi'$$

is an isomorphism (it is already on $\boldsymbol{\mathcal{S}}'$), but also

$$(-\Delta)^\alpha = \Big(u \longmapsto \mathcal{F}^{-1}(|s|^{2\alpha}\,\mathcal{F}u)\Big) : \Phi' \longrightarrow \Phi'.$$

(Use Taylor's formula.) The operator

$$T := (-\Delta)(1 - \Delta)^{-2} = \mathcal{F}^{-1}\left(|s|^2\,(1 + |s|^2)^{-2}\mathcal{F}\right)$$

is an isomorphism on $\Phi'(\mathbb{R}^d)$ and restricts to $A(1 + A)^{-2}$ on $X = \mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d)$. The universal space U may then be identified with

$$U = \bigcup_{n\in\mathbb{N}} T^{-n}X = \bigcup_{n\in\mathbb{N}} (2 - \Delta - \Delta^{-1})^n \mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d).$$

Within this space one finds the **Bessel potential spaces**

$$H_p^{2\alpha}(\mathbb{R}^d) = (1 + \Delta)^{-\alpha}\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d) = \{u \in \Phi' \mid (1 + |s|^2)^{2\alpha}\widehat{u} \in \mathcal{F}\mathbf{L}^{\boldsymbol{p}}\}$$

(which may be identified with subspaces of $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$). The homogeneous fractional domain spaces

$$X^{(\alpha)} = (-\Delta)^{-\alpha}\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d) = \{u \in \Phi' \mid |s|^{2\alpha}\,\widehat{u} \in \mathcal{F}\mathbf{L}^{\boldsymbol{p}}\}$$

are the well-known **Riesz potential spaces**.

Remark 8.3.5. Let a be *any* homogeneous elliptic polynomial, and let A be the associated operator on $\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d)$. Then as for the Laplace operator one can find the universal space of A within $\Psi'(\mathbb{R}^d)$. The homogeneous fractional domain spaces are

$$X^{(\alpha)} = (-A)^{-\alpha}\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d) = \{u \in \Phi' \mid (a(s))^\alpha \widehat{u} \in \mathcal{F}\mathbf{L}^{\boldsymbol{p}}\}.$$

A proof of this fact requires us to show that any function $f \in \mathbf{C}^\infty(\mathbb{R}^d \setminus \{0\})$ such that each derivative $D^\alpha f$ has at most polynomial growth at 0 and at ∞, is a continuous multiplier on the space $\Phi(\mathbb{R}^d)$.

Remark 8.3.6. Let $2 \neq p \in (1, \infty)$. The Laplacian $A = -\Delta_p$ on $X = \mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d)$ has dense domain $\mathcal{D}(-\Delta_p) = \mathbf{W}^{\mathbf{2},\boldsymbol{p}}(\mathbb{R}^d)$ and dense range (X is reflexive). Since the natural H^∞-calculus for A is bounded (Proposition 8.3.4), one has in particular $-\Delta_p \in \mathrm{BIP}(X)$, whence

$$[\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d), \mathbf{W}^{\mathbf{2},\boldsymbol{p}}(\mathbb{R}^d)]_\theta = \mathcal{D}((-\Delta_p)^\theta) = H^{2\theta,p}(\mathbb{R}^d)$$

is the Bessel potential space for all $\theta \in (0, 1)$, by Theorem 6.6.9. Real interpolation yields

$$(\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d), \mathbf{W}^{\mathbf{2},\boldsymbol{p}}(\mathbb{R}^d))_{\theta,q} = (H_p^0, H_p^2)_{\theta,q} = B_{pq}^{2\theta}(\mathbb{R}^d),$$

which is a so-called **Besov space**, see [29, Theorem 6.4.2] and cf. [158, Example 1.3.10]. However, it is known that $H_p^s = B_{p,q}^s$ if and only if $p = q = 2$, see [215, Section 2.3.3, Remark 4] or [214]. As a consequence we obtain an example of a sectorial operator A on a UMD Banach space X, with bounded H^∞-calculus and such that $\mathcal{D}(A^\theta) \neq (X, \mathcal{D}(A))_{\theta,q}$ for all $\theta \in (0, 1), q \in [1, \infty]$.

The Laplace Operator as a Strip-type Operator

Finally, let us specialise to dimension $d = 1$, considering $-\Delta$ on $\mathbf{L}^{\boldsymbol{p}}(\mathbb{R})$, $1 \leq p < \infty$.

Lemma 8.3.7. *Let $\lambda \in \mathbb{C} \setminus [0, \infty)$ and $\mu \in \mathbb{C}$ such that $\mu^2 = -\lambda$ and $\operatorname{Re} \mu > 0$, i.e., $\mu = (-\lambda)^{1/2}$. Then*

$$\mathcal{F}\left(\frac{-1}{2\mu} e^{-\mu|\cdot|}\right)(s) = \frac{1}{\lambda - s^2} \qquad (s \in \mathbb{R}).$$

Proof. We compute

$$\begin{aligned} \int_{\mathbb{R}} \frac{-1}{2\mu} e^{-\mu|t|} e^{-ist}\, dt &= \int_0^\infty \frac{-1}{2\mu} e^{-\mu t} e^{-ist}\, dt + \int_0^\infty \frac{1}{2\mu} e^{-\mu t} e^{ist}\, dt \\ &= \frac{-1}{2\mu}\left[\frac{-1}{\mu + is} + \frac{-1}{\mu - is}\right] = \frac{1}{\lambda - s^2}. \end{aligned}$$ ∎

From the previous lemma it follows that the resolvent of the Laplacian is given by the convolution

$$R(\lambda, -\Delta_p)u = \left(\frac{-1}{2\mu} e^{-\mu|\cdot|}\right) * u \qquad (u \in \mathbf{L}^{\boldsymbol{p}}(\mathbb{R}; X)),$$

independent of $p \in [1, \infty]$. As always, the $\mathbf{L^1}$-norm of the kernel yields a bound for the operator norm. Thus

$$\|R(\lambda, A_p)\| \leq \|R(\lambda, \Delta_1)\| = \int_{\mathbb{R}} \frac{1}{2\,|\mu|} e^{-\operatorname{Re}\mu|t|}\, dt = \frac{1}{|\mu| \operatorname{Re}\mu} = \frac{2\left|\sin\frac{\theta}{2}\right|}{|\operatorname{Im}\lambda|},$$

where $\theta = \arg(-\lambda) \in (-\pi, \pi)$. Since $|\sin(\theta/2)| \leq 1$, $-\Delta_p$ is a (strong) strip-type operator of height 0 on $\mathbf{L^p}(\mathbb{R}; X)$. If $X = H$ is a Hilbert space, then by Plancherel's theorem $i\Delta_2$ generates a unitary group on $\mathbf{L^2}(\mathbb{R}; H)$. However, for $p \neq 2$ things are quite different.

Proposition 8.3.8. *If $p \neq 2$ and $0 \neq r \in \mathbb{R}$, the operator $e^{ir\Delta_p}$ is unbounded on $\mathbf{L^p}(\mathbb{R})$.*

Proof. Fix $p \neq 2$ and $r \neq 0$. By the mere definition of $\mathcal{D}(\Delta_p)$ we see that $\boldsymbol{\mathcal{S}}(\mathbb{R}) \subset \bigcap_{n\geq 0} \mathcal{D}(\Delta_p^n)$. Using the representation of the resolvent by convolution operators (Lemma 8.3.7) we obtain easily that $\mathcal{F}e^{ir\Delta}u = e^{-irs^2}\widehat{u}(s)$ for any $u \in \boldsymbol{\mathcal{S}}(\mathbb{R})$. So the boundedness of $e^{ir\Delta}$ is equivalent to the function e^{-irs^2} being a bounded $\mathbf{L^p}(\mathbb{R})$-Fourier multiplier. Hence in the following we may suppose that $1 \leq p < 2$. Fix $a > 0$, let $u \in \boldsymbol{\mathcal{S}}(\mathbb{R})$ such that $\widehat{u}(s) = e^{-as^2}$, and set $v := e^{ir\Delta_p}u$. Then $\widehat{v}(s) = e^{-(ir+a)s^2}$. Note that u and v can be determined explicitly, namely $u = G_a$ and $v = G_{a+ir}$ (see Proposition 8.3.1). Computing $\mathbf{L^p}$-norms yields

$$\frac{\left\|e^{ir\Delta_p}u\right\|_p}{\|u\|_p} = \left(\frac{r^2 + a^2}{a^2}\right)^{\frac{2-p}{4p}} \to \infty \qquad \text{as } a \searrow 0.$$

□

From Proposition 8.3.8 one derives easily that in the case where $p \neq 2$ the operator $i\Delta_p$ does not generate an exponentially bounded semigroup.

Theorem 8.3.9. *If $p \in (1, \infty)$ and X is a UMD space, then $A_p := e^{\Delta_p}$ is a bounded, injective, sectorial operator of angle 0 on $\mathbf{L^p}(\mathbb{R}; X)$. One has $A_p \in \operatorname{BIP}(\mathbf{L^p}(\mathbb{R}))$ if and only if $p = 2$.*

Proof. We have $A_p = T(1)$ where $T(\lambda) = e^{\lambda\Delta_p}$ is the holomorphic semigroup generated by Δ_p. So A_p is bounded and injective. To prove sectoriality, one must bound the operator family $\left[t(t + e^{\Delta_p})^{-1}\right]_{t>0}$. The operator $t(t + e^{\Delta_p})^{-1}$ is a Fourier-multiplier, with symbol

$$m_t(s) := \frac{t}{t + e^{-s^2}},$$

which is a function of bounded variation $\operatorname{Var}_{-\infty}^{\infty}(m_t)x = 2(t + 1)^{-1} \leq 2$. We may apply Bourgain's version of the Marcinciewicz theorem (Theorem E.6.2 a)) — or rather a simpler version sometimes called *Stečkin's theorem* — and conclude that m_t is an $\mathbf{L^p}$-multiplier with $\|m_t\|_{\mathcal{M}_p} \leq 2c_p$ for some constant $c_p > 0$ independent of $t > 0$. The second assertion follows from Proposition 8.3.8. □

Theorem 8.3.9 shows in particular that the familiar one-dimensional heat semigroup on $\mathbf{L^p}(\mathbb{R})$, $p \in (1, \infty)$, is the semigroup of fractional powers of a sectorial operator.

8.4 The Derivative on the Line

We fix any Banach space X and consider the operator $A = d/dt$ on $\mathbf{TD}(\mathbb{R}; X)$ and its restrictions A_p to the spaces $\mathbf{L^p}(\mathbb{R}; X)$ for $p \in [1, \infty]$. The symbol of this operator is

$$a(s) = is \qquad (s \in \mathbb{R}),$$

which is a homogeneous elliptic polynomial of degree 1. Hence we may in principle employ the results of the previous sections. However, since we can write down the resolvent explicitly, we prefer to give direct arguments wherever possible.

Proposition 8.4.1. *Let $p \in [1, \infty]$. The derivative operator $A_p = d/dt$ on the space $\mathbf{L^p}(\mathbb{R}; X)$ is sectorial of angle $\pi/2$ with domain $\mathcal{D}(A_p) = \mathbf{W^{1,p}}(\mathbb{R}; X)$ and spectrum $\sigma(A_p) = i\mathbb{R}$. It is injective and densely defined if $p \in [1, \infty)$. It has dense range if $p \in (1, \infty)$. Its resolvent is given by*

$$R(\lambda, A)u = r_\lambda * u, \qquad \textit{where} \qquad r_\lambda = \begin{cases} -e^{\lambda t}\mathbf{1}_{(0,\infty)} & (\operatorname{Re}\lambda < 0), \\ e^{\lambda t}\mathbf{1}_{(-\infty,0)} & (\operatorname{Re}\lambda > 0). \end{cases}$$

One has $\overline{\mathcal{R}(A_1)} = \{u \in \mathbf{L^1}(\mathbb{R}; X) \mid \widehat{u}(0) = 0\}$, and A_∞ restricts to a densely defined sectorial operator with dense range A_0 on $\mathbf{C_0}(\mathbb{R}; X)$, with domain $\mathbf{C_0^1}(\mathbb{R}; X)$.

Proof. The operator A_p is defined by $A_p u = u'$ with

$$\mathcal{D}(A_p) = \{u \in \mathbf{L^p}(\mathbb{R}; X) \mid u' \in \mathbf{L^p}(\mathbb{R}; X)\} = \mathbf{W^{1,p}}(\mathbb{R}; X)$$

by definition of the latter space. Given $\operatorname{Re}\lambda < 0$ it is easily proved that

$$\mathcal{F}\left(-e^{\lambda t}\mathbf{1}_{(0,\infty)}\right)(s) = \frac{1}{\lambda - is} \qquad (s \in \mathbb{R}).$$

Hence $\lambda \in \varrho(A_p)$ and $R(\lambda, A_p)$ is just convolution with the kernel $-e^{\lambda t}\mathbf{1}_{(0,\infty)}$. Young's inequality shows that $\|R(\lambda, A_p)\| \le \|-e^{\lambda t}\mathbf{1}_{(0,\infty)}\|_1 = |\operatorname{Re}\lambda|^{-1}$. Analogous arguments apply in the case that $\operatorname{Re}\lambda > 0$. The estimate for the resolvent implies readily the sectoriality of the operator A_p. Injectivity (in the case $p < \infty$ and of the operator A_0) is clear since a distribution u satisfying $u' = 0$ must be a constant function.

We determine the spectrum of A_p. Let φ be any test function with $\varphi(0) = 1$, and consider the sequence

$$f_n(t) := \varphi\left(\frac{t}{n}\right) \cdot \left(n^{\frac{1}{p}}\,\|\varphi\|_{\mathbf{L^p}}\right)^{-1}.$$

Then it is easy to see that $\|f_n\|_{\mathbf{L}^p} = 1$ but $f_n' \to 0$. This shows that 0 is an approximate eigenvalue of A_p for any $p \in [1,\infty]$ and of A_0. (If $p = \infty$, the constants are in the kernel of A_∞, whence here 0 is in fact an eigenvalue.) To deal with the other purely imaginary numbers, just observe that the operators A and $A + ir$ are similar via the isometric isomorphism $(Uf)(s) := e^{-irs} f(s)$.

As to the dense range, observe that if $\varphi \in \boldsymbol{\mathcal{D}}(\mathbb{R}; X)$ with $\int \varphi = 0$ the function $\psi(t) := \int_{-\infty}^t \varphi(r)\, dr$ is a primitive of φ and also contained in $\boldsymbol{\mathcal{D}}(\mathbb{R}; X)$. Hence the inclusion

$$\Big\{\varphi \in \boldsymbol{\mathcal{D}}(\mathbb{R}; X) \mid \int \varphi = 0\Big\} \quad \subset \quad \mathcal{R}(A_p)$$

holds. The space on the left is dense in $\mathbf{L}^p(\mathbb{R}; X)$ if $p \in (1,\infty)$, and in $\mathbf{C_0}(\mathbb{R}; X)$. Indeed, let ψ be any test function. Then $\psi_n(t) := \psi(t) - (1/n)\psi(t/n)$ defines a test function ψ_n with $\int \psi_n = 0$ and $\psi_n \to \psi$ in $\mathbf{L}^p$ if $1 < p \le \infty$. In the $\mathbf{L}^1$-case we note that $\int \varphi' = 0$ for each $\varphi \in \boldsymbol{\mathcal{D}}(\mathbb{R}; X)$. Since test functions are dense within $\mathbf{W}^{1,1}(\mathbb{R}; X)$, we have $\int f' = 0$ for all $f \in \mathcal{D}(A_1) = \mathbf{W}^{1,1}(\mathbb{R}; X)$, whence $\mathcal{R}(A_1) = \{f \in \mathbf{L}^1(\mathbb{R}; X)) \mid \int f = 0\}$. $\square$

From the explicit formula for the resolvent we derive the norm estimate

$$\|R(\lambda, A_p)\| \le \|R(\lambda, A_1)\| = |\operatorname{Re} \lambda|^{-1}.$$

Hence by the Hille–Yosida theorem (Theorem A.8.6) the operator $-A_p$ generates a strongly continuous isometric group on $\mathbf{L}^p(\mathbb{R}; X)$ if $p \in [1,\infty)$. However, we do not have to employ this theorem.

Proposition 8.4.2. *Let $p \in [1,\infty)$. Then $-A_p = -d/dt$ generates the* **right shift group** *on $\mathbf{L}^p(\mathbb{R}; X)$ defined by*

$$[S(t)u](s) \quad := \quad u(s-t) \qquad (s,t \in \mathbb{R}, u \in \mathbf{L}^p(\mathbb{R}; X)).$$

The same is true for $-A_0$ on $\mathbf{C_0}(\mathbb{R}; X)$.

Proof. Clearly $(S(t))_{t\in\mathbb{R}}$ is a strongly continuous group of isometries on the mentioned spaces. Let B denote its generator as defined in Appendix A.8. Then for $\operatorname{Re} \lambda > 0$ and $u \in \boldsymbol{\mathcal{D}}(\mathbb{R})$ we have

$$\begin{aligned} R(\lambda, B)u &= \int_0^\infty e^{-\lambda t} S(t)u\, dt = \int_0^\infty e^{-\lambda t} u(\cdot - t)\, dt = \left[e^{-\lambda t}\mathbf{1}_{(0,\infty)}\right] * u \\ &= -R(-\lambda, A)u = R(\lambda, -A)u. \end{aligned}$$

Hence $R(\lambda, B) = R(\lambda, -A)$, i.e. $B = -A$. $\square$

Let us turn to the functional calculus. The general results of Sections 8.1 and 8.2 show that for $\varphi \in (\pi/2, \pi)$ and $f \in H_0^\infty(S_\varphi)$ the operator $f(A_p)$ is given by the Fourier multiplier operator with symbol $f(is)$, i.e.,

$$f(A_p)u = \mathcal{F}^{-1}\left(f(is)\widehat{u}(s)\right) = g * u \qquad (u \in \mathbf{L}^p(\mathbb{R}, X)),$$

where $g = \mathcal{F}^{-1}(f(is))$. However, there is another way to see this: by Lemma 3.3.1 the function f is the Laplace transform of some function $h \in \mathbf{L^1}(0,\infty)$, and extending h by 0 to the negative real line one sees that $\mathcal{F}h = f(is)$, i.e., $h = g$. By the Phillips calculus (Proposition 3.3.2) one has

$$f(A_p)u = \int_0^\infty g(t)\, S(t)u\, dt = \int_0^\infty g(t)u(\cdot - t)\, dt = g * u$$

for all $u \in \mathbf{L^p}(\mathbb{R}; X)$ since g vanishes on $(-\infty, 0)$. Note that this agrees with the formulae we have for the function g. In fact, writing $r_z = -e^{zt}\mathbf{1}_{(0,\infty)}$ in (8.7) we obtain

$$g(t) \;=\; \frac{1}{2\pi i}\int_\Gamma f(z) r_z\, dz\,(t) \;=\; \frac{-1}{2\pi i}\int_\Gamma f(z)e^{zt}\, dz \qquad (t > 0),$$

which is exactly (3.8). (The contour of course is $\Gamma = \partial S_{\omega'}$ for some $\omega' \in (\pi/2, \varphi)$.) For further results on the functional calculus we refer to the general approach in Propositions 8.1.6 and 8.2.3. More interesting is the following theorem.

Theorem 8.4.3. *Let X be any Banach space, and let $p \in (1,\infty)$. Then the following assertions are equivalent.*

(i) *X is a UMD space.*

(ii) *For one/any $r > 0$ the operators $(d/dt)^{\pm ir}$ are bounded on $\mathbf{L^p}(\mathbb{R}; X)$.*

(iii) *The operator d/dt on $\mathbf{L^p}(\mathbb{R}; X)$ is in* BIP.

(iv) *The natural $H^\infty(S_\varphi)$-calculus for the operator d/dt on $\mathbf{L^p}(\mathbb{R}; X)$ is bounded, for one/any $\varphi \in (\pi/2, \pi)$.*

Proof. The implication (i)⇒(iv) is a consequence of Theorem 8.2.7. The implications (iv)⇒(iii)→(ii) are trivial since A is densely defined and has dense range.

Suppose that (ii) holds. Then by Proposition 8.2.3 the functions $(is)^{ir}$ and $(is)^{-ir}$ are both $\mathbf{L^p}(\mathbb{R}; X)$-Fourier multipliers. Hence also $(-is)^{ir}$ is an $\mathbf{L^p}(\mathbb{R}; X)$-Fourier multiplier (see Section E.1). Since the Fourier multipliers form an algebra, we have that $(-is)^{ir}(is)^{-ir} = e^{\pi \operatorname{sgn} s}$ is an $\mathbf{L^p}(\mathbb{R}; X)$-Fourier multiplier. But then also

$$-i \operatorname{sgn} s = -i\frac{e^{\pi \operatorname{sgn} s} - e^{-\pi}}{e^{\pi} - e^{-\pi}}$$

is an $\mathbf{L^p}(\mathbb{R}; X)$- Fourier multiplier. This is (i), by definition. □

Corollary 8.4.4. *Let $X \neq 0$ be any Banach space, and let $0 \neq r \in \mathbb{R}$ and $\varepsilon \geq 0$. Then the operator $(\varepsilon + d/dt)^{ir}$ is unbounded on $\mathbf{C_0}(\mathbb{R}; X)$ and on $\mathbf{L^1}(\mathbb{R}; X)$.*

Proof. The function $m(s) := (is)^{ir} = e^{ir \ln|s| + (\pi/2)\operatorname{sgn} s}$ is not continuous at 0, whence it cannot be an $\mathbf{L^\infty}$-(resp., $\mathbf{L^1}$-)Fourier multiplier (cf. Corollary 8.1.9 and its proof). □

The Derivative as a Strip-type Operator

Let us write $B := -id/dt$ for the moment. Since iB generates a strongly continuous isometric group, B is a strong strip-type operator of height $\omega_{st}(B) = 0$. Hence we may ask whether e^B is sectorial or, even more, whether the natural H^∞-calculus of B (on strips) is bounded, cf. also Theorem 5.4.1.

We begin with the case $p = 1$, i.e., we consider B as an operator on $\mathbf{L^1}(\mathbb{R})$. By Proposition 8.4.1 the resolvent is given by a convolution $R(\lambda, B)u = s_\lambda * u$ for a certain function $s_\lambda \in \mathbf{L^1}(\mathbb{R})$. Since the mapping

$$\big[\mu \longmapsto (u \longmapsto \mu * u)\big] : \mathbf{M}(\mathbb{R}) \longrightarrow \mathcal{L}(\mathbf{L^1}(\mathbb{R}))$$

is isometric, one sees that for every $f \in \mathcal{F}(H_\varphi)$, $\varphi > 0$, the operator $f(B)$ is given by convolution with some $\mathbf{L^1}$-function g_f. Clearly $\mathcal{F}g_f = f\big|_{\mathbb{R}}$. From this it is easy to derive the following lemma.

Lemma 8.4.5. *Let $\varphi > 0$, and let $f \in H^\infty(H_\varphi)$. Then $f(B)$ is bounded on $\mathbf{L^1}(\mathbb{R})$ if and only if there is $\mu \in \mathbf{M}(\mathbb{R})$ such that $\mathcal{F}\mu = f\big|_{\mathbb{R}}$.*

Proof. By using a regulariser one sees that $\mathcal{F}f(B)u = f(s)\widehat{u}(s)$ for all $u \in \mathcal{D}(B^2)$. Hence boundedness of $f(B)$ is equivalent to f being a bounded $\mathbf{L^1}(\mathbb{R})$-Fourier multiplier. An appeal to Appendix E.4 concludes the proof. ☐

Corollary 8.4.6. *Let $B := -id/dt$ on the space $\mathbf{L^1}(\mathbb{R})$. Then e^B has empty resolvent set. In particular, e^B is not sectorial.*

Proof. Consider the function $f(z) = (\lambda - e^z)^{-1}$ for $\lambda \notin [0, \infty)$. Then $f \in H^\infty(S_\varphi)$ for some small $\varphi > 0$. If $f(B)$ is bounded, by Lemma 8.4.5 there is $\mu \in \mathbf{M}(\mathbb{R})$ with $\mathcal{F}\mu f\big|_{\mathbb{R}}$. But then $\lim_{s\to\infty} \widehat{\mu}(s) = 0 \neq \lambda^{-1} = \lim_{s\to-\infty} \widehat{\mu}(s)$, contradicting Proposition E.4.3. ☐

Let us turn to the case where $p \in (1, \infty)$. Here Monniaux's Theorem 4.4.3 shows that $A := e^B$ is sectorial of angle 0 and $A \in \mathrm{BIP}(0)$. One can show, however, that B cannot have a bounded H^∞-calculus on strips. Indeed, using a theorem of deLeeuw [63] and the interpolation Theorem 9.1.5 in Section 9.1.2 below, one would obtain that every bounded double sequence is a bounded Fourier multiplier on $\mathbf{L^p}(\mathbb{T})$. This amounts to the fact that the trigonometric system $(e^{in\cdot})_{n\in\mathbb{Z}}$ is an unconditional basis of $\mathbf{L^p}(\mathbb{T})$; but this is false, cf. Section 9.1.3 below.

8.5 The Derivative on a Finite Interval

We now examine the restriction of the derivative operator $A = d/dt$ to finite intervals. First, we have to say a few words on the half-line case.

The Derivative on the Half-line

We fix a Banach space X and a parameter $p \in [1, \infty]$, and consider $A = d/dt$ as an operator on $\mathbf{L^p}(\mathbb{R}; X)$ as in the previous section. Then we form the space

$$Y := \begin{cases} \mathbf{L^p}((0,\infty); X) & \text{if } p \in [1,\infty), \\ \mathbf{C_0}((0,\infty); X) & \text{if } p = \infty, \end{cases}$$

which may be regarded in a natural way as a subspace of $\mathbf{TD}(\mathbb{R}; X)$ each member of which vanishes on $\boldsymbol{\mathcal{D}}(-\infty, 0)$, i.e., has support in $[0, \infty)$. (We omit reference to the parameter p as often as possible.)

Clearly, the space Y is invariant under all shift operators $S(t), t \geq 0$, hence this operator family forms a strongly continuous semigroup on Y, called the **right shift semigroup** semigroup. Its generator is the part $-A_Y$ of $-A$ in Y, so for the domain of A_Y we obtain

$$\mathcal{D}(A_Y) = \begin{cases} \mathbf{W_0^{1,p}}((0,\infty); X) = \{f \in \mathbf{W^{1,p}}((0,\infty); X) \mid f(0) = 0\}, \\ \mathbf{C_0^1}((0,\infty); X) = \{f \in \mathbf{C^1}((0,\infty); X) \mid f, f' \in \mathbf{C_0}((0,\infty); X)\}. \end{cases}$$

(Note that $\mathbf{W^{1,p}}(\mathbb{R}; X) \hookrightarrow \mathbf{C_0}(\mathbb{R}; X)$ if $p \in [1, \infty)$.) The space Y is invariant under the operators $R(\lambda, A)$, $\operatorname{Re} \lambda < 0$ (see the representation in Proposition 8.4.1), hence their restrictions to Y form the resolvent of A_Y.

Lemma 8.5.1. *One has $\sigma(A_Y) = \mathbb{C}_+$.*

Proof. Fix λ with $\operatorname{Re} \lambda > 0$. We show that $\lambda - A_Y$ cannot have dense range. In fact, let $u \in \mathcal{D}(A_Y)$; then using integration by parts we compute

$$\int_0^\infty [(\lambda - A_Y)u](t)\, e^{-\lambda t}\, dt = \int_0^\infty \lambda e^{-\lambda t} u(t) - u'(t) e^{-\lambda t}\, dt = u(t) e^{-\lambda t}\big|_0^\infty = 0.$$

Writing $e_\lambda(t) = e^{-\lambda t}$ this implies that $\int_0^\infty v e_\lambda = 0$ for all $v \in \overline{\mathcal{R}(A_Y)}$. If $0 \neq x \in X$ and $0 \neq \psi$ is any positive test function on $(0, \infty)$ such that $\int_0^\infty \psi e_\lambda \neq 0$, we have $\psi \otimes x \notin \overline{\mathcal{R}(A_Y)}$. $\square$

As to the elementary functional calculus, given $\varphi \in (\pi/2, \pi)$ and $e \in H_0^\infty(S_\varphi)$, the space Y is invariant under $e(A)$, whence $e(A_Y) = e(A)\big|_Y$. This means

$$e(A_Y)u = g * u = \int_0^t g(t-s)u(s) \qquad (u \in Y),$$

where $g \in \mathbf{L^1}(0, \infty)$ is such that $\widehat{g}(s) = e(is)$. For more general functions f we obtain the following.

Proposition 8.5.2. *Let $p \in [1, \infty]$ and Y, A_Y be as above, let $\varphi \in (\pi/2, \pi)$, and let $f \in \mathcal{B}(S_\varphi)$, i.e., $f \in \mathcal{O}(S_\varphi)$ is polynomially bounded at 0 and at ∞. Then the following assertions hold.*

a) $\mathcal{D}(f(A_Y)) = \mathcal{D}(f(A)) \cap Y$, *i.e.* $f(A)u \in Y$ *whenever* $u \in \mathcal{D}(f(A)) \cap Y$.

b) $f(A_Y)$ *is bounded if and only if* $f(A)$ *is bounded if and only if* $f(is)$ *is a bounded* $\mathbf{L^p}(\mathbb{R}; X)$*-Fourier multiplier. In this case* $\|f(A_Y)\| = \|f(A)\|$.

Proof. a) The function f is regularised by a power of $\tau(z) = z(1+z)^{-2}$, say $h := \tau^n f \in H_0^\infty$. Let $u \in \mathcal{D}(f(A)) \cap Y$, and define $v := f(A)u$. Then $\tau(A)^n v = h(A)u =: w \in Y$. This means that $\tau(A)^{n-1}v = 2w + Aw + A^{-1}w$. But clearly $2w + Aw = 2w + w'$ vanishes on $(-\infty, 0)$ since w does. Moreover, $A^{-1}w$ must be a constant c on $(-\infty, 0)$ since $w = AA^{-1}w = (A^{-1}w)'$ vanishes there. However, we are on $\mathbf{L^p}$ with $p < \infty$ or on $\mathbf{C_0}$, and this forces $c = 0$. Hence $\tau(A)^{n-1}v \in Y$, and inductively we arrive at $v \in Y$.

b) Suppose first that $f(is)$ is a bounded $\mathbf{L^p}(\mathbb{R}; X)$-Fourier multiplier. By Propositions 8.1.6, 8.2.3 and Corollary 8.2.9, we conclude that $f(A)$ is bounded. Hence by a), $\mathcal{D}(f(A_Y)) = Y \cap \mathcal{D}(f(A)) = Y$, whence $f(A_Y)$ is bounded. Suppose conversely that $f(A_Y)$ is a bounded operator on Y. Let $u \in \boldsymbol{\mathcal{D}}(\mathbb{R}; X)$ be any test function. Find $a \in \mathbb{R}$ such that $\tau_a u = u(\cdot - a)$ is supported in $(0, \infty)$, i.e., $\tau_a u \in Y$. By assumption, there is $v \in Y$ such that $v = f(A)(\tau_a u)$ and one has $\|v\|_p \le \|f(A_Y)\| \, \|\tau_a u\|_p$. This implies that $\widehat{v} = f(is)\widehat{\tau_a u} = f(is)e^{-isa}\widehat{u}$ on $\mathbb{R} \setminus \{0\}$. This yields $\widehat{\tau_{-a} v} = f(is)\widehat{u}$ on $\mathbb{R} \setminus \{0\}$, whence $u \in \mathcal{D}(f(A))$ with $f(A)u = \tau_{-a}v$. Since the $\mathbf{L^p}$-norm is translation invariant, $\|f(A)u\|_p = \|v\|_p \le \|f(A_Y)\| \, \|\tau_a u\|_p = \|f(A_Y)\| \, \|u\|_p$. Now, $f(A)$ is a closed operator and $\boldsymbol{\mathcal{D}}(\mathbb{R}; X)$ is dense, so we may conclude that $f(A)$ is a bounded operator, and this again implies that $f(is)$ is a bounded $\mathbf{L^p}(\mathbb{R}; X)$-Fourier multiplier. The equality of the norms $\|f(A)\| = \|f(A_Y)\|$ is implicit in the above considerations. $\square$

From Proposition 8.5.2 and the result about the derivative on the whole axis we immediately obtain the following.

Corollary 8.5.3. *Let* $X \neq 0$ *be a Banach space.*

a) *Let* $p \in (1, \infty)$. *The space* X *is UMD if and only if for some* $0 \neq r \in \mathbb{R}$ *the operators* $(d/dt)^{\pm ir}$ *are bounded on* $\mathbf{L^p}((0, \infty); X)$. *In this case the operator* d/dt *has a bounded* $H^\infty(S_\varphi)$*-calculus on* $\mathbf{L^p}((0, \infty); X)$, *for each* $\varphi \in (\pi/2, \pi)$.

b) *On* $\mathbf{C_0}((0, \infty); X)$ *and on* $\mathbf{L^1}((0, \infty); X)$ *the operator* $(\varepsilon + d/dt)^{ir}$ *is unbounded for all* $0 \neq r \in \mathbb{R}$, $\varepsilon \ge 0$.

The Derivative on a Finite Interval

We keep the notation X, A, Y from the previous section and turn to the derivative operator on a finite interval. Let us fix $\tau > 0$ and consider the spaces

$$Y_\tau := \begin{cases} \mathbf{L^p}((0, \tau); X) & \text{if } p \in [1, \infty), \\ \mathbf{C_0}((0, \tau]; X) & \text{if } p = \infty. \end{cases}$$

One could describe Y_τ as a factor space of Y, but we refrain from doing so. By

$$R_\tau := \big(u \longmapsto u\big|_{(0,\tau]}\big) : Y \longrightarrow Y_\tau$$

we denote the restriction mapping. For $u \in Y_\tau$ we denote by $v = u^\sim$ *any* extension of u to the half-line, i.e., $v \in Y$ with $R_\tau v = u$. The **right shift** semigroup on Y_τ is defined by

$$S_\tau(t)u := R_\tau[S(t)u^\sim] \qquad (t \geq 0)$$

(which of course does not depend on the actual choice of $u^\sim$). It is easy to see that S_τ is a strongly continuous semigroup on Y_τ and $S_\tau(t) = 0$ whenever $t \geq \tau$. The negative generator of this semigroup is denoted by A_τ. In order to describe its domain we define

$$\mathbf{W_0^{1,p}}((0,\tau);X) := \{u \in \mathbf{W^{1,p}}((0,\tau);X) \mid u(0) = 0\} \tag{8.9}$$

for $p \in [1,\infty]$ and also

$$\mathbf{C_0^1}((0,\tau];X) := \{u \in \mathbf{C^1}([0,\tau];X) \mid u(0) = u'(0) = 0\}.$$

Then we can assert the following facts.

Lemma 8.5.4. *One has $A_\tau = d/dt$ (in the distributional sense) with*

$$\mathcal{D}(A_\tau) = \begin{cases} \mathbf{W_0^{1,p}}((0,\tau);X) & \text{if } p \in [1,\infty), \\ \mathbf{C_0^1}((0,\tau];X) & \text{if } p = \infty. \end{cases}$$

For $\operatorname{Re}\lambda < 0$ *one has*

$$R(\lambda, A_\tau)u = R_\tau[R(\lambda, A_Y)u^\sim] \qquad (u \in Y_\tau)$$

or — equivalently — $R(\lambda, A_\tau)R_\tau = R_\tau R(\lambda, A_Y)$. Moreover, $\varrho(A_\tau) = \mathbb{C}$ and

$$R(\lambda, A_\tau)u\,(s) \;=\; -e^{\lambda s}\int_0^s e^{-\lambda t}\,u(t)\,dt \qquad (s \in [0,\tau])$$

for all $\lambda \in \mathbb{C}, u \in Y_\tau$.

Proof. Suppose that $\operatorname{Re}\lambda < 0$, and let $u \in Y_\tau$. Then

$$\begin{aligned} R(\lambda, A_\tau)u &= -\int_0^\infty e^{\lambda t} S_\tau(t)u\,dt = -\int_0^\infty e^{\lambda t} R_\tau[S_\tau(t)u^\sim]\,dt \\ &= R_\tau\left(-\int_0^\infty e^{\lambda t} S(t)u^\sim\,dt\right) = R_\tau[R(\lambda, A_Y)u^\sim] \end{aligned}$$

by the definition of a generator. Since R_τ is surjective, it follows that $\mathcal{D}(A_\tau) = R_\tau\mathcal{D}(A_Y)$, and this amounts to the characterisation given in the theorem. It also shows that $A_\tau = d/dt$. Now observe that the shift on Y_τ is nilpotent, i.e., $S_\tau(t) = 0$

for all $t \geq \tau$. Hence actually $R(\lambda, A_\tau)u = -\int_0^\tau e^{\lambda t} S_\tau(t)u\,dt$ for all $\operatorname{Re}\lambda < 0$ and this has an obvious holomorphic extension to the entire complex plane. Thus in fact $\varrho(A_\tau) = \mathbb{C}$ and

$$R(\lambda, A_\tau)u = -R_\tau \int_0^\tau e^{\lambda t} S(t)u^\sim\,dt = -\int_0^\tau e^{\lambda t} u^\sim(s-t)\,dt = -e^{\lambda s}\int_0^s e^{-\lambda t}\,u(t)\,dt$$

for *all* $\lambda \in \mathbb{C}, u \in Y_\tau, s \in [0,\tau]$. □

The operator $A_\tau = d/dt$ on Y_τ is called the **Riemann–Liouville operator**. Its inverse $V = A_\tau^{-1} = -R(0, A_\tau)$ is given by

$$(Vu)(s) = \int_0^s u(t)\,dt \qquad (s \in [0,\tau], u \in Y_\tau)$$

and is called the **Volterra operator**. Since A_τ is invertible, the operators $V^\alpha = A_\tau^{-\alpha}$ are bounded for $\operatorname{Re}\alpha > 0$. In fact, one has a nice formula for them.

Proposition 8.5.5. *Let τ, Y_τ, A_τ, V be as above. Then*

$$V^\alpha u\,(s) = A_\tau^{-\alpha} u\,(s) = \frac{1}{\Gamma(\alpha)}\int_0^s (s-t)^{\alpha-1}u(t)\,dt \qquad (s \in [0,\tau])$$

for all $\operatorname{Re}\alpha > 0$. *The operator family $(V^\alpha)_{\operatorname{Re}\alpha>0}$ forms a strongly continuous, exponentially bounded, holomorphic semigroup of angle $\pi/2$ on Y_τ.*

Proof. As $-A_\tau$ generates the nilpotent semigroup S_τ, we may apply Proposition 3.3.5. This yields

$$\begin{aligned} V^\alpha u\,(s) &= \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1} S_\tau(t)u\,dt(s) = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}u(\cdot - t)\,dt(s) \\ &= \frac{1}{\Gamma(\alpha)}\int_0^s t^{\alpha-1}u(s-t)\,dt = \frac{1}{\Gamma(\alpha)}\int_0^s (s-t)^{\alpha-1}u(t)\,dt. \end{aligned}$$

The second assertion follows from the general theory in Chapter 3. □

The operators V^α are called **Riemann–Liouville fractional integrals** and the semigroup $(V^\alpha)_{\operatorname{Re}\alpha>0}$ is called the **Riemann–Liouville semigroup** on $(0,\tau)$.

We now prove an analogue of Theorem 8.4.3 and Corollary 8.5.3. This requires several steps.

Lemma 8.5.6. *Let τ, Y_τ, A_τ, V be as above, and let $\varphi \in (\pi/2, \pi)$.*

a) *One has $f(A_\tau)R_\tau u = R_\tau f(A_Y)u$ for all $f \in \mathcal{E}(S_\varphi), u \in Y$.*

b) *If $f \in \mathcal{B}(S_\varphi)_{A_Y}$, then $R_\tau f(A_Y) \subset f(A_\tau)R_\tau$ (as operators on Y).*

Proof. a) follows immediately from the relation $R(\lambda, A_\tau)R_\tau = R_\tau R(\lambda, A_Y)$ (established in Lemma 8.5.4). Let $e^n f \in \mathcal{E}(S_\varphi)$, where $e(z) = z(1+z)^{-2}$. Let $u \in \mathcal{D}(f(A_Y))$, so $(fe^n)(A_Y)u = e(A_Y)^n f(A_Y)v$. Applying R_τ on both sides of this equation we obtain by a) $(fe^n)(A_\tau)R_\tau u = e(A_\tau)^n R_\tau f(A_Y)u$. This shows that $R_\tau u \in \mathcal{D}(f(A_\tau)$ and $f(A_\tau)R_\tau u = R_\tau f(A_Y)u$. □

Part b) of the previous lemma shows that the boundedness of $f(A_Y)$ implies the boundedness of $f(A_\tau)$, and in this case we have $\|f(A_\tau)\| \le \|f(A_Y)\|$. In particular if X is UMD and $p \in (1, \infty)$, then since A_Y has a bounded $H^\infty(S_\varphi)$-calculus, so does A_τ. There is no reason to believe that for a general function f the boundedness of $f(A_\tau)$ should imply the boundedness of $f(A_Y)$. However, we shall see in the following that it is in fact true for the purely imaginary powers.

Lemma 8.5.7. *Let $\sigma, \tau > 0$ and $0 \ne r \in \mathbb{R}$. Then $A_\tau^{ir} \in \mathcal{L}(Y_\tau)$ if and only if $A_\sigma^{ir} \in \mathcal{L}(Y_\sigma)$, and in this case one has equality of norms $\|A_\tau^{ir}\|_{\mathcal{L}(Y_\tau)} = \|A_\sigma^{ir}\|_{\mathcal{L}(Y_\sigma)}$. Moreover, $(A_Y)^{ir} \in \mathcal{L}(Y)$, and $\|(A_Y)^{ir}\| = \|A_\tau^{ir}\|$.*

Proof. We first prove the independence of the length τ of the interval. Consider the mapping

$$\Phi := (f \longmapsto f(\tau \cdot)) : Y_\tau \longrightarrow Y_1,$$

which clearly is an isomorphism with $\|\Phi u\|_{Y_1} = \tau^{-1/p} \|u\|_{Y_\tau}$ for all $u \in Y_\tau$. This yields $\|\Phi\| \left\|\Phi^{-1}\right\| = 1$. Via Φ the operator A_1 is similar to A_τ, i.e. $\Phi^{-1} A_1 \Phi = \tau A_\tau$. Hence all the operators τA_τ $(\tau > 0)$ are mutually similar. But clearly $(\tau A_\tau)^{ir} = \tau^{ir} (A_\tau)^{ir}$, for example by the composition rule. So A_τ^{ir} is bounded if and only if $(\tau A_\tau)^{ir}$ is bounded and by similarity this holds either for all $\tau > 0$ or for none of them. Suppose that it holds for all. Then we must have

$$\left\|A_\tau^{ir}\right\| = \left\|\tau^{ir} A_\tau^{ir}\right\| = \left\|(\tau A_\tau)^{ir}\right\| = \left\|\Phi^{-1} A_1^{ir} \Phi\right\| \le \left\|\Phi^{-1}\right\| \|\Phi\| \left\|A_1^{ir}\right\| = \left\|A_1^{ir}\right\|$$

and the reverse inequality in a similar fashion. We are left to show that this implies also that $A_Y^{ir} \in \mathcal{L}(Y)$. To this aim, let $u \in Y$. We claim that for $0 < \sigma < \tau < \infty$ we have

$$A_\tau^{ir} R_\tau u\big|_{(0,\sigma)} = A_\sigma^{ir} R_\sigma u.$$

To prove this claim we multiply the left side with the operator $e(A_\sigma)$ and obtain

$$\begin{aligned} e(A_\sigma) \left[A_\tau^{ir} R_\tau u\big|_{(0,\sigma)}\right] &= e(A_\sigma) R_\sigma \left[A_\tau^{ir} R_\tau u\right]^\sim = R_\sigma e(A_Y) \left[A_\tau^{ir} R_\tau u\right]^\sim \\ &= R_\sigma \left[e(A_\tau) A_\tau^{ir} R_\tau u\right]^\sim = R_\sigma \left[(z^{ir} e)(A_\tau) R_\tau u\right]^\sim = R_\sigma (z^{ir} e)(A_Y) u \\ &= (z^{ir} e)(A_\sigma) R_\sigma u = e(A_\sigma) A_\sigma^{ir} R_\sigma u. \end{aligned}$$

Since $e(A_\sigma)$ is injective, the claim is proved. We define $v : (0, \infty) \longrightarrow X$ by

$$v\big|_{(0,\tau)} := A_\tau^{ir} R_\tau u \qquad (\tau > 0),$$

which is a valid definition by the claim above. Since the number $c := \|A_\tau^{ir}\|_{\mathcal{L}(Y_\tau)}$ is independent of τ, we may conclude that $v \in Y$ and $\|v\|_Y \le c \|u\|_Y$ at least in the case where $p < \infty$. One easily shows that $e(A_Y) v = (z^{ir} e)(A_Y) u$ by simply multiplying R_τ on both sides, $\tau > 0$ being arbitrary. Hence we have shown $\mathcal{D}(A_Y^{ir}) = Y$ in the case where $p < \infty$. In the remaining case $p = \infty$, i.e., $Y = \mathbf{C_0}(\mathbb{R}; X)$, we cannot conclude that $v \in Y$, but only that $v : [0, \infty) \longrightarrow$

X is continuous and bounded (by $c\,\|u\|_\infty$) and satisfies $v(0) = 0$. However, if $u \in \mathcal{R}(e(A_Y)) = \mathcal{D}(A_Y) \cap \mathcal{R}(A_Y)$ (which obviously is contained in $\mathcal{D}(A_Y^{ir})$) the argument shows $v = A_Y^{ir}u$ and therefore $\|A_Y^{ir}u\|_Y \le c\,\|u\|_Y$. Now, having $Y = \mathbf{C_0}((0,\infty);X)$, the operator $A_Y = d/dt$ has dense domain and dense range, and since A_Y^{ir} is closed, we may finally conclude that $A_Y^{ir} \in \mathcal{L}(Y)$ and $\|(A_Y)^{ir}\| \le \|A_\tau^{ir}\|$. The converse inequality has already been proved. $\square$

Let us summarise our considerations.

Theorem 8.5.8. *Let $0 \neq X$ be any Banach space, let $\tau > 0$, $p \in [1,\infty]$, and consider the operator d/dt on $\mathbf{L^p}((0,\tau);X)$ ($p < \infty$) or $\mathbf{C_0}((0,\tau];X)$ ($p = \infty$) with domain as specified in Lemma* 8.5.4. *Then the following assertions hold.*

a) *If $p = 1$ or $p = \infty$, then the operator $(\varepsilon + d/dt)^{ir}$ is unbounded for any $0 \neq r, \varepsilon \ge 0$.*

b) *If $p \in (1,\infty)$, the following assertions are equivalent:*
   (i) *X is a UMD space.*
   (ii) *The operators $(d/dt)^{\pm ir}$ are bounded operators on $\mathbf{L^p}((0,\tau);X)$ for some $r > 0$.*
   (iii) *The operator d/dt has bounded $H^\infty(S_\varphi)$-calculus on $\mathbf{L^p}((0,\tau);X)$ for every $\varphi \in (\pi/2,\pi)$.*

Proof. Simply combine Lemmas 8.5.6 and 8.5.7 with the analogous results for the operator A_Y on the half-line (Corollary 8.5.3). $\square$

Remark 8.5.9. Let us point out that our operator A_τ is only *one* possible realisation of the derivative operator on the finite interval $(0,\tau)$. It is determined by the fact that $-A_\tau$ generates the right shift semigroup. In a similar manner one can consider *periodic boundary conditions*, i.e., change the domain to

$$\mathbf{W^{1,p}_{per}}((0,\tau);X) := \{f \in \mathbf{W^{1,p}}((0,\tau);X) \mid f(0) = f(\tau)\},$$

with analogous results. Some authors prefer this approach because the Fourier transform arguments become much easier and the situation still shows all relevant features, cf. [11].

The Volterra operator on $\mathbf{C}[0,\boldsymbol{\tau}]$

Already in the introduction we considered the Volterra operator

$$Vu(s) := \int_0^s u(t)\,dt, \qquad s \in [0,\tau],\ u \in \mathbf{C}([0,\tau];X)$$

as a bounded operator on $\mathbf{C}([0,\tau];X)$, where X is a Banach space. It is injective and its inverse is the derivative operator $A := V^{-1} = d/dt$, but with the domain

$$\mathcal{D}(A) = \mathcal{R}(V) = \{x \in \mathbf{C^1}([0,\tau];X) \mid x(0) = 0\}.$$

With this domain, $-A$ does not generate a semigroup.

Proof. Note that $Z := \mathbf{C}([0,\tau];X) = \mathbf{C_0}((0,\tau];X) \oplus X$ the second summand consisting of all constant functions. Suppose that $-A$ generates an (exponentially bounded) semigroup S on Z. By Proposition A.8.5 the space $Y_\tau = \overline{\mathcal{D}(A)} = \mathbf{C_0}((0,\tau];X)$ is invariant under the semigroup S and the restricted semigroup on Y_τ has the part $-A_\tau$ of $-A$ as its generator. But $\mathcal{D}(A_\tau) = \mathbf{C_0^1}(0,\tau]$ and since we already know that $-A_1$ generates the right shift semigroup on Y_τ, it follows that the original semigroup S coincides with the right shift on $\mathbf{C_0}(0,1]$. Since $\mathcal{D}(A)$ is invariant under the semigroup S, $S(\tau/2)u \in \mathcal{D}(A) \subset \mathbf{C^1}[0,\tau]$, where $u = (s \longmapsto s)$. But this is obviously false. □

In spite of this result, we can estimate the resolvent of A directly, because a short computation yields the formula

$$R(\lambda,A)u\,(s) = -e^{\lambda s}\int_0^s e^{-\lambda t}u(t)\,dt = -\int_0^s e^{\lambda t}u(s-t)\,dt \qquad (s \in [0,\tau])$$

for all $\lambda \in \mathbb{C}$ and all $u \in \mathbf{C}([0,\tau];X)$. This implies readily the estimate

$$\|R(\lambda,A)\| \leq \frac{1}{|\operatorname{Re}\lambda|} \qquad (\operatorname{Re}\lambda < 0).$$

In particular, $-A$ is a *Hille–Yosida operator* in the sense of [10, p.144]. So A is sectorial of angle $\pi/2$ (with $M(A) = 1$) and by general theory (Proposition 2.1.1) also $V = A^{-1} \in \operatorname{Sect}(\pi/2)$. By embedding $\mathbf{C}([0,\tau]);X) \subset \mathbf{L^2}((0,\tau);X)$ we see that the formula for the fractional powers

$$V^\alpha u\,(s) = \frac{1}{\Gamma(\alpha)}\int_0^s (s-t)^{\alpha-1}u(t)\,dt \qquad (s \in [0,\tau])$$

(with $\operatorname{Re}\alpha > 0, u \in \mathbf{C}([0,1];X)$) still holds. The next result is not surprising.

Proposition 8.5.10. *Let $0 \neq X$, $0 \neq r \in \mathbb{R}$, and $\varepsilon > 0$. Then either operator V^{ir}, $\left[V(\varepsilon+V)^{-1}\right]^{ir}$ is unbounded on $\mathbf{C}([0,\tau];X)$.*

Proof. Note that $\overline{\mathcal{D}(A)) \cap \mathcal{R}(A)} = \overline{\mathcal{D}(A)} = \mathbf{C_0}((0,\tau];X) = Y_\tau$. Obviously, $A_{Y_\tau} = A_\tau$ and we may apply Proposition 5.3.1. Namely, if $A^{-ir} = V^{ir} \in \mathcal{L}(\mathbf{C}[0,\tau])$ this proposition shows that also $(A_1)^{-ir} \in \mathcal{L}(Y_\tau)$, and this by Theorem 8.5.8 is not the case for $r \neq 0$. If we assume that $(V(\varepsilon+V)^{-1})^{ir}$ is bounded, by $\varepsilon V(\varepsilon+V)^{-1} = (\varepsilon+V^{-1})^{-1}$ we see that also $(\varepsilon+V^{-1})^{ir}$ is bounded. But V^{-1} is invertible, and the perturbation theorem Proposition 5.5.3 implies that also $V^{-ir} = (V^{-1})^{ir}$ is bounded, contradicting what we have proved before. □

8.6 Comments

8.1 and 8.2. Elliptic Operators. For these sections we used [10, Chapter 8] and [161, Chapter 12] as a basis together with a hand-written manuscript [8] from which we learned essentially how to prove Theorems 8.1.2 and Theorems 8.2.1. The characterisation of $\mathcal{D}(f(A_p))$ given in Propositions 8.1.6 and 8.2.3 are due to the author, but the characterisation of the boundedness of $f(A_p)$ is well known, as well as Theorem 8.2.7, cf. [141, Proposition 13.11]. Corollaries 8.1.7, 8.1.9 and their $\mathbf{L^p}$-versions unify the single results in [161, Chapter 12].

8.3 The Laplace Operator. Most of the results of this section are well known, cf. [10, Example 3.7.6 and Remark 3.7.7] for the semigroup aspects and [141, Example 10.2] for the functional calculus aspects. The identification of the universal extrapolation space for the Laplacian is inspired by [198, §25], where the space $\Phi(\mathbb{R}^d)$ is called the *Lizorkin* space of test functions. There [198, §26, §27] one can find also many formulae employing Riesz potentials, Bessel potentials, Gauss- and Poisson semigroups, as well as the connections with the theory of hypersingular integrals. Proposition 8.3.8 (in a slightly different form) is a classical result due to Hörmander [120], and our proof follows the original. The result was reproved and extended by Arendt, El-Mennaoui and Hieber [13] making use of boundary values of holomorphic semigroups, cf. [10, Theorem 3.9.4]. The proof of the fact that e^{Δ_p} is a sectorial operator is due to the author. It answers in the negative a question of Arendt in [9, 7.2.2 (j)], namely whether the property BIP 'extrapolates' from $\mathbf{L^2}$ to $\mathbf{L^p}$ with $p \neq 2$. We remark that some natural questions remain open here. First, we do not know whether the operator e^{Δ_1} is sectorial on $\mathbf{L^1}(\mathbb{R})$. Second it seems to be unknown whether the assumption that $i\Delta_2$ generates a semigroup on $\mathbf{L^2}(\mathbb{R}; X)$ already forces X to be a Hilbert space.

More General Elliptic Operators. We focussed our presentation on constant-coefficient homogeneous elliptic operators because on the one hand this theory is fundamental for all the results about more general operators, on the other hand the connection with Fourier multiplier operators is particularly nice.

Already from our results we may easily pass to operators of the form

$$Au = \sum_{|\alpha|=m} a_\alpha D^\alpha u + \sum_{|\alpha|<m} a_\alpha(t) D^\alpha u \qquad (u \in \mathbf{L^p}(\mathbb{R}^d; X)),$$

where $a(s) = \sum_{|\alpha|=m} a_\alpha (is)^\alpha$ is a homogeneous polynomial with constant coefficients and $a_\alpha \in \mathbf{L^\infty}(\mathbb{R}^d; \mathcal{L}(X))$ for $|\alpha| < m$. Indeed, the domain inclusion of Theorem 8.2.1 applied to the so-called **principal part** A_m induced by the polynomial a yields that the second part

$$B := \sum_{|\alpha|<m} a_\alpha(t) D^\alpha$$

comprising the 'lower-order terms' is a 'good' perturbation of the principal part. This is to say that one may apply Lemma 5.5.1 to see that $A = A_m + B$ is — after translation by a constant — an invertible sectorial operator, and the functional calculus properties are governed by A_m in the sense of Proposition 5.5.3.

Of course, one may pass from constant (scalar) coefficients in the principal part to variable (operator) coefficients and from the whole space $\mathbb{R}^d$ to some open subset $\Omega \subset \mathbb{R}^d$ adding suitable boundary conditions. The general scheme is to reduce everything to the constant-coefficient case by perturbation results and a localisation procedure. We do not go into details here but refer to [69, Chapter II] and [9, Section 8] for a general introduction and further references. The localisation procedure is detailed in [141, Section 6], see also [141, Theorem 13.13].

Whereas in the above procedure some regularity is needed for the coefficients in the principal part, there is an approach where only $\mathbf{L}^{\boldsymbol{\infty}}$-coefficients are required. Namely, for open $\Omega \subset \mathbb{R}^d$ and $a_{ij} \in \mathbf{L}^{\boldsymbol{\infty}}(\Omega)$ such that

$$\operatorname{Re} \sum_{i,j=1}^{d} a_{ij}(t) s_i \overline{s_j} \geq c\, |s|^2 \qquad (s \in \mathbb{C}^d)$$

for some $c > 0$, we define the quadratic form

$$a(u, v) := \int_\Omega \sum_{i,j=1}^{d} a_{ij}(t) D_i u(t) \overline{D_j v(t)}\, dt$$

for $u, v \in V$, where V is some closed subspace of $\mathbf{W}^{\mathbf{1,2}}(\Omega)$ containing $\mathbf{W}_{\mathbf{0}}^{\mathbf{1,2}}(\Omega)$. As in Section 7.3.2 we see that a is an elliptic form, and in fact for each $\lambda > 0$ the form a_λ is coercive. Under certain conditions on V and the form one can — via the so-called Beurling–Deny criteria — establish that the associated operator A_2 on $\mathbf{L}^{\mathbf{2}}(\Omega)$ extrapolates to a sectorial operator A_p on all $\mathbf{L}^{\boldsymbol{p}}$-spaces, $p \in [1, \infty]$. Moreover, one obtains kernel estimates which, with the help of Calderón–Zygmund theory, allow one to transfer from $\mathbf{L}^{\mathbf{2}}$ to $\mathbf{L}^{\boldsymbol{p}}$ the boundedness of the H^∞-calculus and other properties related to the functional calculus, see [3, Section 7] for the basic ideas, as well as [9, Section 8] and [141, Chapter 8] and the references therein.

8.4 and 8.5. The Derivative and the Volterra Operator. Results on the spectral theory of the derivative and on the shift semigroup are standard, see [85]. Theorem 8.4.3 — which characterises UMD spaces by the derivative on the line — is due to Prüsss [192, Section 8.2]. For its counterparts Corollary 8.5.3 a) (half-line) and Theorem 8.5.8 b) (finite interval) we do not know of a reference. With the appropriate definitions one can prove easily that the derivative on $\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}; X)$ has bounded H^∞-calculus even on **double sectors** of the form $\Sigma_\omega := S_\omega \cap -S_\omega$, $(\omega \in (\pi/2, \pi)$. Hieber and Prüss [118] use this and a transference principle to show that every generator of a bounded group on a UMD space has a bounded H^∞-calculus on such double sectors.

Guerre-Delabriére [99] shows that a space X is UMD if and only if $(-\Delta_p) \in \operatorname{BIP}(\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}; X))$ for some/each $p \in (1, \infty)$. Martinez and Sanz [161, Theorem 12.1.11] show by using concrete functions that the derivative on $\mathbf{L}^{\mathbf{1}}[0, 1]$ does not have any non-trivial bounded imaginary power. We do not know of a direct reference for Proposition 8.5.10. The Riemann–Liouville fractional integrals have been studied thoroughly by Samko, Kilbas and Marichev [198], cf. also [161, Chapter 2]. When one takes the left shift instead of the right shift, one is lead to the so-called *Weyl fractional integrals*, also treated in [161] and [198]. Galé and Pytlik [93] construct a certain functional calculus for holomorphic semigroups, based on the Weyl fractional calculus.

Chapter 9

Mixed Topics

In this chapter we present three different topics related to the functional calculus. The first (Section 9.1) is a review of some counterexamples; the common feature here is the use of spaces with bases that are not unconditional. The second (Section 9.2) is an application of functional calculus methods in theoretical numerical analysis, namely stability and convergence results for rational approximation schemes. The final Section 9.3 is devoted to regularity questions of solutions of inhomogeneous Cauchy problems, in particular to the so-called *maximal regularity problem*.

9.1 Operators Without Bounded H^∞-Calculus

We present a general method for constructing sectorial operators without bounded H^∞-calculus. The construction uses Schauder bases.

9.1.1 Multiplication Operators for Schauder Bases

Suppose that X is a Banach space and $\mathcal{B} = (e_n)_{n\in\mathbb{N}}$ is a Schauder basis for X (see Appendix E.6 for a definition). One can then define the projections

$$P_m : \sum\nolimits_{n=1}^{\infty} x_n e_n \longmapsto \sum\nolimits_{n=1}^{m} x_n e_n$$

and finds that $P_m \in \mathcal{L}(X)$ for each $m \in \mathbb{N}$, and that $M_0 := \sup_m \|P_m\|$ is finite. The number M_0 is called the **basis constant** of the basis $(e_n)_n$. Given a scalar sequence $a = (a_n)_n \subset \mathbb{C}$ we define

$$|\!|\!| a |\!|\!| := \limsup_n |a_n| + \sum\nolimits_{n\geq 1} |a_{n+1} - a_n| \,,$$

which may be infinite. With a we associate a **multiplication operator** A on X by

$$\begin{aligned} \mathcal{D}(A) &:= \Big\{ x = \sum\nolimits_n x_n e_n \in X \;\Big|\; \sum\nolimits_n a_n x_n e_n \text{ converges} \Big\}, \\ Ax &:= \sum\nolimits_n a_n x_n e_n \qquad (x \in \mathcal{D}(A)). \end{aligned}$$

Since the coordinate projections $(x \mapsto x_n) = P_{n+1} - P_n$ are continuous, the operator A is easily seen to be closed. Moreover, the operator A is injective if and only if $a_n \neq 0$ for all $n \in \mathbb{N}$. We call the sequence a a **multiplier** for the basis $(e_n)_n$ if A is a bounded operator.

Lemma 9.1.1. *Let $a = (a_n)_{n\in\mathbb{N}} \in \mathbb{C}$ such that $\|\!|a|\!\| < \infty$. Then a is a multiplier, i.e., $A \in \mathcal{L}(X)$, and one has $\|A\| \leq M_0 \,\|\!|a|\!\|$. If in addition $a_n \to 0$ as $n \to \infty$, A is a compact operator.*

Proof. Take $x \in X$ and write $s_1 := 0$ and $s_n := \sum_{k=1}^{n-1} x_n e_n = P_{n-1}x$ for $n \geq 2$. Then partial summation yields

$$\sum_{n=1}^{m} a_n x_n e_n = \sum_{n=1}^{m} a_n(s_{n+1} - s_n) = a_m s_{m+1} + \sum_{n=1}^{m-1} (a_n - a_{n+1}) s_{n+1}.$$

Since $\|\!|a|\!\| < \infty$, $a_0 := \lim_m a_m$ exists, so the first summand tends to $a_0 x$ as $m \to \infty$. The second can be estimated by

$$\left\| \sum_{n=1}^{m-1} (a_n - a_{n+1}) s_{n+1} \right\| \leq \sum_{n=1}^{m-1} |a_n - a_{n+1}| \, \|P_n x\| \leq M_0 \,\|\!|a|\!\|\; \|x\|,$$

whence the series converges absolutely. The remaining arguments are straightforward, the compactness following from $\|P_n A - A\| \to 0$. ☐

If the sequence $a = (a_n)_n$ is positive and strictly increasing, we obtain a sectorial operator.

Lemma 9.1.2. *Let $a = (a_n)_n \subset (0, \infty)$ be a strictly increasing sequence with $\lim_n a_n = \infty$, and let A be the associated operator. Then A is sectorial of angle $\omega_A = 0$, $\sigma(A) \subset [a_1, \infty)$, and A has compact resolvent. The holomorphic semigroup generated by $-A$ is immediately compact, i.e., e^{-tA} is compact for each $t > 0$.*

Proof. Given $\lambda \notin [a_1, \infty)$, we form the sequence $r_\lambda := (1/(\lambda - a_n))_n$ and compute

$$\|\!|r_\lambda|\!\| = \sum_{k\geq 1} \left| \frac{1}{\lambda - a_{k+1}} - \frac{1}{\lambda - a_k} \right| = \sum_{k\geq 1} \left| \int_{a_k}^{a_{k+1}} \frac{dt}{(\lambda - t)^2} \right| \leq \int_{a_1}^{\infty} \frac{dt}{|\lambda - t|^2}.$$

By Lemma 9.1.1 the operator associated to r_λ is bounded (even compact) and it is easily seen that this operator equals $R(\lambda, A)$. In particular, $\sigma(A) \subset [a_1, \infty)$. For $\lambda = |\lambda| e^{i\varphi}$, $0 < |\varphi| \leq \pi$, we obtain

$$\|\lambda R(\lambda, A)\| \leq M_0 \,|\lambda|\, \|\!|r_\lambda|\!\| \leq M_0 \int_0^\infty \frac{|\lambda|\, dt}{|\lambda - t|^2} = M_0 \int_0^\infty \frac{dt}{|e^{i\varphi} - t|^2}.$$

This shows that $A \in \mathrm{Sect}(0)$. Since each semigroup operator e^{-tA} is obtained by a norm-convergent integral of resolvents (which are known to be compact), it must be compact. ☐

Remark 9.1.3. Using the estimate $\|r_\lambda\| \le \int_0^\infty |\lambda - t|^{-2}\, dt$, one sees easily that A is also a strong strip-type operator with $\omega_{st}(A) = 0$ and $R(\lambda, A) \le \pi M_0 / |\operatorname{Im} \lambda|$ for $\lambda \notin \mathbb{R}$.

The functional calculus for a so-obtained sectorial/strip-type operator A is straightforward.

Proposition 9.1.4. *Let $a = (a_n)_n \subset (0, \infty)$ be a strictly increasing sequence with $\lim_n a_n = \infty$, and let A be the associated multiplication operator on X. Let $\omega \in (0, \pi)$ and $f \in H^\infty(S_\omega)$. Then $f(A)$ is the multiplication operator associated with the sequence $(f(a_n))_n$.*

The same conclusion holds true if one regards A as a strip-type operator and takes a bounded function on a strip $f \in H^\infty(H_\omega)$, $\omega > 0$.

Proof. Take $e \in H_0^\infty(S_\omega)$ first. Then $e(A)$ is defined by the Cauchy integral, and a straightforward application of Cauchy's theorem yields $[e(A)x]_n = e(a_n)x_n$ for all n. Hence $e(A)$ is the multiplication operator associated with $(e(a_n))_n$. The operator $e(A)$ is injective if and only if $e(a_n) \neq 0$ for all n. For general $f \in H^\infty$ take a regulariser e for f. Then

$$
\begin{aligned}
x \in \mathcal{D}(f(A)) &\iff \exists y \in X : (ef)(A)x = e(A)y \\
&\iff \exists y \in X\ \forall n : e(a_n)f(a_n)x_n = (ef)(a_n)x = e(a_n)y_n \\
&\iff \exists y \in X\ \forall n : f(a_n)x_n = y_n,
\end{aligned}
$$

since $e(a_n) \neq 0$ for all n. This proves the claim. In the strip case the arguments are analogous. □

If the basis is unconditional, every bounded sequence is a multiplier. Hence in this case a sectorial multiplication operator A always has a bounded H^∞-calculus. If the basis is conditional, one always finds a bounded sequence b such that the corresponding operator is not bounded, cf. [224, Theorem II.D.2]. The whole problem is now to ensure that this sequence is of the form $b = (f(a_n))_n$. This is the subject of the next section.

9.1.2 Interpolating Sequences

In this section we deal with the problem to find a bounded holomorphic function f which coincides on certain predefined points z_n with certain previously fixed numbers a_n.

Denote for the moment by Z the space of all entire functions $f \in \mathcal{O}(\mathbb{C})$ such that

$$\|f\|_Z := \sup_{z \in \mathbb{C}} e^{-|\operatorname{Im} z|^2} |f(z)| < \infty.$$

Then $\|f\|_Z$ is a norm on Z which turns it into a Banach space. Moreover, Z embeds canonically in to each space $H^\infty(H_\omega)$, $\omega > 0$. Let us denote by

$$R := (f \longmapsto f\big|_{\mathbb{Z}}) : Z \longrightarrow \ell^\infty(\mathbb{Z})$$

the restriction mapping.

Theorem 9.1.5 (Interpolating Sequences). *There exists a bounded linear operator $T : \ell^\infty(\mathbb{Z}) \longrightarrow Z$ with $R \circ T = I$ on $\ell^\infty(\mathbb{Z})$.*

Proof. For a given sequence $a \in \ell^\infty(\mathbb{Z})$ we define $Sa(z) := \sum_{n\in\mathbb{Z}} a(n)e^{-(z-n)^2}$ for $z \in \mathbb{C}$. It is easily seen that the series converges uniformly on compacta and one has the estimate

$$|S(z)| \le \|a\|_\infty \, e^{|\operatorname{Im} z|^2} \sum_{n\in\mathbb{Z}} e^{-(\operatorname{Re} z - n)^2} \qquad (z \in \mathbb{C}).$$

The function $g(x) := \sum_{n\in\mathbb{Z}} e^{-(x-n)^2}$ is continuous and 1-periodic, hence bounded. Thus we have proved that $S : \ell^\infty(\mathbb{Z}) \to Z$ continuously. Next, we claim that $RS = I + H$ where the operator $H \in \mathcal{L}(\ell^\infty(\mathbb{Z}))$ is given by the convolution (on $\mathbb{Z}$) $Ha = h * a$ with

$$h(n) = \begin{cases} e^{-n^2} & n \neq 0, \\ 0 & n = 0. \end{cases}$$

Indeed,

$$Sa(k) = \sum_n a(n)e^{-(k-n)^2} = \sum_n a(k-n)e^{-n^2} = a(k) + \sum_{n\neq 0} a(k-n)e^{-n^2}.$$

Now, $\|H\| \le \|h\|_1 = 2\sum_{n\ge 1} e^{-n^2} < 1$. Therefore the operator $I + H$ is invertible and we can define $T := S(I + H)^{-1}$. □

As a consequence of the theorem, one says that the integers $\mathbb{Z}$ form an **interpolating sequence** for strips. That means, given any bounded (double-)sequence $a \in \ell^\infty(\mathbb{Z})$ there is an entire function f, bounded on every strip H_ω, $\omega > 0$, such that $f(n) = a_n$ for all $n \in \mathbb{Z}$. Using the sector-strip correspondence via the exponential function, one immediately obtains a result for the sector.

Corollary 9.1.6. *Let $c > 0$ be fixed. Then there is a bounded linear operator $T_c : \ell^\infty(\mathbb{Z}) \longrightarrow H^\infty(S_\pi)$ such that $(T_c a)(c^n) = a_n$ for all $n \in \mathbb{Z}$.*

Proof. Define $T_c a(z) := (Ta)(\log z / \log c)$ for $|\arg z| < \pi$, where T is the operator from Theorem 9.1.5 above. □

A common choice is $c = 2$, so that the corollary says that the sequence $(2^n)_{n\in\mathbb{Z}}$ is an interpolating sequence for sectors.

9.1.3 Two Examples

Being prepared by the previous paragraphs, we state the main result.

Theorem 9.1.7. *Let $\mathcal{B} := (e_n)_{n\in\mathbb{N}}$ be a Schauder basis for the Banach space X, and let A be the multiplication operator on X associated with the sequence $(2^n)_{n\in\mathbb{N}}$. Suppose that the basis $\mathcal{B}$ is* not *unconditional. Then the natural $H^\infty(S_\pi)$-calculus for A is not bounded.*

Proof. Since the Schauder basis is conditional, there exists a sequence $b = (b_n)_{n\in\mathbb{N}}$ such that the corresponding multiplication operator B is not bounded. By Corollary 9.1.6 there is a function $f \in H^\infty(S_\pi)$ with $f(2^n) = b_n$ for all $n \in \mathbb{N}$. Proposition 9.1.4 yields $f(A) = B$. □

The previous result becomes even more interesting as it is known that every Banach space with an *unconditional* basis also has a *conditional one* (see [204, Theorem 23.2]). Since most of the classical separable Banach spaces have unconditional bases, they all admit 'nice' operators without bounded H^∞-calculus. In particular, every separable Hilbert space has an unconditional (and therefore also a conditional) basis. When we combine what we have proved so far with McIntosh's theorem (Theorem 7.3.1) we obtain the following.

Corollary 9.1.8 (Le Merdy). *Let H be a separable Hilbert space. Then there exists an operator A on H such that the following statements hold.*

1) *The operator A is invertible and sectorial of angle* 0.
2) *The semigroup generated by $-A$ is immediately compact.*
3) $A \notin \mathrm{BIP}(H)$.

Leaving Hilbert spaces, one can ask for an example of an operator without bounded H^∞-calculus but such that it nevertheless has bounded imaginary powers. In order to construct such an operator, we turn to a prominent example of a conditional Schauder basis.

Fix $p \in (1,\infty)$ and consider the space $\mathbf{L}^{\boldsymbol{p}}(\mathbb{T})$, where $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit circle group, endowed with the normalised Haar measure μ. For $n \in \mathbb{Z}$ we let

$$e_n := (z \longmapsto z^n) = e^{int}$$

be the n-th character of $\mathbb{T}$. As is well known, $(e_n)_{n\in\mathbb{Z}}$ is an orthonormal basis of $\mathbf{L}^{\mathbf{2}}(\mathbb{T})$. For $u \in \mathbf{L}^{\mathbf{1}}(\mathbb{T})$ its n-th Fourier coefficient is given by $\widehat{u}(n) := \int_{\mathbb{T}} u e_{-n}\, d\mu$. Let

$$\mathsf{H}^{\boldsymbol{p}}(\mathbb{T}) := \{u \in \mathbf{L}^{\boldsymbol{p}}(\mathbb{T}) \mid \widehat{u}(n) = 0 \text{ for all } n < 0\}$$

be usual the **Hardy space**. As is well known, the **Riesz projection**

$$R := \Big(\sum\nolimits_{n\in\mathbb{Z}} a_n e_n \longmapsto \sum\nolimits_{n\geq 0} a_n e_n\Big)$$

originally defined on the space $\boldsymbol{\mathcal{T}} := \operatorname{span}\{e_n \mid n \in \mathbb{Z}\}$ of **trigonometric polynomials** extends to a bounded projection on $\mathbf{L}^{\boldsymbol{p}}(\mathbb{T})$ with range $\mathsf{H}^{\boldsymbol{p}}(\mathbb{T})$. As a consequence of this, for each $u \in \mathbf{L}^{\boldsymbol{p}}(\mathbb{T})$ its Fourier series

$$D_N * u = \sum\nolimits_{-N}^{N} \widehat{u}(n) e_n$$

converges to u in $\mathbf{L}^{\boldsymbol{p}}$-norm as $N \to \infty$. (Here D_N denotes the **Dirichlet kernel**.) Hence $(e_n)_{n\geq 0}$ is a Schauder basis of $\mathsf{H}^{\boldsymbol{p}}(\mathbb{T})$. If $p \neq 2$, this basis is not unconditional, as follows from [224, Proposition II.D.9].

Let us now consider the multiplication operator A on $X = \mathsf{H}^{\boldsymbol{p}}(\mathbb{T})$ corresponding to the sequence $(2^n)_{n\geq 0}$. It follows from Lemma 9.1.2 that A is invertible and sectorial of angle 0, and from Theorem 9.1.7 that the natural $H^\infty(S_\pi)$-calculus for A is not bounded. However, $A \in \mathrm{BIP}(X, 0)$, since A^{is} is the multiplication operator associated with the sequence $(2^{ins})_{n\geq 0}$ and

$$A^{is}u = \sum\nolimits_{n\geq 0} 2^{ins}\widehat{u}(n)e_n = \sum\nolimits_{n\geq 0} \widehat{u}(n)e^{in(\cdot+s\log 2)} = U(s\log 2)u,$$

where $(U(t))_{t\in\mathbb{R}}$ is the (isometric) rotation group on $\mathbf{L}^{\boldsymbol{p}}(\mathbb{T})$.

Write $X_p := \ker R$ for the kernel of the Riesz projection. Then $\mathbf{L}^{\boldsymbol{p}}(\mathbb{T}) = X_p \oplus \mathsf{H}^{\boldsymbol{p}}(\mathbb{T})$ and the operator $I \oplus A$ is an invertible sectorial operator of angle 0 on $\mathbf{L}^{\boldsymbol{p}}(\mathbb{T})$ with bounded (even isometric) imaginary powers but without bounded H^∞-calculus.

9.1.4 Comments

The first example of a sectorial operator without bounded imaginary powers was given by Komatsu [131, Section 14, Examples 6 & 8] (reproduced in [161, Example 7.3.3]), modelled on the space $\mathbf{c}_0$. McIntosh and Yagi [169, Theorem 4] gave an example of an invertible operator A on a Hilbert space without bounded H^∞-calculus. The method is to construct A as a direct sum of a sequence of finite-dimensional operators with certain special properties.

The idea of using conditional bases for constructing counterexamples was introduced by Baillon and Clément [23]. Venni [219] subsequently used it to give an example of an operator A with A^{is} being bounded for some but not all $s \in \mathbb{R}$. The method was elaborated further by Lancien, Le Merdy and Simard, cf. [143, 145, 203, 147].

In those papers it is usually referred to Carleson's theorem to obtain that the sequence of powers $(2^n)_{n\geq 1}$ is an interpolating sequence for the halfplane. Our proof of this fact, i.e., essentially Theorem 9.1.5, was abstracted from [51, Lemma 5.3], where an example is given of an operator A on $\mathbf{L}^{\boldsymbol{p}}(\mathbb{R})$ ($p \neq 1, 2, \infty$) that has bounded imaginary powers but not a bounded H^∞-calculus. Actually, this example is more or less the analogue of our second example in Section 9.1.3, due to Lancien [143], cf. [9, Section 4.5.3].

Although the construction is elegant, the examples are artificial in a sense and in fact the natural differential operators on the natural reflexive spaces all do have a bounded H^∞-calculus, cf. Section 8.6. However, if one passes to *pseudo*-differential operators, things are different, as Hieber [117] has pointed out.

9.2 Rational Approximation Schemes

In this section we turn to an application of functional calculus to questions of numerical analysis.

9.2.1 Time-Discretisation of First-Order Equations

Assume that one is given an autonomous first-order initial-value problem

$$x' = F(x), \qquad x(0) = x_0$$

in a state Banach space X. An *exact solution* is a differentiable curve $(u(t))_{t \geq 0}$ starting at x_0 and satisfying

$$\frac{d}{dt} u(t) = F(u(t)) \qquad (t > 0),$$

i.e., at any time $t > 0$ the velocity of u at t coincides with the value of the vector field F at the point $u(t)$. In simulating the dynamical system numerically one tends to work with piecewise affine curves instead of exact solutions. So given a fixed $t > 0$ one might think of dividing the interval $[0, t]$ into $n \in \mathbb{N}$ equal parts $0 = t_0 < t_1 < \cdots < t_n = t$, each of length $h := t_{k+1} - t_k = t/n$, and consider a continuous curve $u : [0, t] \longrightarrow X$ starting at $u(0) = x_0$ and being affine on each interval $[t_{j-1}, t_j]$. Using the notation $u_k = u(t_k)$, the vector $\delta u_k := h^{-1}(u_{k+1} - u_k)$ is the constant velocity vector of u on $[t_k, t_{k+1}]$.

In order that u can count as an approximate solution to our original problem, this velocity vector should have something to do with the vector field F. One — certainly natural — choice is to require

$$\delta u_k = h^{-1}\,(u_{k+1} - u_k) \;=\; F(u_k),$$

that means, at each t_k the curve u starts off with velocity $F(u(t_k))$. This method is commonly called the **forward** (or explicit) **Euler method**. When instead we require $\delta u_k = F(u_{k+1})$, i.e., we want u to *arrive* at t_{k+1} with the right velocity, it is called the **backward** (or implicit) **Euler method**.

Let us now look at the case that $F(x) = -Ax$, where A is a sectorial operator of angle $\omega_A < \pi/2$. The forward Euler method leads to the equation

$$u_{k+1} = u_k - hAu_k = (1 - hA)u_k$$

for all $k \leq n - 1$. Hence $u_n = (1 - hA)^n x_0$. Obviously, if we want to apply this method in an infinite-dimensional setting where the operator A is unbounded, this has some drawbacks since the approximation u will only be defined if the initial value is sufficiently *smooth*, i.e., is contained in $\mathcal{D}(A^N)$ for some (high) power $N \in \mathbb{N}$. Using instead the backward Euler method yields

$$u_{k+1} - u_k = -hAu_{k+1},$$

which amounts to $u_{k+1} = (1 + hA)^{-1} u_k$ and there are no restrictions for the initial value x_0. (Note that our assumptions on A ensure that $(1 + hA)^{-1} \in \mathcal{L}(X)$.) So we are lead to consider

$$u_n = \left(1 + \frac{t}{n} A\right)^{-n} x_0 \; = r\left(\frac{t}{n} A\right)^{n} x_0$$

(with $r(z) = r_{be}(z) := (1+z)^{-1}$) as an approximation of $e^{-tA}x_0$.

Of course, apart from the backward Euler scheme there are other methods which work well in the infinite-dimensional setting. We might, for example, require

$$\delta u_k = h^{-1}\,(u_{k+1} - u_k) \;=\; F\left(\frac{u_{k+1}+u_k}{2}\right) = -\frac{1}{2}A(u_{k+1}+u_k),$$

which amounts to the fact that the straight curve $u\big|_{[t_k,t_{k+1}]}$ has its velocity coinciding with the vector field F exactly at the midpoint between its beginning u_k and its end u_{k+1}. A short computation then yields

$$u_n = \left[\left(1 - \frac{t}{2n}A\right)\left(1 + \frac{t}{2n}A\right)^{-1}\right]^n x_0 = r\left(\frac{t}{n}A\right)^n x_0$$

with

$$r(z) = r_{cn}(z) := \frac{1-\frac{1}{2}z}{1+\frac{1}{2}z}.$$

This method is called the **Crank–Nicolson** scheme.

We cast these examples into a proper definition. A **rational approximation scheme** for the operator A is given by some rational function r with poles outside the spectral sector $\overline{S_{\omega_A}}$. It consists in computing

$$u_n = r(hA)^n u_0, \qquad nh = t$$

for $n \in \mathbb{N}, t > 0$. We call the scheme **accurate of order p** ($p \in \mathbb{N}$) if

$$r(z) - e^{-z} \;=\; O(z^{p+1}) \qquad \text{as} \quad z \to 0.$$

While the backward Euler scheme is accurate of order 1, the Crank–Nicolson scheme is accurate of order 2.

In the following we consider rational functions r and examine the properties of the corresponding approximation schemes, i.e., of the operator family

$$r(hA)^n \qquad (h > 0,\, n \in \mathbb{N}).$$

Of course one may ask for convergence $r(tA/n)^n x \to e^{-tA}x$ in general or, more precisely, for convergence rates depending on the smoothness of x. Apart from that, one may ask for the **stability** of the approximation scheme determined by r with respect to the operator A. By this we mean simply the uniform boundedness

$$K := \sup_{h>0, n\in\mathbb{N}} \|r(hA)^n\| \;<\; \infty, \tag{9.1}$$

which is of ultimate importance: if in the computation of the initial value x_0 one makes an additive error e_0, the total error after computing u is not larger than $K\,\|e_0\|$, independent of the smallness of the time-step h and the number of performed iterations n.

9.2.2 Convergence for Smooth Initial Data

We fix an angle $\varphi \in (0, \pi/2]$ and a rational function r satisfying

$$|r(z)| \le 1 \qquad (z \in \overline{S_\varphi}). \tag{9.2}$$

This condition is sometimes called the $A(\varphi)$**-stability** of the function r. We call a set of paramaters (p, q, a) **adapted** to the function r if it satisfies the following conditions:

1) $p, q \in \mathbb{N}$, $0 \ne a \in \mathbb{C}$ and $|\arg a| + p\varphi \le \pi/2$.
2) $r(z) - r(0)e^{-az^p} = O(z^q) \qquad \text{as} \quad z \to 0.$
3) $p < q$ in the case that $|r(0)| = 1$.

If (p, q, a) is adapted to r, we set

$$s_r := \begin{cases} \frac{q-p}{p} & \text{if } |r(0)| = 1, \\ \infty & \text{if } |r(0)| < 1. \end{cases} \tag{9.3}$$

(This last concept will not be used until Proposition 9.2.5.) The next lemma tells us that adapted triples exist.

Lemma 9.2.1. *Let $\varphi \in (0, \pi/2]$, and let r be an $A(\varphi)$-stable rational function. In the case where $|r(0)| < 1$ the triple $(1, 1, 1)$ is adapted. In the case where $|r(0)| = 1$ there is exactly one adapted triple (p, q, a). In particular, if r is accurate of order $s \in \mathbb{N}$, then $(1, s+1, 1)$ is adapted.*

Proof. By holomorphy one has $r(z) - r(0)e^{-z} = O(z)$, so $(1, 1, 1)$ is adapted if $|r(0)| < 1$ since in this case we do not require $q > p$. Suppose that $|r(0)| = 1$. Then for (p, q, a) to be adapted it is necessary that $r(z) = r(0) - r(0)az^p + O(z^{p+1})$ near $z = 0$. This determines p, a, and finally q. The inequality $|\arg a| + p\varphi \le \pi/2$ is now easily deduced from the fact that $|r(0)| = 1$ and $|r(z)| \le 1$ for $z \in \overline{S_\varphi}$. $\square$

The next lemma describes the asymptotic behaviour of the rational function $r(z)^n$ at 0.

Lemma 9.2.2. *Let $\varphi \in (0, \pi/2]$, let r be an $A(\varphi)$-stable rational function, and let (p, q, a) be adapted to r. Then the following assertions hold.*

a) *If $|r(0)| < 1$, then for each $\theta \in (0, \varphi)$ there are constants $C \ge 0, c \in [0, 1)$ such that*

$$\left| r(z)^n - r(0)^n e^{-naz^p} \right| \le Cc^n \, |z|^q \qquad (z \in \overline{S_\theta},\ |z| \le 1,\ n \in \mathbb{N}).$$

b) *If $|r(0)| = 1$, then for each $\theta \in (0, \varphi)$ there are constants $C \ge 0, c > 0$ such that*

$$\left| r(z)^n - r(0)^n e^{-naz^p} \right| \le C \, |z|^q \, n e^{-nc|z|^p} \qquad (z \in \overline{S_\theta},\ |z| \le 1,\ n \in \mathbb{N}).$$

Proof. Fix a constant $C' \geq 0$ with $|r(z) - r(0)e^{-az^p}| \leq C' \, |z|^q$ for $z \in S_\varphi, |z| \leq 1$, and let $\theta \in (0, \varphi)$

a) Suppose that $|r(0)| < 1$. By the Maximum Principle one has $|r| < 1$ in the interior of S_φ. Hence $0 < d := \sup\{|r(z)| \mid z \in \overline{S_\theta},\ |z| \leq 1\} < 1$. Choose $c \in (d, 1)$. Then

$$\left|r(z)^n - r(0)^n e^{-naz^p}\right| = \left|r(z) - r(0)e^{-az^p}\right| \left|\sum\nolimits_{j=0}^{n-1} r(z)^j r(0)^{n-j-1} e^{-a(n-j-1)z^p}\right|$$

$$\leq C' \, |z|^q \sum\nolimits_{j=0}^{n-1} d^{n-1} e^{-(n-j-1)\operatorname{Re}(az^p)} \leq C' n d^{n-1} \, |z|^q = C c^n \, |z|^q$$

for all $n \in \mathbb{N}$, $|z| \leq 1$ and some other constant $C \geq 0$. This proves a).

In the case $|r(0)| = 1$ we define $c_1 := \inf\{\operatorname{Re} az^p \mid z \in S_\theta,\ |z| = 1\} > 0$ and obtain $|e^{-az^p}| \leq e^{-c_1 |z|^p}$ for all $z \in S_\theta$. Next, we claim that there exists $c_2 > 0$ such that

$$|r(z)| \leq e^{-c_2 |z|^p} \qquad (z \in S_\theta, |z| \leq 1). \tag{9.4}$$

Assume the contrary. Then for some sequence $(z_n)_n$ we have $|r(z_n)| > e^{-|z_n|^p/n}$. After passing to a subsequence we may suppose that $(z_n)_n$ converges to some $z_0 \in \overline{S_\theta}$ with $|z_0| \leq 1$. Hence $|r(z_0)| \geq 1$. The Maximum Principle and (9.2) now imply that $z_0 = 0$. Writing $t_n := |z_n|$ and using that (p, q, a) is adapted it follows that

$$e^{-\frac{1}{n} t_n^p} < C_1 t_n^q + e^{-c_1 t_n^p} \qquad (n \in \mathbb{N}),$$

but this is impossible, since $t_n \to 0$ and $q > p$. Having thus established (9.4), we define $c := \min\{c_1, c_2\}$ and estimate

$$\begin{aligned}\left|r(z)^n - r(0)^n e^{-naz^p}\right| &= \left|r(z) - e^{-az^p}\right| \left|\sum\nolimits_{j=0}^{n-1} r(z)^j r(0)^{n-j-1} e^{-a(n-j-1)z^p}\right| \\ &\leq C' \, |z|^q \sum\nolimits_{j=0}^{n-1} e^{-jc_2|z|^p} e^{-c_1(n-j-1)|z|^p} \\ &\leq C' \, |z|^q \, n e^{-(n-1)c|z|^p} \leq (C' e^c) \, |z|^q \, n e^{-nc|z|^p}. \end{aligned}$$

□

We now use Lemma 9.2.2 in the special case of accurate functions r to obtain optimal approximation results for smooth initial data.

Theorem 9.2.3 (Convergence Theorem). *Let r be a rational function accurate of order $p \in \mathbb{N}$ and satisfying (9.2). Then for every $\alpha \in (0, p]$ and $\theta \in (0, \varphi)$ there is a constant $C = C(r, \theta, \alpha)$ such that for every sectorial operator A on a Banach space X such that $\omega_A \in [0, \theta)$ one has an estimate*

$$\left\|r(hA)^n x - e^{-nhA} x\right\| \leq C \, M(A, \theta) h^\alpha \, \|A^\alpha x\|$$

for all $x \in \mathcal{D}(A^\alpha)$ and all $h > 0, n \in \mathbb{N}$.

Proof. Fix $\theta \in (0, \varphi)$. The triple $(1, p+1, 1)$ is adapted and $r(0) = 1$, hence by Lemma 9.2.2 we find constants $C, c > 0$ such that

$$\left|r(z)^n - e^{-nz}\right| \le C\,|z|^{p+1}\,ne^{-cn|z|} \qquad (z \in \overline{S_\theta}, |z| \le 1, n \in \mathbb{N}).$$

Also, by using the $A(\varphi)$-stability, we find $|r(z)^n - e^{-nz}| \le 2$ for all $z \in S_\varphi, n \in \mathbb{N}$. Define $f_n(z) := z^{-\alpha}(r(z)^n - e^{-nz})$. Since $f_n(z) \in H_0^\infty$ — using the definition of $f_n(hA)$ by means of a Cauchy-integral — we can compute

$$\begin{aligned}
\|f_n(hA)\| &\le \frac{1}{2\pi}\int_{\partial S_\theta} |f_n(z)|\;\|R(z,hA)\|\;|dz| \\
&\le \frac{M(hA,\theta)}{2\pi}\left(C\int_{\partial S_\theta, |z|\le 1} |z|^{p+1-\alpha}\,ne^{-cn|z|}\frac{|dz|}{|z|} + 2\int_{\partial S_\theta, |z|\ge 1} |z|^{-\alpha}\,\frac{|dz|}{|z|}\right) \\
&\le \frac{M(A,\theta)}{2\pi}\left(2C\int_0^1 ne^{-nct}t^{p-\alpha+1}\,\frac{dt}{t} + 4\int_1^\infty \frac{dt}{t^{\alpha+1}}\right) \\
&\le \frac{M(A,\theta)}{2\pi}\left(2C\int_0^\infty t^{p-\alpha}e^{-ct}\,dt \;+\; 4\int_1^\infty t^{-(\alpha+1)}\,dt\right) = C'\,M(A,\theta)
\end{aligned}$$

for a certain constant $C' \ge 0$. Hence for $x \in \mathcal{D}(A^\alpha)$, we have

$$\left\|r(hA)^n x - e^{-nhA}x\right\| = \|(hA)^\alpha f_n(hA)x\| \le C'M(A,\theta)h^\alpha\,\|A^\alpha x\|\,.$$

This concludes the proof. □

9.2.3 Stability

We recall that in Theorem 9.2.3 the case that $\alpha = 0$ is excluded. We shall see in a moment that in fact a norm convergence statement like

$$\left\|r\,(tA/n)^n - e^{-tA}\right\| \;\to\; 0$$

in general is only true if $|r(\infty)| < 1$. However, we are going to show the *stability* of the appproximation scheme, requiring nothing else than $A(\varphi)$-stability of r.

Theorem 9.2.4 (Stability Theorem). *Let $\varphi \in (0, \pi/2]$, and let r be an $A(\varphi)$-stable rational function. Then for every $\theta \in (0, \varphi)$ there is a constant $C = C(\theta, r) \ge 0$ such that*

$$\sup_{n\in\mathbb{N}, h>0} \|r(hA)^n\| \;\le\; CM(A,\theta)$$

for every sectorial operator A with $\omega_A \in [0, \theta)$.

As in the previous section, the proof of this theorem rests on a careful asymptotic analysis of the rational function r. The crucial fact is the following.

Proposition 9.2.5. *Let $\varphi \in (0, \pi/2]$, let r be an $A(\varphi)$-stable rational function, and let (p, q, a) be adapted to r. Then for g_n defined by*

$$g_n(z) := r(z)^n - r(0)^n e^{-anz^p}$$

and $\theta \in (0, \varphi)$ there is $\varepsilon \in [0, 1)$ such that

$$\int_{\partial S_\theta, |z|\le 1} |g_n(z)| \frac{|dz|}{|z|} \;=\; \begin{cases} O(\varepsilon^n) & \text{if } |r(0)| < 1, \\ O(n^{-s}) & \text{if } |r(0)| = 1. \end{cases}$$

(where $s = s_r = (q-p)/p$ is as defined in (9.3)*) and*

$$\int_{\partial S_\theta, |z|\ge 1} \left| r(0)^n e^{-naz^p} \right| \frac{|dz|}{|z|} \;=\; O(\varepsilon^n).$$

Proof. Given an $A(\varphi)$-stable rational function r and an angle $\theta \in (0, \varphi)$ according to Lemma 9.2.2 one can write

$$\left| r(z)^n - r(0)^n e^{-az^p} \right| \;\le\; \begin{cases} C\, c^n \;|z| \\ C\; |z|^q \, n e^{-cn|z|^p} \end{cases} \qquad (n \in \mathbb{N},\; z \in \overline{S_\theta},\; |z| \le 1)$$

(where $c \in [0, 1)$ in the first and $p < q, c > 0$ in the second case). If $|r(0)| < 1$,

$$\int_{\partial S_\theta, |z|\le 1} |g_n(z)| \frac{|dz|}{|z|} \;=\; \int_{\partial S_\theta, |z|\le 1} \left| r(z)^n - r(0)^n e^{-anz^p} \right| \frac{|dz|}{|z|} \le 2Cc^n.$$

If however $|r(0)| = 1$, one has

$$\begin{aligned} \int_{\partial S_\theta, |z|\le 1} \left| r(z)^n - r(0)^n e^{-anz^p} \right| \frac{|dz|}{|z|} &\le C \int_{\partial S_\theta, |z|\le 1} |z|^q \, n e^{-cn|z|^p} \frac{|dz|}{|z|} \\ \le\; 2C \int_0^1 t^q n e^{-cnt^p} \frac{dt}{t} &\le \frac{2C}{p} n^{1-\frac{q}{p}} \int_0^\infty t^{\frac{q}{p}-1} e^{-ct}\, dt. \end{aligned}$$

This proves the first statement. To prove the second, choose $c > 0$ in such a manner that $\left| e^{-az^p} \right| \le e^{-c|z|^p}$ for all $z \in \overline{S_\theta}$ (cf. the proof of Lemma 9.2.2). Then

$$\int_{\partial S_\theta, |z|\ge 1} \left| r(0)^n e^{-anz^p} \right| \frac{|dz|}{|z|} \le 2 \int_1^\infty e^{-cnt^p} \frac{dt}{t} \le \frac{2e^{-cn}}{cnp}. \qquad \square$$

The strategy for the proof of Theorem 9.2.4 is to apply Proposition 9.2.5 not only to the function $r(z)$ but as well to $r(z^{-1})$ (which is also an $A(\varphi)$ stable rational function). We choose (p, q, a) adapted to r and (p', q', b) adapted to $r(z^{-1})$ and consider the functions

$$f_n(z) := r(z)^n - r(0)^n e^{-naz^p} - r(\infty)^n e^{-nbz^{-p'}}. \tag{9.5}$$

(Note that $f_n \in H_0^\infty(S_\theta)$ for every $\theta \in (0, \varphi)$.) We thus can write

$$r(hA)^n = f_n(A) + r(0)^n \left(e^{-anz^p} \right)(hA) + r(\infty)^n \left(e^{-nbz^{-p'}} \right)(hA).$$

The two last summands are unproblematic, as the following lemma shows.

Lemma 9.2.6. *Let $\varphi \in (0, \pi/2]$, and let $p, p' \in \mathbb{N}$ and $a, b \in \mathbb{C} \setminus \{0\}$ such that $|\arg a| + p\varphi$, $|\arg b| + p'\varphi \le \pi/2$. Then for any $\theta \in (0, \varphi)$ there is a constant C such that*

$$\sup_{h>0, n\in\mathbb{N}} \left\| e^{-anz^p}(hA) \right\| + \sup_{h>0, n\in\mathbb{N}} \left\| e^{-bnz^{-p'}}(hA) \right\| \le C\, M(A, \theta)$$

for every sectorial operator A with $\omega_A \in [0, \theta)$.

Proof. This follows immediately from Proposition 2.6.11, since by hypothesis we have $e^{-az^p}, e^{-bz^{-p'}} \in \mathcal{E}(S_{\varphi'})$ for every $\varphi' \in (0, \varphi)$. □

To prove Theorem 9.2.4 it therefore suffices to show the uniform boundedness of the operators $f_n(hA)$. However, even more is true, as the following proposition shows.

Proposition 9.2.7. *Let $\varphi \in (0, \pi/2]$, let r be an $A(\varphi)$-stable rational function, and let (p, q, a) be adapted to $r(z)$ and (p', q', b) be adapted to $w := r(z^{-1})$. Let $s := s_r, s' := s_w$ be attached to these adapted triples as in* (9.3)*. Define $\sigma := \min\{s, s'\}$. Then for every $\theta \in (0, \varphi)$ there are constants $C \ge 0, \varepsilon \in [0, 1)$ such that*

$$\|f_n(hA)\| \le \begin{cases} C\, \varepsilon^n\, M(A, \theta) & \text{if } |r(0)| < 1 \text{ and } |r(\infty)| < 1, \\ C\, n^{-\sigma}\, M(A, \theta) & \text{if } |r(0)| = 1 \text{ or } |r(\infty)| = 1, \end{cases}$$

for every $h > 0, n \in \mathbb{N}$ and every sectorial operator A such that $\omega_A \in [0, \theta)$.

Proof. After pulling out the usual factor $M(hA, \theta)/(2\pi) = M(A, \theta)/(2\pi)$ we are reduced to estimating

$$\int_{\partial S_\theta, |z|\le 1} \left| r(z)^n - r(0)^n e^{-naz^p} \right| \frac{|dz|}{|z|} + \int_{\partial S_\theta, |z|\ge 1} \left| r(0)^n e^{-naz^p} \right| \frac{|dz|}{|z|}$$
$$+ \int_{\partial S_\theta, |z|\ge 1} \left| r(z)^n - r(\infty)^n e^{-nbz^{-p'}} \right| \frac{|dz|}{|z|} + \int_{\partial S_\theta, |z|\le 1} \left| r(\infty)^n e^{-nbz^{-p'}} \right| \frac{|dz|}{|z|}.$$

The first two summands are estimated by means of Proposition 9.2.5 applied to r, the last two summands by applying Proposition 9.2.5 to $r(z^{-1})$ instead. If both numbers $|r(0)|$ and $|r(\infty)|$ are strictly less than 1, all terms converge to 0 exponentially fast. If one of these numbers is equal to 1, then the slowest convergence rate is exactly $n^{-\sigma}$. □

Let us state a nice corollary.

Corollary 9.2.8. *Let $\varphi \in (0, \pi/2]$, and let r be an $A(\varphi)$-stable rational function such that $|r(0)| < 1$ and $|r(\infty)| < 1$. Then for every $\theta \in (0, \varphi)$ there exist constants $C \ge$ and $\varepsilon \in (0, 1)$ such that*

$$\|r(hA)^n\| \le C\, \varepsilon^n\, M(A, \theta) \qquad (h > 0, n \in \mathbb{N})$$

for every sectorial operator A with $\omega_A \in [0, \theta)$.

Proof. Write $r(hA)^n = f_n(A) + r(0)^n e^{-naz^p}(hA) + r(\infty)^n e^{-nbz^{-p'}}(hA)$ and apply Lemma 9.2.6 and Proposition 9.2.7. $\square$

Having proved the Stability Theorem 9.2.4, we return to the question of norm convergence $r(t/nA)^n \to e^{-tA}$ in the case where r is accurate of some order p. If $|r(\infty)| < 1$, there is no problem.

Corollary 9.2.9. *Let $\varphi \in (0, \pi/2]$, and let r be an $A(\varphi)$-stable rational function, accurate of order $p \in \mathbb{N}$. Suppose that $|r(\infty)| < 1$. Then for every $\theta \in (0, \varphi)$ there is a constant $C \geq 0$ such that*

$$\left\| r\left(\frac{t}{n}A\right)^n - e^{tA} \right\| \leq C\, n^{-p}\, M(A, \theta) \qquad (t > 0, n \in \mathbb{N})$$

for every sectorial operator A with $\omega_A \in [0, \theta)$.

Proof. By hypothesis, $r(0) = 1$ and the triple $(1, p+1, 1)$ is adapted to r, with $s = p$. Since $\varepsilon := |r(\infty)| < 1$, we have $s' = \infty$, whence $\sigma = \min\{s, s'\} = p$. Now we write

$$\left\| r(hA)^n - e^{-tA} \right\| \leq \| f_n(hA) \| + \varepsilon^n \left\| e^{-nbz^{-p'}}(hA) \right\| \leq \left(Cn^{-p} + C'\varepsilon^n \right) M(A, \theta)$$

for $n \in \mathbb{N}, h > 0$ by Lemma 9.2.6 and Proposition 9.2.7. Writing $h = t/n$ concludes the proof. $\square$

Example 9.2.10. Consider $r(z) = (1+z)^{-1}$, i.e., the backward Euler scheme. It is $A(\pi/2)$-stable and accurate of order 1, with $r(\infty) = 0$. So given a sectorial operator A on a Banach space X with $\omega_A < \pi/2$ one obtains

$$\left(t/nR(t/n, -A) \right)^n = \left(1 + \frac{t}{n}A \right)^{-n} = e^{-tA} + O(n^{-1})$$

uniformly in $t > 0$. This approximation of the semigroup is called the **Post–Widder Inversion Formula**.

Example 9.2.11. Consider $r(z) = (1 - z/2)(1 + z/2)^{-1}$, i.e., the Crank–Nicolson scheme. Then $r(z^{-1}) = (2z-1)(2z+1)^{-1}$ and $r(\infty) = -1$. Hence $r(z) = -e^{-4z^{-1}} + O(z^{-2})$ as $z \to \infty$. The obstacle against a convergence $r(tA/n)^n \to e^{-tA}$ in norm is therefore $e^{-(4n^2/t)z^{-1}}(A)$. If for example $A = -\Delta$ on $\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d)$, then by Corollary 8.2.4 the norm of the operator $e^{-(4n^2/t)z^{-1}}(A)$ is independent of $n \in \mathbb{N}$. Since at the same time this operator must be injective (its kernel equals the space $A^{-1}0 = \mathcal{N}(A) = 0$), this norm is different from 0 and so norm convergence of the Crank–Nicholson scheme fails. However, since A is densely defined, strong convergence holds, as follows from the Stability Theorem 9.2.4 and the convergence for smooth initial data (Theorem 9.2.3).

9.2.4 Comments

The use of functional calculus methods in order to obtain bounds for time-discretizations of infinite-dimensional Cauchy problems goes back at least to the 1970's. Hersh and Kato [116] and afterwards Brenner and Thomeé [37] use the Phillips calculus, while Le Roux [152] (in a parabolic context) uses the holomorphic (Dunford–Riesz) calculus. The convergence theorem for smooth initial data (Theorem 9.2.3) apparently was known in the 1970's (see [38]), as well as the stability theorem (Theorem 9.2.4) in the case $|r(\infty)| < 1$. In 1993, Crouzeix, Larsson, Piskarev and Thomée [56] gave a proof of Theorem 9.2.4, and our presentation follows its lines. Thomée [212, Chapter 11] instead of integrals over sectors uses the formula for the Taylor functional calculus (Corollary 2.3.5) as does Hansbo in [114] where some modified results are obtained.

Independent of the abovementioned proof for the stability theorem, a different one was given by Palencia [182]. His result is actually a little more general and it allowed him the application of his methods also to so-called *multistep methods* [183]. As a matter of fact, he also uses the holomorphic functional calculus, but via a Möbius transformation he transforms everything to the circle and into a question on bounded operators.

Obviously related to the topics discussed so far is the question for conditions on a *bounded* operator T on a Banach space X that ensure uniform boundedness of its iterates $(T^n)_{n\geq 1}$, i.e., the **power-boundedness** of T. The Stability Theorem 9.2.4 shows how to generate power-bounded operators, but this seems a little unwieldy when actually the operator T is given. In fact, one looks for mere *spectral* conditions on T.

It is easily seen that for a power-bounded operator T one necessarily has $\sigma(T) \subset \overline{\mathbb{D}}$, where $\mathbb{D}$ is the interior of the unit circle. With a little more work one can also establish an inequality of the form

$$\|R(\lambda, T)\| \leq \frac{C}{|\lambda| - 1} \qquad (|\lambda| > 1)$$

for some constant C. This inequality is called the **Kreiss condition**. It was Kreiss who proved that on finite-dimensional spaces, this condition is actually equivalent to the power-boundedness of T. In infinite dimensions a growth of the powers of T as $O(n)$ is the best one can achieve. (See [206] for proofs of these facts and a nice survey.) In recent years, the stronger condition

$$\|R(\lambda, T)\| \leq \frac{C}{|\lambda - 1|} \qquad (|\lambda| > 1),$$

called the **Ritt** or **Tadmor–Ritt condition**, has been extensively studied. By some basic results from Chapter 2 we see that with $A := I - T$ the Ritt condition is equivalent to:

$$A \in \mathcal{L}(X),\ \sigma(A) \subset \{\lambda \mid |\lambda - 1| < 1\} \cup \{0\} \text{ and } A \text{ is sectorial with } \omega_A < \pi/2.$$

The last condition is equivalent to $-A$ generating a bounded holomorphic semigroup, as we know from Section 3.4. NEVANLINNA [175] showed that T being a Ritt-type operator is equivalent to the following two facts:

(i) T is power-bounded.

(ii) The family $\left(n(T^n - T^{n+1})\right)_{n\geq 1}$ is uniformly bounded.

It is fairly elementary to show that (i), (ii) imply the Ritt condition, see [30, Theorem 2.3]. We sketch a proof for the converse implication, using the methods developed in this section. To see that $T = I - A$ is power-bounded, one compares the function $(1-z)^n$ with the exponential e^{-nz} as in Proposition 9.2.5, employing an analogue of Lemma 9.2.2. The uniform boundedness of the family $((1-z)^n - e^{-nz})\,(A)$ is then proved as in Proposition 9.2.7, and the uniform boundedness of $(e^{-nA})_{n\geq 1}$ is already known.

An alternative proof consists in showing that the Cayley transform $B := (1+T)(1-T)^{-1}$ is a (possibly multi-valued) sectorial operator of angle strictly less than $\pi/2$, and — writing $T = r(A)$ for $r = (z-1)(z+1)^{-1}$ — applying the stability theorem.

The proof of (ii) relies on a nice trick that we have learned from TOMILOV, cf. [213]. Instead of A consider the operator matrix

$$\mathcal{A} := \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}, \qquad \mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \oplus \mathcal{D}(A),$$

on the product space $\mathcal{X} = X \oplus X$. It is easy to show that $\varrho(A) \subset \varrho(\mathcal{A})$ with

$$R(\lambda, \mathcal{A}) = \begin{pmatrix} R(\lambda, A) & -AR(\lambda, A)^2 \\ 0 & R(\lambda, A) \end{pmatrix} \qquad (\lambda \in \varrho(A)).$$

In particular, $\mathcal{A}$ is 'of the same type' as A, i.e., also $I - \mathcal{A}$ is a Ritt operator. For the functional calculus one establishes easily the identity

$$f(\mathcal{A}) = \begin{pmatrix} f(A) & (zf')(A) \\ 0 & f(A) \end{pmatrix}.$$

Since for $f(z) = (1-z)^{n+1}$ one has $zf' = (n+1)z(1-z)^n$, the power-boundedness of $I - \mathcal{A}$ implies the uniform boundedness of $(n+1)A(1-A)^n = (n+1)(1-T)T^n$, and this was to prove. (This method obviously leads to the following nice result: if $\mathcal{F} \subset H^\infty$ is a collection of functions such that $\{f(A) \mid f \in \mathcal{F}\}$ is uniformly bounded for *every* sectorial operator, then also the collection $\{zf' \mid f \in \mathcal{F}\}$ has this property.)

Recently, EL-FALLAH and RANSFORD [83, 84] have determined extremal growth of the iterates of T under generalised Ritt conditions. VITSE [220] establishes bounded Besov class functional calculi for Ritt operators and generators of holomorphic semigroups.

9.3 Maximal Regularity

The study of the functional calculus for sectorial operators has two major roots. One is the Kato square root problem (connected with the Calderón problem on Cauchy kernels on Lipschitz curves); this was the reason for McIntosh to develop the functional calculus in the first place. (See Chapter 7, especially the comments in Section 7.5.) The second root is the maximal regularity problem for inhomogeneous Cauchy problems.

9.3.1 The Inhomogeneous Cauchy Problem

In this section, A always denotes a sectorial operator on a Banach space with $\omega_A < \pi/2$. Fix $\tau \in (0, \infty]$ (a 'time-horizon') and consider for $f \in \mathbf{L}^1_{\mathrm{loc}}([0, \tau); X)$ and $x \in X$ the following inhomogeneous problem on $[0, \tau)$:

$$u' + Au = f, \qquad u(0) = x. \tag{9.6}$$

A **mild solution** of this problem is a function $u \in \mathbf{C}([0, \tau); X)$ with $u(0) = x$ and

$$\int_0^t u(s)\, ds \in \mathcal{D}(A), \qquad u(t) + A \int_0^t u(s)\, ds = \int_0^t f(s)\, ds$$

for all $t \in [0, \tau)$. *Mild solutions are unique.*

Proof. Let u_1, u_2 be mild solutions, and let $v(t) := (1 + A)^{-1}(u_1(t) - u_2(t))$. Then $v(0) = 0$, $v \in \mathbf{C}([0, \tau), \mathcal{D}(A))$ and $v(t) + \int_0^t Av(s)\, ds = 0$. Hence $v \in \mathbf{C}^1([0, \tau); X)$ and $v' = -Av$. Thus for $0 < s < t < \tau$ we let $w(s) := e^{-(t-s)A}v(s)$ and obtain $(d/ds)w(s) = Ae^{-(t-s)A}v(s) + e^{-(t-s)A}v'(s) = 0$. So the function w is constant on $(0, t)$, and since $v \in \mathbf{C}([0, \tau); \mathcal{D}(A))$, w is actually continuous up to the endpoints $s = 0, t$. This yields $v(t) = e^{-tA}v(0) = 0$. Consequently, $u = 0$. □

On the other hand, it is easy to show that by the **Variation of Constants** formula (also called **Duhamel's principle**)

$$u(t) = e^{-sA}x + \int_0^t e^{-(t-s)A} f(s)\, ds \qquad (t \in [0, \tau))$$

a mild solution is defined. This is therefore the unique mild solution of (9.6).

As one can imagine, the question of additional regularity of mild solutions is a central topic in the theory of evolution equations. Obviously, the regularity of the solution u will depend on regularity of the initial value $x \in X$ and the regularity of the inhomogeneity f. For the sake of simplicity we confine ourselves to the situation when $x = 0$.

Thus in the following we consider the problem

$$u' + Au = f, \qquad u(0) = 0 \tag{9.7}$$

on $[0, \tau)$ whose (mild) solution is given by

$$S(t)f := u(t) = \int_0^t e^{-(t-s)A} f(s)\, ds = (e^{-sA} * f)(t). \tag{9.8}$$

We choose a Banach space $\mathcal{X} \subset \mathbf{L}^1_{\mathrm{loc}}([0,\tau); X)$ as a reservoir for our inhomogeneities and look for regularity of the solution u. Classically, one chooses $\mathcal{X} := \mathbf{L}^p((0,\tau); X)$ for some $p \in [1, \infty)$ or $\mathcal{X} = \mathbf{C_0}((0,\tau]; X)$ (in case $\tau < \infty$) but other choices are possible. On $\mathcal{X}$ we let $\mathcal{B} := d/dt$ be the derivative operator as in Section 8.5, i.e., $-\mathcal{B}$ generates the right shift semigroup on $\mathcal{X}$. Moreover, A induces an operator $\mathcal{A} = I \otimes A$ on $\mathcal{X}$. In case $\mathcal{X} = \mathbf{L}^p((0,\tau); X)$ the operator $\mathcal{A}$ is given by

$$\mathcal{D}(\mathcal{A}) := \mathbf{L}^p((0,\tau); \mathcal{D}(A)), \qquad (\mathcal{A}\, u)(t) := A[u(t)] \qquad (u \in \mathcal{D}(\mathcal{A})),$$

with an analogous definition in the case $\mathcal{X} = \mathbf{C_0}((0,\tau]; X)$. It is easy to see that spectral and functional calculus properties of A carry over to $\mathcal{A}$. In particular, $\mathcal{A}$ is a sectorial operator on $\mathcal{X}$ of angle $\omega_{\mathcal{A}} = \omega_A < \pi/2$ and $\mathcal{B}$ is a sectorial operator on $\mathcal{X}$ of angle $\omega_{\mathcal{B}} = \pi/2$. Hence the two operators $\mathcal{A}, \mathcal{B}$ satisfy the so-called **parabolicity condition**

$$\omega_{\mathcal{A}} + \omega_{\mathcal{B}} < \pi. \tag{9.9}$$

Moreover, the operators $\mathcal{A}$ and $\mathcal{B}$ commute in the resolvent sense:

$$R(\lambda, \mathcal{A})R(\mu, \mathcal{B}) = R(\mu, \mathcal{B})R(\lambda, \mathcal{A}) \qquad (\lambda \in \varrho(\mathcal{A}),\ \mu \in \varrho(\mathcal{B})).$$

Our mild solution u is — philosophically — a solution to the equation

$$\mathcal{A}\, u + \mathcal{B}\, u = f.$$

It is not literally a solution since we do not know whether $u \in \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})$. However, the solution operator $f \longmapsto u$ is a kind of an inverse of the sum $\mathcal{A} + \mathcal{B}$. Hence it is reasonable to define an appropriate notion of the *sum* of $\mathcal{A}$ and $\mathcal{B}$ as an extension $\mathcal{C} \supset \mathcal{A} + \mathcal{B}$ of the algebraic sum. (The solution operator is then nothing else than $\mathcal{C}^{-1}$.) The idea how to do this stems from functional calculus and is elaborated in the following.

9.3.2 Sums of Sectorial Operators

We pass to a more abstract setting, changing notation towards a more common style. Let X be a Banach space, and let A, B be two sectorial operators on X, commuting in the resolvent sense:

$$R(\lambda, A)R(\mu, B) = R(\mu, B)R(\lambda, A) \qquad (\lambda \in \varrho(A),\ \mu \in \varrho(B)).$$

(Have a look at Proposition A.2.6 to find equivalent formulations.) In general the algebraic sum $A + B$ with the domain $\mathcal{D}(A) \cap \mathcal{D}(B)$ may be a very poor operator since $\mathcal{D}(A) \cap \mathcal{D}(B)$ can reduce to 0. However, one can try to set up a **joint functional calculus** for the pair (A, B) and then define $C := (z+w)(A, B)$ as the actual 'sum'.

The construction is rather simple. Take $\varphi \in (\omega_A, \pi)$ and $\psi \in (\omega_B, \pi)$. By purely algebraic methods one constructs a homomorphism of algebras

$$\Phi : \mathcal{E}(S_\varphi) \otimes \mathcal{E}(S_\psi) \longrightarrow \mathcal{L}(X)$$

such that $\Phi(f \otimes g) = f(A)g(B)$. Note that in a natural way one has an embedding

$$\mathcal{E}(S_\varphi) \otimes \mathcal{E}(S_\psi) \quad \subset \quad \mathcal{O}(S_\varphi \times S_\psi).$$

This yields a proper abstract functional calculus and the function $f(z, w) = z + w$ is regularisable since

$$\frac{z+w}{(1+z)(1+w)} = \left(1 - \frac{1}{1+z}\right) \otimes \frac{1}{1+w} \;+\; \frac{1}{1+z} \otimes \left(1 - \frac{1}{1+z}\right)$$

belongs to $\mathcal{E}(S_\varphi) \otimes \mathcal{E}(S_\psi)$. One can extend this primary functional calculus to larger algebras. For example, let $H_0^\infty(S_\varphi \times S_\psi)$ be the algebra of functions $f \in \mathcal{O}(S_\varphi \times S_\psi)$ such that there exist $C, s > 0$ such that

$$|f(z, w)| \le C \min(|z|^s, |z|^{-s}) \cdot \min(|w|^s, |w|^{-s}) \qquad (z \in S_\varphi,\ w \in S_\psi).$$

For $f \in H_0^\infty(S_\varphi \times S_\psi)$ one defines

$$\Phi(f) = f(A, B) = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} f(z, w) R(z, A) R(w, B)\, dz\, dw,$$

where Γ_1, Γ_2 are obvious sector boundaries. Since

$$H_0^\infty(S_\varphi \times S_\psi) \;\cap\; \big(\mathcal{E}(S_\varphi) \otimes \mathcal{E}(S_\psi)\big) \;=\; H_0^\infty(S_\varphi) \otimes H_0^\infty(S_\psi)$$

(as a little argument reveals), the extension of Φ to the algebra

$$\mathcal{E}(S_\varphi \times S_\psi) := H_0^\infty(S_\varphi \times S_\psi) + \mathcal{E}(S_\varphi) \otimes \mathcal{E}(S_\psi)$$

is well defined. This primary calculus can then be extended as usual to unbounded functions (see Chapter 1 and Section 2.3). Having at hand this joint functional calculus we can finally define $C := (z + w)(A, B)$ as a closed operator. Without further assumptions, however, almost nothing can be proved about C, but it turns out that the *parabolicity condition* $\omega_A + \omega_B < \pi$ (see (9.9)) is sufficient to render C a nice operator.

Theorem 9.3.1. *Let A, B be two resolvent-commuting sectorial operators on the Banach space X such that $\omega_A + \omega_B < \pi$, and let $C := (z + w)(A, B)$ be defined by the joint functional calculus. Then the following statements hold.*

a) *$A + B \subset C$ and $\mathcal{D}(A) \cap \mathcal{D}(C) = \mathcal{D}(A) \cap \mathcal{D}(B) = \mathcal{D}(C) \cap \mathcal{D}(B)$.*

b) *C is a sectorial operator with angle $\omega_C \le \max\{\omega_A, \omega_B\}$.*

c) *$\mathcal{N}(C) = \mathcal{N}(A) \cap \mathcal{N}(B)$.*

d) *$\mathcal{D}(A) \cap \mathcal{D}(B) \ \subset\ \mathcal{D}(C) \ \subset\ \bigcap_{\alpha \in (0,1)} \mathcal{D}(A^\alpha) \cap \mathcal{D}(B^\alpha)$.*

e) *$\bigcup_{\alpha > 0} \mathcal{R}(A^{1+\alpha}) + \mathcal{R}(B^{1+\alpha}) \ \subset\ \mathcal{R}(C) \ \subset \mathcal{R}(A) + \mathcal{R}(B)$.*

Proof. a) We have $A + B = (z)(A, B) + (w)(A, B) \subset (z + w)(A, B) = C$, by general functional calculus rules. Analogously $C - A \subset B$ and $C - B \subset A$. This completely proves a).

b) Fix $\pi > \omega > \max\{\omega_A, \omega_B\}$ and $\varphi \in (\omega_A, \pi), \psi \in (\omega_B, \pi)$ with $\max\{\varphi, \psi\} < \omega$ and $\varphi + \psi < \pi$. Then $S_\varphi + S_\psi = S_{\max\{\varphi,\psi\}}$. For $\lambda \in \mathbb{C}$ such that $|\arg \lambda| \in [\omega, \pi]$ we write

$$\frac{\lambda}{\lambda - (z + w)} = \frac{\lambda^2}{(\lambda - z)(\lambda - w)} + \frac{\lambda z w}{(\lambda - z - w)(\lambda - w)(\lambda - z)},$$

and all functions are in $\mathcal{O}(S_\varphi \times S_\psi)$. The second summand — let us call it $f_\lambda(z, w)$ — is even in $H_0^\infty(S_\varphi \times S_\psi)$, as a moment's reflection shows. This yields directly

$$\lambda R(\lambda, C) = \left[\frac{1}{\lambda - (z + w)}\right](A, B) = \lambda R(\lambda, A)\, \lambda R(\lambda, B) + f_\lambda(A, B) \ \in\ \mathcal{L}(X).$$

Moreover, for $\mu = \lambda / |\lambda|$ we have

$$\begin{aligned}\|f_\lambda(A, B)\| &\lesssim \int_{\Gamma_1} \int_{\Gamma_2} \frac{|\lambda|\, |dz|\, |dw|}{|\lambda - (z + w)|\, |\lambda - z|\, |\lambda - w|} \\ &= \int_{\Gamma_1} \int_{\Gamma_2} \frac{|dz|\, |dw|}{|\mu - (z + w)|\, |\mu - z|\, |\mu - w|}.\end{aligned}$$

c) Let $\alpha \in (0, 1)$. Then

$$\frac{z^{1+\alpha}}{(z + w)(1 + z)(1 + w)} = \frac{z^\alpha}{(1 + z)(1 + w)} - \frac{z^\alpha w}{(z + w)(1 + z)(1 + w)},$$

and the second summand is in $H_0^\infty(S_\varphi \times S_\psi)$ since $z^\varepsilon w^{1-\varepsilon}/(z + w)$ is bounded for each $\varepsilon \in [0, 1]$. Note further that

$$\begin{aligned}&\left(\frac{z^\alpha w}{(z + w)(1 + z)(1 + w)}\right)(A, B) \\ &= \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \frac{z^\alpha w}{(z + w)(1 + z)(1 + w)} R(w, B)\, dw\, R(z, A)\, dz \\ &= (1 + B)^{-1} \frac{-1}{2\pi i} \int_{\Gamma_1} \frac{z^\alpha}{(1 + z)} [B R(-z, B)] R(z, A)\, dz.\end{aligned}$$

It follows that $f_\alpha(A,B) \in \mathcal{L}(X)$, where $f_\alpha(z,w) := z^{1+\alpha}(z+w)^{-1}(1+z)^{-1}$. Therefore,

$$A^{1+\alpha}(1+A)^{-(1+\alpha)} = Cf_\alpha(A,B)(1+A)^{-\alpha} \supset (1+A)^{-\alpha}f_\alpha(A,B)C, \tag{9.10}$$

hence $\mathcal{N}(C) \subset \mathcal{N}(A^{1+\alpha}(1+A)^{-(1+\alpha)}) = \mathcal{N}(A)$, cf. Proposition 3.1.1 d). By symmetry, $\mathcal{N}(C) \subset \mathcal{N}(B)$ as well. But $\mathcal{N}(A) \cap \mathcal{N}(B) \subset \mathcal{N}(C)$ is clear by a), so c) is proved.

d) For $x \in \mathcal{D}(C)$, formula (9.10) shows that $A^{1+\alpha}(1+A)^{-(1+\alpha)}x \in \mathcal{D}(A^\alpha)$, whence necessarily $x \in \mathcal{D}(A^\alpha)$ (cf. Proposition 3.1.1 g)). By symmetry and a), this proves d).

e) Another look on (9.10) yields $\mathcal{R}(A^{1+\alpha}) = \mathcal{R}(A^{1+\alpha}(1+A)^{-(1+\alpha)}) \subset \mathcal{R}(C)$. Employing symmetry again establishes the left-hand inclusion in e). To complete the proof take $x \in \mathcal{D}(C)$. We have to prove $y := Cx \in \mathcal{R}(A) + \mathcal{R}(B)$. By definition we have

$$A(1+A)^{-1}(1+B)^{-1}x + B(1+B)^{-1}(1+A)^{-1}x = (1+A)^{-1}(1+B)^{-1}y,$$

and this is obviously contained in $\mathcal{R}(A) + \mathcal{R}(B)$. Hence

$$\begin{aligned} y &= (1+B)^{-1}(1+A)^{-1}y + B(1+B)^{-1}(1+A)^{-1}y + A(1+A)^{-1}y \\ &\in \mathcal{R}(A) + \mathcal{R}(B) + \mathcal{R}(B) + \mathcal{R}(A) = \mathcal{R}(A) + \mathcal{R}(B). \end{aligned} \qquad \square$$

Let us state some immediate but important consequences.

Corollary 9.3.2. *Let A, B, C as in Theorem* 9.3.1. *Then the following assertions hold.*

a) *If A or B is injective/invertible, so is C.*

b) *If A or B has dense range, so has C.*

c) *C is densely defined if and only if both A and B are densely defined. If this is the case, then $C = \overline{A+B}$.*

Proof. Employing some standard facts about fractional powers (Proposition 3.1.1, Corollary 3.1.11) the statements are more or less direct consequences of Theorem 9.3.1. Suppose that A, B, C are densely defined, and let $(x,y) \in C$. Then

$$x_n := n^2(n+A)^{-1}(n+B)^{-1}x \to x \quad \text{and} \quad y_n := n^2(n+A)^{-1}(n+B)^{-1}y \to y.$$

But $(x_n, y_n) \in C$ and $x_n, y_n \in \mathcal{D}(A) \cap \mathcal{D}(B)$, hence $(x_n, y_n) \in A + B$. $\square$

Remark 9.3.3. Let A, B, C be as in Theorem 9.3.1. An elementary but fairly tedious discussion yields that for $\omega \in (\max\{\omega_A, \omega_B\}, \pi)$ and $f \in \mathcal{E}(S_\omega)$ one has $f(z+w) \in \mathcal{E}(S_\varphi \times S_\psi)$, where the angles φ, ψ have to be close to ω_A, ω_B respectively. Moreover, one can prove the composition rule $f(C) = [f(z+w)](A,B)$ for those f. By an equivalent of Proposition 1.3.6 such a composition rule then extends to more general functions f.

We now introduce a concept which in the special context of inhomogeneous equations is of fundamental importance (see Section 9.3.3 below.) Let A, B, C be as above. A Banach space $E \subset X$ (with continuous inclusion) is called a **space of maximal regularity** for the pair (A, B) if the following holds:

$$\forall\, x \in E\ \exists!\ u \in \mathcal{D}(A) \cap \mathcal{D}(B)\ :\ Au + Bu = x \quad \text{and} \quad Au \in E.$$

Note that one does not require $u \in E$. The pair (A, B) is said to have **(abstract) maximal regularity** if $E = X$ is a space of maximal regularity.

Lemma 9.3.4. *Let A, B, C be as above, and suppose that A is injective. Define*

$$f(A, B) := \left(\frac{z}{z + w}\right)(A, B).$$

For a Banach space $E \subset X$ the following assertions are equivalent:

(i) *$E \subset X$ is a space of maximal regularity for (A, B).*

(ii) *$E \subset \mathcal{R}(C) \cap \mathcal{D}(f(A, B))$ and $f(A, B)E \subset E$.*

(iii) *$E \subset \mathcal{R}(C)$ and $f(A, B)$ restricts to a bounded operator on E.*

In particular, (A, B) has maximal regularity if C is invertible and $f(A, B) \in \mathcal{L}(X)$.

Proof. The equivalence of (ii)⇔(iii) is an easy consequence of the continuity of the embedding $E \subset X$ and the Closed Graph Theorem. By general functional calculus and Theorem 9.3.1 a) one has

$$AC^{-1} \subset f(A, B) \quad \text{and} \quad \mathcal{R}(C) \cap \mathcal{D}(f(A, B)) = \mathcal{D}(AC^{-1}) = \mathcal{R}(A + B).$$

From these identities the equivalence (i)⇔(ii) readily follows. □

Theorem 9.3.5 (Da Prato–Grisvard). *Let A, B be resolvent-commuting, sectorial operators on the Banach space X satisfying the parabolicity condition $\omega_A + \omega_B < \pi$. If A or B is invertible, then each space $E := (X, \mathcal{D}(A^\alpha))_{\theta,p}$, $\theta \in (0, 1), p \in [1, \infty]$, is a space of maximal regularity for (A, B).*

Proof. By hypothesis and Corollary 9.3.2 the operator C is invertible, hence the inclusion $E := (X, \mathcal{D}(A^\alpha))_{\theta,p} \subset \mathcal{R}(C)$ is trivial. So one is left to show that $E \subset \mathcal{D}(f(A, B))$ and $f(A, B)E \subset E$. By Theorem 9.3.1 d) we have $\mathcal{R}(C^{-1}) = \mathcal{D}(C) \subset \mathcal{D}(A^{1-\varepsilon})$ for each $\varepsilon \in (0, 1)$. On the other hand, $E = (X, \mathcal{D}(A^\alpha))_{\theta,p} \subset \mathcal{D}(A^\varepsilon)$ for some (small) $\varepsilon \in (0, 1)$, see Corollary 6.6.3. Hence

$$f(A, B) \supset [A^{1-\varepsilon} C^{-1}]\, A^\varepsilon$$

and $A^{1-\varepsilon} C^{-1} \in \mathcal{L}(X)$. This shows $\mathcal{D}(A^\varepsilon) \subset \mathcal{D}(f(A, B))$, hence $E \subset \mathcal{D}(f(A, B))$.

We now prove the invariance of E under $f(A, B)$. Choose a function $\psi_1 \in \mathcal{E}(S_\varphi)$

satisfying the conditions of the Characterisation Theorem 6.2.9 for the interpolation space E. Let $0 \neq \psi \in H_0^\infty(S_\varphi)$. Then also $\psi_2 := \psi\psi_1$ satisfies the hypotheses of that theorem. For $x \in E$ we have

$$t^{-\theta \operatorname{Re} \alpha} \psi_2(tA) f(A,B)x = (\psi(tz)f(z,w))(A,B)\,[t^{-\theta \operatorname{Re} \alpha}\psi_1(tA)x],$$

hence it suffices to prove that the operator family $(\psi(tz)f(z,w))(A,B), t > 0$, is uniformly bounded. By the same kind of manipulations as in the proof of Theorem 9.3.1 one sees that in fact

$$(\psi(tz)f(z,w))(A,B) = \frac{1}{2\pi i}\int_\Gamma \psi(tz)[zR(-z,B)]R(z,A)\,dz \qquad (t > 0).$$

So the desired uniform boundedness follows by the ususal estimates and change of variables. □

Remark 9.3.6. The above proof still works when neither A nor B is invertible, as long as C is. (This is the case when, e.g., $\mathcal{N}(A) \cap \mathcal{N}(B) = 0$ and, for some $\varepsilon > 0$, $\mathcal{R}(A^{1+\varepsilon}) + \mathcal{R}(B^{1+\varepsilon}) = X$.) However, it is unclear if there are reasonable situations where one can apply the resulting more general version of Theorem 9.3.5.

9.3.3 (Maximal) Regularity

Let us come back to our starting point, namely the inhomogeneous equation (9.7)

$$u' + Au = f, \qquad u(0) = 0.$$

Recall the definition of the operators $\mathcal{A}, \mathcal{B}$ in Section 9.3.1. As we already remarked there, $\mathcal{A}, \mathcal{B}$ are resolvent-commuting sectorial operators satisfying the parabolicity condition. So $\mathcal{C} := (z+w)(\mathcal{A}, \mathcal{B})$ is defined and Theorem 9.3.1 as well as Corollary 9.3.2 are applicable. In particular, when $\tau < \infty$ then $\mathcal{B}$ is invertible and hence $\mathcal{C}$ is invertible. As we intended, *$u := \mathcal{C}^{-1} f$ is indeed the unique mild solution of* (9.7).

Proof. Let $v := (1 + A)^{-1}u = (1 + \mathcal{A})^{-1}\mathcal{C}^{-1} f = \mathcal{C}^{-1}(1 + A)^{-1}f$. Then $v \in \mathcal{D}(\mathcal{C}) \cap \mathcal{D}(\mathcal{A})$, i.e., $v \in \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B})$ and $\mathcal{A}v + \mathcal{B}v = (1 + A)^{-1}f$. This means $v' + Av = (1+A)^{-1}f$, and integrating yields $v(t) + \int_0^t Av(s)\,ds = \int_0^t (1+A)^{-1}f(s)\,ds$ for $t \in [0, \tau]$. Going back to u yields

$$(1 + A)^{-1}u(t) + A(1 + A)^{-1}\int_0^t u(s)\,ds = (1 + A)^{-1}\int_0^t f(s)\,ds.$$

Since $A(1 + A)^{-1} = I - (1 + A)^{-1}$, this shows first that $\int_0^t u(s)\,ds \in \mathcal{D}(A)$ and then that $u(t) + A\int_0^t u(s)\,ds = \int_0^t f(s)\,ds$, i.e., u is a (the) mild solution. □

Part d) of Theorem 9.3.1 yields regularity results as the following.

Proposition 9.3.7. *Let $p \in [1,\infty)$, $\tau < \infty$, and $f \in \mathbf{L}^{\boldsymbol{p}}((0,\tau);X)$. Let u be the unique mild solution of* (9.7)*. Then*

$$u \;\in\; \mathbf{L}^{\boldsymbol{p}}((0,\tau);(X,\mathcal{D}(A))_{\theta,p}) \;\cap\; \mathbf{W}_{\mathbf{0}}^{\boldsymbol{\theta},\boldsymbol{p}}((0,\tau);X)$$

for all $\theta \in (0,1)$.

Here $\mathbf{W}_{\mathbf{0}}^{\boldsymbol{\theta},\boldsymbol{p}}((0,\tau);X) = (\mathbf{L}^{\boldsymbol{p}}((0,\tau);X), \mathbf{W}_{\mathbf{0}}^{\mathbf{1},\boldsymbol{p}}((0,\tau);X))_{\theta,p}$ is a **fractional Sobolev space**.

Proof. This follows from Theorem 9.3.1 d) in noting that for any sectorial operator A on a Banach space one has $\bigcap_{\alpha\in(0,1)} \mathcal{D}(A^\alpha) = \bigcap_{\theta\in(0,1)}(X,\mathcal{D}(A))_{\theta,p}$. $\square$

Remark 9.3.8. Proposition 9.3.7 is designed to be just an *example* of what can be done with Theorem 9.3.1. In particular, one can choose a different interpolation parameter q instead of p, and one obtains a similar result on the space of continuous functions.

In the case that $\tau = \infty$, the operator $\mathcal{C}$ is not surjective in general. In fact it can be shown that if we work on $\mathcal{X} = \mathbf{L}^{\boldsymbol{p}}((0,\infty);X)$, and $\mathcal{C}$ is invertible, then already A must be invertible, i.e., the semigroup $(e^{-tA})_{t\geq 0}$ is exponentially stable. This is known as *Datko's theorem*, see [10, Theorem 5.1.2].

Let us turn to the so-called **maximal regularity problem**. We call a Banach space $E \subset \mathbf{L}^{\mathbf{1}}_{\mathrm{loc}}([0,\tau);X)$ (continuous inclusion) a **space of maximal regularity** for A if for every $f \in E$ the corresponding mild solution u of (9.7) satisfies:

$$u \in \mathbf{W}^{\mathbf{1}}_{\mathrm{loc}}([0,\tau);X) \cap \mathbf{L}^{\mathbf{1}}_{\mathrm{loc}}([0,\tau);\mathcal{D}(A)) \quad \text{and} \quad \mathrm{A}u, u' \in E.$$

(Note that we do not require $u \in E$, although in many cases this is automatically satisfied.) This means that the solution should have the best regularity properties one can expect, with respect to both operators A and d/dt, and the space E is invariant under the mappings $(f \longmapsto Au), (f \longmapsto u')$. (It is this invariance property that renders the whole concept ultimately important for applications, see the comments in Section 9.3.4.)

Theorem 9.3.9 (Da Prato–Grisvard). *Let $\tau < \infty, p \in [1,\infty)$, $q \in [1,\infty], \theta \in (0,1)$, and $\operatorname{Re}\alpha > 0$. Let A be a sectorial operator on the Banach space X with $\omega_A < \pi/2$. Then the spaces*

$$E \;=\; \mathbf{L}^{\boldsymbol{p}}\big((0,\tau);(X,\mathcal{D}(A^\alpha))_{\theta,q}\big), \quad \mathbf{C_0}\big((0,\tau];(X,\mathcal{D}(A^\alpha))_{\theta,q}\big)$$

and

$$E \;=\; \mathbf{W}_{\mathbf{0}}^{\boldsymbol{\theta},\boldsymbol{p}}((0,\tau);X), \quad \mathbf{C}_{\mathbf{0}}^{\theta}((0,\tau];X)$$

are spaces of maximal regularity for A.

Proof. Fix $p \in [1, \infty)$ and consider the usual operators $\mathcal{A}$ and $\mathcal{B} = d/dt$ on $\mathcal{X} = \mathbf{L}^{\boldsymbol{p}}((0,\tau); X)$. By Theorem 9.3.5, $E = \mathbf{L}^{\boldsymbol{p}}((0,\tau); (X, \mathcal{D}(A^\alpha))_{\theta,q}) = (\mathcal{X}, \mathcal{D}(\mathcal{A}^\alpha))_{\theta,q}$ is a space of abstract maximal regularity for the pair $(\mathcal{A}, \mathcal{B})$; this means that E is in fact a space of maximal regularity for A. Interchanging the roles of $\mathcal{A}$ and $\mathcal{B}$ yields that $E = \mathbf{W}_{\mathbf{0}}^{\boldsymbol{\theta},\boldsymbol{p}}((0,\tau); X)$ is a space of maximal regularity for A. The remaining statements are proved analogously by working on $\mathcal{X} = \mathbf{C}_{\mathbf{0}}((0,\tau]; X)$. □

Let τ be finite, for the time being. A theorem of Baillon [22] states that $E = \mathbf{C}_{\mathbf{0}}((0,\tau]; X)$ is a space of maximal regularity for A if and only if either A is bounded or X contains a copy of $\mathbf{c}_0$, cf. [81]. In reasonable concrete situations, e.g. on reflexive spaces, this is never the case, whence continuous maximal regularity is not a promising property to study. Things change when we pass to the so-called $\mathbf{L}^{\boldsymbol{p}}$-maximal regularity for $p \in (1, \infty)$. More precisely, we say that the operator has the $\mathrm{MR}_{p,\tau}$-property, where $p \in (1,\infty), \tau \in (0, \infty]$, if $E = \mathbf{L}^{\boldsymbol{p}}((0,\tau); X)$ is a space of maximal regularity for A. It turns out that many common operators on common Banach spaces do indeed possess this property, but this fact is far from being trivial.

We digress a little on that property. From the Variation of Constants formula (9.8) it follows readily that for finite τ and $\lambda \in \mathbb{R}$ one has $A \in \mathrm{MR}_{\tau,p}$ if and only if $A + \lambda \in \mathrm{MR}_{\tau,p}$. Hence as far as maximal regularity is concerned, one can always suppose that A is invertible. One can further show that the $\mathrm{MR}_{\tau,p}$-property is independent of $\tau \in (0,\infty)$ [71, Theorem 2.5]. If A is invertible, i.e., if the semigroup is even exponentially stable, one can include $\tau = \infty$ in this statement. Therefore, we can omit the reference to τ and write just $A \in \mathrm{MR}_p$. (It turns out that it is even independent of p, see below.)

The problem of maximal regularity is essentially a question of operator-valued Fourier multipliers, i.e., vector-valued singular integrals. To see this, let for simplicity A be invertible and $\tau = \infty$. Loosely speaking, $A \in \mathrm{MR}_p$ is equivalent to

$$f \in \mathbf{L}^{\boldsymbol{p}}((0,\infty); X) \quad \Longrightarrow \quad Ae^{-sA} * f \in \mathbf{L}^{\boldsymbol{p}}((0,\infty); X),$$

i.e., $(f \longmapsto Ae^{-sA} * f)$ (originally defined on the dense set $\boldsymbol{\mathcal{D}}((0,\infty); \mathcal{D}(A)))$ is a bounded operator on the space $\mathbf{L}^{\boldsymbol{p}}((0,\infty); X)$. Since the norm of the kernel Ae^{-sA} grows like s^{-1} as $s \to 0$, this is a singular integral operator.

Lemma 9.3.10. *Let A be a densely defined, invertible, sectorial operator on the Banach space X such that $\omega_A < \pi/2$. Then for $p \in (1,\infty)$ the following assertions are equivalent:*

(i) $A \in \mathrm{MR}_p$.

(ii) *The function $A(is + A)^{-1} \in \mathbf{L}^{\boldsymbol{\infty}}(\mathbb{R}; \mathcal{L}(X))$ is a bounded $\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}; X)$-Fourier multiplier.*

Proof. Denote by $S = (f \longmapsto u = e^{-\,\cdot\, A} * f)$ the solution operator as in (9.8), and define $Y := \mathbf{L}^{\boldsymbol{p}}((0,\infty); \mathcal{D}(A)) \cap \mathbf{W}_{\mathbf{0}}^{\mathbf{1},\boldsymbol{p}}((0,\infty); X)$. Set $m(s) := A(is + A)^{-1}$.

(i)⇒(ii). Take any $f \in \boldsymbol{\mathcal{D}}((0,\infty); \mathcal{D}(A))$. Then $ASf = SAf$ and $\mathcal{F}(ASf)(s) =$

$m(s)\widehat{f}(s)$. By hypothesis, $\|ASf\|_p \le c\,\|f\|_p$ for some constant c independent of f. Hence

$$\left\|\mathcal{F}^{-1}(m\widehat{f})\right\|_p \le c\,\|f\|_p\,. \tag{9.11}$$

By shifting functions this inequality extends to all $f \in \boldsymbol{\mathcal{D}}(\mathbb{R};\mathcal{D}(A))$, and then even to all $f \in \boldsymbol{\mathcal{S}}(\mathbb{R};X)$, because $\mathcal{D}(A)$ is dense in X. Thus (ii) follows, by definition (see Appendix E.4).

(i)⇒(ii). Fix a constant $c > 0$ such that (9.11) is satisfied for all $f \in \boldsymbol{\mathcal{S}}(\mathbb{R};X)$, and take $f \in \boldsymbol{\mathcal{D}}((0,\infty);\mathcal{D}(A))$. Then $(Sf)' = f - ASf$, whence $Sf \in Y$ and

$$\|Sf\|_Y \sim \|(Sf)'\|_p + \|ASf\|_p + \|Sf\|_p \lesssim \|ASf\|_p + \|Sf\|_p \lesssim \|ASf\|_p + \|f\|_p$$

since the semigroup is exponentially stable. However, $ASf = \mathcal{F}(m\widehat{f})$ as above, whence $\|ASf\|_p \le c\,\|f\|_p$. Since $\boldsymbol{\mathcal{D}}((0,\infty);\mathcal{D}(A))$ is dense in $\mathbf{L}^{\boldsymbol{p}}((0,\infty);X)$, we see that in fact S maps $\mathbf{L}^{\boldsymbol{p}}((0,\infty);X)$ boundedly into Y, hence $A \in \mathrm{MR}_p$. ☐

Using an important result of Benedek, Calderón and Panzone from [28] one can prove that if $A \in \mathrm{MR}_p$ for *some* $p \in (1,\infty)$ this actually holds for *all* $p \in (1,\infty)$, see [71, Theorem 4.2.]. Thus we can omit even the reference to p and write $A \in \mathrm{MR}$ instead of $A \in \mathrm{MR}_p$.

From Lemma 9.3.10 and Plancherel's theorem it follows readily that on a Hilbert space $X = H$ *every* sectorial operator A with $\omega_A < \pi/2$ satisfies $A \in \mathrm{MR}_2$. It was proved by Coulhon and Lamberton in [50] that the Poisson semigroup on $X = \mathbf{L}^{\mathbf{2}}(\mathbb{R};Y)$ has maximal regularity if and only if Y is a UMD space. (See Appendix E.6 for the definition of UMD spaces.) This not only showed that there exist operators without $\mathbf{L}^{\boldsymbol{p}}$-maximal regularity, but at the same time stressed the importance of UMD spaces for possible positive results. In 1987 Dore and Venni [75] found the following abstract result which was a milestone at the time.

Theorem 9.3.11 (Dore–Venni). *Let X be a UMD space, and let A, B be resolvent-commuting, sectorial operators on X such that $A, B \in \mathrm{BIP}(X)$ and $\theta_A + \theta_B < \pi$. Then $C := (z+w)(A,B) = A + B$, $C \in \mathrm{BIP}(X)$ with $\theta_C \le \max\{\theta_A, \theta_B\}$, and $f(A,B) \in \mathcal{L}(X)$, where $f(z,w) = z/(z+w)$. In particular, if C is surjective then the pair (A,B) has abstract maximal regularity.*

The Dore–Venni theorem can be used to prove maximal $\mathbf{L}^{\boldsymbol{p}}$-regularity of operators with bounded imaginary powers on UMD spaces.

Corollary 9.3.12. *Let X be a UMD space, and let $A \in \mathrm{BIP}(X)$. Then $A \in \mathrm{MR}_p$ for each $p \in (1,\infty)$.*

Proof. We define as usual the operators $\mathcal{A}, \mathcal{B}$ on $\mathfrak{X} := \mathbf{L}^{\boldsymbol{p}}((0,\tau);X)$. Since X is a UMD space, the space $\mathfrak{X}$ is also a UMD space, and the derivative operator $\mathcal{B}$ has bounded imaginary powers on $\mathfrak{X}$, see Theorem 8.5.8. Hence we can apply the Dore–Venni theorem to conclude that $\mathcal{C} = \mathcal{A} + \mathcal{B}$, i.e., $\mathcal{D}(\mathcal{C}) \subset \mathcal{D}(A)$, whence the assertion follows. ☐

So far, one still could believe that on UMD spaces every sectorial operator A with $\omega_A < \pi/2$ has maximal $\mathbf{L}^p$-regularity. However, this hope is in vain, as Kalton and Lancien [124] showed that among a very large class of Banach spaces, a space X with the property that every negative generator of a holomorphic semigroup on X has maximal regularity, must be isomorphic to a Hilbert space. Around the same time Weis finally succeeded to give a characterisation which more or less settled the question.

Theorem 9.3.13 (Weis). *Let X be a UMD Banach space, and let A be a densely defined, sectorial operator on X with $\omega_A < \pi/2$. Then $A \in \mathrm{MR}$ if and only if the set $\{A(is + A)^{-1} \mid s \neq 0\}$ is R-bounded.*

(See Appendix E.7 for the definition of R-boundedness.) The proof is a combination of Lemma 9.3.10 and Weis' operator-valued Mikhlin multiplier theorem (Theorem E.7.4) together with some simple facts about R-boundedness, cf. [141].

9.3.4 Comments

Basic facts on the inhomogeneous Cauchy problem can be found in practically every book on evolution equations. In [85, Section VI.7.a] one can find a rigorous derivation of Duhamel's principle, including a discussion of regularity. Lunardi [157] extensively treats the Hölder-regularity theory.

The seminal paper for regularity of solutions is [58] by Da Prato and Grisvard who introduced the sum-of-operators point of view. Theorems 9.3.1, 9.3.5, and 9.3.9 are already in their paper. They do not talk explicitly about functional calculus but the basic Cauchy-type integrals are already there. A functional calculus for more than one sectorial operator was introduced for the first time by Albrecht in his 1994 thesis and subsequently studied by Lancien, Lancien and Le Merdy [142] and Albrecht, Franks and McIntosh [4]. (However, injectivity of at least one of the operators was a crucial assumption.) Sums of commuting operators can also be dealt with by a functional calculus for a single operator, but this calculus then has to involve also operator-valued functions, cf. [142], [46]. The material of Chapter 1 can be easily modified to cover this, cf. the comments in Section 1.6. However, the sum operator C is not as easily obtained as by means of the joint functional calculus. Our proof of the Da Prato–Grisvard Theorem 9.3.5 is inspired by [110], cf. also [46].

Maximal $\mathbf{L}^p$-regularity has been an issue so important in the last twenty years that there are numerous texts treating the topic. As an outstanding instruction we recommend Dore's survey [71]. The importance of the concept of maximal regularity lies in the fact that its validity allows to obtain solutions to non-linear equations via the Banach fixed point theorem, see [9, 6.2.10] for the rough structure and [45, 47, 86] for 'real' applications.

Theorem 9.3.11 and its corollary is basically due to Dore and Venni [75], with slight generalisations due to Prüss and Sohr [193]. Its original proof is in the spirit of functional calculus, and in fact MONNIAUX [172] and finally UITERDIJK [218] brought to light the underlying functional calculus ideas, see also [161, Chapter 9]. After important work of CLÉMENT, DE PAGTER, SUKOCHEV and WITVILIET [44] (see Proposition E.7.3) WEIS [223] finally gave the long desired characterisation, a detailed account of which is now in [141]. A discrete approach to Weis' theorem was given by ARENDT and BU [11].

There is also a functional calculus approach to Weis' theorem. Indeed, KALTON and WEIS [125] proved the following result, see also [141, Theorem 12.13].

Theorem 9.3.14 (Kalton–Weis). *Let A, B be two resolvent-commuting, sectorial operators with dense domain and range on a Banach space X. Let $\varphi \in (\omega_A, \pi)$ and $\psi \in (\omega_B, \pi)$ such that $\varphi + \psi < \pi$, and suppose that the following two conditions are satisfied.*

1) *A has a bounded $H^\infty(S_\varphi)$ calculus.*
2) *The set $\{\lambda R(\lambda, B) \mid \lambda \notin \overline{S_\psi}\}$ is R-bounded.*

Then $C = (z + w)(A) = A + B$ and $f(A, B) \in \mathcal{L}(X)$, where $f(z, w) = z/(z + w)$.

One way to prove this theorem is to show that an operator with bounded scalar-valued H^∞-calculus always has a bounded *operator-valued* H^∞-calculus when one restricts to functions with R-bounded range (see [77] for a proof). Theorem 9.3.14 is *not* a generalisation of the Dore–Venni theorem, but it can replace it as long as only maximal regularity is concerned.

Finally, we remark that there are maximal regularity results for non-commuting operators [173, 221, 159]. In view of the prospective applications this seems a promising field for further research. A functional calculus approach in the non-commutative setting — if there is any — is still to be developed.

Appendix A

Linear Operators

This chapter is supposed to be a 'reminder' of some operator theory, including elementary spectral theory and approximation results, rational functional calculus and semigroup theory. There is a slight deviation from the standard literature on operator theory in that we deal with multi-valued operators right from the start.

A.1 The Algebra of Multi-valued Operators

Let X, Y, Z be Banach spaces. A linear **operator** from X to Y is a linear subspace of the direct sum space $X \oplus Y$.

A linear operator may fail to be the graph of a mapping. The subspace

$$A0 := \{x \in X \mid (0, x) \in A\} \subset X$$

is a measure for this failure. If $A0 = 0$, the relation $A \subset X \oplus Y$ is *functional*, i.e., the operator A is the graph of a mapping, and it is called **single-valued**. Since in the main text we deal with single-valued linear operators almost exclusively (not without significant exceptions, of course), we make the following

Agreement: Unless otherwise stated, the term 'operator' always is to be understood as 'single-valued linear operator'. We call an operator **multi-valued** if we wish to stress that it is *not necessarily* single-valued (but it may be).

The **image** of a point x under the multi-valued operator A is the set

$$Ax := \{y \in Y \mid (x, y) \in A\}.$$

This set is either empty (this means, the multi-valued operator A is 'undefined' at x) or it is an affine subspace of Y in the 'direction' of the space $A0$.

With a multi-valued operator $A \subset X \oplus Y$ we associate the spaces

kernel	$\mathcal{N}(A)$	$:=$	$\{x \in X \mid (x, 0) \in A\}$,
domain	$\mathcal{D}(A)$	$:=$	$\{x \in X \mid$ there is $y \in Y$ such that $(x, y) \in A\}$,
range	$\mathcal{R}(A)$	$:=$	$\{y \in Y \mid$ there is $x \in X$ such that $(x, y) \in A\}$.

The multi-valued operator A is called **injective** if $\mathcal{N}(A) = 0$, and **surjective** if $\mathcal{R}(A) = Y$. If $\mathcal{D}(A) = X$, then A is called **fully defined**.

Let $A, B \subset X \oplus Y$ and $C \subset Y \oplus Z$ be multi-valued operators, and let $\lambda \in \mathbb{C}$ be a scalar. We define the **sum** $A + B$, the **scalar multiple** λA, the **inverse** A^{-1}, and the **composite** CA by

$$\begin{aligned} A + B &:= \{(x, y + z) \in X \oplus Y \mid (x, y) \in A,\ (x, z) \in B\}, \\ \lambda A &:= \{(x, \lambda y) \in X \oplus Y \mid (x, y) \in A\}, \\ A^{-1} &:= \{(y, x) \in Y \oplus X \mid (x, y) \in A\}, \\ CA &:= \{(x, z) \in X \oplus Z \mid \exists\, y \in Y : (x, y) \in A \wedge (y, z) \in C\}. \end{aligned}$$

Then we have the following identities:

$$\begin{aligned} \mathcal{D}(A + B) &= \mathcal{D}(A) \cap \mathcal{D}(B), \\ \mathcal{D}(A^{-1}) &= \mathcal{R}(A), \\ \mathcal{D}(\lambda A) &= \mathcal{D}(A), \\ \mathcal{D}(CA) &= \{x \in \mathcal{D}(A) \mid \exists\, y \in \mathcal{D}(C) : (x, y) \in A\}. \end{aligned}$$

The **zero operator** is $0 := \{(x, 0) \mid x \in X\}$ and $I := \{(x, x) \mid x \in X\}$ is the **identity operator**. (This means that in general we have only $0A \subset 0$ but *not* $0A = 0$.) Scalar multiples of the identity operator λI are abbreviated as λ, in particular we write $\lambda + A$ instead of $\lambda I + A$.

Proposition A.1.1. *Let X be a Banach space, and let $A, B, C \subset X \oplus X$ be multi-valued linear operators on X.*

a) *The set of multi-valued operators on X is a* semigroup *with respect to composition, i.e., the* **associative law** *$A(BC) = (AB)C$ holds. The identity operator I is the neutral element in this semigroup. Moreover, the* **inversion law** *$(AB)^{-1} = B^{-1}A^{-1}$ holds.*

b) *The set of multi-valued operators on X is an abelian semigroup with respect to sum, with the zero operator 0 as its neutral element.*

c) *For $\lambda \neq 0$ one has*
$$\lambda A = (\lambda I)A = A(\lambda I).$$

d) *The multi-valued operators A, B, C satisfy the following* **monotonicity laws***:*

$$\begin{aligned} A \subset B &\implies AC \subset BC,\ CA \subset CB, \\ A \subset B &\implies A + C \subset B + C,\ \lambda A \subset \lambda B. \end{aligned}$$

e) *The following* **distributivity inclusions** *hold:*

$$\begin{aligned} (A+B)C &\subset AC+BC, \quad \textit{with equality if } C \textit{ is single-valued;} \\ CA+CB &\subset C(A+B), \quad \textit{with equality if } \mathcal{R}(A) \subset \mathcal{D}(C). \end{aligned}$$

In particular, there is equality in both cases if $C \in \mathcal{L}(X)$ *(see below).*

A multi-valued operator $A \subset X \oplus Y$ is called **closed** if it is closed in the natural topology on $X \oplus Y$. If A is closed, then the multi-valued operators λA (for $\lambda \neq 0$) and A^{-1} are closed as well. Furthermore, the spaces $\mathcal{N}(A)$ and $A0$ are closed. Sum and composition of closed multi-valued operators are *not* necessarily closed.

For each multi-valued operator A one can consider its **closure** $\overline{A}$ in $X \oplus Y$. A *single-valued* operator is called **closable** if $\overline{A}$ is again single-valued.

If a multi-valued operator A is closed, then every subspace $V \subset \mathcal{D}(A)$ such that

$$\overline{\{(x,y) \in A \mid x \in V\}} = A$$

is called a **core** for A.

A single-valued operator A is called **continuous** if there is $c \geq 0$ such that

$$\|Ax\| \leq c\,\|x\| \qquad (x \in \mathcal{D}(A)).$$

Every continuous operator is closable. An operator is called **bounded** if it is continuous and fully defined. We let

$$\mathcal{L}(X,Y) := \{A \subset X \oplus Y \mid A \text{ is a bounded operator}\}$$

be the set of all bounded operators from X to Y. If $X = Y$ we write just $\mathcal{L}(X)$ in place of $\mathcal{L}(X,X)$.

For a single-valued operator A there is a natural norm on $\mathcal{D}(A)$, namely the **graph norm**

$$\|x\|_A := \|x\| + \|Ax\| \qquad (x \in \mathcal{D}(A)).$$

Then A is closed if and only if $(\mathcal{D}(A), \|.\|_A)$ is complete.

Lemma A.1.2. *Let X and Y be Banach spaces. A single-valued operator $A \subset X \oplus Y$ is continuous if and only if $\mathcal{D}(A)$ is a closed subspace of X.*

Proof. The continuity of the operator is equivalent to the fact that the graph norm is equivalent to the original norm. Closedness of the operator is equivalent to the fact that $\mathcal{D}(A)$ is complete with respect to the graph norm. Therefore the assertion follows from the Open Mapping Theorem. $\square$

Lemma A.1.3. *Let A be a closed multi-valued operator on the Banach space X, and let $T \in \mathcal{L}(X)$. Then AT is closed.*

Proof. Let $(x_k, y_k) \in AT$ with $x_k \to x$ and $y_k \to y$. Then there is z_k such that $Tx_k = z_k$ and $(z_k, y_k) \in A$. Since $T \in \mathcal{L}(X)$, one has $z_k = Tx_k \to Tx$. The closedness of A implies that $z \in \mathcal{D}(A)$ and $(z, y) \in A$, hence $(x, z) \in AT$. $\square$

A multi-valued operator $A \subset X \oplus Y$ is called **invertible** if $A^{-1} \in \mathcal{L}(Y, X)$. We denote by

$$\mathcal{L}(X)^\times := \{T \mid T, T^{-1} \in \mathcal{L}(X)\}$$

the set of bounded invertible operators on X.

Lemma A.1.4. *Let A be a closed multi-valued operator on the Banach space X, and let T be an invertible multi-valued operator. Then TA is closed.*

Proof. Suppose that $(x_n, y_n) \in TA$ with $x_n \to x$ and $y_n \to y$. Then there are z_n such that $(x_n, z_n) \in A$ and $(z_n, y_n) \in T$. Since $T^{-1} \in \mathcal{L}(X)$, $z_n = T^{-1}y_n \to T^{-1}y$. The closedness of A implies that $(x, T^{-1}y) \in A$, whence $(x, y) \in TA$. $\square$

Note that the last result in general is false without the assumption that T is invertible.

A.2 Resolvents

In this section A denotes a multi-valued linear operator on the Banach space X. The starting point for the spectral theory is the following lemma.

Lemma A.2.1. *The identity*

$$I - [I + \lambda A^{-1}]^{-1} = \lambda(\lambda + A)^{-1}$$

holds for all $\lambda \in \mathbb{C}$.

Proof. For $\lambda = 0$ the assertion is (almost) trivial. Therefore, let $\lambda \neq 0$. If $x, y \in X$ and $z := (1/\lambda)y$, one obtains

$$\begin{aligned}(x, y) \in \lambda(\lambda + A)^{-1} &\Leftrightarrow (x, z) \in (\lambda + A)^{-1} \Leftrightarrow (z, x) \in (\lambda + A)\\ &\Leftrightarrow (z, x - \lambda z) \in A \Leftrightarrow (x - \lambda z, z) \in A^{-1} \Leftrightarrow (x - \lambda z, \lambda z) \in \lambda A^{-1}\\ &\Leftrightarrow (x - \lambda z, x) \in I + \lambda A^{-1} \Leftrightarrow (x, x - \lambda z) \in [I + \lambda A^{-1}]^{-1}\\ &\Leftrightarrow (x, \lambda z) \in I - [I + \lambda A^{-1}]^{-1} \Leftrightarrow (x, y) \in I - [I + \lambda A^{-1}]^{-1}.\end{aligned}$$ $\square$

We call the mapping

$$\big(\lambda \longmapsto R(\lambda, A) := (\lambda - A)^{-1}\big) : \mathbb{C} \longrightarrow \{\text{multi-valued operators on } X\}$$

the **resolvent** of A. The set

$$\varrho(A) := \{\lambda \in \mathbb{C} \mid R(\lambda, A) \in \mathcal{L}(X)\}$$

is called the **resolvent set**, and $\sigma(A) := \mathbb{C} \setminus \varrho(A)$ is called the **spectrum** of A.

Corollary A.2.2. *For all $\lambda, \mu \in \mathbb{C}$ the identity*

$$I - [I + (\lambda - \mu)R(\mu, A)]^{-1} = (\lambda - \mu)R(\lambda, A)$$

holds true.

Proof. Just replace λ by $(\lambda - \mu)$ and A by $\mu - A$ in Lemma A.2.1. □

Proposition A.2.3. *Let A be a closed multi-valued linear operator on the Banach space X. The resolvent set $\varrho(A)$ is an open subset of $\mathbb{C}$. More precisely, for $\mu \in \varrho(A)$ one has* $\mathrm{dist}(\mu, \sigma(A)) \geq \|R(\mu, A)\|^{-1}$ *and*

$$R(\lambda, A) = \sum_{k=0}^{\infty} (\mu - \lambda)^k R(\mu, A)^{k+1} \quad \textit{for} \quad |\lambda - \mu| < \|R(\mu, A)\|^{-1}.$$

The resolvent mapping $R(., A) : \varrho(A) \longrightarrow \mathcal{L}(X)$ is holomorphic and the **resolvent identity**

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\mu, A)R(\lambda, A)$$

holds for all $\lambda, \mu \in \varrho(A)$.

If $A \in \mathcal{L}(X)$ one has $\emptyset \neq \sigma(A) \subset \{z \in \mathbb{C} \mid |z| \leq \|A\|\}$ and

$$R(\lambda, A) = \sum_{k=0}^{\infty} \lambda^{-(k+1)} A^k \qquad (|\lambda| > \|A\|).$$

Proof. As an abbreviation we write $R(\lambda)$ instead of $R(\lambda, A)$. Take $\mu \in \varrho(A)$ and $\lambda \in \mathbb{C}$ such that $|\lambda - \mu| < \|R(\mu)\|^{-1}$. Then a well-known result from the theory of bounded operators states that $I + (\lambda - \mu)R(\mu)$ is invertible with

$$(I + (\lambda - \mu)R(\mu))^{-1} = \sum_{k \geq 0} (\mu - \lambda)^k R(\mu)^k \in \mathcal{L}(X)$$

being its inverse. Combined with Corollary A.2.2 this gives $R(\lambda) \in \mathcal{L}(X)$ and

$$\begin{aligned} R(\lambda) &= \frac{1}{\lambda - \mu}(I - \sum_{k \geq 0} (\mu - \lambda)^k R(\mu)^k) = \frac{1}{\mu - \lambda} \sum_{k \geq 1} (\mu - \lambda)^k R(\mu)^k \\ &= \sum_{k \geq 0} (\mu - \lambda)^k R(\mu)^{k+1}. \end{aligned}$$

Let $\lambda, \mu \in \varrho(A)$, and let $a \in X$. Set $x := R(\lambda)a$. Then $(x, a) \in (\lambda - A)$ and hence $(x, a + (\mu - \lambda)x) \in (\mu - A)$. This implies that

$$R(\lambda)a = x = R(\mu)(a + (\mu - \lambda)x) = (R(\mu) + (\mu - \lambda)R(\mu)R(\lambda))a.$$

The proofs of the remaining statements are well known. □

If $A \in \mathcal{L}(X)$ we call

$$r_A := r(A) := \inf\{r > 0 \mid \sigma(A) \subset B_r(0)\}$$

the **spectral radius** of A.

Each mapping $R : \Omega \longrightarrow \mathcal{L}(X)$ with $\emptyset \neq \Omega \subset \mathbb{C}$ such that the resolvent identity

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu) \qquad (\lambda,\, \mu \in \Omega)$$

holds, is called a **pseudo-resolvent**. Since we allow multi-valued operators, we obtain the following proposition, which fails to be true for single-valued operators.

Proposition A.2.4. *Let $R : \Omega \longrightarrow \mathcal{L}(X)$ be a pseudo-resolvent. Then there is one and only one multi-valued operator A on X such that $\Omega \subset \varrho(A)$ and $R(\lambda) = R(\lambda, A)$ for all $\lambda \in \Omega$.*

Proof. If $R(\lambda) = R(\lambda, A)$, then $A = \lambda - R(\lambda)^{-1}$. This shows that the operator A is uniquely determined by each single $R(\lambda)$. Thus we define $A_\lambda := \lambda - R(\lambda)^{-1}$. What we have to show is that all the A_λ are equal, i.e., that

$$\lambda + R(\mu)^{-1} = \mu + R(\lambda)^{-1} \qquad (\lambda, \mu \,\in\, \Omega).$$

By interchanging the roles of μ and λ it is clear that we are done as soon as we know one inclusion. Let $(x, y) \in \mu + R(\lambda)^{-1}$. This means that $x = R(\lambda)a$ where $a := y - \mu x$. Then

$$\begin{aligned} R(\mu)(y - \lambda x) &= R(\mu)(a + (\mu - \lambda)x) = R(\mu)(I + (\mu - \lambda)R(\lambda))a \\ &= R(\lambda)a = x. \end{aligned}$$

But this gives $(x, y - \lambda x) \in R(\mu)^{-1}$, whence $(x, y) \in \lambda + R(\mu)^{-1}$. □

Corollary A.2.5. *Let $R_1 : \Omega_1 \longrightarrow \mathcal{L}(X)$ and $R_2 : \Omega_2 \longrightarrow \mathcal{L}(X)$ be pseudo-resolvents. If $R_1(z) = R_2(z)$ for some $z \in \Omega_1 \cap \Omega_2$, then $R_1(z) = R_2(z)$ for all such z.*

Let $T \in \mathcal{L}(X)$. We say that the operator T **commutes** with the multi-valued operator A if

$$(x, y) \in A \quad \Longrightarrow \quad (Tx, Ty) \in A$$

for all $x, y \in X$. This is equivalent to the condition $TA \subset AT$. Obviously, T commutes with A if and only if T commutes with A^{-1}. An immediate consequence of this fact is the following proposition.

Proposition A.2.6. *Let $T \in \mathcal{L}(X)$, and suppose that $\varrho(A) \neq \emptyset$. Then the following assertions are equivalent.*

(i) $[T, R(\lambda, A)] := TR(\lambda, A) - R(\lambda, A)T = 0$ *for some* $\lambda \in \varrho(A)$.

(ii) $[T, R(\lambda, A)] = 0$ *for all* $\lambda \in \varrho(A)$.

(iii) $TA \subset AT$.

Let A, B be closed multi-valued operators, and suppose that $\varrho(A) \neq \emptyset$. We say that B **commutes with the resolvents** of A if B commutes with $R(\lambda, A)$ for *each* $\lambda \in \varrho(A)$. If $\varrho(B) \neq \emptyset$, it is sufficient that this is the case for a single λ. Moreover, it follows that A commutes with the resolvents of B.

One should note that each *single-valued* operator A with $\varrho(A) \neq \emptyset$ commutes with its own resolvents. (This is due to the identity $B^{-1}Bx = x + \mathcal{N}(B)$ for $x \in \mathcal{D}(B)$ which is true for all operators B.)

Proposition A.2.7. *Let A be a multi-valued operator such that $\varrho(A) \neq \emptyset$, and let $T \in \mathcal{L}(X)$ be injective. If T commutes with A, then $T^{-1}AT = A$.*

Proof. Because of $AT \supset TA$ we have $T^{-1}AT \supset T^{-1}TA = A$. The converse inclusion $T^{-1}AT \subset A$ is equivalent to the following statement: If $(Tx, Ty) \in A$, then $(x, y) \in A$, for all $x, y \in X$. Let $\lambda \in \varrho(A)$. From $(Tx, Ty) \in A$ it follows that $(Tx, T(y - \lambda x)) = (Tx, Ty - \lambda Tx) \in (A - \lambda)$. This gives $R(\lambda, A)T(\lambda x - y) = Tx$. Now, T commutes with $R(\lambda, A)$, hence $TR(\lambda, A)(\lambda x - y) = Tx$. By injectivity of T we obtain $R(\lambda, A)(\lambda x - y) = x$, whence $(x, y) \in A$. This completes the proof. $\square$

Let A be a multi-valued operator on the Banach space X, and let Y be another Banach space, continuously embedded in X. We denote by $A_Y := A\big|_Y$ the **part** of A in Y, i.e.,

$$A_Y := A \cap (Y \oplus Y).$$

In the case where A is single-valued this means

$$\mathcal{D}(A_Y) = \{x \in Y \cap \mathcal{D}(A) \mid Ax \in Y\} \quad \text{and} \quad A_Y y = Ay \qquad (y \in \mathcal{D}(A_Y)).$$

Proposition A.2.8. *Let A be any operator on X, and let $Y \subset X$ be another Banach space with continuous inclusion. Then the following assertions hold.*

a) *If A is closed, then also A_Y is closed.*

b) $[A^{-1}]_Y = [A_Y]^{-1}$.

c) $\lambda - A_Y = (\lambda - A)_Y$ *holds for all* $\lambda \in \mathbb{C}$.

d) *If $\lambda \in \varrho(A)$, then the two assertions*

   (i) $\lambda \in \varrho(A_Y)$;

   (ii) *Y is $R(\lambda, A)$-invariant;*

   are equivalent. In this case $R(\lambda, A_Y) = R(\lambda, A)_Y$.

Proof. a) Suppose that A is closed. Let $(y_n, z_n) \in A_Y, (y_n, z_n) \to (y, z)$ within $Y \oplus Y$. Since the inclusion $Y \subset X$ is continuous, $(y_n, z_n) \to (y, z)$ within $X \oplus X$, and since A is closed, it follows that $(y, z) \in A \cap Y \oplus Y = A_Y$. b) and c) are trivial, and d) follows from a), b) and c). $\square$

Corollary A.2.9. *Let X, Y, A be as above. If $\mathcal{D}(A) \subset Y$, then $\varrho(A) \subset \varrho(A_Y)$.*

A.3 The Spectral Mapping Theorem for the Resolvent

We define $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ to be the one-point compactification of $\mathbb{C}$ (**Riemann sphere**) with the usual conventions for computation [1]. For a multi-valued operator A we call the set

$$\tilde{\sigma}(A) := \begin{cases} \sigma(A) & \text{if} \quad A \in \mathcal{L}(X), \\ \sigma(A) \cup \{\infty\} & \text{if} \quad A \notin \mathcal{L}(X), \end{cases}$$

the **extended spectrum** of A.

Proposition A.3.1 (Spectral Mapping Theorem). *Let A be a closed multi-valued operator on the Banach space X. We have*

$$\tilde{\sigma}(\lambda A) = \lambda\tilde{\sigma}(A), \quad \tilde{\sigma}(A+\lambda) = \tilde{\sigma}(A) + \lambda, \quad \textit{and} \quad \tilde{\sigma}(A^{-1}) = [\tilde{\sigma}(A)]^{-1}$$

for all $\lambda \in \mathbb{C}$. In particular, $\tilde{\sigma}(R(\lambda, A)) = [\lambda - \tilde{\sigma}(A)]^{-1}$ for all $\lambda \in \mathbb{C}$.

This follows from Corollary A.3.3 below. Let $\lambda \in \mathbb{C}_\infty$. The **eigenspace** of A at λ is defined as

$$\mathcal{N}(\lambda, A) := \begin{cases} \mathcal{N}(A-\lambda) & \text{for} \quad \lambda \in \mathbb{C}, \\ A0 & \text{for} \quad \lambda = \infty. \end{cases}$$

Similarly, we define the **range space** of A at λ as

$$\mathcal{R}(\lambda, A) := \begin{cases} \mathcal{R}(A-\lambda) & \text{for} \quad \lambda \in \mathbb{C}, \\ \mathcal{D}(A) & \text{for} \quad \lambda = \infty. \end{cases}$$

Using this notation we define the extended **point spectrum**, **approximate point spectrum**, **residual spectrum**, and **surjectivity spectrum** by

$$\begin{aligned}
P\tilde{\sigma}(A) &:= \{\lambda \in \mathbb{C}_\infty \mid \mathcal{N}(\lambda, A) \neq 0\}, \\
A\tilde{\sigma}(A) &:= \{\lambda \in \mathbb{C}_\infty \mid \mathcal{N}(\lambda, A) \neq 0 \text{ or } \mathcal{R}(\lambda, A) \text{ is not closed}\}, \\
R\tilde{\sigma}(A) &:= \{\lambda \in \mathbb{C}_\infty \mid \overline{\mathcal{R}(\lambda, A)} \neq X\}, \\
S\tilde{\sigma}(A) &:= \{\lambda \in \mathbb{C}_\infty \mid \mathcal{R}(\lambda, A) \neq X\}.
\end{aligned}$$

Clearly we have $\tilde{\sigma}(A) = P\tilde{\sigma}(A) \cup S\tilde{\sigma}(A) = A\tilde{\sigma}(A) \cup R\tilde{\sigma}(A)$. The classical spectra are obtained by intersecting the extended spectra with the complex plane, i.e.,

$$P\sigma(A) = \mathbb{C} \cap P\tilde{\sigma}(A), \quad A\sigma(A) = \mathbb{C} \cap A\tilde{\sigma}(A), \quad \dots .$$

Using the extended spectra allows one to think of 'single-valuedness' of an operator as a spectral condition.

[1] such as, e.g., $\infty + \lambda = \infty$, $\lambda \cdot \infty = \infty$ for $0 \neq \lambda \in \mathbb{C}$, and $0 = 0 \cdot \infty = 1/\infty$.

Lemma A.3.2. *Let $\mu \in \mathbb{C}$ and $\lambda \in \mathbb{C}_\infty$. Then*

$$\mathcal{N}(\lambda, \mu - A) = \mathcal{N}(\mu - \lambda, A) \quad \textit{and} \quad \mathcal{N}(\lambda, A^{-1}) = \mathcal{N}(\lambda^{-1}, A), \quad \textit{as well as}$$
$$\mathcal{R}(\lambda, \mu - A) = \mathcal{R}(\mu - \lambda, A) \quad \textit{and} \quad \mathcal{R}(\lambda, A^{-1}) = \mathcal{R}(\lambda^{-1}, A).$$

Proof. We only show the first and the last equality. The other two are proved similarly. Let $y \in X$. In the case where $\lambda = \infty$, one has

$$\begin{aligned} y \in \mathcal{N}(\lambda, \mu - A) &\Leftrightarrow \quad (0, y) \in \mu - A \quad \Leftrightarrow \quad (0, y) \in A \quad \Leftrightarrow \quad y \in \mathcal{N}(\mu - \lambda, A), \\ y \in \mathcal{R}(\lambda, A^{-1}) &\Leftrightarrow \quad \exists x : (y, x) \in A^{-1} \Leftrightarrow \quad \exists x : (x, y) \in A \Leftrightarrow \quad y \in \mathcal{R}(1/\lambda, A), \end{aligned}$$

since $\mu - \lambda = \infty$ and $1/\lambda = 0$. In the case where $\lambda \neq \infty$, we have

$$\begin{aligned} y \in \mathcal{N}(\lambda, \mu - A) &\Leftrightarrow (y, 0) \in (\mu - A) - \lambda \Leftrightarrow (y, 0) \in A - (\mu - \lambda) \\ &\Leftrightarrow y \in \mathcal{N}(\mu - \lambda, A). \end{aligned}$$

Note further that $\mathcal{R}(0, A^{-1}) = \mathcal{R}(A^{-1}) = \mathcal{D}(A) = \mathcal{R}(1/0, A)$. Hence to finish the proof of $R(\lambda, A^{-1}) = R(\lambda^{-1}, A)$ we may suppose that $\lambda \neq 0$. Then

$$\begin{aligned} y \in \mathcal{R}(\lambda, A^{-1}) &\Leftrightarrow \exists x : (x, y) \in A^{-1} - \lambda \Leftrightarrow \exists x : (y + \lambda x, x) \in A \\ &\Leftrightarrow \exists x : (y + \lambda x, -\lambda^{-1} y) \in A - \lambda^{-1} \Leftrightarrow y \in \mathcal{R}(\lambda^{-1}, A). \end{aligned} \qquad \square$$

Corollary A.3.3. *Let $\mu \in \mathbb{C}$. Then*

$$\mathbf{X}\tilde{\sigma}(\mu - A) = \mu - \mathbf{X}\tilde{\sigma}(A) \quad \textit{and} \quad \mathbf{X}\tilde{\sigma}(A^{-1}) = [\mathbf{X}\tilde{\sigma}(A)]^{-1}$$

for $\mathbf{X} \in \{P, A, R, S\}$.

The following is a characterisation of the approximate point spectrum in terms of **approximate eigenvectors**.

Proposition A.3.4. *Let A be a closed multi-valued operator on X, and let $\lambda \in \mathbb{C}$. The following assertions are equivalent.*

(i) $\lambda \in A\sigma(A)$.

(ii) *There is $(x_n, y_n) \in \lambda - A$ such that $\|x_n\| = 1$ for all n and $y_n \to 0$ as $n \to \infty$.*

Proof. Without loss of generality we may suppose that $B := \lambda - A$ is injective. Since B is closed, one can consider the induced (single-valued) operator $\tilde{B} : \mathcal{D}(B) \longrightarrow X/B0$. Clearly B has a closed range if and only if $\tilde{B}$ has closed range. Now, $0 \oplus B0 \subset \tilde{B}$, whence $\tilde{B}$ is a closed operator $\tilde{B} \subset X \oplus (X/B0)$. The inverse $\tilde{B}^{-1} : X/B0 \longrightarrow X$ is therefore a single-valued closed operator. By Lemma A.1.2 this operator is continuous if and only if its domain — which is $\mathcal{R}(\tilde{B})$ — is closed. This means that B has a closed range if and only if there is a constant c such that $\|\tilde{B}x\|_{X/B0} \geq c\,\|x\|$ for all $x \in \mathcal{D}(B)$. From this readily follows the stated equivalence. $\square$

Corollary A.3.5. *Let A be a closed multi-valued operator on the Banach space X. Then $\partial\tilde{\sigma}(A) \subset A\tilde{\sigma}(A)$.*

The boundary $\partial\tilde{\sigma}(A)$ has to be regarded as a boundary in $\mathbb{C}_\infty$.

Proof. Let $\lambda \in \partial\tilde{\sigma}(A)$. Suppose first that $\lambda \in \mathbb{C}$. Then there is a convergent sequence $\lambda_n \to \lambda$ with $\lambda_n \in \varrho(A)$. Since $\tilde{\sigma}(A)$ is closed, $\lambda \in \sigma(A)$. By Proposition A.2.3 one has $\|R(\lambda_n, A)\| \to \infty$. From this follows that there is a sequence $(z_n)_n$ such that $\|z_n\| = 1$ and $\|R(\lambda_n, A)z_n\| \to \infty$. Let $x_n := \|R(\lambda_n, A)z_n\|^{-1} R(\lambda_n, A)z_n$ and $w_n := \|R(\lambda_n, A)z_n\|^{-1} z_n$. Then $\|x_n\| = 1$, $w_n \to 0$, and $(x_n, w_n) \in \lambda_n - A$. Defining $y_n := w_n + (\lambda - \lambda_n)x_n$ we obtain $(x_n, y_n) \in \lambda - A$ and $y_n \to 0$. Hence $\lambda \in A\sigma(A)$, by Proposition A.3.4.

If $\lambda = \infty$, we use the Spectral Mapping Theorem to apply the above to $0 \in \partial\tilde{\sigma}(A^{-1})$. Then we use Corollary A.3.3. □

A.4 Adjoints

We denote the **dual space** of a Banach space X by X'. Note that there is a canonical identification of $(X \oplus X)'$ and $X' \oplus X'$ that we often employ without explicitly mentioning it. The **canonical duality** between X and X' is denoted by

$$\langle \cdot, \cdot \rangle : X \times X' \longrightarrow \mathbb{C}.$$

Given $A \subset X$ and $B \subset X'$ we define

$$\begin{aligned} M^\perp &:= \{x' \in X' \mid \langle x, x' \rangle = 0 \;\forall\, x \in M\}, \\ N^\top &:= \{\, x \in X \;\mid \langle x, x' \rangle = 0 \;\forall\, x' \in N\}. \end{aligned}$$

Identifying X with a closed subspace of X'' we therefore have $N^\perp \cap X = N^\top$.

Let $A \subset X \oplus X$ be a multi-valued operator. The **adjoint** of A — usually denoted by A' — is defined by

$$(x', y') \in A' \quad :\Leftrightarrow \quad \langle v, x' \rangle = \langle u, y' \rangle \text{ for all } (u, v) \in A.$$

If we let $J := [(u, v) \longmapsto (-v, u)] : X \oplus X \longrightarrow X \oplus X$, we may write

$$A' = [JA]^\perp.$$

It is clear from the definition that A' is always a closed operator. The following lemma is an easy consequence of the Hahn–Banach theorem.

Lemma A.4.1. *Let A be a multi-valued operator on the Banach space X. Then*

$$\begin{aligned} \mathcal{D}(A') &= \{x' \in X' \mid x'A \text{ is single-valued and continuous}\}, \\ A'x &= \{y' \in X' \mid y' \supset x'A\} \qquad (x \in \mathcal{D}(A')). \end{aligned}$$

We collect the basic properties of adjoints.

Proposition A.4.2. *Let A, B be multi-valued linear operators on X. Then the following statements hold.*

a) $A' = (\overline{A})'$ *is w^*-closed.*

b) $(A^{-1})' = (A')^{-1}$.

c) $(\lambda A)' = \lambda A'$, *for* $0 \neq \lambda \in \mathbb{C}$.

d) $A'' \cap (X \oplus X) = \overline{A}$, *where we identify $X \oplus X$ canonically with a closed subspace of $X'' \oplus X''$.*

e) $\mathcal{N}(A') = \mathcal{R}(A)^{\perp}$ *and* $\mathcal{N}(\overline{A}) = \mathcal{R}(A')^{\top}$.

f) $A'0 = \mathcal{D}(A)^{\perp}$ *and* $\overline{A}0 = \mathcal{D}(A')^{\top}$.

g) $\mathcal{D}(A') \subset (A0)^{\perp}$ *and* $\mathcal{R}(A') \subset \mathcal{N}(A)^{\perp}$.

h) *If* $A \in \mathcal{L}(X)$, *then* $A' \in \mathcal{L}(X')$ *and* $\|A'\| = \|A\|$.

i) $A \subset B \quad \Rightarrow \quad B' \subset A'$.

j) $A' + B' \subset (A+B)'$ *with equality if* $A \in \mathcal{L}(X)$.

k) $A'B' \subset (BA)'$ *with equality if* $B \in \mathcal{L}(X)$. *If* $A \in \mathcal{L}(X)$ *and B is closed, one has* $\overline{A'B'}^{w^*} = (BA)'$.

Proof. a) Since J is a topological isomorphism, $\overline{A}' = (J\overline{A})^{\perp} = \overline{JA}^{\perp} = (JA)^{\perp}$.
b) This follows from $(JA)^{-1} = J(A^{-1})$ and $(A^{-1})^{\perp} = (A^{\perp})^{-1}$.
c) We have $(x', y') \in (\lambda A') \Leftrightarrow \langle -\lambda v, x\rangle + (u \,|\, y') = 0 \; \forall (u,v) \in A \Leftrightarrow \langle -v, \lambda x'\rangle + \langle u, y'\rangle = 0 \; \forall (u,v) \in A \Leftrightarrow (\lambda x', y') \in A' \Leftrightarrow (\lambda x', \lambda y') \in \lambda A' \Leftrightarrow (x', y') \in A'$.
d) $A'' \cap (X \oplus X) = (J(JA)^{\perp})^{\top} = (JJA)^{\perp\top} = A^{\perp\top} = \overline{A}$.
e) We have $x' \in \mathcal{N}(A') \Leftrightarrow (x', 0) \in A' \Leftrightarrow \langle -v, x'\rangle = 0 \; \forall v \in \mathcal{R}(A) \Leftrightarrow x' \in \mathcal{R}(A)^{\perp}$. Using this and d), we obtain

$$\mathcal{N}(\overline{A}) = \mathcal{N}(A'' \cap (X \oplus X)) = \mathcal{N}(A'') \cap X = \mathcal{R}(A')^{\perp} \cap X = \mathcal{R}(A')^{\top}.$$

f) follows from e) and b).
g) If $(x', y') \in A'$ and $v \in A0$ then $(0, v) \in A$ and $\langle v, x'\rangle = \langle 0, y'\rangle = 0$, whence $x \in (A0)^{\perp}$. The second statement follows from this and b).
h) Let $A \in \mathcal{L}(H)$. Then A^* is closed and single-valued by f). If $x' \in X'$, one easily sees that $(x', x' \circ A) \in A'$, whence $A'x = x' \circ A$. Hence $\mathcal{D}(A') = X'$ and $A' \in \mathcal{L}(X')$. By the Hahn–Banach theorem,

$$\begin{aligned} \|A'\| &= \sup_{\|x'\| \leq 1} \|A'x'\| = \sup_{\|x'\| \leq 1, \, \|x\| \leq 1} |\langle x, A'x'\rangle| = \sup_{\|x'\| \leq 1, \, \|x\| \leq 1} |\langle Ax, x'\rangle| \\ &= \sup_{\|x\| \leq 1} \|Ax\| = \|A\| . \end{aligned}$$

i) Suppose that $A \subset B$. Hence $JA \subset JB$ and this implies that $(JB)^\perp \subset (JA)^\perp$.

j) Let $(x', y') \in A'$, $(x', z') \in B^*$. The generic element of $J(A+B)$ is $(-v-w, u)$, where $(u, v) \in A$ and $(u, w) \in B$. So $(x', y') \perp (-v, u)$ and $(x', z') \perp (-w, u)$, hence $(x', y'+z') \perp (-v-w, u)$. If $A \in \mathcal{L}(X)$, we write $B = (A+B) - A$ and note that $A' \in \mathcal{L}(X')$ by h).

k) Let $(x', y') \in A'B'$. Then there is z' such that $(x', z') \in B'$ and $(z', y') \in A'$. If $(u, v) \in BA$, one has $(u, w) \in A$ and $(w, v) \in B$ for some w. Hence $\langle v, x'\rangle = \langle w, z'\rangle = \langle u, y'\rangle$. Since $(u, v) \in BA$ was arbitrary, we conclude that $(x', y') \in (BA)^*$.

Suppose now that $B \in \mathcal{L}(X)$ and $(x', y') \in (BA)'$. Define $z' := B'x'$. It suffices to show that $(z', y') \in A'$. Take $(u, w) \in A$ and define $v := Bw$. Hence $(u, v) \in BA$. Therefore

$$\langle u, y'\rangle = \langle v, x'\rangle = \langle Bw, x'\rangle = \langle w, B'x'\rangle = \langle w, z'\rangle\,,$$

whence $(z', y') \in A'$.

Finally, suppose that B is closed and $A \in \mathcal{L}(X)$. Since $(BA)'$ is w^*-closed and $A'B' \subset (BA)'$ we have $\overline{A'B'}^{w^*} \subset (BA)'$. To prove the converse inclusion, by the Hahn–Banach theorem it suffices to show $(A'B')^\top \subset (BA)'^\top$. Now, $(BA)'^\top = J(BA)^{\perp\top} = \overline{J(BA)} = J(BA)$ since BA is closed. Let $(u, v) \in (A'B')^\top$, i.e., $\langle u, x'\rangle + \langle v, y'\rangle = 0$ for all $(x', y') \in A'B'$. This is equivalent to $\langle u, x'\rangle + \langle Av, z'\rangle = 0$ $\forall (x', z') \in B'$. This yields $(-Av, u) \in B'' \cap (X \oplus X) = \overline{B} = B$ since B is closed. Hence $J(u, v) \in BA$ and the statement is proved. $\square$

Corollary A.4.3. *Let A be a multi-valued linear operator on X. Then*

$$(\lambda - A)' = (\lambda - A)' \quad \textit{and} \quad R(\lambda, A)' = R(\lambda, A')$$

for every $\lambda \in \mathbb{C}$. For each $\lambda \in \mathbb{C}_\infty$ we have $\mathcal{N}(\lambda, A') = \mathcal{R}(\lambda, A)^\perp$. In particular, $P\tilde{\sigma}(A') = R\tilde{\sigma}(A)$. If A is closed, one has $\varrho(A) = \varrho(A')$.

Let A be a *single-valued* operator on X. From Proposition A.4.2 f) we see that A' is densely defined if and only if A is *closable*, i.e., $\overline{A}$ is still single-valued; and A' is single-valued if and only if A is densely defined.

Remark A.4.4. Using [39, Theorem II.15] one can prove the *Closed Range Theorem* for multi-valued operators in the same way as for single-valued operators, cf. also [52, Theorem III.4.4].

A.5 Convergence of Operators

In this section we consider the following situation: Let $(A_n)_n$ be a sequence of multi-valued operators on X, and let $\lambda_0 \in \mathbb{C}$ such that $\lambda_0 \in \bigcap_n \varrho(A_n)$ for all n. Suppose further that the sequence $(R(\lambda_0, A_n))_n$ converges to a bounded operator

$R_{\lambda_0} \in \mathcal{L}(X)$ (in norm or in the strong sense). We know from Proposition A.2.4 that there is a unique multi-valued operator A with $R_{\lambda_0} = (\lambda_0 - A)^{-1}$. Since $R_{\lambda_0} \in \mathcal{L}(X)$, we have $\lambda_0 \in \varrho(A)$. We are interested in the question, whether there is convergence of the resolvents at other common resolvent points of the operators A_n. To answer this question, we let

$$\Omega := \big\{\lambda \in \mathbb{C} \;\big|\; \exists N \in \mathbb{N} : \lambda \in \varrho(A_n) \;\forall n \geq N, \text{ and } \sup_{n \geq N} \|R(\lambda, A_n)\| < \infty\big\}.$$

It is immediate from the proof of Proposition A.2.3 that Ω is an open subset of $\mathbb{C}$ and that the mapping $n(\lambda) := \min\{n \in \mathbb{N} \mid \lambda \in \varrho(A_k) \;\forall\, k \geq n\}$ defined on Ω locally can only decrease.

Lemma A.5.1. *Let $\lambda \in \mathbb{C}$.*

a) *If $\lambda \in \varrho(A_n)$ for almost all n and if $R(\lambda, A_n)x \to y$, then $(x, y) \in R(\lambda, A)$.*

b) *If $\lambda \in \Omega$ and $(x, y) \in R(\lambda, A)$, then $R(\lambda, A_n)x \to y$.*

Proof. a) Define $Q_n := (\lambda - \lambda_0)R(\lambda_0, A_n)$ and $Q := (\lambda - \lambda_0)R(\lambda_0, A)$. The resolvent identity implies that $(I + Q_n)R(\lambda, A_n) = R(\lambda_0, A_n)$. But $R(\lambda_0, A_n) \to R(\lambda_0, A)$ strongly, hence $Q_n \to Q$ strongly as well. Thus, we have $(I + Q)y = R(\lambda_0, A)x$. This implies readily that $(x, y) \in R(\lambda, A)$.

b) Let $\lambda \in \Omega$ and $(x, y) \in R(\lambda, A)$. By Corollary A.2.2 one has $I - (I + Q_n)^{-1} = (\lambda - \lambda_0)R(\lambda, A_n)$. From $\lambda \in \Omega$ it follows that $(I + Q_n)^{-1} \in \mathcal{L}(X)$ $(n \geq n_0)$ and $\sup_{n \geq n_0} \big\|(I + Q_n)^{-1}\big\| < \infty$ for some $n_0 \in \mathbb{N}$. From $(x, y) \in R(\lambda, A)$ it follows that $(I + Q)y = R(\lambda_0, A)x$. Therefore,

$$\begin{aligned}(I + Q_n)(R(\lambda, A_n)x - y) &= R(\lambda_0, A_n)x - (I + Q_n)y \\ &\to R(\lambda_0, A)x - (I + Q)y = 0.\end{aligned}$$

Applying $(I + Q_n)^{-1}$ yields $R(\lambda, A_n)x - y \to 0$. $\square$

Corollary A.5.2. *For $\lambda \in \Omega$ the multi-valued operator $(\lambda - A)$ is injective and it has a closed range, i.e., $\lambda \notin A\sigma(A)$. In particular, $\sigma(A) \cap \Omega$ is an open subset of $\mathbb{C}$. Furthermore, we have*

$$\Omega \cap \varrho(A) = \{\lambda \mid \lambda \in \varrho(A_n) \text{ almost all } n,\ (R(\lambda, A_n))_n \text{ strongly convergent}\}.$$

Proof. Let $\lambda \in \Omega$ and $K \geq 0$ such that $\|R(\lambda, A_n)\| \leq K$ for almost all n. By the last lemma, if $(x, y) \in R(\lambda, A)$, then $R(\lambda, A_n)x \to y$. Hence $\|y\| \leq K\,\|x\|$. This shows that $R(\lambda, A)$ is single-valued and has a closed domain. From Corollary A.3.5 we know that $\partial\sigma(A) \subset A\sigma(A)$. Hence $\partial\sigma(A) \cap \Omega = \emptyset$. For $\lambda \in \Omega$ we have $\lambda \in \varrho(A)$ if and only if $R(\lambda, A)$ is fully defined, and this is the case if and only if $(R(\lambda, A_n))_n$ is strongly convergent by Lemma A.5.1 $\square$

The interesting question now is whether $\Omega = \varrho(A)$. It holds true for the norm topology, as the following proposition shows.

Proposition A.5.3. *Let A_n, A and Ω be as above. Suppose that $R(\lambda_0, A_n) \to R(\lambda_0, A)$ in* norm *as $n \to \infty$. Then $\Omega = \varrho(A)$ and $R(\lambda, A_n) \to R(\lambda, A)$ in norm for all $\lambda \in \varrho(A)$.*

Proof. Let $\lambda \in \varrho(A)$, and let Q, Q_n be defined as in the proof of Lemma A.5.1. Then $(I+Q)^{-1} \in \mathcal{L}(X)$. Because $(I+Q_n) \to (I+Q)$ in norm, eventually we have $(I+Q_n)^{-1} \in \mathcal{L}(X)$, and $(I+Q_n)^{-1} \to (I+Q)^{-1}$ in norm. This yields $R(\lambda, A_n) \to R(\lambda, A)$ in norm. In particular, $\lambda \in \Omega$.

Conversely, let $\lambda \in \Omega$. Then $T_n := I + Q_n$ is invertible for all large n with $K := \sup_{n \geq n_0} \left\| T_n^{-1} \right\| < \infty$. But $T_n \to T := I + Q$ in norm. Thus

$$\left\| T_n^{-1} - T_m^{-1} \right\| \leq \left\| T_n^{-1} \right\| \left\| T_m - T_n \right\| \left\| T_m^{-1} \right\| \leq K^2 \left\| T_m - T_n \right\|,$$

hence $(T_n^{-1})_n$ is a norm-Cauchy sequence. Obviously, this implies that T is invertible, whence $\lambda \in \varrho(A)$. $\square$

A.6 Polynomials and Rational Functions of an Operator

In this section A denotes a *single-valued* operator on X.

The sequence of **natural powers** $(A^n)_{n \in \mathbb{N}}$ is defined recursively by

$$A^0 := I, \quad A^{n+1} := A^n A \qquad (n \geq 0).$$

A simple induction argument shows the validity of the **law of exponents** $A^{n+m} = A^n A^m$ for all $n, m \in \mathbb{N}$. In particular, $A^{n+1} = A\, A^n$, implying that the sequence of domains $\mathcal{D}(A^n)$ is decreasing, i.e., $\mathcal{D}(A^{n+1}) \subset \mathcal{D}(A^n)$. Another consequence is the inclusion

$$A^n(\mathcal{D}(A^m)) \subset \mathcal{D}(A^{m-n}) \qquad (m \geq n).$$

Let $p(z) = \sum_{k \geq 0} a_k z^k \in \mathbb{C}[z]$ be a **polynomial** and $\deg(p) := \max\{k \mid a_k \neq 0\}$ its **degree**. The operator

$$p(A) := \sum_{k \geq 0} a_k A^k$$

is well defined (by associativity of operator sums) with domain $\mathcal{D}(p(A)) = \mathcal{D}(A^n)$, where $n = \deg(p)$ if $p \neq 0$, and $n = 0$ if $p = 0$. (Note that this is not a definition but a conclusion.)

Lemma A.6.1. *Let $p, q \in \mathbb{C}[z]$. The following statements hold.*

a) *If $p \neq 0$, then $p(A)q(A) = (pq)(A)$. In particular, if $p \neq 0$ and $x \in X$, then*

$$x \in \mathcal{D}(A^{\deg(p)}) \quad \text{and} \quad p(A)x \in \mathcal{D}(A^n) \quad \iff \quad x \in \mathcal{D}(A^{n+\deg(p)}).$$

b) *If $p(A)$ is injective and $q \neq 0$, then*

$$\mathcal{D}(p(A)^{-1}) \cap \mathcal{D}(q(A)) \subset \mathcal{D}(p(A)^{-1}q(A)) \cap \mathcal{D}(q(A)p(A)^{-1})$$

and $p(A)^{-1}q(A)x = q(A)p(A)^{-1}x$ for each $x \in \mathcal{D}(p(A)^{-1}) \cap \mathcal{D}(q(A))$.

c) *One has* $p(A)+q(A) \subset (p+q)(A)$, *and equality holds in the case that* $\deg(p+q) = \max(\deg(p), \deg(q))$.

d) *If* $T \in \mathcal{L}(X)$ *commutes with* A, *then it also commutes with* $p(A)$.

In particular we have $p(A)q(A) = q(A)p(A)$ *for* $p, q \neq 0$, *and* $p(A)$ *commutes with the resolvents of* A.

Proof. We prove a). The assertion is obviously true if p or q are just scalars. Hence by the Fundamental Theorem of Algebra and the associativity of operator multiplication, we can reduce the problem to the case that $\deg(p) = \deg(q) = 1$. This means that we have to establish the identity

$$A^2 - (\mu + \lambda)A + \mu\lambda = (A - \lambda)(A - \mu)$$

for μ, $\lambda \in \mathbb{C}$. Let $y \in X$ be arbitrary. Because $\mathcal{D}(A^2) \subset \mathcal{D}(A)$ we have $y \in \mathcal{D}(A)$ and $(A - \mu)y \in \mathcal{D}(A)$ if and only if $y \in \mathcal{D}(A^2)$. Thus, a) is proved. From a) it follows that $p(A)q(A) = q(A)p(A)$ whenever $p, q \neq 0$. Now, a short argument gives b). The statement in c) is trivial. To prove d) one first shows $TA^n \subset A^nT$ for $n \in \mathbb{N}$ by induction. (Step: Because $TA \subset AT$, one has $TA^{n+1} \subset ATA^n \subset AA^nT = A^{n+1}T$). This yields $Tp(A) \subset p(A)T$ almost immediately. $\square$

Proposition A.6.2. *Let A be an operator on a Banach spaces X such that $\varrho(A) \neq \emptyset$. Then $p(A)$ is a closed operator for each polynomial $p \in \mathbb{C}[z]$. Furthermore, the* **spectral mapping theorem** $\sigma(p(A)) = p(\sigma(A))$ *holds.*

Proof. We prove the first statement by induction on $n := \deg p$. The case $n = 1$ follows from the fact that the norms $\|x\| + \|Ax\|$ and $\|x\| + \|(A - \mu)x\|$ are equivalent norms on $\mathcal{D}(A)$. Now let $n \geq 1$, $\deg p = n + 1$, and $\lambda \in \varrho(A)$. Defining $\mu := p(\lambda)$, there is $r \in \mathbb{C}[z]$ such that $\deg r = n$ and $p = (x - \lambda)r + \mu$. Let $(x_k)_k$ be a sequence in $\mathcal{D}(A^{n+1})$ converging to x (in X) and such that $p(A)x_k \to y$ (in X as well). Then we have $r(A)x_k \to (A - \lambda)^{-1}(y - \mu x)$. By the induction hypothesis, $r(A)$ is closed, hence $x \in \mathcal{D}(A^n)$ with $r(A) = (A - \lambda)^{-1}(y - \mu x) \in \mathcal{D}(A)$. From a) and c) in Lemma A.6.1 it follows that $x \in \mathcal{D}(A^{n+1})$ and $y - \mu x = (A - \lambda)r(A)x = (p - \mu)(A)x = p(A)x - \mu x$.

To prove the spectral mapping theorem it suffices to show that $p(A)$ is invertible if and only if the roots of p are contained in the resolvent set of A. Therefore, let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, and let

$$T := (A - \lambda_1)(A - \lambda_2) \ldots (A - \lambda_n) : \mathcal{D}(A^n) \longrightarrow X$$

be invertible. Then clearly $A - \lambda_1$ is surjective and $A - \lambda_n$ is injective. But all the $A - \lambda_j$ commute with each other, by Lemma A.6.1 a). Hence all $A - \lambda_j$ are bijective. On the other hand, if $\lambda_j \in \varrho(A)$ for all j, then obviously T is invertible. $\square$

We denote by

$$\mathcal{R}_A := \{p/q \mid p, q \in \mathbb{C}[z] \text{ and } \{\lambda : q(\lambda) = 0\} \subset \varrho(A)\}$$

the set of all rational functions having their poles contained in $\varrho(A)$. For a rational function $r = p/q \in \mathcal{R}_A$ we define

$$r(A) := p(A)q(A)^{-1}.$$

(This is independent of the special choice of p and q by Lemma A.6.1.) Note that there is some arbitrariness in this definition. For example, one could have equally defined $r(A) := q(A)^{-1}p(A)$ (this is essentially the same operator but with a smaller domain). With our definition, the domain of $r(A)$ is

$$\mathcal{D}(r(A)) = \mathcal{D}(A^m) \quad \text{where} \quad m = \begin{cases} \deg(p) - \deg(q) & \text{if } \deg(p) \geq \deg(q), \\ 0 & \text{else.} \end{cases}$$

Proposition A.6.3. *Let A be an operator on a Banach space X such that $\varrho(A) \neq \emptyset$. For $0 \neq r = p/q$, $\tilde{r} = \tilde{p}/\tilde{q} \in \mathcal{R}_A$ the following assertions hold.*

a) *$r(A)$ is a closed operator.*

b) *$r(\tilde{\sigma}(A)) \subset \tilde{\sigma}(r(A))$.*

c) *$r(A)\tilde{r}(A) \subset (r\tilde{r})(A)$, and equality holds, e.g., if*

$$(\deg(p) - \deg(q))(\deg(\tilde{p}) - \deg(\tilde{q}) \geq 0.$$

d) *$r(A) + \tilde{r}(A) \subset (r + \tilde{r})(A)$, and equality holds, e.g., if*

$$\deg(p\tilde{q} + \tilde{p}q) = \max\{\deg(p\tilde{q}), \deg(\tilde{p}q)\}.$$

e) *If $T \in \mathcal{L}(X)$ commutes with A, then it commutes also with $r(A)$.*

Proof. Assertion a) is trivial, and assertions c) and d) follow from Lemma A.6.1. To prove b) we note first that $r(A) - \lambda I = [(p - \lambda q)/q](A)$ for $\lambda \in \mathbb{C}$. Hence we are left to show that $r(A)$ is invertible if $\{z : p(z) = 0\} \subset \varrho(A)$. Since $q(A)^{-1}p(A) \subset r(A) = p(A)q(A)^{-1}$, the operator $r(A)$ is invertible if and only if $p(A)$ is. Assertion e) is a consequence of Lemma A.6.1 and of $Tr(A) = Tp(A)q(A)^{-1} \subset p(A)Tq(A)^{-1} = p(A)q(A)^{-1}T = r(A)T$. ☐

Here is an interesting corollary.

Corollary A.6.4. *Let A be a closed operator on a Banach space X. Let the rational function $\tilde{r} = \tilde{p}/\tilde{q} \in \mathcal{R}_A$ be such that* $\deg(\tilde{p}) = \deg(\tilde{q})$. *Then $r(A)\tilde{r}(A) = \tilde{r}(A)r(A)$ for every $r \in \mathcal{R}_A$.*

Proof. Just apply c) of Proposition A.6.3 twice. Another proof rests on the fact that $\tilde{r}$ may be written as a product of operators of the form $\alpha - \beta R(\lambda, A)$ for some numbers $\lambda \in \varrho(A), \alpha, \beta \in \mathbb{C}$ such that $\alpha \neq 0$. For such operators we have

$$x \in \mathcal{D}(A^n) \quad \Longleftrightarrow \quad (\alpha - \beta R(\lambda, A))x \in \mathcal{D}(A^n)$$

for all n and all $x \in X$. ☐

We conclude the section with a result for adjoints.

Proposition A.6.5. *Let A be a densely defined (single-valued) operator on X with $\varrho(A) \neq \emptyset$. For each polynomial $p \in \mathbb{C}[z]$ we have*

$$p(A)' = p(A').$$

The same statement holds if p is a rational function with poles inside $\varrho(A) = \varrho(A')$.

Proof. Let $r = p/q$ be a rational function with poles inside the set $\varrho(A')$. Suppose first that $\deg p \leq \deg q$. Hence $r(A')$ and $r(A)$ are bounded operators. The function r may be written as a product $r = \prod_j r_j$ where each r_j is either of the form $\alpha(\lambda - z)^{-1}$ or of the form $\alpha(\lambda - z)^{-1} + \beta$. Now the claimed formula $[r(A)]' = r(A')$ follows from Proposition A.4.2 k) and Corollary A.4.3. From this and Proposition A.4.2 b) we may infer that $p(A)' = p(A')$ holds for all polynomials p having their roots inside $\varrho(A')$. Now suppose that $\deg p > \deg q$. Since $\varrho(A) \neq \emptyset$ we can find a polynomial q_1 having its roots inside $\varrho(A')$ with $\deg q + \deg q_1 = \deg p$. Define $\tilde{r} := p/(q_1 q)$. Then

$$r(A') = q_1(A')\tilde{r}(A') = q_1(A)'\tilde{r}(A') \overset{(1)}{=} [\tilde{r}(A)q_1(A)]' \overset{(2)}{=} [q_1(A)\tilde{r}(A)]' = [r(A)]',$$

where we have used Proposition A.4.2 k) in (1) and Proposition A.6.3 in (2). $\square$

A.7 Injective Operators

In this section we consider an *injective* single-valued operator A on a Banach space X. The injectivity enables us to extend the insertion mapping $p \longmapsto p(A)$ to the set of polynomials in z and z^{-1}. We begin with a surprising fact.

Lemma A.7.1. *Let A be injective with $\varrho(A) \neq 0$. Then*

$$p(A^{-1})R(\lambda, A) = R(\lambda, A)p(A^{-1})$$

for all $\lambda \in \varrho(A)$ and all polynomials $p \in \mathbb{C}[z]$.

Proof. Since $R(\lambda, A) \in \mathcal{L}(X)$, both distributivity inclusions (see Proposition A.1.1) are actually equalities, hence we can reduce the proof to the case $p(z) = z$. We have $R(\lambda, A)A^{-1} \subset A^{-1}R(\lambda, A)$ since $R(\lambda, A)$ commutes with A, hence with A^{-1}. Let $x \in \mathcal{D}(A^{-1}R(\lambda, A))$. Then $R(\lambda, A)x = Ay$ for some $y \in \mathcal{D}(A)$. But $R(\lambda, A)x \in \mathcal{D}(A)$, whence $y \in \mathcal{D}(A^2)$, and we may apply $\lambda - A$ on both sides to obtain $x = (\lambda - A)Ay = A(\lambda - A)y \in \mathcal{R}(A)$. This implies that $x \in \mathcal{D}(A^{-1}) = \mathcal{D}(R(\lambda, A)A^{-1})$. $\square$

Let $p(z) = \sum_{k \in \mathbb{Z}} a_k z^k \in \mathbb{C}[z, z^{-1}]$ a *polynomial*. The operator

$$p(A) := \sum_{k \in \mathbb{Z}} a_k A^k$$

is well defined (associativity for operator sums and injectivity of A), with domain

$$\mathcal{D}(p(A)) = \mathcal{D}(A^m) \cap \mathcal{R}(A^n),$$

where

$$n = \min\{k \in \mathbb{N}_0 \mid (z^k p(z))|_{z=0} \neq \infty\},$$
$$m = \min\{k \in \mathbb{N}_0 \mid (z^{-k} p(z))|_{z=\infty} \neq \infty\}.$$

One can write $p \in \mathbb{C}[z, z^{-1}]$ in a unique way as

$$p(z) = q(z)z^{-n} \quad n \in \mathbb{Z},\ q \in \mathbb{C}[z],\ q(0) \neq 0.$$

Then we have $p(A) = q(A)A^{-n}$. (This is clear for $n \leq 0$. In the other case only equality of domains is to be shown; this is easy.) Note that a nice product law as in a) of Lemma A.6.1 for polynomials cannot hold in this situation. Simply look at

$$A^{-1}A \subset I \supset AA^{-1},$$

where the inclusions are strict in general. This example also shows that a general commutative law cannot be expected. However, this is not the end of the story.

Lemma A.7.2. *Let $p, q \in \mathbb{C}[z]$ with $q(0) \neq 0$. Then*

$$p(A^{-1})q(A) \subset q(A)p(A^{-1}).$$

Proof. We may suppose without loss of generality that $p(A^{-1}) = A^{-n}$. If $x \in \mathcal{D}(q(A))$ with $q(A)x \in \mathcal{D}(A^{-n})$, then there is $y \in \mathcal{D}(A^n)$ such that $q(A)x = A^n y$. Since $q(0) \neq 0$, this implies that $x \in \mathcal{R}(A)$, say, $x = Ax_1$. But then $Aq(A)x_1 = A^n y$, and this yields $q(A)x_1 = A^{n-1}y$ by injectivity of A. Inductively, it follows that $x \in \mathcal{R}(A^n)$, say, $x = A^n x_0$. Hence we finally arrive at

$$q(A)A^{-n}x = q(A)x_0 = y = A^{-n}q(A)x.$$

This proves the statement. □

The simple example $A^{-1}(1 + A) \neq (1 + A)A^{-1}$ shows that the inclusion in the last lemma is strict in general.

Corollary A.7.3. *Let $p, q \in \mathbb{C}[z]$ with $p(0)q(0) \neq 0$. Then*

$$p(A^{-1})q(A) = (p(z^{-1})q(z))(A) = q(A)p(A^{-1}).$$

Proof. The first equality is immediate from Lemma A.7.2. To prove the second, we may suppose that $\deg(p) = \deg(q) = 1$, employing the Fundamental Theorem of Algebra and Proposition A.6.3. The operator identity $(A^{-1} + \mu)(A + \lambda) = (1+\mu\lambda)+\mu A+\lambda A^{-1}$ then reduces to an almost trivial comparison of domains. □

Proposition A.7.4. *Let A be an injective operator, and let $p, q \in \mathbb{C}[z, z^{-1}]$. Then the following inclusions hold.*

$$p(A)q(A) \subset (pq)(A) \quad \text{and} \quad p(A) + q(A) \subset (p+q)(A).$$

If $T \in \mathcal{L}(X)$ commutes with A, then it also commutes with $p(A)$. If $\varrho(A) \neq \emptyset$, $p(A)$ is a closed operator.

Proof. We leave the proof to the reader. □

A.8 Semigroups and Generators

In this section we review the basic facts of semigroup theory. Our exposition is different from others in that we do not restrict the approach to *strongly continuous* semigroups.

Let X be a Banach space. A **(degenerate) semigroup** on X is a strongly continuous mapping

$$T : (0, \infty) \longrightarrow \mathcal{L}(X)$$

that possesses the **semigroup property**

$$T(t)T(s) = T(t+s) \qquad (t, s > 0).$$

The semigroup T is called **bounded** if

$$\sup_{0<t<\infty} \|T(t)\| < \infty.$$

If the semigroup is just bounded at 0, i.e., if $\sup_{t\leq 1} \|T(t)\| < \infty$, then there are constants $M \geq 1, \omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\omega t}$ for all $t > 0$, cf. [85, Chapter I, Proposition 5.5]. Hence such a semigroup is called **exponentially bounded**. Given an exponentially bounded semigroup T, the number

$$\omega_0(T) := \inf\{\omega \in \mathbb{R} \mid \exists\, M : \|T(t))\| \leq Me^{\omega t} \quad (t > 0)\,\}$$

is called the **growth bound** of T. The semigroup T is said to be **exponentially stable** if $\omega_0(T) < 0$. A semigroup T satisfying $\|T(t)\| \leq 1$ for all $t > 0$ is called **contractive** or a **contraction semigroup**. It is called **quasi-contractive** if there is $\omega \geq 0$ such that the semigroup $e^{\omega \cdot}T(\cdot)$ is contractive.

The space

$$\{x \in X \mid \lim_{t\searrow 0} T(t)x = x\}$$

is called the **space of strong continuity** of the semigroup T. If it is the whole space X, the semigroup is called **strongly continuous** or a **C_0-semigroup**.

Let T be an exponentially bounded semigroup, and choose constants ω, M such that $\|T(t)\| \le Me^{\omega t}$ for $t > 0$. Then the Laplace transform of T exists at least in the half-plane $\{\operatorname{Re}\lambda > \omega\}$, i.e.,

$$\hat{T}(\lambda)x := \int_0^\infty e^{-\lambda t} T(t)x\, dt \qquad (x \in X)$$

defines a bounded operator on X for every λ with $\operatorname{Re}\lambda > \omega$. One can show that in fact $\hat{T}(\cdot)$ is a pseudo-resolvent, cf. [10, p.114]. Hence there is a unique multi-valued operator A such that

$$(\lambda - A)^{-1} = \int_0^\infty e^{-\lambda t} T(t)\, dt.$$

We call A the **generator** of the semigroup T. By the injectivity of the Laplace transform the semigroup is uniquely determined by its generator. The semigroup T is said to be **non-degenerate** if A is single-valued. We obtain

$$A0 = \bigcap_{t>0} \mathcal{N}(T(t)) = \mathcal{N}(R(\lambda, A)) \qquad (\lambda \in \varrho(A)) \tag{A.1}$$

again by the injectivity of the Laplace transform. Obviously, a C_0-semigroup is non-degenerate. If $\mu \in \mathbb{C}$, then $A + \mu$ generates the semigroup $t \longmapsto e^{\mu t}T(t)$. Hence an operator generates an exponentially bounded semigroup if and only if there is $\omega \in \mathbb{R}$ such that $A - \omega$ generates a bounded semigroup.

Proposition A.8.1. *Let T be a semigroup on the Banach space X satisfying an estimate $\|T(t)\| \le Me^{\omega t}$ for all $t > 0$. Then for all $n \in \mathbb{N}$ and all $\operatorname{Re}\lambda > \omega$,*

$$R(\lambda, A)^n = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-\lambda t} T(t)\, dt \quad \textit{and} \tag{A.2}$$

$$\|R(\lambda, A)^n\| \le \frac{M}{(\operatorname{Re}\lambda - \omega)^n}. \tag{A.3}$$

Proof. The proof is the same as in the strongly continuous case, cf. [85, Corollary I.1.11]. ☐

Note that each operator $T(t)$ commutes with the resolvent of A, whence $\mathcal{D}(A)$ is invariant under the semigroup T.

Proposition A.8.2 (Fundamental Identity for Semigroups). *Let T be an exponentially bounded semigroup with generator A on the Banach space X. Define*

$$D_t := \frac{1}{t}(T(t) - I) \quad \textit{and} \quad V_t := \frac{1}{t}\int_0^t T(s)\, ds$$

for $t > 0$. Then

$$(\lambda R(\lambda, A) - I)V_\varepsilon = D_\varepsilon R(\lambda, A) \quad \textit{and} \quad (V_\varepsilon x, D_\varepsilon x) \in A$$

for all $\varepsilon > 0$, $\operatorname{Re}\lambda > \omega_0(T)$, and all $x \in X$.

Proof. We compute

$$\begin{aligned}(\lambda R(\lambda)-I)\int_0^\varepsilon T(s)\,ds &= \int_0^\infty \lambda e^{-\lambda t}T(t)\int_0^\varepsilon T(s)\,ds\,dt - \int_0^\varepsilon T(s)\,ds\\ &= \int_0^\infty \left(\lambda e^{-\lambda t}\int_0^\varepsilon T(t+s)\,ds - \lambda e^{-\lambda t}\int_0^\varepsilon T(s)\,ds\right)dt\\ &= \int_0^\infty \lambda e^{-\lambda t}\left(\int_t^{t+\varepsilon}\cdots - \int_0^\varepsilon \cdots\right)dt\\ &= \int_0^\infty \lambda e^{-\lambda t}\left(\int_\varepsilon^{\varepsilon+t}\cdots - \int_0^t \cdots\right)dt\\ &\overset{I.p.}{=} \int_0^\infty e^{-\lambda t}(T(t+\varepsilon)-T(t))\,dt = (T(\varepsilon)-I)R(\lambda).\end{aligned}$$

Dividing by ε completes the proof of the first statement. Using this we obtain $V_\varepsilon = R(\lambda)(\lambda V_\varepsilon - D_\varepsilon)$. This shows that $(V_\varepsilon x, \lambda V_\varepsilon x - D_\varepsilon x) \in (\lambda - A)$ for every $x \in X$. ☐

Corollary A.8.3. *Let T be an exponentially bounded semigroup on the Banach space X, and let $x \in X$. The following assertions are equivalent.*

(i) $x \in \overline{\mathcal{D}(A)}$.

(ii) $T(t)x \to x$ *as* $t \searrow 0$.

(iii) $\frac{1}{\varepsilon}\int_0^\varepsilon T(s)x\,ds \to x$ *as* $\varepsilon \searrow 0$.

(iv) $\lambda R(\lambda, A)x \to x$ *as* $\lambda \to \infty$.

In particular, $\overline{\mathcal{D}(A)}$ is the space of strong continuity of T. One has the inclusion $T(t)X \subset \overline{\mathcal{D}(A)}$ for each $t > 0$.

Proof. (iv)⇒(i) and (ii)⇒(iii) are obvious.

(iii)⇒(i) follows, since $\mathcal{R}(V_\varepsilon) \subset \mathcal{D}(A)$ by Proposition A.8.2.

(i)⇒(ii). Let $x \in \mathcal{D}(A)$, and pick $y \in (\lambda - A)x$, where $\lambda > \omega_0(T)$. Then

$$T(\varepsilon)x - x = (T(\varepsilon)-I)R(\lambda)y = \varepsilon V_\varepsilon(\lambda R(\lambda)-I)y \to 0$$

as $\varepsilon \to 0$. Now $\sup_{t\le 1}\|T(t)\| < \infty$, and the stated implication follows.

(i)⇒(iv). Note first that $\|\lambda R(\lambda)\|$ is uniformly bounded for $\lambda > \omega$ where $\omega > \omega_0(T)$. This follows from Proposition A.8.1. Given $x \in \mathcal{D}(A)$ we choose $\mu \in \varrho(A)$ and $y \in (\mu - A)x$ to obtain

$$\lambda R(\lambda)x = \lambda R(\lambda)R(\mu)y = \frac{\lambda}{\lambda-\mu}(R(\mu)y - R(\lambda)y) \to R(\mu)y = x$$

as $\lambda \to \infty$. ☐

Corollary A.8.4. *Let $x \in X$. Then we have*

$$x \in \mathcal{D}(A),\ Ax \cap \overline{\mathcal{D}(A)} \neq \emptyset \quad \Leftrightarrow \quad \lim_{t \searrow 0} \frac{1}{t}(T(t)x - x) =: y \quad \textit{exists},$$

and in this case $\{y\} = Ax \cap \overline{\mathcal{D}(A)}$. In particular, $A0 \cap \overline{\mathcal{D}(A)} = 0$, whence the space $X_T := A0 \oplus \overline{\mathcal{D}(A)}$ is a closed subspace of X.

It follows from Corollary A.8.4 that the part $B := A \cap (\overline{\mathcal{D}(A)} \oplus \overline{\mathcal{D}(A)})$ of A in $\overline{\mathcal{D}(A)}$ is single-valued.

Proposition A.8.5. *Let T be an exponentially bounded semigroup with generator A on the Banach space X. Let $Y := \overline{\mathcal{D}(A)}$. The space Y is invariant under the semigroup T. The semigroup T restricts to a strongly continuous semigroup on Y, with generator $B = A \cap (Y \oplus Y)$.*

The following result is one of the cornerstones of the theory of C_0-semigroups. (See [85, Section II.3] or [10, Theorem 3.3.4] for proofs.)

Theorem A.8.6 (Hille–Yosida). *Let A be a (single-valued) linear operator on the Banach space X. Suppose that A has dense domain and there are $M \geq 1$, $\omega \in \mathbb{R}$ such that $(\omega, A) \subset \varrho(A)$ and*

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda - \omega)^n} \tag{A.4}$$

for all $n \in \mathbb{N}$ and all $\lambda > \omega$. Then A generates a C_0-semigroup satisfying the estimate $\|T(t)\| \leq Me^{\omega t}$ for $t \geq 0$.

Remark A.8.7. Unfortunately there is no similar characterisation for generators of general exponentially bounded semigroups. The resolvent condition (A.3) guarantees that A generates a so-called **integrated semigroup** $S(\cdot)$, see [10, Theorem 3.3.1]. For $x \in \overline{\mathcal{D}(A)}$ one has $S(t)x = \int_0^t T(s)x\,ds$, where T is the semigroup generated by the part B of A in $\overline{\mathcal{D}(A)}$. Then A generates an exponentially bounded semigroup if and only if $S(\cdot)x \in C^1((0,\infty), X)$ for each $x \in X$. Employing [7, Theorem 6.2] one can show that this is always true if the Banach space X has the Radon-Nikodym property.

Let us note an important result on bounded perturbations. For a proof see [85, Theorem III.1.3].

Proposition A.8.8. *Let A be the generator of a C_0-semigroup on the Banach space X, and let $B \in \mathcal{L}(X)$ be a bounded operator. Then $A+B$ generates a C_0-semigroup on X.*

Finally we deal with the important case of *groups*.

Proposition A.8.9. *Let T be an exponentially bounded semigroup with generator A. The following assertions are equivalent:*

(i) *There exists $t_0 > 0$ such that $T(t_0)$ is invertible.*

(ii) *Each $T(t)$ is invertible and the mapping $\tilde{T} : \mathbb{R} \to \mathcal{L}(X)^\times$, defined by*

$$\tilde{T}(t) := \begin{cases} T(t) & (t > 0), \\ I & (t = 0), \\ T(-t)^{-1} & (t < 0) \end{cases}$$

is a strongly continuous group homomorphism.

(iii) *The operator $-A$ generates an exponentially bounded semigroup and A is single-valued.*

Proof. (i)⇒(ii)⇒(iii). Let $T(t_0)$ be invertible. Then the semigroup property readily implies that each $T(t)$ is invertible. Moreover, since $\mathcal{R}(T(t)) \subset \overline{\mathcal{D}(A)}$ for each $t > 0$, we have that A is densely defined. Hence A is single-valued and we are in the standard (C_0-) case. So we can refer to [85, Subsection 3.11], or [186, Section 1.6] for the remaining arguments.

(iii)⇒(i). We denote by S the semigroup generated by $-A$. Choose a number $\omega > \max(\omega_0(T), \omega_0(S))$. Let $Y := \overline{\mathcal{D}(A)}$, and let $B := A \cap (Y \oplus Y)$ the part of A in Y. From Proposition A.8.5 we know that B generates the C_0-semigroup obtained by restricting T to Y. Analogously, $-B$ generates the C_0-semigroup obtained by restricting S to Y. (Note that $\mathcal{D}(-A) = \mathcal{D}(A)$.) The theorem from [85, Section 3.11] now yields that $T(t)S(t)y = S(t)T(t)y = y$ for all $y \in Y$. Let $t_0 > 0$, and suppose that $T(t_0)x = 0$ for some $x \in X$. Then

$$R(\lambda, A)x = \int_0^{t_0} e^{-\lambda t} T(t)x \, dt =: f(\lambda)$$

for $\operatorname{Re}\lambda > \omega$. Obviously f has a holomorphic continuation to an entire function which is bounded on every right half-plane.

Claim: $f(\lambda) = R(\lambda, A)$ for all $\operatorname{Re}\lambda < -\omega$.

Proof of Claim. Consider the function $F : \mathbb{C} \to X \oplus X/A$ defined by $F(\lambda) := (f(\lambda), \lambda f(\lambda) - x) + A$. Then F is entire and $F(\lambda) = 0$ for $\operatorname{Re}\lambda > \omega$, since $(f(\lambda), x) \in \lambda - A$ for these λ. By the uniqueness theorem for holomorphic functions, $F(\lambda) = 0$ for *all* $\lambda \in \mathbb{C}$. Hence $(f(\lambda), x) \in \lambda - A$ even for $\operatorname{Re}\lambda < -\omega$. However, these λ are contained in the resolvent set of A, whence the claim is proved.

From the claim we conclude that f is in fact bounded also on some left half-plane, hence it is constant. However, $f(\lambda) \to 0$ as $\operatorname{Re}\lambda \to \infty$. Thus, $f(\lambda) = 0$ for all $\lambda \in \mathbb{C}$. This implies that $R(\lambda, A)x = 0$ for all λ in some left halfplane. Since A is single-valued, $x = 0$.

So we have shown that $T(t_0)$ is injective. We obtain that $T(t_0) : X \longrightarrow Y$ is an isomorphism. But since $T(t_0) : Y \longrightarrow Y$ is also an isomorphism, we must have $X = Y$. $\square$

One usually writes T again instead of $\tilde{T}$ and calls it a **$\mathbf{C_0}$-group**. In general one cannot omit the assumption 'A is single-valued' from (iii). Indeed, let S be a C_0-group on a Banach space Y and let $X := Y \oplus \mathbb{C}$ with $T(t)(y, \lambda) := (S(t), 0)$ for all $t \in \mathbb{R}$. If B generates S, then

$$A = \{((y, 0), (By, \lambda)) \mid y \in \mathcal{D}(B),\ \lambda \in \mathbb{C}\}$$

generates $(T(t))_{t\geq 0}$ and $-A$ generates $(T(-t))_{t\geq 0}$.

Given a C_0-group $(T(t))_{t\in\mathbb{R}}$ we call

$$\theta(T) := \inf\{\theta > 0 \mid \exists\, M \geq 1 : \ \|T(t)\| \leq Me^{\theta|t|} \quad (t \in \mathbb{R})\,\}$$

the **group type** of T. Let us call the semigroups $T_\oplus$ and $T_\ominus$, defined by

$$T_\oplus(t) := T(t) \quad \text{and} \quad T_\ominus(t) := T(-t) \quad (t \geq 0),$$

the **forward semigroup** and the **backward semigroup** corresponding to the group T. Then we obviously have $\theta(T) = \max\{\omega_0(T_\oplus),\, \omega_0(T_\ominus)\}$.

References

The basic results on multi-valued operators, covered by Section A.1 and Section A.2, are contained, e.g., in [89, Chapter I] or the monograph [52]. Proposition A.2.4 and its corollary on pseudo-resolvents is not included in these books and may be new. The same is true for Propositions A.2.7 and parts of Section A.3. An exhaustive treatment of adjoints of multi-valued operators can be found in [52, Chapter III]. The results on convergence in Section A.5 are generalisations of well-known facts for single-valued operators that can be found, e.g., in [130, Theorem IV.2.23 and Chapter VIII,§1]. Polynomials of operators (Section A.6) are studied in [79, Chapter VII.9] including the spectral mapping theorem (Proposition A.6.2). By [52, Theorem VI.5.4], the spectral mapping theorem for polynomials remains valid for multi-valued operators. We provided the results for rational functions and on injective operators (Section A.7) without direct reference, but it is quite likely that these facts have been published somewhere. The results of [52, Chapter 6] let us expect that there is a generalisation of Sections A.6 and A.7 to multi-valued operators. Section A.8 is an adaptation from the standard monographs in semigroup theory, like [186], [85], and [10]. However, if multi-valued operators are involved (as in Proposition A.8.2 and Proposition A.8.9), we do not know of a direct reference. Semigroups in connection with multi-valued operators are studied in [26] and [25].

Appendix B

Interpolation Spaces

In this appendix we present a short survey of basic definitions and results from the interpolation theory of Banach spaces.

B.1 Interpolation Couples

Let X, Y be two normed spaces. A closed linear (single-valued) and injective operator $\iota \subset X \oplus Y$ is called an **(interpolation) coupling** between X and Y. A pair of normed spaces (X, Y) together with an interpolation coupling ι is called an **interpolation couple**. If (X, Y) is an interpolation couple and both X, Y are Banach spaces, then (X, Y) is called a **Banach couple**.

Lemma B.1.1. *Let X, Y be normed spaces. If there is a Hausdorff topological vector space Z and continuous linear injections $\iota_X : X \longrightarrow Z$ and $\iota_Y : Y \longrightarrow Z$, then (X, Y) is an interpolation couple with respect to the coupling ι defined by*

$$(x, y) \in \iota \quad \iff \quad \iota_X(x) = \iota_Y(y) \tag{B.1}$$

for $x \in X, y \in Y$. Conversely, if ι is some coupling between X and Y, then there is a normed space Z and continuous injections $\iota_X : X \longrightarrow Z$ and $\iota_Y : Y \longrightarrow Z$ such that (B.1) *holds.*

Proof. The first assertion is straightforward to prove. Suppose that ι is some coupling between X and Y. Then the operator $-\iota$ is a closed subspace of the normed space $X \oplus Y$. Define the normed space Z as the quotient space $Z := X \oplus Y / -\iota$ and denote by $[x, y]$ the image of (x, y) under the canonical projection of $X \oplus Y$ onto Z. Then let $\iota_X := (x \longmapsto [x, 0]) : X \longrightarrow Z$ and $\iota_Y := (y \longmapsto [0, y]) : Y \longrightarrow Z$. Clearly, ι_X and ι_Y are linear and continuous. Moreover, they are injective. To see this, consider $x \in X$ such that $\iota_X(x) = 0$. This means that $(x, 0) \in (-\iota)$. Since ι is injective, $x = 0$. The injectivity of ι_Y follows similarly from the single-valuedness of ι. Now, for $x \in X, y \in Y$, we have $\iota_X(x) = \iota_Y(y) \iff [x, 0] = [0, y] \iff (x, -y) \in (-\iota) \iff (x, y) \in \iota$. This proves (B.1). $\square$

Given an interpolation couple (X, Y) (with coupling ι) the normed space

$$X + Y := (X \oplus Y)/(-\iota)$$

defined in the proof of Lemma B.1.1 is called the **sum** of the couple (X, Y). Furthermore, the space

$$X \cap Y := \mathcal{D}(\iota)$$

with the norm $\|x\|_X + \|\iota(x)\|_Y$ is called the **intersection** of the couple (X, Y). If the coupling ι is obtained by (B.1) based on imbeddings ι_X, ι_Y into some Hausdorff topological vector space Z, then $X \cap Y$ and $X + Y$ embed canonically in Z.

In fact, the most common situation is when X and Y are already *subspaces* of some space Z with continuous inclusion mappings. Then the coupling is natural and reference to it is often omitted. Also, the spaces $X \cap Y$ and $X + Y$ appear naturally as subspaces of Z. Viewing X and Y as subspaces of $X + Y$, one has

$$\|x\|_{X+Y} = \inf\{\|a\|_X + \|b\|_Y \mid a \in X,\ b \in Y,\ a + b = x\}, \tag{B.2}$$

$$\|z\|_{X\cap Y} = \|z\|_X + \|z\|_Y \tag{B.3}$$

for the norms on $X + Y$ and $X \cap Y$, respectively.

If both X and Y are complete, i.e., if (X, Y) is a Banach couple, then $X + Y$ and $X \cap Y$ are also complete.

The interpolation couples form the objects of a *category* whose morphisms are the pairs (T_1, T_2) of continuous linear mappings

$$T_1 : X \longrightarrow \tilde{X} \quad \text{and} \quad T_2 : Y \longrightarrow \tilde{Y}$$

such that $\tilde{\iota} \circ T_1 = T_2 \circ \iota$. This condition ensures that one can unambiguously define an operator

$$T : X + Y \longrightarrow \tilde{X} + \tilde{Y}$$

with $T|_X = T_1$ and $T|_Y = T_2$. One then has $T \in \mathcal{L}(X + Y, \tilde{X} + \tilde{Y})$ and $T|_{X\cap Y} \in \mathcal{L}(X \cap Y, \tilde{X} \cap \tilde{Y})$. Hence sum and intersection are (covariant) *functors* from the category of interpolation couples into the category of normed spaces. Usually notation is abused a little and one simply writes T instead of T_1, T_2.

Lemma B.1.2. *Let $Y \subset X$, $\tilde{Y} \subset \tilde{X}$, and let $T \in \mathcal{L}(X, \tilde{X})$. If $T(Y) \subset \tilde{Y}$, then $T \in \mathcal{L}(Y, \tilde{Y})$.*

Proof. This is an easy application of the Closed Graph Theorem. □

By Lemma B.1.2, a morphism $T : (X, Y) \longrightarrow (\tilde{X}, \tilde{Y})$ is already characterised by the following properties.

a) $T \in \mathcal{L}(X + Y, X' + Y')$ and

b) $T(X) \subset X'$, $T(Y) \subset Y'$.

A functor $\mathcal{F}$ from the category of interpolation couples into the category of normed spaces is called an **interpolation functor** if the following assertions hold.

1) $X \cap Y \subset \mathcal{F}(X,Y) \subset X + Y$ for all interpolation couples (X,Y).
2) $\mathcal{F}(T) = T|_{\mathcal{F}(X,Y)}$ for every morphism $T : (X,Y) \longrightarrow (\tilde{X},\tilde{Y})$ of interpolation couples.
3) If (X,Y) is a Banach couple, then also $\mathcal{F}(X,Y)$ is a Banach space.

B.2 Real Interpolation by the K-Method

Let (X,Y) be an interpolation couple. We define

$$K(t,x) := K(t,x,X,Y) := \inf\{\|a\|_X + t\,\|b\|_Y \ \mid x = a + b,\ a \in X,\ b \in Y\}$$

for $x \in X + Y$ and $t > 0$.

Lemma B.2.1. *The following assertions hold true.*

a) *For every $t > 0$ the mapping $x \longmapsto K(t,x)$ is an equivalent norm on $X + Y$. More precisely,*
$$\min(t,1)\,\|x\|_{X+Y} \le K(t,x) \le \max(t,1)\,\|x\|_{X+Y}$$
for all $x \in X + Y$.

b) *For fixed $x \in X+Y$ the mapping $t \longmapsto K(t,x)$ is non-decreasing and concave; in particular, it is continuous.*

c) $K(t,x,X,Y) = tK(1/t,x,Y,X)$ *for all $x \in X + Y, t > 0$.*

d) $K(t,x) \le (t/s)K(s,x)$ *for $0 < s < t$, i.e., for fixed x the mapping $t \longmapsto (1/t)K(t,x)$ is non-increasing.*

e) *For $x \in X \cap Y$ one has $K(t,x) \le \min(1,t)\,\|x\|_{X\cap Y}$.*

f) *If $Y \subset X$, then $K(t,x) \le \|x\|_X$ for all $t > 0$, $x \in X$.*

Proof. The proofs are elementary. □

Let us denote by $\mathbf{L}^{\boldsymbol{p}}_{\boldsymbol{*}}(a,b)$ the space of p-integrable functions on the interval $(a,b) \subset (0,\infty)$ with respect to the measure dt/t. We abbreviate the positive real coordinate $(t \mapsto t)$ simply by t. Now we define

$$(X,Y)_{\theta,p} := \{x \in X + Y \mid (t \longmapsto t^{-\theta}K(t,x)) \in \mathbf{L}^{\boldsymbol{p}}_{\boldsymbol{*}}(0,\infty)\}$$

for $\theta \in [0,1]$ and $p \in [1,\infty]$, and endow this space with the norm

$$\|x\|_{\theta,p} = \left\|t^{-\theta}K(t,x)\right\|_{\mathbf{L}^{\boldsymbol{p}}_{\boldsymbol{*}}(0,\infty)}.$$

The space $(X,Y)_{\theta,p}$ is called the **real interpolation space** with parameters θ, p. Here are basic properties.

Proposition B.2.2. *Let (X,Y) be an interpolation couple, $\theta \in [0,1]$, and $p \in [1,\infty]$. Then the following statements hold.*

a) $(X,Y)_{\theta,p} = (Y,X)_{1-\theta,p}$ *with equality of norms.*

b) $(X,Y)_{\theta,p} = 0$ *if $p < \infty$ and $\theta \in \{0,1\}$.*

c) *There is $c = c(\theta,p)$ such that $K(t,x) \le c\,t^\theta \|x\|_{\theta,p}$ for all $t > 0$ and all $x \in (X,Y)_{\theta,p}$. In particular, $(X,Y)_{\theta,p} \subset (X,Y)_{\theta,\infty}$.*

d) *One has $X \cap Y \subset (X,Y)_{\theta,p} \subset X+Y$, except for $p < \infty$ and $\theta \in \{0,1\}$.*

e) *For $x \in X+Y$ the following assertions are equivalent:*
   (i) $x \in (X,Y)_{0,\infty}$*;*
   (ii) *there is $(x_n)_n \subset X$ with $\sup_n \|x_n\|_X < \infty$ and $\|x_n - x\|_{X+Y} \to 0$.*
   In particular, $(X,Y)_{0,\infty} \subset \overline{X}$.

f) *Let $\theta \in (0,1)$ and $x \in (X,Y)_{\theta,\infty}$. Then there are sequences $(x_n)_n \subset X$ and $(y_n)_n \subset Y$ with the following properties:*
   a) $\|x - x_n\|_{X+Y}, \|x - y_n\|_{X+Y} \to 0$ *and*
   b) $\|x_n\|_X = O(n^\theta)$, $\|y_n\|_Y = O(n^{1-\theta})$.
   In particular, $(X,Y)_{\theta,\infty} \subset \overline{X} \cap \overline{Y}$.

Proof. a) follows from the equality $K(t,x,X,Y) = tK(1/t,x,Y,X)$, which holds for all $x \in X+Y$ and $t > 0$.

b) By a) it suffices to consider the case $\theta = 0$. So suppose that $K(.,x) \in \mathbf{L}^p_*$. Because $\|x\|_{X+Y} \le K(t,x)$ for $t \ge 1$, we have $\int_1^\infty \|x\|^p_{X+Y}\,dt/t < \infty$. This implies that $\|x\|_{X+Y} = 0$.

c) In the case where $p = \infty$ one can choose $c = 1$, by definition. Now suppose $1 \le p < \infty$. If $\theta \in \{0,1\}$, then by b) every c does the job. Hence we can suppose $\theta \in (0,1)$ and estimate

$$\begin{aligned} t^{-\theta} K(t,x) &= (\theta p)^{\frac{1}{p}} \left(\int_t^\infty s^{-\theta p} \frac{ds}{s} \right)^{\frac{1}{p}} K(t,x) \le (\theta p)^{\frac{1}{p}} \left(\int_t^\infty s^{-\theta p} K(s,x)^p \frac{ds}{s} \right)^{\frac{1}{p}} \\ &\le (\theta p)^{\frac{1}{p}} \|x\|_{\theta,p} \end{aligned}$$

for $x \in (X,Y)_{\theta,p}$.

d) We let $t = 1$ in c) and obtain $\|x\|_{X+Y} \le c(\theta,p)\|x\|_{\theta,p}$. For $x \in X \cap Y$ we have $K(t,x) \le \min(1,t)\|x\|_{X\cap Y}$. Now the assertion follows from the implication

$$p = \infty \vee \theta \in (0,1) \quad \Longrightarrow \quad t^{-\theta}\min(1,t) \in \mathbf{L}^p_*(0,\infty).$$

e) Suppose first that $x \in (X,Y)_{0,\infty}$, i.e., that there is $c > 0$ such that $K(t,x) < c$ for all $t > 0$. For each $n \in \mathbb{N}$ choose $x = x_n + y_n \in X+Y$ with $\|x_n\|_X + n\|y_n\|_Y < c$. Then $\sup_n \|x_n\|_X < c$ and $\|x - x_n\|_{X+Y} \le \|y\|_Y \le c_n \to 0$. On the other hand, if (ii) holds, then $K(t,x)_n \le \|x_n\|_X$ and $K(t,x_n) \to K(t,x)$, since each $K(t,\cdot)$ is a norm on $X+Y$. Hence (i) follows.

f) is similar to the proof of part e). □

The next result shows in particular that the mapping

$$(X,Y) \longmapsto (X,Y)_{\theta,p}$$

is an interpolation functor whenever $p \in [1,\infty]$, $\theta \in [0,1]$, except for the case where $p<\infty$ and $\theta \in \{0,1\}$.

Theorem B.2.3. *Let (X,Y) and $(\tilde{X},\tilde{Y})$ be interpolation couples. The following assertions hold.*

a) *If (X,Y) is a Banach couple, then $(X,Y)_{\theta,p}$ is a Banach space for all $p \in [1,\infty]$, $\theta \in [0,1]$.*

b) *One has always*
$$(X,Y)_{\theta,p} \subset (X,Y)_{\theta,q}$$
for $1 \le p \le q \le \infty$ and $\theta \in [0,1]$.

c) *Let $T : (X,Y) \longrightarrow (\tilde{X},\tilde{Y})$ be a morphism of interpolation couples. Then it restricts to a bounded linear mapping $T : (X,Y)_{\theta,p} \longrightarrow (\tilde{X},\tilde{Y})_{\theta,p}$ such that*
$$\|T\|_{(X,Y)_{\theta,p}\to(\tilde{X},\tilde{Y})_{\theta,p}} \le \|T\|_{X\to\tilde{X}}^{1-\theta}\,\|T\|_{Y\to\tilde{Y}}^{\theta}$$
for all $p \in [1,\infty]$ and $\theta \in [0,1]$.

Proof. a) is [157, p. 9/10].

b) By Proposition B.2.2 c) we have $(X,Y)_{\theta,p} \subset (X,Y)_{\theta,\infty}$, whence $\|x\|_{\theta,\infty} \le c\,\|x\|_{\theta,p}$ for all $x \in (X,Y)_{\theta,p}$. If $p<q<\infty$, then

$$\begin{aligned}\|x\|_{\theta,q} &= \left(\int_0^\infty t^{-\theta q} K(t,x)^q \,\frac{dt}{t}\right)^{\frac{1}{q}} = \left(\int_0^\infty t^{-\theta p} K(t,x)^p (t^{-\theta}K(t,x))^{(q-p)}\,\frac{dt}{t}\right)^{\frac{1}{q}}\\ &\le \left(\int_0^\infty t^{-\theta p}K(t,x)^p\,\frac{dt}{t}\right)^{\frac{1}{q}} \sup_{t>0}(t^{-\theta}K(t,x))^{\frac{q-p}{q}} = \|x\|_{\theta,p}^{\frac{p}{q}}\,\|x\|_{\theta,\infty}^{\frac{q-p}{q}} \le c\,\|x\|_{\theta,p}\,.\end{aligned}$$

c) is [157, p. 10]. □

Corollary B.2.4. *Let (X,Y) be an interpolation couple, $p \in [1,\infty]$ and $\theta \in (0,1)$. Then there is $c = c(p,\theta)$ such that*

$$\|x\|_{\theta,p} \le c\,\|x\|_X^{1-\theta}\,\|x\|_Y^{\theta} \qquad (x \in X \cap Y).$$

Proof. Take $X=\tilde{X}, Y=\tilde{Y}$ and $T=I$ in Theorem B.2.3 c). □

Remark B.2.5. Let us briefly consider the special case $Y \subset X$. Then clearly (X, Y) is an interpolation couple. Since in this case $K(t, x)$ is bounded, we certainly have

$$t^{-\theta} K(t, x) \in \mathbf{L}^{\boldsymbol{p}}_*(1, \infty)$$

for all $\theta \in (0, 1]$ and all $p \in [1, \infty]$ (and also for $\theta = 0$ and $p = \infty$). Hence one only has to check the condition at 0, and

$$\left\| t^{-\theta} K(t, x) \right\|_{\mathbf{L}^{\boldsymbol{p}}_*(0,1)}$$

is an equivalent norm on $(X, Y)_{\theta,p}$ (in the relevant cases). A similar remark holds in the case that $X \subset Y$, where only the behaviour of $K(t, x)$ at ∞ is relevant, and so $\|t^{-\theta} K(t, x)\|_{\mathbf{L}^{\boldsymbol{p}}_*(1,\infty)}$ is an equivalent norm on $(X, Y)_{\theta,p}$ (in the relevant cases).

Proposition B.2.6. *Let $Y \subset X$, and let $0 < \theta < \sigma < 1$. Then*

$$Y \subset (X, Y)_{1,\infty} \subset (X, Y)_{\sigma,\infty} \subset (X, Y)_{\theta,1} \subset (X, Y)_{0,\infty} = X.$$

Proof. This is [157, p. 9]. □

The following result is often useful.

Proposition B.2.7. *Let (X, Y) be a Banach couple, and let $\theta \in [0, 1], p \in [1, \infty]$. Then the following identities hold.*

$$(X + Y, Y)_{\theta,p} \cap X = (X, Y)_{\theta,p} \cap X = (X, X \cap Y)_{\theta,p}, \tag{B.4}$$
$$(X + Y, X)_{\theta,p} \cap (X + Y, Y)_{\theta,p} = (X + Y, X \cap Y)_{\theta,p}, \tag{B.5}$$
$$(X, X \cap Y)_{\theta,p} + (Y, X \cap Y)_{\theta,p} = (X + Y, X \cap Y)_{\theta,p}, \tag{B.6}$$
$$(X + Y, X)_{1-\theta,p} \cap (X + Y, Y)_{\theta,p} = (X, Y)_{\theta,p}, \tag{B.7}$$
$$(X, X \cap Y)_{\theta,p} + (Y, X \cap Y)_{1-\theta,p} = (X, Y)_{\theta,p}. \tag{B.8}$$

Proof. In (B.4) the chain of inclusions '$\supset$' holds trivially. Let $x \in (X+Y, Y)_{\theta,p} \cap X$. Then $K(t, x, X, X\cap Y) \le \|x\|_X$, and we only have to account for the case $0 < t \le 1$. Let $x = a+b+c$, where $a \in X$ and $b, c \in Y$. Then $b+c = x-a \in X \cap Y$, whence

$$\begin{aligned} K(t, x, X, X \cap Y) &\le \|a\|_X + t\,\|b + c\|_{X\cap Y} = \|a\|_X + t\,\|b + c\|_Y + t\,\|x - a\|_X \\ &\le \|a\|_X + \|b\|_Y + t\,\|x\|_Y + t\,\|x\|_X + \|a\|_X \\ &\le 2\,[\|a\|_X + \|b\|_Y + t\,\|c\|_Y] + t\,\|x\|_X\,. \end{aligned}$$

Taking the infimum first with respect to a, b (with c fixed) and afterwards with respect to c yields $K(t, x, X, X \cap Y) \le 2K(t, x, X + Y, Y) + t\,\|x\|_X$.

For the remaining arguments we are in need of the so-called 'modular law':

$$B \subset C \quad \Longrightarrow (A + B) \cap C = (A \cap C) + B,$$

which holds for all subspaces A, B, C of a common superspace. The two identities (B.5) and (B.6) are proved by using (B.4) and the modular law (several times):

$$\begin{aligned}(X+Y,X\cap Y)_{\theta,p} &\subset (X+Y,X)_{\theta,p} \cap (X+Y,Y)_{\theta,p}\\ &= [(X+Y,X)_{\theta,p}\cap(X+Y)] \cap [(X+Y,Y)_{\theta,p}\cap(X+Y)]\\ &= \{X+[(X+Y,X)_{\theta,p}\cap Y]\} \cap \{Y+[(X+Y,Y)_{\theta,p}\cap X]\}\\ &= \{X+(Y,X\cap Y)_{\theta,p}\} \cap \{Y+(X,X\cap Y)_{\theta,p}\}\\ &= \{X\cap[Y+(X,X\cap Y)_{\theta,p}]\}+(Y,X\cap Y)_{\theta,p}\\ &= (X\cap Y)+(X,X\cap Y)_{\theta,p}+(Y,X\cap Y)_{\theta,p}\\ &= (X,X\cap Y)_{\theta,p}+(Y,X\cap Y)_{\theta,p}\subset(X+Y,X\cap Y)_{\theta,p}.\end{aligned}$$

This kind of proof also works for the last two identities (B.7) and (B.8):

$$\begin{aligned}(X,Y)_{\theta,p} &\subset (X,X+Y)_{\theta,p}\cap(X+Y,Y)_{\theta,p}\\ &= [(X,X+Y)_{\theta,p}\cap(X+Y)]\cap[(X+Y,Y)_{\theta,p}\cap(X+Y)]\\ &= [X+((X,X+Y)_{\theta,p}\cap Y)]\cap[Y+((X+Y,Y)_{\theta,p}\cap X)]\\ &= [X+(X\cap Y,Y)_{\theta,p}]\cap[Y+(X,X\cap Y)_{\theta,p}]\\ &= (X\cap Y,Y)_{\theta,p}+[X\cap(Y+(X,X\cap Y)_{\theta,p})]\\ &= (X\cap Y,Y)_{\theta,p}+[(X\cap Y)_{\theta,p}+(X,X\cap Y)_{\theta,p}]\\ &\subset (X\cap Y,Y)_{\theta,p}+(X,X\cap Y)_{\theta,p}\subset(X,Y)_{\theta,p}.\end{aligned}$$

□

The Reiteration Theorem

Reiteration is one of the most important techniques, making interpolation theory the powerful tool that it is.

Let (X, Y) be a *Banach couple*, and let E be an **intermediate space**, i.e., a space for which the continuous embeddings $X \cap Y \subset E \subset X + Y$ hold. Fix $\theta \in (0, 1)$. We say that E is **of class $\mathbf{J}_{\boldsymbol{\theta}}$** in (X, Y) if there is $c \geq 0$ such that

$$\|x\|_E \leq c\, \|x\|_X^{1-\theta}\, \|x\|_Y^{\theta} \qquad (x \in X \cap Y).$$

We write $E \in J_\theta(X, Y)$ if this holds.

Lemma B.2.8. *Let $X \cap Y \subset E \subset X + Y$, and let $\theta \in [0, 1]$. Then the following assertions are equivalent.*

(i) $E \in J_\theta(X, Y)$.

(ii) $(X, Y)_{\theta,1} \subset E$.

Proof. See [158, p. 28]. □

Note that for $\theta \in \{0,1\}$ the characterisation given in Lemma B.2.8 must fail, since in this case $(X,Y)_{\theta,1} = 0$.

Fix $\theta \in [0,1]$. We say that E is **of class $\mathbf{K}_\theta$** in (X,Y) if there is a $k \geq 0$ such that

$$K(t,x) \leq kt^\theta \left\|x\right\|_E \qquad (x \in E,\, t > 0).$$

One writes $E \in K_\theta(X,Y)$ if this holds. In the case where $0 < \theta < 1$ one has $E \in K_\theta(X,Y)$ if, and only if $E \subset (X,Y)_{\theta,\infty}$.

Let (X,Y) be a Banach couple, and let $X \cap Y \subset E_1, E_2 \subset X + Y$ be two intermediate spaces. Then (E_0, E_1) is another interpolation couple. The reiteration theorem deals with this situation.

Theorem B.2.9 (Reiteration Theorem). *Let (X,Y) be a Banach couple, $0 \leq \theta_0 < \theta_1 \leq 1$, and $\theta \in [0,1]$.*

a) *If $E_i \in K_{\theta_i}(X,Y)$ for $i = 0,1$, then*

$$(E_0,E_1)_{\theta,p} \subset (X,Y)_{\omega,p} \qquad (p \in [1,\infty]).$$

b) *If $E_i \in J_{\theta_i}(X,Y)$ and E_i is complete for $i = 0,1$, then*

$$(X,Y)_{\omega,p} \subset (E_0,E_1)_{\theta,p} \qquad (p \in [1,\infty]).$$

Here, $\omega := (1-\theta)\theta_0 + \theta\theta_1 \in (0,1)$.

A proof can be found in [158], [215] or [29]. It uses at least one method of constructing the real interpolation spaces 'from below', e.g., the so-called *L-method* or the *trace method.*

B.3 Complex Interpolation

The complex method of interpolation is an abstraction of Thorin's proof of the classical Riesz Interpolation Theorem.

Let us denote (for the moment) by S the vertical strip

$$S := \{z \in \mathbb{C} \mid 0 < \operatorname{Re} z < 1\}.$$

We consider vector-valued functions $f : \overline{S} \longrightarrow X$, where X is some Banach space. For $\theta \in [0,1]$ we denote by f_θ the function

$$f_\theta : \mathbb{R} \longrightarrow X, \quad f_\theta(t) := f(\theta + it) \qquad (t \in \mathbb{R}),$$

i.e., the restriction of f to the vertical line $\{\operatorname{Re} z = \theta\}$. The following lemma is at the heart of the whole construction.

Lemma B.3.1 (Three Lines Lemma). *Let X be a Banach space, and let $f : \overline{S} \longrightarrow X$ be continuous, uniformly bounded, and holomorphic on S. Then*

$$\|f(\theta)\|_X \leq \|f_0\|_\infty^{1-\theta} \, \|f_1\|_\infty^\theta \qquad (\theta \in [0,1]).$$

In particular, $\|f\|_{\infty,\overline{S}} \leq \max(\|f_0\|_\infty, \|f_1\|_\infty)$ **(Maximum Modulus Principle)***.*

Proof. See [196, Theorem 12.8]. □

Let (X, Y) be an interpolation couple. We define

$$\mathcal{F}(X,Y) := \{f \in \mathbf{C^b}(\overline{S}; X+Y) \cap \mathcal{O}(S; X+Y) \mid f_0 \in \mathbf{C^b}(\mathbb{R}; X),\ f_1 \in \mathbf{C^b}(\mathbb{R}; Y)\}$$

and a subspace

$$\mathcal{F}_0(X,Y) := \{f \in \mathcal{F}(X,Y) \mid f_0 \in \mathbf{C_0}(\mathbb{R}; X),\ f_1 \in \mathbf{C_0}(\mathbb{R}; Y)\}.$$

On $\mathcal{F}(X,Y)$ a norm(!) is defined by

$$\|f\|_{\mathcal{F}} := \max\{\|f_0\|_{X,\infty} \, \|f_1\|_{Y,\infty}\} \qquad (f \in \mathcal{F}(X,Y)).$$

It is easy to see that $\mathcal{F}(X,Y)$ is a Banach space if (X,Y) is a Banach couple. Furthermore, $\mathcal{F}_0(X,Y)$ is a closed subspace of $\mathcal{F}(X,Y)$. The following technical lemma is quite important.

Lemma B.3.2. *The set*

$$M := \{e^{\delta z^2 + \lambda z} \otimes a \mid \delta > 0,\ \lambda \in \mathbb{R},\ a \in X \cap Y\}$$

is total in $\mathcal{F}_0(X,Y)$.

Proof. This in fact needs an intricate argument a more detailed account of which can be found in [138, Theorem IV.1.1] or [29, Lemma 4.2.3]. We only sketch the proof. Note that $|e^{\delta z^2+\lambda z}| \leq ce^{-\delta|\mathrm{Im}\, z|^2}$ for $c = e^{\delta+|\lambda|}$. Hence the considered set is in fact contained in $\mathcal{F}_0(X,Y)$ and consists of very rapidly decreasing functions. Let $f \in \mathcal{F}_0$ be arbitrary. One has to approximate f by finite sums of elements of M. The proof of this fact consists of several steps.

1) One has $e^{\varepsilon z^2} f \to f$ in $\mathcal{F}_0$ as $\varepsilon \searrow 0$ for each $f \in \mathcal{F}_0$.

2) By Step 1) one may suppose without loss of generality that f decays rapidly. For such f consider $f_n(z) := \sum_{k\in\mathbb{Z}} f(z+i(2\pi kn))$. (The sum converges absolutely, uniformly for z in compacts of $\overline{S}$.) This yields a $2\pi ni$-periodic function $f_n \in \mathcal{F}$.

3) Let $b_{nk}(s)$ be the k-th Fourier coefficient of the $2\pi n$-periodic function $f_n(s+i\,\cdot)$. An intricate argument using Cauchy's theorem shows that $a_{nk}(s) := e^{-sk/n} b_{nk}(s)$ is independent of $s \in (0,1)$. By continuity, $a_{nk}(0) = a_{nk}(1) \in X \cap Y$.

4) Fejér's theorem yields that $f_n(z)$ can be (uniformly in z) approximated by sums of terms of the form $a_{nk}e^{kz/n}$. Multiplying by $e^{\delta z^2}$ shows that each function $e^{\delta z^2} f_n$ is in the closed span of M.

5) Finally, $f_n \to f$ uniformly on compacts, hence with $\delta > 0, n \in \mathbb{N}$ to be chosen appropriately, $\|e^{\delta z^2} f_n - f\|_{\mathcal{F}}$ can be made arbitrarily small. □

We are now in a position to define the **complex interpolation spaces**. Let (X, Y) be an interpolation couple, and let $\theta \in [0, 1]$. We define

$$[X, Y]_\theta := \{f(\theta) \mid f \in \mathcal{F}(X, Y)\}$$

with the norm

$$\|x\|_\theta := \inf\{\|f\|_{\mathcal{F}} \mid f \in \mathcal{F}(X, Y),\ f(\theta) = x\}.$$

In passing to functions of the form $e^{\delta(z-\theta)^2} f$ one can see that in the definition of $[X, Y]_\theta$ one can replace the space $\mathcal{F}$ by the space $\mathcal{F}_0$. We obtain the following theorem.

Theorem B.3.3. *Let (X, Y) and $(\tilde{X}, \tilde{Y})$ be interpolation couples. The following assertions hold.*

a) *The space $[X, Y]_\theta$ is an intermediate space, i.e.,*

$$X \cap Y \subset [X, Y]_\theta \subset X + Y.$$

b) *The space $X \cap Y$ is dense in $[X, Y]_\theta$ for each $\theta \in (0, 1)$.*

c) *If (X, Y) is a Banach couple, then $[X, Y]_\theta$ is a Banach space for all $\theta \in [0, 1]$.*

d) *Let $T : (X, Y) \longrightarrow (\tilde{X}, \tilde{Y})$ be a morphism of interpolation couples. Then it restricts to a bounded linear mapping $T : [X, Y]_\theta \longrightarrow [\tilde{X}, \tilde{Y}]_\theta$ such that*

$$\|T\|_{[X,Y]_\theta \to [\tilde{X},\tilde{Y}]_\theta} \leq \|T\|_{X \to \tilde{X}}^{1-\theta} \|T\|_{Y \to \tilde{Y}}^{\theta}$$

for all $\theta \in (0, 1)$.

Proof. a) The first inclusion is clear by considering constant functions. The second follows from the Maximum Modulus Principle.

b) is immediate from Lemma B.3.2.

c) follows from abstract nonsense, since $[X, Y]_\theta$ can be written as the factor space $\mathcal{F}(X, Y)/\mathcal{N}$, where $\mathcal{N}$ is the kernel of the evaluation mapping $f \longmapsto f(\theta)$.

d) is straightforward. □

Proposition B.3.4. *Let (X, Y) be a Banach couple. Define*

$$\mathcal{V}(X, Y) := \operatorname{span}\{\varphi \otimes x \mid \varphi \in \mathcal{F}_0(\mathbb{C}, \mathbb{C}),\ x \in X \cap Y\}$$

Then

$$\|a\|_{[X,Y]_\theta} = \inf\{\|f\|_{\mathcal{F}(X,Y)} \mid f \in \mathcal{V}(X, Y)\}$$

for any $a \in X \cap Y, \theta \in (0, 1)$.

Proof. We reproduce the proof from [158, Remark 2.1.5]. Take any $g \in \mathcal{F}_0(X,Y)$ such that $f(\theta) = a$. Then define

$$r(z) := \frac{z-\theta}{z+\theta}, \quad \text{and} \quad h(z) := \frac{g(z) - e^{(z-\theta)^2} a}{r(z)}.$$

Then $|r(z)| \leq 1$ for $z \in \mathcal{S}$ with $|r(z)| \to 1$ as $|z| \to \infty$. Hence $h \in \mathcal{F}_0(X,Y)$. By Lemma B.3.2 one can find a function $k \in M$ such that $\|k - h\|_{\mathcal{F}(X,Y)}$ is arbitrarily small. Now define

$$f(z) := e^{(z-\theta)^2} a + r(z)k(z),$$

which is obviously contained in $\mathcal{V}(X,Y)$ and satisfies $f(\theta) = a$. Moreover,

$$\begin{aligned}\|g - f\|_{\mathcal{F}(X,Y)} &= \left\| rh + e^{(z-\theta)^2} a - f \right\|_{\mathcal{F}(X,Y)} = \|r(h-k)\|_{\mathcal{F}(X,Y)} \\ &\leq \|h - k\|_{\mathcal{F}(X,Y)}.\end{aligned}$$

Now the statement follows. □

An important feature of the complex interpolation spaces is their connection with the real interpolation spaces.

Proposition B.3.5. *Let (X,Y) be an interpolation couple, and let $\theta \in [0,1]$. Then $[X,Y]_\theta$ is of the class $K(\theta)$ and of the class $J(\theta)$, i.e., the inclusion*

$$(X,Y)_{\theta,1} \subset [X,Y]_\theta \subset (X,Y)_{\theta,\infty}$$

holds true.

Proof. Choose $X = \tilde{X}, Y = \tilde{Y}$ and $T = I$ in c) of Theorem B.3.3. By definition, this gives $[X,Y]_\theta \in J_\theta(X,Y)$. For the proof of the second inclusion we refer to [158, Proposition 2.1.10]. The key step is, by applying the conformal mapping $e^{\pi i z}$ and the Poisson kernel for the half-plane, to write each $f \in \mathcal{F}(X,Y)$ can be written on the open strip S as a sum $f = f_0 + f_1$, where f_0, f_1 are given by integrals of f on the axes $\operatorname{Re} z = 0$ and $\operatorname{Re} z = 1$, respectively (see [158, Lemma 2.1.9] or [138, Ch. IV.1.2]). □

Let us remark that there is also a reiteration theorem for the complex method, see [29] or [138].

References

Our main references are [157] and [158]. Other expositions are [29], [138] and [215]. We included proofs for the convenience of the reader, omitting those which seem simple or at least straightforward. The Reiteration Theorem B.2.9 is used in the proof of Dore's Theorem 6.1.3, but later we give an alternative proof without employing the Reiteration Theorem (see Corollary 6.5.8). Proposition B.2.7 may be new for some readers, and in fact it is not contained in the mentioned textbooks. However, one can find more results of this kind in [111].

Appendix C

Operator Theory on Hilbert Spaces

This chapter provides some facts on linear operators on Hilbert spaces, including adjoints (of multi-valued operators), numerical range, symmetric and accretive operators, and the Lax–Milgram theorem. The main difference from standard texts lies in the fact that we have avoided employing the Spectral Theorem in dealing with spectral theory of self-adjoint operators (Proposition C.4.3 – Corollary C.4.6).

We take for granted the basic Hilbert space theory as can be found in [49, Chapter I], [194, Chapter II], or [196, Chapter 4]. During the whole chapter the letter H denotes some *complex* Hilbert space. The scalar product on H is denoted by $(\cdot \mid \cdot)$.

C.1 Sesquilinear Forms

Let V be a vector space over the field of complex numbers. We denote by

$$\mathrm{Ses}(V) := \{a \mid a : V \times V \longrightarrow \mathbb{C} \text{ sesquilinear}\}$$

the space of **sesquilinear forms** on V. Given $a \in \mathrm{Ses}(V)$, the **adjoint form** $\overline{a}$ is defined by

$$\overline{a}(u,v) := \overline{a(v,u)} \qquad (u, v \in V).$$

The **real part** and the **imaginary part** of the form $a \in \mathrm{Ses}(V)$ are defined by

$$\operatorname{Re} a := \frac{1}{2}(a + \overline{a}) \quad \text{and} \quad \operatorname{Im} a := \frac{1}{2i}(a - \overline{a}),$$

respectively. Hence $a = (\operatorname{Re} a) + i(\operatorname{Im} a)$ for every $a \in \mathrm{Ses}(V)$. Using the shorthand notation

$$a(u) := a(u,u)$$

for $a \in \mathrm{Ses}(V)$ and $u \in V$, we obtain

$$(\operatorname{Re} a)(u) = \operatorname{Re}(a(u)) \qquad (u \in V).$$

Given $a \in \mathrm{Ses}(V)$ one has the equations

$$\frac{1}{2}(a(u+v) + a(u-v)) = a(u) + a(v), \tag{C.1}$$

$$a(u,v) = \frac{1}{4}(a(u+v) - a(u-v) + i(a(u+iv) - a(u-iv))); \tag{C.2}$$

these are called **parallelogram law** and **polarisation identity**, respectively. (The importance of the polarisation identity lies in the consequence that each sesquilinear form a is already determined by the associated quadratic form $a(\cdot)$.)

A form a is called **real** if $a(u) \in \mathbb{R}$ for all $u \in V$. It is called **symmetric** if $a = \overline{a}$. Note that $\mathrm{Re}\, a$ and $\mathrm{Im}\, a$ are always symmetric forms. A sesquilinear form $a \in \mathrm{Ses}(V)$ is called **positive** (or **monotone**) if $\mathrm{Re}\, a(u) \geq 0$ for all $u \in V$.

Lemma C.1.1. *A form $a \in \mathrm{Ses}(V)$ is real if and only if it is symmetric.*

Proof. Obviously, a symmetric form is real. Let a be a real form. We have

$$a(u,v) + a(v,u) = \frac{1}{2}(a(u+v) - a(u-v)) \in \mathbb{R}$$

for all $u, v \in V$. Replacing u and v by iu and iv, respectively, we obtain also

$$ia(u,v) - ia(v,u) \in \mathbb{R}$$

for all $u, v \in V$. Combining these facts yields $\mathrm{Im}(a(u,v)) = -\mathrm{Im}(a(v,u))$ and $\mathrm{Re}(a(u,v)) = \mathrm{Re}(a(v,u))$. But this is nothing else than $a(u,v) = \overline{a(v,u)}$. ☐

Proposition C.1.2 (Generalised Cauchy–Schwarz Inequality). *Let $a, b \in \mathrm{Ses}(V)$ be symmetric, and let $c \geq 0$ such that*

$$|a(u)| \leq c\, b(u) \qquad (u \in V).$$

(Note that this implies that b is positive.) Then

$$|a(u,v)| \leq c\sqrt{b(u)}\,\sqrt{b(v)}$$

for all $u, v \in V$.

Proof. The simple proof can be found in [199, Chapter XII, Lemma 3.1]. ☐

A positive form a is sometimes called a **semi-scalar product**. It is called a **scalar product** if it is even **positive definite**, i.e., if it is positive and if $a(u) = 0$ implies that $u = 0$ for each $u \in V$. If $a \in \mathrm{Ses}(V)$ is a semi-scalar product on V, then by

$$\|x\|_a := \sqrt{a(u)}$$

a seminorm on V is defined. The form a is continuous with respect to this seminorm. (This follows from the Cauchy–Schwarz inequality.)

Let $\omega \in [0, \pi/2)$. A form $a \in \mathrm{Ses}(V)$ is called **sectorial of angle** ω if

$$a(u) \neq 0 \;\Rightarrow\; |\arg a(u)| \le \omega \qquad (u \in V).$$

The form a is called **sectorial** if it is sectorial of some angle $\omega < \pi/2$. Obviously, if a is sectorial, then $\operatorname{Re} a$ is positive.

Proposition C.1.3. *Let $a \in \mathrm{Ses}(V)$ such that $\operatorname{Re} a \ge 0$. The following assertions are equivalent.*

(i) *The form a is sectorial.*

(ii) *The form a is continuous with respect to the seminorm induced by the semi-scalar product $\operatorname{Re} a$.*

More precisely: If $|a(u,v)| \le M\sqrt{\operatorname{Re} a(u)}\sqrt{\operatorname{Re} a(v)}$ for all $u, v \in V$, then a is sectorial of angle $\arccos M^{-1}$. Conversely, if a is sectorial of angle ω, then

$$|a(u,v)| \le (1 + \tan\omega)\sqrt{\operatorname{Re} a(u)}\sqrt{\operatorname{Re} a(v)}$$

for all $u, v \in V$.

Proof. Suppose that $|a(u,v)| \le M\sqrt{\operatorname{Re} a(u)}\sqrt{\operatorname{Re} a(v)}$ for all $u, v \in V$. Then, letting $u = v$, we have $|a(u)| \le M \operatorname{Re} a(u)$, whence $M \ge 1$ and $a(u) \neq 0 \Rightarrow |\arg a(u)| \le \arccos M^{-1}$. for all $u \in V$.

Conversely, if a is sectorial of angle $\omega < \pi/2$, then $|\operatorname{Im} a(u)| \le (\tan\omega)\operatorname{Re} a(u)$ for all $u \in V$. The generalised Cauchy–Schwarz inequality, applied to $\operatorname{Re} a$ and $\operatorname{Im} a$, yields

$$|a(u,v)| \le |(\operatorname{Re} a)(u,v)| + |(\operatorname{Im} a)(u,v)| \le (1 + \tan\omega)\sqrt{\operatorname{Re} a(u)}\sqrt{\operatorname{Re} a(v)}$$

for all $u, v \in V$. □

C.2 Adjoint Operators

In this section we provide the theory of Hilbert space adjoints of multi-valued linear operators. The results are more or less the same as for Banach space adjoints (cf. Section A.4). We use different notations to distinguish between Banach space adjoints (A') and Hilbert space adjoints (A^*).

Let $A \subset H \oplus H$ be a multi-valued linear operator on H. The **Hilbert space adjoint** of A, usually denoted by A^*, is defined by

$$(x, y) \in A^* \quad :\Longleftrightarrow \quad (\, v \,|\, x \,) = (\, u \,|\, y \,) \text{ for all } (u, v) \in A. \tag{C.3}$$

If for the moment we define $J := ((u,v) \longmapsto (-v, u)) : H \oplus H \longrightarrow H \oplus H$, then we may write

$$A^* = [JA]^{\perp}$$

where the orthogonal complement is taken in the Hilbert space $H \oplus H$. Hence A^* is always a closed operator. We list the basic properties.

Proposition C.2.1. *Let A, B be multi-valued linear operators on H. Then the following statements hold.*

a) $A^* = (\overline{A})^*$.

b) $(A^{-1})^* = (A^*)^{-1}$.

c) $(\lambda A)^* = \overline{\lambda} A^*$, *for* $0 \neq \lambda \in \mathbb{C}$.

d) $A^{**} = \overline{A}$.

e) $\mathcal{N}(A^*) = \mathcal{R}(A)^\perp$ *and* $\mathcal{N}(\overline{A}) = \mathcal{R}(A^*)^\perp$.

f) $A^*0 = \mathcal{D}(A)^\perp$ *and* $\overline{A}0 = \mathcal{D}(A^*)^\perp$.

g) $\mathcal{D}(A^*) \subset (A0)^\perp$ *and* $\mathcal{R}(A^*) \subset \mathcal{N}(A)^\perp$.

h) *If* $A \in \mathcal{L}(H)$, *then* $A^* \in \mathcal{L}(H)$ *and* $\|A^*\| = \|A\| = \sqrt{\|A^*A\|}$.

i) $A \subset B \quad \Rightarrow \quad B^* \subset A^*$.

j) $A^* + B^* \subset (A+B)^*$ *with equality if* $A \in \mathcal{L}(H)$.

k) $A^*B^* \subset (BA)^*$ *with equality if* $B \in \mathcal{L}(H)$. *If* $A \in \mathcal{L}(H)$ *and* B *is closed, one has* $\overline{A^*B^*} = (BA)^*$.

Proof. a) Since J is a topological isomorphism, $\overline{A}^* = (J\overline{A})^\perp = \overline{JA}^\perp = (JA)^\perp$.
b) follows from $(JA)^{-1} = J(A^{-1})$ and $(A^{-1})^\perp = (A^\perp)^{-1}$.
c) We have

$$
\begin{aligned}
(x,y) \in (\lambda A)^* &\Leftrightarrow (-\lambda v \,|\, x) + (u \,|\, y) = 0 \ \forall (u,v) \in A \\
&\Leftrightarrow (-v \,|\, \overline{\lambda} x) + (u \,|\, y) = 0 \ \forall (u,v) \in A \\
&\Leftrightarrow (\overline{\lambda} x, y) \in A^* \ \Leftrightarrow \ (\overline{\lambda} x, \overline{\lambda} y) \in \overline{\lambda} A^* \ \Leftrightarrow \ (x,y) \in A^*.
\end{aligned}
$$

d) $A^{**} = (J(JA)^\perp)^\perp = (JJA)^{\perp\perp} = A^{\perp\perp} = \overline{A}$.
e) We have

$$
x \in \mathcal{N}(A^*) \ \Leftrightarrow \ (x,0) \in A^* \ \Leftrightarrow \ (-v \,|\, x) = 0 \ \forall v \in \mathcal{R}(A) \ \Leftrightarrow \ x \in \mathcal{R}(A)^\perp.
$$

The second satement follows from the first together with d).

f) We have $y \in A^*0 \ \Leftrightarrow \ (0,y) \in A^* \ \Leftrightarrow \ (u \,|\, y) = 0 \ \forall u \in \mathcal{D}(A) \ \Leftrightarrow \ y \in \mathcal{D}(A)^\perp$. The second statement follows from the first together with d).

g) If $(x,y) \in A^*$, $v \in A0$, and $u \in \mathcal{N}(A)$, then $(-v \,|\, x) = 0$ and $(u \,|\, y) = 0$ by (C.3).

h) Let $A \in \mathcal{L}(H)$. Then A^* is closed and single-valued by f). We show that $\mathcal{D}(A^*) = H$. In fact, let $x \in H$. Then $(u \longmapsto (Au \,|\, x))$ is a continuous linear functional on H. By the Riesz–Fréchet theorem [196, Theorem 4.12] there is $y \in H$ such that $(Au \,|\, x) = (u \,|\, y)$ for all $u \in H$. But this means exactly that $(x,y) \in A^*$. The equation $\|A\| = \|A^*\|$ is easily proved by using the identity $\|T\| = \sup\{|(Tu \,|\, v)| \mid \|u\| = \|v\| = 1\}$, which holds for every bounded operator on H. This implies that $\|A^*A\| \le \|A^*\|\,\|A\| = \|A\|^2$. But if $x \in H$ is arbitrary, we have $\|Ax\|^2 = (\,Ax \,|\, Ax\,) = (\,A^*Ax \,|\, x\,) \le \|A^*A\|\,\|x\|^2$ by the Cauchy–Schwarz inequality. Hence $\|A\|^2 \le \|A^*A\|$.

i). We have $A \subset B \Rightarrow JA \subset JB \Rightarrow (JB)^\perp \subset (JA)^\perp$.

j) Let $(x, y) \in A^*$, $(x, z) \in B^*$. The generic element of $J(A + B)$ is $(-v - w, u)$, where $(u, v) \in A$ and $(u, w) \in B$. So $(x, y) \perp (-v, u)$ and $(x, z) \perp (-w, u)$, hence $(x, y + z) \perp (-v - w, u)$.
If $A \in \mathcal{L}(H)$, we write $B = (A + B) - A$ and note that $A^* \in \mathcal{L}(H)$ by h).

k) Let $(x, y) \in A^*B^*$. Then there is z such that $(x, z) \in B^*$ and $(z, y) \in A^*$. If $(u, v) \in BA$, one has $(u, w) \in A$ and $(w, v) \in B$ for some w. Hence $(\,v \,|\, x\,) = (\,w \,|\, z\,) = (\,u \,|\, y\,)$. Since $(u, v) \in BA$ was arbitrary, $(x, y) \in (BA)^*$. Suppose now that $B \in \mathcal{L}(H)$ and $(x, y) \in (BA)^*$. Define $z := B^*x$. It suffices to show that $(z, y) \in A^*$. Take $(u, w) \in A$ and define $v := Bw$. Hence $(u, v) \in BA$. Therefore

$$(\,u \,|\, y\,) = (\,v \,|\, x\,) = (\,Bw \,|\, x\,) = (\,w \,|\, B^*x\,) = (\,w \,|\, z\,)\,,$$

whence $(z, y) \in A^*$ by (C.3). Finally, suppose that B is closed and $A \in \mathcal{L}(H)$. The assertions already proved yield $(A^*B^*)^* = B^{**}A^{**} = (\overline{B})(\overline{A}) = BA$. Hence $\overline{A^*B^*} = (BA)^*$. ☐

Corollary C.2.2. *Let A be a multi-valued linear operator on H. Then*

$$(\lambda - A)^* = (\overline{\lambda} - A^*) \quad \textit{and} \quad R(\lambda, A)^* = R(\overline{\lambda}, A^*)$$

for every $\lambda \in \mathbb{C}$. In particular, we have $\varrho(A^*) = \{\overline{\lambda} \mid \lambda \in \varrho(A)\} = \overline{\varrho(A)}$.

Let A be a *single-valued* operator on H. From Proposition C.2.1 f) we see that A^* is densely defined if and only if A is *closable*, i.e., $\overline{A}$ is still single-valued; and A^* is single-valued if and only if A is densely defined.

Proposition C.2.3. *Let A be a densely defined (single-valued) operator on H with $\varrho(A) \neq \emptyset$. For a polynomial $p \in \mathbb{C}[z]$ define $p^*(z) := \overline{p(\overline{z})}$. (Hence p^* is obtained from p by conjugating all coefficients.) Then we have*

$$[p^*(A)]^* = p(A^*).$$

The same statement holds if p is a rational function with poles inside $\varrho(A^*)$.

Proof. Let $r = p/q$ be a rational function with poles inside the set $\varrho(A^*)$. Suppose first that $\deg p \le \deg q$. Hence $r(A^*)$ and $r^*(A)$ are bounded operators. The function r may be written as a product $r = \prod_j r_j$ where each r_j is either of the form $\alpha(\lambda - z)^{-1}$ or of the form $\alpha(\lambda - z)^{-1} + \beta$. Now the claimed formula $[r^*(A)]^* = r(A^*)$ follows from Proposition C.2.1 k) and Corollary C.2.2. From this and Proposition C.2.1 b) we can infer that $p^*(A)^* = p(A^*)$ holds for all polynomials p having their roots inside $\varrho(A^*)$. Now suppose that $\deg p > \deg q$.

Since $\varrho(A) \neq \emptyset$, we can find a polynomial q_1 having its roots inside $\varrho(A^*)$ with $\deg q + \deg q_1 = \deg p$. Define $\tilde{r} := p/(q_1 q)$. Then

$$\begin{aligned} r(A^*) &= q_1(A^*)\tilde{r}(A^*) = [q_1^*(A)]^*[\tilde{r}^*(A)]^* \overset{(1)}{=} [\tilde{r}^*(A)q_1^*(A)]^* \\ &\overset{(2)}{=} [q_1^*(A)\tilde{r}^*(A)]^* = [r^*(A)]^*, \end{aligned}$$

where we have used Proposition C.2.1 k) in (1) and Proposition A.6.3 in (2). □

C.3 The Numerical Range

From now on, all considered operators are single-valued (cf. the terminological agreement on page 279.)

Given an operator A on H we call

$$W(A) := \{(\, Au \,|\, u\,) \;\mid u \in \mathcal{D}(A),\; \|u\| = 1\} \subset \mathbb{C}$$

the **numerical range** of A. By [199, Chapter XII, Theorem 5.2] the numerical range $W(A)$ is always a convex subset of the plane.

Proposition C.3.1. *Let A be a closed operator on H. Then $P\sigma(A) \subset W(A)$ and $A\sigma(A) \subset \overline{W(A)}$. Furthermore, one has*

$$\|R(\lambda, A)\| \leq \frac{1}{\operatorname{dist}(\lambda, \overline{W(A)})}$$

for $\lambda \in \varrho(A) \setminus \overline{W(A)}$. If $A \in \mathcal{L}(H)$, we have $\sigma(A) \subset \overline{W(A)}$.

Proof. If $\lambda \in P\sigma(A)$, there is $u \in \mathcal{D}(A)$, $\|u\| = 1$, such that $Au = \lambda u$. This yields $(\, Au \,|\, u\,) = (\, \lambda u \,|\, u\,) = \lambda \|u\|^2 = \lambda$. Hence $\lambda \in W(A)$.

Suppose that $\lambda \notin \overline{W(A)}$ and define $\delta := \operatorname{dist}(\lambda, \overline{W(A)})$. By definition, one has $|(\, Ax \,|\, x\,) - \lambda| \geq \delta$ for all $x \in \mathcal{D}(A)$ with $\|x\| = 1$. Hence

$$|(\,(A - \lambda)\, x \,|\, x\,)| = \left|(\, Ax \,|\, x\,) - \lambda \|x\|^2\right| \geq \delta \|x\|^2$$

for *all* $x \in \mathcal{D}(A)$. But $|(\,(A-\lambda)x \,|\, x\,)| \leq \|(A-\lambda)x\| \, \|x\|$, whence

$$\|(\lambda - A)\, x\| \geq \delta \|x\|$$

for all $x \in \mathcal{D}(A)$. Since A is closed, this implies that $(\lambda - A)$ is injective and has closed range, i.e., $\lambda \notin A\sigma(A)$. Moreover, it shows that $\|R(\lambda, A)\| \leq \delta^{-1}$ if $\lambda \in \varrho(A)$.

If $A \in \mathcal{L}(X)$, then $W(A^*) = \{\overline{\lambda} \mid \lambda \in W(A)\}$. Now, if $\lambda \in R\sigma(A)$, then clearly $\overline{\lambda} \in P\sigma(A^*)$, whence $\lambda \in W(A)$. □

Corollary C.3.2. *Let A be a single-valued, closed operator on H, and let the set $U \subset \mathbb{C} \setminus \overline{W(A)}$ be open and connected. If $U \cap \varrho(A) \neq 0$, then $U \subset \varrho(A)$.*

Proof. The statement follows from Proposition C.3.1 and the fact that $\|R(\lambda, A)\|$ blows up as λ approaches a spectral value (cf. Proposition A.2.3). □

C.4 Symmetric Operators

We begin with a lemma.

Lemma C.4.1. *For an operator A on H the following assertions are equivalent:*

(i) $W(A) \subset \mathbb{R}$.

(ii) $(\, Au \,|\, v\,) = (\, u \,|\, Av\,)$ *for all* $u, v \in \mathcal{D}(A)$.

(iii) $A \subset A^*$.

If this is the case, the operator A is called **symmetric**.

Proof. Define the form a on $V := \mathcal{D}(A)$ by $a(u, v) := (\, Au \,|\, v\,)$. The proof is now an easy consequence of Lemma C.1.1 and the definition of the adjoint (see (C.3)). □

An operator A on H is called **self-adjoint** if $A^* = A$. If A is symmetric/self-adjoint and injective, then A^{-1} is symmetric/self-adjoint, by Proposition C.2.1 b).

Proposition C.4.2. *Let A be an operator on H. Then A is self-adjoint if and only if A is symmetric, closed, and densely defined, and $\mathcal{R}(A \pm i)$ is dense. In this case we have $\sigma(A) \subset \mathbb{R}$ and*

$$\|R(\lambda, A)\| \leq \frac{1}{|\mathrm{Im}\, \lambda|}$$

for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Proof. Suppose that $A = A^*$. Then A is closed since A^* is. Moreover, A^* is single-valued since A is, and this implies that A is densely defined, by Proposition C.2.1 f). Since $A = A^*$ we have $W(A^*) = W(A) \subset \mathbb{R}$. Hence $\mathcal{N}(A^* \pm i) = 0$ by Proposition C.3.1. This yields $\overline{\mathcal{R}(A \pm i)} = H$ by Proposition C.2.1 e).

Now suppose that A is symmetric, closed, and densely defined with $\mathcal{R}(A \pm i)$ being dense. By Proposition C.3.1 and its corollary we conclude that $\sigma(A) \subset \mathbb{R}$ and $\mathcal{N}(A^* \pm i) = 0$. But $A - i : \mathcal{D}(A) \longrightarrow H$ is bijective and $A - i \subset A^* - i$, whence $A - i = A^* - i$. This proves $A = A^*$. The norm inequality for the resolvent follows from Proposition C.3.1. □

Let A be an operator on H, and let $\alpha \in \mathbb{R}$. We write $\alpha \leq A$ if A is self-adjoint and $W(A) \subset [\alpha, \infty)$, and we write $A \leq \alpha$ if $-\alpha \leq -A$. The operator A is called **positive** if $0 \leq A$. We obtain the following characterisation.

Proposition C.4.3. *A closed and densely defined operator A on H is positive if and only if $W(A) \subset [0, \infty)$ and $A + \lambda$ is surjective for some/each $\lambda > 0$. In this case,* $\{\mathrm{Re}\, \lambda < 0\} \subset \varrho(A)$ *and*

$$\left\|(\lambda + A)^{-1}\right\| \leq \frac{1}{\mathrm{Re}\, \lambda}$$

for all $\mathrm{Re}\, \lambda > 0$. *Moreover,* $0 \leq t(t + A)^{-1} \leq 1$ *and* $0 \leq A(t + A)^{-1} \leq 1$ *for all* $t > 0$.

Proof. Let A be a single-valued, closed, and densely defined operator such that $W(A) \subset [0, \infty)$. Since $\mathbb{C} \setminus [0, \infty)$ is open and connected, the stated equivalence is a consequence of Proposition C.3.1 and its corollary. Moreover it follows that in this case $\{\operatorname{Re} \lambda < 0\} \subset \varrho(A)$, and the stated norm inequality holds. Since $t + A$ is self-adjoint, we know that $(t+A)^{-1}$ also is. Furthermore $0 \leq A(t+A)^{-1} \leq 1 \quad \Leftrightarrow$ $0 \leq t(t+A)^{-1} \leq 1 \quad \Leftrightarrow \quad (tu \,|\, (t+A)u) \leq ((t+A)u \,|\, (t+A)u) \;\; \forall u \in \mathcal{D}(A)$. However, this is true if and only if $\|Au\|^2 + t\,(Au \,|\, u) \geq 0$, which is always the case. □

Lemma C.4.4. *Let A be a closed and densely defined operator on H.*

a) *If $-\alpha \leq A \leq \alpha$ for some $\alpha \geq 0$, then $A \in \mathcal{L}(H)$ and $\|A\| \leq \alpha$.*

b) *If $A \in \mathcal{L}(H)$ is self-adjoint, then $\|A\| = \sup\{|\lambda| \mid \lambda \in W(A)\}$. In particular, $W(A)$ is a bounded subset of $\mathbb{R}$.*

c) *If $0 \leq A \leq 1$, then $0 \leq A^2 \leq A$.*

Proof. a) Define $a(u, v) := (Au \,|\, v)$ and $b(u, v) := (u \,|\, v)$ on $V := \mathcal{D}(A)$. The hypothesis implies that $|a(u)| \leq \alpha b(u)$ for all $u \in H$. Moreover, a is symmetric. An application of the generalised Cauchy–Schwarz inequality (Proposition C.1.2) yields $|(Au \,|\, v)| \leq \alpha \|u\| \|v\|$ for all $u, v \in \mathcal{D}(A)$. Since A is densely defined, this inequality holds for all $v \in H$, whence we have $\|Au\| \leq \alpha \|u\|$ for all $u \in \mathcal{D}(A)$. Since A is closed, this implies that $\mathcal{D}(A)$ is closed. Hence $A \in \mathcal{L}(H)$ and $\|A\| \leq \alpha$.

b) Let $A \in \mathcal{L}(H)$ be self-adjoint. Then $|(Au \,|\, u)| \leq \|A\| \|u\|^2$ for all $u \in H$. Hence $\sup |W(A)| \leq \|A\|$. If $\sup |W(A)| \leq \alpha$, then $|(Au \,|\, u)| \leq \alpha \|u\|^2$ for all u. Since $W(A) \subset \mathbb{R}$, this is equivalent to $-\alpha \leq A \leq \alpha$. Now a) implies that $\|A\| \leq \alpha$.

c) We have $A - A^2 = A^2(1 - A) + A(1 - A)^2$, and both summands are positive since A and $1 - A$ are. □

The following result is usually proved with the help of the spectral theorem.

Proposition C.4.5. *Let $A \geq \alpha$ for some $\alpha \in \mathbb{R}$, and define $\alpha_0 := \inf W(A)$. Then $\alpha_0 \in \sigma(A)$. If $\alpha_0 \in W(A)$, then even $\alpha_0 \in P\sigma(A)$.*

Proof. Without loss of generality we may suppose that $\alpha_0 = 0$. Hence $A \geq 0$. Let $Q := A(1 + A)^{-1}$. Then $0 \leq Q \leq 1$. Moreover, $Q \leq A$ in the obvious sense since for $x \in \mathcal{D}(A)$ we have

$$\begin{aligned}\big(A(A+1)^{-1}x \,\big|\, x\big) - (Ax \,|\, x) &= \big(A((A+1)^{-1} - I)x \,\big|\, x\big)\\ &= -\big(A(A+1)^{-1}Ax \,\big|\, x\big) = -\big((A+1)^{-1}Ax \,\big|\, Ax\big) \leq 0.\end{aligned}$$

Applying c) of Lemma C.4.4 we obtain

$$0 \leq \|Qx\|^2 \leq (Qx \,|\, x) \leq (Ax \,|\, x)$$

for all $x \in \mathcal{D}(A)$. Now if there is $x \in \mathcal{D}(A)$ such that $\|x\| = 1$ and $(Ax \,|\, x) = 0$ it follows that $0 = Qx = A(A+1)^{-1}x$. But this implies that $0 \neq (A+1)^{-1}x \in \mathcal{N}(A)$, whence $0 \in P\sigma(A)$. Similarly, if $x_n \in \mathcal{D}(A)$ such that $\|x_n\| = 1$ and $(Ax_n \,|\, x_n) \to 0$, then $Qx_n \to 0$. Hence $(x_n)_n$ is a generalised eigenvector for Q and $0 \in \sigma(Q)$. But this immediately implies that $0 \in \sigma(A)$. $\square$

Corollary C.4.6. *Let A be a bounded and self-adjoint operator on A. Then*

$$\sup W(A),\, \inf W(A) \;\in\; \sigma(A).$$

In particular, $\|A\| = r(A)$, where $r(A)$ denotes the spectral radius of A.

Proof. Let $\alpha := \inf W(A)$ and $\beta := \sup W(A)$. Then $0 = \inf W(A - \alpha) = \inf W(\beta - A)$, whence $0 \in \sigma(A - \alpha) \cap \sigma(\beta - A)$ by Proposition C.4.5. From Lemma C.4.4 b) we know that $r(A) \le \|A\| = \max\{|\alpha|, |\beta|\} \le r(A)$. $\square$

Note that positive operators are special cases of m-accretive operators (see Section C.7 below).

Proposition C.4.7. *Let $S, T \in \mathcal{L}(H)$ be self-adjoint, and let $0 \le S \le T$ in the sense that $(Sx \,|\, x) \le (Tx \,|\, x)$ for all $x \in H$. If S is invertible, then T is invertible and $T^{-1} \le S^{-1}$.*

Proof. By Corollary C.4.6, if S is invertible, there is $0 < \alpha \in \mathbb{R}$ such that $\alpha \le S$. Hence $\alpha \le T$, whence T is invertible. Let $x \in H$, and define $y := T^{-1}x$, $z := S^{-1}x$. Then

$$\begin{aligned}\left(T^{-1}x \,\middle|\, x\right)^2 &= (y \,|\, Sz)^2 \le (y \,|\, Sy)\,(z \,|\, Sz) \le (y \,|\, Ty)\,(z \,|\, Sz)\\ &= \left(T^{-1}x \,\middle|\, x\right)\left(S^{-1}x \,\middle|\, x\right)\end{aligned}$$

by Cauchy–Schwarz. Consequently, $\left(T^{-1}x \,\middle|\, x\right) \le \left(S^{-1}x \,\middle|\, x\right)$. $\square$

C.5 Equivalent Scalar Products and the Lax–Milgram Theorem

Let H be a Hilbert space. We denote by H^* the **antidual** of H, i.e., the space of continuous *conjugate-linear* functionals on H, endowed with the norm

$$\|\varphi\|_{H^*} := \sup\{|\varphi(x)| \;\mid\; x \in H,\ \|x\| = 1\} \qquad (\varphi \in H^*).$$

One sometimes writes $\langle \varphi, x \rangle$ instead of $\varphi(x)$, where $x \in H, \varphi \in H^*$.

Let a be a *continuous* sesquilinear form on H. Then we have an induced linear mapping

$$L_a := (u \longmapsto a(u, \cdot)) : H \longrightarrow H^*, \tag{C.4}$$

which is continuous. The Riesz–Fréchet theorem [196, Theorem 4.12] implies that for each $u \in H$ there is a unique $Qu \in H$ such that

$$a(u, \cdot) = L_a(u) = (Qu \,|\, \cdot)\,.$$

Obviously, Q is a linear and bounded operator ($\|Q\| = \|L_a\|$) and a is uniquely determined by Q. On the other hand, given $Q \in \mathcal{L}(H)$, the form a_Q defined by $a_Q(u, v) := (\, Qu \,|\, v\,)$ is sesquilinear and continuous. Hence the mapping

$$(Q \longmapsto a_Q) : \mathcal{L}(H) \longrightarrow \{\text{continuous, sesquilinear forms on } H\}$$

is an isomorphism. We have $\overline{a_Q} = a_{Q^*}$, whence the form a is symmetric if and only if Q is self-adjoint, and the form is positive if and only if $Q \geq 0$.

Proposition C.5.1. *Let $Q \in \mathcal{L}(H)$. Then the form a_Q is a scalar product on H if and only if Q is positive and injective. The norm induced by this scalar product is equivalent to the original one if and only if Q is invertible.*

Proof. The form a_Q is positive semi-definite if and only if $Q \geq 0$. This is clear from the definitions. If Q is not injective, then obviously a_Q is not definite. Let Q be injective and positive. By Proposition C.4.5 we conclude that $0 \notin W(Q)$. But this means that a_Q is definite. Since a_Q is continuous, equivalence of the induced norm is the same as the existence of a $\delta > 0$ such that $\delta \leq Q$. But this is equivalent to Q being invertible by Proposition C.4.5. □

A scalar product on H that induces a norm equivalent to the original one is simply called an **equivalent scalar product**.

Lemma C.5.2. *Let A be a multi-valued linear operator on H, and let $(\,\cdot\,|\,\cdot\,)_\circ := a_Q$ be an equivalent scalar product on H. Denote by A° the adjoint of A with respect to the new scalar product. Then*

$$(x, y) \in A^\circ \quad \Longleftrightarrow \quad (Qx, Qy) \in A^*$$

for all $x, y \in H$. In particular, $A^\circ = Q^{-1}A^*Q$ if A^* is single-valued.

Proof. Fix $x, y \in H$. Since Q is positive, $(x, y) \in A^\circ \Leftrightarrow (\,v\,|\,x\,)_\circ = (\,u\,|\,y\,)_\circ\, \forall (u, v) \in A \Leftrightarrow (\,v\,|\,Qx\,) = (\,u\,|\,Qy\,)\, \forall (u, v) \in A \Leftrightarrow (Qx, Qy) \in A^*$. The rest is straightforward. □

The next theorem states a sufficient condition for the mapping L_a to be an isomorphism.

Theorem C.5.3 (Lax–Milgram). *Let a be a continuous sesquilinear form on H. Suppose that there is a $\delta > 0$ such that*

$$\operatorname{Re} a(u) \geq \delta \|u\|^2 \tag{C.5}$$

for all $u \in H$. Then the mapping $L_a : H \longrightarrow H^*$ defined by (C.4) *is an isomorphism.*

The inequality (C.5) is called a **coercivity condition**, and a form which satisfies (C.5) for some $\delta > 0$ is called **coercive**.

Proof. Take $u \in H$ with $\|u\| = 1$. Then

$$\|L_a(u)\| \geq |L_a(u)(u)| = |a(u)| \geq \operatorname{Re} a(u) \geq \delta.$$

Hence $\delta\,\|u\| \leq \|L_a(u)\|$ for all u, whence L_a is injective and has a closed range. So it remains to show that $E := \mathcal{R}(L_a)$ is dense in H^*. The Riesz–Fréchet theorem implies that

$$\Phi := (x \longmapsto (\,x\,|\,\cdot\,)) : H \longrightarrow H^*$$

is an isomorphism. Hence E is dense in H^* if and only if $\Phi^{-1}(E)$ is dense in H if and only if $\Phi^{-1}(E)^\perp = 0$. Now

$$\begin{aligned} y \in \Phi^{-1}(E)^\perp \;&\Leftrightarrow\; (\,x\,|\,y\,) = 0 \;\forall\, x \in \Phi^{-1}(E) \;\Leftrightarrow\; \varphi(y) = 0 \quad \forall\, \varphi \in E \\ &\Leftrightarrow\; a(x,y) = 0 \quad \forall\, x \in H \end{aligned}$$

for each $y \in H$. In particular, we have $y \in \Phi^{-1}(E)^\perp \Rightarrow a(y,y) = 0$, but this implies that $y = 0$ by coercivity. Thus, the proposition is proved. □

Remark C.5.4. The coercivity condition in the Lax–Milgram theorem can be weakened to $|a(u)| \geq \delta\,\|u\|^2$ for all $u \in H$. This is easily seen from the proof.

C.6 Weak Integration

Let (Ω, μ) be a measure space. A μ-measurable map $F : \Omega \longrightarrow \mathcal{L}(H)$ is called **weakly integrable** if

$$\int_\Omega |(\,F(\omega)x\,|\,y\,)|\;\mu(d\omega) < \infty$$

for all $x, y \in H$. (By the polarisation identity it suffices to know this for all $x = y \in H$.)

Lemma C.6.1. *Let $F : \Omega \longrightarrow \mathcal{L}(H)$ be weakly integrable. Then there is a unique operator $Q \in \mathcal{L}(H)$ such that*

$$(\,Qx\,|\,y\,) = \int_\Omega (\,F(\omega)x\,|\,y\,)\;\mu(d\omega).$$

One usually writes $\int_\Omega F(\omega)\,\mu(d\omega) := Q$.

Proof. Fix $x \in H$. Then the mapping $y \longmapsto (\,F(\cdot)x\,|\,y\,)$ is linear from H to $\mathbf{L^1}(\Omega,\mu)$ by hypothesis, and by the closed graph theorem it must be continuous. The Riesz–Fréchet theorem [196, Theorem 4.12] shows that there is a unique $Qx \in H$ such that $(\,Qx\,|\,y\,) = \int (\,F(\omega)x\,|\,y\,)\;\mu(d\omega)$ for all $y \in H$. Obviously $Q : H \longrightarrow H$ is linear. The same argument as before shows that if y is fixed, the mapping $(x \longmapsto (\,F(\cdot)x\,|\,y\,)) : H \longrightarrow \mathbf{L^1}(\Omega,\mu)$ is continuous. Therefore, Q is weakly continuous, whence continuous. □

The following lemma is almost trivial.

Lemma C.6.2. *Let $F : \Omega \longrightarrow \mathcal{L}(H)$ be weakly integrable. Then also the function $F^* := (\omega \longmapsto F(\omega)^*)$ is weakly integrable with*

$$\left(\int F(\omega)\, \mu(d\omega)\right)^* = \int F(\omega)^*\, \mu(d\omega).$$

In particular, if $F(\omega)$ is self-adjoint for μ-almost all ω, then also $\int F\, d\mu$ is self-adjoint. If $F(\omega) \geq 0$ for μ-almost all ω, then $\int F\, d\mu \geq 0$.

The next result is of fundamental importance in the theory of functional calculus on Hilbert spaces.

Proposition C.6.3. *Let (Ω, μ) be a measure space, and let $S : \Omega \longrightarrow \mathcal{L}(H)$ be μ-measurable such that the mapping $(\omega \longmapsto S(\omega)^* S(\omega))$ is weakly integrable. Then for each $F \in \mathbf{L}^\infty(\Omega, \mu; \mathcal{L}(H))$ the mapping $(\omega \longmapsto S(\omega)^* F(\omega) S(\omega))$ is weakly integrable and*

$$\left\| \int S(\omega)^*\, F(\omega) S(\omega) \mu(d\omega) \right\| \leq \|F\|_\infty \left\| \int S(\omega)^* S(\omega)\, \mu(d\omega) \right\|.$$

Proof. Define $T := \int S^* S\, d\mu$. Then for all $x, y \in H$,

$$\begin{aligned}
&\int_\Omega |(\, S(\omega)^* F(\omega) S(\omega) x \,|\, y)|\; \mu(d\omega) = \int |(\, F(\omega) S(\omega) x \,|\, S(\omega) y)|\; \mu(d\omega) \\
&\quad \leq \|F\|_\infty \int \|S(\omega)x\|\, \|S(\omega)y\|\; \mu(d\omega) \\
&\quad \leq \|F\|_\infty \left(\int \|S(\omega)x\|^2\; \mu(d\omega) \right)^{\frac{1}{2}} \left(\int \|S(\omega)y\|^2\; \mu(d\omega) \right)^{\frac{1}{2}} \\
&\quad = \|F\|_\infty \left(\int (\, S(\omega)^* S(\omega) x \,|\, x\,)\; \mu(d\omega) \right)^{\frac{1}{2}} \left(\int (\, S(\omega)^* S(\omega) y \,|\, y\,)\; \mu(d\omega) \right)^{\frac{1}{2}} \\
&\quad \leq \|F\|_\infty\, (Tx \,|\, x)^{\frac{1}{2}}\, (Ty \,|\, y)^{\frac{1}{2}} \leq \|F\|_\infty\, \|T\|\, \|x\|\, \|y\|\,.
\end{aligned}$$

This shows that $S^* F S$ is weakly integrable. The norm estimate follows readily. $\square$

Corollary C.6.4. *Let $T : \Omega \longrightarrow \mathcal{L}(H)$ be weakly integrable such that $T(\omega) \geq 0$ for μ-almost all ω. Then*

$$\left\| \int f(\omega) T(\omega)\, \mu(d\omega) \right\| \leq \|f\|_\infty \left\| \int T(\omega)\, \mu(d\omega) \right\|$$

for all $f \in \mathbf{L}^\infty(\Omega, \mu)$.

Proof. Apply Proposition C.6.3 with $F := f$ and $S(\omega) := T(\omega)^{1/2}$. There is also a direct proof avoiding square roots: Define $Q := \int T\,d\mu$, and let $x, y \in H$. Then

$$\begin{aligned}\int |(f(\omega)T(\omega)x \,|\, y)|\;\mu(d\omega) &\le \|f\|_\infty \int |(T(\omega)x \,|\, y)|\;\mu(d\omega)\\ &\le \|f\|_\infty \int (T(\omega)x \,|\, x)^{\frac12}\;(T(\omega)y \,|\, y)^{\frac12}\;\mu(d\omega)\\ &\le \|f\|_\infty \left(\int (T(\omega)x \,|\, x)\;\mu(d\omega)\right)^{\frac12}\left(\int (T(\omega)y \,|\, y)\;\mu(d\omega)\right)^{\frac12}\\ &= \|f\|_\infty\,(Qx \,|\, x)^{\frac12}\,(Qy \,|\, y)^{\frac12} \le \|f\|_\infty\,\|Q\|\,\|x\|\,\|y\|\,. \end{aligned}$$ □

C.7 Accretive Operators

Here are the defining properties of accretive operators.

Lemma C.7.1. *Let A be an operator on the Hilbert space H, and let $\mu > 0$. The following assertions are equivalent.*

(i) $\operatorname{Re}(Au \,|\, u) \ge 0$ *for all* $u \in \mathcal{D}(A)$, *i.e.,* $W(A) \subset \{\operatorname{Re} z \ge 0\}$.

(ii) $\|(A+\mu)u\| \ge \|(A-\mu)u\|$ *for all* $u \in \mathcal{D}(A)$.

(iii) $\|(A+\lambda)u\| \ge (\operatorname{Re}\lambda)\,\|u\|$ *for all* $u \in \mathcal{D}(A)$, $\operatorname{Re}\lambda \ge 0$.

(iv) $\|(A+\lambda)u\| \ge \lambda\,\|u\|$ *for all* $u \in \mathcal{D}(A)$ *and all* $\lambda \ge 0$.

An operator A that satisfies the equivalent conditions (i)–(iv) is called **accretive**. An operator A is called **dissipative** if $-A$ is accretive.

Proof. We have $\|(A+\mu)u\|^2 - \|(A-\mu)u\|^2 = 4\operatorname{Re}(Au \,|\, u)$ for all $u \in \mathcal{D}(A)$. This proves (i)⇔(ii). For $\lambda > 0$ we have $\|(A+\lambda)u\|^2 - \lambda^2\|u\|^2 = \|Au\|^2 + 2\lambda(Au \,|\, u)$ for all $u \in \mathcal{D}(A)$. This shows (i)⇒(iv). Dividing by λ and letting $\lambda \to \infty$ yields the reverse implication.

The implication (iii)⇒(iv) is obvious. Suppose that (i) holds, and let $\operatorname{Re}\lambda \ge 0$. Define $\alpha := \operatorname{Im}\lambda$. Then (i) holds with A replaced by $A + i\alpha$. Since we have already established the implication (i)⇒(iv), we know that $\|((A+i\alpha)+\operatorname{Re}\lambda)u\| \ge (\operatorname{Re}\lambda)\,\|u\|$ for all $u \in \mathcal{D}(A)$. But this is (iii). □

Note that an operator A is symmetric if and only if $\pm iA$ both are accretive. An operator A is called **m-accretive** if A is accretive and closed and $\mathcal{R}(A+1)$ is dense in H.

Proposition C.7.2. *Let A be an operator on H, $\alpha \in \mathbb{R}$ and $\lambda > 0$. The following assertions are equivalent.*

(i) *A is m-accretive.*

(ii) $A + i\alpha$ *is m-accretive.*

(iii) $A + \varepsilon$ *is m-accretive for all* $\varepsilon > 0$.

(iv) $-\lambda \in \varrho(A)$ *and* $\left\|(A-\lambda)(A+\lambda)^{-1}\right\| \le 1$.

(v) $\{\operatorname{Re}\lambda < 0\} \subset \varrho(A)$ *and*

$$\|R(\lambda, A)\| \le \frac{1}{|\operatorname{Re}\lambda|} \qquad (\operatorname{Re}\lambda < 0).$$

(vi) $(-\infty, 0) \subset \varrho(A)$ *and* $\sup_{t>0} \left\|t(t+A)^{-1}\right\| \le 1$.

(vii) A *is closed and densely defined, and* A^* *is m-accretive.*

The operator $(A-1)(A+1)^{-1}$ is called the **Cayley transform** of A.

Proof. Let B be a closed accretive operator on H. By Proposition C.3.1 and its corollary, if $\mathcal{R}(B+\mu)$ is dense for some μ with $\operatorname{Re}\mu > 0$, then $\{\operatorname{Re}\mu < 0\} \subset \varrho(B)$. This consideration is of fundamental importance and is used several times in the sequel. In particular, it shows (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii).

The equivalence (iv)$\Leftrightarrow$(i) holds by (ii) of Lemma C.7.1. Similarly, (v)$\Leftrightarrow$(i) and (vi)$\Leftrightarrow$(i) hold by (iii) and (iv) of Lemma C.7.1.

From (vi) it follows that an m-accretive operator is *sectorial* (see Section 2.1), and as such is densely defined since H is reflexive (see Proposition 2.1.1 h)). Moreover, (vi) implies (vi) with A replaced by A^*, whence (vi)$\Rightarrow$(vii) follows.

Finally, suppose that (vii) holds. Since we already have established the implication (i)$\Rightarrow$(vii), we conclude that $A = \overline{A} = A^{**}$ is m-accretive. $\square$

Theorem C.7.3 (Lumer–Phillips). *An operator A is m-accretive if and only if $-A$ generates a strongly continuous contraction semigroup.*

Proof. If A is m-accretive, parts (vi) and (vii) of Proposition C.7.2 show that the Hille–Yosida theorem (Theorem A.8.6) (with $\omega = 0$ and $M = 1$) is applicable to the operator $-A$. Conversely, suppose that $-A$ generates the C_0-semigroup T such that $\|T(t)\| \le 1$ for all $t \ge 0$. Take $x \in \mathcal{D}(A)$. Then the function $t \longmapsto T(t)x$ is differentiable with derivative $t \longmapsto -AT(t)x$. Thus, $t \longrightarrow \|T(t)x\|^2$ is differentiable with

$$\left.\frac{d}{dt}\right|_{t=0} \|T(t)x\|^2 = (\,T(0)x \,|\, -AT(0)x\,) + (\,-AT(0)x \,|\, T(0)x\,) = -2\operatorname{Re}(\,Ax \,|\, x\,).$$

Since T is a contraction semigroup, the mapping $t \longmapsto \|T(t)x\|^2$ is decreasing. This implies that $d/dt|_{t=0} \|T(t)x\|^2 \le 0$. Hence, $\operatorname{Re}(\,Ax \,|\, x\,) \ge 0$, i.e., A is accretive. Since $1 \in \varrho(-A)$ we conclude that A is in fact m-accretive. $\square$

Theorem C.7.4 (Stone). *An operator B on the Hilbert space H generates a C_0- group $(U(t))_{t\in\mathbb{R}}$ of unitary operators if and only if $B = iA$ for some self-adjoint operator A.*

Proof. If $B = iA$, then B and $-B$ are both m-accretive. By the Lumer–Phillips theorem both operators generate C_0-contraction semigroups. Hence B generates a unitary C_0-group. This proof also works in the reverse direction. □

C.8 The Theorems of Plancherel and Gearhart

We state without proof two theorems which are 'responsible' for the fact that life is so much more comfortable in Hilbert spaces.

Theorem C.8.1 (Plancherel). *Let $f \in \mathbf{L^1}(\mathbb{R}, H) \cap \mathbf{L^2}(\mathbb{R}, H)$, and define $\mathcal{F}(f) : \mathbb{R} \longrightarrow H$ by*

$$\mathcal{F}(f)(t) := \int_{\mathbb{R}} f(s)e^{-ist}\,ds \qquad (t \in \mathbb{R}).$$

Then $\mathcal{F}(f) \in \mathbf{C_0}(\mathbb{R}, H) \cap \mathbf{L^2}(\mathbb{R}, H)$ with $\|\mathcal{F}(f)\|_{\mathbf{L^2}} = \sqrt{2\pi}\,\|f\|_{\mathbf{L^2}}$.

A proof can be found in [10, page 46]. More about the Fourier transform is contained in Appendix E.

Let X be a Banach space, and let T be a C_0-semigroup on X with generator A. From Proposition A.8.1 we know that

$$\{\operatorname{Re} z \geq \omega\} \subset \varrho(A) \quad \text{and} \quad \sup\{\|R(z, A)\| \mid \operatorname{Re} z \geq \omega\} \tag{C.6}$$

for every $\omega > \omega_0(T)$. (Recall that $\omega_0(T)$ is the growth bound of T, see Appendix A, Section A.8.) Let us define the **abszissa of uniform boundedness** of A as

$$s_0(A) := \inf\{\omega \in \mathbb{R} \mid \text{(C.6) holds}\}. \tag{C.7}$$

Hence in general we obtain that $s_0(A) \leq \omega_0(T)$. However, on Hilbert spaces one can say more.

Theorem C.8.2 (Gearhart). *Let A be the generator of a C_0-semigroup on a Hilbert space H. Then*

$$s_0(A) = \omega_0(T).$$

One of the many proofs of this theorem uses the Plancherel theorem and Datko's theorem, which states that $\omega_0(T) < 0$ if and only if $T(\cdot)x \in \mathbf{L^2}(\mathbb{R}_+, H)$ for every $x \in H$. It can be found in [10, page 347].

References

The following books underly our presentation: [130] and [199, Chapter XII] for sesquilinear forms and numerical range, [52, Chapter III] for adjoints, [49, Chapter X, §2] and [194, Chapter VIII] for symmetric and self-adjoint operators, [210, Chapter 2] for accretive operators and the Lax–Milgram theorem, and [85, Chapter II.3.b] for the Lumer–Phillips theorem and Stone's theorem.

The idea how to establish part c) of Lemma C.4.4 is from [144, Chapter XVIII, §4].

Theorem C.8.2 goes back to Gearhart [95], but its present form is due to Prüss [191] and Greiner [174, A-III.7].

Appendix D

The Spectral Theorem

The Spectral Theorem for normal operators exists in many versions. Basically one can distinguish the 'spectral measure approach' and the 'multiplicator approach'. Following Halmos's article [112] we give a consequent 'multiplicator' account of the subject matter.

In this form, the Spectral Theorem essentially says that a given self-adjoint operator on a Hilbert space acts 'like' the multiplication of a real function on an $\mathbf{L^2}$-space. In contrast to usual expositions we stress that the underlying measure space can be chosen locally compact and the real function can be chosen continuous.

D.1 Multiplication Operators

A **Radon measure space** is a pair (Ω, μ) where Ω is a locally compact Hausdorff space and μ is a positive functional on $\mathbf{C_c}(\Omega)$. By the Riesz Representation Theorem [196, Theorem 2.14] we can identify μ with a σ-regular Borel measure on Ω. If the measure μ has the property

$$\varphi \in \mathbf{C_c}(\Omega),\ 0 \le \varphi \ne 0 \quad \Rightarrow \quad \int \varphi \, d\mu > 0, \tag{D.1}$$

then the Radon measure space is called a **standard measure space**. Property (D.1) is equivalent to the fact that a non-empty open subset of Ω has positive μ-measure. If this is the case, the natural mapping $\mathbf{C}(\Omega) \longrightarrow \mathbf{L^1_{loc}}(\Omega, \mu)$ is injective.

Let $f \in \mathbf{C}(\Omega)$ be a continuous function on Ω. The **multiplication operator** M_f on $\mathbf{L^2}(\Omega, \mu)$ is defined by

$$\mathcal{D}(M_f) := \{g \in \mathbf{L^2}(\Omega, \mu) \mid gf \in \mathbf{L^2}(\Omega, \mu)\}, \qquad M_f g := fg \qquad (g \in \mathcal{D}(M_f)).$$

The following proposition summarises the properties of this operator.

Proposition D.1.1. *Let $f \in \mathbf{C}(\Omega)$, where (Ω, μ) is a standard measure space. Then the following assertions hold.*

a) *The operator* $(M_f, \mathcal{D}(M_f))$ *is closed.*

b) *The space* $\mathbf{C_c}(\Omega)$ *is a core for* M_f.

c) *One has* $(M_f)^* = M_{\overline{f}}$.

d) *One has* $M_f \in \mathcal{L}(\mathbf{L^2})$ *if and only if* f *is bounded. If this is the case, one has* $\|M_f\|_{\mathcal{L}(\mathbf{L^2})} = \|f\|_\infty$.

e) *The operator* M_f *is injective if and only if* $\mu(\{f = 0\} \cap K) = 0$ *for every compact* $K \subset \Omega$, *i.e.,* $\{f = 0\}$ *is locally* μ*-null.*

f) *One has* $\sigma(M_f) = \overline{f(\Omega)}$. *Furthermore,* $R(\lambda, M_f) = M_{(\lambda - f)^{-1}}$ *for all* $\lambda \in \varrho(M_f)$.

g) *If* $M_f \subset 0$, *then* $f = 0$.

h) *Let also* $g \in \mathbf{C}(\Omega)$. *Then* $M_f M_g \subset M_{fg}$. *One has equality if* g *is bounded or if* M_f *is invertible.*

Proof. a) Suppose that $g_n \in \mathcal{D}(M_f)$ such that $g_n \to g$ and $fg_n \to h$ in $\mathbf{L^2}$. For $\varphi \in \mathbf{C_c}(\Omega)$ we have $\varphi g_n \to \varphi g$ and $\varphi f g_n \to \varphi h$ in $\mathbf{L^2}$, whence $\varphi f g = \varphi h$. Since φ was arbitrary it follows that $fg = h$.

b) Let $g \in \mathcal{D}(M_f)$. Since $g \in \mathbf{L^2}$, it vanishes outside a set $L = \bigcup_n K_n$, where $K_n \subset K_{n+1}$ and each K_n is compact. Then $\mathbf{1}_{K_n} g \to g$ and $\mathbf{1}_{K_n} fg \to fg$. Hence without loss of generality we may suppose that g vanishes outside a compact set K. By Urysohn's lemma [196, Lemma 2.12] one can find a function $\psi \in \mathbf{C_c}(\Omega)$ such that $\mathbf{1}_K \le \psi \le \mathbf{1}$. Since $\mathbf{C_c}(\Omega)$ is dense in $\mathbf{L^2}$, there is a sequence $\varphi_n \in \mathbf{C_c}(\Omega)$ such that $\varphi_n \to g$ in $\mathbf{L^2}$. Then $\varphi_n \psi \to \psi g = g$ and $f \varphi_n \psi g \to \psi f g = fg$.

c) Let $x, y \in \mathbf{L^2}$. Then

$$(x, y) \in (M_f)^* \quad \Longleftrightarrow \quad \int_\Omega u\overline{y}\, d\mu = \int_\Omega u\overline{\overline{f}x}\, d\mu \quad \forall\, u \in \mathcal{D}(M_f).$$

Since $\mathbf{C_c}(\Omega) \subset \mathcal{D}(M_f)$, this is the case if and only if $\overline{f}x \in \mathbf{L^2}$.

d) Let $f \in \mathbf{C^b}(\Omega)$. Then $\|fg\|_2 \le \|f\|_\infty \|g\|_2$ for each $g \in \mathbf{L^2}$. Hence $M_f \in \mathcal{L}(\mathbf{L^2})$ and $\|M_f\| \le \|f\|_\infty$. Suppose that M_f is bounded and choose $\omega \in \Omega$. For every neighborhood U of ω we define $\varphi_U := \mu(U)^{-1/2}\mathbf{1}_U$. (Note that $\mu(U) \neq 0$ since (Ω, μ) is *standard*.) Then

$$\left|\mu(U)^{-1} \int_U f\, d\mu\right| = |(f\varphi_U \,|\, \varphi_U)_{\mathbf{L^2}}| \le \|M_f\|\, \|\varphi_U\|_2^2 = \|M_f\|\,.$$

Since f is continuous, $\mu(U)^{-1} \int_U f\, d\mu \to f(\omega)$ if U shrinks to $\{\omega\}$. Thus, f is bounded and $\|f\|_\infty \le \|M_f\|$.

e) Suppose there is K such that $\mu(A) > 0$ where $A := \{f = 0\} \cap K$. Then $0 \neq \mathbf{1}_A \in \mathbf{L^2}$ and $M_f(\mathbf{1}_A) = f\mathbf{1}_A = 0$. Hence M_f is not injective. Suppose that $\mu(\{f = 0\} \cap K) = 0$ for every compact K. Let $g \in \mathbf{L^2}$ such that $fg = 0$, and fix a compact set K. Then $fg\mathbf{1}_K = 0$, whence $K = (\{f = 0\} \cap K) \cup (\{g = 0\} \cap K)$. From the hypothesis it follows that $\mu(\{g = 0\} \cap K) = \mu(K)$. Since K was arbitrary and $g \in \mathbf{L^2}$, we conclude that $g = 0$ μ-a.e..

f) Obviously we have $\lambda - M_f = M_{\lambda - f}$ for every $\lambda \in \mathbb{C}$. Thus it suffices to consider the case $\lambda = 0$. If $0 \notin \overline{f(\Omega)}$, then $f^{-1} \in \mathbf{C^b}(\Omega)$ and it is easy to see that in this case $(M_f)^{-1} = M_{f^{-1}}$. Let $0 \in f(\Omega)$, say $f(\omega) = 0$. Consider the functions φ_U defined as in the proof of d). Then $\|\varphi_U\|_2 = 1$ and $\|M_f \varphi_U\|_2 = \|f\varphi_U\| = \mu(U)^{-1} \int_U |f|^2 \to |f(\omega)|^2 = 0$ if U shrinks to $\{\omega\}$. Hence $(\varphi_U)_U$ is an approximate eigenvector for 0.

g) Suppose that $M_f \subset 0$. Since $\mathbf{C_c}(\Omega)$ is a core for M_f, we conclude that $M_f = 0$. Hence we have $f = 0$ by d).

h) It is straightforward to prove $M_f M_g \subset M_{fg}$. Suppose that g is bounded. Then $M_f M_g$ is closed by Lemma A.1.3. Since obviously $\mathbf{C_c}(\Omega) \subset \mathcal{D}(M_f M_g)$ and $\mathbf{C_c}(\Omega)$ is a core for M_{fg} we obtain $M_{fg} \subset M_f M_g$. If M_f is invertible, then f is injective and f^{-1} is bounded (by f)). If $\psi \in \mathcal{D}(M_{fg})$, i.e., $fg\psi \in \mathbf{L^2}$, we have also $g\psi = f^{-1} fg\psi \in \mathbf{L^2}$, whence $\psi \in \mathcal{D}(M_f M_g)$. □

Corollary D.1.2. a) *M_f is symmetric if and only if M_f is self-adjoint if and only if f is real-valued.*

b) *M_f is accretive if and only if M_f is m-accretive if and only if* $\operatorname{Re} f \geq 0$.

c) *M_f is positive if and only if $f(\Omega) \subset [0, \infty)$.*

Proof. Just apply the definitions and Proposition D.1.1. □

D.2 Commutative C^*-Algebras. The Cyclic Case

In this section we start with a Hilbert space H and a commutative sub-C^*-algebra with unit $\mathcal{A}$ of $\mathcal{L}(H)$. We assume that the reader is familiar with the basic notions and results of Gelfand theory, as can be found for example in [197, Chapter 11] or [78, Chapters 3 and 4].

Let H' be another Hilbert space, and let $\mathcal{A}'$ be a commutative sub-C^*-algebra with unit of $\mathcal{L}(H')$. We say that $(\mathcal{A}, H)$ and $(\mathcal{A}', H')$ are **unitarily equivalent**, if there is an unitary isomorphism $U : H \longrightarrow H'$ such that the mapping

$$(T \longmapsto UTU^{-1}) : \mathcal{A} \longrightarrow \mathcal{A}'$$

is bijective.

Suppose there is a **cyclic vector** v, i.e., we have $\overline{\{Tv \mid T \in \mathcal{A}\}} = H$. Then the mapping $(T \longmapsto Tv) : \mathcal{A} \longrightarrow H$ *is injective with dense range* $\mathcal{A}v := \{Tv \mid T \in \mathcal{A}\}$.

Proof. To prove injectivity, let $Tv = 0$ for some $T \in \mathcal{A}$. Then $TSv = STv = S0 = 0$ for every $S \in \mathcal{A}$. Hence $T = 0$ on the dense subspace $\mathcal{A}v$. □

Let Ω denote the spectrum (Gelfand space) of $\mathcal{A}$. Then Ω is compact. On Ω we wish to find a Radon measure μ which turns (Ω, μ) into a standard measure space, and a sub-C^*-algebra $\mathcal{B}$ of $\mathbf{C}(\Omega)$ such that $(\mathcal{A}, H)$ is unitarily equivalent to $(\mathcal{B}, \mathbf{L^2}(\Omega, \mu))$.

By the Gelfand–Naimark theorem [197, Theorem 11.18], the canonical embedding $\Phi : \mathcal{A} \longrightarrow C(\Omega)$ is an isomorphism of C^*-algebras. Define the functional μ on $\mathbf{C}(\Omega)$ by

$$\mu(f) = \int f \, d\mu := \left(\Phi^{-1}(f)v \,\middle|\, v \right)_H \qquad (f \in \mathbf{C}(\Omega)).$$

If $f \geq 0$, there is a g such that $g^*g = f$. This implies that $\mu(f) = \mu(g^*g) = \left(\Phi^{-1}(g)v \,\middle|\, \Phi^{-1}(g)v \right)_H = \left\| \Phi^{-1}(g)v \right\|^2 \geq 0$. Hence μ is positive. If in addition $\mu(f) = 0$, we must have $\Phi^{-1}(g)v = 0$. But v is cyclic, so $\Phi^{-1}(g) = 0$. This implies that $g = 0$, whence $f = 0$. Thus we have shown that (Ω, μ) is a standard measure space.

We now construct the unitary operator $U : H \longrightarrow \mathbf{L^2}(\Omega, \mu)$ as follows. For $w = Tv \in \mathcal{A}v$ we define

$$U(w) = U(Tv) := \Phi(T) \in \mathbf{C}(\Omega) \subset \mathbf{L^2}(\Omega, \mu).$$

Because v is a cyclic vector, U is well defined and of course linear. The computation

$$\begin{aligned} (Tv \,|\, Sv)_H = (S^*Tv \,|\, v) &= \int \Phi(S^*T) \, d\mu = \int \overline{\Phi(S)}\Phi(T) \, d\mu \\ &= (U(Tv) \,|\, U(Sv))_{\mathbf{L^2}(\Omega,\mu)}, \end{aligned}$$

where $T, S \in \mathcal{A}$, shows that U is isometric. The range of U is clearly all of $\mathbf{C}(\Omega)$, which is a dense subspace of $\mathbf{L^2}(\Omega, \mu)$. Hence U has a unique extension to an isometric isomorphism from H to $\mathbf{L^2}(\Omega, \mu)$.

It remains to show that in fact U induces a unitary equivalence of $(\mathcal{A}, H)$ and $(\mathbf{C}(\Omega), \mathbf{L^2}(\Omega, \mu))$. Let $T \in \mathcal{A}$. It suffices to show that $UTU^{-1} = M_{\Phi(T)}$. To prove this it is enough to check the action of both operators on the dense subspace $\mathbf{C}(\Omega)$ of $\mathbf{L^2}(\Omega, \mu)$. If $f \in \mathbf{C}(\Omega)$, there is a unique $S \in \mathcal{A}$ such that $\Phi(S) = f$. Hence we have

$$UTU^{-1}(f) = UTU^{-1}\Phi(S) = U(TSv) = \Phi(TS) = \Phi(T)\Phi(S) = M_{\Phi(T)}(f).$$

Thus we have proved the following theorem.

Proposition D.2.1. *Let H be a Hilbert space, and let $\mathcal{A}$ be a commutative sub-C^*-algebra with unit of $\mathcal{L}(H)$. Suppose that H has a cyclic vector with respect to $\mathcal{A}$. Let Ω be the Gelfand space of $\mathcal{A}$. Then there is a standard Radon measure μ on Ω such that $(\mathcal{A}, H)$ and $(\mathbf{C}(\Omega), \mathbf{L^2}(\Omega, \mu))$ are unitarily equivalent.*

Remark D.2.2. If H is separable and $\mathcal{A}$ is a maximal commutative sub-W^*-algebra of $\mathcal{L}(H)$, then there is a cyclic vector, see [78, Theorem 4.65]. Of course, by an application of Zorn's lemma one can show that each commutative self-adjoint subalgebra of $\mathcal{L}(H)$ is contained in a maximal commutative one (which a fortiori must be a W^*-algebra). Hence a bounded normal operator on a separable Hilbert space is unitarily equivalent to multiplication by a continuous function on an $\mathbf{L^2}$-space over a compact space. This is one version of the Spectral Theorem (cf. Corollary D.3.3).

D.3 Commutative C^*-Algebras. The General Case

Suppose we are given a Hilbert space H and a commutative sub-C^*-algebra with unit $\mathcal{A}$ of $\mathcal{L}(H)$, but such that there is no cyclic vector. We then choose any vector $0 \neq v \in H$ and consider the closed subspace $H_v := \overline{\mathcal{A}v}$ of H. This space H_v **reduces** $\mathcal{A}$, in the sense that it is A-invariant (clear) and even *its orthogonal complement* $H_v^\perp$ *is* $\mathcal{A}$-*invariant.*

Proof. Let $w \perp H_v$, $S \in T$, and $x \in H_v$. Then $(\,Sw\,|\,x\,) = (\,w\,|\,S^*x\,) = 0$, because $S^*x \in H_v$ again. $\square$

If we restrict the operators from $\mathcal{A}$ to the space H_v we obtain a self-adjoint subalgebra with unit $\mathcal{A}_v$ of $\mathcal{L}(H_v)$. Moreover, v is a cyclic vector with respect to $\mathcal{A}_v$. Clearly, the whole procedure can be repeated on the Hilbert space $H_v^\perp$. Therefore, a standard application of Zorn's lemma yields the following lemma.

Lemma D.3.1. *Let H be a Hilbert space, and let $\mathcal{A} \subset \mathcal{L}(H)$ be a commutative sub-C^*-algebra with unit. Then there is a decomposition $H = \bigoplus_{\alpha\in I} H_\alpha$ as a Hilbert space direct sum such that each H_α is $\mathcal{A}$-invariant and has a cyclic vector with respect to $\mathcal{A}$.*

Note that if H is separable, the decomposition in Lemma D.3.1 is actually countable.

Theorem D.3.2. *Let H be a Hilbert space, and let $\mathcal{A} \subset \mathcal{L}(H)$ be a commutative self-adjoint subalgebra. Then there is a standard measure space (Ω, μ) and a subalgebra $\mathcal{B}$ of $\mathbf{C^b}(\Omega)$ such that $(\mathcal{A}, H)$ is unitarily equivalent to $(\mathcal{B}, \mathbf{L^2}(\Omega, \mu))$.*

Proof. Without loss of generality we may suppose that $\mathcal{A}$ is a sub-C^*-algebra with unit of $\mathcal{L}(H)$. By Lemma D.3.1 we can decompose $H = \bigoplus_{\alpha\in I} H_\alpha$, where the H_α are $\mathcal{A}$-invariant and have cyclic vectors v_α, say. We let $\mathcal{A}_\alpha := \overline{\mathcal{A}|_{H_\alpha}} \subset \mathcal{L}(H_\alpha)$, and define Ω_α to be the spectrum of $\mathcal{A}_\alpha$. Proposition D.2.1 (cyclic case) yields an unitary isomorphism

$$U_\alpha : (\mathcal{A}_\alpha, H_\alpha) \longrightarrow (\mathbf{C}(\Omega_\alpha), \mathbf{L^2}(\Omega_\alpha, \mu_\alpha))$$

where μ_α is a standard measure on Ω_α. In fact, $U_\alpha(T_\alpha v_\alpha) = \Phi_\alpha(T_\alpha)$ for each $T_\alpha \in \mathcal{A}_\alpha$, where $\Phi_\alpha : \mathcal{A}_\alpha \longrightarrow \mathbf{C}(\Omega_\alpha)$ is the Gelfand isomorphism (cf. the proof of Proposition D.2.1).

We now let $\Omega := \bigcup \Omega_\alpha$ the (disjoint) topological direct sum of the Ω_α. Clearly, Ω is a locally compact Hausdorff space, and each Ω_α is an open subset of Ω.

If $f \in \mathbf{C}(\Omega_{\alpha_0})$ for some particular α_0, we can extend f continuously to the whole of Ω by letting $f|_{X_\alpha} = 0$ for every other α. Hence we can identify the continuous functions on Ω_{α_0} with the continuous functions on Ω having support within Ω_{α_0}.

Note that for each $\varphi \in \mathbf{C_c}(\Omega)$ there are only finitely many α such that $\varphi|_{\Omega_\alpha} \neq 0$. Hence by

$$\mu(\varphi) = \int_\Omega \varphi \, d\mu := \sum_\alpha \mu_\alpha(\varphi|_{\Omega_\alpha})$$

a positive functional on $\mathbf{C_c}(\Omega)$ is defined. Obviously, (Ω, μ) is a standard measure space, since each $(\Omega_\alpha, \mu_\alpha)$ is one.

We now construct the unitary operator U. The subspace

$$H_0 := \operatorname{span}\{T_\alpha v_\alpha \mid \alpha \in I,\ T_\alpha \in \mathcal{A}_\alpha\}$$

is dense in H. We define

$$U := (\sum_{\alpha \in F} T_\alpha v_\alpha) \longmapsto \sum_{\alpha \in F} \Phi_\alpha(T_\alpha) : H_0 \longrightarrow \mathbf{C_c}(\Omega),$$

where $F \subset I$ is a finite subset, and $T_\alpha \in \mathcal{A}_\alpha$ for all α. Note that $\Phi_\alpha(T_\alpha)$ is a continuous function on Ω_α, hence can be viewed as a continuous function on Ω. It is clear that U is linear and surjective. Take $x, y \in H_0$, say $x = \sum_{\alpha \in F} T_\alpha v_\alpha$ and $y = \sum_{\beta \in G} S_\beta v_\beta$. Then

$$\begin{aligned} (x \,|\, y)_H &= \Big(\textstyle\sum_{\alpha \in F} T_\alpha v_\alpha \,\Big|\, \textstyle\sum_{\beta \in G} S_\beta v_\beta \Big) = \textstyle\sum_{\alpha \in F \cap G} (T_\alpha V_\alpha \,|\, S_\alpha V_\alpha) \\ &= \sum_{\alpha \in F \cap G} \int_{X_\alpha} \Phi(T_\alpha)\overline{\Phi(S_\alpha)} \, d\mu_\alpha = \int_\Omega \Big(\sum_{\alpha \in F} \Phi_\alpha(T_\alpha) \Big) \Big(\overline{\sum_{\beta \in G} \Phi_\beta(S_\beta)} \Big) \, d\mu \\ &= (Ux \,|\, Uy)_{\mathbf{L^2}(\Omega,\mu)}, \end{aligned}$$

and this shows that U is isometric. Since H_0 is dense in H, U extends to a unitary isomorphism $U : H \longrightarrow \mathbf{L^2}(\Omega, \mu)$. To conclude the proof we show that $UTU^{-1} = M_f$, where $T \in \mathcal{A}$ and $f \in \mathbf{C}(\Omega)$ is defined by $f|_{X_\alpha} = \Phi_\alpha(T|_{H_\alpha}) \in \mathbf{C}(\Omega_\alpha)$ for all α. Note that, if this is true, it follows that $f \in \mathbf{C^b}(\Omega)$ because the operator M_f is bounded on $\mathbf{L^2}(\Omega, \mu)$ (cf. Proposition D.1.1 d)). To show the claim, we only have to check the action of both operators on the dense subspace $\mathbf{C_c}(\Omega)$ of $\mathbf{L^2}(\Omega, \mu)$. Therefore, let $\varphi \in \mathbf{C_c}(\Omega)$ and define $\varphi_\alpha := \varphi|_{\Omega_\alpha}$. Then $\varphi = \sum_\alpha \varphi_\alpha$, where the sum is actually finite. We let $S_\alpha := \Phi^{-1}(\varphi_\alpha)$. Then we have $\varphi = Ux$ with $x = \sum_\alpha S_\alpha v_\alpha \in H_0$, and so

$$\begin{aligned} UTU^{-1}(\varphi) &= UTx = UT(\sum_\alpha S_\alpha v_\alpha) = U(\sum_\alpha T_\alpha S_\alpha v_\alpha) = \sum_\alpha \Phi_\alpha(T_\alpha S_\alpha) \\ &= \sum_\alpha \Phi_\alpha(T_\alpha)\Phi_\alpha(S_\alpha) = \sum_\alpha f|_{X_\alpha}\Phi_\alpha(S_\alpha) = f(\sum_\alpha \Phi_\alpha(S_\alpha)) = f\varphi = M_f(\varphi). \end{aligned}$$ □

Corollary D.3.3 (Spectral Theorem I). *Let H be a Hilbert space, and let $(T_j)_{j\in J}$ be a family of commuting bounded normal operators on H. Then there is a standard measure space (Ω, μ) and a family of bounded continuous functions $(f_j)_{j\in J}$ on Ω such that $((T_j)_{j\in J}, H)$ is unitarily equivalent to $((f_j)_{j\in J}, \mathbf{L^2}(\Omega, \mu))$.*

Proof. By Fugledge's theorem [49, Chapter IX, Theorem 6.7] $T_k^*T_j = T_jT_k^*$ for all indices $j, k \in J$. Hence the $*$-algebra $\mathcal{A}$ generated by $(T_j)_{j\in J}$ is commutative. Now we can apply Theorem D.3.2. □

Remark D.3.4. In the case of a single operator T, Fugledge's theorem is not needed for the proof of Corollary D.3.3 (see also Section D.4 below).

D.4 The Spectral Theorem: Bounded Normal Operators

Let H be a Hilbert space, and let $T \in \mathcal{L}(H)$ be a bounded normal operator on H. Denote by $\mathcal{A}$ the sub-C^*-algebra of $\mathcal{L}(H)$ that is generated by T. Because T is normal, $\mathcal{A}$ is commutative. We have $\sigma(\mathcal{A}) = \sigma(T)$ and $\hat{T}$ (i.e. the Gelfand transform of T) is just the coordinate function $(z \longmapsto z)$. Moreover, $R(\lambda, T) \in \mathcal{A}$ for every $\lambda \in \varrho(T)$.

Proof. From elementary Gelfand theory it follows that $\hat{T} : \sigma(\mathcal{A}) \longrightarrow \sigma(T)$ is surjective. But it is also injective, since T generates $\mathcal{A}$. Hence $\hat{T}$ is a homeomorphism, identifying $\sigma(\mathcal{A})$ and $\sigma(T)$. With this identification $\hat{T}$ becomes the coordinate function $(z \longmapsto z)$. If $\lambda \in \varrho(T)$, $r_\lambda := (\lambda - z)^{-1}$ is a continuous function on $\sigma(T)$, and the Gelfand–Naimark theorem implies that there is an operator $R_\lambda \in \mathcal{A}$ that corresponds to r_λ. But obviously we have $R_\lambda = R(\lambda, T)$. □

In the following we review the construction of the proof of Theorem D.3.2. For this we need a lemma the proof of which is straightforward.

Lemma D.4.1. *Let H_0 be a closed subspace of H, with orthogonal projection $P : H \longrightarrow H_0$. The subspace H_0 is $\mathcal{A}$-invariant if and only if $TP = PT$. In this case one has $\sigma(T|_{H_0}) \subset \sigma(T)$ and $R(\lambda, T|_{H_0}) = R(\lambda, T)|_{H_0}$ for each $\lambda \in \varrho(T)$. The C^*-closure $\mathcal{A}_0$ of $\{S|_{H_0} \mid S \in \mathcal{A}\}$ is generated by $T|_{H_0}$ and $\sigma(\mathcal{A}_0) = \sigma(T|_{H_0})$.*

Proceeding along the lines of the proof of Theorem D.3.2, we decompose the Hilbert space $H = \bigoplus_{\alpha \in I} H_\alpha$ into $\mathcal{A}$-invariant and cyclic subspaces H_α. Define $T_\alpha := T|_{H_\alpha}$ and $\Omega_\alpha := \sigma(\mathcal{A}_\alpha) = \sigma(T_\alpha)$. Each Ω_α carries a standard measure μ_α such that (T_α, H_α) is unitarily equivalent to $(M_z, \mathbf{L^2}(\Omega_\alpha, \mu_\alpha))$, where M_z is multiplication by the coordinate function (see above). The locally compact Hausdorff space Ω is the disjoint union of the $\sigma(T_\alpha)$, and the standard measure μ on Ω is defined by $\mu|_{\Omega_\alpha} = \mu_\alpha$. Finally, (T, H) and $(M_f, \mathbf{L^2}(\Omega, \mu))$ are unitarily equivalent, where $f \in \mathbf{C^b}(\Omega)$ is the coordinate function on each $\sigma(T_\alpha)$.

By Lemma D.4.1, $\sigma(T_\alpha)$ is a closed subset of $\sigma(T)$ for each α. Therefore Ω can be viewed as a closed subset of $\sigma(T) \times I$. The measure μ on Ω extends canonically to a measure on $\sigma(T) \times I$ with support Ω. We denote this extension again by μ, but we remark that it might not be a standard measure any more. However, there is clearly a unitary equivalence of $(M_f, \mathbf{L^2}(\Omega, \mu))$ and $(M_h, \mathbf{L^2}(\sigma(T) \times I, \mu))$, where $f : \sigma(T) \times I \longrightarrow \mathbb{C}$ is defined by $h(z, \alpha) = z$ for $(z, \alpha) \in \sigma(T) \times I$. Thus we have proved the following theorem.

Theorem D.4.2 (Spectral Theorem II). *Let H be a Hilbert space, and let $T \in \mathcal{L}(H)$ be a bounded normal operator on H. Then there is a discrete set I and a (not necessarily standard) Radon measure μ on $\sigma(T) \times I$ such that (T, H) is unitarily equivalent to $(M_h, \mathbf{L^2}(\sigma(T) \times I, \mu))$, where h is defined by*

$$h = ((z, \alpha) \longmapsto z) : \sigma(T) \times I \longrightarrow \mathbb{C}.$$

If H is separable, $I = \mathbb{N}$.

D.5 The Spectral Theorem: Unbounded Self-adjoint Operators

Let H be a Hilbert space, and let A be a (not necessarily bounded) self-adjoint operator on H. Then $\sigma(A) \subset \mathbb{R}$. The Spectral Theorem for A states that one can find a standard measure space (Ω, μ) and a real-valued continuous function f on Ω such that (A, H) and $(M_f, \mathbf{L^2}(\Omega, \mu))$ are **unitarily equivalent**. This means that there is a unitary isomorphism $U : H \longrightarrow \mathbf{L^2}(\Omega, \mu)$ which takes the graph of A bijectively onto the graph of M_f, i.e., it holds

$$(x, y) \in A \quad \text{if and only if} \quad (Ux, Uy) \in M_f.$$

The situation can be reduced to the bounded case by taking resolvents. Indeed, let $T := (i - A)^{-1}$. Then T clearly is a bounded normal operator on H. By Corollary D.3.3 we find a standard measure space (Ω_1, μ_1) and a bounded continuous function g_1 on Ω such that (T, H) and $(M_{g_1}, \mathbf{L^2}(\Omega_1, \mu_1))$ are unitarily equivalent. Now T is injective, hence $(g_1 = 0)$ is a closed subset of Ω_1, locally μ_1-null (see Proposition D.1.1 e)). If we set $\Omega := \Omega_1 \setminus (g_1 = 0)$ and $\mu := \mu_1|_\Omega$, we obtain a standard measure space (Ω, μ). It is easy to see that $(M_{g_1}, \mathbf{L^2}(\Omega_1, \mu_1))$

is unitarily equivalent to $(M_g, \mathbf{L^2}(\Omega, \mu))$, where $g := g_1|_\Omega$. But g does not vanish, hence $f := i - g^{-1}$ defines a continuous function on Ω. Clearly, (A, H) is unitarily equivalent to $(M_f, \mathbf{L^2}(\Omega, \mu))$.

If we use Theorem D.4.2 instead of Corollary D.3.3, we obtain a discrete set I and a Radon measure ν on $\sigma(T) \times I$ such that (T, H) is unitarily equivalent to $(h, \mathbf{L^2}(\sigma(T) \times I, \nu))$, where $h : \sigma(T) \times I \longrightarrow \mathbb{C}$ is the projection onto the first coordinate. Thanks to the spectral mapping theorem for resolvents (Proposition A.3.1) we have $\sigma(T) = \varphi(\sigma(A))$, with $\varphi(w) = (i - w)^{-1}$. We define the Radon measure μ on $\sigma(A) \times I$ by

$$\int_{\sigma(A)\times I} f(w, \alpha)\, d\mu(w, \alpha) := \int_{\sigma(T)\times I} f(\varphi^{-1}(z), \alpha)\, d\nu(z, \alpha)$$

for $f \in \mathbf{C_c}(\sigma(A) \times I)$. Then it is immediate that (A, H) is unitarily equivalent to $(M_f, \mathbf{L^2}(\sigma(A) \times I, \mu))$ where $f : \sigma(A) \times I \longrightarrow \mathbb{R}$ is the projection onto the first coordinate.

We summarise our considerations in the next theorem.

Theorem D.5.1 (Spectral Theorem III). *Let H be a Hilbert space, and let A be a self-adjoint operator on H.*

a) *There is a standard measure space (Ω, μ) and a function $f \in \mathbf{C}(\Omega, \mathbb{R})$ such that (A, H) is unitarily equivalent to $(M_f, \mathbf{L^2}(\Omega, \mu))$.*

b) *There is a discrete set I and a positive Radon measure μ on $\sigma(A) \times I$ such that (A, H) is unitarily equivalent to $(M_f, \mathbf{L^2}(\sigma(A) \times I, \mu))$, where f is given by*

$$f = ((z, \alpha) \longmapsto z) : \sigma(A) \times I \longrightarrow \mathbb{R}.$$

If H is separable, $I = \mathbb{N}$.

Remark D.5.2. Our considerations are focussed on self-adjoint operators but with the same proofs one can obtain similar results for unbounded, normal operators.

D.6 The Functional Calculus

The Spectral Theorem allows us to define a functional calculus for a normal operator on a Hilbert space. Let Ω be a locally compact space, μ a positive Radon measure on Ω, and $f \in \mathbf{C}(\Omega)$ a continuous function on Ω. We let $X := \overline{f(\Omega)} \subset \mathbb{C}$. Denote by $\mathbf{B}(X)$ the bounded Borel measurable functions on X. If $g \in \mathbf{B}(X)$, then $g \circ f \in \mathbf{B}(\Omega)$, hence $M_{g\circ f}$ is a bounded operator on $\mathbf{L^2}(\Omega, \mu)$ satisfying $\|M_{g\circ f}\|_{\mathcal{L}(\mathbf{L^2})} \le \|g \circ f\|_\infty \le \|g\|_\infty$. Obviously, the mapping

$$(g \longmapsto g(M_f) := M_{g\circ f}) : \mathbf{B}(X) \longrightarrow \mathcal{L}(\mathbf{L^2}(\Omega, \mu))$$

is a homomorphism of C^*-algebras. Moreover, if g_n is a unifomly bounded sequence in $\mathbf{B}(X)$ converging pointwise to $g \in \mathbf{B}(X)$, then Lebesgue's theorem yields that $g_n(M_f) \to g(M_f)$ strongly. If we put this together with the Spectral Theorem(s), we obtain the following result.

Theorem D.6.1. *Let A be a self-adjoint operator on a Hilbert space H. Then there exists a unique mapping $\Psi : \mathbf{B}(\sigma(A)) \longrightarrow \mathcal{L}(H)$ with the following properties.*

1) *Ψ is a $*$-homomorphism.*
2) *$\Psi((\lambda - z)^{-1}) = R(\lambda, A)$ for all $\lambda \notin \mathbb{R}$.*
3) *If $(g_n)_n \subset \mathbf{B}(\sigma(A))$ is uniformly bounded and $g_n \to g$ pointwise, then $\Psi(g_n) \to \Psi(g)$ strongly.*

Proof. Existence is clear from the remarks above and the Spectral Theorem D.5.1. We show uniqueness. Observe that the (self-adjoint) algebra which is generated by the set $\{(\lambda - z)^{-1} \mid \lambda \notin \mathbb{R}\}$ is uniformly dense in $\mathbf{C_0}(\mathbb{R})$ by the Stone–Weierstrass theorem. The sequence of functions $g_n(z) := in(in - z)^{-1}$ is uniformly bounded on $\mathbb{R}$ and converges pointwise to the constant $\mathbf{1}$. On the other hand it is clear that $inR(in, A) \to I$ strongly. Hence $\Psi(\mathbf{1}) = I$, and so Ψ is determined on $\mathbf{C^b}(\mathbb{R})$. By Tietze's theorem we know that each bounded continuous function on $\sigma(A)$ is the restriction of a bounded continuous function on $\mathbb{R}$. Hence Ψ is determined on $\mathbf{C^b}(\sigma(A))$. Therefore, Ψ is determined on the smallest class $\mathcal{M}$ of functions that contains the bounded continuous ones and is closed under bounded and pointwise convergence. Now, $\sigma(A)$ is a metric, separable, locally compact space, whence by Urysohn's lemma [196, Lemma 2.12] the characteristic functions of compact sets are contained in $\mathcal{M}$. The class $\mathcal{A} := \{M \subset \sigma(A) \mid \mathbf{1}_M \in \mathcal{M}\}$ is easily seen to be a σ-algebra on $\sigma(A)$ that contains the compact subsets. Hence $\mathcal{A} = \mathfrak{B}(\sigma(A))$, the Borel σ-algebra on $\sigma(A)$. But a standard result from measure theory says that each bounded Borel measurable function can be approximated uniformly by a sequence of Borel simple functions. Altogether this implies that $\mathbf{B}(\sigma(A)) = \mathcal{M}$. □

References

A highly readable account of the Spectral Theorem including also the 'spectral measure version' is [194, Chaper VII and Section VIII.3]. One may also profit from [197, p.321 and p. 368] and [49, Chapter IX and ChapterX], where the stress is on the spectral measures. We have based our exposition mainly on the book [201], especially its Chapter IX, from which we learned about standard measure spaces and the 'continuous multiplicator version' of the Spectral Theorem. Theorem D.4.2 in the separable case is [62, Theorem 2.5.1], but it is proved with different methods. For historical remarks on the Spectral Theorem see Ricker [195].

Appendix E

Fourier Multipliers

Here we provide some definitions and results from Harmonic Analysis. As general background we recommend the books [197] for distributions and [80] and [205] for the multiplier theory.

E.1 The Fourier Transform on the Schwartz Space

Let X be any Banach space, and fix $d \in \mathbb{N}$. For an open subset $\Omega \subset \mathbb{R}^d$ the space of **test functions** on Ω is

$$\boldsymbol{\mathcal{D}}(\Omega; X) := \{f \in \mathbf{C}^\infty(\Omega; X) \mid \operatorname{supp} f \text{ is compact}\}.$$

Convergence of sequences within $\boldsymbol{\mathcal{D}}(\Omega; X)$ is defined as in the scalar case, cf. [197, Chapter 6]. For each $k \in \mathbb{N}$ we define

$$\mathbf{C}_\mathbf{0}^\mathbf{k}(\Omega; X) := \left\{f \in \mathbf{C}^\mathbf{k}(\Omega; X) \;\middle|\; D^\alpha f \in \mathbf{C}_\mathbf{0}(\Omega; X),\ \forall\ |\alpha| \le k\right\}$$

where as usual $D^\alpha = \prod_{j=1}^d D_j^{\alpha_j}$, and $D_j = d/dt_j$ is the partial derivative operator in the t_j-direction. Each $\mathbf{C}_\mathbf{0}^\mathbf{k}(\Omega; X)$ is a Banach space under the norm

$$\|f\|_{\mathbf{C}_\mathbf{0}^\mathbf{k}} := \max_{|\alpha| \le k} \|D^\alpha f\|_\infty .$$

The **Schwartz space** $\boldsymbol{\mathcal{S}}(\mathbb{R}^d; X)$ of **rapidly decreasing** X-valued functions on $\mathbb{R}^d$ is defined by

$$\boldsymbol{\mathcal{S}}(\mathbb{R}^d; X) := \left\{f \in \mathbf{C}^\infty(\mathbb{R}^d; X) \;\middle|\; t^\alpha D^\beta f \in \mathbf{L}^\infty(\mathbb{R}^d; X) \text{ for all } \alpha, \beta \in \mathbb{N}^d\right\}.$$

Here $t^\alpha := \prod_{j=1}^d t_j^{\alpha_j}$ for $\alpha \in \mathbb{N}^d$. Convergence of sequences within $\boldsymbol{\mathcal{S}}(\mathbb{R}^d; X)$ is defined as in the scalar case by the family of norms

$$|f|_m := \max_{|\alpha| \le m} \|(1 + |t|)^m D^\alpha f\|_\infty \qquad (m \in \mathbb{N}).$$

In addition we define the space of **functions of tempered growth** as

$$\boldsymbol{\mathcal{P}}(\mathbb{R}^d; X) := \big\{ f \in \mathbf{C}^\infty(\mathbb{R}^d; X) \;\big|\; \forall\, \alpha \in \mathbb{N}^d\, \exists n \in \mathbb{N} : (1+|t|)^{-n} D^\alpha f \;\in\; \mathbf{L}^\infty \big\}.$$

The following facts are well known:

1. $\boldsymbol{\mathcal{D}}(\mathbb{R}^d; X)$ is sequentially dense in $\boldsymbol{\mathcal{S}}(\mathbb{R}^d; X)$.
2. $\boldsymbol{\mathcal{D}}(\mathbb{R}^d) \otimes X$ is a dense subspace of each space $\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X)$, $p \in [1, \infty)$, and of each space $\mathbf{C}_0^k(\mathbb{R}^d; X)$, $k \in \mathbb{N}$.
3. Multiplication $(g, f) \longmapsto gf$ is a bilinear mapping

$$\boldsymbol{\mathcal{P}}(\mathbb{R}^d; \mathcal{L}(X, Y)) \times \boldsymbol{\mathcal{S}}(\mathbb{R}^d; X) \longrightarrow \boldsymbol{\mathcal{S}}(\mathbb{R}^d, Y)$$

with $(f \longmapsto gf)$ being continuous for each fixed g.

Let us denote by $\mathbf{M}(\mathbb{R}^d; X)$ the set of all X-valued Borel-measures μ on $\mathbb{R}^d$ of bounded variation, endowed with the total variation norm $\|\mu\|_{\mathbf{M}}$. For such a measure $\mu \in \mathbf{M}(\mathbb{R}^d; X)$ we define its **Fourier transform** by

$$(\mathcal{F}\mu)(s) := \widehat{\mu}(s) := \int_{\mathbb{R}^d} e^{-it\cdot s}\, \mu(dt).$$

Then $\mathcal{F} : \mathbf{M}(\mathbb{R}^d; X) \longrightarrow \mathbf{BUC}(\mathbb{R}^d; X)$ is linear with $\|\mathcal{F}\mu\|_\infty \le \|\mu\|_{\mathbf{M}}$. The map $(f \longmapsto f(t)dt)$ takes $\mathbf{L}^1(\mathbb{R}^d; X)$ isometrically onto a closed subspace of $\mathbf{M}(\mathbb{R}^d; X)$. Restricting $\mathcal{F}$ to this space yields the formula

$$(\mathcal{F} f)(s) := \widehat{f}(s) := \int_{\mathbb{R}^d} f(t) e^{-it\cdot s}\, dt \qquad (s \in \mathbb{R}^d).$$

The Riemann–Lebesgue lemma asserts that actually $\mathcal{F} : \mathbf{L}^1(\mathbb{R}^d; X) \longrightarrow \mathbf{C}_0(\mathbb{R}^d; X)$. Restricting even further, the Fourier transform is a topological isomorphism

$$\mathcal{F} : \boldsymbol{\mathcal{S}}(\mathbb{R}^d; X) \longrightarrow \boldsymbol{\mathcal{S}}(\mathbb{R}^d; X)$$

with inverse

$$(\mathcal{F}^{-1} f)(t) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(s)\, e^{is\cdot t}\, ds \qquad (t \in \mathbb{R}^d).$$

If X is a Hilbert space, then the Fourier transform extends to an isomorphism on $\mathbf{L}^2(\mathbb{R}^d; X)$ with $\|\mathcal{F} f\|_{\mathbf{L}^2} = (2\pi)^{d/2}\, \|f\|_{\mathbf{L}^2}$. (This is Plancherel's theorem, cf. Theorem C.8.1.)

E.2 Tempered Distributions

Let $\Omega \subset \mathbb{R}^d$ be open. The space of X-valued **distributions** on Ω is defined by

$$\mathbf{D}(\Omega; X) := \{u \mid u : \boldsymbol{\mathcal{D}}(\Omega) \longrightarrow X, \text{ linear and (sequentially) continuous}\}.$$

For any multiindex $\alpha \in \mathbb{N}^d$ and any distribution $u \in \mathbf{D}(\Omega; X)$ one defines the derivative $D^\alpha u \in \mathbf{D}(\Omega; X)$ by

$$\langle D^\alpha u, \varphi \rangle = (-1)^{|\alpha|} \langle u, D^\alpha \varphi \rangle \qquad (\varphi \in \boldsymbol{\mathcal{D}}(\Omega)).$$

The space of X-valued **tempered distributions** is given by

$$\mathbf{TD}(\mathbb{R}^d; X) := \{u \mid u : \boldsymbol{\mathcal{S}}(\mathbb{R}^d) \longrightarrow X, \text{ linear and continuous}\}.$$

Since $\boldsymbol{\mathcal{D}}(\mathbb{R}^d) \subset \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ is dense, one can view $\mathbf{TD}(\mathbb{R}^d; X)$ in a natural way as a subspace of $\mathbf{D}(\mathbb{R}^d; X)$. The equation

$$\langle D^\alpha u, f \rangle = (-1)^{|\alpha|} \langle u, D^\alpha f \rangle$$

remains true for $u \in \mathbf{TD}(\mathbb{R}^d; X), f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$. The **support** of a distribution $T \in \mathbf{D}(\mathbb{R}^d; X)$ is the set

$$\operatorname{supp}(T) := \mathbb{R}^d \setminus \bigcup \{\Omega \subset \mathbb{R}^d \mid \Omega \text{ is open and } \boldsymbol{\mathcal{D}}(\Omega) \subset \ker T\}.$$

Lemma E.2.1. *Let $T \in \mathbf{TD}(\mathbb{R}^d; X)$. Then $\operatorname{supp} T \subset \{0\}$ if, and only if there exist $m \in \mathbb{N}$ and $(a_\alpha)_{\alpha \in \mathbb{N}^d, |\alpha| \le m}$ such that $T = \sum_{|\alpha| \le m} a_\alpha D^\alpha \delta_0$.*

Proof. In the case where $X = \mathbb{C}$, this is [197, Theorem 6.25]. The vector-valued case is proved from the scalar-valued case by using linear functionals. □

As in the scalar-valued case one has a natural embedding

$$\mathbf{L}^{\mathbf{1}}_{\mathrm{loc}}(\Omega; X) \hookrightarrow \mathbf{D}(\Omega; X).$$

Its image is the space of so-called **regular** distributions. For each $p \in [1, \infty]$ and $k \in \mathbb{N}$ one defines the **Sobolev space**

$$\mathbf{W}^{\boldsymbol{k,p}}(\Omega; X) = \left\{ f \in \mathbf{D}(\Omega; X) \;\middle|\; D^\alpha f \in \mathbf{L}^{\boldsymbol{p}}(\Omega; X),\ |\alpha| \le k \right\} \subset \mathbf{L}^{\boldsymbol{p}}(\Omega; X),$$

which is a Banach space with respect to the norm

$$\|f\|_{\mathbf{W}^{\boldsymbol{k,p}}} := \max_{|\alpha| \le k} \|D^\alpha f\|_p .$$

If a regular distribution on $\mathbb{R}^d$ extends to a tempered one, it is called **tempered regular**. Clearly, any $\mathbf{L}^{\mathbf{1}}_{\mathrm{loc}}$-function of tempered growth at ∞ is tempered regular. In particular one has an embedding

$$\mathbf{L}^{\mathbf{1}}(\mathbb{R}^d; X) + \mathbf{L}^{\boldsymbol{\infty}}(\mathbb{R}^d; X) \hookrightarrow \mathbf{TD}(\mathbb{R}^d; X).$$

Many natural operations on functions can be induced on tempered distribution via *adjoint action*. For example, let $\mathcal{S}f(t) := f(-t)$ be the **reflection mapping**, defined on $\mathbf{L}^1_{\mathrm{loc}}(\mathbb{R}^d; X)$. Then $\mathcal{S}$ restricts to a topological isomorphism on $\boldsymbol{\mathcal{S}}(\mathbb{R}^d; X)$ and one defines

$$\langle \mathcal{S}u, f\rangle = \langle u, \mathcal{S}f\rangle \qquad (f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d))$$

for any tempered distribution $u \in \mathbf{TD}(\mathbb{R}^d; X)$. In this way an isomorphism

$$\mathcal{S} : \mathbf{TD}(\mathbb{R}^d; X) \longrightarrow \mathbf{TD}(\mathbb{R}^d; X)$$

is defined, coherent with the orginal definition on $\mathbf{L}^1_{\mathrm{loc}}(\mathbb{R}^d; X)$. In a similar manner, the **Fourier transform** extends to an isomorphism

$$\mathcal{F} : \mathbf{TD}(\mathbb{R}^d; X) \longrightarrow \mathbf{TD}(\mathbb{R}^d; X),$$

defined by:

$$\langle \mathcal{F}u, f\rangle = \langle u, \mathcal{F}f\rangle \qquad (f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d), u \in \mathbf{TD}(\mathbb{R}^d; X)).$$

This definition is coherent with the embedding $\mathbf{M}(\mathbb{R}^d; X) \subset \mathbf{TD}(\mathbb{R}^d; X)$, by Fubini's theorem. Combining both reflection and Fourier transform we obtain

$$\mathcal{F}\mathcal{S}u = \mathcal{S}\mathcal{F}u = (2\pi)^{\frac{d}{2}}\mathcal{F}^{-1}u = (2\pi)^{-\frac{d}{2}}\mathcal{F}^3 u.$$

By adjoint operation $\langle gu, f\rangle = \langle u, gf\rangle$ we obtain also a multiplication

$$\boldsymbol{\mathcal{P}}(\mathbb{R}^d) \times \mathbf{TD}(\mathbb{R}^d; X) \longrightarrow \mathbf{TD}(\mathbb{R}^d; X).$$

Fourier transform, multiplication and derivatives fit nicely together, as the formulae

$$\mathcal{F}D^\alpha u = (is)^\alpha \mathcal{F}u \quad \text{and} \quad D^\alpha \mathcal{F}u = \mathcal{F}((-it)^\alpha u)$$

show. For $\varepsilon > 0$ we define the **dilation** operator U_ε on functions by

$$(U_\varepsilon f)(t) = f(\varepsilon t) \qquad (t \in \mathbb{R}^d).$$

Then $U_\varepsilon : \boldsymbol{\mathcal{S}}(\mathbb{R}^d; X) \longrightarrow \boldsymbol{\mathcal{S}}(\mathbb{R}^d; X)$ is an isomorphism with inverse $U_\varepsilon^{-1} = U_{\varepsilon^{-1}}$. One has

$$\|U_\varepsilon f\|_p = \varepsilon^{-\frac{d}{p}} \|f\|_p \qquad (f \in \mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X),\ p \in [1, \infty]),$$

as well as

$$\mathcal{F}U_\varepsilon f = \varepsilon^{-d} U_\varepsilon^{-1} \mathcal{F}f \quad \text{and} \quad \mathcal{F}^{-1}U_\varepsilon f = \varepsilon^{-d} U_\varepsilon^{-1} \mathcal{F}^{-1} f,$$

whenever $f \in \mathbf{L}^1(\mathbb{R}^d; X)$. The adjoint operator of U_ε is

$$U_\varepsilon' : \mathbf{TD}(\mathbb{R}^d; X) \longrightarrow \mathbf{TD}(\mathbb{R}^d; X),$$

defined by

$$\langle U'_\varepsilon u, f\rangle = \langle u, U_\varepsilon f\rangle \qquad (u \in \mathbf{TD}(\mathbb{R}^d; X),\ f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)).$$

Then $U'_\varepsilon u = \varepsilon^{-d} U_{\varepsilon^{-1}} u$ whenever u is a regular distribution, i.e., is induced by a function. Moreover, we have

$$\|U'_\varepsilon g\|_p = \varepsilon^{-\frac{d}{p'}} \|g\|_p \qquad (g \in \mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X)),$$

i.e., U'_ε is almost isometric on $\mathbf{L}^{\boldsymbol{p}}$ and isometric on $\mathbf{L}^{\mathbf{1}}$. We have the formulae

$$\varepsilon^d U'_\varepsilon \mathcal{F} = \mathcal{F} U'^{-1}_\varepsilon \quad \text{and} \quad \varepsilon^d U'_\varepsilon \mathcal{F}^{-1} = \mathcal{F}^{-1} U'^{-1}_\varepsilon.$$

E.3 Convolution

Let $\mu \in \mathbf{M}(\mathbb{R}^d; \mathcal{L}(X, Y))$. Then for each $f \in \mathbf{C_0}(\mathbb{R}^d; X)$ we define the **convolution**

$$\mu * f(s) := \int f(s-t)\, \mu(dt).$$

Since f is uniformly continuous and μ is more or less concentrated on a compact set, $\mu * f \in \mathbf{C_0}(\mathbb{R}^d; Y)$ with $\|\mu * f\|_\infty \le \|\mu\|_{\mathbf{M}} \|f\|_\infty$. Fubini's theorem implies that

$$\|\mu * f\|_1 \le \|\mu\|_{\mathbf{M}} \|f\|_1$$

for all $f \in \mathbf{C_0}(\mathbb{R}^d; X) \cap \mathbf{L^1}(\mathbb{R}^d; X)$. Hence the mapping $(f \longmapsto \mu * f)$ can be extended to all of $\mathbf{L^1}(\mathbb{R}^d; X)$, obtaining

$$(f \longmapsto \mu * f) : \mathbf{L^1}(\mathbb{R}^d; X) \longrightarrow \mathbf{L^1}(\mathbb{R}^d; Y).$$

More generally, let $p \in [1, \infty)$. Then Minkowski's inequality for integrals yields

$$\|\mu * f\|_p \le \|\mu\|_{\mathbf{M}} \|f\|_p \qquad (f \in \mathbf{C_0}(\mathbb{R}^d; X) \cap \mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X)),$$

i.e., **Young's inequality**. Hence an extension of the convolution to all of $\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X)$ is defined, and Young's inequality is then true for *all* $f \in \mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X)$.

Lemma E.3.1. *The mapping*

$$(f \longmapsto (g \longmapsto f * g)) : \mathbf{L^1}(\mathbb{R}^d) \longrightarrow \mathcal{L}(\mathbf{L^1}(\mathbb{R}^d))$$

is an isometric homomorphism of Banach algebras.

Proof. The homomorphism property is nothing else than associativity of convolution, i.e., $f*(g*h) = (f*g)*h$. This follows from Fubini's theorem. Contractivity is Young's inequality above, and isometry is proved using an approximate identity of norm 1. □

Given $\mu \in \mathbf{M}(\mathbb{R}^d; \mathcal{L}(X,Y))$ and $f \in \mathbf{L^1}(\mathbb{R}^d; X)$ we have

$$\mathcal{F}(\mu * f) = \widehat{\mu} \cdot \widehat{f} \ \in \ \mathbf{C_0}(\mathbb{R}^d; Y)$$

by an easy computation. Defining $m := \widehat{\mu}$ and $T_m := (f \longmapsto \mu * f)$ we have

$$T_m f = \mathcal{F}^{-1}(m\widehat{f})$$

for all Schwartz functions $f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$. Hence the operator T_m acts as multiplication on the Fourier transform side and is therefore an example of a so-called *Fourier multiplier operator*. It is practically impossible to give this term a precise mathematical meaning covering all cases where intuitively it should apply. In the next section we give a definition which is enough for our purposes.

E.4 Bounded Fourier Multiplier Operators

Let X, Y be Banach spaces, and let $d \in \mathbb{N}$. Let $m \in \mathbf{L^\infty}(\mathbb{R}^d; \mathcal{L}(X,Y))$. We consider the map

$$T_m := \Big(f \longmapsto \mathcal{F}^{-1}[m\widehat{f}]\Big) : \boldsymbol{\mathcal{S}}(\mathbb{R}^d; X) \longrightarrow \mathbf{C_0}(\mathbb{R}^d; Y),$$

write $T_m \sim m$, and call T_m a **Fourier multiplier operator** and m the **symbol** of T_m.

The function m is called a (bounded) $\mathbf{L^p}(X,Y)$**-Fourier multiplier** if there is a constant $c = c_p$ such that

$$\|T_m f\|_{\mathbf{L^p}} \le c_p \, \|f\|_{\mathbf{L^p}} \qquad (f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d; X)).$$

(It suffices to have such an estimate for f taken from a dense subspace of $\boldsymbol{\mathcal{S}}$, e.g. from $\boldsymbol{\mathcal{D}}(\mathbb{R}^d; X)$.) In this case, T_m extends uniquely to a bounded (translation-invariant) operator

$$T_m : \begin{cases} \mathbf{L^p}(\mathbb{R}^d; X) \longrightarrow \mathbf{L^p}(\mathbb{R}^d; Y) & \text{in the case where } p \in [1, \infty), \\ \mathbf{C_0}(\mathbb{R}^d; X) \longrightarrow \mathbf{C_0}(\mathbb{R}^d; Y) & \text{in the case where } p = \infty. \end{cases}$$

Let us define

$$\mathcal{M}_p(\mathbb{R}^d; X, Y) := \{m \in \mathbf{L^\infty}(\mathbb{R}^d; \mathcal{L}(X,Y)) \mid m \text{ is a } \mathbf{L^p}(X,Y)\text{-Fourier multiplier}\}$$

together with the norm

$$\|m\|_{\mathcal{M}_p(X,Y)} := \|T_m\|_{\mathcal{L}(\mathbf{L^p}(\mathbb{R}^d;X), \mathbf{L^p}(\mathbb{R}^d;Y))} \, .$$

We list some properties without giving proofs.

1. If $m(s)$ is an $\mathbf{L^p}(X,Y)$-multiplier, then for all $x \in X, y \in Y'$, $\langle y', m(s)x\rangle$ is an $\mathbf{L^p}(\mathbb{R}^d)$-multiplier with $\|\langle y', m(s)x\rangle\|_{\mathcal{M}_p} \le \|m\|_{\mathcal{M}_p(X,Y)} \|x\| \|y'\|$.

2. If $m(s)$ is an $\mathbf{L^p}(X,Y)$-multiplier, then the identity
$$\mathcal{F}T_m f = m\widehat{f}$$
holds for all functions $f \in \mathbf{L^1}(\mathbb{R}^d;X) \cap \mathbf{L^p}(\mathbb{R}^d;X)$ (in the case where $p < \infty$) and for all functions $f \in \mathbf{L^1}(\mathbb{R}^d;X) \cap \mathbf{C_0}(\mathbb{R}^d;X)$ (in the case where $p = \infty$).

3. The space $\mathcal{M}_p(X,Y)$ is a Banach algebra.

4. If X, Y are Hilbert spaces, then
$$\mathcal{M}_2(\mathbb{R}^d;X,Y) = \mathbf{L^\infty}(\mathbb{R}^d;\mathcal{L}(X,Y)) \quad \text{with} \quad \|m\|_{\mathcal{M}_2(X,Y)} = \|m\|_{\mathbf{L^\infty}}$$
due to Plancherel's theorem. If $p \ne 1,2$ or if $p = 2$ and Y is not a Hilbert space, there exists no nice characterisation.

5. If $m(s)$ is a scalar-valued $\mathbf{L^p}(\mathbb{R}^d)$-Fourier multiplier, then also $\overline{m(s)}$ is one.

6. If a scalar-valued function $m \in \mathbf{L^\infty}(\mathbb{R}^d)$ is an $\mathbf{L^p}(\mathbb{R}^d)$-Fourier multiplier, then it is also an $\mathbf{L^{p'}}(\mathbb{R}^d)$-Fourier multiplier, where p' is the conjugated exponent. (See the proof of Theorem E.5.4 below.) Moreover, $\|m\|_{\mathcal{M}_p} = \|m\|_{\mathcal{M}_{p'}}$. This means
$$\mathcal{M}_1(\mathbb{R}^d) = \mathcal{M}_\infty(\mathbb{R}^d) \subset \mathcal{M}_p(\mathbb{R}^d) = \mathcal{M}_{p'}(\mathbb{R}^d) \subset \mathcal{M}_2(\mathbb{R}^d) = \mathbf{L^\infty}(\mathbb{R}^d)$$
for $p \in [1,\infty]$, with contractive inclusions.

7. $\mathcal{M}_p(\mathbb{R}^d;X,Y) \subset \mathbf{L^\infty}(\mathbb{R}^d;\mathcal{L}(X,Y))$ with contractive inclusion. (See the proof of Corollary E.5.5 below.)

8. The Fourier transform yields an isometric isomorphism $\mathcal{F} : \mathbf{M}(\mathbb{R}^d) \longrightarrow \mathcal{M}_1(\mathbb{R}^d)$. In particular, $m \in \mathbf{L^\infty}(\mathbb{R}^d)$ is an $\mathbf{L^1}(\mathbb{R}^d)$-multiplier if, and only if there is $\mu \in \mathbf{M}(\mathbb{R}^d)$ such that $T_m f = \mu * f$ for all $f \in \mathbf{L^1}(\mathbb{R}^d)$. (See the proof of Theorem E.5.4 below.)

9. A scalar function $m \in \mathbf{L^\infty}$ is an $\mathbf{L^1}(\mathbb{R}^d)$-multiplier if, and only if it is an $\mathbf{L^1}(\mathbb{R}^d;X)$-multiplier.

The following lemma is often useful.

Lemma E.4.1. *Let $p \in [1,\infty]$, and let X, Y be Banach spaces.*

a) *If $m = m(s) \in \mathcal{M}_p(\mathbb{R}^d;X,Y)$, then for all $\varepsilon > 0, a \in \mathbb{R}^d$ one has*
$$m(-s),\ m(\varepsilon s),\ e^{ia\cdot s}m(s) \ \in\ \mathcal{M}_p(\mathbb{R}^d;X,Y)$$
and $$\|m(-s)\|_{\mathcal{M}_p} = \|m(\varepsilon s)\|_{\mathcal{M}_p} = \left\|e^{ia\cdot s}m(s)\right\|_{\mathcal{M}_p} = \|m\|_{\mathcal{M}_p}.$$

b) *If* $(m_n)_n \subset \mathcal{M}_p(\mathbb{R}^d; X, Y)$ *such that* $m_n \to m$ *pointwise almost everywhere and* $\sup_{n\in\mathbb{N}} \|m_n\|_{\mathcal{M}_p} < \infty$*, then also* $m \in \mathcal{M}_p(\mathbb{R}^d; X, Y)$*, and one has the estimate* $\|m\|_{\mathcal{M}_p} \le \sup_{n\in\mathbb{N}} \|m_n\|_{\mathcal{M}_p}$.

Proof. a) For $f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d; X)$ we compute

$$\mathcal{F}^{-1}[(\mathcal{S}m)\widehat{f}] = \mathcal{F}^{-1}\mathcal{S}\left(m \cdot \mathcal{F}(\mathcal{S}f)\right) = \mathcal{S}\mathcal{F}^{-1}\left(m \cdot \mathcal{F}(\mathcal{S}f)\right),$$

whence $\left\|\mathcal{F}^{-1}[(\mathcal{S}m)\widehat{f}]\right\|_p = \left\|\mathcal{F}^{-1}\left(m \cdot \mathcal{F}(\mathcal{S}f)\right)\right\|_p \le \|m\|_{\mathcal{M}_p} \|\mathcal{S}f\|_p = \|m\|_{\mathcal{M}_p} \|f\|_p$. Similarly

$$\begin{aligned}\mathcal{F}^{-1}[m(\varepsilon s)\widehat{f}] &= \mathcal{F}^{-1}[(U_\varepsilon m)\widehat{f}] = \mathcal{F}^{-1}U_\varepsilon[mU_\varepsilon^{-1}\widehat{f})] = \varepsilon^{-d}U_\varepsilon^{-1}\mathcal{F}^{-1}[m\varepsilon^d\mathcal{F}(U_\varepsilon f)]\\ &= U_\varepsilon^{-1}\mathcal{F}^{-1}[m\mathcal{F}(U_\varepsilon f)].\end{aligned}$$

This implies that

$$\left\|\mathcal{F}^{-1}[m(\varepsilon s)\widehat{f}]\right\|_p = \varepsilon^{\frac{d}{p}}\left\|\mathcal{F}^{-1}[m\mathcal{F}(U_\varepsilon f)]\right\|_p \le \varepsilon^{\frac{d}{p}}\|m\|_{\mathcal{M}_p}\|U_\varepsilon f\|_p = \|m\|_{\mathcal{M}_p}\|f\|_p .$$

Finally, $\mathcal{F}^{-1}(e^{ia\cdot s}g(s)) = \mathcal{F}^{-1}(g)(\cdot + a)$ and the translations are isometric on $\mathbf{L^p}$. Hence the assertion for $e^{ia\cdot s}m(s)$ follows.

b) The hypotheses imply in particular that $\sup_n \|m_n\|_\infty < \infty$. Hence $m \in \mathbf{L^\infty}$ and $\mathcal{F}^{-1}(m_n\widehat{f}) \to \mathcal{F}^{-1}(m\widehat{f})$ in $\mathbf{C_0}$, for any $f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d; X)$. If $p = \infty$, there is nothing more to do. If $p < \infty$ one employs Fatou's theorem to conclude that

$$\left\|\mathcal{F}^{-1}(m\widehat{f})\right\|_p^p \le \liminf_n \left\|\mathcal{F}^{-1}(m_n\widehat{f})\right\|_p^p \le [\sup_n \|m_n\|_{\mathcal{M}_p}]^p \|f\|_p^p . \qquad \square$$

Although we have the nice characterisation $\mathcal{M}_1(\mathbb{R}^d) = \mathcal{F}\mathbf{M}(\mathbb{R}^d)$, it is often very difficult to establish that a given continuous function m is in fact an $\mathbf{L^1}$-multiplier. The next theorem is very useful in this context. It is based on the so-called *Bernstein lemma* [10, Lemma 8.2.1].

Theorem E.4.2. *Let* $d \in \mathbb{N}$*, and define* $k := \min\{j \in \mathbb{N} \mid j > d/2\}$*. For* $\delta > 0$ *define*

$$\mathcal{M}^\delta := \{m \in \mathbf{C^k}(\mathbb{R}^d) \mid |m|_{\mathcal{M}^\delta} < \infty\},$$

where

$$|m|_{\mathcal{M}^\delta} := \max_{|\alpha|\le k} \sup_{t\in\mathbb{R}^d} |t|^{|\alpha|+\delta} |D^\alpha m(t)| .$$

Then $\mathcal{M}^\delta \hookrightarrow \mathcal{F}\mathbf{L^1} \subset \mathcal{M}_1(\mathbb{R}^d)$*.*

Proof. The proof can be found in [10, Proposition 8.2.3]. $\square$

Finally, let us turn to a result which often helps to see that a given function m can*not* be an $\mathbf{L^1}$-Fourier multiplier. We restrict ourselves to the one-dimensional case.

Proposition E.4.3. *Let $m = \widehat{\mu}$ for some $\mu \in \mathbf{M}(\mathbb{R})$. Then m is uniformly continuous and bounded, and the Cesaro-limits*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T m(s)\,ds \qquad \textit{and} \qquad \lim_{T\to\infty} \frac{1}{T}\int_{-T}^0 m(s)\,ds$$

both exist and are equal to $\mu\{0\}$. In particular, if both limits $\lim_{s\to\infty} m(s)$ and $\lim_{s\to-\infty} m(s)$ exist, these limits are the same.

Proof. We use Fubini's theorem to compute

$$\begin{aligned}\frac{1}{T}\int_0^T m(s)\,ds &= \int_{\mathbb{R}} \frac{1}{T}\int_0^T e^{-ist}\,ds\,\mu(dt) = \int_{t\neq 0} \frac{1}{T}\int_0^T e^{-ist}\,ds\,\mu(dt) + \mu\{0\}\\ &= \mu\{0\} + \int_{t\neq 0} \frac{1-e^{-itT}}{itT}\,\mu(dt).\end{aligned}$$

The second summand tends to 0 as $T \to \infty$, by the Dominated Convergence Theorem. $\square$

E.5 Some Pseudo-singular Multipliers

Very often one is given an operator T which can be represented as $Tf = \mathcal{F}^{-1}m\widehat{f}$ only for a very small subset of functions f. What is more, it may be unknown in the beginning whether m is bounded or not, and m may be allowed to have certain singularities, at least in principle. We examine such a situation in the following, namely allowing m to have a singularity at 0. We start with the appropriate 'small set' of functions and define

$$W(X) := \{f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d; X) \mid \operatorname{supp}\widehat{f} \text{ is compact and } 0 \notin \operatorname{supp}\widehat{f}\}.$$

If we intend $X = \mathbb{C}$ we simply write W in the following. It is clear that $W(X)$ is a subspace of

$$\mathbf{L^1_0}(\mathbb{R}^d; X) := \{f \in \mathbf{L^1}(\mathbb{R}^d; X) \mid \widehat{f}(0) = 0\},$$

which in turn is a closed (translation invariant) subspace of $\mathbf{L^1}(\mathbb{R}^d; X)$. To be able to prove that $W(X)$ is dense in $\mathbf{L^1_0}(\mathbb{R}^d; X)$, we need the following preparatory lemma.

Lemma E.5.1. *Let $f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d; X)$. Then $\widehat{f}(0) = 0$ if, and only if there exist $(f_j)_{j=1}^d \subset \boldsymbol{\mathcal{S}}(\mathbb{R}^d; X)$ such that $f = \sum_{j=1}^d D_j f_j$.*

Proof. If $f = \sum_j D_j f_j$ one has $\widehat{f}(s) = \sum_j is_j \widehat{f_j}(s)$, implying $\widehat{f}(0) = 0$. To prove the reverse implication, let $g := \widehat{f} \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d; X)$. Then $g(0) = 0$ and

$$g(s) = \int_0^1 \frac{d}{dr} f(rs)\,dr = \sum_j s_j \int_0^1 (D_j g)(rs)\,dt = \sum_j s_j g_j(s)$$

with bounded $\mathbf{C}^\infty$-functions g_j. Choose $\varphi \in \boldsymbol{\mathcal{D}}(\mathbb{R}^d)$ with $\varphi \equiv 1$ near 0. Then

$$g = \varphi g + (1-\varphi)g = \sum_j s_j \varphi g_j + \sum_j s_j \left\{ \frac{s_j(1-\varphi)}{|s|^2} g \right\}.$$

The functions $s_j(1-\varphi)/|s|^2$ are easily seen to be of tempered growth, whence we have found $h_j \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d; X)$ such that $g = \sum_j i s_j h_j$. Taking inverse Fourier transforms and writing $f_j = \mathcal{F}^{-1} h_j$ yields $f = \sum_j D_j f_j$. $\square$

We can now construct an approximate identity in $\mathbf{L}^1_0(\mathbb{R}^d)$ as follows. Choose $\psi \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ with $\widehat{\psi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^d)$ and $\widehat{\psi} \equiv 1$ near 0, and define $\psi_\varepsilon(t) := \varepsilon^{-d}\psi(t/\varepsilon)$ for $\varepsilon > 0$. It is well known that $(\psi_\varepsilon)_\varepsilon$ is an approximate identity in $\mathbf{L}^1(\mathbb{R}^d)$, as $\varepsilon \to 0$. We define

$$\varphi_n := \psi_{1/n} - \psi_n.$$

Then $\widehat{\varphi_n}(s) = \widehat{\psi}(s/n) - \widehat{\psi}(ns)$, hence $\varphi_n \in W$.

Lemma E.5.2. *Let $f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d; X)$ and $f_n := \varphi_n * f \in W(X)$. Then*

$$\|f_n - f\|_p \to 0 \quad (1 < p \leq \infty) \quad \textit{and} \quad \|\widehat{f_n} - \widehat{f}\|_p \to 0 \quad (1 \leq p < \infty).$$

If moreover $\widehat{f}(0) = 0$, one has in addition

$$\|f_n - f\|_1 \to 0 \quad \textit{and} \quad \|\widehat{f_n} - \widehat{f}\|_\infty \to 0.$$

Hence the space $W(X)$ is dense in $\mathbf{L}^1_0(\mathbb{R}^d; X)$, in $\mathbf{L}^p(\mathbb{R}^d; X)$ for each $p \in (1, \infty)$, and in $\mathbf{C}_0(\mathbb{R}^d; X)$.

Proof. We examine the behaviour of $\psi_\varepsilon * f$ and of $\mathcal{F}(\psi_\varepsilon * f) = \widehat{\psi_\varepsilon}\widehat{f}$ as $\varepsilon \to 0$ and as $\varepsilon \to \infty$.

As $\varepsilon \to 0$, $\|\psi_\varepsilon * f - f\|_p \to 0$ for *all* $p \in [1, \infty]$. (This is a well known fact about approximate identities in $\mathbf{L}^1$.) On the Fourier side we obtain $\widehat{\psi_\varepsilon}(s) = \widehat{\psi}(\varepsilon s) \to 1$ uniformly on compact subsets of $\mathbb{R}^d$; this implies that $\|\widehat{\psi_\varepsilon}\widehat{f} - \widehat{f}\|_p \to 0$ for all $p \in [1, \infty]$.

What happens as $\varepsilon \to \infty$? On the Fourier side we see that $\widehat{\psi_\varepsilon}(s) = \widehat{\psi}(\varepsilon s) \to 0$ uniformly on compact subsets of $\mathbb{R}^d \setminus \{0\}$. This yields $\|\widehat{\psi_\varepsilon}\widehat{f}\|_p \to 0$ whenever $1 \leq p < \infty$. However, if $\widehat{f}(0) = 0$, then the convergence holds also for $p = \infty$.

Considering as before the case that $\varepsilon \to \infty$, we have

$$\|\psi_\varepsilon * f\|_p \leq \|\psi_\varepsilon\|_p \|f\|_1 = \varepsilon^{-d/p'} \|\psi\|_p \|f\|_1 \to 0$$

in the case where $1 < p \le \infty$. If $\widehat{f}(0) = 0$, by Lemma E.5.1 we can find $(f_j)_{j=1}^d \subset \boldsymbol{\mathcal{S}}(\mathbb{R}^d; X)$ such that $f = \sum_j D_j f_j$. Hence

$$\begin{aligned} \|\psi_\varepsilon * f\|_1 &\le \sum_j \|\psi_\varepsilon * D_j f_j\|_1 = \sum_j \|D_j \psi_\varepsilon * f_j\|_1 \\ &\le \sum \|D_j \psi_\varepsilon\|_1 \|f_j\|_1 = \varepsilon^{-1} \sum_j \|D_j \psi\|_1 \|f\|_1 \to 0. \end{aligned}$$

The remaining statements follow from the density of $\boldsymbol{\mathcal{S}}(\mathbb{R}^d; X)$ in $\mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X)$, $p \in (1, \infty)$, and in $\mathbf{C_0}(\mathbb{R}^d; X)$, and the fact that $\boldsymbol{\mathcal{S}}(\mathbb{R}^d; X) \cap \mathbf{L_0^1}(\mathbb{R}^d; X)$ is dense in $\mathbf{L_0^1}(\mathbb{R}^d; X)$. This may not be clear on first glance, so we give a reason. Let $f \in \mathbf{L_0^1}$, and take any sequence $(f_n) \subset \boldsymbol{\mathcal{S}}$ such that $\|f - f_n\|_1 \to 0$. Then $x_n := \int f_n \to \int f = 0$. Choose $h \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ such that $\int h = 1$. Then $g_n := f_n - h \otimes x_n \to f$ and obviously $g_n \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d; X)$ with $\int g_n = 0$. $\square$

Corollary E.5.3. *The sequence $(\varphi_n)_n$ constructed above is an approximate identity in $\mathbf{L_0^1}(\mathbb{R}^d)$. Moreover, $\|\varphi * f - f\|_p \to 0$ whenever $f \in \mathbf{L}^{\boldsymbol{p}}(\mathbb{R}^d; X)$, $p \in (1, \infty)$, or $f \in \mathbf{C_0}(\mathbb{R}^d; X)$, $p = \infty$.*

We return to the main theme. Each function $m \in \mathbf{L}^{\mathbf{1}}_{\text{loc}}(\mathbb{R}^d \setminus \{0\}; \mathcal{L}(X, Y))$ determines an operator $T_m : W(X) \longrightarrow \mathbf{C_0}(\mathbb{R}^d; X)$ by

$$T_m f := \mathcal{F}^{-1}(m\widehat{f}) \qquad (f \in W(X)).$$

The next results show what happens if T_m is $\mathbf{L}^{\boldsymbol{p}}$-bounded. We begin with the scalar-valued case.

Theorem E.5.4. *Let $m \in \mathbf{L}^{\mathbf{1}}_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$, and let $p \in [1, \infty]$. Suppose that for some constant $c > 0$ we have*

$$\|\mathcal{F}^{-1} m\widehat{f}\|_p \le c\,\|f\|_p \tag{E.1}$$

for all $f \in W$. Then $m \in \mathbf{L}^{\boldsymbol{\infty}}$ and $m \in \mathcal{M}_p(\mathbb{R}^d)$. In the case where $p > 1$ one has $\|m\|_{\mathbf{L}^{\boldsymbol{\infty}}} \le c$ and (E.1) *holds for all $f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$.*

Proof. For $f, g \in W$ a short computation yields $\langle \mathcal{F}^{-1}(m\widehat{g}), f\rangle = \langle \mathcal{S}g, \mathcal{F}^{-1}(m\mathcal{S}f)\rangle$. This implies that

$$\left|\langle \mathcal{F}^{-1}(m\widehat{g}), f\rangle\right| \le c\,\|g\|_{p'}\,\|f\|_{p'}\,.$$

In the case where $1 < p \le \infty$, by the density of W in the corresponding space, we obtain

$$\left\|\mathcal{F}^{-1}(m\widehat{g})\right\|_{p'} \le c\,\|g\|_{p'} \qquad (g \in W).$$

Let us consider now the case where $1 < p < \infty$. Since W is simultaneously dense in $\mathbf{L}^{\boldsymbol{p}}$ and $\mathbf{L}^{\boldsymbol{p'}}$, the operator $T_m := \mathcal{F}^{-1} m \mathcal{F}$ extends to bounded operators on $\mathbf{L}^{\boldsymbol{p}}$ and on $\mathbf{L}^{\boldsymbol{p'}}$ in such a way that these extensions coincide on the intersection. By the Riesz–Thorin interpolation theorem, one then has also a bound on $\mathbf{L^2}$, i.e.

$$\left\|\mathcal{F}^{-1} m\widehat{f}\right\|_2 \le c\,\|f\|_2 \qquad (f \in W).$$

However, since the Fourier transform is an almost isometric isomorphism on $\mathbf{L^2}$, one obtains

$$\|m\varphi\|_2 \le c\,\|\varphi\|_2$$

for all $\varphi \in \boldsymbol{\mathcal{D}}(\mathbb{R}^d \setminus \{0\})$. This shows that the (usual) multiplier operator induced by m on $\mathbf{L^2}$ is bounded, whence $\|m\|_\infty \le c$. With this information one can now use Lemma E.5.2 to prove that (E.1) holds actually for all $f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$.

Let $p = 1$, and consider the sequence $g_n := T_m(\varphi_n)$, $n \in N$, where (φ_n) is the approximate identity in W that has been constructed above. Clearly $(g_n)_n$ is a bounded sequence in $\mathbf{L^1}(\mathbb{R}^d) \subset \mathbf{M}(\mathbb{R}^d)$. Since $\mathbf{M}(\mathbb{R}^d)$ is the dual of $\mathbf{C_0}(\mathbb{R}^d)$, which is a separable Banach space, the closed unit ball of $\mathbf{M}(\mathbb{R}^d)$ with the weak*-topology is compact and metrisable. Hence there is subsequence $(n_k)_k$ and some measure $\mu \in \mathbf{M}(\mathbb{R}^d)$ such that $g_{n_k} \to \mu$ (weak*). It is easily seen by taking Fourier transforms that for every $f \in W$ we have $T_m(\varphi_{n_k} * f) = g_{n_k} * f \to \mu * f$ in some weak sense. But $\varphi_n * f = f$ for large n, hence $T_m f = \mu * f$. By choosing different $f \in W$ we conclude that $m = \widehat{\mu}$, i.e. m is an $\mathbf{L^1}$-Fourier multiplier.

Only the case where $p = \infty$ is left to prove. We have seen in the beginning of this proof that the inequality (E.1) for $p = \infty$ implies the same inequality for $p = 1$. However, we have proved above that this implies that $m = \widehat{\mu}$ for some $\mu \in \mathbf{M}(\mathbb{R}^d)$. Employing Lemma E.5.2 once more yields that (E.1) actually holds for all $f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, i.e., $m \in \mathcal{M}_\infty(\mathbb{R}^d)$. $\square$

Corollary E.5.5. *Let $m \in \mathbf{L^1_{loc}}(\mathbb{R}^d \setminus \{0\}; \mathcal{L}(X,Y))$, and let $p \in (1,\infty]$. Suppose that for some constant $c > 0$ we have*

$$\left\|\mathcal{F}^{-1} m \widehat{f}\right\|_p \le c\,\|f\|_p \tag{E.2}$$

for all $f \in W(X)$. Then $m \in \mathbf{L^\infty}$ and $m \in \mathcal{M}_p(\mathbb{R}^d; X, Y)$.

Proof. Apply Theorem E.5.4 to the scalar multipliers of the form $\langle y', m(\cdot)x\rangle$ to obtain the estimate $|\langle y', m(\cdot)x\rangle| \le c\,\|x\|\,\|y'\|$ for all $x \in X, y' \in Y'$. This shows $\|m\|_{\mathbf{L^\infty}} \le c$. Now use Lemma E.5.2 to obtain the estimate $\|\mathcal{F}^{-1}(m\widehat{f})\|_p \le c\,\|f\|_p$ for *all* $f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d; X)$. $\square$

E.6 The Hilbert Transform and UMD Spaces

One of the 'simplest' non-trivial symbols is given by the function

$$h(s) := -i\,\mathrm{sgn}(s) \qquad (t \in \mathbb{R}).$$

The associated operator $\mathcal{H} := T_h : \boldsymbol{\mathcal{S}}(\mathbb{R}; X) \longrightarrow \mathbf{C_0}(\mathbb{R}; X)$ is called the **Hilbert transform**. It can be computed as

$$\mathcal{H}f(s) = \lim_{\varepsilon \searrow 0} \frac{1}{\pi} \int_{|t| \ge \varepsilon} \frac{f(s-t)}{t}\,dt \qquad (f \in \boldsymbol{\mathcal{S}}(\mathbb{R}; X))$$

(see [80, Chapter 3]). Since the symbol h is not continuous, it cannot be neither an $\mathbf{L^1}$- nor an $\mathbf{L^\infty}$-Fourier multiplier. However, a classical result states that it is an $\mathbf{L^p}(\mathbb{R})$-Fourier multiplier for all $1 < p < \infty$ (cf. [80, Chapter 4]). If X is an arbitrary Banach space, then in general the functon h is not an $\mathbf{L^p}(\mathbb{R}, X)$-Fourier multiplier for *any* $p \in (1, \infty)$. However, if it is an $\mathbf{L^p}(\mathbb{R}; X)$-Fourier multiplier for *some* $p \in (1, \infty)$, then even for *all* $p \in (1, \infty)$. In this case we call the space X a **UMD** space (or an **HT** space).

The name 'UMD' derives from an equivalent characterisation involving so-called *u*nconditional *m*artingale *d*ifferences in X. The independence of p is a consequence of the fact that there is another equivalent characterisation which is 'purely geometric' and does not make reference to p. These results are due to Burkholder [40, 41] and Bourgain [31].

By the Plancherel theorem, each Hilbert space is UMD. Each UMD space X is reflexive (see [187]) and also its dual Y' is UMD. Moreover, if X is UMD, then also $\mathbf{L^p}(\Omega, \mu; X)$ is UMD whenever (Ω, μ) is a measure space and $p \in (1, \infty)$. In particular, each scalar $\mathbf{L^p}$-space is UMD.

Let X be a Banach space. A **Schauder decomposition** of X is a sequence $(\Delta_n)_{n\in\mathbb{Z}}$ of bounded operators on X such that

1) $\Delta_n \Delta_m = \delta_{nm} \Delta_n$ for all $n, m \in \mathbb{Z}$;
2) $x = \sum_{n\in\mathbb{Z}} \Delta_n x$ for every $x \in X$.

If $\mathcal{R}(\Delta_n) =< e_n >$ is one dimensional for each $n \in \mathbb{Z}$, then $(e_n)_{n\in\mathbb{Z}}$ is called a **Schauder basis** of X. If the convergence in 2) is unconditional for every $x \in X$, then the Schauder decomposition/basis is called **unconditional**.

One of the most important properties of a UMD space X is that the spaces $\mathbf{L^p}(\mathbb{R}; X)$ admit of a nice Schauder decomposition, a result which goes back to Bourgain [32].

Theorem E.6.1 (Bourgain). *Let X be a UMD space, and let $p \in (1, \infty)$. Define $I_n := \{t \mid 2^n < |x| \le 2^{n+1}\}$ and $\Delta_n \sim \mathbf{1}_{I_n}$ for all $n \in \mathbb{Z}$. Then $(\Delta_n)_{n\in\mathbb{Z}}$ is an unconditional Schauder decomposition of $\mathbf{L^p}(\mathbb{R}; X)$.*

The decomposition from Theorem E.6.1 is called the **dyadic** or the **Paley–Littlewood** decomposition. This is due to the fact that the case where $X = \mathbb{C}$ is contained in a classical result in Harmonic Analysis due to Paley and Littlewood. Bourgain [32] used the dyadic decomposition to obtain vector-valued multiplier results.

Theorem E.6.2. *Let X be a UMD Banach space.*

a) **(Marcinkiewicz)** *Let $m \in \mathbf{L^\infty}(\mathbb{R})$ such that $\sup_{n\in\mathbb{Z}} \mathrm{Var}(I_n, m) < \infty$. Then for all $p \in (1, \infty)$ the function m is an $\mathbf{L^p}(\mathbb{R}; X)$-Fourier multiplier such that $\|m\|_{\mathcal{M}_p} \le \sup_{n\in\mathbb{Z}} \mathrm{Var}(I_n, m)$.*

b) **(Mikhlin)** *Let* $m \in \mathbf{C^1}(\mathbb{R} \setminus \{0\})$ *such that*

$$c_m := \sup_{s \in \mathbb{R} \setminus \{0\}} |m(s)| + \sup_{s \in \mathbb{R} \setminus \{0\}} |sm'(s)| < \infty.$$

Then for all $p \in (1, \infty)$ *the function* m *is an* $\mathbf{L^p}(\mathbb{R}; X)$*-Fourier multiplier such that* $\|m\|_{\mathcal{M}_p} \leq c_m$.

c) **(Mikhlin, multidimensional)** *Let* $d \in \mathbb{N}$, *and let* $k = \min\{l \in \mathbb{N} \subset k > d/2\}$. *Let* $m \in \mathbf{C^k}(\mathbb{R}^d \setminus \{0\})$ *such that*

$$c_m := \max_{|\alpha| \leq k} \sup_{s \in \mathbb{R}^d \setminus \{0\}} |s|^{|\alpha|} |D^\alpha m(s)| < \infty.$$

Then for all $p \in (1, \infty)$ *the function* m *is an* $\mathbf{L^p}(\mathbb{R}; X)$*-Fourier multiplier such that* $\|m\|_{\mathcal{M}_p} \leq c_m$.

For $X = \mathbb{C}$, part a) is due to Marcinkiewicz, cf. [82], and parts b), c) to Mikhlin, cf. [170]. The UMD-version of c) was proved by Zimmermann [231]. For many years it was an open question how to generalise this theorem to operator-valued symbols. This was finally done by Weis using the concept of R-boundedness.

E.7 R-Boundedness and Weis' Theorem

The **Cantor group** is the set $G := \{-1, 1\}^{\mathbb{Z}}$, i.e., the $\mathbb{Z}$-fold direct product of the multiplicative discrete group $Z_2 \cong \{-1, 1\}$. By Tychonoff's theorem, G is a compact topological group. We denote by μ the normalised Haar measure on G. The projections

$$r_k := ((g_n)_n \longmapsto g_k) : G \longrightarrow \{-1, 1\} \quad (k \in \mathbb{Z})$$

are called **Rademacher functions**. As the Rademachers obviously are continuous characters of the compact group G, they form an orthonormal set in $\mathbf{L^2}(G, \mu)$, i.e.,

$$\int_G r_n r_m \, d\mu = \delta_{nm}$$

for all $n, m \in \mathbb{Z}$. (One can show that the set of Rademachers actually generates the character group of G.) Given any Banach space X, the space $\mathrm{Rad}(X)$ is defined by

$$\mathrm{Rad}(X) := \overline{\mathrm{span}}\{r_n \otimes x \mid n \in \mathbb{Z},\ x \in X\} \subset \mathbf{L^2}(G; X).$$

The so-called **Khintchine–Kahane inequality** asserts, that the norms on $\mathrm{Rad}(X)$ induced by the different embeddings $\langle r_n \mid n \in \mathbb{Z} \rangle \otimes X \subset \mathbf{L^p}(G, X)$ for $1 \leq p < \infty$ are all equivalent, see [153, Part I, Theorem 1.e.13] or [70, 11.1]. We endow $\mathrm{Rad}(X)$ with the $\mathbf{L^p}$-norm which suits best the respective context. If $X = H$ is a Hilbert space we may take $p = 2$ of course.

Lemma E.7.1. *Let $X = H$ be a Hilbert space. Then the identity*

$$\sum\nolimits_n \|x_n\|_H^2 = \Big\| \sum\nolimits_n r_n \otimes x_n \Big\|_{\mathrm{Rad}(H)}^2 \tag{E.3}$$

holds for every finite *two-sided sequence $(x_n)_{n\in\mathbb{Z}} \subset H$.*

Proof. In fact,

$$\begin{aligned} \Big\| \sum\nolimits_n r_n \otimes x_n \Big\|_{\mathrm{Rad}(H)}^2 &= \int_G \Big\| \sum\nolimits_n r_n(g) x_n \Big\|_H^2 \; \mu(dg) \\ &= \sum\nolimits_{n,m} \int_G r_n(g) r_m(g) \,(\, x_n \,|\, x_m \,)\; \mu(dg) \\ &= \sum\nolimits_{n,m} \delta_{nm} \,(\, x_n \,|\, x_m \,) \;=\; \sum\nolimits_n \|x_n\|_H^2 \,. \end{aligned}$$

□

The space (G, μ) is a probability space and the Rademachers $(r_n)_{n\in\mathbb{Z}}$ form a sequence of independent, symmetric, $\{-1, 1\}$-valued random variables on G. The notion of R-boundedness introduced below uses only this feature of the Rademachers, and the actual underlying probability space is irrelevant. One therefore often writes

$$\mathbb{E}\Big\| \sum\nolimits_j r_j \otimes x_j \Big\| \quad \text{instead of} \quad \int_G \Big\| \sum\nolimits_j r_j \otimes x_j \Big\| \; d\mu.$$

Let X, Y be Banach spaces, and let $\mathcal{T}$ be a set of operators in $\mathcal{L}(X, Y)$. The set $\mathcal{T}$ is called *R–bounded* if there is a constant C such that for all finite sets $J \subset \mathbb{N}$ and sequences $(T_j)_{j\in J} \subset \mathcal{T}$ and $(x_j)_{j\in J} \subset X$ one has

$$\mathbb{E} \, \Big\| \sum\nolimits_{j\in J} r_j T_j x_j \Big\|_Y \le C \, \mathbb{E} \, \Big\| \sum\nolimits_{j\in J} r_j x_j \Big\|_X . \tag{E.4}$$

The infimum of all such constants C is called the *R–bound* of the set $\mathcal{T}$, and is denoted by $[\![\, \mathcal{T} \,]\!]^{\mathcal{R}}_{X\to Y}$. If the reference to the spaces is clear, one simply writes $[\![\, \mathcal{T} \,]\!]^{\mathcal{R}}$ for the R-bound.

Remarks E.7.2. 1) By Kahane's inequality [70, 11.1] one obtains an equivalent definition when instead of (E.4) one requires the inequality

$$\left(\mathbb{E} \, \Big\| \sum\nolimits_{j\in J} r_j T_j x_j \Big\|_Y^p \right)^{\frac{1}{p}} \le C \, \left(\mathbb{E} \, \Big\| \sum\nolimits_{j\in J} r_j x_j \Big\|_X^p \right)^{\frac{1}{p}}$$

for a fixed $p \in (1, \infty)$ (cf., e.g., [141]). Of course, the actual R-bound changes with different values of p.

2) The so-called *Kahane's contraction principle* states that for each $c > 0$ the set $\{zI \mid |z| \le c\}$ is R-bounded in $\mathcal{L}(X)$ with R-bound $\le 2c$. (One can remove the 2 if only real scalars are considered, see [141, Proposition 2.5].)

3) The R–boundedness of $\mathcal{T}$ in $\mathcal{L}(X)$ implies the uniform boundedness of $\mathcal{T}$ in $\mathcal{L}(X)$, but the converse holds only in Hilbert spaces. However, if X has cotype 2 and Y has type 2, then the R–boundedness and boundedness of $\mathcal{T}$ in $\mathcal{L}(X,Y)$ are equivalent.

One of the main features of R-bounded sets of operators is the following result, due to de Pagter, Clément, Sukochev and Witvliet [44].

Proposition E.7.3. *Let $(\Delta_n)_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$ be an unconditional Schauder decomposition of the Banach space X, and let $\mathcal{T} \subset \mathcal{L}(X)$ be R-bounded with $T\Delta_n = \Delta_n T$ for all $n \in \mathbb{Z}, T \in \mathcal{T}$. Then each sequence $(T_n)_{n\in\mathbb{Z}} \subset \mathcal{T}$ induces via*

$$x \longmapsto \sum\nolimits_{n\in\mathbb{Z}} T_n \Delta_n x$$

a bounded operator on X.

If one combines Proposition E.7.3 with the dyadic Schauder decomposition one can prove the following result, due to Weis [223].

Theorem E.7.4 (Weis). *Let X, Y be UMD Banach spaces, and let $p \in (1,\infty)$. Then there is constant C with the following property. Whenever $m \in \mathbf{C^1}(\mathbb{R}\setminus\{0\})$ is such that*

$$\begin{aligned} c_1 &:= [\![\, m(t), \mid 0 \neq t \in \mathbb{R} \,]\!]^{\mathcal{R}}_{X\to Y} < \infty \quad \textit{and} \\ c_2 &:= [\![\, tm'(t), \mid 0 \neq t \in \mathbb{R} \,]\!]^{\mathcal{R}}_{X\to Y} < \infty, \end{aligned}$$

then m is an $\mathbf{L^p}$-Fourier multiplier for all $p \in (1,\infty)$ with $\|m\|_{\mathcal{M}_p} \leq C \max(c_1, c_2)$.

Note that part b) of Theorem E.6.2 is a consequence of Theorem E.7.4, by the contraction principle.

References

The theory of scalar-valued symbols on scalar $\mathbf{L^p}$-spaces is classical. One can consult the article [120] of Hörmander (or his books) or [205]. See also [10, Appendix E] for a survey.

More details and also higher dimensional results on the recent work on operator-valued symbols can be found in [141]. An exposition with complete proofs is Hytönens Master's thesis [121].

Appendix F

Approximation by Rational Functions

In this appendix we provide some results from approximation theory. The objective is to approximate a given continuous function f on a compact subset K of the Riemann sphere $\mathbb{C}_\infty$ in some sense by rational functions. Unfortunately, there is not enough room to develop the necessary complex function theory. Hence we have to refer to the literature. However, we could not find any account of the topic that served our purposes perfectly. Therefore, we shall take two results from the book [94] of Gamelin as a starting point and modify them according to our needs.

Note that a subset $K \subset \mathbb{C}_\infty$ is called **finitely connected** if $\mathbb{C}_\infty \setminus K$ has a finite number of connected components. If $K \subset \mathbb{C}_\infty$ is compact, we consider

$$A(K) := \{f \in \mathbf{C}(K) \mid f \text{ is holomorphic on } \mathring{K}\}.$$

The set of *all* rational functions is denoted by $\mathbb{C}(z)$. We view a rational function $r \in \mathbb{C}(z)$ as a continuous (or holomorphic) function from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$. A point $\lambda \in \mathbb{C}_\infty$ is called a **pole** of r if $r(\lambda) = \infty$. Given *any* subset $K \subset \mathbb{C}_\infty$ we define

$$\mathcal{R}(K) := \{r \in \mathbb{C}(z) \mid r(K) \subset \mathbb{C}\}$$

to be the set of rational functions with poles lying outside M. If K is compact, we denote by $R(K)$ the closure of $\mathcal{R}(K)$ in $\mathbf{C}(K)$. Then $A(K)$ is a closed subalgebra of $\mathbf{C}(K)$ with $R(K) \subset A(K)$.

Proposition F.1. [94, Chapter II, Theorem 10.4] *Let $K \subset \mathbb{C}$ be compact and finitely connected. Then $A(K) = R(K)$, i.e., each function $f \in \mathbf{C}(K)$ that is holomorphic on $\mathring{K}$ can be approximated uniformly on K by rational functions r_n which have poles outside K.*

The other result we need is concerned with pointwise bounded approximation. We say that a sequence f_n of functions on a set $\Omega \subset \mathbb{C}_\infty$ converges **boundedly and pointwise** on Ω to a function f, if $\sup_n \sup_{z\in\Omega} |f_n(z)| < \infty$ and $f_n(z) \to f(z)$ for all $z \in \Omega$. For $\Omega \subset \mathbb{C}_\infty$ open we let

$$H^\infty(\Omega) := \{f : \Omega \to \mathbb{C} \mid f \text{ is bounded and holomorophic}\}$$

be the Banach algebra of bounded holomorphic functions on Ω. We occasionally write $\|f\|_\Omega = \|f\|_{\infty,\Omega}$ to denote the supremum norm of $f \in H^\infty(\Omega)$.

Proposition F.2. [94, Chapter VI, Theorem 5.3] *Let $K \subset \mathbb{C}$ be compact and finitely connected. Then for every $f \in H^\infty(\mathring{K})$ there is a sequence of rational functions r_n with poles outside K such that $\|r_n\|_K \le \|f\|_{\mathring{K}}$ and $r_n \to f$ pointwise on $\mathring{K}$. In particular, $r_n \to f$ pointwise boundedly.*

Propositions F.1 and F.2 refer only to subsets K of the plane $\mathbb{C}$. But one can easily extend these results to (strict) subsets $K \subset \mathbb{C}_\infty$ of the Riemann sphere by a rational change of coordinates, a so-called **Möbius transformation**. These are the mappings

$$m(a,b,c,d) := \left(z \longmapsto \frac{az+b}{cz+d} \right) : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$$

with complex numbers $a,b,c,d \in \mathbb{C}$, $ad-bc \neq 0$. It is well known that each Möbius transformation is invertible, its inverse being again a Möbius tranformation. In particular, they are homeomorphisms of $\mathbb{C}_\infty$.

Let $K \subset \mathbb{C}_\infty$ be compact. If $\infty \in K \subset \mathbb{C}_\infty$ and K is not the whole sphere, there is some $p \in \mathbb{C} \setminus K$. The transformation $\varphi := m(0,1,1,-p)$ has the property that $\varphi(K)$ is a compact subset of $\mathbb{C}$. Because φ is a homeomorphism, $\varphi(\mathring{K}) = \varphi(K)^\circ$. Moreover, K is finitely connected if, and only if $\varphi(K)$ is. If r is a rational function with poles outside $\varphi(K)$, $r \circ \varphi$ is a rational function with poles outside K, and one has $\|r\|_{\varphi(K)} = \|r \circ \varphi\|_K$. Finally, if $f : K \to \mathbb{C}$, then $f \in A(K)$ if, and only if $f \circ \varphi^{-1} \in A(\varphi(K))$ and $f \in H^\infty(\mathring{K})$ if, and only if $f \circ \varphi^{-1} \in H^\infty(\varphi(K)^\circ)$.

These considerations show that Propositions F.1 and F.2 remain true for subsets $K \subset \mathbb{C}_\infty$ with $K \neq \mathbb{C}_\infty$.

We now deal with some special sets K. Let $\Omega \subset \mathbb{C}$ be open. We denote by K the closure of Ω in $\mathbb{C}_\infty$, while we keep the notation $\overline{\Omega}$ for the closure of Ω in $\mathbb{C}$. We suppose that

$$K \neq \mathbb{C}_\infty, \quad \infty \in K, \quad \Omega = \mathring{K}, \quad \text{and} \quad K \text{ is finitely connected.}$$

For example, Ω can be anything from the following list.

- $S_\omega = \{z \mid z \neq 0,\ |\arg z| < \omega\}$.
- $H_\omega = \{z \mid |\operatorname{Im} z| < \omega\}$ the horizontal strip of height 2ω, symmetric about the real line, where $\omega > 0$ is arbitrary.
- $\Sigma_\omega = S_\omega \cup -S_\omega$ a double sector, where $\omega < \pi/2$.
- $\Pi_\omega = \{z \mid (\operatorname{Im} z)^2 < 4\omega^2 \operatorname{Re} z\}$ a horizontal parabola, where $\omega > 0$ is arbitrary.

We define $\mathcal{R}^\infty(\Omega) := \mathcal{R}(\Omega) \cap H^\infty(\Omega)$ and $\mathcal{R}_0^\infty(\Omega) := \{r \in \mathcal{R}^\infty(\Omega) \mid r(\infty) = 0\}$. Then it is clear that $\mathcal{R}(K) = \mathcal{R}^\infty(\Omega)$ and

$$\begin{aligned}\mathcal{R}_0^\infty(\Omega) &= \mathcal{R}^\infty(\Omega) \cap \mathbf{C_0}(\overline{\Omega}) \subset \mathcal{R}^\infty(\Omega)\\ &\subset A(K) = \{f \in H^\infty(\Omega) \cap \mathbf{C}(\overline{\Omega}) \mid \lim_{z\to\infty} f(z) \text{ ex.}\}\\ &\subset \{f \in \mathbf{C}(\overline{\Omega}) \mid \lim_{z\to\infty} f(z) \text{ ex.}\} = \mathbf{C}(K).\end{aligned}$$

Proposition F.3. *We have*

$$\mathcal{R}^\infty(\Omega) = \{\frac{p}{q} \mid p, q \in \mathbb{C}[z],\ (q = 0) \cap \overline{\Omega} = \emptyset,\ \deg(p) \le \deg(q)\} \tag{F.1}$$

and

$$\mathcal{R}_0^\infty(\Omega) = \{\frac{p}{q} \mid p, q \in \mathbb{C}[z],\ (q = 0) \cap \overline{\Omega} = \emptyset,\ \deg(p) < \deg(q)\}. \tag{F.2}$$

The algebra $\mathcal{R}_0^\infty(\Omega)$ is generated by the elementary rationals $(\lambda - z)^{-1}$ $(\lambda \notin \overline{\Omega})$. The algebra $\mathcal{R}^\infty(\Omega)$ is generated by the elementary rationals $(\lambda - z)^{-1}$ $(\lambda \notin \overline{\Omega})$ together with the constant $\mathbf{1}$ function. The closure of $\mathcal{R}_0^\infty(\Omega)$ with respect to $\|\cdot\|_\Omega$ is $H^\infty(\Omega) \cap \mathbf{C_0}(\overline{\Omega})$. The closure of $\mathcal{R}^\infty(\Omega)$ with respect to $\|\cdot\|_\Omega$ is $A(K)$.

Proof. Let $r = p/q \in C(z)$ be a rational function. Then r is bounded on Ω if, and only if it is bounded on K (view r as a continuous function from C_∞ to itself). So its poles lie outside K and $r(\infty) \in \mathbb{C}$. This implies that $\deg p \le \deg q$. If $r(\infty) = 0$ it follows that $\deg p < \deg q$. The other inclusions are clear.

Obviously, every elementary rational $(\lambda - z)^{-1}$ with $\lambda \notin \overline{\Omega}$ is contained in $\mathcal{R}_0^\infty(\Omega)$. Since we can write

$$\alpha\frac{\mu - z}{\lambda - z} = \alpha(\frac{\mu - \lambda}{\lambda - z} + \mathbf{1}) \qquad (\alpha, \mu, \lambda \in \mathbb{C},\ \lambda \notin \overline{\Omega})$$

it follows from (F.2) and the Fundamental Theorem of Algebra that the elementary rationals generate $\mathcal{R}_0^\infty(\Omega)$. From (F.1) it is clear that $\mathcal{R}^\infty(\Omega) = \mathcal{R}_0^\infty(\Omega) \oplus \mathbf{1}$.

From Proposition F.1 we know that $R(K) = A(K)$. Let $f \in A(K)$ such that $f(\infty) = 0$. We can find $r_n \in \mathcal{R}^\infty(\Omega)$ such that $\|r_n - f\|_\Omega \to 0$. Since ∞ is in the closure of Ω in $\mathbb{C}_\infty$, this implies that $r_n(\infty) \to f(\infty) = 0$. Hence $\|(r_n - r_n(\infty)) - f\|_\Omega \to 0$ and $r_n - r_n(\infty) \in \mathcal{R}_0^\infty(\Omega)$. $\square$

Proposition F.4. *Let Ω and K be as above, and let $f \in H^\infty(\Omega)$. Then there is a sequence of rational functions $r_n \in \mathcal{R}^\infty(\Omega)$ such that $\|r_n\|_\Omega \le \|f\|_\Omega$ for all n and $r_n \to f$ pointwise on Ω.*

Proof. The statement is just a reformulation of Proposition F.2 combined with the remarks immediately after. $\square$

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