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Continuum Mechanics

Advanced Topics and Research Trends

Antonio Romano
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Preface

In the companion book (*Continuum Mechanics Using Mathematica*®) to this volume, we explained the foundations of continuum mechanics and described some basic applications of fluid dynamics and linear elasticity. However, deciding on the approach and content of this book, *Continuum Mechanics: Advanced Topics and Research Trends*, proved to be a more difficult task. After a long period of reflection, we made the decision to direct our efforts into drafting a book that demonstrates the flexibility and great potential of continuum physics to describe the wide range of macroscopic phenomena that we can observe. It is the opinion of the authors that this is the most stimulating way to learn continuum mechanics. However, it is also quite evident that this aim cannot be fully realized in a single book. Consequently, in this book we chose to present only the basics of interesting continuum mechanics models, along with some important applications of them.

We assume that the reader is familiar with all of the basic principles of continuum mechanics: the general balance laws, constitutive equations, isotropy groups for materials, the laws of thermodynamics, ordinary waves, etc. All of these concepts can be found in *Continuum Mechanics Using Mathematica* and many other books.

We believe that this book gives the reader a sufficiently wide view of the "boundless forest" of continuum mechanics, before focusing his or her attention on the beauty and complex structure of single trees within it (indeed, we could say that *Continuum Mechanics Using Mathematica* provides only the fertile humus on which the trees of this forest take root!).

The topics that we have selected for this book in order to show the power of continuum mechanics to characterize the experimental behavior of real bodies, and the order in which these topics are discussed here, are described below.

In Chap. 1, we discuss some interesting aspects of nonlinear elasticity. We start with the equilibrium equations and their variational formulation and discuss some peculiarities of the boundary value problems of nonlinear elasticity. We then analyze the homogeneous equilibrium solutions of isotropic materials together with the universal equilibrium solutions of Ericksen for compressible elastic materials. Moreover, some experimental results for constitutive equations in nonlinear elasticity are briefly explored. The existence and uniqueness theorems of Van Buren and Stoppelli, as well as Signorini's method, are presented with some recent extensions to live loads. Finally, the chapter concludes with a survey of the propagation of acceleration waves in an elastic body, and a new perturbation method for the analysis of these waves is presented.

In Chap. 2, we discuss the theory of continua with directors, which was proposed at the beginning of the twentieth century by the Cosserat brothers and was subsequently developed by many other authors. In this model, a continuous system S is no longer considered a collection of simple points defined by their coordinates in a frame of reference; instead, S is regarded as a set of complex particles that also possess a certain number of vectors that move independently of the particles with which they are associated. Such a model provides a better description of aggregates of microcrystals, polarized dielectrics, ferromagnetic substances, and one-dimensional and two-dimensional bodies. It can also be applied whenever the system contains a length that: (i) is less than the limit considered in continuum mechanics; (ii) characterizes the dimensions of microscopic regions that influence the macroscopic behavior of the body through their internal evolutions.

In Chap. 3, we consider a simplified model of a continuum with a *nonmaterial* moving surface across which the bulk fields can exhibit discontinuities. The general balance equations of this model are formulated together with the associated local field equations and jump conditions. In Chap. 4, this model is used to describe the phase equilibrium of two different phases. In particular, Maxwell's rule and Clapeyron's equation are derived.

The same model is applied in Chap. 5 to describe dynamical phase changes like melting and evaporation. The related difficult free-boundary problems are stated together with some numerical results.

Chapter 6 introduces the principles of mixture theory. This model, which allows us to describe the evolution of each constituent of a mixture as well as the whole mixture, is very useful in chemistry, biology, and mineralogy (alloys). This chapter contains a proof for the Gibbs rule, together with an analysis of phase equilibrium in a binary mixture.

Chapters 7 and 8 describe the interactions of electric and magnetic fields with matter using a continuum model with a nonmaterial interface. After a general discussion of the different properties resulting from a change of reference frame for the mechanical and electromagnetic equations, the approximations of quasi-electrostatics and quasi-magnetostatics are discussed. In particular, by adopting a continuum mechanics approach, we show that various physical models that have been proposed to explain the behavior of dielectrics and magnetic bodies are actually equivalent from a macroscopic perspective. In other words, different microscopic models can lead to the same macroscopic behavior.

In Chap. 9, we present the macroscopic approach to micromagnetism together with the very difficult mathematical problems associated with this model. Among other things, it is shown that the model of a continuum with a nonmaterial interface can be used to determine the form of Weiss' domains for some crystals and geometries.

Chapter 10 provides an introduction to continua in special relativity. After a brief analysis of the historical motivations of this theory, Minkowski's geometrical model of spacetime is presented. The relativistic balance equations are then formulated in terms of the symmetric momentum–energy four-tensor. After an accurate description of Fermi transport, the intrinsic deformation gradient is introduced, in order to define elastic materials by extending the objectivity principle to special relativity. We then justify the different transformation formulae adopted in the literature for the total work, the total energy and the total heat of an homogeneous system through a wide-ranging discussion of the absolute and relative viewpoints. At the end of this chapter, the fundamental problem of the interaction between matter and electromagnetic fields is analyzed, together with the different models that have been adopted to describe it. Finally, we prove the equivalence of all of these proposals.

There are only a few notebooks written in ***Mathematica***® for this book (which can be downloaded from the publisher's website at http:// www.birkhauser.com/978-0-8176-4869-5), since the topics here discussed are more theoretical in nature than those treated in *Continuum Mechanics Using Mathematica*. However, many of the notebooks associated with that book can also be applied to the topics covered here.

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A. Marasco

Chapter 1

Nonlinear Elasticity

1.1 Preliminary Considerations

In this chapter we focus on the basics of nonlinear elasticity in order to show its interesting mathematical and physical aspects. Readers who are interested in delving deeper into this subject should refer to the many existing books on it (see, for instance, [1]–[15]). We start by listing the main difficulties associated with this subject:

- The equations governing the equilibrium and the motion of an elastic body are nonlinear.
- Instead of being expressed by given functions assigned to the boundary of the region occupied by the elastic body, the boundary conditions are generally functions of the *unknown* deformation.
- Finding the forms of the constitutive equations of an elastic isotropic material is a very complex experimental task. We must determine unknown functions instead of the two Lamé constants that characterize a linearly elastic material.

The nonlinearity of the basic equations of nonlinear elasticity make it difficult to determine explicit solutions for both the equilibrium equations and the motion equations, except in simple cases. For the same reason, it is also an arduous task to prove existence and uniqueness theorems for boundary value problems that can be applied to equilibrium or dynamical problems in nonlinear elasticity. In particular, wave propagation analysis is much more complex than in linear elasticity. In this chapter, we try to analyze all of the above problems. When the subject requires a deeper analysis, references will be suggested.

We assume that the reader is familiar with the foundations of continuum mechanics. Therefore, all of the basic concepts (such as the balance equations) are provided without explanations. If necessary, the reader can

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consult other books on this subject (see, for instance, [1]–[15]); in particular, [16] utilizes the same notation as we have adopted here.

1.2 The Equilibrium Problem

Let S be a homogeneous elastic system in the reference configuration C_*. From now on, S is assumed to be at a constant and uniform temperature. The system S adopts an equilibrium configuration C in the presence of body forces acting on the region C and surface tensions across the whole boundary ∂C or to a part Σ of ∂C. The task of *elastostatics* is to determine the *finite deformation* $\mathbf{x} = \mathbf{x}(\mathbf{X})$, where $\mathbf{X} \in C_*$, $\mathbf{x} \in C$, or, equivalently, the ***displacement*** $\mathbf{u}(\mathbf{X}) = \mathbf{x}(\mathbf{X}) - \mathbf{X}$ that S undergoes when moving from C_* to C under the influence of the applied forces mentioned above. We denote the *deformation gradient* by $\mathbf{F} = (\partial x_i / \partial X_L)$, the *displacement gradient* by $\mathbf{H} = (\partial u_i / \partial X_L) = \mathbf{F} - \mathbf{I}$, and the *right Cauchy–Green tensor* by $\mathbf{C} = \mathbf{F}\mathbf{F}^T$.

The equilibrium equations, the jump conditions across a surface Σ_1 separating two different materials, and the boundary conditions are, respectively:

$$\nabla_{\mathbf{x}} \cdot \mathbf{T} + \rho \mathbf{b} = \mathbf{0}, \qquad \text{in } C - \Sigma_1, \tag{1.1}$$

$$[[\mathbf{T} \cdot \mathbf{n}]] = \mathbf{0}, \qquad \text{on } \Sigma_1, \tag{1.2}$$

$$\mathbf{T} \cdot \mathbf{N} = \mathbf{t}, \qquad \text{on } \Sigma, \tag{1.3}$$

where ρ is the mass density in C, $\mathbf{T}$ is the *Cauchy stress tensor*, $\mathbf{n}$ is the unit vector normal to Σ_1, and $\mathbf{N}$ is the unit vector normal to the part (denoted Σ) of ∂C where surface forces act with a density of $\mathbf{t}$.

It is convenient to use the *Lagrangian equilibrium conditions*, since the unknown function $\mathbf{x} = \mathbf{x}(\mathbf{X})$ depends on the point $\mathbf{X}$ in C_*. Another reason to use these equations is that the forces acting on the part Σ of the boundary ∂C cannot be assigned because ∂C is unknown. The Lagrangian formulation corresponding to (1.1)–(1.3) is expressed by the following equations:[1]

$$\nabla_{\mathbf{X}} \cdot \mathbf{T}_* + \rho_* \mathbf{b} = \mathbf{0}, \qquad \text{in } C_* - \Sigma_{*1}, \tag{1.4}$$

$$[[\mathbf{T}_* \cdot \mathbf{n}_*]] = \mathbf{0}, \qquad \text{on } \Sigma_{*1}, \tag{1.5}$$

$$\mathbf{T}_* \cdot \mathbf{N}_* = \mathbf{t}_*, \qquad \text{on } \Sigma_*, \tag{1.6}$$

where $\mathbf{T}_*$ is the *Piola–Kirchhoff tensor* and ρ_*, $\mathbf{n}_*$, σ_*, $\mathbf{N}_*$, Σ_*, Σ_{*1}, and $\mathbf{t}_*$ are the Lagrangian quantities corresponding to $\mathbf{T}$, ρ, $\mathbf{n}$, σ, $\mathbf{N}$, Σ, Σ_1, and $\mathbf{t}$,

[1]See [16], p. 148.

respectively. In this chapter we will frequently use the following relations:[2]

$$\mathbf{T}_* = J\mathbf{T}(\mathbf{F}^{-1})^T, \tag{1.7}$$

$$d\sigma = J\sqrt{\mathbf{N}_* \cdot \mathbf{C}^{-1}\mathbf{N}_*}\, d\sigma_*, \tag{1.8}$$

$$\mathbf{N} = \frac{(\mathbf{F}^{-1})^T}{\sqrt{\mathbf{N}_* \cdot \mathbf{C}^{-1}\mathbf{N}_*}}\mathbf{N}_*, \tag{1.9}$$

$$\mathbf{t} = \frac{1}{J}\sqrt{\mathbf{N}_* \cdot \mathbf{C}^{-1}\mathbf{N}_*}\,\mathbf{t}_*. \tag{1.10}$$

In a hyperelastic material, the Piola–Kirchhoff stress tensor $\mathbf{T}_*$ is expressed in terms of the specific *elastic potential* ψ by the relation (see [16], p. 161)

$$\mathbf{T}_* = \rho_* \frac{\partial \psi(\mathbf{F})}{\partial \mathbf{F}} = \rho_* \frac{\partial \widehat{\psi}(\mathbf{H})}{\partial \mathbf{H}}, \tag{1.11}$$

where we have introduced the notation $\psi(\mathbf{F}) = \psi(\mathbf{I} + \mathbf{H}) \equiv \widehat{\psi}(\mathbf{H})$.

Substituting (1.11) into (1.4) and introducing the fourth-order *elasticity tensor*

$$A_{ijLM}(\mathbf{H}) = \rho_* \frac{\partial T_{*iL}}{\partial H_{jM}} = \rho_* \frac{\partial^2 \widehat{\psi}}{\partial H_{iL} \partial H_{jM}}, \tag{1.12}$$

we obtain the following second-order quasi-linear partial differential system

$$A_{ijLM}(\mathbf{H}) \frac{\partial^2 u_j}{\partial X_L \partial X_M} + \rho_* b_i = 0, \tag{1.13}$$

whose unknowns are the components u_i, $i = 1, 2, 3$, of the displacement.

One of the main aims of elasticity is to verify that the system (1.13) and the boundary conditions (1.2)–(1.3) allow us to determine (at least in principle) the finite deformation $\mathbf{x} = \mathbf{x}(\mathbf{X})$; i.e., the equilibrium configuration of the body S to which a given load is applied. In other words, we need to establish the conditions for the unknown displacement field that make it possible to prove existence and uniqueness theorems for the boundary value problem obtained by associating the boundary conditions (1.2)–(1.3) with the equilibrium equations (1.13).

In the next section, some specific difficulties of this boundary value problem will be highlighted.

[2] See [16], p. 82, p. 148.

1.3 Remarks About Equilibrium Boundary Problems

We assume that the fields that appear in the equilibrium equations and the boundary conditions are smooth enough to allow us to perform all of the differentiation operations required. Moreover, the boundary part $\partial C_* - \Sigma_*$ is assumed to be fixed or deformed in a known manner. Formally, we write

$$\mathbf{x}(\mathbf{X}) = \mathbf{x}_0(\mathbf{X}) \quad \text{on } \partial C_* - \Sigma_*. \tag{1.14}$$

If $\Sigma_* = \emptyset$, the corresponding boundary value problem (BVP) is said to be one of *place*; if $\partial C_* = \Sigma_*$, then the BVP is one of *traction*. Finally, the BVP is said to be *mixed* when $\Sigma_* \subset \partial C_*$.

We can make the following remarks about these BVPs.

Remark The boundary data of a BVP are *given functions* of the boundary of the domain in which the solution must be found. For instance, to solve the Laplace equation in a domain Ω, we can provide either the values of the unknown solution u on $\partial\Omega$ (Dirichlet's BVP) or the values of its normal derivative (Neumann's BVP). The following examples show that a different situation occurs in nonlinear elasticity.

- Let S be an elastic body at equilibrium, with a uniform pressure p_0 acting on the boundary ∂C of the equilibrium configuration. The Eulerian formulation of the corresponding BVP is expressed by the equations:

$$\begin{aligned} \nabla_{\mathbf{x}} \cdot \mathbf{T} &= \mathbf{0} \qquad \text{in } C, \\ \mathbf{T} \cdot \mathbf{N} &= -p_0 \mathbf{N} \qquad \text{on } \partial C. \end{aligned}$$

  In this formulation, the pressure p_0 is assigned to the unknown boundary ∂C. Using (1.4), (1.6), (1.9), and (1.10), this BVP can be formulated in the following Lagrangian form:

$$\begin{aligned} &\nabla_{\mathbf{X}} \cdot \mathbf{T}_* = \mathbf{0} \qquad \text{in } C_*, \\ &\mathbf{T}_* \cdot \mathbf{N}_* = -p_0 J (\mathbf{F}^{-1})^T \mathbf{N}_* \equiv \mathbf{t}_*(\mathbf{X}) \qquad \text{on } \partial C_*. \end{aligned}$$

  Consequently, $\mathbf{t}_*$ is not a known function of $\mathbf{X} \in \partial C_*$ since it depends on the gradient of the unknown deformation. In other words, the function $\mathbf{t}_*(\mathbf{X})$ cannot be assigned completely because we only know how it depends on the deformation.

- Analogously, consider the elastic system S in Fig. 1.1, and suppose that the specific force $\mathbf{t} = ks(\mathbf{x})\mathbf{i}$ acts on the part Σ of its boundary.

In the above expression, $s(\mathbf{x})$ is the lengthening of the spring at the point $\mathbf{x}$, k is its elastic constant, and $\mathbf{i}$ is the unit vector orthogonal to the wall l.

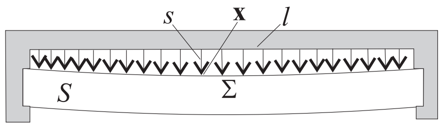

Fig. 1.1 A surface live load

In view of (1.10), the boundary condition to assign to the corresponding part Σ_* of ∂C_* is

$$\mathbf{t}_* = -J\sqrt{\mathbf{N}_* \cdot \mathbf{C}^{-1} \cdot \mathbf{N}_*}\, k\, s(\mathbf{x}(\mathbf{X}))\mathbf{i},$$

which again depends on the unknown deformation.

Any load which depends on the deformation in C_* is called a *live load*, whereas a load that is a known function of $\mathbf{X} \in \Sigma_*$ is said to be a *dead load*.

Dead loads have received a great deal of attention in the literature, but they are actually very difficult to realize. In fact, taking into account the condition that follows from (1.8) and (1.10)

$$\mathbf{t}_*(\mathbf{X}) = \frac{d\sigma}{d\sigma_*}\,\mathbf{t}(\mathbf{x}),$$

we see that the traction $\mathbf{t}$ at the boundary Σ must be given in such a way that $\mathbf{t}_*$ depends on $\mathbf{X}$ but not on the deformation. For instance, we could apply a specific force to a part of the boundary of the Eulerian equilibrium configuration given by

$$\mathbf{t} = \frac{d\mathbf{p}}{d\sigma},$$

where the force $d\mathbf{p}$ that acts on the elementary boundary area $d\sigma$ is constant. Clearly, it is not an easy task to achieve such a load experimentally. Even in the case of a uniform deformation ($\mathbf{F} = const$) under the action of a constant traction $\mathbf{t}$, the corresponding Lagrangian traction does not correspond to a dead load.

Remark A uniqueness theorem cannot hold for a BVP associated with nonlinear elasticity. Three classic examples illustrate this statement.

- There are deformations that coincide at the boundary but assume different values inside the body. For instance, John noted that if either the external or internal boundary of a spherical shell S is rotated by a multiple of 2π about an axis passing through its center without modifying the other boundary, the whole boundary of S will assume the same position but the internal state will be greatly modified.

- $\mathbf{u} = \mathbf{0}$ is an equilibrium solution of a thin hemispherical shell with zero surface traction. However, there is a second solution corresponding to the everted shell.

- Ericksen noted that in a pure traction problem with dead loads, a bar S that is subjected to equal and opposite axial forces at its ends should have at least two equilibrium configurations. In one of these, the forces are tractions; in the other, the bar is subjected to compressions after a rotation of π. Moreover, let S be at equilibrium in the Eulerian configuration C under the action of traction forces $\mathbf{t}$ acting at its ends σ_1 and σ_2 (see Fig. 1.2). If $\mathbf{t}_*$ is the traction per unit area in the reference configuration C_*, then it is easy to verify that S is still at equilibrium in the rotated configuration C' under the action of the compression $\mathbf{t}' = \mathbf{t}$. Let C'' denote the Eulerian equilibrium configuration corresponding to the Lagrangian equilibrium problem starting from the reference configuration C'.

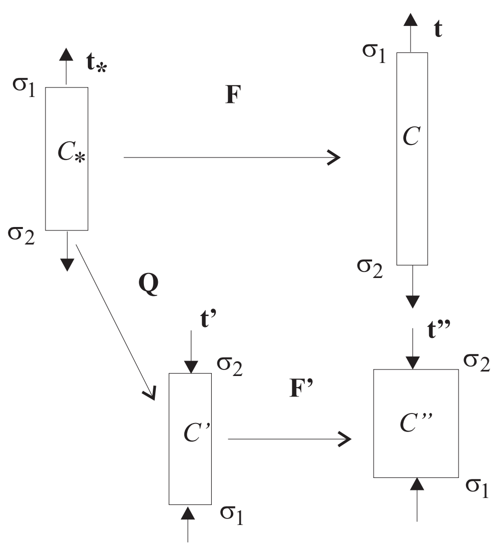

Fig. 1.2 Two possible equilibrium solutions of the same boundary problem

By applying the objectivity principle, and recalling that the loads are dead, we can easily prove that C'' is another possible equilibrium configuration corresponding to the Lagrangian boundary problem associated with C_*.

Remark The *local* equilibrium of any elementary volume dc of S is described by conditions (1.1)–(1.3), which do not imply the *global* equilibrium of S. If we denote by $\mathbf{\Phi}$ and $\mathbf{M}_O$, respectively, the total force and torque of the reactions due to the constraints necessary to satisfy the displacement datum (1.14), then the following global equilibrium conditions hold:

$$\int_{C_*} \rho_* \mathbf{b}\, dc_* + \int_{\Sigma_*} \mathbf{t}_*\, d\sigma_* + \mathbf{\Phi} = \mathbf{0}, \tag{1.15}$$

$$\int_{C_*} \rho_* \mathbf{r} \times \mathbf{b}\, dc_* + \int_{\Sigma_*} \mathbf{r} \times \mathbf{t}_*\, d\sigma_* + \mathbf{M}_O = \mathbf{0}. \tag{1.16}$$

These conditions state that the resultant and the total torque (with respect to the pole O) of all of the forces acting on S vanish. It is clear that, if $\partial C_* - \Sigma_* \neq \emptyset$, the reaction fields $\mathbf{\Phi}$ and $\mathbf{M}_O$ satisfy (1.15)–(1.16). However, in a traction BVP, conditions (1.15) and (1.16) become

$$\int_{C_*} \rho_* \mathbf{b}\, dc_* + \int_{\partial C_*} \mathbf{t}_*\, d\sigma_* = \mathbf{0}, \tag{1.17}$$

$$\int_{C_*} \rho_* \mathbf{r} \times \mathbf{b}\, dc_* + \int_{\partial C_*} \mathbf{r} \times \mathbf{t}_*\, d\sigma_* = \mathbf{0}, \tag{1.18}$$

so that, due to the presence of $\mathbf{t}_*$ and $\mathbf{r} = \mathbf{x}(\mathbf{X}) - \mathbf{x}_0$, they depend on the deformation. Consequently, it is no longer possible to establish whether they are satisfied a priori. In other words, (1.17) and (1.18) represent *equilibrium compatibility conditions* for the data that can only be verified a posteriori.

1.4 Variational Formulation of Equilibrium

The equilibrium BVPs of an elastic system can also be formulated in variational terms. This means that the equilibrium solutions of the BVPs minimize suitable functionals. In this section, the deformation functions are assumed to be of class $C^2(C_*)$, since they must satisfy (1.4)–(1.6). However, if a weak solution is searched for,[3] then the deformation functions are assumed to belong to suitable Sobolev spaces.

[3] See Appendix A.

In order to apply this approach to the equilibrium problems, we introduce the Banach space

$$W = \{\mathbf{u}(\mathbf{X}) \in C^2(C_*) : \mathbf{u}(\mathbf{X}) = \mathbf{0}, \text{on}\, \partial C_* - \Sigma_*\} \tag{1.19}$$

with the norm

$$\|\mathbf{u}(\cdot)\| = \mathrm{Max}_{\mathbf{X}\in C_*} \left\{ \left|u^i(\mathbf{X})\right|, \left|\frac{\partial u^i(\mathbf{X})}{\partial X_L}\right|, \left|\frac{\partial^2 u^i(\mathbf{X})}{\partial X_L \partial X_M}\right| \right\}. \tag{1.20}$$

If we denote the *elastic energy functional* defined on W by

$$\Psi[\mathbf{u}(\cdot)] = \int_{C_*} \rho_* \psi(\mathbf{H})\, dc_*, \tag{1.21}$$

then the following theorem holds.

Theorem 1.1
The displacement $\mathbf{u}_0(\mathbf{X})$ is an equilibrium displacement—i.e., it is a solution of the BVP (1.4)–(1.6)—if and only if it obeys the variational equality[4]

$$D\Psi\left[\mathbf{u}_0(\cdot)|\mathbf{h}(\cdot)\right] = \int_{C_*} \rho_* \mathbf{b}\cdot\mathbf{h}(\mathbf{X})\, dc_* + \int_{\Sigma_*} \mathbf{t}_*\cdot\mathbf{h}(\mathbf{X})\, d\sigma_*, \quad \forall \mathbf{h}(\cdot) \in W, \tag{1.22}$$

where $D\Psi$ is the Frechét differential of the functional (1.21)*.*

PROOF We have

$$\Psi[\mathbf{u}(\cdot)+\mathbf{h}(\cdot)] - \Psi[\mathbf{u}(\cdot)] = \int_{C_*} \rho_* \frac{\partial \psi}{\partial H_{iL}} \frac{\partial h_i}{\partial X_L} dc_* + O(\|\mathbf{h}(\cdot)\|),$$

so that, considering (1.11) and recalling that $\mathbf{h} = \mathbf{0}$ on $C_* - \Sigma_*$, we find that

$$D\Psi[\mathbf{u}_0(\cdot)|\mathbf{h}(\cdot)] = \int_{C_*} \rho_* \frac{\partial \psi}{\partial H_{iL}} \frac{\partial h_i}{\partial X_L}\, dc_*$$

[4]The operator $\mathcal{F} : F \longrightarrow F'$ between two Banach spaces F and F' is *Fréchet differentiable* at $\mathbf{u} \in F$ if

$$\mathcal{F}(\mathbf{u}+\mathbf{h}) = \mathcal{F}(\mathbf{u}) + D\mathcal{F}(\mathbf{u} \mid \mathbf{h}) + O(\| \mathbf{h} \|)$$

$\forall \mathbf{h} \in F$, where

$$D\mathcal{F}(\mathbf{u} \mid \cdot) : F \longrightarrow F'$$

is a linear continuous operator called the *Fréchet differential* of $\mathcal{F}$. The notation

$$D\mathcal{F}(\mathbf{u} \mid \cdot) = D_{\mathbf{u}}\mathcal{F}\cdot\mathbf{h},$$

defines the *Fréchet derivative* of $\mathcal{F}$ at $\mathbf{u}$.

$$= -\int_{C_*} \frac{\partial}{\partial X_L}\left(\rho_* \frac{\partial \psi}{\partial H_{iL}}\right) h_i \, dc_* + \int_{\Sigma_*} \rho_* \frac{\partial \psi}{\partial H_{iL}} h_i N_{*L} \, d\sigma_*$$
$$= -\int_{C_*} \mathbf{h} \cdot \nabla_{\mathbf{X}} \cdot \mathbf{T}_* \, dc_* + \int_{\Sigma_*} \mathbf{h} \cdot \mathbf{T}_* \mathbf{N}_* \, d\sigma_* .$$

It is now straightforward to show that (1.22) is equivalent to the equilibrium conditions (1.4)–(1.7). $\blacksquare$

Theorem 1.2
If $\mathbf{b} = \mathbf{b}(\mathbf{X})$ and $\mathbf{t}_* = \mathbf{t}_*(\mathbf{X})$, then a displacement $\mathbf{u}_0(\mathbf{X}) \in W$ is an equilibrium displacement if and only if it is an extremal of the functional

$$F[\mathbf{u}(\cdot)] = \int_{C_*} \rho_* \psi[\mathbf{H}] \, dc_* - \int_{C_*} \rho_* \mathbf{b}(\mathbf{X}) \cdot \mathbf{u}(\mathbf{X}) \, dc_* - \int_{\Sigma_*} \mathbf{t}_*(\mathbf{X}) \cdot \mathbf{u}(\mathbf{X}) \, d\sigma_* ; \tag{1.23}$$

i.e., if and only if the following condition holds:

$$DF[\mathbf{u}_0(\cdot)|\mathbf{h}(\cdot)] = 0, \quad \forall \mathbf{h}(\mathbf{X}) \in W. \tag{1.24}$$

PROOF If we note that

$$D\left(\int_{C_*} \rho_* \mathbf{b} \cdot \mathbf{u}(\mathbf{X}) \, dc_* + \int_{\Sigma_*} \mathbf{t}_* \cdot \mathbf{u}(\mathbf{X}) \, d\sigma_*\right)$$
$$= \int_{C_*} \rho_* \mathbf{b} \cdot \mathbf{h}(\mathbf{X}) \, dc_* + \int_{\Sigma_*} \mathbf{t}_* \cdot \mathbf{h}(\mathbf{X}) \, d\sigma_* ,$$

then the proof follows from Theorem 1.1. $\blacksquare$

Theorem 1.3
If $\mathbf{b} = -\nabla_{\mathbf{x}} \varphi(\mathbf{x})$ and the body is subjected to a uniform pressure p_e, then $\mathbf{u}_0(\mathbf{X}) \in W$ is an equilibrium displacement if and only if it is an extremal of the functional

$$\bar{F}[\mathbf{u}(\cdot)] = \int_{C_*} (\rho_* \psi(\mathbf{H}) + \rho_* \varphi(\mathbf{u}) + p_e J) \, dc_*$$
$$= \int_{C} \rho \left(\psi(\mathbf{H}) + \varphi(\mathbf{u}) + \frac{p_e}{\rho}\right) dc; \tag{1.25}$$

i.e., if and only if

$$D\bar{F}[\mathbf{u}_0(\cdot)|\mathbf{h}(\cdot)] = 0, \quad \forall \mathbf{h}(\mathbf{X}) \in W. \tag{1.26}$$

PROOF First, taking into account the results of Theorem 1.1, we have

$$D\int_{C_*} (\rho_* \psi(\mathbf{H}) + \rho_* \varphi(\mathbf{u})) \, dc_*$$

$$= -\int_{C_*} \mathbf{h} \cdot (\nabla_{\mathbf{X}} \mathbf{T}_* - \rho_* \nabla_{\mathbf{x}} \varphi)\, dc_* + \int_{\Sigma_*} \mathbf{h} \cdot \mathbf{T}_* \mathbf{N} d\sigma_*$$
$$= -\int_{C_*} \mathbf{h} \cdot (\nabla_{\mathbf{X}} \mathbf{T}_* + \rho_* \mathbf{b})\, dc_* + \int_{\Sigma_*} \mathbf{h} \cdot \mathbf{T}_* \mathbf{N}_*\, d\sigma_*.$$

If we prove that

$$D \int_{C_*} p_e J\, dc_* = -\int_{\Sigma_*} p_e J (\mathbf{F})^{-1} \mathbf{N}_* \cdot \mathbf{h}\, d\sigma_*,$$

then the condition $D\bar{F} = 0$ supplies the equilibrium equations and boundary conditions in the Lagrangian form. To this end, we note that (see (3.50) in [16])

$$\begin{aligned} D \int_{C_*} p_e J\, dc_* &= \int_{C_*} p_e dJ\, dc_* = \int_{C_*} p_e \frac{\partial J}{\partial F_{iL}} \frac{\partial h_i}{\partial X_L}\, dc_* \\ &= \int_{C_*} p_e J (F^{-1})_{iL} \frac{\partial h_i}{\partial X_L}\, dc_* \\ &= \int_{C_*} \frac{\partial}{\partial X_L} \left[p_e J (F^{-1})_{iL} h_i \right]\, dc_* \\ &\quad - \int_{C_*} \frac{\partial}{\partial X_L} \left[p_e J (F^{-1})_{iL} \right] h_i\, dc_* = -\int_{\Sigma_*} \mathbf{t}_* \cdot \mathbf{h}\, d\sigma_*, \end{aligned}$$

since

$$\frac{\partial}{\partial X_L} \left(J (F^{-1})_{iL} \right) = 0. \tag{1.27}$$

In fact, from the identity

$$0 = \frac{\partial}{\partial X_M} \left[\frac{J}{J} F_{iL} (F^{-1})_{Mj} \right],$$

when (3.49) of [16] is taken into account, we derive

$$\begin{aligned} 0 &= J (F^{-1})_{Mi} \frac{\partial}{\partial X_M} \left[\frac{1}{J} F_{iL} \right] + \frac{F_{iL}}{J} \frac{\partial}{\partial X_M} [J (F^{-1})_{Mi}] \\ &= \frac{1}{J} F_{iL} \frac{\partial}{\partial X_M} [J (F^{-1})_{Mi}], \end{aligned}$$

so that

$$\begin{aligned} 0 &= (F^{-1})_{Lj} F_{iL} \frac{\partial}{\partial X_M} [J (F^{-1})_{Mi}] \\ &= \delta_{ij} \frac{\partial}{\partial X_M} [J (F^{-1})_{Mi}] = \frac{\partial}{\partial X_M} [J (F^{-1})_{Mj}], \end{aligned}$$

and the theorem is proved. ∎

1.5 Isotropic Elastic Materials

Let S be an elastic body that is homogeneous and isotropic in the reference configuration C_*. In Sect. 7.2 of [16], it is shown that the elastic potential ψ of S is a function of the principal invariants I, II, and III of the left Cauchy–Green tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$

$$\psi = \psi(I, II, III), \tag{1.28}$$

and that the Cauchy stress tensor $\mathbf{T}$ can be written as follows:

$$\mathbf{T} = f_0\mathbf{I} + f_1\mathbf{B} + f_2\mathbf{B}^2, \tag{1.29}$$

where

$$f_0 = 2\rho\, III\frac{\partial\psi}{\partial III}, \tag{1.30}$$

$$f_1 = 2\rho\left(\frac{\partial\psi}{\partial I} + I\frac{\partial\psi}{\partial II}\right), \tag{1.31}$$

$$f_2 = -2\rho\frac{\partial\psi}{\partial II}. \tag{1.32}$$

On the other hand, from the Cayley–Hamilton theorem,[5]

$$\mathbf{B}^3 - I\mathbf{B}^2 + II\mathbf{B} - III\mathbf{I} = \mathbf{0}. \tag{1.33}$$

Multiplying by $\mathbf{B}^{-1}$ yields

$$\mathbf{B}^2 = I\mathbf{B} - II\mathbf{I} + III\mathbf{B}^{-1}.$$

For this relation, we can write (1.29) in the equivalent form

$$\mathbf{T} = \varphi_0\mathbf{I} + \varphi_1\mathbf{B} + \varphi_2\mathbf{B}^{-1}, \tag{1.34}$$

where

$$\varphi_0 = f_0 - II\, f_2, \tag{1.35}$$

$$\varphi_1 = f_1 + I\, f_2, \tag{1.36}$$

$$\varphi_2 = III\, f_2, \tag{1.37}$$

and

$$\varphi_0 = 2\rho\left(II\frac{\partial\psi}{\partial II} + III\frac{\partial\psi}{\partial III}\right), \tag{1.38}$$

$$\varphi_1 = 2\rho\frac{\partial\psi}{\partial I}, \tag{1.39}$$

$$\varphi_2 = -2\rho III\frac{\partial\psi}{\partial II}. \tag{1.40}$$

[5] See p. 92 of [16].

For an incompressible elastic body, we have $III = \det \mathbf{B} = 1$, and the above formulae become (see p. 165 of [16])

$$\psi = \psi(I, II), \tag{1.41}$$

$$\mathbf{T} = -p\mathbf{I} + 2\rho\frac{\partial\psi}{\partial I}\mathbf{B} - 2\rho\frac{\partial\psi}{\partial II}\mathbf{B}^{-1}, \tag{1.42}$$

where p is an undetermined pressure that depends on the point $\mathbf{x}$.

In view of some of the problems that we consider later, it is useful to introduce the *elastic energy per unit volume* of the reference configuration

$$\Psi = \rho_*\psi, \tag{1.43}$$

through which the relation (1.34) for an elastic compressible material becomes

$$\mathbf{T} = \frac{2}{J}\left(II\frac{\partial\Psi}{\partial II} + III\frac{\partial\Psi}{\partial III}\right)\mathbf{I} + \frac{2}{J}\frac{\partial\Psi}{\partial I}\mathbf{B} - 2\frac{\partial\Psi}{\partial II}\mathbf{B}^{-1}, \tag{1.44}$$

since $\rho J = \rho\sqrt{III} = \rho_*$. For an incompressible elastic material ($III = 1$), (1.42) can be written as follows:

$$\mathbf{T} = -p\mathbf{I} + 2\frac{\partial\Psi}{\partial I}\mathbf{B} - 2J\frac{\partial\Psi}{\partial II}\mathbf{B}^{-1}. \tag{1.45}$$

1.6 Homogeneous Deformations

A deformation $C_* \to C$ of the elastic body S is said to be a *homogeneous deformation* if it has the form

$$\mathbf{x} = \mathbf{F}\,\mathbf{X} + \mathbf{c}, \tag{1.46}$$

where the deformation gradient $\mathbf{F}$ and the vector $\mathbf{c}$ are constant.

When the material is *compressible*, the stress tensor $\mathbf{T}$ is given by (1.29). Consequently, it is constant in any homogeneous deformation, and the equilibrium equation (1.4) is obeyed if and only if there is no body force. In other words, in the absence of body forces, *any* homogeneous deformation obeys the equilibrium equation (1.4) for *any* isotropic elastic material. However, the boundary condition (1.6) depends on both the material and the chosen homogeneous deformation.

When the material is *incompressible*, a homogeneous deformation obeys the equilibrium equation (1.4), even in the presence of body forces, due to the presence of the undetermined function $p(\mathbf{x})$. If, in particular, $\mathbf{b} = 0$, then the pressure p is constant. Again, the boundary condition depends on the material and the homogeneous deformation chosen.

We derive a very important conclusion from these remarks. At least in principle, *it is possible to determine the constitutive relations* (1.44) *and* (1.45) *using homogeneous deformations and surface forces.*

In the following sections we describe some important homogeneous deformations as well as some famous experiments to determine the forms of the constitutive equations of the stress tensor for particular isotropic elastic materials.

1.7 Homothetic Deformation

A *homothetic deformation* of an elastic system S is expressed by the equations

$$x_i = \lambda_i X_i, \quad i = 1, 2, 3, \tag{1.47}$$

where the constants λ_i are nonzero. If $\lambda_i > 1$, then the system S exhibits an extension along the axis X_i; if $0 < \lambda_i < 1$, then the system S exhibits a compression along the axis x_i. The deformation gradient of (1.47) is given by the matrix

$$\mathbf{F} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{1.48}$$

so that the coordinate axes are the principal axes of deformation. From (1.48), we derive

$$J \equiv \det \mathbf{F} = \lambda_1 \lambda_2 \lambda_3; \tag{1.49}$$

moreover, the left Cauchy–Green tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ and its inverse can, respectively, be written as

$$\mathbf{B} = \begin{pmatrix} \lambda_1^2 & 0 & \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix}, \quad \mathbf{B}^{-1} = \begin{pmatrix} \frac{1}{\lambda_1^2} & 0 & \\ 0 & \frac{1}{\lambda_2^2} & 0 \\ 0 & 0 & \frac{1}{\lambda_3^2} \end{pmatrix}. \tag{1.50}$$

Finally, the principal invariants of $\mathbf{B}$ are

$$I = \mathrm{tr}\mathbf{B} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \tag{1.51}$$

$$II = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2, \tag{1.52}$$

$$III = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{1.53}$$

Starting from (1.50)–(1.53) and (1.44), we derive the following expressions for the Cauchy stress tensor components in a compressible elastic material:

$$T_{11} = 2\lambda_1 \left\{ \frac{1}{\lambda_2\lambda_3} \left[\frac{\partial \Psi}{\partial I} + (\lambda_2^2 + \lambda_3^2) \frac{\partial \Psi}{\partial II} \right] + \lambda_2\lambda_3 \frac{\partial \Psi}{\partial III} \right\}, \tag{1.54}$$

$$T_{22} = 2\lambda_2 \left\{ \frac{1}{\lambda_1\lambda_3} \left[\frac{\partial \Psi}{\partial I} + (\lambda_1^2 + \lambda_3^2) \frac{\partial \Psi}{\partial II} \right] + \lambda_1\lambda_3 \frac{\partial \Psi}{\partial III} \right\}, \tag{1.55}$$

$$T_{33} = 2\lambda_3 \left\{ \frac{1}{\lambda_1\lambda_2} \left[\frac{\partial \Psi}{\partial I} + (\lambda_1^2 + \lambda_2^2) \frac{\partial \Psi}{\partial II} \right] + \lambda_1\lambda_2 \frac{\partial \Psi}{\partial III} \right\}, \tag{1.56}$$

$$T_{ij} = 0, \quad i \neq j. \tag{1.57}$$

These relations prove that the state of tension inside the body S is uniform, so the equilibrium equations are obeyed.

Let us denote the parametric equations of the boundary of S in the reference configuration C_* by

$$\mathbf{X} = \mathbf{X}(u_1, u_2).$$

The parametric equations of the boundary ∂C in the deformed equilibrium configuration C are then

$$\mathbf{x} = \lambda_i \mathbf{X}(u_1, u_2). \tag{1.58}$$

It remains to evaluate the surface force

$$\mathbf{t} = \mathbf{T} \cdot \mathbf{N} \tag{1.59}$$

that must be applied to the unit surface of ∂C in order to make the deformation (1.47) possible. In the above equation, $\mathbf{N}$ denotes the unit vector normal to the known boundary surface (1.58).

We now apply the above considerations to the parallelepiped S shown in Fig. 1.3. We note that the faces of S remain parallel to each other under the deformation (1.47). Therefore, the unit vector $\mathbf{N}$ orthogonal to the face $ABCD$ after the deformation becomes

$$\mathbf{N} = (0, 1, 0),$$

whereas the unit vector ν tangent to the same face $ABCD$ has the components

$$\nu = (\alpha, 0, \beta),$$

where

$$\alpha^2 + \beta^2 = 1.$$

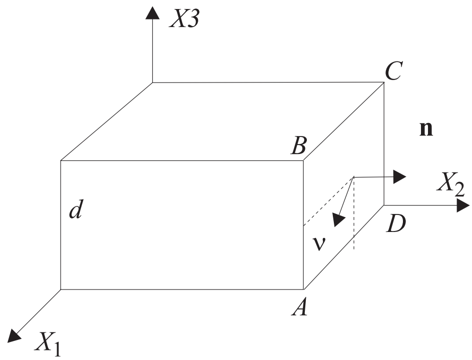

Fig. 1.3 Homothetic deformation of a parallelepiped

Consequently, the normal and tangential forces acting on this face are

$$\mathbf{t}_n = (\mathbf{n} \cdot \mathbf{T}\,\mathbf{n})\mathbf{n} = T_{22}\mathbf{n}, \quad \mathbf{t}_\nu = (\nu \cdot \mathbf{T}\,\mathbf{n})\nu = \mathbf{0}, \tag{1.60}$$

respectively. Applying the same considerations to the other faces, we conclude that the forces are orthogonal to the faces of the parallelepiped on which they act.

When S is incompressible $J = \lambda_1\lambda_2\lambda_3 = 1$, and, in view of (1.45) and (1.50), we can state that the stress tensor has the following components:

$$T_{ii} = -p + 2\frac{\partial \Psi}{\partial I}\lambda_i^2 - 2\frac{\partial \Psi}{\partial II}\frac{1}{\lambda_i^2}, \; i = 1, 2, 3, \tag{1.61}$$

$$T_{ij} = 0, \quad i \neq j. \tag{1.62}$$

In the absence of body force, equilibrium equation (1.1) is verified if the undetermined pressure p satisfies the equation

$$\frac{\partial p}{\partial x_i} = 0; \tag{1.63}$$

i.e., if it is equal to a constant p_0.

Finally, since $\lambda_1\lambda_2\lambda_3 = 1$, we have the following for an incompressible material:

$$T_{ii} = -p_0 + 2\frac{\partial \Psi}{\partial I}\lambda_i^2 - 2\frac{\partial \Psi}{\partial II}\frac{1}{\lambda_i^2}, \quad i = 1, 2, \tag{1.64}$$

$$T_{33} = -p_0 + 2\frac{\partial \Psi}{\partial I}\frac{1}{\lambda_1^2\lambda_2^2} - 2\frac{\partial \Psi}{\partial II}\lambda_1^2\lambda_2^2, \tag{1.65}$$

$$T_{ij} = 0, \quad i \neq j, \tag{1.66}$$

and the principal invariants of $\mathbf{B}$ become:

$$I = \lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2\lambda_2^2}, \tag{1.67}$$

$$II = \lambda_1^2\lambda_2^2 + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}, \tag{1.68}$$

$$III = 1. \tag{1.69}$$

We conclude this section by noting that, if there is no surface force on the face $X_3 = 0$ or on the face $X_3 = d$, $T_{33} = 0$ and (1.65) gives the following value for the pressure p_0:

$$p_0 = 2\frac{\partial\Psi}{\partial I}\frac{1}{\lambda_1^2\lambda_2^2} - 2\frac{\partial\Psi}{\partial II}\lambda_1^2\lambda_2^2. \tag{1.70}$$

Introducing this value of p_0 into (1.64), we obtain

$$T_{11} = 2\left(\lambda_1^2 - \frac{1}{\lambda_1^2\lambda_2^2}\right)\left(\frac{\partial\Psi}{\partial I} + \lambda_2^2\frac{\partial\Psi}{\partial II}\right), \tag{1.71}$$

$$T_{22} = 2\left(\lambda_2^2 - \frac{1}{\lambda_1^2\lambda_2^2}\right)\left(\frac{\partial\Psi}{\partial I} + \lambda_1^2\frac{\partial\Psi}{\partial II}\right). \tag{1.72}$$

1.8 Simple Extension of a Rectangular Block

The particular homothetic deformation

$$x_1 = \alpha X_1, \quad x_2 = \beta X_2, \quad x_3 = \beta X_3, \tag{1.73}$$

where α and β are positive real numbers, is termed a *simple extension*. The tensors $\mathbf{B}$ and $\mathbf{B}^{-1}$ that correspond to this deformation are

$$\mathbf{B} = \begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & \beta^2 & 0 \\ 0 & 0 & \beta^2 \end{pmatrix}, \qquad \mathbf{B}^{-1} = \begin{pmatrix} \dfrac{1}{\alpha^2} & 0 & 0 \\ 0 & \dfrac{1}{\beta^2} & 0 \\ 0 & 0 & \dfrac{1}{\beta^2} \end{pmatrix}. \tag{1.74}$$

If S is compressible, the stress tensor is given by (1.54)–(1.57):

$$T_{11} = 2\alpha\left(\frac{1}{\beta^2}\frac{\partial\Psi}{\partial I} + 2\frac{\partial\Psi}{\partial II} + \beta^2\frac{\partial\Psi}{\partial III}\right), \tag{1.75}$$

$$T_{22} = T_{33} = 2\left[\frac{1}{\alpha}\left(\frac{\partial\Psi}{\partial I} + (\alpha^2+\beta^2)\frac{\partial\Psi}{\partial II}\right) + \alpha\beta^2\frac{\partial\Psi}{\partial III}\right], \tag{1.76}$$

$$T_{ij} = 0, \quad i \neq j. \tag{1.77}$$

Let S be a rectangular block with edges that are parallel to the coordinate axes. Also let $\mathbf{u}_i$, $i = 1, 2, 3$, be the unit vectors along these axes. If ∂S_i is the face with $\mathbf{u}_i$ as its unit normal vector, and $\partial S'_i$ is the face with $-\mathbf{u}_i$ as its unit normal vector, then the surface forces $\mathbf{t}_i$ and $\mathbf{t}'_i$ that must be applied to ∂S_i and $\partial S'_i$, respectively, in order to achieve the above deformation are

$$\mathbf{t}_i = T_{ii}\mathbf{u}_i, \quad \mathbf{t}'_i = -T_{ii}\mathbf{u}_i. \tag{1.78}$$

It is quite natural to wonder if a simple extension can be obtained by the action of normal forces on the faces ∂S_1 and $\partial S'_1$. In order to achieve this, we first apply the forces $\mathbf{t}_1$ and $\mathbf{t}'_1$ obtained from (1.78) for $i = 1$ to these faces; moreover, due to (1.77), we must satisfy the following condition if the forces acting on the other faces are to be eliminated:

$$T_{22} = \frac{1}{\alpha\beta}\frac{\partial\Psi}{\partial I} + \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)\frac{\partial\Psi}{\partial II} + \alpha\beta\frac{\partial\Psi}{\partial III} = 0. \tag{1.79}$$

For a given α (i.e., for an assigned extension or contraction along $\mathbf{u}_1$), the following three cases are possible:

1. Equation 1.79 allows a unique real positive solution β and the requested extension can be achieved.
2. Equation 1.79 does not permit real solutions, and so the assigned extension cannot be achieved in this material.
3. Equation 1.79 allows a number of real positive solutions $(\beta_1, \beta_2, \ldots)$. Then, by substituting the pairs (α, β_1), $(\alpha, \beta_2), \ldots$ into (1.75), we can derive the different forces that can be applied to ∂S_1 and $\partial S'_1$ to give the same extension.

The last case could not be verified for linear elasticity. In fact, in this approximation, when we denote Lamé's coefficients (see p. 176 of [16]) by λ and μ, (1.79) reduces to the condition

$$\lambda\alpha + 2\beta(\lambda + \mu) - 3\lambda - 2\mu = 0,$$

which is a first-degree equation. Consequently, for a given α, it allows one positive real solution β at most.

Again, we refer this deformation to an incompressible elastic parallelepiped S. For a simple extension that preserves the volume, we have $\beta^2 = 1/\alpha$, and the matrices $\mathbf{F}$, $\mathbf{B}$, and $\mathbf{B^{-1}}$ (see 1.74) become

$$\mathbf{F} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \dfrac{1}{\sqrt{\alpha}} & 0 \\ 0 & 0 & \dfrac{1}{\sqrt{\alpha}} \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & \dfrac{1}{\alpha} & 0 \\ 0 & 0 & \dfrac{1}{\alpha} \end{pmatrix}, \ \mathbf{B}^{-1} = \begin{pmatrix} \dfrac{1}{\alpha^2} & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}. \tag{1.80}$$

From (1.45), which defines the Cauchy stress tensor for such a material, and from (1.80), we obtain

$$T_{11} = -p + 2\frac{\partial\Psi}{\partial I}\alpha^2 - 2\frac{\partial\Psi}{\partial II}\frac{1}{\alpha^2}, \tag{1.81}$$

$$T_{22} = T_{33} = -p + 2\frac{\partial\Psi}{\partial I}\frac{1}{\alpha} - 2\frac{\partial\Psi}{\partial II}\alpha, \tag{1.82}$$

$$T_{ij} = 0,\ i \neq j. \tag{1.83}$$

Now, in the absence of body forces, it is possible to achieve the simple extension without any surface forces on the faces parallel to the coordinate planes Ox_1x_2 and Ox_1x_3. According to these conditions we have $T_{22} = T_{33} = 0$, so the uniform pressure is given by the relation

$$p = \frac{2}{\alpha}\frac{\partial\Psi}{\partial I} - 2\alpha\frac{\partial\Psi}{\partial II}. \tag{1.84}$$

Substituting this expression into (1.81), we finally obtain

$$T_{11} = 2\left(\alpha^2 - \frac{1}{\alpha}\right)\frac{\partial\Psi}{\partial I} + 2\left(\alpha - \frac{1}{\alpha^2}\right)\frac{\partial\Psi}{\partial II}. \tag{1.85}$$

1.9 Simple Shear of a Rectangular Block

Let S be a rectangular block. The deformation

$$x_1 = X_1 + KX_2, \quad x_2 = X_2, \quad x_3 = X_3 \tag{1.86}$$

is called a *simple shear* of S. In this deformation, each plane $X_2 = const.$ slides on itself. Any plane $X_3 = const.$ undergoes a similar deformation. Finally, each plane $X_1 = const.$ rotates by the *shear angle* α, and $K = \arctan\alpha$ (see Fig. 1.4) is said to be the *amount of shear*.

The deformation gradient F and the left Cauchy–Green tensor are given by the matrices

$$\mathbf{F} = \begin{pmatrix} 1 & K & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1+K^2 & K & 0 \\ K & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.87}$$

Since

$$\det\mathbf{F} = 1, \tag{1.88}$$

the deformation preserves the volume.

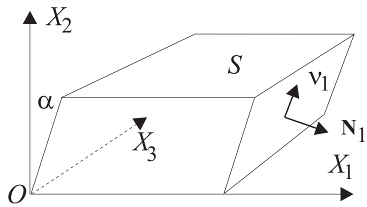

Fig. 1.4 Simple shear of a parallelepiped

The principal invariants of $\mathbf{B}$ are

$$I = 3 + K^2, \tag{1.89}$$

$$II = 3 + K^2, \tag{1.90}$$

$$III = 1, \tag{1.91}$$

and the matrix $\mathbf{B^{-1}}$ is

$$\mathbf{B}^{-1} = \begin{pmatrix} 1 & -K & 0 \\ -K & 1+K^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.92}$$

Using the equations of the two bent faces π_1 and π_2 of S,

$$x_1 + Kx_2 = 0, \quad x_1 - Kx_2 = a, \tag{1.93}$$

where a is the length of the edge between π_1 and π_2, we can derive the unit vectors normal to them:

$$\mathbf{N}_{1,2} = \left(\pm\frac{1}{\sqrt{1+K^2}}, \pm\frac{K}{\sqrt{1+K^2}}, 0 \right). \tag{1.94}$$

Consequently, the vectors tangent to π_1 and π_2 and parallel to the plane Ox_1x_2 are

$$\nu_{1,2} = \left(\pm\frac{K}{\sqrt{1+K^2}}, \pm\frac{1}{\sqrt{1+K^2}}, 0 \right). \tag{1.95}$$

Introducing (1.87)–(1.92) into (1.29)–(1.40), we derive

$$T_{11} = 2\left((1+K^2)\frac{\partial\Psi}{\partial I} + (2+K^2)\frac{\partial\Psi}{\partial II} + \frac{\partial\Psi}{\partial III} \right), \tag{1.96}$$

$$T_{22} = 2\left(\frac{\partial\Psi}{\partial I} + 2\frac{\partial\Psi}{\partial II} + \frac{\partial\Psi}{\partial III} \right), \tag{1.97}$$

$$T_{33} = \frac{\partial \Psi}{\partial I} + (2+K^2)\frac{\partial \Psi}{\partial II} + \frac{\partial \Psi}{\partial III}, \tag{1.98}$$

$$T_{12} = 2K\left(\frac{\partial \Psi}{\partial I} + \frac{\partial \Psi}{\partial II}\right), \tag{1.99}$$

$$T_{13} = T_{23} = 0. \tag{1.100}$$

Since the stress tensor is constant inside S and there is no body force, equilibrium equation (1.1) is obeyed. It remains to evaluate the surface forces that must be applied to the faces π_1 and π_2 of S in order to make (1.87) an equilibrium deformation.

The normal forces to apply are derived by taking into account (1.60) and (1.96)–(1.100):

$$\mathbf{t}_{n1,2} = (\pm\gamma, \pm\gamma K, 0)\begin{pmatrix} T_{11} & T_{12} & 0 \\ T_{12} & T_{22} & 0 \\ 0 & 0 & T_{33} \end{pmatrix}\begin{pmatrix} \pm\gamma \\ \pm\gamma K \\ 0 \end{pmatrix}\mathbf{n}_{1,2},$$

where $\gamma = 1/\sqrt{1+K^2}$. Finally, noting that (1.96)–(1.100) imply $T_{11} - T_{22} = KT_{12}$, we have

$$\mathbf{t}_{n1,2} = \frac{T_{11} - 2KT_{12} + K^2 T_{22}}{1+K^2}\mathbf{n}_{1,2} = \left(T_{22} - K\frac{T_{12}}{1+K^2}\right)\mathbf{n}_{1,2}. \tag{1.101}$$

The tangential forces to apply to π_1 and π_2 are again derived from (1.60) and (1.96)–(1.100):

$$\mathbf{t}_{\nu 1,2} = \frac{KT_{11} - K^2 T_{12} + T_{12} - KT_{22}}{1+K^2}\nu_{1,2} = \frac{T_{12}}{1+K^2}\nu_{1,2}. \tag{1.102}$$

The same procedure can be used to determine the normal and tangential forces to apply to the other faces of S.

In linear elasticity ($K \simeq 0$), deformation (1.86) can be realized by applying only *tangential* forces to the faces of S. In this approximation, the stress tensor is $\mathbf{T} = \lambda I_E \mathbf{I} + 2\mu\mathbf{E}$, where λ and μ are the Lamé coefficients (see [16]) and $\mathbf{E}$ is the infinitesimal deformation tensor.

However, in nonlinear elasticity, (1.86) is an equilibrium deformation if *normal* forces are applied to *all* of the faces of the block S. It is possible to prove that the volume of the block increases or reduces if these normal forces are not applied, meaning that the deformation cannot be isochoric. This behavior of S, which tends to alter in volume under the action of tangential forces, is called *Kelvin's effect*. Moreover, since $T_{11} \neq T_{22}$, we can presume that the expansion of the block assumes different values along the axes if these components are equal (*Poynting's effect*).

1.10 Universal Static Solutions

In the above sections we analyzed some particular homogeneous deformations. In the absence of body force, equilibrium condition (1.1) is satisfied in any homogeneous deformation. Furthermore, these deformations can be realized by applying only the surface forces $\mathbf{t} = \mathbf{T} \cdot \mathbf{N}$, where $\mathbf{N}$ is the unit vector normal to the surface ∂C of the equilibrium configuration C. Finally, these forces obey the global equilibrium conditions (1.17) and (1.18). In fact, we get

$$\int_{\partial C} \mathbf{t}\, d\sigma = \int_{\partial C} \mathbf{T} \cdot \mathbf{N}\, d\sigma = \int_{C} \nabla_{\mathbf{x}} \mathbf{T}\, dc = \mathbf{0},$$

whereas we have the following for the torques:

$$\begin{aligned} \epsilon_{ijl} \int_{\partial C} x_j t_l\, d\sigma &= \epsilon_{ijl} \int_{\partial C} x_j T_{lk} N_k\, d\sigma = \epsilon_{ijl} T_{lk} \int_{\partial C} x_j N_k\, d\sigma \\ &= \epsilon_{ijl} T_{lk} \int_{C} \frac{\partial x_j}{\partial x_k}\, dc = \epsilon_{ijl} T_{lj} \mathrm{vol(C)} = 0. \end{aligned}$$

We have already noted that these surface forces depend on both the homogeneous deformation chosen and the nature of the material. Consequently, we can only hope to determine the stress-deformation relation $\mathbf{T}(\mathbf{F})$ using surface forces and homogeneous deformations. In the following sections we analyze some experiments based on this idea.

An equilibrium solution $\mathbf{x}(\mathbf{X})$ for *any* material S belonging to a given isotropy class that can be obtained by applying only surface forces is said to be a *static universal solution*.

Ericksen [20] proved the following fundamental theorem.

Theorem 1.4
Any static universal solution of a hyperelastic, compressible, isotropic solid is a homogeneous deformation.

PROOF First, from (1.7) and (1.11) we have:

$$\mathbf{T} = 2\rho \mathbf{B} \frac{\partial \psi}{\partial \mathbf{B}};$$

i.e.,

$$T_k^m = -2\rho (B^{-1})_{kp} \frac{\partial \psi}{\partial (B^{-1})_{mp}}.$$

On the other hand, $\rho J = \rho_*$, $J = \det \mathbf{F} = \sqrt{\det \mathbf{C}} = \sqrt{III_C} = \sqrt{III_B}$ $= (III_{B^{-1}})^{-1/2}$. Consequently, the above relation becomes

$$T_k^m = -2\rho_* \sqrt{III_{B^{-1}}} (B^{-1})_{kp} \frac{\partial \psi}{\partial (B^{-1})_{mp}}.$$

Denoting the principal invariants of $\mathbf{B}^{-1}$ by I_1, I_2, and I_3, we have

$$T_k^m = -2\rho\sqrt{I_3}\frac{\partial \psi}{\partial I_i}\frac{\partial I_i}{\partial (B^{-1})_{mp}}(B^{-1})_{kp}, \tag{1.103}$$

and equilibrium condition (1.1) can be written as follows:

$$\begin{aligned} &\frac{\partial \psi}{\partial I_i}\left(\sqrt{I_3}\frac{\partial I_i}{\partial (B^{-1})_{mp}}(B^{-1})_{kp}\right)_{,m} \\ &+ \ \sqrt{I_3}\frac{\partial^2 \psi}{\partial I_i \partial I_j} I_{j,m}\frac{\partial I_i}{\partial (B^{-1})_{mp}}(B^{-1})_{kp} = 0, \end{aligned} \tag{1.104}$$

where the comma denotes partial differentiation with respect to the spatial variables x_i. This equation is obeyed for any isotropic material if and only if the following equations are individually satisfied:

$$\left(\sqrt{I_3}\frac{\partial I_i}{\partial (B^{-1})_{mp}}(B^{-1})_{kp}\right)_{,m} = 0, \tag{1.105}$$

$$\left(I_{j,m}\frac{\partial I_i}{\partial (B^{-1})_{mp}} + I_{i,m}\frac{\partial I_j}{\partial (B^{-1})_{mp}}\right)(B^{-1})_{kp} = 0. \tag{1.106}$$

Not all of the solutions $(B^{-1})_{mp}$ of equations (1.105)-(1.106) are acceptable, since $\mathbf{B}^{-1}$ is related to a deformation $\mathbf{x} = \mathbf{x}(\mathbf{X})$ by the relation

$$(B^{-1})_{ij} = \frac{\partial X^L}{\partial x_i}\frac{\partial X^L}{\partial x_j} = \delta_{LM}\frac{\partial X^L}{\partial x_i}\frac{\partial X^L}{\partial x_j}. \tag{1.107}$$

We note that δ_{LM} are the components of the metric tensor $\mathbf{G}_*$ along the coordinate (X^L) adopted in the reference configuration C_*. Relations (1.107) then show that $(B^{-1})_{ij}$ coincide with the corresponding components of $\mathbf{G}_*$ when the coordinates x^i are adopted for C_*. Consequently, system (1.105)–(1.106) is integrable if and only if there is a coordinate transformation $X^L(x^i)$ such that the metric coefficients δ_{LM} transform into $(B^{-1})_{ij}$. This happens if and only if $\mathbf{B}^{-1}$ is a Euclidean metric tensor, or (equivalently) if and only if the curvature tensor $\mathbf{R}$ associated with $\mathbf{B}^{-1}$ vanishes:

$$\mathbf{R} = \mathbf{0}. \tag{1.108}$$

For $i = 3$, (1.105) gives:

$$\left(\sqrt{I_3}\,\frac{\partial I_3}{\partial (B^{-1})_{mp}}(B^{-1})_{kp}\right)_{,m} = 0.$$

Noting that the term inside the parentheses of the above expression coincides with the cofactor, we have

$$(\sqrt{I_3}\,A^{mp}(B^{-1})_{mp})_{,m} = 0,$$

and, recalling Laplace's rule for evaluating a determinant, we can write

$$((\sqrt{I_3})^3)_{,k} = 0,$$

so that

$$I_3 = const. \tag{1.109}$$

For $i = 1$, and taking (1.109) into account, Eq. 1.105 gives

$$\left(\frac{\partial I_1}{\partial (B^{-1})_{mp}}(B^{-1})_{kp}\right)_{,m} = 0.$$

However, $I_1 = \mathrm{tr}\mathbf{B}^{-1}$, and so we have

$$(B^{-1})^m_{k,m} = 0. \tag{1.110}$$

Finally, inserting $i = 3$ and $j = 1$ into (1.106), we derive the condition

$$\left(I_{1,m}\frac{\partial I_3}{\partial (B^{-1})_{mp}} + I_{3,m}\frac{\partial I_1}{\partial (B^{-1})_{mp}}\right)(B^{-1})_{kp} = 0,$$

which can also be written in the form[6]

$$I_{1,m}A^{mp}(B^{-1})_{kp} = 0.$$

Equivalently, we have

$$I_3 I_{1,k} = 0,$$

and we conclude that

$$I_1 = const. \tag{1.111}$$

It remains to analyze condition (1.108), which can be explicitly written as follows:

$$\begin{aligned}&(B^{-1})_{km,pq} + (B^{-1})_{qp,km} - (B^{-1})_{kp,mq} - (B^{-1})_{qm,kp}\\&+2B^{rs}(\Gamma_{qpr}\Gamma_{kms} - \Gamma_{qmr}\Gamma_{kps}) = 0,\end{aligned} \tag{1.112}$$

[6]In Chap. 3 of [16], we proved the following relation for any nonsingular matrix $\mathbf{a}$:

$$\frac{\partial a}{\partial a^i_j} = a(a_{-1})^j_i,$$

where $a = \det \mathbf{a}$.

where

$$2\Gamma_{kmp} = (B^{-1})_{km,p} + (B^{-1})_{mp,k} - (B^{-1})_{pk,m}. \tag{1.113}$$

Putting $k = p$ and $m = q$ into (1.111), we obtain the condition

$$\begin{aligned}
&(B^{-1})^{k,p}_{k,p} + (B^{-1})^{p,k}_{p,k} - (B^{-1})^{kp}_{,kp} - (B^{-1})^{kp}_{,kp} \\
&+ 2B^{rs}[(B^{-1})^{p}_{r,p} + (B^{-1})^{p}_{r,p} - (B^{-1})^{p}_{p,r}] \\
&\times [(B^{-1})^{k}_{s,k} + (B^{-1})^{k}_{s,k} - (B^{-1})^{k}_{s,k}] \\
&- 2B^{rs}\Gamma^{pk}_{r}\Gamma_{kps} = 0,
\end{aligned} \tag{1.114}$$

from which, taking into account (1.110) and (1.111), we derive

$$B^{rs}\Gamma^{pk}_{r}\Gamma_{kps} = 0.$$

If we put $\mathbf{V} = \mathbf{B}^{1/2}$, this last equation can also be written as follows:

$$(V^{rs}\Gamma_{kpr})^2 = 0.$$

However, $\mathbf{V}$ is positive definite, so

$$\Gamma_{kpr} = 0, \tag{1.115}$$

and we conclude that $\mathbf{B}$ is a constant tensor, the coordinate transformation $x^i = x^i(X^L)$ is linear, and that $\mathbf{F}$ is constant. ∎

Finally, recall that the universal static solutions of an incompressible, isotropic and elastic material are not completely known.

1.11 Constitutive Equations in Nonlinear Elasticity

The experimental determination of the elastic potential function is a very difficult task. It calls for the use of particular devices that allow us to satisfy very restrictive requests, such as the possibility of realizing a uniform state of deformation in the specimen. Moreover, even in the case of an elastic material, its behavior under the action of the applied forces must be evaluated in many directions.

The constitutive equations (1.44) and (1.45) show that the stress tensor of an isotropic elastic body S is completely determined by the elastic deformation energy Ψ, which depends on the three invariants if the material is compressible, and only on the invariants I and II when S is incompressible. In this latter case there are fewer experimental difficulties, and this is why most experiments on finite elasticity refer to incompressible materials. Vulcanized rubber exhibits this behavior for a wide range of deformations.

Let S be an incompressible isotropic elastic material. In its reference configuration C_*, $\mathbf{B}_* = \mathbf{I}$ and so $I_* = II_* = 3$. If we accept the hypothesis that Ψ permits a power expansion close to these values, we can write

$$\Psi = \sum_{n,m=1}^{\infty} A_{nm}(I-3)^m(II-3)^n, \quad A_{00} = 0, \tag{1.116}$$

where A_{nm} are constants.

The simplest form of Ψ was proposed by Treolar ([17], [18]), and it is given by the expression

$$\Psi = A_{10}(I-3). \tag{1.117}$$

It is based on a statistical model in which the rubber is described as a network of long chain molecules. Bodies that obey (1.117) are called *neo-Hookian materials*. A more accurate expression of Ψ, as suggested by Mooney [19], is

$$\Psi = A_{10}(I-3) + A_{01}(II-3). \tag{1.118}$$

It is possible to proceed experimentally in two different ways:

- Given a certain form for the elastic deformation energy, the surface forces that are needed to produce an assigned deformation can be evaluated. An experiment utilizing these forces is then performed, and the measured deformation is compared with the theoretical one.
- Experimental data can be used to deduce information on the form of the deformation elastic energy.

In the following sections, we will describe Treolars' experiments and that of Rivlin and Saunders as examples of the above procedures.

1.12 Treolar's Experiments

Treolar carried out many experiments with vulcanized rubber to verify the reliability of the functions (1.117) and (1.118) (see [17]–[18]). In a first experiment he subjected a rubber specimen to a simple extension. Using (1.117) and (1.85) we can derive

$$\frac{\partial \Psi}{\partial I} = A_{10}, \quad \frac{\partial \Psi}{\partial II} = 0, \tag{1.119}$$

$$T_{11} = 2A_{10}\left(\alpha^2 - \frac{1}{\alpha}\right). \tag{1.120}$$

T_{11} is the force per unit area in the deformed configuration; due to (1.7) and (1.80), the force per unit area in the reference configuration is

$$T_{*11} = \frac{T_{11}}{\alpha} = 2A_{10}\left(\alpha - \frac{1}{\alpha^2}\right). \tag{1.121}$$

Treolar obtained the curve in Fig. 1.5 by increasing the loads on the surface of the specimen and measuring the corresponding extension.

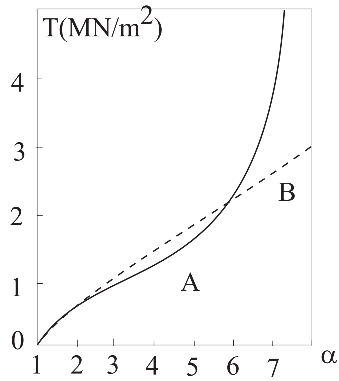

Fig. 1.5 Treolar's curve: A, experimental curve; B, theoretical curve

The figure shows that the theoretical curve (B in Fig. 1.5) only fits the experimental curve (A) well for values of α that are less than 1.5. For $1.5 < \alpha \simeq 6$, A lies below B, while it quickly increases when $\alpha \geq 6$. These results, which have been confirmed by many other experiments, show that the neo-Hookean form (1.117) is only acceptable for small extensions.

1.13 Rivlin and Saunders' Experiment

In this experiment, a thin square sheet of rubber was subjected to homothetic deformations (1.47) (see [19]). The specimen (see Fig. 1.6) had five lugs on each side to which loads were applied. Two sets of orthogonal lines were drawn on the surface of the sheet. The stretched sheet is shown in Fig. 1.7.

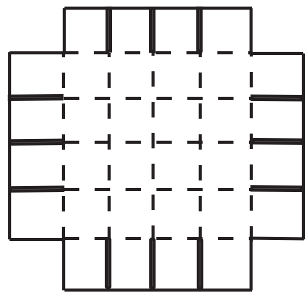

Fig. 1.6 Rubber sheet before the deformation

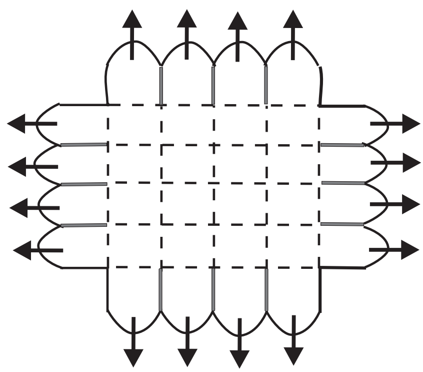

Fig. 1.7 Rubber sheet after the deformation

From (1.118), we obtain

$$\frac{\partial \Psi}{\partial I} = A_{10}, \ \frac{\partial \Psi}{\partial II} = A_{01}, \tag{1.122}$$

so that (1.71) and (1.72) become

$$T_{11} = 2\left(\lambda_1^2 - \frac{1}{\lambda_1^2\lambda_2^2}\right)\left(A_{10} + \lambda_2^2 A_{01}\right), \tag{1.123}$$

$$T_{22} = 2\left(\lambda_2^2 - \frac{1}{\lambda_1^2\lambda_2^2}\right)\left(A_{10} + \lambda_1^2 A_{01}\right). \tag{1.124}$$

For a neo-Hookian material (1.117), the above equations assume the form

$$T_{11} = 2A_{10}\left(\lambda_1^2 - \frac{1}{\lambda_1^2\lambda_2^2}\right), \tag{1.125}$$

$$T_{22} = 2A_{10}\left(\lambda_2^2 - \frac{1}{\lambda_1^2\lambda_2^2}\right), \tag{1.126}$$

so that T_{11} and T_{22} are linear functions of the variables

$$\xi = \lambda_1^2 - \frac{1}{\lambda_1^2\lambda_2^2}, \tag{1.127}$$

$$\eta = \lambda_2^2 - \frac{1}{\lambda_1^2\lambda_2^2}. \tag{1.128}$$

1.14 Nondimensional Analysis of Equilibrium

In the next section we present Signorini's method, which is essentially an application of the regular perturbation method to nonlinear elasticity. In this section, to introduce it, we write the equations of nonlinear elasticity in nondimensional form.

First, we suppose that in the absence of acting forces, the elastic system S assumes an unstressed, homogeneous and isotropic equilibrium configuration C_*. The new equilibrium configuration of S obtained when surface and body forces act on it will be denoted by C. We also suppose that the deformation of S does not differ very much from the deformation we would get if S were a linear elastic material. In other words, the deformed state is assumed to be close to the state assumed by S if it behaved as a linear elastic material. Consequently, in order to write the equilibrium equations of nonlinear elasticity in nondimensional form, we introduce the following comparison quantities:

$$\tilde{T}, \quad l, \quad L, \quad \tilde{b}, \quad \tilde{t},$$

where T has the dimensions of stress, L is a length, $\tilde{b}$ has the dimensions of force per unit mass, and $\tilde{t}$ has the dimensions of force per unit surface. These quantities measure the stress state in C, the magnitude of the displacement, the size of S, as well as the intensity of the body and surface forces, respectively. If the nondimensional quantities are denoted by the same symbols used to denote the corresponding dimensional quantities, then the elasticity equations and the boundary conditions can be written in the form

$$\frac{\tilde{T}}{L}\nabla_* \cdot \mathbf{T}_* = -\tilde{b}\,\mathbf{b}, \tag{1.129}$$

$$\tilde{T}\,\mathbf{T}_*\mathbf{N}_* = \tilde{t}\,\mathbf{t}_*. \tag{1.130}$$

On the other hand, if we adopt the stress tensor of linear elasticity as a measure of the stress state of S, then from the following constitutive relation of linear elasticity:

$$\mathbf{T} = \lambda \,\mathrm{tr}\mathbf{E}\,\mathbf{I} + 2\mu\mathbf{E},$$

where $\mathbf{E}$ is the infinitesimal deformation tensor and λ and μ are the Lamé coefficients, we obtain

$$\tilde{T} \simeq \Gamma \frac{l}{L}.$$

Here, $\Gamma = \mathrm{Max}\{\lambda, \mu\}$ and l represents a length used to evaluate the displacements. Substituting this value of $\tilde{T}$ into (1.130), we obtain the system

$$\nabla_* \cdot \mathbf{T}_* = -\frac{L^2}{l\Gamma}\tilde{b}\mathbf{b}, \tag{1.131}$$

$$\mathbf{T}_*\mathbf{N}_* = \frac{L\tilde{t}}{l\Gamma}\mathbf{t}, \tag{1.132}$$

where all quantities are nondimensional. We recognize that the state C_* is unstressed when

$$\epsilon \equiv \frac{L^2}{l\Gamma}\tilde{b} \simeq \frac{L\tilde{t}}{l\Gamma} = 0.$$

This parameter allows us to evaluate the magnitude of the body and surface forces acting on S that we must apply to an elastic body when we know the nature of S through the coefficient Γ, its dimension L, and the magnitude l of the required displacements.

1.15 Signorini's Perturbation Method for Mixed Problems

In some of the above sections we analyzed some particular solutions for nonlinear elasticity, showing how it is possible to derive the elastic deformation energy from them. In this section we describe Signorini's method [21, 22, 23], which is based on regular perturbation theory and allows us to obtain approximate solutions for boundary value problems associated with nonlinear elasticity.

This method can be used when the following conditions are satisfied:

- The elastic system is subjected to dead loads (see Sect. 1.3)
- The Piola–Kirchhoff stress tensor $\mathbf{T}_*$ depends *analytically* on the displacement gradient $\mathbf{H}$
- The body forces $\mathbf{b}(\epsilon, \mathbf{X})$, the surface forces $\mathbf{t}_*(\epsilon, \mathbf{X})$, and the solution $\mathbf{u}(\epsilon, \mathbf{X})$ depend *analytically* on a perturbation parameter ϵ, which can be identified as the parameter introduced in the previous section

- The boundary value problem of linear elasticity permits one and only one solution.

In the following sections we present some existence and uniqueness theorems for the boundary value problems of nonlinear elasticity that clarify the hypotheses for which the above conditions are obeyed and Signorini's method is applicable.

If the function $\mathbf{T}_* = \mathbf{A}(\mathbf{H})$ is analytic, then we can write

$$\mathbf{T}_* = \mathbf{A}(\mathbf{H}) = \sum_{n=1}^{\infty} \mathbf{A}_n(\mathbf{H}), \quad \mathbf{A}(\mathbf{0}) = \mathbf{0}, \tag{1.133}$$

where the functions $\mathbf{A}_n(\mathbf{H})$ are homogeneous polynomials of degree n in the variable $\mathbf{H}$.[7] Similar expansions hold for the body force $\mathbf{b}(\epsilon, \mathbf{X})$, the surface force $\mathbf{t}_*(\epsilon, \mathbf{X})$, and the displacement $\mathbf{u}(\epsilon, \mathbf{X})$:

$$\mathbf{b}(\epsilon, \mathbf{X}) = \sum_{n=1}^{\infty} \epsilon^n \mathbf{b}_n, \tag{1.134}$$

$$\mathbf{t}_*(\epsilon, \mathbf{X}) = \sum_{n=1}^{\infty} \epsilon^n \mathbf{t}_{*n}, \tag{1.135}$$

$$\mathbf{u}(\epsilon, \mathbf{X}) = \sum_{n=1}^{\infty} \epsilon^n \mathbf{u}_n. \tag{1.136}$$

From (1.136) we derive

$$\mathbf{H} = \sum_{i=1}^{\infty} \epsilon^n \mathbf{H}_n, \tag{1.137}$$

where $\mathbf{H}_n = \nabla_{\mathbf{X}} \mathbf{u}_n$, and by substituting this expression into (1.133) we obtain

$$\mathbf{T}_* = \sum_{n=1}^{\infty} \epsilon^n (\mathbf{C}_{(1)} \mathbf{H}_n + \mathbf{B}_n(\mathbf{H}_1, \dots, \mathbf{H}_{n-1})). \tag{1.138}$$

Here, $\mathbf{C}_1$ is a fourth-order tensor which, considering (1.133), can be identified as the linear elasticity tensor $\mathbf{C}$, $\mathbf{B}_1 = \mathbf{0}$, and $\mathbf{B}_n$ is a polynomial of degree n in the variables $\mathbf{H}_1, \dots, \mathbf{H}_{n-1}$. Using this information, (1.138) becomes:

$$\mathbf{T}_* = \sum_{n=1}^{\infty} \epsilon^n (\mathbf{C}\mathbf{E}_n + \mathbf{B}_n(\mathbf{H}_1, \dots, \mathbf{H}_{n-1})), \tag{1.139}$$

[7] Note that

$$T_{*iL} = A_{iL}(\mathbf{H}) = C_{(1)iLjM} H_{jM} + C_{(2)iLjMhN} H_{jM} H_{hN} + \cdots .$$

with $\mathbf{E}_n = (\mathbf{H}_n + \mathbf{H}_n^T)/2$.

Now we consider the mixed boundary value problem (1.4)–(1.6). Introducing the expansions (1.134), (1.135), and (1.139) into (1.4) and (1.6), we derive the following sequence of *linear* problems:

$$\nabla_{\mathbf{X}} \cdot (\mathbf{C}\mathbf{E}_n) + \rho_* \widehat{\mathbf{b}}_n = \mathbf{0} \text{ in } \mathrm{C}_*, \tag{1.140}$$

$$(\mathbf{C}\mathbf{E}_n) \cdot \mathbf{N}_* = \widehat{\mathbf{t}}_{*n} \text{ on } \partial\Sigma_*, \tag{1.141}$$

$$\mathbf{u}_n = \mathbf{0}, \text{ on } \partial\mathrm{C}_* - \Sigma_*, \tag{1.142}$$

where we have used the notations:

$$\rho_* \widehat{\mathbf{b}}_n \equiv \rho_* \mathbf{b}_n + \nabla_{\mathbf{X}} \cdot \mathbf{B}_n(\mathbf{H}_1, \dots, \mathbf{H}_{n-1}), \tag{1.143}$$

$$\widehat{\mathbf{t}}_{*n} \equiv \mathbf{t}_{*n} - \mathbf{B}_n(\mathbf{H}_1, \dots, \mathbf{H}_{n-1})\mathbf{N}_*. \tag{1.144}$$

For $n = 1$, the above system describes a mixed boundary value problem of linear elasticity with loads of $\widehat{\mathbf{b}}_1 = \mathbf{b}_1$ and $\widehat{\mathbf{t}}_{*1} = \mathbf{t}_1$. More generally, suppose that the system (1.144) has been solved for $n = 1, \dots, m-1$. Then, for $n = m$, we have a new mixed boundary value problem for the *same material and the same domain* with external loads that depend in a known way on the displacements $\mathbf{u}_1, \dots, \mathbf{u}_{m-1}$. In other words, the mth term of the series (1.136) is obtained by solving $m-1$ linear mixed boundary value problems in C_* with different loads.

1.16 Signorini's Method for Traction Problems

The application of Signorini's method to boundary value problems of pure traction is more complex, since we must satisfy the global equilibrium conditions (1.17) and (1.18). As we remarked in Sect. 1.3, these conditions must be interpreted as compatibility conditions; i.e., the solution of the pure traction problem is acceptable if and only if it verifies (1.17) and (1.18).

First, we denote by $\mathbf{r}_*$ the position vector of $\mathbf{X} \in C_*$ with respect to an arbitrary origin O. The position vector $\mathbf{r}$ of the point $\mathbf{x} = \mathbf{x}(\mathbf{X})$ in the deformed equilibrium configuration can then be written as $\mathbf{r} = \mathbf{r}_* + \mathbf{u}$. Consequently, (1.17) and (1.18) assume the form

$$\int_{C_*} \rho_* \mathbf{b} \, dc_* + \int_{\partial C_*} \mathbf{t}_* \, d\sigma_* = \mathbf{0}, \tag{1.145}$$

$$\int_{C_*} \rho_* (\mathbf{r}_* + \mathbf{u}) \times \mathbf{b} \, dc_* + \int_{\partial C_*} (\mathbf{r}_* + \mathbf{u}) \times \mathbf{t}_* \, d\sigma_* = \mathbf{0}, \tag{1.146}$$

We note that, since the loads are dead, (1.145) is *a restriction on the data* $\mathbf{b}$ and $\mathbf{t}_*$. Moreover, the following theorem holds.

Theorem 1.5
(**Da Silva**) *Let $\mathcal{F} = (\rho_* \mathbf{b}, \mathbf{t}_*)$ be a given system of forces acting on a body S. Then the total momentum of $\mathcal{F}$ with respect to an arbitrary pole O can always be reduced to zero by a convenient rigid rotation of S about O, without modifying the direction of the forces.*

PROOF If the *astatic load*

$$\mathbf{A} = \int_{C_*} \mathbf{r}_* \otimes \rho_* \mathbf{b}\, dc_* + \int_{\partial C_*} \mathbf{r}_* \otimes \mathbf{t}_*\, d\sigma_* \tag{1.147}$$

is introduced, then the total momentum of $\mathcal{F}$ vanishes; i.e.,

$$\int_{C_*} \mathbf{r}_* \times \rho_* \mathbf{b}\, dc_* + \int_{\partial C_*} \mathbf{r}_* \times \mathbf{t}_*\, d\sigma_* = \mathbf{0}, \tag{1.148}$$

if and only if $\mathbf{A} = \mathbf{A}^T$. On the other hand, if we denote the orthogonal matrix that defines a rotation about O by $\mathbf{Q}$, we have

$$\int_{C_*} \mathbf{Q}\mathbf{r}_* \times \rho_* \mathbf{b}\, dc_* + \int_{\partial C_*} \mathbf{Q}\mathbf{r}_* \times \mathbf{t}_*\, d\sigma_* = \mathbf{Q}\mathbf{A},$$

since it is assumed that the forces do not change their directions. Thus, the total momentum vanishes after the rotation $\mathbf{Q}$ if and only if

$$\mathbf{Q}\mathbf{A} - \mathbf{A}^T\mathbf{Q}^T = \mathbf{0}. \tag{1.149}$$

To prove the theorem we must now verify that equation (1.149) has a solution $\mathbf{Q}$ for any given matrix $\mathbf{A}$. However, $\mathbf{A}$ can always be represented as the product of an orthogonal matrix $\mathbf{R}$ and a symmetric matrix $\mathbf{S}$, which is positive semidefinite. Thus, it is sufficient to take $\mathbf{Q} = \mathbf{R}^T$ if $\mathbf{Q}$ is a proper rotation, or $\mathbf{Q} = -\mathbf{R}^T$ in the opposite case, and the theorem is proved. ∎

In conclusion, the compatibility condition (1.146) reduces to the equation

$$\int_{C_*} \rho_* \mathbf{u} \times \mathbf{b}\, dc_* + \int_{\partial C_*} \mathbf{u} \times \mathbf{t}_*\, d\sigma_* = \mathbf{0}. \tag{1.150}$$

Using (1.134)–(1.136), this condition becomes

$$\begin{aligned}
&\epsilon^2 \left(\int_{C_*} \rho_* \mathbf{u}_1 \times \mathbf{b}_1\, dc_* + \int_{\partial C_*} \mathbf{u}_1 \times \mathbf{t}_{*1}\, d\sigma_* \right) \\
&+ \epsilon^3 \left(\int_{C_*} \rho_* \mathbf{u}_2 \times \mathbf{b}_1\, dc_* \right) + \int_{\partial C_*} \mathbf{u}_2 \times \mathbf{t}_{*1}\, d\sigma_* \\
&+ \int_{C_*} \rho_* \mathbf{u}_1 \times \mathbf{b}_2\, dc_* + \int_{\partial C_*} \mathbf{u}_1 \times \mathbf{t}_{*2}\, d\sigma_* \Big) + \cdots = \mathbf{0}.
\end{aligned}$$

This can be written in a more compact form as follows:

$$\sum_{n,m=1}^{\infty} \epsilon^{n+m} \left(\int_{C_*} \rho_* \mathbf{u}_n \times \mathbf{b}_m \, dc_* + \int_{\partial C_*} \mathbf{u}_n \times \mathbf{t}_{*m} \, d\sigma_* \right) = \mathbf{0}, \quad (1.151)$$

and we conclude that *if the series* (1.136) *is a solution of the traction boundary value problem*

$$\nabla_{\mathbf{X}} \cdot \mathbf{T}_* + \rho_* \mathbf{b} = \mathbf{0} \ \text{ in } \ C_*, \quad (1.152)$$

$$\mathbf{T}_* \mathbf{N}_* = \mathbf{t}_* \ \text{ on } \ \partial C_*, \quad (1.153)$$

then $\mathbf{u}_n, \dots, \mathbf{u}_n$ *must be solutions of the linear boundary problems*

$$\nabla_{\mathbf{X}} \cdot \mathbf{C}_n \mathbf{E}_n + \rho_* \widehat{\mathbf{b}}_n = \mathbf{0} \ \text{ in } \ C_*, \quad (1.154)$$

$$(\mathbf{C}\mathbf{E}_n)\mathbf{N}_* = \widehat{\mathbf{t}}_n \ \text{ on } \ \partial C_*, \quad (1.155)$$

and they must verify the condition

$$\sum_{m=1}^{n-1} \left(\int_{C_*} \rho_* \mathbf{u}_{n-m} \times \mathbf{b}_m \, dc_* + \int_{\partial C_*} \mathbf{u}_{n-m} \times \mathbf{t}_{*m} \, d\sigma_* \right) = \mathbf{0}. \quad (1.156)$$

Let $\overline{\mathbf{u}}_1$ be a solution of the first-order boundary value problem of traction obtained for $n = 1$ from (1.140)–(1.141):

$$\nabla_{\mathbf{X}} \cdot (\mathbf{C}\mathbf{E}_1) + \rho_* \widehat{\mathbf{b}}_1 = \mathbf{0} \ \text{ in } \ C_*, \quad (1.157)$$

$$(\mathbf{C}\mathbf{E}_1) \cdot \mathbf{N}_* = \widehat{\mathbf{t}}_{*1} \ \text{ on } \ \partial C_*. \quad (1.158)$$

Since $\mathbf{u}_1 = \overline{\mathbf{u}}_1 + \mathbf{W}_1 \mathbf{r}_*$ is still a solution of (1.157)–(1.158) for any skew-symmetric tensor $\mathbf{W}_1$, we can try to determine $\mathbf{W}_1$ so as to eliminate the coefficient of ϵ in (1.151). In other words, we want to determine the skew-symmetric tensor $\mathbf{W}_1$ that satisfies the equation

$$\int_{C_*} \rho_* \mathbf{W}_1 \mathbf{r}_* \times \mathbf{b}_1 \, dc_* + \int_{\partial C_*} \mathbf{W}_1 \mathbf{r}_* \times \mathbf{t}_{*1} \, d\sigma_* = -\mathbf{R}_1, \quad (1.159)$$

where

$$\mathbf{R}_1 = \int_{C_*} \rho_* \overline{\mathbf{u}}_1 \times \mathbf{b}_1 \, dc_* + \int_{\partial C_*} \overline{\mathbf{u}}_1 \times \mathbf{t}_{*1} \, d\sigma_*.$$

Recalling the expression (1.147) for the astatic load, the above equation becomes

$$\epsilon_{ijk} W_{(1)jh} A_{(1)hk} = -R_{(1)i}. \quad (1.160)$$

If $\omega_{(1)i} = \epsilon_{ijh} W_{(1)jh}/2$ is the adjoint vector of $W_{(1)jh}$, we have $W_{(1)jh} = 2\epsilon_{jhi}\omega_i$, and noting that $\epsilon_{ikj}\epsilon_{jhl} = \delta_{ih}\delta_{kl} - \delta_{il}\delta_{kh}$, (1.160) assumes the form

$$(A_{(1)ik} - A_{(1)hh}\delta_{ik})\omega_{(1)k} = 2R_{(1)i},$$

which can also be written as

$$(\mathbf{A}_1 - (\mathrm{tr}\mathbf{A}_1)\mathbf{I})\omega_1 = 2\mathbf{R}_1. \tag{1.161}$$

In conclusion, (1.161) has one and only one solution ω_1 if and only if

$$\det(\mathbf{A}_1 - (\mathrm{tr}\mathbf{A}_1)\mathbf{I}) \neq 0. \tag{1.162}$$

If this condition is satisfied and ω_1 is the solution of (1.161), then the displacement $\mathbf{u}_1 = \overline{\mathbf{u}}_1 + \mathbf{W}_1\mathbf{r}_*$ is a solution of the boundary value problem (1.157)–(1.158) for which the coefficient of ϵ in (1.151) vanishes.

More generally, let $\mathbf{u}_1, \ldots, \mathbf{u}_{p-1}$ be solutions of the boundary value problem (1.154)–(1.155) for $N = 1, \ldots, p-1$ that satisfy (1.151) for $n = 2, \ldots, p-1$. If $\overline{\mathbf{u}}_p$ is a solution of (1.154)–(1.155) for $n = p$, then we can choose the skew-symmetric tensor $\mathbf{W}_p$ in such a way that $\overline{\mathbf{u}}_p + \mathbf{W}_p\mathbf{r}_*$ satisfies (1.151) for $n = p$. In other words, $\mathbf{W}_p$ has to be a solution of the equation

$$\int_{C_*} \rho_* \mathbf{W}_p \mathbf{r}_* \times \mathbf{b}_p \, dc_* + \int_{\partial C_*} \mathbf{W}_p \mathbf{r}_* \times \mathbf{t}_{*p} \, d\sigma_* = -\mathbf{R}_p, \tag{1.163}$$

where

$$\begin{aligned} \mathbf{R}_p = & \int_{C_*} \rho_* \overline{\mathbf{u}}_p \times \mathbf{b}_p \, dc_* + \int_{\partial C_*} \overline{\mathbf{u}}_p \times \mathbf{t}_{*p} \, d\sigma_* \\ + & \sum_{m=1}^{p-1} \left(\int_{C_*} \rho_* \mathbf{u}_m \times \mathbf{b}_{p+1-m} \, dc_* + \int_{\partial C_*} \mathbf{u}_m \times \mathbf{t}_{*p+1-m} \, d\sigma_* \right). \end{aligned} \tag{1.164}$$

Proceeding as we did for (1.159), using the hypothesis (1.162) we find that $\mathbf{W}_p$ is a solution of (1.161).

All of the above results prove *Signorini's existence and uniqueness theorem*:

Theorem 1.6
If the dead load $(\rho\mathbf{b}_1, \mathbf{t}_{*1})$ *satisfies condition* (1.162) *and the pure traction boundary value problem of linear elasticity permits a solution, then the compatibility condition allows us to determine one and only one solution* $\mathbf{u}_n$ *of* (1.154)–(1.155) *for any* n.

1.17 Loads with an Equilibrium Axis

The dead load $(\rho\mathbf{b}_1, \mathbf{t}_{*1})$ is said to have an *equilibrium axis* if

$$\det(\mathbf{A}_1 - (\mathrm{tr}\mathbf{A}_1)\mathbf{I}) = 0. \tag{1.165}$$

If this condition is obeyed, then Signorini's procedure cannot be applied. In this section we analyze the meaning of (1.165).

First, we note that (1.165) is equivalent to saying that $\mathrm{tr}\mathbf{A}_1$ is an eigenvalue of the astatic load $\mathbf{A}_1$. Considering this, the following theorem holds.

Theorem 1.7
$\alpha \equiv \mathrm{tr}(\mathbf{A}_1)$ is an eigenvalue of $\mathbf{A}_1$ if and only if

$$\mathbf{A}_1 = \beta(\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2) + \alpha \mathbf{e}_3 \otimes \mathbf{e}_3, \tag{1.166}$$

where $\mathbf{e}_3$ is the eigenvector corresponding to α, and β is a real number.

PROOF Since $(\rho_* \mathbf{b}, \mathbf{t}_*)$ satisfies the condition (1.148), the corresponding astatic load $\mathbf{A}_1$ is symmetric. Therefore, it permits a basis of eigenvectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ in which the representative matrix of $\mathbf{A}_1$ is diagonal

$$\begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix},$$

where $\beta_1, \beta_2, \beta_3$ are the eigenvalues of $\mathbf{A}_1$. If $\alpha = \mathrm{tr}\mathbf{A}_1$ is an eigenvalue, then we have

$$\beta_1 + \beta_2 + \alpha = \alpha;$$

i.e., $\beta_1 = -\beta_2$ and (1.166) is proved. On the other hand, if (1.166) holds, $\alpha = \mathrm{tr}\mathbf{A}_1$ is an eigenvalue of $\mathbf{A}_1$. ∎

Theorem 1.8
The astatic load $\mathbf{A}_1$ permits representation (1.166) *if and only if*

$$\mathbf{Q}\mathbf{A}_1 - \mathbf{A}_1 \mathbf{Q}^T = \mathbf{0}, \tag{1.167}$$

for any rotation $\mathbf{Q}$ about the axis $\mathbf{e}_3$.

PROOF First, in the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, any rotation about $\mathbf{e}_3$ is described by the matrix

$$\begin{pmatrix} \cos\theta & \sin\theta 0 \\ -\sin\theta & \cos\theta 0 \\ 0 & 0 \quad 1 \end{pmatrix}, \tag{1.168}$$

where θ is the angle of rotation about $\mathbf{e}_3$ starting from $\mathbf{e}_1$. It is then easy to verify that conditions (1.166) and (1.168) are equivalent to (1.167). ∎

When (1.149) is taken into account, the above theorems show that Signorini's method fails when there is an axis $\mathbf{e}_3$ such that the dead load $(\rho_* \mathbf{b}_1, \mathbf{t}_{*1})$ is equilibrated for any rotation about $\mathbf{e}_3$.

1.18 Second-Order Hyperelasticity

In this section we introduce second-order hyperelasticity in order to analyze a simple but interesting application of Signorini's method (see [31]–[37]).

Let S be a hyperelastic isotropic body, and let C_* be a homogeneous unstressed reference configuration. The elastic potential ψ is a function of the three principal invariants I_B, II_B, and III_B of the right Cauchy–Green tensor $\mathbf{B}$, or, equivalently, a function of the three principal invariants of the Saint–Venant deformation tensor G (see Chap. 3 of [16]):

$$\rho_* \widetilde{\psi}(I_B, II_B, III_B) = \rho_* \psi(I_G, II_G, III_G), \tag{1.169}$$

where the invariants I_G, II_G, III_G are infinitesimal and are of the same order of the deformation gradient $\mathbf{H}$. The expansion of ψ at $(0,0,0)$, up to third-order terms in the components of $\mathbf{H}$, is

$$\rho_* \psi \simeq \alpha_1 I_G + \alpha_2 II_G + \alpha_3 III_G + \alpha_4 I_G^2 + \alpha_5 I_G II_G + \alpha_6 I_G^3, \tag{1.170}$$

where $\alpha_1, \ldots, \alpha_6$ are suitable numeric coefficients. In view of (1.11), we have

$$\mathbf{T}_* = (\mathbf{I} + \mathbf{H}) \left[\frac{\partial \psi}{\partial I_G} \frac{\partial I_G}{\partial \mathbf{G}} + \frac{\partial \psi}{\partial II_G} \frac{\partial II_G}{\partial \mathbf{G}} + \frac{\partial \psi}{\partial III_G} \frac{\partial III_G}{\partial \mathbf{G}} \right].$$

If we take into account formulae (3.52) and (3.53) in [16], which give the derivatives of the principal invariants of a matrix with respect to the matrix itself, the above equation becomes

$$\begin{aligned} \mathbf{T}_* = (\mathbf{I} + \mathbf{H}) \bigg[& \frac{\partial \psi}{\partial I_G} \mathbf{I} + \frac{\partial \psi}{\partial II_G} (I_G \mathbf{I} - \mathbf{G}) \\ & + \frac{\partial \psi}{\partial III_G} (\mathbf{G}^2 - I_{\mathbf{G}} \mathbf{G} + II_{\mathbf{G}} \mathbf{I}) \bigg]. \end{aligned} \tag{1.171}$$

When we recall the hypothesis that $\mathbf{T}(\mathbf{0}) = \mathbf{0}$ in the absence of deformation, then we obtain $\alpha_1 = 0$ from (1.170) and (1.171). Consequently, since $\mathbf{H} = 2\mathbf{E} - \mathbf{H}^T$, we can write (1.171) in the following form:

$$\begin{aligned} \mathbf{T}_* = (\mathbf{I} + 2\mathbf{E} - \mathbf{H}^T) & \left[(2\alpha_4 I_{\mathbf{G}} + \alpha_5 II_{\mathbf{G}} + 3\alpha_6 I_{\mathbf{G}}^2) \mathbf{I} \right. \\ & \left. + (\alpha_2 + \alpha_5 I_{\mathbf{G}})(I_{\mathbf{G}} \mathbf{I} - \mathbf{G}) + \alpha_3 (\mathbf{G}^2 - I_{\mathbf{G}} \mathbf{G} + II_{\mathbf{G}} \mathbf{G} \right], \end{aligned}$$

so that we have

$$\begin{aligned} \mathbf{T}_* = (\mathbf{I} + 2\mathbf{E} - \mathbf{H}^T) & \left\{ \left[(2\alpha_4 + \alpha_2) I_{\mathbf{G}} + (3\alpha_6 + \alpha_5) I_{\mathbf{G}}^2 + (\alpha_3 + \alpha_5) II_{\mathbf{G}} \right] \mathbf{I} \right. \\ & \left. - (\alpha_2 + (\alpha_3 + \alpha_5) I_{\mathbf{G}}) \mathbf{G} + \alpha_3 \mathbf{G}^2 \right\}. \end{aligned} \tag{1.172}$$

Remembering that

$$\mathbf{G} = \mathbf{E} + \frac{1}{2}\mathbf{H}\mathbf{H}^T, \tag{1.173}$$

we can easily prove the following relations with our approximations:

$$I_{\mathbf{G}} = I_{\mathbf{E}} + \frac{1}{2} I_{\mathbf{H}\mathbf{H}^T}, \quad I_{\mathbf{G}}^2 = I_{\mathbf{E}}^2, \tag{1.174}$$

$$II_{\mathbf{G}} = II_{\mathbf{E}}, \quad \mathbf{G}^2 = \mathbf{E}^2, \tag{1.175}$$

The above relations allow us to write the Piola–Kirchhoff stress tensor in the form

$$\begin{aligned} \mathbf{T}_* = \lambda I_{\mathbf{E}}\mathbf{I} + 2\mu\mathbf{E} + \left(\frac{\lambda}{2} I_{\mathbf{H}\mathbf{H}^T} + \beta_1 I_{\mathbf{E}}^2 + \beta_2 II_{\mathbf{E}}\right)\mathbf{I} \\ + (2\lambda - \beta_2) I_{\mathbf{E}}\mathbf{E} + \beta_4\mathbf{E}^2 - \lambda I_{\mathbf{E}}\mathbf{H}^T - \mu(\mathbf{H}^T)^2, \end{aligned} \tag{1.176}$$

where the five coefficients λ, μ, β_1, β_2, β_4 are related to the coefficients $\alpha_2, \ldots, \alpha_6$ by the relations

$$\lambda = 2\alpha_4 + \alpha_2, \quad 2\mu = -\alpha_2, \tag{1.177}$$

$$\beta_1 = 3\alpha_6 + \alpha_5, \quad \beta_2 = \alpha_3 + \alpha_5, \quad \beta_4 = \alpha_3 - 2\alpha_2. \tag{1.178}$$

If we suppose that a nondimensional parameter ϵ is introduced and we recall Signorini's hypotheses (1.134), (1.135) and (1.136) about the displacement and the loads, then we obtain, up to second-order terms in the parameter ϵ,

$$I_{\mathbf{E}} = \epsilon I_{\mathbf{E}_1} + \epsilon^2 I_{\mathbf{E}_2}, \quad \mathbf{E} = \epsilon\mathbf{E}_1 + \epsilon^2\mathbf{E}_2, \quad I_{\mathbf{E}}^2 = \epsilon^2 I_{\mathbf{E}_1}^2, \tag{1.179}$$

$$I_{\mathbf{H}\mathbf{H}^T} = \epsilon^2 I_{\mathbf{H}_1\mathbf{H}_1^T}, \quad II_{\mathbf{E}} = II_{\mathbf{E}_1}, \quad I_{\mathbf{E}}\mathbf{E} = \epsilon^2 I_{\mathbf{E}_1}\mathbf{E}_1, \tag{1.180}$$

$$\mathbf{E}^2 = \epsilon^2\mathbf{E}_1^2, \quad I_{\mathbf{E}}\mathbf{H}^T = \epsilon^2 I_{\mathbf{E}_1}\mathbf{H}_1^T, \quad (\mathbf{H}^T)^2 = \epsilon^2(\mathbf{H}_1^T)^2. \tag{1.181}$$

When we take into account these relations, the Piola–Kirchhoff stress tensor (1.176) becomes

$$\mathbf{T}_* = \epsilon\mathbb{C}\mathbf{E}_1 + \epsilon^2(\mathbb{C}\mathbf{E}_2 + \mathbf{B}_1(\mathbf{H}_1)), \tag{1.182}$$

where $\mathbb{C}$ is the linear elasticity stress tensor, so that

$$\mathbb{C}\mathbf{E}_i = \lambda I_{\mathbf{E}_i}\mathbf{I} + 2\mu\mathbf{E}_i, \quad i = 1, 2, \tag{1.183}$$

and

$$\begin{aligned} \mathbf{B}_1(\mathbf{H}_1) = \left(\frac{1}{2} I_{\mathbf{H}_1\mathbf{H}_1^T} + \beta_1 I_{\mathbf{E}_1}^2 + \beta_2 II_{\mathbf{E}_1}\right)\mathbf{I} \\ + (2\lambda - \beta_2) I_{\mathbf{E}_1}\mathbf{E}_1 + \beta_4\mathbf{E}_1^2 - \lambda I_{\mathbf{E}_1}\mathbf{H}_1^T - \mu(\mathbf{H}_1^T)^2. \end{aligned} \tag{1.184}$$

We conclude this section by noting that, if the material is elastic but not hyperelastic, then, instead of (1.176), the approximate expression for the Piola–Kirchhoff tensor is

$$\mathbf{T}_* = \lambda I_{\mathbf{E}}\mathbf{I} + 2\mu\mathbf{E} + \left[\frac{\lambda}{2}\left(I_{\mathbf{H}\mathbf{H}^T} + 2I_{\mathbf{E}}^2\right) + \beta_1 I_{\mathbf{E}}^2 + \beta_2 II_{\mathbf{E}}\right]\mathbf{I}$$
$$+ \beta_3 I_{\mathbf{E}}\mathbf{E} + \beta_4\mathbf{E}^2 - \lambda I_{\mathbf{E}}\mathbf{H}^T - \mu(\mathbf{H}^T)^2. \tag{1.185}$$

By comparing (1.185) and (1.176), we see that a second-order elastic body is a second-order hyperelastic body if and only if

$$\beta_3 = 2\lambda - \beta_2. \tag{1.186}$$

1.19 A Simple Application of Signorini's Method

In this section we show a simple application of Signorini's method. Let S be a cube of a homogeneous, hyperelastic material in the reference configuration C_* (see Fig. 1.8). We assume that face a is fixed, whereas face b is acted upon by a constant force $\mathbf{t}$ parallel to b.

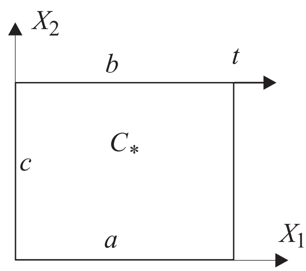

Fig. 1.8 Simple shear of a parallelepiped

If we denote the small angle that face c forms with axis OX_2 after the deformation by ϵ, then, up to second-order terms, we have

$$\mathbf{t} = \epsilon\mathbf{t}_1 + \epsilon^2\mathbf{t}_2, \quad \mathbf{H} = \epsilon\mathbf{H}_1 + \epsilon^2\mathbf{H}_2. \tag{1.187}$$

It is very easy to verify that the system of linear elasticity

$$\nabla\cdot(\mathbb{C}\mathbf{E}_1) = \mathbf{0} \quad \text{in}\, C_* \tag{1.188}$$
$$\mathbb{C}\mathbf{E}_1 = \mathbf{t}_1 \quad \text{on}\, b, \tag{1.189}$$

permits the solution

$$\mathbf{u}_1 = (X_2, 0, 0), \tag{1.190}$$

corresponding to a small simple shear. In fact, we have

$$\mathbf{H}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.191}$$

$$\mathbf{E} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.192}$$

so that the first-order stress tensor $\mathbf{T}_{*,1} = 2\mu\mathbf{E}_1$ is constant and (1.188) is satisfied. Moreover, the first-order traction acting on face b is given by

$$\mathbf{t}_1 = (\mu, 0, 0). \tag{1.193}$$

To evaluate the displacement $\mathbf{u}_2$, we must consider the system

$$\nabla \cdot (\mathbb{C}\mathbf{E}_2) = -\nabla \cdot (\mathbf{B}_1(\mathbf{H}_1)) \quad \text{in}\, C_* \tag{1.194}$$

$$\mathbb{C}\mathbf{E}_2 = \mathbf{t}_2 - \mathbf{B}_1(\mathbf{H}_1)\mathbf{N} \quad \text{on}\, b. \tag{1.195}$$

On the other hand, we get

$$\mathbf{H}_1^T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.196}$$

$$\mathbf{H}_1\mathbf{H}_1^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.197}$$

$$\mathbf{E}_1^2 = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\mathbf{H}_1^T)^2 = \mathbf{0}, \tag{1.198}$$

$$I_{\mathbf{H}_1\mathbf{H}_1^T} = 1, \quad II_{\mathbf{E}_1} = \frac{1}{4}, \quad I_{\mathbf{E}_1}^2 = 0. \tag{1.199}$$

Finally, we obtain the expression for the right-hand side of (1.195):

$$\mathbf{B}_1(\mathbf{H}_1) = \frac{1}{4}(2\lambda - \beta_2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{4}\beta_4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.200}$$

and the second-order system assumes the form

$$\nabla \cdot (\mathbb{C}\mathbf{E}_2) = \mathbf{0} \quad \text{in}\, C_* \tag{1.201}$$

$$\mathbb{C}\mathbf{E}_2 = \mathbf{t}_2 - \mathbf{B}_1(\mathbf{H}_1)\mathbf{N} \quad \text{on}\, b. \tag{1.202}$$

We note that the equilibrium equation in C_* is identical to (1.188), whereas the boundary condition on b is different since any face of the cube is acted upon by a normal force. We will not solve this more complex problem here; we will simply remark that in order to get a second-order deformation corresponding to a simple shear, we must apply a normal force that balances the force $\mathbf{B}_1(\mathbf{H}_1)\mathbf{N}$ to any face. This result has already been proved for finite elasticity (see Sect. 1.9).

1.20 Van Buren's Theorem

In this section, we sketch out the proof of an existence and uniqueness theorem for a mixed boundary value problem that was developed by Van Buren (see [24]). At the end of this section we will cite some results obtained by Stoppelli that refer to the existence and uniqueness of the solutions of pure traction boundary value problems. All of these theorems have *local* character since they only ensure the existence and uniqueness of the solutions for deformations that are not too far from those to which the linear elasticity approximation can be applied. Moreover, they require that an existence and uniqueness theorem holds for the corresponding problems of linear elasticity. All of the theorems cited here are based on the *inverse mapping theorem*, (Banach–Caccioppoli) which gives sufficient conditions for the invertibility of a map between two Banach spaces.

Let F and F' be two Banach spaces and $f : F \longrightarrow F'$ be a Frechét differentiable function whose differential $Df(x, h)$ is a continuous function of $x \in F$. Moreover, let us suppose that at the point $x_0 \in F$ the differential

$$Df(x_0, h) : F \longrightarrow F'$$

is an *isomorphism*. In these hypotheses, the inverse mapping theorem states that it is possible to find a neighborhood $I(x_0)$ of x_0 and a neighborhood $I(F(x_0))$ of $F(x_0)$ in which F can be inverted.

Let C_* be a *natural* reference configuration of an elastic body S (i.e., a configuration in which the stress tensor vanishes), and let ϵ be a nondimensional parameter that is characteristic of the problem (see the above sections). We suppose that S is acted upon by specific body forces $\mathbf{b}(\mathbf{X}, \epsilon)$ and surface forces $\mathbf{t}_*(\mathbf{X}, \epsilon)$. The functions $\mathbf{b}(\mathbf{X}, \epsilon)$ and $\mathbf{t}_*(\mathbf{X}, \epsilon)$, which are respectively defined on $C_* \times I_0$ and $\partial C'_* \times I_0$ (where I_0 is in the neighborhood of the origin of $\Re$ and $\partial C'_*$ is the part of ∂C_* upon which the surface forces act) are assumed to vanish for $\epsilon = 0$ and to be differentiable with respect to ϵ around $\epsilon = 0$:

$$\mathbf{b}(\mathbf{X}, \epsilon) = \epsilon \mathbf{b}_1(\mathbf{X}) + \mathbf{B}(\mathbf{X}, \epsilon), \tag{1.203}$$

$$\mathbf{t}_*(\mathbf{X}, \epsilon) = \epsilon \mathbf{t}_1(\mathbf{X}) + \mathbf{T}_*(\mathbf{X}, \epsilon). \tag{1.204}$$

We want to prove that, under suitable hypotheses, the nonlinear mixed boundary value

$$\nabla_{\mathbf{X}} \cdot \mathbf{T}_*(\mathbf{H_0}) + \rho_* \mathbf{b}(\mathbf{X},\epsilon) = \mathbf{0}, \quad \forall \mathbf{X} \in C_* \tag{1.205}$$

$$\mathbf{T}_*(\mathbf{H}_0) \cdot \mathbf{N}_* = \mathbf{t}_*(\mathbf{X}, \epsilon) \quad \forall \mathbf{X} \in \partial C'_*, \tag{1.206}$$

$$\mathbf{u}_0(\mathbf{X}) = \mathbf{0}, \quad \forall \mathbf{X} \in \partial C_* - \partial C'_*. \tag{1.207}$$

has one and only one solution for any value of ϵ in the neighborhood of the origin. To prove this, we make use of the inverse mapping theorem.

Let us introduce the pair of vector functions

$$\mathbf{h} = \nabla_{\mathbf{X}} \cdot \mathbf{T}_*(\mathbf{H}) + \rho_* \mathbf{b}(\mathbf{X}, \epsilon), \tag{1.208}$$

$$\mathbf{g} = -\mathbf{T}_*(\mathbf{H})\mathbf{N}_* + \mathbf{t}_*(\mathbf{X}, \epsilon), \tag{1.209}$$

where $\mathbf{H} = \nabla_{\mathbf{X}} \mathbf{u}(\mathbf{X})$ and $\mathbf{u}(\mathbf{X})$ is an arbitrary displacement field such that $\mathbf{u}(\mathbf{X}) = \mathbf{0}$ on $\partial C_* - \partial C'_*$.

We denote by F the vector space of the functions $\mathbf{u}(\mathbf{X})$, which are suitably regular in $\mathbf{X} \in \mathbf{C}_*$ and vanish on $\partial C_* - \partial C'_*$. We also suppose that it is possible to introduce a norm $\parallel \mathbf{u}(\mathbf{X}) \parallel$ into F such that F becomes a Banach space. Similarly, we consider the Banach space F' of the pairs $(\mathbf{h}(\mathbf{X}), \mathbf{g}(\mathbf{X}))$, as equipped with a convenient norm $\parallel (\mathbf{h}(\mathbf{X}), \mathbf{g}(\mathbf{X})) \parallel$. In this way, we can associate a *nonlinear* operator with the system (1.205), (1.206) and (1.207):

$$\mathbb{F} : F \longrightarrow F', \tag{1.210}$$

such that

$$\mathbb{F}(\mathbf{u}(.)) = (\mathbf{h}(.), \mathbf{g}(.)). \tag{1.211}$$

We note that to the displacement $\mathbf{0}(.) \in F$ corresponds the pair $(\rho_* \mathbf{b}(., \epsilon), \mathbf{t}_*(., \epsilon))$. Moreover, if $\mathbb{F}$ is of class C^1 in the neighborhood of $\mathbf{0}(.)$, with respect to the norms of F and F', and if the Fréchet derivative $D_{\mathbf{0}}\mathbb{F} : F \longrightarrow F'$ is an isomorphism, then, due to the inverse mapping theorem, there is a neighborhood N of $\mathbf{0}(.)$ such that the correspondence between N and $\mathbb{F}(N)$ is a diffeomorphism. Consequently, if $(\mathbf{0}, 0) \in \mathbb{F}(N)$, then there is one and only one displacement $\mathbf{u}_0(\mathbf{X}, \epsilon)$ such that $\mathbb{F}(\mathbf{u}_0) = (\mathbf{0}, 0)$. When we recall how the pair $(\mathbf{h}, \mathbf{g})$ has been defined, we conclude that $\mathbf{u}_0(\mathbf{X}, \epsilon)$ represents the solution of the boundary value problem (1.205)–(1.207).

In conclusion, in order to prove an existence and uniqueness theorem for the boundary value problem (1.205)–(1.207) by the inverse mapping theorem, we need:

- To define the Banach vector spaces F and F'
- To verify that the functional (1.210) is of class C^1
- To prove that $D_0\mathbb{F}$ is an isomorphism
- To prove that there is a neighborhood N of $\mathbf{0}(.)$ such that $\mathbb{F} : N \to \mathbb{F}(N)$ is a one-to-one map and $(\mathbf{0}, 0) \in \mathbb{F}(N)$.

To realize all of the above conditions, we suppose that

1. The region C_* is compact.
2. ∂C_* is a surface of class $C^{2+\lambda}$, $\lambda > 0$; i.e., this requires that in the neighborhood of any of its points it is possible to introduce local coordinates $(x_1, x_2, x_3) \in \Re^3$ such that the transformation functions $X_L = X_L(x_i)$ have Hölder-continuous second derivatives with exponent λ;[8] moreover, $(x_1, x_2, x_3) \in C_*$ if $x_3 < 0$, whereas $(x_1, x_2, x_3) \in \partial C_*$ if and only if $x_3 = 0$. In other words, (x_1, x_2) are local coordinates on ∂C_*.
3. The function $\mathbf{T}_*(\mathbf{F})$ is of class C^3 in the region defined by the inequality $\| \mathbf{F} - \mathbf{I} \| < \gamma$, where $\gamma > 0$.
4. $\mathbf{b}(., \epsilon)$ is of class $C^{0+\lambda}(C_*)$ and $\mathbf{t}_*(., \epsilon)$ is of class $C^{0+\lambda}(\partial C''_*)$.
5. The linear elasticity tensor $\mathbb{C} = (\partial \mathbf{T}_*/\partial \mathbf{H})_{\mathbf{H}=\mathbf{0}}$ satisfies all of the properties that ensure that the corresponding boundary value problem of linear elasticity has one and only one solution.

When these conditions are obeyed, the Banach space F is the vector space of all vector fields $\mathbf{u}(.)$ of class $C^{2+\lambda}(C_*)$ equipped with the norm

$$\| (\mathbf{h}, \mathbf{g}) \| = \| \mathbf{h} \|_{C^{0+\lambda}(\overline{C}_*)} + \| \mathbf{g} \|_{C^{0+\lambda}(\partial C''_*)}, \tag{1.212}$$

where

$$\| \mathbf{h} \|_{C^{0+\lambda}(\overline{C}_*)} = \sum_{i=1}^{3} (\max_{\mathbf{X} \in C_*} | h_i(\mathbf{X}) | + A_i),$$

$$\| \mathbf{g} \|_{\mathbf{C^{0+\lambda}(\partial C''_*)}} = \sum_{\mathbf{i=1}}^{\mathbf{3}} (\max_{\mathbf{X} \in \partial \mathbf{C}''_*} | \mathbf{g_i}(\mathbf{X}) | + \mathbf{B_i}).$$

[8] In a metric space with metrics d, the function f is Hölder-continuous with exponent $\lambda > 0$ in a region D if

$$| f(x) - f(x') | \leq M\, d(x, x')^\lambda,$$

$\forall x,\, x' \in D$. M is called the Hölder coefficient of f in D.

Here, A_i and B_i are the Hölder coefficients of h_i e g_i in C_* and $\partial C_*''$, respectively.

Finally, let D be the open set F of the vector fields $\mathbf{u} \in \mathbf{F}$ for which $\| \mathbf{u} \| < \gamma$, where γ is the positive number that appears in condition 3.

We can now sketch the proof. To verify that $\mathbb{F}$ is of class C^1 together with its inverse, we put

$$(\Delta\mathbf{h}, \Delta\mathbf{g}) = \mathbb{F}(\mathbf{u} + \Delta\mathbf{u}) - \mathbb{F}(\mathbf{u}).$$

Consequently, we have

$$\Delta\mathbf{h} = \nabla_{\mathbf{X}} \cdot (\mathbf{T}_*(\mathbf{F} + \Delta\mathbf{F}) - \mathbf{T}_*(\mathbf{F})) \quad \text{in}\, C_*, \tag{1.213}$$

$$\Delta\mathbf{g} = -(\mathbf{T}_*(\mathbf{F} + \Delta\mathbf{F}) - \mathbf{T}_*(\mathbf{F})) \cdot \mathbf{N}_* \quad \text{on}\, \partial C_*'', \tag{1.214}$$

where $\Delta\mathbf{F} = \nabla_{\mathbf{X}} \Delta\mathbf{u}$.

On the other hand,

$$\Delta T_{*iL} \equiv T_{*iL}(\mathbf{F} + \Delta\mathbf{F}) - T_{*iL}(\mathbf{F}) = \frac{\partial T_{*iL}}{\partial F_{jM}} \Delta F_{jM} + O(\| \Delta\mathbf{F} \|),$$

so that, if we introduce the notation

$$A_{ijLM} = \frac{\partial T_{*iL}}{\partial F_{jM}}, \quad B_{ijkLMN} = \frac{\partial^2 T_{*iL}}{\partial F_{jM} \partial F_{kN}},$$

the system (1.213)–(1.214) becomes

$$\begin{aligned} \Delta h_i = {} & \left(A_{ijLM} \frac{\partial^2}{\partial X_L \partial X_M} + B_{ijkLMN} \frac{\partial^2 u_k}{\partial X_L \partial X_M} \frac{\partial}{\partial X_N} \right) \Delta u_j \\ & + O(\| \Delta\mathbf{u} \|), \end{aligned} \tag{1.215}$$

$$\Delta g_i = -A_{ijLM} \frac{\partial \Delta u_j}{\partial X_M} N_{*L} + O(\| \Delta\mathbf{u} \|). \tag{1.216}$$

(1.215) and (1.216) show that the nonlinear operator $\mathbb{F}$ is Fréchet differentiable $\forall \mathbf{u} \in F$, since, due to the norm (1.212) and condition 3, the operator

$$D_{\mathbf{u}}\mathbb{F} = \left(A_{ijLM} \frac{\partial^2}{\partial X_L \partial X_M} + B_{ijkLMN} \frac{\partial^2 u_k}{\partial X_L \partial X_M} \frac{\partial}{\partial X_N}, -A_{ijLM} \frac{\partial \Delta u_j}{\partial X_M} N_{*L} \right) \tag{1.217}$$

is a linear map ($F \to F'$) that can be regarded as the Fréchet derivative of $\mathbb{F}$ at the point $\mathbf{u} \in F$.

In order to verify that $\mathbb{F}$ is a diffeomorphism of class C^1, we have to show that $D_{\mathbf{u}}\mathbb{F}$ is continuous with respect to $\mathbf{u} \in F$. Now, if $\overline{\mathbf{u}}, \mathbf{u} \in F$, the difference

$$(D_{\overline{\mathbf{u}}}\mathbb{F} - D_{\mathbf{u}}\mathbb{F})\Delta\mathbf{u}$$

becomes

$$
\begin{aligned}
&\left(\left(\overline{A}_{ij}^{LM} - A_{ij}^{LM}\right)\frac{\partial^2 \Delta u^j}{\partial X^L \partial X^M}\right.\\
&+\left(\overline{B}_{ijk}^{LMN}\frac{\partial^2 \overline{u}^k}{\partial X^L \partial X^M} - B_{ijk}^{LMN}\frac{\partial^2 u^k}{\partial X^L \partial X^M}\right)\frac{\partial \Delta u^j}{\partial X^N};\\
&-\left.\left(\overline{A}_{ij}^{LM} - A_{ij}^{LM}\right)\frac{\partial \Delta u^j}{\partial X^M}N_{*L}\right),
\end{aligned}
\tag{1.218}
$$

and its norm goes to zero when $\overline{\mathbf{u}} \to \mathbf{u}$ in F. Finally,

$$
D_{\mathbf{0}}\mathbb{F} = \left(\mathbb{C}_{ij}^{LM}\frac{\partial^2}{\partial X^L \partial X^M}, -\mathbb{C}_{ij}^{LM}N_{*M}\right), \tag{1.219}
$$

where $\mathbb{C}$ is the linear elasticity tensor evaluated at the natural configuration $\mathbf{u} = \mathbf{0}$. This means that the equation

$$
D_{\mathbf{0}}\mathbb{F}(\Delta\mathbf{u} = (\Delta\mathbf{h}, \Delta\mathbf{g}) \tag{1.220}
$$

coincides with the mixed boundary value problem of linear elasticity. Therefore, if an existence and uniqueness theorem holds for this problem, then $D_{\mathbf{0}}\mathbb{F}$ is an isomorphism. Due to the inverse mapping theorem, the nonlinear operator (1.211) is a diffeomorphism of class C^1 between a neighborhood N of $(\mathbf{0}, 0) \in F$ and a neighborhood $\mathbb{F}(N)$ of the image $(\rho_*\mathbf{b}(., \epsilon), \mathbf{t}_*(., \epsilon)) \in F'$. If $|\epsilon|$ is sufficiently small, then $(\mathbf{0}, \mathbf{0}) \in \mathbb{F}(N)$ and there is only one point $\mathbf{u}_0 \in N$ such that $\mathbb{F}(\mathbf{u}_0) \in \mathbb{F}(N)$.

We can state the following theorem (from Van Buren):

Theorem 1.9
Under hypotheses 1–5, it is possible to find two positive numbers ξ and ζ such that, for any $\epsilon > 0$ and $|\epsilon| < \xi$, the boundary value problem (1.205)–(1.207) permits one and only one solution $\mathbf{u}_0$ that satisfies the condition

$$
\|\mathbf{u}_0\| < \zeta.
$$

It is possible to prove that, if the functions $\mathbf{b}(\mathbf{X}, \epsilon)$, $\mathbf{t}_*(\mathbf{X}, \epsilon)$, and $\mathbf{T}_*(\mathbf{F}, \epsilon)$ are analytic functions of the variable ϵ, then (at least for $\epsilon| < \xi$) the solution $\mathbf{u}_0$ is an analytic function of ϵ and Signorini's method remains valid.

In [26]–[58] (see also [2] and [3]), again using the inverse mapping theorem, Stoppelli proves an existence and uniqueness theorem for the pure traction equilibrium boundary value problem. This case is more difficult than the above case for two reasons. First, we must impose the condition that the acting forces are equilibrated:

$$
\int_{C_*} \rho_*\mathbf{b}\, dc_* + \int_{\partial C_*} \mathbf{t}_*\, d\sigma_* = \mathbf{0}, \tag{1.221}
$$

$$\int_{C_*} \mathbf{r} \times \rho_* \mathbf{b} \, dc_* + \int_{\partial C_*} \mathbf{r} \times \mathbf{t}_* \, d\sigma_* = \mathbf{0}, \tag{1.222}$$

where $\mathbf{r}$ is the position vector.

Moreover, the pure traction boundary value problem of linear elasticity does not allow a uniqueness theorem (see Sect. 10.3 of [16]) since the solution is determined up to an arbitrary infinitesimal rigid displacement.

We conclude this section by mentioning that a wide-ranging and detailed analysis of equilibrium boundary value problems of nonlinear elasticity can be found in [30]. In this book, which collects together many results obtained by the author, theorems of existence and uniqueness and the analytic dependence of the solutions on the nondimensional parameter ϵ are proven for many boundary value problems of finite elasticity with dead loads and some special types of live loads.

1.21 An Extension of Signorini's Method to Live Loads

At the beginning of this chapter we remarked that the acting force are almost always described by live loads. However, these loads introduce many mathematical difficulties into the analysis of elastostatics and elastodynamics (see [28, 29]).

In this section we present an extension of Signorini's method to traction boundary value problems of equilibrium with live loads *in which the prescribed surface traction is parallel to the vector normal to the boundary of the elastic body.* An existence, uniqueness and analytic dependence on ϵ was demonstrated in [30] for this particular type of live load when:

1. The Piola–Kirchooff stress tensor $\mathbf{T}_* = J\mathbf{T}(\mathbf{F}^-1)^T$ depends analytically on the displacement gradient $\mathbf{H}$
2. The loads $\mathbf{b}(\epsilon, \mathbf{X}, \mathbf{u}, \mathbf{H})$ and $\mathbf{t}_*(\epsilon, \mathbf{X}, \mathbf{u}, \mathbf{H})$ are analytic functions of ϵ.

We now consider the following boundary value problem of pure traction for an elastic material:

$$\nabla_{\mathbf{X}} \cdot \mathbf{T}_* = \mathbf{0} \quad \text{in}\, C_*, \tag{1.223}$$

$$\mathbf{T}_* \mathbf{N}_* = \epsilon \mathbf{t}_* \quad \text{on}\, \partial C_*, \tag{1.224}$$

where the loads must be globally equilibrated; i.e., they must obey the conditions:

$$\int_{\partial C_*} \mathbf{t}_* \, d\sigma_* = \mathbf{0}, \tag{1.225}$$

$$\int_{\partial C_*} \mathbf{r} \times \mathbf{t}_* \, d\sigma_* = \mathbf{0}. \tag{1.226}$$

We wish to solve the above problems for the second-order elasticity approximation. We provide the basic results here; they are explored in more detail in [37].

Since we are interested in the second-order elasticity, we must consider the approximate expression (1.185) for the Piola–Kirchhoff stress tensor. Using Signorini's method, we must assume that $\mathbf{u} = \epsilon\mathbf{u}_1 + \epsilon^2\mathbf{u}_2$. Following the usual procedure, we obtain

$$\mathbf{T}_* = \epsilon\mathbf{T}_{*1} + \epsilon^2(\mathbf{T}_{*2} + \mathbf{B}_{*1}) \tag{1.227}$$

from (1.185), where

$$\mathbf{T}_{*i} = \lambda I_{\mathbf{E}}\mathbf{I} + 2\mu\mathbf{E}_i, \quad i = 1, 2, \tag{1.228}$$

$$\begin{aligned}\mathbf{B}_{*1} = &\left[\left(\frac{1}{2}\left(I_{\mathbf{H}_1\mathbf{H}_1^T} + 2I_{\mathbf{E}_1}^2\right) + \beta_1 I_{\mathbf{E}_1}^2 + \beta_2 II_{\mathbf{E}_1}\right)\right]\mathbf{I} \\ &+ \beta_3 I_{\mathbf{E}_1}\mathbf{E}_1 + \beta_4\mathbf{E}_1^2 - \lambda I_{\mathbf{E}_1}\mathbf{H}_1^T - \mu(\mathbf{H}_1^T)^2.\end{aligned} \tag{1.229}$$

Similarly, we assume that the live loads in the actual equilibrium configuration can be written in the following way:

$$\epsilon\mathbf{t}(\mathbf{X}, \mathbf{u}, \mathbf{H}) = \epsilon\mathbf{t}_1(\mathbf{X}) + \epsilon^2[(\nabla_{\mathbf{u}}\mathbf{t})_{\mathbf{0}}\mathbf{u}_1 + (\nabla_{\mathbf{H}}\mathbf{t})_{\mathbf{0}}\mathbf{H}_1]. \tag{1.230}$$

In order to find an approximate formula for the surface force $\mathbf{t}_*$ acting on the boundary of the reference configuration, we start by recalling that $\mathbf{t}_* = J\mathbf{t}\sqrt{\mathbf{N}_* \cdot \mathbf{C}^{-}1\mathbf{N}_*}$ (see Eq. 1.10), where $\mathbf{C}$ is the right Cauchy–Green tensor. On the other hand, it is straightforward to prove that the following approximate formula holds for $J = \det\mathbf{F}$:

$$J = 1 + \epsilon I_{\mathbf{H}_1} + \epsilon^2(I_{\mathbf{H}_2} + II_{\mathbf{H}_1}). \tag{1.231}$$

Moreover, from $\mathbf{C} = \mathbf{I} + 2\mathbf{E} + \mathbf{H}\mathbf{H}^T$, it is possible to prove (see [37]) that

$$\sqrt{\mathbf{N}_* \cdot \mathbf{C}^{-1}\mathbf{N}_*} = 1 - \epsilon\frac{a}{2} - \epsilon^2\frac{1}{2}\left(\frac{a^2}{4} + b\right), \tag{1.232}$$

where

$$a = \mathbf{N}_* \cdot \mathbf{E}_1\mathbf{N}_*, \tag{1.233}$$

$$b = \mathbf{N}_* \cdot (2\mathbf{E}_2 - \mathbf{H}_1^2 - \mathbf{H}_1\mathbf{H}_1^T - (\mathbf{H}_1^T)^2)\mathbf{N}_*. \tag{1.234}$$

The above results allow us to write the following approximate expressions for the surface loads:

$$\epsilon\mathbf{t}_* = \epsilon\mathbf{t}_{*1} + \epsilon^2\mathbf{t}_{*2}, \tag{1.235}$$

where

$$\mathbf{t}_{*1} = \mathbf{t}_1, \tag{1.236}$$

$$\mathbf{t}_{*2} = I_{\mathbf{H}_1}\mathbf{t}_1 - \mathbf{t}_1\mathbf{N}_* \cdot \mathbf{E}_1\mathbf{N}_* + (\nabla_{\mathbf{u}}\mathbf{t})_{\mathbf{0}}\mathbf{u}_1 + (\nabla_{\mathbf{H}}\mathbf{t})_{\mathbf{0}}\mathbf{H}_1. \tag{1.237}$$

Finally, the approximate formulation of the pure traction boundary value problem (1.224) and the compatibility conditions is given by the following equations:

$$\nabla_{\mathbf{X}} \cdot \mathbf{T}_{*1} = \mathbf{0} \quad \text{in}\, C_*, \tag{1.238}$$
$$\mathbf{T}_{*1}\mathbf{N}_* = \mathbf{t}_{*1} \quad \text{on}\, \partial C_*, \tag{1.239}$$

$$\nabla_{\mathbf{X}} \cdot (\mathbf{T}_{*2} + \mathbf{B}_{*1}) = \mathbf{0} \quad \text{in}\, C_*, \tag{1.240}$$
$$(\mathbf{T}_{*2} + \mathbf{B}_{*1})\mathbf{N}_* = \epsilon \mathbf{t}_{*2} \quad \text{on}\, \partial C_*. \tag{1.241}$$

$$\int_{\partial C_*} \mathbf{t}_{*1}\, d\sigma_* = \mathbf{0}, \tag{1.242}$$
$$\int_{\partial C_*} (\mathbf{X} + \mathbf{u}_1) \times \mathbf{t}_{*1}\, d\sigma_* = \mathbf{0}, \tag{1.243}$$
$$\int_{\partial C_*} \mathbf{t}_{*2}\, d\sigma_* = \mathbf{0}, \tag{1.244}$$
$$\int_{\partial C_*} ((\mathbf{X} + \mathbf{u}_1) \times \mathbf{t}_{*2} + \mathbf{u}_2 \times \mathbf{t}_{*1})\, d\sigma_* = \mathbf{0}. \tag{1.245}$$

We note that only the first of the compatibility conditions listed above is a restriction on the applied loads that can be controlled a priori.

An interesting example of a live load is given by a constant pressure in the actual configuration; i.e., $\mathbf{t} = -p_0\mathbf{N}$. For this particular surface force, the relations (1.236) and (1.237) reduce to the following ones:

$$\mathbf{t}_{*1} = -p_0\mathbf{N}_*, \tag{1.246}$$
$$\mathbf{t}_{*2} = -p_0 I_{\mathbf{H}_1}\mathbf{N}_* - p_0(\mathbf{N}_* \cdot \mathbf{E}_1\mathbf{N}_*)\mathbf{N}_*. \tag{1.247}$$

We can see that in the reference configuration we must add a further pressure (that depends on the first-order deformation) to the pressure p_0. Also, in this simple case, the compatibility conditions (except for the first one) must be verified after the solution to the specific problem has been found.

The above equations are solved in [37] for elastic bodies with a simple geometry. The corresponding solutions, while referring to particular cases, are used to propose experiments that allow us to determine the second-order elastic constants $\beta_1, \ldots, \beta_4$.

1.22 Second-Order Singular Surfaces

One interesting subject in the field of nonlinear elasticity is wave propagation, which is difficult to analyze due to the nonlinearity of the equations governing the propagation and evolution of waves. Due to its importance in geophysics, engineering, nondestructive testing of materials, electronic signal processing devices, etc., wave propagation in nonlinear elastic, isotropic, and homogeneous media has been investigated in many papers (see, for instance, [38, 57]).

Important results are known for ordinary principal waves in isotropic elastic materials. We recall that principal waves propagate along the principal axes of deformation, which, due to the isotropy of the material, are also the principal axes of stress (see Sects. 1.23 and 1.24). These waves, when they exist, can only be longitudinal or transverse. Furthermore, Ericksen's formulae supply unique values for their speeds provided that *the form of the stress relation is known*. Conversely, if these speeds are known functions of the three principal stretches (obtained experimentally for example), then the response coefficients are uniquely determined. In spite of the strong relation between wave propagation and the constitutive equations in nonlinear elasticity, it is not an easy task to deduce the constitutive equations of an isotropic elastic material experimentally, since we must determine (using particular static deformations or wave propagation for example) functions that depend on the principal invariants of the left Cauchy–Green tensor $\mathbf{B}$. For these reasons, many authors have analyzed wave propagation in special classes of materials (Mooney–Rivlin, Blatz–Ko, neo-Hookean, St. Venant–Kirchhoff materials, etc.) that are described by simple constitutive relations, and for particular deformations (see, for instance, [44]–[51]). Alternatively, for arbitrary, sufficiently small deformations, it is possible to analyze the above problem for second-order elasticity. In this case, the constitutive equations are determined by only a few material constants (five for hyperelastic compressible materials, two for incompressible bodies). On the other hand, second-order constitutive relations (which are usually proposed for isotropic materials) yield a good description of the mechanical response of an elastic body for sufficiently small deformations. Furthermore, again for second-order elasticity, the speeds of the waves depend on both the direction of the propagation and on the deformation. In the final sections of this chapter we present some results relating to the propagation of ordinary waves in nonlinear elastic bodies. The topics discussed here are based on Sects. 4.5 and 8.9 of [16].

In another approach, wave propagation has been studied by considering the evolution of waves of small amplitude in prestressed nonlinear elastic materials. Finally, there are many papers in which the propagation of plane waves is advantageously considered by reducing the corresponding problem to one spatial dimension (see, for instance, [53]-[54]).

Let $g(\mathbf{x}, t) = 0$ be the equation of an oriented moving surface $\Sigma(t)$. We suppose that $\Sigma(t)$ divides the region $C(t)$ (which is instantaneously occupied by S) into two parts $C^-(t)$ and $C^+(t)$, where $C^+(t)$ is the region containing the exterior unit normal $\mathbf{N}$ to $\Sigma(t)$. If at least one of the second-order derivatives of the displacement field $\mathbf{u}(\mathbf{X},t)$ exhibits a finite discontinuity across $\Sigma(t)$, then $\Sigma(t)$ is said to be a *second-order singular surface*. Moreover, the regions $C^-(t)$ and $C^+(t)$ are called the *perturbed* and *undisturbed regions*, respectively. The surface $\Sigma(t)$ is also called the *wavefront*.

The unit normal $\mathbf{N}$ to $\Sigma(t)$ is given by

$$N_i = \frac{1}{|\nabla_{\mathbf{x}} g|}\frac{\partial g}{\partial x_i}, \tag{1.248}$$

whereas the *normal speed* is

$$c_N = -\frac{1}{|\nabla_{\mathbf{x}} g|}\frac{\partial g}{\partial t}. \tag{1.249}$$

Finally, if $\mathbf{v}$ is the velocity of the particles of S that occupy the surface $\Sigma(t)$ at the instant t, then the *local speed of propagation*, which is defined by the relation

$$U_N = c_N - \mathbf{v}\cdot\mathbf{N}, \tag{1.250}$$

denotes the relative velocity of $\Sigma(t)$ along the normal $\mathbf{N}$ with respect to the particles S located on $\Sigma(t)$ at the instant t.

We shall see that it also is useful to describe the evolution of $\Sigma(t)$ in the reference configuration C_*. To this end, let $\Sigma_*(t)$ be the surface $G(\mathbf{X},t) = g(\mathbf{x}(\mathbf{X},t),t) = 0$ image in C_* of $g(\mathbf{x},t) = 0$. This surface, which moves within C_* with a normal speed of

$$U_{N_*} = -\frac{1}{|\nabla_{\mathbf{X}} G|}\frac{\partial G}{\partial t}, \tag{1.251}$$

has the vector given by

$$N_{*L} = \frac{1}{|\nabla_{\mathbf{X}} G|}\frac{\partial G}{\partial X_L} \tag{1.252}$$

as its unit normal vector $\mathbf{N}_*$.

In order to identify the relations among the quantities (1.248), (1.250) in $C(t)$ and the quantities (1.251), (1.252) in C_*, we start by noting that

$$N_{*L} = \frac{1}{|\nabla_{\mathbf{X}} G|}\frac{\partial g}{\partial x_i}F_{iL} = \frac{|\nabla_{\mathbf{x}} g|}{|\nabla_{\mathbf{X}} G|}F_{iL}N_i \equiv \Gamma F_{iL}N_i. \tag{1.253}$$

Since $\mathbf{N}_*$ and $\mathbf{N}$ are unit vectors, we can use the above relation to evaluate the function Γ

$$\Gamma = \frac{1}{\sqrt{B_{ij}N_iN_j}}, \tag{1.254}$$

$$\frac{1}{\Gamma} = \sqrt{C^{-1}_{LM}N_{*L}N_{*M}}\,, \tag{1.255}$$

where **B** and **C** denote the left and right Cauchy–Green tensors, respectively.

Moreover, (1.253) indicates that the relation between U_{N_*} and U is given by

$$U_{N_*} = -\frac{1}{|\nabla_{\mathbf{X}} G|}\frac{\partial G}{\partial t} = -\Gamma \frac{1}{|\nabla_{\mathbf{x}} g|}\left(\frac{\partial g}{\partial t} + v_i \frac{\partial g}{\partial x_i}\right) = \Gamma U_N.$$

We now summarize the main formulae we derived above:

$$N_{*L} = \Gamma F_{iL} N_i, \tag{1.256}$$

$$N_i = \frac{1}{\Gamma} F_{Li}^{-1} N_{*L}, \tag{1.257}$$

$$\Gamma = \frac{1}{\sqrt{B_{ij}N_iN_j}} = \frac{1}{\sqrt{C_{LM}^{-1} N_{*L} N_{*M}}}, \tag{1.258}$$

$$U_{N_*} = \Gamma U, \tag{1.259}$$

and we conclude this section by recalling (see [16]) that if $\Sigma(t)$ is a second-order singular surface, the jumps in the second-order derivatives of the equations of motion $\mathbf{x}(\mathbf{X}, t)$ across $\Sigma_*(t)$ are expressed by the formulae

$$\left[\left[\frac{\partial^2 x_i}{\partial X_L \partial X_M}\right]\right] = a_{*i} N_{*L} N_{*M}, \tag{1.260}$$

$$\left[\left[\frac{\partial^2 x_i}{\partial X_M \partial t}\right]\right] = -a_{*i} U_{N_*} N_{*M}, \tag{1.261}$$

$$\left[\left[\frac{\partial^2 x_i}{\partial t^2}\right]\right] = a_{*i} U_N^2 N_*, \tag{1.262}$$

where a_{*i} is called the amplitude of the discontinuity across Σ_*.

To evaluate the jumps in the second-order derivatives of the equation of motion across the moving surface $\Sigma(t)$, we write (1.260)–(1.261) in the equivalent form:

$$\left[\left[\frac{\partial F_{iL}}{\partial X_M}\right]\right] = a_{*i} N_{*L} N_{*M}, \tag{1.263}$$

$$\left[\left[\dot{F}_{iM}\right]\right] = -a_{*i} U_{N_*} N_{*M}, \tag{1.264}$$

$$[[\ddot{x}_i]] = a_{*i} U_{N_*}^2. \tag{1.265}$$

If the first of the above equations is multiplied by F_{Mj}^{-1} and (1.256)–(1.259) are taken into account, then we obtain the following jump conditions across $\Sigma(t)$:

$$\left[\left[\frac{\partial F_{iL}}{\partial x_j}\right]\right] = a_i F_{hL} N_h N_j, \tag{1.266}$$

$$\left[\left[\frac{\partial v_i}{\partial x_j}\right]\right] = -a_i U_N N_j, \tag{1.267}$$

$$[[\ddot{x}_i]] = a_i U^2, \tag{1.268}$$

where

$$a_i = \Gamma^2 a_{*i}. \tag{1.269}$$

1.23 Singular Waves in Nonlinear Elasticity

Let S be an elastic continuous system that is homogeneous, compressible, and isotropic in the reference configuration C_*. If we denote the mass density in the actual configuration $C(t)$ by ρ, the displacement field from C_* to $C(t)$ by $\mathbf{u}(\mathbf{X}, t)$, the Cauchy stress tensor by $\mathbf{T}$, and the specific body force by $\mathbf{b}$, the Eulerian local momentum balance is expressed by the following equation:

$$\rho\ddot{\mathbf{u}} = \nabla \cdot \mathbf{T} + \rho\mathbf{b}. \tag{1.270}$$

If $\mathbf{H} = \nabla\mathbf{u}$ is the displacement gradient, let $\mathbf{T}(\mathbf{H})$ be the constitutive equation of the elastic material S. If we write (1.270) in the region $C^+(t)$ and $C^-(t)$, evaluate the limits of the corresponding equations when $\mathbf{x}$ approaches $\Sigma(t)$ from $C^-(t)$ and $C^+(t)$, and subtract the obtained results, we obtain the *jump system* associated with (1.270):

$$\rho\left[\left[\frac{\partial^2 u_i}{\partial t^2}\right]\right] = \frac{\partial T_{ij}}{\partial H_{lM}}\left[\left[\frac{\partial H_{lM}}{\partial x_j}\right]\right]. \tag{1.271}$$

Introducing (1.266) and (1.268) into (1.271), we get the equation

$$(Q_{ij}(\mathbf{H}, \mathbf{N}) - \rho U_N^2 \delta_{ij})a_j = 0, \tag{1.272}$$

where

$$Q_{ij}(\mathbf{H}, \mathbf{N}) = \frac{\partial T_{ik}}{\partial H_{jM}} F_{hM} N_k N_h \tag{1.273}$$

is called the *acoustic tensor.*

Algebraic condition (1.272) expresses *Hadamard's theorem*, described below.

Theorem 1.10
Given an undisturbed state $\mathbf{u}^+(\mathbf{X}, t)$ towards which the ordinary wave $\Sigma(t)$ propagates, then, due to the continuity of $\mathbf{u}(\mathbf{X}, t)$ and its first derivatives across $\Sigma(t)$, the matrix $\mathbf{Q}$ is a known function of t and $\mathbf{r} \in \Sigma(t)$. Furthermore, given a propagation direction $\mathbf{N}$, the local speeds of propagation

U_N are the square roots of the eigenvalues of the acoustic tensor, and the amplitudes of singularity $\mathbf{a}$ *are its eigenvectors.*

If the amplitude of singularity $\mathbf{a}$ satisfies the condition

$$\mathbf{a} \cdot \mathbf{N} = 0, \tag{1.274}$$

the ordinary wave is said to be *transverse*. Moreover, when $\mathbf{a}$ is parallel to $\mathbf{N}$, the wave is said to be *longitudinal*.

For a given material, the acoustic tensor $\mathbf{Q}$ depends on both the deformation gradient $\mathbf{H}$ in the undisturbed region and the unit vector $\mathbf{N}$. Therefore, in general, we cannot say anything about the existence of the eigenvalues and eigenvectors of $\mathbf{Q}$. In order to derive some properties of the acoustic tensor, we adopt an equivalent formulation of (1.272) in the reference configuration C_*. This new version of the eigenvalue problem (1.272) will be very useful for introducing a perturbation method for the analysis of wave propagation, which will be presented in the following sections. Since the mass densities in C_* and $C(t)$ are related by the equation $\rho_* = J\rho$, where $J = \det \mathbf{F}$, when we multiply (1.272) by Γ^4 and take (1.260)–(1.262) and (1.269) into account, we obtain

$$(Q_{*ij} - \rho_* U_{N_*}^2 \delta_{ij}) a_{*ij} = 0, \tag{1.275}$$

with

$$Q_{*ij} = J\Gamma^2 Q_{ij}. \tag{1.276}$$

To understand the meaning of (1.275), we start by proving that

$$J\frac{\partial T_{il}}{\partial H_{jM}} = \frac{\partial T_{*iL}}{\partial H_{jM}} F_{lL}, \tag{1.277}$$

where T_{*iL} is the Piola–Kirchhoff stress tensor. In fact, from the definition (1.7) of the tensor $\mathbf{T}_*$, when we recall (1.27), we have the identity

$$\frac{\partial T_{*iL}}{\partial X_N} = \frac{\partial}{\partial X_N}\left(J T_{ij} F_{Nj}^{-1}\right) = J\frac{\partial T_{ij}}{\partial X_N} F_{Nj}^{-1},$$

which can also be written as

$$\frac{\partial F_{lM}}{\partial X_N}\frac{\partial T_{*iL}}{\partial H_{lM}} = J\frac{\partial F_{lM}}{\partial X_N}\frac{\partial T_{ij}}{\partial H_{lM}} F_{Nj}^{-1}.$$

Since the above relation is identically satisfied for any deformation, we reach the result

$$\frac{\partial T_{*iL}}{\partial H_{lM}} = J\frac{\partial T_{ij}}{\partial H_{lM}} F_{Nj}^{-1}$$

and (1.277) is proved.

We now consider the local balance of momentum in the reference configuration

$$\rho_* \ddot{u}_i = \frac{\partial T_{*iL}}{\partial H_{lM}} \frac{\partial^2 u_l}{\partial X_M \partial X_L}. \tag{1.278}$$

The jump system associated with (1.278) on the surface $\Sigma_*(t)$ is

$$\left(\frac{\partial T_{*iL}}{\partial H_{lM}} N_{*L} N_{*M} - \rho_* U_{N_*}^2 \delta_{il} \right) a_{*l}, \tag{1.279}$$

which, in view of (1.276), (1.277) and (1.256), coincides with (1.275), thus yielding the formula

$$Q_{*ij} \equiv \Gamma^2 J Q_{ij} = \frac{\partial T_{*iL}}{\partial H_{lM}} N_{*l} N_{*M}. \tag{1.280}$$

We conclude this section by noting that if the material is hyperelastic, we have (see (1.11)

$$T_{*iL} = \rho_* \frac{\partial \psi}{\partial H_{iL}} \tag{1.281}$$

$$Q_{*ij} = \rho_* \frac{\partial \psi}{\partial H_{iL} \partial H_{jM}}, \tag{1.282}$$

so that Q_{*ij} is symmetric. Bearing (1.276) in mind, the acoustic tensor is also *symmetric*.[9]

If Q_{ij} is symmetric and *strongly elliptic* (i.e., if

$$Q_{ij} \xi_i \xi_j > 0, \tag{1.283}$$

for any vector ξ, any deformation gradient $\mathbf{H}$, and any unit vector $\mathbf{N}$), then its eigenvalues are real and positive and there is at least a basis of orthogonal eigenvectors.

1.24 Principal Waves in Isotropic Compressible Elastic Materials

The Cauchy stress tensor of a compressible, homogeneous, elastic, and isotropic body is given by (see (1.29))

$$\mathbf{T} = f_0 \mathbf{I} + f_1 \mathbf{B} + f_2 \mathbf{B}^2, \tag{1.284}$$

[9]In [3] it is proven that the material is hyperelastic if the acoustic tensor is symmetric.

where the functions f_i, $i = 0, 1, 2$ depend on the principal invariants I_B, II_B, and III_B of the left Cauchy–Green tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$. Inserting (1.284) into the momentum balance

$$\rho \ddot{x}_i = \frac{\partial T_{ik}}{\partial x_k} + \rho b_i, \tag{1.285}$$

we obtain

$$\rho \ddot{x}_i = \frac{\partial T_{ih}}{\partial B_{pq}} \frac{\partial B_{pq}}{\partial x_h} + \rho b_i. \tag{1.286}$$

If $\Sigma(t)$ is a wavefront of a second-order singular wave, then the jump system associated with (1.285) is

$$\rho\, [[\ddot{x}_i]] = \frac{\partial T_{ih}}{\partial B_{pq}} \left[\left[\frac{\partial B_{pq}}{\partial x_h}\right]\right]. \tag{1.287}$$

In order to evaluate the jump appearing on the right-hand side of (1.287), we note that (1.266) leads to

$$\begin{aligned}
\left[\left[\frac{\partial B_{pq}}{\partial x_h}\right]\right] &= \left[\left[\frac{\partial F_{pL}}{\partial x_h}\right]\right] F_{qL} + F_{pL} \left[\left[\frac{\partial F_{qL}}{\partial x_h}\right]\right] \\
&= (a_p F_{qL} F_{kL} + a_q F_{pL} F_{kL}) N_k N_h \\
&= (a_p B_{qk} + a_q B_{pk}) N_k N_h.
\end{aligned} \tag{1.288}$$

Since $\partial T_{ih}/\partial B_{pq}$ is symmetric with respect to the indices p and q, the jump system (1.287) assumes the final form

$$(Q_{iq} - \rho U_N^2 \delta_{iq}) a_q = 0, \tag{1.289}$$

where the acoustic tensor Q_{iq} is given by

$$Q_{iq} = 2 \frac{\partial T_{ih}}{\partial B_{pq}} B_{pk} N_k N_h. \tag{1.290}$$

In general, it is impossible to establish if there are second-order ordinary waves along a given direction $\mathbf{N}$. In order to obtain some concrete results, we suppose that $\mathbf{N}$ is a *principal axis of deformation*; that is, an eigenvector of $\mathbf{B}$

$$\mathbf{B}\mathbf{N} = v^2 \mathbf{N}. \tag{1.291}$$

From (1.284) we derive that a principal axis of deformation is also a *principal axis of stress*; that is, an eigenvector of $\mathbf{T}$ corresponding to the eigenvalue

$$\Lambda_T = f_0 + f_1 v^2 + f_2 v^4. \tag{1.292}$$

We define a *principal wave* as a wave that propagates along a principal axis of stress (or of strain). The following theorem holds for these waves.

Theorem 1.11
The principal axes of stress are acoustic axes. Moreover, the principal waves, when they exist, are longitudinal or transverse.

PROOF In order to prove this result, we must evaluate the acoustic tensor corresponding to a principal axis of strain. After simple but tedious calculations, it is possible to prove that (see Eq. 1.290)

$$\begin{aligned}\frac{\partial T_{ih}}{\partial B_{pq}} &= \frac{1}{2}\left(\delta_{ip}\delta_{hq} + \delta_{hp}\delta iq\right) \\ &+ \frac{1}{2}f_2\left(\delta_{ip}B_{kq} + \delta_{iq}B_{kp} + \delta_{kp}B_{iq} + \delta_{kq}B_{ip}\right) \\ &+ \delta_{ik}\left[\frac{\partial f_0}{\partial I_B}\delta_{pq} + \frac{\partial f_0}{\partial II_B}\left(I_B\delta_{pq} - B_{pq}\right) + III_B\frac{\partial f_0}{\partial III_B}B^{-1}_{pq}\right] \\ &+ B_{ik}\left[\frac{\partial f_1}{\partial I_B}\delta_{pq} + \frac{\partial f_1}{\partial II_B}\left(I_B\delta_{pq} - B_{pq}\right) + III_B\frac{\partial f_1}{\partial III_B}B^{-1}_{pq}\right] \\ &+ B_{il}B_{lk}\left[\frac{\partial f_2}{\partial I_B}\delta_{pq} + \frac{\partial f_2}{\partial II_B}\left(I_B\delta_{pq} - B_{pq}\right) + III_B\frac{\partial f_2}{\partial III_B}B^{-1}_{pq}\right].\end{aligned} \tag{1.293}$$

A principal eigenvector of deformation $\mathbf{N}_1$ obeys the equations

$$B_{pq}N_{1p} = v_1^2 N_{1q}, \quad (B^{-1})_{pq} = v_1^{-2}N_{1q}. \tag{1.294}$$

Substituting (1.294) into (1.290), we obtain the expression for the acoustic tensor corresponding to the principal axis of strain:

$$\begin{aligned}v_1^{-2}Q_{km}(\mathbf{N}_1,\mathbf{B}) &= (f_1 + v_1^2 f_2)\delta_{lm} + f_2 B_{km} \\ &+ 2N_{1k}N_{1m}\left\{\frac{1}{2}f_1 + v_1^2 f_2 + \right. \\ &\left.\sum_{i=0}^{2}(v_1^2)^i\left[\frac{\partial f_i}{\partial I_B} + (v_2^2 + v_3^2)\frac{\partial f_i}{\partial II_B} + v_2^2 v_3^2\frac{\partial f_i}{\partial III_B}\right]\right\},\end{aligned} \tag{1.295}$$

where v_2^2 and v_3^2 are the eigenvalues of $\mathbf{B}$ corresponding to two eigenvectors $\mathbf{N}_2$ and $\mathbf{N}_2$ of $\mathbf{B}$, which are orthogonal to $\mathbf{N}_1$. We note that the acoustic tensor $\mathbf{Q}$ is *symmetric* along the principal axes of strain. It is now evident that $\mathbf{N}_1$, $\mathbf{N}_2$ and $\mathbf{N}_3$ are eigenvectors of $\mathbf{Q}$, and so the theorem is proved. ∎

It is possible to prove the following results relating to principal waves (see [4]).

Theorem 1.12
Transverse principal waves. *If* $\mathbf{f_{-1}} = \mathbf{0}$, *then both of the transverse principal waves traveling along a given principal axis have the same absolute*

speed of propagation. If $\mathbf{f_{-1}} \neq \mathbf{0}$*, then these waves travel at the same speed if and only if the corresponding principal stretches are equal. In this case any amplitude is possible and all of these transverse waves have the same propagation speeds.*

Theorem 1.13

Longitudinal principal waves. *The propagation speeds of longitudinal waves in the directions of two equal principal stretches are equal.*

Theorem 1.14

Principal wave speed. *The squared propagation speeds of principal longitudinal waves are positive if and only if each principal tension is an increasing function of the corresponding principal stretch (provided that the other principal stretches are held constant). The squared propagation speeds of principal transverse waves are positive if and only if the greatest principal tension occurs always in the direction of greatest principal stretch.*

Finally, in [4], C. Truesdell determines the first-order effects due to second-order elasticity on the speeds of principal waves in isotropic elastic bodies.

We conclude this section with some remarks about the above theorems.

Remark The propagation speeds of principal waves are expressed in terms of the functions f_0, f_1, f_2 and their derivatives, but it is well known that it is only possible to determine these functions experimentally for very simple classes of materials.

Remark In the above theorems there is no information about the propagation speeds along directions that are not principal axes of strain.

1.25 A Perturbation Method for Waves in Compressible Media

In this section we propose a perturbation method for investigating the propagation of ordinary waves in second-order elastic, isotropic, compressible, and homogeneous materials (see [55, 56]). This method allows us to determine the first-order terms of the speeds and the amplitudes of the waves *along any arbitrary direction of propagation.* In particular, it reduces the eigenvalue problem for the acoustic tensor to several simpler problems: the first coincides with the eigenvalue problem for the acoustic tensor of linear elasticity; the second supplies the corrective first-order terms of the speeds and amplitudes.

Since this method is based on the formulation of wave propagation in the reference configuration, we recall expression (1.185) for the Piola–Kirchhoff tensor for second-order elasticity:

$$\mathbf{T}_* = \lambda I_{\mathbf{E}}\mathbf{I} + 2\mu\mathbf{E} + \left[\frac{\lambda}{2}\left(I_{\mathbf{H}\mathbf{H}^T} + 2I_{\mathbf{E}}^2\right) + \beta_1 I_{\mathbf{E}}^2 + \beta_2 I I_{\mathbf{E}}\right]\mathbf{I}$$
$$+ \beta_3 I_{\mathbf{E}}\mathbf{E} + \beta_4\mathbf{E}^2 - \lambda I_{\mathbf{E}}\mathbf{H}^T - \mu(\mathbf{H}^T)^2. \tag{1.296}$$

In [56] it is demonstrated that the Lagrangian acoustic tensor $\mathbf{Q}_*$ (see Sect. 1.23) in the reference configuration corresponding to the stress tensor (1.296) is

$$\begin{aligned} Q_{ij}\left(\mathbf{H}, \mathbf{N}_*\right) = {} & \left(\lambda + \mu\right) N_{*i}N_{*j} + \mu\delta_{ij}N_{*L}N_{*L} \\ & + \left\{\lambda H_{jM} + \left[2\left(\lambda + \beta_1\right) + \beta_2\right]\delta_{jM}H_{ll}\right\} N_{*i}N_{*M} \\ & - \frac{1}{2}\beta_2\left(H_{Mj} + H_{jM}\right) N_{*i}N_{*M} \\ & + \left[\frac{1}{2}\beta_3\left(H_{iL} + H_{Li}\right) - \lambda H_{Li}\right] N_{*L}N_{*j} \\ & + \frac{1}{2}\left[\beta_3\delta_{ij}H_{ll} + \frac{1}{2}\beta_4\left(H_{ij} + H_{ji}\right)\right] N_{*L}N_{*L} \\ & + \frac{1}{4}\beta_4\delta_{ij}\left(H_{ML} + H_{LM}\right) N_{*L}N_{*M} \\ & + \left[\frac{1}{4}\beta_4\left(H_{iM} + H_{Mi}\right) - \mu H_{Mi}\right] N_{*j}N_{*M} \\ & + \left(\frac{1}{2}\beta_3 - \lambda\right) H_{ll}N_{*j}N_{*i} \\ & + \left[\frac{1}{4}\beta_4\left(H_{jL} + H_{Lj}\right) - \mu H_{Lj}\right] N_{*L}N_{*i} \\ & \equiv Q_{ij}^{(0)} + Q_{ij}^{(1)}, \end{aligned} \tag{1.297}$$

where $Q_{ij}^{(0)}$ is the acoustic tensor of linear elasticity corresponding to the direction $\mathbf{N}_*$, and $Q_{ij}^{(1)}$ is the remaining part of Q_{ij}, which is a linear function of $\mathbf{H}$.

Let ϵ be a small parameter related to the problem we are considering. According to Signorini's method, we can write the displacement $\mathbf{u}$ in the form

$$\mathbf{u} = \epsilon\mathbf{u}^{(1)} + O\left(1\right), \tag{1.298}$$

so that we also have

$$\mathbf{H} = \epsilon\mathbf{H}^{(1)} + O\left(1\right), \tag{1.299}$$

where $\mathbf{H}^{(1)}$ is the nondimensional displacement gradient referred to $\mathbf{u}^{(1)}$.

To apply Signorini's method to the dynamic compatibility conditions (1.251), which we now write in the nondimensional form

$$(Q_{*ij} - \Lambda\delta_{ij})\, a_{*j} = 0, \quad i = 1, 2, 3, \tag{1.300}$$

we recall that the quantities $Q_{*ij}\left(\mathbf{N}_*\left(\epsilon\right), \mathbf{H}\left(\epsilon\right)\right), \Lambda\left(\epsilon\right)$, and $a_{*j}\left(\epsilon\right)$ are intended to be analytic functions of ϵ.

The expansion of $Q_{*ij}\left(\epsilon\right)$ is obtained by introducing (1.299) into (1.297):

$$Q_{*ij}\left(\epsilon\right) = Q^{(0)}_{*ij}(\mathbf{N}_*\left(\epsilon\right), \mathbf{0}) + \epsilon A_{ijlP} H^{(1)}_{lP} + O\left(1\right), \tag{1.301}$$

where A_{ijlP} denotes the coefficient of H_{lP} in $Q^{(1)}_{*ij}$.

Moreover, we have

$$\mathbf{a}_* = \mathbf{a}^{(0)}_* + \epsilon\mathbf{a}^{(1)}_* + O\left(1\right), \quad \Lambda = \lambda^{(0)} + \epsilon\lambda^{(1)} + O\left(1\right). \tag{1.302}$$

We now consider the expansion of $\mathbf{N}_*\left(\epsilon\right)$ up to first-order terms:

$$\mathbf{N}_*\left(\epsilon\right) = \mathbf{N}^{(0)}_* + \epsilon\mathbf{N}^{(1)}_*. \tag{1.303}$$

Consequently, (1.301) assumes the form

$$\begin{aligned} Q^{(0)}_{*ij}(\mathbf{N}_*\left(\epsilon\right), \mathbf{0}) &= \left(\lambda + \mu\right) N^{(0)}_{*i} N^{(0)}_{*j} + \mu\delta_{ij} N^{(0)}_{*L} N^{(0)}_{*L} \\ &\quad + \epsilon\left[\left(\lambda + \mu\right)\left(N^{(0)}_{*i} N^{(1)}_{*j} + N^{(1)}_{*i} N^{(0)}_{*j}\right)\right. \\ &\quad \left. + \mu\delta_{ij}\left(N^{(0)}_{*L} N^{(1)}_{*L} + N^{(1)}_{*L} N^{(0)}_{*L}\right)\right] \\ &\equiv Q^{(0,0)}_{*ij} + \epsilon Q^{(0,\epsilon)}_{*ij}. \end{aligned} \tag{1.304}$$

Finally, due to (1.301)–(1.304), when we neglect terms of a higher order than 1, the dynamic compatibility equation (1.300) becomes

$$\begin{aligned} &\left[Q^{(0,0)}_{*ij} + \epsilon Q^{(0,\epsilon)}_{*ij} + \epsilon A^{(0)}_{ijlP} H^{(1)}_{lP}\right]\left(a^{(0)}_{*j} + \epsilon a^{(1)}_{*j}\right) \\ &= \left(\lambda^{(0)} + \epsilon\lambda^{(1)}\right)\left(a^{(0)}_{*j} + \epsilon a^{(1)}_{*j}\right), \end{aligned} \tag{1.305}$$

where $A^{(0)}_{ijlP}$ is the value of A_{ijlP} when we substitute $\mathbf{N}^{(\mathbf{0})}_*$ for $\mathbf{N}_*$. Using this result, we can derive the following system:

$$Q^{(0,0)}_{*ij} a^{(0)}_{*j} = \lambda^{(0)} a^{(0)}_{*i}, \tag{1.306}$$

$$Q^{(0,0)}_{*ij} a^{(1)}_{*j} + \left(Q^{(0,\epsilon)}_{*ij} + A^{(0)}_{ijlP} H^{(1)}_{lP}\right) a^{(0)}_{*j} = \left(\lambda^{(0)} a^{(1)}_{*i} + \lambda^{(1)} a^{(0)}_{*i}\right). \tag{1.307}$$

We note that (1.306) is the usual eigenvalue equation for the acoustic tensor of linear elasticity. Consequently, $\lambda^{(0)}/\rho_*$ must be the square of the propagation speed and $\mathbf{a}^{(0)}_* = \left(a^{(0)}_{*j}\right)$ the amplitude of the discontinuity across the singular surface in linear elasticity. The three equations (1.307) in the four unknowns $a_{*j}^{(1)}$, $j = 1, 2, 3$, and $\lambda^{(1)}$, determine (at least in principle) the direction of $\mathbf{a}_*^{(1)}$ and the value of $\lambda^{(1)}$.

Remark We recall that in linear elasticity the acoustic tensor is always symmetric. In general, this result does not hold in nonlinear elasticity. In particular, $\mathbf{Q}$ is symmetric if and only if $\mathbf{Q}_*$ is symmetric (see Eq. 1.269). Moreover, we have seen that $\mathbf{Q}_*$ is symmetric when $\mathbf{N}_*$ is a direction that corresponds to a principal direction of strain in C_*.

If we identify $\mathbf{N}_*$ as a principal axis of stress, and we denote the eigenvector of the tensor $\mathbf{Q}_*$ up to first-order terms belonging to the eigenvalue $\lambda^{(0,h)} + \epsilon\lambda^{(1,h)}, h = 1, 2, 3$ as $\mathbf{a}_*^{(0,h)} + \epsilon\mathbf{a}_*^{(1,h)}$, then we must satisfy the following orthogonality conditions:

$$\left(\mathbf{a}_*^{(0,h)} + \epsilon\mathbf{a}_*^{(1,h)}\right) \cdot \left(\mathbf{a}_*^{(0,k)} + \epsilon\mathbf{a}_*^{(1,k)}\right) = 0, \quad h \neq k,$$

which in turn are equivalent to

$$\mathbf{a}_*^{(0,h)}\mathbf{a}_*^{(0,k)} = 0, \tag{1.308}$$

$$\mathbf{a}_*^{(0,h)}\mathbf{a}_*^{(1,k)} + \mathbf{a}_*^{(1,h)}\mathbf{a}_*^{(0,k)} = 0, \quad h \neq k. \tag{1.309}$$

We start analyzing the principal waves and we prove that the speeds we obtain with the perturbation method coincide with the speeds obtained by Truesdell in [4] using a different approach.[10] Let $\Lambda_1, \Lambda_2, \Lambda_3$ be the eigenvalues of the left Cauchy–Green tensor $\mathbf{B}$. In the corresponding basis of eigenvectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, the tensor $\mathbf{B}$ is diagonal and the directions of the principal axes of strain are

$$\mathbf{N}_1 \equiv (1, 0, 0)\,, \quad \mathbf{N}_2 \equiv (0, 1, 0)\,, \quad \mathbf{N}_3 \equiv (0, 0, 1)\,. \tag{1.310}$$

In this new basis, up to first–order terms, $\Lambda_i = 1 + \epsilon\Lambda_i^{(1)}, i = 1, 2, 3$, and, since $\mathbf{B} = \mathbf{I} + 2\epsilon\mathbf{E}^{(1)}$, we have

$$\mathbf{E}^{(1)} = \begin{pmatrix} \Lambda_1^{(1)}/2 & 0 & 0 \\ 0 & \Lambda_2^{(1)}/2 & 0 \\ 0 & 0 & \Lambda_3^{(1)}/2 \end{pmatrix},$$

[10] This can be done provided that we relate the constitutive constants $\lambda, \mu, \beta_1, \ldots \beta_4$ to the constants $\alpha_1, \ldots \alpha_6$ used in [4] by the formulae $\alpha_1 = \lambda/\mu, \alpha_2 = 1, \mu\alpha_3 = \beta_1, \mu\alpha_4 = \beta_2, \mu\,(\alpha_5 + 2) = \beta_3, \mu\alpha_6 = \beta_4$.

$$\mathbf{H}^{(1)} = \begin{pmatrix} \Lambda_1^{(1)}/2 & H_{12}^{(1)} & H_{13}^{(1)} \\ -H_{12}^{(1)} & \Lambda_2^{(1)}/2 & H_{23}^{(1)} \\ -H_{13}^{(1)} & -H_{23}^{(1)} & \Lambda_3^{(1)}/2 \end{pmatrix}.$$

Let $\mathbf{N}(\epsilon) = \mathbf{N}^{(0)} + \epsilon\mathbf{N}^{(1)}$ be a unit vector normal to $\Sigma(t)$. This means that, in our approximation, $\mathbf{N}^{(0)} \cdot \mathbf{N}^{(0)} = 1$ and $\mathbf{N}^{(0)} \cdot \mathbf{N}^{(1)} = \mathbf{0}$. Moreover, due to (1.256) and (1.258), we have

$$\Gamma = 1 - \epsilon E_{ij}^{(1)} N_i^{(0)} N_j^{(0)} \tag{1.311}$$

in the same approximation, and

$$N_{*L} = N_L^{(0)} + \epsilon\left(N_L^{(1)} + H_{iL}^{(1)} N_i^{(0)} - E_{hk}^{(1)} N_h^{(0)} N_k^{(0)} N_L^{(0)}\right). \tag{1.312}$$

From (1.25) and (1.25), we obtain the vectors that correspond to (1.310) in C_*:

$$\mathbf{N}_{*1} \equiv (1,0,0) + \epsilon\left(H_{12}^{(1)}, H_{13}^{(1)}, 0\right) \tag{1.313}$$

$$\mathbf{N}_{*2} \equiv (0,1,0) + \epsilon\left(H_{21}^{(1)}, 0, H_{33}^{(1)}\right), \tag{1.314}$$

$$\mathbf{N}_{*3} \equiv (0,0,1) + \epsilon\left(H_{33}^{(1)}, H_{32}^{(1)}, 0\right). \tag{1.315}$$

Using the above formulae we can explicitly write the two eigenvalue problems (1.305) and (1.307). The speeds of the principal waves in C_* are determined in [55] and [56], as well as the speeds for any direction of propagation $\mathbf{N}$. Moreover, an analysis of the wave propagation along any direction is carried out when the undisturbed region is subjected to a simple extension or a simple shear in [55] and [56]. Note that all of these results can be derived using the notebook ***Chapter1.nb***, written in the software ***Mathematica***®. This notebook can be downloaded via the internet. ***Mathematica***® notebooks from this chapter are available for download at http://www.birkhauser.com/978-0-8176-4869-5.

1.26 A Perturbation Method for Analyzing Ordinary Waves in Incompressible Media

In this section we propose a perturbation method for investigating the propagation of (necessarily) transverse waves in second-order elastic, isotropic, incompressible, and homogeneous materials (see [57]). As in [55, 56], this method allows us to reduce the eigenvalue problem for the acoustic tensor (equipped with the orthogonality conditions between the amplitudes and the propagation directions) to two simpler problems: the first of these coincides with the eigenvalue problem for the acoustic tensor in linear elasticity, whereas the second supplies the first-order terms to add to the speeds and amplitudes of linear elasticity.

First, we briefly describe the theory of the ordinary waves in *incompressible* elastic materials and we recall the main results for the propagation of principal transverse waves.

Let S be an elastic, homogeneous, incompressible, and isotropic continuous system in the reference configuration C_*. In what follows, we neglect all of the thermal phenomena associated with the evolution of S. If $\mathbf{X}$ is any point in C_* and $\mathbf{x}$ denotes the corresponding position in the actual configuration $C(t)$, then the finite deformation of S, in going from C_* to $C(t)$, can be equivalently expressed by the finite deformation $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ or by the displacement $\mathbf{u} = \mathbf{u}(\mathbf{X}, t)$. It is well known that the constitutive equation of the Cauchy stress tensor $\mathbf{T}$ takes the following form (see (1.42)) for incompressible materials (in which only isochoric deformations are possible):

$$\mathbf{T} = -p\mathbf{I} + f_1\mathbf{B} + f_{-1}\mathbf{B}^{-1} \equiv -p\mathbf{I} + \widetilde{\mathbf{T}}, \tag{1.316}$$

where $\widetilde{\mathbf{T}}$ depends on the deformation gradient $\mathbf{F}$, f_1, f_{-1} are functions of the two principal invariants $I_{\mathbf{B}}$ and $II_{\mathbf{B}}$ of $\mathbf{B}$, and p is an indeterminate pressure.

The local momentum balance is given by the equation

$$\rho_*\ddot{\mathbf{x}} = -\nabla p + \nabla \cdot \widetilde{\mathbf{T}} + \rho_*\mathbf{b} \quad \text{in}\, C(t), \tag{1.317}$$

where $\rho = \rho_*$ is the constant mass density of S, and $\mathbf{b}$ is the specific body force.

Let $\Sigma(t)$ be a moving surface with equation $g(\mathbf{x}, t) = 0$ in the actual configuration and equation $G(\mathbf{X}, t) = 0$ in the reference configuration. If the solution $\mathbf{x}(\mathbf{X}, t)$ of the second-order equation (1.317) exhibits a discontinuity in some or all of its second-order derivatives across the surface $\Sigma(t)$, then $\mathbf{x}(\mathbf{X}, t)$ is said to represent an ordinary wave and $\Sigma(t)$ is called the wavefront that divides the region $C(t)$ into the perturbed region $C_(t)$ and undisturbed region $C^+(t)$. Furthermore, the vector $\mathbf{a}$, which characterizes the strength of the discontinuity of the second-order derivatives of $\mathbf{x}(\mathbf{X}, t)$ across $\Sigma(t)$, is called the amplitude of the singularity. We denote the normal speed of propagation of $\Sigma(t)$ by $c_N = -\frac{\partial g/\partial t}{|\nabla g|}$, and the local speed of the wavefront by $U_N = c_N - \dot{x}_N$. Finally, since the constraint of incompressibility rules out longitudinal waves, all the singular waves in an incompressible material are necessarily transverse; i.e., $\mathbf{a} \cdot \mathbf{N} = 0$.

If equation (1.317) is written in both regions ($C^-(t)$ and $C^+(t)$), its limits when $\mathbf{x}$ goes to a point $\mathbf{r} \in \Sigma(t)$ are considered and the obtained results are subtracted, then we obtain the jump system associated with (1.317):

$$\rho_*[[\ddot{x}_i]]_{\mathbf{r}} = -\left[\left[\frac{\partial p}{\partial x_i}\right]\right]_{\mathbf{r}} + \frac{\partial \widetilde{T}_{ij}}{\partial H_{lM}}\left[\left[\frac{\partial H_{lm}}{\partial x_j}\right]\right]_{\mathbf{r}}, \quad i = 1, 2, 3. \tag{1.318}$$

Recalling the kinematic relations of the second-order singular surfaces (see p. 119 of [16]),

$$[[\ddot{x}_i]]_{\mathbf{r}} = U_N^2 a_i, \quad \left[\left[\frac{\partial H_{lm}}{\partial x_j}\right]\right]_{\mathbf{r}} = F_{hM} N_j N_h a_l, \tag{1.319}$$

and the expression for the pressure jump across $\Sigma(t)$,

$$-\left[\left[\frac{\partial p}{\partial x_i}\right]\right]_{\mathbf{r}} = A N_i,$$

the jump system (1.318) then becomes

$$\rho_* U_N^2 a_i = A N_i + \widetilde{Q}_{il} a_l, \quad i = 1, 2, 3, \tag{1.320}$$

where

$$\widetilde{Q}_{il} = \frac{\partial \widetilde{T}_{ij}}{\partial H_{lM}} F_{hM} N_j N_h.$$

Taking the scalar product of (1.320) with $\mathbf{N}$ and recalling that $\mathbf{a} \cdot \mathbf{N} = 0$, we have

$$A = -N_j \widetilde{Q}_{jl} a_l.$$

Finally, we obtain the following equation for the amplitudes and speeds:

$$(Q_{il} - \rho_* U_N^2 \delta_{il}) a_l = 0, \tag{1.321}$$

where

$$Q_{il} = \widetilde{Q}_{il} - N_i N_j \widetilde{Q}_{jl} \tag{1.322}$$

is the *acoustic tensor*.

Algebraic conditions (1.321) express Hadamard's well-known result (see Theorem 1.10). For isotropic incompressible materials, the acoustic axes of principal waves are principal axes of deformation. Moreover, the principal waves are necessarily transverse, and the classical compatibility conditions of these waves hold only for stretches where $\det \mathbf{B} = 1$. Ericksen's formulae on the propagation speeds of transverse principal waves are expressed in terms of the functions f_1, f_{-1} and their derivatives (see p. 293 of [4]). However, since it is practically impossible to determine these functions experimentally, Truesdell (see [4]) found the first terms of the expansion of the exact formulae for the local speed only for transverse principal waves in a eigenvector basis of $\mathbf{B}$ (see p. 293 of [4]). Therefore, even to a first-order approximation, there is no information on the speed of propagation along directions that are not principal axes of strain.

We now propose a perturbation method that is similar to the method we discussed in the previous section in order to overcome these difficulties, at least for second-order elasticity. This approach allows us to obtain the first-order speeds *along any direction and for all "sufficiently small" deformations*.

We start by recalling the following relations (see Sect. 1.22), which are rewritten here for the convenience of the reader:

$$U_{*N} = \frac{\nabla g}{\nabla_{*G}} \equiv \Gamma U_N, \quad U_N = \frac{\nabla_* G}{\nabla_g} \equiv \frac{1}{\Gamma} U_{*N}, \tag{1.323}$$

$$a_{*i} = \frac{1}{\Gamma^2} a_i, \quad a_i = \Gamma^2 a_{*i}, \tag{1.324}$$

where U_{*N} and a_{*i} denote, respectively, the local speed and the amplitude of the singularity of the surface $\Sigma_*(t)$ in the reference configuration C_*. We also recall that the relations between the two unit normal vectors $\mathbf{N}$ and N_* are expressed by the formulae

$$N_i = \frac{1}{\Gamma} F^{-1}_{Li} N_{*L}, \quad N_{*L} = \Gamma F_{iL} N_i. \tag{1.325}$$

Due to (1.323)–(1.325), instead of analyzing (1.321) it is more convenient to study the following equation in C_*:

$$(Q_{*il} - \rho_* U^2_{*N} \delta_{il}) a_{*l} = 0, \quad i = 1, 2, 3, \tag{1.326}$$

where $Q_{*il} = \widetilde{Q}_{*il} - \hat{Q}_{*il}$, and

$$\widetilde{Q}_{*il} = \frac{\partial \widetilde{T}_{ij}}{\partial H_{lM}} F^{-1}_{Lj} N_{*L} N_{*M}, \quad \hat{Q}_{*il} = \frac{1}{\Gamma^2} F^{-1}_{Li} F^{-1}_{Mj} N_{*L} N_{*M} \widetilde{Q}_{*jl}.$$

It is well known that the Cauchy stress tensor of an elastic, isotropic, incompressible and homogeneous body, up to second-order terms, takes the form (see p. 241 of [4])

$$\mathbf{T}(\mathbf{H}) = -p\mathbf{I} + 2\mu \mathbf{E} + \mu \mathbf{H}\mathbf{H}^T + \beta_1 \mathbf{E}^2, \tag{1.327}$$

where μ is a Lamé's coefficient and β_1 is a second-order constitutive constant. In [57], simple but tedious calculations show that the first-order Lagrangian acoustic tensor $\mathbf{Q}^*$ corresponding to the propagation direction $\mathbf{N}_*$ and the small deformation $\mathbf{H}$ is

$$\begin{aligned} Q_{*ij}(\mathbf{H}, \mathbf{N}_*) = &-\mu(N_{*i}N_{*j} - \delta_{ij}) \\ &+ \frac{1}{2}\beta_1 E_{ij} + \frac{1}{2}[(4\mu + \beta_1)E_{ih} + 2\mu H_{hi}] N_{*j} N_{*h} \\ &- \frac{1}{2}(\beta_1 E_{jh} - 2\mu H_{hj}) N_{*i} N_{*h} \\ &- \frac{1}{2} H_{hk}[2N_{*i}N_{*j}(4\mu + \beta_1) - \beta_1 \delta_{ij}] N_{*h} N_{*k}. \end{aligned} \tag{1.328}$$

In order to reduce problem (1.326)—into which we have introduced (1.328)—to a family of simple problems, we suppose that a suitable nondimensional analysis of the local momentum balance causes us to introduce a *small parameter* ϵ that depends on the acting forces, the material characteristics, and the geometry of C_*. We then suppose that the displacement $\mathbf{u}$ and the pressure p can be written in the following form:

$$\mathbf{u} = \epsilon \mathbf{u}^{(1)} + O(1), \quad p = \epsilon p^{(1)} + O(1), \tag{1.329}$$

so that we also have

$$\mathbf{H} = \epsilon \mathbf{H}^{(1)} + O(1), \tag{1.330}$$

where $\mathbf{H}^{(1)}$ is the nondimensional displacement gradient relative to $\mathbf{u}^{(1)}$. We suppose that the quantities $Q_{*ij}(\mathbf{H}(\epsilon), \mathbf{N}_*(\epsilon))$, $a_{*i}(\epsilon)$, $\lambda(\epsilon)$ and $\mathbf{N}_*(\epsilon)$ are analytic functions of the small parameter ϵ, so that, up to first-order terms, we obtain

$$\mathbf{a}_*(\epsilon) = \mathbf{a}_*^{(0)} + \epsilon \mathbf{a}_*^{(1)}, \quad \lambda(\epsilon) = \lambda^{(0)} + \epsilon \lambda^{(1)}, \quad \mathbf{N}_*(\epsilon) = \mathbf{N}_*^{(0)} + \epsilon \mathbf{N}_*^{(1)}. \tag{1.331}$$

Baed on these results, (1.328) assumes the following form:

$$\begin{aligned}
Q_{*ij}(\mathbf{H}(\epsilon), \mathbf{N}_*(\epsilon)) &= -\mu \left(N_{*i}^{(0)} N_{*j}^{(0)} - \delta_{ij} \right) - \epsilon \mu \left(N_{*i}^{(0)} N_{*j}^{(1)} + N_{*i}^{(1)} N_{*j}^{(0)} \right) \\
&\quad + \frac{1}{2} \epsilon \beta_1 E_{ij}^{(1)} + \frac{1}{2} \epsilon \left[(4\mu + \beta_1) E_{ih}^{(1)} + 2\mu H_{hi}^{(1)} \right] N_{*j}^{(0)} N_{*h}^{(0)} \\
&\quad - \frac{1}{2} \left(\beta_1 E_{jh}^{(1)} - 2\mu H_{hj}^{(1)} \right) N_{*i}^{(0)} N_{*h}^{(0)} \\
&\quad - \frac{1}{2} \epsilon H_{hk}^{(1)} \left[2(4\mu + \beta_1) N_{*i}^{(0)} N_{*j}^{(0)} - \beta_1 \delta_{ij} \right] N_{*h}^{(0)} N_{*k}^{(0)} \\
&\equiv Q_{ij}^{(0)} + \epsilon Q_{ij}^{(1)}.
\end{aligned} \tag{1.332}$$

We recall that the waves are necessarily transverse in incompressible materials; i.e., $\mathbf{a} \cdot \mathbf{N} = 0$. In view of (1.324) and (1.325), this condition becomes

$$a_{*i} F_{Li}^{-1} N_{*L} = 0,$$

so that, in the first-order approximation, we have

$$\mathbf{a}_*^{(0)} \cdot \mathbf{N}_*^{(0)} = 0, \tag{1.333}$$

$$\mathbf{a}_*^{(1)} \cdot \mathbf{N}_*^{(0)} + \mathbf{a}_*^{(0)} \cdot \left(\mathbf{N}_*^{(1)} - \mathbf{N}_*^{(0)} \mathbf{H}^{(1)} \right) = 0. \tag{1.334}$$

Finally, when we impose conditions (1.333) and (1.334) and neglect terms of a higher order than 1, we can derive the following relations from the dynamical conditions (1.326):

$$(\mu - \lambda^{(0)}) a_{*i}^{(0)} = 0, \tag{1.335}$$

$$(\mu - \lambda)a_{*i}^{(1)} - \lambda^{(1)}a_{*i}^{(0)} + \frac{1}{2}a_{*i}^{(0)}H_{hk}^{(1)}N_{*h}^{(0)}N_{*k}^{(0)} + \frac{1}{2}\beta_1 a_{*j}^{(0)}\left(E_{ij}^{(1)} + E_{jh}^{(1)}N_{*i}^{(0)}N_{*h}^{(0)}\right) = 0. \qquad (1.336)$$

We note that, when we associate the conditions (1.333) with system (1.335), we obtain the usual eigenvalue equations of linear elasticity. Consequently, we find that μ/ρ_* is the square of the propagation speed, and that condition (1.335) supplies the amplitude of the discontinuity $\mathbf{a}_*^{(0)}$. Similarly, when we identify $\lambda^{(0)}$ with μ and we consider only first-order isochoric deformations (i.e., $I_{\mathbf{H}^{(1)}} = 0$), then equations (1.336) determine (at least in principle) the direction of $\mathbf{a}_*^{(1)}$, $j = 1, 2, 3$ and the value of $\lambda^{(1)}$.

We recall that in nonlinear elasticity the tensor $\mathbf{Q}_*$ cannot be symmetric for any direction and any deformation. In particular, in our approximation, we get

$$Q_{*12} - Q_{*21} = \epsilon(2\mu + \beta 1)\left[\left(E_{11}^{(1)} - E_{22}^{(1)}\right)N_{*1}^{(0)}N_{*2}^{(0)} - E_{12}^{(1)}\left(N_{*1}^{(0)}2 - N_{*2}^{(0)}2\right)\right] + \epsilon(2\mu + \beta_1)\left[E_{13}^{(1)}N_{*2}^{(0)}N_{*3}^{(0)} - E_{23}^{(1)}N_{*1}^{(0)}N_{*3}^{(0)}\right],$$

$$Q_{*13} - Q_{*31} = \epsilon(2\mu + \beta 1)\left[\left(E_{11}^{(1)} - E_{33}^{(1)}\right)N_{*1}^{(0)}N_{*3}^{(0)} + E_{12}^{(1)}N_{*2}^{(0)}N_{*3}^{(0)}\right] - \epsilon(2\mu + \beta_1)\left[\left(N_{*1}^{(0)}2 - N_{*3}^{(0)}2\right)E_{13}^{(1)} + E_{23}^{(1)}N_{*1}^{(0)}N_{*2}^{(0)}\right],$$

$$Q_{*23} - Q_{*32} = \epsilon(2\mu + \beta 1)\left[\left(E_{22}^{(1)} - E_{33}^{(1)}\right)N_{*2}^{(0)}N_{*3}^{(0)} + E_{12}^{(1)}N_{*1}^{(0)}N_{*3}^{(0)}\right] - \epsilon(2\mu + \beta_1)\left[E_{13}^{(1)}N_{*1}^{(0)}N_{*2}^{(0)} + \left(N_{*2}^{(0)}2 - N_{*3}^{(0)}2\right)E_{23}^{(1)}\right],$$

where the components of the infinitesimal strain tensor must satisfy the condition $I_{\mathbf{E}^{(1)}} = 0$. When $\mathbf{Q}_*$ is symmetric and we denote the eigenvector of the acoustic tensor belonging to the eigenvalue $\lambda^{(0,h)} + \epsilon\lambda^{(1,h}$, $h = 1, 2, 3$ by $\mathbf{a}_*^{(0,h)} + \epsilon\mathbf{a}_*^{(1,h)}$, we must satisfy the following orthogonality conditions:

$$\left(\mathbf{a}_*^{(0,h)} + \epsilon\mathbf{a}_*^{(1,h)}\right)\cdot\left(\mathbf{a}_*^{(0,k)} + \epsilon\mathbf{a}_*^{(1,k)}\right) = 0, \quad h \neq k,$$

which are equivalent to the conditions

$$\mathbf{a}_*^{(0,h)} \cdot \mathbf{a}_*^{(0,k)} = 0, \qquad (1.337)$$

$$\mathbf{a}_*^{(0,h)} \cdot \mathbf{a}_*^{(1,k)} + \mathbf{a}_*^{(1,h)} \cdot \mathbf{a}_*^{(0,k)} = 0, \quad h \neq k. \qquad (1.338)$$

Using (1.335) and (1.336), the first-order terms of the speeds and the amplitudes of the principal waves and of the waves in any propagation direction when the undisturbed region is subjected to an arbitrary isochoric deformation are determined in [57]. Another application of this method is also presented when the undisturbed region is subjected to a simple shear, and some results are provided for Mooney–Rivlin materials. Finally, we conclude by mentioning that all of the results of this section can be derived using the notebook ***Chapter1.nb***.

Chapter 2

Micropolar Elasticity

2.1 Preliminary Considerations

The model of an elastic body S presented in the previous chapter cannot always be used to describe the behavior of a real body in a satisfactory way. In some cases, it can be usefully replaced by a more sophisticated model in which a set of one or more vectors, called *directors*, are associated with any point of S. We will now list some physical situations in which this model appears to be meaningful.

- Let S be a *narrow body*; i.e., a body in which two dimensions are much smaller than the remaining one, or one dimension is negligible with respect to the others. For instance, we may consider a cylinder C whose diameter is much smaller than its length or a block B in which the length δ of one of its edges is negligible with respect to the others. We are tempted to describe the body with a one-dimensional continuum in the first case and with a two-dimensional continuum in the second case. In other words, we could assume that knowledge of the configuration of the axis of the cylinder C yields an acceptable localization of C. Similarly, we could obtain a sufficient description of the position of the block B when the configuration of any surface σ present within its thickness δ is determined.

  Although this assumption leads to a more simplified description of narrow bodies, we at once understand that it cannot be completely satisfactory. In fact, by limiting our considerations to the cylinder C, we will be able to describe its bending or stretching but not its resistance to torsion, since we have erased the dimensions to which it is related. In order to retrieve the erased dimensions, we can associate any point on the axis of C with a pair of vectors whose orientations can describe the torsion of C. Similarly, in the case of block B, we can associate any point on the surface σ with a normal vector in order to preserve a trail to the erased dimension.

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- It may be that the behavior of a three-dimensional body S depends on an internal structure with a characteristic dimension that is below the typical length of continuum physics. For instance, if an element of volume dc of S is filled by a small crystal, we can take the structure of this crystal into account in the usual three-dimensional model by simply changing the scale of observation. Alternatively, we can associate the element dc with three mutually orthogonal vectors that remember its internal structure.

- In some cases, the behavior of the continuous system is not sufficiently described by the macroscopic motion since the small variations of velocity inside the volume element dc, although not macroscopically detectable, influence the evolution of the system. This means that, together with the macroscopic motion, we must consider a micromotion that describes the internal evolution of the considered volume element. In particular, if this micromotion is assumed to be rigid, the continuum is said to be micropolar. The micromotion can then be equivalently described either by an orthogonal matrix or by three *arbitrary* unit vectors defined at any point in the continuum. In other words, we can say that any particle of a micropolar continuum has three extra degrees of freedom.

- To describe the behavior of a polarized elastic dielectric, we need to take into account the polarization field (see Chap. 4). A similar situation occurs in ferromagnetic substances or in liquid crystals.

In this chapter, which is only an introduction to this subject, we consider *micropolar continuous systems*.[1] As we have seen above, these continuous systems are defined by one of the following equivalent conditions:

- *The micromotion is described by an orthogonal matrix* χ

- *Any particle* $\mathbf{x}$ *carries three unit orthonormal vectors* $\mathbf{d}_{(L)}$, $L = 1, 2, 3$, *which rotate independently of* $\mathbf{x}$. These vectors, which can be chosen arbitrarily, are called *directors*.

2.2 Kinematics of a Micropolar Continuum

Let S be a micropolar continuous system, and let C_* and C be the reference configuration and the actual configuration of S, respectively. We

[1] Readers interested in this topic should consult references [58]–[70].

denote a Cartesian frame of reference by $Ox_1x_2x_3$, and the unit vectors along the axes of $Ox_1x_2x_3$ by $(\mathbf{e}_i)$, $i = 1, 2, 3$. In order to describe the evolution of S, we need to determine both the equation of motion $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, $\mathbf{X} \in C_*$, $\mathbf{x} \in C$, and the orthogonal matrix $\chi(\mathbf{X}, t)$ that describes the micromotion. With the usual notation, $\mathbf{F}$ denotes the deformation gradient relative to the motion $\mathbf{x}(\mathbf{X}, t)$. An arbitrary vector $\mathbf{V}$ at the point $\mathbf{X} \in C_*$ is transformed into the vector

$$\widehat{\mathbf{v}} = \mathbf{F}\mathbf{V} \quad (\widehat{v}_i = F_{iL}V_L)$$

of the actual configuration by the motion $\mathbf{x}(\mathbf{X}, t)$, and into the vector

$$\mathbf{v} = \chi\mathbf{V} \quad (v_i = \chi_{iL}V_L) \tag{2.1}$$

by the micromotion. In particular, we choose three arbitrary unit directors $\mathbf{D}_{(L)}$ at the point $\mathbf{X}$ that coincide with the unit vectors along the axes of a Cartesian frame of reference $O_*X_1X_2X_3$ in C_*. If we denote the corresponding vectors in the actual configuration by $\mathbf{d}_{(L)} = \chi\mathbf{D}_L$, using (2.1) we get

$$d_{(L)i} = \chi_{iL}. \tag{2.2}$$

Then, from (2.2) we derive that

$$d_{(L)j}d_{(L)h} = \delta_{jh}, \quad d_{(L)h}d_{(M)h} = \delta_{LM}. \tag{2.3}$$

Equation 2.2 shows that the micromotion can also be described by providing the vector functions $\mathbf{d}_{(L)} = \mathbf{d}_{(L)}(\mathbf{X}, t)$, $i = 1, 2, 3$. Note that these functions are not independent because of the six orthogonality conditions (2.3).

Since the local deformation of any element of volume dc is obtained from the equation $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ in the usual way (see Chap. 3 of [16]), we now need to describe the behavior of the directors.

There are two ways to realize this objective. First, we can describe the motion of the directors using the results for the dynamics of rigid bodies, which allow us to write the evolution equations for the directors in the form

$$\dot{\mathbf{d}}_{(L)} = \omega \times \mathbf{d}_{(L)},$$

where the vector function $\omega(t)$ is given by the relation

$$\omega = \frac{1}{2}\mathbf{d}_{(L)} \times \dot{\mathbf{d}}_{(L)}.$$

The axial vector $\omega(t)$, which is called the *microgyration vector* or the *micropolar vector*, has the components

$$\omega_i = \frac{1}{2}\epsilon_{ijk}d_{(L)j}\dot{d}_{(L)k} \equiv \Lambda_{ikL}\dot{d}_{(L)k}, \tag{2.4}$$

and it represents the angular velocity of the triad $\mathbf{d}_{(L)}$ with respect to the reference frame $Ox_1x_2x_3$.

Alternatively, we can use three new *independent* variables that completely define the vectors $\mathbf{d}_{(L)}$ and the orthogonal matrix χ (see Eq. 2.1). In order to introduce these new variables, which were proposed by Eringen in [58]–[61], we note that the orthogonality of the matrix χ allows us to write it in the exponential form

$$\chi = e^{\mathbf{\Phi}} = \mathbf{I} + \mathbf{\Phi} + \frac{1}{2!}\mathbf{\Phi}^2 + \frac{1}{3!}\mathbf{\Phi}^3 + \cdots,$$

where $\mathbf{\Phi}$ is a skew-symmetric matrix. If

$$\varphi_i = \frac{1}{2}\epsilon_{ijl}\Phi_{jl} \tag{2.5}$$

denotes the adjoint of $\mathbf{\Phi}$, then we can write

$$\mathbf{\Phi} = \begin{pmatrix} 0 & \varphi_3 & -\varphi_2 \\ -\varphi_3 & 0 & \varphi_1 \\ \varphi_2 & -\varphi_1 & 0 \end{pmatrix}. \tag{2.6}$$

From the Hamilton–Cayley theorem (see [16])

$$\mathbf{\Phi}^3 - I\mathbf{\Phi}^2 + II\mathbf{\Phi} - III\,\mathbf{I} = \mathbf{0},$$

where I, II and III are the principal invariants of $\mathbf{\Phi}$:

$$\begin{aligned} I &= \mathrm{tr}\mathbf{\Phi} = 0, \\ II &= \frac{1}{2}[(\mathrm{tr}\mathbf{\Phi})^2 - \mathrm{tr}\mathbf{\Phi}^2] = \varphi_1^2 + \varphi_2^2 + \varphi_3^2, \\ III &= \det\mathbf{\Phi} = 0, \end{aligned}$$

we can derive the relation

$$\mathbf{\Phi}^3 = -II\mathbf{\Phi},$$

which, in turn, implies the sequence of equations

$$\begin{aligned} &\mathbf{\Phi}^4 = \mathbf{\Phi}\mathbf{\Phi}^3 = -II\mathbf{\Phi}^2, \\ &\mathbf{\Phi}^5 = \mathbf{\Phi}\mathbf{\Phi}^4 = -II\mathbf{\Phi}^3 = II^2\mathbf{\Phi}, \\ &\mathbf{\Phi}^6 = \mathbf{\Phi}\mathbf{\Phi}^5 = \mathbf{\Phi}II^2\mathbf{\Phi} = II^2\mathbf{\Phi}^2, \\ &\mathbf{\Phi}^7 = \mathbf{\Phi}\mathbf{\Phi}^6 = \mathbf{\Phi}II^2\mathbf{\Phi}^2 = -II^3\mathbf{\Phi}, \\ &\ldots \\ &\ldots \end{aligned}$$

and we can write the matrix χ as follows:

$$\begin{aligned} \chi = e^{\mathbf{\Phi}} &= \mathbf{I} + \mathbf{\Phi} + \frac{1}{2!}\mathbf{\Phi}^2 + \frac{1}{3!}\mathbf{\Phi}^3 + \cdots \\ &= \mathbf{I} + \mathbf{\Phi} + \frac{1}{2!}\mathbf{\Phi}^2 - \frac{1}{3!}II\mathbf{\Phi} - \frac{1}{4!}II\mathbf{\Phi}^2 \\ &+ \frac{1}{5!}II^2\mathbf{\Phi} + \frac{1}{6!}II^2\mathbf{\Phi}^2 + \cdots. \end{aligned}$$

Finally, introducing the notation

$$\alpha_1 = 1 - \frac{1}{3!}II + \frac{1}{5!}II^2 - \cdots, \tag{2.7}$$

$$\alpha_2 = \frac{1}{2!} - \frac{1}{4!}II + \frac{1}{6!}II^2 - \cdots, \tag{2.8}$$

we have

$$\chi = \mathbf{I} + \alpha_1 \mathbf{\Phi} + \alpha_2 \mathbf{\Phi}^2. \tag{2.9}$$

Noting (2.6) and (2.9), we can write

$$\chi = \begin{pmatrix} -\alpha_2(\varphi_2^2 + \varphi_3^2) & \alpha_2\varphi_1\varphi_2 + \alpha_1\varphi_3 & -\alpha_1\varphi_2 + \alpha_2\varphi_1\varphi_3 \\ \alpha_2\varphi_1\varphi_2 - \alpha_1\varphi_3 & -\alpha_2(\varphi_1^2 + \varphi_3^2) & \alpha_1\varphi_1 + \alpha_2\varphi_2\varphi_3 \\ \alpha_1\varphi_2 + \alpha_2\varphi_1\varphi_3 & -\alpha_1\varphi_1 + \alpha_2\varphi_2\varphi_3 & -\alpha_2(\varphi_1^2 + \varphi_2^2) \end{pmatrix}.$$

An orthogonal matrix always permits the eigenvalue $\lambda = 1$ and the corresponding eigenvector $\mathbf{n}$ obeys the vector equation

$$(\chi - \mathbf{I})\mathbf{n} = \mathbf{0},$$

which in our case is equivalent to the system

$$\begin{aligned}
&-\alpha_2(\varphi_2^2 + \varphi_3^2)n_1 + (\alpha_2\varphi_1\varphi_2 + \alpha_1\varphi_3)n_2 + (-\alpha_1\varphi_2 + \alpha_2\varphi_1\varphi_3)n_3 = 0,\\
&(\alpha_2\varphi_1\varphi_2 - \alpha_1\varphi_3)n_1 - \alpha_2(\varphi_1^2 + \varphi_3^2)n_2 + (\alpha_1\varphi_1 + \alpha_2\varphi_2\varphi_3)n_3 = 0,\\
&(\alpha_1\varphi_2 + \alpha_2\varphi_1\varphi_3)n_1 + (-\alpha_1\varphi_1 + \alpha_2\varphi_2\varphi_3)n_2 - \alpha_2(\varphi_1^2 + \varphi_2^2)n_3 = 0.
\end{aligned}$$

It is well known that these equations are not independent. From the first two of them, we can easily derive the relations

$$n_1 = \frac{\varphi_1}{\varphi_3}n_3, \quad n_2 = \frac{\varphi_2}{\varphi_3}n_3,$$

and, by imposing the condition $n_1^2 + n_2^2 + n_3^2 = 1$, we finally find that

$$\mathbf{n} = \frac{1}{\theta}\varphi, \tag{2.10}$$

where

$$\varphi = (\varphi_1, \varphi_2, \varphi_3), \quad \theta = \sqrt{\varphi_1^2 + \varphi_2^2 + \varphi_3^2}. \tag{2.11}$$

In conclusion, the matrix χ defines a rotation about the axis $\mathbf{n}$ by an angle θ. We are now capable of expressing the matrix χ in terms of θ and $\mathbf{n}$. From (2.9) we have

$$\chi_{lm} = \delta_{lm} + \alpha_1\Phi_{lm} + \alpha_2\Phi_{ln}\Phi_{nm},$$

and, taking into account (2.5) and (2.9), we can write

$$\begin{aligned}\chi_{lm} &= \delta_{lm} + \alpha_1 \epsilon_{lmn}\varphi_n - \alpha_2 \epsilon_{lpn}\epsilon_{nmq}\varphi_p\varphi_q \\ &= \delta_{lm} + \alpha_1 \epsilon_{lmn}\varphi_n - \alpha_2(\delta_{lm}\delta_{pq} - \delta_{lq}\delta_{pm})\varphi_p\varphi_q \\ &= \delta_{lm} + \alpha_1 \epsilon_{lmn}\theta n_n - \alpha_2\delta_{lm}\theta^2 + \alpha_2\theta^2 n_l n_m \\ &= (1-\alpha_2\theta^2)\delta_{lm} + \alpha_1\theta\epsilon_{lmn}n_n + \alpha_2\theta^2 n_l n_m.\end{aligned}$$

On the other hand, $II = \varphi_1^2 + \varphi_2^2 + \varphi_3^2$, so we can derive the following relations from (2.7), (2.8) and (2.11):

$$1 - \alpha_2\theta^2 = \cos\theta, \quad \alpha_1\theta = \sin\theta, \quad \alpha_2\theta^2 = 1 - \cos\theta,$$

which allow us to write χ in the final form

$$\chi_{lm} = \cos\theta\delta_{lm} + \sin\theta\epsilon_{lmn}n_n + (1-\cos\theta)n_l n_m. \tag{2.12}$$

We define the *angular microvelocity tensor* as follows:

$$\boldsymbol{\Omega} = \chi\dot{\chi}^T, \quad \Omega_{jk} = \chi_{jL}\dot{\chi}_{kL} = d_{(L)j}\dot{d}_{(L)k}. \tag{2.13}$$

Note that this tensor is skew symmetric, as deriving the orthogonality condition $\chi\chi^T = \mathbf{I}$ leads to the relation

$$\dot{\chi}\chi^T + \chi\dot{\chi}^T = \mathbf{0},$$

which can also be written

$$\chi\dot{\chi}^T = -(\chi\dot{\chi}^T)^T.$$

Due to (2.13), the adjoint vector of $\boldsymbol{\Omega}$

$$\omega_i = \frac{1}{2}\epsilon_{ijk}\Omega_{jk}, \quad \Omega_{jk} = \epsilon_{jki}\omega_i, \tag{2.14}$$

can be written as

$$\omega_i = \Lambda_{ijh}\dot{d}_{(L)j}, \quad \Lambda_{ijL} = \frac{1}{2}\epsilon_{ikj}d_{(L)k}. \tag{2.15}$$

A comparison between (2.15) and (2.4) shows that ω is simply the angular microgyration vector.

Starting from (2.14) and (2.12), it is possible to prove the other formula

$$\omega_i = \widetilde{\Lambda}_{ij}\dot{\varphi}_j, \tag{2.16}$$

where $\widetilde{\Lambda}_{ij}$ is a suitable matrix. Note that (2.15) expresses the angular microvelocity as a function of the director fields, whereas (2.16) uses the variables φ (i.e., the axis and the angle of rotation determined by χ).

In the next section we will associate an intrinsic angular momentum $\mathbf{k}$ with any point $\mathbf{x}$ in the micropolar continuous system S. We will assume that, as in the mechanics of rigid bodies, this angular momentum depends linearly on the angular microvelocity. In other words, we assume that there is a symmetric *microinertia tensor density* $\mathbf{J} = (J_{ij})$ such that

$$\mathbf{k} = \mathbf{J}\omega, \quad \mathbf{J} = \mathbf{J}^T. \tag{2.17}$$

In this section we determine an evolution equation for the microinertia tensor $\mathbf{J}$. If we denote the value of $\mathbf{J}$ in the reference configuration C_* by $\mathbf{J}(0)$, we have the relation

$$J_{ij} = \chi_{iL}\chi_{jM}J_{LM}(0).$$

Differentiating this with respect to time and utilizing (2.13) leads to

$$\begin{aligned}\dot{J}_{ij} &= (\dot{\chi}_{iL}\chi_{jM} + \chi_{iL}\dot{\chi}_{jM})J_{LM}(0)\\ &= [(\chi_{pL}\Omega_{pi})\chi_{jM} + \chi_{iL}(\chi_{pM}\Omega_{pj})]J_{LM}(0)\\ &= \chi_{pL}\chi_{jM}J_{LM}(0)\Omega_{pi} + \chi_{iL}\chi_{pM}J_{LM}(0)\Omega_{pj};\end{aligned}$$

that is, the *conservation of microinertia*:

$$\dot{J}_{ij} = J_{pj}\Omega_{pi} + J_{ip}\Omega_{pj} = (J_{pj}\epsilon_{pil} + J_{ip}\epsilon_{pjl})\omega_l, \tag{2.18}$$

where (2.14) has been taken into account.

Remark Since the quantities φ_i are *independent*, it is quite natural to adopt them as fundamental variables to describe the evolution of a micropolar system, which is exactly the point of view assumed in [59]. This reduces the unknowns and the equations for the problems, making it easier to derive the restrictions imposed by the dissipation principle on the constitutive equations. However, it is much more complex to derive the consequences of the objectivity principle and to introduce suitable deformation tensors. For this reason, from now on, we will use the *dependent* quantities $d_{(L)h}$ and take their dependence into account in the dissipation principle by introducing suitable Lagrangian multipliers.

2.3 Mechanical Balance Equations

We now state the mechanical balance laws governing the evolution of a continuous system S with directors $(\mathbf{d}_{(L)})$, $L = 1, 2, 3$, under the condition that they form an orthogonal triad. We begin from the general case in which *the angular microvelocity of any triad* $(\mathbf{d}_{(L)})$ *differs from the angular velocity of the point of* S *located at the origin of* $(\mathbf{d}_{(L)})$.

Before all this, we must provide a picture of the acting forces. In doing this, we generalize the model proposed in Chap. 5 of [16]. We still suppose that:

- The external actions on any material volume c of S can be divided into *mass forces* that are continuously distributed over c, and *contact forces* that act on the boundary ∂c of c
- The contact forces have a molecular origin, so they can be represented by surface vector fields.

However, we now suppose that their action on any material volume c is described by a *force field* and a *couple field*. In particular, the total action of the mass forces is expressed by a total force

$$\mathbf{F}_m(c) = \int_c \rho \mathbf{b}\, dc, \tag{2.19}$$

where $\mathbf{b}$ is the *specific mass force* defined on c, and a total momentum

$$\mathbf{M}_0(c) = \int_c \rho(\mathbf{r} \times \mathbf{b} + \mathbf{l})\, dc, \tag{2.20}$$

where $\mathbf{r}$ is the position vector of any point on c with respect to the pole O, and $\mathbf{l}$ is the *specific body couple*.

Similarly, the action of the contact forces is given by a total force

$$\mathbf{F}_\sigma(\partial c) = \int_{\partial c} \mathbf{t}\, d\sigma, \tag{2.21}$$

where $\mathbf{t}$ is the *stress per unit area* defined on ∂c, and a resultant momentum

$$\mathbf{M}_{\sigma,0}(c) = \int_{\sigma c} (\mathbf{r} \times \mathbf{t} + \mathbf{m})\, d\sigma, \tag{2.22}$$

where $\mathbf{m}$ is the *surface couple stress*.

According to these assumptions, we can still accept the *mass conservation* of any material volume c

$$\frac{d}{dt}\int_c \rho\, dc = 0, \tag{2.23}$$

where ρ is the mass density of S, and the momentum balance

$$\frac{d}{dt}\int_c \rho \mathbf{v}\, dc = \int_{\partial c} \mathbf{t}\, d\sigma + \int_c \rho \mathbf{b}\, dc, \tag{2.24}$$

where $\mathbf{v}$ is the velocity field.

For the angular momentum balance, we can postulate the equation

$$\frac{d}{dt}\int_c \rho(\mathbf{r}\times\mathbf{v}+\mathbf{k})\,dc = \int_{\partial c}(\mathbf{r}\times\mathbf{t}+\mathbf{m})\,d\sigma + \int_c \rho(\mathbf{r}\times\mathbf{b}+\mathbf{l})\,dc, \qquad (2.25)$$

where

$$\mathbf{k} = \mathbf{J}\omega \qquad (2.26)$$

is the *spin density*, $\mathbf{J}$ is the microinertia tensor, and ω denotes the angular microvelocity.

If $\mathbf{n}$ is the exterior unit vector normal to ∂c, and Cauchy's hypothesis (see Chap. 5 of [16])

$$\mathbf{t} = \mathbf{t}(\mathbf{x}, t, \mathbf{n})$$

is accepted, then it is possible to prove (see Theorem 5.1 of [16]) the existence of Cauchy's stress tensor $\mathbf{T}$ such that

$$\mathbf{t} = \mathbf{T}\mathbf{n}. \qquad (2.27)$$

Consequently, due to the arbitrariness of the region c and the continuity of the functions under the integrals, we can derive the usual local forms of (2.23) and (2.24):

$$\dot{\rho} + \rho v_{i,i} = 0, \qquad (2.28)$$

$$\rho\dot{v}_i = T_{ij,j} + \rho b_i. \qquad (2.29)$$

From now on, a subscript comma will denote partial differentiation. In order to find the local form of balance equation (2.25), we first note that the transport theorem allows us to write (2.25) as follows:

$$\begin{aligned}\int_c \rho\frac{d}{dt}(\epsilon_{ijk}r_j v_k)\,dc &+ \int_c \rho\dot{k}_i\,dc \\ &= \int_{\partial c}\epsilon_{ijk}r_j T_{kh}n_h\,d\sigma + \int_{\partial c} m_i\,d\sigma \\ &+ \int_c \rho\epsilon_{ijk}r_j b_k\,dc + \int_c \rho l_i\,dc. \qquad (2.30)\end{aligned}$$

If we now apply the divergence theorem to the first term on the right-hand side of (2.30), note that

$$(\epsilon_{ijk}r_j T_{kh})_{,h} = \epsilon_{ijk}T_{kj} + \epsilon_{ijk}r_j T_{kh,h},$$

and consider (2.29), we change (2.30) into the form

$$\int_c \rho\dot{\mathbf{k}}\,dc = \int_{\partial c}\mathbf{m}\,d\sigma + \int_c(\tau + \rho\mathbf{l})\,dc, \qquad (2.31)$$

where τ is the opposite of the adjoint of the skew-symmetric part of the stress tensor $\mathbf{T}$; i.e.,

$$\tau_i = \epsilon_{ijk}T_{kj} = -\epsilon_{ikj}T_{kj}. \qquad (2.32)$$

If we extend Cauchy's hypothesis (2.27) to **m**; i.e., if we suppose that

$$\mathbf{m} = \mathbf{m}(\mathbf{x}, t, \mathbf{n}),$$

and if we again apply Theorem 5.1 of [16], we can show the existence of the *stress couple tensor* **M** such that

$$\mathbf{m} = \mathbf{M}\mathbf{n}, \tag{2.33}$$

and (2.31) can be written in the local form

$$\rho \dot{k}_i = M_{ij,j} + \tau_i + \rho l_i. \tag{2.34}$$

2.4 Energy and Entropy

In order to describe the exchanges between mechanical energy and thermal energy, we postulate the following *energy balance equation*:

$$\begin{aligned}\frac{d}{dt}\int_c \rho\left(\frac{1}{2}v^2 + \frac{1}{2}\mathbf{k}\cdot\omega + \epsilon\right)dc = &\int_{\partial c}(\mathbf{v}\cdot\mathbf{T}\mathbf{n} + \omega\cdot\mathbf{M}\mathbf{n} - \mathbf{h}\cdot\mathbf{n})\,d\sigma \\ &+ \int_c \rho(\mathbf{b}\cdot\mathbf{v} + \mathbf{l}\cdot\omega + r)\,dc,\end{aligned} \tag{2.35}$$

where ϵ is the *specific internal energy*, **h** is the *heat flux vector*, and r is the *external power supply per unit volume*.

Applying the transport theorem to the integral on the left-hand side and the Gauss theorem to the first integral on the right-hand side, we can write (2.35) in the form

$$\begin{aligned}&\int_c (\rho\dot{v}_i - \rho b_i - T_{il,l})v_i\,dc \\ &+ \int_c \left[\left(\frac{1}{2}\rho\dot{k}_i - \rho l_i - M_{il,l}\right)\omega_i + \frac{1}{2}\rho k_i\dot{\omega}_i\right]dc + \int_c \rho\dot{\epsilon}\,dc \\ &= \int_c (\rho r + T_{il}v_{i,l} + M_{il}\omega_{i,l} - h_{l,l})\,dc.\end{aligned} \tag{2.36}$$

The first integral vanishes for (2.29), whereas the function under the second integral can be transformed as follows, taking into account (2.34):

$$\begin{aligned}&\left(\frac{1}{2}\rho\dot{k}_i - \rho l_i - M_{il,l}\right)\omega_i + \frac{1}{2}\rho k_i\dot{\omega}_i \\ &= \left(\rho\dot{k}_i - \rho l_i - M_{il,l}\right)\omega_i - \frac{1}{2}\rho\dot{k}_i\omega_i + \frac{1}{2}\rho k_i\dot{\omega}_i \\ &= \tau_i\omega_i + \frac{1}{2}\rho(k_i\dot{\omega}_i - \dot{k}_i\omega_i).\end{aligned}$$

Using conditions (2.17) and (2.18), we derive the result

$$\begin{aligned} k_i\dot{\omega}_i - \dot{k}_i\omega_i &= -\dot{J}_{ij}\omega_i\omega_j \\ &= (\epsilon_{ikl}\omega_l J_{kj} + \epsilon_{jkl}\omega_l J_{ki})\omega_i\omega_j = 0, \end{aligned}$$

since ϵ_{ilk} is skew symmetric while $\omega_i\omega_l$ is symmetric. In conclusion, we can give (2.35) the following local form:

$$\rho\dot{\epsilon} = T_{il}v_{i,l} + M_{il}\omega_{i,l} - \tau_l\omega_l - h_{l,l} + \rho r. \tag{2.37}$$

We postulate that the entropy principle takes the form we stated in Sect. 5.6 of [16]:

$$\rho\dot{\eta} \geq -\left(\frac{h_l}{\theta}\right)_{,l} + \rho\frac{r}{\theta}, \tag{2.38}$$

where η is the *specific entropy* and θ is the absolute temperature.

Retrieving the terms $-\nabla\cdot\mathbf{h} + \rho r$ from (2.38) and substituting the result into (2.37), we obtain *Clausius–Duhem's inequality*

$$-\rho(\dot{\psi} + \eta\dot{\theta}) + T_{ij}v_{i,j} + M_{ij}\omega_{i,j} - \tau_l\omega_l - \frac{h_l\theta_{,l}}{\theta} \geq 0, \tag{2.39}$$

where

$$\psi = \epsilon - \theta\eta$$

is the *specific free energy*.

We conclude this section by writing the local equations (2.28), (2.29), (2.34), (2.37) and (2.39) in the corresponding Lagrangian form. This result could be obtained by applying the same procedure we used in Sect. 5.7 of [16]. However, that approach starts with the *integral* balance laws written in the reference configuration, meaning that we would repeat all of the calculations that led us to the Eulerian local balance equations. Therefore, we will instead place these last equations directly into their corresponding Lagrangian form using the formula

$$(JF^{-1}_{Lj})_{,L} = 0, \tag{2.40}$$

where $\mathbf{F} = (F_{iL})$ is the deformation gradient and $J = \det\mathbf{F}$.[2]

[2] Using (3.50) of [16], we have

$$\frac{\partial F^{-1}_{Lk}}{\partial F_{iM}} = -F^{-1}_{Li}F^{-1}_{Mk}, \quad \frac{\partial J}{\partial F_{iM}} = JF^{-1}_{Mi}.$$

Therefore,

$$\begin{aligned} (JF^{-1}_{Lj})_{,L} &= F_{hN,L}\left(\frac{\partial J}{\partial F_{hN}}F^{-1}_{Lj} + J\frac{\partial F^{-1}_{Lj}}{\partial F_{hN}}\right) \\ &= J\left(F^{-1}_{Nh}F^{-1}_{Lj} - F^{-1}_{Lh}F^{-1}_{Nj}\right)_{,LN} = 0. \end{aligned}$$

Starting from the Lagrangian equation of mass conservation $\rho_* = \rho J$ and (2.40), it is easy to verify that (2.29), (2.34), (2.37) and (2.39) can be written in the form

$$\rho_* v_i = T_{*iL},_L + \rho_* b_i, \tag{2.41}$$

$$\rho_* \dot{k}_i = \tau_{*i} + M_{*iL},_L + \rho_* l_i, \tag{2.42}$$

$$\rho_* \dot{\epsilon} = T_{*iL} v_i,_L + M_{*iL} \omega_i,_L - \tau_{*i}\omega_i - h_{*L},_L + \rho_* r, \tag{2.43}$$

$$-\rho_*(\dot{\psi} + \eta\dot{\theta}) + T_{*iL} v_i,_L + M_{*iL}\omega_i,_L - \tau_{*i}\omega_i - \frac{1}{\theta} h_{*L}\theta,_L \geq 0, \tag{2.44}$$

where we have introduced the Lagrangian fields

$$T_{*iL} = J T_{ij} F_{Lj}^{-1}, \tag{2.45}$$

$$M_{*iL} = J M_{ij} F_{Lj}^{-1}, \tag{2.46}$$

$$\tau_{*i} = \epsilon_{ijk} F_{jL} T_{*kL}, \tag{2.47}$$

$$h_{*L} = J h_i F_{Li}^{-1}. \tag{2.48}$$

2.5 Elastic Micropolar Systems

We say that the continuum S is an *elastic micropolar system* if its behavior is described by the constitutive equations

$$\mathbf{T} = \mathbf{T}(\mathbf{F}, \mathbf{d}_{(L)}, \nabla_{\mathbf{X}}\mathbf{d}_{(L)}, \theta, \nabla_{\mathbf{X}}\theta), \tag{2.49}$$

$$\mathbf{M} = \mathbf{M}(\mathbf{F}, \mathbf{d}_{(L)}, \theta, \nabla_{\mathbf{X}}\mathbf{d}_{(L)}, \nabla_{\mathbf{X}}\theta), \tag{2.50}$$

$$\mathbf{h} = \mathbf{h}(\mathbf{F}, \mathbf{d}_{(L)}, \theta, \nabla_{\mathbf{X}}\mathbf{d}_{(L)}, \theta, \nabla_{\mathbf{X}}\theta), \tag{2.51}$$

$$\psi = \psi(\mathbf{F}, \mathbf{d}_{(L)}, \theta, \nabla_{\mathbf{X}}\mathbf{d}_{(L)}, \theta, \nabla_{\mathbf{X}}\theta), \tag{2.52}$$

$$\eta = \eta(\mathbf{F}, \mathbf{d}_{(L)}, \theta, \nabla_{\mathbf{X}}\mathbf{d}_{(L)}, \theta, \nabla_{\mathbf{X}}\theta). \tag{2.53}$$

It is well known that these constitutive equations cannot be assigned arbitrarily, since they must satisfy the constitutive axioms. In this section we derive the restrictions imposed by the dissipation principle (see Chap. 6 of [16]).

If the fields $\mathbf{F}, \mathbf{d}_{(P)}, \theta$ are assumed to depend on the variables $\mathbf{X}, t$, then we have:

$$\begin{aligned}\dot{\psi} = {} & \frac{\partial\psi}{\partial F_{iL}} F_{jL} v_{i,j} + \frac{\partial\psi}{\partial d_{(P)j}} \dot{d}_{(P)j} \\ & + \frac{\partial\psi}{\partial d_{(P)j,L}} F_{hL} \dot{d}_{(P)j,h} + \frac{\partial\psi}{\partial\theta}\dot{\theta} + \frac{\partial\psi}{\partial\theta_{,L}} F_{jL}\dot{\theta}_{,j}.\end{aligned} \tag{2.54}$$

The introduction of this relation into (2.39) leads to the inequality

$$\begin{aligned}
&-\rho\left(\frac{\partial\psi}{\partial\theta}+\eta\right)\dot{\theta}+\left(T_{ij}-\rho\frac{\partial\psi}{\partial F_{iL}}F_{jL}\right)v_{i,j}\\
&-\rho\frac{\partial\psi}{\partial\theta_{,L}}F_{jL}\dot{\theta}_{,j}+\left(M_{lh}\Lambda_{lji}-\rho\frac{\partial\psi}{\partial(d_{(P)j,L})}F_{hL}\right)\dot{d}_{(P)j,h}\\
&+\left(-\rho\frac{\partial\psi}{\partial d_{(P)j}}+M_{lh}\Lambda_{lji,h}-\epsilon_{lhk}T_{kh}\Lambda_{lji}\right)\dot{d}_{(P)j}-\frac{h_{*i}}{\theta}\theta_{,i}\geq 0. \quad (2.55)
\end{aligned}$$

The dissipation principle states that this inequality must be satisfied in any thermodynamic process $\mathbf{F}(\mathbf{X},t)$, $\mathbf{d}_i(\mathbf{X},t)$, and $\theta(\mathbf{X},t)$ *provided that conditions* (2.1) *are verified*. In order to account for the consequences of (2.1), we note that the following can be derived from $d_{(P)j}d_{(P)}=\delta_{jk}$:

$$\dot{d}_{(P)j}d_{(P)k}+d_{(P)j}\dot{d}_{(P)k}=0,$$
$$\dot{d}_{(P)j,h}\,d_{(P)k}+\dot{d}_{(P)j}d_{(P)k,h}+d_{(P)j,h}\,\dot{d}_{(P)k}+d_{(P)j}\dot{d}_{(P)k,h}=0,$$

where the right-hand sides are *symmetric* with respect to the indices j and k. Consequently, the quantities $d_{(P)j}$, $\dot{d}_{(P)j}$ and $\dot{d}_{(P)j,h}$ that appear in (2.55) can be considered independent if the following terms are added to (2.55) (see [13]):

$$\lambda_{kj}(\dot{d}_{(P)j}d_{(P)k}+d_{(P)j}\dot{d}_{(P)k})=2\lambda_{kj}\dot{d}_{(P)j}d_{(P)k},$$

$$\begin{aligned}
&\mu_{kjh}(\dot{d}_{(P)j,h}\,d_{(P)k}+\dot{d}_{(P)j}d_{(P)k,h}+d_{(P)j,h}\,\dot{d}_{(P)k}+d_{(P)j}\dot{d}_{(P)k,h})\\
&=2\mu_{kjh}(\dot{d}_{(P)j,h}\,d_{(P)k}+\dot{d}_{(P)j}d_{(P)k,h}),
\end{aligned}$$

where the unknown Lagrangian multipliers λ_{kj} and μ_{kjh} can be assumed to be symmetric with respect to the indices j and k.

With the introduction of these terms into (2.55), we have

$$\begin{aligned}
&-\rho\left(\frac{\partial\psi}{\partial\theta}+\eta\right)\dot{\theta}+\left(T_{ij}-\rho\frac{\partial\psi}{\partial F_{iL}}F_{jL}\right)v_{i,j}\\
&-\rho\frac{\partial\psi}{\partial\theta_{,L}}F_{jL}\dot{\theta}_{,j}+\left(M_{lh}\Lambda_{lji}-\rho\frac{\partial\psi}{\partial(d_{(P)j,L})}F_{hL}+2\mu_{kjh}d_{(P)k}\right)\dot{d}_{(P)j,h}\\
&+\left(-\rho\frac{\partial\psi}{\partial d_{(P)j}}+M_{lh}\Lambda_{lji,h}-\epsilon_{lhk}T_{kh}\Lambda_{lji}+2\mu_{kjh}d_{(P)k,h}+2\lambda_{kj}d_{(P)k}\right)\dot{d}_{(P)j}\\
&-\frac{h_{*i}}{\theta}\theta_{,i}\geq 0. \qquad (2.56)
\end{aligned}$$

Consequently (see Chap. 6 of [16]), the constitutive equations (2.49)–(2.53) must identically obey the relations

$$\psi=\psi(\mathbf{F},\mathbf{d}_i,\nabla_{\mathbf{X}}\mathbf{d}_i), \qquad (2.57)$$

$$\eta = -\frac{\partial \psi}{\partial \theta}, \tag{2.58}$$

$$T_{ij} = \rho \frac{\partial \psi}{\partial F_{iL}} F_{jL}, \tag{2.59}$$

$$M_{lh} \Lambda_{lji} = \rho \frac{\partial \psi}{\partial (d_{(P)j},_L)} F_{hL} - 2\mu_{kjh} d_{(P)k}, \tag{2.60}$$

$$\begin{aligned} \rho \frac{\partial \psi}{\partial d_{(P)j}} &- M_{lh} \Lambda_{lji},_h + \epsilon_{lhk} T_{kh} \Lambda_{lji} \\ &- 2\mu_{kjh} d_{(P)k},_h - 2\lambda_{kj} d_{(P)k} = 0, \end{aligned} \tag{2.61}$$

$$h_i \theta,_i \leq 0. \tag{2.62}$$

When (2.15) is taken into account, relation (2.60) can also be written as

$$M_{lh} \epsilon_{lnj} d_{(P)n} = 2\rho \frac{\partial \psi}{\partial (d_{(P)j},_L)} F_{hL} - 4\mu_{kjh} d_{(P)k}.$$

Multiplying this equation by $d_{(P)r}$, and then by ϵ_{nrj}, and noting that (see Eq. 2.2)

$$d_{(P)n} d_{(P)r} = \delta_{nr}, \quad \epsilon_{lrj} \epsilon_{nrj} = 2\delta_{ln},$$

we obtain

$$M_{lh} = \rho \epsilon_{lrj} d_{(P)r} \frac{\partial \psi}{\partial (d_{(P)j},_L)} F_{hL}, \tag{2.63}$$

since the term $\mu_{rjk} \epsilon_{nrj}$ vanishes due to the symmetry of μ_{rjk} and the skew symmetry of ϵ_{nrj} with respect to the indices r and j.

Equation 2.61 can be written in a more expressive form that highlights its meaning. Multiplying (2.61) by $d_{(P)n}$, we have

$$\begin{aligned} &\rho \ \frac{\partial \psi}{\partial d_{(P)j}} d_{(P)n} - \frac{1}{2} M_{lh} \epsilon_{lkj} \, d_{(P)n} d_{(P)k,h} + \frac{1}{2} \epsilon_{lhk} T_{kh} \epsilon_{lpj} d_{(P)p} d_{(P)n} \\ &- 2\mu_{kjh} d_{(P)k},_h \, d_{(P)n} - 2\lambda_{kj} d_{(P)k} d_{(P)n} = 0. \end{aligned} \tag{2.64}$$

On the other hand,

$$\begin{aligned} d_{(P)k} d_{(P)n} &= \delta_{kn}, \\ d_{(P)n} d_{(P)k},_h &= (d_{(P)n} d_{(P)k}),_h - d_{(P)k} d_{(P)n},_h \\ &= -d_{(P)k} d_{(P)n},_h = -(F^{-1})_{Mh} d_{(P)k} d_{(P)n},_M , \end{aligned}$$

and so (2.64) becomes

$$\begin{aligned} &\rho \ \frac{\partial \psi}{\partial d_{(P)j}} \, d_{(P)n} + M_{lh} \Lambda_{lji} d_{(P)n},_M \, (F^{-1})_{Mh} + \frac{1}{2} \epsilon_{lhk} T_{kh} \epsilon_{lnj} \\ &- 2\mu_{kjh} d_{(P)n} d_{(P)k},_h - 2\lambda_{nj} = 0. \end{aligned}$$

Due to (2.60), we can write this equation as follows:

$$\rho\frac{\partial\psi}{\partial d_{(P)j}}\,d_{(P)n}+\left(\rho\frac{\partial\psi}{\partial(d_{(P)j},_L)}F_{hL}-2\mu_{kjh}d_{(P)k}\right)d_{(P)n},_M\,(F^{-1})_{Mh}$$
$$+\frac{1}{2}\epsilon_{lhk}T_{kh}\epsilon_{lnj}-2\mu_{kjh}d_{(P)n}d_{(P)k},_h-2\lambda_{nj}=0;$$

i.e.,

$$\rho\frac{\partial\psi}{\partial d_{(P)j}}\,d_{(P)n}+\rho\frac{\partial\psi}{\partial(d_{(P)j},_L)}d_{(P)n},_L-2\mu_{kjh}d_{(P)k}d_{(P)n},_h$$
$$+\frac{1}{2}\epsilon_{lhk}T_{kh}\epsilon_{lnj}-2\mu_{kjh}d_{(P)n}d_{(P)k},_h-2\lambda_{nj}=0.$$

However, $d_{(P)k}d_{(P)n},_h=-d_{(P)n}d_{(P)k},_h$. Therefore, multiplying the above equation by ϵ_{pnj}, noting that $\epsilon_{pnj}\epsilon_{lnj}=2\delta_{pl}$, and taking (2.59) and (2.60) into account, we can write the above equation in the form

$$\epsilon_{pnj}\left(\frac{\partial\psi}{\partial d_{(P)j}}d_{(P)n}+\frac{\partial\psi}{\partial(d_{(P),j},_L)}d_{(P)n},_L+\frac{\partial\psi}{\partial F_{jL}}F_{nL}\right)=0. \tag{2.65}$$

In the next section we prove that (2.65) is equivalent to the objectivity principle.

In conclusion, for an elastic micropolar continuous system, the dissipation principle implies that

$$\psi=\psi(F_{iL},d_{(P)j},d_{(P)j},_L,\theta), \tag{2.66}$$
$$\eta=-\frac{\partial\psi}{\partial\theta}, \tag{2.67}$$
$$T_{ij}=\rho\frac{\partial\psi}{\partial F_{iL}}F_{jL}, \tag{2.68}$$
$$M_{ij}=\rho\epsilon_{ipq}d_{(P)p}\frac{\partial\psi}{\partial d_{(P)q},_L}F_{jL}, \tag{2.69}$$
$$h_i(F_{iL},d_{(P)j},d_{(P)j},_L,\theta,\theta,_L)\theta,_i\le 0. \tag{2.70}$$

Moreover, condition (2.65) is equivalent to the objectivity principle.

2.6 The Objectivity Principle

In this section we analyze the consequences of the objectivity principle. This principle requires that the constitutive equations are invariant with respect to any change of rigid frame of reference. In particular, the free energy ψ of an elastic micropolar continuum has to satisfy the condition

$$\psi(\mathbf{F}, \mathbf{d}_{(P)}, \mathbf{d}_{(P)},_L, \theta) = \psi(\mathbf{QF}, \mathbf{Qd}_{(P)}, \mathbf{Qd}_{(P)},_L, \theta) \tag{2.71}$$

for any orthogonal matrix $\mathbf{Q}$.

Theorem 2.1
The free energy $\psi(\mathbf{F}, \mathbf{d}_{(P)}, \mathbf{d}_{(P)},_L, \theta)$ satisfies the objectivity principle if and only if it has the form

$$\psi = \widetilde{\psi}(\mathbf{C}, \mathbf{F}^T\mathbf{d}_{(P)}, \mathbf{F}^T\mathbf{d}_{(P)},_L, \theta), \tag{2.72}$$

where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ is the left Cauchy–Green tensor.

PROOF Let $\mathbf{F} = \mathbf{RU}$ be the polar decomposition of the deformation gradient, where $\mathbf{R}$ is orthogonal and $\mathbf{U}$ is symmetric (see Chap. 3 of [16]). If (2.71) is satisfied for any orthogonal matrix $\mathbf{Q}$, then, upon choosing $\mathbf{Q} = \mathbf{R}^T$, we have

$$\psi(\mathbf{F}, \mathbf{d}_{(P)}, \mathbf{d}_{(P)},_L, \theta) = \psi(\mathbf{U}, \mathbf{R}^T\mathbf{d}_{(P)}, \mathbf{R}^T\mathbf{d}_{(P)},_L, \theta).$$

However, it follows from $\mathbf{F} = \mathbf{RU}$ that $\mathbf{R}^T = \mathbf{U}^{-1}\mathbf{F}^T$ and $\mathbf{C} = \mathbf{U}^2$, so we have proved (2.72).

Conversely, under a change of rigid frame of reference $\mathcal{R} \to \mathcal{R}'$ determined by the orthogonal matrix $\mathbf{Q}$, we have $\mathbf{F}' = \mathbf{QF}$, $\mathbf{d}'_{(P)} = \mathbf{Qd}_{(P)}$, $\mathbf{C}'_{(P)} = \mathbf{F}'^T\mathbf{F}' = \mathbf{C}$, and $\mathbf{F}'^T\mathbf{d}'_{(P)} = \mathbf{F}^T\mathbf{d}_{(P)}$. Then, from (2.72), we derive

$$\begin{aligned} \psi(\mathbf{F}', \mathbf{d}'_{(P)}, \mathbf{d}'_{(P)},_L, \theta) &= \widetilde{\psi}(\mathbf{C}', \mathbf{F}'^T\mathbf{d}'_{(P)}, \mathbf{F}'^T\mathbf{d}'_{(P)},_L, \theta) \\ \widetilde{\psi}(\mathbf{C}, \mathbf{F}^T\mathbf{d}_{(P)}, \mathbf{F}^T\mathbf{d}_{(P)},_L, \theta) &= \psi(\mathbf{F}, \mathbf{d}_{(P)}, \mathbf{d}_{(P)},_L, \theta), \end{aligned}$$

and the theorem is proved. ∎

The objectivity of the constitutive equations of η, $\mathbf{T}$, and $\mathbf{M}$ is guaranteed by the following theorem.

Theorem 2.2
If the constitutive equations of ψ, η, $\mathbf{T}$, and $\mathbf{M}$ satisfy the dissipation principle and ψ is objective, then the constitutive relations of η, $\mathbf{T}$, and $\mathbf{M}$ given by (2.67)–(2.69) are also objective.

PROOF We limit ourselves to proving that the constitutive equation of $\mathbf{T}$ is objective. In fact, (2.68) and (2.72) lead us to

$$\begin{aligned}
T'_{ij} &\equiv f_{ij}(\mathbf{F}', \mathbf{d}'_{(P)}, \mathbf{d}'_{(P)},_L, \theta) = \rho F'_{jL}\frac{\partial\widetilde{\psi}}{\partial F'_{iL}}(\mathbf{C}', \mathbf{F}'^T\mathbf{d}'_{(P)}, \mathbf{F}'^T\mathbf{d}'_{(P)},_L, \theta) \\
&= \rho Q_{jh}Q_{ik}F_{hL}\frac{\partial\widetilde{\psi}}{\partial F_{kL}}(\mathbf{C}, \mathbf{F}^T\mathbf{d}_{(P)}, \mathbf{F}^T\mathbf{d}_{(P)},_L, \theta) \\
&= Q_{ik}Q_{jh}f_{ij}(\mathbf{F}, \mathbf{d}_{(P)}, \mathbf{d}_{(P)},_L, \theta),
\end{aligned}$$

and we have proved the objectivity. ∎

We now want to prove the following theorem.

Theorem 2.3
Equation (2.65) *is equivalent to the objectivity of* $\psi(\mathbf{F}, \mathbf{d}_{(P)}, \mathbf{d}_{(P)},_L, \theta)$ *provided that the dissipation principle is satisfied.*

PROOF Let $\mathbf{Q}(\epsilon)$ be an arbitrary family of orthogonal matrices that depend on the real variable ϵ such that $\mathbf{Q}(0) = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. The objectivity of $\psi(\mathbf{F}, \mathbf{d}_{(P)}, \mathbf{d}_{(P)},_L, \theta)$ is then equivalent to requiring that the relation

$$\psi(\mathbf{Q}(\epsilon)\mathbf{F}, \mathbf{Q}(\epsilon)\mathbf{d}_{(P)}, \mathbf{Q}(\epsilon)\mathbf{d}_{(P)},_L, \theta) = \psi(\mathbf{F}, \mathbf{d}_{(P)}, \mathbf{d}_{(P)},_L, \theta) \tag{2.73}$$

is satisfied for any value of ϵ and for any function $\mathbf{Q}(\epsilon)$.

Equation (2.73) can also be written as

$$\psi(Q_{hj}(\epsilon)F_{jL}, Q_{hj}(\epsilon)d_{(P)j}, Q_{hj}(\epsilon)d_{(P)j},_L, \theta) = \psi(F_{jL}, d_{(P)j}, d_{(P)j},_L, \theta). \tag{2.74}$$

Differentiating (2.73) with respect to ϵ and evaluating the result at $\epsilon = 0$, we obtain

$$W_{hj}\left(\frac{\partial\psi}{\partial F_{jL}}F_{hL} + \frac{\partial\psi}{\partial d_{(P)j}}d_{(P)h} + \frac{\partial\psi}{\partial d_{(P)j},_L}d_{(P)h},_L\right) = 0, \tag{2.75}$$

where $W_{hj} = Q'_{hj}(0)$. On the other hand, Taylor's expansion of $\mathbf{Q}(\epsilon)$ at $\epsilon = 0$ is

$$\mathbf{Q}(\epsilon) = \mathbf{W}\epsilon + O(\epsilon),$$

and from the orthogonality condition $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$ we find that $\mathbf{W}$ is a skew-symmetric tensor. If $w_n = \frac{1}{2}\epsilon_{hij}W_{ij}$ is the adjoint vector of W_{ij} then we have $W_{hj} = \epsilon_{hjp}w_p$, and (2.74) becomes:

$$\epsilon_{hjp}w_p\left(\frac{\partial\psi}{\partial F_{jL}}F_{hL} + \frac{\partial\psi}{\partial d_{(P)j}}d_{(P)h} + \frac{\partial\psi}{\partial d_{(P)j},_L}d_{(P)h},_L\right) = 0.$$

Finally, we obtain (2.65) from the above relation when the arbitrariness of w_i is taken into account. $\blacksquare$

Further simplification of the constitutive equations (2.66)–(2.70) is achieved by reducing the number of independent variables. First, we introduce the material tensor

$$\Pi_{PL} = F_{hL} d_{(P)h} \quad P, L = 1, 2, 3, \tag{2.76}$$

and the material pseudotensor

$$\Gamma_{PL} = \frac{1}{2} \epsilon_{PQR} d_{(R)h} d_{(Q)h},_L \quad P, L = 1, 2, 3, \tag{2.77}$$

where ϵ_{PQR} are the components of the Levi–Civita tensor in Cartesian coordinates. Relation (2.77) implies that

$$\begin{aligned} \epsilon_{MNP} \Gamma_{PL} &= \epsilon_{MNP} \epsilon_{PQR} d_{(R)h} d_{(Q)h},_L \\ &= \frac{1}{2} (\delta_{MQ} \delta_{NR} - \delta_{MR} \delta_{NQ}) d_{(R)h} d_{(Q)h},_L \\ &= \frac{1}{2} (d_{(N)h} d_{(M)h},_L - d_{(M)h} d_{(N)h},_L). \end{aligned}$$

However, from $d_{(N)h} d_{(M)h} = \delta_{NM}$ we can derive that $d_{(N)h} d_{(M)h},_L = -d_{(M)h} d_{(N)h},_L$, so the above relation assumes the form

$$\epsilon_{MNP} \Gamma_{PL} = d_{(N)h} d_{(M)h},_L \,. \tag{2.78}$$

Now, starting from the condition $d_{(P)h} d_{(P)k} = \delta_{hk}$ and (2.78), it is easy to show that all of the independent variables appearing in (2.72) can be expressed in terms of $\mathbf{\Pi}$ and $\mathbf{\Gamma}$. In fact,

$$\begin{aligned} C_{LM} &= F_{hL} F_{hM} = F_{hL} d_{(P)h} d_{(P)k} F_{kM} = \Pi_{PL} \Pi_{PM}, \\ F_{hM} d_{(P)h},_L &= F_{hM} d_{(Q)h} d_{(Q)k} d_{(P)k},_L = \Pi_{QM} \epsilon_{PQR} \Gamma_{RL}. \end{aligned}$$

It is worth highlighting the use of the tensor $\mathbf{\Pi}$ to describe the deformation of the micropolar elastic system. First, we recall that a vector $d\mathbf{X}$, which has its origin at a point $\mathbf{X}$ of the reference configuration, is transformed into the vector $d\mathbf{x}$,

$$dx_h = F_{hL} dX_L, \tag{2.79}$$

which is applied at the point $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ of the actual configuration.

On the other hand, we also have

$$\Pi_{ML} dX_L = F_{hL} d_{(M)h} dX_L = d\mathbf{x} \cdot \mathbf{d}_{(M)},$$

so that

$$d\mathbf{x} = (d\mathbf{x} \cdot \mathbf{d}_{(M)}) \mathbf{d}_{(M)}. \tag{2.80}$$

We conclude that *the tensor* $\mathbf{\Pi}$ *describes the local deformation with respect to the directors*. Consequently, if the evolutions of both the directors and the tensor $\mathbf{\Pi}$ are known, the whole deformation of the micropolar system is obtained.

Finally, we can write (2.72) in the form

$$\psi = \widehat{\psi}(\mathbf{\Pi}, \mathbf{\Gamma}, \theta). \tag{2.81}$$

On the other hand,

$$\frac{\partial \Pi_{PL}}{\partial F_{hM}} = \delta_{LM} d_{(P)h}, \tag{2.82}$$

$$\frac{\Gamma_{PL}}{\partial d_{(R)h,M}} = \frac{1}{2}\epsilon_{PRQ} d_{(Q)h} \delta_{LM}, \tag{2.83}$$

so that (2.68) becomes

$$T_{ij} = \rho \frac{\partial \psi}{\partial F_{iL}} F_{jL} = \rho d_{(R)i} \frac{\partial \widehat{\psi}}{\partial \Pi_{RL}} F_{jL} = \rho \chi_{iR} \frac{\partial \widehat{\psi}}{\partial \Pi_{RL}} F_{jL}. \tag{2.84}$$

In order to express (2.69) in terms of $\mathbf{\Pi}$ and $\mathbf{\Gamma}$, we observe that (2.83) allows us to write

$$\begin{aligned} M_{ij} &= \rho \epsilon_{ipq} d_{(R)p} \frac{\partial \psi}{\partial d_{(R)q,L}} F_{jL} \\ &= \frac{1}{2} \rho \epsilon_{ipq} \epsilon_{RNK} d_{(N)q} d_{(K)p} \frac{\partial \widehat{\psi}}{\partial \Gamma_{RL}} F_{jL} \\ &= \frac{1}{2} \epsilon_{RNK} (\mathbf{d}_{(N)} \times \mathbf{d}_{(K)})_i \frac{\partial \widehat{\psi}}{\partial \Gamma_{RL}} F_{jL}, \end{aligned}$$

where $(\mathbf{d}_{(N)} \times \mathbf{d}_{(K)})_i$ denotes the ith component of the cross product $(\mathbf{d}_{(N)} \times \mathbf{d}_{(K)})$. On the other hand, it is easy to verify that $\epsilon_{RNK} \mathbf{d}_{(N)} \times \mathbf{d}_{(K)} = 2\mathbf{d}_{(R)}$, $R = 1, 2, 3$, so that we finally obtain

$$M_{ij} = \rho d_{(R)i} \frac{\partial \widehat{\psi}}{\partial \Gamma_{(R)L}} F_{jL} = \rho \chi_{iR} \frac{\partial \widehat{\psi}}{\partial \Gamma_{(R)L}} F_{jL}. \tag{2.85}$$

In particular, if we assume that the micropolar system is isotropic, its free energy $\psi(\mathbf{\Pi}, \mathbf{\Gamma})$ is an isotropic function of its variables:

$$\psi(\mathbf{\Pi}, \mathbf{\Gamma}) = \psi(\mathbf{Q}\mathbf{\Pi}\mathbf{Q}^T, \mathbf{Q}\mathbf{\Gamma}\mathbf{Q}^T) \tag{2.86}$$

for any orthogonal matrix $\mathbf{Q}$. Equivalently, we can say that ψ is a function (see Appendix A of [59])

$$\psi = \widehat{\psi}(I_1, \ldots, I_{15}) \tag{2.87}$$

of the 15 invariants

$$
\begin{gathered}
I_1 = \mathrm{tr}\mathbf{\Pi}, \quad I_2 = \frac{1}{2}\mathrm{tr}\mathbf{\Pi}^2, \quad I_3 = \frac{1}{3}\mathrm{tr}\mathbf{\Pi}^3, \\
I_4 = \frac{1}{2}\mathrm{tr}\mathbf{\Pi}\mathbf{\Pi}^T, \quad I_5 = \mathrm{tr}\mathbf{\Pi}^2\mathbf{\Pi}^T, \quad I_6 = \frac{1}{2}\mathrm{tr}\mathbf{\Pi}^2(\mathbf{\Pi}^T)^2, \\
I_7 = \mathrm{tr}\mathbf{\Pi}\mathbf{\Gamma}, \quad I_8 = \mathrm{tr}\mathbf{\Pi}\mathbf{\Gamma}^2, \quad I_9 = \mathrm{tr}\mathbf{\Pi}^2\mathbf{\Gamma}, \\
I_{10} = \mathrm{tr}\mathbf{\Gamma}, \quad I_{11} = \frac{1}{2}\mathrm{tr}\mathbf{\Gamma}^2, \quad I_{12} = \frac{1}{3}\mathrm{tr}\mathbf{\Gamma}^3, \\
I_{13} = \frac{1}{2}\mathrm{tr}\mathbf{\Gamma}\mathbf{\Gamma}^T, \quad I_{14} = \mathrm{tr}\mathbf{\Gamma}^2\mathbf{\Gamma}^T, \quad I_{15} = \frac{1}{2}\mathrm{tr}\mathbf{\Gamma}^2(\mathbf{\Gamma}^T)^2.
\end{gathered}
\tag{2.88}
$$

For isotropic microelastic bodies, the relations (2.84) and (2.85) become

$$
T_{ij} = \rho d_{(R)i} \sum_{h=1}^{15} \frac{\partial\widehat{\psi}}{\partial I_h} \frac{\partial I_h}{\partial \Pi_{(R)}} F_{jL}, \tag{2.89}
$$

$$
M_{ij} = \rho d_{(R)i} \sum_{h=1}^{15} \frac{\partial\widehat{\psi}}{\partial I_h} \frac{\partial I_h}{\partial \Gamma_{(R)L}} F_{jL}. \tag{2.90}
$$

2.7 Some Remarks on Boundary Value Problems

In this section we analyze the mathematical problem posed by the equations that describe the evolution of a microelastic system. First, it is worth listing all of the equations involved:

$$
\begin{aligned}
&\dot{\rho} + \rho v_{i,i} = 0, \\
&\dot{J}_{ij} = (J_{pj}\epsilon pil + J_{ip}\epsilon pjl)\omega_l, \\
&\rho\dot{v}_i = T_{ij,j} + \rho b_i, \\
&\rho\dot{k}_i = M_{ij,j} + \rho l_i, \\
&\rho\dot{\epsilon} = T_{il}v_{i,l} + M_{il}\omega_{i,l} - \tau_l\omega_l - h_{l,l} + \rho r.
\end{aligned}
$$

If the evolution of the microelastic system is described by the variables $x_i(\mathbf{X}, t)$, $\varphi_i(\mathbf{X}, t)$, $\rho(\mathbf{X}, t)$, $J_{ij}(\mathbf{X}, t)$, $\omega_i(\mathbf{X}, t)$, and $\theta(\mathbf{X}, t)$, and the constitutive relations (2.84), (2.85) are taken into account, then we have to consider the kinematic relation (2.17) in order to balance the number of equations and unknowns:

$$
\omega_i = \widetilde{\Lambda}_{ij}\dot{\varphi}_j.
$$

This is the viewpoint adopted by Eringen and Kafadar in [58]. However, if the variables φ_i that describe the micromotion are replaced with the orthogonal matrix $\chi = (\chi_{iL})$, we must add the equations (2.3) and (2.4):

$$\chi_{iL}\chi_{jL} = \delta_{ij},$$
$$\omega_i = \Lambda_{ikL}\dot{d}_{Lk}.$$

We may hope to attain a well-posed boundary value problem by giving the initial data for the unknown fields as well as suitable boundary conditions for the stress and the stress couple. For instance, we could assign the conditions

$$\mathbf{T}\cdot\mathbf{N} = \widehat{\mathbf{t}},$$
$$\mathbf{M}\cdot\mathbf{N} = \widehat{\mathbf{m}},$$

where $\widehat{\mathbf{t}}$ and $\widehat{\mathbf{m}}$ are given functions of the boundary ∂C of the region C occupied by the continuous system, and $\mathbf{N}$ is the outward unit vector normal to ∂C. It is important to note that the second condition requires a deeper discussion concerning its physical meaning (see also Sect. 2.8).

2.8 Asymmetric Elasticity

In this section we analyze the asymmetric elasticity case, where

- The directors move with the angular velocity of the macroscopic motion
$$\omega_i = \frac{1}{2}\epsilon_{ikh}v_{h,k} \tag{2.91}$$
- The intrinsic angular momentum $\mathbf{k}$ vanishes.

The analysis developed in the previous sections does not hold for the above condition, for two main reasons:

- We do not need to determine the motion of the directors
- The quantities $v_{i,j}$, $\omega_{i,j}$, and ω_i are not independent, so the dissipation principle does not lead us to the same restrictions on the constitutive equations.

In order to analyze the consequences of these remarks in detail, we start by noting that the first of them implies that

$$\mathbf{d}_{(P)}(\mathbf{X},t) = \mathbf{R}\mathbf{d}_{(P)}(\mathbf{X},0), \tag{2.92}$$

where $\mathbf{R}$ is the orthogonal matrix in the polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$. Since $\mathbf{R} = \mathbf{F}(\sqrt{\mathbf{F}^T\mathbf{F}})^{-1}$ (see Chap. 3 of [16]), we conclude that

$$\psi(\mathbf{F}, \mathbf{d}_{(P)}, \mathbf{d}_{(P)},_L, \theta) = \widehat{\psi}(\mathbf{F}, \mathbf{F},_L, \theta). \tag{2.93}$$

Differentiating the right-hand side of this equation with respect to time, we derive

$$\dot{\widehat{\psi}} = \frac{\partial \widehat{\psi}}{\partial F_{iL}} \dot{F}_{iL} + \frac{\partial \widehat{\psi}}{\partial F_{iL,M}} \dot{F}_{iL,M} + \frac{\partial \widehat{\psi}}{\partial \theta} \dot{\theta}.$$

On the other hand, we also have

$$\begin{aligned} \dot{F}_{iL,M} &= (v_{i,L})_{,M} = (v_{i,h} F_{hL})_{,M} \\ &= v_{i,h} F_{hL,M} + v_{i,hM} F_{hL} \\ &= v_{i,h} F_{hL,M} + v_{i,hj} F_{hL} F_{jM}, \end{aligned} \tag{2.94}$$

and consequently $\dot{\widehat{\psi}}$ can also be written as follows:

$$\dot{\widehat{\psi}} = \left(\frac{\partial \widehat{\psi}}{\partial F_{iL}} F_{jL} + \frac{\partial \widehat{\psi}}{\partial F_{iL,M}} F_{jL,M} \right) v_{i,j} + \frac{\partial \widehat{\psi}}{\partial F_{iL,M}} F_{hL} F_{jM} v_{i,hj} + \frac{\partial \widehat{\psi}}{\partial \theta} \dot{\theta}. \tag{2.95}$$

Finally, the dissipation inequality (2.39) becomes

$$\begin{aligned} &- \rho \left(\frac{\partial \widehat{\psi}}{\partial \theta} + \eta \right) \dot{\theta} + \left(T_{ij} - \rho \frac{\partial \widehat{\psi}}{\partial F_{iL}} F_{jL} - \rho \frac{\partial \widehat{\psi}}{\partial F_{iL,M}} F_{jL,M} \right) v_{i,j} \\ &+ M_{ij} \omega_{i,j} - \rho \frac{\partial \widehat{\psi}}{\partial F_{kL,M}} F_{hL} F_{jM} v_{k,hj} \\ &- \epsilon_{lji} T_{ij} \omega_l - \frac{h_i}{\theta} \theta_{,i} \geq 0. \end{aligned} \tag{2.96}$$

From (2.91), we derive that

$$\epsilon_{lji} \omega_l = v_{[i,j]}, \tag{2.97}$$

$$v_{[k,h]j} = \epsilon_{khi} \omega_{i,j}, \tag{2.98}$$

and the dissipation inequality (2.96) can be written as follows:

$$\begin{aligned} &- \rho \left(\frac{\partial \widehat{\psi}}{\partial \theta} + \eta \right) \dot{\theta} + \left(T_{ij} - \rho \frac{\partial \widehat{\psi}}{\partial F_{iL}} F_{jL} - \rho \frac{\partial \widehat{\psi}}{\partial F_{iL,M}} F_{jL,M} \right) v_{i,j} \\ &+ \left(M_{ij} - \rho \epsilon_{khi} \frac{\partial \widehat{\psi}}{\partial F_{[hL,M}} F_{k]L} F_{jM} \right) \omega_{i,j} \\ &- T_{[ij]} v_{[i,j]} - \left(\rho \frac{\partial \widehat{\psi}}{\partial F_{(hL,M}} F_{k)L} F_{jM} \right) v_{(i,h)j} - \frac{h_i}{\theta} \theta_{,i} \geq 0. \end{aligned} \tag{2.99}$$

In this inequality, the quantities $\omega_{i,j}$ cannot be chosen arbitrarily since $\omega_{i,i} = (1/2) \epsilon_{ihk} v_{k,hi} = 0$ due to the symmetry of $v_{k,hi}$ with respect to the indices h, i as well as the skew symmetry of ϵ_{ihk} with respect to the same indices. Therefore, we can choose the quantities $\omega_{i,j}$ arbitrarily if we add the term $\lambda\delta_{ij}\omega_{i,j}$ to the inequality (2.99), where λ is a Lagrangian multiplier.

In conclusion, we obtain the final form of the dissipation inequality:

$$-\rho\left(\frac{\partial\widehat{\psi}}{\partial\theta}+\eta\right)\dot{\theta}+\left(T_{(ij)}-\rho\frac{\partial\widehat{\psi}}{\partial F_{(iL}}F_{j)L}-\rho\frac{\partial\widehat{\psi}}{\partial F_{(iL,M}}F_{j)L,M}\right)v_{(i,j)}$$
$$-\left(\rho\frac{\partial\widehat{\psi}}{\partial F_{[iL}}F_{j]L}-\rho\frac{\partial\widehat{\psi}}{\partial F_{[iL,M}}F_{j]L,M}\right)v_{[i,j]} \tag{2.100}$$
$$+\left(M_{ij}+\lambda\delta_{ij}-\rho\epsilon_{khi}\frac{\partial\widehat{\psi}}{\partial F_{[hL,M}}F_{k]L}F_{jM}\right)\omega_{i,j}$$
$$-\left(\rho\frac{\partial\widehat{\psi}}{\partial F_{(hL,M}}F_{k)L}F_{jM}\right)v_{(i,h)j}-\frac{h_i}{\theta}\theta_{,i}\geq 0, \tag{2.101}$$

from which we derive

$$\eta=\frac{\partial\widehat{\psi}}{\partial\theta}, \tag{2.102}$$
$$T_{(ij)}=\rho\frac{\partial\widehat{\psi}}{\partial F_{(iL}}F_{j)L}+\rho\frac{\partial\widehat{\psi}}{\partial F_{(iL,M}}F_{j)L,M}, \tag{2.103}$$
$$M_{ij}=-\lambda\delta_{ij}+\rho\epsilon_{khi}\frac{\partial\widehat{\psi}}{\partial F_{[hL,M}}F_{k]L}F_{jM}, \tag{2.104}$$

$$\frac{\partial\widehat{\psi}}{\partial F_{[iL}}F_{j]L}+\rho\frac{\partial\widehat{\psi}}{\partial F_{[iL,M}}F_{j]L,M}=0, \tag{2.105}$$
$$\frac{\partial\widehat{\psi}}{\partial F_{(hL,M}}F_{k)L}F_{jM}=0, \tag{2.106}$$
$$-\frac{h_i}{\theta}\theta_{,i}\geq 0. \tag{2.107}$$

Concerning the above relations, note that:

- The free energy is a potential for the entropy, the symmetric part of the stress $\mathbf{T}$, and the stress couple tensor $\mathbf{M}$
- The skew-symmetric part of $\mathbf{T}$ is not determined
- The symmetric part of the stress couple tensor is not determined owing to the presence of the Lagrangian multiplier
- Condition (2.105) is equivalent to the objectivity principle (see Sect. 2.6)

- Condition (2.106) represents a further restriction on the constitutive equation of ψ.

It is important to understand that the evolutions of all of the parts of $\mathbf{T}$ and $\mathbf{M}$ that are not determined by the free energy ψ are unaffected. In order to demonstrate the validity of this statement, we first note that the equation of momentum balance can be written as follows:

$$\rho\dot{v}_i = T_{(ij),j} + T_{[ij],j} + \rho b_i. \tag{2.108}$$

On the other hand, the angular momentum equation is used to define the skew-symmetric part of the stress tensor

$$T_{[ij],j} = -\frac{1}{2}\epsilon_{hij}M_{hk,k}. \tag{2.109}$$

Consequently, (2.108) becomes

$$\rho\dot{v}_i = T_{(ij),j} - \frac{1}{2}\epsilon_{hij}M_{hk,k} + \rho b_i. \tag{2.110}$$

Moreover, the undetermined part $\lambda\delta_{h,k}$ of $M_{h,k}$ does not contribute to (2.110) since

$$\epsilon_{hij}(\lambda\delta_{hk})_{,k} = \epsilon_{hij}\lambda_{,hj} = 0.$$

When the constitutive equations (2.102)–(2.104) are given, the mass continuity equation and (2.11) allow us (at least in principle), together with suitable boundary and initial data, to determine the unknowns $\rho(\mathbf{X}, t)$ and $\mathbf{x}(\mathbf{X}, t)$.

Chapter 3

Continuous System with a Nonmaterial Interface

3.1 Introduction

In this chapter, we propose a *macroscopic* model of phase transitions. It is essential to note that any macroscopic model of these phenomena does not describe why a phase transition takes place nor the modifications it produces in the matter at a microscopic level. It is only able to describe *how* it takes place.

In proposing a macroscopic model of phase transitions in a continuous system $\mathbb{S}$, it is possible to adopt either of two points of view. We can start from the experimental evidence that the different phases are separated by very narrow boundary layers across which the fields associated with $\mathbb{S}$ vary continuously but sharply. Due to the high values assumed by the gradients of the fields in these layers, the ordinary constitutive equations of the body undergoing the phase transition are assumed to depend very weakly on the higher-order gradients of the above fields. Consequently, the higher-order derivatives in the local equations of balance are multiplied by small coefficients. It is well known that this circumstance implies the existence of boundary layers whose localization and form depend on both the equations and the boundary data (see [71, 72]). While this approach describes the phase transitions in a sufficiently realistic way, it exhibits so many mathematical difficulties that only a few simple problems can be solved.

In the other macroscopic approach, a model of two or more continuous media separated by interfaces is adopted (see, for instance, [73]–[79]). The basic idea of this approach is that we can replace the narrow boundary layers between the phases with surfaces of discontinuity for the volume fields. However, we must also associate some physical attributes with these surfaces that, in a certain way, evoke the complex structure of the fields in the layers used as their substitutes. This is achieved by associating with the interface some surface fields obtained by suitable averaging of the volume fields in the boundary layers.

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We adopt this second approach in this chapter. Consequently, we give the balance equations of a continuous system in which two phases C_1 and C_2 are separated by a *nonmaterial* interface Σ. This characteristic of Σ results from the fact that, during a phase change, Σ consists of various different particles at any given instant. Consequently, we must describe the evolution of a nonmaterial surface at which physical fields are defined at any instant. Here, we adopt a simplified version of the model proposed in [75]–[79], which allows us to describe some interesting physical situations that are analyzed in subsequent chapters.

3.2 Velocity of a Moving Surface

Let

$$\mathbf{r} = \mathbf{r}(u^1, u^2, t), \quad (u^1, u^2) \in \Omega \subset \Re^2 \tag{3.1}$$

be the parametric equation of a surface $\Sigma(t)$ moving in the Euclidean three-dimensional space $\Re^3$ (see Appendix B).

It is important to note that, if the moving surface $\Sigma(t)$ is *nonmaterial* (i.e., if it does not consist of the same particles throughout), then the surface coordinates (u^1, u^2) only have a geometric meaning. Consequently, there is nothing to prevent us from adopting new surface coordinates (U^1, U^2) at any instant t such that

$$u^\alpha = u^\alpha(U^1, U^2, t), \tag{3.2}$$

where the functions (3.2) are invertible for any t.

If $\Sigma(t)$ is a moving surface and (U^1, U^2) and (u^1, u^2) are two arbitrary surface coordinates on $\Sigma(t)$ related by the transformation (3.2), then we can write

$$\mathbf{r} = \mathbf{r}(u^1, u^2, t) = \hat{\mathbf{r}}(U^1, U^2, t). \tag{3.3}$$

It is possible to associate with the points of $\Sigma(t)$ a velocity *that depends on the parametrization* via the definitions

$$\mathbf{c} = \left(\frac{\partial \mathbf{r}}{\partial t}\right)_{u^\alpha}, \quad \hat{\mathbf{c}} = \left(\frac{\partial \hat{\mathbf{r}}}{\partial t}\right)_{U^\Delta}. \tag{3.4}$$

From (3.2)–(3.4), we find that

$$\hat{\mathbf{c}} = \mathbf{c} + \left(\frac{\partial u^\alpha}{\partial t}\right)_{U^\Delta} \mathbf{a}_\alpha, \tag{3.5}$$

where $(\mathbf{a}_\alpha) = (\partial \mathbf{r}/\partial u^\alpha)$ is the surface basis relative to the coordinate curves u^α (see Appendix B). From (3.5) we derive that *the component of the* *velocity along the unit vector* $\mathbf{n}$*, normal to* $\Sigma(t)$*, is independent of the parametrization*

$$\hat{c}_n = \hat{\mathbf{c}} \cdot \mathbf{n} = \mathbf{c} \cdot \mathbf{n} = c_n. \tag{3.6}$$

Remark Definitions (3.4) are only meaningful for a material surface. In fact, in this case, two pairs of curvilinear coordinates (u^α), (U^α) characterize the same material particles of Σ if (3.2) does not depend on t. Consequently, (3.4) supplies a velocity that is independent of the *material* coordinates (u^α). However, for a nonmaterial moving surface, a physical meaning can only be attributed to the normal velocity.

If the surface $\Sigma(t)$ is given by the equation $f(\mathbf{r}, t) = 0$, where $\mathbf{r} = \mathbf{r}(u^1, u^2, t)$, then we have

$$\left(\frac{\partial f}{\partial t}\right)_{\mathbf{r}} + \nabla f \cdot \left(\frac{\partial \mathbf{r}}{\partial t}\right)_{u^\alpha} = \left(\frac{\partial f}{\partial t}\right)_{\mathbf{r}} + |\nabla f| \mathbf{n} \cdot \mathbf{c} = 0,$$

and the normal speed of Σ can be written as follows:

$$c_n = -\frac{\partial f}{\partial t} \frac{1}{|\nabla f|}. \tag{3.7}$$

Remark Let we suppose that physical considerations lead us to associate a velocity $\mathbf{V}$ with the particles, which instantaneously occupy the nonmaterial moving surface $\Sigma(t)$. Then, it is always possible to choose the parameter (u^α) on the surface in such a way that the component $\mathbf{c}_s$ of the velocity $\mathbf{c}$ along $\Sigma(t)$ in this parametrization is equal to the corresponding component $\mathbf{V}_s$ of $\mathbf{V}$. In fact, under a change of parameters given by (3.2), we have the following identity according to (3.5) and (3.6):

$$\hat{\mathbf{c}}_s(U^\Delta, t) = \mathbf{c}_s(u^\alpha, t) + \left(\frac{\partial u^\alpha}{\partial t}\right)_{U^\Delta} \mathbf{a}_\alpha,$$

which, in components, becomes

$$\hat{c}_s^\Delta(U^\Delta, t)\hat{\mathbf{a}}_\Delta = c_s^\alpha(u^\alpha, t)\mathbf{a}_\alpha + \left(\frac{\partial u^\alpha}{\partial t}\right)_{U^\Delta} \mathbf{a}_\alpha.$$

However, we also have $\hat{\mathbf{a}}_\Delta = \frac{\partial \mathbf{r}}{\partial U^\Delta} = \frac{\partial u^\alpha}{\partial U^\Delta}\mathbf{a}_\alpha$. Therefore, to satisfy the above condition, it is sufficient for the functions (3.2) to be solutions of the differential system

$$\frac{\partial u^\alpha}{\partial t} = \hat{c}_s^\Delta \frac{\partial u^\alpha}{\partial U^\Delta} - V_s^\alpha. \tag{3.8}$$

In what follows, when we can attribute a physical meaning to the velocity of the particles that instantaneously lie on the nonmaterial surface $\Sigma(t)$, the equation $\mathbf{r}(u^\alpha, t)$ is referred to these coordinates, so that

$$\mathbf{c}_s = \mathbf{V}_s. \tag{3.9}$$

3.3 Velocity of a Moving Curve

Let $\Gamma(t)$ be a moving nonmaterial curve on $\Sigma(t)$. If $\Sigma(t)$ has the parametric representation $\mathbf{r} = \hat{\mathbf{r}}(U^\Delta, t)$, the parametric equation of $\Gamma(t)$ becomes

$$\mathbf{r} = \hat{\mathbf{r}}(U^\Delta(\mu, t), t) = \hat{\varphi}(\mu, t). \tag{3.10}$$

The velocity of $\Gamma(t)$ in the parametrization we have chosen is given by

$$\hat{\mathbf{C}} = \left(\frac{\partial \hat{\varphi}}{\partial t}\right)_\mu = \left(\frac{\partial \hat{\mathbf{r}}}{\partial t}\right)_{U^\Delta} + \frac{\partial \hat{\mathbf{r}}}{\partial U^\Delta}\left(\frac{\partial U^\Delta}{\partial t}\right)_\mu,$$

so that, considering (3.5), we can write

$$\hat{\mathbf{C}} = \hat{\mathbf{c}} + \left(\frac{\partial U^\Delta}{\partial t}\right)_\mu \hat{\mathbf{a}}_\Delta. \tag{3.11}$$

If we introduce different parameters for $\Sigma(t)$ and $\Gamma(t)$ (i.e., if we use the new parameters $u^\alpha = u^\alpha(U^\Delta, t)$ for $\Sigma(t)$ and $\mu = \mu(\Lambda, t)$ for $\Gamma(t)$), then we have a different equation for $\Gamma(t)$:

$$\mathbf{r} = \mathbf{r}(u^\alpha(U^\Delta(\mu(\lambda, t), t), t) = \hat{\mathbf{r}}(U^\Delta(\mu, t), t), \tag{3.12}$$

and the velocity $\mathbf{C}$ of $\Gamma(t)$ becomes

$$\begin{aligned}\mathbf{C} = \left(\frac{\partial \mathbf{r}}{\partial t}\right)_\lambda &= \left(\frac{\partial \mathbf{r}}{\partial t}\right)_{u^\alpha} + \\ &\frac{\partial \mathbf{r}}{\partial u^\alpha}\left[\left(\frac{\partial u^\alpha}{\partial t}\right)_{U^\Delta} + \frac{\partial u^\alpha}{\partial U^\Delta}\left(\left(\frac{\partial U^\Delta}{\partial t}\right)_\mu + \left(\frac{\partial U^\Delta}{\partial \mu}\right)\left(\frac{\partial \mu}{\partial t}\right)_\lambda\right)\right] \\ &= \mathbf{c} + \left(\frac{\partial u^\alpha}{\partial t}\right)_{U^\Delta}\mathbf{a}_\alpha + \left[\left(\frac{\partial U^\Delta}{\partial t}\right)_\mu + \frac{\partial U^\Delta}{\partial \mu}\left(\frac{\partial \mu}{\partial t}\right)_\lambda\right]\hat{\mathbf{a}}_\Delta.\end{aligned}$$

If we note that $(\partial U^\Delta/\partial \mu)\hat{\mathbf{a}}_\Delta$ is a tangent vector $\hat{\tau}$ to the curve $\Gamma(t)$ in the parametrization $U^\Delta = U^\Delta(\mu, t)$, then we obtain the relation

$$\mathbf{C} = \mathbf{c} + \left(\frac{\partial u^\alpha}{\partial t}\right)_{U^\Delta}\mathbf{a}_\alpha + \left(\frac{\partial U^\Delta}{\partial t}\right)_\mu \hat{\mathbf{a}}_\Delta + \left(\frac{\partial \mu}{\partial t}\right)_\lambda \hat{\tau}. \tag{3.13}$$

By comparing (3.13) and (3.11), and taking into account (3.5) and (3.6), we deduce that

$$\mathbf{C} - \hat{\mathbf{C}} = \left(\frac{\partial \mu}{\partial t}\right)_\lambda \hat{\tau}. \tag{3.14}$$

This result shows that *the projections of* $\mathbf{C}$ *along the unit vector* ν *(which is tangent to* $\Sigma(t)$ *and orthogonal to* $\Gamma(t)$*) and along the unit normal* $\mathbf{n}$ *to* $\Sigma(t)$ *are independent of the parametrization of* $\Sigma(t)$ *and* $\Gamma(t)$:

$$\mathbf{C} \cdot \nu = \hat{\mathbf{C}} \cdot \nu, \tag{3.15}$$

$$\mathbf{C} \cdot \mathbf{n} = \hat{\mathbf{C}} \cdot \mathbf{n}. \tag{3.16}$$

3.4 Thomas' Derivative and Other Formulae

In this section, we define the Thomas derivative of a field $\mathbf{F}(\mathbf{r}, t)$ assigned to the moving surface $\Sigma(t)$, and prove some useful differentiation formulae. The *Thomas derivative* of $\mathbf{F}(\mathbf{r}, t)$, which is defined by the limit

$$\frac{\delta \mathbf{F}}{\delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{F}(\mathbf{r} + c_n \mathbf{n} \Delta t, t + \Delta t) - \mathbf{F}(\mathbf{r}, t)}{\Delta t}, \tag{3.17}$$

denotes the rate change of the field $\mathbf{F}$ with respect to an observer moving along the unit normal $\mathbf{n}$ to $\Sigma(t)$ with normal speed c_n. It is evident that we can write

$$\frac{\delta \mathbf{F}}{\delta t} = \left(\frac{\partial \mathbf{F}}{\partial t}\right)_{\mathbf{r}} + c_n \mathbf{n} \cdot \nabla_{\mathbf{r}} \mathbf{F}. \tag{3.18}$$

To find the expression for the Thomas derivative when the field $\mathbf{F}(\mathbf{r}, t)$ is expressed as a function $\widetilde{\mathbf{F}}(u^\alpha, t)$ of the surface parameters, we note that

$$\dot{\mathbf{F}} = \left(\frac{\partial \mathbf{F}}{\partial t}\right)_{\mathbf{r}} + \mathbf{c} \cdot \nabla_{\mathbf{r}} \mathbf{F} = \left(\frac{\partial \widetilde{\mathbf{F}}}{\partial t}\right)_{u^\alpha}.$$

Consequently, (3.18) becomes

$$\frac{\delta \mathbf{F}}{\delta t} = \left(\frac{\partial \widetilde{\mathbf{F}}}{\partial t}\right)_{u^\alpha} - \mathbf{c}_s \cdot \nabla_{\mathbf{r}} \mathbf{F}. \tag{3.19}$$

On the other hand, if (x^i) is any curvilinear system of spatial coordinates, we have (see (2.28) of [16])

$$\begin{aligned} \nabla_{\mathbf{r}} \mathbf{F} \cdot \mathbf{a}_\alpha &= (\mathbf{F}_{,i} \otimes \mathbf{e}^i) \cdot \mathbf{a}_\alpha \\ &= \mathbf{F}_{,i}\, \mathbf{e}^i \cdot \left(\frac{\partial x^j}{\partial u^\alpha} \mathbf{e}_j\right) = \widetilde{\mathbf{F}}_{,\alpha}\,, \end{aligned} \tag{3.20}$$

where $(\mathbf{e}_i)$ and $(\mathbf{e}^i)$ denote the natural basis and the dual basis, respectively, associated with the coordinates (x^i). (3.19) can then also be written as follows (see B.52):

$$\frac{\delta \mathbf{F}}{\delta t} = \left(\frac{\partial \widetilde{\mathbf{F}}}{\partial t}\right)_{u^\alpha} - c_s^\alpha \widetilde{\mathbf{F}}_{,\alpha} = \left(\frac{\partial \widetilde{\mathbf{F}}}{\partial t}\right)_{u^\alpha} - \mathbf{c}_s \cdot \nabla_s \widetilde{\mathbf{F}}. \tag{3.21}$$

We conclude this section by evaluating the time derivatives of the metric coefficients $a_{\alpha\beta}(u^{\alpha,t})$. First, from the evident relations

$$\dot{\mathbf{a}}_\alpha = \dot{\mathbf{r}}_{,\alpha} = \mathbf{c}_{,\alpha} \, ,$$

we derive that

$$\dot{a}_{\alpha\beta} = \mathbf{c}_{,\alpha} \cdot \mathbf{a}_\beta + \mathbf{a}_\alpha \cdot \mathbf{c}_{,\beta} \, .$$

When we take into account (B.42), the above equation assumes the form

$$\dot{a}_{\alpha\beta} = 2(c_{(\alpha;\beta)} - b_{\alpha\beta} c_n) \equiv 2\eta_{\alpha\beta}. \tag{3.22}$$

From this result, we immediately obtain

$$\dot{a} = \frac{\partial a}{\partial a_{\alpha\beta}} \dot{a}_{\alpha\beta} = a a^{\alpha\beta} \dot{a}_{\alpha\beta} = 2a\eta_\alpha^\alpha. \tag{3.23}$$

Finally, if the hypotheses discussed in the second remark at the end of Sect. 3.2 are satisfied, and we adopt the coordinates that lead to (3.9) on the moving surface, then (3.19) assumes the form

$$\frac{\delta \mathbf{F}}{\delta t} = \left(\frac{\partial \widetilde{\mathbf{F}}}{\partial t}\right)_{u^\alpha} - V_s^\alpha \widetilde{\mathbf{F}}_{,\alpha} \, , \tag{3.24}$$

whereas in (3.22) it is

$$\eta_{\alpha\beta} = 2(V_{s(\alpha;\beta)} - b_{\alpha\beta} c_n). \tag{3.25}$$

3.5 Differentiation Formulae

In this section we prove an important differentiation formula which allows us to formulate the equations of balance for a continuous system with an interface.

Let $\mathbf{f}(\mathbf{x}, t)$ be a tensor field defined in a moving region $V(t) \subset \Re^3$ for any moment in time $t \in [t_0, t_1]$. Suppose that the region $V(t)$ is divided into two parts $V^-(t)$ and $V^+(t)$ by a moving regular surface $\Sigma(t)$, where

$V^+(t)$ is the part containing the unit vector $\mathbf{n}$ normal to $\Sigma(t)$ (see Fig. 3.1). The field $\mathbf{f}(\mathbf{x}, t)$ is regular, at any t, in $V(t) - \Sigma(t)$, but may exhibit finite discontinuities across $\Sigma(t)$. We denote the finite limit values of $\mathbf{f}(\mathbf{x}, t)$ from $V^-(t)$ and $V^+(t)$, respectively, at any point $\mathbf{r} \in \Sigma(t)$ by $\mathbf{f}^-(\mathbf{x}, t)$ and $\mathbf{f}^+(\mathbf{x}, t)$. Finally, we assume that these limits are smooth functions of their arguments.

Let

$$g(\mathbf{x}, t) = 0 \tag{3.26}$$

and

$$\mathbf{r} = \mathbf{G}(u^\alpha, t), \quad \alpha = 1, 2, \tag{3.27}$$

be the implicit and parametric representations of $\Sigma(t)$, respectively. In the above section we have already noted that the normal speed (3.7) of $\Sigma(t)$ does not depend on the parametrization.

Let $V(t)$ be a moving volume whose boundary $\partial V(t)$ is represented by one of the following equations:

$$p(\mathbf{x}, t) = 0, \tag{3.28}$$

$$\mathbf{r} = \mathbf{P}(u^\alpha, t). \tag{3.29}$$

If $\mathbf{N}$ is the unit normal vector to $\partial V(t)$ and

$$C_N = \left(\frac{\partial \mathbf{P}}{\partial t}\right)_{u^\alpha} \cdot \mathbf{N} = -\frac{1}{|\nabla p|}\frac{\partial p}{\partial t} \tag{3.30}$$

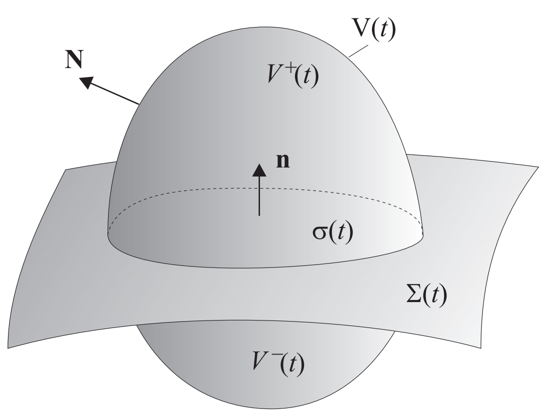

Fig. 3.1 A moving surface $\Sigma(t)$ that intersects a moving volume $V(t)$

is the normal speed of $\partial V(t)$, then the following differentiation formula holds (see Sect. 3.6 of [16]):

$$\frac{d}{dt}\int_{V(t)} \mathbf{f}\, dv = \int_{V(t)} \frac{\partial \mathbf{f}}{\partial t}\, dv + \int_{\partial V(t)} \mathbf{f} C_N\, d\sigma - \int_{\sigma(t)} [[\mathbf{f}]] c_n\, d\sigma, \tag{3.31}$$

where $[[\mathbf{f}]] = \mathbf{f}^+ - \mathbf{f}^-$ is the jump in $\mathbf{f}$ across the singular surface $\Sigma(t)$, and $\sigma(t) = \Sigma(t) \bigcap V(t)$ is the part of $\Sigma(t)$ that is instantaneously contained in $\partial V(t)$.

Now we prove another differentiation formula that is related to moving nonmaterial surfaces (see [77]). Let $\Sigma^*(t)$ be a moving surface whose points satisfy one of the following equations:

$$\varphi(\mathbf{r}, t) = 0, \quad \mathbf{r} = \mathbf{\Phi}(u^\alpha, t), \tag{3.32}$$

and let $\Gamma(t)$ be a closed curve

$$u^\alpha = \psi^\alpha(s, t) \tag{3.33}$$

that moves on $\Sigma^*(t)$ with a velocity (see (3.11))

$$\mathbf{W} = \frac{\partial \psi^\alpha}{\partial t}\mathbf{a}_\alpha + \frac{\partial \mathbf{\Phi}}{\partial t}, \tag{3.34}$$

where $\mathbf{a}_\alpha = (\partial \mathbf{\Phi}/\partial u^\alpha)$ is the holonomic basis associated with the coordinates (u^α).

Let $\gamma(t)$ be a moving curve on $\Sigma(t)$ along which the surface $\Sigma(t)$ is not regular. We suppose that the part $\Sigma(t)$ of $\Sigma^*(t)$ that is inside $\Gamma(t)$ is divided into two parts ($\Sigma^-(t)$ and $\Sigma^+(t)$) by the moving curve $\gamma(t)$ (see Fig. 3.2). In order to define $\Sigma^+(t)$, we introduce at any point on $\Gamma(t)$ the unit vector ν_Γ, which is orthogonal to $\Gamma(t)$, tangential to $\Sigma(t)$ and oriented towards the exterior of $\Sigma(t)$. Further, we consider the unit vector $\mathbf{N}$ that is orthogonal to $\Sigma(t)$, and we choose the unit tangent vector τ to $\Gamma(t)$ in such a way that the frame $\tau, \nu_{\mathbf{\Gamma}}, \mathbf{N}$ is counterclockwise. This choice introduces an orientation on $\Gamma(t)$, which, in turn, determines two opposite orientations along $\gamma(t)$ according to whether $\gamma(t)$ is considered part of the boundary of either of the two parts into which Σ is divided by $\gamma(t)$. We can arbitrarily define one of them as a positive orientation and the other as a negative orientation. Then we identify $\Sigma^+(t)$ with the part of $\Sigma(t)$ that contains ν_γ^+ (see Fig. 3.2).

Let $\mathbf{F}(\mathbf{x}, t)$ be a field on the surface $\Sigma^*(t)$, possibly the restriction on $\Sigma^*(t)$ of a three-dimensional field $\mathbf{f}(\mathbf{x}, t)$ assigned in a volume containing $\Sigma^*(t)$. The field $\mathbf{F}(\mathbf{x}, t)$ is assumed to be regular in $\Sigma(t) - \gamma(t)$, with finite discontinuities across $\gamma(t)$. The notations $\mathbf{F}^-(\mathbf{x}, t)$ and $\mathbf{F}^+(\mathbf{x}, t)$ denote the finite values of the limit values of $\mathbf{F}(\mathbf{x}, t)$ upon going from $\Sigma^-(t)$ and $\Sigma^+(t)$, respectively, to a point $\mathbf{r} \in \gamma(t)$.

We can now prove the following formula:

$$\frac{d}{dt}\int_{\Sigma(t)} \mathbf{F}(\mathbf{r},t)\,d\sigma = \int_{\Sigma(t)} \left(\frac{\delta \mathbf{F}}{\delta t} - 2\mathbf{F}HC_N\right) d\sigma + \int_{\Gamma(t)} \mathbf{F}W_\Gamma\,ds - \int_{\gamma(t)} [[\mathbf{F}]]\,w_\gamma\,ds. \tag{3.35}$$

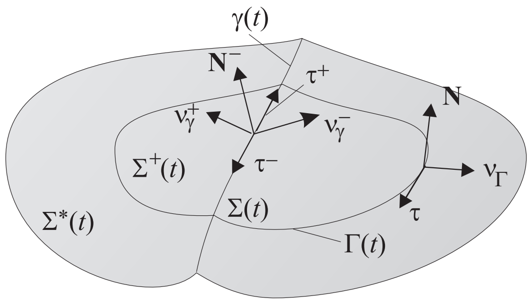

Fig. 3.2 A moving curve $\gamma(t)$ of singular points on the moving surface $\Sigma^*(t)$

In (3.35), $W_\Gamma = \mathbf{W}\cdot\nu_\Gamma$, $w_\gamma = \mathbf{w}\cdot\nu_\gamma$, where $\mathbf{W}$ and $\mathbf{w}$ denote the velocities of the curves $\Gamma(t)$ and $\gamma(t)$, respectively. Moreover, H is the mean curvature of $\Sigma(t)$ (see Appendix B). We explicitly note that the velocities $\mathbf{W}$ and $\mathbf{w}$ depend on the parametrization of $\Sigma(t)$, $\Gamma(t)$, and $\gamma(t)$. However, in Sects. 3.2 and 3.3 we showed that W_Γ and w_γ, and consequently the right-hand side of (3.35), do not depend on the parametrization.

In order to prove (3.35), we first suppose that $\Sigma(t)\bigcup\gamma(t) = \emptyset$. Then we denote by $\Sigma_0(t+\Delta t)$ the surface whose points $\mathbf{R}$ are defined by the following equation:

$$\mathbf{R} = \mathbf{r} + C_N\Delta t\mathbf{N}, \quad \forall\mathbf{r}\in\Sigma(t); \tag{3.36}$$

i.e., the surface obtained by moving the points of $\Sigma(t)$ along the unit normal $\mathbf{N}$ of the quantity $C_N\Delta t$. These points obey the equation of the surface $\Sigma(t+\Delta t)$ to within second-order terms in Δt since, due to (3.32), we have

$$\varphi(\mathbf{r} + C_N\Delta t\mathbf{N}, t+\Delta t) = C_N\Delta t\mathbf{N}\cdot\nabla\varphi + \frac{\partial\varphi}{\partial t}\Delta t + O(\Delta t), \tag{3.37}$$

and the sum of the first two terms on the right-hand side of the above relation vanishes due to the definition of the normal speed of $\Sigma(t)$:

$$C_N = -\frac{1}{|\nabla\varphi|}\frac{\partial\varphi}{\partial t}.$$

On the other hand, to within second-order terms, the surface $\Sigma(t+\Delta t)$ is obtained from $\Sigma_0(t+\Delta t)$ by taking into account the new position of $\Gamma(t+\Delta t)$ on it. All of these remarks and the formula $d\sigma = \sqrt{a(\mathbf{r},t)}\, du^1 du^2$ (see Appendix B) lead us to the relation

$$\int_{\Sigma(t+\Delta t)} \mathbf{F}(\mathbf{r}(t+\Delta t), t+\Delta t)\, d\sigma = \int_{\Gamma(t)} \mathbf{F} W_\Gamma \Delta t\, ds \tag{3.38}$$

$$+ \int_{\Sigma(t)} \mathbf{F}(\mathbf{r}(t) + C_N \Delta t \mathbf{N}(t), t+\Delta t) \sqrt{a(\mathbf{r}(t) + C_N \Delta t \mathbf{N}(t), t+\Delta t)}\, du^1 du^2.$$

Subtracting the term

$$\int_{\Sigma(t)} \mathbf{F}(\mathbf{r},t) \sqrt{a(\mathbf{r},t)}\, du^1 du^2$$

from both sides of (3.38) and dividing by Δt in the limit $\Delta t \to 0$, we obtain

$$\frac{d}{dt}\int_{\Sigma(t)} \mathbf{F}(\mathbf{r},t)\, d\sigma = \int_{\Sigma(t)} \left(\sqrt{a}\frac{\delta \mathbf{F}}{\delta t} + \mathbf{F}\frac{\delta}{\delta t}\sqrt{a} \right) du^1 du^2 + \int_{\Gamma(t)} \mathbf{F} W_\Gamma\, ds. \tag{3.39}$$

On the other hand, if we denote the vectors of the holonomic basis associated with the coordinates u^α by $\mathbf{a}_\alpha = \partial \mathbf{\Phi}/\partial u^\alpha$, we have

$$\frac{\delta}{\delta t}\sqrt{a} = \frac{1}{2\sqrt{a}}\frac{\delta a}{\delta t} = \frac{1}{2\sqrt{a}} a a^{\alpha\beta} \frac{\delta a_{\alpha\beta}}{\delta t} = \sqrt{a} a^{\alpha\beta} \mathbf{a}_\alpha \cdot \frac{\delta \mathbf{a}_\alpha}{\delta t}. \tag{3.40}$$

It then remains to evaluate the Thomas derivative $\delta \mathbf{a}_\alpha/\delta t$. From Eq. 3.37 for the surface $\Sigma_0(t+\Delta t)$ we obtain the relation

$$\mathbf{a}_\alpha(\mathbf{R}, t+\Delta t) = \mathbf{a}_\alpha(\mathbf{r},t) + (C_N \mathbf{N})_{,\alpha}\, \Delta t + O(\Delta t),$$

which, taking into account the Gauss–Weingarten formulae (see Appendix B), implies that

$$\frac{\delta \mathbf{a}_\alpha}{\delta t} = (C_N)_{,\alpha}\, \mathbf{N} - C_N b_\alpha^\lambda \mathbf{a}_\lambda, \tag{3.41}$$

where $b_{\lambda\alpha}$ denote the coefficients of the second quadratic form of the surface $\Sigma(t)$ (see Appendix B). In view of (3.41), (3.40) becomes

$$\frac{\delta}{\delta t}\sqrt{a} = -\sqrt{a} a^{\alpha\beta} C_N b_{\alpha\beta} = -2\sqrt{a} H C_N. \tag{3.42}$$

This last result allows us to write (3.39) in the following form:

$$\frac{d}{dt}\int_{\Sigma(t)} \mathbf{F}(\mathbf{r},t)\, d\sigma = \int_{\Sigma(t)} \left(\frac{\delta \mathbf{F}}{\delta t} - 2\mathbf{F} H C_N \right) \sqrt{a}\, du^1 du^2$$

$$+ \int_{\Gamma(t)} \mathbf{F} W_\Gamma\, ds. \tag{3.43}$$

It is now sufficient to apply (3.43) to both of the surfaces $\Sigma^-(t)$ and $\Sigma^+(t)$ to obtain (3.35).

3.6 Balance Laws

In this section we consider the general form of a *balance law* for a continuous system $\mathbb{B}$ with a nonmaterial interface $\Sigma(t)$, which is singular for the volume fields associated with $\mathbb{B}$. This surface is assumed to carry *regular* surface fields that describe the material properties of $\Sigma(t)$.

A general balance law for $\mathbb{B}$ has the form

$$\begin{aligned}\frac{d}{dt}\int_{V(t)} \mathbf{f}\, dv + \frac{d}{dt}\int_{\sigma(t)} \mathbf{F}\, d\sigma = &- \int_{\partial V(t)} [\mathbf{f}\otimes(\mathbf{v}-\mathbf{C}) + \varphi]\cdot \mathbf{N}\, d\sigma \\ &- \int_{\partial\sigma(t)} [\mathbf{F}\otimes(\mathbf{V}_s - \mathbf{W}_s) + \mathbf{\Psi}]\cdot \nu_\sigma\, ds \\ &+ \int_{V(t)} \mathbf{r}\, dv + \int_{\sigma(t)} \mathbf{R}\, d\sigma, \end{aligned} \tag{3.44}$$

where $\sigma(t) = V(t)\bigcup\Sigma(t)$, ν_σ is the unit vector that is tangent to $\sigma(t)$ and normal to $\partial\sigma(t)$, $\mathbf{v}$ and $\mathbf{V}$ are the transport velocities of the fields $\mathbf{f}$ and $\mathbf{F}$, respectively, $\mathbf{C}$ is the velocity of the boundary $\partial V(t)$, and $\mathbf{W}$ denotes the speed of the boundary $\partial\sigma(t)$. The fields φ and $\mathbf{\Psi}$ are the nonconvective fluxes of $\mathbf{f}$ and $\mathbf{F}$, respectively. Finally, $\mathbf{r}$ is the supply per unit volume of $\mathbf{f}$, and $\mathbf{R}$ represents the supply per unit area of the field $\mathbf{F}$.

In the balance law (3.44), the volume $V(t)$ can move in an arbitrary way. The most familiar choices correspond to a material volume (i.e., to a volume moving with the velocity $\mathbf{v}$ of the particles of $\mathbb{B}$) or to a fixed volume. In the first case $\mathbf{v}\cdot\mathbf{N} = \mathbf{C}\cdot\mathbf{N}$, and the balance law becomes

$$\begin{aligned}\frac{d}{dt}\int_{V(t)} \mathbf{f}\, dv + \frac{d}{dt}\int_{\sigma(t)} \mathbf{F}\, d\sigma = &- \int_{\partial V(t)} \varphi\cdot \mathbf{N}\, d\sigma \\ &- \int_{\partial\sigma(t)} [\mathbf{F}\otimes(\mathbf{V}_s - \mathbf{W}_s) + \mathbf{\Psi}]\cdot \nu_\sigma\, ds \\ &+ \int_{V(t)} \mathbf{r}\, dv + \int_{\sigma(t)} \mathbf{R}\, d\sigma. \end{aligned} \tag{3.45}$$

On the other hand, for a fixed volume ($\mathbf{C} = \mathbf{0}$), we have

$$\begin{aligned}\frac{d}{dt}\int_{V} \mathbf{f}\, dv + \frac{d}{dt}\int_{\sigma(t)} \mathbf{F}\, d\sigma = &- \int_{\partial V} [\mathbf{f}\otimes\mathbf{v} + \varphi]\cdot \mathbf{N}\, d\sigma \\ &- \int_{\partial\sigma(t)} [\mathbf{F}\otimes(\mathbf{V}_s - \mathbf{W}_s) + \mathbf{\Psi}]\cdot \nu_\sigma\, ds \\ &+ \int_{V} \mathbf{r}\, dv + \int_{\sigma(t)} \mathbf{R}\, d\sigma. \end{aligned} \tag{3.46}$$

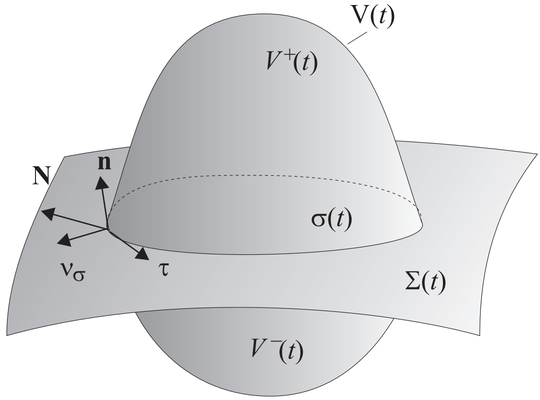

Fig. 3.3 A moving surface $\Sigma(t)$ intersecting the moving volume $V(t)$

It is important to note that the velocity $\mathbf{W}$ of the curve $\partial\sigma(t)$ depends on both the velocity $\mathbf{C}$ of the boundary $\partial V(t)$ and the velocity $\mathbf{c}$ of the singular surface $\Sigma(t)$. This dependence can be made explicit by imposing that the curve $\partial\sigma(t)$ belongs to $\partial V(t)$ and $\Sigma(t)$. In fact, from this condition, we have

$$\mathbf{W}\cdot\mathbf{N} = C_N, \quad \mathbf{W}\cdot\mathbf{n} = c_n. \tag{3.47}$$

Introducing the basis $(\tau, \mathbf{n}, \nu_\sigma)$ along the curve $\partial\sigma(t)$ (see Fig. 3.3), the velocity $\mathbf{W}$ of the points of $\partial\sigma$ can be written as follows:

$$\mathbf{W} = W_\tau \tau + W_\nu \nu_\sigma + W_n \mathbf{n}, \tag{3.48}$$

and from (3.47) we obtain

$$W_\nu \nu_\sigma \cdot \mathbf{N} + W_n \mathbf{n}\cdot\mathbf{N} = C_N, \quad W_n = c_n.$$

On the other hand, since $\mathbf{N} = (\nu_\sigma \cdot \mathbf{N})\nu_\sigma + (\mathbf{n}\cdot\mathbf{N})\mathbf{n}$, we have

$$\nu\cdot\mathbf{N} = \sqrt{1-(\mathbf{n}\cdot\mathbf{N})^2},$$

and the vector $\mathbf{W}$ permits the following representation:

$$\mathbf{W} = W_\tau \tau + \frac{C_N - c_n \mathbf{n}\cdot\mathbf{N}}{\sqrt{1-(\mathbf{n}\cdot\mathbf{N})^2}}\nu_\sigma + c_n \mathbf{n}, \tag{3.49}$$

where $C_N = v_n$ for material volumes and $C_N = 0$ for fixed volumes.

In order to localize the general balance laws, we must take into account the differentiation formulae (3.31) and (3.35), as well as the generalized

Gauss theorems for the integrals over $\partial V(t)$ and $\partial\sigma(t)$. Further, when applying the Gauss theorem to the integral over $\partial\sigma(t)$, we can use (B.58) since, without any loss of generality, we can assume that $\mathbf{\Psi}\cdot\mathbf{n}=\mathbf{0}$. Finally, again with the aim of simplifying the model, we suppose that $\Sigma(t)$ is regular, together with the fields defined on it. The balance law, expressed in any of the forms (3.44), (3.45), and (3.46), leads to the following local equations and jump conditions via a standard method:

$$\frac{\partial\mathbf{f}}{\partial t}+\nabla\cdot(\mathbf{f}\otimes\mathbf{v}+\varphi)-\mathbf{r}=\mathbf{0},\quad \text{in C}-\Sigma, \tag{3.50}$$

$$\begin{aligned}\frac{\delta\mathbf{F}}{\delta t}&+\nabla_s\cdot(\mathbf{F}\otimes\mathbf{V}_s+\mathbf{\Psi})-2Hc_n\mathbf{F}-\mathbf{R}\\ &=[[\mathbf{f}(c_n-v_n)-\varphi\cdot\mathbf{n}]],\quad \text{on }\Sigma.\end{aligned} \tag{3.51}$$

We can write the above equations in a more convenient form. First, we note that

$$\nabla_s\cdot(\mathbf{F}\otimes\mathbf{V}_s)=(\mathbf{F}\otimes\mathbf{V}_s)_{,\alpha}\cdot\mathbf{a}^\alpha=\mathbf{F}_{,\alpha}V_s^\alpha+\mathbf{F}\nabla_s\cdot\mathbf{V}_s. \tag{3.52}$$

Let us introduce onto the moving nonmaterial interface $\Sigma(t)$ the coordinates (u^α), which lead to (3.9). Recalling (3.24) and (3.52), and introducing the notation $\widetilde{\mathbf{F}}=\mathbf{F}(u^\alpha,t)$, we transform the above equations into the following:

$$\frac{\partial\mathbf{f}}{\partial t}+\nabla\cdot(\mathbf{f}\otimes\mathbf{v}+\varphi)-\mathbf{r}=\mathbf{0},\quad \text{in C}-\Sigma, \tag{3.53}$$

$$\begin{aligned}\left(\frac{\partial\widetilde{\mathbf{F}}}{\partial t}\right)_{u^\alpha}&+\widetilde{\mathbf{F}}\nabla_s\cdot\mathbf{V_s}+\nabla_s\cdot\mathbf{\Psi}-2Hc_n\widetilde{\mathbf{F}}-\mathbf{R}\\ &=[[\mathbf{f}(c_n-v_n)-\varphi\cdot\mathbf{n}]],\quad \text{on }\Sigma.\end{aligned} \tag{3.54}$$

A balance law, in integral or local form, has no physical meaning if we do not provide explicit expressions for the quantities that appear in it. Sufficiently general balance equations for a continuous system with an interface are given by (3.24), (3.27), (3.31), (3.34) and (3.35) in [78] and [79]. We consider balance equations based on the terms listed in Table 3.1 here.

The columns of Table 3.1 refer to mass conservation, momentum balance, angular momentum balance, and energy balance, respectively. Consequently, $\rho,\mathbf{v},\mathbf{t},\mathbf{b},e$, and $\mathbf{h}$ denote the mass density, the velocity, the stress tensor, the body force, and the heat flux in the bulk regions, respectively. Finally, ρ_s is the surface mass density, $\mathbf{T}$ is the surface stress tensor, E is the specific energy of the interface Σ, $\mathbf{V}$ is the velocity of the particles that lie instantaneously on Σ, and $\mathbf{a}_\alpha$ is a vector of the holonomic basis tangent to Σ (see Appendix B). The term that appears in the last row of the column, related to the angular momentum balance, is due to the average processes at the boundary layer that allow us to associate surface quantities with the interface (see p. 57 of [78]).

$\mathbf{f}$	ρ	$\rho\mathbf{v}$	$\rho\mathbf{r}\times\mathbf{v}$	$\rho(\frac{1}{2}v^2+e)$
$\mathbf{F}$	ρ_s	$\rho_s\mathbf{V}$	$\rho_s\mathbf{r}\times\mathbf{V}$	$\rho_s(\frac{1}{2}V^2+E)$
φ	0	$-\mathbf{t}$	$-\mathbf{r}\times\mathbf{t}$	$-\mathbf{v}\cdot\mathbf{t}+\mathbf{h}$
$\mathbf{\Psi}$	0	$-\mathbf{T}$	$-\mathbf{r}\times\mathbf{T}$	$-\mathbf{V}\cdot\mathbf{T}$
$\mathbf{r}$	0	$\rho\mathbf{b}$	$\rho\mathbf{r}\times\mathbf{b}$	$\rho\mathbf{v}\cdot\mathbf{b}$
$\mathbf{R}$	0	0	$\rho_s(c_n-V_n)\mathbf{V}\times\mathbf{n}$	0

Table 3.1 Fields included in the balance laws

If we introduce the terms present in the first column of Table 3.1 into the general equations (3.53) and (3.54), we obtain the local equation and the jump condition relating to **mass conservation**:

$$\dot{\rho}+\rho\nabla\cdot\mathbf{v}=0,\quad \text{in}\,\mathrm{C}-\Sigma, \tag{3.55}$$

$$\left(\frac{\partial\rho_s}{\partial t}\right)_{u^\alpha}+\rho_s\nabla_s\cdot\mathbf{V}_s-2Hc_n\rho_s=[[\rho(c_n-v_n)]]\,,\quad \text{on}\,\Sigma. \tag{3.56}$$

Similarly, after introducing the data from the second column into (3.53) and (3.54) and some simple calculations in which we take (3.55) and (3.56) into account, we deduce the following form of the **momentum balance**:

$$\rho\dot{\mathbf{v}}-\nabla\cdot\mathbf{t}-\rho\mathbf{b}=\mathbf{0},\quad \text{in}\,\mathrm{C}-\Sigma, \tag{3.57}$$

$$\rho_s\left(\frac{\partial\mathbf{V}}{\partial t}\right)_{u^\alpha}-\nabla_s\cdot\mathbf{T}=[[\rho(\mathbf{v}-\mathbf{V})(c_n-v_n)+\mathbf{t}\cdot\mathbf{n}]]\,,\quad \text{on}\,\Sigma. \tag{3.58}$$

Proceeding in the same way with the data in the third column, and again taking into account (3.55)–(3.58), we derive the local equations and jump conditions relating to the **angular momentum and energy balance**:

$$\mathbf{r}\times\mathbf{t}=\mathbf{0},\quad \text{in}\,\mathrm{C}-\Sigma, \tag{3.59}$$

$$\mathbf{a}_\alpha\times\mathbf{T}\cdot\mathbf{a}^\alpha=\mathbf{0},\quad \text{on}\,\Sigma. \tag{3.60}$$

We note that the proof for the existence of the surface stress tensor $\mathbf{T}$ is similar to the proof for the existence of the Cauchy stress tensor $\mathbf{t}$. Moreover, the surface stress tensor allows us to write the force $\mathbf{t}_\sigma$ acting along any curve γ on Σ in the following way:

$$\int_\gamma \mathbf{T}\cdot\nu_\sigma\,ds,$$

where ν_σ is the unit normal to γ. From the above relation, it follows that we can always assume that

$$\mathbf{T} \cdot \mathbf{n} = \mathbf{0}. \tag{3.61}$$

Moreover, it is well known that condition (3.59), which can be written in components as

$$\epsilon_{ijl} t_{jl} = 0,$$

where ϵ_{ijl} is the Levi–Civita tensor, is equivalent to the symmetry of the stress tensor in the bulk regions:

$$\mathbf{t} = \mathbf{t}^T. \tag{3.62}$$

On the other hand, in view of (3.61), in the basis $(\mathbf{a}_\alpha, \mathbf{n})$, where $(\mathbf{a}_\alpha)$ is the holonomic basis associated with the curvilinear coordinates (u^α) on Σ, the condition (3.60) can also be written in the form

$$\mathbf{a}_\alpha \times (T^{\lambda\beta}\mathbf{a}_\lambda \otimes \mathbf{a}_\beta + T^{3\lambda}\mathbf{n} \otimes \mathbf{a}_\lambda) \cdot \mathbf{a}^\alpha = \mathbf{0},$$

which leads us to the condition

$$\mathbf{a}_\alpha \times T^{\lambda\alpha}\mathbf{a}_\lambda + \mathbf{a}_\alpha \times T^{3\alpha}\mathbf{n} = \mathbf{0}. \tag{3.63}$$

Since the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{n}$ form a counterclockwise basis, the above equation becomes

$$(T^{12} - T^{21})\mathbf{n} - T^{31}\mathbf{a}_2 + T^{32}\mathbf{a}_1 = \mathbf{0},$$

so that

$$T^{12} = T^{21}, \quad T^{31} = T^{32} = 0. \tag{3.64}$$

In other words, the stress tensor is symmetric and it only generates forces that are tangential to Σ.

In view of (3.64) and (B.51), the surface divergence of $\mathbf{T}$ is

$$\nabla_s \cdot \mathbf{T} = T^{\alpha\gamma}_{;\alpha}\mathbf{a}_\gamma + T^{\alpha\gamma}b_{\alpha\gamma}\mathbf{n}, \tag{3.65}$$

so that (3.57) is equivalent to the following equations along the plane tangential to Σ and along the unit normal $\mathbf{n}$, respectively:

$$\rho_s \left(\frac{\partial \mathbf{V}}{\partial t}\right)^\alpha - T^{\alpha\gamma}_{;\alpha} = [[\rho(v^\gamma - V^\gamma)(c_n - v_n) + (\mathbf{tn})^\gamma]]\,, \tag{3.66}$$

$$\rho_s \left(\frac{\partial \mathbf{V}}{\partial t}\right)^n - T^{\alpha\gamma}b_{\alpha\gamma} = [[\rho(v_n - V_n)(c_n - v_n) - p]]\,, \tag{3.67}$$

where we have introduced the pressure

$$p = -\mathbf{n} \cdot \mathbf{tn}. \tag{3.68}$$

Before moving on, it is convenient to prove that

$$\nabla_s \cdot (\mathbf{V} \cdot \mathbf{T}) = T^{\alpha\beta}\sigma_{\alpha\beta} + \mathbf{V} \cdot \nabla_s \cdot \mathbf{T}. \tag{3.69}$$

Due to (B.49) and (3.65), we have

$$\nabla_s \cdot (\mathbf{V} \cdot \mathbf{T}) = (\mathbf{V} \cdot \mathbf{T})_{,\alpha} \cdot \mathbf{a}^\alpha = \mathbf{V}_{,\alpha} \cdot \mathbf{T}\mathbf{a}^\alpha + \mathbf{V} \cdot \nabla_s \cdot \mathbf{T}. \tag{3.70}$$

On the other hand, we also have

$$\mathbf{V}_{,\alpha} \cdot \mathbf{T}\mathbf{a}^\alpha = T^{\alpha\beta}(V_{\alpha;\beta} - b_{\alpha\beta}V_n) \equiv T^{\alpha\beta}\sigma_{\alpha\beta}, \tag{3.71}$$

and (3.69) is proved.

Finally, if we introduce the data from the fourth column of the table into (3.53) and (3.54) and take into account the conservation of mass, the momentum balance and (3.69), we can change the **energy balance** into the following form:

$$\rho\dot{\epsilon} - \mathrm{tr}(\mathbf{t} \otimes \mathbf{v}) + \nabla \cdot \mathbf{h} = 0, \quad \text{in}\, \mathrm{C} - \Sigma, \tag{3.72}$$

$$\rho_s \left(\frac{\partial E}{\partial t}\right)_{u^\alpha} - T^{\alpha\beta}\sigma_{\alpha\beta} = \left[\left[\rho\left(\frac{1}{2}(v - V)^2 + e\right)(c_n - v_n) + (\mathbf{v} - \mathbf{V}) \cdot \mathbf{t}\mathbf{n} - \mathbf{h} \cdot \mathbf{n}\right]\right], \quad \text{on}\, \Sigma. \tag{3.73}$$

3.7 Entropy Inequality and Gibbs Potential

Together with the balance equations, we must take into account the second law of thermodynamics (see [16]), which leads us to the following local inequality in the bulk phases:

$$\rho\theta\dot{\eta} + \nabla \cdot \mathbf{h} - \frac{1}{\theta}\mathbf{h} \cdot \nabla\theta \geq 0, \tag{3.74}$$

where η is the specific entropy and θ the absolute temperature. Moreover, at the interface Σ, when θ is continuous across Σ, the second law implies the following jump conditions:

$$\theta\rho_s \left(\frac{\partial S}{\partial t}\right)_{u^\alpha} - [[\rho\theta(\eta - S)(c_n - v_n) - \mathbf{h} \cdot \mathbf{n}]] \geq 0, \tag{3.75}$$

where S is the surface specific entropy.

It is well known (see [16]) that the local energy balance (3.72) and the associated jump condition (3.73) allow us to transform (3.74) into the *reduced dissipation inequality*:

$$-\rho(\dot{\psi}+\eta\dot{\theta})+\mathrm{tr}(\mathbf{t}\otimes\mathbf{v})-\frac{1}{\theta}\mathbf{h}\cdot\nabla\theta\geq 0, \tag{3.76}$$

where

$$\psi=e-\theta\eta \tag{3.77}$$

is the specific free energy in the bulk phases. On the other hand, by eliminating the term $[[\mathbf{h}\cdot\mathbf{n}]]$ via (3.73) and (3.75), we can derive the following jump condition:

$$\begin{aligned}&-\rho_s\left(\dot{\Psi}+S\dot{\theta}\right)+T^{\alpha\beta}V_{\alpha;\beta}\\&+\left[\left[\rho\left(\frac{1}{2}(\mathbf{v}-\mathbf{V})^2+\psi-\Psi\right)(c_n-v_n)+(\mathbf{v}-\mathbf{V})\cdot\mathbf{tn}\right]\right]\geq 0,\end{aligned} \tag{3.78}$$

where

$$\Psi=E-\theta S \tag{3.79}$$

is the free energy per unit area, and $\dot{A}=(\partial A/\partial t)_{u^\alpha}$.

We now assign the following particular constitutive equations to the surface quantities Ψ, S, and $\mathbf{T}$:

$$\Psi=\Psi(a_{\alpha\beta},\theta), \tag{3.80}$$

$$S=S(a_{\alpha\beta},\theta), \tag{3.81}$$

$$\mathbf{T}=\mathbf{T}(a_{\alpha\beta},\theta), \tag{3.82}$$

and we impose the condition that they must satisfy the dissipation principle (i.e., inequality (3.78)) in any process. In order to recognize the consequences of the dissipation principle, we use (3.22) and (3.25) to write the time derivative of Ψ as follows:

$$\begin{aligned}\dot{\Psi}&=\frac{\partial\Psi}{\partial\theta}\dot{\theta}+\frac{\partial\Psi}{\partial a_{\alpha\beta}}(V_{\alpha;\beta}-b_{\alpha\beta}c_n)\\&=\frac{\partial\Psi}{\partial\theta}\dot{\theta}+\frac{\partial\Psi}{\partial a_{\alpha\beta}}\sigma_{\alpha\beta}+\frac{\partial\Psi}{\partial a_{\alpha\beta}}b_{\alpha\beta}(V_n-c_n).\end{aligned} \tag{3.83}$$

Introducing this expression into (3.78), and requiring that the corresponding inequality is satisfied in any process, we obtain the following results:

$$S=-\frac{\partial\Psi}{\partial\theta}, \tag{3.84}$$

$$T^{\alpha\beta}=\frac{\partial\Psi}{\partial a_{\alpha\beta}}, \tag{3.85}$$

$$
\begin{aligned}
&\left[\left[\rho\left(\frac{1}{2}(\mathbf{v}-\mathbf{V})^2+\psi-\Psi\right)(c_n-v_n)+(\mathbf{v}-\mathbf{V})\cdot\mathbf{tn}\right]\right]\\
&-\frac{\partial\Psi}{\partial a_{\alpha\beta}}b_{\alpha\beta}(V_n-c_n)\geq 0. \qquad (3.86)
\end{aligned}
$$

We now deduce other interesting consequences of the residual inequality (3.86). To this end, we decompose the velocities $\mathbf{v}$, $\mathbf{V}$ in the components $\mathbf{v}_s$, $\mathbf{V}_s$ tangent to $\Sigma(t)$, and the components $v_n\mathbf{n}$, $V_n\mathbf{n}$ along the normal $\mathbf{n}$ to $\Sigma(t)$. Then, in view of (3.68), we have

$$
\begin{aligned}
(\mathbf{v}-\mathbf{V})\cdot\mathbf{tn} &= (\mathbf{v}_s-\mathbf{V}_s)\cdot\mathbf{tn}+(v_n-V_n)\\
&= (\mathbf{v}_s-\mathbf{V}_s)\cdot\mathbf{tn}-p(v_n-V_n)\\
&= (\mathbf{v}_s-\mathbf{V}_s)\cdot\mathbf{tn}+p(c_n-V_n)+p(V_n-c_n).
\end{aligned}
$$

Consequently, (3.86) becomes

$$
\begin{aligned}
&\left[\left[\rho\left(\frac{1}{2}(\mathbf{v}-\mathbf{V})^2+g-\Psi\right)(c_n-v_n)-(\mathbf{v}_s-\mathbf{V}_s)\cdot\mathbf{tn}\right]\right]\\
&-\left(T^{\alpha\beta}b_{\alpha\beta}-[[p]]\right)(V_n-c_n)\geq 0, \qquad (3.87)
\end{aligned}
$$

where

$$
g=\psi+\frac{p}{\rho} \qquad (3.88)
$$

is the *specific Gibbs potential.*.

If we assume that the constitutive equations for the bulk quantities g and p are functions of some field variables like the density ρ, the deformation gradient $\mathbf{F}$, etc., then the jumps that appear in (3.87) will depend on the limits $\rho^{\pm}$, $\mathbf{F}^{\pm}$, etc. of these fields on $\Sigma(t)$. Moreover, if we suppose that the above jumps also depend on the variables $(c_n-v_n)^{\pm}$ and $(V_n-c_n)^{\pm}$, then we can state that the right-hand side f of (3.88) reaches its minimum when these variables are equal to zero. Consequently, the first derivatives of f are equal to zero at equilibrium:

$$
(g-\Psi)_0^- = 0, \qquad (3.89)
$$

$$
(g-\Psi)_0^+ = 0, \qquad (3.90)
$$

$$
(T^{\alpha\beta}b_{\alpha\beta}-[[p]])_0 = 0. \qquad (3.91)
$$

Equation 3.91 coincides with (3.67) at equilibrium, while (3.89) and (3.90) imply that on the interface, at equilibrium, we have

$$
[[g]]_0 = 0. \qquad (3.92)
$$

3.8 Other Balance Equations

If the continuous system $\mathbb{S}$ with a moving interface has electromagnetic fields, then—besides the thermomechanical balance laws of the above section—we need to consider the Maxwell equations, which are expressed by integral laws that take one of the two following forms (see [77]):

$$\int_{\partial V(t)} \mathbf{u}\cdot\mathbf{N}\,d\sigma = \int_{V(t)} r\,dv + \int_{\sigma(t)} R\,d\sigma, \tag{3.93}$$

$$\begin{aligned}\frac{d}{dt}\int_{S(t)} \mathbf{u}\cdot\mathbf{N}\,d\sigma &= \int_{\partial S(t)} \mathbf{a}\cdot\tau_s\,ds \\ &+ \int_{S(t)} \mathbf{g}\cdot\mathbf{N}\,d\sigma + \int_{\gamma(t)} \mathbf{k}\cdot\nu_\Sigma\,ds,\end{aligned} \tag{3.94}$$

where $\mathbf{u}$, $\mathbf{a}$, $\mathbf{g}$, $\mathbf{k}$ are vector fields, r and R are scalar fields, $V(t)$ is any material volume, $\Sigma(t)$ is any open material surface, $\gamma(t)$ is the intersection between $V(t)$ and a singular moving surface $\Sigma(t)$, $\sigma(t) = V(t)\bigcup\Sigma(t)$, $\mathbf{N}$ is the unit normal to $\partial V(t)$ or to $\Sigma(t)$, ν_Σ is the unit vector orthogonal to $\gamma(t)$, and τ_s is the unit vector tangent to $\gamma(t)$.

In the local form, the above equations become

$$\nabla\cdot\mathbf{u} = r, \quad \text{in}\, V(t)-\Sigma(t) \tag{3.95}$$

$$[[\mathbf{u}]]\cdot\mathbf{n} = R, \quad \text{on}\,\Sigma(t) \tag{3.96}$$

$$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{v}\nabla\cdot\mathbf{u} = \nabla\times(\mathbf{a}+\mathbf{v}\times\mathbf{u}) + \mathbf{g}, \quad \text{in}\, V(t)-\Sigma(t), \tag{3.97}$$

$$(\mathbf{n}\times[[\mathbf{u}\times(\mathbf{w}-\mathbf{v})-\mathbf{a}]]-\mathbf{k})\cdot\nu_\Sigma = 0, \quad \text{on}\,\Sigma(t). \tag{3.98}$$

In these formulae, $\mathbf{v}$ is the velocity of the particles of $\mathbb{S}$, $\mathbf{w}$ is the velocity of $\gamma(t)$, and $\mathbf{n}$ is the unit vector normal to $\Sigma(t)$.

It is possible to localize the integral equation (3.94) according to the following derivation formula, which holds for any *material* surface $\Sigma(t)$ intersecting a singular nonmaterial interface $\Sigma(t)$:

$$\begin{aligned}\frac{d}{dt}\int_{S(t)} \mathbf{u}\cdot\mathbf{N}\,d\sigma &= \int_{S(t)} \left[\frac{\partial\mathbf{u}}{\partial t} + \nabla\times(\mathbf{u}\times\mathbf{v}) + \mathbf{v}\nabla\cdot\mathbf{u}\right]\cdot\mathbf{N}\,d\sigma \\ &+ \int_{\gamma(t)} \mathbf{n}\times[[\mathbf{u}\times(\mathbf{w}-\mathbf{v})]]\cdot\nu_\Sigma\,ds.\end{aligned} \tag{3.99}$$

We can prove this formula in two different ways. First, since the surface $\Sigma(t)$ is material and $\mathbf{F} = \mathbf{u}\cdot\mathbf{N}$, we can verify[1] that

$$\mathbf{W} = \mathbf{C} = \mathbf{v} \tag{3.100}$$

[1] See p. 37 of [78].

and

$$\frac{\delta \mathbf{F}}{\partial t} + \nabla_s \cdot (\mathbf{F} \otimes \mathbf{C}_s) - 2HC_N \mathbf{F} = \left[\frac{\partial \mathbf{u}}{\partial t} + \nabla \times (\mathbf{u} \times \mathbf{v}) + \mathbf{v} \nabla \cdot \mathbf{u} \right] \cdot \mathbf{N}.$$

A different proof of (3.99) is given on p. 121 of [77].

We note that (3.98) is a consequence of the application of the Stokes theorem to a material surface $\Sigma(t)$ containing the curve $\gamma(t) = S(t) \bigcup \Sigma(t)$, along which the field $\mathbf{u}$ and the bulk fields can exhibit finite discontinuities. Therefore, in the integral balance law there is the integral

$$\int_{\gamma(t)} (\mathbf{n} \times [[\mathbf{u} \times (\mathbf{w} - \mathbf{v}) - \mathbf{a}]] - \mathbf{k}) \cdot \nu_\Sigma \, ds,$$

which vanishes along the arbitrary curve $\gamma(t) = S(t) \bigcup \Sigma(t)$, thus proving (3.98). However, this result does not allow us to conclude that

$$\mathbf{n} \times [[\mathbf{u} \times (\mathbf{w} - \mathbf{v}) - \mathbf{a}]] - \mathbf{k} = \mathbf{0}. \tag{3.101}$$

In fact, since $\Sigma(t)$ is material, we have $\mathbf{w} \cdot \mathbf{N} = v_N$ and $\mathbf{w} \cdot \mathbf{n} = c_n$. Consequently, (3.98) becomes

$$([[(c_n - v_n)\mathbf{u} - u_n(\mathbf{w} - \mathbf{v}) + \mathbf{n} \times \mathbf{a}]] + \mathbf{k}) \cdot \nu_\Sigma = 0. \tag{3.102}$$

If the basis $(\tau, \mathbf{n}, \mathbf{N})$ is introduced, then

$$\mathbf{w} - \mathbf{v} = (w_\tau - v_\tau)\tau + \frac{c_n - v_n}{\sin^2 \alpha}\mathbf{n} - \frac{(c_n - v_n)\cos\alpha}{\sin^2 \alpha}\mathbf{N}, \tag{3.103}$$

where $\cos\alpha = \mathbf{n} \cdot \mathbf{N}$. This relation allows us to write (3.98) as follows:

$$([[(c_n - v_n)\mathbf{u} - \mathbf{n} \times \mathbf{a}]] - \mathbf{k}) \cdot \nu_\Sigma + [[\mathbf{n} \cdot \mathbf{u}(c_n - v_n)]] \cot\alpha = 0. \tag{3.104}$$

The arbitrariness of the material surface $\Sigma(t)$ requires that the above equation must be identically satisfied for every value of the angle α. For $\alpha = \pi/2$ we get

$$[[(c_n - v_n)\mathbf{u} - \mathbf{n} \times \mathbf{a}]] - \mathbf{k} = \mathbf{0}, \tag{3.105}$$

$$[[u_n(c_n - v_n)]] = 0. \tag{3.106}$$

Finally, we can conclude that (3.98) is equivalent (3.102) if and only if (3.105) and (3.106) are satisfied.

In particular, from (3.105) and (3.106), we can deduce the following:

1. The condition $[[u_n]] = 0$ implies $[[v - n]] = 0$

2. If the singular surface $\Sigma(t)$ is material, then $c_n = v_n$ and (3.106) reduces to

$$[[\mathbf{n} \times \mathbf{a}]] + \mathbf{k} = \mathbf{0}. \tag{3.107}$$

3.9 Integral Form of Maxwell's Equations

In order to apply the results obtained in this section, we consider a continuous system $\mathbb{S}$ that moves while carrying charges and currents. If we denote any material volume by $V(t)$, an arbitrary material surface by $S(t)$, a nonmaterial singular surface by $\Sigma(t)$, and a *material* surface that carries a surface charge with density ω_e and a surface current with density $\mathbf{K}$ by $\Sigma^*(t)$, then the charge conservation takes the integral form

$$\frac{d}{dt}\left(\int_{V(t)} \rho_e \, dv + \int_{\sigma^*(t)} \omega_e \, d\sigma\right) = -\int_{\partial V(t)} \mathbf{J} \cdot \mathbf{N} \, d\sigma - \int_{\partial\sigma^*(t)} \mathbf{K} \cdot \nu_\Sigma \, ds, \tag{3.108}$$

where $\sigma^*(t) = \Sigma^*(t) \bigcup V(t)$, $\mathbf{J}$ is the conductive current density in the space, and ν_Σ is the unit vector that is tangent to $\sigma^*(t)$ and normal to $\partial\sigma^*(t)$.

Moreover, the integral balance laws of electromagnetism are:

$$\int_{\partial V(t)} \mathbf{D} \cdot \mathbf{N} \, d\sigma = \int_{V(t)} \rho_e \, dv + \int_{\sigma^*(t)} \omega_e \, d\sigma, \tag{3.109}$$

$$\int_{\partial V(t)} \mathbf{B} \cdot \mathbf{N} \, d\sigma = 0, \tag{3.110}$$

$$\begin{aligned} \frac{d}{dt}\int_{S(t)} \mathbf{D} \cdot \mathbf{N} \, d\sigma = & \int_{\partial S(t)} (\mathbf{H} - \mathbf{v} \times \mathbf{D}) \cdot \tau \, ds \\ & - \int_{S(t)} \mathbf{J} \cdot \mathbf{N} \, d\sigma - \int_{\gamma(t)} \mathbf{K} \cdot \nu_\Sigma \, ds, \end{aligned} \tag{3.111}$$

$$\frac{d}{dt}\int_{S(t)} \mathbf{B} \cdot \mathbf{N} \, d\sigma = -\int_{\partial S(t)} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \tau \, ds, \tag{3.112}$$

where $\gamma(t) = \sigma^*(t) \bigcap S(t)$, and $\mathbf{E}$ and $\mathbf{H}$ denote the electric field and the magnetic field, respectively. Finally, $\mathbf{D}$ and $\mathbf{B}$ are the electric and magnetic induction fields.

Applying the results of the preceding section to Maxwell's equations, we are led to the conditions

$$\nabla \cdot \mathbf{D} = \rho_e, \quad \nabla \cdot \mathbf{B} = 0, \quad \text{in}\, V(t) - \Sigma(t), \tag{3.113}$$

$$[[\mathbf{D}]] \cdot \mathbf{n} = \omega_e, \quad [[\mathbf{B}]] \cdot \mathbf{n} = 0, \quad \text{on}\, \Sigma(t). \tag{3.114}$$

$$\frac{\partial \mathbf{D}}{\partial t} = \nabla \times \mathbf{H} + \mathbf{J}, \quad \text{in}\, V(t) - \Sigma(t), \tag{3.115}$$

$$[[(c_n - v_n)\mathbf{D} - \mathbf{n} \times (\mathbf{H} - \mathbf{v} \times \mathbf{D})]] - \mathbf{k} = \mathbf{0}, \tag{3.116}$$

$$[[D_n(c_n - v_n)]] = \omega_e c_n - [[D_n v_n]] = 0, \quad \text{on}\, \Sigma(t). \tag{3.117}$$

In Chap. 7 we analyze Maxwell's equations in matter in detail.

Chapter 4

Phase Equilibrium

4.1 Boundary Value Problems in Phase Equilibrium

In this chapter we analyze some phase equilibrium problems using the model of a continuous system with an interface, which we explored in the previous chapter. We consider a system S consisting of two phases (that fill the regions C_1 and C_2) and an interface Σ. The body force $\mathbf{b}$ is assumed to derive from a potential energy $U(\mathbf{x})$, so that $\mathbf{b} = -\nabla U$.

Starting from the local balance equations and the jump conditions (3.55)–(3.72), we obtain the set of phase equilibrium equations

$$\nabla \cdot \mathbf{t} - \rho \nabla U = \mathbf{0}, \text{ in } C_1 \cup C_2 - \Sigma, \tag{4.1}$$

$$T^{\alpha\beta}_{;\beta} = [[t^{\alpha}_{s}]]\,, \tag{4.2}$$

$$T^{\alpha\beta} b_{\alpha\beta} = [[p]]\,, \tag{4.3}$$

$$[[g]] = 0, \text{ on } \Sigma - \Gamma, \tag{4.4}$$

where we have introduced the notations $p = -\mathbf{n} \cdot \mathbf{t} \cdot \mathbf{n}$ and $\mathbf{t}_s = \mathbf{t} \cdot \mathbf{n} + p\mathbf{n}$ to denote, respectively, the pressure on the interface Σ and the tangential stress, both of which are due to the bulk phases. Further, $\mathbf{T}$ is the surface stress tensor, $b_{\alpha\beta}$ are the coefficients of the second fundamental form of the interface (see Appendix B), and g is the specific Gibbs potential.

The boundary conditions are given by the prescribed external pressure p_e on a part $\partial C' \subset \partial(C_1 \cup \partial C_2) \equiv \partial C$ and by the contact force γ on the line $\partial\Sigma' = \partial\Sigma \cap \partial C$ such that (see Fig. 4.1)

$$\mathbf{t} \cdot \mathbf{n} = -p_e \mathbf{N}, \text{ on } \partial C', \quad \mathbf{T} \cdot \nu_\Sigma = \gamma, \text{ on } \partial\Sigma'. \tag{4.5}$$

We start by analyzing the phase equilibrium of perfect fluids. To this end, we recall some results from elementary thermodynamics in the next section. The case in which one of the phases is filled with an elastic solid

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is then considered. Finally, we describe the equilibrium of a crystal in its melt or vapor, stating the Gibbs principle and the Wulff law.

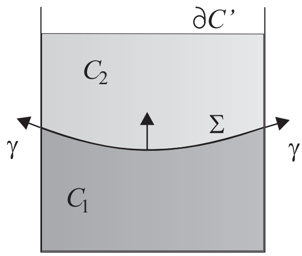

Fig. 4.1 Equilibrium of two fluid phases

4.2 Some Phenomenological Results of Changes in State

In elementary thermodynamics [80]–[83], the homogeneous equilibrium states of a pure substance $\mathbb{B}$ are described by a state equation of the form

$$p = p(v, \theta), \tag{4.6}$$

where p is the pressure, $v = 1/\rho$ is the *specific volume* and θ is the absolute temperature. Since the states are homogeneous, all of these quantities are uniform and constant in all of the regions occupied by $\mathbb{B}$. Moreover, all of the transformations that lead the system from an equilibrium state to another equilibrium state are assumed to be quasi-static.

It can be shown experimentally that (4.6) is not defined for all positive values of p and θ. More precisely, the qualitative behavior of the surface (4.6) in the space p, v, θ is represented in Fig. 4.2, where the forbidden states are indicated by the darkest shading. The figure shows that the substance can be solid, liquid or vapor. It can pass from one state to another in transformations that are represented geometrically by curves on the surface S. Of particular interest are the isobaric, isovolumic and isothermal transformations. They are obtained by intersecting S with the planes $p = const$, $v = const$ and $\theta = const$, respectively.

It is very interesting to note that (4.6) can only be written in the form $v = v(p, \theta)$ in the regions of S that are internal to Σ_S, Σ_L and Σ_v. Along the curves $1'$-$1''$, $2'$-$2''$, $3'$-$3''$, the function $v = v(p, \theta)$ assumes two values that represent the specific volumes of two coexisting phases at a given pressure and temperature. The curves that are obtained by projecting the above curves onto the plane p, θ are termed Clapeyron's curves of sublimation, melting and vaporization, respectively. In particular, three phases coexist on the curve b with assigned values of specific volumes.

Another remarkable aspect of (4.6) is represented by the existence of three values p_c, θ_c and v_c: the coordinates of the point A for which the liquid and vapor phases become indistinguishable (the opalescence phenomenon). Moreover, when $\theta > \theta_c$, we cannot obtain the liquid phase by increasing the pressure. The values p_c, θ_c, and v_c are called critical values, and the isotherm $\theta = \theta_c$ is the critical isotherm. This curve exhibits a horizontal inflexion at the point (p_c, θ_c, v_c).

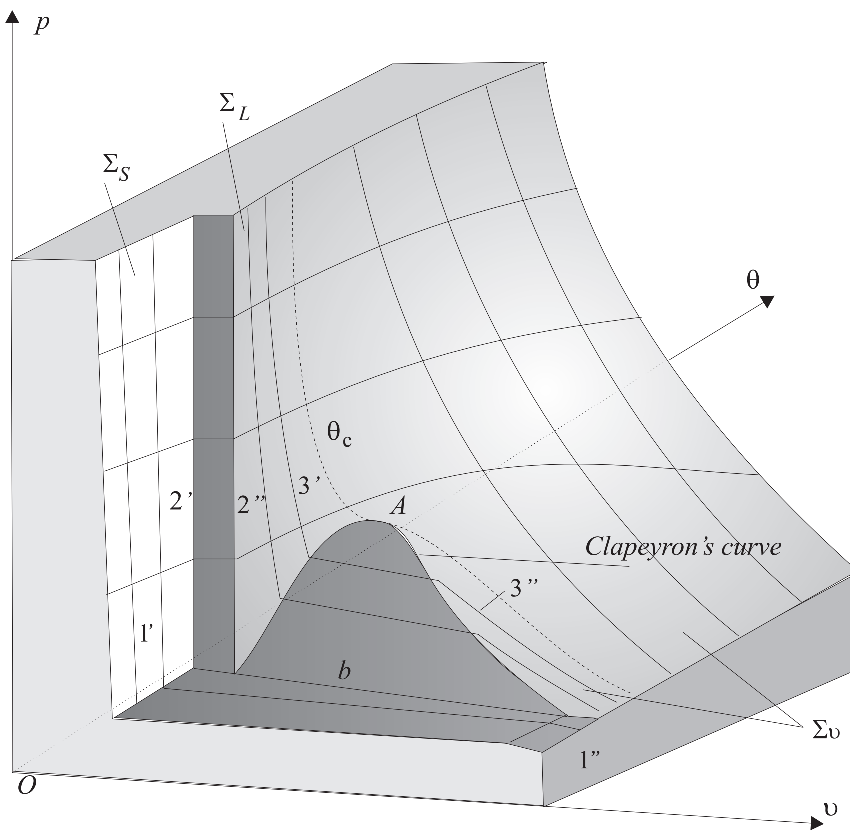

Fig. 4.2 The surface $p = p(v, \theta)$

The first theoretical attempt to determine the analytic form of (4.6) was made by Van der Waals, who derived the following equation by statistical considerations:

$$p = \frac{r\theta}{v - b} - \frac{a}{v^2}, \tag{4.7}$$

where $r = R/M$ is the ratio between the universal constant R of gases and the molar mass M, while a and b are two constants that depend on the substance. b arises due to the finite dimensions of the molecules, whereas the term a/v^2 reflects the molecular forces of cohesion. If $a = b = 0$, then the equation of a perfect gas is obtained:

$$p = \frac{r\theta}{v}. \tag{4.8}$$

Relation 4.7 can be written as a third-degree equation in the unknown v:

$$pv^3 - (pb + r\theta)v^2 + av - ab = 0. \tag{4.9}$$

Now, if we suppose that this equation yields a triple root v_c for suitable values p_c and θ_c of p, θ, then it can be written in the form

$$p_c(v - v_c)^3 = 0. \tag{4.10}$$

On the other hand, if we put $p = p_c$, $\theta = \theta_c$, and $v = v_c$ into (4.9) and compare the resulting equation with (4.10), we find that

$$v_c = 3b, \quad p_c = \frac{a}{27b^2}, \quad \theta_c = \frac{8}{27}\frac{a}{rb}; \tag{4.11}$$

i.e., we determine the relations between p_c, θ_c, v_c and a, b, r.

We can convert (4.7) into a form that is independent of the particular substance. In this way we can more easily compare the equation with the experimental results. Upon introducing the nondimensional quantities

$$P = \frac{p}{p_c}, \quad \tau = \frac{\theta}{\theta_c}, \quad V = \frac{v}{v_c}, \tag{4.12}$$

(4.7) becomes

$$P = \frac{8\tau}{3V - 1} - \frac{3}{V^2}. \tag{4.13}$$

This equation takes a form that does not depend on the particular substance since it contains only numerical constants (the principle of corresponding states). The isotherms we obtain from (4.13) are represented in Fig. 4.3.

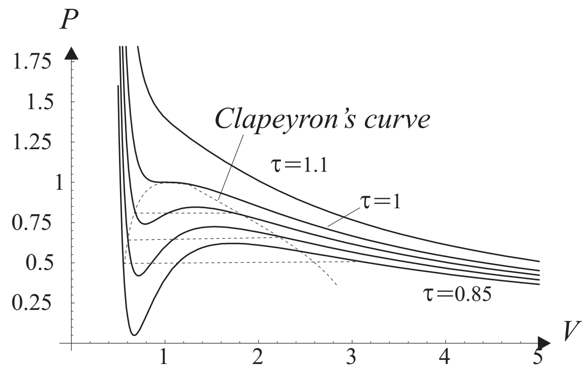

Fig. 4.3 Isotherm curves

These curves are in good agreement with the experimental behavior of the real isotherms, with two exceptions. First, they enter the forbidden region, which is bounded by Clapeyron's curve, and there is no criterion to define this region. We show that this problem can be solved if the Gibbs potential of the material is known. Moreover, the states defined by the triplets p, θ, v when one of the phases consists of very small regions ($10^{-1} - 1$ mm) can belong to the forbidden region. This circumstance will be explained by supposing that the interface between the phases is able to exert a surface tension. In this situation, the liquid drops can be at equilibrium with the vapor, which is at a higher pressure than the pressure $\overline{p}$ corresponding to equilibrium with the plane interface (superheated liquid). Similarly, the vapor bubbles can be at a pressure higher than $\overline{p}$ without becoming water.

4.3 Equilibrium of Fluid Phases with a Planar Interface

In this section we suppose that the bulk phases are filled with perfect fluids. More precisely, we assume that:

- The stress tensor in the phases C_1 and C_2 takes the form

$$\mathbf{t} = -p(\rho, \theta)\mathbf{I} \tag{4.14}$$

- The surface stress tensor of the interface Σ is

$$\mathbf{T} = \gamma(a, \theta_s)\mathbf{I}_s, \tag{4.15}$$

where γ is the surface tension and $\mathbf{I}_s$ is the unit tensor on Σ

- There is no body force
- The interface is planar.

We recall that the pressure p is expressed in terms of the specific free energy ψ by the relation:

$$p = \rho^2 \frac{\partial \psi}{\partial \rho} = -\frac{\partial \psi}{\partial \upsilon}, \tag{4.16}$$

where $p(\upsilon, \theta)$ is assumed to be invertible with respect to υ. Therefore, from the Gibbs potential $g(p, \theta) = \psi + p/\rho = \psi(p, \theta) + p\upsilon(p, \theta)$ and (4.15), we obtain

$$\upsilon(p, \theta) = \frac{\partial g}{\partial p}. \tag{4.17}$$

Finally, the phase equilibrium equations (4.1)–(4.3) become

$$p = const, \Rightarrow g(p, \theta) = const, \quad \text{in } \mathrm{C}_1 \cup \mathrm{C}_2, \tag{4.18}$$
$$\gamma = const, \quad \text{on } \Sigma, \tag{4.19}$$
$$[[p]] = 0, \tag{4.20}$$
$$[[g(p, \theta)]] = 0, \quad \text{on } \Sigma - \Gamma \tag{4.21}$$

with the boundary conditions

$$p = p_e, \ \text{ on } \partial \mathrm{C}', \quad \gamma \nu_\Sigma = \gamma, \ \text{ on } \partial\Sigma \cap \partial \mathrm{C}. \tag{4.22}$$

In other words, the pressure is uniform in $C_1 \cup C_2$, together with $g(p, \theta)$, and γ is uniform on Σ. The equilibrium value p_e of the external pressure is determined by the condition that the Gibbs potential must be continuous across Σ for any value of the temperature that is compatible with the coexistence of two phases.

We now determine *Maxwell's rule*, which supplies a very expressive geometrical formulation of the equilibrium conditions (4.20) and (4.21). Since the temperature, the specific volume, and the pressure are uniform in any phase, we do not distinguish between the values of these quantities in the phases and those at the interface. In particular, $p^- = p^+ = p_1 = p_2 = p_e$. Therefore, we can write (4.21) in the form

$$\psi_2 - \psi_1 = -p_e(\upsilon_2 - \upsilon_1),$$

or, equivalently,

$$\int_{\upsilon_1}^{\upsilon_2} \frac{\partial \psi}{\partial \upsilon}\, d\upsilon = -p_e(\upsilon_2 - \upsilon_1).$$

This relation, in view of (4.16), becomes

$$\int_{\upsilon_1}^{\upsilon_2} p(\upsilon, \theta)\, d\upsilon = p_e(\upsilon_2 - \upsilon_1), \tag{4.23}$$

and it can be read in the following way. *At the phase equilibrium, the area under the curve $p = p(\upsilon, \theta)$ is equal to the area of a rectangle with a basis of $\upsilon_2 - \upsilon_1$ and a height of p_e, or (equivalently) the shaded areas in Fig. 4.4 between the straight line $p = p_e$ and the curve $p = p(\upsilon, \theta)$ are equal.*

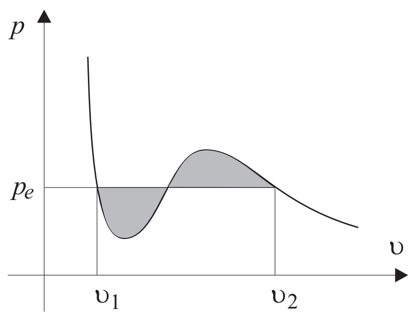

Fig. 4.4 Graphical representation of Maxwell's rule

4.4 Equilibrium of Fluid Phases with a Spherical Interface

Now we will analyze, in the absence of body forces, the phase equilibrium of fluids whose phases are separated by a spherical interface with the structure of a membrane. From the phase equilibrium equations (4.1)–(4.3), we derive

$$p_i = c_i, \quad \text{in} \;\; \mathrm{C}_1 \cup \mathrm{C}_2, \tag{4.24}$$

$$\frac{2\gamma}{R} = c > 0, \tag{4.25}$$

$$[[p]] = c, \tag{4.26}$$

$$[[g(p)]] = 0, \quad \text{on}\, \Sigma - \Gamma \tag{4.27}$$

where R is the radius of curvature of the interface, and c_i, $i = 1, 2$, c and γ are constant. Since the pressure is uniform in each phase, from now on we will use the notation $p_1 = p_l$ and $p_2 = p_\upsilon$.

Bearing in mind the behavior of the isothermal curves derived using the Van der Waals equation (see Fig. 4.3), we assume that the function $p(v,\theta)$ satisfies the following conditions:

- There is a critical value θ_c of θ such that, for every $\theta > \theta_c$, the function $p(.,\theta) \in C^1(b,\infty)$, where $b>0$; moreover, $\partial p/\partial v < 0$ in (b,∞) and

$$\lim_{v\to b} p(v,\theta) = \infty, \qquad \lim_{v\to\infty} p(v,\theta) = 0. \tag{4.28}$$

- For every $\theta \in (0,\theta_c)$, the function $p(.,\theta) \in C^1\left[(b,\overline{v}_l(\theta)) \cup (\overline{v}_v(\theta),\infty)\right]$, $\overline{v}_l(\theta) < \overline{v}_v(\theta)$, and the following relation holds:

$$\lim_{\theta\to\theta_c} \overline{v}_l(\theta) = \lim_{\theta\to\theta_c} \overline{v}_v(\theta). \tag{4.29}$$

Moreover, $\partial p/\partial v < 0$ in $(b,\overline{v}_l(\theta)) \cup (\overline{v}_v(\theta),\infty)$, and

$$\overline{p}_l = p(\overline{v}_l,\theta) < \overline{p}_v = p(\overline{v}_v,\theta), \tag{4.30}$$

$$\lim_{v\to\overline{v}_1} \frac{\partial p}{\partial v} = \lim_{v\to\overline{v}_v} \frac{\partial p}{\partial v} = 0. \tag{4.31}$$

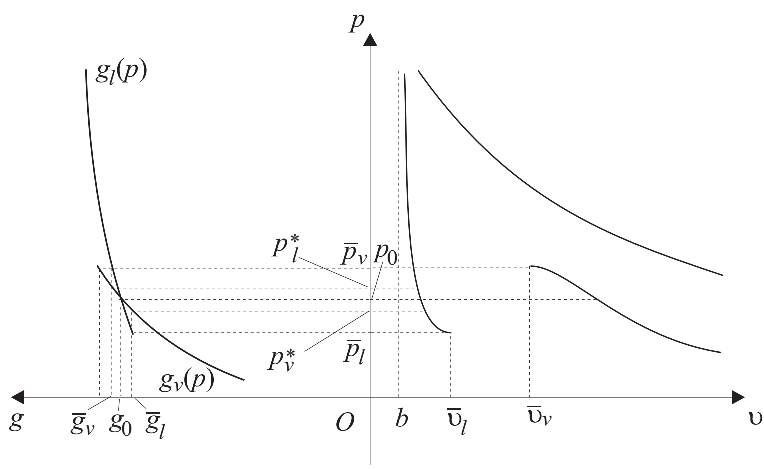

Fig. 4.5 Comparison between the curves $g(p)$ and $p(v)$ at phase equilibrium for a bubble

From now on, the temperature will be omitted from all of the formulae to simplify the notation. The two hypotheses given above imply that the function $p(v)$ is invertible on (b,∞) for $\theta > \theta_c$ and that the inverse function is decreasing. Similarly, when $\theta < \theta_c$, by inverting the decreasing function $p(v)$ on $(b, \overline{v}_l)$ and on $(\overline{v}_v, \infty)$, respectively, we obtain two functions:

$$v_l(p) : (\overline{p}_l, \infty) \to (b, \overline{v}_l), \quad v_v(p) : (0, \overline{p}_v) \to (\overline{v}_v, \infty).$$

Using the relation $\partial g / \partial p = v$, we can derive the Gibbs potentials in the liquid and the vapor:

$$g_l(p) = \int v_l(p)\, dp + \phi_l(\theta), \quad p \in (p_l, \infty), \tag{4.32}$$

$$g_v(p) = \int v_v(p)\, dp + \phi_v(\theta), \quad p \in (0, p_v), \tag{4.33}$$

where the functions ϕ_l and ϕ_v are undetermined. When we take into account the relations

$$\frac{\partial g}{\partial p} = v(p) > 0, \quad \frac{\partial^2 g}{\partial p^2} = \frac{\partial v}{\partial p} = \left(\frac{\partial p}{\partial v}\right)^{-1} < 0,$$

we can conclude that the functions $g_l(p)$ and $g_v(p)$ are always increasing and that they exhibit upward convexity. Moreover, since

$$v_l(p') < v_v(p''), \quad p' \in (\overline{p}_l, \infty),\ p'' \in (0, \overline{p}_v),$$

the function $g_l(p)$ has shallower slopes than those of the function $g_v(p)$ (see Fig. 4.5).

It is not possible to localize the curves $g_l(p)$ and $g_v(p)$ in the plane (p, g) due to the presence of the arbitrary functions $\phi_l(\theta)$ and $\phi_v(\theta)$. In order to reduce this indetermination, we also assume that:

- For every $\theta < \theta_c$, there is a value $p_0 \in (\overline{p}_l, \overline{p}_v)$ such that

$$g_l(p_0) = g_v(p_0). \tag{4.34}$$

It is evident that (4.34) is equivalent to requiring the existence of a solution of the equation $[[g(p)]] = 0$ with a planar interface. Moreover, the condition (4.34) determines the difference $\phi_l - \phi_v$. Finally, from (4.34) and all of the other properties we have already deduced, we can say that (see Fig. 4.5)

$$g_v(p) < g_l(p), \quad \text{if } p \in (\overline{p}_l, p_0) \tag{4.35}$$

$$g_v(p) > g_l(p), \quad \text{if } p \in (p_0, \overline{p}_v). \tag{4.36}$$

Now we are in a position to justify the existence of drops of liquid or bubbles of vapor at phase equilibrium. First, we note that the surface tension increases the pressure inside the spherical phase so that, at equilibrium, the pressure inside the bubble or the drop is greater than the pressure outside it. Then (see Fig. 4.5), for any value of the Gibbs potential g in the interval $(\overline{g}_l, g_0)$ this is a corresponding pair (p_l, p_v) of values for the pressure corresponds in the interval $(\overline{p}_l, p_0)$ such that the difference $c = p_v - p_l > 0$ belongs in the interval $(0, p_v^* - \overline{p}_l)$. Consequently, bubbles of vapor with a radius $R = 2\gamma/c$ can exist. The pressure inside the bubbles is less than the pressure $p_l < p_0$ of the liquid phase.

Similarly, a pair $(p_v, p_l) \in (p_0, \overline{p}_v)$ corresponds to any value of $g \in (g_0, \overline{g}_v)$ such that $c = p_l - p_v > 0$ and $c \in (0, \overline{p}_l - P_l^*)$. Therefore, there are drops of liquid with a radius $R = 2\gamma/c$ in the presence of vapor at a pressure that is higher than p_0 but less than the pressure of the liquid.

4.5 Variational Formulation of Phase Equilibrium

In this section we discuss the phase equilibrium from a variational point of view. This means that we do not use the equilibrium system (4.1) but instead resort to the Gibbs principle, according to which *the equilibrium configurations are extremals of the total free energy with respect to any variation at constant mass* (see, for instance, [84]–[86]).

Let us consider, in the absence of body forces, the total free energy

$$\Psi = \int_{C_S} \rho_S \psi_S \, dv + \int_{C_F} \rho_F \psi_F \, dv + \int_{\Sigma} \Psi_s \, d\sigma + \int_{\sigma_e} p_e \, d\sigma, \tag{4.37}$$

of a system $\mathbb{S}$ that consists of a solid phase of volume C_S, a fluid phase of volume C_F, and an interface Σ that separates the bulk regions. In (4.37), ψ_S denotes the specific free energy of the solid phase, ψ_F the specific free energy of the liquid phase, and Ψ_s the free energy for a unit area of the interface Σ. Finally, p_e is a uniform external pressure acting on the region σ_e of the boundary of the fluid phase. For the sake of simplicity, we suppose that:

- The solid phase is elastic, so that $\psi_S = \psi_S(\mathbf{F})$, where $\mathbf{F}$ is the deformation gradient of the displacement $\mathbf{u}(\mathbf{X})$ and $\mathbf{X}$ belongs to a reference configuration C_* of the solid phase
- The region C_F is filled with a perfect fluid, so that $\psi_F = \psi_F(\rho_F)$
- The surface free energy Ψ_s of the interface is constant.

Let us consider the following families of functions:

- $K = \{\delta\mathbf{k} : \Sigma \to \Re^3 | \delta\mathbf{k}(\Sigma)$ is a regular surface, $\mathbf{k}(\partial\Sigma) = \Sigma\}$; H, whose elements are all the functions $\delta\mathbf{h} : C \to C$, where $C = C_S \cup C_F - \Sigma$

such that:

- $\delta\mathbf{h}$ is regular and exhibits finite discontinuities together with its derivatives across Σ
- $\delta\mathbf{h}(\partial C) = \mathbf{0}$ on $\partial C - \sigma_e$, where σ_e is a part of the boundary of the fluid phase at which a given external pressure p_e is applied
- $\delta\mathbf{h}_{|C_S}$ and $\delta\mathbf{h}_{|C_F}$ are diffeomorphisms that locally conserve the mass.

The Gibbs principle states that $(\mathbf{u}, \rho_F, \mathbf{r})$ is a phase equilibrium configuration if and only if it is an extremal of Ψ with respect to all of the above variations $(\delta\mathbf{h}, \delta\mathbf{k})$ for which the total mass M of the system $\mathbb{S}$ does not vary; i.e., for all of the variations $(\delta\mathbf{h}, \delta\mathbf{k})$ that satisfy the global constraint

$$\varphi = \int_{C_S} \rho_S\, dv + \int_{C_F} \rho_F\, dv - M = 0. \tag{4.38}$$

It is well known that this is equivalent to searching for the extremal of the functional

$$\Phi = \Psi + \lambda\varphi, \tag{4.39}$$

where λ is a constant Lagrangian multiplier. In order to evaluate the Fréchet differential $d\Phi$ of Φ, we start by noting that the variations of C_S under $(\delta\mathbf{h}, \delta\mathbf{k})$ result from the variation due to $\delta\mathbf{h}_{|C_S} \equiv \delta\mathbf{h}_S$, where the mass remains constant, and from the variation due to $\delta\mathbf{k}$. Then, if $\mathbf{n}$ is the unit normal vector to Σ that points toward C_F, we have

$$\begin{aligned} d\Psi_S &\equiv \int_{C'_S} \rho'_S \psi_S(\mathbf{F})\, dv' - \int_{C_S} \rho_S \psi_S(\mathbf{F})\, dv \\ &= \int_{C_S} [\psi_S(\mathbf{F}') - \psi_S(\mathbf{F})]\, dv - \int_\Sigma \rho_S \psi_S(\delta h_{Sn} - \delta K_n)\, d\sigma \\ &= \int_{C_S} \rho_S \frac{\partial \psi_S}{\partial F_{iL}} \delta h_{Si,j}\, dv - \int_\Sigma \rho_S \psi_S(\delta h_{Sn} - \delta k_n)\, d\sigma, \end{aligned} \tag{4.40}$$

where the meaning of the notation is evident. By applying the Gauss theorem and recalling that $\delta\mathbf{h} = \mathbf{0}$ on ∂C_S, we obtain

$$\begin{aligned} d\Psi_S = &-\int_{C_S} \left(\rho_S \frac{\partial \psi_S}{\partial F_{iL}} F_{jL}\right)_{,j} \delta h_{Si}\, dv + \int_\Sigma \rho_S \frac{\partial \psi_S}{\partial F_{iL}} F_{jL} N_j \delta h_{Si}\, d\sigma \\ &- \int_\Sigma \rho_S \psi_S(\delta h_{Sn} - \delta k_n)\, d\sigma, \end{aligned} \tag{4.41}$$

where $\mathbf{N}$ is the unit vector that is normal to $\partial C_S - \Sigma$.

Now we introduce the quantities

$$\mathbf{t} = \rho_S \frac{\partial \psi_S}{\partial \mathbf{F}} \mathbf{F}^T \text{ in } C_S, \tag{4.42}$$

$$p_S = -\mathbf{n} \cdot \mathbf{t} \cdot \mathbf{n} \text{ on } \Sigma, \tag{4.43}$$

$$\mathbf{t}_s = \mathbf{t} \cdot \mathbf{n} + p_S \mathbf{n} \text{ on } \Sigma, \tag{4.44}$$

$$g_S = \psi_S + \frac{p}{\rho_S} \text{ on } \Sigma, \tag{4.45}$$

which define the stress tensor $\mathbf{t}$ in C_S, the pressure p_S, the tangential stress $\mathbf{t}_s$, and the Gibbs potential g_S on Σ, respectively. Equation 4.41 can then be written in the form

$$\begin{aligned} d\Psi_S = &-\int_{C_S} \nabla \cdot \mathbf{t} \cdot \delta \mathbf{h} \, dv - \int_\Sigma \rho_S g_S (\delta h_{Sn} - \delta k_n) \, d\sigma \\ &+ \int_\Sigma \mathbf{t}_s \cdot (\delta \mathbf{h}_s - \delta \mathbf{k}_s) \, d\sigma - \int_\Sigma p_S \delta k_n \, d\sigma + \int_\Sigma \mathbf{t}_s \cdot \delta \mathbf{k}_s \, d\sigma. \end{aligned} \tag{4.46}$$

Similarly, noting that $\delta \rho_F = -\rho_F (\delta h_F)^i{}_{,i}$, it is possible to prove that

$$\begin{aligned} d\Psi_F = &\int_{C_F} \nabla p \cdot \delta \mathbf{h}_F \, dv - \int_{\sigma_e} (p - p_e) \mathbf{n} \delta h_{Fn} \, d\sigma \\ &+ \int_\Sigma \rho_F g_F (\delta h_{Fn} - \delta k_n) \, d\sigma + \int_\Sigma p \delta k_n \, d\sigma, \end{aligned} \tag{4.47}$$

where

$$p = -\rho_F^2 \frac{\partial \psi_F}{\partial \rho_F}, \text{ in } C_F, \tag{4.48}$$

$$g_F = \psi_F + \frac{p}{\rho_F}, \text{ on } \Sigma. \tag{4.49}$$

It remains to evaluate the differential $d\Psi_s$ of the third integral appearing on the right-hand side of (4.37). First, from (B.17), we have

$$\delta(d\sigma) = \frac{1}{2a} \delta a d\sigma.$$

Consequently,

$$\delta(d\sigma) = \frac{1}{2a} \frac{\partial a}{\partial a_{\alpha\beta}} \delta a_{\alpha\beta} \, d\sigma = \frac{1}{2} a^{\alpha\beta} \delta a_{\alpha\beta} \, d\sigma. \tag{4.50}$$

To evaluate the quantities $\delta a_{\alpha\beta}$, due to the variation $\delta \mathbf{k}$ of the interface Σ, we note that

$$\delta a_{\alpha\beta} = \delta(\mathbf{a}_\alpha \cdot \mathbf{a}_\beta) = \delta \mathbf{a}_\alpha \cdot \mathbf{a}_\beta + \mathbf{a}_\alpha \cdot \delta \mathbf{a}_\beta.$$

On the other hand, after the variation $\delta\mathbf{k}$, the equation of the interface Σ becomes $\mathbf{r}+\delta\mathbf{k}$, so that $\delta\mathbf{a}_\alpha = \delta\mathbf{k}_{,\alpha}$. Recalling (B.42), we have

$$\delta a_{\alpha\beta} = \delta k_{\beta;\alpha} + \delta k_{\alpha;\beta} - 2b_{\alpha\beta}\delta k_n, \tag{4.51}$$

so that, by substituting (4.51) into (4.50), we obtain

$$\delta(d\sigma) = (a^{\alpha\beta}\,\delta k_{\alpha;\beta} - 2H\delta k_n)d\sigma. \tag{4.52}$$

This formula allows us to write

$$d\int_\Sigma \Psi_s\, d\sigma = \int_\Sigma \Psi_s a^{\alpha\beta}\delta k_{\alpha;\beta}\, d\sigma - \int_\Sigma 2H\Psi_s\delta k_n\, d\sigma. \tag{4.53}$$

Applying the Gauss theorem to the first integral on the right-hand side of (4.53) and recalling that $a^{\alpha\beta}_{;\beta} = 0$, $\Psi_s = const$ and $\delta\mathbf{k} = \mathbf{0}$ on $\partial\Sigma$, we finally derive the relation

$$d\int_\Sigma \Psi_s\, d\sigma = -\int_\Sigma 2H\Psi_s\delta k_n\, d\sigma. \tag{4.54}$$

On the other hand, it is very easy to verify that

$$d\varphi = -\int_\Sigma \rho_S(\delta h_{Sn} - \delta k_n)\, d\sigma + \int_\Sigma \rho_F(\delta h_{Fn} - \delta k_n)\, d\sigma. \tag{4.55}$$

By combining the results (4.46), (4.47), (4.54) and (4.55), we can write the condition $d\Phi = 0$ as follows:

$$\begin{aligned}
&-\int_{C_S} \nabla\cdot\mathbf{t}\cdot\delta\mathbf{h}_S\, dv + \int_{C_F} \nabla p\cdot\mathbf{h}_F\, dv \\
&-\int_\Sigma \rho_S(g_S+\lambda)(\delta h_{Sn} - \delta k_n)\, d\sigma + \int_\Sigma \rho_F(g_F+\lambda)(\delta h_{Fn} - \delta k_n)\, d\sigma \\
&+\int_\Sigma \mathbf{t}_s\cdot(\delta\mathbf{h}_s - \delta\mathbf{k}_s)\, d\sigma - \int_\Sigma (p_S - p + 2H\Psi_s)\delta k_n\, d\sigma = 0.
\end{aligned}$$

The arbitrariness of the variations $\delta\mathbf{h}$ and $\delta\mathbf{k}$ again lead us to phase equilibrium system (4.1), provided that we identify Ψ_s with the constant surface tension γ and note that $T^{\alpha\beta} = \gamma a^{\alpha\beta}$ (for a more general case, see [78]).

4.6 Phase Equilibrium in Crystals

In this section we present some classical results, obtained by Gibbs and Wulff (see [86, 87]), regarding the phase equilibrium of a crystal in its melt. Other important results relating to this subject are obtained in [88]–[97].

Let C be a polyhedric crystal with a volume V, n faces σ_i, $i = 1, \ldots, n$, and m edges. We denote the length of the edge between the faces σ_i and σ_j by l_{ij}, and the unit vector that is normal to l_{ij} and lies in the face σ_i by ν_{ij}. Finally, a uniform surface energy E_i is associated with the face σ_i of C. This means that E_i is assumed to be dependent only on the orientation of σ_i with respect to the crystal.

Let $[C]$ be the class of all polyhedrons with the same volume V of C, and the same number of faces, edges of C and faces that are parallel to the faces of C. Moreover, we suppose that any polyhedron belonging to $[C]$ is convex with respect to an internal point. This means that there is an internal point $\mathbf{r}_0$ such that the perpendicular from $\mathbf{r}_0$ to any face σ_i intersects σ_i at an internal point. The Gibbs principle postulates that a polyhedron $C \in [C]$ is an equilibrium configuration in the presence of its melt if the total surface energy

$$\Psi_s = \sum_{i=1}^{n} E_i \sigma_i \tag{4.56}$$

has an extremum at C in the set $[C]$. Moreover, this configuration is stable if it corresponds to a minimum in $[C]$. To determine the conditions that characterize the polyhedron C, we start by providing the procedure used to pass from C to any other polyhedron in the set $[C]$. To this end, we introduce the normal displacements δh_{ni}, $i = 1, \ldots, n$, along the unit normal $\mathbf{n}_i$ to the face σ_i, as well as the displacements $\delta \mathbf{K}_{ij}$ of any edge l_{ij} of C. Of course, these displacements will define a new polyhedron in $[C]$ if the following conditions are satisfied (see Fig. 4.6):

$$\delta \mathbf{K}_{ij} \cdot \mathbf{n}_i = \delta k_{ni}, \quad \delta \mathbf{K}_{ij} \cdot \mathbf{n}_j = \delta k_{nj}. \tag{4.57}$$

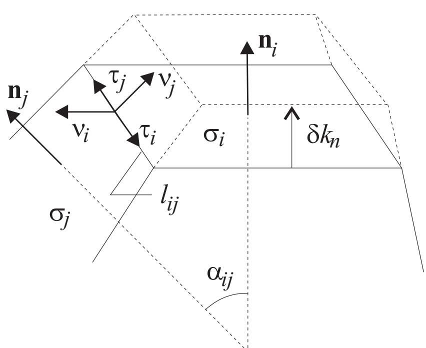

Fig. 4.6 Notation for a polyhedric crystal

It is evident that the elements of the class $[C]$ are obtained by arbitrarily varying δk_{ni} and $\delta \mathbf{K}_{ij}$ provided that these displacements leave the volume

V constant and obey (4.57). According to the Gibbs principle, we state that C is an equilibrium configuration for the crystal if the total surface energy

$$\Psi_s(\delta k_{ni}, \delta \mathbf{K}_{ij}) = \sum_{i=1}^{n} E_i \sigma_i \tag{4.58}$$

has an extremum at $\mathbf{0}$ for any $\delta k_{ni}, \delta \mathbf{K}_{ij}$ that satisfies (4.57) and the global condition

$$\Phi(\delta k_{ni}, \delta \mathbf{K}_{ij}) = \int_{C'} dv - V = 0, \tag{4.59}$$

$\forall C' \in [C]$. This is equivalent to saying that $\mathbf{0}$ is an extremum for $\Psi_s - \hat{\lambda}\Phi$, where $\hat{\lambda}$ is a Lagrangian multiplier. In other words, we must satisfy the condition

$$d(\Psi_s - \hat{\lambda}\Phi) = 0, \tag{4.60}$$

for any choice of $\delta k_{ni}, \delta \mathbf{K}_{ij}$ that obeys (4.57).

From the Gauss theorem, we have

$$\begin{aligned} d\Phi &= \int_{C'} dv - \int_C dv \\ &= \frac{1}{3}\left(\int_{\partial C} (\mathbf{r} + \delta k_n \mathbf{n}) \cdot \mathbf{n}\, d\sigma - \int_{\partial C} \mathbf{r} \cdot \mathbf{n}\, d\sigma\right) \\ &= \frac{1}{3}\int_{\partial C} \delta k_n\, d\sigma, \end{aligned}$$

where $\mathbf{r}$ is the position vector of any point on the crystal surface. By recalling that δk_n is uniform on any crystal face σ_i, we can write

$$d\Phi = \frac{1}{3}\sum_{i=1}^{n} \delta k_{ni}\sigma_i. \tag{4.61}$$

In view of (4.61), condition (4.60) becomes

$$\sum_p (E_i \nu_{ij} + E_j \nu_{ji}) \cdot \delta \mathbf{K}_{ij} l_{ij} - \lambda \sum_{i=1}^{n} \delta k_{ni}\sigma_i, \tag{4.62}$$

for any choice of $\delta k_{ni}, \delta \mathbf{K}_{ij}$ that satisfies (4.57). In relation (4.62), $\lambda = \hat{\lambda}/3$ and the first summation is extended over all of the edges of the crystal. If we introduce the notation $\cos\alpha_{ij} = \mathbf{n}_i \cdot \mathbf{n}_j$, then it is easily proven that

$$\nu_{ij} = -\cot\alpha_{ij}\mathbf{n}_i + \csc\alpha_{ij}\mathbf{n}_j. \tag{4.63}$$

By taking (4.57) and (4.63) into account, we can write (4.63) as follows:

$$\sum_p (A_{ij}\delta k_{ni} + A_{ji}\delta k_{nj}) l_{ij} - \lambda \sum_{i=1}^{n} \sigma_i \delta k_{ni} = 0, \tag{4.64}$$

where

$$A_{ij} = E_j \csc \alpha_{ij} - E_i \cot \alpha_{ij}. \tag{4.65}$$

The first summation in (4.64) contains the same number of terms as there are pairs of adjacent faces, and it becomes

$$\sum_{i=1}^{n} \left({\sum_j}' A_{ij} l_{ij} - \lambda \sigma_i \right) \delta k_{ni} = 0. \tag{4.66}$$

In the above equation, $\sum_j'$ denotes a summation extended only to the edges that form the boundary of the face σ_i. From the arbitrariness of δk_{ni}, we finally obtain the *Gibbs rule* for the equilibrium configuration of a crystal in its melt (see Eq. 666 of [86])

$${\sum_j}' A_{ij} l_{ij} = \lambda \sigma_i, \quad i = 1, \ldots, n. \tag{4.67}$$

We aim to recognize the restrictions that lead from (4.67) to the equilibrium form of the crystal C in our hypothesis that C is convex with respect to the internal point $\mathbf{r}_0$. To this end, we denote the distances of $\mathbf{r}_0$ from the adjacent faces σ_i and σ_j by h_i and h_j, respectively. It is then easy to verify that (see Fig. 4.7 and [97])

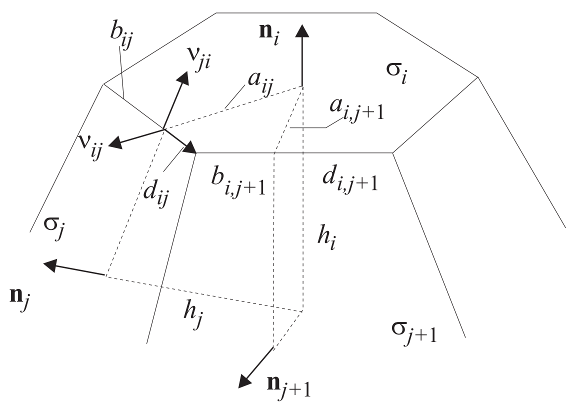

Fig. 4.7 A complete set of notations in a polyhedric crystal

$$h_i \mathbf{n}_i + a_{ij} \nu_{ij} - a_{ji} \nu_{ji} - h_j \mathbf{n}_j = \mathbf{0}. \tag{4.68}$$

This condition implies the relations

$$\begin{aligned} a_{ij} + a_{ji} \cos \alpha_{ij} &= h_j \sin \alpha_{ij}, \\ a_{ji} + a_{ij} \cos \alpha_{ij} &= h_i \sin \alpha_{ij}, \end{aligned}$$

from which we derive that

$$a_{ij} = h_j \csc \alpha_{ij} - h_i \cot \alpha_{ij} \geq 0, \tag{4.69}$$

$$a_{ji} = h_i \csc \alpha_{ij} - h_j \cot \alpha_{ij} \geq 0. \tag{4.70}$$

The above equations tell us that the inequalities

$$\cos \alpha_{ij} \leq \frac{h_j}{h_i}, \quad \cos \alpha_{ij} \leq \frac{h_i}{h_j} \tag{4.71}$$

must be satisfied if two adjacent faces are formed. The conditions (4.69) and (4.70) allows us to write the area of the surface σ_i in the form

$$\sigma_i = \frac{1}{2} \sum_j{}' a_{ij} l_{ij} = \frac{1}{2} \sum_j{}' (h_j \csc \alpha_{ij} - h_i \cot \alpha_{ij}) l_{ij}. \tag{4.72}$$

Similarly, it can be proven (see [97]) that l_{ij} is a linear combination of h_i, h_j, h_{j-1}, and h_{j+1}. Introducing these quoted relations and (4.72) into (4.67), we get a system of n equations in the unknowns h_i that, in principle, yield the equilibrium configuration of a crystal provided that we know the number of faces n and the angles α_{ij} between the normals to any pair of faces S_i and S_j.

Moreover, by introducing (4.72) into (4.67), we obtain

$$\sum{}' \left[\left(E_j - \frac{\lambda}{2} \csc \alpha_{ij} h_j \right) - \left(E_i - \frac{\lambda}{2} h_i \right) \cot \alpha_{ij} \right] l_{ij} = 0, \tag{4.73}$$

and we can conclude that, for any crystal, there are possible equilibrium configurations that obey *Wulff's law*

$$\frac{2E_i}{h_i} = \lambda, \quad i = 1, \ldots, n. \tag{4.74}$$

The constant λ that appears in (4.74) can easily be determined from ({3.58) and (4.74), which leads us to

$$\Psi = \frac{\lambda}{2} \sum_{i=1}^{n} h_i \sigma_i = \frac{3}{2} \lambda V, \tag{4.75}$$

so that

$$\lambda = \frac{2\Psi}{3V}$$

has the dimensions of a pressure.

The equilibrium configurations of a crystal can also be obtained by minimizing the total free energy of the crystal and bulk phases at constant mass. For the sake of simplicity, we suppose that the crystal is completely surrounded by its melt or vapor and that the pressure is uniform everywhere.

The total free energy Ψ can then be written as (see 4.37)

$$\Psi = \sum_{i=1}^{n} E_i\sigma_i + \int_C \rho\psi(\mathbf{F})\, dv + \int_{C_F} \rho_F\psi_F(\rho_F)\, dv, \tag{4.76}$$

where C is the region occupied by the crystal, C_F is the region of the melt or vapor, ψ is the free energy of the bulk crystal phase (which depends on the deformation gradient $\mathbf{F}$), and ψ_F is the free energy of the fluid phase. Proceeding as in the above section, it is possible to prove that an equilibrium configuration of the whole system is characterized by the following conditions:

$$p = const, \quad \text{in}\, C, \tag{4.77}$$

$$p_F = p_e, \quad \text{in}\, C_F, \tag{4.78}$$

$$\psi + p = \psi_F + p_F, \quad \text{on}\, \partial C \tag{4.79}$$

$$\frac{1}{\sigma_i}\sum_{j}' A_{ij} l_{ij} = p - p_F, \quad i, j = 1, \dots, n. \tag{4.80}$$

In particular, the Wulff law becomes

$$\frac{2E_i}{h_i} = p - p_F, \quad i = 1, \dots, n. \tag{4.81}$$

We conclude this section with the following remarks.

Remark In the formulae found in classical papers or textbooks, the surface tension γ_i of the crystal surface σ_i does not differ from its surface energy E_i. This is due to the hypothesis that σ_i is supposed to have the structure of a soap film.

Remark Due to the very small values of surface energy E_i, appreciable pressure jumps require very small crystals (see (4.81)).

4.7 Wulff's Construction

It is important to note that we do not know the number of faces n of a crystal at equilibrium with its melt or vapor. Moreover, when the pressure jump is known, there are many possible equilibrium configurations, depending on n. Wulff suggested that there is only one equilibrium state that is in accordance with (4.81) and that also corresponds to an absolute minimum in the surface energy. This configuration can be obtained with Wulff's construction, as follows. Consider the function $E(\mathbf{n})$, which gives the surface energy per unit area of the face σ whose unit normal is $\mathbf{n}$. If we put $f(\mathbf{n}) = 2E(\mathbf{n})/(p - p_F)$, then the equilibrium configuration of the crystal is given by the spatial region Ω, which is defined by the intersection of all of the half-planes that satisfy the following condition:

$$\mathbf{r} \cdot \mathbf{n} \leq f(\mathbf{n}), \quad \forall \mathbf{n} \in S^2, \tag{4.82}$$

where S^2 is the unit sphere.

If we consider an ideal planar crystal, this construction is shown in Figs. 4.8–4.9 for $f(\mathbf{n}) = 4 + \cos(4\theta)$ and in Figs. 4.10–4.11 for $f(\mathbf{n}) = 4 + \cos(8\theta)$.

These figures can be obtained by referring to the notebook ***Chapter4.nb***, written in the software ***Mathematica***®. Moreover, this notebook also allows us to derive the Wulff shapes of three-dimensional crystals. The notebook can be downloaded via the Internet. ***Mathematica***® notebooks from this chapter are available for download at http://www.birkhauser.com/978-0-8176-4869-5.

Note that it has not been proven that Wulff's construction corresponds to a global minimum of the surface energy; it is simply a reasonable rule for obtaining an equilibrium configuration of a crystal.

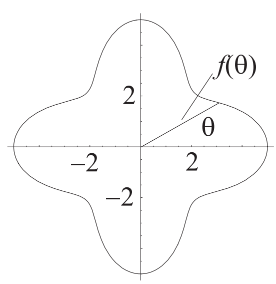

Fig. 4.8 A possible form for the surface energy

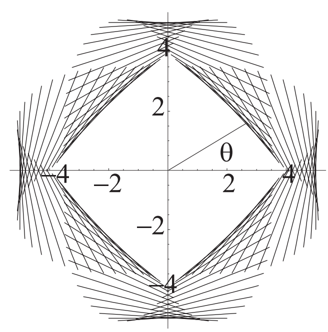

Fig. 4.9 Wulff construction relating to the surface energy of Fig. 4.8

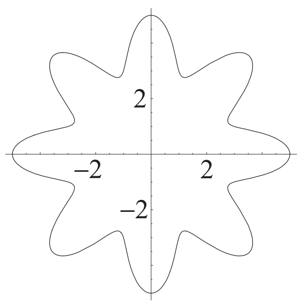

Fig. 4.10 Another possible form for the surface energy

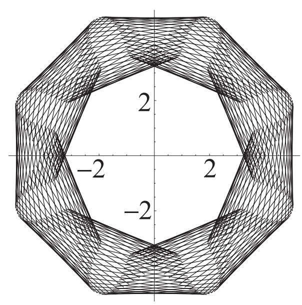

Fig. 4.11 Wulff construction relating to the surface energy of Fig. 4.10

Therefore, a more correct formulation of crystal equilibrium involves finding the closed surface Σ that represents a minimum of the functional

$$\Psi(\Sigma) = \int_{\Sigma} E(\mathbf{n})\, d\sigma,$$

under the constraint that the volume V internal to Σ is assigned and the energy $E(\mathbf{n}) : S^2 \to \Re^+$ per unit area is a continuous function that is differentiable almost everywhere (see [88]–[92]). This approach to crystal equilibrium attributes a dominant role to surface tension in crystallization, but it does not relate this process to temperature and pressure. A more complete variational formulation of equilibrium involves requiring that the displacement $\mathbf{u}$ in the elastic crystal C, its boundary Σ, and the density ρ in the fluid phase C_F represent a minimum for the functional

$$\Psi(\mathbf{u}, \Sigma, \rho) = \int_{C} \psi(\nabla\mathbf{u})\, dv + \int_{C_F} \psi_F(\rho)\, dv + \int_{\Sigma} e(\mathbf{n})\, d\sigma,$$

under the constraint that the total mass is constant. This formulation should imply a pressure jump on the surface of the crystal, even if it is planar. For a different formulation of crystal equilibrium and evolution based on the nonlocal theories of continua, see [93]–[100].

Chapter 5

Stationary and Time-Dependent Phase Changes

5.1 The Problem of Continuous Casting

In the preceding chapter we used the model of a continuous system with a nonmaterial interface to analyze some phase equilibrium problems. In this chapter we show that it is possible to describe some stationary and time-dependent phase changes by again adopting suitable models of a continuous system with a nonmaterial interface.

Let us consider the introduction of liquid metal into a long cylindrical pipe T, where it is cooled in order to be converted into the solid state before being extracted from the other side of T. If the liquid is introduced at a constant velocity, we say that this process is *continuous casting* if we can continuously extract the solid metal in such a way that the surface Σ that separates the two phases remains fixed inside T. This problem involves determining the correct thermal boundary conditions on ∂T when the entrance velocity is given together with the geometry of T.

We introduce a model that is appropriate to describe stationary continuous casting. More precisely, we consider a system S comprising a solid bulk phase C_s and a fluid bulk phase C_l that are separated by a nonmaterial interface Σ (see Fig. 5.1). Bearing in mind that the surface separating the fluid and solid phases must be at rest in continuous casting, we assume that:

1. The interface Σ is a fixed surface that does not have mechanical quantities and reduces to a surface of discontinuity for the bulk fields

2. The process is stationary

3. The materials that fill the two regions C_s and C_l are incompressible

A. Romano, A. Marasco, *Continuum Mechanics*, Modeling and Simulation in Science, Engineering and Technology, DOI 10.1007/978-0-8176-4870-1_5,

and inviscid[1]

4. The only specific body force acting is the force of gravity $\rho\mathbf{g}$

5. In the bulk phases, the Cauchy stress tensor $\mathbf{t}$ reduces to a pressure, and the specific internal energy e is proportional to the absolute temperature θ:
$$\mathbf{t}_i = -p_i\mathbf{I}, \quad e_i = c_i\theta, \quad i = l, s, \tag{5.1}$$
where c_i are the constant specific heats of the two phases

6. The heat current vector is given by
$$\mathbf{h}_i = -k_i\nabla\theta_i, \quad i = l, s, \tag{5.2}$$
where k_i is the constant thermal conductivity of the phase C_i;

7. The velocity fields in C_l and C_s are uniform and directed along the the vertical axis Oz; that is, if $\mathbf{u}$ denotes the downward unit vector along Oz, then
$$\mathbf{v}_i = v_i\mathbf{u}, \quad i = l, s \tag{5.3}$$

8. The absolute temperature θ is continuous across Σ.

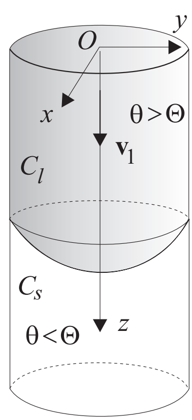

Fig. 5.1 Stationary two-phase system for continuous casting

[1]This hypothesis is justified because the velocity of entry is very low.

If we account for hypotheses 1–7 in the balance equations and the jump conditions (4.48)–(4.53) and (3.92), we obtain the following equations in $C_l \bigcup C_s - \Sigma$:

$$\nabla \cdot \mathbf{v} = 0, \tag{5.4}$$

$$\mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \rho \mathbf{g}, \tag{5.5}$$

$$c\rho \mathbf{v} \cdot \nabla \theta = k\Delta\theta, \tag{5.6}$$

and the following jump conditions on Σ:

$$[[\rho v_n]] = 0, \tag{5.7}$$

$$[[\rho \mathbf{v} v_n - p\mathbf{n}]] = \mathbf{0}, \tag{5.8}$$

$$\left[\left[\rho\left(\frac{1}{2}v^2 + e\right) v_n - p v_n + k\nabla\theta \cdot \mathbf{n}\right]\right] = 0, \tag{5.9}$$

$$\left[\left[\psi + \frac{p}{\rho}\right]\right] = 0. \tag{5.10}$$

We note that (5.10) is equivalent to saying that, if the evolution of the system is close to equilibrium, then inequality (3.92) can be replaced by Eq. 5.10 (see Sect. 4.7).

If we consider the remaining hypotheses, 8 and 9, then (5.4)–(5.10) assume the following final forms:

in $C_l \bigcup C_s - \Sigma$:

$$v_i = v_i^0, \tag{5.11}$$

$$p_i = \rho_i g z + p_i^0, \tag{5.12}$$

$$\rho_i c_i v_i \frac{\partial \theta_i}{\partial z} = k_i \Delta\theta_i, \quad i = l, s, \tag{5.13}$$

on Σ:

$$\rho_s v_{sn} = \rho_l v_{ln}, \tag{5.14}$$

$$[[v]]\, \rho_l v_{ln} \mathbf{u} = [[p]]\, \mathbf{n}, \tag{5.15}$$

$$\left[\left[\frac{1}{2}v^2 + e\right]\right] \rho_l v_{ln} - [[pv]]\, \mathbf{n} \cdot \mathbf{u} = -\,[[k\nabla\theta]] \cdot \mathbf{n}, \tag{5.16}$$

$$[[\psi]] + \frac{p_s}{\rho_s} - \frac{p_l}{\rho_l} = 0. \tag{5.17}$$

In (5.11) and (5.12), v_i^0 and p_i^0 are constants that can assume different values in C_l and C_s. In C_l they coincide, respectively, with the velocity of entry of the melt and the pressure it is subjected to.

Using nondimensional analysis, it is possible to show that the terms that appear in the above system have different weights. To this end, let us introduce suitable scaling quantities (L, U, P, Θ) for length, velocity, pressure, and temperature. We use the known velocity of entry v_l of the melt present in C_l as the scaling velocity U, whereas P and Θ will be identified with the atmospheric pressure and the melting temperature. Moreover, if $c = \min\{c_l, c_s\}$, we assume that

$$v_l^2 \ll \frac{P}{\rho_l} \ll \hat{c}\Theta. \tag{5.18}$$

In other words, the kinetic energy is assumed to be much smaller than the specific energy associated with the pressure, and the latter, in turn, is assumed to be much smaller than the specific internal energy.

Finally, we introduce the notation

$$\mu = \frac{\rho_s}{\rho_l}, \quad \alpha = \frac{k_l}{\rho_l v_l c_l\, L}, \tag{5.19}$$

$$\hat{c} = \frac{c_s}{c_l}, \quad \hat{k} = \frac{k_s}{k_l}. \tag{5.20}$$

After labeling the nondimensional quantities with the same symbols as the corresponding dimensional quantities and denoting $\theta_{,x}$ as the partial derivative of θ with respect to x, as well as some tedious but simple calculations, equations (5.13) become

$$\theta_{l,z} = \alpha\Delta\theta_l, \tag{5.21}$$

$$\theta_{s,z} = \alpha\frac{\hat{k}}{\mu\hat{c}}\Delta\theta_s, \tag{5.22}$$

whereas (5.14) and (5.15) lead us to the following results at the interface Σ for the nondimensional velocities and pressures:

$$v_s = \frac{1}{\mu}, \quad v_l = 1, \tag{5.23}$$

$$v_{sn} = \frac{1}{\mu}\mathbf{u}\cdot\mathbf{n}, \quad v_{ln} = \mathbf{u}\cdot\mathbf{n}, \tag{5.24}$$

$$[[p]] = 0. \tag{5.25}$$

Finally, if we recall that the temperature is continuous across Σ and coincides with the melting temperature Θ, (5.16) and (5.17) reduce to the following conditions on Σ:

$$\left(\hat{c}\theta_s^+ - \theta_l^-\right)\mathbf{u}\cdot\mathbf{n} = -\alpha(\hat{k}(\nabla\theta_s)^+ - (\nabla\theta_l)^-)\cdot\mathbf{n}, \tag{5.26}$$

$$[[\psi]] = 0. \tag{5.27}$$

If $z = \varphi(x, y)$ is the unknown equation for the interface, then the vector $(-\varphi_{,x}, -\varphi_{,y}, 1)$ is orthogonal to Σ, so we have

$$\mathbf{u} \cdot \mathbf{n} = \frac{1}{(1 + \varphi^2{}_{,x} + \varphi^2{}_{,y})^{1/2}}, \tag{5.28}$$

and (5.26) can also be written as

$$\left(\hat{c}\theta_s^+ - \theta_l^-\right) = \alpha(\hat{k}(\nabla\theta_l)^+ - (\nabla\theta_s)^-) \cdot (\nabla\varphi - \mathbf{u}). \tag{5.29}$$

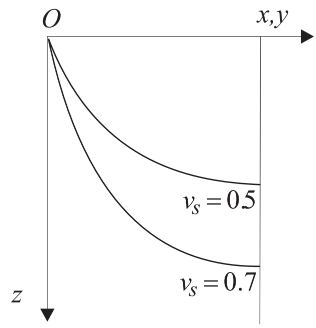

Fig. 5.2 Interface profiles in continuous casting

Finally, in this model of continuous casting, we obtain the following boundary value problems:

in C_s:

$$\theta_{l,z} = \alpha \Delta \theta_l, \tag{5.30}$$

in C_l,

$$\theta_{s,z} = \alpha \frac{\hat{k}}{\mu \hat{c}} \Delta \theta_s, \tag{5.31}$$

on Σ:

$$\left(\hat{c}\theta_s^+ - \theta_l^-\right) = \alpha(\hat{k}(\nabla\theta_l)^+ - (\nabla\theta_s)^-) \cdot (\nabla\varphi + \mathbf{u}), \tag{5.32}$$

on $\partial(C_l \bigcup C_s)$:

$$\begin{aligned} \theta(x, y, 0) = \theta_1, \quad \theta(x, y, L) = \theta_2, \\ k\frac{d\theta}{dn} = f(x, y, z). \end{aligned} \tag{5.33}$$

A variational formulation as well as a weak existence theorem for boundary value problem (5.30)–(5.33) are presented in [101]. In [102], L. Faria and J. Rodrigues analyze the above problem from a numerical point of view by assuming rotational symmetry about the $z-$axis. By supposing that the heat extracted from the lateral boundary is expressed by the formula $f = \alpha[\theta - h(z)]$, they obtain the profiles shown in Fig. 5.2 for the interface. From these profiles, we can deduce that the maximum depth of the interface with respect to the plane xy is affected by both the extraction velocity v_s and the lateral cooling.

5.2 On the Evolution of the Solid–Liquid Phase Change

In this section and the next, we analyze the dynamical evolutions of state changes under the following hypotheses (see [103]):

1. The interface $\Sigma(t)$ is a moving surface of discontinuity for the bulk fields
2. The fields depend on only one spatial variable x
3. One of the phases is filled with an incompressible substance at rest, whose specific energy is expressed by the function
$$e = c\,\theta, \tag{5.34}$$
where θ is the absolute temperature and c the constant specific heat
4. The one-dimensional heat flux in both phases takes the form
$$h = -k\theta,_x\,, \tag{5.35}$$
where the conductivity k is a constant that depends on the phase.

We start with the *solid–liquid* state change; i.e, with a system S consisting of two phases C_s and C_l of the same substance in the solid and liquid states, respectively. These phases fill the layers $0 \le x < s(t)$ and $s(t) < x \le L(t)$, where $s(t)$ denotes the location of the *planar* interface $\Sigma(t)$, which, due to hypothesis 1, does not have material fields. Moreover, in view of hypothesis 2, all of the bulk fields are functions of the spatial variable x and of the time t. Taking into account conditions 1–4, the balance equations and the jump conditions (4.48)–(4.53) become:

in the solid phase C_s, i.e., for $(x,t) \in [0, s(t)) \times [0, \infty)$:
$$\rho_s c_s \theta,_t = k_s \theta,_{xx}\,, \tag{5.36}$$

in the liquid phase C_l, i.e., for $(x,t) \in (s(t), L(t)] \times [0,\infty)$:

$$\rho,_t + (\rho v),_x = 0, \tag{5.37}$$
$$\rho(v,_t + v\, v,_x) = -p,_x\,, \tag{5.38}$$
$$\rho(e,_t + ve,_x) = -pv,_x + k\theta,_{xx}\,, \tag{5.39}$$

on the planar interface $\Sigma(t)$:

$$\rho(\dot{s} - v) = \rho_s \dot{s}, \tag{5.40}$$
$$v\rho(\dot{s} - v) - [[p]] = 0, \tag{5.41}$$
$$\left[\left[\frac{1}{2}(v - \dot{s})^2 + e\right]\right] \rho_s \dot{s} - pv = -\,[[k\theta,_x]]\,, \tag{5.42}$$
$$\left[\left[\psi + \frac{p}{\rho}\right]\right] = 0. \tag{5.43}$$

We note that (5.43) holds at equilibrium, but we suppose here that it also remains valid under dynamic conditions (see Sect. 4.7). To simplify the notation, we have not affixed a subscript l to all quantities referring to the liquid.

The above equations also hold if C_s is filled with a liquid and C_l with a gas. This case will be analyzed in the next section. For the case we are considering here, we can further simplify (5.36)–(5.43) by recalling the incompressibility of the liquid phase, which is expressed by the condition:

$$\rho = const. \tag{5.44}$$

From (5.37) and (5.44), we immediately deduce that v depends only on t. Consequently, (5.40) leads us to the relation

$$v = \alpha \dot{s}, \tag{5.45}$$

where

$$\alpha = \frac{\rho - \rho_s}{\rho}. \tag{5.46}$$

It is easy to verify that (5.36)–(5.43) become:

in the solid phase C_s, i.e., for $(x,t) \in [0, s(t)) \times [0,\infty)$:

$$\theta,_t = \frac{k_s}{\rho_s c_s}\theta,_{xx} \equiv a_s \theta,_{xx}\,, \tag{5.47}$$

in the liquid phase C_l, i.e., for $(x,t) \in (s(t), L(t)] \times [0,\infty)$:

$$\alpha\rho\ddot{s} = -p,_x\,, \tag{5.48}$$
$$\theta,_t + \alpha\dot{s}\theta,_x = \frac{k}{\rho c}\theta,_{xx} \equiv a\theta,_{xx}\,, \tag{5.49}$$

on the planar interface $\Sigma(t)$:

$$[[p]] = \alpha\rho_s\dot{s}^2, \tag{5.50}$$

$$[[e]]\,\rho_s\dot{s} - \frac{1}{2}\rho_s\alpha\beta\dot{s}^3 - \alpha p\dot{s} = -\,[[k\theta,_x]]\,, \tag{5.51}$$

$$[[\psi]] - \frac{1}{2}\alpha\mu\dot{s}^2 - \alpha\frac{p_s}{\rho_s} = 0, \tag{5.52}$$

where

$$\beta = \frac{\rho+\rho_s}{\rho} \simeq 2, \quad \mu = \frac{\rho}{\rho_s} \simeq 1. \tag{5.53}$$

We explicitly remark that the incompressibility of the liquid forces the boundary $L(t)$ to move with the velocity v of the particles of C_l; i.e.,

$$\dot{L}(t) = v(t) = \alpha\dot{s}(t), \tag{5.54}$$

so that the motion of the free boundary of C_l is determined by the evolution of the interface $\Sigma(t)$. Usually, the densities of the two phases have almost the same value, so $\alpha \ll 1$ and consequently the linear dimensions of the whole system almost remain constant during the phase change. The boundary value problem (5.47)–(5.52), although highly simplified, is still very difficult to solve. However, a nondimensional analysis shows that not all of the terms in the above equations have the same weight.

Let X be a length comparable to the dimension of the system S, for instance $X = L(0)$. As is well known, the quantity $T = X^2/a_s$ denotes a time that is characteristic of the conduction phenomena. This means that the velocity of the interface can be assumed to be comparable with the rate $V = X/T$. Finally, let us define the initial and boundary data for the temperature, and let Θ be the melting temperature at the ordinary pressure P. By introducing the nondimensional variables

$$\begin{aligned} &x^\star = \frac{x}{X}, \quad t^\star = \frac{t}{T}, \quad p^\star = \frac{p}{P}, \\ &v^\star = \frac{v}{V}, \quad \theta^\star = \frac{\theta}{\Theta}, \\ &\dot{s}^\star = \frac{\dot{s}}{V}, \quad e^\star = \frac{e}{c_s\Theta}, \quad \psi^\star = \frac{\psi}{c_s\Theta}, \end{aligned} \tag{5.55}$$

we obtain the following nondimensional system in which we have again used the same symbols for the nondimensional fields:

$$\theta,_t = \theta,_{xx}\,, \quad (x,t) \in [0, s(t)) \times [0, \infty), \tag{5.56}$$

$$p,_x = -\alpha A\mu\ddot{s}, \quad (x,t) \in (s(t), L(t)] \times [0, \infty), \tag{5.57}$$

$$\theta,_t + \alpha\dot{s}\theta,_x = \hat{a}\theta,_{xx}\,, \tag{5.58}$$

$$[[p]] = \alpha A\dot{s}^2, \tag{5.59}$$

$$[[e]]\,\dot{s} + \frac{1}{2}\alpha\beta B\dot{s}^3 - \alpha\frac{B}{A}p\dot{s} = -\left[\left[\hat{k}\theta_{,x}\right]\right], \tag{5.60}$$

$$[[\psi]] - \frac{1}{2}\mu\alpha B\dot{s}^2 - \alpha\frac{B}{A}p_s = 0, \tag{5.61}$$

where

$$\hat{k} = \frac{k}{k_s}, \quad \hat{k_s} = 1,$$
$$\hat{a} = \frac{a}{a_s}, \quad A = \frac{\rho_s V^2}{P}, \quad B = \frac{V^2}{c_s\Theta}. \tag{5.62}$$

Tables 5.1 and 5.2 contain numerical values of the physical quantities we are considering and show that the values of A, B, and B/A are negligible with respect to α.

		ρ (g/cm^3)	c (10^6erg/gK)	k (/10^6erg/cms K)	$\theta_{\mathbf{M}}$	λ/10
Fe	*sol.*	7.36	6.91	2.91	1808	2.7
	liq.	6.9	8.66	2.33		
Cu	*sol.*	8.62	4.81	30.9	1356	2.1
	liq.	8.36	5.44	30.9		
Al	*sol.*	2.55	11.39	24.2	930	3.9
	liq.	2.38	10.47	24.2		
H_2O	*sol.*	0.91	19.26	0.22	273	3.35
	liq.	1	41.86	0.05		

Table 5.1 Thermodynamic data for some materials

	α	$\hat{\mathbf{k}}$	$\hat{\mathbf{a}}$	λ	A	B	B/A
Fe	−0.067	0.80	0.68	2.18	2.4×10^{-12}	2.6×10^{-16}	1.1×10^{-4}
Cu	−0.031	1	0.91	3.27	4.7×10^{-10}	8.5×10^{-14}	1.8×10^{-4}
Al	−0.071	1	1.17	3.72	1.7×10^{-10}	5.6×10^{-14}	3.8×10^{-4}
H_2O	0.087	0.25	0.11	6.37	1.4×10^{-14}	3×10^{-17}	2.1×10^{-3}

Table 5.2 Numerical values of the nondimensional quantities defined in (5.62)

In Table 5.2, $\lambda = [[e]]/c_s\Theta$ denotes the nondimensional latent heat. The numerical results listed in these tables tell us that we can neglect terms containing the factors A, B, or A/B, meaning that (5.56)–(5.61) reduce to the following:

$$\theta_{,t} = \theta_{,xx}\,, \quad (x,t) \in [0, s(t)) \times [0,\infty), \tag{5.63}$$

$$p = p_e, \quad (x,t) \in (s(t), l(t)] \times [0,\infty), \tag{5.64}$$

$$\theta_{,t} + \alpha\dot{s}\theta_{,x} = \hat{a}\theta_{,xx}\,, \tag{5.65}$$

$$[[p]] = 0, \tag{5.66}$$

$$[[e]]\,\dot{s} = -\left[\left[\hat{k}\theta_{,x}\right]\right], \tag{5.67}$$

$$[[\psi]] = 0. \tag{5.68}$$

where $l(t) = L(t)/X$.

Equation 5.68 gives the melting temperature for a given pressure. This is the classical *Stefan problem*, about which much has been written (see for instance [104]–[105]).

5.3 On the Evolution of the Liquid–Vapor Phase Change

The liquid–vapor phase change is a very complex phenomenon produced by a combination of conduction and convection. Vapor bubbles form inside the liquid, and these bubbles can merge, producing larger ones. These reach the free surface of the liquid, where they release their vapor. However, when the external temperature is only a few degrees from the evaporation temperature, the phenomenon occurs through simpler modalities.

Let us consider a liquid mass inside a rigid container with a freely moving piston, which is subjected to a pressure that is less than the critical one. Moreover, let us suppose that the liquid is at rest at the saturation temperature. If the temperature of the piston is increased by a few degrees, a vaporization process begins across the liquid surface near the piston (see p. 491 of [82]), and a vapor phase forms that has a density much less than the liquid density. Consequently, the total volume occupied by the system increases in proportion to the quantity of matter supporting the phase change. There are the following fundamental differences from the solid–liquid phase change (analyzed in the previous section):

1. The vapor phase is not incompressible

2. The densities of the two phases are so different from each other that the vapor density is negligible compared to the density of the liquid.

In other words, the two cases differ with respect to the state equations that describe the system and the approximations we can introduce.

We again limit ourselves to fields that depend on the spatial variable x and the time t. The interface is still planar and its motion is described by the function $s(t)$. Finally, the liquid is assumed to be incompressible and at rest. Under these assumptions, the phenomenon is again described by the system (5.40)–(5.43), except that the equations that originally referred to the solid now refer to the liquid, and the equations that originally referred to the liquid now refer to the vapor. We now need to find further simplifications deriving from condition 2.

We find from (5.40) that

$$v = \frac{\rho - \rho_l}{\rho}\dot{s}, \tag{5.69}$$

where the fields without suffixes refer to the vapor phase. On the other hand, the vapor density is negligible with respect to the liquid density: for instance, at atmospheric pressure and a temperature of 100°C, the water density is 0.958 g/cm^3, whereas the vapor density is 0.596 $\times 10^{-3}$ g/cm^3. Therefore, we can replace (5.69) with the approximate equation

$$v = -\frac{\rho_l}{\rho}\dot{s}. \tag{5.70}$$

This relation allows us to eliminate v from the jump conditions (5.41)–(5.43), which become

$$[[p]] = -\rho_l \frac{\rho_l}{\rho}\dot{s}^2, \tag{5.71}$$

$$\left(\frac{1}{2}\left(\frac{\rho_l}{\rho}\right)^2 \dot{s}^2 + [[e]] + \frac{p}{\rho}\right)\rho_l \dot{s} = -\left[[k\theta_{,x}]\right], \tag{5.72}$$

$$[[\psi]] + \frac{p}{\rho} - \frac{\rho_l}{\rho}\dot{s}^2 = 0. \tag{5.73}$$

Again, we can simplify the problem by resorting to nondimensional analysis. When the scaling quantities are chosen, the large difference between the densities of the two phases is the determining factor. In fact, condition 2 implies that, when a given quantity of matter passes from one phase to the other, there are large variations in:

- The volume occupied
- The particle velocity.

These remarks constrain the scaling quantities that can be chosen for the length, density, and velocity according to the phase. In particular, we choose the constant value ρ_l to scale the density of the liquid, and the value $\overline{\rho}$ of the vapor density at atmospheric pressure and the vaporization temperature to scale the vapor density. These choices lead to

$$\mu = \frac{\overline{\rho}}{\rho_l} \ll 1. \tag{5.74}$$

For water (as an example), $\mu = 0.622 \times 10^{-3}$. Assuming that, when a given mass of liquid becomes a vapor, its volume increases such that its linear dimension increases by about $1/\mu$ times, we introduce a scaling length X_l in the liquid phase and a scaling length $X = X_l/\mu$ in the vapor phase.

If we use the quantity $T = X_l^2/a_l$ as the timescale, where $a_l = k_l/\rho_l c_l$ is the diffusivity of the liquid, we obtain two velocities $V_l = X_l/T$ and $V = X/T$. We refer to the former as the vapor velocity to the latter as the interface velocity. Finally, we obtain the following nondimensional variables:

$$
\begin{aligned}
&t^\star = \frac{t}{T}, \quad x^\star = \frac{x}{X_l}, \quad x \in [0, s(t)),\\
&x^\star = \frac{x}{X}, \quad x \in (s(t), L(t)],
\end{aligned} \tag{5.75}
$$

$$
\begin{aligned}
&\rho^\star = \frac{\rho}{\bar{\rho}}, \quad \dot{s}^\star = \frac{\dot{s}}{V_l}, \quad v^\star = \frac{v}{V},\\
&\theta^\star = \frac{\theta}{\Theta}, \quad p^\star = \frac{p}{P}, \quad e^\star = \frac{e}{c_l\Theta}, \quad \psi^\star = \frac{\psi}{c_l\Theta},
\end{aligned} \tag{5.76}
$$

where Θ is, for instance, the highest temperature at the boundary, and c_l is the specific heat of the liquid.

Simple calculations lead us to the following equations (for the sake of simplicity we have not assigned asterisks to the nondimensional fields):
in the liquid phase, i.e., for $(x, t) \in [0, s(t)) \times [0, \infty)$:

$$p_l = p_l(t), \tag{5.77}$$

$$\theta_{l,t} = \theta_{l,xx}\,, \tag{5.78}$$

in the vapor phase, i.e., for $(x, t) \in (s(t), l(t)] \times [0, \infty)$:

$$\rho_{,t} + (\rho v)_{,x} = 0, \tag{5.79}$$

$$A\rho(v_{,l} + v\, v_{,x}) = -p_{,x}\,, \tag{5.80}$$

$$\rho(e_{,t} + v e_{,x}) = -B p v_{,x} + \hat{k}\mu\theta_{,xx}\,, \tag{5.81}$$

on the interface:

$$v = -\frac{\dot{s}}{\rho}, \tag{5.82}$$

$$[[p]] = -A\frac{\dot{s}^2}{\rho}, \tag{5.83}$$

$$\frac{1}{2}AB\frac{\dot{s}^2}{\rho^2} + \left([[e]] + B\frac{p}{\rho}\right)\dot{s} = -\left(\mu\hat{k}\theta_{,x} - \theta_{l,x}\right), \tag{5.84}$$

$$[[\psi]] + B\frac{p}{\rho} - AB\frac{1}{\rho_l}\frac{\dot{s}^2}{\rho^2} = 0, \tag{5.85}$$

where $l(t) = L(t)/X_l$, and

$$\hat{k} = \frac{k}{k_l}, \quad A = \frac{\overline{\rho}V^2}{P}, \quad B = \frac{P}{\overline{\rho}c_l\Theta}. \tag{5.86}$$

In order to evaluate the orders of magnitude of the nondimensional quantities A and B, we assume that the vapor phase is a perfect gas. This hypothesis implies the following constitutive equation for the pressure:

$$p = \frac{R}{M}\rho\theta, \tag{5.87}$$

where $R = 8.31 \times 10^7$ erg/(mol K) is the universal gas constant and M is the molar mass of the substance undergoing the phase transition. In particular, for $\theta = \Theta$, (5.87) implies the following relation involving the reference quantities $\overline{\rho}$, Θ, and P:

$$P = \frac{R}{M}\overline{\rho}\Theta. \tag{5.88}$$

Consequently,

$$B = \frac{P}{\overline{\rho}c_l\Theta} = \frac{R}{Mc_l}. \tag{5.89}$$

For water, $M = 18$ g and $c_l \simeq 50 \times 10^6$ erg/K, so $B \simeq 0.1$.

On the other hand, from (5.86) and (5.87), we have

$$A = \frac{\overline{\rho}V^2}{P} = \frac{MV^2}{R}. \tag{5.90}$$

If we take $X = 10^2$ cm and recall that $\overline{\rho} \simeq 10^{-3}$ g/cm^3, then it is easy to verify that $V = 10^{-1}$ cm/s and $A \simeq 10^{-9} \ll 1$. It is worth noting that, since B is on the order of unity, we can use $\overline{\rho}c_l\Theta$ as the reference pressure. If we do this, the evolution equations are obtained from (5.78)–(5.86) by putting $B = 1$. Finally, we note that, although $\mu \simeq 10^{-3}$, this factor multiplies the higher-order derivatives in (5.81) and (5.84). Therefore, these terms cannot be neglected as they express the presence of a boundary layer at the interface.

From (5.80) and (5.83), we derive that the pressure is uniform throughout the whole system and that it coincides with the external pressure p_e. Finally, the above equations assume the following simplified forms:
in the liquid phase, i.e.,for $(x,t) \in [0, s(t)) \times [0, \infty)$:

$$\theta_{,t} = \theta_{,xx}\,, \tag{5.91}$$

in the vapor phase, i.e., for $(x,t) \in (s(t), l(t)] \times [0, \infty)$:

$$\rho_t + (\rho v)_{,x} = 0, \tag{5.92}$$

$$\rho(e_{,t} + ve_{,x}) = -pv_{,x} + \hat{k}\mu\theta_{,xx}\,, \tag{5.93}$$

on Σ:

$$v = -\frac{\dot{s}}{\rho}, \tag{5.94}$$

$$\left([[e]] + B\frac{p}{\rho}\right)\dot{s} = -\left(\hat{k}\mu\theta_{,x} - \theta_{l,x}\right), \tag{5.95}$$

$$[[\psi]] + B\frac{p}{\rho} = 0, \tag{5.96}$$

This system should be supplemented with the appropriate initial data for the density, velocity and temperature, as well as the boundary data for the temperature and the need for field continuity across the interface. When we assign constitutive equations for the liquid and the vapor, we obtain a boundary value problem in the unknowns $\rho(x,t)$, $v(x,t)$, $\theta(x,t)$, and $\dot{s}(x,t)$.

5.4 The Case of a Perfect Gas

In this section we analyze the system (5.91)–(5.96) assuming that the vapor phase is a perfect gas. In other words, we suppose that the specific internal energies and free energies are given by the following constitutive equations:

$$e_l = c_l\theta + e_0, \quad \psi_l = -c_l\theta\ln\frac{\theta}{\Theta} + \psi_0, \tag{5.97}$$

$$e = c\,\theta, \quad \psi = \frac{R}{M}\theta\ln\frac{\rho}{\overline{\rho}} - c\theta\ln\frac{\theta}{\Theta}, \tag{5.98}$$

where e_0 and ψ_0 are constants, Θ is a reference temperature and $\overline{\rho}$ is the corresponding value for the vapor density. The additive constants are omitted from (5.98) since it is always possible to assume that they vanish in one of the two phases. Moreover, the pressure p in the vapor is given by the constitutive equation

$$p = \frac{R}{M}\rho\theta. \tag{5.99}$$

In nondimensional form, the above equations become

$$e_l = \theta + e_\star, \quad \psi_l = -\theta\ln\theta + \psi_\star, \tag{5.100}$$

$$e = \hat{c}\,\theta, \quad \psi = \frac{R}{M}\theta\ln\rho - c\theta\ln\theta, \tag{5.101}$$

where $e_\star = c_0/c_l\Theta$, $\psi_\star = \psi_0/c_l\Theta$, $\hat{c} = c/c_l$, and ρ, θ are nondimensional quantities. Finally, using (5.87) and (5.88), we obtain the nondimensional form for the pressure in the vapor:

$$p = \rho\theta. \tag{5.102}$$

Introducing (5.100), (5.101) and (5.102) into (5.96), we obtain the following equation:

$$\frac{R\theta}{M}\ln\rho - \hat{c}\theta\ln\theta + \theta\ln\theta + B\frac{p}{\rho} = 0,$$

which, in view of (5.102), assumes the form

$$\frac{R\theta}{M}\ln p - \frac{R\theta}{M}\ln\theta - \hat{c}\theta\ln\theta + \theta\ln\theta + B\theta = 0.$$

This equation gives us the value of the pressure on the interface for any vaporization temperature $\overline{\theta}$:

$$p = e^{-\frac{BM}{R}}\overline{\theta}^{\frac{R+(\hat{c}-1)M}{R}}. \tag{5.103}$$

The inverse function

$$\overline{\theta} = e^{\frac{BM}{R}}p^{\frac{R}{R+M(\hat{c}-1)}}, \tag{5.104}$$

supplies the vaporization temperature for any pressure. Since in our problem the pressure is uniform in both phases and continuous across the interface, p coincides with the external pressure p_e. Consequently, (5.104) gives the vaporization temperature for any external pressure. Finally, the latent heat $\lambda = [[e]] + Bp/\rho$ is given by the relation

$$\lambda = (\hat{c} - 1 + B)\overline{\theta}. \tag{5.105}$$

Now we suppose that the initial and boundary temperatures in the liquid phase coincide with the vaporization temperature corresponding to the given external pressure p_e. Under these conditions, the temperature field in the liquid has a constant value throughout the liquid phase. An important consequence of this remark is obtained from (5.101) and (5.102):

$$\rho e = \frac{p}{\theta}\,\hat{c}\,\theta = p_e\hat{c} = const. \tag{5.106}$$

Consequently, (5.93) leads us to the result

$$(\rho e + p_e)v - \mu\hat{k}\theta_{,x} = p_e(\hat{c}+1)v - \mu\hat{k}\theta_{,x} = f(t), \tag{5.107}$$

where $f(t)$ is an arbitrary function of time. Since the expression on the left-hand side of (5.107) assumes, at a given instant, the same value throughout the vapor phase, we can determine $f(t)$ by evaluating the left-hand side at the interface. Then, taking into account (5.94), (5.95) and (5.105), and recalling that $(\theta,_x)^- = 0$, we obtain

$$p_e(\hat{c}+1)v - \mu\hat{k}\theta,_x = -(\hat{c}+1)\overline{\theta}\dot{s} + (\hat{c}-1+B)\overline{\theta}\dot{s}, \tag{5.108}$$

so that

$$v = C\theta,_x + D\dot{s}, \tag{5.109}$$

where

$$C = \frac{\mu\hat{k}}{p_e(\hat{c}+1)}, \quad D = \frac{B-2}{p_e(\hat{c}+1)}\overline{\theta} \tag{5.110}$$

and $\overline{\theta}$ is a known function of p_e (see (5.104)).

It remains to consider mass conservation (5.92), which, in view of (5.102) and (5.109), can be written as follows:

$$\left(\frac{1}{\theta}\right)_{,t} + \left(\frac{C}{\theta}\theta,_x + \frac{D}{\theta}\dot{s}\right)_{,x} = 0,$$

or, equivalently, in the form

$$\theta,_t + C(\theta,_x)^2 + D\dot{s}\theta,_x - C\theta\theta,_{xx} = 0. \tag{5.111}$$

Finally, the following equation holds at the interface:

$$\lambda(p_e)\dot{s} = -\mu\hat{k}\theta,_{xx}\,. \tag{5.112}$$

It is important to note that (5.111) and (5.112) must be solved in the region $(s(t), l(t)) \times [0, \infty)$, where the upper bound $l(t)$ is not known. Therefore, we need a further condition relating to this function, which can be obtained by requiring the conservation of the total mass M_t of the whole system:

$$\rho_l s(t) + \int_{s(t)}^{L(t)} \rho dx = \rho_l s(t) + \int_{s(t)}^{L(t)} \frac{p_e M}{R\theta} dx = M_t. \tag{5.113}$$

When this condition is written in nondimensional form, we obtain a new condition that allows us to complete the formulation of the free boundary value problem.

Chapter 6

An Introduction to Mixture Theory

We have already remarked that the simplified models of continuum mechanics (perfect and viscous fluids, elastic systems, etc.) do not always accurately describe the complex phenomenology exhibited by real materials. In Chap. 2 we discussed a nonstandard model that includes (along with the usual elastic properties) microrotation, revealing internal microstructure. There are other situations in which we must derive more complex models to recover some phenomenological features related to internal structure that is erased by the continuous model. For instance, in Chap. 3 the model of a continuum with an interface was proposed in order to produce a macroscopic description of phase transitions in simple materials.

Another more complex model is required to address *mixtures*, and we will introduce this model here in order to explain another class of phenomena.

There is no doubt that mixing two or more constituents macroscopically could lead to a continuum that could be described by a simple continuum mechanics model. However, recovering its constitution allows us to supply a unique description for all continua that differ in either their percentages of constituents or their conditions of motion. On the other hand, the model that unifies all of these continua is not a simple one, as it requires the introduction of internal variables like the concentrations and the diffusion velocities of the constituents inside the mixture.

Furthermore, we would very much like to obtain the answer to the following question: is it possible to determine the properties of a mixture when the properties of each constituent of the mixture are known?

In this chapter we limit ourselves to sketching out the foundations of this complex and controversial subject, about which many papers have been written. For the sake of simplicity, we will only present the theory of *nonreacting fluid mixtures*. More particularly, we will apply the model of *simple or classical mixtures* to the phase change of a constituent in a binary mixture. Readers interested in delving deeper into this subject should refer to [108]–[113].

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This brief exposition starts from Truesdell's approach ([106, 107]), which is based on the assumption that the balance laws hold for each constituent and the whole mixture. In this way, it is possible to identify the relationship between the characteristic quantities of the mixture and those of its constituents. We then analyze the model proposed by Gurtin and Vargas [108] for fluid mixtures with low diffusion velocities, and that given by Gurtin [109] for nonreacting fluid mixtures with arbitrary diffusion velocities.

6.1 Balance Laws

Let S be a mixture of m constituents S_a, $a = 1, \dots, m$. Concerning the motion of each constituent, we assume that:

- The equations of motion $\mathbf{x}_a = \mathbf{x}_a(\mathbf{X}_a, t)$ of each constituent are smooth functions, where $\mathbf{X}_a$ varies in a unique reference configuration C_* and t belongs to the time interval $[0, T]$
- For any $t \in [0, T]$, the functions $\mathbf{x}_a(\mathbf{X}_a, t)$ define a one-to-one correspondence between C_* and the actual configuration $C_a = \mathbf{x}_a(C_*, t)$
- The region C_a of the actual configuration of each constituent at any instant coincides with the actual configuration C of the whole mixture
- The balance equations hold during the motions of each constituent and the whole mixture.

We note that the first three assumptions imply that all of the constituents are always present at any point of the actual configuration C. In other words, we are excluding from our scheme any case where a constituent, upon diffusing into a mixture, occupies regions that change in volume over time. When this happens, there is a moving surface inside the actual configuration across which the fields exhibit jumps due to the absence of one or more constituents on one side of it.

If we denote the equation of motion of S by $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, we can define the velocity $\mathbf{v}_a$ of S_a and the velocity $\mathbf{v}$ of S as follows:

$$\mathbf{v}_a = \frac{\partial \mathbf{x}_a}{\partial t}, \qquad \mathbf{v} = \frac{\partial \mathbf{x}}{\partial t}. \tag{6.1}$$

Since we are considering a nonreacting mixture, the local mass balance of S_a can be written as follows:

$$\frac{\partial \rho_a}{\partial t} + \nabla \cdot (\rho_a \mathbf{v}_a) = 0, \tag{6.2}$$

where ρ_a is the mass density of S_a. Moreover, the local balance of mass for S is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{6.3}$$

where ρ is the mass density of S.

In order to introduce a reasonable relation between the quantities ρ_a, $\mathbf{v}_a$ and ρ, $\mathbf{v}$, we note that adding (6.2) for $a = 1, \ldots, m$, leads to the condition

$$\frac{\partial}{\partial t} \sum_{a=1}^{m} \rho_a + \nabla \cdot \sum_{a=1}^{m} \rho_a \mathbf{v}_a = 0,$$

which, compared with (6.3), leads to the natural (but not unique) quantities

$$\rho = \sum_{a=1}^{m} \rho_a, \qquad \mathbf{v} = \frac{1}{\rho} \sum_{a=1}^{m} \rho_a \mathbf{v}_a. \tag{6.4}$$

The first of these equations is quite obvious, while the second identifies the velocity of any element of S with the velocity of its center of mass.

The momentum balance for any S_a can be written in the following form:

$$\frac{\partial}{\partial t}(\rho_a \mathbf{v}_a) + \nabla \cdot (\rho_a \mathbf{v}_a \otimes \mathbf{v}_a) = \nabla \cdot \mathbf{T}_a + \rho_a \mathbf{b}_a, \tag{6.5}$$

where $\mathbf{T}_a$ and $\mathbf{b}_a$ denote the stress tensor and the body force density of S_a, respectively.

Similarly, the momentum balance of S is:

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) = \nabla \cdot \mathbf{T} + \rho \mathbf{b}, \tag{6.6}$$

where the meanings of $\mathbf{T}$ and $\mathbf{b}$ are as one would expect considering (6.5).

If we add all of the equations (6.5) and take (6.4) into account, we obtain

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot \sum_{a=1}^{m} \rho_a \mathbf{v}_a \otimes \mathbf{v}_a = \nabla \cdot \sum_{a=1}^{m} \mathbf{T}_a + \sum_{a=1}^{m} \rho_a \mathbf{b}_a. \tag{6.7}$$

In order to express the partial velocities $\mathbf{v}_a$ in terms of $\mathbf{v}$, we introduce the *diffusion velocities*

$$\mathbf{u}_a = \mathbf{v}_a - \mathbf{v}, \tag{6.8}$$

from which, in view of (6.4), we derive the condition

$$\sum_{a=1}^{m} \rho_a \mathbf{u}_a = \mathbf{0}. \tag{6.9}$$

Using (6.8) and (6.9), we can now place (6.7) in the form of (6.6), provided that we set

$$\mathbf{T} = \sum_{a=1}^{m} (\mathbf{T}_a - \rho_a \mathbf{u}_a \otimes \mathbf{u}_a), \qquad \rho \mathbf{b} = \sum_{a=1}^{m} \rho_a \mathbf{b}_a. \tag{6.10}$$

We note again that the introduction of quantities (6.4) and (6.10) is spontaneous but not unique.

Taking into account the angular momentum balance of any constituent, we conclude that any stress tensor $\mathbf{T}_a$ is symmetric. Finally, due to (6.10), we can derive the symmetry of the whole stress tensor $\mathbf{T}$.

It remains to consider the energy balance equation for each constituent:

$$\frac{\partial}{\partial t}\left(\rho_a \epsilon_a + \frac{1}{2}\rho_a v_a^2\right) + \nabla \cdot \left(\left(\rho_a \epsilon_a + \frac{1}{2}\rho_a v_a^2\right) \mathbf{v}_a\right) =$$

$$\nabla \cdot (\mathbf{T}_a \cdot \mathbf{v}_a) - \nabla \cdot \mathbf{j}_a + \rho_a \mathbf{b}_a \cdot \mathbf{v}_a + \rho_a r_a, \tag{6.11}$$

where ϵ_a is the specific internal energy of S_a, $\mathbf{j}_a$ is the energy flux vector, and r_a is the specific energy supply. It is worth noting that, in the absence of any other form of energy, the vector flux $\mathbf{j}_a$ reduces to the heat flux $\mathbf{h}_a$.

On the other hand, the energy balance for the whole mixture is

$$\frac{\partial}{\partial t}\left(\rho \epsilon + \frac{1}{2}\rho v^2\right) + \nabla \cdot \left(\left(\rho \epsilon + \frac{1}{2}\rho v^2\right) \mathbf{v}\right) =$$

$$\nabla \cdot (\mathbf{T} \cdot \mathbf{v}) - \nabla \cdot \mathbf{j} + \rho \mathbf{b} \cdot \mathbf{v} + \rho r, \tag{6.12}$$

where the quantities involved here have analogous meanings to those above. Taking into account (6.8) and (6.9), we easily obtain

$$\frac{1}{2}\sum_{a=1}^{m} \rho_a v_a^2 = \frac{1}{2}\rho v^2 + \frac{1}{2}\sum_{a=1}^{m} \rho_a u_a^2. \tag{6.13}$$

By adding together the equations (6.11) for $a = 1, \ldots, m$ and considering (6.13), (6.4) and (6.10), we derive (6.12) provided that we put

$$\rho \epsilon = \sum_{a=1}^{m} \left(\rho_a \epsilon_a + \frac{1}{2}\rho_a u_a^2\right) \tag{6.14}$$

$$\mathbf{j} = \sum_{a=1}^{m} \left(\mathbf{j}_a - \left(\rho_a \epsilon_a + \frac{1}{2}\rho_a u_a^2\right) \mathbf{u}_a - \mathbf{T}_a \cdot \mathbf{u}_a\right), \tag{6.15}$$

$$\rho r = \sum_{a=1}^{m} \rho_a (\mathbf{b}_a \cdot \mathbf{u}_a + r_a). \tag{6.16}$$

Note that, even if each flux vector $\mathbf{j}_a$ reduces to the heat flux vector $\mathbf{h}_a$, the total flux vector $\mathbf{j}$ does not coincide with the sum of all of the heat flux vectors $\mathbf{h}_a$, $a = 1, \ldots, m$, in the presence of a convective term.

Finally, the entropy inequality for each constituent S_a is

$$\frac{\partial}{\partial t}\rho_a\eta_a + \nabla \cdot \rho_a\eta_a\mathbf{v}_a \geq -\nabla \cdot \left(\frac{\mathbf{h}_a}{\theta_a}\right) + \rho_a\frac{r_a}{\theta_a},$$

while we have

$$\frac{\partial}{\partial t}\rho\eta + \nabla \cdot \rho\eta\mathbf{v} \geq -\nabla \cdot \left(\frac{\mathbf{h}}{\theta}\right) + \rho\frac{r_*}{\theta}, \tag{6.17}$$

for the whole mixture, where the specific entropy η, the heat flux vector $\mathbf{h}$, and the specific entropy supply r_* are given by

$$\rho\eta = \sum_{a=1}^{m} \rho_a\eta_a, \tag{6.18}$$

$$\frac{\mathbf{h}}{\theta} = \sum_{a=1}^{m} \left(\frac{\mathbf{h}_a}{\theta_a} + \rho_a\eta_a\mathbf{u}_a\right), \tag{6.19}$$

$$\frac{\rho r_*}{\theta} = \sum_{a=1}^{m} \frac{\rho_a r_a}{\theta_a}, \tag{6.20}$$

where η_a is the specific entropy of S_a, and θ is the temperature of the mixture, which we will assume to be equal to the temperature θ_a of each constituent S_a from now on. We remark that, even in this case, the total entropy flux vector does not reduce to $\sum_a \mathbf{h}_a/\theta$ in the presence of the convective term $\sum_a \rho_a\eta_a\mathbf{u}_a$. Furthermore, the entropy flux vector does not coincide with the vector $\mathbf{j}/\theta$.

We have used the balance equations in the form

$$\frac{\partial \mathbf{F}}{\partial t} + \nabla \cdot (\mathbf{F} \otimes \mathbf{v}) = \nabla \cdot \mathbf{\Phi} + \mathbf{g},$$

where the meanings of the quantities involved are evident (see also Sect. 5.1 of [16]) to make it easier to compare the quantities related to each constituent and the corresponding global quantities. However, it is well known that, if the mass density is introduced together with the mass balance and the specific density $\mathbf{f} = \mathbf{F}/\rho$, the balance equations can be written in the equivalent form

$$\rho\dot{\mathbf{f}} = \nabla \cdot \mathbf{\Phi} + \mathbf{g},$$

which will be used in the following sections.

In order to write the balance equations in this form, we introduce the *concentration*

$$c_a = \frac{\rho_a}{\rho}, \tag{6.21}$$

of the constituent S_a. From (6.4), we have

$$\sum_{a=1}^{m} c_a = 1. \tag{6.22}$$

Consequently, the mass balance (6.2) becomes

$$\rho \frac{\partial c_a}{\partial t} + c_a \frac{\partial \rho}{\partial t} + \nabla \cdot \rho_a \mathbf{v}_a = 0.$$

This equation, taking into account (6.3) and (6.8), assumes the final form

$$\rho \dot{c}_a + \nabla \cdot (\rho c_a \mathbf{u}_a) = 0, \tag{6.23}$$

whereas the conservation of mass (6.3) for the whole mixture is

$$\dot{\rho} + \rho \nabla \cdot \mathbf{v} = 0. \tag{6.24}$$

Moreover, bearing in mind (6.8), and noting that

$$\frac{\partial \mathbf{v}_a}{\partial t} + \mathbf{v}_a \cdot \nabla \mathbf{v}_a = \dot{\mathbf{v}}_a + \mathbf{u}_a \cdot \nabla \mathbf{v}_a, \tag{6.25}$$

then we can use (6.5) and (6.6) to derive the new forms of the momentum balance for S_a:

$$\rho_a \dot{\mathbf{v}}_a + \rho_a \mathbf{u}_a \cdot \nabla \mathbf{v}_a = \nabla \cdot \mathbf{T}_a + \rho_a \mathbf{b}_a, \tag{6.26}$$

and S:

$$\rho \dot{\mathbf{v}} = \nabla \cdot \mathbf{T} + \rho \mathbf{b}, \tag{6.27}$$

respectively.

Similarly, the local energy balance for S assumes the form

$$\rho \dot{\epsilon} = \mathbf{T} : \nabla \mathbf{v} - \nabla \cdot \mathbf{j} + \rho_a \mathbf{b}_a \cdot \mathbf{v}_a + \rho_a r_a. \tag{6.28}$$

Finally, by eliminating $\rho_a r_a$ from (6.17) and (6.28), we obtain the reduced dissipation inequality for the whole mixture:

$$-\rho(\dot{\psi} + \eta \dot{\theta}) + \mathbf{T} : \nabla \mathbf{v} + \nabla \cdot (\mathbf{j} - \mathbf{h}) + \rho_a \mathbf{b}_a \cdot \mathbf{u}_a - \frac{1}{\theta} \mathbf{h} \cdot \nabla \theta \geq 0. \tag{6.29}$$

Remark We conclude this section by showing a useful relation for comparing the local balance equations for each constituent and the corresponding equations for the whole mixture when they are written in the form

$$\frac{\partial \mathbf{F}}{\partial t} + \nabla \cdot (\mathbf{F} \otimes \mathbf{v}) = \nabla \cdot \mathbf{\Phi} + \mathbf{g}.$$

More precisely, we wish to prove the fundamental formula

$$\rho\dot{\mathbf{\Psi}} = \sum_{a=1}^{m} \rho_a \mathbf{\Psi}'_a - \sum_{a=1}^{m} \nabla \cdot (\rho_a \mathbf{u}_a \otimes \mathbf{\Psi}_a), \tag{6.30}$$

where

$$\rho\mathbf{\Psi} = \sum_{a=1}^{m} \rho_a \mathbf{\Psi}_a, \tag{6.31}$$

and

$$\mathbf{\Psi}'_a = \frac{\partial \mathbf{\Psi}_a}{\partial t} + \mathbf{v}_a \cdot \nabla \mathbf{\Psi}_a \tag{6.32}$$

denotes the material derivative in the direction of the motion of S_a.

In order to prove (6.30), we start by noting that (6.31), (6.23) and (6.32) imply the equation

$$\dot{\mathbf{\Psi}} = -\frac{1}{\rho}\sum_{a=1}^{m} \nabla \cdot (\rho c_a \mathbf{u}_a)\mathbf{\Psi}_a + \sum_{a=1}^{m} c_a \mathbf{\Psi}'_a$$
$$- \sum_{a=1}^{m} c_a \mathbf{v}_a \cdot \nabla \mathbf{\Psi}_a + \mathbf{v} \cdot \sum_{a=1}^{m} c_a \nabla \mathbf{\Psi}_a,$$

from which, bearing in mind relation (6.9) and noting that

$$\sum_{a=1}^{m} \nabla \cdot (\rho c_a \mathbf{u}_a)\mathbf{\Psi}_a = \nabla \cdot \sum_{a=1}^{m} (\rho c_a \mathbf{u}_a \otimes \mathbf{\Psi}_a) - \rho \sum_{a=1}^{m} c_a \mathbf{u}_a \cdot \nabla \mathbf{\Psi}_a,$$

we can easily derive (6.30).

6.2 Classical Mixtures

In the classical theory of fluid mixtures, the behavior of the mixture S is described by the fields of mass density ρ, velocity $\mathbf{v}$, and temperature θ, as well as by the concentrations c_a of the constituents. The fields $\rho, \mathbf{v}$, and θ satisfy the balance laws for momentum, energy and entropy for the whole mixture. On the other hand, the concentrations c_a fulfill the mass conservation equations (6.23). However, due to the presence of the diffusion velocities $\mathbf{u}_a$, we have more unknowns than equations. This implies that we need to assign constitutive relations for the diffusion fluxes $\mathbf{\Phi}_a = \rho_a \mathbf{u}_a$. In conclusion, the system of equations that governs the evolution of a classical mixture S of m constituents is:

$$\dot{\rho} = -\nabla \cdot \rho \mathbf{v}, \tag{6.33}$$

$$\rho \dot{\mathbf{v}} = \nabla \cdot \mathbf{T} + \rho \mathbf{b}, \tag{6.34}$$

$$\rho \dot{\epsilon} = \mathbf{T} : \nabla \mathbf{v} - \nabla \cdot (\mathbf{h} + \mathbf{j}) + \rho r, \tag{6.35}$$

$$\rho \dot{c}_a = -\nabla \cdot \mathbf{\Phi}_a, \quad a = 1, \ldots, m-1. \tag{6.36}$$

Note that we consider the balance equations relating to the concentrations of $m-1$ constituents, since $\sum_a c_a = 1$. It should also be noted that the energy flux vector in (6.35) is written as the sum of the heat flux vector $\mathbf{h}$ and an extra energy flux $\mathbf{j}$ (we will clarify the meaning of this later).

Remark In the literature, the local energy balance (6.35) is usually written in the form

$$\rho \dot{\epsilon} = \mathbf{T} : \nabla \mathbf{v} - \nabla \cdot \mathbf{h}' + \rho r,$$

whereas the entropy flux vector is expressed as follows in the entropy inequality:

$$\nabla \cdot \left(\frac{\mathbf{h}}{\theta} + \mathbf{j}' \right).$$

It is evident that the formulation adopted for the energy balance equation and the entropy inequality here is fully equivalent to the formulation usually adopted in the literature (see [108]).

We assume that the constitutive equations take the form

$$\mathbf{A} = \mathbf{F}(\mathbf{B}), \tag{6.37}$$

where

$$\mathbf{A} = (\psi, \eta, \mathbf{T}, \mathbf{h}, \mathbf{j}, \mathbf{\Phi}_a), \quad \mathbf{B} = (\rho, c_a, \theta, \rho,_L, c_a,_L, \theta,_L), \tag{6.38}$$

and they must satisfy the following entropy inequality in every process:

$$-\rho(\dot{\psi} + \eta \dot{\theta}) + T_{ij} v_{i,j} - j_{i,i} - \frac{h_i \theta,_i}{\theta} \geq 0. \tag{6.39}$$

In (6.38) we used the notation $u,_L = \partial u / \partial X_L$, where (X_L) are the Cartesian coordinates in the material configuration C_*.

In order to find the restrictions on the constitutive equations (6.37) derived from the dissipation principle, we write the time derivative of the specific free energy ψ while taking (6.33) and (6.36) into account:

$$\begin{aligned} \dot{\psi} = {} & \frac{\partial \psi}{\partial \theta} \dot{\theta} - \rho \frac{\partial \psi}{\partial \rho} \delta_{ij} \frac{\partial v_i}{\partial x_j} - \frac{1}{\rho} \frac{\partial \psi}{\partial c_a} \Phi_{a_i},_i \\ & + \frac{\partial \psi}{\partial \rho,_L} \dot{\rho},_L + \frac{\partial \psi}{\partial c_a,_L} \dot{c}_a,_L + \frac{\partial \psi}{\partial \theta,_L} \dot{\theta},_L . \end{aligned} \tag{6.40}$$

If we substitute (6.40) into (6.39), and note that the quantities $\dot{\theta}$, $\dot{\rho}_{,L}$, $\dot{c}_a$ and $\dot{\theta}_{,L}$ do not appear in the list $\mathbf{B}$, so they can be arbitrarily taken at any point $\mathbf{X}$ and at any instant t, we have

$$\psi = \psi(\rho, \theta, c_a), \tag{6.41}$$

$$\eta = -\frac{\partial \psi}{\partial \theta}, \tag{6.42}$$

$$T_{ij} = -\rho^2 \frac{\partial \psi}{\partial \rho} \delta_{ij}, \tag{6.43}$$

meaning that inequality (6.39) reduces to the expression

$$\left(j_i - \frac{\partial \psi}{\partial c_a} \Phi_{a\,i} \right)_{,i} + \left(\frac{\partial \psi}{\partial c_a} \right)_{,i} \Phi_{a\,i} + \frac{h_i \theta_{,i}}{\theta} \leq 0. \tag{6.44}$$

Introducing the notation

$$\mu_a = \frac{\partial \psi}{\partial c_a}, \tag{6.45}$$

$$k_i = j_i - \mu_a \Phi_{a\,i}, \tag{6.46}$$

inequality (6.44) becomes

$$k_{i,i} + \mu_{a,i} \Phi_{a\,i} + \frac{h_i \theta_{,i}}{\theta} \leq 0. \tag{6.47}$$

The quantities μ_a, $a = 1, \ldots, m-1$, are called the *chemical potentials* of the $m-1$ components.

In the next theorem we prove the result

$$\mathbf{k} = \mathbf{0}, \tag{6.48}$$

meaning that (6.46) and (6.47) become

$$\mathbf{j} = \mu_a \boldsymbol{\Phi}_a, \quad \mu_{a,i} \Phi_{a\,i} + \frac{h_i \theta_{,i}}{\theta} \leq 0. \tag{6.49}$$

We first note that (6.47) can be written as follows:

$$\frac{\partial k_i}{\partial \rho_{,j}} \rho_{,ji} + \frac{\partial k_i}{\partial c_{a,j}} c_{a,ji} + \frac{\partial k_i}{\partial \theta_{,j}} \theta_{,ji} + \Lambda(\rho, c_a, \theta, \rho_{,i}, c_{a,i}, \theta_{,i}) \leq 0,$$

where Λ is a suitable function of its variables. This inequality implies that

$$\frac{\partial k_i}{\partial \rho_{,j}} = -\frac{\partial k_j}{\partial \rho_{,i}} \tag{6.50}$$

$$\frac{\partial k_i}{\partial c_{a,j}} = -\frac{\partial k_j}{\partial c_{a,i}} \tag{6.51}$$

$$\frac{\partial k_i}{\partial \theta_{,j}} = -\frac{\partial k_j}{\partial \theta_{,i}}. \tag{6.52}$$

Finally, we remark that the function $\mathbf{k}(\rho, c_a, \theta, \rho_{,i}, c_{a,i}, \theta_{,i})$ is isotropic, since the mixture consists of fluids (see Sect. 7.3 of [16]).

The following theorem proves (6.48).

Theorem 6.1
Let $\mathbf{f}(\mathbf{w}_{(1)}, \ldots, \mathbf{w}_n)$ be an isotropic vector function of the vector variables $\mathbf{w}_{(1)}, \ldots, \mathbf{w}_{(n)}$. If

$$\frac{\partial f_i}{\partial w_{(h)j}} = -\frac{\partial f_j}{\partial w_{(h)i}}, \quad h = 1, \ldots, n, \tag{6.53}$$

then $\mathbf{f} = \mathbf{0}$.

PROOF From (6.53) it follows that

$$\frac{\partial f_i}{\partial w_{(h)i}} = 0, \quad h = 1, \ldots, n. \tag{6.54}$$

Moreover, by differentiating (6.53) and taking into account (6.54), we obtain the conditions

$$\frac{\partial^2 f_i}{\partial w_{(h)j} \partial w_{(k)j}} = \frac{\partial^2 f_j}{\partial w_{(h)i} \partial w_{(k)j}} = 0,$$

which imply that

$$\frac{\partial^3 f_i}{\partial w_{(h)j} \partial w_{(k)l} \partial w_{(m)p}} = 0 \tag{6.55}$$

when two of the indices i, j, l are equal. This result allows us to write the Taylor expansion of $\mathbf{f}$ in the form

$$\mathbf{f} = \mathbf{f}(\mathbf{0}, \ldots, \mathbf{0}) + \mathbf{F}_{(h)} \cdot \mathbf{w}_{(h)} + \mathbf{G}_{(hk)} \mathbf{w}_{(h)} \mathbf{w}_{(k)}, \tag{6.56}$$

where $\mathbf{F}_{(h)}$ and $\mathbf{G}_{(hk)}$ are constant tensors. In fact, from (6.55) we can conclude that the higher-order terms of the expansion vanish, since two indices relating to the components of the vectors $\mathbf{w}_{(h)}$ of the higher-order derivatives are always equal.

On the other hand, the isotropy condition

$$\mathbf{f}(\mathbf{Q}\mathbf{w}_{(1)}, \ldots, \mathbf{Q}\mathbf{w}_{(n)}) = \mathbf{Q}\mathbf{f}(\mathbf{w}_{(1)}, \ldots, \mathbf{w}_{(n)}), \tag{6.57}$$

where $\mathbf{Q}$ is any orthogonal matrix, implies that

$$\mathbf{f}(\mathbf{0}, \ldots, \mathbf{0}) = \mathbf{0}, \quad \mathbf{G}_{(hk)} = \mathbf{0},$$

when we put $\mathbf{Q} = -\mathbf{I}$, and finally we conclude that

$$\mathbf{f} = \mathbf{F}_{(h)} \cdot \mathbf{w}_{(h)}.$$

Due to hypothesis (6.53), the tensors $\mathbf{F}_{(h)}$ are skew symmetric, whereas the isotropy of $\mathbf{f}$ implies that

$$\mathbf{QF}_{(h)} = \mathbf{F}_{(h)}\mathbf{Q}$$

for any orthogonal matrix $\mathbf{Q}$, meaning that $\mathbf{F}_{(h)} = \mathbf{0}$ and so the theorem is proved. ∎

6.3 Nonclassical Mixtures

We call a mixture S whose evolution equations are provided for each constituent a *nonclassical mixture*.

The motion of S is governed by the following field equations (see Eqs. (6.23), (6.26) and (6.28)):

$$\dot{\rho}_a + \rho_a \nabla \cdot \mathbf{v}_a + \mathbf{u}_a \cdot \nabla \rho_a = 0. \tag{6.58}$$

$$\rho_a \dot{\mathbf{v}}_a + \rho_a \mathbf{u}_a \cdot \nabla \mathbf{v}_a = \nabla \cdot \mathbf{T}_a + \rho_a \mathbf{b}_a, \tag{6.59}$$

$$\rho\dot{\epsilon} - \mathbf{T} : \nabla \mathbf{v} + \nabla \cdot \mathbf{j} = \rho_a \mathbf{b}_a \cdot \mathbf{v}_a + \rho_a r_a. \tag{6.60}$$

Note that we only need the energy balance for the whole mixture due to the hypothesis that the temperature of each constituent is the same.

In order to find the restrictions imposed by the reduced dissipation inequality, it is convenient to write (6.29) in an equivalent form. To this end, we start by noting that (see Eqs. (6.8) and (6.10)):

$$\begin{aligned}\mathbf{T} : \nabla \mathbf{v} = T_{aij} v_{a_i},_j - &\left[\left(T_{aij} - \frac{1}{2}\rho_a u_a^2 \delta_{ij}\right) u_{ai}\right],_j \\ &+ T_{aij},_j\, u_{ai} - \frac{1}{2} u_a^2 (\rho_a u_{aj}),_j - \rho_a u_{ai} u_{aj} v_{ai},_j\,.\end{aligned}$$

Taking into account (6.58) and (6.59), the above relation can be written as

$$\begin{aligned}\mathbf{T} : \nabla \mathbf{v} = T_{aij} v_{a_i},_j - &\left[\left(T_{aij} - \frac{1}{2}\rho_a u_a^2 \delta_{ij}\right) u_{ai}\right],_j \\ &+ \rho_a u_{ai} \dot{v}_{ai} + \frac{1}{2}\rho u_a^2 \dot{c}_a - \rho_a b_{ai} u_{ai}.\end{aligned} \tag{6.61}$$

Substituting (6.61) into (6.29), we obtain the inequality

$$-\rho(\dot{\psi} + \eta\dot{\theta}) + T_{aij} v_{ai},_j - k_i,_i + \rho_a u_{ai} \dot{v}_{ai} + \frac{1}{2}\rho u_a^2 \dot{c}_a - \frac{1}{\theta} h_i \theta,_i \geq 0, \tag{6.62}$$

where

$$k_i = j_i - h_i + \left(T_{aji} - \frac{1}{2}\rho_a u_a^2 \delta_{ij}\right) u_{aj}. \tag{6.63}$$

Note that from (6.9) we have

$$\frac{1}{2}\rho\frac{d}{dt}(c_a u_a^2) = \frac{1}{2}\rho\dot{c}_a u_a^2 + \rho c_a u_{ai}(\dot{v}_{ai} - \dot{v}_i)$$
$$= \rho_a u_{ai}\dot{v}_{ai} + \frac{1}{2}\rho u_a^2 \dot{c}_a, \tag{6.64}$$

and the inequality (6.62) assumes the final form

$$-\rho(\dot{\psi}_I + \eta\dot{\theta}) + \mathbf{T}_a : \nabla\mathbf{v}_a - \nabla\cdot\mathbf{k} - \frac{\mathbf{h}}{\theta}\cdot\nabla\theta \geq 0, \tag{6.65}$$

where

$$\psi_I = \psi - \frac{1}{2}c_a u_a^2. \tag{6.66}$$

Now we assume that the constitutive equations take the form

$$\mathbf{A} = \mathbf{F}(\mathbf{B}), \tag{6.67}$$

where

$$\mathbf{A} = (\psi_I, \eta, \mathbf{T}_a, \mathbf{h}, \mathbf{k}), \quad \mathbf{B} = (\rho_a, \mathbf{v}_a, \theta, \rho_a,_L, \theta,_L), \tag{6.68}$$

and that they must satisfy inequality (6.65) in every process.

In (6.68) we used the notation $u,_L = \partial u/\partial X_L$, where (X_L) are the Cartesian coordinates in the material configuration C_*.

In order to find the restrictions on the constitutive equations (6.68) derived from the dissipation principle, we write the time derivative of the specific free energy ψ while taking into account (6.68):

$$\dot{\psi}_I = \frac{\partial\psi_I}{\partial\theta}\dot{\theta} + \frac{\partial\psi_I}{\partial v_{ai}}\dot{v}_{ai} + \frac{\partial\psi_I}{\partial\rho_a}\dot{\rho}_a$$
$$+ \frac{\partial\psi_I}{\partial\rho_a,_L}\dot{\rho}_a,_L + \frac{\partial\psi_I}{\partial\theta,_L}\dot{\theta},_L. \tag{6.69}$$

In view of (6.58), the above relation can also be written as follows:

$$\dot{\psi}_I = \frac{\partial\psi_I}{\partial\theta}\dot{\theta} + \frac{\partial\psi_I}{\partial v_{ai}}\dot{v}_{ai}$$
$$- \frac{\partial\psi_I}{\partial\rho_a}(\rho_a\delta_{ij}v_{ai},_j + \rho_a,_j u_a,_j)$$
$$+ \frac{\partial\psi_I}{\partial\rho_a,_L}\dot{\rho}_a,_L + \frac{\partial\psi_I}{\partial\theta,_L}\dot{\theta},_L. \tag{6.70}$$

On the other hand, we have

$$k_i,_i = \frac{\partial k_i}{\partial\rho_a}\rho_a,_i + \frac{\partial k_i}{\partial v_a,_j}v_{aj},_i$$
$$+ \frac{\partial k_i}{\partial\rho_a,_L}(F^{-1})_{Mi}\rho_a,_{LM} + \frac{\partial k_i}{\partial\theta}\theta,_i$$
$$+ \frac{\partial k_i}{\partial\theta,_i}(F^{-1})_{Mi}\theta,_{LM}, \tag{6.71}$$

where $(F^{-1})_{Mi} = \partial X_M/\partial x_i$.

If we substitute (6.71) and (6.70) into (6.66) and note that the quantities $\dot{\theta}, \dot{v}_{ai}, \rho,_{LM}, \theta_{LM}, \dot{\rho}_a,_L$ can be chosen arbitrarily at any point $\mathbf{X}$ and at any instant t, we have

$$\psi_I = \psi_I(\rho_a, \theta), \tag{6.72}$$

$$\eta = -\frac{\partial \psi_I}{\partial \theta}, \tag{6.73}$$

$$T_{aij} = -\rho\rho_a \frac{\partial \psi_I}{\partial \rho_a}\delta_{ij} + \frac{\partial k_{(j}}{\partial v_{ai}}, \tag{6.74}$$

$$\frac{\partial k_i}{\partial \rho_{a,(L}}(F^{-}1)_{M)i} = 0, \qquad \frac{\partial k_i}{\partial \theta_{,(L}}(F^{-}1)_{M)i} = 0. \tag{6.75}$$

Moreover, the residual inequality is

$$\left(\rho\frac{\partial \psi_I}{\partial \rho_a}u_{aj} - \frac{\partial k_j}{\partial \rho_a}\right)\rho_{a,j} - \left(\frac{\partial k_i}{\partial \theta} + \frac{h_i}{\theta}\right)\theta_{,i} \geq 0. \tag{6.76}$$

6.4 Balance Equations of Binary Fluid Mixtures

In the remaining part of this chapter we propose to extend the results obtained in Chap. 3 to binary mixtures of fluids. This further generalization of a continuous system with an interface will permit us to derive the Gibbs rule for phase equilibrium in a mixture, to describe the evaporation of a component of a binary mixture into a gas, etc.

We would now like to formulate a model to describe the phase transition in a binary mixture. This means that our system consists of two phases, C_1 and C_2, separated by an interface Σ. Moreover, each phase can be occupied by one or two constituents of the mixture. To formulate an appropriate model of a continuum with an interface, we must modify the balance equations we proposed in Chap. 3, since now we are in the presence of a binary mixture. However, if we limit ourselves to classical mixtures, we need only add the equation which gives the evolution of the concentration c of one constituent of the mixture to the equation of mass conservation, the momentum balance, and the energy balance for the whole system. It is also important to recall that the total energy balance for classical mixtures differs from the corresponding energy balance for simple continua, since the energy flux vector for mixtures is the sum of the heat flux vector and an extra flux vector $\mu\mathbf{\Phi}$, where μ is the chemical potential of the constituent considered and $\mathbf{\Phi}$ is its diffusive flux (see Eqs. (6.35) and (6.49)). We must also account for the fact that the interface Σ itself is a material system that can influence the behavior of the mixture. Consequently, we must introduce all of the contributions from the interface into the balance equations, as we did in Chap. 3. Bearing in mind all of the above remarks, we adopt the following balance equations for a binary classical mixture (see Eqs. (3.55–3.60) and (3.73)):

mass conservation:

$$\dot{\rho} + \rho \nabla \cdot \mathbf{v} = 0, \quad \text{in C} - \Sigma, \tag{6.77}$$

$$\left(\frac{\partial \rho_s}{\partial t}\right)_{u^\alpha} + \rho_s \nabla_s \cdot \mathbf{V}_s - 2H c_n \rho_s = [[\rho U]], \quad \text{on}\, \Sigma. \tag{6.78}$$

momentum balance:

$$\rho \dot{\mathbf{v}} - \nabla \cdot \mathbf{t} - \rho \mathbf{b} = \mathbf{0}, \quad \text{in C} - \Sigma, \tag{6.79}$$

$$\rho_s \left(\frac{\partial \mathbf{V}}{\partial t}\right)_{u^\alpha} - \nabla_s \cdot \mathbf{T} = [[\rho(\mathbf{v} - \mathbf{V})U + \mathbf{tn}]], \quad \text{on}\, \Sigma. \tag{6.80}$$

energy balance:

$$\rho \dot{\epsilon} - \text{tr}(\mathbf{t} \otimes \mathbf{v}) + \nabla \cdot (\mathbf{h} + \mu \mathbf{\Phi}) = 0, \quad \text{in C} - \Sigma, \tag{6.81}$$

$$\begin{aligned} &\rho_s \left(\frac{\partial E}{\partial t}\right)_{u^\alpha} - T^{\alpha\beta}\sigma_{\alpha\beta} + \nabla_s \cdot (\mathbf{h}_s + \mu_s \mathbf{\Phi}_s) \\ &= \left[\left[\rho\left(\frac{1}{2}(\mathbf{v} - \mathbf{V})^2 + e - E\right)U + (\mathbf{v} - \mathbf{V}) \cdot \mathbf{tn}\right]\right] \\ &- [[(\mathbf{h} + \mu \mathbf{\Phi}) \cdot \mathbf{n}]] \quad \text{on}\, \Sigma. \end{aligned} \tag{6.82}$$

mass balance for one constituent:

$$\rho \dot{c} + \nabla \cdot \mathbf{\Phi} = 0, \quad \text{in C}_\text{i} - \Sigma \text{ for i} = 1, 2, \text{on}\, \Sigma \tag{6.83}$$

$$\rho_s \left(\frac{\partial c_s}{\partial t}\right)_{u^\alpha} + \nabla_s \cdot \mathbf{\Phi}_s = [[\rho(c - c_s)U - \mathbf{\Phi} \cdot \mathbf{n}]]. \tag{6.84}$$

Here $U = c_n - v_n$, and the rest of the notation used in the above equations is the same as that used in Chapt. 3. We also note that the interface is considered a classical mixture, as shown by the presence of the concentration c_s and the conductive flux $\mathbf{\Phi}_s$ in the above equations.

The form of the partial mass balance (6.84) presented above requires justification. Due to the general balance law (3.51), mass conservation for the first constituent S_1 at the interface should be written in the form

$$\begin{aligned} \frac{\delta \rho_{s1}}{\delta t} + \nabla_s \cdot (\rho_{s1} \mathbf{V}_{s1}) - 2H c_n \rho_{s1} &= [[\rho_1 (c_n - v_{1n})]] \\ &= [[\rho_1 (c_n - v_n)]] - [[\rho_1 (v_{1n} - v_n)]] \\ &= [[\rho_1 U]] - [[\mathbf{\Phi} \cdot \mathbf{n}]], \end{aligned}$$

where ρ_1 is the mass density in the bulk phases of S_1, $\mathbf{v}_1$ is the velocity of S_1, ρ_{s1} is its surface mass density, and $\mathbf{V}_{s1}$ is its surface velocity. Now, recall that the concentrations of S_1 in the volume and at the interface, c and c_s, are given by $\rho_1 = c\rho$ and $\rho_{s1} = c_s\rho_s$. If we take into account (6.78), then the above equation can be placed in the form

$$-c_s\nabla_s \cdot (\rho_s \mathbf{V}_s) + \nabla_s \cdot (\rho_{s1}\mathbf{V}_{s1}) + \rho_s \frac{\delta c_s}{\delta t} = [[\rho(c - c_{s1})U]] - [[\mathbf{\Phi} \cdot \mathbf{n}]] \,,$$

or in the other equivalent form

$$\rho_s \frac{\delta c_s}{\delta t} + \rho_s V_s^\alpha c_{s,\alpha} + \nabla_s \cdot (\rho_{1s}(\mathbf{V}_{s1} - \mathbf{V}_s)) = [[\rho(c - c_{s1})U]] - [[\mathbf{\Phi} \cdot \mathbf{n}]] \,.$$

It is sufficient to recall that $\mathbf{\Phi}_s = \rho_{1s}(\mathbf{V}_{s1} - \mathbf{V}_s)$ and property (3.9) of the parametrization (u^α) to obtain (6.84).

In order to develop the theory of phase changes in binary mixtures, we need to assign the constitutive equations of the mixture as a whole, as well as the constitutive equations of the diffusive fluxes of the first constituent in the volume and at the interface.

6.5 Constitutive Equations

In the theory of classical mixtures, the variables that appear in the constitutive equations are the velocity gradient $\nabla\mathbf{v}$, the density ρ, the concentration c, the temperature θ, and the gradients of density, concentration, and temperature. For the sake of simplicity, we assume that the constitutive relations depend only on the first gradients of density, concentration, and temperature. Similarly, we suppose that the constitutive equations for fluid mixtures at the interface depend on $\sigma_{\alpha\beta}$, $a = \det(a_{\alpha\beta})$, ρ_s, c_s and θ_s, as well as on $\nabla_s\rho_s$, $\nabla_s c_s$ and $\nabla_s\theta_s$. The dependence on a is introduced to account for any adsorption that takes place at the interface. Moreover, we distinguish between the equilibrium stress tensor and the dynamic stress tensor, both in the bulk phases and on the interface, using the notation

$$\mathbf{t} = -p\mathbf{I} + \mathbf{t}_d, \tag{6.85}$$

$$\mathbf{T} = \gamma\mathbf{I}_s + \mathbf{T}_d, \tag{6.86}$$

where $\mathbf{I}_s = (a^{\alpha\beta})$ is the identity tensor on Σ.

In order to derive the restrictions on the constitutive equations imposed by the dissipation principle, we must first formulate the reduced dissipation inequality. By eliminating $\nabla \cdot \mathbf{h}$ between (6.81) and the entropy principle

$$\rho\theta\dot{\eta} + \nabla \cdot \mathbf{h} - \frac{1}{\theta}\mathbf{h} \cdot \nabla\theta \geq 0, \tag{6.87}$$

we obtain

$$-(\dot{\psi}+\eta\dot{\theta})-p\nabla\mathbf{v}+\mathrm{tr}(\mathbf{t}_d\otimes\nabla\mathbf{v})-\nabla\cdot(\mu\mathbf{\Phi})-\frac{1}{\theta}\mathbf{h}\cdot\nabla\theta\geq 0. \tag{6.88}$$

On the other hand, using (6.77) and (6.83), we can derive

$$\nabla\cdot\mathbf{v}=-\frac{\dot{\rho}}{\rho},$$
$$\nabla\cdot(\mu\mathbf{\Phi})=\mathbf{\Phi}\cdot\nabla\mu+\mu\nabla\cdot\mathbf{\Phi}=\mathbf{\Phi}\cdot\nabla\mu-\rho\mu\dot{c},$$

and (6.88) becomes

$$\begin{aligned}&-\rho(\dot{\psi}+\eta\dot{\theta}-\frac{p}{\rho^2}\dot{\rho}-\mu\dot{c})+\mathrm{tr}(\mathbf{t}_d\otimes\nabla\mathbf{v})\\&-\mathbf{\Phi}\cdot\nabla\mu-\frac{1}{\theta}\mathbf{h}\cdot\nabla\theta\geq 0.\end{aligned} \tag{6.89}$$

From our constitutive assumptions, it follows that

$$\begin{aligned}\dot{\psi}=&\frac{\partial\psi}{\partial\theta}\dot{\theta}+\frac{\partial\psi}{\partial\rho}\dot{\rho}+\frac{\partial\psi}{\partial c}\dot{c}+\frac{\partial\psi}{\partial\nabla\mathbf{v}}\dot{\nabla\mathbf{v}}\\&+\frac{\partial\psi}{\partial\nabla\rho}\dot{\nabla\rho}+\frac{\partial\psi}{\partial\nabla c}\dot{\nabla c}+\frac{\partial\psi}{\partial\nabla\theta}\dot{\nabla\theta},\end{aligned} \tag{6.90}$$

and when we substitute this relation into (6.89) we obtain an inequality from which it is possible to derive (using a standard procedure) the following relations:

$$\psi=\psi(\rho,c,\theta), \tag{6.91}$$
$$\eta=\eta(\rho,c,\theta)=-\frac{\partial\psi}{\partial\theta}, \tag{6.92}$$
$$p=p(\rho,c,\theta)=\rho^2\frac{\partial\psi}{\partial\rho}, \tag{6.93}$$
$$\mu=\mu(\rho,c,\theta)=\frac{\partial\psi}{\partial c}, \tag{6.94}$$
$$\mathrm{tr}(\mathbf{t}_d\otimes\nabla\mathbf{v})-\mathbf{\Phi}\cdot\nabla\mu-\frac{1}{\theta}\mathbf{h}\cdot\nabla\theta\geq 0. \tag{6.95}$$

We can regard the term on the left-hand side of (6.95) as being a function f of $\nabla\mathbf{v}$, ∇c, $\nabla\rho$, and $\nabla\theta$ (see Eq. 6.94), which reaches its minimum when all of the variables vanish. Consequently, all of the partial derivatives with respect to the above variables vanish for $\nabla\mathbf{v}=\nabla c=\nabla\rho=\nabla\theta=\mathbf{0}$, and so

$$(\mathbf{t}_d)^0=\mathbf{\Phi}^0=\mathbf{h}^0=\mathbf{0}. \tag{6.96}$$

We can conclude that there is no friction, no convective flux, and no heat conduction when the gradients of the above fields vanish.

We must now proceed in a similar way with the energy balance (6.82) and the entropy inequality at the interface (3.75). For the sake of simplicity, we suppose that $T_d^{\alpha\beta} = 0$ and $\mathbf{h}_s = \mathbf{0}$. By eliminating $[[\mathbf{h}\cdot\mathbf{n}]]$ between these equations, and recalling (6.86), we then obtain

$$\begin{aligned} &-\rho_s(\dot{\Psi} + S\dot{\theta}) + \gamma a^{\alpha\beta}\sigma_{\alpha\beta} - \nabla_s \cdot (\mu_s \mathbf{\Phi}_s) \\ &+ \left[\left[\rho\left(\frac{1}{2}(\mathbf{v}-\mathbf{v})^2 + \psi - \Psi\right)U + (\mathbf{v}-\mathbf{V})\cdot\mathbf{tn}\right]\right] \\ &- [[\mu\mathbf{\Phi}]]\cdot\mathbf{n} \geq 0, \end{aligned} \tag{6.97}$$

where $\dot{A} = (\partial A/\partial t)_{u^\alpha}$ and $\Psi = E - \theta S$.

On the other hand, taking (6.84) into account, we have

$$\begin{aligned} -\nabla_s(\mu_s\mathbf{\Phi}_s) &= -\mathbf{\Phi}_s\cdot\nabla_s\mu_s - \mu_s\nabla_s\cdot\mathbf{\Phi}_s \\ &= -\mathbf{\Phi}_s\cdot\nabla_s\mu_s + \mu_s\rho_s\dot{c}_s - \mu_s[[\rho(c-c_s)U - \mathbf{\Phi}\cdot\mathbf{n}]]. \end{aligned}$$

Moreover,

$$\begin{aligned} (\mathbf{v}-\mathbf{V})\cdot\mathbf{tn} &= (v_n - c_n + c_n - V_n)\mathbf{n}\cdot\mathbf{tn} + (\mathbf{v}_\tau - \mathbf{V}_\tau)\cdot\mathbf{tn} \\ &= pU - p(c_n - V_n) + (\mathbf{v}_\tau - \mathbf{V}_\tau)\cdot\mathbf{tn}, \end{aligned}$$

where $\mathbf{v}_\tau$ denotes the component along the interface.

Bearing these results in mind, we can write (6.97) in the following way:

$$\begin{aligned} &-\rho_s(\dot{\Psi} + S\dot{\theta} - \mu_s\dot{c}_s) + \gamma a^{\alpha\beta}\sigma_{\alpha\beta} - \mathbf{\Phi}_s\cdot\nabla_s\mu_s \\ &+ \left[\left[\rho\left(\frac{1}{2}(\mathbf{v}-\mathbf{v})^2 + g - \Psi - \mu_s(c-c_s)\right)U\right]\right] \\ &+ [[(\mathbf{v}_\tau - \mathbf{V}_\tau)\cdot\mathbf{tn} + (\mu_s - \mu)\mathbf{\Phi}\cdot\mathbf{n}]] - [[p]]\,(c_n - V_n) \geq 0, \end{aligned} \tag{6.98}$$

where $g = \psi + p/\rho$ is the Gibbs potential in the bulk phases.

Finally, from (6.78) we can deduce the relation

$$\begin{aligned} \dot{\rho}_s &= -\rho_s\nabla_s\cdot\mathbf{V}_s + 2Hc_n\rho_s + [[\rho U]] \\ &= -\rho_s a^{\alpha\beta}V_{\alpha;\beta} + \rho_s a^{\alpha\beta}b_{\alpha\beta}c_n + [[\rho U]]. \end{aligned} \tag{6.99}$$

Moreover,

$$\gamma a^{\alpha\beta}(V_{\alpha;\beta} - b_{\alpha\beta}V_n). \tag{6.100}$$

Now we adopt the following constitutive equation for Ψ:

$$\Psi = \Psi(\rho_s, c_s, \theta, \nabla\rho_s, \nabla c_s). \tag{6.101}$$

Similar constitutive equations will be adopted for the other surface fields.

As a consequence of (6.99) and (6.101), we have

$$-\rho_s\dot{\Psi} = \rho_s\frac{\partial\Psi}{\partial\rho_s}a^{\alpha\beta}V_{\alpha;\beta} - \rho_s^2\frac{\partial\Psi}{\partial\rho_s}a^{\alpha\beta}b_{\alpha\beta}c_n - \rho_s\frac{\partial\Psi}{\partial\rho_s}[[\rho U]] - \rho_s\frac{\partial\Psi}{\partial c_s}\dot{c}_s - \rho_s\frac{\partial\Psi}{\partial\theta}\dot{\theta} - \rho_s\frac{\partial\Psi}{\partial\nabla\rho_s}\dot{\overline{\nabla\rho_s}} - \rho_s\frac{\partial\Psi}{\partial\nabla c_s}\dot{\overline{\nabla c_s}}. \quad (6.102)$$

Introducing this expression into the inequality (6.98), we obtain

$$\begin{aligned}
&- \rho_s\left(S + \frac{\partial\Psi}{\partial\theta}\right)\dot{\theta} + \rho_s\left(\mu_s - \frac{\partial\Psi}{\partial c_s}\right)\dot{c}_s + \left(\gamma + \rho_s^2\frac{\partial\Psi}{\partial\rho_s}\right)a^{\alpha\beta}V_{\alpha;\beta}\\
&- \rho_s\frac{\partial\Psi}{\partial\nabla\rho_s}\dot{\overline{\nabla\rho_s}} - \rho_s\frac{\partial\Psi}{\partial\nabla c_s}\dot{\overline{\nabla c_s}} - \left(\gamma V_n + \rho_s^2\frac{\partial\Psi}{\partial\rho_s}c_n\right)a^{\alpha\beta}b_{\alpha\beta}\\
&- \rho_s\frac{\partial\Psi}{\partial\rho_s}[[\rho U]] - [[p]](c_n - V_n) - \mathbf{\Phi_s}\cdot\nabla\mu_s\\
&+ \left[\left[\rho\left(\frac{1}{2}(\mathbf{v} - \mathbf{V})^2 + g - \Psi - \mu_s(c - c_s)\right)U\right]\right]\\
&+ [[(\mathbf{v}_\tau - \mathbf{V}_\tau)\cdot\mathbf{tn} + (\mu_s - \mu)\mathbf{\Phi}\cdot\mathbf{n}]] \geq 0. \qquad (6.103)
\end{aligned}$$

Using a standard procedure, we deduce from (6.103) the following properties of the constitutive equations:

$$\Psi = \Psi(\rho_s, c_s, \theta), \quad (6.104)$$

$$S = -\frac{\partial\Psi}{\partial\theta} = S(\rho_s, c_s, \theta), \quad (6.105)$$

$$\mu_s = \frac{\partial\Psi}{\partial c_s} = \mu_s(\rho_s, c_s, \theta), \quad (6.106)$$

$$\gamma = -\rho_s^2\frac{\partial\Psi}{\partial\rho_s}, \quad (6.107)$$

$$\begin{aligned}
&\frac{\gamma}{\rho_s}[[\rho U]] + [[2\gamma H - p]](c_n - V_n) - \mathbf{\Phi_s}\cdot\nabla\mu_s\\
&+ \left[\left[\rho\left(\frac{1}{2}(\mathbf{v} - \mathbf{V})^2 + g - \Psi - \mu_s(c - c_s)\right)U\right]\right]\\
&+ [[(\mathbf{v}_\tau - \mathbf{V}_\tau)\cdot\mathbf{tn} + (\mu_s - \mu)\mathbf{\Phi}\cdot\mathbf{n}]] \geq 0. \qquad (6.108)
\end{aligned}$$

This last inequality can also be written as follows:

$$\begin{aligned}
&[[2\gamma H - p]](c_n - V_n) - \mathbf{\Phi_s}\cdot\nabla\mu_s\\
&+ \left[\left[\rho\left(\frac{1}{2}(\mathbf{v} - \mathbf{V})^2 + g - G - \mu_s(c - c_s)\right)U\right]\right]\\
&+ [[(\mathbf{v}_\tau - \mathbf{V}_\tau)\cdot\mathbf{tn} + (\mu_s - \mu)\mathbf{\Phi}\cdot\mathbf{n}]] \geq 0, \qquad (6.109)
\end{aligned}$$

where $G = \Psi - \gamma/\rho_s$ is the Gibbs potential at the surface.

Moreover, in view of (6.96), we have

$$\boldsymbol{\Phi} = \mathbf{A} \cdot \nabla\rho + \mathbf{B} \cdot \nabla c + \mathbf{C},$$

where $\mathbf{C}$ is a higher order than $\nabla\rho$ and ∇c. We assume that at least one of the tensors $\mathbf{A}$ and $\mathbf{B}$ is different from zero.

In order to derive other significant consequences of the inequality (6.109), we regard its left-hand side as a function f of $(c_n - V_n)$, $U^{\mp}$, $(\mathbf{v}_\tau - \mathbf{V}_\tau)^{\mp}$, of the gradients of ρ and c on both sides of the interface, and of the surface gradients of ρ_s and c_s (see Eqs. 6.96 and 6.106).

The function f reaches its minimum when all of the above variables vanish. The derivatives of f with respect to these variables also vanish when they are evaluated at zero. Therefore, in particular, we have

$$[[p]]_0 - 2\gamma H = 0, \tag{6.110}$$
$$(g^+ - G - \mu_s(c^+ - c_s))_0 = 0, \tag{6.111}$$
$$(g^- - G - \mu_s(c^- - c_s))_0 = 0, \tag{6.112}$$
$$(\boldsymbol{\Phi}_s)_0 = 0. \tag{6.113}$$
$$\mu^+ - \mu_s = 0, \tag{6.114}$$
$$\mu^- - \mu_s = 0. \tag{6.115}$$

Equivalently, we can say that the following conditions hold at the interface:

$$[[p]]_0 - 2\gamma H = 0, \tag{6.116}$$
$$[[g - G - \mu c]] = 0. \tag{6.117}$$
$$[[\mu]] = 0, \tag{6.118}$$
$$(g^+ - G - \mu_s(c^+ - c_s))_0 = 0, \tag{6.119}$$
$$\mu^+ - \mu_s = 0, \tag{6.120}$$

where the index 0 has been omitted. These relations are valid at equilibrium, but we can also use them for processes that are not too far from equilibrium.

6.6 Phase Equilibrium and Gibbs' Principle

In this section we analyze a simple case of phase equilibrium in a fluid binary mixture in order to prove the Gibbs rule for phase equilibrium. We suppose that there are no body forces, that the temperature is uniform in both phases, that the other fields are uniform in each phase, that the stress tensors in the bulk phases reduce to a uniform pressure, and that the interface is planar. Using these hypotheses, the equilibrium equations, which we derive from (6.77)–(6.84),

$$\nabla \cdot \mathbf{t} + \rho \mathbf{b} = 0, \tag{6.121}$$

$$-\nabla_s \cdot \mathbf{T}_s = [[\mathbf{t} \cdot \mathbf{n}]], \tag{6.122}$$

$$\nabla \cdot \mathbf{\Phi} = 0, \tag{6.123}$$

$$\nabla_s \cdot \mathbf{\Phi}_s = -[[\mathbf{\Phi} \cdot \mathbf{n}]], \tag{6.124}$$

in view of the uniformity of the fields p, ρ, c, ρ_s, c_s as well as (6.96) and (6.113), are identically satisfied. Consequently, in order to determine the equilibrium conditions, we must find, for a given temperature and assigned constitutive equations for g, G, μ and μ_s, the six numerical unknowns $\rho^{\pm}$, $c^{\pm}$, ρ_s and c_s that satisfy the five equations (6.116)–(6.120). In other words, we need assign two variables, for instance c and θ, to determine the equilibrium state. This result is in full agreement with the *Gibbs rule*, which states that the degree of freedom F of the equilibrium configuration of a compound system is given by the formula

$$F = C + 2 - \pi, \tag{6.125}$$

where C is the number of components in the mixture and π is the number of phases.

The above considerations relate to a system of two phases separated by a *planar* interface. If we assume that the interface Σ is spherical, then we have a further degree of freedom, i.e., the radius R of Σ in the relation

$$2\frac{\gamma}{R} = [[p]],$$

which replaces (6.116).

The analysis of the equilibrium when the fields are not uniform is much more complex.

6.7 Evaporation of a Fluid into a Gas

In this section we analyze the evaporation of a fluid into a gas (see [114]). The system we consider consists of a pure liquid phase C_1 and a gaseous phase C_2, which is a binary mixture of the vapor arising from the pure phase C_1 and another gas. The evaporation of water into air is an example of the situation we are considering.

We analyze the process under the following hypotheses:

- The interface, which is planar, has no material characteristics

- The phase C_2 is unbounded
- The fields depend only on the spatial variable x and the time t.

Under these conditions, the phase C_1 is represented by the interval $[0, s(t))$, where $s(t)$ is the equation of the planar interface, whereas the phase C_2 is represented by the interval $(s(t), \infty)$.

For $(x, t) \in [0, s(t)) \times (0, \infty)$ we have the equations

$$v = 0, \tag{6.126}$$

$$p = const, \tag{6.127}$$

$$\theta_{,t} = a_l \theta_{,xx}\,, \tag{6.128}$$

where a_l is the thermal diffusivity of the liquid.

In the phase $(x, t) \in [s(t), \infty) \times (0, \infty)$, the equations are

$$\rho_{,t} + (\rho v)_{,x} = 0, \tag{6.129}$$

$$\rho(c_{,t} + v c_{,x}) = -\Phi_{,x}\,, \tag{6.130}$$

$$\rho(v_{,t} + v v_{,x}) = -p_{,x}\,, \tag{6.131}$$

$$\rho(e, t + v e_{,x}) = -p v_{,x} + k\theta_{,xx} - (\mu\Phi)_{,x}\,, \tag{6.132}$$

where the standard notation is used.

At the interface, we have

$$\rho(\dot{s} - v) = \rho_l \dot{s}, \tag{6.133}$$

$$\rho c(\dot{s} - v) - \rho_l \dot{s} = \Phi, \tag{6.134}$$

$$\rho(\dot{s} - v) = [[p]]\,, \tag{6.135}$$

$$\rho\left(\frac{1}{2}v^2 + e + \frac{p}{\rho}\right)(\dot{s} - v) - \rho_l\left(e + \frac{p}{\rho_l}\right)\dot{s} = -\,[[k\theta_{,x}]] + \mu\Phi, \tag{6.136}$$

$$\frac{1}{2}v^2 + \left[\left[\psi + \frac{p}{\rho}\right]\right] - \mu c = 0. \tag{6.137}$$

Since $\rho \ll \rho_l$, from (6.133) we have

$$v \simeq -\frac{\rho_l}{\rho}\dot{s}, \tag{6.138}$$

and (6.134)–(6.137) become

$$\Phi = \rho_l \dot{s}(c - 1), \tag{6.139}$$

$$[[p]] = -\frac{\rho_l^2}{\rho}\dot{s}^2, \tag{6.140}$$

$$\left(\frac{1}{2}\frac{\rho_l^2}{\rho^2}\dot{s}^2 + [[e+]] + \frac{p}{\rho}\right)\rho_l \dot{s} = -\,[[k\theta_{,x}]] + \mu\rho_l\dot{s}(c - 1), \tag{6.141}$$

$$\frac{1}{2}\frac{\rho_l^2}{\rho^2}\dot{s}^2 + [[\psi]] + \frac{p}{\rho} - \mu c = 0. \tag{6.142}$$

This system can be simplified by repeating the same nondimensional analysis as that found in Sect. 5.4. For a more detailed discussion of the resulting system and its consequences, see [114].

Chapter 7

Electromagnetism in Matter

7.1 Integral Balance Laws

Let $\mathcal{S}$ be a continuous system, and let $C(t)$ be the region occupied by $\mathcal{S}$ at the instant t. Generally, $C(t)$ is the union of the disjoint regions $C_1(t), \ldots$, $C_\nu(t)$, and $\mathcal{S}$ exhibits the same physical properties in each of these regions. For instance, if $\mathcal{S}$ consists of two adjacent dielectrics occupying the regions $C_1(t)$ and $C_2(t)$ in the presence of a fixed conductor of volume C_3, then we have $C(t) = C_1(t) \cup C_2(t) \cup C_3 \cup C_4(t)$, where $C_4(t)$ is the space around the dielectrics and the conductor.

We also assume that some fields f associated with $\mathcal{S}$ may exhibit finite discontinuities $[[f]] = f^+ - f^-$ across the oriented *material* boundary $\partial C_i(t)$ of region $C_i(t)$. Here f^+ denotes the limit value of f on $\partial C_i(t)$ obtained when this surface is approached from the region containing the vector $\mathbf{n}$ normal to $\partial C_i(t)$, and f^- is the limit obtained when $\partial C_i(t)$ is approached from the other side.

Moreover, we assume the existence of a *nonmaterial* surface $\pm$, possibly consisting of the disjoint parts $\pm_\infty(\sqcup), \ldots, \pm_\backslash(\sqcup)$, which is a surface of discontinuity for one or more fields of $\mathcal{S}$ (shock wave or phase surface).

Finally, we assume that the electromagnetic field produced by S during its evolution is described by the following balance equations:

$$-\frac{d}{dt}\int_{s(t)} \mathbf{B}\cdot\mathbf{N}ds = \int_{\partial s(t)} (\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\tau dl, \tag{7.1}$$

$$\frac{d}{dt}\int_{s(t)} \mathbf{D}\cdot\mathbf{N}ds = \int_{\partial s(t)} (\mathbf{H} - \mathbf{v}\times\mathbf{D})\cdot\tau dl - I(s(t)), \tag{7.2}$$

$$\int_{\partial c(t)} \mathbf{D}\cdot\mathbf{N}ds = Q_f(c(t)), \tag{7.3}$$

$$\int_{\partial c(t)} \mathbf{B}\cdot\mathbf{N}ds = 0. \tag{7.4}$$

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In (7.1)–(7.4), $s(t)$ and $c(t)$ denote a material surface and a material volume, respectively. Moreover, $\mathbf{N}$ is the external unit vector normal to $s(t)$ or to $\partial c(t)$, and τ is the unit vector tangent to $\partial s(t)$. Finally,

$$\begin{aligned}
\mathbf{v} &= \textit{velocity field},\\
\mathbf{E} &= \textit{electric field},\\
\mathbf{H} &= \textit{magnetic field},\\
\mathbf{D} &= \textit{electric induction field},\\
\mathbf{B} &= \textit{magnetic induction field}.
\end{aligned}$$

Also, $I(s(t))$ is the invariant current across the material surface $s(t)$, and $Q_f(c(t))$ is the free charge present in $c(t)$. It is clear that when matter is absent or at rest $\mathbf{v} = \mathbf{0}$.

The following considerations lead us to the explicit expressions of $I(s(t))$ and $Q_f(c(t))$. If the charge is distributed with a volume density ρ_f in the regions $\widehat{c}_i$, $i = 1, 2, \ldots, p$, and with a surface density ω_f on the material surfaces $\widehat{s}_i$, $i = 1, 2, \ldots q$, then we have

$$Q_f(c(t)) = \sum_{i=1}^{p} \int_{\widehat{c}_i \cap c(t)} \rho_f dc + \sum_{i=1}^{q} \int_{\widehat{s}_i \cap c(t)} \omega_f ds. \tag{7.5}$$

Similarly, if $\mathbf{j}$ is the current density in three-dimensional conductors and $\mathbf{k}$ is the current density on material conducting surfaces $\widetilde{s}_i, i = 1, 2, \ldots, r$, then we have

$$I(s(t)) = \int_{s(t)} (\mathbf{j} - \rho_f \mathbf{v}) \cdot \mathbf{N} ds + \sum_{i=1}^{q} \int_{\widetilde{s}_i \cap s(t)} (\mathbf{k} - \omega_f \mathbf{V}) \cdot \nu dl, \tag{7.6}$$

where $\mathbf{V}$ is the velocity of the charge with a surface density ω_f, and ν is the unit vector normal to the curve $\widetilde{s}_i \cup s(t)$ and tangent to $\widetilde{s}_i$.

The local equations, which derive from (7.1)–(7.4) when we take into account (5.12), (5.13) and (5.16) of [16] or Sect. 3.8, are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{7.7}$$

$$\nabla \times \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}, \tag{7.8}$$

$$\nabla \cdot \mathbf{D} = \rho_f, \tag{7.9}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{7.10}$$

where the fields are regular.

On *material* singular surfaces, where the normal velocity c_n along the normal $\mathbf{n}$ is equal to v_n, the following jump conditions hold:

$$[[\mathbf{n} \times (\mathbf{E} + \mathbf{v} \times \mathbf{B})]] = \mathbf{0}, \tag{7.11}$$

$$[[\mathbf{n} \times (\mathbf{H} - \mathbf{v} \times \mathbf{D})]] = \mathbf{k} - \omega_f \mathbf{V}, \tag{7.12}$$
$$[[\mathbf{D} \cdot \mathbf{n}]] = \omega_f, \tag{7.13}$$
$$[[\mathbf{B} \cdot \mathbf{n}]] = 0. \tag{7.14}$$

Finally, on *nonmaterial* singular surfaces, where $c_n \neq v_n$, $\omega_f = 0$, and $\mathbf{k} = \mathbf{0}$, we have

$$[[(c_n - v_n)\mathbf{B} - \mathbf{n} \times (\mathbf{E} + \mathbf{v} \times \mathbf{B})]] = \mathbf{0}, \tag{7.15}$$
$$[[(c_n - v_n)\mathbf{D} + \mathbf{n} \times (\mathbf{H} - \mathbf{v} \times \mathbf{D})]] = \mathbf{0}, \tag{7.16}$$
$$[[v_n]] = 0, \tag{7.17}$$
$$[[\mathbf{D} \cdot \mathbf{n}]] = 0, \tag{7.18}$$
$$[[\mathbf{B} \cdot \mathbf{n}]] = 0. \tag{7.19}$$

If we recall that $\nabla \cdot \nabla \times \mathbf{a} = \mathbf{0}$ for any vector field $\mathbf{a}$, and apply the operator $\nabla \cdot$ to (7.7), we can derive (7.10). Moreover, applying the operator $\nabla \cdot$ to (7.8) and taking (7.9) into account leads us to the *charge conservation law*:

$$\frac{\partial \rho_f}{\partial t} + \nabla \cdot \mathbf{j} = 0. \tag{7.20}$$

In conclusion, when the source ρ_f is given, the fields $\mathbf{E}(\mathbf{x}, t)$, $\mathbf{H}(\mathbf{x}, t)$, $\mathbf{D}(\mathbf{x}, t)$, $\mathbf{B}(\mathbf{x}, t)$, and $\mathbf{j}(\mathbf{x}, t)$ must satisfy the independent equations (7.7), (7.8), and (7.20). Consequently, these equations are not sufficient to determine the above fields, even when the initial data and boundary conditions are given. As is usual in continuum mechanics, we must assign *constitutive equations* in order to get the same number of equations as unknowns.

For instance, a *linear and isotropic* dielectric (e.g., a vacuum) is defined by the following constitutive relations:

$$\mathbf{D} = \epsilon \mathbf{E}, \tag{7.21}$$
$$\mathbf{B} = \mu \mathbf{H}, \tag{7.22}$$
$$\mathbf{j} = \mathbf{0}, \tag{7.23}$$

where ϵ and μ are the electric and magnetic permeabilities, respectively.

For a nonlinear rigid conductor, the constitutive equations assume the form

$$\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{H}), \tag{7.24}$$
$$\mathbf{B} = \mathbf{B}(\mathbf{E}, \mathbf{H}), \tag{7.25}$$
$$\mathbf{j} = \mathbf{j}(\mathbf{E}, \mathbf{H}). \tag{7.26}$$

As a further example, we consider a nonlinear elastic dielectric that is defined by the relations

$$\mathbf{D} = \mathbf{D}(\mathbf{F}, \mathbf{E}, \mathbf{H}), \tag{7.27}$$
$$\mathbf{B} = \mathbf{B}(\mathbf{F}, \mathbf{E}, \mathbf{H}), \tag{7.28}$$

where $\mathbf{F}$ is the deformation gradient. In the next section we study the general theory of constitutive equations for a continuous system in the presence of an electromagnetic field.

7.2 Electromagnetic Fields in Rigid Bodies at Rest

In this section we analyze the electromagnetic fields in a system S of rigid bodies that are at rest in a given frame of reference. We note that Maxwell's equations maintain the form (7.7)–(7.10), whereas the jump conditions (7.11)–(7.14) become

$$\mathbf{n} \times [[\mathbf{E}]] = \mathbf{0}, \tag{7.29}$$

$$\mathbf{n} \times [[H]] = \mathbf{k}, \tag{7.30}$$

$$[[\mathbf{D}]] \cdot \mathbf{n} = \omega_f, \tag{7.31}$$

$$[[\mathbf{B}]] \cdot \mathbf{n} = 0. \tag{7.32}$$

It is well known that circulating current inside a conductor always produces heat in the body (the *Joule effect*). Therefore, to completely describe the phenomenology of the interaction between electromagnetic fields and matter, we are also compelled to consider the energy balance and the entropy inequality. The appropriate form of the energy balance of S is (see [115, 116])

$$\frac{d}{dt}\int_C \epsilon\, dc = -\int_{\partial C} (\mathbf{E} \times \mathbf{H} + \mathbf{h}) \cdot \mathbf{N}\, d\sigma + \int_C r\, dc, \tag{7.33}$$

where C is any fixed volume whose boundary is ∂C, $\mathbf{N}$ is the external unit normal to ∂C, ϵ is the internal energy per unit volume, $\mathbf{h}$ is the heat flux vector, and r is the external energy supply per unit volume. This principle states that the variations in the internal energy present in a region C are due to the flux of thermal and electromagnetic energy across ∂C. The vector $\mathbf{E} \times \mathbf{H}$ is termed *Poynting's vector*. When the fields are regular, (7.33) implies the following local equation:

$$\dot{\epsilon} = -\nabla \cdot (\mathbf{E} \times \mathbf{H} + \mathbf{h}) + r, \tag{7.34}$$

whereas the jump condition on a surface of discontinuity σ for the fields is (see Eq. 5.3 of [16])

$$[[\mathbf{E} \times \mathbf{H} + \mathbf{h}]] \cdot \mathbf{n} = 0, \tag{7.35}$$

where $\mathbf{n}$ is the unit vector normal to σ.

In view of the vector identity

$$\nabla(\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b}$$

and Maxwell's equations (7.7)–(7.10), (7.34) can also be written as

$$\dot{\epsilon} = \mathbf{E} \cdot \dot{\mathbf{D}} + \mathbf{H} \cdot \dot{\mathbf{B}} + \mathbf{E} \cdot \mathbf{j} - \nabla \mathbf{h}. \tag{7.36}$$

Moreover, the jump condition (7.35) assumes the explicit form

$$[[\mathbf{E}]] \times \mathbf{H}^{+} \cdot \mathbf{n} + \mathbf{E}^{-} \times [[\mathbf{H}]] + [[\mathbf{h}]] \cdot \mathbf{n} = \mathbf{0},$$

where $a^{\pm}$ denote the limit values on the surface of discontinuity for the quantity a arising from the region containing the unit normal $\mathbf{n}$ and in the other one, respectively. Bearing in mind (7.29) and (7.30) and using the cyclic property of the mixed product, the above relation becomes

$$[[\mathbf{h}]] \cdot \mathbf{n} = -\mathbf{k} \cdot \mathbf{E}^{-}. \tag{7.37}$$

We note that in the absence of surface currents, the heat flux vector and Poynting's vector are continuous across σ.

Together with the energy balance, we must consider the entropy inequality[1]

$$\frac{d}{dt} \int_C \eta dc \geq - \int_{\partial C} \frac{\mathbf{h}}{\theta} \cdot \mathbf{N} d\sigma + \int_C \frac{r}{\theta}\, dc, \tag{7.38}$$

where η is the entropy per unit volume and θ is the absolute temperature. When the temperature is continuous everywhere, the integral inequality (7.38) is equivalent to the following local condition:

$$\dot{\eta} \geq -\nabla \cdot \frac{\mathbf{h}}{\theta} + \frac{r}{\theta}. \tag{7.39}$$

If we eliminate the term $-\nabla \mathbf{h} + r$ between (7.36) and (7.39), then we obtain the reduced dissipation inequality

$$-(\dot{\psi} + \eta\dot{\theta}) + \mathbf{E} \cdot \dot{\mathbf{D}} + \mathbf{H} \cdot \dot{\mathbf{B}} + \mathbf{E} \cdot \mathbf{j} - \frac{\mathbf{h} \cdot \nabla\theta}{\theta} \geq 0, \tag{7.40}$$

where

$$\psi = \epsilon - \theta\eta \tag{7.41}$$

is the free energy per unit volume. Now we consider the following class of constitutive equations:

$$\mathbf{u} = \mathbf{F}(\mathbf{v}), \tag{7.42}$$

[1]For a more general principle of entropy, see [117, 134].

where

$$\mathbf{u} = (\psi, \eta, \mathbf{E}, \mathbf{H}, \mathbf{j}, \mathbf{h}), \quad \mathbf{v} = (\mathbf{D}, \mathbf{B}, \theta, \nabla\theta). \tag{7.43}$$

From (7.43) we have

$$\dot{\psi} = \frac{\partial\psi}{\partial\theta}\dot{\theta} + \frac{\partial\psi}{\partial\nabla\theta} \cdot \nabla\dot{\theta} + \frac{\partial\psi}{\partial\mathbf{D}} \cdot \dot{\mathbf{D}} + \frac{\partial\psi}{\partial\mathbf{B}} \cdot \dot{\mathbf{B}},$$

so that (7.40) becomes

$$\begin{aligned} &-\left(\frac{\partial\psi}{\partial\theta} + \eta\right)\dot{\theta} + \frac{\partial\psi}{\partial\nabla\theta} \cdot \nabla\dot{\theta} + \left(\mathbf{E} - \frac{\partial\psi}{\partial\mathbf{D}}\right) \cdot \dot{\mathbf{D}} \\ &+ \left(\mathbf{H} - \frac{\partial\psi}{\partial\mathbf{B}}\right) \cdot \dot{\mathbf{B}} + \mathbf{E} \cdot \mathbf{j} - \frac{\mathbf{h} \cdot \nabla\theta}{\theta} \geq 0. \end{aligned} \tag{7.44}$$

It is easy to verify that the quantities $\theta, \dot{\theta}$, $\nabla\theta$, $\mathbf{D}$, $\dot{\mathbf{B}}$, $\mathbf{D}$ and $\dot{\mathbf{B}}$ can be chosen arbitrarily at any point and at any instant.[2] Therefore, we can derive the following from (7.44):

$$\psi = \psi(\mathbf{D}, \mathbf{B}, \theta), \tag{7.45}$$

$$\eta = -\frac{\partial\psi}{\partial\theta} = \eta(\mathbf{D}, \mathbf{B}, \theta), \tag{7.46}$$

$$\mathbf{E} = \frac{\partial\psi}{\partial\mathbf{D}} = \mathbf{E}(\mathbf{D}, \mathbf{B}, \theta), \tag{7.47}$$

$$\mathbf{H} = \frac{\partial\psi}{\partial\mathbf{B}} = \mathbf{H}(\mathbf{D}, \mathbf{B}, \theta), \tag{7.48}$$

$$\mathbf{E} \cdot \mathbf{j} - \frac{\mathbf{h} \cdot \nabla\theta}{\theta} \geq 0. \tag{7.49}$$

The above relations show that, for a material described by the constitutive equations (7.43), the free energy, which is a function of $\mathbf{D}$, $\mathbf{B}$ and θ, is a potential for η, $\mathbf{E}$ and $\mathbf{H}$. Finally, the constitutive relations for $\mathbf{j}$ and $\mathbf{h}$ must satisfy inequality (7.49).

We also note that (7.47)–(7.48) lead to the identities

$$\frac{\partial E_i}{\partial D_j} = \frac{\partial E_j}{\partial D_i}, \tag{7.50}$$

[2] In fact, it is sufficient to consider the thermoelectromagnetic fields

$$\begin{aligned} \mathbf{D}(\mathbf{x}, t) &= \mathbf{D}_0 + \mathbf{F}(\mathbf{x} - \mathbf{x}_0) + \mathbf{g}(t - t_0), \\ \mathbf{B}(\mathbf{x}, t) &= \mathbf{B}_0 + \mathbf{G}(\mathbf{x} - \mathbf{x}_0) + \mathbf{h}(t - t_0), \\ \theta(\mathbf{x}, t) &= \theta_0 + \mathbf{a} \cdot (\mathbf{x} - \mathbf{x}_0) + b(t - t_0), \end{aligned}$$

where $\mathbf{F}$, $\mathbf{G}$ are constant tensors, $\mathbf{D}_0$, $\mathbf{B}_0$, $\mathbf{g}$, $\mathbf{h}$ and $\mathbf{a}$ are constant vectors, and b is a scalar, and to note that these quantities can be determined such that the Maxwell equations are satisfied when the constitutive relations (7.42) are assigned.

$$\frac{\partial H_i}{\partial B_j} = \frac{\partial H_j}{\partial B_i}, \tag{7.51}$$

$$\frac{\partial E_i}{\partial B_j} = \frac{\partial H_i}{\partial D_j}, \tag{7.52}$$

as well as the *Gibbs relation*

$$d\psi = \mathbf{E} \cdot d\mathbf{D} + \mathbf{H} \cdot d\mathbf{B} - \eta d\theta. \tag{7.53}$$

We finally remark that, if the functions (7.47) and (7.48) can be inverted with respect to the variables $\mathbf{D}$ and $\mathbf{B}$, then we can write the relations

$$\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{H}, \theta), \tag{7.54}$$

$$\mathbf{B} = \mathbf{B}(\mathbf{E}, \mathbf{H}, \theta), \tag{7.55}$$

which imply that $\mathbf{E}$ and $\mathbf{H}$ can be considered fundamental fields. In this hypothesis, the dissipation inequality (7.40) assumes the following equivalent form:

$$-(\dot{\zeta} + \eta\dot{\theta}) + \mathbf{D} \cdot \dot{\mathbf{E}} + \mathbf{B} \cdot \dot{\mathbf{H}} + \mathbf{E} \cdot \mathbf{j} - \frac{\mathbf{h} \cdot \nabla\theta}{\theta} \geq 0, \tag{7.56}$$

where we have introduced the *enthalpy* per unit volume

$$\zeta = \psi - \mathbf{D} \cdot \mathbf{E} - \mathbf{H} \cdot \mathbf{B}. \tag{7.57}$$

Using (7.56), we can derive the following thermodynamic restrictions:

$$\zeta = \zeta(\mathbf{E}, \mathbf{H}, \theta), \tag{7.58}$$

$$\eta = -\frac{\partial \zeta}{\partial \theta} = \eta(\mathbf{E}, \mathbf{H}, \theta), \tag{7.59}$$

$$\mathbf{D} = -\frac{\partial \zeta}{\partial \mathbf{E}}, \tag{7.60}$$

$$\mathbf{B} = -\frac{\partial \zeta}{\partial \mathbf{H}}, \tag{7.61}$$

$$\mathbf{E} \cdot \mathbf{j} - \frac{\mathbf{h} \cdot \nabla\theta}{\theta} \geq 0. \tag{7.62}$$

Finally, the Gibbs relation becomes

$$d\zeta = -\mathbf{D} \cdot d\mathbf{E} - \mathbf{B} \cdot d\mathbf{H} - \eta d\theta, \tag{7.63}$$

and the identities (7.50)–(7.52) assume the form

$$\frac{\partial D_i}{\partial E_j} = \frac{\partial D_j}{\partial E_i}, \tag{7.64}$$

$$\frac{\partial B_i}{\partial H_j} = \frac{\partial B_j}{\partial H_i}, \tag{7.65}$$

$$\frac{\partial D_i}{\partial H_j} = \frac{\partial B_i}{\partial E_j}. \tag{7.66}$$

We conclude this section with some remarks. First, we note that the function $f(\mathbf{E}, \nabla\theta) = \mathbf{E} \cdot \mathbf{J} - (\mathbf{h} \cdot \nabla\theta)/\theta \geq 0$ reaches its minimum at $\mathbf{E} = \nabla\theta = \mathbf{0}$, so that

$$\left(\frac{\partial f}{\partial \mathbf{E}}\right)_{\mathbf{0}} = \mathbf{j}(\mathbf{0}, \mathbf{H}, \theta, \mathbf{0}) = \mathbf{0},$$
$$\left(\frac{\partial f}{\partial \nabla\theta}\right)_{\mathbf{0}} = \mathbf{h}(\mathbf{0}, \mathbf{H}, \theta, \mathbf{0}) = \mathbf{0}.$$

These conditions tell us that *there is neither heat flux nor current in the absence of an electric field and a temperature gradient.* Finally, when $\nabla\theta$ vanishes, we have the inequality

$$\mathbf{E} \cdot \mathbf{j} \geq 0, \tag{7.67}$$

which expresses the Joule effect. If $\mathbf{E} = \mathbf{0}$, then we obtain the condition

$$\mathbf{h} \cdot \nabla\theta \leq 0, \tag{7.68}$$

which expresses Fourier's law.

7.3 Constitutive Equations for Isotropic Rigid Bodies

In this section we analyze the forms of the constitutive equations for isotropic media in the presence of electromagnetic fields. Since the electric field $\mathbf{E}$ is a polar vector while the magnetic field $\mathbf{H}$ is an axial vector (see Sect. 1.5 of [16]), we are faced with functions that depend on polar and axial vectors. To simplify our analysis, we note that the axial vector $\mathbf{H}$ can be replaced by the skew-symmetric tensor $\mathbf{W}$, the adjoint of which is $\mathbf{H}$:

$$W_{ij} = \epsilon_{ijh} H_h, \quad H_i = \frac{1}{2}\epsilon_{ijh} W_{jh}. \tag{7.69}$$

In view of (7.58)–(7.62), we see that the constitutive equations

$$\zeta = \zeta(\mathbf{E}, \mathbf{W}, \theta), \tag{7.70}$$
$$\mathbf{h} = \mathbf{h}(\mathbf{E}, \mathbf{W}, \theta, \nabla\theta), \tag{7.71}$$
$$\mathbf{j} = \mathbf{j}(\mathbf{E}, \mathbf{W}, \theta, \nabla\theta). \tag{7.72}$$

must be isotropic functions of their variables. By applying the representation theorem for isotropic scalar functions (see [14]) to ζ, we obtain

$$\zeta = \zeta(E^2, \mathrm{tr}(\mathbf{W})^2, \mathbf{E} \cdot \mathbf{W}^2\mathbf{E}). \tag{7.73}$$

Also, due to (7.69), we have

$$\mathrm{tr}(\mathbf{W})^2 = W_{lh}W_{hl} = \epsilon_{lhi}\epsilon_{hlj}H_iH_j = \delta_{ij}H_iH_j,$$

so that

$$\mathrm{tr}(\mathbf{W})^2 = H^2. \tag{7.74}$$

Moreover,

$$\begin{aligned}\mathbf{E}\cdot\mathbf{W}^2\mathbf{E} &= E_iW_{ih}W_{hk}E_k\\ &= E_i\epsilon_{ihl}H_l\epsilon_{hkm}H_mE_k\\ &= (\delta_{im}\delta_{lk}-\delta_{ik}\delta_{lm})E_iE_kH_lH_m,\end{aligned}$$

and finally we have

$$\mathbf{E}\cdot\mathbf{W}^2\mathbf{E} = (\mathbf{E}\cdot\mathbf{H})^2 - E^2H^2. \tag{7.75}$$

Thus, the constitutive equation for ζ assumes the final form

$$\zeta = \zeta(E^2, H^2, (\mathbf{E}\cdot\mathbf{H})^2). \tag{7.76}$$

Again resorting to the representation theorem of a vector isotropic function, and introducing the notation $\mathbf{g} = \nabla\theta$, we have

$$\begin{aligned}\mathbf{f}(\mathbf{E},\mathbf{W},\theta,\nabla\theta) &= f_1(\pi)\mathbf{E} + f_2(\pi)\mathbf{g} + f_3(\pi)\mathbf{W}\mathbf{E} + f_4(\pi)\mathbf{W}\mathbf{g}\\ &\quad + f_5(\pi)\mathbf{W}^2\mathbf{E} + f_6(\pi)\mathbf{W}^2\mathbf{g},\end{aligned} \tag{7.77}$$

where the functions $f_1,\ldots,f_6$ are isotropic functions of

$$\pi = (E^2, g^2, \mathrm{tr}\mathbf{W}^2, \mathbf{E}\cdot\mathbf{g}, \mathbf{E}\cdot\mathbf{W}^2\mathbf{E}, \mathbf{E}\cdot\mathbf{W}\mathbf{g}, \mathbf{E}\cdot\mathbf{W}^2\mathbf{g}, \mathbf{g}\cdot\mathbf{W}^2\mathbf{g}). \tag{7.78}$$

Noting that

$$\mathbf{E}\cdot\mathbf{W}\mathbf{g} = E_iW_{ih}g_h = \mathbf{E}\cdot\mathbf{g}\times\mathbf{H}, \tag{7.79}$$

$$\mathbf{E}\cdot\mathbf{W}^2\mathbf{g} = E_iW_{ih}W_{hl}g_l = -(\mathbf{E}\times\mathbf{H})\cdot(\mathbf{g}\times\mathbf{H}) \tag{7.80}$$

$$(\mathbf{W}\mathbf{E})_i = \epsilon_{ihl}H_lE_h = (\mathbf{E}\times\mathbf{H})_i \tag{7.81}$$

$$(\mathbf{W}^2\mathbf{E})_i = \epsilon_{ihl}H_l\epsilon_{hjp}H_pE_j = (\mathbf{H}\times(\mathbf{H}\times\mathbf{E}))_i, \tag{7.82}$$

and taking into account (7.74) and (7.75), we attain the final forms of the constitutive equations of $\mathbf{h}$ and $\mathbf{j}$:

$$\begin{aligned}\mathbf{h} &= F_1(\pi)\mathbf{E} + F_2(\pi)\mathbf{g} + F_3(\pi)\mathbf{E}\times\mathbf{H} + F_4(\pi)\mathbf{g}\times\mathbf{H}\\ &\quad + F_5(\pi)\mathbf{H}\times(\mathbf{H}\times\mathbf{E}) + F_6(\pi)\mathbf{H}\times(\mathbf{H}\times\mathbf{g}),\end{aligned} \tag{7.83}$$

$$\begin{aligned}\mathbf{j} &= G_1(\pi)\mathbf{E} + G_2(\pi)\mathbf{g} + G_3(\pi)\mathbf{E}\times\mathbf{H} + G_4(\pi)\mathbf{g}\times\mathbf{H}\\ &\quad + G_5(\pi)\mathbf{H}\times(\mathbf{H}\times\mathbf{E}) + G_6(\pi)\mathbf{H}\times(\mathbf{H}\times\mathbf{g}),\end{aligned} \tag{7.84}$$

where

$$\pi = (E^2, g^2, \mathbf{H}^2, \mathbf{E}\cdot\mathbf{g}, \mathbf{E}\cdot\mathbf{H}^2, \mathbf{E}\cdot(\mathbf{g}\times\mathbf{H}), (\mathbf{E}\times\mathbf{H})\cdot(\mathbf{g}\times\mathbf{H}), (\mathbf{g}\cdot\mathbf{H})^2). \tag{7.85}$$

7.4 Approximate Constitutive Equations for Isotropic Bodies

In this section we provide approximate expressions for the constitutive equations stated in the above section.

Let us consider the linear approximation of (7.60), (7.61), (7.83) and (7.84). We have

$$\zeta = \zeta_0(\theta) - \frac{1}{2}\epsilon(\theta)E^2 - \frac{1}{2}\mu(\theta)H^2, \tag{7.86}$$

and so

$$\mathbf{D} = \epsilon(\theta)\mathbf{E}, \tag{7.87}$$
$$\mathbf{B} = \mu(\theta)\mathbf{H}. \tag{7.88}$$

Moreover, (7.83) and (7.84) become:

$$\mathbf{h} = \beta(\theta)\mathbf{E} - k(\theta)\mathbf{g}, \tag{7.89}$$
$$\mathbf{j} = \sigma(\theta)\mathbf{E} - \alpha(\theta)\mathbf{g}, \tag{7.90}$$

where α, k, β and σ are positive functions of the temperature.

In order to account for nonlinear effects, we consider second-order terms in the variables that appear in the constitutive equations. We note that (7.87) is still valid, so (7.87) and (7.88) hold up to third-order terms in $\mathbf{E}$ and $\mathbf{H}$. Moreover, the relations (7.83) and (7.84) assume the form

$$\mathbf{h} = \beta\mathbf{E} - k\mathbf{g} + k_1(\mathbf{E} \times \mathbf{H}) + k_2(\mathbf{g} \times \mathbf{H}), \tag{7.91}$$
$$\mathbf{j} = \sigma\mathbf{E} - \alpha\mathbf{g} + \alpha_1(\mathbf{E} \times \mathbf{H}) + \alpha_2(\mathbf{g} \times \mathbf{H}), \tag{7.92}$$

where all of the coefficients that appear in the above equations depend on θ.

These equations show that, beside the linear effects, the electromagnetic fields influence thermal and electrical conduction through second-order terms that represent the following physical phenomena:

- The Ettingshausen effect, due to $k_1(\mathbf{E} \times \mathbf{H})$
- The Leduc–Righi effect, due to $k_2(\mathbf{g} \times \mathbf{H})$
- The Hall effect, due to $\alpha_1(\mathbf{E} \times \mathbf{H})$
- The Nernst effect, due to $\alpha_2(\mathbf{g} \times \mathbf{H})$.

7.5 Maxwell's Equations and the Principle of Relativity

In this section we will introduce some relativistic concepts which will be analyzed in greater detail in Chap. 10.

Before considering a theory for a deformable continuum S in the presence of electromagnetic fields, it is essential to determine the transformation group under which all of the equations describing the evolution of S are covariant. It is easy to verify that Maxwell's equations are covariant under Lorentz transformations by resorting to the four-dimensional Minkowski formulation of spacetime and writing these equations as tensor relations in this space (see Chap. 10). However, we will instead follow the classical approach Einstein presented in his famous paper from 1905 on special relativity.

First, we note that verifying the covariance of a set of equations under a given transformation group is not an easy and purely formal task. In fact, we shall see that we can only use the transformation formulae to derive how the spatial and temporal coordinates change together with the corresponding differentiation operators. In other words, we have no information about the transformation of any physical quantity that appears in the evolution equations. This implies that we must assume that these quantities behave in a way that makes the field equations covariant. However, it is possible that the required mathematical behavior is not compatible with the experimental evidence.

It is also important to note that the field equations are not sufficient to determine the evolution of S, since we need to introduce the constitutive relations into them in order to obtain a closed system in the unknowns given by the fundamental fields. Consequently, both the field equations and the constitutive relations must be covariant.

We start by proving that Maxwell's equations are not covariant under Galilean transformations.

Consider the Galilean transformation

$$x'_i = x_i - u_i t, \quad i = 1, 2, 3, \tag{7.93}$$

$$t' = t, \tag{7.94}$$

where (x_1, x_2, x_3, t) are the spacetime coordinates of an event in the inertial frame I, and (x'_1, x'_2, x'_3, t') are the spacetime coordinates of the *same* event in another inertial frame I' that is moving at a constant velocity (u_1, u_2, u_3) with respect to I. It is easy to verify that (7.93)–(7.94) imply the following transformation formulae for the differential operators:

$$\frac{\partial}{\partial x_i} = \frac{\partial x'_j}{\partial x_i}\frac{\partial}{\partial x'_j} + \frac{\partial t'}{\partial x_i}\frac{\partial}{\partial t'} = \frac{\partial}{\partial x'_i}, \tag{7.95}$$

$$\frac{\partial}{\partial t} = \frac{\partial x'_j}{\partial t}\frac{\partial}{\partial x'_j} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = -u_j\frac{\partial}{\partial x'_j} + \frac{\partial}{\partial t'}. \tag{7.96}$$

By using (7.95) and (7.96) in the first component of (7.7), we obtain

$$\frac{\partial E_3}{\partial x'_2} - \frac{\partial E_2}{\partial x'_3} = u_1\frac{\partial B_1}{\partial x'_1} + u_2\frac{\partial B_1}{\partial x'_2} + u_3\frac{\partial B_1}{\partial x'_3} - \frac{\partial B_1}{\partial t'},$$

which, taking into account (4.10), assumes the form

$$\frac{\partial}{\partial x'_2}(E_3 + (u_1B_2 - u_2B_1)) - \frac{\partial}{\partial x'_3}(E_2 + (u_3B_1 - u_1B_3)) = -\frac{\partial B_1}{\partial t'}.$$

Using the same approach for the other components of (7.7), we then derive the vector equation

$$\nabla' \times (\mathbf{E} + \mathbf{u} \times \mathbf{B}) = -\frac{\partial \mathbf{B}}{\partial t'}. \tag{7.97}$$

Relation (7.97) is a *mixed* equation since it contains derivatives with respect to the new variables x'_i and t' but the electromagnetic fields still relate to the old inertial frame I. Therefore, this equation will have the same form in I' if and only if the fields $\mathbf{E}$ and $\mathbf{B}$ transform according to the rules

$$\mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B}, \qquad \mathbf{B}' = \mathbf{B}. \tag{7.98}$$

Applying the same arguments to (7.8), we obtain the equation

$$\nabla' \times (\mathbf{H} - \mathbf{u} \times \mathbf{D}) = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t'}, \tag{7.99}$$

which becomes a Maxwell equation (7.8) in I' if and only if

$$\mathbf{H}' = \mathbf{H} - \mathbf{u} \times \mathbf{D}, \quad \mathbf{D}' = \mathbf{D}, \quad \mathbf{j}' = \mathbf{j}. \tag{7.100}$$

We note that (7.95), (7.98) and (7.100) imply that the remaining Maxwell equations (7.9) and (7.10) are still covariant. However, these results do not allow us to conclude that electrodynamic theory is invariant for Galilean transformations. In fact, due to the transformation rules (7.98) and (7.100), the constitutive equations are not invariant. It sufficient to consider the constitutive equations for a vacuum to verify this statement.

We now consider the Lorentz transformations[3]

$$x_1' = \gamma(x_1 - ut), \tag{7.101}$$
$$x_2' = x_2, \tag{7.102}$$
$$x_3' = x_3, \tag{7.103}$$
$$t' = \gamma(t - \frac{u}{c^2}x_1), \tag{7.104}$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}.$$

Instead of (7.95) and (7.96) we have

$$\frac{\partial}{\partial x_1} = \frac{\partial x_j'}{\partial x_1}\frac{\partial}{\partial x_j'} + \frac{\partial t'}{\partial x_1}\frac{\partial}{\partial t'} = \gamma\frac{\partial}{\partial x_1'} - \gamma\frac{u}{c^2}\frac{\partial}{\partial t'}, \tag{7.105}$$
$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_2'}, \tag{7.106}$$
$$\frac{\partial}{\partial x_3} = \frac{\partial}{\partial x_3'}, \tag{7.107}$$
$$\frac{\partial}{\partial t} = \frac{\partial x_j'}{\partial t}\frac{\partial}{\partial x_j'} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = -\gamma u\frac{\partial}{\partial x_1'} + \gamma\frac{\partial}{\partial t'}. \tag{7.108}$$

On the other hand, (7.7) and (7.10) are equivalent to the system

$$\begin{aligned}
\frac{\partial E_3}{\partial x_2} - \frac{\partial E_2}{\partial x_3} &= -\frac{\partial B_1}{\partial t},\\
\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1} &= -\frac{\partial B_2}{\partial t},\\
\frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2} &= -\frac{\partial B_3}{\partial t},\\
\frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3} &= 0,
\end{aligned}$$

which, due to the transformation rules (7.105)–(7.108), become

$$\begin{aligned}
\frac{\partial E_3}{\partial x_2'} - \frac{\partial E_2}{\partial x_3'} &= \gamma u\frac{\partial B_1}{\partial x_1'} - \gamma\frac{\partial B_1}{\partial t'},\\
\frac{\partial E_1}{\partial x_3'} - \gamma\frac{\partial E_3}{\partial x_1'} + \gamma\frac{u}{c^2}\frac{\partial E_3}{\partial t'} &= \gamma u\frac{\partial B_2}{\partial x_1'} - \gamma\frac{\partial B_2}{\partial t},
\end{aligned}$$

[3]For the sake of simplicity, we only consider special transformations here. To explore the general case, refer to Chap. 10.

$$
\begin{aligned}
&\gamma\frac{\partial E_2}{\partial x_1'} - \gamma\frac{u}{c^2}\frac{\partial E_2}{\partial t'} - \frac{\partial E_1}{\partial x_2'} = \gamma u\frac{\partial B_3}{\partial x_1'} - \gamma\frac{\partial B_3}{\partial t},\\
&\gamma\frac{\partial B_1}{\partial x_1'} - \gamma\frac{u}{c^2}\frac{\partial B_1}{\partial t'} + \frac{\partial B_2}{\partial x_2'} + \frac{\partial B_3}{\partial x_3'} = 0.
\end{aligned}
$$

Substituting the value of $\gamma\frac{\partial B_1}{\partial x_1'}$ obtained from the fourth equation into the first one and then substituting the value of $\gamma\frac{\partial B_1}{\partial t'}$ derived from the first equation into the fourth one, we arrive at the system

$$
\begin{aligned}
&\frac{\partial}{\partial x_2'}\gamma(E_3 + uB_2) - \frac{\partial}{\partial x_3'}\gamma(E_2 - uB_3) = -\frac{\partial B_1}{\partial t'},\\
&\frac{\partial E_1}{\partial x_3'} - \frac{\partial}{\partial x_1'}(E_3 + uB_2) = -\frac{\partial}{\partial t'}\gamma(B_2 + \frac{u}{c^2}E_3),\\
&\frac{\partial}{\partial x_1'}\gamma(E_2 - uB_3) - \frac{\partial E_1}{\partial x_2'} = -\frac{\partial}{\partial t'}\gamma(B_3 - \frac{u}{c^2}E_2),\\
&\frac{\partial B_1}{\partial x_1'} + \frac{\partial}{\partial x_2'}\gamma(B_2 + \frac{u}{c^2}E_3) + \frac{\partial}{\partial x_3'}\gamma(B_3 - \frac{u}{c^2}E_2) = 0,
\end{aligned}
$$

which assumes the form of the Maxwell equations (7.7) and (7.9) in the inertial frame I' if and only if

$$
\begin{aligned}
B_1' &= B_1,\\
B_2' &= \gamma(B_2 + \frac{u}{c^2}E_3),\\
B_3' &= \gamma(B_3 - \frac{u}{c^2}E_2),\\
E_1' &= E_1,\\
E_2' &= \gamma(E_2 - uB_3),\\
E_3' &= \gamma(E_3 + uB_2).
\end{aligned}
$$

If we introduce the notations $\mathbf{a}_\parallel$ and $\mathbf{a}_\perp$ to denote the projections of the vector $\mathbf{a}$ along $\mathbf{u}$ and orthogonal to $\mathbf{u}$, respectively, then the above transformation formulae can be written in the vector forms

$$\mathbf{B}_\parallel' = \mathbf{B}_\parallel, \tag{7.109}$$

$$\mathbf{B}_\perp' = \gamma\left(\mathbf{B}_\perp - \frac{1}{c^2}\mathbf{u}\times\mathbf{E}_\perp\right), \tag{7.110}$$

$$\mathbf{E}_\parallel' = \mathbf{E}_\parallel, \tag{7.111}$$

$$\mathbf{E}_\perp' = \gamma(\mathbf{E}_\perp + \mathbf{u}\times\mathbf{B}_\perp). \tag{7.112}$$

Using the same arguments it is possible to verify that the other two Maxwell equations (7.8 and 7.9) as well as the conservation of charge (7.20) lead us to the following transformation rules:

$$\mathbf{H}'_{\parallel} = \mathbf{H}_{\parallel}, \tag{7.113}$$
$$\mathbf{H}'_{\perp} = \gamma(\mathbf{H}_{\perp} - \mathbf{u} \times \mathbf{D}_{\perp}), \tag{7.114}$$
$$\mathbf{D}'_{\parallel} = \mathbf{D}_{\parallel}, \tag{7.115}$$
$$\mathbf{D}'_{\perp} = \gamma\left(\mathbf{D}_{\perp} + \frac{1}{c^2}\mathbf{u} \times \mathbf{H}_{\perp}\right), \tag{7.116}$$
$$\mathbf{j}'_{\parallel} = \gamma(\mathbf{j}_{\parallel} - \rho_f \mathbf{u}), \tag{7.117}$$
$$\mathbf{j}'_{\perp} = \mathbf{j}_{\perp}, \tag{7.118}$$
$$\rho'_f = \gamma\left(\rho_f - \frac{\mathbf{u} \cdot \mathbf{j}}{c^2}\right). \tag{7.119}$$

It is easy to verify that the transformation rules (7.109)–(7.119) imply that the constitutive equations are invariant upon Lorentz transformations. [4] For instance, for the electric induction vector $\mathbf{D}$, if $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$, we have

$$\mathbf{D}' = \mathbf{D}'_{\parallel} + \mathbf{D}'_{\perp} = \mathbf{D}_{\parallel} + \gamma(\mathbf{D}'_{\perp} + \frac{1}{c^2}\mathbf{u} \times \mathbf{H}_{\perp}) = \epsilon\mathbf{E}_{\parallel} + \gamma(\epsilon\mathbf{E}_{\perp} + \frac{1}{c^2}\mathbf{u} \times \mathbf{H}_{\perp}),$$

so that, using (7.112)–(7.119) and recalling that $c^2 = 1/\epsilon\mu$, we obtain

$$\mathbf{D}' = \epsilon\mathbf{E}'.$$

7.6 Quasi-electrostatic and Quasi-magnetostatic Approximations

If we wish to study the evolution of a continuous system that includes charges and currents, we need to consider both the balance equations from continuum mechanics and Maxwell's equations. This approach is meaningful if Maxwell's equations are form invariant or *covariant* for the same transformation group under which the equations of continuum mechanics are covariant. It is well known that the former equations are covariant with respect to Lorentz transformations, whereas the latter equations are covariant with respect to Galilean transformations. Consequently, in order to obtain a coherent theory it is necessary to propose a relativistic formulation

[4] Note that, for the total current density $\mathbf{j}$, we only need to consider the conduction current density $\mathbf{j}_0$ in the proper frame and analyze the behavior of the constitutive equation $\mathbf{j}_0 = \sigma\mathbf{E}_0$ under a Lorentz transformation. We recall that the proper frame for a point $\mathbf{x} \in \mathbf{S}$ at the instant t is a reference in which $\mathbf{v} = \mathbf{0}$.

for continuum mechanics. However, we will prove that, in the presence of acoustic frequencies (less than 10^4 Hz) and nonrelativistic velocities (less than 10^5 km/s), it is possible to find approximate forms of Maxwell's equations that are covariant for Galilean transformations.

We start with the *quasi-electrostatic approximation*. First, we introduce the reference quantities

$$E^*, D^*, H^*, B^*, J^*, L, T$$

for the electric field, electrical induction, the magnetic field, magnetic induction, the electric current, the characteristic length, and the characteristic time, respectively. Consequently, the Maxwell equations in nondimensional form can be written as follows:

$$\nabla \times \mathbf{E} = -\frac{LB^*}{TE^*}\frac{\partial \mathbf{B}}{\partial t} = -\frac{UB^*}{E^*}\frac{\partial \mathbf{B}}{\partial t}, \tag{7.120}$$

$$\nabla \times \mathbf{H} = \frac{J^*L}{H^*}\mathbf{j} + \frac{LD^*}{TH^*}\frac{\partial \mathbf{D}}{\partial t} = \frac{J^*L}{H^*}\mathbf{j} + \frac{UD^*}{H^*}\frac{\partial \mathbf{D}}{\partial t}, \tag{7.121}$$

where we have used the same symbols for the nondimensional fields and operators; moreover, $U = L/T$ is the reference velocity.

The quasi-electrostatic approximation is based on the following assumptions:

- The continuum S is a *dielectric* ($\mathbf{j} = \mathbf{0}$).
- There is no free charge ($\rho_f = 0$).
- S is magnetically linear; in other words, in the proper frame I_0, the following constitutive equation holds:

$$\mathbf{B_0} = \mu \mathbf{H}_0.$$

- The magnetic field vanishes in the proper frame I_0 (i.e., there is no magnet). This means that magnetic fields that appear in other frames are due to the motion of S. Noting that the rules for transforming from any frame of reference I to the proper frame I_0 are obtained by inserting $\mathbf{u} = \mathbf{v}$ into (7.109)–(7.119), we have

$$\mathbf{0} \simeq (\mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E}_\perp) \Longrightarrow B^* \simeq \frac{UE^*}{c^2}. \tag{7.122}$$

- Points in S have nonrelativistic velocities; i.e.,

$$\frac{U^2}{c^2} \ll 1. \tag{7.123}$$

Conditions (7.122) and (7.123) imply that the right-hand side of (7.12) can be neglected. Consequently, in Maxwell's equations (7.7) and (7.9), the magnetic fields do not appear:

$$\nabla \times \mathbf{E} = \mathbf{0}, \tag{7.124}$$

$$\nabla \cdot \mathbf{D} = 0. \tag{7.125}$$

On singular material surfaces, the conditions (7.29) and (7.31) become

$$[[\mathbf{n} \times \mathbf{E}]] = \mathbf{0}, \tag{7.126}$$

$$[[\mathbf{D} \cdot \mathbf{n}]] = 0, \tag{7.127}$$

whereas conditions (7.16)–(7.18) assume the following form on nonmaterial singular surfaces:

$$[[(c_n - v_n)\mathbf{D} + \mathbf{n} \times (\mathbf{H} - \mathbf{v} \times \mathbf{D})]] = \mathbf{0}, \tag{7.128}$$

$$[[v_n]] = 0. \tag{7.129}$$

$$[[\mathbf{D} \cdot \mathbf{n}]] = 0. \tag{7.130}$$

The remaining two Maxwell equations

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \tag{7.131}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{7.132}$$

can be used to determine the magnetic field $\mathbf{H}$, which, in the present approximation, is a secondary effect induced by the electric fields.

It is an easy exercise to verify that (7.124) and (7.125), as well as the jump conditions (7.126) and (7.127), are covariant for Galilean transformations if and only if

$$\mathbf{E}' = \mathbf{E}, \tag{7.133}$$

$$\mathbf{D}' = \mathbf{D}, \tag{7.134}$$

$$\mathbf{H}' = \mathbf{H} - \mathbf{u} \times \mathbf{D}, \tag{7.135}$$

$$\mathbf{B}' = \mathbf{B}. \tag{7.136}$$

In other words, the Maxwell equations that describe the evolutions of the fields $\mathbf{E}$ and $\mathbf{D}$ are invariant with respect to Galilean transformations together with the constitutive equation for $\mathbf{D}$. However, the remaining Maxwell equations, which determine the fields $\mathbf{H}$ and $\mathbf{B}$ when the electric fields are known, are covariant under Galilean transformations to within very small terms. In fact, in the linear case we have

$$\mathbf{B}' = \mathbf{B} = \mu\mathbf{H} = \mu\mathbf{H}' + \mu\epsilon\mathbf{u} \times \mathbf{E} = \mu\mathbf{H}' + \frac{1}{c^2}\mathbf{u} \times \mathbf{E}.$$

The *quasi-magnetostatic approximation* is based on the following assumptions:

- The continuum S is a conductor.
- S is electrically and magnetically linear:
$$\mathbf{D} = \epsilon\mathbf{E}, \quad \mathbf{B} = \mu\mathbf{H}. \tag{7.137}$$
- The electric fields in the proper frame I_0 vanish. This implies that there is no free charge ($\rho_f = 0$) and no current in I_0. Moreover, the electric fields are due to the motion of S. Based on this hypothesis and the transformation formulae (7.111) and (7.112) when $\mathbf{u} = \mathbf{v}$, we obtain
$$\mathbf{0} \simeq (\mathbf{E} + \mathbf{v} \times \mathbf{B}_{\perp}) \Longrightarrow E^* \simeq UB^*. \tag{7.138}$$
- The points of S have nonrelativistic velocities:
$$\frac{U^2}{c^2} \ll 1. \tag{7.139}$$

Conditions (7.138) and (7.139) imply that the second term on the right-hand side of (7.121) can be neglected, so the fundamental equations of quasi-magnetostatics become

$$\nabla \times \mathbf{H} = \mathbf{j}, \tag{7.140}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{7.141}$$

for all points where the fields involved are regular. On material singular surfaces, the jump conditions (7.12) can be written as

$$[[\mathbf{n} \times \mathbf{H}]] = \mathbf{k}, \tag{7.142}$$
$$[[\mathbf{B} \cdot \mathbf{n}]] = 0, \tag{7.143}$$

whereas the corresponding jump conditions on nonmaterial surfaces are

$$[[(c_n - v_n)\mathbf{B} + \mathbf{n} \times (\mathbf{D} - \mathbf{v} \times \mathbf{H})]] = \mathbf{0}, \tag{7.144}$$
$$[[v_n]] = 0, \tag{7.145}$$
$$[[\mathbf{D} \cdot \mathbf{n}]] = 0. \tag{7.146}$$

The remaining two equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{7.147}$$
$$\nabla \cdot \mathbf{D} = 0 \tag{7.148}$$

can be used to determine the electric field $\mathbf{E}$, which is a secondary effect induced by the magnetic field in this approximation.

Moreover, we remark that the magnetic fields are invariant under Galilean transformations:

$$\mathbf{H}' = \mathbf{H}, \qquad \mathbf{B}' = \mathbf{B}, \tag{7.149}$$

whereas the electric fields are invariant to within a very small amount. In fact,

$$\mathbf{D}' = \mathbf{D} = \epsilon\mathbf{E} = \epsilon\mathbf{E}' - \mu\epsilon\mathbf{u}\times\mathbf{H} = \mu\mathbf{E}' - \frac{1}{c^2}\mathbf{u}\times\mathbf{H}.$$

Finally, charge conservation law (7.20) becomes

$$\nabla\cdot\mathbf{j} = 0, \tag{7.150}$$

and it is invariant with respect to Galilean transformation if and only if

$$\mathbf{j}' = \mathbf{j}. \tag{7.151}$$

To complete the model of quasi-magnetostatics, we must assign the constitutive equation for the current density $\mathbf{j}$. Limiting our analysis to the case of a linear isotropic conductor, in the proper frame I_0 we have

$$\mathbf{j}_0 = \sigma\mathbf{E}_0. \tag{7.152}$$

Since the electric field is a secondary effect in quasi-magnetostatics, the current is only relevant when the conductivity is high. In this case, the equation $\mathbf{E}_0 = \mathbf{E} + \mathbf{v}\times\mathbf{B}$ leads to the constitutive relation

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v}\times\mathbf{B}). \tag{7.153}$$

7.7 Balance Equations for Quasi-electrostatics

In order to obtain a model of a moving continuum in the presence of electromagnetic fields, we need to write the balance equations for momentum, angular momentum, and energy. Considering what was said in the above sections, the electromagnetic equations we must associate with these balance laws are the equations of quasi-magnetostatics or quasi-electrostatics in order to ensure that the whole set of equations is covariant for the Galilean group.

We start by analyzing an elastic dielectric. To do this we can adopt one of the following approaches:

- Specify the body forces and torques in the usual equations of continuum mechanics based on a model of the interaction between the matter and the electromagnetic field. This approach, which has been widely adopted in the literature, requires increasingly sophisticated models that lead to increasingly involved electromagnetic forces.

- Adopt a phenomenological approach that does not require a model.

We will adopt the second approach and show that it encompasses any possible model.

The local forms of the momentum equation and the associated jump conditions are (see Eqs. (5.30–5.31) of [16])

$$\rho\dot{\mathbf{v}} = \nabla\cdot\mathbf{T} + \rho\mathbf{b}, \tag{7.154}$$

$$[[\rho\mathbf{v}(c_n - v_n) + \mathbf{T}\mathbf{n}]] = \mathbf{0}, \tag{7.155}$$

where ρ is the mass density, $\mathbf{v}$ is the velocity field, $\mathbf{T}$ is the symmetric total stress tensor, $\mathbf{b}$ is the external body force, and c_n denotes the velocity at which the discontinuity surface advances.

The local forms of the energy balance and the corresponding jump conditions are (see Eq. (5.46) of [16])

$$\rho\dot{\varepsilon} = \mathbf{T} : \nabla\mathbf{v} - \nabla\cdot(\mathbf{h} + \mathbf{E}_0\times\mathbf{H}_0) + \rho r, \tag{7.156}$$

$$[[\rho(\frac{1}{2}v^2 + \epsilon)(c_n - v_n) + \mathbf{v}\cdot\mathbf{T}\mathbf{n} - (\mathbf{h} + \mathbf{E}_0\times\mathbf{H}_0)\cdot\mathbf{n}]] = 0, \tag{7.157}$$

where ϵ is the specific internal energy and the energy flux vector is obtained by adding the Poynting vector $\mathbf{E}_0\times\mathbf{H}_0$, which is evaluated in the proper frame of any particle in the continuum, to the heat current vector. This approach allows this vector quantity to be invariant under a change of reference frame.

Finally, for the reduced dissipation inequality, we have the conditions (see Eqs. (5.61) and (5.63) of [16])

$$-\rho(\dot{\psi} + \eta\dot{\theta}) + \mathbf{T} : \nabla\mathbf{v} - \frac{\mathbf{h}\cdot\nabla\theta}{\theta} \geq 0, \tag{7.158}$$

$$[[\rho(\frac{1}{2}v^2 + \psi)(c_n - v_n) + \mathbf{v}\cdot\mathbf{T}\mathbf{n}]] \leq 0, \tag{7.159}$$

where $\psi = \epsilon - \eta\theta$ is the specific free energy, η is the specific entropy, and θ is the absolute temperature.

We now use the Maxwell equations of quasi-electrostatics to write the above relations in an equivalent form. Using the vector identity

$$\nabla\cdot(\mathbf{E}_0\times\mathbf{H}_0) = \mathbf{H}_0\cdot\nabla\times\mathbf{E}_0 - \mathbf{E}_0\cdot\nabla\times\mathbf{H}_0,$$

and (7.133)–(7.136), we obtain

$$-\nabla\cdot(\mathbf{E}_0\times\mathbf{H}_0) = \mathbf{E}\cdot\nabla\times(\mathbf{H} - \mathbf{v}\times\mathbf{D}) - (\mathbf{H} - \mathbf{v}\times\mathbf{D})\cdot\nabla\times\mathbf{E}.$$

If we take into account (7.124) and (7.125), we can write

$$\begin{aligned}-\nabla\cdot(\mathbf{E}_0\times\mathbf{H}_0) = {} & \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} - \mathbf{D}\cdot\nabla\mathbf{v}\cdot\mathbf{E} \\ & + \mathbf{v}\cdot\nabla\mathbf{D}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{v}(\nabla\cdot\mathbf{D}) + \mathbf{D}\cdot\mathbf{E}(\nabla\cdot\mathbf{v}),\end{aligned}$$

which, noting that

$$\begin{aligned}
\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} &= \mathbf{E}\cdot\dot{\mathbf{D}} - \mathbf{v}\cdot\nabla\mathbf{D}\cdot\mathbf{E} \\
&= \frac{d}{dt}\left(\rho\frac{\mathbf{D}\cdot\mathbf{E}}{\rho}\right) - \mathbf{D}\cdot\dot{\mathbf{E}} - \mathbf{v}\cdot\nabla\mathbf{D}\cdot\mathbf{E} \\
&= \rho\frac{d}{dt}\left(\frac{\mathbf{D}\cdot\mathbf{E}}{\rho}\right) - (\mathbf{D}\cdot\mathbf{E})\nabla\cdot\mathbf{v} - \mathbf{D}\cdot\dot{\mathbf{E}} - \mathbf{v}\cdot\nabla\mathbf{D}\cdot\mathbf{E},
\end{aligned}$$

becomes

$$-\nabla\cdot(\mathbf{E}_0\times\mathbf{H}_0) = \rho\frac{d}{dt}\left(\frac{\mathbf{D}\cdot\mathbf{E}}{\rho}\right) - \mathbf{D}\cdot\dot{\mathbf{E}} - \mathbf{D}\otimes\mathbf{E} : \nabla\mathbf{v}. \tag{7.160}$$

Introducing this relation into (7.156), we obtain

$$\rho\frac{d}{dt}\left(\epsilon - \frac{\mathbf{D}\cdot\mathbf{E}}{\rho}\right) = -\mathbf{D}\cdot\dot{\mathbf{E}} + (\mathbf{T} - \mathbf{D}\otimes\mathbf{E}) : \nabla\mathbf{v} - \nabla\cdot\mathbf{h} + \rho r, \tag{7.161}$$

whereas the reduced dissipation inequality (7.158) assumes the form

$$-\rho(\dot{\zeta} + \eta\dot{\theta}) - \mathbf{D}\cdot\dot{\mathbf{E}} + (\mathbf{T} - \mathbf{D}\otimes\mathbf{E}) : \nabla\mathbf{v} - \frac{\mathbf{h}\cdot\nabla\theta}{\theta} \geq 0, \tag{7.162}$$

where

$$\zeta = \psi - \frac{\mathbf{D}\cdot\mathbf{E}}{\rho} \tag{7.163}$$

is the specific enthalpy.

A *thermoelastic dielectric* is defined by the following constitutive equations:

$$\zeta = \zeta(\mathbf{F}, \mathbf{E}, \theta, \mathbf{g}), \tag{7.164}$$

$$\eta = \eta(\mathbf{F}, \mathbf{E}, \theta, \mathbf{g}), \tag{7.165}$$

$$\mathbf{T} = \mathbf{T}(\mathbf{F}, \mathbf{E}, \theta, \mathbf{g}), \tag{7.166}$$

$$\mathbf{D} = \mathbf{D}(\mathbf{F}, \mathbf{E}, \theta, \mathbf{g}), \tag{7.167}$$

$$\mathbf{h} = \mathbf{h}(\mathbf{F}, \mathbf{E}, \theta, \mathbf{g}), \tag{7.168}$$

where $\mathbf{g} = \nabla\theta$. Substituting these relations into (7.162) and bearing in mind that $\nabla\mathbf{v} = (\mathbf{F}^{-1})^T\dot{\mathbf{F}}$, we obtain the inequality

$$\begin{aligned}
&-\rho\frac{\partial\zeta}{\partial\mathbf{g}}\cdot\mathbf{g} - \rho\left(\eta + \frac{\partial\zeta}{\partial\theta}\right)\dot{\theta} + \left(\mathbf{D} + \rho\frac{\partial\zeta}{\partial\mathbf{E}}\right)\dot{\mathbf{E}} \\
&+ \left((\mathbf{T} - \mathbf{D}\otimes\mathbf{E})(\mathbf{F}^{-1})^T - \rho\frac{\partial\zeta}{\partial\mathbf{F}}\right)\dot{\mathbf{F}} - \frac{\mathbf{h}\cdot\nabla\theta}{\theta} \geq 0,
\end{aligned} \tag{7.169}$$

from which, based on the usual arguments (see Chap. 6 of [16]), we derive the thermodynamic restrictions on the constitutive equations

$$\zeta = \zeta(\mathbf{F}, \mathbf{E}, \theta), \tag{7.170}$$

$$\eta = -\frac{\partial \zeta}{\partial \theta} = \eta(\mathbf{F}, \mathbf{E}, \theta), \tag{7.171}$$

$$\mathbf{T} = \mathbf{D} \otimes \mathbf{E} + \rho \frac{\partial \zeta}{\partial \mathbf{F}} \mathbf{F}^T = \mathbf{T}(\mathbf{F}, \mathbf{E}, \theta), \tag{7.172}$$

$$\mathbf{D} = -\rho \frac{\partial \zeta}{\partial \mathbf{E}} = \mathbf{D}(\mathbf{F}, \mathbf{E}, \theta), \tag{7.173}$$

$$\mathbf{h} \cdot \nabla \theta \leq 0. \tag{7.174}$$

In other words, *the specific enthalpy is a thermodynamic potential for the entropy, the stress tensor and the electric induction field*, which consequently depend only on $\mathbf{F}, \mathbf{E}$ and θ. The heat current vector still depends on the temperature gradient and must satisfy inequality (7.174).

Moreover, the principle of material frame indifference implies that the constitutive equations assume the following final forms (see Chap. 6 of [16]):

$$\zeta = \zeta(\mathbf{C}, \mathbf{F}^T \mathbf{E}, \theta), \tag{7.175}$$

$$\eta = -\frac{\partial \zeta}{\partial \theta} = \eta(\mathbf{C}, \mathbf{F}^T \mathbf{E}, \theta), \tag{7.176}$$

$$\mathbf{T} = \mathbf{D} \otimes \mathbf{E} + 2\rho \mathbf{F} \frac{\partial \zeta}{\partial \mathbf{C}} \mathbf{F}^T, \tag{7.177}$$

$$\mathbf{D} = -\rho \frac{\partial \zeta}{\partial \mathbf{E}} = \mathbf{F} \hat{\mathbf{D}}(\mathbf{C}, \mathbf{F}^T \mathbf{E}, \theta), \tag{7.178}$$

$$\mathbf{h} = \mathbf{F} \hat{\mathbf{h}}(\mathbf{C}, \mathbf{F}^T \mathbf{E}, \mathbf{F}^T \mathbf{g}). \tag{7.179}$$

7.8 Isotropic and Anisotropic Constitutive Equations

In this section we start considering isotropic dielectrics. To do this, we utilize the definitions and theorems given in Sect. 7.2 of [16].

First we introduce the material vectors

$$\mathcal{E} = \mathbf{F}^T \mathbf{E}, \quad \boldsymbol{\Theta} = \mathbf{F}^T \mathbf{g} \tag{7.180}$$

and we omit the nonessential dependence on θ of the constitutive equations (7.164)–(7.168). A dielectric material S is isotropic if and only if

$$\zeta(\mathbf{F}\mathbf{Q}, \mathbf{Q}^T \mathcal{E}) = \zeta(\mathbf{F}, \mathcal{E}), \tag{7.181}$$

$$\eta(\mathbf{FQ},\mathbf{Q}^T\mathcal{E})=\eta(\mathbf{F},\mathcal{E}),\tag{7.182}$$
$$\mathbf{QT}(\mathbf{FQ},\mathbf{Q}^T\mathcal{E})\mathbf{Q}^T=\mathbf{T}(\mathbf{F},\mathcal{E}),\tag{7.183}$$
$$\mathbf{Q}\hat{\mathbf{D}}(\mathbf{FQ},\mathbf{Q}^T\mathcal{E})=\hat{\mathbf{D}}(\mathbf{F},\mathcal{E}),\tag{7.184}$$
$$\mathbf{Q}\hat{\mathbf{h}}(\mathbf{FQ},\mathbf{Q}^T\mathcal{E},\mathbf{Q}^T\boldsymbol{\Theta})=\hat{\mathbf{h}}(\mathbf{F},\mathcal{E},\boldsymbol{\Theta}).\tag{7.185}$$

Recalling that the objectivity principle leads to (7.175)–(7.179), the above equations can also be written in the form

$$\zeta(\mathbf{Q}^T\mathbf{CQ},\mathbf{Q}^T\mathcal{E})=\zeta(\mathbf{C},\mathcal{E}),\tag{7.186}$$
$$\eta(\mathbf{Q}^T\mathbf{CQ},\mathbf{Q}^T\mathcal{E})=\eta(\mathbf{C},\mathcal{E}),\tag{7.187}$$
$$\mathbf{QT}(\mathbf{Q}^T\mathbf{CQ},\mathbf{Q}^T\mathcal{E})\mathbf{Q}^T=\mathbf{T}(\mathbf{C},\mathcal{E}),\tag{7.188}$$
$$\mathbf{QD}(\mathbf{Q}^T\mathbf{CQ},\mathbf{Q}^T\mathcal{E})=\mathbf{D}(\mathbf{C},\mathcal{E}),\tag{7.189}$$
$$\mathbf{Q}\hat{\mathbf{h}}(\mathbf{Q}^T\mathbf{CQ},\mathbf{Q}^T\mathcal{E},\mathbf{Q}^T\boldsymbol{\Theta})=\hat{\mathbf{h}}(\mathbf{C},\mathcal{E},\boldsymbol{\Theta}).\tag{7.190}$$

If we replace the arbitrary orthogonal matrix $\mathbf{Q}$ with $\mathbf{Q}^T$, then we conclude that the constitutive equations are expressed by isotropic functions of their variables. Using the usual representation theorems of isotropic functions and limiting our attention to the isothermal case, we get

$$\zeta=\zeta(I,II,III,\mathcal{E}^2,\mathcal{E}^T\mathbf{C}\mathcal{E},\mathcal{E}^T\mathbf{C}^2\mathcal{E}),\tag{7.191}$$
$$\eta=\eta(I,II,III,\mathcal{E}^2,\mathcal{E}^T\mathbf{C}\mathcal{E},\mathcal{E}^T\mathbf{C}^2\mathcal{E}),\tag{7.192}$$
$$\begin{aligned}\mathbf{T}=\mathbf{F}(&k_0\mathbf{I}+k_1\mathbf{C}+k_2\mathbf{C}^2\\&+k_3\mathcal{E}\otimes\mathcal{E}+k_4(\mathcal{E}\otimes\mathbf{C}\mathcal{E}+\mathbf{C}\mathcal{E}\otimes\mathcal{E})\\&+k_5(\mathcal{E}\otimes\mathbf{C}^2\mathcal{E}+\mathbf{C}^2\mathcal{E}\otimes\mathcal{E}))\mathbf{F}^T,\end{aligned}\tag{7.193}$$
$$\mathbf{D}=\mathbf{F}(h_0\mathbf{I}+h_1\mathbf{C}+h_2\mathbf{C}^2)\mathcal{E},\tag{7.194}$$

where I, II and III are the principal invariants of $\mathbf{C}$, and the functions h_i and k_i depend on the same variables as ζ.

Bearing in mind (7.180), recalling the definitions of the right and left Cauchy–Green tensors (see Sect. 3.2 of Vol. I) $\mathbf{C}=\mathbf{F}^T\mathbf{F}$, $\mathbf{B}=\mathbf{FF}^T$, and remembering the Cayley–Hamilton theorem and that the principal invariants of $\mathbf{C}$ and $\mathbf{B}$ coincide, we can easily verify the identities

$$\begin{aligned}\mathcal{E}^2&=\mathbf{E}^T\mathbf{BE},\\ \mathcal{E}^T\mathbf{C}\mathcal{E}&=\mathbf{E}^T\mathbf{B}^2\mathbf{E},\\ \mathcal{E}^T\mathbf{C}^2\mathcal{E}&=\mathbf{E}^T\mathbf{B}^3\mathbf{E}\\ &=\mathbf{E}^T(I\mathbf{B}^2-II\mathbf{B}+III\mathbf{I})\mathbf{E}.\end{aligned}$$

Consequently, relations (7.191)–(7.194) can also be written in the form

$$\zeta=\zeta(I,II,III,\mathbf{E}^2,\mathbf{E}^T\mathbf{BE},\mathbf{E}^T\mathbf{B}^2\mathbf{E}),\tag{7.195}$$
$$\eta=\eta(I,II,III,\mathbf{E}^2,\mathbf{E}^T\mathbf{BE},\mathbf{E}^T\mathbf{B}^2\mathbf{E}),\tag{7.196}$$

$$\begin{aligned}\mathbf{T} &= (K_0\mathbf{I} + K_1\mathbf{B} + K_2\mathbf{B}^2 \\ &\quad + K_3\mathbf{E}\otimes\mathbf{E} + K_4(\mathbf{E}\otimes\mathbf{BE} + \mathbf{BE}\otimes\mathbf{E}) \\ &\quad + K_5(\mathbf{E}\otimes\mathbf{B}^2\mathbf{E} + \mathbf{B}^2\mathbf{E}\otimes\mathbf{E})), \qquad (7.197)\\ \mathbf{D} &= (H_0\mathbf{I} + H_1\mathbf{B} + H_2\mathbf{B}^2)\mathbf{E}, \qquad (7.198)\end{aligned}$$

where K_i and H_i are new functions of the invariants $I, II, III, \mathbf{E}^2, \mathbf{E}^T\mathbf{BE}$ and $\mathbf{E}^T\mathbf{B}^2\mathbf{E}$.

It is very interesting to note that, in the linear theory of *isotropic* elastic dielectrics, the electric field has no influence on the stress and the deformation has no influence on the electrical induction. These influences, which appear when *second-order* terms in the deformation and electric field are considered in (7.197) and (7.198), are called *electrostriction*.

The situation is different for anisotropic elastic dielectrics; in such dielectrics these influences can appear as a first-order effect. In fact, if we denote the displacement gradient by $\mathbf{H} = \mathbf{F} - \mathbf{I}$, in the linear approximation we have $\mathbf{C} - \mathbf{I} \simeq 2\mathbf{S}$, where $\mathbf{S}$ is the infinitesimal strain tensor.

Moreover, the quadratic approximation for the enthalpy ζ can be written as follows:

$$\zeta = \frac{1}{2\rho_*}\left(\mathbb{C}_{ijkl}S_{ij}S_{kl} + 2\chi_{ijk}S_{ij}E_k + \frac{1}{2}\epsilon_{ij}E_iE_j\right), \qquad (7.199)$$

where the linear elasticity tensor $\mathbf{S}$, the piezoelectric tensor χ and the dielectric tensor ϵ verify the symmetry conditions

$$\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{klij}, \qquad (7.200)$$

$$\chi_{ijk} = \chi_{jik}, \quad \epsilon_{ij} = \epsilon_{ji}. \qquad (7.201)$$

Finally, from (7.177) and (7.178) we obtain the linear constitutive equations for the stress tensor $\mathbf{T}$ and the electrical induction $\mathbf{D}$:

$$T_{ij} = \mathbb{C}_{ijkl}S_{kl} + \chi_{ijk}E_k, \qquad (7.202)$$

$$D_i = \chi_{jki}S_{jk} + \epsilon_{ij}E_j. \qquad (7.203)$$

7.9 Polarization Fields and the Equations of Quasi-electrostatics

In the above section we adopted a phenomenological approach in order to describe the interaction between the electric field and matter. In other words, no particular physical model was proposed to justify the balance equations for momentum, angular momentum, and energy. In this section we adopt the polarization vector to describe the electric field. We thus rewrite these balance equations in a new form which is interpreted by resorting to a simple model of dielectrics.

First, we introduce the *polarization vector*

$$\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E},$$

as well as the *specific polarization vector*

$$\mathbf{p} = \frac{\mathbf{P}}{\rho}.$$

We then have

$$\rho \frac{d}{dt}\left(\frac{\mathbf{D}\cdot\mathbf{E}}{\rho}\right) = \rho \frac{d}{dt}\left(\epsilon_0 \frac{E^2}{\rho}\right) + \rho\dot{\mathbf{p}}\cdot\mathbf{E} + \rho\mathbf{p}\cdot\dot{\mathbf{E}}, \tag{7.204}$$

$$-\mathbf{D}\cdot\dot{\mathbf{E}} = -\rho\mathbf{p}\cdot\dot{\mathbf{E}} - \frac{1}{2}\epsilon_0\rho\left(\frac{d}{dt}\frac{E^2}{\rho} + E^2\nabla\cdot\mathbf{v}\right), \tag{7.205}$$

$$-\mathbf{D}\otimes\mathbf{E} = -\epsilon_0\mathbf{E}\otimes\mathbf{E} - \rho\mathbf{p}\otimes\mathbf{E}, \tag{7.206}$$

and the balance equation for energy (7.156) can be placed in the form

$$\rho\dot{e}_m = \rho\mathbf{E}\cdot\dot{\mathbf{p}} + \mathbf{t}_m : \nabla\mathbf{v} - \nabla\cdot\mathbf{h} + \rho r, \tag{7.207}$$

where e_m and $\mathbf{t}_m$ are defined as follows:

$$e_m = \epsilon - \epsilon_0 \frac{E^2}{2\rho}, \tag{7.208}$$

$$\mathbf{t}_m = \mathbf{T} + \frac{1}{2}\epsilon_0 E^2 \mathbf{I} - \epsilon_0\mathbf{E}\otimes\mathbf{E} - \rho\mathbf{p}\otimes\mathbf{E}. \tag{7.209}$$

On the other hand, Maxwell's equations (7.124) and (7.125) become

$$\nabla\times\mathbf{E} = \mathbf{0},$$
$$\epsilon_0\nabla\cdot\mathbf{E} = -\nabla\cdot\mathbf{P} \equiv \rho_P.$$

Since the following identity holds:

$$\nabla\cdot\left(\frac{1}{2}\epsilon_0 E^2\mathbf{I} - \epsilon_0\mathbf{E}\otimes\mathbf{E} - \rho\mathbf{p}\otimes\mathbf{E}\right) = -\mathbf{P}\cdot\nabla\mathbf{E}, \tag{7.210}$$

we obtain the balance equations and Maxwell's equations in the following final forms:

$$\rho\dot{\mathbf{v}} = \nabla\cdot\mathbf{t}_m + \mathbf{P}\cdot\nabla\mathbf{E}, \tag{7.211}$$

$$\rho\dot{e}_m = \rho\mathbf{E}\cdot\dot{\mathbf{p}} + \mathbf{t}_m : \nabla\mathbf{v} - \nabla\cdot\mathbf{h} + \rho r, \tag{7.212}$$

$$\nabla\times\mathbf{E} = \mathbf{0}, \tag{7.213}$$

$$\epsilon_0\nabla\cdot\mathbf{E} = -\nabla\cdot\mathbf{P} \equiv \rho_P. \tag{7.214}$$

A simple physical description of these equations can be obtained. It is known that a continuous distribution of dipoles with a volume density $\mathbf{P}$ is equivalent to a continuous distribution of charges with a density $\rho_P = -\nabla \cdot \mathbf{P}$. Moreover, a force per unit volume given by $-\mathbf{P}\cdot\nabla\mathbf{E}$ acts on these dipoles. Finally, a power $\rho\mathbf{E}\cdot\dot{\mathbf{p}}$ per unit volume is associated with their motion. Consequently, the dielectric can be regarded as a continuous distribution of dipoles, and $\mathbf{t}_m$ is the *mechanical* stress tensor.

Together with the above equations we must also consider the reduced dissipation inequality, which can also be written

$$-\rho(\dot{\psi}_m + \eta\dot{\theta}) + \rho\mathbf{E}\cdot\dot{\mathbf{p}} + \mathbf{t}_m : \nabla\mathbf{v} - \frac{\mathbf{h}\cdot\nabla\theta}{\theta} \geq 0, \tag{7.215}$$

where

$$\psi_m = e_m - \eta\theta. \tag{7.216}$$

Inequality (7.215) suggests that the form we should choose for the constitutive equations is

$$A = A(\mathbf{F}, \mathbf{p}, \theta, \nabla\theta), \tag{7.217}$$

where A denotes one of the fields $\psi_m, \eta, \mathbf{E}, \mathbf{h}$. Using the usual procedure, we derive the following restrictions on the constitutive equations from the dissipation principle:

$$\psi_m = \psi_m(\mathbf{F}, \mathbf{p}, \theta), \tag{7.218}$$

$$\eta = -\frac{\partial\psi_m}{\partial\theta} = \eta(\mathbf{F}, \mathbf{p}, \theta), \tag{7.219}$$

$$\mathbf{t}_m = \rho\frac{\partial\psi_m}{\partial\mathbf{F}}\mathbf{F}^T = \mathbf{t}_m(\mathbf{F}, \mathbf{p}, \theta), \tag{7.220}$$

$$\mathbf{E} = -\rho\frac{\partial\zeta}{\partial\mathbf{p}} = \mathbf{E}(\mathbf{F}, \mathbf{p}, \theta), \tag{7.221}$$

$$\mathbf{h}\cdot\nabla\theta \leq 0. \tag{7.222}$$

We have provided a physical interpretation of the equations (7.211)–(7.214) by replacing the dielectric with a continuous dipole distribution. This model can be regarded as a proper description of the real situation if the mechanical stress tensor $\mathbf{t}_m$ depended only on mechanical variables. However, (7.220) shows that this is not the case. This implies that, if we suppose that $\mathbf{t}_m = \mathbf{t}_1(\mathbf{F}, \theta) + \mathbf{t}_2(\mathbf{F}, \mathbf{p}, \theta)$, then $\nabla\cdot\mathbf{t}_2(\mathbf{F}, \mathbf{p}, \theta)$ can be regarded as a further force produced by the polarization field, which calls for a new and more complex physical interpretation of the interaction between matter and electric polarization.

7.10 More General Constitutive Equations

The theory presented in the above sections is not able to predict all of the phenomena observed in elastic dielectrics. In particular, the class of constitutive equations considered does not allow us to explain some experimental features, like the ferroelectricity exhibited by some dielectrics and the anomalous behavior of capacity for thin condensers (see [130, 131]).

For these reasons, we consider the following wider class of constitutive equations:

$$A = A(\mathbf{F}, \mathbf{p}, \nabla_{\mathbf{X}}\mathbf{p}, \theta, \nabla\theta), \tag{7.223}$$

where A still denotes one of the fields ψ_m, η, and $\mathbf{E}, \mathbf{h}$, and $\nabla_{\mathbf{X}}$ denotes the material gradient; i.e., the gradient with respect to the coordinates (X_L) in the reference configuration. In order to derive the restrictions on the constitutive equations (7.223) due to the dissipation principle, we assume that equations (7.211), (7.213) and (7.214) still hold, while we introduce into (7.212) an extra-flux of energy

$$\mathbf{s} = \tau \cdot \dot{\mathbf{p}}, \tag{7.224}$$

where $\tau = \tau(\mathbf{F}, \mathbf{p}, \nabla_{\mathbf{X}}\mathbf{p}, \theta, \nabla\theta)$. Under these hypotheses, dissipation inequality (7.215) becomes

$$-\rho(\dot{\psi}_m + \eta\dot{\theta}) + \rho\mathbf{E} \cdot \dot{\mathbf{p}} + \mathbf{t}_m : \nabla\mathbf{v} + \nabla \cdot (\tau \cdot \dot{\mathbf{p}}) - \frac{\mathbf{h} \cdot \nabla\theta}{\theta} \geq 0. \tag{7.225}$$

We now note that

$$\dot{\psi}_m = \frac{\partial\psi_m}{\partial F_{iL}}\dot{F}_{iL} + \frac{\partial\psi_m}{\partial p_i}\dot{p}_i + \frac{\partial\psi_m}{\partial p_{i,L}}\dot{p}_{i,L} + \frac{\partial\psi_m}{\partial\theta}\dot{\theta} + \frac{\partial\psi_m}{\partial\theta_{,L}}\dot{\theta}_{,L}. \tag{7.226}$$

Therefore, substituting (7.226) into (7.225) and applying the dissipation principle, which requires that the inequality obtained holds for any process, we find that

$$\psi_m = \psi_m(\mathbf{F}, \mathbf{p}, \nabla\mathbf{p}, \theta), \tag{7.227}$$

$$\eta = -\frac{\partial\psi_m}{\partial\theta}, \tag{7.228}$$

$$\mathbf{t}_{(m)ij} = \rho\frac{\partial\psi_{(m)}}{\partial F_{iL}}F_{jL}, \tag{7.229}$$

$$\tau_{ij} = \rho\frac{\partial\psi_{(m)}}{\partial p_{j,L}}F_{iL}, \tag{7.230}$$

$$E_j = \frac{\partial\psi_{(m)}}{\partial p_j} - \frac{1}{\rho}\left(\rho\frac{\partial\psi_{(m)}}{\partial p_{j,L}}F_{iL}\right)_{,i}, \tag{7.231}$$

$$\mathbf{h} \cdot \nabla\theta \leq 0. \tag{7.232}$$

It is worth noting that, in the absence of the extra-flux $\mathbf{s}$, the internal free energy ψ_m cannot depend on the polarization gradient and so we obtain the theory presented in the above section.

We conclude by recalling that the results presented in this section can be applied to describe the behavior of a thin dielectric material placed in a plane parallel capacitor (see [131]).

7.11 Lagrangian Formulation of Quasi-electrostatics

It is well known that the Lagrangian formulation (see Sect. 5.7 of [16]) is needed to analyze the behavior of an elastic material. Consequently, we must write all of the equations governing the evolution of an elastic dielectric in the Lagrangian form. In the quasi-static approximation, the Maxwell equations (7.1)–(7.4) for a dielectric become

$$\int_{\partial s(t)} \mathbf{E} \cdot \tau dl = 0, \tag{7.233}$$

$$\frac{d}{dt}\int_{s(t)} \mathbf{D} \cdot \mathbf{N} ds = \int_{\partial s(t)} (\mathbf{H} - \mathbf{v} \times \mathbf{D}) \cdot \tau dl, \tag{7.234}$$

$$\int_{\partial c(t)} \mathbf{D} \cdot \mathbf{N} ds = 0, \tag{7.235}$$

where $s(t)$ and $c(t)$ are an arbitrary material surface and volume, respectively.

Moreover, the balance equations for momentum and energy in the integral form are

$$\frac{d}{dt}\int_{c(t)} \rho \mathbf{v}\, dc = \int_{\partial c(t)} \mathbf{T} \cdot \mathbf{N}\, ds + \int_{c(t)} \rho \mathbf{b}\, dc, \tag{7.236}$$

$$\frac{d}{dt}\int_{c(t)} \rho \left(\frac{1}{2}v^2 + \epsilon\right) dc =$$

$$\int_{\partial c(t)} (\mathbf{T} \cdot \mathbf{v} - \mathbf{E}_0 \times \mathbf{H}_0 - \mathbf{h}) \cdot \mathbf{N}\, ds + \int_{c(t)} \rho \mathbf{b} \cdot \mathbf{v}\, dc. \tag{7.237}$$

Here $\mathbf{E}_0$ and $\mathbf{H}_0$ denote the electric and magnetic fields in the proper frame, respectively (see Eqs. 7.133 and 7.135).

Finally, the entropy inequality takes the following form:

$$\frac{d}{dt}\int_{c(t)} \rho \eta\, dc = \int_{\partial c(t)} \frac{\mathbf{h}}{\theta} \cdot \mathbf{N}\, ds. \tag{7.238}$$

In order to find the Lagrangian formulation for the above equations, we must transform them into integral relations over volumes and surfaces present in the reference configuration C_*. Recalling formulae (3.5) and (3.19) of [16] (see also Sect. 1.1), we can write (7.233)–(7.235) in the form

$$\int_{\partial s_*} \mathcal{E} \cdot \tau_* \, dl_* = 0, \tag{7.239}$$

$$\frac{d}{dt} \int_{s_*} \mathcal{D} \cdot \mathbf{N}_* \, ds = \int_{\partial s_*} \mathcal{H} \cdot \tau_* \, dl_*, \tag{7.240}$$

$$\int_{\partial c_*} \mathcal{D} \cdot \mathbf{N}_* \, ds_* = 0, \tag{7.241}$$

where we have introduced the *Lagrangian electromagnetic fields*

$$\mathcal{E}_L = F_{iL} E_i, \tag{7.242}$$

$$\mathcal{H}_L = F_{iL} H_{0,i}, \tag{7.243}$$

$$\mathcal{D}_L = J(F^{-1})_{Li} D_i, \tag{7.244}$$

and we have denoted the magnetic field in the proper frame (see Eq. 7.135) by $\mathbf{H}_0 = \mathbf{H} - \mathbf{v} \times \mathbf{D}$.

We can now write (7.239)–(7.241) in the following local form:

$$\nabla_{\mathbf{X}} \mathcal{E} = \mathbf{0}, \tag{7.245}$$

$$\nabla_{\mathbf{X}} \mathcal{H} = \dot{\mathcal{D}}, \tag{7.246}$$

$$\nabla_{\mathbf{X}} \cdot \mathcal{D} = 0. \tag{7.247}$$

In order to obtain the Lagrangian form of the energy balance, we prove that

$$\int_{\partial c(t)} \mathbf{E}_0 \times \mathbf{H}_0 \cdot \mathbf{N} \, ds = \int_{\partial c_*} \mathcal{E} \times \mathcal{H} \cdot \mathbf{N}_* \, ds_*, \tag{7.248}$$

where $\mathbf{N}_*$ is the outward unit vector normal to ∂C_*. From (7.245), (7.246) and the formula proved in Sect. 3.3 of [16],

$$N_i d\sigma = J(F^{-1})_{Ki} N_{*k} d\sigma_{*k},$$

we find that the right-hand side of (7.248) can also be written as

$$\int_{\partial c_\star} \epsilon_{ijl} (F^{-1})_{Lj} (F^{-1})_{Ml} \mathcal{E}_L \mathcal{H}_M J(F^{-1})_{Ki} N_{*K} \, d\sigma_{*K}.$$

Therefore, we can prove (7.248) if we consider the definition of the determinant of the matrix $\mathbf{A}$:

$$\det \mathbf{A} \epsilon_{KLM} = \epsilon_{ijh} A_{iK} A_{jL} A_{hM}.$$

Taking into account this result, and applying the procedure described in Sect. 5.7 of [16], we obtain the local forms for the balance equations of momentum and energy:

$$\rho_* \dot{\mathbf{v}} = \nabla_{\mathbf{X}} \cdot \mathbf{T}_* + \rho_* \mathbf{b}, \tag{7.249}$$

$$\rho_* \dot{e} = \mathbf{T}_* \cdot \dot{\mathbf{F}}^T - \mathcal{D} \cdot \dot{\mathcal{E}} - \nabla_{\mathbf{X}} \cdot \mathbf{h}_*, \tag{7.250}$$

as well as the entropy inequality

$$\rho_* \theta \dot{\eta} \geq -\nabla_{\mathbf{X}} \cdot \mathbf{h}_* + \frac{\mathbf{h}_* \cdot \nabla_{\mathbf{X}} \theta}{\theta}, \tag{7.251}$$

where $\rho_* = J\rho$ is the mass density in the reference configuration, and

$$e = \epsilon - \frac{\mathcal{D} \cdot \mathcal{E}}{\rho_*}. \tag{7.252}$$

Eliminating $\nabla_{\mathbf{X}} \cdot \mathbf{h}_*$ between (7.250) and (7.251) leads to the Lagrangian form of the reduced dissipation inequality:

$$-\rho_* (\dot{\zeta} + \eta \dot{\theta}) + \mathbf{T}_* \cdot \dot{\mathbf{F}}^T - \mathcal{D} \cdot \dot{\mathcal{E}} - \frac{\mathbf{h}_* \cdot \nabla_{\mathbf{X}} \theta}{\theta} \geq 0, \tag{7.253}$$

where

$$\zeta = e - \theta \eta, \tag{7.254}$$

is the specific enthalpy.

In order to make the number of equations the same as the number of unknowns, we have to assign the set of fields $\mathbf{A} = (\mathbf{T}_*, \mathcal{D}, \mathbf{h}_*, \zeta)$ in terms of the displacement field $\mathbf{u}(\mathbf{X}, t)$ and the temperature field $\theta(\mathbf{X}, t)$, $(\mathbf{X}, t) \in C_* \times \Re$. For an elastic dielectric we have

$$\mathbf{A} = \mathbf{A}(\mathbf{S}, \mathcal{E}, \theta, \nabla \theta), \tag{7.255}$$

where $S_{iL} = u_{i,L}$. Substituting (7.255) into (7.253) and applying the dissipation principle, the following restrictions on the constitutive equations can be obtained:

$$\zeta = \zeta(\mathbf{S}, \mathcal{E}, \theta), \tag{7.256}$$

$$\mathbf{T}_* = \rho_* \frac{\partial \zeta}{\partial \mathbf{S}}, \tag{7.257}$$

$$\mathcal{D} = -\rho_* \frac{\partial \zeta}{\partial \mathcal{E}}, \tag{7.258}$$

$$\eta = -\frac{\partial \zeta}{\partial \theta}, \tag{7.259}$$

$$\mathbf{h}_* \cdot \nabla_{\mathbf{X}} \theta \leq 0. \tag{7.260}$$

7.12 Variational Formulation for Equilibrium in Quasi-electrostatics

In this section, extending the results obtained in Sect. 1.4, we give the variational formulation for the equilibrium problem of an elastic dielectric in the absence of thermal phenomena. This means that the solutions of the equilibrium boundary value problem are extremals of a suitable functional (see also Appendix A).

In this hypothesis, (7.245), (7.247) and (7.249) become

$$\nabla_{\mathbf{X}} \times \mathcal{E} = \mathbf{0}, \tag{7.261}$$

$$\nabla_{\mathbf{X}} \cdot \mathcal{D} = 0, \tag{7.262}$$

$$\nabla_{\mathbf{X}} \cdot \mathbf{T}_* + \rho_* \mathbf{b} = \mathbf{0}. \tag{7.263}$$

Since (7.261) implies the existence of an electric potential $\varphi(\mathbf{X})$ such that

$$\mathcal{E} = -\nabla_{\mathbf{X}} \varphi, \tag{7.264}$$

the constitutive equation (7.256) assumes the form

$$\zeta = \zeta(u_{i,L}, \varphi,_L), \tag{7.265}$$

whereas the equilibrium equations (7.262) and (7.263), taking into account (7.257), (7.258), become

$$\left(\rho_* \frac{\partial \zeta}{\partial u_{i,L}}\right)_{,L} + \rho_* b_i = 0, \tag{7.266}$$

$$\left(\rho_* \frac{\partial \zeta}{\partial \varphi,_L}\right)_{,L} = 0. \tag{7.267}$$

In order to avoid nonessential problems with convergence at infinity, we assume that the dielectric is in contact with a rigid conductor across the whole boundary ∂C_* of the reference configuration of the material. In this case, the mechanical boundary data are

$$\rho_* \frac{\partial \zeta}{\partial u_{i,L}} N_{*L} = t_{*i}, \quad \partial C'_* \subset \partial C_*, \tag{7.268}$$

$$u_i = 0, \quad \partial C_* - \partial C'_*, \tag{7.269}$$

while the electric boundary conditions can be written as

$$\int_{\partial C''_*} \rho_* \frac{\partial \zeta}{\partial \varphi,_L} N_{*L}\, ds_* = Q_0, \quad \partial C''_* \subset \partial C_*, \tag{7.270}$$

$$\varphi = 0, \quad \partial C_* - \partial C''_*. \tag{7.271}$$

In other words, we assign the value of the potential φ on $\partial C_* - \partial C''_*$ and the total charge Q_0 on the other conductors.

We now denote by $W_0^1(C_*)$ the Sobolev space (see Appendix A) whose elements are the functions $k = (v_i(\mathbf{X}), \lambda(\mathbf{X})) : C_* \to \Re^4$ that belong to $L^2(C_*)$ together with their generalized derivatives, and verify the conditions

$$v_i = 0 \text{ on } \partial C_* - \partial C'_*, \quad \lambda = 0 \text{ on } \partial C_* - \partial C''_*. \tag{7.272}$$

Moreover, we recall that W_0^1 is complete with respect to the ordinary Sobolev norm

$$||k||_W^1 0(C_*) = \sum_{\alpha=1}^{4} \left(\int_{C_*} (k_\alpha)^2 \, dc_* + \sum_{L=1}^{3} \int_{C_*} (k_{\alpha,L})^2 \, dc_* \right). \tag{7.273}$$

Finally, let $\mathbb{K} = (u_i, \varphi) \in W_0^1$ be an equilibrium solution of the boundary value problem (7.266)–(7.271).

If we multiply (7.266) by v_i and (7.267) by λ and add the relations obtained, we get the following scalar relation after integrating over C_*:

$$\int_{C_*} \left[\left(\rho_* \frac{\partial \zeta}{\partial k_{\alpha,L}} \right)_{,L} k_\alpha + \rho_* b_i v_i \right] dc_*. \tag{7.274}$$

By applying Gauss' theorem, the above equation assumes the equivalent form

$$\begin{aligned} -\int_{C_*} \rho_* \frac{\partial \zeta}{\partial k_{\alpha,L}} k_{\alpha,L} \, dc_* &+ \int_{C_*} \rho_* b_i v_i \, dc_* \\ &+ \int_{\partial C_*} \rho_* \frac{\partial \zeta}{\partial k_{\alpha,L}} k_\alpha N_{*L} \, ds_*. \end{aligned} \tag{7.275}$$

Taking into account the boundary conditions (7.270) and (7.271), we obtain the weak formulation for the equilibrium problem (7.266)–(7.269):

$$\begin{aligned} \int_{C_*} \rho_* \frac{\partial \zeta}{\partial k_{\alpha,L}} k_{\alpha,L} \, dc_* = &\int_{C_*} \rho_* b_i v_i \, dc_* \\ &+ \int_{\partial C_*} t_{*i} v_i \, ds_* + Q_0 \lambda, \end{aligned} \tag{7.276}$$

where λ is the constant value of k_4 on $\partial C''_*$.

It is a simple exercise to verify that (7.276) coincides with Frechèt's differential of the functional

$$\begin{aligned} \mathcal{F}[k_\alpha] = &\int_{C_*} \rho_* \zeta(u_{i,L}, \varphi_L) \, dc_* \\ &- \int_{C_*} \rho_* b_i u_i \, dc_* - \int_{\partial C_*} t_{*i} u_i \, ds_* - Q_0 \varphi_{\partial C''_*}, \end{aligned} \tag{7.277}$$

where $\varphi_{\partial C''_*}$ is the unknown constant value of the potential on those conductors to which the total charge has been assigned.

If the potential on the conductors has a nonzero value φ_0, it will be sufficient to search for the extremals of the functional (7.277) in the affine space $k_1 + W_0^1(C_*)$, where $k_1 = (0, \lambda)$ and λ is a square-summable function with first derivatives such that

$$\lambda = \begin{cases} \varphi_0 \text{ on } \partial C_* - \partial C''_* \\ \text{undetermined constant value on } \partial C''_*. \end{cases} \tag{7.278}$$

In the above variational formulation we used the displacement $\mathbf{u}$ and the electric field $\mathbf{E}$. However, we could choose the pair $(\mathbf{u}, \mathcal{D})$ as independent variables. To do this, it is sufficient to recall (7.252) and write the functional (7.277) in the equivalent form

$$\begin{aligned} \Psi[\mathbf{u}, \mathcal{D}] = &\int_{C_*} \rho_* e(\nabla \mathbf{u}, \mathcal{D})\, dc_* - \int_{C_*} \mathcal{E}_L \mathcal{D}_L\, dc_* \\ &- \int_{C_*} \rho_* b_i u_i\, dc_* - \int_{\partial C_*} t_{*i} u_i\, ds_* - Q_0 \varphi_{\partial C''_*}. \end{aligned} \tag{7.279}$$

Taking into account (7.252) and (7.262), we have

$$\begin{aligned} -\int_{C_*} \mathcal{E}_L \mathcal{D}_L\, dc_* &= \int_{C_*} \varphi_{,L}\, \mathcal{D}_L\, dc_* \\ &= \int_{C_*} (\varphi \mathcal{D}_L)_{,L}\; dc_* - \int_{C_*} \varphi \mathcal{D}_{L,L}\, dc_* \\ &= \int_{\partial C_*} \varphi \mathcal{D}_L n_{*L}\, ds_* = \varphi_{\partial C''_*} Q_0. \end{aligned}$$

Consequently, (7.279) becomes

$$\Psi[\mathbf{u}, \mathcal{D}] = \int_{C_*} \rho_* e(\nabla \mathbf{u}, \mathcal{D})\, dc_* - \int_{C_*} \rho_* b_i u_i\, dc_* - \int_{\partial C'_*} t_{*i} u_i\, ds_*. \tag{7.280}$$

In order to verify that the extremals of the functional (7.280) are solutions of the right equilibrium equations and boundary conditions, we recall that the field $\mathcal{D}$ must verify the local condition

$$\mathcal{D}_{L,L} = 0, \tag{7.281}$$

as well as the global condition

$$\int_{\partial C''_*} \mathcal{D}_L n_{*L}\, ds_* = Q_0. \tag{7.282}$$

In other words, we have to find the conditioned extremals of the functional (7.280); i.e., the extremals of the functional

$$\Psi[\mathbf{u}, \mathcal{D}] = \int_{C_*} \rho_* e(\nabla \mathbf{u}, \mathcal{D})\, dc_* - \int_{C_*} \rho_* b_i u_i \, dc_* - \int_{\partial C'_*} t_{*i} u_i \, ds_* \\ + \int_{C_*} \lambda \mathcal{D}_{L,L}\, dc_* + \mu \left(\int_{\partial C''_*} \mathcal{D}_L n_{*L}\, ds_* - Q_0 \right), \tag{7.283}$$

where λ is a Lagrangian multiplier that depends on $\mathbf{X} \in C_*$ and μ is a constant Lagrangian multiplier.

It is again a simple exercise to verify that the extremals of (7.283) satisfy the conditions

$$\left(\rho_* \frac{\partial \zeta}{\partial u_{i,L}} \right)_{,L} + \rho_* b_i = 0, \tag{7.284}$$

$$\lambda_{,L} = \rho_* \frac{\partial e}{\partial \mathcal{D}_L}, \tag{7.285}$$

$$\lambda_{\partial C''_*} = \mu. \tag{7.286}$$

Identifying the Lagrangian multiplier with the electric potential φ and consequently λ_L with the Lagrangian electric field $\mathcal{E}_L$, we obtain the right equilibrium and boundary conditions. After this identification, we note that (7.285) implies the equation $\nabla_{\mathbf{X}} \mathcal{E} = 0$.

Chapter 8

Introduction to Magnetofluid Dynamics

8.1 An Evolution Equation for the Magnetic Field

In this section we consider the equations of quasi-magnetostatics for a fluid S that is a *perfect* conductor.

In Sect. 7.6 we saw that in quasi-magnetostatics the Maxwell equations for the fundamental fields $\mathbf{B}$, $\mathbf{H}$ and $\mathbf{j}$ assume the forms

$$\nabla \times \mathbf{H} = \mathbf{j}, \tag{8.1}$$
$$\nabla \cdot \mathbf{B} = 0, \tag{8.2}$$
$$\nabla \cdot \mathbf{j} = 0, \tag{8.3}$$

where

$$\mathbf{B} = \mu \mathbf{H}, \tag{8.4}$$
$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \sigma \mathbf{E}^{\mathbf{0}}. \tag{8.5}$$

Moreover, the electric field $\mathbf{E}$ satisfies the equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{8.6}$$

Finally, the fields $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{j}$ verify the following transformation rules under the Galilean group:

$$\mathbf{B}' = \mathbf{B}, \tag{8.7}$$
$$\mathbf{H}' = \mathbf{H}, \tag{8.8}$$
$$\mathbf{j}' = \mathbf{j}. \tag{8.9}$$

We note that, when (8.4) and (8.5) are taken into account, (8.1) becomes

$$\nabla \times \mathbf{B} = \mu\sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

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Due to (8.6), the above equation can also be written as follows:

$$\nabla \times \nabla \times \mathbf{B} = -\mu\sigma \frac{\partial \mathbf{B}}{\partial t} + \mu\sigma \nabla \times (\mathbf{v} \times \mathbf{B}). \tag{8.10}$$

From a known vector identity and (8.2) we obtain

$$\nabla \times \nabla \times \mathbf{B} = \nabla(\nabla \cdot \mathbf{B}) - \triangle \mathbf{B} = -\triangle \mathbf{B}.$$

Consequently, from (8.10), we can derive the second-order partial differential equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{\mu\sigma} \triangle \mathbf{B} \tag{8.11}$$

in the unknown $\mathbf{B}$ when the velocity field $\mathbf{v}$ is given. In Sect. 8.3 we associate the mechanical balance equations needed to obtain a closed system with (8.11).

We conclude this section with a nondimensional analysis of (8.11). Introducing the reference quantities L and T, which were defined in Sect. 7.3, and denoting the nondimensional quantities with the same symbols, (8.11) assumes the form

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{R_m} \triangle \mathbf{B}, \tag{8.12}$$

where

$$R_m = \frac{\mu\sigma L^2}{T} \tag{8.13}$$

is the *magnetic Reynolds number*.

It is interesting to evaluate the contributions of the different terms in (8.13). To do this, we note that, in a good conductor with $\sigma = 10^8$ ohm^{-1}/m and $\mu = 10^{-6}$ henry/m,

$$R_m \simeq \frac{10^2 L^2}{T}.$$

Therefore, as long as the quantities L and T (which are strongly related to the nature of the problem considered) assume values such that $R_m \ll 1$, then (8.12) reduces to the equation

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\mu\sigma} \triangle \mathbf{B}. \tag{8.14}$$

In the opposite case (i.e., when $R_m \gg 1$), (8.12) becomes

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}). \tag{8.15}$$

Equation 8.14 describes the diffusion of the magnetic field inside a conductor, and this process is accompanied by a decay in $\mathbf{B}$. Dimensional analysis shows that the decay time is on the order of $L^2\mu\sigma$, where L is a reference length comparable with the size of the conductor. For ordinary conductors the decay time is short, but for celestial bodies, because of their much larger sizes, the decay time is very large. For instance, it has been shown that for the magnetic field of a sun spot, the time of decay is at least 300 years. In this chapter we will analyze some consequences of (8.14) and (8.15).

8.2 Balance Equations in Magnetofluid Dynamics

Equation 8.12, when equipped with suitable boundary and initial conditions, allows us to evaluate the magnetic field provided that the velocity field is known. To obtain a closed system of field equations, we need to introduce the dynamical balance equations for a moving fluid conductor. As we saw in the preceding chapter, coupling Maxwell's equations with the classical balance equations does not lead to contradictions for low velocities and low frequencies (the quasi-static approximation).

Also, in this new situation, we prefer not to start with a particular model for the interaction between the conductor S and the magnetic field. In other words, again we postulate that both mass conservation and momentum balance hold in the standard form

$$\dot{\rho} + \rho\nabla\cdot\mathbf{v} = 0, \tag{8.16}$$

$$\rho\dot{\mathbf{v}} - \nabla\cdot\mathbf{T} + \rho\mathbf{b} = \mathbf{0}, \tag{8.17}$$

where ρ is the mass density, $\mathbf{T}$ is the *total symmetric* stress tensor, and $\mathbf{b}$ is the specific *nonmagnetic* external body force acting on S.

The set of equations (8.12), (8.16) and (8.17) form a closed system in the unknown fields ρ, $\mathbf{v}$ and $\mathbf{B}$, provided that the constitutive equation for $\mathbf{T}$ is given as a function of these fields. However, this model is not satisfactory, since the electric current in the conductor S produces heat inside S (the Joule effect). Therefore, we must consider the temperature as a new unknown field. In order to determine this field we introduce the energy balance in the form

$$\frac{d}{dt}\int_{c(t)} \rho\left(\frac{1}{2}v^2 + \epsilon\right) dc = \int_{\partial c(t)} \left(\mathbf{v}\cdot\mathbf{T} - \mathbf{h} - \mathbf{E}^0 \times \mathbf{H}^0\right)\cdot\mathbf{N}\, d\sigma + \int_{c(t)} \rho(\mathbf{b}\cdot\mathbf{v} + r)\, dc. \tag{8.18}$$

In (8.18), ϵ is the specific internal energy, $\mathbf{h}$ is the heat flux vector, $\mathbf{E}^0 \times \mathbf{H}^0$ is the Poynting vector, evaluated in the proper frame of a generic particle of S, and r is the external supply of energy.

As is usual in continuum mechanics, the intention is for (8.18) to be valid for any material volume $c(t)$. Therefore, if the fields under the integrals are regular, then (8.18) is equivalent to the following local energy balance:

$$\rho\dot{\epsilon} = \mathbf{T} : \nabla\mathbf{v} - \nabla \cdot \mathbf{h} - \nabla \cdot (\mathbf{E}^0 \times \mathbf{H}^0) + \rho r. \tag{8.19}$$

8.3 Equivalent Form of the Balance Equations

In this section, we will place the balance equations stated in the above section into a new form, since this formulation enables an interesting physical interpretation. This result is obtained by manipulating the divergence of Poynting's vector.

From (8.5) and (8.7), we have

$$\begin{aligned}\mathbf{E}^0 \times \mathbf{H}^0 &= (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \times \frac{\mathbf{B}}{\mu} \\ &= \mathbf{E} \times \frac{\mathbf{B}}{\mu} + (\mathbf{B} \cdot \mathbf{v})\frac{\mathbf{B}}{\mu} - \frac{B^2}{\mu}\mathbf{v},\end{aligned}$$

so that

$$\begin{aligned}\nabla \cdot (\mathbf{E}^0 \times \mathbf{H}^0) &= \nabla \cdot (\mathbf{E} \times \frac{\mathbf{B}}{\mu}) + \nabla \cdot \left[(\mathbf{B} \cdot v)\frac{\mathbf{B}}{\mu} - \frac{B^2}{\mu}\mathbf{v}\right] \\ &= \frac{\mathbf{B}}{\mu} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \frac{\mathbf{B}}{\mu} + \frac{\mathbf{B}}{\mu} \cdot \nabla(\mathbf{B} \cdot \mathbf{v}) \\ &\quad + \mathbf{B} \cdot \mathbf{v}\nabla \cdot \mathbf{B} - \frac{B^2}{\mu}\nabla \cdot \mathbf{v} - \mathbf{v} \cdot \nabla\left(\frac{B^2}{\mu}\right).\end{aligned}$$

Taking into account (8.6) and (8.1), the above relation becomes

$$\begin{aligned}\nabla \cdot (\mathbf{E}^0 \times \mathbf{H}^0) &= -\frac{\partial \mathbf{B}}{\partial t} \cdot \frac{\mathbf{B}}{\mu} - \mathbf{E} \cdot \mathbf{J} + \frac{\mathbf{B}}{\mu} \cdot (\mathbf{B} \times \nabla \times \mathbf{v}) \\ &\quad + \mathbf{B} \cdot \mathbf{v} \times \mathbf{J} + \mathbf{v} \cdot \nabla\mathbf{B} \cdot \frac{\mathbf{B}}{\mu} + \mathbf{B} \cdot \nabla\mathbf{v} \cdot \frac{\mathbf{B}}{\mu} \\ &\quad - \frac{B^2}{\mu}\nabla \cdot \mathbf{v} - \mathbf{v} \cdot \nabla\frac{B^2}{\mu}.\end{aligned}$$

The third term on the right-hand side of the above relation vanishes since the two vectors are parallel. Moreover, $\mathbf{B} \cdot (\mathbf{v} \times \mathbf{J}) = -\mathbf{J} \cdot (\mathbf{v} \times \mathbf{B})$, so, taking into account (8.5), we finally get

$$\nabla \cdot (\mathbf{E}^0 \times \mathbf{H}^0) = -\frac{\partial \mathbf{B}}{\partial t} \cdot \frac{\mathbf{B}}{\mu} - \mathbf{E}^0 \cdot \mathbf{J} + \mathbf{v} \cdot \nabla \mathbf{B} \cdot \frac{\mathbf{B}}{\mu} + \mathbf{B} \cdot \nabla \mathbf{v} \cdot \frac{\mathbf{B}}{\mu} - \frac{B^2}{\mu} \nabla \cdot \mathbf{v} - \mathbf{v} \cdot \nabla \frac{B^2}{\mu}. \tag{8.20}$$

On the other hand, we also have

$$\begin{aligned}\frac{\mathbf{B}}{\mu} \cdot \frac{\partial \mathbf{B}}{\partial t} &= \frac{\mathbf{B}}{\mu} \cdot \dot{\mathbf{B}} - \mathbf{v} \cdot \nabla \mathbf{B} \cdot \frac{\mathbf{B}}{\mu} \\ &= \frac{1}{2} \frac{d}{dt} \left(\frac{B^2}{\mu} \right) - \mathbf{v} \cdot \nabla \mathbf{B} \cdot \frac{\mathbf{B}}{\mu} \\ &= \frac{\rho}{2\rho} \frac{d}{dt} \left(\frac{B^2}{\mu} \right) - \mathbf{v} \cdot \nabla \mathbf{B} \cdot \frac{\mathbf{B}}{\mu} \\ &= \frac{\rho}{2} \frac{d}{dt} \frac{B^2}{\mu\rho} - \frac{\rho}{2} \frac{B^2}{\mu} \frac{d}{dt} \left(\frac{1}{\rho} \right) - \mathbf{v} \cdot \nabla \mathbf{B} \cdot \frac{\mathbf{B}}{\mu},\end{aligned}$$

and taking into account the equation of mass conservation, we have the relation

$$-\frac{\mathbf{B}}{\mu} \cdot \frac{\partial \mathbf{B}}{\partial t} = -\frac{1}{2} \rho \frac{d}{dt} \left(\frac{B^2}{\mu\rho} \right) + \frac{1}{2} \frac{B^2}{\mu} \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{B} \cdot \frac{\mathbf{B}}{\mu}. \tag{8.21}$$

Inserting (8.21) into (8.20), we obtain the following expression of the divergence of Poynting's vector

$$\nabla \cdot (\mathbf{E}^0 \times \mathbf{H}^0) = -\rho \frac{d}{dt} \left(\frac{B^2}{2\mu\rho} \right) + \frac{1}{2} \frac{B^2}{\mu} \nabla \cdot \mathbf{v} + 2\mathbf{v} \cdot \nabla \mathbf{B} \cdot \frac{\mathbf{B}}{\mu} - \mathbf{E}^0 \cdot \mathbf{J} + \mathbf{B} \cdot \nabla \mathbf{v} \cdot \frac{\mathbf{B}}{\mu} - \frac{B^2}{\mu} \nabla \cdot \mathbf{v} - \mathbf{v} \cdot \nabla \frac{B^2}{\mu}. \tag{8.22}$$

Since

$$2\mathbf{v} \cdot \nabla \mathbf{B} \cdot \frac{\mathbf{B}}{\mu} - \mathbf{v} \cdot \nabla \frac{B^2}{\mu} = \frac{2}{\mu} v_i \frac{\partial H_j}{\partial x_i} H_j - \frac{v_i}{\mu} \frac{\partial}{\partial x_i} H_i H_j = 0, \tag{8.23}$$

we can convert (8.22) into the following significant form:

$$-\nabla \cdot (\mathbf{E}^0 \times \mathbf{H}^0) = \rho \frac{d}{dt} \left(\frac{B^2}{2\mu\rho} \right) + \frac{1}{2} \frac{B^2}{\mu} \nabla \cdot \mathbf{v} - (\mathbf{B} \otimes \frac{\mathbf{B}}{\mu}) : \nabla \mathbf{v} + \mathbf{E}^0 \cdot \mathbf{J}. \tag{8.24}$$

If we introduce this expression into the local energy balance (8.19) and recall (8.5), then we obtain

$$\rho \dot{e} = \mathbf{t} : \nabla \mathbf{v} - \nabla \cdot \mathbf{h} + \frac{J^2}{\sigma} + \rho r, \tag{8.25}$$

where

$$e = \epsilon - \frac{1}{2\mu\rho}B^2 \equiv \epsilon - e_{(m)}, \tag{8.26}$$

$$\mathbf{t} = \mathbf{T} - \left[-\frac{1}{2\mu}B^2\mathbf{I} + \frac{1}{\mu}\mathbf{B}\otimes\mathbf{B}\right] \equiv \mathbf{T} - \mathbf{t}_{(m)}. \tag{8.27}$$

Since $e_{(m)}$ and $\mathbf{t}_{(m)}$ coincide with the magnetic energy and the Maxwell magnetic stress tensor of a pure magnetic field, (8.25) suggests that e and $\mathbf{t}$ should be regarded as the *mechanical* energy and the *mechanical* stress tensor, respectively. In this way, the total specific energy is obtained by adding the mechanical energy and the magnetic energy, and the total stress tensor is the sum of the mechanical and magnetic stress tensors. It is evident that such an interpretation is reasonable if and only if e and $\mathbf{t}$ depend only on mechanical variables. In the next section we prove, via the dissipation principle, that this statement is not always valid.

We now determine the form assumed by the momentum balance (8.17). Taking into account the decomposition (8.27), Eq. 8.17 becomes

$$\rho\dot{\mathbf{v}} = \nabla\cdot\mathbf{t} + \nabla\cdot\left[-\frac{1}{2\mu}B^2\mathbf{I} + \frac{1}{\mu}\mathbf{B}\otimes\mathbf{B}\right] + \rho\mathbf{b}. \tag{8.28}$$

On the other hand,

$$\begin{aligned}
&-\frac{1}{2\mu}\frac{\partial}{\partial x_i}(B_jB_j) + \frac{1}{\mu}\frac{\partial}{\partial x_j}(B_iB_j)\\
&= -\frac{1}{\mu}B_j\frac{\partial B_j}{\partial x_i} + \frac{1}{\mu}B_j\frac{\partial B_i}{\partial x_j} + \frac{1}{\mu}B_i\frac{\partial B_j}{\partial x_j}\\
&= B_j\left(\frac{\partial H_i}{\partial x_j} - \frac{\partial H_j}{\partial x_i}\right) = (\nabla\times\mathbf{H})\times\mathbf{B}
\end{aligned} \tag{8.29}$$

since $\nabla\cdot\mathbf{B} = 0$.

Finally, the momentum balance assumes the form

$$\rho\dot{\mathbf{v}} = \nabla\cdot\mathbf{t} + \mathbf{J}\times\mathbf{B} + \rho\mathbf{b}. \tag{8.30}$$

This equation can be interpreted by stating that the motion of the fluid conductor is determined by both the external force $\rho\mathbf{b}$ and the Lorentz force $\mathbf{J}\times\mathbf{B}$. For this interpretation, the remarks relating to (8.25) also hold.

Collecting together (8.15), (8.25) and (8.30), we can say that the motion of a liquid that is a good conductor is governed by the following equations:

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}), \tag{8.31}$$

$$\rho\dot{\mathbf{v}} = \nabla\cdot\mathbf{t} + \frac{1}{\mu}\nabla\times\mathbf{B}\times\mathbf{B} + \rho\mathbf{b}, \tag{8.32}$$

$$\rho\dot{e} = \mathbf{t} : \nabla\mathbf{v} - \nabla\cdot\mathbf{h} + \frac{1}{\mu^2\sigma}(\nabla\times\mathbf{B})^2 + \rho r, \tag{8.33}$$

where the unknowns are the fundamental fields $\mathbf{v}$, $\mathbf{B}$ and θ, provided that the constitutive equations of e, $\mathbf{t}$ and $\mathbf{h}$ are given. It is evident that we must also consider mass conservation since ρ is an unknown when the conductor is a fluid.

8.4 Constitutive Equations

We will omit the standard proof that the second law of thermodynamics, together with the energy balance (8.33), leads to the following dissipation inequality:

$$-\rho(\dot{\psi} + \eta\dot{\theta}) + \mathbf{t} : \nabla\mathbf{v} + \frac{1}{\mu^2\sigma}(\nabla \times \mathbf{B})^2 - \frac{\mathbf{h} \cdot \nabla\theta}{\theta} \geq 0, \tag{8.34}$$

where ψ is the specific free energy and η is the specific entropy.

It is well known that the dissipation principle requires that the constitutive equations for ψ, η, $\mathbf{t}$ and $\mathbf{h}$ must satisfy the inequality (8.34) in any thermodynamical process $\mathbf{v}(\mathbf{x}, t)$, $\mathbf{B}(\mathbf{x}, t)$, $\theta(\mathbf{x}, t)$. Denoting one of the fields ψ, η, $\mathbf{t}$ and $\mathbf{h}$ by A, we assume that the constitutive equations for these variables take the form

$$A = A(\rho, \nabla\mathbf{v}, \mathbf{B}, \theta, \nabla\theta). \tag{8.35}$$

Before applying the dissipation principle to these constitutive equations, we note from (8.31) that

$$\frac{\partial \mathbf{B}}{\partial t} = \mathbf{B} \cdot \nabla\mathbf{v} - \mathbf{v} \cdot \nabla\mathbf{B} - \mathbf{B}\nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \cdot \mathbf{B}.$$

Since $\nabla \cdot \mathbf{B} = 0$, the above relation can also be written as

$$\dot{\mathbf{B}} = \mathbf{B} \cdot \nabla\mathbf{v} - \mathbf{B}\nabla \cdot \mathbf{v}. \tag{8.36}$$

Moreover, we introduce the notation

$$\mathbf{t}_d = \mathbf{t} - \mathbf{t}(\rho, \mathbf{0}, \mathbf{B}, \mathbf{0}) \equiv \mathbf{t} - \mathbf{t}_e, \tag{8.37}$$

which defines the dynamic stress $\mathbf{t}_d$ and the stress at equilibrium $\mathbf{t}_e$.

Taking into account (8.36) and mass conservation, we have

$$\begin{aligned}\dot{\psi} = -\rho\frac{\partial\psi}{\partial\rho}\mathbf{I} : \nabla\mathbf{v} + \frac{\partial\psi}{\partial\theta}\dot{\theta} + \left(\frac{\partial\psi}{\partial\mathbf{B}} \otimes \mathbf{B} - \frac{\partial\psi}{\partial\mathbf{B}} \cdot \mathbf{BI}\right) : \nabla\mathbf{v} \\ + \frac{\partial\psi}{\partial\nabla\mathbf{v}}\dot{\overline{\nabla\mathbf{v}}} + \frac{\partial\psi}{\partial\nabla\theta}\dot{\overline{\nabla\theta}}.\end{aligned} \tag{8.38}$$

If this expression is inserted into (8.34), we obtain the following inequality:

$$\begin{aligned}
&-\rho\left(\eta+\frac{\partial\psi}{\partial\theta}\right)\dot{\theta}+\left(\mathbf{t}_e-\rho^2\frac{\partial\psi}{\partial\rho}\mathbf{I}-\frac{\partial\psi}{\partial\mathbf{B}}\otimes\mathbf{B}+\frac{\partial\psi}{\partial\mathbf{B}}\cdot\mathbf{BI}\right):\nabla\mathbf{v}\\
&+\frac{\partial\psi}{\partial\nabla\mathbf{v}}\dot{\nabla\mathbf{v}}+\frac{\partial\psi}{\partial\nabla\theta}\dot{\nabla\theta}\\
&+\mathbf{t}_d:\nabla\mathbf{v}+\frac{1}{\mu^2\sigma}(\nabla\times\mathbf{B})^2-\frac{\mathbf{h}\cdot\nabla\theta}{\theta}\geq 0,
\end{aligned}\tag{8.39}$$

which is satisfied in any process if and only if

$$\psi=\psi(\rho,\mathbf{B},\theta),\tag{8.40}$$

$$\eta=-\frac{\partial\psi}{\partial\theta}=\eta(\rho,\mathbf{B},\theta),\tag{8.41}$$

$$\mathbf{t}_e=\rho^2\frac{\partial\psi}{\partial\rho}\mathbf{I}+\rho\frac{\partial\psi}{\partial\mathbf{B}}\otimes\mathbf{B}-\rho\frac{\partial\psi}{\partial\mathbf{B}}\cdot\mathbf{BI},\tag{8.42}$$

$$\mathbf{t}_d:\nabla\mathbf{v}+\frac{1}{\mu^2\sigma}(\nabla\times\mathbf{B})^2+\frac{\mathbf{h}\cdot\nabla\theta}{\theta}\geq 0.\tag{8.43}$$

These relations show that, in general, decomposing the system into a mechanical part and a magnetic part is not a valid approach. The only way to achieve this is to take $\psi(\rho,\theta)$, $\mathbf{t}_d(\rho,\theta,\nabla\mathbf{v})$, and $\mathbf{h}(\rho,\theta,\nabla\mathbf{v})$. This hypothesis, which is not natural, does not satisfy the principle of equipresence and implies that there is no influence of the magnetic field on thermal phenomena. However, under these conditions, the above relations become

$$\psi=\psi(\rho,\theta),\tag{8.44}$$

$$\eta=-\frac{\partial\psi}{\partial\theta}=\eta(\rho,\theta),\tag{8.45}$$

$$\mathbf{t}_e=\rho^2\frac{\partial\psi}{\partial\rho}\mathbf{I},\tag{8.46}$$

$$\mathbf{t}_d:\nabla\mathbf{v}-\frac{\mathbf{h}\cdot\nabla\theta}{\theta}\geq 0.\tag{8.47}$$

8.5 Ordinary Waves in Magnetofluid Dynamics

In this section we analyze the existence of ordinary waves in a moving good conductor S carrying a magnetic field. We recall that an ordinary wave is a moving surface $\Sigma(t)$ across which the highest derivatives that appear in the partial differential equations we are analyzing have finite discontinuities (see [16] and Sect. 1.22).

First, we consider the case of an *incompressible conducting liquid S*. For this system, (8.31)–(8.33) become

$$\nabla \cdot \mathbf{v} = 0, \tag{8.48}$$

$$\rho \dot{\mathbf{v}} = -\nabla p + \frac{1}{\mu} \nabla \times \mathbf{B} \times \mathbf{B}, \tag{8.49}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}). \tag{8.50}$$

Before considering the jump system associated with the above equations, we transform (8.50) into a more convenient form. Due to (8.48) and the condition $\nabla \cdot \mathbf{B} = 0$, we have

$$\nabla \times (\mathbf{v} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{B},$$

so (8.50) can equivalently be written as follows:

$$\dot{\mathbf{B}} = \mathbf{B} \cdot \nabla \mathbf{v}. \tag{8.51}$$

Then we recall the fundamental formulae expressing the jumps in the first derivatives that appear in Eqs. 8.48, 8.49, and 8.51:

$$\left[\left[\frac{\partial v_i}{\partial x_j}\right]\right] = \lambda_i n_j, \tag{8.52}$$

$$[[\dot{v}_i]] = -\lambda_i U, \tag{8.53}$$

$$\left[\left[\frac{\partial B_i}{\partial x_j}\right]\right] = b_i n_j, \tag{8.54}$$

$$\left[\left[\dot{B}_i\right]\right] = -b_i U, \tag{8.55}$$

$$\left[\left[\frac{\partial p}{\partial x_j}\right]\right] = \mathbb{P} n_j, \tag{8.56}$$

where $\mathbf{n}$ is the unit vector normal to the wavefront $\Sigma(t)$, λ_i are the components of the amplitude of the discontinuity of $\nabla \mathbf{v}$, b_i are the components of the amplitude of the discontinuity of $\nabla \mathbf{B}$, and $\mathbb{P}$ is the amplitude of the discontinuity of ∇p. Finally, $U = c_n - v_n$ is the local speed of propagation of $\Sigma(t)$.

Finally, the jump system in relation to (8.48), (8.49), and (8.51) is

$$\lambda \cdot \mathbf{n} = 0, \tag{8.57}$$

$$-\rho \lambda U = -\mathbb{P}\mathbf{n} - \frac{1}{\mu}(\mathbf{b} \times \mathbf{n}) \times \mathbf{B}, \tag{8.58}$$

$$-\mathbf{b} U = (\mathbf{B} \cdot \mathbf{n}) \lambda. \tag{8.59}$$

From (8.57) and (8.59), we have

$$\mathbf{b} \cdot \mathbf{n} = 0. \tag{8.60}$$

Therefore, upon expanding the double vector product that appears in (8.58) according to the formula

$$(\mathbf{b} \times \mathbf{n}) \times \mathbf{B} = (\mathbf{b} \cdot \mathbf{B})\mathbf{n} - (\mathbf{B} \cdot \mathbf{n})\mathbf{b},$$

taking the scalar product of (8.58) by $\mathbf{n}$, and considering (8.57) and (8.60), we obtain

$$\mathbb{P} = -\frac{1}{\mu}\mathbf{b} \cdot \mathbf{B}. \tag{8.61}$$

This result leads us to the system

$$\begin{aligned} \rho U \lambda &= -\frac{\mathbf{B} \cdot \mathbf{n}}{\mu}\mathbf{b}, \\ \mathbf{b} &= -\frac{\mathbf{B} \cdot \mathbf{n}}{U}\lambda, \end{aligned} \tag{8.62}$$

from which we derive

$$U = \mp\frac{|\mathbf{B} \cdot \mathbf{n}|}{\sqrt{\mu\rho}}. \tag{8.63}$$

We conclude that transverse ordinary waves that propagate with a speed given by (8.63) are possible in a highly conducting liquid.

We now consider the case of a compressible conducting fluid. In this case, the equations to consider are (see Eqs. (8.31)–(8.33)):

$$\dot{\rho} = -\rho\nabla \cdot \mathbf{v}, \tag{8.64}$$

$$\rho\dot{\mathbf{v}} = -p'\nabla\rho + \frac{1}{\mu}\nabla \times \mathbf{B} \times \mathbf{B}, \tag{8.65}$$

$$\dot{\mathbf{B}} = \mathbf{B} \cdot \nabla\mathbf{v} - \mathbf{B}\nabla\mathbf{v}, \tag{8.66}$$

and the related jump system is:

$$-\sigma U = -\rho\lambda \cdot \mathbf{n}, \tag{8.67}$$

$$-\rho U\lambda = -p'\sigma\mathbf{n} - \frac{1}{\mu}[(\mathbf{b} \cdot \mathbf{B})\mathbf{n} - (\cdot\mathbf{B} \cdot \mathbf{n})\mathbf{b}], \tag{8.68}$$

$$-U\mathbf{b} = (\mathbf{B} \cdot \mathbf{n})\lambda - \mathbf{B}(\lambda \cdot \mathbf{n}), \tag{8.69}$$

where the notation used has the same meaning as that used above, and σ is the amplitude of discontinuity in the mass density.

From (8.67) and (8.69) we have

$$\sigma = \rho\frac{\lambda \cdot \mathbf{n}}{U}, \quad \mathbf{b} = -\frac{1}{U}[(\mathbf{B} \cdot \mathbf{n})\lambda - \mathbf{B}(\lambda \cdot \mathbf{B})], \tag{8.70}$$

and so, after some calculation, we deduce the following result from (8.67):

$$(\mathbf{Q} - U^2\mathbf{I})\lambda = \mathbf{0}, \tag{8.71}$$

which shows that λ must be eigenvector of the symmetric *acoustic tensor*

$$\mathbf{Q} = \left(p' + \frac{B^2}{\mu\rho}\right)\mathbf{n}\otimes\mathbf{n} + \frac{(\mathbf{B}\cdot\mathbf{n})^2}{\mu\rho}\mathbf{I} - \frac{\mathbf{B}\cdot\mathbf{n}}{\mu\rho}(\mathbf{B}\otimes\mathbf{n}+\mathbf{n}\otimes\mathbf{B}). \tag{8.72}$$

If we denote the unit vector along $\mathbf{B}$ by $\mathbf{u}$ and put

$$U_s^2 = p', \quad U_a^2 = \frac{B^2}{\mu\rho}, \tag{8.73}$$

then the acoustic tensor assumes the form

$$\mathbf{Q} = \left(U_s^2 + U_a^2\right)\mathbf{n}\otimes\mathbf{n} + U_a^2\cos^2\theta\mathbf{I} - U_a^2\cos\theta(\mathbf{n}\otimes\mathbf{u}+\mathbf{u}\otimes\mathbf{n}). \tag{8.74}$$

We note that U_s is the usual speed of ordinary waves in compressible fluids in the absence of magnetic fields. The symmetry of $\mathbf{Q}$ implies that its eigenvalues are real; however, they must also be positive in order to have a real speed of propagation. We recall that $\mathbf{n}$ is a unit vector that assigns the direction of propagation of the ordinary wave. In order to simplify the analysis of the eigenvalue equation (8.71), we introduce a frame of reference $Ox_1x_2x_3$ such that Ox_1 is along $\mathbf{n}$ and the coordinate plane Ox_1x_2 contains the vector $\mathbf{B}$. In the above frame of reference we have $\mathbf{n} = (1,0,0)$ and $\mathbf{u} = (\cos\theta, \sin\theta, 0)$, where θ is the angle between $\mathbf{B}$ and the Ox_2-axis.

Using this notation, the characteristic equation of the matrix $\mathbf{Q}$ assumes the following coordinate form:

$$\det\begin{pmatrix} U_s^2 + U_a^2\sin^2\theta - U^2 & -U_a^2\sin\theta\cos\theta & 0 \\ -U_a^2\sin\theta\cos\theta & U_a^2\cos^2\theta - U^2 & 0 \\ 0 & 0 & U_a^2\cos^2\theta - U^2 \end{pmatrix} = 0. \tag{8.75}$$

Developing the determinant on the left-hand side of the above equation, we obtain the algebraic equation

$$(U_a^2\cos^2\theta - U^2)(U^4 - (U_s^2+U_a^2)U^2 + U_a^2U_s^2\cos^2\theta) = 0, \tag{8.76}$$

whose solutions are

$$U = \pm U_a\cos\theta, \tag{8.77}$$

$$U = \pm\frac{1}{\sqrt{2}}\sqrt{U_s^2 + U_a^2 \pm \sqrt{U_s^4 + U_a^4 + 2(1-2\cos^2\theta}}. \tag{8.78}$$

The eigenvectors corresponding to the above eigenvalues are

$$(0,0,1), \quad (a+b,1,0), \quad (a-b,1,0), \tag{8.79}$$

where

$$a = \frac{1}{U_a^2}\csc 2\theta(U_a^2\cos 2\theta - U_s^2), \tag{8.80}$$

$$b = \frac{1}{U_a^2}\csc 2\theta\sqrt{U_s^2 + U_a^2 - 2U_s^2U_a^2\cos 2\theta}. \tag{8.81}$$

Waves traveling at a speed of U_a are called *Alfven waves*. In view of (8.79), such a wave is transverse, whereas the other two waves are oblique. Figures 8.1 and 8.2 show polar plots of the speeds given by (8.77) and (8.78) when $U_a < U_s$ and $U_a > U_s$, respectively, and when the value of U_s is normalized to 1.

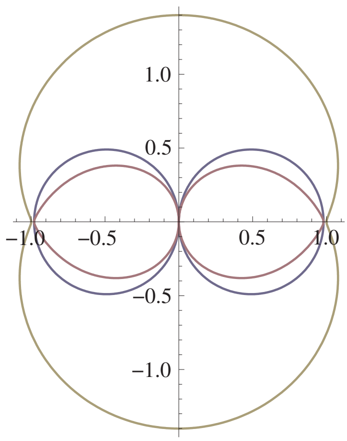

Fig. 8.1 Polar plot of speed for $U_a < U_s$

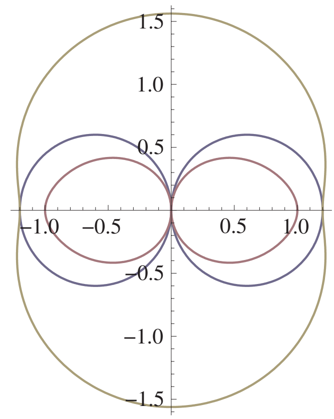

Fig. 8.2 Polar plot of speed for $U_a > U_s$

8.6 Alfven's Theorems

Equation 8.15 has significant consequences that are represented by two theorems, known as *Alfven's theorems*. Before we define and prove these theorems, we first define a *magnetic line of force* as a curve whose tangent at any point has the same direction as the vector $\mathbf{B}$ at that point.

Theorem 8.1
The magnetic flux across any material surface is constant.

PROOF From Eq. (4.48) in [16], we conclude that the vector identity

$$\frac{d}{dt}\int_{S(t)} \mathbf{B}\cdot\mathbf{N}\,d\sigma = \int_{S(t)}\left[\frac{\partial \mathbf{B}}{\partial t} + \nabla\times(\mathbf{B}\times\mathbf{v}) + \mathbf{v}\nabla\cdot\mathbf{B}\right]\cdot\mathbf{N}\,d\sigma \qquad (8.82)$$

holds for an arbitrary material surface $S(t)$. Then, from (8.2) and (8.15), we derive that

$$\frac{d}{dt}\int_{S(t)} \mathbf{B}\cdot\mathbf{N}\,d\sigma = 0. \tag{8.83}$$

∎

Theorem 8.2
The magnetic lines of force are material lines.

PROOF Let T_0 be a magnetic tube formed from magnetic lines of force starting from the point of a curve γ_0 in configuration C_0 of the conducting fluid S at the instant t (see Fig. 8.3). σ_0 denotes any surface lying on T_0. It is evident that the flux of $\mathbf{B}$ across σ_0 is zero. Let T_t and σ_t be the images at the instant t, according to the equations of motion, of the surfaces T_0 and σ_0, respectively. Since T_t and σ_t are material surfaces, then, due to the above theorem, the flux of $\mathbf{B}$ at the instant t across σ_t is still zero. However, σ_t is arbitrary, so $\mathbf{B}$ is orthogonal to the surface T_t, which is therefore a tube of magnetic lines of force. ∎

Essentially, this theorem states that magnetic lines of force are carried during the motion.

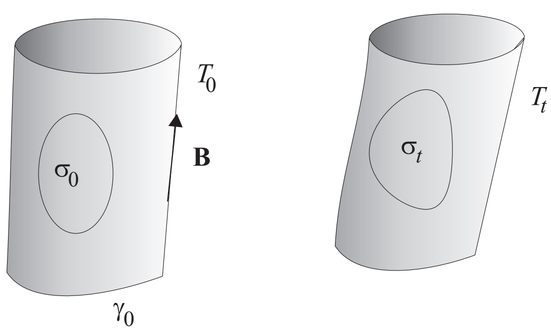

Fig. 8.3 Evolution of a magnetic tube

8.7 Laminar Motion Between Two Parallel Plates

In this section we analyze the stationary laminar motion of a conducting viscous *liquid* S that flows between two parallel fixed plates π_1 and π_2

under the influence of a uniform magnetic field $\mathbf{B}_0/\mu$ which is orthogonal to the plates (see Fig. 8.4). S is incompressible and the motion is stationary. Therefore, if we neglect the action of external forces, then (8.31) and (8.32) become

$$\mathbf{0} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{\mu\sigma}\Delta\mathbf{B}, \tag{8.84}$$

$$\rho\mathbf{v} \cdot \nabla\mathbf{v} = -\nabla p + \frac{1}{\mu}\nabla \times \mathbf{B} \times \mathbf{B} + \rho\nu\Delta\mathbf{v}, \tag{8.85}$$

where ρ is the constant mass density and ν the viscosity coefficient of S (see p. 272 of [16]).

We must add the following other equations to those given above:

$$\nabla \cdot \mathbf{v} = 0, \quad \nabla \cdot \mathbf{B} = 0. \tag{8.86}$$

These state the incompressibility of the liquid S and the solenoidal character of $\mathbf{B}$.

Physical evidence suggests that we should search for a solution with the following form:

$$\mathbf{v} = v(z)\mathbf{i}, \quad \mathbf{B} = b(x)\mathbf{i} + B_0\mathbf{k}, \quad p = p(x, z). \tag{8.87}$$

First, we note that these fields satisfy both of the equations in (8.86). Further, it is staightforward to verify that

$$\nabla \times (\mathbf{v} \times \mathbf{B} = B_0\frac{\partial v}{\partial z}\mathbf{i}, \quad \Delta\mathbf{B} = \frac{\partial^2 b}{\partial z^2}\mathbf{i}, \tag{8.88}$$

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \frac{\partial b}{\partial z}\mathbf{i} - b\frac{\partial b}{\partial z}\mathbf{k}, \tag{8.89}$$

$$\mathbf{v} \cdot \nabla\mathbf{v} = \mathbf{0}, \quad \Delta\mathbf{v} = \frac{\partial^2 v}{\partial z^2}\mathbf{k}. \tag{8.90}$$

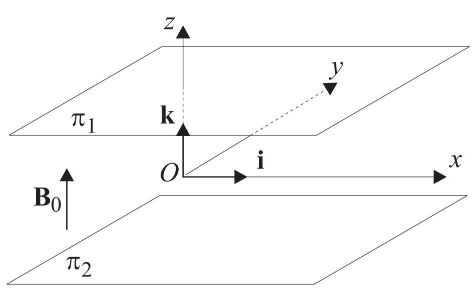

Fig. 8.4 Flow between parallel plates

Consequently, the components of the above equations along the axis Ox are

$$\mu\sigma B_0 \frac{\partial v}{\partial z} + \frac{\partial^2 b}{\partial z^2} = 0, \tag{8.91}$$

$$-\frac{\partial p}{\partial x} + \frac{B_0}{\mu}\frac{\partial b}{\partial z} + \rho\nu\frac{\partial^2 v}{\partial z^2} = 0, \tag{8.92}$$

whereas their projections along the axis Oz give the following condition:

$$\frac{\partial p}{\partial z} + \frac{1}{2\mu}\frac{\partial b^2}{\partial z}. \tag{8.93}$$

We are now faced with a system of three differential equations in the three unknowns $v(z)$, $b(z)$ and $p(x,z)$. Since the second and the third terms on the left-hand side of (8.92) depend only on the variable z, and the first term is a function of x and z, we deduce that

$$\frac{\partial p}{\partial z} = -\mathbb{P}, \tag{8.94}$$

where $\mathbb{P}$ is a constant. Integrating (8.93), we obtain the following expression for p:

$$p(x,z) = -\mathbb{P}x + p_0 - \frac{1}{2\mu}b^2(z), \tag{8.95}$$

where p_0 is constant. We can say that $p(x,z)$ will be known when $b(z)$ is given together with two values of $p(x,z)$ along the axis Ox.

In view of (8.94), Eqs. (8.91) and (8.92) lead us to the system

$$\mu\sigma B_0 \frac{dv}{dz} + \frac{d^2 b}{dz^2} = 0, \tag{8.96}$$

$$\frac{B_0}{\mu}\frac{db}{dz} + \rho\nu\frac{d^2 v}{dz^2} = -\mathbb{P} \tag{8.97}$$

in the unknowns $v(z)$ and $b(z)$. First integration of (8.96) yields

$$\frac{db}{dz} = \mathbb{A} - \mu\sigma B_0 v, \tag{8.98}$$

where $\mathbb{A}$ is a constant. Introducing this result into (8.97), we derive a second-order linear equation with constant coefficients:

$$\frac{d^2 v}{dz^2} - \frac{L^2}{M^2}v + \frac{1}{\rho\nu}\left(\mathbb{P} + \frac{B_0}{\mu}\mathbb{A}\right) = 0, \tag{8.99}$$

where L is the distance between the plates and

$$M = B_0 L\sqrt{\frac{\sigma}{\rho\nu}} \tag{8.100}$$

is the nondimensional *Hartmann number*. Since the fluid S is assumed to be viscous, the natural boundary conditions to associate with (8.99) are

$$v(-L) = 0, \quad v(L) = 0. \tag{8.101}$$

We can easily verify that the solution of the boundary value problem corresponding to (8.99) and (8.101) is given by the function

$$v(z) = \left(\mathbb{P} + \frac{\mathbb{A}B_0}{\mu}\right) \frac{\cosh(M) - \cosh\left(\frac{Mz}{L}\right)}{\sigma B_0^2 \cosh(M)}, \tag{8.102}$$

which still contains the constant $\mathbb{A}$. Integrating (8.98) and taking into account (8.102), we obtain the result

$$b(z) = c + \mathbb{A}z - \left(\mathbb{P} + \frac{\mathbb{A}B_0}{\mu}\right) \mu L \frac{\frac{z}{L}\cosh(M) - \frac{1}{M}\sinh\left(\frac{Mz}{L}\right)}{B_0 \cosh(M)}. \tag{8.103}$$

It now remains to determine the constants $\mathbb{A}$ and c in the expressions (8.102) and (8.103). To this end, we must assign suitable boundary conditions for the function $b(z)$. The starting equation (8.84) is more general than (8.50), which describes the evolution of the magnetic field for conductors with very high conductivities. Consequently, Alfen's theorems do not hold for the problem we are considering. However, due to the presence of the term $\nabla \times (\mathbf{v} \times \mathbf{B})$, we can assume that the magnetic lines of force are partially transported by the flow and that there is no transport when the velocity vanishes. Based on this hypothesis, we can derive the following boundary conditions:

$$b(-L) = 0, \quad b(L) = 0, \tag{8.104}$$

which allow us to place (8.102) and (8.103) in the following final forms:

$$v(z) = M\mathbb{P}\frac{\cosh(M) - \cosh\left(\frac{Mz}{L}\right)}{B_0^2 \sigma \sinh(M)} \tag{8.105}$$

$$b(z) = \mathbb{P}\mu L \frac{\sinh\left(\frac{Mz}{L}\right) - \frac{z}{L}\sinh(M)}{B_0 \sinh(M)}. \tag{8.106}$$

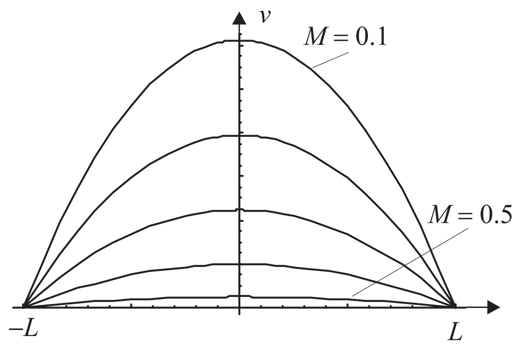

Fig. 8.5 Flow plot for $0.1 \leq M \leq 0.5$

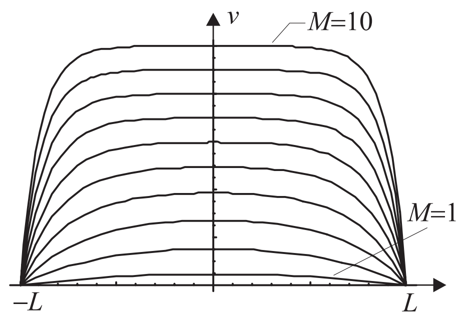

Fig. 8.6 Flow plot for $1 \leq M \leq 10$

Figures 8.6 and 8.7 show flow plots for various Hartmann numbers. We can see that, for low Hartmann numbers, the velocity tends to the parabolic profile given by the formula

$$v(z) = \frac{M^2 \mathbb{P}(L^2 - z^2)}{2B_0^2 L^2 \sigma}, \tag{8.107}$$

which coincides with the first term in the Taylor series for (8.105). For large values of M, the profile of $v(z)$ shows the presence of boundary layers next to the plates.

8.8 Law of Isorotation

A common problem encountered in astrophysics involves analyzing a fluid mass S that has a high electrical conductivity, carries a magnetic field, and rotates about a fixed axis a with an angular velocity ω that does not change over time. To investigate this problem, we denote a system of cylindrical coordinates about the axis a by (r, φ, z), and we assume that the angular velocity ω is a function of r and z:

$$\omega = \omega(r, z), \tag{8.108}$$

i.e., we suppose that the velocity field displays cylindrical symmetry about the axis a. If we denote the velocity components along the axes of a unit holonomic basis $(\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{e}_z)$ relative to the cylindrical coordinates by (v_r, v_φ, v_z), then it follows that the velocity $\mathbf{v}$ of the point $(r, \varphi, z) \in S$ can be written as follows:

$$\mathbf{v} = r^2\omega(r, z)\mathbf{e}_\varphi \equiv v_\varphi(r, z)\mathbf{e}_\varphi, \tag{8.109}$$

and the flow will exhibit cylindrical symmetry about a. Further, we have[1]

$$\nabla \cdot \mathbf{v} = \frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z} = 0. \tag{8.110}$$

Finally, due to the stationarity hypothesis and (8.110), the continuity equation for mass becomes

$$\mathbf{v} \cdot \nabla\rho = r\omega\frac{\partial \rho}{\partial \varphi} = 0, \tag{8.111}$$

so that $\rho = \rho(r, z)$. If we assign the pressure p using the state equation $p = p(\rho)$, then we can conclude that the pressure field also exhibits cylindrical symmetry about the axis of rotation a.

It now remains to analyze the Maxwell equations (8.50), which in our hypotheses reduce to

$$\nabla \times (\mathbf{v} \times \mathbf{B}) = \mathbf{0}. \tag{8.112}$$

If the region occupied by S is simply connected, then the above equation implies the existence of a smooth function $\Phi(r, \varphi, z)$ such that

$$\mathbf{v} \times \mathbf{B} = \nabla\Phi. \tag{8.113}$$

This condition can be explicitly written in the form

$$r\omega\mathbf{e}_\varphi \times (B_r\mathbf{e}_r + B_\varphi\mathbf{e}_\varphi + B_z\mathbf{e}_z) = \frac{\partial\Phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\varphi}\mathbf{e}_\varphi + \frac{\partial\Phi}{\partial z}\mathbf{e}_z,$$

[1]For the expression for the divergence of a vector field in cylindrical coordinates, see p. 65 of [16].

which is equivalent to the equation

$$r\omega(B_z\mathbf{e}_r - B_r\mathbf{e}_z) = \frac{\partial\Phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\varphi}\mathbf{e}_\varphi + \frac{\partial\Phi}{\partial z}\mathbf{e}_z,$$

from which we derive

$$\Phi = \Phi(r,z), \tag{8.114}$$

$$B_r = -\frac{1}{r\omega}\frac{\partial\Phi}{\partial z} = B_r(r,z), \tag{8.115}$$

$$B_z = \frac{1}{r\omega}\frac{\partial\Phi}{\partial r} = B_z(r,z). \tag{8.116}$$

We conclude that the components B_r and B_z of the induced magnetic field do not depend on the angular variable φ. Moreover, due to (8.113), we obtain the relations

$$\nabla\Phi\cdot\mathbf{B} = 0, \quad \nabla\Phi\cdot\mathbf{v} = 0, \tag{8.117}$$

which allow us to state that both the current and the magnetic lines lie on the surfaces of revolution $\Phi(r,z) = const.$ Writing (8.115) and (8.116) in the forms $r\omega B_r = -\partial\Phi/\partial z$ and $r\omega B_z = \partial\Phi/\partial r$, differentiating the former with respect to r and the latter with respect to z, and adding the results obtained, we derive the condition

$$\frac{\partial}{\partial r}(r\omega B_r) + \frac{\partial}{\partial z}(r\omega B_z) = 0. \tag{8.118}$$

On the other hand, the equation $\nabla\cdot\mathbf{B} = 0$ gives us

$$\frac{\partial}{\partial r}(rB_r) + \frac{\partial}{\partial z}(rB_z) + \frac{\partial B_\varphi}{\partial\varphi} = 0. \tag{8.119}$$

Multiplying both the sides of this equation by ω and subtracting (8.118) from the result, we obtain

$$rB_r\frac{\partial\omega}{\partial r} + rB_z\frac{\partial\omega}{\partial z} - \omega\frac{\partial B_\varphi}{\partial\varphi} = 0. \tag{8.120}$$

Since the first two terms of this equation and ω do not depend on φ, we can say that B_φ does not depend on φ. Therefore, (8.119) and (8.120) reduce to

$$\frac{\partial}{\partial r}(rB_r) + \frac{\partial}{\partial z}(rB_z) = 0, \tag{8.121}$$

$$rB_r\frac{\partial\omega}{\partial r} + rB_z\frac{\partial\omega}{\partial z} = 0. \tag{8.122}$$

The first of the above equations implies the existence of a function $V(r,z)$ such that

$$B_r = -\frac{1}{r}\frac{\partial V}{\partial z}, \quad B_z = \frac{1}{r}\frac{\partial V}{\partial r}. \tag{8.123}$$

Consequently, (8.122) becomes

$$\frac{\partial V}{\partial r}\frac{\partial \omega}{\partial z} - \frac{\partial V}{\partial z}\frac{\partial \omega}{\partial r} = 0, \tag{8.124}$$

and the angular velocity ω is a function of V:

$$\omega = \omega(V). \tag{8.125}$$

Due to (8.124), we can introduce the meridian magnetic induction field

$$\mathbf{B}_m = \frac{1}{r}\left(-\frac{\partial V}{\partial z}\mathbf{e}_r + \frac{\partial V}{\partial r}\mathbf{e}_z\right). \tag{8.126}$$

An integral curve $(r(\lambda), \varphi(\lambda), z(\lambda))$ of this field, containing a point P, obeys the equations

$$\frac{dr}{d\lambda} = -\frac{1}{r}\frac{\partial V}{\partial z}, \quad \frac{d\varphi}{d\lambda} = 0, \quad \frac{dz}{d\lambda} = \frac{1}{r}\frac{\partial V}{\partial r}, \tag{8.127}$$

meaning that it lies in a meridian plane containing P and its tangent vector is orthogonal to ∇V at any point. In other words, it is a meridian curve of the surface of revolution $V = const$ that contains P.

Finally, due to (8.125), we can state that *the angular velocity* ω *is constant on any surface* $V = const$ *obtained by rotating a meridian integral curve of* $\mathbf{B}_m$ *about the rotation axis of* S. This result is known as *Ferraro's law of isorotation* (1937).

Chapter 9

Continua with an Interface and Micromagnetism

9.1 Ferromagnetism and Micromagnetism

In the presence of an external magnetic field, a ferromagnetic substance[1] exhibits a behavior that is quite different from the behavior of a paramagnetic body. Under the same conditions, the former shows an induced magnetization that is much greater than the corresponding magnetization exhibited by the the latter. Moreover, the functional relation between the magnetic field and the magnetization is nonlinear in a ferromagnetic body and linear in a paramagnetic one. Finally, in a ferromagnetic body the magnetization depends not only on the actual value of the magnetic field but its history (i.e., hysteresis).

Weiss [136] explained the anomalous behavior of ferromagnetic bodies by supposing that, even in the absence of an external magnetic field, the magnetic state of a ferromagnetic crystal C is described by a vector field $\mathbf{M} = M_0\mathbf{m}$, where M_0 is a constant scalar that is characteristic of C and denotes the magnetization per unit volume, and $\mathbf{m}$ is a smooth unit magnetic field. The latter is piecewise constant in almost all of the volume D of the crystal, except in thin layers, across which it varies between one constant direction and another. The regions D_a, $a = 1, \ldots, m$, in which $\mathbf{M}$ is constant are called *Weiss domains*.[2] A Weiss domain has at least one microscopic dimension (a few hundred microns), whereas its other dimensions are comparable to those of the crystal. On the other hand, transition layers, which are called *domain walls* or *Bloch walls* [139], have a thickness

[1] What is said in this chapter about ferromagnetic crystals also holds for ferroelectric crystals; see, for instance, [140, 141].

[2] Experimental proof of the existence of Weiss domains was found by Barkhauser [137], whereas the physical explaination for ferromagnetism was provided by Heisenberg [138].

A. Romano, A. Marasco, *Continuum Mechanics*, Modeling and Simulation in Science, Engineering and Technology, DOI 10.1007/978-0-8176-4870-1_9,

of a few microns. The theoretical justification for this complex distribution derives from the combined effects of three factors: the magnetic energy related to the anisotropy of the crystal, dipole–dipole exchange quantum forces, and the form of the crystal. The presence of an external magnetic field modifies the weights of the effects of these three factors on the domain distribution. In particular, for a weak external magnetic field $\mathbf{H}$, domains in which $\mathbf{M} \cdot \mathbf{H} > 0$ increase in volume whereas domains that obey the opposite condition reduce in volume. For a strong external magnetic field, the magnetization in any domain tends to become parallel to $\mathbf{H}$.

Micromagnetism is the most natural attempt to describe the above situation by adopting a continuum physics perspective (see, for instance, [144]–[154]). Since we are interested in demonstrating the difficulties of this theory here, we will limit ourselves to the case of a *rigid* ferromagnetic crystal.

Micromagnetism is based on the following assumptions:

- The energy per unit volume of the crystal is a function $e(\mathbf{m}, \nabla\mathbf{m})$ of the unit vector $\mathbf{m}$ and its gradient. In particular, in uniaxial crystals, it usually takes the form

$$e = M_0 \frac{1}{2}\alpha \sum_{i=1}^{3} (\nabla m_i)^2 + \frac{1}{2}\beta \sum_{i=1}^{2} m_i^2, \tag{9.1}$$

  where α and β are constants that depend on the crystal.

- Denoting the magnetic potential by ϕ, the possible equilibrium configurations of the crystal correspond to minima of the total energy

$$F(\mathbf{m}, \phi) = \int_D M_0 e(\mathbf{m}, \nabla\mathbf{m})\, dv - \frac{1}{2}\int_{\Re^3} |\nabla\phi|^2\, dv + M_0 \int_D \mathbf{m} \cdot \nabla\phi\, dv, \tag{9.2}$$

  under the constraint

$$\mathbf{m} \cdot \mathbf{m} = 1. \tag{9.3}$$

It is not difficult to verify that the Lagrange equations of the functional (9.2) under the constraint (9.3) are

$$\begin{aligned} e,_{m_i} - (e,_{m_{i,j}}),_j - \phi,_i + \lambda m_i &= 0, \\ \mu_0 \phi,_i - M_0 m_{i,i} &= 0, \end{aligned} \tag{9.4}$$

where μ_0 denotes the permeability of a vacuum, λ is a Lagrangian multiplier arising from condition (9.3), and we have used the notation $f,_a = \partial f/\partial a$.

In order to see that (9.4), in principle, provides for the presence of Bloch walls, it is sufficient to recall that the coefficients that multiply the second-order derivatives in (9.4) are very small. For instance, the coefficient α in (9.1) is on the order of 10^{-13}. In other words, the equations (9.4) make it possible to have boundary layers inside the volume D of the crystal. However, it is very difficult to find their locations in D, even for the simplest geometry of D. Information about the domains can only be obtained by resorting to rather drastic approximations (see [142, 143, 153, 155]).[3]

In the next few sections we show that the model of a continuum with an interface allows us to determine the form and the magnitude of the Weiss domains, at least when the volume D of the crystal has a simple geometry.

9.2 A Ferromagnetic Crystal as a Continuum with an Interface

In order to obtain a sufficiently accurate description of the domains and to simplify the task of determining them, we replace the domain walls with surfaces of discontinuity S_b, $b = 1, \dots, q$ for $\mathbf{m}$, and take into account the energy contained in a transition layer with a surface energy of $e_\sigma(\mathbf{n}, \mathbf{m}^-, \mathbf{m}^+)$, which depends a priori on the orientation of the unit normal $\mathbf{n}$ of S_b with respect to the crystallographic axes and on the constant vectors $\mathbf{m}$ in two domains adjacent to the domain wall under consideration. However, in addition to this surface energy, we will still consider an energy per unit magnetization and per unit volume of the crystal that depends on $\nabla\mathbf{m}$ as well as $\mathbf{m}$. Actually, except for very particular forms of D associated with the symmetry class of the crystal, satisfying the boundary conditions on ∂D may require the formation of small regions that adhere to the external walls of the ferromagnetic crystal where the magnetization field $\mathbf{m}$ is not uniform. The equilibrium configuration of a *rigid* ferromagnetic crystal in the presence of an external magnetic field is then derived by imposing the requirement that the total energy becomes stationary with respect to variations in the magnetization, the external magnetic field and the surfaces S_b, which engender the Weiss domains under the constraints that $\mathbf{m}$ is a unit vector and that the volume of D remains constant (see [156]).

Starting from these equations, we prove in the next sections that:

- The domain walls inside D are necessarily planar
- The resulting magnetic field $\mathbf{H}$ in each domain is uniform
- In some cases, $\mathbf{p}$ and $\mathbf{H}$ are parallel in each domain.

[3] It would be interesting to analyze this problem using numerical methods.

9.3 Variations in Surfaces of Discontinuity

We consider a domain D in which there is a given field $f(\xi)$ where $\xi = \xi(\mathbf{x})$ denotes a collection of variables that depend on the Cartesian coordinates $\mathbf{x}$. We assume that D is the union of m disjoint open subdomains D_a, $a = 1, \ldots, m$, i.e., $D = \bigcup_{a=1}^{m} D_a$, $D_a \bigcap D_b = \emptyset$. The subdomains are separated by q surfaces S_b, $b = 1, \ldots, q$ (see Fig. 9.1). In what follows, $S = \bigcup_{b=1}^{q} S_b$ and we assume that f exhibits a discontinuity at S.

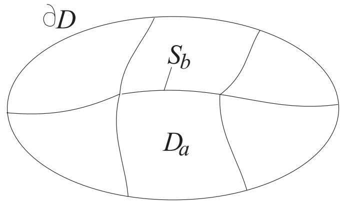

Fig. 9.1 Weiss domains and Bloch walls

We now evaluate the variation in the integral

$$I = \int_D f(\xi)\, dv, \tag{9.5}$$

due to the variations δ in the field variables and the variations $\delta\mathbf{r}$ in the surfaces of discontinuity S_b, while the shape of D is retained. To simplify the analysis, we start with a single surface of discontinuity S.

From (9.5), we have

$$\delta I = \delta \int_{D^-} f(\xi)\, dv + \delta \int_{D^+} f(\xi)\, dv,$$

where

$$\delta \int_{D^-} f(\xi)\, dv = \int_{D^- + \delta D^-} f(\xi + d\xi)\, dv + \int_{D^-} f(\xi)\, dv,$$

and an analogous formula holds for D^+. On the other hand, ∂D is fixed and D^- only changes with variations in S, so that

$$\delta \int_{D^-} f(\xi)\, dv = \int_{D^-} f_{,\xi} \cdot \delta\xi\, dv + \int_S f^- \delta r_n\, da, \tag{9.6}$$

where δr_n denote the displacement of S along the unit normal $\mathbf{n}$, which is assumed to belong to D^+. Finally, we obtain

$$\delta \int_D f(\xi)\, dv = \int_D f_{,\xi} \cdot \delta\xi\, dv - \int_S [[f]] \delta r_n\, da. \tag{9.7}$$

More generally, if S is the union of some surfaces, we have the requested formula

$$\delta \int_D f(\xi)\, dv = \int_D f,_\xi \cdot \delta\xi \, dv - \sum_{q=1}^{q} \int_S [[f]] \delta r_n \, da. \tag{9.8}$$

It now remains to evaluate the variation in the integral

$$F = \int_S \phi(\xi)\, da,$$

when the surfaces of discontinuity S_b are modified. The following formula is proved in [156]:

$$\begin{aligned}\sum_{b=1}^{q} \int_{S_b} \phi(\xi)\, da &= \sum_{b=1}^{q} \int_{S_b} (\phi,_\xi \cdot \delta\xi - \phi b_\alpha^\alpha \delta r_n - \phi_{;\alpha} \delta r^\alpha)\, da \\ &\quad - \sum_{c=1}^{r} \int_{\Gamma_c} [[\phi \mathbf{n}]] \times \mathbf{t} \cdot \delta \mathbf{r}\, ds, \end{aligned} \tag{9.9}$$

where b_α^α is the trace of the second fundamental form on S (see Appendix B), δr^α are the components of the variation $\delta\mathbf{r}$ in the surface S along the natural basis $\mathbf{a}_\alpha$, Γ_c, $c = 1, \ldots, r$ are the curves along which the surfaces S_b intersect each other, and $\mathbf{t}$ is the unit tangent vector to the curves Γ_c.

9.4 Variational Formulation of Weiss Domains

Let us consider a rigid ferromagnetic crystal C that occupies a fixed region D of the space $\Re^3$. We assume that C is in an external static magnetic field, and we denote the sum of the external magnetic field and the intrinsic magnetic field generated by the crystal by $\mathbf{H}$. Since $\nabla \times \mathbf{H} = 0$, we denote the total magnetic potential by ϕ. We are essentially interested in the formation of Weiss domains in which the polarization is uniform. The walls of Weiss domains are in essence boundary layers across which there are large magnetization gradients. We approximate the boundary layers by surfaces, but we take into account the effect of the boundary layers by introducing appropriate surface energies on the walls of the Weiss domains. As a consequence, the magnetization field in D is only piecewise differentiable.

We will first consider a more general case in which the magnetization vector $M_0\mathbf{m}$ has a constant magnitude but may change in direction. We now state a variational principle which allows us to derive all of the field equations and the jump and boundary conditions. Then we will show that the Weiss domains actually exist almost throughout D, except in a very thin layer adjacent to the exterior boundary ∂D, which may or may not form depending on the shape of ∂D.

We will consider the following functional to give the total energy of the crystal C in the external magnetic field:

$$F = \int_D M_0 e(\mathbf{m}, \nabla \mathbf{m})\, dv + \frac{1}{2}\mu_0 \int_{\Re^3} \phi,_i \phi,_i \, dv - M_0 \int_D \phi,_i m_i \, dv$$
$$+ \sum_{b=1}^{q} M_0 \int_{S_b} e_\sigma(\mathbf{n}, \mathbf{m}^-, \mathbf{m}^+)\, da, \tag{9.10}$$

where $e(\mathbf{m}, \nabla \mathbf{m})$ is the magnetic energy per unit volume of the crystal, the second integral denotes the magnetic energy in $\Re^3$, the third integral is the interaction energy in D between the magnetic field and magnetization, and e_σ is the energy per unit area of S_b, which depends on the orientation of S_b and on the values taken by $\mathbf{m}$ when it approaches S_b from either side.

We postulate that the following variational principle holds: *the equilibrium configurations* $\mathbf{m}, \phi, S_b$ *of the ferromagnetic crystal* C *are extremals of the functional* (9.10) *with respect to all of the variations in* $\mathbf{m}, \phi, S_b$ *for which* D *is constant and* $\mathbf{m}$ *is a unit vector field.*

Applying (9.8) and (9.9) to (9.10), we obtain the following local conditions after tedious calculations:

- In D_a, $a = 1, \ldots, m$,

$$e,_{m_i} - (e,_{m_{i,j}}),_j - \phi,_i + \lambda m_i = 0, \tag{9.11}$$
$$\mu_0 \phi,_{ii} - M_0 m_{i,i} = 0, \tag{9.12}$$

- In $\Re^3$,

$$\phi,_{ii} = 0, \tag{9.13}$$

- On ∂D,

$$\mu_0 [[\phi,_i]] n_i + M_0 m_i n_i = 0, \tag{9.14}$$
$$e,_{p_{i,j}} n_j = 0, \tag{9.15}$$

- On S_b, $b = 1, \ldots, q$,

$$\left[\left[M_0 e + \frac{1}{2}\mu_0 \phi,_i \phi,_i - M_0 \phi,_i m_i \right]\right]$$
$$+ M_0 (V^\alpha_{;\alpha} + b^\alpha_\alpha e_\sigma) = 0, \tag{9.16}$$
$$[[\mu_0 \phi,_i - M_0 m_i]]\, n_i = 0, \tag{9.17}$$
$$e_{\sigma, m_i^+} m^+_{i;\alpha} + e_{\sigma, m_i^-} m^-_{i;\alpha} = 0 \tag{9.18}$$
$$e_{\sigma, m_i^+} - e^+,_{m_{i,j}} n_j = 0, \tag{9.19}$$
$$e_{\sigma, m_i^-} + e^+,_{m_{i,j}} n_j = 0, \tag{9.20}$$

- On Γ_c, $c = 1, \ldots, r$,

$$[[(\mathbf{V} \times \mathbf{n} \cdot \mathbf{t})\mathbf{n} - e_\sigma \mathbf{n} \times \mathbf{t}]] = \mathbf{0}. \tag{9.21}$$

In the above equations, λ and Λ are the two Lagrangian multipliers related, respectively, to the condition $\mathbf{m} \cdot \mathbf{m} = 1$ and the invariance of the total volume of D. Finally, $\mathbf{V}$ is the projection onto S_b of the vector $\partial e_\sigma / \partial \mathbf{n}$.

9.5 Weiss Domain Structure

We now assume that $\mathbf{m}$ is a constant vector field. Equations 9.11–9.14 then become:

- In D_a, $a = 1, \ldots, m$,

$$e_{,m_i} - \phi_{,i} + \lambda m_i = 0, \tag{9.22}$$
$$\mu_0 \phi_{,ii} = 0, \tag{9.23}$$

- On S_b, $b = 1, \ldots, q$,

$$\mu_0 [[\phi_{,i}]] n_i + M_0 m_i n_i = 0. \tag{9.24}$$

It might be wise to remark here that the Lagrange multiplier $\lambda(\mathbf{x}$ is generally different for each domain. Since p_i is assumed to be constant, $e_{,p_i}$ is also constant, and from (9.22) we have

$$\phi_{,i} = \lambda(\mathbf{x}) m_i + e_{,m_i}. \tag{9.25}$$

From (9.23) and (9.25), we see that

$$\lambda_{,i} \, m_i = 0. \tag{9.26}$$

On the other hand, the integrability conditions of (9.25) are

$$\lambda_{,i} \, m_j = \lambda_{,j} \, m_i.$$

Multiplying this relation by m_i, recalling that $\mathbf{m}$ is a unit vector, and recalling (9.26), we then find that

$$\lambda_{,j} = \lambda_i m_i m_j = 0.$$

Hence, the Lagrangian multiplier $\lambda(\mathbf{x})$ can only be a constant λ_a in each domain D_a, where the magnetic potential can now be written as

$$\phi_a = \left[\lambda_a m_i^{(a)} + \frac{\partial e}{\partial m_i^{(a)}} \right] x_i + d_a, \tag{9.27}$$

and the magnetic field in D_a is constant:

$$H_i^{(a)} = \lambda_a m_i^{(a)} + \frac{\partial e}{\partial m_i^{(a)}}. \tag{9.28}$$

We note that the direction of this field is generally different from that of the magnetization, depending on the isotropic class of the crystal.

It now remains to consider relation (9.26), which holds on one of the Weiss walls between two adjacent Weiss domains. Due to (9.28), this can be written as

$$[[(\mu_0\lambda - M_0)m_i + \mu_0 e_{,m_i}]]n_i = 0. \tag{9.29}$$

If we introduce the *constant* vector

$$b_i = [[(\mu_0\lambda - M_0)m_i + \mu_0 e_{,m_i}]], \tag{9.30}$$

(9.29) becomes

$$b_i n_i = 0. \tag{9.31}$$

This relation implies that the unit normal vector to a wall separating two Weiss domains should remain perpendicular to a constant vector determined by the magnetization vectors and Lagrangian multipliers in those domains. If the equation of the domain wall is given by

$$f(\mathbf{x}) = 0, \tag{9.32}$$

then (9.31) requires that f must satisfy the following first-order linear partial differential equation:

$$b_i f_{,i} = 0, \tag{9.33}$$

whose general solution is

$$f(\mathbf{x}) = b_1 x_3 - b_3 x_1 - b_1 g(b_2 x_1 - b_1 x_2) = 0, \tag{9.34}$$

where g is an arbitrary function of its argument. It is clear that (9.34) defines a ruled surface with generating lines that are parallel to the vector $\mathbf{b}$. The principal curvature of this surface is

$$k_1 = 0, \quad k_2 = |\mathbf{b}|^2 g'' \left[1 + \left(\frac{b_3}{b_1} + b_2 g'\right)^2 + b_1^2 g'^2\right]^{-3/2}, \tag{9.35}$$

where the primes denote differentiation with respect to the argument on which g depends.

We now impose the continuity of magnetic potential across the domain wall. To simplify the notation, we introduce new constants

$$c_i = \lambda m_i + e_{,m_i} \tag{9.36}$$

for each Weiss domain and denote the values of these constants in the two Weiss domains adjacent to the domain wall S_b by c_i^+ and c_i^-. Hence, the continuity of ϕ across S_b is expressed as (see Eq. (9.27)):

$$c_i^+ x_i|_{S_b} + d^+ = c_i^- x_i|_{S_b} + d^-,$$

or, employing (9.34),

$$\begin{aligned} &c_1^+ x_1 + c_2^+ x_2 + c_3^+ \left[\frac{b_3}{b_1} x_1 + g(b_2 x_1 - b_1 x_2)\right] + d^+ \\ &= c_1^- x_1 + c_2^- x_2 + c_3^- \left[\frac{b_3}{b_1} x_1 + g(b_2 x_1 - b_1 x_2)\right] + d^-, \end{aligned}$$

which leads us to

$$\begin{aligned} &(c_3^- - c_3^+) g(b_2 x_1 - b_1 x_2) \\ &= \left[c_1^+ - c_1^- + \frac{b_3}{b_1}(c_3^+ - c_3^-)\right] x_1 + (c_2^+ - c_2^-) x_2 + d^+ - d^-. \end{aligned} \quad (9.37)$$

This relation implies that g can only be a linear function of its argument, such as

$$g = A(b_2 x_1 - b_1 x_2) + B, \quad (9.38)$$

where A and B are constants. This in turn implies via (9.33) that only planar domain walls are admissible. Introducing (9.36) and (9.37), we also find that

$$c_1^+ - c_1^- + \frac{b_3}{b_1}(c_3^+ - c_3^-) = -Ab_2(c_3^+ - c_3^-), \quad (9.39)$$

$$c_2^+ - c_2^- = -Ab_1(c_3^+ - c_3^-), \quad (9.40)$$

$$d^+ - d^- = -B(c_3^+ - c_3^-), \quad (9.41)$$

which impose certain restrictions on the position of the domain wall. The relations (9.39) and (9.40) clearly imply that

$$b_i [[c_i]] = 0 \quad (9.42)$$

on S_b. Introducing (9.30) into the above relation and rearranging the terms, we find that it can be expressed as

$$\begin{aligned} &\mu_0 |[[\lambda \mathbf{m}]]|^2 - M_0(\lambda^+ + \lambda^-)(1 - \mathbf{m}^+ \cdot \mathbf{m}^-) \\ &- M_0 [[\mathbf{m}]] \cdot [[e_{,\mathbf{p}}]] - \mu_0 |[[e_{,\mathbf{p}}]]|^2 = 0 \end{aligned} \quad (9.43)$$

on S_b, $b = 1, \ldots, q$.

Let us consider the conditions (9.18)–(9.20). If we suppose that

$$e_\sigma(\mathbf{n}, \mathbf{m}^+, \mathbf{m}^-) = e_\sigma(\mathbf{n}, [[\mathbf{m}]]), \quad (9.44)$$

then

$$e_{\sigma,m_i^+} + e_{\sigma,m_i^-} = 0$$

and (9.19) and (9.20) lead to the condition

$$e_{,m_{i,j}}^{+}\, n_j = e_{,m_{i,j}}^{-}\, n_j. \tag{9.45}$$

When we recall that $\mathbf{m}$ is constant in each domain, that the domain walls are planes, and (9.44), we find that

$$e_{\sigma,\alpha} = 0, \tag{9.46}$$

i.e., the surface energy is constant on each domain wall.

Finally, taking into account (9.46) and the definition of $\mathbf{V} = \partial e_\sigma / \partial \mathbf{n}$, condition (9.18) becomes

$$\left[\!\left[M_0 e + (\mu_0\lambda - M_0) m_i e_{,m_i} + \frac{1}{2}\mu_0 e_{,m_i}\, e_{,m_i} + \frac{1}{2}\mu_0\lambda^2 - M_0\lambda \right]\!\right] = 0 \tag{9.47}$$

on S_b, $b = 1, \ldots, q$.

We now observe that, in each Weiss domain D_a, $a = 1, \ldots, m$, we have the set of unknown constants $\lambda_a, \mathbf{m}^{(a)}, d_a$ that determine the magnetization field and the magnetic potential in each domain, the Lagrangian multiplier Λ and the parameters A_b and B_b that characterize the bth domain wall S_b, $b = 1, \ldots, q$. Of course, since $\mathbf{m}^{(a)}$ is a unit vector, it gives rise to only two unknowns in each domain. Moreover, the magnetic potential is determined up to an arbitrary constant, meaning that the unknowns b_a are essentially $m-1$. In any case, we have many more unknowns than equations. Therefore, a solution to this problem should not be expected. Moreover, the equations at hand are nonlinear, so real values for the unknowns may not exist at all in certain cases. It is clear that this depends heavily on the symmetry group of the crystal under consideration. If a crystal class does not permit a real solution to be obtained for the above problem, then we must infer that such a crystal class cannot exhibit ferromagnetic properties.

9.6 Weiss Domains in the Absence of a Magnetic Field

In this section we will analyze the case in which the magnetic field vanishes identically throughout the space and the magnetization field $\mathbf{m}$ is piecewise constant. According to these hypotheses, (9.22) and (9.24) become

$$e_{,m_i} = -\lambda m_i, \tag{9.48}$$

$$[[m_i]]\, n_i = 0. \tag{9.49}$$

Finally, in view of (9.48), relation (9.47) can be written as

$$[[M_0 e]] = 0. \tag{9.50}$$

Condition (9.49), which must be obeyed at the domain walls S_b, again implies that each S_b is a ruled surface. Since we cannot use the continuity of the magnetic field across domain walls, we cannot prove that S_b is a planar surface. However, if we note that the walls remain planar provided that even a very weak magnetic field is present, then for continuity reasons we can assume that they remain planar in the absence of a magnetic field.

The boundary condition (9.14) on ∂D requires that

$$m_i n_i = 0; \tag{9.51}$$

that is, the magnetization vector should be in the plane tangent to the boundary of the ferromagnetic crystal. If ∂D does not consist of the union of planar surfaces that can carry the admissible magnetization vector, then it is obvious that the internal Weiss domains cannot be extended to the boundary. Therefore, for any arbitrary form of the crystal, we are compelled to assume the existence of a thin layer adjacent to the boundary of the crystal in which the magnetization vector cannot be assumed to be piecewise uniform. We expect that the solution for $\mathbf{m}$ in this layer D_L approaches the constant values that prevail inside the crystal at points sufficiently far from the boundary ∂D. Therefore, we assume that there is a piecewise uniform magnetization field that is again the solution of (9.48) in the neighborhoods of Weiss domains that are close to the boundary. However, the field $\mathbf{m}$ must vary very rapidly from these constant states to the field tangent to the boundary. Hence, we must assume that the magnetization gradients in a very thin boundary layer adjacent to ∂D are very large, but that they asymptotically approach a piecewise constant state in the rest of D_L. Thus, in this layer we must consider the general equations

$$e,_{m_i} - (e,_{m_{i,j}}),_j + \lambda m_i = 0, \tag{9.52}$$

$$m_{i,i} = 0. \tag{9.53}$$

By resorting to appropriate nondimensional analysis, we expect to transform (9.52) into equations that have a small parameter in front of the second-order derivatives of m_i. Hence, we are faced with a singular perturbation problem. We only remark here that the approximate boundary layer equations that can be obtained by the usual singular perturbation technique should be solved by satisfying the boundary conditions

$$m_i n_i = 0, \quad e,_{m_{i,j}} n_j = 0, \quad \text{on } \partial D, \tag{9.54}$$

and the resulting equations should be matched with the constant states that satisfy the following jump conditions on the wall domains:

$$[[m_i]] n_i = 0 \tag{9.55}$$

using the usual matching techniques.

We conclude by noting that such an approach would produce several admissible configurations for Weiss domains. The only way to obtain the actual configuration is to resort to the energy functional and to determine which one of the admissible configurations leads to an absolute minimum in this functional.

9.7 Weiss Domains in Uniaxial Crystals

In order to illustrate the approach presented in the above sections, we now treat a very simple case. We consider a uniaxial crystal C whose axis z corresponds to the axis of easiest magnetization. In the reference frame $Oxyz$, where z is the crystal axis, the energy per unit volume is

$$e = \frac{M_0}{2}\left\{\alpha\left[(\nabla m_x)^2 + (\nabla m_y)^2 + (\nabla m_z)^2\right] + \beta(m_x^2 + m_y^2)\right\}. \tag{9.56}$$

We further consider a rectangular specimen of such a crystal, as shown in Fig. 9.2. Since we would like to investigate the case in which there is only a pure magnetization field, we must consider equations (9.48)–(9.49) in order to construct Weiss domains in the examined crystal. We then have

$$M_0\beta m_x = -\lambda m_x, \quad M_0\beta m_x = -\lambda m_x, \quad 0 = \lambda m_z. \tag{9.57}$$

We now assume that $m_y = 0$. We then have the following admissible directions for the magnetization vector:

$$\lambda = 0, \quad m_z = \mp 1, \quad m_x = 0; \tag{9.58}$$

$$\lambda = -M_0\beta, \quad m_x = \mp 1, \quad m_z = 0. \tag{9.59}$$

When we take into account the boundary conditions

$$m_x = 0 \quad \text{on } x = 0, l_1 \quad \text{and } m_z = 0 \quad \text{on } z = 0, l, \tag{9.60}$$

we immediately see that the boundary layer vanishes in this case and that the Weiss domains have the form shown in Fig. 9.2. We can verify that all of the jump conditions at the domain walls are satisfied. Of course, the parameter d, which defines individual Weiss domains, still cannot be determined. If there are m domains, then we can only state that

$$dm = l_1. \tag{9.61}$$

We note that, essentially, we have two different Weiss domains, labeled 1 and 2. For piecewise constant fields, (9.56) yields

$$e = 0 \quad \text{in } D_1 \quad \text{and} \quad e = \frac{1}{2}M_0\beta \quad \text{in } D_2. \tag{9.62}$$

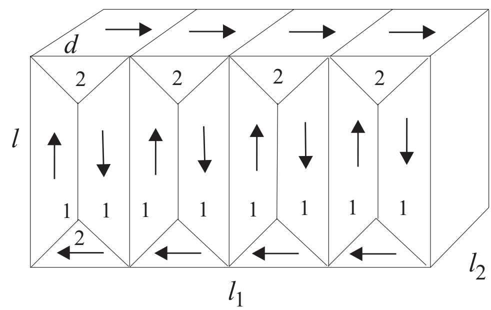

Fig. 9.2 Distribution of Weiss domains in a uniaxial crystal

The surface energies at the domains walls are of course constant, and they are given by

$$e_\sigma = e_1 \quad \text{on vertical walls;} \quad e_\sigma = e_2 \quad \text{on slanted walls.} \tag{9.63}$$

In order to determine d, we take into account the energy functional (9.10), which, due to (9.56) and the above results, can be written as

$$F^{(1)} = \left[\frac{M_0}{2}\beta\frac{d^2}{4} + e_2\sqrt{2}\,d^2 m + e_1(l-d)m + e_1 l(m-1)\right] l_2. \tag{9.64}$$

If we use (9.61) to eliminate m in the above expression, we find that

$$F^{(1)} = \left[\frac{M_0}{4}\beta l_1 d + \frac{2e_1}{d} l l_1 + 2\sqrt{2} e_2 l_1 - e_1(l_1 + l)\right] l_2. \tag{9.65}$$

The minimum of this function corresponds to

$$\frac{M_0}{4}\beta = 2\frac{e_1}{d^2} l;$$

that is, to

$$d_1 = 2\sqrt{2l}\sqrt{\frac{e_1}{M_0\beta}}. \tag{9.66}$$

Of course, we cannot expect that the expression (9.66) for d satisfies (9.61) for an integer value of m. However, in common crystals, d is very small and m is quite a large number, so (9.66) can be interpreted without any great error as the integer number closest to the ratio l/d_1.

We wish to underline that the configuration shown in Fig. 9.2 is not the unique solution to the problem posed. Indeed, the configuration shown in Fig. 9.3 also satisfies all of the boundary and jump conditions. In this case, the broken lines are not domain walls, since there is no discontinuity in $\mathbf{m}$ across them.

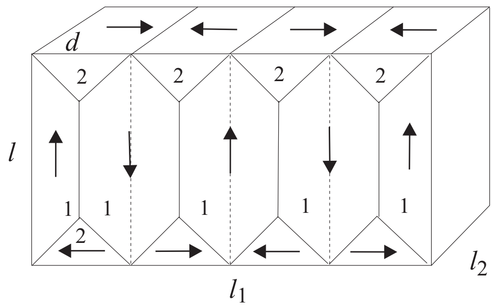

Fig. 9.3 More favorable distribution of Weiss domains in a uniaxial crystal

Hence, the total energy does not have the last term in (9.64), and instead of (9.65) we obtain the expression

$$F^{(2)} = \left[\frac{1}{4}M_0\beta l_1 d + \frac{e_1}{d}ll_1 + 2\sqrt{2}e_2 l_1 - e_1 l_1\right] l_2, \tag{9.67}$$

which is minimized for

$$d_2 = 2\sqrt{l}\sqrt{\frac{e_1}{M_0\beta}} = \frac{1}{\sqrt{2}}d_1. \tag{9.68}$$

If we evaluate the difference between the minimum values of F for the two different Weiss domain configurations, we find that

$$F^{(2)} < F^{(1)}. \tag{9.69}$$

Therefore, the configuration shown in Fig. 9.2 is more favorable.

We note that we could perform our calculations so as to include a different size for each Weiss domain, such as $d_1, \ldots, d_m$, subject to the condition $d_1 + \cdots + d_m = l_1$. It is a simple exercise to show that the absolute minimum of F corresponds to the case in which all d_i are equal to each other.

In [142], the value of the parameter d in ferromagnetic uniaxial crystals is determined by considering the energy in the very thin layers corresponding to domain walls. The value obtained is

$$d = 2\sqrt{2l}\sqrt[4]{\frac{\alpha}{\beta}}. \tag{9.70}$$

Comparing (9.70) with (9.68), we see that the surface energy density is given by

$$e_1 = 2M_0\sqrt{\alpha\beta}. \tag{9.71}$$

We conclude this section by recalling some extensions of the above results (see [157]). We have already shown that the approach used to solve the problem of Weiss domains greatly depends on whether a magnetic field is present or not. In the paper cited, it is proven that the configuration proposed by Landau and Lifshitz in [142] in the absence of magnetic fields is impossible, even in the presence of an external magnetic field. Moreover, a new configuration is shown to be admissible in the presence of a particular magnetic field. The crystals considered are still uniaxial but their geometries differ. Instead of a parallelepiped, the case where the upper and lower planar boundaries are not perpendicular to the z-axis is investigated. Finally, some approximate expressions for the surface energy of the Weiss walls and the surface of the crystal are proposed in [158], based on a numerical analysis of the equations of micromagnetism.

9.8 A Variational Principle for Elastic Ferromagnetic Crystals

Up to now, our analysis has focused on *rigid* ferromagnetic crystals. In this section we include the possibility of elastic deformations. To this end, we replace the variational principle (9.10) with the following (see [159]):

$$F = \int_D \rho M_0 e(\mathbf{m}, \nabla\mathbf{m}, \mathbf{F})\, dv + \frac{1}{2}\mu_0 \int_{\Re^3} \phi_{,i}\, \phi_{,i}\; dv - M_0 \int_D \phi_{,i}\, m_i\, dv$$
$$+ \sum_{b=1}^{q} M_0 \int_{S_b} e_\sigma(\mathbf{n}, \mathbf{m}^-, \mathbf{m}^+)\, da, \tag{9.72}$$

where ρ is the mass density, e is the specific internal energy, and $\mathbf{F}$ is the deformation gradient. We define the equilibrium configurations as the solutions of the following variational equation:

$$\delta F = \int_D \rho f_k \delta k\, dv + \int_{\partial D} t_{(\mathbf{n})k} \delta x_k\, da, \tag{9.73}$$

with the constraints

$$\mathbf{p}\cdot\mathbf{p} = 1, \quad \sum_{a=1}^{m} \int_{D_a} \rho\, dv - M = const, \tag{9.74}$$

which express, respectively, that $\mathbf{p}$ has unit magnitude and that the total mass M is conserved under all admissible variations. Equivalently, we must find the extremals of the functional

$$F_1 = F + \frac{1}{2}\rho M_0 \lambda \int_D (p_i p_i - 1)\, dv + \mu M_0 \left(\sum_{a=1}^{m} \int_D \rho\, dv - M \right) dv. \tag{9.75}$$

Here, $\rho M_0\lambda$ and μM_0 are the Lagrangian multipliers related to the constraints (9.74), and the factors M_0 and ρ are introduced for convenience.

Again using (9.8) and (9.9), and the same notation as that used in Sect. 9.4, it is possible to prove after tedious calculations (see [159]) that the extremals of (9.75) are the solutions of the following local and jump equations:

- In D

$$\rho e,_{m_i} - \pi_{ji,j} - \rho\phi_i + \rho\lambda p_i = 0, \tag{9.76}$$
$$_L t_{lk,l} - {}_M t_{lk,l} + \rho f_k = 0, \tag{9.77}$$
$$\mu_0\phi,_{ii} - M_0(\rho m_i),_i = 0, \tag{9.78}$$

- In $\Re^3 - D$

$$\phi,_{ii} = 0, \tag{9.79}$$

- On ∂D

$$({}_L t_{lk,l} - {}_M t_{lk,l}) n_i = 0, \tag{9.80}$$
$$[[\phi]] = 0, \tag{9.81}$$
$$\pi_{ij} n_i = 0, \tag{9.82}$$
$$([[\mu_0\phi,_i]] + \rho M_0 m_i) n_i = 0, \tag{9.83}$$

- On S_b

$$[[\mu_0\phi,_i - \rho M_0 m_i]]\, n_i = 0, \tag{9.84}$$
$$[[\phi]] = 0, \tag{9.85}$$
$$e_{\sigma,m_i^+} - \pi_{ji}^+ n_j = 0, \tag{9.86}$$
$$e_{\sigma,m_i^-} - \pi_{ji}^- n_j = 0, \tag{9.87}$$

- On S_b

$$e_{\sigma,m_i^+} m_{i;\alpha}^+ + e_{\sigma,m_i^-} m_{i;\alpha}^- = 0, \tag{9.88}$$

$$\left[\left[\rho M_0 e + \frac{1}{2}\mu_0\phi,_k\,\phi,_k - \rho M_0\phi,_k\, m_k + M_0\mu\rho\right]\right] + M_0(V^\alpha_{;\alpha} + b^\alpha_\alpha e_\sigma) = 0 \tag{9.89}$$

$$\left[\rho M_0 e + \frac{1}{2}\mu_0\phi,_k\,\phi,_k - \rho M_0\phi,_k\, m_k + M_0\mu\rho\right]^{+,-} + [({}_L t_{lk} - {}_M t_{lk}) n_k n_l]^{+,-} = 0, \tag{9.90}$$

$$[({}_L t_{lk} - {}_M t_{lk}) n_l x_{k\alpha}]^{+,-} = 0, \tag{9.91}$$

- On Γ_c

$$[[\mathbf{V} \times \mathbf{n} \cdot \mathbf{t})\mathbf{n} - e_\sigma \mathbf{n} \times \mathbf{t}]] = \mathbf{0}. \tag{9.92}$$

In the above equations we have introduced the following notation:

$$\pi_{ji} = \rho \frac{\partial e}{\partial m_{i,j}}, \tag{9.93}$$

$$_L t_{ki} = \rho M_0 \frac{\partial e}{\partial F_{lK}} F_{lK}, \tag{9.94}$$

$$\begin{aligned} _M t_{ji} = {} & \mu_0 \left(\phi_{,i} \, \phi_{,j} - \frac{1}{2} \phi_{,k} \, \phi_k \delta_{ij} \right) \\ & - \rho M_0 m_j \phi_i + M_0 \pi_{jk} m_{k,i}. \end{aligned} \tag{9.95}$$

In the literature, $_M\mathbf{t}$ and $_L\mathbf{t}$ are called, respectively, the *generalized Maxwell stress tensor* and the *local stress tensor*. Finally, if $\mathbf{a}_\alpha$, $\alpha = 1, 2$, are the vectors of the natural basis associated with the coordinates u^α on S_b, then $x_{i\alpha}$ denote the components of $\mathbf{a}_\alpha$ with respect to the Cartesian basis $\mathbf{u}_i$.

9.9 Weiss Domains in Elastic Uniaxial Crystals

We are interested in studying Weiss domains with planar walls. In rigid ferroelectric crystals, we have proved that this situation is realized when $\mathbf{m}$ is uniform in each domain D_a and the magnetic potential is continuous across the domain walls. In an elastic ferromagnetic crystal the internal energy e also depends on the deformation gradient $\mathbf{F}$, meaning that, if we wish to retain planar domain walls, we must assume that:

- There is no body force f_i

- The deformation gradient is constant in each domain D_a:

$$\mathbf{F} = \mathbf{F}_a \quad \text{in } D_a. \tag{9.96}$$

Since the mass density ρ in D_a is

$$\rho = \frac{\rho_\star}{|\det \mathbf{F}|}, \tag{9.97}$$

where ρ_0 is the constant mass density in the unstressed reference configuration, we see that ρ is constant in each D_a. However, $\mathbf{m}$ is also constant in D_a, so the governing equations reduce to

- In D_a

$$e_{,m_i} - \phi_{,i} + \lambda m_i = 0, \tag{9.98}$$

$$\phi_{,ii} = 0, \tag{9.99}$$

- On S_b

$$[[\mu_0 \phi_{,i} - \rho M_0 m_i]]\, n_i = 0. \tag{9.100}$$

As shown in Sect. 9.5, this set of equations leads to a magnetic potential given by

$$\phi_a = \left[\lambda_a m_i^{(a)} + \frac{\partial e}{\partial p_i^{(a)}} \right] x_i + b_a, \tag{9.101}$$

where λ_a is the Lagrangian multiplier related to D_a and b_a is an arbitrary constant. Again, by employing the continuity of ϕ across the domain walls, we can prove that the domain walls remain planar in the presence of a homogeneous deformation. Since a Weiss domain has at least one microscopic dimension, and the deformation experienced by the crystal is usually very small, we can reasonably approximate a nonhomogeneous deformation in the crystal by a piecewise one. In this way, the results obtained will hold for nonhomogeneous deformations.

As we saw in Sect. 9.5, we can prove that e_σ is constant on any domain wall for which

$$[[\mu_0 \phi_{,i} - \rho M_0 m_i]] = 0, \tag{9.102}$$

$$[[\phi]] = 0, \tag{9.103}$$

$$\left[\left[\rho M_0 e + \frac{1}{2} \mu_0 \phi_{,k}\, \phi_{,k} - \rho M_0 \phi_{,k}\, m_k + M_0 \mu \rho \right]\right] = 0, \tag{9.104}$$

$$\left[\rho M_0 e + \frac{1}{2} \phi_{,k}\, \phi_{,k}\, m_k + M_0 \mu \rho \right]^{+,-} - \left[({}_L t_{lk} - {}_M t_{lk}) n_k n_l \right]^{+,-} = 0, \tag{9.105}$$

$$\left[({}_L t_{lk} - {}_M t_{lk}) n_l x_{k\alpha} \right]^{+,-} = 0. \tag{9.106}$$

Moreover, the boundary conditions on ∂D become

$$({}_L t_{ij} - {}_M t_{ij}) n_j = t_{(\mathbf{n})i}, \tag{9.107}$$

$$[[\phi]] = 0, \tag{9.108}$$

$$([[\mu_0 \phi_{,i}\,]] + \rho \mu_0 m_i) n_i = 0. \tag{9.109}$$

By taking the difference between of (9.106) and using (9.105), we arrive at the following equivalent conditions on S_b:

$$[[\mu_0 \phi_{,i} - \rho M_0 m_i]] = 0, \tag{9.110}$$

$$[[\phi]] = 0, \tag{9.111}$$

$$\left[\left[\rho M_0 e + \frac{1}{2}\mu_0 \phi,_k \phi,_k - \rho M_0 \phi,_k m_k + M_0 \mu \rho\right]\right] = 0, \tag{9.112}$$

$$[[{}_L t_{lk} - {}_M t_{lk}]]\, n_l n_k = 0 \tag{9.113}$$

$$({}_L t_{lk} - {}_M t_{lk})^{+,-} n_l x_{k,\alpha} = 0, \tag{9.114}$$

$$\begin{aligned} -M_0 \rho^+ e^+ + \rho^+ M_0 \phi,_k^+ m_k^+ + ({}_L t_{lk}^+ - {}_M t_{lk}^+) n_k n_l \\ -\frac{1}{2}\mu_0 \phi,_k^+ \phi_k^+ - M_0 \rho^+ \mu^+ = 0. \end{aligned} \tag{9.115}$$

Also, in this new situation, all of the equations we will use to determine a solution are nonlinear. Consequently, a solution may not exist, or many solutions may exist and we need to choose the most energetically favorable. In the absence of deformation, we have already remarked that if the boundary ∂D is not planar we are compelled to introduce a boundary layer adjacent to ∂D.

9.10 A Possible Weiss Domain Distribution in Elastic Uniaxial Crystals

To illustrate the general approach we have developed so far, we consider a uniaxial, transversally isotropic ferromagnetic crystal that is rectangular in shape and extends to infinity in the y-direction. The crystal is assumed to be subjected to uniform tractions along its boundaries. The energy of the crystal is assumed to be given by the relation

$$\begin{aligned} \rho_0 M_0 e = \frac{1}{2} M_0 \left\{\alpha \left[(p_{x,x})^2 + (p_{x,z})^2 + (p_{z,x})^2 + (p_{z,z})^2\right] + \beta p_x^2\right\} \\ + \frac{1}{2}(c_1 e_{xx}^2 + c_2 e_{zz}^2 + 2c_3 e_{xz}^2 + 2c_4 e_{xx} e_{zz}) \end{aligned} \tag{9.116}$$

for small deformations. Here e_{ij} are the components of the infinitesimal strain tensor, and we have assumed that the deformation is planar ($e_{yy} = e_{yz} = e_{yx} = 0$) and the magnetization vector $\mathbf{m}$ is in the xz-plane. The form of the strain energy for transversely isotropic bodies can be found, for instance, in [141]. Finally, ρ_0 denotes the constant mass density in the unstressed reference configuration.

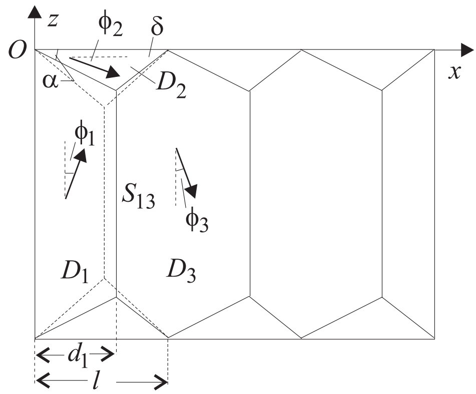

Fig. 9.4 Changes in the Weiss domains in a loaded uniaxial crystal

Now we intend to analyze the change in the Weiss domain distribution, analyzed in Sect. 9.6, due to a homogeneous deformation and the induced magnetic field. If the crystal is rigid, we have shown that the distribution shown in Fig. 9.3 and (by dashed lines) in Fig. 9.4 is possible. We now assume that the crystal undergoes a small deformation due to suitable uniform tractions on its boundaries. Because of the symmetry of the Weiss domain distribution, it is sufficient to consider only the domains D_1, D_2 and D_3 (see Fig. 9.4) in order to elucidate basic properties of the fields. After many involved calculations, it is shown in [159] that all of the equations and jump and boundary conditions in the previous section are satisfied by the configuration shown in Fig. 9.4. In other words, in any domain D_a, $a = 1, 2, 3$, the magnetization rotates in the Oxy-plane by a small angle ϕ_a. Moreover, a magnetic field is produced that is parallel to the Ox-axis. Surfaces like S_{13} undergo a small translation along the Ox-axis, whereas surfaces S_{12} and S_{23} rotate by small angles, so that $\alpha, \delta < \pi/4$. Finally, the whole configuration requires the application of suitable uniform tractions at the boundary. It is this particular choice of applied tractions which, once again, forces us to conjecture the existence of a boundary layer at the boundary.

9.11 A More General Variational Principle

We have reiterated many times that the volume of a rigid ferromagnetic crystal C is the union of a number of subregions (the Weiss domains) in which the magnetization $\mathbf{M} = M_0\mathbf{m}$ is constant. Two adjacent domains are separated by a very thin layer, the domain wall, across which the unit vector $\mathbf{m}$ undergoes sharp variations. A similar but generally thicker layer appears near the crystal boundary. We have already noted in Sect. 9.1 that, in order to justify this complex structure inside the crystal, the micromagnetism assumes that the equilibrium configurations are extremals of the energy functional (9.2). In principle, this approach leads to the existence of boundary layers due to the presence of the second derivatives in (9.3), which are multiplied by the very small parameter α. However, the boundary value problem we are faced with is very difficult to solve.

In the model of a continuum with an interface presented in the above sections, the domain walls are replaced with interfaces across which the magnetic potential is continuous but the magnetization exhibits finite discontinuities. However, the magnetization gradient still appears in the constitutive equations in order to describe the behavior of the fields ϕ and $\mathbf{m}$ near the surface of the crystal. It is evident that this circumstance does not agree with the main goal of the model of a continuum with an interface, which involves replacing the layers with surfaces of discontinuity. Therefore, we will analyze the consequences of replacing *both* the domain walls *and* the boundary layer on the crystal boundary ∂D with interfaces across which ϕ can also be discontinuous (see [160]). Moreover, we assume different constitutive equations for the surface energy of the domain walls and the boundary ∂D in order to take into account the physical differences between these two types of layers. This new model is then applied to analyze Kittel's structure [144].

As usual, we assume that the ferromagnetic crystal D is the union of the regions D_n, $n = 1, \ldots, m$, representing the Weiss domains. The magnetization field takes the form $\mathbf{M} = M_0\mathbf{m}$, where M_0 is constant and the unit vector $\mathbf{m}$ exhibits discontinuities across *both* the domain walls S_b, $b = 1, \ldots, q$, and the boundary ∂D of the crystal. Finally, we denote the union of all of the domain walls by

$$\Sigma = \bigcup_{b=1}^{q} S_b.$$

For a rigid crystal, we assume that the functional of the total energy, in a Gaussian CGS system, has the form

$$\begin{aligned} F(\mathbf{m}, \phi, \mathbf{k}) = & \int_D e(\mathbf{m})\, dv - \frac{1}{8\pi} \int_{\Re^3} \phi_{,i}\, \phi_{,i}\, dv \\ & + M_0 \int_D m_{,i}\, \phi_{,i}\, dv + \sum_{b=1}^{q} \int_{S_b} e_\sigma(\mathbf{m}^+, \mathbf{m}^-, \tau, \mathbf{n})\, da \\ & + \int_{\partial D} \hat{e}_\sigma(\mathbf{m}^+, \mathbf{m}^-, \tau, \mathbf{N}, \nabla_\sigma \phi^+, \nabla_\sigma \phi^+)\, da, \end{aligned} \qquad (9.117)$$

where $\hat{e}_\sigma$ is the energy per unit area of ∂D, $\nabla_\sigma \phi$ is the surface gradient of ϕ, $\mathbf{n}$ is the unit vector orthogonal to S_b, $\mathbf{N}$ is the unit vector orthogonal to ∂D, and $\tau = [[\phi]]$. Moreover, $\mathbf{k}(u^\alpha)$, $\alpha = 1, 2$, is the equation of the surface $\Sigma \cup \partial D$ in terms of the surface parameters u^α. The rest of the notation is the same as that we used in Sect. 9.8.

The fields ϕ, $\mathbf{m}$ and $\mathbf{k}$, which characterize the equilibrium configurations of the crystal, are extremals of the functional (9.17) under the constraints

$$\mathbf{m} \cdot \mathbf{m} = 1 \text{ in } D, \tag{9.118}$$

$$\mathbf{k}(u^\alpha) \qquad \text{is assigned on } \partial D. \tag{9.119}$$

The calculations needed to evaluate the first variation of the functional (9.117) are very tedious, and we refer the reader to [160] for further details. Here, we limit ourselves to quoting the final results, which are collected together in the following equations:

In D_n, $n = 1, \ldots, m$,

$$e,_{m_i} + \lambda m_i + M_0 \phi,_i = 0, \tag{9.120}$$

$$-\phi,_{ii} + 4\pi M_0 m_{i,i} = 0. \tag{9.121}$$

In $\Re^3 - D$,

$$\phi,_{ii} = 0. \tag{9.122}$$

On ∂D,

$$4\pi[\hat{e}_{\sigma,\tau} - (\hat{e}_{\sigma,\phi,^+_\alpha})_{;\alpha}] = (-\phi,_i)^+ N_i, \tag{9.123}$$

$$4\pi[\hat{e}_{\sigma,\tau} - (\hat{e}_{\sigma,\phi,^-_\alpha})_{;\alpha}] = (-\phi,_i + 4\pi M_0 m_i)^- N_i, \tag{9.124}$$

$$\hat{e}_{\sigma,m_i^-} + \lambda^- m_i^- = 0. \tag{9.125}$$

On Σ,

$$4\pi_{\sigma,\tau} = (-\phi,_i + 4\pi M_0 m_i)^+ n_i, \tag{9.126}$$

$$4\pi_{\sigma,\tau} = (-\phi,_i + 4\pi M_0 m_i)^- n_i, \tag{9.127}$$

$$e_{\sigma,m_i^\mp} + \lambda^\mp m_i^\mp = 0, \tag{9.128}$$

$$\left[\left[e - \frac{1}{8\pi}\phi,_i \phi,_i + M_0 m_i \phi_i\right]\right] + e_\sigma b_{\alpha\alpha} - \Pi^\alpha_{;\alpha} = 0. \tag{9.129}$$

On Γ_p, $p = 1, \ldots, r$,

$$\sum_{b=1}^{s(p)} [e_{\sigma_b} \nu_b - (\mathbf{\Pi}_b \cdot \nu_b)\mathbf{n}_b] = \mathbf{0}. \tag{9.130}$$

In (9.129) and (9.130), Γ_p is the edge formed by $s(p)$ domain walls S_b that intersect with each other, ν is the unit vector tangent to the wall S_b along Γ_p, $b_{\alpha\alpha}$ is the mean curvature of Σ, and

$$\Pi^\alpha = a^{\alpha\beta} k_{i,\beta} e_{\sigma,n_i}, \tag{9.131}$$

where $a^{\alpha\beta}$ are the reciprocal metric coefficients on the surface Σ in the coordinates (u^α). Finally, using (9.123), (9.124), (9.123) and (9.124), we have

$$[[-\phi,_i + 4\pi M_0 m_i]]\, n_i = 0 \quad \text{on}\, \Sigma, \tag{9.132}$$

$$[[-\phi,_i + 4\pi M_0 m_i]]\, N_i = -4\pi[(\hat{e}_{\sigma,\phi,^+_\alpha})_{;\alpha} + (\hat{e}_{\sigma,\phi,^-_\alpha})_{;\alpha}, \;\; \text{on}\, \partial D. \tag{9.133}$$

Now we wish to apply the above equations to a planar slab S (of thickness L) of a uniaxial crystal that has the z-axis of easiest magnetization orthogonal to the faces of S (see Fig. 9.5).

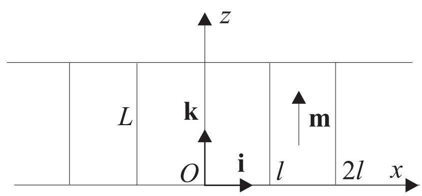

Fig. 9.5 Kittel's structure

Kittel proved (see [144, 145, 152]) that the distribution shown in Fig. 9.5 is possible if the crystal is highly anisotropic and no external magnetic field is applied. This result is derived by assuming that:

- The magnetization is uniform up to the crystal boundary ∂S
- The magnetic potential ϕ satisfies the Laplace equation
- ϕ is continuous across ∂S, but its normal derivatives on ∂S are discontinuous and differ in sign.

These hypotheses are not derived from micromagnetism; they are justified by assuming that the magnetic field near ∂S is equivalent to the magnetic field produced by an alternating magnetic mass distribution with a surface density $\mp M_0$.

In the reference frame of Fig. 9.5, the magnetic anisotropy energy of a uniaxial ferromagnetic crystal can be written as

$$e(\mathbf{m}) = \frac{1}{2}\beta M_0^2 (m_x^2 + m_y^2), \tag{9.134}$$

where β is the anisotropy constant. On the other hand, in our configuration, we have $m_x = m_y = 0$ and then $e(\mathbf{m}) = 0$. From (9.120) we obtain the following potential in each domain D_n:

$$\phi^{(n)} = -\frac{\lambda^{(n)}}{M_0} m_z^{(n)} z + c^{(n)}, \tag{9.135}$$

where $\lambda^{(n)}$, $c^{(n)}$ are constants and $m_z = \mp 1$. By imposing potential continuity across the domain walls, we find that

$$\lambda^{(1)} = -\lambda^{(2)} = \ldots \equiv \lambda, \quad c^{(1)} = c^{(2)} = \ldots \equiv c,$$

so that

$$\phi^{(n)} = -\frac{\lambda}{M_0} z + c. \tag{9.136}$$

Due to the symmetries of the configuration, all of the fields that characterize the equilibrium configuration must be invariant under the coordinate transformations $x' = -x$, $z' = -z + L$, so that $\lambda = 0$. However, the potential is defined up to an arbitrary constant, and then

$$\phi^{(n)} = 0, \text{ in } D_n, \quad \phi^- = 0 \text{ on } \partial D. \tag{9.137}$$

Moreover, from (9.124) and (9.137) it follows that

$$\hat{e}_\sigma = \pm M_0 \phi^+ + \widetilde{e}_\sigma(\mathbf{m}^-, \mathbf{N}, \phi,_x^+).$$

Based on the analysis performed in [158], we assume that the above relation can be written in the form

$$\hat{e}_\sigma = \pm M_0 \phi^+ + \frac{l}{16\pi} (\phi,_x^+)^2, \text{ on } \partial D. \tag{9.138}$$

In [160] it is proven that, if we take into account all of the above results, the total energy functional (9.117) assumes the form

$$F(\mathbf{m}, \phi, \mathbf{k}) = \int_\Sigma e_\sigma(\mathbf{m}^+, \mathbf{m}^-, \mathbf{m})\, d\sigma \pm \frac{1}{2} M_0 \int_{\partial D} \phi^+ \, d\sigma. \tag{9.139}$$

Therefore, to evaluate the total energy of Kittel's configuration, we must determine the magnetic potential outside the crystal S. The solution $\phi(x, z)$ we are looking for is periodic with respect to x (period: $2l$), and, because of (9.133), it must obey the boundary condition

$$-(\mathbf{N} \cdot \nabla\phi)^+ = 4\pi M_0 (\mathbf{m} \cdot \mathbf{N})^- - 4\pi (\hat{e}_{\sigma,\phi,_x^+}),_x \,, \text{ on } \partial D.$$

In conclusion, taking into account (9.138), the magnetic potential ϕ must be a solution of the following boundary value problem:

$$\phi_{,xx} + \phi_{,zz} = 0, \text{ in } \Re^3 - D, \tag{9.140}$$

$$\phi_{,z}(x, L) + \frac{l}{2}\phi_{,xx}(x, L) = \begin{cases} -4\pi M_0, & 0 \leq x \leq l, \\ 4\pi M_0, & l \leq x \leq 2l, \end{cases} \tag{9.141}$$

$$\phi_{,z}(x, 0) - \frac{l}{2}\phi_{,xx}(x, 0) = \begin{cases} -4\pi M_0, & 0 \leq x \leq l, \\ 4\pi M_0, & l \leq x \leq 2l, \end{cases} \tag{9.142}$$

$$\lim_{z \to \mp\infty} \phi(x, z) = 0. \tag{9.143}$$

In [160], using elementary methods, the solution ϕ of the above boundary value problem is determined for $z > L$ and $z < 0$. In particular, we have:

$$\phi(x, L) = \frac{32 M_0 l}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2(2+(2n+1)\pi)} \sin\frac{(2n+1)\pi}{l} x,$$

$$\phi(x, 0) = -\frac{32 M_0 l}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2(2+(2n+1)\pi)} \sin\frac{(2n+1)\pi}{l} x.$$

Substituting these expressions into (9.139), we determine the total energy of the configuration as a function of l:

$$F(l) = e_\sigma \frac{L}{l} + 12.7 \frac{M_0^2 l}{\pi^2}. \tag{9.144}$$

In order to determine the equilibrium configuration, we find the value of l that minimizes the total energy (9.144)

$$l = \frac{\pi}{M_0}\sqrt{L\frac{e_\sigma}{12.7}}. \tag{9.145}$$

Finally, we need to derive the value of the surface energy on the domain walls if we want to compare the value of (9.145) with that determined by Kittel. To do this, it is sufficient to use the expression for e_σ we found in [158]:

$$e_\sigma = 2\gamma M_0^2 \sqrt{\alpha\beta}, \tag{9.146}$$

where α is the inhomogeneity constant and $\gamma = 1.18$. Finally, introducing (9.146) into (9.145), we obtain the requested Weiss domain width:

$$l = 0.9\sqrt{2L\sqrt{\alpha\beta}}, \tag{9.147}$$

which is in good agreement with Kittel's result (see [144])

$$l = 0.8\sqrt{2L\sqrt{\alpha\beta}}.$$

9.12 Weiss Domain Branching

In this section we analyze an interesting phenomenon that occurs in ferromagnetic crystals: the *branching of Weiss domains* (see [144, 145, 152]).

Let D be a planar unbounded slab of a uniaxial crystal where the easy magnetization axis is orthogonal to the face of S (see Fig. 9.6). In the absence of external magnetic fields, and when the crystal is strongly anisotropic, different equilibrium configurations can occur. An equilibrium configuration with a single domain in which the magnetization field is parallel to the axis of easy magnetization is possible provided that the thickness L of S is very small. In this case, the energy F_s present in the length L is given by the formula

$$F_s = 2\pi M_0^2 L. \tag{9.148}$$

As L increases, the Kittel configuration (see Fig. 9.6) with alternating domains becomes more favorable. The magnetic energy present in the length L is (see [144])

$$F_k = 3.5 M_0^2 \sqrt{L\sqrt{\alpha\beta}}, \tag{9.149}$$

where α is the inhomogeneity constant. In fact, for larger values of L, the values of F_s are greater than the corresponding values of F_k. By comparing (9.148) and (9.149), we obtain the critical value L_c of L, which marks the passage from one configuration to the other:

$$L_c = 0.3\sqrt{\alpha\beta}. \tag{9.150}$$

Nevertheless, at sufficiently large values of L, this configuration also becomes unstable, and a branching phenomenon occurs in the crystal: large wedges that penetrate deep into the crystal appear and are accompanied by many other smaller wedges (see Fig. 9.6). In [161] it is proven that the following expression for the energy present in the length L holds for this more complex domain distribution:

$$F_b = 3\sqrt[3]{M_0^2 \mu \beta_\sigma^2 L}, \tag{9.151}$$

where the energy per unit area e_σ of the slanted walls of the wedges is (see 9.146)

$$e_\sigma = 2M_0^2 \gamma \sqrt{\alpha\beta}, \tag{9.152}$$

and γ is a constant that depends on the crystal. By comparing (9.150) and (9.151), we obtain the second critical value L_b of the thickness of S, which corresponds to the branching

$$L_b = 100\gamma^4 \mu^2 \beta^2 \sqrt{\alpha\beta}. \tag{9.153}$$

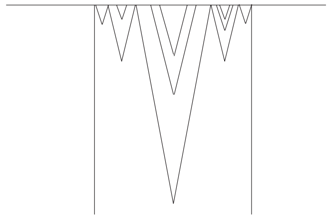

Fig. 9.6 Branching

We only sketch out the procedure followed in [161] to prove (9.153) here. Starting from the variational principle, which we explored in the above section, the authors assume that:

- The magnetization is uniform in the largest wedges
- The effect of the smallest wedges is described by the structural properties of ∂D.

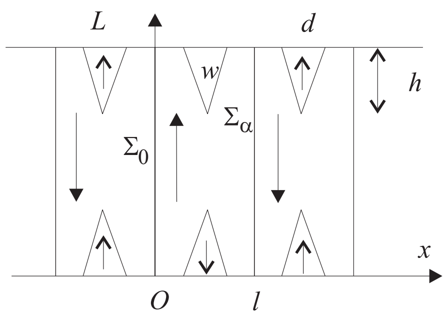

Fig. 9.7 Simplified branching in a slab of uniaxial crystal

Let w be the union of the wedge-shaped domains, Σ_α be the union of the planar oblique interfaces between w and the Kittel domains, and Σ_0 be the union of the planar interfaces that separate the Kittel domains (see Fig. 9.7). The first assumption made above simplifies the description of the largest domains, but it also introduces a new difficulty. Since the magnetization inside w is directed along the Oz-axis, $M_0\mathbf{m} \cdot \mathbf{n} \neq 0$, so fictitious magnetic poles appear on Σ_α, and the energy associated with these poles must be considered.

After some calculations, the energy $F(h, l, L)$ of the configuration is determined as a function of the height of the wedges, the thickness of the

Kittel domains, and the thickness L of the slab S. The values of l and h corresponding to a stable configuration for a given L are then obtained by minimizing the energy $F(h, l, L)$ with respect to these variables. For magnetoplumbite, the critical value of L is found to be about 1 cm, which is in good agreement with experimental results.

9.13 Weiss Domains in an Applied Magnetic Field

In this section, the variational principle proposed in Sect. 9.11 is applied to describe how a uniform external magnetic field $\mathbb{H}$ modifies the Weiss domains of a Kittel configuration (see Fig. 9.5) in a uniaxial crystal D. Considering the results presented in the above section, the thickness L of the slab must lie between the values given by (9.150) and (9.153). In [162] it is proven that the variational principle and the uniformity of the magnetic field in each Weiss domain imply that the following jump conditions hold at each domain wall:

$$[[\phi]] = 0, \quad [[\mathbf{H}]] = \mathbf{0}. \tag{9.154}$$

Consequently, the magnetic potential ϕ and the total magnetic field $\mathbf{H}$ are uniform across the whole volume of the crystal D. While the application of an external magnetic field could vary the thickness of the Weiss domains, it is clear that the distribution will remain periodic and that the period will be still represented by two adjacent domains that could now have different thicknesses, l_1 and l_2.

For the reasons we described in the above sections, the authors then chose the following constitutive equations:

$$e(\mathbf{m}) = \frac{1}{2}\beta M_0^2 (m_x^2 + m_y^2), \tag{9.155}$$

$$e_\sigma(\mathbf{m}^+, \mathbf{m}^-, \mathbf{n}, \phi^+, \phi^-) = \widetilde{e}_\sigma(\mathbf{m}^+, \mathbf{m}^-, \mathbf{n}), \tag{9.156}$$

$$\begin{aligned} \hat{e}_\sigma(\mathbf{m}^+, \mathbf{m}^-, \mathbf{n}, \phi^+, \phi^-, \phi^+{}_{,x}) = {} & M_0 \mathbf{m}^- \cdot \mathbf{N}\,[[\phi]] \\ & - \frac{l_1 + l_2}{16\pi}(\phi^+{}_{,x})^2. \end{aligned} \tag{9.157}$$

Starting from these constitutive equations as well as (9.154) and the variational principle, the following relations are proven in [162]:

$$m_x^{(1)} = m_x^{(2)} = \sin\theta, \tag{9.158}$$

$$m_z^{(1)} = m_z^{(2)} = \cos\theta, \tag{9.159}$$

$$H_x^{(1)} = H_x^{(2)} = H_x = \beta M_0 \sin\theta, \tag{9.160}$$

$$H_z^{(1)} = H_z^{(2)} = H_z = 0, \tag{9.161}$$

where θ is the angle between $\mathbf{m}$ and the Oz-axis, and the index (i) refers to two adjacent domains. Moreover, the magnetic potential ϕ in the region $\Re^3 - D$ is a solution of the Laplace equation

$$\phi,_{xx} + \phi,_{yy} = 0, \tag{9.162}$$

which satisfies the following boundary conditions at the faces of the slab:

- $0 \leq x \leq l_1$

$$-\phi,_z(x, L) - \frac{l_1 + l_2}{2\pi}\phi,_{xx}(x, L) = 4\pi M_0 \cos\theta, \tag{9.163}$$

- $l_1 \leq x \leq l_1 + l_2$

$$-\phi,_z(x, L) - \frac{l_1 + l_2}{2\pi}\phi,_{xx}(x, L) = -4\pi M_0 \cos\theta, \tag{9.164}$$

- $0 \leq x \leq l_1$

$$\phi,_z(x, 0) - \frac{l_1 + l_2}{2\pi}\phi,_{xx}(x, 0) = -4\pi M_0 \cos\theta, \tag{9.165}$$

- $l_1 \leq x \leq l_1 + l_2$

$$-\phi,_z(x, L) - \frac{l_1 + l_2}{2\pi}\phi,_{xx}(x, L) = 4\pi M_0 \cos\theta; \tag{9.166}$$

the asymptotic conditions

$$\lim_{z \to \mp\infty} (-\phi,_z) = \mathbb{H}_z, \tag{9.167}$$

$$\lim_{z \to \mp\infty} (-\phi,_x) = \mathbb{H}_x; \tag{9.168}$$

as well as the periodic conditions

$$\phi(0, z) - (l_1 + l_2)\mathbb{H}_s = \phi(l_1 + l_2, z), \tag{9.169}$$

$$\phi(0, 0) = 0, \quad \phi(0, L) = -\mathbb{H}_z L. \tag{9.170}$$

The solution to this boundary value problem is found in [162], where, in particular, it is proven that

$$l_1 = \frac{1 + \gamma}{1 - \gamma}\, l_2, \tag{9.171}$$

where

$$\gamma = \frac{\mathbb{H}_z}{4\pi M_0 \cos\theta}. \tag{9.172}$$

To complete the analysis, all of the above results are inserted into (9.155)–(9.157). The resulting expressions would then be substituted into the variational principle to determine the total energy of the configuration, which, in view of (9.171) and (9.172), is a function of l_2 and θ. The values of these quantities could be determined by requiring that the first derivatives of the total energy with respect to these variables vanish. Unfortunately, these calculations are too difficult to carry out. Therefore, a shortcut is adopted. It is assumed that the *average* values of ϕ, $\mathbf{H}$ and $\mathbf{B} = \mu_0 \mathbf{H} + M_0 \mathbf{m}$ in the part $0 \leq l_1 + l_2$, $z = L$, of the boundary ∂D obey the standard condition

$$\mathbf{N} \times [[\mathbf{H}]] = \mathbf{0}.$$

It is then easily proven that (see [162])

$$\sin\theta = \frac{\mathbb{H}_x}{\beta M_0}. \tag{9.173}$$

This relation, together with (9.171) and (9.172), lead to the following results:

- If $\mathbb{H}_x = 0$ then $\theta = 0$ and $l_1 > l_2$ (see Fig. 9.8)
- If $\mathbb{H}_z = 0$ then $\theta \neq 0$ and $l_1 = l_2$ (see Fig. 9.9).

  The total energies of both configurations are determined in [162] as functions of l_1, as well as the magnetization curves for the slab.

  We conclude this chapter by recalling another application (see [163]) of the general variational principle presented in Sect. 9.11 and applied in subsequent sections.

  In [144] and [152] it is proven that the configuration shown in Fig. 9.5 is also possible in the absence of magnetic fields in a *cubic* crystal, since all of the boundary conditions and field equations that derive from the variational principle (9.10) are obeyed. Moreover, if a, b and L are the dimensions of the parallelepiped, the total energy E_T of the crystal is given by the formula

$$\frac{E_T}{ab} = \left(\frac{L}{l} - 1\right) e_\sigma^{(\pi)} + 2\sqrt{2} e_\sigma^{(\pi/2)}, \tag{9.174}$$

  where l is the width of the vertical domains, and $e_\sigma^{(\pi)}$ and $e_\sigma^{(\pi/2)}$ are the energies per unit area of the vertical and slanted walls, respectively. However, this configuration cannot be accepted even though it is observed experimentally, since there is no value of l for which the total energy has a minimum. In [163] it is proven that this configuration becomes possible when it is assumed that the crystal is not rigid. The boundary equations and boundary conditions that follow from the variational principle when crystal deformability is introduced are too complex to analyze. For this reason, a perturbation method is proposed in [163] that allows us to perform an approximate analysis of these equations. This makes it possible to verify that the pyramidal domains are slightly deformed and that the total energy is minimized at equilibrium.

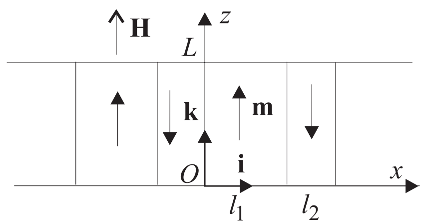

Fig. 9.8 Kittel's domains in the presence of a magnetic field orthogonal to the faces of the slab

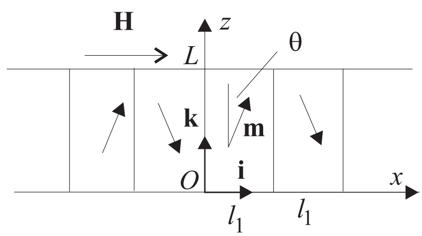

Fig. 9.9 Kittel's domains in the presence of a magnetic field parallel to the faces of the slab

Chapter 10

Relativistic Continuous Systems

10.1 Lorentz Transformations

In this section, for the reader's convenience, we briefly recall the physical foundations upon which special relativity is built. This introduction will be useful when we present relativistic continuum mechanics.

The wave character of the propagation of light was established during the eighteenth century, when scientists were convinced that all physical phenomena could be described by mechanical models. Consequently, it appeared natural to the researchers of that time to assume that empty space is filled with an isotropic and transparent medium, the *ether*, which supports light waves. This hypothesis seemed to be confirmed by the fact that forces acting on charges and currents could be evaluated by assuming that electromagnetic fields generate a deformation state in the ether, which is described by the Maxwell stress tensor. However, this description was found to be unacceptable for the following reasons:

- The high value of the speed of light required a very high ether density
- Such a high density implied the existence of both longitudinal and transverse waves, whereas Maxwell's equations only showed the transverse character of electromagnetic waves.

Another problem with the hypothesis that the ether was a material medium was the fact that the Maxwell equations were not covariant (i.e., invariant in form) under Galilean transformations. Consequently, electromagnetic phenomena did not obey the principle of relativity: they were assumed to hold only in *one* frame of reference, the *optically isotropic frame*. All attempts to localize this frame of reference (by Michelson, Morley, Kennedy, Fitzgerald, etc.) had failed, so researchers were faced with a significant physical inconsistency.

Einstein provided a brilliant and revolutionary solution to the above problem by accepting the existence of the optically isotropic system and

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exploiting the consequences of this assumption. More precisely, he postulated that

There is at least one optically isotropic reference frame.

Using the above postulate, it is possible to find a reference frame I in which light propagates in empty space at a constant speed and in a straight line in any direction. In particular, in this frame I, it is possible to define a global time t by choosing an arbitrary value c for the speed of light in empty space. In fact, the time measured by a clock located at a point $O \in I$ will be accepted if a light signal sent from O at $t = t_0$ to an arbitrary point P, where it is reflected by a mirror back toward O, returns to O at the instant $t = t_0 + 2OP/c$ for any t_0. Let us suppose that a set of identical clocks are distributed at different points in space. A clock at any point P can be synchronized with the clock located at the fixed point O by sending a light signal from O at the instant t_0 and imposing that it arrives at P at the instant $t = t_0 + OP/c$. (We will not prove that the time variable defined by the above procedure is independent of the initial point O here.)

If we accept Galilean transformations in passing from one reference frame I to another, I', which is moving at a constant velocity $\mathbf{u}$ with respect to I, then there is, at most, one isotropic optical frame. On the other hand, if we drop the assumptions upon which Galilean transformations are based, then it is possible to prove (see p. 403 of [164]) that there are an infinite number of optically isotropic reference frames. More precisely, if (x^1, x^2, x^3, t) and (x'^1, x'^2, x'^3, t') are the spatial and temporal coordinates associated with the same event by two observers I and I', then the *finite* relations between these coordinates are expressed by the following *Lorentz transformations*:

$$x'^i = x'^i_O + Q^i_j \left(\delta^j_h + (\gamma - 1) \frac{u^j}{u^2} u_h \right) x^h - Q^i_j \gamma u^j t, \tag{10.1}$$

$$t' = t_0 - \gamma \frac{u_h x^h}{c^2} + \gamma t, \tag{10.2}$$

where x^i_O, t_0, and u^i are constant, Q^i_j are the coefficients of a constant orthogonal matrix, $u = |\mathbf{u}|$, and

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{u}{c}\right)^2}} \equiv \frac{1}{\sqrt{1 - \beta^2}}. \tag{10.3}$$

Remark It is not an easy task to prove (10.1) and (10.2) starting from the above postulate. However, if we add the hypothesis that the required transformation is linear, then it becomes very simple to derive these Lorentz transformations, as shown in most books on special relativity. *This assumption is equivalent to requiring that the optically isotropic frames are also inertial frames.*

Remark Let us consider any event which has constant spacetime coordinates in the frame I'. Then, differentiating (10.1), we obtain

$$\left(\delta^i_j + (\gamma - 1)\,\frac{u^i}{u^2}u_j\right)\frac{dx^j}{dt} = \gamma u^i. \tag{10.4}$$

It is an easy exercise to verify that the above system allows the following solution:

$$\frac{dx^i}{dt} = u^i, \tag{10.5}$$

so we can say that any point in I' moves with a constant velocity (u^i) with respect to I. Therefore, the relative motion of I' with respect to I is a uniform translatory motion with velocity (u^i). Similarly, we can prove that any point in I moves with a constant velocity $-\mathbf{u}$ with respect to I'. In particular, x'^i_O denotes the coordinates of the origin of I with respect to I' when $t = 0$.

Remark In the limit $c \to \infty$, the Lorentz transformations reduce to the Galilean ones.

We will not discuss the well-known consequences of the Lorentz transformations here (such the contraction of moving lengths, the retardation of moving clocks, stellar aberration, Fizeau's formula for the velocity of light in an optical medium, the transverse Doppler effect, etc.). We will limit ourselves to recalling an important result about the cause–effect relation that derives from Lorentz transformations. Let us suppose that an event that occurs at point P_1 at the instant t_1 in an optically isotropic frame I produces an effect at the distant point P_2 at the instant $t_2 > t_1$. The temporal order between the two events is not modified in any optically isotropic frame if and only if the signal originating from P_1 at the instant t_1 travels from P_1 to P_2 at a speed that is less than that of light in a vacuum.

We now introduce some particular Lorentz transformations. We define a *Lorentz transformation without rotation* as any Lorentz transformation obtained from (10.1) by assuming that the 3×3 orthogonal matrix (Q^i_j) reduces to the identity matrix:

$$x'^i = x'^i_O + \left(\delta^i_h + (\gamma - 1)\,\frac{u^i}{u^2}u_h\right)x^h - \gamma u^i t, \tag{10.6}$$

$$t' = t_0 - \gamma\frac{u_h x^h}{c^2} + \gamma t. \tag{10.7}$$

In order to understand the meaning of the above definition, we denote by (S^i_j) the constant 3×3 orthogonal matrix defining the rotation $\overline{x}^i = S^i_j x^j$

of the spatial axes of I, in which the uniform velocity $\mathbf{u}$ of I' with respect to I becomes parallel to the axis $O\overline{x}^1$:

$$\begin{pmatrix} \overline{u}^1 \\ 0 \\ 0 \end{pmatrix} = (S^i_j)(u^j). \tag{10.8}$$

If we note that $u^2 = u_h u^h$ and $u_h x^h$ are invariant with respect to a change in spatial axes, then, if we apply the rotation (S^i_j) to both sides of (10.6) (by performing the *same* rotation on the spatial axes of I' and I), we obtain

$$\overline{x}'^i = \overline{x}'^i_O + \left(\delta^i_h + (\gamma - 1)\frac{\overline{u}^i}{\overline{u}^2}\overline{u}_h\right)\overline{x}^h - \gamma\overline{u}^i t, \tag{10.9}$$

$$t' = t_0 - \gamma\frac{\overline{u}_h\overline{x}^h}{c^2} + \gamma t. \tag{10.10}$$

If we recall (10.8), assume that $x'^i_O = t_0 = 0$, and (for the sake of simplicity) omit the overline, these formulae lead us to the *special Lorentz transformations*

$$x'^i = \gamma\left(x^1 - ut\right), \tag{10.11}$$

$$x'^2 = x^2, \tag{10.12}$$

$$x'^3 = x^3, \tag{10.13}$$

$$t' = \gamma(t - \frac{u}{c^2}x^1). \tag{10.14}$$

These transformations include all significant relativistic aspects, since the most general Lorentz transformations can be obtained by arbitrary rotations of the spatial axes of the frames I and I'.

Remark From (10.2) we obtain that

$$\frac{dt}{dt'} = \left[\gamma\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)\right]^{-1}. \tag{10.15}$$

Using the above relation, simple but tedious calculations allow us to derive the transformation formulae for the velocity and the acceleration under a Lorentz transformation from (10.1). For instance, starting from the special Lorentz transformations, we obtain the following special formulae for the velocities:

$$v'^1 = \frac{v^1 - u}{1 - \dfrac{uv^1}{c^2}}, \tag{10.16}$$

$$v'^2 = \frac{1}{\gamma}\frac{v^2}{1 - \dfrac{uv^1}{c^2}}, \tag{10.17}$$

$$v'^3 = \frac{1}{\gamma} \frac{v^3}{1 - \frac{uv^1}{c^2}}. \tag{10.18}$$

Remark We conclude this section by noting that the linearity of the Lorentz transformations does not modify the rectilinear and uniform character of any motion. Consequently, if the principle of inertia is accepted in special relativity, the optically isotropic frames are also *inertial frames*.

10.2 The Principle of Relativity

In classical mechanics, a relativity principle holds for the class of inertial frames that are related to each other by the Galilean transformations. Einstein extended this principle to any field of physics. More precisely, he assumed that

The fundamental equations of physics take the same form for all inertial frames. Analytically, the fundamental equations of physics must be covariant under Lorentz transformations.

To clarify the meaning of this important principle, let us consider a physical law that is expressed by the following *differential* relation in the inertial frame I:

$$F\left(A, B, \ldots, \frac{\partial A}{\partial x^i}, \frac{\partial B}{\partial x^i}, \ldots, \frac{\partial A}{\partial t}, \frac{\partial B}{\partial t}, \ldots\right) = 0, \tag{10.19}$$

where A, $B, \ldots$ are physical fields that depend on the spatial coordinates and on time. If we denote the corresponding fields evaluated by the inertial observer I' by A', $B', \ldots$, then this law satisfies the principle of relativity if it assumes the form

$$F\left(A', B', \ldots, \frac{\partial A'}{\partial x'^i}, \frac{\partial B'}{\partial x'^i}, \ldots, \frac{\partial A'}{\partial t'}, \frac{\partial B'}{\partial t'}, \ldots\right) = 0, \tag{10.20}$$

in the new frame I', where A', $B', \ldots$, are the quantities A, $B, \ldots$ evaluated by the inertial observer I'.

It is important to understand that the principle of relativity does not state that the evolution of a physical phenomenon does not differ between inertial frames. It only states that the fundamental physical laws, which are expressed as *differential equations*, take the same form in any inertial frame. As a consequence, two inertial observers who perform an experiment under the *same* initial and boundary conditions will obtain the same results.

We note that our ability to verify the covariance of physical law (10.19) is related to our knowledge of the transformation law for the quantities A,

$B, \ldots$ in going from the inertial frame I to the inertial frame I'. In other words, from a mathematical viewpoint, a physical law could be covariant with respect to more transformation groups, provided that the transformed quantities A', $B', \ldots$ are suitable defined. Therefore, we can only establish the covariance of a physical law by verifying that the assumed transformation laws for the quantities A, $B, \ldots$ are valid experimentally.

In order to clarify this aspect of the theory, we consider the continuity equation for electric charge in empty space,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0, \tag{10.21}$$

where ρ is the charge density and $\mathbf{J}$ is the current vector. If, for the sake of simplicity, we consider the special Galilean transformation

$$\mathbf{x}' = \mathbf{x} - \mathbf{u}t, \tag{10.22}$$
$$t' = t, \tag{10.23}$$

then we obtain the following relations between the primed and unprimed derivatives:

$$\nabla = \nabla', \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \mathbf{u} \cdot \nabla. \tag{10.24}$$

Consequently, (10.21) can also be written in the form

$$\frac{\partial \rho}{\partial t'} + \nabla' \cdot (\mathbf{J} - \rho \mathbf{u}) = 0. \tag{10.25}$$

We can conclude that, in empty space, the continuity equation for electric charge is invariant under Galilean transformations if and only if

$$\rho' = \rho, \quad \mathbf{J}' = \mathbf{J} - \rho \mathbf{u}. \tag{10.26}$$

However, if we start from the special Lorentz transformations (10.12)–(10.14) instead of (10.24), we derive the formulae

$$\frac{\partial}{\partial x^1} = \gamma \left(\frac{\partial}{\partial x'^1} - \frac{u}{c^2} \frac{\partial}{\partial t'} \right), \tag{10.27}$$
$$\frac{\partial}{\partial x^2} = \frac{\partial}{\partial x'^2}, \tag{10.28}$$
$$\frac{\partial}{\partial x^3} = \frac{\partial}{\partial x'^3}, \tag{10.29}$$
$$\frac{\partial}{\partial t} = \gamma \left(\frac{\partial}{\partial t'^1} - u \frac{\partial}{\partial x'^1} \right). \tag{10.30}$$

Then, under a Lorentz transformation, (10.21) becomes

$$\begin{aligned}&\frac{\partial}{\partial t'}\left[\gamma\left(\rho-\frac{u}{c^2}J^1\right)\right]+\frac{\partial}{\partial x'^1}\left[\gamma\left(J^1-\rho u\right)\right]\\&+\frac{\partial}{\partial x'^2}J^2+\frac{\partial}{\partial x'^3}J^3=0.\end{aligned}\tag{10.31}$$

We see that the continuity equation is covariant under Lorentz transformations if and only if

$$\rho'=\gamma\left(\rho-\frac{u}{c^2}J^1\right),\quad J'^1=\gamma(J^1-\rho u),\quad J'^2=J^2,\quad J'^3=J^3.\tag{10.32}$$

We conclude that, a priori (i.e., from a mathematical point of view), the continuity equation for charge could be covariant under both Galilean and Lorentz transformations provided that we assume different transformation properties for the charge density and the current vector. However, experience will compel us to decide which of (10.26) and (10.32) should be chosen. We emphasize that it is possible that (again from a mathematical point of view) the equations considered could be covariant under only one of the two transformation groups. For instance, the Maxwell equations in a vacuum

$$\nabla\times\mathbf{H}=\epsilon_0\frac{\partial\mathbf{E}}{\partial t},\tag{10.33}$$

$$\nabla\times\mathbf{E}=-\mu_0\frac{\partial\mathbf{H}}{\partial t}\tag{10.34}$$

are not covariant under Galilean transformations. In fact, due to (10.24), the above equations can also be written as follows:

$$\nabla'\times\mathbf{H}=\epsilon_0\frac{\partial\mathbf{E}}{\partial t'}-\epsilon_0\mathbf{u}\cdot\nabla'\mathbf{E},\tag{10.35}$$

$$\nabla'\times\mathbf{E}=\mu_0\frac{\partial\mathbf{H}}{\partial t'}+\mu_0\mathbf{u}\cdot\nabla'\mathbf{H}.\tag{10.36}$$

If we recall the vector identity

$$\nabla\times(\mathbf{a}\times\mathbf{b})=\mathbf{a}\cdot\nabla\mathbf{v}-\mathbf{b}\cdot\nabla\mathbf{a},\tag{10.37}$$

(10.35)–(10.36) become

$$\nabla'\times(\mathbf{H}+\epsilon_0\mathbf{u}\times\mathbf{E})=\epsilon_0\frac{\partial\mathbf{E}}{\partial t'},\tag{10.38}$$

$$\nabla'\times(\mathbf{E}-\mu_0\mathbf{u}\times\mathbf{H})=-\mu_0\frac{\partial\mathbf{H}}{\partial t'}.\tag{10.39}$$

It is evident that covariance of the first equation under Galilean transformations requires that

$$\mathbf{H}'=\mathbf{H}+\epsilon_0\mathbf{u}\times\mathbf{E},\qquad\mathbf{E}'=\mathbf{E},\tag{10.40}$$

whereas covariance of the second equation leads to the condition

$$\mathbf{E}' = \mathbf{E} - \mu_0 \mathbf{u} \times \mathbf{H}, \qquad \mathbf{H}' = \mathbf{H}, \tag{10.41}$$

which contradicts (10.40).

10.3 Minkowski Spacetime

It is well known that Minkowski supplied an elegant and useful geometric formulation of the special theory of relativity. This formulation represents the bridge that Einstein crossed to reach the geometric formulation of gravitation.

We start by analyzing the geometric structure of the four-dimensional space underlying this model. We denote by V_4 a generalized Euclidean four-dimensional space in which an orthonormal reference frame $(O, \mathbf{e}_\alpha)$, $\alpha = 1, \ldots, 4$, can be found such that the coefficients $\eta_{\alpha\beta}$ of the scalar product, which is defined by a symmetric covariant 2-tensor $\mathbf{g}$, are given by the following matrix:

$$(\eta_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{10.42}$$

The space V_4 is called Minkowski *spacetime*, and any point $P \in V_4$ is an *event*. Moreover, any vector $\mathbf{v}$ in the vector space associated with V_4 is said to be a *four-vector* or *4-vector*.

Any reference frame V_4 in which the coefficients $\eta_{\alpha\beta}$ of the scalar product assume the form (10.41) is called a *Lorentz frame*. Finally, a *Lorentz transformation* is any (linear) coordinate transformation

$$x'^\alpha = x_0'^\alpha + A^\alpha_\beta x^\beta, \tag{10.43}$$

relating the coordinates (x^α) of any event in the Lorentz frame $(O, \mathbf{e}_\alpha)$ to the coordinates $(O', \mathbf{e}'_\alpha)$ of the same event in the Lorentz frame $(O', \mathbf{e}'_\alpha)$. The definition of the Lorentz frame requires that the matrix A^α_β has to satisfy the condition

$$\eta_{\alpha\beta} = A^\lambda_\alpha A^\mu_\beta \eta_{\lambda\mu}. \tag{10.44}$$

Since, in any Lorentz frame, the square of the distance s between two events $(x^\alpha_{(1)})$ and $(x^\alpha_{(2)})$ assumes the form

$$s^2 = \eta_{\alpha\beta}(x^\alpha_{(2)} - x^\alpha_{(1)})(x^\beta_{(2)} - x^\beta_{(1)}) = \sum_{i=1}^{3} (x^i_{(2)} - x^i_{(1)})^2 - (x^4_{(2)} - x^4_{(1)})^2, \tag{10.45}$$

we can say that Lorentz transformations are orthogonal transformations of V_4, provided that the orthogonality is evaluated by the scalar product $\eta_{\alpha\beta}$. From (10.44) or (10.45), we can conclude that the set of all of these transformations is a *group*.

The square $\mathbf{v} \cdot \mathbf{v} = \mathbf{g}(\mathbf{v}, \mathbf{v})$ of the norm of a 4-vector $\mathbf{v}$ of V_4 can be positive, negative or zero. In the first case, we say that $\mathbf{v}$ is a *space-like* 4-vector. In the second case, $\mathbf{v}$ is said to be a *time-like* 4-vector. Finally, if $\mathbf{v} \cdot \mathbf{v} = 0$, then $\mathbf{v}$ is a *null* 4-vector. At any point $O \in V_4$, the set of events $P \in V_4$ such that the 4-vectors $\overrightarrow{OP} = \mathbf{v}$ are null, i.e., the set of the 4-vectors for which

$$\overrightarrow{OP} \cdot \overrightarrow{OP} = 0, \tag{10.46}$$

is a cone C_O that has its vertex at O. This is called the *light cone* at O. The condition (10.46) defines a cone since, if $\overrightarrow{OP} \in C_O$, then $\lambda\overrightarrow{OP} \in C_O$. In the Lorentz frame $(O, \mathbf{e}_\alpha)$, the cone (10.46) is represented by the equation

$$\sum_{i=1}^{3} (x^i)^2 - (x^4)^2 = 0. \tag{10.47}$$

The 4-vectors $\mathbf{v} = \overrightarrow{OP}$ which correspond to the events P that are internal to C_O are time-like 4-vectors, whereas those which correspond to the events P that are external to C_O are space-like 4-vectors.

Let us assign an arbitrary time-like 4-vector $\hat{\mathbf{e}}$, and let $(O, \hat{\mathbf{e}})$ be a uniform vector field of V_4. We say that, at any $O \in V_4$, the 4-vector $\hat{\mathbf{e}}$ defines the *direction of the future* at O. Moreover, the internal region C_O^+ of C_O, which $\hat{\mathbf{e}}$ belongs to, is said to be the *future* of O, whereas the remaining internal region C_O^- of C_O is the *past* of O. Finally, the set of events external to C_O is defined as the *present* of O.

We note that the first three axes of a Lorentz frame $(O, \mathbf{e})$ are space-like 4-vectors, whereas the fourth axis $\mathbf{e}_4$ is a time-like 4-vector. Moreover, let $\Sigma_{O,\mathbf{e}_4}$ be the three-dimensional space formed by all of the 4-vectors generated by linear combinations of the three 4-vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. It is easy to verify that $\Sigma_{O,\mathbf{e}_4}$ is a properly Euclidean space with respect to the scalar product $\mathbf{g}$, and that its elements are space-like 4-vectors.

Before we can clarify the physical meanings of the all of the definitions provided above, we must prove the following propositions.

PROPOSITION 10.1

Let $\mathbf{u}$ *be any time-like 4-vector, and let us denote the three-dimensional space of all 4-vectors that are orthogonal to* $\mathbf{u}$ *by* $\Sigma_{O,\mathbf{u}}$. *Then, any* $\mathbf{v} \in \Sigma_{O,\mathbf{u}}$ *is a space-like 4-vector.*

PROOF Let $(O, \mathbf{e}_\alpha)$ be a Lorentz frame, and let $\mathbf{u} = u^i\mathbf{e}_i + u^4\mathbf{e}_4 \equiv \mathbf{u}_\perp + u^4\mathbf{e}_4$ be the decomposition of the 4-vector $\mathbf{u}$ in the basis $(\mathbf{e}_\alpha)$. Since

$\mathbf{u}_\perp \cdot \mathbf{u}_\perp = \sum_1^3 (u^i)^2$, the 4-vector $\mathbf{u}_\perp$ is space-like. Using the same notation, we put $\mathbf{v} = \mathbf{v}_\perp + v^4\mathbf{e}_4$, so that the following relations hold:

$$\mathbf{u}\cdot\mathbf{u} = \mathbf{u}_\perp \cdot \mathbf{u}_\perp - (u^4)^2 < 0,$$
$$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}_\perp \cdot \mathbf{v}_\perp - u^4 v^4 = 0,$$
$$\mathbf{v}\cdot\mathbf{v} = \mathbf{v}_\perp \cdot \mathbf{v}_\perp - (v^4)^2.$$

Consequently,

$$\mathbf{v}\cdot\mathbf{v} = \mathbf{v}_\perp \cdot \mathbf{v}_\perp - \frac{(\mathbf{u}_\perp \cdot \mathbf{v}_\perp)^2}{(u^4)^2},$$

so that, by applying Schwartz's inequality to $\mathbf{u}_\perp \cdot \mathbf{v}_\perp$, we obtain

$$\mathbf{v}\cdot\mathbf{v} \geq \mathbf{v}_\perp \cdot \mathbf{v}_\perp - \frac{(\mathbf{u}_\perp \cdot \mathbf{u}_\perp)(\mathbf{v}_\perp \cdot \mathbf{v}_\perp)}{(u^4)^2}.$$

Finally, this inequality implies that

$$\mathbf{v}\cdot\mathbf{v} \geq -\frac{\mathbf{v}_\perp \cdot \mathbf{v}_\perp}{(u^4)^2}[\mathbf{u}_\perp \cdot \mathbf{u}_\perp - (u^4)^2] > 0,$$

and the proposition is proved. ∎

Remark Based on the above proposition, we can say that any time-like 4-vector $\mathbf{u}$ defines infinite Lorentz frames at the event O. In fact, it is sufficient to consider the frame $(O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{u}/(|\mathbf{u}|))$ where the mutually orthogonal unit vectors $\mathbf{e}_i$, $i = 1, 2, 3$, belong to $\Sigma_{O,\mathbf{u}}$.

PROPOSITION 10.2

If $\mathbf{v}$ is a space-like 4-vector at $O \in V_4$, then it is possible to find at least a time-like 4-vector such that $\mathbf{u}\cdot\mathbf{v} = 0$.

PROOF Let $(O, \mathbf{e}_\alpha)$ be a Lorentz frame at O. Then, by adopting the same notation used in the above proposition, we have

$$\mathbf{v}_\perp \cdot \mathbf{v}_\perp - (v^4)^2 > 0. \tag{10.48}$$

Now we must prove that there is at least a 4-vector $\mathbf{u}$ such that

$$\mathbf{u}\cdot\mathbf{u} = \mathbf{u}_\perp \cdot \mathbf{u}_\perp - (u^4)^2 < 0, \tag{10.49}$$
$$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}_\perp \cdot \mathbf{v}_\perp - u^4 v^4 = 0. \tag{10.50}$$

From (10.49), we derive that the 4-vector $\mathbf{u}$ is orthogonal to $\mathbf{v}$ if we arbitrarily choose the components u^1, u^2, u^3 (i.e., the space-like 4-vector $\mathbf{u}_\perp$), provided that the component u^4 is given by

$$u^4 = \frac{\mathbf{u}_\perp \cdot \mathbf{v}_\perp}{v^4}.$$

For this choice of v^4, condition (10.49) becomes

$$\mathbf{u} \cdot \mathbf{u} = \mathbf{u}_\perp \cdot \mathbf{u}_\perp - \frac{(\mathbf{u}_\perp \cdot \mathbf{v}_\perp)^2}{(v^4)^2}.$$

Again by applying Schwartz's inequality, we can write the above equation as follows:

$$\mathbf{u} \cdot \mathbf{u} = \mathbf{u}_\perp \cdot \mathbf{u}_\perp \left(1 - \frac{\mathbf{v}_\perp \cdot \mathbf{v}_\perp}{(v^4)^2}\right). \tag{10.51}$$

Finally, since $\mathbf{u}_\perp \cdot \mathbf{u}_\perp > 0$, (10.49) and (10.51) indicate that $\mathbf{u} \cdot \mathbf{u} < 0$. ∎

Remark For any space-like vector $\mathbf{v}$ at the event O, we can find infinite Lorentz frames $(O, \mathbf{e}_\alpha)$ for which $\mathbf{e}_1 = \mathbf{v}/|\mathbf{v}|$. In fact, it is sufficient to take one of the existing time-like 4-vectors $\mathbf{e}_4$ orthogonal to $\mathbf{v}$ and choose another two unit vectors that are orthogonal to each other and to $\mathbf{v}$ in the three-dimensional space $\Sigma_{O,\mathbf{e}_4}$ to which $\mathbf{v}$ belongs.

Let us suppose that we have introduced a direction of the future using the uniform 4-vector $\hat{\mathbf{u}}$, so it is possible to define the cones C_O^- and C_O^+ at any event O.

PROPOSITION 10.3

Let $\mathbf{u} \in C_O^+$. Then $\mathbf{v} \in C_O^+$ if and only if

$$\mathbf{u} \cdot \mathbf{v} < 0. \tag{10.52}$$

PROOF In fact, $\mathbf{u} \cdot \mathbf{u} < 0$ since $\mathbf{u} \in C_O^+$. However, $\mathbf{u} \cdot \mathbf{w}$ is a continuous function of $\mathbf{w} \in C_O^+$, which is a connected set. Consequently, if $\mathbf{u} \cdot \mathbf{v} > 0$ for a 4-vector $\mathbf{v} \in C_O^+$, a time-like 4-vector $\mathbf{u}^\star \in C_O^+$ would exist such that $\mathbf{u} \cdot \mathbf{u}^\star = 0$. However, this is impossible for Proposition 10.1.

On the other hand, because of the remark about Proposition 10.1, we can find a Lorentz frame at O that has $\mathbf{e}_4 = \mathbf{u}/|\mathbf{u}|$ as a time-like axis. In this frame, $\mathbf{u} \cdot \mathbf{u} = -u^4 v^4$. On the other hand, since $\mathbf{u}, \mathbf{u} \in C_O^+$, we have $u^4 > 0$, $v^4 > 0$ and the proposition is proved. ∎

We denote the set of Lorentz frames $(O, \mathbf{e}_\alpha)$ at O whose axes $\mathbf{e}_4$ belong to the positive cone C_O^+ by L_O^+. The Lorentz transformation between two Lorentz frames in L_O^+ is said to be *orthochronous*. It is evident that the totality of these transformations is a group.

PROPOSITION 10.4

Let $(O, \mathbf{e}_\alpha) \in L_O^+$ be a Lorentz frame. Then the Lorentz frame $(O, \mathbf{e}'_\alpha)$

belongs to L_O^+ if and only if

$$A_4^4 > 0. \tag{10.53}$$

PROOF Since $\mathbf{e}'_4 = A_4^\alpha \mathbf{e}_\alpha$, we have $\mathbf{e}'_4 \cdot \mathbf{e}_4 = A_4^4$. The proposition is proved if we consider Proposition 10.3. ∎

10.4 Physical Meaning of Minkowski Spacetime

In order to attribute physical meanings to *some* of the geometrical objects associated with Minkowski spacetime, we begin with the following remark. Let I and I' be two inertial reference frames. We have already said that the relation between the coordinates (x_A^i, t_A) and $(x_A'^i, t'_A)$, which the observers I and I' associate with the same event A, is a Lorentz transformation. It is a very simple exercise to verify that the following quadratic form:

$$s^2 \equiv \sum_{i=1}^{3}(x_A^i - x_B^i)^2 - c^2(t_A - t_B)^2 = \sum_{i=1}^{3}(x_A'^i - x_B'^i)^2 - c^2(t'_A - t'_B)^2, \tag{10.54}$$

is invariant under any Lorentz transformation relating the spacetime coordinates (x_A^i, t_A), (x_B^i, t_B) of the event A in the Lorentz frame I and the corresponding coordinates $(x_A^{i\prime}, t'_A)$, $(x_B^{i\prime}, t'_B)$ of the same event in I'.

Moreover, from (10.1) or (10.7), we find that

$$\frac{dt'}{dt} = \gamma > 0. \tag{10.55}$$

Furthermore, if we introduce the notation $x^4 = ct$, then the quadratic form (10.54) becomes identical to (10.45).

Let us introduce a direction of the future at any point of V_4 via the uniform time-like field $\mathbf{e}_4$. Due to Proposition 10.2 and its associated remark, we can define a Lorentz frame $(O, \mathbf{e}_\alpha) \in L_O^+$ at any point in V_4. Let us introduce a one-to-one correspondence among the inertial frames I in physical space and the orthogonal or Lorentz frames in L_O^+ in the following way. First, we associate a fixed inertial frame I with the Lorentz frame $(O, \mathbf{e}_\alpha)$ in L_O^+. Let I' be any inertial frame whose relation with I is expressed by (10.1) and (10.2). With I' we then associate the Lorentz frame $(O', \mathbf{e}'_\alpha)$, whose transformation formulae (10.43) with respect to $(O, \mathbf{e}_\alpha)$ are given by (see Eqs. 10.1, 10.2 and 10.43)

$$(x_{O'}^\alpha) = (x_{O'}^i, ct_0), \tag{10.56}$$

$$(A^\alpha_\beta) = \begin{pmatrix} Q^i_j\left(\delta^j_h + (\gamma-1)\dfrac{u^j u_h}{u^2}\right) & -\gamma Q^i_j \dfrac{u^j}{c} \\ -\gamma \dfrac{u_i}{c} & \gamma \end{pmatrix}. \tag{10.57}$$

Since

$$A^4_4 = \gamma > 0, \tag{10.58}$$

the Lorentz frame $(O', \mathbf{e}'_\alpha)$ belongs to L^+_O.

On the other hand, let us assign the Lorentz frame $(O', \mathbf{e}'_\alpha) \in L^+_O$ by (10.43), where $A^4_4 > 0$. In order to determine the corresponding inertial frame I', we must determine the right-hand sides of (10.56) and (10.57); i.e., the quantities t_0, u^i, Q^i_j, and $x^i_{O'}$. Due to the orthogonality of the matrix (Q^i_j), we must evaluate ten quantities starting from the four coordinates $(x^\alpha_{O'})$ and the six independent coefficients A^α_β (see (10.44)).

In particular, if I and I' are related by a special Lorentz transformation, then (10.56) and (10.57) reduce to the following formulae:

$$(x^\alpha_{O'}) = (0, 0), \tag{10.59}$$

$$(A^\alpha_\beta) = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\dfrac{u}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\dfrac{u}{c} & 0 & 0 & \gamma \end{pmatrix}. \tag{10.60}$$

The above considerations allow us to clarify the physical meanings of the definitions we provided in the preceding section. For any point $P \in V_4$ with coordinates (x^α) in a Lorentz frame $(O, \mathbf{e}_\alpha) \in L^+_O$, there is a corresponding event with coordinates $(x^i, x^4/c)$ in the inertial frame I. In particular, all of the events belonging to the light cone C_O have coordinates that obey (10.47). These points in V_4 correspond to all of the events with coordinates in the inertial frame I that satisfy the equation

$$\sum_{i=1}^{3}(x^i)^2 - c^2t^2 \equiv r^2 - c^2t^2 = 0 \tag{10.61}$$

or, equivalently, the two equations $r - ct = 0$ and $r + ct = 0$, which represent a spherical light wave expanding from O and a spherical light wave contracting toward O, respectively.

Moreover, due to Proposition 10.1 and its related remark, for any point $P \in C^+_O$ it is possible to find a Lorentz frame in L^+_O and then an inertial frame I such that the event P has the coordinates $(0, 0, 0, t)$ $(t > 0)$ in I, meaning that it appears to happen after the event at the origin. However, if $P \in C^-_O$, then the corresponding event appears to happen before the event at O for the observer I. In other words, any event belonging to C^+_O happens after the event at O for some inertial frames, and any event in C_O^- happens before the event at O for some inertial frames.

Also, due to Proposition 10.2 and its related remark, an event that belongs to the present of O appears to happen at the same time to some inertial observer I.

We conclude this section with a very important remark that is a consequence of the correspondence between the inertial frames and the Lorentz frames in V_4. *If we succeed in formulating the physical laws by tensor relations in V_4, they will be covariant with respect to Lorentz transformations; in other words, they will satisfy the principle of relativity.*

10.5 Four-Dimensional Equation of Motion

The way in which Einstein arrived at a new formulation for the dynamics of a single particle P moving in an inertial frame I is well known. The Newtonian equations were replaced with the following:

$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{F}, \tag{10.62}$$

$$\frac{d}{dt}(mc^2) = \mathbf{F}\cdot\mathbf{v}, \tag{10.63}$$

where

$$m = \frac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}} \tag{10.64}$$

is the *relativistic mass* of P and m_0 is its *rest mass*. It is also well known that (10.63) states the equivalence of mass and energy. In this section we review the four-dimensional formulation of (10.62) and (10.63) in the spacetime V_4.

Let

$$x^i = x^i(t) \tag{10.65}$$

be the equation of motion of a particle with respect to an inertial frame I. With this trajectory we can associate a curve σ of V_4, which has the following equations in the Lorentz frame $T \equiv (O, \mathbf{e}_\alpha)$ corresponding to I:

$$x^i = x^i(t), \quad x^4 = ct. \tag{10.66}$$

The curve σ is called *world trajectory* or *world line* of P. It defines a curve of V_4 that does not depend on the Lorentz frame adopted. The square norm of the vector $\mathbf{t} = (\dot{x}^i, c)$ tangent to σ is

$$\sum_{i=1}^{3} (\dot{x}^i)^2 - c^2 < 0, \tag{10.67}$$

and it is negative since the velocity of P in any inertial frame is less than the velocity c of light in a vacuum. In other words, the curve σ is time-like and its tangent vector at any point P lies within the nap C_O^+ of the light cone at P.

Let $x^i(\bar{t})$ be the position of P at the instant $\bar{t}$ in the inertial frame I. We call the inertial frame that has its origin at $x^i(\bar{t})$ and moves with a velocity $\overline{\mathbf{v}}$ with respect to I the *rest frame* or *proper frame* $\overline{I}$ of P at the instant $\bar{t}$. The corresponding Lorentz frame $((x^i(\bar{t}), c\bar{t}), \mathbf{e}_0) \equiv \overline{T}$ has a time axis $\mathbf{e}_4$ that is tangent to σ at the point $x^i(\bar{t})$, since the vector $\mathbf{e}_4$ in $\overline{T}$ must have the components $(0, 0, 0, 1)$. In going from T to $\overline{T}$, the infinitesimal distance between two events on σ is invariant, so that

$$ds^2 = \left(\sum_{i=1}^{3} (\dot{x}^i)^2 - c^2 \right) dt^2 = -c^2 d\tau^2, \tag{10.68}$$

where τ is the *proper time*; i.e., the time evaluated by the observer $I(\bar{t})$. Using (10.68), we derive the relation

$$\frac{dt}{d\tau} = \gamma. \tag{10.69}$$

If we adopt this time along σ, the parametric equations (10.66) become

$$x^\alpha = x^\alpha(\tau). \tag{10.70}$$

We define the *world velocity* or 4-*velocity* as the 4-vector

$$U^\alpha = \frac{dx^\alpha}{d\tau}, \tag{10.71}$$

which, in view of (10.69), has the following components in the Lorentz frame T:

$$U^\alpha = (\gamma \mathbf{v}, \gamma c). \tag{10.72}$$

Moreover,

$$U^\alpha U_\alpha = \gamma^2 (\mathbf{v}^2 - c^2) = -c^2 < 0. \tag{10.73}$$

We must now verify that we can write the equations (10.62) and (10.63) in the covariant form

$$m_0 \frac{dU^\alpha}{d\tau} = \Phi^\alpha, \tag{10.74}$$

where the 4-*force* is given by

$$(\Phi^\alpha) = \gamma \left(\mathbf{F}, \frac{\mathbf{F} \cdot \mathbf{v}}{c} \right). \tag{10.75}$$

10.6 Integral Balance Laws

In this section we formulate the relativistic balance laws of continuum thermomechanics. These equations:

- Are covariant under Lorenz transformations
- Extend the equivalence of mass and energy to any field carried by S
- Associate a momentum with any moving energy.

Many ways have been proposed to formulate these equations in the literature (see, for instance, [165]). Here, we prefer to follow the approach proposed in [166]–[170], since it represents the most natural extension of the ideas of classical continua to special relativity.

Let S be a continuous system that is moving with respect to the inertial reference frame I, and let $C(t)$ be the actual configuration of S in I. We make the following three assumptions:

- In any inertial frame there exist

$$\mathbf{g}(\mathbf{x},t), \mathbf{t}(\mathbf{x},t), h(\mathbf{x},t), \mathbf{p}(\mathbf{x},t), \tag{10.76}$$

  where $\mathbf{g}(\mathbf{x},t)$ is the momentum density, $\mathbf{t}(\mathbf{x},t)$ is the momentum current tensor, $h(\mathbf{x},t)$ is the energy density, and $\mathbf{p}(\mathbf{x},t)$ is the energy current vector, which satisfy the classical balance laws

$$\frac{d}{dt}\int_{c(t)} \mathbf{g}(\mathbf{x},t)\, dv = \int_{\partial c(t)} \mathbf{t}(\mathbf{x},t)\cdot \mathbf{n}\, d\sigma, \tag{10.77}$$

$$\frac{d}{dt}\int_{c(t)} h(\mathbf{x},t)\, dv = -\int_{\partial c(t)} \mathbf{p}(\mathbf{x},t)\cdot \mathbf{n}\, d\sigma, \tag{10.78}$$

$$\frac{d}{dt}\int_{c(t)} \mathbf{x}\times\mathbf{g}(\mathbf{x},t)\, dv = \int_{\partial c(t)} \mathbf{x}\times\mathbf{t}(\mathbf{x},t)\cdot \mathbf{n}\, d\sigma, \tag{10.79}$$

  where $c(t)\subset C(t)$ is an arbitrary material volume, $\mathbf{n}$ is the exterior unit vector normal to $\partial c(t)$, and $\mathbf{x}$ is the position vector of any particle with respect to the origin of the spatial axes of I.

- If we denote one of the fields in (10.76) by ψ^i, $i = 1,\ldots,4$, then the following is obtained upon moving from the inertial frame I to any inertial frame I':

$$\psi'^i = f^i(\psi^1,\ldots,\psi^4), \tag{10.80}$$

  where the functions f^i satisfy the condition

$$f^i(\mathbf{0},\mathbf{0},0,\mathbf{0}) = \mathbf{0}. \tag{10.81}$$

That is, if the continuous system does not contain matter or energy in a given inertial frame, then this is also observed by any other inertial observer.

- The amount of energy present in an elementary volume in the proper frame is independent of the observer.

The arbitrariness of the material volume $c(t)$ and the regularity of the fields under the integrals allow us to write the balance equations (10.77)–(10.79) in the following local forms:

$$\frac{\partial g^i}{\partial t} + \frac{\partial}{\partial x^j}(g^i v^j - t^{ij}) = 0, \tag{10.82}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x^j}(hv^j + p^j) = 0, \tag{10.83}$$

$$\epsilon_{ijl}(v^j g^l - t^{lj}) = 0. \tag{10.84}$$

If we introduce the notation $x^4 = ct$ and the matrix

$$(T^{\alpha\beta}) = \begin{pmatrix} g^i v^j - t^{ij} & cg^i \\ \frac{1}{c}(hv^i + p^i) & h \end{pmatrix}, \tag{10.85}$$

where $\alpha, \beta = 1, \ldots, 4$, then (10.82)–(10.84) assume the forms

$$\frac{\partial T^{\alpha\beta}}{\partial x^\beta} = 0, \tag{10.86}$$

$$T^{[ij]} = 0. \tag{10.87}$$

Let us denote the Lorentz frame of the spacetime V_4 associated with the arbitrary inertial frame I by T. We can now prove the following theorem.

Theorem 10.1
Under the three assumptions made above, the quantities (10.85) *are the components of a symmetric tensor of* V_4*.*

PROOF The tensor character of $(T^{\alpha\beta})$ is proven in Appendix D. In order to prove the symmetry of $T^{\alpha\beta}$, we show that the symmetry of $T^{\alpha\beta}$ with respect to spatial indices implies that it is symmetric with respect to all indices. Let $\overline{I}$ be the proper inertial frame for the particle $\mathbf{x}$ at the instant t. If we denote the coefficient of Lorentz transformation $\overline{I} \to I$ by $\overline{A}^\alpha_\beta$, then

$$T^{ij} = \overline{A}^i_l \overline{A}^j_m \overline{T}^{lm} + \overline{A}^i_4 \overline{A}^j_l \overline{T}^{4l} + \overline{A}^i_l \overline{A}^j_4 \overline{T}^{l4} \overline{A}^i_4 + \overline{A}^j_4 \overline{T}^{44}, \tag{10.88}$$

where $\overline{T}^{\alpha\beta}$ are the components of $T^{\alpha\beta}$ in $\overline{I}$. Subtracting from (10.88) the relation obtained by exchanging i with j in (10.88), we obtain

$$0 = (\overline{A}_4^i \overline{A}_l^j - \overline{A}_l^j \overline{A}_4^i)\mu^l, \tag{10.89}$$

where $\mu^l = \overline{T}^{4l} - \overline{T}^{l4}$. From (10.57), if $Q_j^i = \delta_j^i$ and we identify the relative velocity (u^i) with the opposite of the velocity (v^i) of the particle, we get

$$\left[\frac{v^i}{c}\left(\delta_j^i + \frac{v^j v_l}{v^2}(\gamma - 1)\right) - \frac{v^j}{c}\left(\delta_l^i + \frac{v_l^v}{v^2}(\gamma - 1)\right)\right]\mu^l = 0;$$

that is,

$$v^i \mu^j - v^j \mu^i = 0. \tag{10.90}$$

Since these equations must be satisfied for any choice of (v^i), we have $\mu^i = 0$ and the theorem is proved. $\blacksquare$

We can therefore conclude that:

- The balance equations (10.82)–(10.84) are valid in any inertial frame if and only if the quantities $T^{\alpha\beta}$ are the components of a symmetric tensor
- The global symmetry of $T^{\alpha\beta}$ implies that

$$g^i v^j - t^{ij} = g^j v^i - t^{ji}, \tag{10.91}$$

$$g^i = \frac{h}{c^2} v^i + \frac{p^i}{c^2}. \tag{10.92}$$

The second of the above equations shows that a momentum g^i is associated with the energy hv^i carried during the motion and with any energy current p^i. This result is the most general form of the equivalence of momentum and energy.

10.7 The Momentum–Energy Tensor

The symmetric tensor $T^{\alpha\beta}$ upon which the final forms of the balance equations depend is called the *momentum–energy tensor*. In this section we determine its general form in the absence of electromagnetic fields.

In the inertial frame I, let us consider an arbitrary particle $\overline{\mathbf{x}}$ in the continuous system S at the instant $\overline{t}$. Using our notation, we denote the proper frame of $\overline{\mathbf{x}}$ at the instant $\overline{t}$ by $\overline{I}$, and the corresponding Lorentz frame in V_4 by $\overline{T}$. In this frame, the components of $(T^{\alpha\beta})$ are given by the following matrix:

$$\begin{pmatrix} -\overline{t}^{ij} & c\overline{g}^i \\ \dfrac{\overline{p}^i}{c} & h \end{pmatrix}, \tag{10.93}$$

and they obey the symmetry conditions

$$\overline{t}^{ij} = \overline{t}^{ji}, \quad \overline{g}^i = \frac{\overline{p}^i}{c^2}. \tag{10.94}$$

Let (v^i) be the velocity of the particle $\overline{\mathbf{x}}$ with respect to I, and let $(\overline{A}^{\alpha}_{\beta})$ be the coefficients of the matrix relative to the transformation $\overline{I} \to I$. Then, from (10.57), in which we put $Q^i_j = \delta^i_j$, $u^i = -v^i$, and from (10.72), we derive that

$$\overline{A}^{\alpha}_4 = \frac{U^\alpha}{c}. \tag{10.95}$$

Now we define the three symmetric tensors $\Theta^{\alpha\beta}$, $\Pi^{\alpha\beta}$ and $Q^{\alpha\beta}$, which have the following components in the rest frame $\overline{I}$:

$$(\overline{\Theta}^{\alpha\beta}) = \begin{pmatrix} \mathbf{0} & 0 \\ 0 & \overline{h} \end{pmatrix}, \tag{10.96}$$

$$(\overline{\Pi}^{\alpha\beta}) = \begin{pmatrix} -\overline{t}^{ij} & 0 \\ 0 & 0 \end{pmatrix}, \tag{10.97}$$

$$(\overline{Q}^{\alpha\beta}) = \begin{pmatrix} \mathbf{0} & -\dfrac{\overline{p}^i}{c} \\ \dfrac{\overline{p}^i}{c} & 0 \end{pmatrix}. \tag{10.98}$$

It is evident that

$$\overline{T}^{\alpha\beta} = \overline{\Theta}^{\alpha\beta} + \overline{\Pi}^{\alpha\beta} + \overline{Q}^{\alpha\beta}. \tag{10.99}$$

On the other hand, the components of the above tensors in any Lorentz frame are

$$\overline{\Theta}^{\alpha\beta} = \overline{A}^{\alpha}_4 \overline{A}^{\beta}_4 \overline{h} = \frac{\overline{h}}{c^2} U^\alpha U^\beta, \tag{10.100}$$

$$\overline{\Pi}^{\alpha\beta} = \overline{A}^{\alpha}_i \overline{A}^{\beta}_j \overline{t}^{ij}, \tag{10.101}$$

$$\overline{Q}^{\alpha\beta} = -U^\alpha \overline{A}^{\beta}_i \frac{\overline{p}^i}{c^2} - \overline{A}^{\alpha}_i U^\beta \frac{\overline{p}^i}{c^2}. \tag{10.102}$$

If we introduce the 4-vector (q^α), the components of which in the rest frame are

$$(\overline{q}^\alpha) = \left(-\frac{\overline{p}^i}{c^2} \right), \tag{10.103}$$

then the momentum–energy tensor assumes the form

$$T^{\alpha\beta} = \rho_0 U^\alpha U^\beta + \Pi^{\alpha\beta} + U^\alpha q^\beta + q^\alpha U^\beta, \tag{10.104}$$

where $\rho_0 = \overline{h}/c^2$ is the rest density mass.

The momentum–energy tensor describes the physical nature of the continuous system. In this section we consider three simple examples of continuous relativistic systems that correspond to three different choices of the momentum–energy tensor:

1. When $\Pi^{\alpha\beta} = q^\alpha = 0$, we have

$$T^{\alpha\beta} = \rho_0 U^\alpha U^\beta, \tag{10.105}$$

and the continuous system is said to be made of *incoherent matter.*

2. If $q^\alpha = 0$ and we get

$$(\overline{\Pi}^{\alpha\beta}) = \begin{pmatrix} p_0\delta^{ij} & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix} = \begin{pmatrix} p_0\delta^{ij} & \mathbf{0} \\ \mathbf{0} & -p_0 \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & p_0 \end{pmatrix} \tag{10.106}$$

in the proper frame, then, from (10.101) and (10.95), we have

$$\Pi^{\alpha\beta} = p_0 \eta^{\alpha\beta} + \frac{p_0}{c^2} U^\alpha U^\beta. \tag{10.107}$$

Finally, the momentum–energy tensor assumes the form

$$T^{\alpha\beta} = \left(\rho_0 + \frac{p_0}{c^2}\right) U^{\alpha\beta} + p_0 \eta^{\alpha\beta}, \tag{10.108}$$

and the continuous system described by this momentum–energy tensor is said to be a *perfect fluid.* Usually, the rest pressure p_0 is given as a function of the rest mass ρ_0:

$$p_0 = p_0(\rho_0). \tag{10.109}$$

3. Finally, if $\overline{p}^i$ is identified with the heat current vector in the rest frame, then momentum–energy tensor (10.104) defines a *heat-conducting perfect fluid.*

It is worth noting that it is possible to postulate balance equations that are more general than (10.77)–(10.79) (see [165] and [168]). Such balance laws assume the existence of an internal angular momentum $\mathbf{k}$ and an internal mass momentum $\mathbf{n}$ per unit volume. Because of these assumptions, (10.79) is replaced with the following new equation for the angular momentum balance:

$$\frac{d}{dt}\int_{c(t)} (\mathbf{x} \times \mathbf{g}(\mathbf{x}, t) + \mathbf{k})\, dv = \int_{\partial c(t)} \mathbf{x} \times \mathbf{t}(\mathbf{x}, t) \cdot \mathbf{n}\, d\sigma. \tag{10.110}$$

Moreover, a mew balance equation must be added, which takes the form

$$\frac{d}{dt}\int_{c(t)}\left(\frac{h(\mathbf{x})}{c^2}+\mathbf{n}(\mathbf{x},t)\right)dv = -\frac{1}{c^2}\int_{\partial c(t)}\mathbf{xp}(\mathbf{x},t)\cdot\mathbf{n}\,d\sigma + \int_{c(t)}\mathbf{g}(\mathbf{x},t)\,dv.$$

The physical motivation for the presence of a mass momentum together with an internal angular momentum is given in [165]. In this more general theory, the momentum–energy tensor is no longer symmetric.

10.8 Fermi and Fermi–Walker Transport

In order to determine the expressions for the momentum–energy tensors of more general materials in special relativity, we must introduce some more kinematic ideas.

In view of (10.57), a Lorentz transformation without rotation is determined by the following transformation matrix:

$$(A^{\alpha}_{\beta}) = \begin{pmatrix} \left(\delta^i_j + (\gamma-1)\dfrac{u^i u_j}{u^2}\right) & -\gamma\dfrac{u^i}{c} \\ -\gamma\dfrac{u_i}{c} & \gamma \end{pmatrix}. \tag{10.111}$$

Let us suppose that the components u^i of the uniform velocity $\mathbf{u}$ of the inertial frame I' with respect to the inertial frame I are infinitesimal quantities ϵ^i. To within second-order terms in the variables μ^i, we have

$$\gamma = 1 + O(\mu), \quad \mu = \sqrt{\sum_{i=1}^{3}(\mu^i)^2}.$$

Consequently, in the same approximation, and using the notation $\epsilon^i = \mu^i/c$, (10.111) can be written as follows:

$$(A^{\alpha}_{\beta}) = \left(\delta^{\alpha}_{\beta}\right) + \begin{pmatrix} \mathbf{0} & -\epsilon^i \\ -\epsilon^i & 0 \end{pmatrix}. \tag{10.112}$$

A Lorentz transformation defined by matrix (10.112) is said to be an *infinitesimal Lorentz transformation without rotation*.

Let us consider a field of unit time-like vectors $\gamma(\mathbf{x})$ in an open set of V_4, and let Γ be the congruence of the time-like integral curves of $\gamma(\mathbf{x})$. We denote the three-dimensional space of the spatial 4-vectors that are orthogonal to $\gamma(\mathbf{x})$ at the point $\mathbf{x} \in W$ by $\Sigma_{\mathbf{x}}$ (see Propositions 10.1 and 10.2 in Sect. 10.3). If $(\mathbf{e}_i(\mathbf{x}))$, $i = 1, 2, 3$, is an orthonormal basis in $\Sigma_{\mathbf{x}}$, then $(\mathbf{x}, \mathbf{e}_i(\mathbf{x}, \gamma(\mathbf{x})$ is a Lorentz frame at $\mathbf{x}$, and the following conditions are satisfied:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \tag{10.113}$$

$$\mathbf{e}_i \cdot \gamma = 0, \tag{10.114}$$

$$\gamma \cdot \gamma = -1. \tag{10.115}$$

We now consider a point $\mathbf{x} + d\mathbf{x} \in W$ that is next to $\mathbf{x}$, and the vector $\gamma(\mathbf{x}) + d\gamma$ at $\mathbf{x} + d\mathbf{x}$. Since $\gamma + d\gamma$ must be a time-like 4-vector, we have $(\gamma + d\gamma) \cdot (\gamma + d\gamma) = -1$, so

$$\gamma \cdot d\gamma = 0. \tag{10.116}$$

We want to determine a spatial orthonormal basis $\mathbf{e}_i(\mathbf{x}) + d\mathbf{e}_i$ in $\Sigma_{\mathbf{x}+d\mathbf{x}}$ (see Fig. 10.1) in such a way that the Lorentz transformation between the two frames $(\mathbf{x}, \mathbf{e}_i(\mathbf{x}), \gamma(\mathbf{x}))$ and $(\mathbf{x}+d\mathbf{x}, \mathbf{e}_i(\mathbf{x})+d\mathbf{e}_i, \gamma(\mathbf{x})+d\gamma)$ is an infinitesimal Lorentz transformation without rotation. To achieve this aim, we start by noting that the inverse matrix of (10.112) is

$$((A^{-1})^\alpha_\beta) = (\delta^\alpha_\beta) + \begin{pmatrix} \mathbf{0} & \epsilon^i \\ \epsilon^i & 0 \end{pmatrix} \equiv (\delta^\alpha_\beta) + (\epsilon^\alpha_\beta). \tag{10.117}$$

Therefore, if we make $\mathbf{e}_4 = \gamma$, we have

$$\mathbf{e}_\alpha + d\mathbf{e}_\alpha = (A^{-1})^\beta_\alpha \mathbf{e}_\beta = \mathbf{e}_\alpha + \epsilon^\beta_\alpha \mathbf{e}_\beta; \tag{10.118}$$

i.e.,

$$d\mathbf{e}_i = \epsilon^4_i \mathbf{e}_4. \tag{10.119}$$

From (10.119), it follows that

$$\epsilon^4_i = \epsilon^i_4 = d\mathbf{e}_4 \cdot \mathbf{e}_i, \tag{10.120}$$

and finally we have

$$d\mathbf{e}_i = (\mathbf{e}_i \cdot d\mathbf{e}_4)\mathbf{e}_4 = (\mathbf{e}_i \cdot d\gamma)\gamma. \tag{10.121}$$

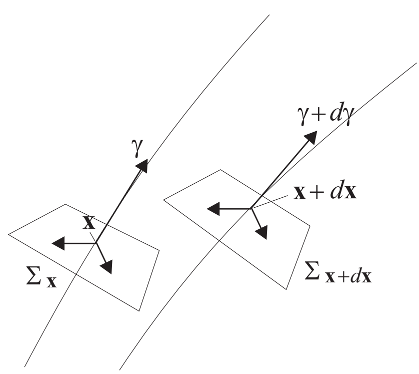

Fig. 10.1 Two Lorentz frames

These relations supply the vectors $(\mathbf{e}_i + d\mathbf{e}_i)$ that we must associate with $\gamma + d\gamma$, where $d\gamma$ satisfies (10.116), in order to obtain the requested frame $(\mathbf{x} + d\mathbf{x}, \mathbf{e}_i + d\mathbf{e}_i, \gamma + d\gamma)$, which is related to $(\mathbf{x}, \mathbf{e}_i, \gamma)$ by an infinitesimal Lorentz transformation without rotation. Finally, it is very simple to verify that

$$(\mathbf{e}_i + d\mathbf{e}_i) \cdot (\mathbf{e}_i + d\mathbf{e}_i) = \delta_{ij}, \tag{10.122}$$

$$(\mathbf{e}_i + d\mathbf{e}_i) \cdot (\gamma + d\gamma) = 0, \tag{10.123}$$

$$(\gamma + d\gamma) \cdot (\gamma + d\gamma) = -1. \tag{10.124}$$

Let $\mathbf{x}(\lambda)$ be a curve ρ of V_4 in the region W for which the time-like congruence Γ is given. We consider a Lorentz frame $(\mathbf{x}(0), \mathbf{e}_{0,i}, \gamma_0)$, where γ is the time-like 4-vector that is tangent to the curve $\rho \in \Gamma$ at the point $\mathbf{x}(0)$, and the 4-vectors $(\mathbf{e}_i)$ belong to $\Sigma_{\mathbf{x}(0)}$ (see Fig. 10.2).

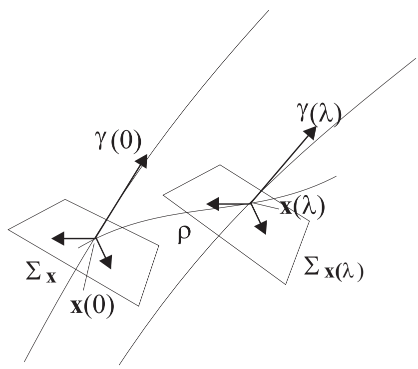

Fig. 10.2 Fermi–Walker transport of a Lorentz frame

Let γ denote the unit time-like vector that is tangent to the curve σ of Γ at the point where ρ intersects σ. Now consider the system of ordinary differential equations

$$\frac{d\mathbf{e}_i}{d\lambda} = \left(\mathbf{e}_i \cdot \frac{d\gamma}{d\lambda}\right)\gamma, \tag{10.125}$$

with the initial data

$$\mathbf{e}_i(0) = \mathbf{e}_{0,i}, \quad \mathbf{e}_{0,i} \cdot \mathbf{e}_{0,j} = \delta_{ij}, \quad \mathbf{e}_{0,i} \cdot \gamma_0 = 0. \tag{10.126}$$

Due to (10.126), we can easily verify that

$$\frac{d}{d\lambda}(\mathbf{e}_i \cdot \gamma) = \frac{d}{d\lambda}(\mathbf{e}_i \cdot \mathbf{e}_j) = 0. \tag{10.127}$$

Consequently, the solution $(\mathbf{e}_i(\lambda))$ of system (10.125) that obeys the initial data comprises unit vectors that are orthogonal to each other and to γ. Moreover, they satisfy (10.121); thus, at any point of $\mathbf{x}(\lambda) \in \rho$, the frame $(\mathbf{x}(\lambda), \mathbf{e}_i(\lambda), \gamma(\lambda))$ is a Lorentz frame that is related to the frame $(\mathbf{x}(\lambda+d\lambda), \mathbf{e}_i(\lambda+d\lambda), \gamma(\lambda+d\lambda))$ by an infinitesimal Lorentz transformation without rotation. The above solution is called the *Fermi–Walker transport* of the vectors $(\mathbf{e}_i(0))$ along ρ and the congruence Γ. In particular, if ρ belongs to Γ, then the solution $(\mathbf{e}_i(\lambda))$ of (10.125)–(10.126) is said to be the *Fermi transport* of $(\mathbf{e}_i(0))$ along ρ.

We conclude this section with the following theorem:

Theorem 10.2
Let $\mathbf{X}(\lambda)$ be a time-like curve. We denote its time-like unit tangent vector by $\gamma(\lambda)$ and the three-dimensional space of the space-like vectors orthogonal to $\gamma(\lambda)$ at $\mathbf{X}(\lambda)$ by $\Sigma_{\mathbf{X}(\lambda)}$. A 4-vector $\widetilde{\mathbf{u}}(\lambda) \in \Sigma_{\mathbf{X}(\lambda)}$ undergoes Fermi–Walker transport along ρ if and only if the spatial basis $(\mathbf{e}_i)$ in $\Sigma_{\mathbf{X}(\lambda)}$ is obtained by Fermi–Walker transport along ρ of the spatial basis $(\mathbf{e}_i(0))$ in $\Sigma_{\mathbf{X}(0)}$ and the components $\widetilde{u}^i$ of $\widetilde{\mathbf{u}}$ with respect to the basis $(\mathbf{e}_i)$ are constant.

PROOF If $\widetilde{\mathbf{u}} = \widetilde{u}^i(\lambda)\mathbf{e}_i$ undergoes Fermi–Walker transport along ρ, we have

$$\frac{d\widetilde{\mathbf{u}}}{d\lambda} = \left(\widetilde{\mathbf{u}} \cdot \frac{d\gamma}{d\lambda}\right)\gamma;$$

i.e.,

$$\frac{d\widetilde{u}^i}{d\lambda}\mathbf{e}_i + \widetilde{u}^i\frac{d\mathbf{e}_i}{d\lambda} = \widetilde{u}^i\left(\mathbf{e}_i \cdot \frac{d\gamma}{d\lambda}\right)\gamma.$$

Since the vectors $\mathbf{e}_i(\lambda)$ are unit vectors that are orthogonal to γ and to each other, we have $\gamma \cdot \mathbf{e}_i = 0$ and $\mathbf{e}_i \cdot d\mathbf{e}_i/d\lambda = 0$. Therefore, the scalar product of the above relation with $\mathbf{e}_i$ gives

$$\frac{d\widetilde{u}^i}{d\lambda} = 0.$$

Consequently, we also have

$$\frac{d\mathbf{e}_i}{d\lambda} = (\mathbf{e}_i \cdot \gamma)\gamma,$$

and the basis $\mathbf{e}_i(\lambda)$ undergoes Fermi transport along ρ.

It is evident that the last two relations imply the Fermi transport of $\widetilde{\mathbf{u}}(\lambda)$ along the curve ρ. ∎

In the next section we show the fundamental role of Fermi transport in relativistic continuum mechanics.

10.9 The Space Projector

Let

$$x^i = x^i(X^L, t) \tag{10.128}$$

be the equations of motion of a continuous system S with respect to an inertial frame I. The Lagrangian coordinates can be identified with the coordinates of the points of S at the instant $t = 0$ in I. Starting from (10.128), we can define the usual kinematic quantities of continuum mechanics, such as deformation gradient, velocity gradient, angular velocity, and so on. However, it is evident that the transformation properties of these quantities under a Lorentz transformation differ from those of the corresponding classical quantities. For instance, let us suppose that in the inertial frame I the spatial velocity deformation gradient $\mathbf{D}$ vanishes. The motion is then rigid with respect to I. However, upon shifting to another inertial frame I', the corresponding spatial velocity deformation gradient does not vanish, so the motion is no longer rigid.

Instead of considering this approach to relativistic kinematics, which depends on the observer in a complex way, we will analyze a different approach that involves observers located in the rest frames of the different points of S. To this end, we make use of the geometrical representation of the motion of S in the Lorentz frame T, which corresponds to I in V_4. We first note that the world trajectories of the points of S define a congruence Γ of time-like curves[1] in a region W of V_4 that will be called the *universe tube* of S. The vector field of the 4-velocity (U^α) of the different particles of S is defined in W.

Let us consider a particle $\mathbf{x} \in S$ at the instant t for the inertial observer I, and let I_0 be the rest frame of $\mathbf{x}$ at the instant t. The event $\mathbf{X} \equiv (\mathbf{x}, t) \in V_4$ belongs to the world trajectory $\rho \in \Gamma$ of the particle $\mathbf{x}$, and the Lorentz frame T_0, with the time-axis tangent to ρ at $\mathbf{X}$, is the four-dimensional representation of I_0 (see Fig. 10.3). In this frame, the 4-velocity $\mathbf{U}$ has components of $(0, 0, 0, c)$.

We now determine the infinitesimal space vector $\widetilde{\delta\mathbf{x}}$, which connects (at the same instant for the rest observer T_0) two adjacent particles in the continuum S. This vector belongs to the three-dimensional space $\Sigma_{\mathbf{X}}$ of the space-like vectors orthogonal to the unit 4-vector $\gamma = \mathbf{U}/c$, which is

[1]Actually, the velocity of any particle is less than c.

tangent to the world trajectory of the particle $\mathbf{x} \in S$ at $\mathbf{X}$.

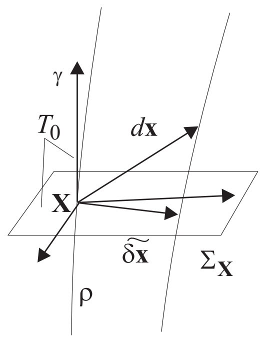

Fig. 10.3 Space and time decomposition of a 4-vector

It is evident that

$$d\mathbf{x} = \widetilde{\delta\mathbf{x}} + \lambda\gamma, \tag{10.129}$$

where the component λ of $d\mathbf{x}$ along γ is given by

$$\lambda = -(d\mathbf{x} \cdot \gamma)\gamma = -\frac{1}{c^2}(d\mathbf{x} \cdot \mathbf{U})\mathbf{U}. \tag{10.130}$$

Finally, the requested space-like 4-vector $\widetilde{\delta\mathbf{x}}$ is

$$\widetilde{\delta\mathbf{x}} = (\mathbf{I} + \frac{1}{c^2}\mathbf{U} \otimes \mathbf{U})d\mathbf{x}, \tag{10.131}$$

where $\mathbf{I} = (\delta^\alpha_\beta)$ is the identity tensor. In any Lorentz frame, the components of $\widetilde{\delta\mathbf{x}}$ are

$$\widetilde{\delta x}^\alpha = (\delta^\alpha_\beta + \frac{1}{c^2}U^\alpha U_\beta)dx^\beta, \tag{10.132}$$

$$\widetilde{\delta x}_\alpha = (\eta_{\alpha\beta} + \frac{1}{c^2}U_\alpha U_\beta)dx^\beta, \tag{10.133}$$

and the double 4-tensor

$$P_{\alpha\beta} = (\eta_{\alpha\beta} + \frac{1}{c^2}U_\alpha U_\beta), \tag{10.134}$$

is called the *space projector* in the three-dimensional space $\Sigma_{\mathbf{X}}$. It is possible to extend the projection on $\Sigma_{\mathbf{X}}$ to any tensor. For instance, if $t^{\alpha\beta}$ is a 4-tensor in V_4, we define its space projection $\widetilde{t}_{\alpha\beta}$ by the formula

$$\widetilde{t}_{\alpha\beta} = P_{\alpha\mu}P_{\beta\nu}t^{\mu\nu}. \tag{10.135}$$

10.10 Intrinsic Deformation Gradient

We now derive some useful consequences of (10.131). First, we introduce in the region W (which is formed by the world trajectories of the particles of S) a system of coordinates (y^i, τ) that is adapted to the congruence Γ. This means that any curve $\rho \in \Gamma$ is determined by the triad (y^i), whereas τ is a parameter along ρ which, from now on, we will identify with the proper time. The variables (y^i) will be called *material coordinates* since they define a particle in S. Then, in any Lorentz frame, the equations for the world trajectories of the particles in S can be written as follows:

$$x^\alpha = x^\alpha(y^i, \tau), \quad det\left(\frac{\partial x^i}{\partial y^j}\right) > 0, \tag{10.136}$$

so that

$$U^\alpha = \frac{\partial x^\alpha}{\partial \tau}. \tag{10.137}$$

We explicitly note that the variable τ assumes the meaning of the proper time provided that events belonging to the same curve of Γ are considered. Along such a curve we have

$$d\tau = -\frac{1}{c} U_\alpha dx^\alpha. \tag{10.138}$$

However, the proper time τ can be defined for the whole region W if and only if the differential form (10.138) is exact.

Let us consider an event $\mathbf{X}_0$ on the world line $\rho \in \Gamma$, and let $T_0(0)$ be the rest frame at this event. Since $dx^\alpha = (\partial x^\alpha / \partial y^i) dy^i + U^\alpha d\tau$ due to (10.136), we can use (10.133) to obtain

$$\widetilde{\delta x}^\alpha = P_{\alpha\beta} \frac{\partial x^\beta}{\partial y^i} dy^i \equiv \widetilde{F}_i^\alpha dy^i, \tag{10.139}$$

where $(\widetilde{F}_i^\alpha)$ is called the *intrinsic deformation gradient*.

In order to interpret the physical meaning of (10.139), we consider the spatial vector $d\mathbf{y} \in \Sigma_{\mathbf{X}_0}$. If $\mathbf{e}_{0,i}(0)$ is an orthonormal basis in $\Sigma_{\mathbf{X}_0}$, we have $d\mathbf{y} = dy^i \mathbf{e}_{0,i}(0)$.

At the event $\mathbf{X}(\tau) \in \rho$, corresponding to the value τ for the proper time, we consider the triad of the spatial vectors $\mathbf{e}_{0,i}(\tau)$ obtained by $\mathbf{e}_{0,i}(0)$ through Fermi transport along ρ (see Fig. 10.4). Due to the properties of this transport, the three vectors $\mathbf{e}_{0,i}(\tau)$ form an orthonormal basis in the three-dimensional space $\Sigma_{\mathbf{X}}(\tau)$ formed by the vectors orthogonal to the unit vector $\mathbf{e}_{0,4}(\tau)$ tangent to ρ at $\mathbf{X}(\tau)$. Moreover, the components dy^i of

$d\mathbf{y}$ remain constant during the transport (see Theorem 10.2), and $dy^i\mathbf{e}_{0,i}$ is the transported vector.

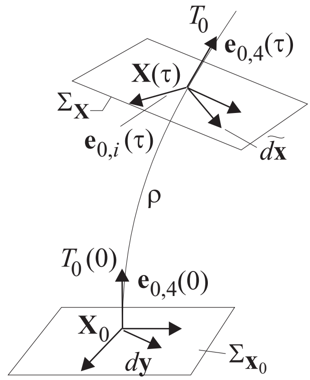

Fig. 10.4 Fermi transport of a Lorentz frame

If we refer (10.37) to the basis $\mathbf{e}_{0,i}$ of $\Sigma_{\mathbf{X}}$, then we have

$$\widetilde{\delta x}^i\,\mathbf{e}_{0,i}(\tau) = \widetilde{F}^i_j dy^j\,\mathbf{e}_{0,i}(\tau), \tag{10.140}$$

which, considering (10.34), defines an isomorphism in the three-dimensional space $\Sigma_{\mathbf{X}}$.

If we introduce the notation

$$(\widetilde{F}^{-1})^i_\alpha = \frac{\partial y^i}{\partial x^\alpha}, \tag{10.141}$$

it is easy to verify that the following relations hold:

$$\widetilde{F}^\alpha_j U_\alpha = 0, \tag{10.142}$$

$$(\widetilde{F}^{-1})^i_\alpha U^\alpha = 0, \tag{10.143}$$

$$(\widetilde{F}^{-1})^i_\alpha \widetilde{F}^\alpha_j = \delta^i_j, \tag{10.144}$$

$$(\widetilde{F}^{-1})^i_\alpha \widetilde{F}^\beta_i = P^\alpha_\beta. \tag{10.145}$$

In fact, from (10.135) and (10.140) we have

$$\widetilde{F}^\alpha_i U_\alpha = \left(\frac{\partial x^\alpha}{\partial y^i} + \frac{1}{c^2}U^\alpha U_\beta \frac{\partial x^\beta}{\partial y^i}\right) U_\alpha = 0,$$

since $U^\alpha U_\alpha = -c^2$. Moreover,

$$(\widetilde{F}^{-1})^i_\alpha U^\alpha = \frac{\partial y^i}{\partial x^\alpha} U^\alpha = \frac{\partial y^i}{\partial \tau} = 0,$$

since y^i and τ are independent variables. It is then very simple to verify (10.144) and (10.145).

In what follows, we use the notation (10.135) to denote the projection onto $\Sigma_{\mathbf{X}}$ of any 4-tensor $\mathbf{t}$.

We now prove the following theorem.

Theorem 10.3
The following identities hold:

$$\widetilde{\frac{d\widetilde{F}_i^\rho}{d\tau}} = \widetilde{F}_i^\lambda \widetilde{\frac{\partial U^\rho}{\partial x^\lambda}}, \tag{10.146}$$

$$\widetilde{\frac{\partial U^\rho}{\partial x^\mu}} = (\widetilde{F}^{-1})^i_\mu \widetilde{\frac{d\widetilde{F}_i^\rho}{d\tau}}. \tag{10.147}$$

PROOF From (10.139) and (10.134), we have

$$\begin{aligned}
\frac{d\widetilde{F}_i^\alpha}{d\tau} &= \frac{\partial}{\partial \tau}\left(\frac{\partial x^\alpha}{\partial y^i} + \frac{1}{c^2}U^\alpha U_\beta \frac{\partial x^\beta}{\partial y^i}\right) \\
&= \frac{\partial U^\alpha}{\partial y^i} + \frac{A^\alpha}{c^2}U_\beta \frac{\partial x^\beta}{\partial y^i} + \frac{U^\alpha}{c^2}A_\beta \frac{\partial x^\beta}{\partial y^i} + \frac{U^\alpha}{c^2}U_\beta \frac{\partial U^\beta}{\partial y^i} \\
&= \frac{\partial x^\mu}{\partial y^i} P^\alpha_\beta \frac{\partial U^\beta}{\partial x^\mu} + \frac{1}{c^2}(A^\alpha U_\beta + U^\alpha A_\beta)\frac{\partial x^\beta}{\partial y^i},
\end{aligned} \tag{10.148}$$

where $A^\alpha = dU^\alpha/d\tau$ is the 4-acceleration. On the other hand, from (10.139) we obtain

$$\frac{\partial x^\mu}{\partial y^i} = \widetilde{F}_i^\mu - \frac{1}{c^2}U^\mu U_\nu \frac{\partial x^\nu}{\partial y^i},$$

so that (10.148) becomes

$$\begin{aligned}
\frac{d\widetilde{F}_i^\alpha}{d\tau} = {} & \widetilde{F}_i^\mu P^\alpha_\beta \frac{\partial U^\beta}{\partial x^\mu} + \frac{1}{c^2}(A^\alpha U_\beta + U^\alpha A_\beta)\frac{\partial x^\beta}{\partial y^i} \\
& - P^\alpha_\beta \frac{U^\mu}{c^2} U_\nu \frac{\partial x^\nu}{\partial y^i}\frac{\partial U^\beta}{\partial x^\mu}.
\end{aligned}$$

However, we also have

$$U^\mu \frac{\partial U^\beta}{\partial x^\mu} = A^\beta, \quad A^\alpha U_\alpha = 0, \quad P^\alpha_\beta A^\beta = A^\alpha, \quad \widetilde{F}_i^\mu = P^\mu_\lambda \widetilde{F}_i^\lambda,$$

so the above relation can be written as follows:

$$\frac{d\widetilde{F}_i^\alpha}{d\tau} = \widetilde{F}_i^\lambda P^\alpha_\beta P^\mu_\lambda \frac{\partial U^\beta}{\partial x^\mu} + \frac{1}{c^2}U^\alpha A_\beta \frac{\partial x^\beta}{\partial y^i}. \tag{10.149}$$

Applying the projector P^ρ_α to both sides of (10.149), and noting that $P^\rho_\alpha P^\alpha_\beta = P^\rho_\beta$, $P^\rho_\alpha U^\alpha = 0$, we finally obtain

$$P^\rho_\alpha \frac{d\widetilde{F}^\alpha_i}{d\tau} = \widetilde{F}^\lambda_i P^\rho_\beta P^\mu_\lambda \frac{\partial U^\beta}{\partial x^\mu},$$

and (10.146) is proved. Identity (10.147) follows from (10.146) and (10.145). ∎

Remark Equation 10.146 describes the evolution of the intrinsic deformation gradient with respect to any Lorentz frame T. If we denote the reference frame in T by $(O, \mathbf{e}_\alpha)$, and the family of frames that are obtained by Fermi transport along the world trajectory ρ of a particle in the continuum system S by $(\mathbf{X}(\lambda), \mathbf{e}'_i, \mathbf{U}/c)$, then, due to the spatial character of $\widetilde{F}^\alpha_i$, we have

$$F'^j_i \mathbf{e}'_j = F^\alpha_i \mathbf{e}_\alpha. \tag{10.150}$$

On the other hand, since $\mathbf{e}'_i$ is obtained by Fermi transport along ρ, we also have

$$\frac{d}{d\tau}(\widetilde{F}'^j_i \mathbf{e}'_j) = \frac{\widetilde{F}'^j_i}{d\tau}\mathbf{e}'_j + \widetilde{F}'^j_i \frac{1}{c^2}\left(\mathbf{e}'_j \cdot \frac{d\mathbf{U}}{d\tau}\right)\mathbf{U}, \tag{10.151}$$

and the projection of (10.151) onto $\Sigma_{\mathbf{X}(\lambda)}$ coincides with (10.146). In conclusion, (10.146) gives the spatial evolution of the intrinsic deformation gradient with respect to proper observers that assume spatial axes in $\Sigma_{\mathbf{X}(\lambda)}$ which satisfy Fermi transport along the world trajectory ρ.

We conclude this section by defining the *intrinsic Cauchy–Green tensor*:

$$\widetilde{C}_{ij} = \eta_{\alpha\beta}\widetilde{F}^\alpha_i \widetilde{F}^\beta_j. \tag{10.152}$$

10.11 Relativistic Dissipation Inequality

From the results of Sect. 10.7, we know that the relativistic balance equations for momentum and energy can be written as follows:

$$\frac{\partial T^{\alpha\beta}}{\partial x^\beta} = f^\alpha, \tag{10.153}$$

where

$$T^{\alpha\beta} = \rho_0 U^\alpha U^\beta + \Pi^{\alpha\beta} + U^\alpha q^\beta + q^\alpha U^\beta \tag{10.154}$$

is the symmetric momentum–energy 4-tensor, and

$$(f^\alpha) = \left(\mathbf{f}, \frac{1}{c}(\mathbf{f}\cdot\mathbf{v} + r)\right), \quad f^\alpha U_\alpha = 0 \tag{10.155}$$

is the 4-force acting per unit volume of the continuous system S. In (10.155), $\mathbf{f}$ is the force per unit volume with respect to an inertial frame, $\mathbf{v}$ is the velocity field of S, and r is the energy supplied per unit volume.

It is well known that the position

$$\rho_0 = \mu_0\left(1 + \frac{\epsilon_0}{c^2}\right) \equiv \mu_0\chi \tag{10.156}$$

defines the *proper density of matter* as well as the *proper internal energy* ϵ_0 per unit proper mass.. Moreover, in the absence of proper mass variation, μ_0 satisfies the equation

$$\frac{\partial}{\partial x^\alpha}(\mu_0 U^\alpha) = 0. \tag{10.157}$$

We also assume the following local form for the entropy inequality (see [171]):

$$\frac{\partial}{\partial x^\alpha}(s_0 U^\alpha) \geq -c^2\frac{\partial}{\partial x^\alpha}\left(\frac{q^\alpha}{\theta_0}\right) + \frac{r_0}{\theta_0}, \tag{10.158}$$

where s_0 is the proper entropy per unit volume and θ_0 is the proper absolute temperature. If we define the *proper specific entropy* η_0 is introduced as follows:

$$s_0 = \mu_0\eta_0, \tag{10.159}$$

the above inequality assumes the form

$$\mu_0\dot{\eta}_0 \geq -c^2\frac{\partial}{\partial x^\alpha}\left(\frac{q^\alpha}{\theta_0}\right) + \frac{r_0}{\theta_0}, \tag{10.160}$$

where the dot denotes the derivative with respect to the proper time τ along a world line of any particle of S.

In order to derive an important inequality from (10.153) and (10.160), we start by noting that (10.156) and (10.157) lead to

$$\begin{aligned}\frac{\partial}{\partial x^\alpha}(\rho_0 U^\alpha U^\beta) &= \frac{\partial}{\partial x^\alpha}(\mu_0\chi U^\alpha U^\beta)\\ &= \chi U^\alpha\frac{\partial}{\partial x^\beta}(\mu_0 U^\beta) + \mu_0 U^\beta\frac{\partial}{\partial x^\alpha}(\chi U^\alpha)\\ &= \mu_0\frac{d}{d\tau}(\chi U^\alpha) = \frac{\dot{\epsilon}_0}{c^2}U^\alpha + \mu_0\chi A^\alpha,\end{aligned} \tag{10.161}$$

where $A^\alpha = dU^\alpha/d\tau$ is the 4-acceleration. In view of (10.161), we can derive the following from the balance equations (10.153): derive that

$$\mu_0\frac{\dot{\epsilon}_0}{c^2}U^\alpha + \mu_0\chi A^\alpha + \frac{\partial\Pi^{\alpha\beta}}{\partial x^\beta} + \frac{\partial}{\partial x^\beta}(q^\alpha U^\beta + q^\beta U^\alpha) = f^\alpha. \tag{10.162}$$

On the other hand, it is easy to verify that (10.97) and (10.103) imply that

$$\Pi^{\alpha\beta}U_\beta = \Pi^{\beta\alpha}U_\beta = 0, \tag{10.163}$$

$$q^\alpha U_\alpha = 0. \tag{10.164}$$

Moreover, from $U^\alpha U_\alpha = -c^2$, we obtain

$$U_\alpha A^\alpha = 0, \tag{10.165}$$

$$U_\alpha \frac{\partial U^\alpha}{\partial x^\beta} = 0. \tag{10.166}$$

Finally, if we multiply (10.162) by U_α and consider (10.163)–(10.166), the condition $U_\alpha U^\alpha = -c^2$, and (10.155), we get

$$-\mu_0\dot{\epsilon}_0 + U_\alpha \frac{\partial \Pi^{\alpha\beta}}{\partial x^\beta} + U_\alpha U^\beta \frac{\partial q^\alpha}{\partial x^\beta} - c^2 \frac{\partial q^\alpha}{\partial x^\alpha} = r_0. \tag{10.167}$$

Again taking into account (10.163) and (10.164), we can write the above equation in the following form:

$$\mu_0\dot{\epsilon}_0 = \Pi_\alpha^\beta \widetilde{\frac{\partial U^\alpha}{\partial x^\beta}} - q^\alpha A_\alpha - c^2 \frac{\partial q^\alpha}{\partial x^\alpha} + r_0, \tag{10.168}$$

which, due to (10.148), becomes

$$\mu_0\dot{\epsilon}_0 = S_\alpha^i \widetilde{\frac{d\widetilde{F}_i^\alpha}{d\tau}} - q^\alpha A_\alpha - c^2 \frac{\partial q^\alpha}{\partial x^\alpha} + r_0, \tag{10.169}$$

where

$$S_\alpha^i = \Pi_\alpha^\beta (\widetilde{F}^{-1})_\beta^i. \tag{10.170}$$

By eliminating r_0 between (10.160) and (10.169), and noting that $q^\alpha U_\alpha = 0$ (see (10.103)), we obtain the inequality

$$-\mu_0(\dot{\psi}_0 + \eta_0\dot{\theta}_0) - S_\alpha^i \widetilde{\frac{d\widetilde{F}_i^\alpha}{d\tau}} - q^\alpha A_\alpha - \frac{c^2}{\theta_0} q^\alpha g_\alpha, \tag{10.171}$$

where

$$\psi_0 = \epsilon_0 - \theta_0\eta_0 \tag{10.172}$$

is the *specific free energy* and

$$\widetilde{g}_\alpha = \widetilde{\frac{\partial \theta_0}{\partial x^\alpha}} = P_\alpha^\beta \frac{\partial \theta_0}{\partial x^\beta}. \tag{10.173}$$

On the other hand, if we recall (10.141), we also find that

$$\begin{aligned} \widetilde{g}_\alpha &= P_\alpha^\beta \left[(\widetilde{F}^{-1})_\beta^i \frac{\partial \theta_0}{\partial y^i} + \frac{\partial \tau}{\partial x^\beta}\dot{\theta}_0 \right] \\ &\equiv P_\alpha^\beta \left[(\widetilde{F}^{-1})_\beta^i G_i + \tau_\alpha \dot{\theta}_0 \right]. \end{aligned} \tag{10.174}$$

Consequently $q^\alpha P^\beta_\alpha = q^\beta$ since q^α lies in the three-dimensional space orthogonal to U^α, and so we have

$$q^\alpha \widetilde{g}_\alpha = q^\alpha (\widetilde{F}^{-1})^i_\alpha G_i + \tau_\alpha \dot{\theta}_0. \tag{10.175}$$

Finally, the inequality (10.171), or its equivalent form

$$\begin{aligned} &-\mu_0 \left[\dot{[\psi]}_0 + \left(\eta_0 + c^2 \frac{q^\alpha \tau_\alpha}{\theta_0} \right) \dot{\theta}_0 \right] \\ &-S^i_\alpha \widetilde{\frac{d\widetilde{F}^\alpha_i}{d\tau}} - q^\alpha A_\alpha - \frac{1}{\theta_0} q^\alpha (\widetilde{F}^{-1})^i_\alpha G_i \geq 0, \end{aligned} \tag{10.176}$$

is called the relativistic *reduced dissipation inequality*.

10.12 Thermoelastic Materials in Relativity

Just as in the classical theory of continua, we call the pair of functions $x^\alpha(y^i, \tau), \theta_0(y^i, \tau)$ a *thermokinetic process*, while a *thermodynamic process* is a thermokinetic process together with the following other functions:

$$\psi_0 = \psi_0(y^i, \tau), \tag{10.177}$$
$$\eta_0 = \eta_0(y^i, \tau), \tag{10.178}$$
$$\Pi^{\alpha\beta} = \Pi^{\alpha\beta}(y^i, \tau), \tag{10.179}$$
$$q^\alpha = q^\alpha(y^i, \tau). \tag{10.180}$$

In this section we analyze *relativistic thermoelastic materials*; i.e., materials that are described by the following constitutive equations:

$$\psi_0 = \psi_0(\widetilde{F}^\alpha_i, A^\alpha, \theta_0, G_i), \tag{10.181}$$
$$\eta_0 = \eta_0(\widetilde{F}^\alpha_i, A^\alpha, \theta_0, G_i), \tag{10.182}$$
$$\Pi^{\alpha\beta} = \Pi^{\alpha\beta}(\widetilde{F}^\alpha_i, A^\alpha, \theta_0, G_i), \tag{10.183}$$
$$q^\alpha = q^\alpha(\widetilde{F}^\alpha_i, A^\alpha, \theta_0, G_i). \tag{10.184}$$

It is reasonable to extend the *dissipation principle* (see Chap. 6 of [16]) to special relativity while requiring that *the constitutive equations* (10.181)–(10.184) *must satisfy the reduced dissipation inequality* (10.176) *in any thermokinetic process.*

In order to derive the restrictions on the constitutive equations (10.181)–(10.184) resulting from this principle, we first prove the following statement:

Theorem 10.4
Let $\mathbf{X} = (y^i, \tau)$ be an event on the world line of the particle (y^i) in the continuous system S. It is always possible to find, at least in the neighborhood of $\mathbf{X}$ in V_4, a motion $x^\alpha(y^i, \tau)$ such that the following quantities have arbitrary values at $\mathbf{X}$:

$$\widetilde{F}_i^\alpha, \widetilde{\frac{d\widetilde{F}_i^\alpha}{d\tau}}, A^\alpha, \theta_0, \dot{\theta}_0, G_i, \dot{G}_i. \tag{10.185}$$

PROOF The equations

$$x^i(y^j, t) = a^i(t) + F_j^i(t) y^j \tag{10.186}$$

represent a motion of a continuous system S with respect to the inertial frame I if and only if the 3×3 matrix (F_j^i) is not singular and the velocity of any point is less than c. The velocity $u^i(t)$ of the particle $(y^i) = \mathbf{0}$ is less than c provided that the arbitrary functions $a^i(t)$ are chosen in such a way that $\sum_{i=1}^3 [(\dot{a}^i(t))^2] < c^2$. When this condition is satisfied, the squared modulus of the velocity (v^i) of the other particles of S,

$$v^2 = u^2 + 2u_i \dot{F}_j^i y^j + \dot{F}_j^i \dot{F}_h^i y^j y^h, \tag{10.187}$$

will be less than c^2, at least when the point (y^i) belongs to a suitable neighborhood $\mathbb{I}$ of $\mathbf{0}$. Under this condition, we can define the congruence

$$x^i(y^j, t) = a^i(t) + F_j^i(t) y^j, \tag{10.188}$$

$$x^4 = ct \tag{10.189}$$

of the world trajectories of the particles of S in the neighborhood $\mathbb{I} \times \Re$ of V_4. In this way, it is possible to arbitrarily define the 4-velocity and the 4-acceleration of any particle $\mathbf{0}$ of S. Now we prove that, starting from (10.188), we can also arbitrarily assign the quantities $\widetilde{F}_i^\alpha$, $\widetilde{\frac{d\widetilde{F}_i^\alpha}{d\tau}}$. Let

$$d\hat{x}^i = \frac{\partial x^i}{\partial y^j} dy^i = F_j^i(t) dy^j \tag{10.190}$$

be the infinitesimal vector between the two events $(\mathbf{0}, t)$ and (dy^i, t), which are simultaneous with respect to the observer I. In the Lorentz frame corresponding to I, we can define the 4-vector $(d\hat{x}^\alpha) = (d\hat{x}^i, 0)$. Moreover, for the event $\mathbf{X} = (\mathbf{0}, t)$, we consider the 4-velocity U^α and the associated three-dimensional space $\Sigma_{\mathbf{X}}$ of the vectors that are orthogonal to U^α. In this case,

$$d\widetilde{x}^\alpha = P_\beta^\alpha d\hat{x}^\beta,$$

so that

$$\widetilde{F}_i^\alpha = P_\beta^\alpha \hat{F}_i^\beta. \tag{10.191}$$

This relation shows that an arbitrary choice of $\hat{F}_j^i(t)$ corresponds to an arbitrary choice of $\widetilde{F}_\beta^\alpha$. By differentiating with respect to time, we can also verify that $\widetilde{\frac{d\widetilde{F}_i^\alpha}{d\tau}}$ can be chosen in an arbitrary way starting from $F_j^i(t)$. A similar line of reasoning can be applied to the proper field of temperature,

$$\theta_0(y^i, t) = a(t) + b_j(t)y^i.$$

$\blacksquare$

In order to derive the restrictions on the constitutive equations due to the reduced dissipation inequality, we assume that this inequality is evaluated at an event $\mathbf{X} \in V_4$, and we denote the 4-velocity of the particle of the continuous system S at $\mathbf{X}$ by U^α and the three-dimensional space of the 4-vectors that are orthogonal to U^α by $\Sigma_{\mathbf{X}}$.

If we differentiate (10.181) with respect to the proper time τ,

$$\dot{\psi}_0 = \frac{\partial \psi_0}{\partial \widetilde{F}_i^\alpha} \frac{d\widetilde{F}_i^\alpha}{d\tau} + \frac{\partial \psi_0}{\partial \theta_0}\dot{\theta}_0 + \frac{\partial \psi_0}{\partial G_i}\dot{G}_i + \frac{\partial \psi_0}{\partial A_\alpha}\dot{A}^\alpha + \frac{\partial \psi_0}{\partial \dot{\theta}_0}\ddot{\theta}_0, \tag{10.192}$$

and we denote the projection of $\partial\psi_0/\partial\widetilde{F}_i^\alpha$ on $\Sigma_{\mathbf{X}}$ and U^α by

$$\widetilde{\left(\frac{\partial \psi_0}{\partial \widetilde{F}_i^\alpha}\right)} = P_\alpha^\beta \left(\frac{\partial \psi_0}{\partial \widetilde{F}_i^\alpha}\right), \tag{10.193}$$

$$\left(\frac{\partial \psi_0}{\partial \widetilde{F}_i^\alpha}\right)^{\|} = -\frac{1}{c^2}U_\alpha U^\beta \left(\frac{\partial \psi_0}{\partial \widetilde{F}_i^\alpha}\right), \tag{10.194}$$

then the inequality (10.184) can also be written as follows:

$$\begin{aligned}
&-\mu_0 \left[\left(\eta_0 + c^2 \frac{q^\alpha \tau_\alpha}{\theta_0}\right)\right]\dot{\theta}_0 - \left[S_\alpha^i + \mu_0 \widetilde{\left(\frac{\partial \psi_0}{\partial \widetilde{F}_i^\alpha}\right)}\right] \widetilde{\frac{d\widetilde{F}_i^\alpha}{d\tau}} \\
&-\mu_0 \left(\frac{\partial \psi_0}{\partial \widetilde{F}_i^\alpha}\right)^{\|} (\dot{\widetilde{F}}_i^\alpha)^{\|} - \mu_0 \frac{\partial \psi_0}{\partial G_i}\dot{G}_i - \mu_0 \frac{\partial \psi_0}{\partial \dot{\theta}_0}\ddot{\theta}_0 - \mu_0 \frac{\partial \psi_0}{\partial A^\alpha}\dot{A}^\alpha \\
&-q_\alpha A^\alpha - c^2 \frac{q^\alpha (\widetilde{F}^{-1})_\alpha^i}{\theta_0} G_i \geq 0.
\end{aligned} \tag{10.195}$$

Since this inequality must be satisfied for any choice of the quantities in (10.185), we necessarily find that

$$\psi_0 = \psi_0(\widetilde{F}_i^\alpha, \theta_0), \tag{10.196}$$

$$\eta_0^{(e)} = -\frac{\partial_0}{\partial\theta_0}, \tag{10.197}$$

$$S^i_\alpha = -\mu_0 \left(\frac{\widetilde{\partial\psi_0}}{\partial \widetilde{F}^\alpha_i} \right) = S^i_\alpha(\widetilde{F}^\alpha_i, \theta_0), \tag{10.198}$$

$$\left(\frac{\partial\psi_0}{\partial \widetilde{F}^\alpha_i} \right)^{\parallel} = 0, \tag{10.199}$$

$$\mu_0 \eta_0^{(d)} \dot{\theta} - q^\alpha A_\alpha - c^2 \frac{q^\alpha (\widetilde{F}^{-1})^i_\alpha}{\theta_0} G_i \geq 0, \tag{10.200}$$

where

$$\eta_0^{(e)}(\widetilde{F}^\alpha_i, \theta_0) \equiv \eta_0(\widetilde{F}^\alpha_i, \theta_0, \dot{\theta}_0, G_i, A^\alpha) - \eta_0(\widetilde{F}^\alpha_i, \theta_0, 0, 0, 0) \tag{10.201}$$

is the specific entropy at equilibrium and

$$\eta_0^{(d)}(\widetilde{F}^\alpha_i, \theta_0, \dot{\theta}_0, G_i, A^\alpha) = \eta_0(\widetilde{F}^\alpha_i, \theta_0, \dot{\theta}_0, G_i, A^\alpha) - \eta_0^{(e)}(\widetilde{F}^\alpha_i, \theta_0) \tag{10.202}$$

is the remaining part of the specific entropy. In particular, (10.200) leads to the following inequalities:

$$\eta_0^{(d)}(\widetilde{F}^\alpha_i, \theta_0, \dot{\theta}_0, G_i, 0)\dot{\theta}_0 - c^2 \frac{q^\alpha (\widetilde{F}^{-1})^i_\alpha}{\theta_0} G_i \geq 0, \tag{10.203}$$

$$q^\alpha(\widetilde{F}^\alpha_i, \theta_0, 0, 0, A^\alpha) A_\alpha \leq 0. \tag{10.204}$$

The following remarks hold.

Remark The dependence on A^α in the constitutive equations is essential. Indeed, if we omitted it, then (10.204) would lead to $q^\alpha = 0$. Moreover, it implies the presence of heat conduction even in the absence of a temperature gradient.

Remark The inequality (10.122) is a relativistic extension of a result obtained by Bogy and Naghdi [172] which implies that thermal waves have a finite propagation velocity.

Remark Condition (10.199) is a consequence of the objectivity principle in special relativity, as formulated in [173, 174, 175, 176].[2] It is possible to

[2] For a deeper analysis of the objectivity principle in special and general relativity, see [177, 178, 179].

prove that this form of the objectivity principle in special relativity leads to

$$\psi_0 = \psi_0(\widetilde{C}_{ij}, \theta_0). \tag{10.205}$$

Consequently,

$$\frac{\partial \psi_0}{\partial \widetilde{F}_i^\alpha} = \frac{\partial \psi_0}{\partial \widetilde{C}_{lm}} \frac{\partial \widetilde{C}^{lm}}{\partial \widetilde{F}_i^\alpha} = 2\eta_{\alpha\beta} \widetilde{F}_l^\beta \frac{\partial \psi_0}{\partial \widetilde{C}_{li}},$$

and (10.199) is proved if we take into account (10.142).

10.13 About the Physical Meanings of Relative Quantities

In the above sections we started from the assumption that the balance equations for momentum, energy and angular momentum also hold during the evolution of a continuous system S in special relativity. However, we did not accept the classical expressions for the quantities that appear in the balance laws a priori; instead, we imposed the covariance of these equations under Lorentz transformations, proving that the above balance equations can be written as the divergence of the symmetric momentum–energy tensor $T^{\alpha\beta}$ of the spacetime V_4. As a consequence, the description of a continuous system in special relativity requires that we are able to express its components in terms of the thermodynamic process. Some examples of this tensor have been supplied together with the nature of the continua which they describe. In any case, the proposed momentum–energy tensors $T^{\alpha\beta}$ were defined by giving their components $\overline{T}^{\alpha\beta}$ in the proper frame $\overline{I}$, where their physical meanings were more evident. In this way, the whole description of S is realized in spacetime; i.e., it is a geometrical description in V_4 of the evolution of S. It is evident that, by resorting to the transformation formulae

$$T^{\alpha\beta} = A^\alpha_\lambda A^\beta_\mu \overline{T}^{\lambda\mu}, \tag{10.206}$$

it is possible to express the components $T^{\alpha\beta}$ in terms of the physical quantities $\overline{T}^{\lambda\mu}$ and then, using (10.85), to obtain the *relative quantities* $\mathbf{g}, \mathbf{t}, h, \mathbf{p}$ (that appear in the balance equations) as functions of the proper components of $T^{\alpha\beta}$. These formulae, which are proved in [180], take the following forms:

$$\mathbf{g} = \left[\gamma^2 \overline{h} - \gamma(\gamma - 1)\frac{\mathbf{v} \cdot \overline{\mathbf{t}}\mathbf{v}}{v^2} + \left(2\gamma^2 \frac{v^2}{c^2} - \gamma + 1\right) \frac{\mathbf{v} \cdot \overline{\mathbf{p}}}{v^2}\right] \frac{\mathbf{v}}{c^2} - \gamma \frac{\mathbf{v} \cdot \overline{\mathbf{t}}}{c^2} + \gamma \frac{\overline{\mathbf{p}}}{c^2}, \tag{10.207}$$

$$h = \gamma^2 \left(\overline{h} - \frac{\mathbf{v} \cdot \overline{\mathbf{t}}\mathbf{v}}{c^2} + 2\frac{\mathbf{v} \cdot \overline{\mathbf{p}}}{c^2} \right), \tag{10.208}$$

$$\begin{aligned} \mathbf{t} &= \overline{\mathbf{t}} - (\gamma - 1)\frac{\mathbf{v} \otimes (\mathbf{v} \cdot \overline{\mathbf{t}})}{v^2} + \frac{\gamma - 1}{\gamma}\frac{\overline{\mathbf{t}} \cdot \mathbf{v}}{v^2} \\ &\quad + \frac{(\gamma - 1)^2}{\gamma}\frac{\mathbf{v} \cdot \overline{\mathbf{t}}\mathbf{v}}{v^4}\mathbf{v} \otimes \mathbf{v} + \frac{\gamma}{c^2}\mathbf{v} \otimes \left[\overline{\mathbf{p}} - \frac{\gamma - 1}{\gamma}\frac{\mathbf{v} \cdot \overline{\mathbf{p}}\mathbf{v}}{v^2} \right], \end{aligned} \tag{10.209}$$

$$\mathbf{p} = -\gamma \mathbf{v} \cdot \overline{\mathbf{t}} + (\gamma - 1)\frac{\mathbf{v} \cdot \overline{\mathbf{t}}\mathbf{v}}{v^2} + \gamma \left[\overline{\mathbf{p}} - \frac{\gamma - 1}{\gamma}\frac{\mathbf{v}(\mathbf{v} \cdot \overline{\mathbf{p}})}{v^2} \right]. \tag{10.210}$$

If the above expressions are introduced into the balance equations, and the constitutive equations of the overlined quantities in the rest frame are provided, then we obtain a set of partial differential equations that, at least in principle, allow us to determine the evolution of the continuous system when they are equipped with suitable initial and boundary conditions.

However, the balance equations in relation to the inertial observer I only involve quantities that refer to the rest frame instead of quantities that relate to the observer I. In other words, it is not yet clear how we can define the stress, the specific energy, the heat current vector, etc. relative to I. It is evident that there are many possible ways to define these quantities by reasonable definitions supported by suitable experimental procedures to measure them. In the literature there are many different proposals for defining the relative stress, the heat current vector, etc. In [180], the following definitions for the relative stress tensor $\mathbf{T}$ and the relative heat current vector $\mathbf{s}$ are considered:

$$\mathbf{t} = \mathbf{T} - \frac{\alpha}{c^2}\mathbf{v} \otimes \mathbf{s}, \tag{10.211}$$

$$\mathbf{p} = \mathbf{v} \cdot \mathbf{T} - \mathbf{s}, \tag{10.212}$$

where α is an unknown real number. Then, still in [180], homogeneous thermodynamic processes are considered in order to define the global energy U, the total work L, and the total heat Q that the system exchanges with the external world. Further, it is proved that, for $\alpha = 0$, the transformation formulae for U, L, Q that should be applied when shifting from the rest frame to any inertial frame coincide with the transformation formulae proposed by Einstein, Planck and von Laue (see [181, 182]). For $\alpha = 1$, the transformation formulae reduce to the formulae proposed by Kibble and Møller. Finally, for $\alpha = c^2(\gamma - 1)/(\gamma v^2)$, it is possible to derive the transformation formulae proposed by Landsberg.

In conclusion, *all of the above options and many others are acceptable, since each is associated with a particular arbitrary definition of the relative quantities, together with a corresponding measuring process.*

10.14 Maxwell's Equation in Matter

In the above sections we presented a relativistic theory of thermodynamic continua. We now wish to account for the case of a continuum that contains charges and currents, since the Maxwell equations are covariant under Lorentz transformations. However, there is an important question to solve. Electrodynamics in a vacuum in the presence of charges and currents is a well-established theory, and there is only one formulation for it. In contrast, in terms of electrodynamics in matter, many formulations have been proposed for the interaction between electromagnetic fields and moving matter. The first of them, which is due to Minkowski, is purely phenomenological, and it assumes that Maxwell's equations take the same form in moving bodies as in bodies at rest. In all other formulations, matter is replaced by fictitious distributions of charges and currents that are derived using a model of magnetization and polarization. In this way, it is possible to derive the Maxwell equations in moving media from the well-established equations in vacuum using the charges and currents supplied by the adopted model. Each model corresponds to a different expression for the 4-force $f^{\alpha}_{(e)}$ acting on the matter, and thus a different momentum–energy tensor $T^{\alpha\beta}_{(e)}$ for the electromagnetic field defined by the condition

$$f^{\alpha}_{(e)} = -\frac{\partial T^{\alpha\beta}_{(e)}}{\partial x^{\beta}}. \tag{10.213}$$

In any case, the balance equations assume the form

$$\frac{\partial T^{\alpha\beta}}{\partial x^{\beta}} = 0, \tag{10.214}$$

where

$$T^{\alpha\beta} = T^{\alpha\beta}_{(m)} + T^{\alpha\beta}_{(e)} \tag{10.215}$$

is the total momentum–energy tensor, whereas $T^{\alpha\beta}_{(m)}$ and $T^{\alpha\beta}_{(e)}$ denote its *mechanical* part and its *electromagnetic* part, respectively. In some cases, the choice of $f^{\alpha}_{(e)}$ leads to an electromagnetic tensor $T^{\alpha\beta}_{(e)}$, which is not symmetric, together with the total momentum energy tensor. For a review of the many proposals for the interaction between matter and an electromagnetic field, together with their physical motivations, see [183]– [193] for instance.

Choosing one of the proposed models is a difficult task, and it appears to be largely a matter of preference. In [191] it is suggested that all of these models could be equivalent. In line with this suggestion, the following result is proved in [193].

Let S be a charged continuous system, and let $\mathbf{x}$ be a particle of S that is moving with respect to the inertial observer I at the instant t. We, denote the 4-velocity at the point $\mathbf{X} = (\mathbf{x}, t) \in V_4$ and the three-dimensional space of the spatial vectors that are orthogonal to U^α at $\mathbf{X}$ by U^α and $\Sigma_{\mathbf{X}}$, respectively. The electromagnetic field in matter is then described by six fields, $d^\alpha, e^\alpha, b^\alpha, h^\alpha, p^\alpha$ and m^α, that belong to $\Sigma_{\mathbf{X}}$, which represent the electric induction, the electric field, the magnetic induction, the magnetic field, the polarization, and the magnetization in the rest frame $\overline{T}$ of V_4 at $\mathbf{X}$. The following relations hold between these fields:

$$p^\alpha = d^\alpha - \chi_0 e^\alpha, \tag{10.216}$$

$$m^\alpha = b^\alpha - \omega_0 h^\alpha, \tag{10.217}$$

where χ_0 is the dielectric constant and ω_0 the magnetic permeability of vacuum. In other words, only four of the above fields are independent. On the other hand, there are only two independent Maxwell equations. Therefore, taking into account (10.216) and (10.217), we must give the constitutive equations for two of the above fields in terms of the remaining two fields. It is then possible to prove that, *whichever of the four fields we decide to use to describe the electromagnetic fields in matter, there is a particular corresponding model*. Consequently, all of the proposed interaction models are fully equivalent, and they correspond to particular selections of the fundamental variables adopted in order to describe the interaction between matter and fields.

In this section we will only sketch out the proof of the above statement; it is fully proved in [193].

In a mechanical theory, the 4-vector $c^2 q^\alpha$ in (10.104) is defined by the condition that its components in the rest frame must coincide with the heat current vector. In the presence of electromagnetic fields, we assume that the vector $c^2 q^\alpha$ must be replaced by the 4-vector $c^2(q^\alpha + \sigma^\alpha_{(e)}) \in \Sigma_{\mathbf{X}}$, where the components of $c^2 \sigma^\alpha_{(e)}$ in the rest frame are

$$c^2 \overline{\sigma}^\alpha_{(e)} = (\overline{\mathbf{E}} \times \overline{\mathbf{H}}, 0). \tag{10.218}$$

In other words, the energy current vector in the rest frame is obtained by adding the heat current vector and Poynting's vector. In any Lorentz frame we have[3]

$$c^2 \sigma^\alpha_{(e)} = -\frac{1}{c} \epsilon^{\alpha\beta\lambda\mu} e_\beta h_\mu U_\lambda, \tag{10.219}$$

where $\epsilon^{\alpha\beta\lambda\mu}$ is the Levi–Civita symbol, and (10.168) becomes

$$\mu_0 \dot{\epsilon}_0 = \Pi^\beta_\alpha \widetilde{\frac{\partial U^\alpha}{\partial x^\beta}} - (q^\alpha + \sigma^\alpha_{(e)}) A_\alpha - c^2 \frac{\partial q^\alpha}{\partial x^\alpha} - c^2 \frac{\partial \sigma^\alpha_{(e)}}{\partial x^\alpha} + r_0. \tag{10.220}$$

[3]It is sufficient to verify that (10.219) reduces to (10.218) in the rest frame.

We must now evaluate the 4-divergence $\partial\sigma^{\alpha}_{(e)}/\partial x^{\alpha}$. All that we have said so far is based on assumption (10.218), and we have not used either Maxwell's equations or any model for the interaction between matter and electromagnetic fields yet.

10.15 Minkowski's Description

In order to evaluate the 4-divergence of σ_e, we must adopt a description of the interaction between matter and electromagnetic fields. We start from the phenomenological model of Minkowski, in which the basic variables are e^{α} and h^{α}, whereas d^{α} and b^{α} are expressed as functions of e^{α} and h^{α}. In this description, we replace the notation $\sigma^{\alpha}_{(e)}$ with σ^{α}_{M}.

The Maxwell equations (see Sects. 3.9 and 7.5)

$$\nabla \times \mathbf{H}_M = \mathbf{J} + \frac{\partial \mathbf{D}_M}{\partial t}, \tag{10.221}$$

$$\nabla \cdot \mathbf{D}_M = \rho_f, \tag{10.222}$$

$$\nabla \times \mathbf{E}_M = -\frac{\partial \mathbf{B}_M}{\partial t}, \tag{10.223}$$

$$\nabla \cdot \mathbf{B}_M = 0, \tag{10.224}$$

where the subscript M refers to the Minkowski approach, can be written in the following four-dimensional forms:

$$\frac{\partial H^{\alpha\beta}_M}{\partial x^{\beta}} = J^{\alpha}_M, \tag{10.225}$$

$$\frac{\partial E^{*\alpha\beta}_M}{\partial x^{\beta}} = 0. \tag{10.226}$$

In the above equations, the components of the skew-symmetric tensors $H^{\alpha\beta}_M$ and $E^{*\alpha\beta}_M$ of V_4 are given by the matrices

$$\begin{pmatrix} 0 & H^3_M & -H^2_M & -cD^1_m \\ -H^3_M & 0 & H^1_M & -cD^2_M \\ H^2_M & -H^1_M & 0 & -cD^3_M \\ cD^1_M & cD^2_M & cD^3_M & 0 \end{pmatrix}, \tag{10.227}$$

$$\begin{pmatrix} 0 & E^3_M & -E^2_M & -cB^1_m \\ -E^3_M & 0 & E^1_M & -cB^2_M \\ E^2_M & -E^1_M & 0 & -cB^3_M \\ cB^1_M & cB^2_M & cB^3_M & 0 \end{pmatrix}, \tag{10.228}$$

and

$$J^{\alpha}_{M} = (\mathbf{J}, c\rho_f). \tag{10.229}$$

In [193] it is proved that

$$H^{\alpha\beta}_{M} = \frac{1}{c}\epsilon^{\alpha\beta\lambda\mu}U_{\mu}h_{\lambda} + (U^{\alpha}d^{\beta} - U^{\beta}d^{\alpha}), \tag{10.230}$$

$$E^{*\alpha\beta}_{M} = \frac{1}{c}\epsilon^{\alpha\beta\lambda\mu}U_{\mu}e_{\lambda} + (U^{\alpha}b^{\beta} - U^{\beta}b^{\alpha}), \tag{10.231}$$

$$-c^2\frac{\partial\sigma^{\alpha}_{M}}{\partial x^{\alpha}} = e_{\alpha}\dot{d}^{\alpha} + h_{\alpha}\dot{b}^{\alpha} + \Pi^{\alpha\beta}_{M}\frac{\partial\widetilde{U^{\alpha}}}{\partial x^{\beta}} + \sigma^{\alpha}_{M}A_{\alpha} + e_{\alpha}c^{\alpha}, \tag{10.232}$$

where

$$\Pi^{\alpha\beta} = (e_{\lambda}d^{\lambda} + h_{\lambda}b^{\lambda})\eta^{\alpha\beta} - (e^{\alpha}d^{\beta} + h^{\alpha}b^{\beta}), \tag{10.233}$$

$$\sigma^{\alpha}_{M} = -\frac{1}{c}\epsilon^{\alpha\beta\lambda\mu}e_{\beta}h_{\mu}U_{\lambda}, \tag{10.234}$$

$$c^{\alpha} = P^{\alpha}_{\beta}J^{\beta}_{M}. \tag{10.235}$$

When expression (10.232) is introduced into (10.220), and the Minkowski electromagnetic specific energy ξ_M in the rest frame is defined by the condition

$$\mu_0\frac{d\xi_M}{d\tau} = e_{\alpha}\dot{d}^{\alpha} + h_{\alpha}\dot{b}^{\alpha}, \tag{10.236}$$

we obtain the expression for the *electromagnetic momentum–energy tensor* in the Minkowski description:

$$T^{\alpha\beta}_{M} = \mu_0\xi_M U^{\alpha}U^{\beta} + \Pi^{\alpha\beta}_{M} + \sigma^{\alpha}_{M}U^{\beta} + \sigma^{\beta}_{M}U^{\alpha}. \tag{10.237}$$

10.16 Ampere's Model

We conclude this chapter by describing Ampere's model.[4] In view of what we proved in the above sections, this model can be obtained by choosing suitable fundamental variables to describe the electromagnetic field. To derive this model, we use the fundamental variables b^{α}, e^{α}, p^{α} and m^{α}, where the last two variables are given by constitutive equations that depend on at least b^{α} and e^{α}.

The first step involves expressing the electromagnetic current vector $\sigma^{\alpha}_{(e)}$, which now we denote by σ^{α}_{A}, in terms of the fields b^{α}, e^{α}, p^{α} and m^{α}. Then,

[4] For the models of Chu and Boffi, see [193].

from (10.219), we obtain

$$\sigma_A^\alpha = \sigma_M^\alpha = -\frac{1}{c^3}\epsilon^{\alpha\beta\lambda\mu}e_\beta\left(\frac{b_\mu}{\omega_0} - m_\mu\right)U_\lambda. \tag{10.238}$$

On the other hand, when we introduce the 4-tensors (see Eqs. 10.230 and 10.231)

$$H_A^{\alpha\beta} = \frac{1}{c}\epsilon^{\alpha\beta\lambda\mu}U_\mu\frac{b_\lambda}{\omega_0} + (U^\alpha e^\beta - U^\beta e^\alpha), \tag{10.239}$$

$$E_A^{*\alpha\beta} = E_M^{*\alpha\beta} = \frac{1}{c}\epsilon^{\alpha\beta\lambda\mu}U_\mu e_\lambda + (U^\alpha b^\beta - U^\beta b^\alpha), \tag{10.240}$$

Maxwell's equations assume the following forms:

$$\frac{\partial H^{\alpha\beta}}{\partial x^\beta} = J_A^\alpha, \tag{10.241}$$

$$\frac{\partial E^{*\alpha\beta}}{\partial x^\beta} = 0, \tag{10.242}$$

where the current vector J_A^α is given by

$$J_A^\alpha = J^\alpha + \frac{1}{c}\epsilon^{\alpha\beta\lambda\mu}\frac{\partial}{\partial x^\beta}(U_\mu m_\lambda) - \frac{\partial}{\partial x^\beta}(U^\alpha p^\beta - U^\beta p^\alpha). \tag{10.243}$$

In [193] it is proved that, in any Lorentz frame, the Maxwell equations (10.241) and (10.242) take the following spatial forms:

$$\begin{aligned}\nabla \times \frac{\mathbf{B}_A}{\mu_0} - \chi_0\frac{\partial \mathbf{E}_A}{\partial t} = {} & \frac{\partial \mathbf{P}_A}{\partial t} + \nabla \times (\mathbf{P}_A \times \mathbf{v}) \\ & + \nabla \times \mathbf{M}_A - \frac{1}{c^2}\frac{\partial}{\partial t}(\mathbf{M}_A \times \mathbf{v}) + \mathbf{J},\end{aligned} \tag{10.244}$$

$$\chi_0 \nabla \cdot \mathbf{E}_A = -\nabla \cdot \mathbf{P}_A + \frac{1}{c^2}\nabla \cdot (\mathbf{M}_A \times \mathbf{v}) + \rho_f, \tag{10.245}$$

$$\nabla \times \mathbf{E}_A = -\frac{\partial \mathbf{B}_A}{\partial t}, \tag{10.246}$$

$$\nabla \cdot \mathbf{B}_A = 0. \tag{10.247}$$

In [193], all of the terms that appear in the above equations are justified by replacing the matter with a suitable distribution of charges and currents and accounting for relativistic effects. Then, starting from these Maxwell equations, the divergence of σ_A^α is evaluated with the corresponding electromagnetic momentum–energy tensor.

Appendix A

Brief Introduction to Weak Solutions

A.1 Weak Derivative and Sobolev Spaces

Let Ω be a bounded and open subset of $\Re^n$. Let $D(\Omega)$ be the vector space of $C^\infty(\Omega)$-functions with a compact support S_φ present in Ω:[1]

$$D(\Omega) = \{\varphi \in C^\infty(\Omega),\ S_\varphi \subset \Omega\}\,.$$

If $f \in C^1(\overline{\Omega})$, then it is always possible to write

$$\int_\Omega \frac{\partial f}{\partial x^i}\varphi\, d\Omega = \int_\Omega \frac{\partial}{\partial x^i}(f\varphi)\, d\Omega - \int_\Omega f\frac{\partial \varphi}{\partial x^i}\, d\Omega. \tag{A.1}$$

If the first integral on the right-hand side is transformed into an integral over the boundary $\partial\Omega$ by Gauss's theorem, and the hypothesis $\varphi \in D(\Omega)$ is taken into account, then the previous identity becomes

$$\int_\Omega \frac{\partial f}{\partial x^i}\varphi\, d\Omega = -\int_\Omega f\frac{\partial \varphi}{\partial x^i}\, d\Omega. \tag{A.2}$$

Conversely, if a function $\chi \in C^0(\overline{\Omega})$ exists for any $f \in C^1(\overline{\Omega})$ such that

$$\int_\Omega \chi\varphi\, d\Omega = -\int_\Omega f\frac{\partial \varphi}{\partial x^i}\, d\Omega \quad \forall\varphi \in D(\Omega), \tag{A.3}$$

then, subtracting (A.2) and (A.3), we obtain the condition

$$\int_\Omega \left(\chi - \frac{\partial f}{\partial x^i}\right)\varphi\, d\Omega = 0 \quad \forall\varphi \in D(\Omega), \tag{A.4}$$

[1]The support of φ is the subset of Ω in which $\varphi \neq 0$.

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which implies that $\chi = \partial f/\partial x^i$ because of the continuity of both of these functions. All of the above considerations lead us to introduce the following definition, which relates to the Hilbert space $L_2(\Omega)$ of all the square-summable functions in the set Ω with finite Lebesgue measure.

Definition *Any $f \in L_2(\Omega)$ is said to have a weak or generalized derivative if there is a function $\chi \in L_2(\Omega)$ that satisfies condition* (A.3).

It is possible to prove that the generalized derivative has the following properties:

- If $f \in C^1(\overline{\Omega})$, then its weak derivative coincides with the ordinary one
- The weak derivative of $f \in L_2(\Omega)$ is defined almost everywhere by condition (A.3); i.e., it belongs to $L_2(\Omega)$, and is uniquely determined
- Under certain auxiliary restrictions,[2] if χ is the weak derivative of the product $f_1 f_2$, where $f_1, f_2 \in L_2(\Omega)$, then

$$\int_\Omega \chi\varphi \, d\Omega = \int_\Omega (\chi_1 f_2 + f_1 \chi_2)\, \varphi \, d\Omega,$$

  where χ_i is the generalized derivative of f_i, $i = 1, 2$.

From now on, the weak derivative of $f(x_1, \dots x_n) \in L_2(\Omega)$ with respect to x_i will be denoted by $\partial f/\partial x_i$.

Example A.1

The weak derivative of the function $f(x) = |x|$, $x \in [-1, 1]$, is given by the following function of $L_2([-1, 1])$:

$$\frac{df}{dx} = \begin{cases} -1, & x \in [-1, 0), \\ \ \ 1, & x \in (0, 1]. \end{cases}$$

This is proven by the following chain of identities:

$$\begin{aligned} \int_{-1}^{1} \frac{df}{dx}\varphi \, dx &= -\int_{-1}^{1} |x| \frac{d\varphi}{dx} \, dx = \int_{-1}^{0} x \frac{d\varphi}{dx} \, dx - \int_{0}^{1} x \frac{d\varphi}{dx} \, dx \\ &= [x\varphi]_{-1}^{0} - \int_{-1}^{0} \varphi \, dx - [x\varphi]_0^1 + \int_0^1 \varphi \, dx \\ &= -\int_{-1}^{0} \varphi dx + \int_0^1 \varphi \, dx. \end{aligned}$$

[2] See Sect. 109 of [194].

Example A.2
The function

$$f(x) = \begin{cases} 0, & x \in [-1, 0), \\ 1, & x \in [0, 1] \end{cases}$$

does not have a weak derivative. In fact, we have

$$\int_{-1}^{1} \frac{df}{dx} \varphi \, dx = -\int_{-1}^{1} f \frac{d\varphi}{dx} \, dx = -[\varphi]_0^1 = \varphi(0),$$

and no function can satisfy this equality for any φ.

Definition *The vector space*

$$W_2^1(\Omega) = \left\{ f : \ f \in L_2(\Omega), \ \frac{\partial f}{\partial x_i} \in L_2(\Omega), \ i = 1, \ldots, n \right\} \tag{A.5}$$

of the functions that belong to $L_2(\Omega)$, together with their first weak derivatives, is called a Sobolev space.

It becomes a normed space if the following norm is introduced:

$$\|f\|_{1,2} = \left(\int_\Omega f^2 \, d\Omega + \sum_{i=1}^{n} \int_\Omega \left(\frac{\partial f}{\partial x_i} \right)^2 d\Omega \right)^{1/2}. \tag{A.6}$$

Recalling that in $L_2(\Omega)$ we usually adopt the norm

$$\|f\|_{L_2(\Omega)} = \left(\int_\Omega f^2 \, d\Omega \right)^{1/2},$$

we see that (A.6) can also be written as

$$\|f\|_{1,2}^2 = \|f\|_{L_2(\Omega)}^2 + \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_{L_2(\Omega)}^2. \tag{A.7}$$

The following theorem is given without proof.

Theorem A.1
The space $W_2^1(\Omega)$, equipped with the Sobolev norm (A.6), *is a Banach space (i.e., it is complete). More precisely, it coincides with the completion $H_2^1(\Omega)$ of the space*

$$\{f \in C^1(\Omega), \ \|f\|_{1,2} < \infty\}.$$

This theorem states that, for all $f \in W_2^1(\Omega)$, there is a sequence $\{f_k\}$ of functions $f_k \in C^1(\Omega)$ with a finite norm (A.7) such that

$$\lim_{k\to\infty} \|f - f_k\|_{1,2} = 0. \tag{A.8}$$

In turn, in view of (A.6), this condition can be written as

$$\lim_{k\to\infty} \left\{ \int_\Omega (f - f_k)^2 \, d\Omega + \sum_{i=1}^n \int_\Omega \left(\frac{\partial f}{\partial x_i} - \frac{\partial f_k}{\partial x_i} \right)^2 d\Omega \right\} = 0,$$

or equivalently

$$\lim_{k\to\infty} \int_\Omega (f - f_k)^2 d\Omega = 0,$$

$$\lim_{k\to\infty} \int_\Omega \left(\frac{\partial f}{\partial x_i} - \frac{\partial f_k}{\partial x_i} \right)^2 d\Omega = 0, \qquad i = 1, \ldots, n.$$

In other words, each element $f \in W_2^1(\Omega)$ is the limit in $L_2(\Omega)$ of a sequence $\{f_k\}$ of $C^1(\Omega)$-functions, and its weak derivatives are also the limits in $L_2(\Omega)$ of the sequences of ordinary derivatives $\{\partial f_k / \partial x_i\}$.

The preceding theorem makes it possible to define the weak derivatives of a function as the limits in $L_2(\Omega)$ of sequences of derivatives of functions belonging to $C^1(\Omega)$, as well as to introduce the Sobolev space as the completion $H_2^1(\Omega)$ of $\{f \in C^1(\Omega), ||f||_{1,2} < \infty\}$.

Let $C_0^1(\Omega)$ denote the space of all the C^1-functions that have both a compact support present in Ω and a finite norm (A.6). Another important functional space is $\hat{H}_2^1$, which is the completion of $C_0^1(\Omega)$. In such a space, if the boundary $\partial\Omega$ is regular in a suitable way, the following *Poincaré inequality* holds:

$$\int_\Omega f^2 \, d\Omega \leq c \sum_{i=1}^n \int_\Omega \left(\frac{\partial f}{\partial x_i} \right)^2 d\Omega, \qquad \forall f \in \hat{H}_2^1(\Omega), \tag{A.9}$$

where c denotes a positive constant that depends on the domain Ω. If we consider the other norm

$$||f||_{\hat{H}_2^1(\Omega)} = \left(\sum_{i=1}^n \int_\Omega \left(\frac{\partial f}{\partial x_i} \right)^2 d\Omega \right)^{1/2}, \tag{A.10}$$

then it is easy to verify the existence of a constant c^1 such that

$$||f||_{\hat{H}_2^1(\Omega)} \leq ||f||_{1,2} \leq c^1 ||f||_{\hat{H}_2^1(\Omega)},$$

so that the norms (A.6) and (A.9) are equivalent.

We conclude this section by introducing the concept of the *trace*. If $f \in C^1(\overline{\Omega})$, then it is possible to consider the restriction of f over $\partial\Omega$. Conversely, if $f \in H_2^1(\Omega)$, it is not possible to consider the restriction of f over $\partial\Omega$, since the measure of $\partial\Omega$ is zero and f is defined almost everywhere; i.e., up to a set with a vanishing measure. In order to attribute a meaning to the trace of f, it is sufficient to remember that, if $f \in H_2^1$, then there is a sequence of $C^1(\overline{\Omega})$-functions f_k such that

$$\lim_{k\to\infty} ||f - f_k||_{1,2} = 0. \tag{A.11}$$

Next, consider the sequence $\{\overline{f}_k\}$ of restrictions on $\partial\Omega$ of the functions f_k, and assume that, *as a consequence of* (A.11), this converges to the function $\overline{f} \in L_2(\Omega)$. In this case, it would be quite natural to call $\overline{f}$ the *trace* of f on $\partial\Omega$.

More precisely, the following theorem can be proven.

Theorem A.2
A unique linear and continuous mapping

$$\gamma : H_2^1(\Omega) \to L_2(\Omega) \tag{A.12}$$

exists such that $\gamma(f)$ coincides with the restriction on $\partial\Omega$ of any function $f \in C^1(\overline{\Omega})$. Moreover,

$$\hat{H}_2^1(\Omega) = \{f \in H_2^1(\Omega), \gamma(f) = 0\}. \tag{A.13}$$

A.2 A Weak Solution of a PDE

Consider the following classical boundary value problems related to Poisson's equation in the bounded domain $\Omega \subset \Re$ that has a regular boundary $\partial\Omega$:

$$\begin{aligned} \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega; \end{aligned} \tag{A.14}$$

$$\begin{aligned} \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} &= f \quad \text{in } \Omega, \\ \frac{du}{dn} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} n_i &= 0 \quad \text{on } \partial\Omega; \end{aligned} \tag{A.15}$$

where f is a given $C^0(\Omega)$-function. The previous boundary problems are called the *Dirichlet boundary value problem* and the *Neumann boundary value problem*, respectively.

Both of these problems allow a solution in the set $C^2(\Omega)\bigcap C^0(\overline{\Omega})$; this is unique for the first problem, whereas it is defined up to an arbitrary constant for the second boundary value problem.

By multiplying $(\mathrm{A.14})_1$ for any $v \in H_2^1(\Omega)$ and integrating over Ω, we obtain

$$\int_\Omega v \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}\, d\Omega = \int_\Omega f v\, d\Omega. \tag{A.16}$$

Recalling the identity

$$v\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(v\frac{\partial u}{\partial x_i}\right) - \frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i},$$

and using Gauss's theorem, boundary value problem (A.16) can be placed in the following form:

$$\sum_{i=1}^n \int_{\partial\Omega} v\frac{\partial u}{\partial x_i} n_i\, d\sigma - \sum_{i=1}^n \int_\Omega \frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i}\, d\Omega = \int_\Omega f v\, d\Omega, \tag{A.17}$$

where (n_i) is the unit vector normal to $\partial\Omega$.

In conclusion:

- If u is a smooth solution of Dirichlet's boundary value problem (A.14) and v is any function in $\hat{H}(\Omega)_2^1$, then the integral relation (A.17) becomes

$$-\sum_{i=1}^n \int_\Omega \frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i}\, d\Omega = \int_\Omega f v\, d\Omega \qquad \forall v \in \hat{H}_2^1(\Omega). \tag{A.18}$$

- If u is a smooth solution of Neumann's boundary value problem (A.15) and v is any function in $H(\Omega)_2^1$, then from (A.17) we can derive

$$-\sum_{i=1}^n \int_\Omega \frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i}\, d\Omega = \int_\Omega f v\, d\Omega \qquad \forall v \in H_2^1(\Omega). \tag{A.19}$$

Conversely, it is easy to verify that if u is a smooth function, the integral relations (A.18) and (A.19) imply that u is a regular solution of the boundary value problems (A.14) and (A.15), respectively. All of the above considerations suggest the following definitions:

- A function $u \in \hat{H}_2^1(\Omega)$ is a *weak solution* of the boundary value problem (A.14) if it satisfies the integral relation (A.18)

- A function $u \in H_2^1(\Omega)$ is a *weak solution* of the boundary value problem (A.15) if it satisfies the integral relation (A.19).

Of course, a weak solution is not necessarily a smooth (or strong) solution to the above boundary value problems, but it is possible to prove its existence under very general hypotheses. Moreover, by resorting to regularization procedures that can be applied when the boundary data are suitably regular, a weak solution can be proven to be smooth.

More generally, instead of (A.14) and (A.15), let us consider the following mixed boundary value problem:

$$\begin{aligned}
\sum_{i=1}^{n} \frac{\partial}{\partial x_i} A_{Li}(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) = f_L \qquad \forall \mathbf{x} \in \Omega, \\
\mathbf{u} = \mathbf{0} \qquad \forall \mathbf{x} \in \partial\Omega' \subset \partial\Omega, \\
A_{Li}(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) = g_L(\mathbf{x}) \qquad \forall \mathbf{x} \in \partial\Omega - \partial\Omega',
\end{aligned} \tag{A.20}$$

where $\mathbf{u}(\mathbf{x})$ is a p-dimensional vector field that depends on $\mathbf{x} = (x_1, \ldots, x_n)$, $L = 1, \ldots, p$, $\Omega \subset \Re^n$, and $\mathbf{n}$ is the unit vector normal to $\partial\Omega$.

By proceeding as before, it is easy to verify that the weak formulation of the boundary value problem (A.20) can be written as

$$-\int_\Omega \mathbf{A}(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) \cdot \nabla \mathbf{v} \, d\Omega = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, d\Omega - \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{v} \, d\sigma \qquad \forall \mathbf{v} \in \mathbb{U}, \tag{A.21}$$

where $\mathbb{U} = (\hat{H}_2^1(\Omega))^L$ is the Banach space of all the vector functions that vanish on $\partial\Omega'$.

Remark It is very important to note that if the assigned value of the unknown at the boundary is not zero in one of the problems (A.14) or (A.21), then the weak solution cannot belong to $H_2^1(\Omega)$ or to $\mathbb{U}$, respectively. This difficulty can be overcome by introducing any auxiliary function $\hat{g}(\mathbf{x})$ that extends the values of the boundary value g to Ω. The usual weak formulation can then be applied to the new unknown $\hat{\mathbf{u}} - \hat{\mathbf{g}}$. However, it is not easy to find the function $\hat{g}(\mathbf{x})$, especially when g or the boundary $\partial\Omega$ is not regular.

A.3 The Lax–Milgram Theorem

Let us suppose that the vector function $\mathbf{A}$ that appears under the integral on the left-hand side of (A.21) depends linearly on $\mathbf{u}$ and $\nabla \mathbf{u}$. It is then convenient to formulate the boundary value problem (A.21) in an abstract way. In the above linearity hypothesis, the left-hand side of (A.21) is a bilinear form of $\mathbf{u}$ and $\mathbf{v}$ on the Hilbert space $\mathbb{U}$:

$$B : \mathbb{U} \times \mathbb{U} \to \Re.$$

On the other hand, the right-hand side of (A.21) defines a linear form F on $\mathbb{U}$:

$$F : \mathbb{U} \to \Re.$$

Consequently, the weak formulation (A.21) of the boundary value problem (A.20) can be written as

$$B(\mathbf{u}, \mathbf{v}) = F(\mathbf{v}) \qquad \forall \mathbf{v} \in \mathbb{U}. \tag{A.22}$$

There is a remarkable theorem from Lax and Milgram for this kind of equation. Before stating it, some definitions are necessary.

Definition *A bilinear form $B : \mathbb{U} \times \mathbb{U} \to \Re$ is $\mathbb{U}$-elliptic with respect to the norm induced by the scalar product of the Hilbert space $\mathbb{U}$ if*

$$B(\mathbf{u}, \mathbf{u}) \geq \alpha ||\mathbf{u}||_{\mathbb{U}}, \tag{A.23}$$

where α is a positive real number.

Definition *A bilinear form $B : \mathbb{U} \times \mathbb{U} \to \Re$ is $\mathbb{U}$-continuous if there is a positive constant M such that*

$$|B(\mathbf{u}, \mathbf{v})| \leq M ||\mathbf{u}||_{\mathbb{U}} ||\mathbf{v}||_{\mathbb{U}}. \tag{A.24}$$

Definition *If $F : \mathbb{U} \to \Re$ is a linear form on the Hilbert space $\mathbb{U}$, the $\mathbb{U}$-norm of F is defined by the relation*

$$||F||_{\mathbb{U}} = \frac{|F(\mathbf{u})|}{||\mathbf{u}||_{\mathbb{U}}} \qquad \forall \mathbf{u} \in \mathbb{U}. \tag{A.25}$$

Theorem A.3

Let $B : \mathbb{U} \times \mathbb{U} \to \Re$ be a continuous and $\mathbb{U}$-elliptic bilinear form on the Hilbert space $\mathbb{U}$. There is then one and only one solution $\mathbf{u}$ of (A.22) *that depends continuously on the boundary data; i.e, such that*

$$||\mathbf{u}||_{\mathbb{U}} \leq \frac{1}{\alpha} ||F||_{\mathbb{U}}. \tag{A.26}$$

Appendix B

Elements of Surface Geometry

B.1 Regular Surfaces

Let $\Re^3$ be Euclidean three-dimensional space and $(O, \mathbf{e}_i)$, $i = 1, 2, 3$, be an orthonormal frame of reference in $\Re^3$. If $\mathbf{r} = x^i \mathbf{e}_i$ denotes the position vector of any point in $\Re^3$, then a *regular surface* S is defined by a vector equation

$$\mathbf{r} = \mathbf{r}(u^1, u^2) \tag{B.1}$$

such that its components

$$x^i = x^i(u^1, u^2) \tag{B.2}$$

are functions of class C^2, the Jacobian matrix of which has a rank equal to 2.

The relations

$$\mathbf{a}_\alpha \equiv \mathbf{r},_\alpha = \frac{\partial \mathbf{r}}{\partial u^\alpha} = \frac{\partial x^i}{\partial u^\alpha}\mathbf{e}_i, \tag{B.3}$$

$\alpha = 1, 2$, define two vectors that are tangent to the coordinate curves on S. These vectors are linearly independent at any point on S, since the above hypotheses imply that

$$|\mathbf{a}_1 \times \mathbf{a}_2| \neq 0. \tag{B.4}$$

Consequently, at any point $\mathbf{r} \in S$, they form a basis for the space $T_\mathbf{r}$ tangent to S at $\mathbf{r}$. This basis is called a *coordinate basis* or *holonomic basis* at $\mathbf{r}$ associated with the curvilinear coordinates u^α.

From (B.1) and (B.3) we derive the square of the line element ds that connects the points $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$:

$$ds^2 \equiv (d\mathbf{r})^2 = a_{\alpha\beta} du^\alpha du^\beta, \tag{B.5}$$

where

$$a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta = \sum_{i=1}^{3} \frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^i}{\partial u^\beta} = a_{\alpha\beta} \tag{B.6}$$

are the *metric coefficients*.

Since

$$
\begin{aligned}
|\mathbf{a}_1 \times \mathbf{a}_2| &= \sqrt{(\mathbf{a}_1 \times \mathbf{a}_2)\cdot(\mathbf{a}_1 \times \mathbf{a}_2)} \\
&= \sqrt{(\mathbf{a}_1 \cdot \mathbf{a}_1)(\mathbf{a}_2 \cdot \mathbf{a}_2) - (\mathbf{a}_1 \cdot \mathbf{a}_2)^2} \\
&= \sqrt{a_{11}a_{22} - a_{12}^2},
\end{aligned}
$$

we have

$$|\mathbf{a}_1 \times \mathbf{a}_2| = \sqrt{a}, \tag{B.7}$$

where

$$a = \det a_{\alpha\beta} > 0. \tag{B.8}$$

If we introduce the reciprocal metric coefficients via the relations

$$a^{\alpha\beta} = \frac{A^{\alpha\beta}}{a}, \quad (A^{\alpha\beta} = \text{ cofactor of } a_{\alpha\beta}), \tag{B.9}$$

then

$$a^{\alpha\lambda}a_{\lambda\beta} = \delta^{\alpha}_{\beta}. \tag{B.10}$$

We note that the vectors $\mathbf{a}_\alpha$ are neither unit vectors nor mutually orthogonal; when they form an orthonormal basis at a point, the coordinates u^α are said to be *orthogonal* at that point.

It is often useful to consider, besides the basis $(\mathbf{a}_\alpha)$, the *reciprocal basis* formed by the independent vectors

$$\mathbf{a}^\alpha = a^{\alpha\beta}\mathbf{a}_\beta, \tag{B.11}$$

which satisfy the conditions

$$\mathbf{a}^\alpha \cdot \mathbf{a}_\beta = \delta^{\alpha}_{\beta}. \tag{B.12}$$

Due to (B.12), if $\mathbf{v}$ is a vector that is tangent to S, then we can write

$$\mathbf{v} = v^\alpha \mathbf{a}_\alpha = v_\alpha \mathbf{a}^\alpha, \tag{B.13}$$

where

$$v_\alpha = a_{\alpha\beta}v^\beta, \quad v^\alpha = a^{\alpha\beta}v_\beta. \tag{B.14}$$

We can conclude that the contravariant components of $\mathbf{v}$ with respect to the reciprocal basis $\mathbf{a}^\alpha$ coincide with the covariant components of $\mathbf{v}$ with respect to the basis $\mathbf{a}_\alpha$.

Starting from the quadratic form (B.5), which represents the metric on S or the *first fundamental form* of S, we can deduce all of the metric properties of the surface S. Thus, if a curve γ is given on S by the parametric equations

$$u^\alpha = u^\alpha(t), \quad t \in [a, b], \tag{B.15}$$

its length $l(\gamma)$ is given by

$$l(\gamma) = \int_a^b \sqrt{a_{\alpha\beta} \frac{du^\alpha}{dt} \frac{du^\beta}{dt}}\, dt. \tag{B.16}$$

Similarly, consider a region $\sigma \subset S$ obtained by varying the curvilinear coordinates over the set $\Omega = [a_1, b_1] \times [a_2, b_2] \subset \Re^2$. The area of the surface element $d\sigma$ is defined by the relation

$$d\sigma = |du^1 \mathbf{a}_1 \times du^2 \mathbf{a}_2| = \sqrt{a} du^1 du^2,$$

so we have

$$\sigma = \int_\Omega \sqrt{a}\, du^1 du^2. \tag{B.17}$$

B.2 The Second Fundamental Form

On a regular surface S, the position

$$\mathbf{n} = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|} \tag{B.18}$$

defines (at least locally) a unit vector field that is orthogonal to S. We say that S is *locally oriented* when it is equipped with this vector field. Since the functions (B.2) are of class C^2 and $\mathbf{a}_\alpha \cdot \mathbf{n} = 0$, we can define the following quantities (see B.3):

$$b_{\alpha\beta} = -\mathbf{a}_\alpha \cdot \mathbf{n}_{,\beta} = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{n} = b_{\beta\alpha}, \tag{B.19}$$

$$\Gamma_{\alpha\beta\gamma} = \mathbf{a}_\alpha \cdot \mathbf{a}_{\beta,\gamma} = \Gamma_{\alpha\gamma\beta}, \tag{B.20}$$

$$\Gamma^\alpha_{\beta\gamma} = a^{\alpha\lambda} \Gamma_{\lambda\beta\gamma}. \tag{B.21}$$

The quadratic form $b_{\alpha\beta} du^\alpha du^\beta$ is called the *second fundamental form* of S, whereas $\Gamma_{\alpha\beta\gamma}$ and $\Gamma^\alpha_{\beta\gamma}$ represent the *Christoffel symbols of the first and second kind*, respectively. It is now possible to prove the *Gauss–Weingarten equations*

$$\mathbf{a}_{\beta,\alpha} = \Gamma^\gamma_{\beta\alpha} \mathbf{a}_\gamma + b_{\alpha\beta} \mathbf{n}, \tag{B.22}$$

$$\mathbf{n}_{,\alpha} = -b^\gamma_\alpha \mathbf{a}_\gamma, \tag{B.23}$$

where

$$\Gamma^\gamma_{\beta\alpha} = \frac{1}{2} a^{\gamma\lambda} (a_{\lambda\beta,\alpha} + a_{\alpha\lambda,\beta} - a_{\alpha\beta,\lambda}). \tag{B.24}$$

In fact, we can write

$$\mathbf{n}_{,\alpha} = c^\beta_\alpha \mathbf{a}_\beta + d_\alpha \mathbf{n}, \tag{B.25}$$

since $(\mathbf{a}_\alpha, \mathbf{n})$ is a basis of the three-dimensional space $\Re^3$. Using (B.25) and the condition $\mathbf{n} \cdot \mathbf{n} = 1$, we at once derive (B.23) by noting that $\mathbf{a}_\beta \cdot \mathbf{n} = 0$ and $\mathbf{n} \cdot \mathbf{n}_\alpha = 0$. Similarly, due to (B.19) and (B.20), we only have to prove (B.24). To this end, we differentiate the relations $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ with respect to u^λ and use (B.20) to obtain

$$\Gamma_{\beta\alpha\lambda} + \Gamma_{\alpha\beta\lambda} = a_{\alpha\beta},_\lambda \,. \tag{B.26}$$

Cyclic permutations of the indices lead to the equations

$$\Gamma_{\lambda\beta\alpha} + \Gamma_{\beta\lambda\alpha} = a_{\beta\lambda},_\alpha \,, \tag{B.27}$$

$$\Gamma_{\alpha\lambda\beta} + \Gamma_{\lambda\alpha\beta} = a_{\lambda\alpha},_\beta \,. \tag{B.28}$$

By adding (B.27) to (B.26) and subtracting (B.28) from the result, we derive (B.24) when the symmetry properties of the Christoffel symbols are taken into account.

Let γ be a curve on the surface S, and let $\mathbf{r} = \mathbf{r}(u^\alpha(s))$ be its equation (s is the arc length on γ). If ξ is the curvature of γ and μ is its principal normal unit vector, the curvature vector $\mathbf{k}$ is expressed by the Frenet formula

$$\mathbf{k} = \xi\mu = \frac{d\lambda}{ds}, \tag{B.29}$$

where $\lambda = \lambda^\alpha \mathbf{a}_\alpha$ is the unit vector tangent to γ. Using (B.22), the above relation becomes

$$\mathbf{k} = \left(\frac{d\lambda^\alpha}{ds} + \Gamma^\alpha_{\nu\beta}\lambda^\nu\lambda^\beta \right) \mathbf{a}_\alpha + b_{\alpha\beta}\lambda^\alpha\lambda^\beta \mathbf{n}. \tag{B.30}$$

Let us define the *normal curvature* of γ at (u^α) as the quantity

$$\xi_n(\lambda) = \mathbf{k} \cdot \mathbf{n} = b_{\alpha\beta}\lambda^\alpha\lambda^\beta. \tag{B.31}$$

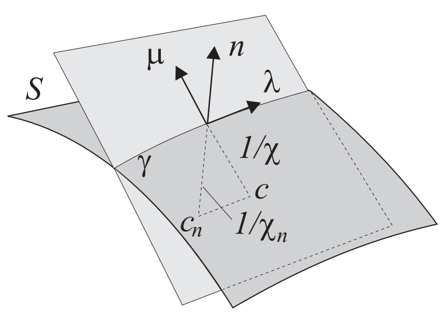

Fig. B.1 Normal curvature

This formula shows that all of the surface curves that pass through a point $\mathbf{r}(u^\alpha)$ have the same normal curvature. If γ_n is a curve whose osculating plane is determined by λ and $\mathbf{n}$, then we have

$$\xi_n \equiv \mathbf{k} \cdot \mathbf{n} = \chi\mu \cdot \mathbf{n} = \chi, \tag{B.32}$$

and therefore the normal curvature of γ_n coincides with the normal curvature. Moreover, from the obvious inequality

$$|\xi_n| = |\chi\mu \cdot \mathbf{n}| \leq |\chi|, \tag{B.33}$$

we can conclude that, *among all of the curves on S with the same unit tangent vector λ, the curve γ_n has the least curvature; moreover, the center of curvature c_n for γ_n coincides with the projection onto its osculating plane of the centers of curvature c of all of the curves tangent to λ* (see Fig. B.1).

Let S be a regular surface of class C^2 and $T_2(\mathbf{r})$ be its tangent plane at the point $\mathbf{r}$. The first fundamental form, which is positive definite, allows us to regard $T_2(\mathbf{r})$ as a Euclidean two-dimensional vector space. A basis for it is given by $(\mathbf{a}_1, \mathbf{a}_2)$, and the scalar product is defined by the metric coefficients $a_{\alpha\beta}$ at $\mathbf{r}$. The tensor $b_{\alpha\beta}$ is symmetric, and so there are two real eigenvalues, φ_1 and φ_2 (which could also be equal), together with an orthonormal basis comprising two eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ belonging to the above eigenvalues. These eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are solutions of the homogeneous linear system

$$(b_{\alpha\beta} - \varphi a_{\alpha\beta})v^\beta = 0, \tag{B.34}$$

whereas the eigenvalues φ_1 and φ_2 satisfy the characteristic equation

$$\det(b_{\alpha\beta} - \varphi a_{\alpha\beta}) = 0, \tag{B.35}$$

which can also be put into the form

$$\varphi^2 - 2H\varphi + K = 0, \tag{B.36}$$

where H and K are the two principal invariants of $(b_{\alpha\beta})$:

$$H = \frac{1}{2} b^\alpha_\alpha, \quad K = \frac{1}{a} \det(b_{\alpha\beta}) = \frac{b}{a}. \tag{B.37}$$

In the basis $(\mathbf{v}_1, \mathbf{v}_2)$, the tensor $b_{\alpha\beta}$ is represented by the diagonal matrix

$$\begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix}, \tag{B.38}$$

and the quadratic form (B.31) can be written as

$$\chi_n(\lambda) = \varphi_1(\lambda^1)^2 + \varphi_2(\lambda^2)^2, \tag{B.39}$$

where (λ^1, λ^2) are the components of the tangent vector λ with respect to the basis $(\mathbf{v}_1, \mathbf{v}_2)$. If $\chi_n(\lambda)$ does not vanish identically, then three cases can be distinguished:

1. φ_1, $\varphi_2 \neq 0$ have the same sign. The quadratic form $\chi_n(\lambda)$ then has a definite sign at $\mathbf{r}$ and so the normal curvature always has the same sign upon varying λ. All of the points of S lie on the same side of the plane $T_2(\mathbf{r})$. In this case, $\mathbf{r}$ is said to be an *elliptic point* (see Fig. B.2).

2. φ_1, $\varphi_2 \neq 0$ have opposite signs. This means that the curvatures along the two orthogonal lines, defined by $\mathbf{v}_1$ and $\mathbf{v}_2$, have opposite signs. Therefore, the tangent plane $T_2(\mathbf{r})$ intersects S in a neighborhood of $\mathbf{r}$ known as a *hyperbolic point*. Moreover, there are two directions λ_1 and λ_2 along which the curvature vanishes. These lines, called *asymptotic lines*, divide $T_2(\mathbf{r})$ into four regions where the curvature is positive or negative in turn (see Fig. B.3).

3. $\varphi_1 \neq 0$, $\varphi_2 = 0$. In this case, $\chi_n(\lambda) = \varphi_1(\lambda_1)^2$. Therefore, the normal curvature has the same sign along any direction that differs from $\mathbf{v}_2$, and it vanishes when λ is parallel to $\mathbf{v}_2$. The point $\mathbf{r}$ is said to be a *parabolic point.* (see Fig. B.4).

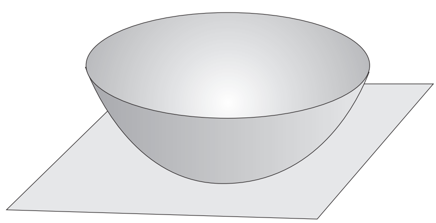

Fig. B.2 Elliptic point

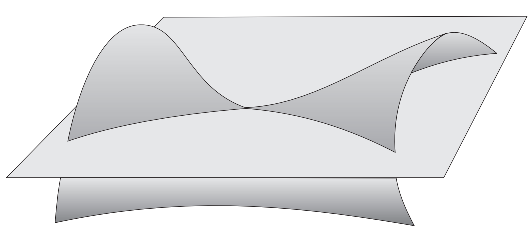

Fig. B.3 Hyperbolic point

Fig. B.4 Parabolic point

The eigenvalues of $b_{\alpha\beta}$, which we will henceforth denote by $\chi_{(1)}$ and $\chi_{(2)}$, are called the *principal curvatures* of S at $\mathbf{r}$, whereas the eigenvectors of $b_{\alpha\beta}$ are the *principal directions* of S at $\mathbf{r}$. From Cartesio's rule, we obtain the following relations:

$$\chi_{(1)} + \chi_{(2)} = 2H, \quad \chi_{(1)}\chi_{(2)} = K. \tag{B.40}$$

By definition, the scalar quantity H is the *mean curvature* of S and K is the *Gaussian curvature.*

A curve γ on S is said to be a *line of curvature* if its tangent vector at any point is a principal direction. In particular, if $\chi_{(1)} = \chi_{(2)}$ at each point, then all of the coordinate curves are lines of curvature (planes and spheres). It is possible to prove that, in the neighborhood of any point $\mathbf{r}$ on a surface of class C^k, $k \geq 2$, there is a local system of coordinates such that the corresponding coordinate curves at $\mathbf{r}$ are principal lines. It is also evident that the coordinate curves are principal lines if and only if

$$a_{12} = b_{12} = 0. \tag{B.41}$$

In such a system of coordinates, the following relations hold:

$$ds^2 = a_{11}(du^1)^2 + a_{22}(du^2)^2,$$

$$(a^{\alpha\beta}) = \begin{pmatrix} \dfrac{1}{a_{11}} & 0 \\ 0 & \dfrac{1}{a_{22}} \end{pmatrix}, \quad (b_{\alpha\beta}) = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix},$$

$$\chi_{(1)} = \frac{b_{11}}{a_{11}}, \quad \chi_{(2)} = \frac{b_{22}}{a_{22}},$$

$$H = \frac{1}{2}\left(\frac{b_{11}}{a_{11}} + \frac{b_{22}}{a_{22}}\right), \quad K = \frac{b_{11}b_{22}}{a_{11}a_{22}}.$$

B.3 Surface Gradient and the Gauss Theorem

Let $\mathbf{u} = u^\alpha \mathbf{a}_\alpha + u^3 \mathbf{n}$ be a vector field on the surface S that is not necessarily tangent to S. From (B.22) and (B.23), we derive the relation

$$\mathbf{u},_\alpha = (u^\gamma_{;\alpha} - b^\gamma_\alpha u^3)\mathbf{a}_\gamma + (u,^3_\alpha + b_{\alpha\beta}u^\beta)\mathbf{n}, \tag{B.42}$$

where

$$u^\gamma_{;\alpha} = u,^\gamma_\alpha + \Gamma^\gamma_{\alpha\beta}u^\beta \tag{B.43}$$

are the components of the *covariant derivative* of the tangent field $\mathbf{v}_s = u^\alpha \mathbf{a}_\alpha$ obtained by projecting $\mathbf{u}$ onto the tangent plane $T_2(\mathbf{r})$ at any point $\mathbf{r} \in S$. In particular, if $\mathbf{u}$ is tangent to S, then $u^3 = 0$ and (B.42) reduces to the following formula:

$$\mathbf{u},_\alpha = u^\gamma_{;\alpha}\mathbf{a}_\gamma + b_{\alpha\beta}u^\beta \mathbf{n}. \tag{B.44}$$

Similarly, if we denote a double tensor that obeys the condition $\mathbf{T} \cdot \mathbf{n} = \mathbf{0}$ by

$$\mathbf{T} = T^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta + T^{3\alpha}\mathbf{n} \otimes \mathbf{a}_\alpha,$$

then, instead of (B.44), we have

$$\begin{aligned}\mathbf{T},_\alpha = {} & (T^{\beta\gamma}_{;\alpha} - T^{3\beta}b^\gamma_\alpha)\mathbf{a}_\beta \otimes \mathbf{a}_\gamma + (T^{3\beta}_{;\alpha} + T^{\beta\gamma}b_{\gamma\alpha}) \otimes \mathbf{n}\mathbf{a}_\beta \\ & + T^{\beta\gamma}b_{\gamma\alpha}\mathbf{a}_\beta \otimes \mathbf{n} + T^{3\beta}b_{\alpha\beta}\mathbf{n} \otimes \mathbf{n},\end{aligned} \tag{B.45}$$

where

$$T^{\beta\gamma}_{;\alpha} = T,^{\beta\gamma}_\alpha + \Gamma^\beta_{\alpha\delta}T^{\delta\gamma} + \Gamma^\gamma_{\alpha\delta}T^{\beta\delta}, \tag{B.46}$$

$$T^{\beta 3}_{;\alpha} = T,^{\beta 3}_\alpha + \Gamma^\beta_{\alpha\gamma}T^{\gamma 3}. \tag{B.47}$$

Let us define the *surface gradient* of $\mathbf{T}$ as follows:

$$\nabla_s \mathbf{T} = \mathbf{T},_\alpha \otimes \mathbf{a}^\alpha, \tag{B.48}$$

and the *surface divergence* as

$$\nabla_s \cdot \mathbf{T} = \mathbf{T},_\alpha \cdot \mathbf{a}^\alpha. \tag{B.49}$$

From (B.42) and (B.37), we have

$$\nabla_s \cdot \mathbf{u} = u^\gamma_{;\gamma} - 2Hu^3, \tag{B.50}$$

whereas, when $\mathbf{n} \cdot \mathbf{T} = \mathbf{0}$, (B.45) yields

$$\nabla_s \cdot \mathbf{T} = (T^{\alpha\gamma}_{;\alpha} - T^{\alpha 3}b^\gamma_\alpha)\mathbf{a}_\gamma + (T^{\alpha 3}_{;\alpha} + T^{\alpha\gamma}b_{\gamma\alpha})\mathbf{n}. \tag{B.51}$$

Consider a rectilinear system of coordinates (x^i) in $\Re^3$ and the associated frame of reference $(O, \mathbf{e}_i)$. Since we have

$$\begin{aligned}\mathbf{T},_\alpha &= (T^{ij}\mathbf{e}_i \otimes \mathbf{e}_j),_\alpha = (T^{ij}\mathbf{e}_i \otimes \mathbf{e}_j),_h x^h,_\alpha \\ &= (T^{ij},_h \mathbf{e}_i \otimes \mathbf{e}_j)(\mathbf{e}_h \cdot \mathbf{a}_\alpha) = \mathbf{a}_\alpha \cdot \nabla \mathbf{T}\end{aligned} \tag{B.52}$$

for any double tensor $\mathbf{T}$ from (B.3), we can conclude that $\mathbf{u},_\alpha$ denotes the gradient of $\mathbf{u}$ along the vector $\mathbf{a}_\alpha$. We also note that, if $\mathbf{u}$ is a constant vector field in the coordinates (x^i), then $\nabla \mathbf{u} = \mathbf{0}$, $\mathbf{u},_\alpha = \mathbf{0}$ and $\nabla_s \cdot \mathbf{u} = 0$.

We want to verify that the definition (B.50) leads us to a generalization of the Gauss theorem. In fact, if $\mathbf{u}$ is a vector field on S that is not necessarily tangent to S, we have

$$\int_S u^\alpha_{;\alpha}\, d\sigma = \int_{\partial S} \mathbf{u} \cdot \nu\, dl, \tag{B.53}$$

where ν is a unit vector that is tangent to S and orthogonal to the boundary ∂S. Using (B.50), we obtain

$$\int_S (\nabla_s \cdot \mathbf{u} + 2H\mathbf{u} \cdot \mathbf{n})\, d\sigma = \int_{\partial S} \mathbf{u} \cdot \nu\, dl. \tag{B.54}$$

By applying this result to the vector $\mathbf{v} \cdot \mathbf{T}$, where $\mathbf{T}$ is a double tensor field and $\mathbf{v}$ an *arbitrary constant* vector field, we get

$$\int_S [\nabla_s \cdot (\mathbf{v} \cdot \mathbf{T}) + 2H\mathbf{v} \cdot \mathbf{T} \cdot \mathbf{n}]\, d\sigma = \int_{\partial S} \mathbf{v} \cdot \mathbf{T} \cdot \nu\, dl. \tag{B.55}$$

Taking into account the arbitrariness of the constant vector field $\mathbf{v}$, we obtain the formula

$$\int_S [\nabla_s \cdot \mathbf{T} + 2H\mathbf{T} \cdot \mathbf{n}]\, d\sigma = \int_{\partial S} \mathbf{T} \cdot \nu\, dl, \tag{B.56}$$

which reduces to the following:

$$\int_S \nabla_s \cdot \mathbf{T}\, d\sigma = \int_{\partial S} \mathbf{T} \cdot \nu\, dl, \tag{B.57}$$

since we have supposed that

$$\mathbf{T} \cdot \mathbf{n} = \mathbf{0}. \tag{B.58}$$

We need to generalize the above formulae to the case in which the surface S is not regular along a curve Γ and the vector or tensor fields undergo a jump across Γ. We assume that S is divided by Γ into two parts S^- and S^+ (see Fig. B.5). By applying (B.54) separately to each of the two parts S^- and S^+ and adding the results, we find that

$$\int_S [\nabla_s \cdot \mathbf{u} + 2H\mathbf{n} \cdot \mathbf{u}]\, d\sigma = \int_{\partial S} \nu \cdot \mathbf{u}\, dl + \int_\Gamma \{\nu \cdot \mathbf{u}\}\, dl, \tag{B.59}$$

where

$$\{\nu \cdot \mathbf{u}\} = \nu^+ \cdot \mathbf{u}^+ - \nu^- \cdot \mathbf{u}^- \tag{B.60}$$

and the rest of the notation is self-explanatory.

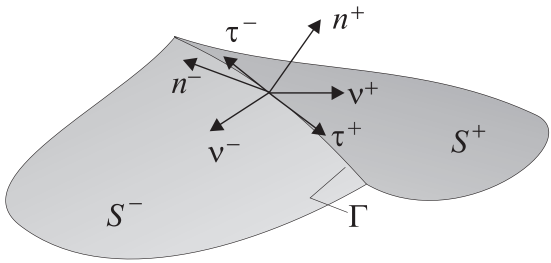

Fig. B.5 The Gauss theorem for a singular surface

If we denote the unit vector that is tangent to Γ and which determines the orientation of Γ by $\tau = \mathbf{n} \times \nu$, then we have $\nu = \tau \times \mathbf{n}$. Consequently, (B.60) becomes

$$\{\nu \cdot \mathbf{u}\} = \tau^+ \cdot \mathbf{n}^+ \times \mathbf{u}^+ + \tau^- \cdot \mathbf{n}^- \times \mathbf{u}^- = -\tau \cdot [[\mathbf{n} \times \mathbf{u}]], \tag{B.61}$$

since $\tau^+ = \tau^- \equiv \tau$, provided that the sense of τ is the same as that of the second term in the jump. Finally, we can put (B.56) into the form below:

$$\int_S [\nabla_s \cdot \mathbf{u} + 2H\mathbf{n} \cdot \mathbf{u}]\, d\sigma = \int_{\partial S} \nu \cdot \mathbf{u}\, dl - \int_\Gamma \tau \cdot [[\mathbf{n} \times \mathbf{u}]]\, dl. \tag{B.62}$$

Similarly, if $\mathbf{T}$ is a double tensor that obeys the condition $\mathbf{n} \cdot \mathbf{T} = \mathbf{0}$, then it can be proven that

$$\int_S \nabla_s \cdot \mathbf{T}\, d\sigma = \int_{\partial S} \nu \cdot \mathbf{T}\, dl - \int_\Gamma \tau \cdot [[\mathbf{n} \times \mathbf{T}]]\, dl. \tag{B.63}$$

Appendix C

First-Order PDE

C.1 Monge's Cone

In Chap. 9 of [16] and Chap. 1 of this volume, it was shown that the evolution of the wavefront of an ordinary wave is governed by a first-order PDE called the *eikonal equation*. This coincides with the characteristic equation associated with the hyperbolic first-order PDE that describes the phenomenon under examination. In this Appendix we sketch out the method proposed by Monge, Ampere and Cauchy to reduce the integration of a first-order PDE to the integration of a system of ordinary equations.

Let $F(\mathbf{x}, u, \mathbf{p})$ be a function of class $C^2(\Re^{2n+1})$ that obeys the following conditions:

1. The set $\mathbb{F} = \{(\mathbf{x}, u, \mathbf{p}) \in \Re^{2n+1},\ F(\mathbf{x}, u, \mathbf{p}) = 0\}$ is not empty
2. $\sum_{i=1}^{n} F_{p_i}^2 \neq 0,\ \forall (\mathbf{x}, u, \mathbf{p}) \in \mathbb{F}$.

Under these hypotheses, we consider the following general first-order PDE

$$F(\mathbf{x}, u, \nabla u) = 0 \tag{C.1}$$

in the unknown $u = u(\mathbf{x}) \in C^2(D)$, with $D \subset \Re^n$.

We first note that any solution $u = u(\mathbf{x})$ of (C.1) defines a surface Σ that is called an *integral surface* of the equation (C.1). Moreover, the vector $\mathbf{N} \equiv (\nabla u, -1) \equiv (\mathbf{p}, -1)$ of $\Re^{n+1}$ is normal to the integral surface Σ. Consequently, the equation (C.1) expresses the relation between the vectors normal to all of the integral surfaces at any point $(\mathbf{x}, u)$.

To understand the geometrical meaning of this relation, we start by noting that the quantities $(\mathbf{x}_0, u_0, \mathbf{p}_0)$ completely define a plane that contains the point $(\mathbf{x}_0, u_0)$ and has a normal vector with components $(\mathbf{p}_0, -1)$. Then, for a fixed point $(\mathbf{x}_0, u_0)$, the equation

$$F(\mathbf{x}_0, u_0, \mathbf{p}_0) = 0, \tag{C.2}$$

defines a set Π_0 of planes that contain $(\mathbf{x}_0, u_0)$ and are tangent to the integral surfaces of (C.1) to which $(\mathbf{x}_0, u_0)$ belongs. Due to conditions 1 and 2, Π_0 is not empty, and we can assume that $F_{p_n}(\mathbf{x}_0, u_0, \mathbf{p}_0) \neq 0$. In turn, this result implies that the equation (C.2), at least locally, can be written in the form

$$p_n = p_n(\mathbf{x}_0, u_0, p_1, \ldots, p_{n-1}).$$

Consequently, the set Π_0 consists of a family of planes that depend on the parameters $(p_1, \ldots, p_{n-1})$. Any plane $\pi \in \Pi_0$ contains the point $(\mathbf{x}_0, u_0)$ and has $(p_1, \ldots, p_{n-1}, p_n(p_1, \ldots, p_{n-1}), -1)$ as a normal vector; i.e., it is represented by the equation

$$f(\mathbf{X}, U, p_1, \ldots, p_{n-1}) \equiv \sum_{\alpha=1}^{n-1} p_\alpha (X_\alpha - x_{0\alpha}) + p_n(p_\alpha)(U - u_0) = 0, \quad \text{(C.3)}$$

where $(\mathbf{X}, U)$ are the coordinates of any point on π.

The envelope of all of these planes is defined by the system comprising the equation (C.3) and the following other equation:

$$(X_\alpha - x_{0\alpha}) + \frac{\partial p_n}{\partial p_\alpha}(X_n - x_{0n}) = 0, \quad \alpha = 1, \ldots, n-1, \quad \text{(C.4)}$$

which is obtained by differentiating (C.3) with respect to $p_\alpha,\ \alpha = 1, \ldots, n-1$. Using Dini's theorem, $\partial p_n / \partial p_\alpha = -F_{p_\alpha}/F_{p_n}$ and (C.4) becomes

$$F_{p_n}(X_\alpha - x_{0\alpha}) - F_{p_\alpha}(X_n - x_{0n}) = 0, \quad \alpha = 1, \ldots, n-1. \quad \text{(C.5)}$$

Equations (C.3) and (C.5) constitute a linear system of n equations in the unknowns $((X_\alpha - x_{0\alpha}), (X_n - x_{0n}))$. The determinant of the matrix of the coefficients of this system is

$$\Delta = \begin{pmatrix} p_1 & p_2 & \ldots & p_{n-1} & p_n \\ F_{p_n} & 0 & \ldots & 0 & -F_{p_1} \\ 0 & F_{p_n} & \ldots & 0 & -F_{p_2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & F_{p_n} & -F_{p_{n-1}} \end{pmatrix} = (-1)^{n-1} F_{p_n}^{n-2} (\mathbf{p} \cdot \mathbf{F_p}) \neq 0.$$

Consequently, the parametric equations of the envelope of Π_0 become

$$X_i - x_{0i} = \frac{F_{p_i}}{\mathbf{p} \cdot \mathbf{F_p}}(U - u_0), \quad i = 1, \ldots, n. \quad \text{(C.6)}$$

These equations define a cone C_0, since, if the vector $\mathbf{V} = (\mathbf{X} - \mathbf{x}_0, U - u_0)$ is a solution of (C.6), the vector $\lambda \mathbf{V}$ (where $\lambda \in \Re$) is also a solution. This cone C_0 is called *Monge's cone* at $(\mathbf{x}_0, u_0)$.

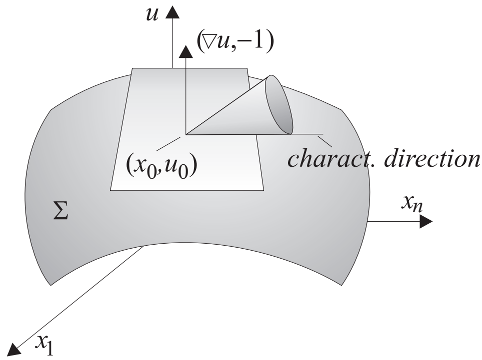

Fig. C.1 Monge's cone

C.2 Characteristic Strips

The above considerations allow us to state that, if $u = u(\mathbf{x})$ is an integral surface Σ of (C.1), then the plane tangent to Σ at the point $(\mathbf{x}_0, u_0)$ belongs to Π_0; i.e., it is tangent to Monge's cone at that point (see Fig. C.1). Moreover, the characteristic directions of Monge's cone, which are tangent to Σ, define a field of tangent vectors on Σ whose integral curves are called *characteristic curves*.

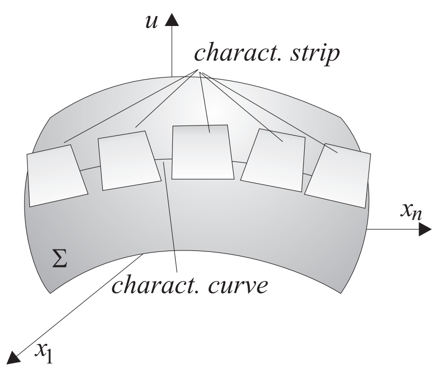

Fig. C.2 Characteristic curve and strip

We can associate a plane tangent to Σ that has director cosines $(\mathbf{p}, -1) \equiv (\nabla u(\mathbf{x}(s)), -1)$ with any point on this curve $(\mathbf{x}(s), u(s))$. The one-parameter family of these planes along a characteristic curve is said to be a *characteristic strip* (see Fig. C.2).

We now write the system of ordinary differential equations in the unknowns $(\mathbf{x}(s), u(s), \mathbf{p}(s))$ that define a characteristic strip on the integral surface Σ. Since the tangent to a characteristic curve at a point belongs to Monge's cone at that point, from (C.6) we have

$$\frac{d\mathbf{x}}{ds} = F_{\mathbf{p}}, \tag{C.7}$$

$$\frac{du}{ds} = \mathbf{p} \cdot F_{\mathbf{p}}. \tag{C.8}$$

Moreover, $\mathbf{p}(s) = \nabla u(\mathbf{x}(s))$ along a characteristic curve $(\mathbf{x}(s), u(s))$, and so we get

$$\frac{dp_i}{ds} = u_{x_i x_j} F_{p_j}.$$

On the other hand, the differentiation of (C.1) with respect to x_i gives

$$F_{x_i} + u_{x_i} F_u + u_{x_i x_j} F_{p_j} = 0.$$

Comparing the latter two relations, we obtain the following system of $2n+1$ equations:

$$\frac{d\mathbf{x}}{ds} = F_{\mathbf{p}}, \tag{C.9}$$

$$\frac{d\mathbf{p}}{ds} = -(F_{\mathbf{x}} + F_u \mathbf{p}), \tag{C.10}$$

$$\frac{du}{ds} = \mathbf{p} \cdot F_{\mathbf{p}} \tag{C.11}$$

in the $2n+1$ unknowns $(\mathbf{x}(s), u(s), \mathbf{p}(s))$, which is called the *characteristic system* of (C.1).

We have proven that, if $u(\mathbf{x})$ is an integral surface of (C.1), then its characteristic strips satisfy system (C.9)–(C.11). Moreover, any solution of (C.9)–(C.11) is a characteristic strip of an integral surface of (C.1), as proven by the following theorem.

Theorem C.1
Let $(\mathbf{x}_0, u_0, \mathbf{p}_0 = \nabla u(\mathbf{x}_0))$ be the plane tangent to an integral surface Σ at the point $(\mathbf{x}_0, u_0)$. The solution of (C.9)–(C.11) corresponding to the initial datum $(\mathbf{x}_0, u_0, \mathbf{p}_0)$ is then a characteristic strip of Σ.

PROOF It is sufficient to recall what we have already proven and to remark that the characteristic strip determined by the initial datum $(\mathbf{x}_0, u_0, \mathbf{p}_0)$ is unique for the uniqueness theorem. ∎

Taking into account the function $F(\mathbf{x}(s), u(s), \mathbf{p}(s))$, and recalling (C.9)–(C.11), we verify that

Theorem C.2

$$F(\mathbf{x}(s), u(s), \mathbf{p}(s)) = const. \tag{C.12}$$

Remark If (C.1) is quasi-linear,

$$F(\mathbf{x}, u, \nabla u) = \mathbf{a}(\mathbf{x}, u) \cdot \nabla u - b(\mathbf{x}, u) = 0, \tag{C.13}$$

then the system (C.9) and (C.11) becomes

$$\frac{d\mathbf{x}}{ds} = \mathbf{a}(\mathbf{x}, u), \tag{C.14}$$

$$\frac{du}{ds} = b(\mathbf{x}, u). \tag{C.15}$$

This is a system of $n+1$ equations in the unknowns $(\mathbf{x}(s), u(s))$, which can be solved without the help of (C.10). More particularly, if (C.13) is linear, then $\mathbf{a}$ and $\mathbf{b}$ depend only on $\mathbf{x}$. Therefore, the equations (C.14) supply the projection of the characteristic curves in $\Re^n$, and (C.15) gives the remaining unknown $u(s)$.

Remark Let us suppose that (C.1) takes the form

$$F(\mathbf{x}, \nabla u) = 0. \tag{C.16}$$

Denoting the variable x_n by t, the derivative F_{u_t} by p, the vector $(x_1, \dots, x_{n-1})$ by $\mathbf{x}$, and the vector $(p_1, \dots, p_{n-1})$ by $\mathbf{p}$, the above equation becomes

$$F(\mathbf{x}, t, \mathbf{p}, p) = 0, \tag{C.17}$$

If $F_p \neq 0$, then (C.17) can be written as the Hamilton–Jacobi equation

$$p + H(\mathbf{x}, t, \mathbf{p}) = 0, \tag{C.18}$$

the characteristic system of which is

$$\frac{d\mathbf{x}}{ds} = H_{\mathbf{p}}, \tag{C.19}$$

$$\frac{d\mathbf{p}}{ds} = -H_{\mathbf{x}}, \tag{C.20}$$

$$\frac{du}{ds} = \mathbf{p} \cdot H_{\mathbf{p}} + p, \tag{C.21}$$

$$\frac{dt}{ds} = 1, \tag{C.22}$$

$$\frac{dp}{ds} = -H_t. \tag{C.23}$$

Due to (C.22), we can identify s with t, and the above system becomes

$$\frac{d\mathbf{x}}{dt} = H_{\mathbf{p}}, \tag{C.24}$$
$$\frac{d\mathbf{p}}{dt} = -H_{\mathbf{x}}, \tag{C.25}$$
$$\frac{du}{dt} = \mathbf{p} \cdot H_{\mathbf{p}} - H, \tag{C.26}$$
$$\frac{dp}{ds} = -H_t. \tag{C.27}$$

We note that (C.24) and (C.25) are Hamiltonian equations that can be solved without needing to solve the other equations.

C.3 Cauchy's Problem

The Cauchy problem relating to (C.1) can be formulated as follows.

Let Γ *be a* $(n-1)$*-dimensional manifold present in a region* D *of* $\Re^n$*, and let* $u_0(\mathbf{x}) \in C^2(\Gamma)$ *be an assigned function on* Γ*. Determine a solution* $u(\mathbf{x})$ *of (C.1) whose restriction to* Γ *coincides with* $u_0(\mathbf{x})$.

In other words, if

$$\mathbf{x}_0 = \mathbf{x}_0(v_\alpha), \quad (v_\alpha) \in V \subset \Re^{n-1}, \tag{C.28}$$

is a parametric representation of Γ, Cauchy's problem consists of finding a solution $u(\mathbf{x})$ of (C.1) such that

$$u(\mathbf{x}_0(v_\alpha)) = u_0(\mathbf{x}_0(v_\alpha)). \tag{C.29}$$

Geometrically, we can say that we want to determine an $n-$dimensional manifold $u(\mathbf{x})$ that satisfies (C.1) and contains the initial $(n-1)$-dimensional manifold Γ.

We prove that the requested solution can be obtained as the envelope of a suitable family of characteristic strips. To this end, we first note that the points of Γ given by the initial datum (C.29) supply the initial data for the unknowns $\mathbf{x}(s), u(s)$ of (C.9) and (C.11). However, so far we have no initial data for the unknown $\mathbf{p}(s)$. We show that these data can also be obtained by (C.29). In fact, we recall that the variable $\mathbf{p}$, together with -1, defines the vector normal to the plane tangent to the integral surface Σ at the point $(\mathbf{x}, u(\mathbf{x})) \in \Sigma$. Moreover, Σ must be tangent to the Monge cone along a characteristic direction at any point, and, in particular, at any point of Γ (see Fig. C.3).

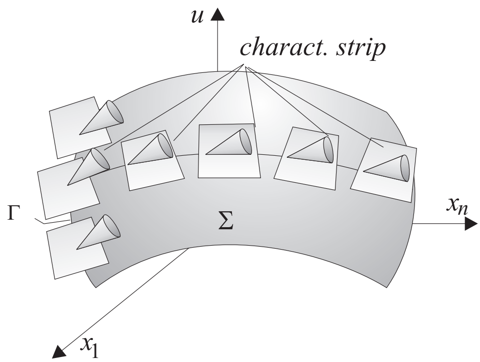

Fig. C.3 Cauchy data for (C.9)–(C.11)

Consequently, we take $\mathbf{p}_0$ in such a way that, at any point on Γ, $(\mathbf{p}_0, -1)$ is orthogonal to the vector $(\partial \mathbf{x}/\partial v_\alpha, \partial u/\partial v_\alpha)$, which is tangent to Γ. In formulae, we have

$$\frac{\partial \mathbf{x}_0}{\partial v_\alpha} \cdot \mathbf{p}_0 - \frac{\partial u_0}{\partial v_\alpha} = 0, \tag{C.30}$$

$$F(\mathbf{x}_0(v_\alpha), u_0(v_\alpha), \mathbf{p}_0(v_\alpha)) = 0, \tag{C.31}$$

$\alpha = 1, \ldots, n-1$.

If $(\overline{\mathbf{x}}_0(\overline{v}_\alpha), \overline{u}(\overline{v}_\alpha), \overline{\mathbf{p}}_0(\overline{v}_{\alpha_0}) \in \Gamma$ verifies the above system and the following condition is satisfied at this point:

$$J = \det \left(\frac{\partial \mathbf{x}_0}{\partial v_\alpha} \; F_{\mathbf{p}} \right) \neq 0, \tag{C.32}$$

the system (C.30) and (C.31) can be solved with respect $\mathbf{p}_0(v_\alpha)$ in a neighborhood of the point $(\overline{\mathbf{x}}_0(\overline{v}_\alpha), \overline{u}(\overline{v}_\alpha), \overline{\mathbf{p}}_0(\overline{v}_{\alpha_0}))$. In this way, we obtain the initial data for (C.9)–(C.11).

The proof of the following theorem can be found in any book on PDEs:

Theorem C.3
If it is possible to complete the initial datum (C.29) by solving the system (C.30) and (C.31) with respect to $\mathbf{p}_0(v_\alpha)$, then there is one and only one solution of the Cauchy problem.

Appendix D

The Tensor Character of Some Physical Quantities

In this Appendix we complete the proof of the theorem stated in Sect. 10.6.

PROPOSITION D.1

Let S be a continuous system, and let (U^α) be the 4-velocity field of its particles. We denote fields associated with S that satisfy the conditions

$$\frac{\partial T^{\alpha\beta}}{\partial x^\beta} = 0 \tag{D.1}$$

in any Lorentz frame $(O, \mathbf{e}_\alpha)$ by $T^{\alpha\beta}$, $\alpha, \beta = 1, \ldots, 4$. Moreover, if $T'^{\alpha\beta}$ are the corresponding fields in the other Lorentz frame $(O', \mathbf{e}'_\alpha)$, we have

$$T'^{\alpha\beta} = T'^{\alpha\beta}(T^{\lambda\mu}), \tag{D.2}$$

$$T^{\lambda\mu} U_\lambda U_\mu = c^2 \overline{T}^{44}, \tag{D.3}$$

where $\overline{T}^{44}$ is evaluated in the proper frame of a particle of S. If $T'^{\alpha\beta}(0) = 0$ and $\overline{T}^{44}$ is invariant, then

$$T'^{\alpha\beta} = A^\alpha_\lambda A^\beta_\mu T^{\lambda\mu}, \tag{D.4}$$

where (A^α_β) is the matrix of the frame change $(O, \mathbf{e}_\alpha) \to (O', \mathbf{e}'_\alpha)$.

PROOF We first prove that the functions (D.2) are linear. From (D.1) and (D.2), we derive the condition

$$\frac{\partial T'^{\alpha\beta}}{\partial T^{\lambda\mu}} (A^{-1})^\nu_\beta \frac{\partial T^{\lambda\mu}}{\partial x^\nu} = 0, \tag{D.5}$$

which must be satisfied for all of the quantities $\partial T^{\lambda\mu}/\partial x^\nu$ such that

$$\partial T^{\lambda\mu}/\partial x^\mu = 0. \tag{D.6}$$

Since the above relation can equivalently be written as

$$\delta_\mu^\nu \frac{\partial T^{\lambda\mu}}{\partial x^\nu} = 0, \tag{D.7}$$

we conclude that the relation

$$\left(\frac{\partial T'^{\alpha\beta}}{\partial T^{\lambda\mu}} (A^{-1})_\beta^\nu - \Gamma_\lambda^\alpha \delta_\mu^\nu \right) \frac{\partial T^{\lambda\mu}}{\partial x^\nu} = 0, \tag{D.8}$$

where Γ_λ^α are suitable Lagrangian multipliers, must be satisfied for any choice of the quantities $\partial T^{\lambda\mu}/\partial x^\nu$. Consequently, we have

$$\frac{\partial T'^{\alpha\beta}}{\partial T^{\lambda\mu}} = \Gamma_\lambda^\alpha A_\mu^\rho. \tag{D.9}$$

We now prove that the multipliers Γ_λ^α do not depend on $T^{\lambda\mu}$. We have

$$\frac{\partial^2 T'^{\alpha\rho}}{\partial T^{\tau\nu} \partial T^{\lambda\mu}} = \frac{\partial \Gamma_\lambda^\alpha}{\partial T^{\tau\nu}} A_\mu^\rho,$$

and by changing the derivation order we get

$$\frac{\partial \Gamma_\lambda^\alpha}{\partial T^{\tau\nu}} A_\mu^\rho = \frac{\partial \Gamma_\tau^\alpha}{\partial T^{\lambda\mu}} A_\nu^\rho.$$

Multiplying the above relation for $(A^{-1})_\rho^\beta$ leads to

$$\frac{\partial \Gamma_\lambda^\alpha}{\partial T^{\tau\nu}} \delta_\mu^\beta = \frac{\partial \Gamma_\tau^\alpha}{\partial T^{\lambda\mu}} \delta_\nu^\beta.$$

For $\beta = \mu \neq \nu$, this relation implies that

$$\frac{\partial \Gamma_\lambda^\alpha}{\partial T^{\tau\nu}} = 0. \tag{D.10}$$

Finally, taking into account the condition $T'^{\alpha\beta}(0) = 0$, we can write

$$T'^{\alpha\beta} = \Gamma_\lambda^\alpha A_\mu^\rho T^{\lambda\mu}. \tag{D.11}$$

Let W_α denote any 4-vector field of spacetime. If we introduce the quantities

$$S^\beta = W_\alpha T^{\alpha\beta}, \tag{D.12}$$

then, from (D.11) and (D.1), we obtain

$$S'^\beta = S'^\beta(S^\lambda), \tag{D.13}$$

$$\frac{\partial S^\beta}{\partial x^\beta} = 0 \tag{D.14}$$

in every Lorentz frame. Proceeding as above, we prove that

$$S'^{\beta} = \lambda A^{\beta}_{\alpha} S^{\alpha}. \tag{D.15}$$

Taking into account (D.15), (D.12) and (D.11), we reach the conclusion that

$$T'^{\alpha\beta} = \lambda A^{\alpha}_{\lambda} A^{\beta}_{\mu} T^{\lambda\mu}, \tag{D.16}$$

where λ is constant. From the invariance of $\overline{T}^{44}$, it follows that $\lambda = 1$. ∎

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