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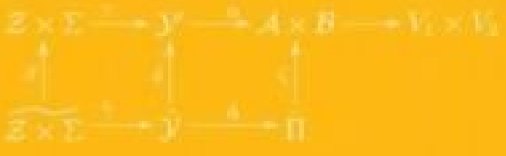
Jan Christian Rohde

Cyclic Coverings, Calabi-Yau Manifolds and Complex Multiplication



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Preface

Calabi-Yau manifolds have been an object of extensive research during the last two decades. One of the reasons is the importance of Calabi-Yau 3-manifolds in modern physics - notably string theory. An interesting class of Calabi-Yau manifolds is given by those with complex multiplication (CM). Calabi-Yau manifolds with CM are also of interest in theoretical physics, e.g. in connection with mirror symmetry and black hole attractors.

It is the main aim of this book to construct families of Calabi-Yau 3-manifolds with dense sets of fibers with complex multiplication. Most examples in this book are constructed using families of curves with dense sets of fibers with CM. The contents of this book can roughly be divided into two parts. The first six chapters deal with families of curves with dense sets of CM fibers and introduce the necessary theoretical background. This includes among other things several aspects of Hodge theory and Shimura varieties. Using the first part, families of Calabi-Yau 3-manifolds with dense sets of fibers with CM are constructed in the remaining five chapters. In the appendix one finds examples of Calabi-Yau 3-manifolds with complex multiplication which are not necessarily fibers of a family with a dense set of CM fibers.

The author hopes to have succeeded in writing a readable book that can also be used by non-specialists. On the other hand the expert will find new results about variations of Hodge structures and new examples of families of curves and Calabi-Yau manifolds with dense sets of fibers with CM. The author believes that this book will also be interesting for physicists.

This book is based on the authors doctoral thesis at Universität Duisburg-Essen. The author wishes to thank his former adviser Eckart Viehweg for his excellent guidance and support. The text has been revised for publication at the Graduiertenkolleg “Analysis, Geometrie und String Theorie” at Leibniz Universität Hannover.

Hannover, February 2009 *J. C. Rohde*

Contents

Introduction

These lecture notes deal with construction methods of Calabi-Yau manifolds with a special arithmetic property. In these methods we use curves with a similar arithmetic property, namely, complex multiplication. In the case of abelian varieties complex multiplication has been well studied by number theorists. The first six chapters describe how this theory for abelian varieties can be applied to the construction of curves with complex multiplication. The remaining five chapters and the appendix are devoted to the construction methods of Calabi-Yau manifolds with a similarly defined arithmetic property.

We give new examples of families of curves with dense sets of complex multiplication fibers and new examples of families of Calabi-Yau manifolds with a dense set of fibers with a similar arithmetic property. Moreover we will acquaint the reader with Mumford-Tate groups, which we use as a main tool for the study of Hodge structures and of variations of Hodge structures. The generic Mumford-Tate groups of families of cyclic covers of the projective line will be computed for a large class of examples.

Let us consider curves and Hodge structures on curves. In particular elliptic curves are both Calabi-Yau manifolds and abelian varieties. In general the points on a curve C of genus g generate a commutative group, which can be endowed with the structure of an abelian variety of dimension g, which is the Jacobian $\mathrm{Jac}(C)$ of C. The curve C can be obtained from $\mathrm{Jac}(C)$ and the principal polarization on $\mathrm{Jac}(C)$. In order to study the curve C and its properties one can also study $\mathrm{Jac}(C)$. Abelian varieties and their arithmetic properties have been well-studied by number theorists.

By Riemann's theorem, a polarized abelian variety with symplectic basis corresponds to a pure polarized integral Hodge structure of weight 1. Thus curves are determined by their Hodge structures. Therefore curves satisfy a Torelli Theorem. For Calabi-Yau manifolds one has also a local Torelli theorem. Thus one can study curves and Calabi-Yau manifolds in terms of their Hodge structures.

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Let $\mathbb{Z} \subseteq R \subseteq \mathbb{R}$ be a ring. Recall that an R-Hodge structure on an R-module V is given by a decomposition of $V_{\mathbb{C}}$ into subvector spaces $V^{p,q}$ with $\overline{V^{p,q}} = V^{q,p}$. We will see that each R-Hodge structure on V can also be given by a corresponding representation

$$h : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$$

of the Deligne torus $\mathbb{S}$, which is the algebraic subgroup of $\mathrm{GL}(\mathbb{R}^2)$ given by the matrices

$$M(x,y) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

If V and h yield a $\mathbb{Q}$-Hodge structure, we use the representation h for the definition of the Mumford-Tate group $\mathrm{MT}(V,h)$. The Mumford-Tate group $\mathrm{MT}(V,h)$ is the smallest subgroup of $\mathrm{GL}(V_{\mathbb{R}})$ defined over $\mathbb{Q}$ such that $h(\mathbb{S})$ is contained in $\mathrm{MT}(V,h)$. For a rational Hodge structure (V,h) of weight k one can replace $\mathbb{S}$ by its subgroup S^1 given by the matrices $M(x,y)$ with

$$\det M(x,y) = 1.$$

In this case one can also replace $\mathrm{MT}(V,h)$ by the analogously defined Hodge group $\mathrm{Hg}(V,h)$. The Hodge group $\mathrm{Hg}(V,h)$ coincides with the Zariski connected component of the identity in $\mathrm{MT}(V,h) \cap \mathrm{SL}(V)$. For any field F with $\mathbb{Q} \subseteq F \subseteq \mathbb{R}$ one can also consider F-Hodge structures (V,h) and define $\mathrm{MT}_F(V,h)$ and $\mathrm{Hg}_F(V,h)$ in an analogous way.

Let us consider the information which can be obtained from $\mathrm{MT}(V,h)$: for example one says that an elliptic curve E has complex multiplication, if E has a non-trivial endomorphism. This name is motivated by the fact that in this case the endomorphism ring of E is a CM field. In general an abelian variety X of dimension g is of CM type, if its endomorphism algebra contains a commutative $\mathbb{Q}$-algebra of dimension $2g$. The Mumford-Tate group of the Hodge structure on $H^1(X,\mathbb{Q})$ is a torus, if and only if X is of CM type. We say that a rational Hodge structure (V,h) has complex multiplication (CM), if $\mathrm{MT}(V,h)$ is a torus. For a curve C the Hodge structures on $H^1(C,\mathbb{Q})$ and $H^1(\mathrm{Jac}(C),\mathbb{Q})$ are isomorphic. Hence we say that a curve has CM, if the Mumford-Tate group of the Hodge structure on $H^1(C,\mathbb{Q})$ is a torus algebraic group.

Remark 1. *One can also study families of compact Kähler manifolds and their variations of Hodge structures in terms of Mumford-Tate groups. Let D be a connected complex manifold and $\mathcal{V}$ be a polarized variation of $\mathbb{Q}$-Hodge structures of weight k over D. Then over a dense subset D^0 of D the Mumford-Tate groups of all Hodge structures coincide. Let* $\mathrm{MT}(\mathcal{V})$ *denote the common Mumford-Tate group. The Hodge structures over the points of the complement of D^0 have a Mumford-Tate group contained in* $\mathrm{MT}(\mathcal{V})$. *The group* $\mathrm{MT}(\mathcal{V})$ *is called the generic Mumford-Tate group.*

We will introduce Shimura data, which consist of a reductive $\mathbb{Q}$-algebraic group G and a representation $h : \mathbb{S} \to G_{\mathbb{R}}$ satisfying certain conditions. Again consider an abelian variety X. For example the pair consisting of the Mumford-Tate group of the Hodge structure on $H^1(X,\mathbb{Q})$ and the representation h given by this Hodge structure yields a Shimura datum. By using the conditions which a Shimura datum has to satisfy we obtain:

Theorem 2. *Let (G,h) be a Shimura datum and W be a finite dimensional real vector space. Then the conjugacy class of h in $G_{\mathbb{R}}$ can be endowed with the structure of a complex manifold D. Moreover each closed embedding $G_{\mathbb{R}} \to \mathrm{GL}(W)$ yields a variation of Hodge structures over D such that over a dense set of points $p \in D$ one has Hodge structures with complex multiplication.*

Note that in the case of the Hodge structure on $H^1(X,\mathbb{Q})$ given by h and the closed embedding

$$\mathrm{id} : \mathrm{MT}(H^1(X,\mathbb{Q}),h) \hookrightarrow \mathrm{GL}(H^1(X,\mathbb{Q}))$$

the assumptions of the previous Theorem are satisfied, if X is an abelian variety.

We will give a definition of complex multiplication for arbitrary compact Kähler manifolds. Due to their application in theoretical physics we are especially interested in Calabi-Yau 3-manifolds. In theoretical physics one is also interested in complex multiplication (see [37], [38]).

Here a Calabi-Yau manifold X of dimension n is a compact Kähler manifold of dimension n such that $\Gamma(\Omega^i_X) = 0$ for all $i = 1, \ldots, n-1$ and $\omega_X \cong \mathcal{O}_X$.

For odd dimensional compact Kähler manifolds one has the intermediate Jacobians as a generalization of the Jacobians of curves. In general the intermediate Jacobian J is not an abelian variety, but only a complex torus. In the case of an arbitrary complex torus complex multiplication is defined as for an abelian variety. It can occur that the intermediate Jacobian J is constant for a family of Calabi-Yau 3-manifolds (see Example 1.6.9). Hence one intermediate Jacobian is not sufficient for an accurate description of Calabi-Yau 3-manifolds and their Hodge structures. Nevertheless the intermediate Jacobian of the manifold X of odd dimension k is of CM type, if $\mathrm{Hg}(H^k(X,\mathbb{Q}),h)$ is a torus. Moreover the endomorphism algebra of a Hodge structure (V,h) contains a commutative subalgebra of dimension equal to $\dim V$, if $\mathrm{MT}(V,h)$ is a torus. Thus we say that a compact Kähler manifold X of dimension n has CM over a totally real number field F, if $\mathrm{Hg}_F(H^n(X,F))$ is a torus. It would be very interesting to get mirror pairs of Calabi-Yau 3-manifolds with complex multiplication (see [23]).

One can also consider the Hodge groups of the Hodge structures $H^k(X,\mathbb{Q})$ for some $k \neq \dim X$. In the case of a Calabi-Yau manifold X of dimension $n > 3$, it may occur that the Hodge structure on $H^n(X,\mathbb{Q})$ has CM and the Hodge structure on $H^{n-1}(X,\mathbb{Q})$ has not CM for example. By considering the Hodge diamond of a Calabi-Yau manifold X of dimension $n \leq 3$, one concludes that this can not occur for $\dim X \leq 3$. In this case the condition of complex multiplication is equivalent to the property that for all k the Hodge group of $H^k(X, \mathbb{C})$ is commutative. We will call any family of Calabi-Yau n-manifolds, which has a dense set of fibers X satisfying the property that for all k the Hodge group of the Hodge structure on $H^k(X, \mathbb{Q})$ is commutative, a $CMCY$ family of n-manifolds. Here we will give some examples of $CMCY$ families of 3-manifolds and explain how to construct $CMCY$ families of n-manifolds in an arbitrarily high dimension. Moreover we will explicitly determine some fibers with complex multiplication (see Example 7.3.1, Section 7.4, Remark 8.3.6, Remark 9.4.1 and Remark 11.3.13).

Example 3. *The first example of a $CMCY$ family of 3-manifolds was given by C. Borcea [8]. This example uses the family $\mathcal{E}$ of elliptic curves given by*

$$\mathbb{P}^2 \supset V(y^2 x_0 + x_1(x_1 - x_0)(x_1 - \lambda x_0)) \to \lambda \in \mathbb{A}^1 \setminus \{0, 1\}.$$

By $y \to -y$, one has a global involution ι on $\mathcal{E}$. Now let $\mathcal{E}_i$ with involution ι_i be a copy of $\mathcal{E}$ for $i = 1, 2, 3$. We construct the family

$$\mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3 / \langle (\iota_1, \iota_2), (\iota_2, \iota_3) \rangle \to (\mathbb{A}^1 \setminus \{0, 1\})^3.$$

By blowing up the singular sections, we obtain a $CMCY$ family of Calabi-Yau 3-manifolds.

In a similar way one can use n copies of $\mathcal{E}$ and construct a $CMCY$ family of n-manifolds (see [56]). Similar to the previous example, we will use involutions on $CMCY$ families to obtain new $CMCY$ families of manifolds in higher dimension. The other main tool of construction which we use is motivated by the following example:

Example 4. *Starting with a family of cyclic covers of $\mathbb{P}^1$ with a dense set of CM fibers, E. Viehweg and K. Zuo [58] have constructed a CMCY family of 3-manifolds. This construction is given by a tower of projective algebraic manifolds starting with a family $\mathcal{F}_1$ of cyclic covers of $\mathbb{P}^1$ given by*

$$\mathbb{P}^2 \supset V(y_1^5 + x_1(x_1 - x_0)(x_1 - \alpha x_0)(x_1 - \beta x_0)x_0) \to (\alpha, \beta) \in \mathcal{M}_2,$$

which has a dense set of CM fibers. Since each of these covers given by the fibers of the family can be embedded into $\mathbb{P}^2$, the fibers of $\mathcal{F}_1$ are the branch loci of the fibers of a family $\mathcal{F}_2$ of cyclic covers of $\mathbb{P}^2$ of degree 5. Moreover the fibers of $\mathcal{F}_2$, which can be embedded into $\mathbb{P}^3$, are the branch loci of the fibers of a family $\mathcal{F}_3$ of cyclic covers of $\mathbb{P}^3$, which can be embedded into $\mathbb{P}^4$. The family $\mathcal{F}_3$ is given by

$$\mathbb{P}^4 \supset V(y_3^5 + y_2^5 + y_1^5 + x_1(x_1 - x_0)(x_1 - \alpha x_0)(x_1 - \beta x_0)x_0) \to (\alpha, \beta) \in \mathcal{M}_2.$$

By the adjunction formula, the fibers of $\mathcal{F}_3$ are Calabi-Yau 3-manifolds.

Let $q \in \mathcal{M}_2$. The fiber $(\mathcal{F}_3)_q$ has CM, if $(\mathcal{F}_2)_q$ has CM and $(\mathcal{F}_2)_q$ has CM, if $(\mathcal{F}_1)_q$ has CM. Because of this argument, the family $\mathcal{F}_3$ has a dense set of CM fibers which lie over the same points as the CM fibers of the family of curves we have started with.

The previous example contains a deformation of the Fermat quintic in $\mathbb{P}^4$, which is a well-studied example of a Calabi-Yau manifold with complex multiplication (see [38]). In the appendix we will give some examples of Calabi-Yau 3-manifolds which are not necessarily a fiber of a family with infinitely many CM fibers.

By the previous example, we are led to be interested in the examples of families of curves with a dense set of CM fibers for our search for $CMCY$ families of n-manifolds. There is an other motivation given by an open question in the theory of curves, too. In [11] R. Coleman formulated the following conjecture:

Conjecture 5. *Fix an integer $g \geq 4$. Then there are only finitely many complex algebraic curves C of genus g such that $\mathrm{Jac}(C)$ is of CM type.*

Let $\mathcal{P}_n$ denote the configuration space of $n+3$ points in $\mathbb{P}^1$. One can endow these $n+3$ points in $\mathbb{P}^1$ with local monodromy data and use these data for the construction of a family $\mathcal{C} \to \mathcal{P}_n$ of cyclic covers of $\mathbb{P}^1$ (see Construction 3.2.1).

The action of $\mathrm{PGL}_2(\mathbb{C})$ on $\mathbb{P}^1$ yields a quotient $\mathcal{M}_n = \mathcal{P}_n/\mathrm{PGL}_2(\mathbb{C})$. By fixing 3 points on $\mathbb{P}^1$, the quotient $\mathcal{M}_n$ can also be considered as a subspace of $\mathcal{P}_n$.

Remark 6. *In [29] J. de Jong and R. Noot gave counterexamples for $g = 4$ and $g = 6$ to the conjecture above. In [58] E. Viehweg and K. Zuo gave an additional counterexample for $g = 6$. The counterexamples are given by families $\mathcal{C} \to \mathcal{P}_n$ of cyclic covers of $\mathbb{P}^1$ with dense sets of CM fibers. Here we will find additional families $\mathcal{C} \to \mathcal{P}_n$ of cyclic genus 5 and genus 7 covers of $\mathbb{P}^1$ with dense sets of complex multiplication fibers, too.*

All new examples $\mathcal{C} \to \mathcal{P}_n$ of the preceding remark have a variation $\mathcal{V}$ of Hodge structures similar to the examples of J. de Jong and R. Noot [29], and of E. Viehweg and K. Zuo [58], which we call pure $(1,n) - VHS$. Let $\mathrm{Hg}(\mathcal{V})$ denote the generic Hodge group of $\mathcal{V}$ and let K denote an arbitrary maximal compact subgroup of $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{V})(\mathbb{R})$. In Section 4.4 we prove that a pure $(1,n) - VHS$ induces an open (multivalued) period map to the symmetric domain associated with $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{V})(\mathbb{R})/K$, which yields the dense sets of complex multiplication fibers. We obtain the following result in Chapter 6:

Theorem 7. *There are exactly 19 families $\mathcal{C} \to \mathcal{P}_n$ of cyclic covers of $\mathbb{P}^1$ which have a pure $(1,n) - VHS$ (including all known and new examples).*

We will use the fact that the monodromy group $\mathrm{Mon}^0(\mathcal{V})$ is a subgroup of the derived group $\mathrm{Hg}^{\mathrm{der}}(\mathcal{V})$ and we will study $\mathrm{Mon}^0(\mathcal{V})$. Let ψ be a generator of the Galois group of $\mathcal{C} \to \mathcal{P}_n$ and $C(\psi)$ be the centralizer of ψ in the symplectic group with respect to the intersection pairing on an arbitrary fiber of $\mathcal{C}$. In Chapter 4 we obtain the result, which will be useful for our study of $\mathrm{Hg}^{\mathrm{der}}(\mathcal{V})$ and $\mathrm{Mon}^0(\mathcal{V})$:

Lemma 8. *The monodromy group* $\mathrm{Mon}^0(\mathcal{V})$ *and the Hodge group* $\mathrm{Hg}(\mathcal{V})$ *are contained in* $C(\psi)$.

We will not be able to determine $\mathrm{Mon}^0(\mathcal{V})$ for all families $\mathcal{C} \to \mathcal{P}_n$ of cyclic covers of $\mathbb{P}^1$. But we will obtain for example the following results in Chapter 5:

Proposition 9. *Let* $\mathcal{C} \to \mathcal{P}_n$ *be a family of cyclic covers of degree* m *onto* $\mathbb{P}^1$. *Then one has:*

1. *If the degree* m *is a prime number* ≥ 3, *the algebraic groups* $C^{\mathrm{der}}(\psi)$, $\mathrm{Mon}^0(\mathcal{V})$ *and* $\mathrm{Hg}^{\mathrm{der}}(\mathcal{V})$ *coincide.*
2. *If* $\mathcal{C} \to \mathcal{P}_{2g+2}$ *is a family of hyperelliptic curves, one obtains*

$$\mathrm{Mon}^0(\mathcal{V}) = \mathrm{Hg}(\mathcal{V}) \cong \mathrm{Sp}_{\mathbb{Q}}(2g).$$

3. *In the case of a family of covers of* $\mathbb{P}^1$ *with 4 branch points, we need a pure* $(1,1)-VHS$ *to obtain an open period map to the symmetric domain associated with* $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{V})(\mathbb{R})/K$.

By our new examples of Viehweg-Zuo towers, we will only obtain $CMCY$ families of 2-manifolds. C. Voisin [60] has described a method to obtain Calabi-Yau 3-manifolds by using involutions on $K3$ surfaces. C. Borcea [9] has independently arrived at a more general version of the latter method, which allows to construct Calabi-Yau manifolds in arbitrary dimension. By using this method, we obtain in Section 7.2:

Proposition 10. *For* $i = 1, 2$ *assume that* $\mathcal{C}^{(i)} \to V_i$ *is a* $CMCY$ *family of* n_i*-manifolds endowed with the* V_i*-involution* ι_i *such that for all* $p \in V_i$ *the ramification locus* $(R_i)_p$ *of* $\mathcal{C}_p^{(i)} \to \mathcal{C}_p^{(i)}/\iota_i$ *consists of smooth disjoint hypersurfaces. In addition assume that* V_i *has a dense set of points* $p \in V_i$ *such that for all* k *the Hodge groups* $\mathrm{Hg}(H^k(\mathcal{C}_p^{(i)}, \mathbb{Q}))$ *and* $\mathrm{Hg}(H^k((R_i)_p, \mathbb{Q}))$ *are commutative. By blowing up the singular locus of the family* $\mathcal{C}^{(1)} \times \mathcal{C}^{(2)}/\langle(\iota_1, \iota_2)\rangle$, *one obtains a* $CMCY$ *family of* n_1+n_2*-manifolds over* $V_1 \times V_2$ *endowed with an involution satisfying the same assumptions as* ι_1 *and* ι_2.

Remark 11. *By the preceding proposition, one can apply the construction of C. Borcea and C. Voisin for families to obtain an infinite tower of* $CMCY$ *families of* n*-manifolds, which we call a Borcea-Voisin tower.*

Example 12. *The family* $\mathcal{C} \to \mathcal{M}_1$ *given by*

$$\mathbb{P}^2 \supset V(y_1^4 - x_1(x_1 - x_0)(x_1 - \lambda x_0)x_0) \to \lambda \in \mathcal{M}_1$$

has a pure $(1,1)-VHS$*. Hence by the construction of Viehweg and Zuo [58], one concludes that the family* $\mathcal{C}_2$ *given by*

$$\mathbb{P}^3 \supset V(y_2^4 + y_1^4 - x_1(x_1 - x_0)(x_1 - \lambda x_0)x_0) \to \lambda \in \mathcal{M}_1 \tag{1}$$

is a $CMCY$ *family of 2-manifolds.*

This family has many $\mathcal{M}_1$*-automorphisms. The quotients by some of these automorphisms yield new examples of* $CMCY$ *families of 2-manifolds. Moreover there are some involutions on* $\mathcal{C}_2$ *which make this family and its quotient families of* $K3$*-surfaces suitable for the construction of a Borcea-Voisin tower (see Section* 7.4 *for the construction of* $\mathcal{C}_2$*, and for the automorphism group and the quotient families of* $\mathcal{C}_2$ *see Section* 9.3*, Section* 9.4 *and Section* 9.5*).*

Example 13. *The family* $\mathcal{C} \to \mathcal{M}_3$ *given by*

$$\mathbb{P}(2,1,1) \supset V(y_1^3 - x_1(x_1-x_0)(x_1-ax_0)(x_1-bx_0)(x_1-cx_0)x_0) \to (a,b,c) \in \mathcal{M}_3$$

has a pure $(1,3)-VHS$*. The desingularization* $\tilde{\mathbb{P}}(2,2,1,1)$ *of the weighted projective space* $\mathbb{P}(2,2,1,1)$ *is given by blowing up the singular locus. By a modification of the construction of Viehweg and Zuo, the family* $\mathcal{W}$ *given by*

$$\begin{aligned}\tilde{\mathbb{P}}(2,2,1,1) \supset \tilde{V}(y_2^3 + y_1^3 - x_1(x_1 - x_0)(x_1 - ax_0)(x_1 - bx_0)(x_1 - cx_0)x_0)\\ \to (a,b,c) \in \mathcal{M}_3\end{aligned} \tag{2}$$

is a $CMCY$ *family of 2-manifolds. The family* $\mathcal{W}$ *has a degree 3 quotient, which yields a* $CMCY$ *family of 2-manifolds. Moreover it has an involution, which makes it and its degree 3 quotient suitable for the construction of a Borcea-Voisin tower (see Chapter 8 for the construction of* $\mathcal{W}$ *and Section* 9.1 *for its degree 3 quotient).*

By using the preceding example, we will obtain (see Section 9.2 for the construction and Section 10.3 for the maximality):

Theorem 14. *Let* $\mathbb{F}_3$ *be the Fermat curve of degree 3 and* $\alpha_{\mathbb{F}_3}$ *denote a generator of the Galois group of the degree 3 cover* $\mathbb{F}_3 \to \mathbb{P}^1$*. The family* $\mathcal{W}$ *has two* $\mathcal{M}_3$*-automorphism* α' *and* α'' *of order 3 such that the quotients* $\mathcal{W} \times \mathbb{F}_3/\langle(\alpha', \alpha_{\mathbb{F}_3})\rangle$ *and* $\mathcal{W} \times \mathbb{F}_3/\langle(\alpha'', \alpha_{\mathbb{F}_3})\rangle$ *have desingularizations, which are* $CMCY$ *families of 3-manifolds. Moreover one of these families is maximal.*

By using the V. V. Nikulins classification of involutions on $K3$ surfaces [51] and the construction of C. Voisin [60], we obtain in Chapter 11:

Theorem 15. *For each integer* $1 \leq r \leq 11$ *there exists a maximal holomorphic* $CMCY$ *family of algebraic 3-manifolds with Hodge number* $h^{2,1} = r$*.*

This book is organized as follows. The first three chapters explain well-known facts and yield an introduction of the notations. Chapter 1 is an introduction to Hodge Theory and Shimura varieties with a special view towards complex multiplication. We consider cyclic covers of $\mathbb{P}^1$ in Chapter 2. Moreover Chapter 3 introduces the remaining facts, which we need for the description of families of cyclic covers of $\mathbb{P}^1$ and their variations of Hodge structures.

In Chapter 4 we consider the Galois group action of a cyclic cover of $\mathbb{P}^1$ and we state first results for the generic Hodge group of a family $\mathcal{C} \to \mathcal{P}_n$. Moreover we will give a sufficient criterion for the existence of a dense set of CM fibers given by the pure $(1,n) - VHS$. In Chapter 5 we compute $\mathrm{Mon}^0(\mathcal{V})$, which provides much information about $\mathrm{Hg}(\mathcal{V})$. We will see that $\mathrm{Mon}^0(\mathcal{V})$ coincides with $C^{\mathrm{der}}(\psi)$ in infinitely many cases. In Chapter 6 we classify the examples of families of cyclic covers of $\mathbb{P}^1$ providing a pure $(1,n) - VHS$.

The basic methods of the construction of $CMCY$-families in higher dimension are explained in Chapter 7. We introduce the Borcea-Voisin tower and the Viehweg-Zuo tower and realize that only a small number of families of cyclic covers of $\mathbb{P}^1$ are suitable to start the construction of a Viehweg-Zuo tower. We will also discuss some methods to find concrete CM fibers at the end of this chapter. In Chapter 8 we will give a modified version of the method of E. Viehweg and K. Zuo to construct the $CMCY$ family of 2-manifolds given by (2). We consider the automorphism groups of our examples given by (1) and (2) in Chapter 9. This yields the further quotients of the families given by (1) and (2) which are $CMCY$ families of 2-manifolds. We will see that these quotients are endowed with involutions, which make them suitable for the construction of a Borcea-Voisin tower. Moreover we will construct the families $\mathcal{Q}$ and $\mathcal{R}$ of Theorem 14 in Chapter 9. The next chapter is devoted to the *length* of the Yukawa coupling of our examples families (motivated by the question of rigidity) and the Hodge numbers of their fibers. We finish this chapter with an outlook onto the possibilities to construct $CMCY$ families of 3-manifolds by quotients of higher order. In Chapter 11 we use directly the mirror construction of C. Voisin to obtain maximal holomorphic $CMCY$ families of 2-manifolds, which are suitable for the construction of a holomorphic Borcea-Voisin tower.

Throughout this book we use the conventions of Algebraic Geometry as in [26]. Most of the results and conventions about Hodge theory which we need can be found in [61].

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Chapter 1
An introduction to Hodge structures and Shimura varieties

In this chapter we recall the general facts about Hodge structures and Shimura varieties, which are needed in the sequel. We will explain that a Shimura datum consisting of a $\mathbb{Q}$-reductive group G and a homomorphism $h : \mathbb{S} \to G_{\mathbb{R}}$ satisfying certain conditions allows the construction of a Hermitian symmetric domain D. We will also give a definition of complex multiplication (CM), give a criterion for complex multiplication and discuss some conjectures concerning complex multiplication.

Shimura varieties and complex multiplication are closely related. One can construct a variation of Hodge structures on a Hermitian symmetric domain obtained from a Shimura datum. This variation of Hodge structures yields Hodge structures with complex multiplication over a dense set of points. Due to the André-Oort conjecture, one assumes that every variation of Hodge structures which contains infinitely many Hodge structures with complex multiplication is of this kind.

In the first two sections we recall the basic definitions of Hodge theory and consider polarized integral Hodge structures of type $(1,0),(0,1)$, which correspond to isomorphism classes of polarized abelian varieties with symplectic basis by Riemann's theorem. We define Shimura data and construct Hermitian symmetric domains by using Shimura data in Section 1.3 and Section 1.4 respectively. The construction of Shimura varieties from the Hermitian symmetric domains obtained by Shimura data is sketched in Section 1.5.

In Section 1.6 we motivate our definition of complex multiplication and write it down. Section 1.7 contains the theorem that a Shimura datum yields a Hermitian symmetric domain D and a VHS on D, which yields Hodge structures with CM over a dense set of points. In this Section we discuss some examples and conjectures about families with dense sets of complex multiplication fibers, too.

J.C. Rohde, *Cyclic Coverings, Calabi-Yau Manifolds and Complex Multiplication*, Lecture Notes in Mathematics 1975, DOI: 10.1007/978-3-642-00639-5_2,

1.1 The basic definitions

Definition 1.1.1. Let R be a ring such that $\mathbb{Z} \subseteq R \subseteq \mathbb{R}$. An R-Hodge structure is given by an R-module V and a decomposition

$$V \otimes_R \mathbb{C} = \bigoplus_{p,q \in \mathbb{Z}} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$.

We will always assume that $V_{\mathbb{R}}$ has finite dimension.

1.1.2. Let the Deligne torus $\mathbb{S}$ be the $\mathbb{R}$-algebraic group given by the matrices

$$M(x,y) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \quad \text{with} \quad x^2 + y^2 > 0, \quad x, y \in \mathbb{R}.$$

We identify the complex number $z = x + iy \in \mathbb{C}^*$ with $M(x,y) \in \mathbb{S}(\mathbb{R})$. One checks easily that this yields an isomorphism between $\mathbb{C}^*$ and $\mathbb{S}(\mathbb{R})$. Let $t := (\det M(x,y))^{-1}$. By using this identification, one sees easily that the Deligne torus $\mathbb{S}$ is given by the affine variety

$$V(t(x^2+y^2) - 1) \subset \mathbb{A}^3_{\mathbb{R}}.$$

The Deligne torus $\mathbb{S}$ is also given by the Weil restriction

$$\mathbb{S} = Res_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}.$$

Proposition 1.1.3. *Let V be an $\mathbb{R}$-vector space. Each real Hodge structure on V defines by*

$$z \cdot \alpha^{p,q} = z^p \bar{z}^q \alpha^{p,q} \quad (\forall \alpha^{p,q} \in V^{p,q}, z \in \mathbb{C}^* \cong \mathbb{S}(\mathbb{R}))$$

an action of $\mathbb{S}$ on $V \otimes \mathbb{C}$ such that one has an $\mathbb{R}$-algebraic homomorphism $h : \mathbb{S} \to \mathrm{GL}(V)$. Moreover by the eigenspace decomposition of $V_{\mathbb{C}}$ with respect to the characters of $\mathbb{S}$, any representation given by an algebraic homomorphism $h : \mathbb{S} \to \mathrm{GL}(V)$ corresponds to a real Hodge structure on V.

Proof. (see [16], 1.1.1[1]) □

---

[1] Note that P. Deligne writes

$$z \cdot \alpha^{p,q} = z^{-p} \bar{z}^{-q} \alpha^{p,q} \quad \text{instead of} \quad z \cdot \alpha^{p,q} = z^p \bar{z}^q \alpha^{p,q}$$

in [16]. But this is only a matter of the chosen conventions.

From now on, unless stated otherwise, let V be a $\mathbb{Q}$-vector space and let

$$h : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$$

be the algebraic homomorphism corresponding to a Hodge structure on V. The algebraic subgroup $S^1 \subset \mathbb{S}$ is given by

$$V(x^2 + y^2 - 1) \subset \mathbb{A}^2_{\mathbb{R}}.$$

This yields

$$S^1(\mathbb{R}) = \{z \in \mathbb{C} : z\bar{z} = 1\} \subset \mathbb{C}^*.$$

We consider the exact sequence

$$0 \to \mathbb{R}^* \overset{\mathrm{id}}{\hookrightarrow} \mathbb{C}^* \xrightarrow{z \to z/\bar{z}} S^1(\mathbb{R}) \to 0,$$

which can be obtained by the exact sequence

$$0 \to \mathbb{G}_{m,\mathbb{R}} \overset{w}{\hookrightarrow} \mathbb{S} \to S^1 \to 0$$

of $\mathbb{R}$-algebraic groups.

Remark 1.1.4. The homomorphism given by $h \circ w$ is called weight homomorphism. There exists a $k \in \mathbb{Z}$ such that $V^{p,q} = 0$ for all $p + q \neq k$, if and only if $h \circ w$ is given by $r \to r^k$. In this case the Hodge structure (V, h) is of weight k.

Remark 1.1.5. By Proposition 1.1.3, any (real) Hodge structure on $V_{\mathbb{R}}$ of weight k determines a unique morphism $h_1 : S^1 \to GL(V_{\mathbb{R}})$ given by

$$S^1 \hookrightarrow \mathbb{S} \xrightarrow{h} \mathrm{GL}(V_{\mathbb{R}}).$$

Since $\mathbb{S} = \mathbb{G}_{m,\mathbb{R}} \cdot S^1$, one can reconstruct h from $h|_{S^1}$ and the weight homomorphism. By using Proposition 1.1.3 again, one can easily see that there is a correspondence between Hodge structures of weight k on $V_{\mathbb{R}}$ and representations $h_1 : S^1 \to \mathrm{GL}(V_{\mathbb{R}})$ given by

$$z \cdot \alpha^{p,q} = z^p \bar{z}^q \alpha^{p,q}$$

for all $\alpha^{p,q} \in V^{p,q}$, which must satisfy $p + q = k$ for all $V^{p,q} \neq 0$.

We call an R-Hodge structure (V, h) pure, if $V^{p,q} = 0$ for all $p, q < 0$.

Example 1.1.6. A pure integral Hodge structure of weight k is given by

$$H^k(X, \mathbb{C}) = H^k(X, \mathbb{Z}) \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}(X) \quad \text{with} \quad H^{p,q}(X) = H^q(X, \Omega^p_X)$$

for any compact Kähler manifold X.

1.1.7. Let X be a compact Kähler manifold. The Hodge numbers $h^{p,q} = \dim H^{p,q}(X)$ are often visualized by Hodge diamonds. For example assume that X is a Calabi-Yau manifold. We say that X is a Calabi-Yau manifold, if X is Kähler manifold of dimension n such that $\omega_X \cong \mathcal{O}_X$ and $h^{k,0} = 0$ for $k = 1, \dots n-1$.

By [6], **VIII**. Proposition 3.4, one has $h^{1,1} = 20$ for a $K3$ surface resp., a Calabi-Yau 2-manifold. Thus by Hodge symmetry and Serre duality, the Hodge diamond of a $K3$ surface is given by:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Moreover by Hodge symmetry and Serre duality, the Hodge diamond of a Calabi-Yau 3-manifold is given by:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Definition 1.1.8. Let R be a ring such that $\mathbb{Z} \subseteq R \subseteq \mathbb{R}$. A polarized R-Hodge structure of weight k is given by an R-Hodge structure of weight k on an R-module V and a bilinear form $Q : V \times V \to R$, which is symmetric, if k is even, alternating otherwise, and whose extension on $V \otimes_R \mathbb{C}$ satisfies:

1. The Hodge decomposition is orthogonal for the Hermitian form $i^k Q(\cdot, \bar{\cdot})$.
2. For all $\alpha \in V^{p,q} \setminus \{0\}$ one has

$$i^{p-q}(-1)^{\frac{k(k-1)}{2}} Q(\alpha, \bar{\alpha}) > 0.$$

Example 1.1.9. Let X be a compact Kähler manifold. Recall that for $k \leq dim(X)$ the primitive cohomology $H^k(X, \mathbb{R})_{\text{prim}}$ is the kernel of the Lefschetz operator

$$L^{n-k+1} : H^k(X, \mathbb{R}) \to H^{2n-k+2}(X, \mathbb{R})$$

given by

$$\alpha \to \wedge^{n-k+1}(\omega) \wedge \alpha,$$

where $n := dim(X)$, the chosen Kähler form is denoted by ω and $\alpha \in H^k(X, \mathbb{R})$. By

$$(\alpha, \beta) := \int_X \wedge^{n-k}(\omega) \wedge \alpha \wedge \beta,$$

one obtains a polarization on $H^k(X, \mathbb{Z})_{\text{prim}}$ and hence a polarized integral Hodge structure on $H^k(X, \mathbb{Z})_{\text{prim}}$, if $[\omega] \in H^2(X, \mathbb{Z})$ (see [61], 7.1.2)[2].

Definition 1.1.10. Let $\mathbb{Q} \subseteq F \subset \mathbb{R}$ be a field and V be a F-vector space. The Hodge group $\text{Hg}_F(V, h)$ of a F Hodge structure (V, h) is the smallest F-algebraic subgroup G of $\text{GL}(V)$ such that

$$h(S^1) \subseteq G \times_F \mathbb{R}.$$

The Mumford-Tate group $\text{MT}_F(V, h)$ of a F Hodge structure (V, h) is the smallest F-algebraic subgroup G of $\text{GL}(V)$ such that

$$h(\mathbb{S}) \subseteq G \times_F \mathbb{R}.$$

For simplicity we will write $\text{Hg}(V, h)$ instead of $\text{Hg}_{\mathbb{Q}}(V, h)$ and $\text{MT}(V, h)$ instead of $\text{MT}_{\mathbb{Q}}(V, h)$.[3]

We will mainly consider rational Hodge structures. Nevertheless we often take a view towards K Hodge structures, where $\mathbb{Q} \subseteq F \subseteq \mathbb{R}$ is a field. This case can also be interesting (for example see [2] and [42]).

Next we define variations of Hodge structures (VHS). Consider a smooth family $f : X \to Y$ of algebraic manifolds. We use the variation of Hodge structures of such a family for the motivation of the general definition of variations of Hodge structures. First we need to recall the definition of the higher direct image sheaf:

Definition 1.1.11. Let $f : A \to B$ be a continuous map of topological spaces and $\mathcal{F}$ be a sheaf of abelian groups on A. The higher direct image sheaf is the sheaf associated to the presheaf given by

$$V \to H^1(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$$

for all open subsets $V \subset B$.

Remark 1.1.12. The higher direct image sheaf $R^k f_*(\mathbb{C})$ is a local system i.e. a locally constant sheaf of stalk G, where G is an abelian group. In our case G is given by the complex numbers. This follows from the fact that the fibers are

[2] There is a more general definition of a polarized Hodge structure (see [16], 1.1.10). But here we will mainly consider Hodge structures given by the primitive cohomology on a Kähler manifold. Moreover we obtain $H^n(X, \mathbb{R})_{\text{prim}} = H^n(X, \mathbb{R})$, if X is a curve or if X is a Calabi-Yau 3-manifold. Hence in these two cases of interest $H^n(X, \mathbb{R}_{\text{prim}})$ is independent by the chosen Kähler form. Moreover by its definition, the corresponding polarization is independent of the Kähler form, if $k = n$. Thus in these two cases the integral polarized Hodge structure depends only on the isomorphism class of X.

[3] In [17], Section 3 one finds an alternative definition of the Mumford-Tate group.

diffeomorphic such that the corresponding family of differentiable manifolds is locally constant (see [61] 9.1.1). The variation of Hodge structures will be given by a filtration of

$$\mathcal{H}^k := R^k f_*(\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_Y$$

by holomorphic subbundles. Thus let us first explain that a filtration can give the Hodge structure of a fiber:

1.1.13. Let $\mathbb{Z} \subseteq R \subseteq \mathbb{R}$ be a ring and V be an R-module. Recall that we have two equivalent definitions of an R-Hodge structure on V. A Hodge structure can be defined by a certain direct sum decomposition of $V_{\mathbb{C}}$ into the subvector spaces $V^{p,q}$ (see Definition 1.1.1) or by a representation $h : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$.

One needs a third equivalent definition of Hodge structures of weight k to understand how a filtration of subbundles yields Hodge structures on the fibers of $\mathcal{H}^k$, which will be the respective Hodge structures of weight k on the fibers of f. A pure Hodge structure of weight k on V can be given by a decreasing filtration $F^\bullet$ on $V_{\mathbb{C}}$ such that

$$V_{\mathbb{C}} = F^0 V_{\mathbb{C}} \supset F^1 V_{\mathbb{C}} \supset \ldots \supset F^{k+1} V_{\mathbb{C}} = 0.$$

The filtration satisfies

$$V_{\mathbb{C}} = F^p V_{\mathbb{C}} \oplus \overline{F^{k-p+1} V_{\mathbb{C}}}$$

for all p. The direct summand $V^{p,q}$ is given by

$$V^{p,q} = F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}}.$$

It is an easy exercise to check that such a filtration yields a Hodge structure of weight k and a Hodge structure of weight k yields such a filtration.

Proposition 1.1.14. *Let $X \to Y$ be a smooth morphism of algebraic manifolds and*

$$\mathcal{H}^k := R^k f_*(\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_Y.$$

One has a filtration $F^\bullet$ of $\mathcal{H}^k$ by holomorphic subbundles $F^p\mathcal{H}^k$ such that for all $y \in Y$ one has $F^p H^k(X_y, \mathbb{C}) = F^p_y \mathcal{H}^k$. Moreover one can define bundles $\mathcal{H}^{p,k-p} = F^p\mathcal{H}^k / F^{p+1}\mathcal{H}^k$ such that $\mathcal{H}^{p,k-p}_y = H^{p,k-p}(X_y)$.

Proof. (see [61], 10.2.1) □

Remark 1.1.15. The $\mathcal{H}^{p,k-p}$ are <u>not</u> subbundles of $\mathcal{H}^k$. This motivates the definition of the variation of Hodge structures by a filtration.

Next we need to construct the Gauss-Manin connection:

Construction 1.1.16. We endow $\mathcal{H}^k$ with a connection in the following way:

Let U be a simply connected open subset of Y. Over U the local system $R^k f_*(\mathbb{C})$ can be considered as a locally constant sheaf. Moreover let

$$\sigma = \sum \alpha_i \otimes \sigma_i \in \mathcal{H}^k(U),$$

where $\alpha_i \in R^k f_*(\mathbb{C})(U)$ and $\sigma_i \in \mathcal{O}(U)$. The Gauss-Manin connection $\nabla : \mathcal{H}^k \to \mathcal{H}^k \otimes \Omega_Y$ is locally defined by

$$\mathcal{H}^k(U) \ni \sigma = \sum \alpha_i \otimes \sigma_i \stackrel{\nabla}{\to} \sum \alpha_i \otimes d\sigma_i \in (\mathcal{H}^k \otimes \Omega_Y)(U).$$

By gluing, the connection is defined for each open subset of Y. Thus the connection is globally defined.

Remark 1.1.17. The Gauss-Manin connection satisfies the Griffiths transversality condition. That is

$$\nabla(F^p\mathcal{H}^k) \subseteq F^{p-1}\mathcal{H}^k \otimes \Omega_Y \quad \text{and} \quad \nabla(F^{p+1}\mathcal{H}^k) \subseteq F^p\mathcal{H}^k \otimes \Omega_Y.$$

Thus by using quotients, we can define the map

$$\bar{\nabla}^{p,k-p} : \mathcal{H}^{p,k-p} \to \mathcal{H}^{p-1,k-p+1} \otimes \Omega_Y.$$

The Gauss-Manin connection is obviously not linear, but $\bar{\nabla}^{p,k-p}$ is a morphism of $\mathcal{O}_Y$-modules (see [61] 10.2.2).

For the motivation of the definition of a polarized variation of Hodge structures note one additional fact:

Remark 1.1.18. Let R be a ring such that $\mathbb{Z} \subseteq R \subseteq \mathbb{R}$. Recall that a family of algebraic manifolds provides a locally constant family in the category of differentiable manifolds. Thus the sheaf $R^k f_*(R)$ is locally constant. Therefore the polarization on $R^k f_*(R)$ obtained by the polarization of the Hodge structures of the fibers is locally constant, too.

Definition 1.1.19. Let D be a complex manifold and R be a ring such that $\mathbb{Z} \subseteq R \subseteq \mathbb{R}$. A variation $\mathcal{V}$ of R-Hodge structures of weight k over D is given by a local system $\mathcal{V}_R$ of R-modules of finite rank and a filtration $\mathcal{F}^\bullet$ of $\mathcal{V}_{\mathcal{O}_D}$ by holomorphic subbundles such that:

1. Griffiths transversality condition holds.
2. $(\mathcal{V}_{R,p}, \mathcal{F}_p^\bullet)$ is an R-Hodge structure of weight k for all $p \in D$.

The variation $\mathcal{V}$ of Hodge structures is polarized, if there is a flat (i.e. locally constant) bilinear form Q on $\mathcal{V}_R$ such that $(\mathcal{V}_{R,p}, \mathcal{F}_p^\bullet, Q_p)$ is a polarized R-Hodge structure of weight k for all $p \in D$.

Next we need to introduce and construct the parametrizing spaces of Hodge structures. We start by the construction of the Grassmannian:

Construction 1.1.20. Let W be a complex vector space of dimension N and $0 < k < N$. The Grassmannian $\mathrm{Grass}(W,k)$ is the manifold, which parametrizes all complex subvector spaces of W of codimension k.[4] Let $K \subset W$ be a subvector space of codimension k and $L \subset W$ be an other subvector space such that

$$K \oplus L = W.$$

Moreover let

$$\pi_K : W \to K \ \text{ resp., } \ \pi_L : W \to L$$

denote the respective projection onto K resp., L with kernel L resp., K. Now let $Z \subset W$ be an other vector space of codimension k such that $Z \cap L = \{0\}$. This vector space Z is identified with

$$h_Z = \pi_L \circ (\pi_K)|_Z^{-1} : K \to L.$$

By $Z \leftrightarrow h_Z$, an open neighborhood of the point $p_K \in \mathrm{Grass}(W,k)$, which represents K, can be given by $\mathrm{Hom}(K,L)$. Since each subvector space $Z \subset W$ of codimension k corresponds to exactly one point $p_Z \in \mathrm{Grass}(W,k)$, one obtains gluing isomorphisms between the open sets. Thus $\mathrm{Grass}(W,k)$ is a complex manifold of dimension $k(N-k)$.

Moreover $\mathrm{Grass}(W,k)$ is projective and the tangent space $T_K\mathrm{Grass}(W,k)$ is given by

$$T_K\mathrm{Grass}(W,k) \cong \mathrm{Hom}(K, W/K).$$

(see [61], 10.1).

1.1.21. Let (V,h) be an R-Hodge structure of weight k. For simplicity we assume that (V,h) is pure. The Hodge structure is given by the Hodge filtration

$$V_{\mathbb{C}} = F^0V_{\mathbb{C}} \supset F^1V_{\mathbb{C}} \supset \ldots \supset F^kV_{\mathbb{C}} \supset F^{k+1}V_{\mathbb{C}} = 0.$$

Moreover the flag space $\mathcal{F}_{h^{k,0},\ldots,h^{p,k-p},\ldots,h^{0,k}}$, which parametrizes the filtrations by subvector spaces of the respective codimensions, satisfies

$$\mathcal{F}_{h^{k,0},\ldots,h^{p,k-p},\ldots,h^{0,k}} \subset \prod_p \mathrm{Grass}(F^pV_{\mathbb{C}}, h^{0,k} + h^{1,k-1} + \ldots + h^{p,k-p}).$$

---

[4] Many authors define the Grassmannian $\mathrm{Grass}(W,k)$ as the manifold, which parametrizes all subvector spaces of W of dimension k and not of codimension k. We use this abbreviation, since it makes our notations below easier.

Recall that

$$V^{p,k-p} = F^p V_{\mathbb{C}}/F^{p+1}V_{\mathbb{C}} \ \text{ resp., } \ h^{p,k-p} = \operatorname{codim}_{F^pV_{\mathbb{C}}}(F^{p+1}V_{\mathbb{C}}).$$

The pure Hodge structures of weight k with the Hodge numbers $h^{k,0}\dots$, $h^{p,k-p}\dots h^{0,k}$ are classified by an open subset $\mathcal{D}'$ of the flag space. This open set $\mathcal{D}'$ is defined by the condition

$$F^pV_{\mathbb{C}} \oplus \overline{F^{k-p+1}V_{\mathbb{C}}} = V_{\mathbb{C}}.$$

Now assume that (V, h, Q) is a polarized pure R-Hodge structure. Thus Q yields a fixed polarization. The set $\mathcal{D} \subset \mathcal{D}'$, which parametrizes the Hodge structures (V, h') such that (V, h', Q) is a polarized Hodge structure, is an open subset of an analytic subspace of $\mathcal{D}'$.

The space $\mathcal{D}$ is called the period domain. Let $q \in D$ denote the point corresponding to our polarized Hodge structures (V, h, Q). The tangent space T_qD is a subvector space of

$$T_q\mathcal{F}_{h^{k,0},\dots,h^{p,k-p},\dots,h^{0,k}}$$

given by

$$T_q\mathcal{F}_{h^{k,0},\dots,h^{p,k-p},\dots,h^{0,k}} = \bigoplus_p \operatorname{Hom}(F^{p+1}V_{\mathbb{C}}, F^pV_{\mathbb{C}}/F^{p+1}V_{\mathbb{C}}).$$

(see [61], 10.1).

Now one constructs easily the period map of a family:

1.1.22. A variation $\mathcal{V}$ of pure polarized R-Hodge structures over a simply connected complex manifold S yields a holomorphic map $p : S \to \mathcal{D}$, where $\mathcal{D}$ is a suitable period domain. This map depends on the choice of a trivialization of $\mathcal{V}$ over the simply connected manifold S. Now assume that $f : X \to Y$ is a holomorphic family of Kähler manifolds and $\mathcal{V}$ its variation of integral Hodge structures. In this case one can define a multivalued holomorphic map $p : Y \to \mathcal{D}$, which is called the period map.

1.2 Jacobians, Polarizations and Riemann's Theorem

Let X be a Kähler manifold. Consider the following exact sequence:

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

This yields the complex torus

$$\mathrm{Pic}^0(X) = H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}),$$

which parametrizes the line bundles of degree 0. For a curve one has the following construction of a complex torus:

Construction 1.2.1. Let C be a curve. Moreover let

$$Z = \sum_{i=1}^{m} n_i P_i \quad \text{with} \quad \sum_{i=1}^{m} n_i = 0$$

be a cycle of points $P_i \in C$. There exists a differentiable chain Γ such that $\partial\Gamma = Z$. Let ω be a holomorphic 1-form on C. The value of the integral $\int_\Gamma \omega$ depends on Z up to the homology $H_1(C, \mathbb{Z})$. Thus Z yields an unique point of the Jacobian

$$\mathrm{Jac}(C) = H^{1,0}(C)^*/H_1(C, \mathbb{Z}),$$

which is a complex torus. There exists a holomorphic map

$$C \to \mathrm{Jac}(C),$$

which is called the Abel-Jacobi map. We fix a point $p_0 \in C$ and send each $p \in C$ to the unique class in $\mathrm{Jac}(C)$ of the path integral over an arbitrary path from p_0 to p. (see any good book about Riemann surfaces)

We will see that $\mathrm{Pic}^0(C) \cong \mathrm{Jac}(C)$ for any curve C. Moreover we will check that $\mathrm{Jac}(C)$ is an abelian variety. The theory of abelian varieties, their Hodge structures and their parametrizing spaces contains several features and motivates the definition of Shimura data, which we will need in the sequel.

Let R be a ring such that $\mathbb{Z} \subseteq R \subseteq \mathbb{C}$ and C be a curve. The homology

$$H_1(C, R) := H_1(C, \mathbb{Z}) \otimes_{\mathbb{Z}} R$$

and the cohomology $H^1(C, R)$ are canonical duals (see [61], Théorème 4.47). On $H_1(C, \mathbb{Z})$ one defines the dual Hodge structure of weight -1 of the Hodge structure on $H^1(C, \mathbb{Z})$ given by the Hodge filtration

$$0 \subset H^{0,-1}(C) \subset H_1(C, \mathbb{C}) \quad \text{such that} \quad H^{0,-1}(C) = H^{0,1}(C)^* \quad \text{and}$$
$$H^{-1,0}(C) = H^{1,0}(C)^*.$$

In the sequel we will also need the following relations between the Hodge structure of weight 1 on $H^1(C, \mathbb{Z})$ and the Hodge structure of weight -1 on the homology $H_1(C, \mathbb{Z})$:

1.2.2. For each ring $\mathbb{Z} \subset R \subset \mathbb{C}$ we have $H^1(C, R) \cong H_1(C, R)^*$. By integration over $\mathbb{R}$-valued paths, we obtain an isomorphism

$$\phi : H^1(C,\mathbb{R}) \to H_1(C,\mathbb{R})^* \to H^1_{\mathrm{DR}}(C,\mathbb{R}).$$

The integral classes in the de Rham cohomology $H^1_{\mathrm{DR}}(C,\mathbb{R})$ are given by $\phi(H^1(C,\mathbb{Z}))$.

On the homology $H_1(C,\mathbb{Z})$ of a curve C one can define an intersection pairing $(\cdot,\cdot)$, which is an alternating bilinear form. The intersection form on $H_1(C,\mathbb{Z})$ can be given by the matrix

$$\begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}$$

with respect to a fixed symplectic basis (for example see [7], 11.1). Thus the intersection form yields an isomorphism $\sigma_R : H_1(C,R) \to H_1(C,R)^*$ for all rings $\mathbb{Z} \subset R \subset \mathbb{C}$. In terms of the de Rham cohomology it assigns to each $\alpha \in H_1(C,R)$ the $\eta_\alpha \in H^1_{\mathrm{DR}}(C,\mathbb{C})$, which has the property that

$$(\gamma,\alpha) = \int_\gamma \eta_\alpha$$

for all $\gamma \in H_1(C,\mathbb{C})$. By this definition, one has $\eta_\alpha \in H_1(C,R)^* = H^1(C,R)$. In addition one has

$$(\gamma,\alpha) = \int_C \eta_\gamma \wedge \eta_\alpha$$

(compare [30], Section 5.1).[5]

Moreover one has

$$\sigma \circ h_{-1}(z) = h_1(z) \circ \sigma \quad \text{for all} \quad z \in S^1(\mathbb{R}),$$

where h_{-1} and h_1 denote the corresponding embeddings

$$h_{-1} : S^1 \to \mathrm{GL}(H_1(C,\mathbb{R})) \quad \text{and} \quad h_1 : S^1 \to \mathrm{GL}(H^1(C,\mathbb{R}))$$

of the respective Hodge structures. Thus the Hodge groups of these Hodge structures on $H_1(C,\mathbb{Z})$ and $H^1(C,\mathbb{Z})$ are isomorphic. Hence for a study of the Hodge structure on $H^1(C,\mathbb{Z})$, it is sufficient to consider the dual Hodge structure on $H_1(C,\mathbb{Z})$.

Recall that the Hodge decomposition of the Hodge structure on $H^1(C,\mathbb{Q})$ is orthogonal with respect to the Hermitian form

$$iQ(\cdot,\bar{\cdot}) = i\int_C \cdot \wedge \bar{\cdot}.$$

[5] In [30] the last equation is written down only for $R = \mathbb{Z}$. By $H^1(C,R) = H^1(C,\mathbb{Z}) \otimes_{\mathbb{Z}} R$ and by $H_1(C,R) = H_1(C,\mathbb{Z}) \otimes_{\mathbb{Z}} R$, one obtains the last equation for each ring $\mathbb{Z} \subseteq R \subseteq \mathbb{C}$.

Thus by the polarization, $H^{0,1}(C)$ is canonical isomorphic to $H^{1,0}(C)^*$. Since 1.2.2 yields a corresponding canonical isomorphism $\sigma : H_1(C,\mathbb{Z}) \to H^1(C,\mathbb{Z})$, one concludes:

Corollary 1.2.3. *Let C be a curve. Then* $\mathrm{Pic}^0(C)$ *and* $\mathrm{Jac}(C)$ *are isomorphic.*

Next we consider polarizations on abelian varieties:

Remark 1.2.4. Let $A = W/L$ be a complex g-dimensional torus. There is a canonical isomorphism between $H^2(A,\mathbb{Z})$ and $\mathbb{Z}$-valued alternating forms on $L = H_1(A,\mathbb{Z})$. Moreover for an alternating integral form E on L, there is a line bundle $\mathcal{L}$ on A with $c^1(\mathcal{L}) = E$, if and only if $E(i\cdot, i\cdot) = E(\cdot,\cdot)$. By

$$H(u,v) = E(iu,v) + iE(u,v),$$

we get the corresponding Hermitian form H from E and conversely, given H we obtain E by $E = \Im H$. (See [7], Proposition 2.1.6 and Lemma 2.1.7)

A polarization on an abelian variety is given by a line bundle $\mathcal{L}$, whose Hermitian form H, which corresponds to its first Chern class E, is positive definite. The alternating form E of the polarization can be given by the matrix

$$\begin{pmatrix} 0 & D_g \\ -D_g & 0 \end{pmatrix}$$

with respect to a symplectic basis of L, where $D_g = \mathrm{diag}(d_1,\ldots,d_g)$ with $d_i | d_{i+1}$ (see [7], 3.1). The matrix D_g depends on the polarization, and it is called the type of the polarization. The polarization E on A is principal, if $D_g = E_g$.

A positive definite Hermitian form H on W, which has the property that $\Im H$ is an integral alternating form on L, satisfies that $\Im H(i\cdot, i\cdot) = \Im H(\cdot,\cdot)$ resp., is a polarization. Since the Chern class of a line bundle $\mathcal{L}$ is a polarization, if and only if $\mathcal{L}$ is ample (see [7], Proposition 4.5.2.), H yields an ample line bundle. By the Theorem of Chow, A is algebraic in this case. Moreover if A is an abelian variety, there is a positive definite Hermitian form H on W such that $\Im H$ is integral on L (see [48], **I**. 3).

Example 1.2.5. Let X be a Kähler manifold. On

$$\mathrm{Pic}^0(X) = H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z})$$

one has a negative definite Hermitian form given by the polarization $iQ(\cdot,\bar{\cdot})$ of the weight one Hodge structure. Hence by $-iQ(\cdot,\bar{\cdot})$, one has a positive definite Hermitian form. Since $Q(\cdot,\cdot)$ is integral on $H^1(X,\mathbb{Z})$, the same holds true for the projection of $H^1(X,\mathbb{Z})$ to $H^1(X,\mathcal{O}_X)$ with respect to $\Im(-iQ(\cdot,\bar{\cdot}))$ as one can check easily. Hence $\mathrm{Pic}^0(X)$ has a polarization and it is an abelian variety.

Assume that X is a curve C. The intersection form on $H_1(C,\mathbb{Z})$ can be given by the matrix

$$\begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}$$

with respect to a fixed symplectic basis (see 1.2.2). Hence the polarization on $\mathrm{Jac}(C)$ is principal.

We repeat the consideration of the preceding example in a more general setting, which will allow us to construct the moduli space $\mathfrak{h}_g$ of abelian varieties of dimension g with extra structure explained below. This space will be our first motivating example to use Shimura data. A Shimura datum will endow $\mathfrak{h}_g$ with the structure of a Hermitian symmetric domain such that the holomorphic universal family of abelian varieties has a dense set of fibers of CM type.

Now let V denote a $\mathbb{Q}$-vector space of dimension $2g$, let Q be a rational alternating bilinear form on V, and let J be a complex structure on $V_{\mathbb{R}}$ (i.e. an automorphism J with $J^2 = -\mathrm{id}$). Moreover a Hodge structure of type $(1,0),(0,1)$ on V is given by a decomposition

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}.$$

In an analogue way one defines the type of an arbitrary Hodge structure given by a decomposition of $V_{\mathbb{C}}$.

Remark 1.2.6. It is very easy to see that there is a correspondence between Hodge structures h on V of type $(1,0),(0,1)$ and complex structures J on $V_{\mathbb{R}}$ via $h(i) = J$.

Lemma 1.2.7. *The complex structure J on $V_{\mathbb{R}}$ corresponds to a polarized Hodge structure (V,h,Q) of type $(1,0),(0,1)$, if and only if it satisfies*

$$Q(J\cdot, J\cdot) = Q(\cdot,\cdot) \quad \textit{and} \quad Q(J\tilde{v},\tilde{v}) > 0$$

for all $\tilde{v} \in V_{\mathbb{R}}$.

Proof. Let the complex structure J on $V_{\mathbb{R}}$ be given by a polarized Hodge structure of type $(1,0),(0,1)$ on V. Any $\tilde{v},\tilde{w} \in V_{\mathbb{R}}$ can be given by

$$\tilde{v} = v + \bar{v} \quad \textit{and} \quad \tilde{w} = w + \bar{w}$$

for some $v,w \in H^{1,0}$, where $H^{1,0}$ and $H^{0,1}$ are totally isotropic with respect to Q. Hence:

$$Q(J\tilde{v}, J\tilde{w}) = Q(iv, -i\bar{w}) + Q(-i\bar{v}, iw) = Q(v,\bar{w}) + Q(\bar{v},w) = Q(\tilde{v},\tilde{w})$$

Since the Hermitian form given by $iQ(v, \bar{v})$ is positive definite on $H^{1,0}$, one concludes:

$$Q(J\tilde{v}, \tilde{v}) = Q(iv - i\bar{v}, v + \bar{v}) = Q(iv, \bar{v}) + Q(-i\bar{v}, v) = 2iQ(v, \bar{v}) > 0 \quad (1.1)$$

Conversely assume that $Q(J\cdot, \cdot) > 0$ and $Q(\cdot, \cdot) = Q(J\cdot, J\cdot)$. Thus one has

$$Q(v_1, v_2) = Q(Jv_1, Jv_2) = Q(iv_1, iv_2) = -Q(v_1, v_2)$$
$$\text{resp., } Q(v_1, v_2) = Q(Jv_1, Jv_2) = Q(-iv_1, -iv_2) = -Q(v_1, v_2)$$

for all $v_1, v_2 \in H^{1,0} := Eig(J, i)$ resp., for all $v_1, v_2 \in H^{0,1} := Eig(J, -i)$. Hence $H^{1,0}$ resp., $H^{0,1}$ is isotropic with respect to Q. The same calculation as in (1.1) implies that $iQ(\cdot, \bar{\cdot})$ is positive definite on $H^{1,0}$ and negative definite on $H^{0,1}$. Hence one gets a polarized Hodge structure of type $(1,0), (0,1)$ by Remark 1.2.6. □

By the preceding lemma and an easy calculation using that $z = a + ib \in S^1(\mathbb{R})$ implies $a^2 + b^2 = 1$,[6] we obtain:

Proposition 1.2.8. *A polarized Hodge structure* (V, h, Q) *of type* $(1,0), (0,1)$ *induces a faithful symplectic representation*

$$h : S^1 \to \mathrm{Sp}(V_{\mathbb{R}}, Q).$$

Corollary 1.2.9. *Let* (V, h, Q) *be a polarized Hodge structure of type* $(1,0), (0,1)$. *Then*

$$\mathrm{Hg}(V, h) \subseteq \mathrm{Sp}(V, Q).$$

Theorem 1.2.10 (Riemann). *There is a correspondence between polarized abelian varieties of dimension* g *and polarized Hodge structures* (L, h, Q) *of type* $(1,0), (0,1)$ *on a torsion-free lattice* L *of rank* $2g$.

Proof. Let (L, h, Q) be a polarized Hodge structure on a torsion-free lattice L of rank $2g$. By

$$L \otimes \mathbb{R} \hookrightarrow L \otimes \mathbb{C} \to H^{0,1},$$

one has an isomorphism f of $\mathbb{R}$-vector spaces. The complex structure of the Hodge structure turns $L_{\mathbb{R}}$ into a $\mathbb{C}$-vector space. One has $f(\lambda v) = \bar{\lambda} f(v)$ for all complex numbers λ. By f, the alternating form Q may be considered as (real) alternating form on $H^{0,1}$. But it satisfies $Q(iv, v) < 0$ for all $v \in H^{0,1}$. Hence let $E = -Q$. Lemma 1.2.7 implies that $E(i\cdot, i\cdot) = E(\cdot, \cdot)$ and $E(iv, v) > 0$ for

[6] Let $v, w \in V_{\mathbb{R}}$. The calculation is given by:

$$Q(zv, zw) = a^2 Q(v, w) + b^2 Q(v, w) + ab(Q(Jv, w) + Q(v, Jw)) =$$

$$= Q(v, w) + ab(Q(Jv, w) + Q(Jv, J(Jw))) = Q(v, w) + ab(Q(Jv, w) + Q(Jv, -w)) = Q(v, w)$$

all $v \in H^{0,1}$. Thus the corresponding Hermitian form is positive definite (see Remark 1.2.4) and we have a polarization on the complex torus $H^{0,1}/L$ and hence an abelian variety.

Conversely take a polarized abelian variety (A, E), where $A = W/L$. Let $Q := -E$. By $J = -i$, one has similar to Lemma 1.2.7 a complex structure corresponding to a polarized Hodge structure of type $(1,0), (0,1)$ on L. Thus we have obviously obtained the desired correspondence. □

Since a polarized rational Hodge structure can be considered as polarized integral Hodge structure with respect to a fixed lattice, if the polarization on this lattice is integral, one concludes by Lemma 1.2.7 and Theorem 1.2.10:

Corollary 1.2.11. *There is a bijection between the sets of polarized abelian varieties $A = W/L$ and complex structures on $L \otimes \mathbb{R}$ satisfying*

$$Q(J\cdot, J\cdot) = Q(\cdot, \cdot) \quad \textit{and} \quad Q(Jv, v) > 0$$

for all $v \in L \otimes \mathbb{R}$ with respect to an integral alternating form Q on L.

Remark 1.2.12. In order to obtain isomorphism classes of certain objects corresponding to the polarized integral Hodge structures (L, h, Q) one can fix a basis B of L. Usually this basis B is symplectic with respect to the polarization E of A. Hence a polarized abelian variety with symplectic basis consists of the triple (A, E, B). The conditions

$$Q(J\cdot, J\cdot) = Q(\cdot, \cdot) \quad \text{and} \quad Q(Jv, v) > 0$$

of the previous corollary are called Riemann conditions. Hence by Theorem 1.2.10, we have proved that a complex structure on $L \otimes \mathbb{R}$ corresponds to the isomorphism class of a polarized abelian variety with symplectic basis, if and only if it satisfies the Riemann conditions.

Remark 1.2.13. Two curves are isomorphic, if their Jacobians are isomorphic as principally polarized abelian varieties (see [7], Theorem 11.1.7). Since we have proved that polarized abelian varieties correspond to polarized integral Hodge structures, one concludes that two curves C and C' are isomorphic, if and only if there is an isomorphism between the polarized Hodge structures on $H^1(C, \mathbb{Z})$ and $H^1(C', \mathbb{Z})$. This yields the Torelli theorem for curves.

1.3 The definition of the Shimura datum

We will endow the set of principally polarized abelian varieties with symplectic basis with the structure of a Hermitian symmetric domain. Such a domain can be obtained from a Shimura datum. Let G be a $\mathbb{Q}$-algebraic reductive group. A Shimura datum is given by a homomorphism $h : \mathbb{S} \to G_{\mathbb{R}}$ of algebraic groups, which satisfies some conditions, which we explain here.

For the definition of the Shimura datum and the construction of Hermitian symmetric domains we need to recall some facts about algebraic groups. We can assume that our algebraic groups are defined over a field F of characteristic 0. Thus our groups are defined over perfect fields.

Remark 1.3.1. Let G be an algebraic group. The adjoint group G^{ad} is the quotient of G obtained by the adjoint representation of G on its Lie algebra $\mathfrak{g}$. It is a well-known fact that G has the following algebraic subgroups:

By G^0, we denote the Zariski connected component of identity. The derived group G^{der} of G is the subgroup of G generated by its commutators. By $Z(G)$, we denote the center of G. The Radical $R(G)$ is the maximal connected normal solvable subgroup of G. The unipotent radical $R_u(G)$ of G is given by

$$R_u(G) := \{g \in R(G) | g \text{ is unipotent}\}.^7$$

Definition 1.3.2. Let G be an algebraic group. Then one says:

- The group G is a reductive, if
$$R_u(G) = \{e\}.$$
- The group G semisimple, if
$$R(G) = \{e\}.$$
- The group G is simple, if $\{G\}$ and $\{e\}$ are the only normal connected subgroups of G.

There exists an alternative description of semisimple groups. By the following proposition, one sees that a semisimple algebraic group G is isogeneous to the fiberproduct of simple groups.

Proposition 1.3.3. *Let $\{e\} \neq G$ be a semisimple algebraic group. Then G is isogeneous to the fiberproduct of its minimal nontrivial normal subgroups.*

Proof. (follows from [10], **IV**. Proposition 14.10) □

By comparing the definition of reductive algebraic groups and semisimple algebraic groups, one sees easily that semisimple groups are reductive. The following proposition yields an additional relation between reductive groups and semisimple groups.

---

[7] Many authors (for example see [1], [10]) define $R(G)$ and $R_u(G)$ only for groups over algebraically closed fields. However, these subgroups are defined over our field F of characteristic 0 (see [53], Subsection 2.1.9).

Proposition 1.3.4. *Let G be a connected algebraic group. It is reductive, if and only if it is the almost direct product of a torus and a semisimple group. These groups can be given by $Z(G)^0$ and G^{der}.*

Proof. (see [54], Chapter **I**. §3 for the first statement and [10], **IV**. 14.2 for the second statement) □

Example 1.3.5. For technical and historical reasons we need to introduce the general symplectic group $\mathrm{GSp}(V, Q)$. The general symplectic group $\mathrm{GSp}(V, Q)$ is given by the automorphisms of the $\mathbb{Q}$-vector space V, which preserve alternating bilinear form Q up to a scalar. From this definition, it is clear that $\mathrm{GSp}(V, Q)$ is given by the almost direct product

$$\mathrm{GSp}(V, Q) = \mathbb{G}_{m,\mathbb{Q}} \cdot \mathrm{Sp}(V, Q).$$

The complex symplectic group $\mathrm{Sp}(V, Q)(\mathbb{C})$ is given by one of the classical simple Lie groups. Therefore $Sp(V, Q)$ is simple. The center of $\mathrm{GSp}(V, Q)$ is given by the torus $\mathbb{G}_{m,\mathbb{Q}}$ (see [40], page 66). Thus $\mathrm{GSp}(V, Q)$ is reductive by the previous proposition.

Remark 1.3.6. Let G be a reductive $\mathbb{Q}$-algebraic group with largest commutative quotient T. In this case we obtain (see [15], 1.1):

1. One has the exact sequences:

$$1 \to G^{\mathrm{der}} \to G \to T \to 1$$
$$1 \to Z(G) \to G \to G^{\mathrm{ad}} \to 1$$
$$1 \to Z(G^{\mathrm{der}}) \to Z(G) \to T \to 1$$

2. The exact sequences induce a natural isogeny $G^{\mathrm{der}} \to G^{\mathrm{ad}}$ with kernel $Z(G^{\mathrm{der}})$.

Assume that G is reductive $\mathbb{Q}$-algebraic. From Proposition 1.3.4 and the fact that G^{der} and G^{ad} are isogeneous (see Remark 1.3.6), we conclude:

Corollary 1.3.7. *Let G be a reductive $\mathbb{Q}$-algebraic group. Then G^{ad} is semisimple.*

Assume that G be a reductive $\mathbb{Q}$-algebraic group. By the previous Corollary and Remark 1.3.6, one concludes that G^{ad} is a semisimple group with trivial center. Moreover $\mathbb{R}$-algebraic groups can be considered in terms of Lie groups, since they yield Lie groups by their $\mathbb{R}$-rational points. The following lemma concerns in particular $G^{\mathrm{ad}}(\mathbb{R})$.

Lemma 1.3.8. *If G is a semisimple connected Lie group with trivial center, then it is isomorphic to a direct product of simple groups with trivial centers.*

Proof. By [27], **II**. Corollary 5.2, the group G coincides with $G^{\mathrm{ad}} \cong G/Z(G)$. Since the Lie algebra $\mathfrak{g}$ of G is the direct sum of simple Lie algebras, $\mathfrak{g}$ is the Lie algebra of a certain direct product of simple groups, too. Without loss of generality one can assume that these simple Lie groups have trivial centers. Recall that the adjoint group depends only on the Lie algebra. Thus this product of simple groups is isomorphic to its adjoint, which is the adjoint of G coinciding with G. □

In the definition of the Shimura datum one demands that the group G is reductive, since $\mathbb{R}$-algebraic reductive groups have Cartan involutions, which will be important:

Definition 1.3.9. Let G be a connected $\mathbb{R}$ algebraic group. An involutive automorphism θ of G is a Cartan involution, if the Lie subgroup

$$G^{\theta}(\mathbb{R}) := \{g \in G(\mathbb{C}) | g = \theta(\bar{g})\}$$

of $G(\mathbb{C})$ is compact.

Proposition 1.3.10. *A connected $\mathbb{R}$-algebraic group is reductive, if and only if it has a Cartan involution. Any two Cartan involutions are conjugate by an inner automorphism.*

Proof. By [54], **I**. Corollary 4.3, each connected $\mathbb{R}$-algebraic reductive group has a Cartan involution and the Cartan involutions are conjugate. Let θ be a Cartan involution on the connected $\mathbb{R}$ algebraic group G. Thus $G^{\theta}(\mathbb{R})$ is compact. By [54], **I**. Proposition 3.3, the group $G^{\theta}(\mathbb{R})$ is reductive. Thus one concludes that $G_{\mathbb{C}}$ and G are reductive as in the proof of [54], **I**. Theorem 4.2(i). □

Note that $\mathrm{id}^2 = \mathrm{id}$. Hence it can be considered as an involutive automorphism. This leads to the following examples of reductive groups:

Example 1.3.11. Let K be a connected $\mathbb{R}$-algebraic group such that $K(\mathbb{R})$ is a compact Lie group. One has

$$G^{\mathrm{id}}(\mathbb{R}) := \{g \in G(\mathbb{C}) | g = \bar{g}\} = G(\mathbb{R}),$$

which is compact by our assumption. Hence each compact $\mathbb{R}$-algebraic group $K \subset \mathrm{GL}(W)$ is reductive and has a Cartan involution given by id. Since any two Cartan involutions of K are conjugate and id is fixed by conjugation, the identity map id is the only Cartan involution of K.

Example 1.3.12. Let V be an $\mathbb{R}$-vector space of dimension N. The Group $\mathrm{GL}(V)$ has an involution given by $\theta : M \to (M^t)^{-1}$. On $\mathrm{GL}(V)(\mathbb{C})$ one has that $M = \theta(\bar{M})$, if and only if $M\bar{M}^t = E_{2g}$ resp., if and only if $M \in \mathrm{U}(N)$. It is a well-known fact that $\mathrm{U}(N)$ is compact. Thus θ is a Cartan involution.

Remark 1.3.13. Assume that G is an $\mathbb{R}$-algebraic reductive group with Cartan involution θ and $G' \subset G$ is a Zariski closed subgroup such that $\theta(G) = G$. Since a closed subgroup of a compact Lie group is a compact Lie group, one concludes easily that $\theta|_{G'}$ is a Cartan involution on G'.

From now on let (L, h, Q) be a polarized integral Hodge structure of type $(1,0),(0,1)$ on a torsion-free lattice L of rank $2g$ and $V := L \otimes \mathbb{Q}$. For simplicity we assume that Q is given by

$$J_0 = \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix} \tag{1.2}$$

with respect to a symplectic basis of L.

We use the preceding remark to find a Cartan involution on $\mathrm{Sp}(V, Q)$:

Example 1.3.14. Let $M \in \mathrm{Sp}(V,Q)(\mathbb{R})$. Then the matrix M is a $2g \times 2g$ matrix with

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, D are $g \times g$ matrices. Since $M \in \mathrm{Sp}(V,Q)(\mathbb{R})$, one has

$$M^t J_0 M = J_0 \Leftrightarrow M^{-1} = J_0^{-1} M^t J_0.$$

Hence

$$M = J_0^{-1}(M^t)^{-1} J_0 \Leftrightarrow J_0 M J_0^{-1} = (M^t)^{-1}.$$

Recall that a Cartan involution of $GL(V_{\mathbb{R}})$ is given by $M \to (M^t)^{-1}$ (see Example 1.3.12). Thus the conjugation by J_0 coincides with the restriction of this Cartan involution to $\mathrm{Sp}(V,Q)_{\mathbb{R}}$. Hence by the preceding remark, this yields a proof for the reductivity of $\mathrm{Sp}(V,Q)_{\mathbb{R}}$, which implies that $\mathrm{Sp}(V,Q)$ is reductive, too.

Example 1.3.15. The Hodge group $\mathrm{Hg}(V, h)$ contains the complex structure $J = h(i)$, which acts by the multiplication with z on $H^{1,0}$ and by the multiplication with $\bar{z}$ on $H^{0,1}$.

Let $\{b_1, \ldots, b_g\}$ be a basis of $H^{1,0}$ and

$$\Re b_k = b_k + \bar{b}_k, \quad \Im b_k = i(b_k - \bar{b}_k).$$

One has that

$$\mathcal{B} = \{\Im b_1, \ldots, \Im b_g, \Re b_1, \ldots, \Re b_g | k = 1, \ldots, g\}$$

is a basis of $V_{\mathbb{R}}$. From the fact that $J(\Re b_k) = \Im b_k$ and $J(\Im b_k) = -\Re b_k$, one concludes that J can be given by J_0 (see (1.2)) with respect to the basis $\mathcal{B}$. Hence by the same arguments as in Example 1.3.14 and the fact that $\mathrm{Hg}(V, h) \subset \mathrm{Sp}(V, Q)$, the group $\mathrm{Hg}(V, h)_{\mathbb{R}}$ is reductive.

Note that $\mathbb{S} = \mathbb{G}_{m,\mathbb{R}} \cdot S^1$ and the weight one Hodge structure (V, h) satisfies $h|_{\mathbb{G}_{m,\mathbb{R}}} = \mathrm{id}$. Thus from the fact that

$$h(S^1) \subset \mathrm{Sp}(V, Q) \quad \text{and} \quad \mathrm{GSp}(V, Q) = \mathbb{G}_{m,\mathbb{Q}} \cdot \mathrm{Sp}(V, Q)$$

one concludes that

$$\mathrm{MT}(V, h) \subset \mathrm{GSp}(V, Q).$$

Later we will see that $\mathrm{MT}(V, h)$ is also reductive.

We see that the result of the previous example holds true in general:

Theorem 1.3.16. *Let $\mathbb{Q} \subseteq F \subset \mathbb{R}$ be a field and (V, h, Q) be a polarized F-Hodge structure of weight k. Then*

$$g \to h(i) g h^{-1}(i)$$

yields a Cartan involution of $\mathrm{Hg}_F(V, h)_{\mathbb{R}}$ and $\mathrm{Hg}_F(V, h)$ is reductive.

Proof. Let $C = h(i)$. Since $h(-1)$ yields either id or $-$id, the inner automorphism θ of $\mathrm{Hg}_F(V, h)$ given by

$$g \to CgC$$

is an involution. Note that C acts by the multiplication with i^{p-q} on $V^{p,q}$. By the definition of the polarization of a Hodge structure of weight k,

$$H'_{(p,q)} := i^{p-q}(-1)^{\frac{k(k-1)}{2}} Q(\cdot, \bar{\cdot})$$

is a positive definite Hermitian form on $V^{p,q}$. Thus we define the Hermitian form

$$H_{(p,q)} := Q(\cdot, \overline{C \cdot})$$

on $V^{p,q}$. Since the Hodge decomposition is orthogonal for the Hermitian form $i^k Q(\cdot, \bar{\cdot})$, the different Hermitian forms $H_{(p,q)}$ give a Hermitian form H on $V_{\mathbb{C}}$, which is either positive definite or negative definite. Thus the unitary group $\mathrm{U}(V_{\mathbb{C}}, H)(\mathbb{R})$ is a compact Lie group. We show that

$$\mathrm{Hg}_F(V, h)^{\theta}(\mathbb{R}) \subseteq \mathrm{Hg}_F(V, h)(\mathbb{C}) \cap \mathrm{U}(V_{\mathbb{C}}, H)(\mathbb{R}),$$

which implies that $\mathrm{Hg}_F(V, h)^{\theta}(\mathbb{R})$ is compact resp., θ is a Cartan involution. From this result one concludes that $\mathrm{Hg}_F(V, h)$ is reductive.

Let $G(V, Q) = \mathrm{Sp}(V, Q)$, if k is odd, and $G(V, Q) = \mathrm{O}(V, Q)$, if k is even. Note that for each polarized polarized F-Hodge structure of weight k one has

$$\mathrm{Hg}_F(V, h) \subseteq G(V, Q).$$

This follows from the fact that the Hodge decomposition is orthogonal for $i^kQ(\cdot,\bar{\cdot})$. Assume that $g \in \mathrm{Hg}_F(V,h)^\theta(\mathbb{R})$. Thus one has

$$Q(v,\overline{Cu}) = Q(gv, g\overline{Cu}) = Q(gv, \overline{CgC^{-1}Cu}) = Q(gv, \overline{Cgu}).$$

Hence g is contained in $\mathrm{Hg}_F(V,h)(\mathbb{C}) \cap \mathrm{U}(V_{\mathbb{C}}, H)(\mathbb{R})$. $\square$

We need to show that $\mathrm{MT}(V,h)$ is reductive, since $h : \mathbb{S} \to \mathrm{MT}(V,h)_{\mathbb{R}}$ is a Shimura datum instead of $h : S^1 \to \mathrm{Hg}(V,h)_{\mathbb{R}}$ in the case of a rational Hodge structure. The definition of the Shimura datum will demand that $\mathrm{MT}(V,h)$ is reductive. For the proof that $\mathrm{MT}(V,h)$ is reductive, we compare $\mathrm{Hg}(V,h)$ and $\mathrm{MT}(V,h)$:

Lemma 1.3.17. *Let F be a field such that $\mathbb{Q} \subseteq F \subset \mathbb{R}$ and (V,h) be an F-Hodge structure. Then one has*

$$\mathrm{Hg}_F(V,h) = (\mathrm{MT}_F(V,h) \cap \mathrm{SL}(V))^0.$$

Moreover $\mathrm{MT}_F(V,H)$ *is the almost direct product of* $\mathrm{Hg}_F(V,h)$ *and* $\mathbb{G}_{m,F}$.

Proof. Since $\overline{V^{p,q}} = V^{q,p}$, one concludes $\dim V^{p,q} = \dim V^{q,p}$. By this fact and the fact that each $z \in S^1(\mathbb{R})$ acts by the multiplication with $z^p\bar{z}^q$ on $V^{p,q}$, one has $h(z) \in \mathrm{SL}(V)(\mathbb{R})$ for each $z \in S^1(\mathbb{R})$. Hence $\mathrm{Hg}_F(V,h) \subset \mathrm{SL}(V)$.

By the natural multiplication, we have a morphism

$$m : \mathrm{Hg}_F(V,h) \times \mathbb{G}_{m,F} \to \mathrm{MT}_F(V,h)$$

with finite kernel, since $\mathrm{Hg}_F(V,h) \subset \mathrm{SL}(V)$. Thus the Zariski closure Z of

$$m(\mathrm{Hg}_F(V,h) \times \mathbb{G}_{m,F}) \subseteq \mathrm{MT}_F(V,h)$$

is an F-algebraic subgroup of $\mathrm{MT}_F(V,h)$. Moreover one has that

$$h(\mathbb{S}) \subseteq Z_{\mathbb{R}} \subseteq \mathrm{MT}_F(V,h)_{\mathbb{R}}.$$

Hence $Z = \mathrm{MT}_F(V,h)$.

Since all homomorphisms $f : G \to G'$ of algebraic groups over algebraically closed fields satisfy $f(G) = \overline{f(G)}$ (see [1], Satz 2.1.8), we have the equality

$$\mathrm{Hg}_F(V,h)_{\bar{F}} \cdot \mathbb{G}_{m,\bar{F}} = Z_{\bar{F}} = \mathrm{MT}_F(V,h)_{\bar{F}}.$$

Now let $M \in \mathrm{MT}_F(V,h)(\bar{F}) \cap \mathrm{SL}(V)(\bar{F})$. It is given by a product $N \cdot M_1$ with $N \in \mathbb{G}_m(\bar{F})$ and $M_1 \in \mathrm{Hg}_F(V,h)(\bar{F})$. Since $\mathrm{Hg}_F(V,h)(\bar{F}) \subset \mathrm{SL}(V)(\bar{F})$, one concludes

$$N \in \mathbb{G}_m(\bar{F}) \cap \mathrm{SL}(V)(\bar{F}) = \mu_n(\bar{F}),$$

where $\dim V = n$. One has $M \in \mathrm{Hg}_F(V,h)(\bar{K})$, if and only if $N \in Hg_F(V,h)(\bar{F})$. Hence by the fact that $\mu_n(\bar{F})$ is finite, one obtains the statement. $\square$

Remark 1.3.18. For the polarized Hodge structure of weight 1 of a curve of genus g, we have a natural embedding $\mathrm{Hg}(V,h) \subset \mathrm{Sp}(V,Q)$. Since $\mu_{2g}(\mathbb{Q})$ is not a subgroup of $\mathrm{Sp}(V,Q)$ for $g > 1$ and for $g = 1$ one has $\mu_2 \subset h(S^1)$, we obtain the equality

$$\mathrm{Hg}(V,h) = \mathrm{MT}(V,h) \cap \mathrm{SL}(V)$$

only in the case of a genus one curve.

We conclude by the previous lemma:

Corollary 1.3.19. *Let $\mathbb{Q} \subseteq F \subset \mathbb{R}$ be a field and (V,h,Q) be a polarized F-Hodge structure of weight k. Then*

$$\mathrm{MT}_F^{\mathrm{der}}(V,h) = \mathrm{Hg}_F^{\mathrm{der}}(V,h) \quad \textit{and} \quad \mathrm{MT}_F^{\mathrm{ad}}(V,h) = \mathrm{Hg}_F^{\mathrm{ad}}(V,h).$$

Moreover one concludes by Lemma 1.3.17 that the center of $\mathrm{MT}_F(V,h)$ is the almost direct product of $\mathbb{G}_{m,F}$ and the center of $\mathrm{Hg}_F(V,h)$. Since $\mathrm{Hg}_F(V,h)$ is reductive by Theorem 1.3.16, one concludes by Proposition 1.3.4 that $\mathrm{MT}_F(V,h)$ is the almost direct product of its center and $\mathrm{Hg}_F^{\mathrm{der}}(V,h)$. Again we apply Proposition 1.3.4 and obtain:

Corollary 1.3.20. *Let $\mathbb{Q} \subseteq F \subset \mathbb{R}$ be a field and (V,h,Q) be a polarized F-Hodge structure of weight k. Then the $\mathrm{MT}_F(V,h)$ is reductive.*

By Corollary 1.3.19, we will see later that the homomorphism $h : S^1 \to \mathrm{Sp}(V,Q)_{\mathbb{R}}$ given by one of our polarized integral Hodge structures of type $(1,0),(0,1)$ can be considered as a Shimura datum. In literature the Shimura datum is given by $h : \mathbb{S} \to \mathrm{GSp}(V,\mathbb{Q})_{\mathbb{R}}$, where h is the corresponding homomorphism of our Hodge structure in the sense of Proposition 1.1.3.

The Hermitian symmetric domains obtained from Shimura data (G,h) are given by homogeneous spaces $G^{\mathrm{der}}(\mathbb{R})/K$, where K is the compact group fixed by the Cartan involution restricted to G^{der}. At present we can construct the homogeneous space in the case of our examples:

Construction 1.3.21. An embedding $h : S^1 \to \mathrm{Sp}(V,Q)_{\mathbb{R}}$ obtained by a polarized integral Hodge structure (L,h,Q) of type $(1,0),(0,1)$ corresponds to the complex structure $J := h(i)$, which satisfies $Q(J\cdot, J\cdot) = Q(\cdot,\cdot)$ and $Q(Jv,v) > 0$. By the definition of $\mathrm{Sp}(V,Q)_{\mathbb{R}}$, one has $J \in \mathrm{Sp}(V,Q)_{\mathbb{R}}$. Moreover one checks easily that gJg^{-1} satisfies the same conditions for all $g \in \mathrm{Sp}(V,Q)_{\mathbb{R}}$.

The complex structure J can be given by the same matrix as J_0 with respect to a symplectic basis. Thus there exists a $g \in \mathrm{GL}(V)(\mathbb{R})$ such that $J = gJ_0g^{-1}$. Therefore J yields a Cartan involution of $\mathrm{Sp}(V,Q)_{\mathbb{R}}$. By Proposition 1.3.10, one can assume that $g \in \mathrm{Sp}(V,Q)(\mathbb{R})$. Therefore $\mathrm{Sp}(V,Q)(\mathbb{R})$ acts by conjugation transitively on the set of complex structures $J \in \mathrm{Sp}(V,Q)(\mathbb{R})$ satisfying $Q(Jv,v) > 0$.

Let K be the subgroup of $\mathrm{Sp}(V,Q)(\mathbb{R})$, which leaves a fixed $h(S^1)$ stable by conjugation. The set of points of the homogeneous space $\mathfrak{h}_g := \mathrm{Sp}(V,Q)(\mathbb{R})/K$ can be identified with the set of complex structures $J \in \mathrm{Sp}(V,Q)(\mathbb{R})$ satisfying $Q(Jv,v) > 0$. By the preceding section, the points of $\mathfrak{h}_g$ can be identified with the polarized integral Hodge structures (L,h,Q) of type $(1,0),(0,1)$ resp., principally polarized abelian varieties of dimension g with symplectic basis.

In the same way one can construct a homogeneous subspace of $\mathfrak{h}_g$ using the Hodge group $\mathrm{Hg}(L_{\mathbb{Q}},h')$ of a polarized integral Hodge structure (L,h',Q) instead of $\mathrm{Sp}(V,Q)$. This space parametrizes all polarized integral Hodge structures (L,h,Q) with a Hodge group contained in $\mathrm{Hg}(L_{\mathbb{Q}},h')$

It is quite easy to see that a corresponding construction runs well in the case of a representation $h : \mathbb{S} \to \mathrm{GSp}(V,Q)_{\mathbb{R}}$ obtained by a polarized integral Hodge structure (L,h',Q) of type $(1,0),(0,1)$. In this case we obtain a homogeneous space, which parametrizes all complex structures $J \in \mathrm{GSp}(V,Q)(\mathbb{R})$. In a similar way one obtains a subspace parametrizing all integral Hodge structures (L,h) with a Mumford-Tate group contained in $\mathrm{MT}(L_{\mathbb{Q}},h')$.

Now let us define the Shimura datum:

Definition 1.3.22. A Shimura datum (G,h) is given by a reductive $\mathbb{Q}$-algebraic group G and a conjugacy class of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$ of algebraic groups satisfying:

1. The restriction of the inner automorphism of $h(i)$ on $G_{\mathbb{R}}$ to $G^{\mathrm{der}}_{\mathbb{R}}$ is a Cartan involution.
2. The adjoint group G^{ad} does not have any direct $\mathbb{Q}$-factor H such that $H(\mathbb{R})$ is a compact Lie group.
3. The representation $(\mathrm{ad} \circ h)(\mathbb{S})$ on $Lie(G)_{\mathbb{C}}$ corresponds to a Hodge structure of the type $(1,-1) \oplus (0,0) \oplus (-1,1)$.

In the cases of our examples we have already seen that Condition (1) in the definition of a Shimura datum allows the construction of a homogeneous space, which parametrizes the conjugacy class of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$. In the next section we will see that Condition (2) and Condition (3) allow one to endow the connected components of this homogeneous space with the structure of a Hermitian symmetric domain.

We will also accept a pair (G,h) as Shimura datum, if the representation $(\mathrm{ad} \circ h)(\mathbb{S})$ on $Lie(G)_{\mathbb{C}}$ is trivial resp., corresponds to a Hodge structure of the type $(0,0)$. In this case we obtain a homogeneous space consisting of only one point.

Remark 1.3.23. If one compares our definition of a Shimura datum with other definitions used in literature, one finds some different formulations (for example compare [16], [40]). This happens, since one can replace the Conditions (1) and (2) by equivalent conditions:

1. Since G^{der} and G^{ad} are isogeneous, the compact subgroups of $G^{\mathrm{der}}_{\mathbb{R}}$ and $G^{\mathrm{ad}}_{\mathbb{R}}$ correspond. Moreover the inner automorphism of $(\mathrm{ad} \circ h)(i)$ on $G^{\mathrm{ad}}_{\mathbb{R}}$ is well-defined. Thus the inner automorphism of $(\mathrm{ad} \circ h)(i)$ on $G^{\mathrm{ad}}_{\mathbb{R}}$ is a Cartan involution, if and only if the inner automorphism of $h(i)$ on $G_{\mathbb{R}}$ to $G^{\mathrm{der}}_{\mathbb{R}}$ is a Cartan involution. Thus one can replace Condition (1) by the condition that the inner automorphism of $(\mathrm{ad} \circ h)(i)$ on $G^{\mathrm{ad}}_{\mathbb{R}}$ is a Cartan involution.
2. By Example 1.3.11, Condition (2) is equivalent to the condition that the adjoint group G^{ad} does not have any direct $\mathbb{Q}$-factor H, which satisfies $\theta|_H = \mathrm{id}_H$ for a Cartan involution θ of G^{ad}. Usually one writes that G^{ad} does not have any direct $\mathbb{Q}$-factor H such that the inner automorphism of $(\mathrm{ad} \circ h)(i)$ restricted to H is trivial. This is equivalent to the condition that G^{ad} does not have a direct $\mathbb{Q}$-factor H such that $\mathrm{pr}_H \circ \mathrm{ad} \circ h$ is trivial.

Now we give our first example of a Shimura datum:

Proposition 1.3.24. *Assume that (V, h, Q) is a polarized rational Hodge structure of type $(1,0), (0,1)$. Then $(\mathrm{GSp}(V, Q), h)$ is a Shimura datum.*

Proof. We have seen that $\mathrm{GSp}(V, Q)$ is reductive.

By Construction 1.3.21, we have a conjugacy class of complex structures, which corresponds to a conjugacy class of homomorphisms $h : \mathbb{S} \to \mathrm{GSp}(V, Q)_{\mathbb{R}}$ satisfying the condition (1) in the definition of the Shimura datum.

Recall that $\mathrm{Sp}_{2g}(\mathbb{C})$ is a classical simple Lie group. Thus $\mathrm{GSp}(V, Q)^{\mathrm{ad}} = \mathrm{Sp}(V, Q)^{\mathrm{ad}}$ has only one direct simple factor, which is not compact. Hence condition (2) of the Shimura datum is satisfied.

Since the center of $\mathrm{GSp}(V, Q)_{\mathbb{R}}$ is given by $\mathbb{G}_{m,\mathbb{R}}$ (see [40], page 66), the kernel of the adjoint representation on $Lie(\mathrm{GSp}(V, Q)_{\mathbb{R}})$ of any $h(\mathbb{S})$ in the conjugacy class is given by $\mathbb{G}_{m,\mathbb{R}}$. Since $h(a + ib) = aE_{2g} + bJ$, each $g \in \mathrm{GSp}(V, Q)(\mathbb{R})$ commutes with J, if and only if it commutes with each element of $\mathbb{S}(\mathbb{R})$. Hence on the complexified eigenspace $(\mathfrak{p}_0)_{\mathbb{C}}$ with eigenvalue -1 with respect to the Cartan involution, $\mathbb{S}$ acts by the characters $z/\bar{z}$ and $\bar{z}/z$. This corresponds to a Hodge structure of the type $(1, -1) \oplus (0, 0) \oplus (-1, 1)$ on $Lie(\mathrm{GSp}(V, Q)_{\mathbb{R}})$. Hence condition (3) is satisfied. □

Definition 1.3.25. A Shimura datum (G, h) is of Hodge type, if there is a closed embedding $\rho : G \hookrightarrow \mathrm{GSp}(V, Q)$ such that one has the Shimura datum of Example 1.3.24 by

$$\mathbb{S} \stackrel{h}{\hookrightarrow} G_{\mathbb{R}} \stackrel{\rho_{\mathbb{R}}}{\hookrightarrow} \mathrm{GSp}(V, Q)_{\mathbb{R}}.$$

In the next section we use Shimura data to construct complex manifolds, which will be used for the construction of quasi-projective varieties, which are the Shimura varieties. A Shimura variety is of Hodge type, if it is obtained by a Shimura datum (G, h) of Hodge type. We will use the examples of Shimura data of Hodge type, which are given the following proposition:

Proposition 1.3.26. *Let* (V, h, Q) *be a polarized* $\mathbb{Q}$*-Hodge structure of type* $(1, 0), (0, 1)$*. Then* $(\mathrm{MT}(V.h), h)$ *is a Shimura datum.*

Proof. By Corollary 1.3.20, the Mumford-Tate group $\mathrm{MT}(V, h)$ is reductive. The inner automorphism given by

$$g \to h(i)gh^{-1}(i)$$

descends to a Cartan involution θ on $\mathrm{MT}^{\mathrm{ad}}(V, h)_{\mathbb{R}} = \mathrm{Hg}^{\mathrm{ad}}(V, h)_{\mathbb{R}}$. Hence condition (1) in the definition of the Shimura datum, is satisfied.

Next we have to show that any direct $\mathbb{Q}$-factor with trivial Cartan involution is isomorphic to $\{e\}$. Let H be a simple direct $\mathbb{Q}$-factor of $\mathrm{MT}(V, h)^{\mathrm{ad}}$ with trivial Cartan involution. We have a surjection

$$s : \mathrm{MT}(V, h) \xrightarrow{\mathrm{ad}} \mathrm{MT}(V, h)^{\mathrm{ad}} \xrightarrow{pr_H} H,$$

which is obviously a homomorphism of $\mathbb{Q}$-algebraic groups. Hence the kernel $\tilde{K}$ of s is a $\mathbb{Q}$-algebraic group. The complex structure J, which satisfies that the conjugation by $\mathrm{ad}(J)$ is the Cartan involution, satisfies that all elements of the adjoint group $H_{\mathbb{R}}$ commute with $\mathrm{ad}(J)$. Thus J is contained in $\tilde{K}_{\mathbb{R}}$. Hence $h(\mathbb{S}) \subset \tilde{K}_{\mathbb{R}}$, which implies $\tilde{K} = \mathrm{MT}(V, h)$ resp., $H = \{e\}$.

The conjugacy class of the representation $h : \mathbb{S} \to \mathrm{MT}_{\mathbb{R}}(V, h) \hookrightarrow \mathrm{GSp}_{\mathbb{R}}(V, Q)$ is the Shimura datum of Proposition 1.3.24. Hence the adjoint representation of $\mathbb{S}$ on $Lie(\mathrm{MT}(V, h))_{\mathbb{C}} \subset Lie(\mathrm{GSp}(V, Q))_{\mathbb{C}}$ induces a Hodge structure of the same type (or of the type $(0, 0)$). □

1.4 Hermitian symmetric domains

In this section we construct Hermitian symmetric domains by using Shimura data. These domains will later be our restricted period domains, which parametrize Hodge structures (V, h) such that $h(\mathbb{S})$ is contained in a given reductive group. Siegel's upper half plane remains to be our illustrating example. In the preceding section we have constructed Siegel's upper half plane $\mathfrak{h}_g$ as homogeneous space, which parametrizes polarized integral Hodge structures of type $(1, 0), (0, 1)$. Here we see that Siegel's upper half plane can be endowed with the structure of a Hermitian symmetric domain.

Let G be an $\mathbb{R}$-algebraic group. Moreover let G^0 denote the Zariski connected component of identity and let $G^+(\mathbb{R})$ denote the connected component of identity for the Lie group $G(\mathbb{R})$. In general the Lie group $G^0(\mathbb{R})$ is not a connected manifold, which will be a reason to be careful in this section. For example $\mathbb{G}_m(\mathbb{R})$ is Zariski connected, but has two connected components given by the real numbers larger than 0 and the real numbers smaller than 0. Only the inclusion $G^0(\mathbb{R}) \supset G^+(\mathbb{R})$ holds true in general.

1.4.1. Let (G, h) be a Shimura datum. The elements of the conjugacy class of h are given by the points of the homogeneous space $D = G(\mathbb{R})/K$, where K is the isotropy group of h, i. e. the subgroup of $G(\mathbb{R})$ such that:

$$ghg^{-1} = h \quad (\forall g \in K)$$

Since the multiplication by $g \in G(\mathbb{R})$ is a diffeomorphism of the Lie group $G(\mathbb{R})$, all connected components yield manifolds isomorphic to the space given by $G^+(\mathbb{R})$. Note that $G^{\mathrm{ad}}(\mathbb{R})$ is a connected Lie group, since it can be obtained as the quotient

$$G^{\mathrm{ad}}(\mathbb{R}) = G^+(\mathbb{R})/(Z \cap G^+(\mathbb{R})).$$

Assume that $G(\mathbb{R})$ has r connected components. Since the center Z of $G(\mathbb{R})$ fixes all representations h by conjugation, D can be considered as

$$D = \dot{\bigcup_{i=1,\ldots,r}} G^{\mathrm{ad}}(\mathbb{R})/\mathrm{ad}_{G(\mathbb{R})}(K).$$

Now we start to endow the connected components of the homogeneous space $D = G(\mathbb{R})/K$ obtained from a Shimura datum (G, h) with the structure of a Hermitian symmetric domain.

Let M be a $\mathcal{C}^\infty$ manifold. An almost complex structure on M is a smoothly varying family $(J_p)_{p \in M} : T_pM \to T_pM$ of automorphisms of the respective tangent spaces T_pM for all $p \in M$, which satisfies the condition $J_p^2 = -1$ for all $p \in M$. Thus J_p is a complex structure on the vector space T_pM for all $p \in M$. Such a pair (M, J) is called an almost complex manifold. For example the affine complex line given by $\mathbb{C}$ with the almost complex structure

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial y} \to -\frac{\partial}{\partial x} \tag{1.3}$$

is an almost complex manifold.

As everyone should know, the complex line $\mathbb{C}$ is not only an almost complex manifold, but a complex manifold. An almost complex structure J on a $\mathcal{C}^\infty$ manifold M is called integrable, if M is endowed with the structure of a complex manifold, which induces the almost complex structure J. Let

$$S_J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY],$$

where X and Y are vector fields.

Theorem 1.4.2. *An almost complex complex structure J is integrable, if and only if it satisfies*

$$S_J = 0.$$

Proof. (see [50]) □

In order to get some sense for the criterion of Theorem 1.4.2, one can use it to verify that the almost complex structure on $\mathbb{C}$ given by (1.3) is integrable.

For our construction of a Hermitian symmetric domain by a Shimura datum we need the following definition:

Definition 1.4.3. A smooth 2-tensor field g on a $\mathcal{C}^\infty$ manifold M is a family of bilinear maps $g_p : T_pM \times T_pM \to T_pM$ such that for all smooth vector fields X, Y the map $p \to g_p(X,Y)$ is smooth. The 2-tensor field g is a Riemannian structure, if for all $p \in M$ the bilinear form g_p is symmetric and positive definite.

Now we endow a homogeneous space obtained from a Shimura datum with a Riemannian structure:

Example 1.4.4. Let $h : \mathbb{S} \to G_{\mathbb{R}}$ be a Shimura datum and $D = G(\mathbb{R})/K_h(\mathbb{R})$ be the homogeneous space parametrizing the elements of the conjugacy class of h. By [27], **II**. §4, the homogeneous space D is a $\mathcal{C}^\infty$ manifold and the elements of $G(\mathbb{R})$ act as diffeomorphisms on D. We construct a $G(\mathbb{R})$ invariant Riemannian form in the following way:

On the real vector space

$$T_hD = Lie(G(\mathbb{R}))/Lie(K_h(\mathbb{R}))$$

one finds easily a symmetric and positive definite bilinear form $(\cdot,\cdot)_h$. Let $u, v \in T_hD$. Moreover let for all $g \in K_h(\mathbb{R})$ the homomorphism $u \to dg(u)$ be given by the differential of the diffeomorphism obtained from g. Since $K_h(\mathbb{R})$ is compact, the function given by

$$K_h(\mathbb{R}) \ni g \to (dg(u), dg(v))_h \in \mathbb{R}$$

reaches a maximal value and a minimal value over $K_h(\mathbb{R})$. In addition the compactness of $K_h(\mathbb{R})$ implies that $K_h(\mathbb{R})$ has a finite Haar measure dx. Thus the bilinear form

$$(u,v)'_h = \int_{K_h(\mathbb{R})} (dg(u), dg(v))dx$$

is well-defined. Since $(\cdot,\cdot)$ is symmetric and positive definite, one concludes easily that $(\cdot,\cdot)'$ is symmetric and positive definite, too. Moreover it is fixed by the action of K_h.

Let $h' \in D$. There exists a $g \in G(\mathbb{R})$ with $g(h') = h$. Let dg denote the differential of the diffeomorphism on D obtained from g. On the tangent space of h' we define the positive definite and symmetric bilinear form $(\cdot,\cdot)'_{h'}$ given by

$$(\cdot,\cdot)'_{h'} = (dg(\cdot), dg(\cdot))'_h.$$

Note that for all $g, g' \in G(\mathbb{R})$ with

$$g'(h') = g(h') = h$$

there exists a $k \in K_h(\mathbb{R})$ with $g' = k \circ g$. Since $(\cdot,\cdot)'_h$ is $K_h(\mathbb{R})$-invariant, $(\cdot,\cdot)'_{h'}$ is independent of the choice of g. Thus the action of $G(\mathbb{R})$ on D as a transitive diffeomorphism group yields a well-defined $G(\mathbb{R})$-invariant Riemannian structure on each connected component of D. (see also the proof of [27], **IV**. Proposition 3.4)

Definition 1.4.5. Let M be a connected $\mathcal{C}^\infty$ manifold with an almost complex structure J. A Riemannian structure g on M is a Hermitian structure, if

$$g(J\cdot, J\cdot) = g(\cdot,\cdot).$$

Example 1.4.6. From the definition of the Shimura datum, $Lie(G_{\mathbb{C}})$ has an eigenspace decomposition of the type

$$(1,-1), \quad (0,0), \quad (-1,1)$$

with respect to the action of $h(\mathbb{S})$. The intersection of the $(0,0)$ eigenspace with $Lie(G_{\mathbb{R}})$ coincides with $Lie((K_h)_{\mathbb{R}})$. Thus on the vector space

$$T_h(D) = Lie(G(\mathbb{R}))/Lie(K_h(\mathbb{R}))$$

we have a complex structure J_h obtained from the eigenspace decomposition of $Lie(G_{\mathbb{C}})$. Note that $h(\mathbb{S})$ is contained in the center of $(K_h)_{\mathbb{R}}$ and the complex structure J_h is given by the differential of the map obtained from some root $h(\sqrt{i})$. Let $g \in G(\mathbb{R})$ and $h = g(h')$. Now let dg denote the differential of the diffeomorphism of D given by g. By

$$J_{h'} = dg^{-1} \circ J_h \circ dg,$$

one defines a complex structure on $T_{h'}D$. Note that for all $g, g' \in G(\mathbb{R})$ with

$$g'(h') = g(h') = h$$

there exists a $k \in K_h(\mathbb{R})$ with $g' = k \circ g$. Since $h(\sqrt{i})$ commutes with all $k \in K_h(\mathbb{R})$, one obtains

$$d(g')^{-1} \circ J_h \circ d(g') = dg^{-1} \circ dk^{-1} \circ J_h \circ dk \circ dg = dg^{-1} \circ J_h \circ dg.$$

Thus $J_{h'}$ is independent of the choice of g and we obtain a well-defined $G(\mathbb{R})$-invariant almost complex structure J on D.

In Example 1.4.4 we have constructed a $G(\mathbb{R})$-invariant Riemannian structure on each connected component of D. By the construction of this Riemannian structure $(\cdot,\cdot)'_{h'}$, one sees easily that

$$(J_{h'}(\cdot), J_{h'}(\cdot))'_{h'} = (\cdot,\cdot)'_{h'}.$$

Thus we have a $G(\mathbb{R})$-invariant Hermitian structure on each connected component of D.

Definition 1.4.7. A Hermitian symmetric space is a connected complex manifold M endowed with an Hermitian structure such that each point $p \in M$ is an isolated fixed point of an involutive holomorphic isometry of M.

We consider the Riemannian structure on each connected component of D, where D is the homogeneous space obtained from a Shimura datum (G, h). Since the almost complex structure J is $G(\mathbb{R})$-invariant and D is a homogeneous space, it is sufficient to consider J at one arbitrary point $h \in D$. By using the criterion $S_J = 0$ of Theorem 1.4.2, one can show that the almost complex structure J on a homogeneous space D obtained from a Shimura datum is integrable (use the results of Example 1.4.4 and Example 1.4.6 and compare to [27], **VIII**. Proposition 4.2 and its proof).

Since the Hermitian structure is $G(\mathbb{R})$-invariant, the Cartan involution obtained from $h(i)$ acts on D as an involutive isometry with isolated fixed point representing h. By the fact that J is integrable, we conclude:

Proposition 1.4.8. *Each connected component of the homogeneous space D obtained from a Shimura datum is a Hermitian symmetric space.*

Let D^+ denote a connected component of D. Note that the group of holomorphic isometries $\mathrm{Hol}(D^+, g)$ of the Hermitian symmetric space (D^+, g) is endowed with the structure of a Lie group instead of the structure of an algebraic group. Thus one is not able to define a Cartan involution of $\mathrm{Hol}(D^+, g)$ as we have done for $\mathbb{R}$-algebraic groups. In this case one considers the complexified Lie algebra $Lie_{\mathbb{C}}(\mathrm{Hol}(D^+, g))$ and defines a Cartan involution for Lie algebras (for details see [27], **III**. §7).

In our case the Cartan involution on $G^{\mathrm{ad}}_{\mathbb{R}}$ induces a Cartan involution on the Lie algebra of $G^{\mathrm{ad}}(\mathbb{R})$ in the sense of [27]. Condition (2) in the definition of the Shimura datum guarantees that G^{ad} is not compact. Moreover the action of $G(\mathbb{R})^+$ on a connected component D^+ of D descends to an action of $G^{\mathrm{ad}}(\mathbb{R})$ on D^+, since the center $Z(G)(\mathbb{R})$ acts trivial on the conjugacy class and $G^{\mathrm{ad}} \cong G/Z(G)$.

1.4.9. We will see that the quotient of $G^{\mathrm{ad}}(\mathbb{R})$ by its direct compact factors is the connected component of the group of holomorphic isometries $(\mathrm{Hol}(D^+, g))^+$ of the Hermitian symmetric space D^+ obtained from a Shimura datum (G, h). By Condition (2) in the definition of the Shimura datum, one concludes that $\mathrm{Hol}(D^+, g)^+$ is a noncompact semisimple Lie group. This Lie group is endowed with an involution ι, which induces by its differential a Cartan involution on $Lie(\mathrm{Hol}(D^+, g))$. Let $K_\iota \subset \mathrm{Hol}(D^+, g)^+$ be the subgroup, on which ι acts as id. The isotropy group K of one point $p \in D^+$ satisfies $K_\iota^+ \subseteq K \subseteq K_\iota$. Such a Hermitian symmetric space D^+ is called a Hermitian symmetric domain.

1.4.10. Hermitian symmetric domains have the following properties:

- Each Hermitian symmetric domain D is biholomorphic to an open bounded connected complex submanifold D' of $\mathbb{C}^N$. Moreover each $p \in D'$ is an isolated fixed point of an involutive holomorphic diffeomorphism $\phi : D' \to D'$, which is induced from a corresponding involutive isometry of D. Such a domain D' is called bounded symmetric domain. Conversely each bounded symmetric domain D' can be equipped with a Hermitian metric (called Bergman metric), which turns D' into a Hermitian symmetric domain (see [27], **VIII**. Theorem 7.1).

For example the upper half plane $\mathfrak{h}_1$ given by the complex numbers $x+iy$ with $y > 0$ is biholomorphic to the ball $\mathbb{B}_1 = \{z \in \mathbb{C} : |z| < 1\}$, which is a bounded symmetric domain. The biholomorphic map $\phi : \mathfrak{h}_1 \to \mathbb{B}_1$ is given by

$$\phi(z) = \frac{z-1}{z+1}.$$

- Each holomorphic diffeomorphism between bounded symmetric domains is an isometry for the Bergman metrices (see [27], **VIII**. Proposition 3.5).
- Let $\mathrm{Is}(D, g)$ denote the group of $\mathcal{C}^\infty$-isometries of the Hermitian symmetric domain (D, g) and $\mathrm{Hol}(D)$ denote the group of holomorphic diffeomorphisms acting on D. Then one has

$$\mathrm{Is}(D, g)^+ = \mathrm{Hol}(D, g)^+ = \mathrm{Hol}(D)^+$$

(see [40], Proposition 1.6).
- A Hermitian symmetric domain D is irreducible, if $\mathrm{Hol}(D, g)^+$ is simple. Each Hermitian symmetric domain D is a product

$$D = D_1 \times \ldots \times D_k$$

of irreducible Hermitian symmetric domains $D_1, \ldots, D_k$ (follows from [27], **VIII**. Proposition 5.5). By the classification of simple Lie groups, one obtains a classification of irreducible Hermitian symmetric domains (use [27], **VIII**. Theorem 6.1 and [27], **X**. Table **V**).

Theorem 1.4.11. *Let $h : \mathbb{S} \to G$ be a Shimura datum, W be a real vector space and K denote the centralizer of $h(\mathbb{S})$. Then each connected component D^+ of $D = G(\mathbb{R})/K(\mathbb{R})$ has a unique structure of a Hermitian symmetric domain. These domains are isomorphic, where the connected component of the group of holomorphic isometries is given by the quotient of $G^{\mathrm{ad}}(\mathbb{R})$ by its direct compact factors. Each representation $\rho : G_{\mathbb{R}} \to \mathrm{GL}(W)$ yields a holomorphic variation $(W, \rho \circ h)_{h \in D}$ of Hodge structures on D.*

Proof. (See [16], 2.1.1.) □

By the preceding considerations of this section, we have already proved that the connected components D^+ of $D = G(\mathbb{R})/K(\mathbb{R})$ have a unique structure of a Hermitian symmetric space. The proof of the remaining statement about variations of Hodge structures can be found in the same essay [16] of P. Deligne in Proposition 1.1.14.(i).

Now let us consider our main example:

Example 1.4.12. The Lie group $\mathrm{GSp}(V,Q)(\mathbb{R})$ has two connected components. One component consists of matrices with positive determinant and the other consists of matrices with negative determinant. Hence the corresponding homogeneous space D parametrizing the elements of the conjugacy class of a Hodge structure of an abelian variety given by h has two connected components. Note that $\mathrm{GSp}(V,Q)(\mathbb{R})^+$ is a product of $\mathrm{Sp}(V,Q)(\mathbb{R})$ and $\mathbb{G}_m^+(\mathbb{R})$. Since $\mathbb{G}_m(\mathbb{R})^+$ is contained in the stabilizers of all points, the corresponding connected homogeneous space coincides with $\mathfrak{h}_g$ such that the preceding Theorem endows $\mathfrak{h}_g$ with the structure of a Hermitian symmetric domain. By the representation of $\mathrm{GSp}(V,Q)_\mathbb{R}$ given by the identical embedding $\mathrm{GSp}(V,Q)_\mathbb{R} \hookrightarrow \mathrm{GL}(V)_\mathbb{R}$, the upper half plane $\mathfrak{h}_g$ is endowed with the natural holomorphic variation of polarized integral Hodge structures of type $(1,0),(0,1)$.

Assume that (V,h,Q) is a polarized $\mathbb{Q}$-Hodge structure of type $(1,0),(0,1)$. By Proposition 1.3.26, the pair $(\mathrm{MT}(V,h),h)$ is a Shimura datum. Lemma 1.3.17 ensures that the connected components of the conjugacy class of $h: S^1 \to \mathrm{Hg}(V,h)$ are given by connected components of $\mathrm{MT}(V,Q)(\mathbb{R})/K(\mathbb{R})$ contained in the upper half plane $\mathfrak{h}_g$. Thus we rather work with $\mathrm{Hg}(V,h)$ than with $\mathrm{MT}(V,h)$. By Corollary 1.3.19, the pair $(\mathrm{Hg}(V,h),h)$ can be considered as Shimura datum, too:

Remark 1.4.13. Assume that (V,h,Q) is a polarized integral Hodge structure of type $(1,0),(0,1)$. By Corollary 1.3.19, one has that $\mathrm{MT}^{\mathrm{ad}}(V,h) = \mathrm{Hg}^{\mathrm{ad}}(V,h)$. Thus one has that $\mathrm{MT}^{\mathrm{ad}}(V,h)(\mathbb{R}) = \mathrm{Hg}^{\mathrm{ad}}(V,h)(\mathbb{R})$. Hence by the preceding construction, $\mathrm{Hg}(V,h)^{\mathrm{ad}}(\mathbb{R})$ is the identity component of the holomorphic isometry group of the Hermitian symmetric domain D^+, where D^+ is a connected component of the conjugacy class of h. The isotropy group of the point representing h is given by the compact subgroup of $\mathrm{Hg}^{\mathrm{ad}}(V,h)(\mathbb{R})$ fixed by the Cartan involution on $\mathrm{Hg}^{\mathrm{ad}}(V,h)_\mathbb{R}$ obtained from the inner automorphism of the complex structure $J = h(i)$. Hence one can consider the pair consisting of V and $h|_{S^1}: S^1 \to \mathrm{Hg}(V,h)_\mathbb{R}$ as Shimura datum, too.

Note that D. Mumford and J. Tate have originally constructed families of abelian varieties over Hermitian symmetric domains by using the Hodge group instead of the Mumford-Tate group (see [46] and [47]) as we will do in a similar way. The Mumford-Tate group was later introduced by number theorists, who work with Shimura varieties, for technical reasons.

Now we construct the holomorphic family of principally polarized abelian varieties over $\mathrm{Hg}(V,h)(\mathbb{R})/K$ corresponding to the VHS induced by the closed embedding

$$\mathrm{id}: \mathrm{Hg}(V,h) \hookrightarrow \mathrm{Sp}(V,h),$$

where (V,h,Q) is a polarized rational Hodge structure of type $(1,0),(0,1)$.

Construction 1.4.14. Let (L, h, Q) be a polarized $\mathbb{Z}$-Hodge structure of type $(1,0),(0,1)$ with $V := L_{\mathbb{Q}}$ as before and $\{v_1, \ldots, v_g, w_1, \ldots, w_g\}$ be a symplectic basis of L with respect to Q. For example it may be given on $L := H^1(C, \mathbb{Z})$, where C is a curve of genus g. Moreover let $[v_i]$ resp., $[w_i]$ denote the image of v_i resp., w_i by the map

$$L \to L \otimes \mathbb{C} \to H^{0,1}.$$

One has that $\mathrm{Hg}(V, h) \subset \mathrm{Sp}(V, Q)$. Let $K \subset \mathrm{Hg}(V, h)^+(\mathbb{R})$ be the centralizer of $h(S^1(\mathbb{R}))$. Thus $\mathrm{Hg}(V, h)^+(\mathbb{R})/K$ is a Hermitian symmetric domain as we have seen. Consider the linearly independent set $B = \{[w_1], \ldots, [w_g]\} \subset H^{0,1}$, which generates the real subvector space W. Now iW is obviously generated by $\{[Jw_1], \ldots, [Jw_g]\}$. The principal polarization H of the abelian variety $A = H^{0,1}/L$ is given by the corresponding alternating form $E = -Q$ as in the proof of Theorem 1.2.10. Since E vanishes on W, the principal polarization H given by $H = E(i., .) + iE(., .)$ vanishes on the complex vector space $W \cap iW$, too. Hence $W \cap iW = 0$. Thus the fact that $Span_{\mathbb{R}}(v, Jv)$ is mapped to $Span_{\mathbb{C}}([v])$ implies that B is a $\mathbb{C}$-basis of $H^{0,1}$. Hence the period matrix of the corresponding abelian variety may be given by (Z, E_g), where the columns of Z are given by the $[v_i]$ in their coordinates with respect to B.

Thus the embedding $H^{1,0} \hookrightarrow V_{\mathbb{C}}$ is given by the matrix $(Z^t, -E_g)^t$. Since we have a holomorphic variation of Hodge structures, this matrix varies holomorphically. Thus the period matrices of the corresponding abelian varieties vary holomorphically, too. Hence the corresponding action of L on $H^{0,1} \times \mathrm{Hg}(V, h)^+(\mathbb{R})/K$ is holomorphic and we obtain a holomorphic family of abelian varieties over $\mathrm{Hg}(V, h)^+(\mathbb{R})/K$.

By the previous construction, the period matrices of the fibers of our holomorphic family of abelian varieties over $\mathrm{Hg}(V, h)^+(\mathbb{R})/K$ are given by (Z, E_g). Recall that Siegel's upper half plane $\mathfrak{h}_g$ parametrizes the principally polarized abelian varieties with symplectic basis. Hence for each $p \in \mathfrak{h}_g$ one finds exactly one matrix Z_p such that (Z_p, E_g) is the period matrix of the given principally polarized abelian variety with symplectic basis. Thus the mapping $\varphi : p \to Z_p$ is injective and well-defined. By the previous construction, φ is holomorphic, too.

Proposition 1.4.15. *A matrix (Z, E_g) is the period matrix of a principally polarized abelian variety with respect to a symplectic basis, if and only if*

$$Z^t = Z \quad \text{and} \quad \Im Z > 0.$$

Proof. (see [7], Proposition 8.1.1) □

Since the set of matrices Z satisfying the conditions of the preceding Proposition has the structure of a smooth complex manifold, φ is a holomorphic diffeomorphism. By φ, one can endow the set of these matrices Z

with the structure of a Hermitian symmetric domain. Hence we obtain the often used description of the upper half plane:

Proposition 1.4.16.

$$\mathfrak{h}_g \cong \{M \in M_g(\mathbb{C}) | Z^t = Z, \Im Z > 0\}$$

Especially in the case $g = 1$ one obtains

$$\mathfrak{h}_1 = \{x + iy \in \mathbb{C} | y > 0\}.$$

1.5 The construction of Shimura varieties

In the preceding section we have seen that a Shimura datum yields a bounded symmetric domain. This is the first step of the construction of a Shimura variety. For completeness we sketch the construction of a Shimura variety in this section. Later we will only need to use the language of Shimura data and bounded symmetric domains obtained from these data.

Definition 1.5.1. Let G be a $\mathbb{Q}$-algebraic group. An arithmetic subgroup Γ of $G(\mathbb{Q})$ is a group, which is commensurable with $G(\mathbb{Z})$.

A subgroup Γ of a connected Lie group H is arithmetic, if there is a $\mathbb{Q}$-algebraic group G, an arithmetic subgroup Γ_0 of $G(\mathbb{Q})$ and a surjective homomorphism $\eta : G(\mathbb{R})^+ \to H$ of Lie groups with compact kernel such that $\eta(\Gamma_0) = \Gamma$.

The second step of the construction of a Shimura variety is given by the following theorem:

Theorem 1.5.2 (of Baily and Borel). *Let D be a bounded symmetric domain, and Γ be an arithmetic subgroup of* $\mathrm{Hol}(D)^+$*. Then the quotient $\Gamma \backslash D$ can be endowed with a structure of a complex quasi-projective variety. This structure is unique, if Γ is torsion-free.*

Proof. (see [16], 2.1.2. and for the construction of the structure of a complex variety see [5]) □

Next one needs the ring of finite adèles,[8] which is given by

$$\mathbb{A}^f = \mathbb{Q} \otimes_{\mathbb{Z}} \prod_p \mathbb{Z}_p,$$

[8] One reason for the introduction of adèle rings is given by the fact that one wants to have canonical models of Shimura varieties over number fields in number theory. We will not need canonical models of Shimura varieties over number fields. For completeness we write it down.

where p runs over all prime numbers. Hence $\mathbb{A}^f$ is the subring of $\prod \mathbb{Q}_p$ consisting of the (a_p) such that $a_p \in \mathbb{Z}_p$ for almost all a_p. Now let (G, h) be a Shimura datum, which gives the bounded symmetric domain D^+ by a connected component of the conjugacy class D of h, and K be a compact open subgroup of $G(\mathbb{A}^f)$.

Definition 1.5.3. Let G be a $\mathbb{Q}$-algebraic group. A principal congruence subgroup of $G(\mathbb{Q})$ is

$$\Gamma(n) := \{g \in G(\mathbb{Z}) | g \equiv E_g \bmod n\}$$

for some $n \in \mathbb{N}$. A congruence subgroup of $G(\mathbb{Q})$ is a subgroup Γ containing $\Gamma(n)$ such that $[\Gamma : \Gamma(n)] < \infty$ for some $n \in \mathbb{N}$.

Lemma 1.5.4. *Let K be a compact open subgroup of $G(\mathbb{A}^f)$. Then $\Gamma := K \cap G(\mathbb{Q})$ is a congruence subgroup of $G(\mathbb{Q})$.*

Proof. (see [40], Proposition 4.1) □

The Shimura variety $\mathrm{Sh}_K(G, h)$ is given by the double quotient

$$\mathrm{Sh}_K(G, h) := G(\mathbb{Q})\backslash D \times G(\mathbb{A}^f)/K := G(\mathbb{Q})\backslash(D \times (G(\mathbb{A}^f)/K)).$$

Proposition 1.5.5. *Let K be a compact open subgroup of $G(\mathbb{A}^f)$, $C := G(\mathbb{Q})\backslash G(\mathbb{A}^f)/K$, and $\Gamma_{[g]} = gKg^{-1} \cap G(\mathbb{Q})^+$ for some $[g] \in C$. Then one has*

$$\mathrm{Sh}_K(G, h) = \bigsqcup_{[g] \in C} \Gamma_{[g]} \backslash D^+.$$

Proof. (see [40], Lemma 5.13) □

Hence the preceding proposition and the Theorem of Baily and Borel endow $\mathrm{Sh}_K(G, h)$ with the structure of an algebraic variety. By [40], Proposition 3.2, the surjection $G \to G^{\mathrm{ad}}$ maps a congruence subgroup of G onto an arithmetic subgroup of G^{ad}. Now we consider compact open subgroups with the property that the resulting arithmetic subgroups on

$$G^{\mathrm{ad}}(\mathbb{R}) = \mathrm{Hol}(D^+, g)^+ = \mathrm{Hol}(D^+)^+$$

are torsion-free. Recall that the structure of a complex quasi-projective variety on the quotient of a bounded symmetric domain by a torsion-free arithmetic group is unique. If $K' \subset K$, we have a natural morphism

$$\mathrm{Sh}_{K'}(G, h) \to \mathrm{Sh}_K(G, h). \tag{1.4}$$

By the projective limit running over all compact open $K \subset G(\mathbb{A}^f)$ proving a torsion-free arithmetic group on $G^{ad}(\mathbb{R})$, which is given via (1.4), we obtain the Shimura variety[9]

$$\mathrm{Sh}(G,h) = \varprojlim \mathrm{Sh}_K(G,h).$$

1.6 The definition of complex multiplication

One says that an elliptic curve has complex multiplication, if its endomorphism ring is a complex multiplication field (CM field), i.e. a totally imaginary quadratic extension of a totally real number field. For an arbitrary abelian variety we define:

Definition 1.6.1. An abelian variety A is of CM type, if it is isogeneous to a fiberproduct of simple abelian varieties X_i $(i = 1, \ldots, n)$ such that there are fields $K_i \subset \mathrm{End}(X_i) \otimes_{\mathbb{Z}} \mathbb{Q}$, which satisfy

$$[K_i : \mathbb{Q}] \geq 2 \cdot dim(X_i).$$

Proposition 1.6.2. *If the abelian variety A is of CM type, the fields K_i are CM fields and satisfy*

$$[K_i : \mathbb{Q}] = 2 \cdot dim(X_i).$$

Proof. (see [35], Chapter 1, Theorem 3.1 and see [35], Chapter 1, Lemma 3.2.) □

Many authors say that an abelian variety X has complex multiplication, if there exists a skew field F and an embedding $F \hookrightarrow \mathrm{End}_{\mathbb{Q}}(X)$ of $\mathbb{Q}$-algebras (see [7], [34]). This definition can be used in many applications.

However, we will use a much stronger definition of complex multiplication for arbitrary Kähler manifolds, which is motivated by the previous definition of abelian varieties of CM type. We consider complex multiplication as a property, which characterizes the Hodge group. Recall the following facts:

Remark 1.6.3. By a principal polarization on the abelian variety X, we have an isomorphism between X and its dual abelian variety $\hat{X}$ given by

$$\hat{X} = H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}).$$

Thus for each curve C the Hodge structures on $H^1(C, \mathbb{Z})$ and $H^1(\mathrm{Jac}(C), \mathbb{Z})$ are isomorphic. Moreover each polarization yields an isogeny $X \to \hat{X}$ (compare [7], 2.4).

[9] Some authors denote only $\mathrm{Sh}(G,h)$ as Shimura variety.

Proposition 1.6.4. *An abelian variety A is of CM type, if and only if* $\mathrm{Hg}(H^1(A,\mathbb{Q}))$ *is a torus algebraic group.*

Proof. (follows from [47]) □

By Remark 1.6.3 and Proposition 1.6.4, one concludes:

Corollary 1.6.5. *Let C be a curve. Then* $\mathrm{Hg}(H^1(C,\mathbb{Q}),h_C)$ *is a torus, if and only if* $\mathrm{Jac}(C)$ *is of CM type.*

Now let F denote a totally real number field, (V,h) be an F-Hodge structure and

$$\mathrm{End}_F(V,h) := \{M \in \mathrm{Hom}_F(V,V)|gh = hg\}$$

be its endomorphism algebra. Note that an abelian variety X is isogeneous to its dual abelian variety

$$\hat{X} \cong H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z})$$

(see Remark 1.6.3). Thus the endomorphism algebra of X given by $\mathrm{End}(X)\otimes_{\mathbb{Z}} \mathbb{Q}$ can be identified with $\mathrm{End}_{\mathbb{Q}}(H^1(X,\mathbb{Q}),h_X)$. Proposition 1.6.4 tells us that an abelian variety X has a commutative endomorphism algebra of rank equal to $\dim H^1(X,\mathbb{Q})$, if $\mathrm{Hg}(H^1(X,\mathbb{Q}),h_X)$ is a torus. Thus the endomorphism algebra $\mathrm{End}_{\mathbb{Q}}(H^1(X,\mathbb{Q}),h_X)$ of the Hodge structure contains a commutative endomorphism algebra of rank equal to $\dim H^1(X,\mathbb{Q})$, if $\mathrm{Hg}(H^1(X,\mathbb{Q}),h_X)$ is a torus algebraic group. We give a generalization of this version of Proposition 1.6.4, which will motivate our definition of complex multiplication:

Proposition 1.6.6. *Let F denote a totally real number field and (V,h) be an F-Hodge structure. The endomorphism algebra* $\mathrm{End}_F(V,h)$ *contains a commutative subalgebra of dimension $n = \dim V$, if the Mumford-Tate group* $\mathrm{MT}_F(V,h)$ *is a torus.*

Proof. Assume that $\mathrm{MT}_F(V,h)$ is a torus. Thus it is contained in a maximal torus T of $\mathrm{GL}(V)$. Up to conjugation $T_{\mathbb{C}}$ is given by the torus of diagonal matrices. Thus

$$\dim T = \dim V.$$

Now let $W(T)$ denote the subvector space of $\mathrm{Hom}_F(V,V)$, which is generated by the elements of $T(F)$. It is a Zariski closed subset of $\mathrm{Hom}_F(V,V)$ and a commutative subalgebra of $End_F(V,h)$. Moreover it contains the torus T of dimension n. Hence

$$\dim W(T) \geq n.$$

On the other hand, each conjugation by an invertible matrix is an automorphism of the algebra $End_{\mathbb{C}}(V,h)$. Thus each element of $W(T)_{\mathbb{C}}$ is up to conjugation a diagonal matrix, which implies

$$\dim W(T) = n.$$

□

Let us first consider intermediate Jacobians and afterwards discuss definitions of complex multiplication, which use one intermediate Jacobian. These definitions have some interesting applications as we will see. However, we will see that one intermediate Jacobian does not accurately describe the Hodge structure of a Calabi-Yau 3-manifold. Therefore we consider two intermediate Jacobians in the case of a Calabi-Yau 3-manifold:

1.6.7. Let X be a Calabi-Yau 3-manifold. The Hodge structure on $H^3(X,\mathbb{Z})$ is given by the decomposition

$$H^3(X,\mathbb{C}) = H^{3,0}(X) \oplus H^{2,1}(X) \oplus H^{1,2}(X) \oplus H^{0,3}(X).$$

The Calabi-Yau 3-manifold X has the following intermediate Jacobians:

- The Griffiths intermediate Jacobian $\mathrm{J}_G(X)$ is the complex torus corresponding to the Hodge structure of type $(1,0),(0,1)$ on $H^3(X,\mathbb{Z})$, which is given by the direct sum decomposition

$$H^{1,0} := H^{3.0}(X) \oplus H^{2,1}(X), \quad H^{0,1} := H^{1,2}(X) \oplus H^{0,3}(X).$$

- The Weil intermediate Jacobian $\mathrm{J}_W(X)$ is the abelian variety corresponding to the Hodge structure of type $(1,0),(0,1)$ on $H^3(X,\mathbb{Z})$, which is given by the direct sum decomposition

$$H^{1,0} := H^{2,1}(X) \oplus H^{0,3}(X), \quad H^{0,1} := H^{3,0}(X) \oplus H^{1,2}(X).$$

The Weil intermediate Jacobian is $\mathrm{J}_W(X)$ is a principally polarized abelian variety. But it does not vary holomorphically in general.

The Griffiths intermediate Jacobian $\mathrm{J}_G(X)$ varies holomorphically. But it is not algebraic in general. (see [8])

Remark 1.6.8. For each Kähler manifold X of dimension $2n-1$ we can define the intermediate Jacobian

$$\mathrm{J}(X) := H^{2n-1}(X,\mathbb{C})/(F^n(H^{2n-1}(X,\mathbb{C})) \oplus H^{2n-1}(X,\mathbb{Z})),$$

which coincides with the Griffiths intermediate Jacobian J_G in the case of a Calabi-Yau 3-manifold.

Some possible definitions of complex multiplication for a Kähler manifold X of dimension $2n-1$ use the intermediate Jacobian $\mathrm{J}(X)$. Many authors say that X has complex multiplication, if $\mathrm{J}(X)$ is of CM type or $\mathrm{End}_{\mathbb{Q}}\mathrm{J}(X)$ contains a skew field F. This leads to definitions, which are often used in many applications in mathematics and theoretical physics.

For example such a definition is used by S. Gukov and C. Vafa [23]. Mirror pairs of Calabi-Yau 3-manifolds with Griffiths intermediate Jacobians, which are respectively of CM-type over a number field F with $[F:\mathbb{Q}] = 2(h^{2,1}+1)$, correspond to *rational conformal field theories.*

However for an accurate description of the VHS the Griffiths intermediate Jacobian does not give enough information in general. Let us consider the following example. It uses methods, which will be explained later. Thus the reader is suggested to return to this example after he has read the rest of this book, if he does not understand it now.

Example 1.6.9. There exists a $K3$ surface S with an involution ι_S such that ι_S acts on $H^{1,1}(S)$ by the character 1 and on $H^{2,0}(S) \oplus H^{0,2}(S)$ by the character -1, which yields an eigenspace decomposition over $\mathbb{Q}$. This is the last example in the table of 11.3.11. Note that t is the rank of the sublattice $\mathrm{Pic}(S)_0$ of the Picard lattice, which is fixed by ι_S. In this case we have $t = 20$. This implies that

$$\mathrm{Pic}(S)_0 \otimes_{\mathbb{Z}} \mathbb{C} = H^{1,1}(S)$$

such that ι_S acts on $H^{1,1}(S)$ by the character 1 as we have claimed.

The restricted Hodge structure (V_-, h_-) of the eigenspace with eigenvalue -1 satisfies

$$\mathrm{Hg}(V_-, h_-)_{\mathbb{R}} \subseteq \mathrm{SO}(2)$$

(see Section 11.2). Since $\mathrm{SO}(2)$ is commutative, $\mathrm{Hg}(V_-, h_-)$ is commutative, too.

Each elliptic curve E has an involution ι_E such that $E/\langle \iota_E \rangle \cong \mathbb{P}^1$. By the Borcea-Voisin construction, which we explain in Section 7.2, we obtain a Calabi-Yau 3-manifold X by blowing up the singularities of

$$S \times E/\langle (\iota_S, \iota_E) \rangle.$$

The integral Hodge structure on the third cohomology of $S \times E$ is up to torsion given by

$$(H^3(S \times E, \mathbb{Z}), h) = (H^2(S, \mathbb{Z}), h_S) \otimes (H^1(E, \mathbb{Z}), h_E)$$

(follows from [61], Théorème 11.38). Since the points fixed by ι_S are given by rational curves, one concludes with respect to the blowing up of these rational curves

$$H^3(\widetilde{S \times E}, \mathbb{Z}) = H^3(S \times E, \mathbb{Z}).$$

Due to [61], 7.3.2, one concludes that $(H^3(X, \mathbb{Z}), h_X)$ is the sub-hodge structure of $(H^3(S \times E, \mathbb{Z}), h)$ given by

$$(H^3(X, \mathbb{Z}), h_X) = (V_- \cap H^2(S, \mathbb{Z}), h_-) \otimes (H^1(E, \mathbb{Z}), h_E).^{10}$$

[10] In this situation one may ask for torsion. Since the kernel of the natural homomorphism

$$H^3(X, \mathbb{Z}) \to H^3(X, \mathbb{Z}) \otimes \mathbb{Q} = H^3(X, \mathbb{Q})$$

is given by the torsion elements, the weight one Hodge structure corresponding to the Jacobian can be defined over the torsion-free lattice $H^3(X, \mathbb{Z})/torsion$. Thus we can disregard the torsion.

Due to the fact that

$$(V_-)_{\mathbb{C}} = H^{2,0}(S) \oplus H^{0,2}(S),$$

the Griffiths intermediate Jacobian $\mathrm{J}_G(X)$ of our Calabi-Yau manifold X has a corresponding integral Hodge structure given by

$$H^{1,0} = H^{2,0}(S) \otimes H^1(E,\mathbb{C}), \quad H^{0,1} = H^{0,2}(S) \otimes H^1(E,\mathbb{C}). \tag{1.5}$$

Note that for all elliptic curves the vector space $H^1(E,\mathbb{C})$ is given by

$$H^1(E,\mathbb{C}) = \Lambda \otimes_{\mathbb{Z}} \mathbb{C},$$

where $\Lambda \cong \mathbb{Z}^2$ does not depend on the respective elliptic curve. Thus the Griffiths intermediate Jacobian $\mathrm{J}_G(X)$ and its corresponding integral Hodge structure do not depend on the chosen elliptic curve. Therefore the different Calabi-Yau 3-manifolds obtained from different elliptic curves have the same Griffiths intermediate Jacobian. By the description of the corresponding weight one Hodge structure (1.5), the Hodge group of the intermediate Jacobian is isomorphic to $\mathrm{Hg}(V_-, h_-)$. Therefore one concludes that the Hodge group of the intermediate Jacobian is a torus. Thus $\mathrm{J}_{\mathrm{G}}(X)$ is of CM type.

Thus we use a stronger term of complex multiplication:

Definition 1.6.10. Let F be a totally real number field. A compact Kähler manifold X of dimension n has complex multiplication (CM) over F, if the Hodge group of the F Hodge structure on $H^n(X,F)$ is a torus. We say that X has complex multiplication, if it has complex multiplication over $\mathbb{Q}$.

Proposition 1.6.11. *A Calabi-Yau 3-manifold X has CM, if and only if its Griffiths intermediate Jacobian $\mathrm{J}_G(X)$ is of CM type, its Weil intermediate Jacobian $\mathrm{J}_W(X)$ is of CM type and the Hodge groups of the corresponding weight one Hodge structures commute.*

Proof. ([8], Theorem 2.3) □

Remark 1.6.12. Let X be the Calabi-Yau 3-manifold X of Example 1.6.9. The Hodge structure on $H^3(X,\mathbb{Q})$ is given by the tensor product of the Hodge structures (V_-, h_-) and $(H^1(E,\mathbb{Q}), h_E)$. By Proposition 7.1.4, the Hodge structure $(H^3(X,\mathbb{Q}), h_X)$ has a commutative Hodge group resp., X has CM, if and only if (V_-, h_-) and $(H^1(E,\mathbb{Q}), h_E)$ have CM. Recall that $\mathrm{J}_G(X)$ is of CM type for all elliptic curves E. It follows that if E does not have complex multiplication, $\mathrm{J}_G(X)$ is of CM type and X does not have CM. Hence the fact that the intermediate Jacobian $\mathrm{J}_G(X)$ is of CM type does not imply that X has CM.

Now we note that a corresponding implication holds true in the case of every odd dimensional Kähler manifold:

Proposition 1.6.13. *Let X be a Kähler manifold of dimension $2n-1$. The intermediate Jacobian*

$$\mathrm{J}(X) := H^{2n-1}(X,\mathbb{C})/(F^n(H^{2n-1}(X,\mathbb{C})) \oplus H^{2n-1}(X,\mathbb{Z}))$$

is of CM type, if X has CM.

Proof. Let

$$h_{\mathrm{J}(X)} : S^1 \to \mathrm{GL}(H^{2n-1}(X,\mathbb{R}))$$

denote the representation, which yields the weight one Hodge structure corresponding to $\mathrm{J}(X)$. Assume that X has CM. Thus $\mathrm{Hg}(H^{2n-1}(X,\mathbb{Q}), h_X)$ is a torus. It is contained in a maximal torus T of $\mathrm{GL}(H^{2n-1}(X,\mathbb{Q}))$. The fact that $h_X(\mathbb{S})(\mathbb{R})$ commutes with $T(\mathbb{R})$ is equivalent to the fact that for each $g \in T(\mathbb{R})$ one has

$$g(H^{k,2n-1-k}(X)) = H^{k,2n-1-k}(X) \quad \text{with} \quad k = 0, 1, \ldots 2n-1.$$

From this fact one concludes that $h_{\mathrm{J}(X)}(S^1)$ is contained in the centralizer of T. Since $\mathrm{GL}(H^{2n-1}(X,\mathbb{Q}))$ is reductive, the maximal torus T is its own centralizer. This follows from the fact that the centralizers of the maximal tori (i. e. the Cartan subgroups) of a reductive group are the maximal tori (see [10], **IV**. 13.17). Hence one concludes that $h_{\mathrm{J}(X)}(S^1) \subset T$, which implies that $\mathrm{J}(X)$ is of CM type. □

1.7 Criteria and conjectures for complex multiplication

We have introduced the theory of Shimura varieties, which we will use for the construction of families with a dense set of CM points defined below:

Definition 1.7.1. Let D be a complex manifold and $\mathcal{V}$ be a holomorphic variation of rational Hodge structures on D. A point $p \in D$ is a CM point with respect to $\mathcal{V}$, if $\mathcal{V}_p$ has CM.

Let $\mathcal{X} \to D$ be a holomorphic family of complex manifolds. A point $p \in D$ is a CM point with respect to $\mathcal{X}$, if $\mathcal{X}_p$ is a CM fiber resp., $\mathcal{X}_p$ has a complex multiplication.

By the next theorem, we give a criterion for dense sets of CM points, which implies that the family of abelian varieties over $\mathrm{Hg}(V,h)(\mathbb{R})/K$ of Construction 1.4.14 has a dense set of CM fibers. We only need to understand the definition of Shimura data and Hermitian symmetric domains. The construction of Shimura varieties by bounded symmetric domains has been written down for completeness.

Recall that a Shimura datum (G, h) gives a Hermitian symmetric domain D and a representation of G gives a variation of Hodge structures over D. Now consider the following theorem:

Theorem 1.7.2. *Let (G, h) denote a Shimura datum. The set of CM points with respect to the VHS induced by some closed embedding $G \to \mathrm{GL}(W)$ for some $\mathbb{Q}$-vector space W is dense in $G(\mathbb{R})/K(\mathbb{R})$.*

Proof. By the following lemma, we have only to show that there exists one CM point on $G(\mathbb{R})/K$. Since we have the closed embedding $G \to \mathrm{GL}(W)$, each $\mathbb{Q}$-algebraic torus of G can be identified with a $\mathbb{Q}$-algebraic torus of $\mathrm{GL}(W)$. Thus the existence of a CM point is equivalent to the statement that there is a

$$h : \mathbb{S} \to G_{\mathbb{R}} \to \mathrm{GL}(W)$$

in this VHS, which factors through a $\mathbb{Q}$-algebraic torus of G.

Now let T be a maximal ($\mathbb{Q}$-algebraic) torus of G. The centralizers of the maximal tori (i. e. the Cartan subgroups) of a reductive group are the maximal tori (see [10], IV. 13.17.). The torus $T_{\mathbb{R}}$ is contained in a maximal torus T_M of $G_{\mathbb{R}}$, which has the property that each point of T_M is contained in the centralizer of $T_{\mathbb{R}}$ resp., in the centralizer of T. Thus the torus $T_{\mathbb{R}}$ is in fact maximal in $G_{\mathbb{R}}$.

Recall that $K^0 \subset G_{\mathbb{R}}$ denotes the Zariski connected component of the centralizer of $h(\mathbb{S})$. It yields the compact Lie group $K^0(\mathbb{R})$. Hence Example 1.3.11 tells us that K^0 is reductive. Moreover $h(\mathbb{S})$ is contained in the center of K^0 and the center of K^0 is a torus (see Proposition 1.3.4). Thus there exists a maximal torus T_0 of $G_{\mathbb{R}}$ which contains $h(\mathbb{S})$. Recall that an element of G is regular, if its centralizer is a Cartan subgroup and that the regular elements in $G_{\mathbb{C}}$ resp., $(T_0)_{\mathbb{C}}$ contain a Zariski open dense subset of $G_{\mathbb{C}}$ resp. $(T_0)_{\mathbb{C}}$ (see [10], **IV**. 12.2 and [10], **IV**. Theorem 12.3). Let $t \in T_0(\mathbb{R})$ be regular. The centralizer of t is the maximal torus T_0, since the Cartan subgroups coincide with the maximal tori in the case of a reductive group. The proof of the fact that the regular elements of G contain a Zariski open dense subset uses the fact that the morphism

$$G_{\mathbb{C}} \times T_{\mathbb{C}} \to G_{\mathbb{C}} \ \text{ via } \ (g, x) \to gxg^{-1}$$

is dominant (see the proof of [10], **IV**. Theorem 12.3). Since this morphism is defined over $\mathbb{R}$, the differential over $\mathbb{R}$ is also surjective at any point. By the Real Approximation Theorem, $G(\mathbb{Q})$ lies dense in the manifold $G(\mathbb{R})$. Hence there exists a regular $\mathbb{Q}$-rational element of G near to $t \in T_0(\mathbb{R})$, which is conjugate to t and whose centralizer is a maximal torus. This torus is defined over $\mathbb{Q}$ and contains an element of the conjugacy class of $h(\mathbb{S})$. □

Lemma 1.7.3. *Let (G, h) denote a Shimura datum. Assume that $G(\mathbb{R})/K$ contains a CM point with respect to a VHS induced by some closed* *embedding* $G \to \mathrm{GL}(W)$ *for some* $\mathbb{Q}$*-vector space* W*. Then the set of* CM *points of the same type with respect to the same* VHS *is dense in* $G(\mathbb{R})/K$*.*

Proof. We have two cases. Assume that G is a $\mathbb{Q}$-algebraic torus. In this case $G(\mathbb{R})/K$ consists of one point. The fact that we have a closed embedding $G \hookrightarrow \mathrm{GL}(W)$ implies that the Hodge group of the Hodge structure over this point is a subtorus of the torus G.

In the other case G is not a $\mathbb{Q}$-algebraic torus. By the assumptions, we have a CM point in $G(\mathbb{R})/K$ with respect to the VHS, which is induced by some closed embedding $G \to \mathrm{GL}(W)$. This implies that G contains a $\mathbb{Q}$-algebraic torus T such that the conjugacy class of $h : \mathbb{S} \to G_{\mathbb{R}}$ contains an element, which factors through $T_{\mathbb{R}}$. By our preceding construction, the stabilizer of the CM point $[s_0]_K \in G(\mathbb{R})/K$ is given by $s_0 K s_0^{-1}$. Thus one can replace K by $s_0 K s_0^{-1}$. In this case the fact that the VHS is induced by an embedding $G \hookrightarrow \mathrm{GL}(W)$ implies that the Hodge group of the Hodge structure over $[e]$ is a subtorus of T. Hence $[e]$ is a CM point with respect to this VHS, and any $s \in G(\mathbb{Q}) \subset G(\mathbb{R})$ has the property that it is mapped to a CM point, too. By the Real Approximation Theorem, $G(\mathbb{Q})$ lies dense in the manifold $G(\mathbb{R})$ for all connected affine $\mathbb{Q}$-algebraic groups G. Since the quotient map is continuous, the set of CM points in $G(\mathbb{R})/K$ is dense. $\square$

Remark 1.7.4. Let F denote a totally real number field and (V, h, Q) be a pure polarized F-Hodge structure of weight k. Moreover let K denote the centralizer of h in $\mathrm{Hg}_F(V,h)(\mathbb{R})$. We can relax the assumptions of Theorem 1.7.2 and show that the conjugacy class of h in $\mathrm{Hg}_F(V,h)$ given by the homogeneous space $\mathrm{Hg}_F(V,h)(\mathbb{R})/K$ contains a dense set of F-Hodge structures with CM over F. The arguments are very similar:

A maximal torus of $\mathrm{Hg}_F(V,h)$ yields also a maximal torus of $\mathrm{Hg}_F(V,h)_{\mathbb{R}}$. Note that $\mathrm{Hg}_F(V,h)(F)$ is dense in $\mathrm{Hg}_F(V,h)(\mathbb{R})$ (see [53], Theorem 7.7). Since $\mathrm{Hg}_F(V,h)$ is reductive, the same methods as above yield an F-rational maximal torus, which contains $h(\mathbb{S})$ up to conjugation. Due to the fact that $\mathrm{Hg}_F(V,h)(F)$ is dense in $\mathrm{Hg}_F(V,h)(\mathbb{R})$, one concludes that the set of points, which represent Hodge structures with CM over F, is dense in $\mathrm{Hg}_F(V,h)(\mathbb{R})/K$.

Now we apply Theorem 1.7.2 to the following example.

Example 1.7.5. Let X be a curve. We have the rational Hodge structure $(H^1(X,\mathbb{Q}), h_X)$ of weight 1. The Shimura datum $(\mathrm{MT}(H^1(X,\mathbb{Q}), h_X), h_X)$ and the representation

$$\mathrm{id} : \mathrm{MT}(H^1(X,\mathbb{Q}), h_X) \hookrightarrow \mathrm{GL}(H^1(X,\mathbb{Q}))$$

give a variation of Hodge structures. This variation of Hodge structures contains exactly all Hodge structures $(H^1(X,\mathbb{Q}), h)$, which are conjugated to $(H^1(X,\mathbb{Q}), h_X)$ and have a Mumford-Tate group satisfying

$$\mathrm{MT}(H^1(X,\mathbb{Q}),h) \subseteq \mathrm{MT}(H^1(X,\mathbb{Q}),h_X).$$

Especially $(H^1(X,\mathbb{Q}),h_X)$ occurs in this variation of Hodge structures. By Theorem 1.7.2, over a dense set of points the occurring Hodge structures have CM.

1.7.6. Recall that we want to find infinitely many fibers in a family $f : \mathcal{X} \to Y$ of curves or Calabi-Yau manifolds, which have CM. Assume that f is a family of curves. Moreover recall that $p : Y \to \mathfrak{h}_g$ denotes the period map. In the case of curves we have a Torelli theorem, which implies that the isomorphism class of the curve X of genus g is determined by a point of $\mathfrak{h}_g$.

Let D denote the subdomain of the upper half plane $\mathfrak{h}_g$ given by the Shimura datum of the preceding example. Assume that X does not have CM. Otherwise D would consist of only one point. If $(H^1(X,\mathbb{Q}),h_X) \in D$ has a neighborhood U in D such that $U \subset p(Y)$, the restricted holomorphic family $\mathcal{X}_{p^{-1}(U)} \to p^{-1}(U)$ contains infinitely many CM fibers. This follows from Theorem 1.7.2.

In Section 3.1 we will see that the family $f : \mathcal{X} \to Y$ has a generic Mumford-Tate group MT such that the Mumford-Tate groups $\mathrm{MT}(H^n(X,\mathbb{Q}), h_X)$ are contained in the generic Mumford-Tate group MT for all fibers X. Moreover assume that the period map is generically finite. Let D be bounded symmetric domain obtained from the Shimura datum (MT, h_X), where X is some fiber of f. If one can prove that the Hermitian symmetric domain D satisfies

$$\dim D \leq \dim Y,$$

one concludes from the generic finiteness of the period map that

$$\dim D = \dim Y$$

and the family has a dense set of CM fibers.

Next one can ask for a necessary condition for the existence of CM fibers of a family. The André-Oort conjecture (compare [3], [52]) concerns this question for a necessary condition.

Conjecture 1.7.7. *Assume that S is a Shimura variety and $Z \subset S$ is an irreducible algebraic subvariety. Then Z contains a dense set of CM points, only if it is a Shimura subvariety of S.*

Remark 1.7.8. Let (G,h) denote a Shimura datum obtained from the generic Mumford-Tate group of a family of curves and the Hodge structure of one fiber given by h, which satisfies the conditions of 1.7.6. In this case Theorem 1.7.2 yields only a discrete set of CM points. Due to the André-Oort conjecture, one can conject that any nonconstant family has at most a discrete set of CM fibers.

1.7.9. By Proposition 1.4.16, one concludes that

$$\dim \mathfrak{h}_g = \frac{g(g+1)}{2}.$$

Moreover by [14], the moduli space of curves of genus $g \geq 2$ is a quasi-projective variety of the dimension $3g-3$. For an introduction to moduli of curves we refer to [25]. Thus for $g \leq 3$ the dimension of the moduli space of curves of genus g and $\dim \mathfrak{h}_g$ coincide. Hence the moduli space of curves of genus $g \leq 3$ contains a dense set of points representing curves with CM.

By the same arguments as in 1.7.9, one can see that $\dim \mathfrak{h}_g$ is larger than the dimension of the moduli space of curves of dimension g for $g > 3$. In this case the subspace of $\mathfrak{h}_g$, whose points represent the Jacobians of the curves of genus g has a smaller dimension than $\mathfrak{h}_g$. Hence the existence of non-trivial families with dense sets of CM fibers and the André-Oort conjecture imply that there are subsets of this locus, which have the structure of a Hermitian symmetric domain. R. Coleman [11] thought that each of these domains would at most consist of one point. He formulated the following conjecture:

Conjecture 1.7.10. *Fix an integer $g \geq 4$. Then there are only finitely many complex algebraic curves C of genus g such that* $\mathrm{Jac}(C)$ *is of CM type.*

In [29] J. de Jong and R. Noot gave counterexamples to the previous conjecture for $g = 4$ and $g = 6$. In [58] E. Viehweg and K. Zuo gave an additional counterexample for $g = 6$. In Chapter 6 we will give counterexamples for $g = 5$ and $g = 7$, which occur in the lists of Section 6.3. All counterexamples are given by families of curves, which are parametrized over a Shimura variety, which is given by a ball quotient. Note that the complex n-ball $\mathbb{B}_n$ is a bounded symmetric domain. A ball quotient is a quotient of $\mathbb{B}_n$ by an arithmetic subgroup of the identity component of the group of holomorphic isometries of $\mathbb{B}_n$ for some n. Let us consider the complex ball $\mathbb{B}_n$ in detail:

1.7.11. The complex n-ball $\mathbb{B}_n$ is the domain contained in $\mathbb{P}^n$ given by the points

$$p = (p_0 : p_1 : \ldots : p_n),$$

which satisfy

$$|\frac{p_1}{p_0}|^2 + \ldots + |\frac{p_1}{p_0}|^2 < 1.$$

This is equivalent to the condition

$$0 < |p_0|^2 - |p_1|^2 - \ldots - |p_n|^2.$$

As one can easily see the Lie group $\mathrm{PU}(1,n)(\mathbb{R})$ acts on $\mathbb{B}_n$ and the stabilizer of the point

$$p = (1 : 0 : \ldots : 0)$$

is the subgroup $\mathrm{P}(\mathrm{U}(1) \times \mathrm{U}(n))(\mathbb{R})$. The group $\mathrm{PU}(1, n)$ is not a $\mathbb{C}$-algebraic group, since complex conjugation is not $\mathbb{C}$-linear. Since one can consider $\mathbb{C}^{n+1}$ as real vector space and the complex conjugation is $\mathbb{R}$-linear, $\mathrm{U}(1, n)$ and $\mathrm{PU}(1, n)$ are $\mathbb{R}$-algebraic. Due to the remarks below [27], **X**. Table **V**, the homogeneous space $\mathrm{PU}(1, n)(\mathbb{R})/\mathrm{P}(\mathrm{U}(1) \times \mathrm{U}(n))(\mathbb{R})$ is a Hermitian symmetric domain. By [31], Volume **II**. Example 10.7, one has

$$\mathbb{B}_n \cong \mathrm{PU}(1, n)(\mathbb{R})/\mathrm{P}(\mathrm{U}(1) \times \mathrm{U}(n))(\mathbb{R}).$$

Due to its counterexamples, the Coleman conjecture has to be reformulated in the following way:

Conjecture 1.7.12. *There exists an integer $g' > 7$ such that for all fixed $g \geq g'$ there are only finitely many complex algebraic curves C with CM type of genus g.*

The Coleman conjecture motivates similar conjectures for manifolds of other kinds. For example consider the weight one Hodge structures of the Weil intermediate Jacobian $\mathrm{J}_W(X)$ of a Calabi Yau 3-manifold X with polarization Q. This intermediate Jacobian can be given by a point of the upper half plane

$$\mathfrak{h}_{1+h^{2,1}} \cong \mathrm{Sp}(H^3(X, \mathbb{R}), Q)/\mathrm{U}(1 + h^{2,1}).$$

By 1.7.9, one has that

$$\dim \mathfrak{h}_{1+h^{2,1}} = \frac{(h^{2,1} + 1)(h^{2,1} + 2)}{2}.$$

On the other hand the universal deformation of a Calabi-Yau 3-manifold is a family over a basis of dimension $h^{2,1}$ (see [61] 10.3.2). Hence one can conject that for almost all fixed $h^{1,1}$ and $h^{2,1}$ there are only finitely many Calabi-Yau 3-manifolds with CM, which have the Hodge numbers $h^{1,1}$ and $h^{2,1}$. This conjecture has been formulated by S. Gukov and C. Vafa [23].

Here we give some examples of families of Calabi-Yau 3-manifolds with dense sets of CM fibers. Thus for some fixed $h^{1,1}$ and $h^{2,1}$ there are infinitely many Calabi-Yau 3-manifolds with CM, which have the Hodge numbers $h^{1,1}$ and $h^{2,1}$. There are known examples of families of Calabi-Yau 3-manifolds, which contain a dense set of CM fibers, too:

Example 1.7.13. By C. Borcea [8], two examples of families with complex multiplication fibers have been constructed. The first example uses the family $\mathcal{E}$ of elliptic curves given by

$$\mathbb{P}^2 \supset V(y^2 x_0 + x_1(x_1 - x_0)(x_1 - \lambda x_0)) \to \lambda \in \mathbb{A}^1 \setminus \{0, 1\}.$$

By $y \to -y$, one has a global involution ι on $\mathcal{E}$. Now let $\mathcal{E}_i$ with involution ι_i be a copy of $\mathcal{E}$ for $i = 1, 2, 3$. We obtain the family

$$\mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3 / \langle (\iota_1, \iota_2), (\iota_2, \iota_3) \rangle \to (\mathbb{A}^1 \setminus \{0, 1\})^3.$$

By blowing up the singular sections, we obtain a family of Calabi-Yau 3-manifolds with a dense set of complex multiplication fibers.

The other example of C. Borcea uses the family $\mathcal{C}$ of degree 2 covers of $\mathbb{P}^2$ ramified over six lines in general position. By the Galois group action, one has an involution ι_C on $\mathcal{C}$. By blowing up the intersection loci of these lines, one obtains the family $\tilde{\mathcal{C}}$ of $K3$ surfaces. The involution ι_C acts on $\tilde{\mathcal{C}}$, too. By blowing up the singular locus of $\tilde{\mathcal{C}} \times \mathcal{E} / \langle (\iota_C, \iota_E) \rangle$, we obtain a family of Calabi-Yau 3-manifolds with a dense set of complex multiplication fibers.

Later E. Viehweg and K. Zuo [58] have constructed a deformation of the Fermat quintic in $\mathbb{P}^4$, which is a well-studied Calabi-Yau 3-manifold with complex multiplication:

Example 1.7.14. We will later see that the VHS of the family $\mathcal{F}_1$ given by

$$\mathbb{P}^2 \supset V(y_1^5 + x_1(x_1 - x_0)(x_1 - \alpha x_0)(x_1 - \beta x_0)x_0) \to (\alpha, \beta) \in \mathcal{M}_2$$

allows to consider its basis as ball quotient. Thus this family has a dense set of CM fibers. Since each of these covers given by the fibers of the family can be embedded into $\mathbb{P}^2$, the fibers of $\mathcal{F}_1$ are the branch loci of the fibers of a family $\mathcal{F}_2$ of cyclic covers of $\mathbb{P}^2$ of degree 5. Moreover the fibers of $\mathcal{F}_2$, which can be embedded into $\mathbb{P}^3$, are the branch loci of the fibers of a family $\mathcal{F}_3$ of cyclic covers of $\mathbb{P}^3$, which can be embedded into $\mathbb{P}^4$. The family $\mathcal{F}_3$ is given by

$$\mathbb{P}^4 \supset V(y_3^5 + y_2^5 + y_1^5 + x_1(x_1 - x_0)(x_1 - \alpha x_0)(x_1 - \beta x_0)x_0) \to (\alpha, \beta) \in \mathcal{M}_2.$$

By the adjunction formula, the fibers of $\mathcal{F}_3$ are Calabi-Yau 3-manifolds.

Let $q \in \mathcal{M}_2$. The fiber $(\mathcal{F}_3)_q$ has CM, if $(\mathcal{F}_2)_q$ has CM and $(\mathcal{F}_2)_q$ has CM, if $(\mathcal{F}_1)_q$ has CM. Because of this argument, the family $\mathcal{F}_3$ has a dense set of CM fibers, which lie over the same points as the CM fibers of the family of curves we have started with.

We will use, combine and modify the methods of the previous two examples in order to obtain new examples. It is our main topic to explain these methods.

An other method to obtain Calabi-Yau manifolds with complex multiplication was suggested by Y. Zhang. Due to the André-Oort conjecture, he conjects that the Basis of a family of Calabi-Yau 3-manifolds with a dense set of CM fibers has the structure of a Shimura (sub)variety (see [64], page 20).

A Shimura subvariety of a Shimura variety can be obtained from an embedding of Shimura data $(G_1, h_1) \to (G_2, h_2)$. An embedding of Shimura data is given by a closed embedding $j : G_1 \to G_2$ of $\mathbb{Q}$-algebraic groups such that the conjugacy class of h_2 coincides with the conjugacy class of

$$\mathbb{S} \xrightarrow{h_1} (G_1)_{\mathbb{R}} \xrightarrow{j_{\mathbb{R}}} (G_2)_{\mathbb{R}}.$$

Thus (G_1, h_1) yields a Hermitian symmetric subdomain D_1 of the Hermitian symmetric domain D_2 obtained by (G_2, h_2). Note that not all Shimura subvarieties of $\mathrm{Sh}(G_2, h_2)$ are of that type (see [43], Remark 2.6). A complex submanifold M of D_2 is totally geodesic, if for all $p \in M$ each geodesic in D_2, which is tangent to M at p is contained in M. Thus one says that an irreducible subvariety V of the Shimura variety $\mathrm{Sh}(G_2, h_2)$ is totally geodesic, if it is obtained from a totally geodesic submanifold M of D_2.

In [43] B. Moonen has proved the following Theorem:

Theorem 1.7.15. *An irreducible subvariety V of a Shimura variety $\mathrm{Sh}(G, h)$ is a Shimura subvariety, if and only if it contains a CM point and it is totally geodesic.*

Chapter 2
Cyclic covers of the projective line

Recall that we will study variations of Hodge structures of families of cyclic coverings of the projective line. Moreover some families of such covers are suitable for the construction of families of Calabi-Yau manifolds with dense sets of complex multiplication fibers. In order to understand variations of Hodge structures of such families of cyclic coverings we need to understand the Hodge structure of a cyclic covering $C \to \mathbb{P}^1$.

A cyclic cover $\pi : C \to \mathbb{P}^1$ is given by

$$y^m = (x - a_1)^{d_1} \cdot \ldots \cdot (x - a_n)^{d_n}, \tag{2.1}$$

where each d_k is an integer satisfying $1 \leq d_k \leq m - 1$. The numbers d_k are not uniquely determined by the isomorphism class of a cover. However, these numbers determine the isomorphism class of a cover and we will use them for the computation of the variation of Hodge structures in the following chapters.

In Section 2.1 we give a general description of cyclic covers of $\mathbb{P}^1$ and explain which tuples $(d_1, \ldots, d_n)$ yield equivalent covers. We will see that the Galois group action of the cyclic covering yields an eigenspace decomposition of $\pi_*(\mathbb{C})$ over the complement of the branch points. In Section 2.2 we use the branch indices d_k for the description of the monodromy representations of these eigenspaces. We have also an eigenspace decomposition of $H^1(C, \mathbb{C})$ by the Galois group action, which can also be described by using the branch indices d_k, as we will do in Section 2.3. In the next chapter this eigenspace decomposition will be extended to an eigenspace decomposition of the VHS of our families of cyclic coverings of $\mathbb{P}^1$. In Section 2.4 we cover certain curves C given by (2.1) by a Fermat curve, which implies that each of these certain curves C has CM.

J.C. Rohde, *Cyclic Coverings, Calabi-Yau Manifolds and Complex Multiplication*, Lecture Notes in Mathematics 1975, DOI: 10.1007/978-3-642-00639-5_3,

2.1 Description of a cyclic cover of the projective line

Let us first repeat some known facts about Galois covers of $\mathbb{P}^1$.

Definition 2.1.1. Let T_1, T_2, and S be topological spaces resp., complex manifolds resp., algebraic varieties. The coverings $f_1 : T_1 \to S$ and $f_2 : T_2 \to S$, which are morphisms in the respective category, are called equivalent, if there is an isomorphism $g : T_1 \to T_2$ in the respective category such that $f_1 = f_2 \circ g$.

Proposition 2.1.2. *Let G be a finite group, and $S := \{a_1, \ldots, a_n\} \subset \mathbb{A}^1 \subset \mathbb{P}^1$. There is a correspondence between the following objects:*

1. *The isomorphism classes of Galois extensions of $\mathbb{C}(\mathbb{P}^1) = \mathbb{C}(x)$ with Galois group G and branch points contained in S.*
2. *The equivalence classes of (non-ramified) Galois coverings $f : R \to \mathbb{P}^1 \setminus S$ of topological spaces with deck transformation group isomorphic to G.*
3. *The normal subgroups in the fundamental group $\pi_1(\mathbb{P}^1 \setminus S)$ with quotient isomorphic to G.*

Proof. (see [62], Theorem 5.14) □

Remark 2.1.3. We will need to understand the correspondence of the preceding Proposition. The correspondence between (1) and (2) is given by the facts that a Galois covering $f : R \to \mathbb{P}^1 \setminus S$ (of topological spaces) yields a covering $f : \bar{R} \to \mathbb{P}^1$ of compact Riemann surfaces, and any morphism of compact Riemann surfaces corresponds to an embedding of their function fields.

The correspondence between (2) and (3) is given by the path lifting properties of coverings of Hausdorff spaces. Take $b \in R$. Let $p = f(b)$, and $\gamma \in \pi_1(\mathbb{P}^1 \setminus S, p)$, and $f^*(\gamma(0)) = b$. Then $f^*(\gamma(1)) = g \cdot b$ for some $g \in G \cong \mathrm{Deck}(R/(\mathbb{P}^1 \setminus P))$. This induces a homomorphism $\Phi_b : \pi_1(\mathbb{P}^1 \setminus S, p) \to G$ and a kernel of this homomorphism, which is a normal subgroup G.

Remark 2.1.4. Let $f : R \to \mathbb{P}^1$ be a Galois covering with branch points $a_1, \ldots, a_n$. One can choose $\gamma_1, \ldots, \gamma_n \in \pi_1(\mathbb{P}^1 \setminus P)$ such that each γ_k is given by a loop running counterclockwise "around" exactly one a_k. Hence one has that

$$\gamma_n = \gamma_1^{-1} \ldots \gamma_{n-1}^{-1}$$

and we conclude that

$$\Phi_b(\gamma_n) = \Phi_b(\gamma_1)^{-1} \ldots \Phi_b(\gamma_{n-1})^{-1}.$$

From now on we consider only irreducible cyclic covers of $\mathbb{P}^1$. An irreducible cyclic cover can be given by a prime ideal

$$(y^m - (x - a_1)^{d_1} \cdot \ldots \cdot (x - a_n)^{d_n}) \subset \mathbb{C}[x, y].$$

First this ideal defines only an affine curve in $\mathbb{A}^2$, which has singularities, if there are some $d_k > 1$. But there exists a unique smooth projective curve C birationally equivalent to this affine curve. By the natural projection onto the x-axis, one obtains a cyclic cover of the smooth curve C onto $\mathbb{P}^1$.

Remark 2.1.5. Let us consider the cover given by

$$y^m = (x - a_1)^{d_1} \cdot \ldots \cdot (x - a_n)^{d_n},$$

and fix a $k_0 \in \{1, \ldots n\}$. By an automorphism of $\mathbb{P}^1$, one can put a_{k_0} onto 0. Let $\mu_{k_0} = \frac{d_{k_0}}{m} \in \mathbb{Q}$, and D a small disc centered in 0, which does not contain any other a_k with $k \neq k_0$. Take any point $p \in \partial D$ and remove the segment $[0, p]$. The topological space $D \setminus [0, p]$ is simply connected. Hence one can define root functions $z \to z^{\mu_{k_0}}$ on this space, which are given by:

$$z^{\mu_{k_0}} = |z|^{\mu_{k_0}} \exp(\frac{2\pi i t d_{k_0}}{m} + 2\pi i \frac{\ell}{m}) \text{ (with } \ell = 0, 1, \ldots, m-1 \text{ and } z = |z| \exp(2\pi i t))$$

Since the cover is given by $y^m = x^{d_{k_0}}$ resp., $y = x^{\mu_{k_0}}$ over a small disc around 0, we may lift a closed path around 0 to some path with starting point $(z, z^{\mu_{k_0}})$ and ending point $(z, e^{2\pi i \mu_{k_0}} z^{\mu_{k_0}})$.

Definition 2.1.6. Let $e^{2\pi i \mu_{k_0}}$ and d_{k_0} be given by Remark 2.1.5. Then $e^{2\pi i \mu_{k_0}}$ is the local monodromy datum of d_{k_0}.

Lemma 2.1.7. *Assume that $d_1, \ldots, d_n < m$. Let the (non-singular projective) curve C be given by*

$$y^m = (x - a_1)^{d_1} \cdot \ldots \cdot (x - a_n)^{d_n}.$$

Then the Galois group G is $\mathbb{Z}/(m)$, and the covering $C \to \mathbb{P}^1$ is given by the kernel of the homomorphism Φ given by $\gamma_k \to d_k \in \mathbb{Z}/(m)$. The point ∞ is a branch point and

$$\Phi(\gamma_\infty) = -\sum_{k=1}^{n} d_k \mod m,$$

if and only if m does not divide $\sum_{k=1}^{n} d_k$.

Proof. The last statement of the lemma follows by the preceding rest of the lemma and the Remark 2.1.4.

The Galois group and $\mathbb{Z}/(m)$ are obviously isomorphic. Let us remove the ramification points of C. Then we obtain a Riemann surface R. Now take a small loop γ_k around p_k, which starts and ends in $p \in \mathbb{P}^1$. Moreover take a point $b \in R$ with $f(b) = p$. The definition of R and Remark 2.1.5 imply that the lifting $f^*(\gamma_k)$ of the path γ_k starting in b ends in the point $d_k \cdot b$. Hence the statement follows from Proposition 2.1.2 and Remark 2.1.3. $\square$

Let $d \in \mathbb{Z}$ and $1 < m \in \mathbb{N}$. The residue class of d in $\mathbb{Z}/(m)$ is denoted by $[d]_m$.

Remark 2.1.8. Let $G = \mathbb{Z}/(m)$, and $[d]_m \in \mathbb{Z}/(m)^*$. We consider the kernels of the monodromy representations of the covers locally given by

$$y^m = (x - a_1)^{d_1} \cdot \ldots \cdot (x - a_n)^{d_n}$$

and

$$y^m = (x - a_1)^{[dd_1]_m} \cdot \ldots \cdot (x - a_n)^{[dd_n]_m}.$$

By the preceding lemma, these kernels coincide. Hence we conclude that both covers are equivalent.

2.2 The local system corresponding to a cyclic cover

Now let us assume that our cover $\pi : C \to \mathbb{P}^1$ is given by

$$y^m = (x - a_1)^{d_1} \cdot \ldots \cdot (x - a_n)^{d_n},$$

where m divides $d_1 + \ldots + d_n$ and ∞ is not a branch point. Moreover let

$$S := \{a_1, \ldots, a_n\}.$$

First let us consider the construction of a cyclic cover of an arbitrary algebraic manifold:

Remark 2.2.1. Let X be a complex algebraic manifold, $\mathcal{L}$ an invertible sheaf on X and

$$D = \sum b_k D_k$$

a normal crossing divisor on X, where $\mathcal{L}^m = \mathcal{O}(D)$ and $0 < b_k < m$ for each k. Then by $\mathcal{L}$ and D, one can construct a cyclic cover of degree m onto X (see [20], §3).

Definition 2.2.2. Let b_k and D_k be given by the previous remark. The number b_k is called the branch index of D_k with respect to this cyclic cover.

Example 2.2.3. In the case of

$$X = \mathbb{P}^1, \quad D = \sum_{k=1}^{n} d_k a_k, \quad \mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(\frac{1}{m} \sum_{k=1}^{n} d_k),$$

the cyclic cover of Remark 2.2.1 is given by

$$y^m = (x - a_1)^{d_1} \cdot \ldots \cdot (x - a_n)^{d_n}.$$

Next we describe the local system $\pi_*(\mathbb{C})|_{\mathbb{P}^1\setminus S}$ and its monodromy.

Lemma 2.2.4. *Let V be a $\mathbb{C}$-vector space of dimension n, and X be an arcwise connected and locally simply connected topological space with $x \in X$. Then the monodromy representation provides a bijection between the set of isomorphism classes of local systems of stalk V on X and the set of representations*

$$\pi_1(X, x) \to \mathrm{GL}_n(\mathbb{C}),$$

modulo the action of $\mathrm{Aut}_{\mathbb{C}}(V)$ *by conjugation.*

Proof. (see [61], Remarque 15.12) □

Since $\mathrm{GL}_1(\mathbb{C}) \cong \mathbb{C}^*$ is commutative, we can conclude:

Corollary 2.2.5. *The monodromy yields a bijection between the set of isomorphism classes of rank one local systems on $\mathbb{P}^1 \setminus S$ and the set of representations*

$$\pi_1(\mathbb{P}^1 \setminus S) \to \mathrm{GL}_1(\mathbb{C}).$$

The Galois group of our covering curve is isomorphic to $\mathbb{Z}/(m)$ and generated by a map ψ, which is given by $(x, y) \to (x, e^{2\pi i \frac{1}{m}} y)$ with respect to the above affine curve contained in $\mathbb{A}^2$, which is birationally equivalent to the covering curve. Hence a character χ of this group is determined by $\chi(\psi)$ with $\chi(\psi) \in \{e^{2\pi i \frac{j}{m}} | j = 0, 1, \ldots, m-1\}$. Thus the character group is isomorphic $\mathbb{Z}/(m)$ and we identify the character, which maps ψ to $e^{2\pi i \frac{j}{m}}$, with $j \in \mathbb{Z}/(m)$.[1]

Let D be an arbitrary disc contained in $\mathbb{P}^1 \setminus S$. The preimage of D is given by the disjoint union of discs D_r with $r = 0, 1, \ldots, m-1$ such that $\psi(D_r) = D_{[r+1]_m}$. The vector space $\pi_*\mathbb{C}_C|_{\mathbb{P}^1\setminus S}(D)$ has the basis $\{v_j | j = 0, 1, \ldots, m-1\}$, where

$$v_j := (e^{\frac{2\pi j(m-1)}{m}}, \ldots, e^{\frac{2\pi j}{m}}, 1),$$

and the r-th. coordinate denotes the value of the corresponding section of $\pi^{-1}(D)$ on D_r. By the push-forward action, each v_j is an eigenvector with respect to the character given by j. Since D is arbitrary, one can glue the local eigenspaces, and obtain an eigenspace decomposition

$$\pi_*\mathbb{C}_C|_{\mathbb{P}^1\setminus S} = \bigoplus_{j=0}^{m-1} \mathbb{L}_j$$

[1] These two identifications with $\mathbb{Z}/(m)$ are obviously not canonical, but useful for the description of $\pi_*\mathbb{C}_C|_{\mathbb{P}^1\setminus S}$ by using our explicit equation for $\pi : C \to \mathbb{P}^1$ as we will see a little bit later.

into rank 1 local systems, where $\mathbb{L}_j$ is the eigenspace with respect to the character given by $j \in \mathbb{Z}/(m)$. Hence the monodromy representation $\rho : \pi_1(\mathbb{P}^1 \setminus S) \to GL_m(\mathbb{C})$ has the corresponding decomposition

$$\rho = (\rho_0, \rho_1, \dots, \rho_{m-1}) : \pi_1(\mathcal{X}) \to \prod_{i=0}^{m-1} \mathrm{GL}_1(\mathbb{C}),$$

where

$$\rho_j : \pi_1(\mathbb{P}^1 \setminus S) \to \mathrm{GL}_1(\mathbb{C})$$

is the monodromy representation of $\mathbb{L}_j$ for all $j = 0, 1, \dots, m-1$.

Let us recall that our cyclic cover C is given by

$$y^m = (x - a_1)^{d_1} \dots (x - a_n)^{d_n},$$

where ∞ is not a branch point. Now let $x \in \mathbb{P}^1 \setminus S$, and $x \in D$, where D is a sufficiently small open disc as above. Take a counterclockwise loop γ_k around a_k and cover the loop with a finite number of (sufficiently) small discs. The continuation of $\tilde{s}$ on the unification of these discs leads to a multisection. By Remark 2.1.5, the possible liftings $\gamma_k^{(r)}$ of the loop γ_k are paths with starting point $\gamma_k^{(r)}(0) = y_r$, where $y_r \in D_r$ and ending point $\gamma_k^{(r)}(1) = y_{[d_k+r]_m}$. This implies that the monodromy representation of $\mathbb{L}_j$ maps γ_k to $e^{\frac{2\pi j d_k}{m}}$. Hence we conclude:

Theorem 2.2.6. *Let the cyclic cover $\pi : C \to \mathbb{P}^1$, which is not branched over ∞, be given by*

$$y^m = (x - a_1)^{d_1} \dots (x - a_n)^{d_n}. \tag{2.2}$$

Then the local system $\pi_* \mathbb{C}|_{\mathbb{P}^1 \setminus S}$ is given by the monodromy representation

$$\gamma_k \to \{(x_j)_{j=0,1\dots,m-1} \to (e^{\frac{2\pi i j d_k}{m}} x_j)_{j=0,1\dots,m-1}\}.$$

Remark 2.2.7. One can consider $\pi_*(\mathbb{Q}(e^{2\pi i \frac{1}{m}}))|_{\mathbb{P} \setminus S}$, too. Since a generator ψ of $\mathrm{Gal}(C; \mathbb{P}^1)$ satisfies $\psi^m = 1$, the minimal polynomial of its action on $\pi_*(\mathbb{Q}(e^{2\pi i \frac{1}{m}}))|_{\mathbb{P} \setminus S}$ decomposes into linear factors contained in $\mathbb{Q}(e^{2\pi i \frac{1}{m}})[x]$. Hence the eigenspace decomposition is defined over $\mathbb{Q}(e^{2\pi i \frac{1}{m}})$.

Each local system L of $\mathbb{C}$-vector spaces on any topological space X has a dual local system $L^\vee$ given by the sheafification of the presheaf

$$U \to \mathrm{Hom}_{\mathbb{C}}(L, \mathbb{C}).$$

Proposition 2.2.8. *One has*

$$\mathbb{L}_j^\vee = \bar{\mathbb{L}}_j.$$

Furthermore the monodromy representation $\mu_{\mathbb{L}_j^\vee}$ *of* $\mathbb{L}_j^\vee$ *is given by* $\mu_{\mathbb{L}_j^\vee}(\gamma_s) = \overline{\mu_{\mathbb{L}_j}(\gamma_s)}$ *for all* $s \in S$.

Proof. (see [19], Proposition 2) □

Hence by the respective monodromy representations, we obtain for all $j = 1, \ldots, m-1$:

Corollary 2.2.9.

$$\mathbb{L}_j^\vee = \mathbb{L}_{m-j}$$

Let $r|m$. We consider the $\mathbb{C}$-algebra endomorphism Φ_r of $\mathbb{C}[x,y]$ given by $x \to x$ and $y \to y^r$. The (non-singular) curve C is birationally equivalent to the affine variety given by $\mathrm{Spec}(\mathbb{C}[x,y]/I)$, where

$$I = (y^m - (x-a_1)^{d_1} \ldots (x-a_n)^{d_n}).$$

By Φ_r, we obtain the prime ideal

$$\Phi_r^{-1}(I) = (y^{\frac{m}{r}} - (x-a_1)^{d_1} \ldots (x-a_n)^{d_n}).$$

Let C_r be the irreducible projective non-singular curve birationally equivalent to the affine variety given by $\mathrm{Spec}(\mathbb{C}[x,y]/\Phi_r^{-1}(I))$.

Remark 2.2.10. By the equation above, we have a cover $\pi_r : C_r \to \mathbb{P}^1$ of degree $\frac{m}{r}$. The homomorphism Φ_r induces a cover $\phi_r : C \to C_r$ of degree r such that

$$\pi = \pi_r \circ \phi_r.$$

Proposition 2.2.11.

$$(\pi_r)_*\mathbb{C}_{C_r}|_{\mathbb{P}^1 \setminus S} = \bigoplus_{j=0}^{\frac{m}{r}-1} \mathbb{L}_{r \cdot j} \subset \pi_* \mathbb{C}_C|_{\mathbb{P}^1 \setminus S}.$$

Proof. Let $m_0 := \frac{m}{r}$. By Theorem 2.2.6, the monodromy representation of the local system $(\pi_r)_*\mathbb{C}_{C_r}|_{\mathbb{P}^1 \setminus \{a_1,\ldots,a_n\}}$ is given by

$$\gamma_k \to \{(x_j)_{j=0,1\ldots,\frac{m}{r}-1} \to (e^{\frac{2\pi i j d_k}{m_0}} x_j)_{j=0,1\ldots,\frac{m}{r}-1} = (e^{\frac{2\pi i j r d_k}{m}} x_j)_{j=0,1\ldots,\frac{m}{r}-1}\}.$$

By the respective monodromy representations of the local systems $\mathbb{L}_j$, this yields the statement. □

2.3 The cohomology of a cover

In this section we discuss some known facts about the eigenspace decomposition of the Hodge structure of a curve C with respect to a cyclic cover $\pi : C \to \mathbb{P}^1$. The main reference for this section is given by §3 of the book [20] of H. Esnault and E. Viehweg. Section 2 of the essay [18] of P. Deligne and G. D. Mostow contains additional information about our case.

Let $\pi : C \to \mathbb{P}^1$ be given by

$$y^m = (x - a_1)^{d_1} \cdot \ldots \cdot (x - a_n)^{d_n}$$

such that ∞ is not a branch point,

$$S = \{a_1, \ldots, a_n\}, \;\; D = d_1 a_1 + \ldots + d_n a_n \;\text{ and }\; \mathcal{L}^{(j)} = \mathcal{O}_{\mathbb{P}^1}(j\frac{d_1 + \ldots + d_n}{m} - \sum_{k=1}^{n+3}[\frac{j}{m} \cdot d_k]).$$

Moreover let the generator ψ of the Galois group of π be given by $(x, y) \to (x, e^{2\pi i \frac{1}{m}} y)$ with respect to the explicit equation above, which yields π.

We fix some new notation: Let $q \in \mathbb{Q}$ and $[q]$ denote the largest integer, which is smaller than q. Then we define $[q]_1 := q - [q]$. Moreover we define

$$S_j := \{a \in S | [j\mu_a]_1 \neq 0\}.$$

Proposition 2.3.1. *The sheaves $\pi_*(\mathcal{O})$ and $\pi_*(\omega)$ have a decomposition into eigenspaces with respect to the Galois group representation, which are given by the sheaves ${\mathcal{L}^{(j)}}^{-1}$ and*

$$\omega_j := \omega_{\mathbb{P}^1}(\log D^{(j)}) \otimes {\mathcal{L}^{(j)}}^{-1} \quad \text{with} \quad D^{(j)} := \sum_{a \in S_j} a$$

for $j = 0, 1, \ldots, m - 1$ such that ψ acts via pull-back by the character $e^{2\pi i \frac{j}{m}}$ on ${\mathcal{L}^{(j)}}^{-1}$ resp., ω_j.

Proof. The eigenspace decomposition of $\pi_*(\mathcal{O})$ follows by [20], Corollary 3.11. Moreover [20], Lemma 3.16, d) yields the decomposition of $\pi_*(\omega)$ into the claimed sheaves. Since ${\mathcal{L}^{(j)}}^{-1}$ is an eigenspace with respect to the Galois group representation, ω_j is an eigenspace of the same eigenvalue. □

Remark 2.3.2. One has obviously $h^0(\omega_0) = 0$. By [20], $2.3, c$), one concludes that

$$\omega_{\mathbb{P}^1}(\log D^{(j)}) = \omega_{\mathbb{P}^1}(D^{(j)})$$

for $j = 1, \ldots, m - 1$. Hence for $j = 1, \ldots, m - 1$ we obtain

$$h^0(\omega_j) = h^0(\mathcal{O}_{\mathbb{P}^1}(-2 + deg(D^{(j)}) - j\frac{d_1 + \ldots + d_{n+3}}{m} + \sum_{k=1}^{n+3}[\frac{j}{m} \cdot d_k]))$$
$$= -1 + |S_j| + \sum_{a \in S_j}(-j\mu_a + [j\mu_a]) = -1 + \sum_{a \in S_j}(1 - [j\mu_a]_1).$$

But here we want to determine our eigenspaces on $\pi_*(\omega_C)$ with respect to the push-forward action. Thus we put $\omega^{(j)} := \omega_{[m-j]_m}$, and we obtain

$$h_j^{1,0}(C) := h^0(\omega^{(j)}) = h^0(\omega_{[m-j]_m}) = -1 + \sum_{a \in S_j}(1 - [(m-j)\mu_a]_1) = -1 + \sum_{a \in S_j}[j\mu_a]_1.$$

Moreover let $H_j^{0,1}(C)$ denote the vector space of antiholomorphic 1-forms on C with respect to the corresponding character of the Galois group action. Since the push-forward action of the Galois group respects the alternating form of the polarization of the Hodge structure on $H^1(C, \mathbb{Z})$, one concludes that $H^{0,1}_{[m-j]_m}(C)$ is the dual of $H_j^{1,0}(C)$. Thus:

Proposition 2.3.3. *We have the eigenspace decomposition*

$$H^1(C, \mathbb{C}) = \bigoplus_{j=1}^{m-1} H_j^1(C, \mathbb{C}) \quad \textit{with} \quad H_j^{1,0}(C) \oplus H_j^{0,1}(C) = H_j^1(C, \mathbb{C}).$$

Moreover by $h_j^{0,1}(C) = h^{1,0}_{[m-j]_m}(C)$ and the preceding calculations, one concludes:

Proposition 2.3.4. *We have*

$$h_j^{1,0}(C) = \sum_{s \in S_j}[j\mu_s]_1 - 1, \quad \textit{and} \quad h_j^{0,1}(C) = \sum_{s \in S_j}(1 - [j\mu_s]_1) - 1.$$

The preceding two propositions imply:

Corollary 2.3.5.

$$h_j^1(C, \mathbb{C}) = |S_j| - 2$$

2.4 Cyclic covers with complex multiplication

Let us now search for examples of covers of $\mathbb{P}^1$ with complex multiplication. The family given by

$$\mathbb{P}^2 \supset V(y^m - x_1(x_1 - x_0)(x_1 - a_1x_0) \ldots (x_1 - a_{m-3}x_0))$$
$$\to (a_1, \ldots, a_{m-3}) \in (\mathbb{A}^1 \setminus \{0, 1\})^{m-3} \setminus \{a_i = a_j | i \neq j\}$$

has obviously a fiber isomorphic to the Fermat curve $\mathbb{F}_m$, which is given by $V(y^m+x^m+1)$ and has complex multiplication (see [22] and [32]). For another family with a fiber with complex multiplication, we must work a little bit.

Lemma 2.4.1. *If (V,h_1) and (W,h_2) are two $\mathbb{Q}$-Hodge structures of weight k, then*

$$\mathrm{Hg}(V\oplus W,h_1\oplus h_2)\subset \mathrm{Hg}(V,h_1)\times \mathrm{Hg}(W,h_2)\subset \mathrm{GL}(V)\times \mathrm{GL}(W)\subset \mathrm{GL}(V\oplus W),$$

and the projections

$$\mathrm{Hg}(V\oplus W)\to \mathrm{Hg}(V), \quad \textit{and} \quad \mathrm{Hg}(V\oplus W)\to \mathrm{Hg}(W)$$

are surjective.

Proof. (see [58], Lemma 8.1) □

Lemma 2.4.2. *Let $V\subset W$ be a rational sub-Hodge structure of a polarized Hodge structure W. Then we have a direct sum decomposition*

$$W = V\oplus V',$$

where V' is also a rational sub-Hodge structure of W.

Proof. (see [61], Lemme 7.26) □

Lemma 2.4.3. *A curve C, which is covered by the Fermat curve $\mathbb{F}_m$ given by $V(x^m+y^m+z^m)\subset \mathbb{P}^2$ for some $1\le m\in\mathbb{N}$, has complex multiplication.*

Proof. A covering $\mathbb{F}_m\to C$ yields an injective vector space homomorphism

$$H^1(C,\mathbb{Q})\to H^1(\mathbb{F}_m,\mathbb{Q}),$$

which extends to an embedding of Hodge structures (see [61], 7.3.2 for more details). This embedding induces a direct sum decomposition into two rational sub-Hodge structures of $H^1(\mathbb{F}_m,\mathbb{Q})$ (see Lemma 2.4.2). Hence by Lemma 2.4.1 and the fact that $\mathbb{F}_m$ has complex multiplication, one obtains the statement. □

Theorem 2.4.4. *Let $0<d_1,d<m$, and ξ_k denote a primitive k-th. root of unity for all $k\in\mathbb{N}$. Then the curve C, which is given by*

$$y^m = x^{d_1}\prod_{i=1}^{n-2}(x-\xi_{n-2}^i)^d,$$

is covered by the Fermat curve $\mathbb{F}_{(n-2)m}$ given by $V(y^{(n-2)m}+x^{(n-2)m}+1)$ and has complex multiplication.

Proof. Let C be the curve, which is given by

$$y^m = x^{d_1} \prod_{i=1}^{n-2} (x - \xi_{n-2}^i)^d,$$

and $\phi : \mathbb{A}^2 \to \mathbb{A}^2$ be the morphism, which is given by $y \to yx^{d_1}$ and $x \to x^m$. By a little abuse of notation, we denote by $C \cap \mathbb{A}^2$ the singular affine curve given by the equation above, which is birationally equivalent to C. The corresponding homomorphism $\phi^* : \mathbb{C}[x,y] \to \mathbb{C}[x,y]$ sends the ideal, which defines $C \cap \mathbb{A}^2$, to the ideal generated by

$$y^m x^{m \cdot d_1} - x^{m \cdot d_1} \prod_{i=1}^{n-2} (x^m - \xi_{n-2}^i)^d.$$

This is contained in the ideal generated by

$$y^m - \prod_{i=1}^{n-2} (x^m - \xi_{n-2}^i)^d. \tag{2.3}$$

Let $m_0 := \frac{m}{gcd(m,d)}$, and $d_0 := \frac{d}{gcd(m,d)}$. It is obvious that

$$y^m - \prod_{i=1}^{n-2} (x^m - \xi_{n-2}^i)^d = \prod_{j=0}^{gcd(m,d)-1} (y^{m_0} - \xi_{gcd(m,d)}^j \prod_{i=1}^{n-2} (x^m - \xi_{n-2}^i)^{d_0}).$$

Now we take the curve C_1, which is given by

$$y^{m_0} = \prod_{i=1}^{n-2} (x^m - \xi_{n-2}^i)^{d_0}.$$

By the definitions of m_0 and d_0, and Remark 2.1.8, the curve C_1 is given by

$$y^{m_0} = \prod_{i=1}^{n-2} (x^m - \xi_{n-2}^i),$$

too. Hence this curve irreducible, and ϕ induces a cover $C_1 \to C$ resp., ϕ^* induces a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[C \cap \mathbb{A}^2] \to \mathbb{C}[C_1 \cap \mathbb{A}^2]$. By $x \to x$ and $y \to y^{n-2\frac{m}{m_0}}$, we get a cover of the Fermat curve $\mathbb{F}_{(n-2)m}$ given by $V(y^{(n-2)m} + x^{(n-2)m} + 1)$ onto C_1. Now we use the composition of these covers $\mathbb{F}_{(n-2)m} \to C_1$ and $C_1 \to C$, and Lemma 2.4.3. This yields the statement. $\square$

Chapter 3
Some preliminaries for families of cyclic covers

In this chapter we collect the remaining preparations for the computations concerning the VHS of our families $\pi : \mathcal{C} \to \mathcal{P}_n$ of cyclic covering of $\mathbb{P}^1$, which we construct in this chapter.

Let $\mathcal{V}$ denote the VHS of the family $\mathcal{X} \to Y$ of curves and $\mathrm{Mon}^0(\mathcal{V})$ denote the identity component of the Zariski closure of the monodromy group of $\mathcal{V}$. In Section 3.1 we introduce the generic Hodge group $\mathrm{Hg}(\mathcal{V})$, which is the maximum of the Hodge groups of all occurring Hodge structures in $\mathcal{V}$. Moreover $\mathrm{Hg}(\mathcal{V})$ coincides with the Hodge groups of the Hodge structures in $\mathcal{V}$ over the complement of a unification of countably many submanifolds of Y. Our families $\pi : \mathcal{C} \to \mathcal{P}_n$ are constructed in Section 3.2. We will also make some general remarks about the monodromy representation of $\mathcal{V}$ including the fact that the Galois group action yields an eigenspace decomposition in Section 3.2. In Section 3.3 we make some explicit computations of the monodromy representations of these eigenspaces. These computations are motivated from the fact that $\mathrm{Mon}^0(\mathcal{V})$ is a normal subgroup of the derived group $\mathrm{Hg}^{\mathrm{der}}(\mathcal{V})$ of the generic Hodge group! as we see in Section 3.1.

3.1 The generic Hodge group

We want to study the variations of Hodge structures (VHS) of the families of cyclic covers of $\mathbb{P}^1$, which will be constructed in the next section. Hence let us first make some general observations about the relation between their monodromy groups and Hodge groups resp., Mumford-Tate groups. These observation lead to the definition of the generic Hodge group defined below.

Proposition 3.1.1. *Let W be a connected complex manifold and $\mathcal{V}$ be a polarized variation of rational Hodge structures of weight k over W. Then there is a countable union $W' \subset W$ of submanifolds such that all $\mathrm{MT}(\mathcal{V}_p)$ coincide (up to conjugation by integral matrices) for all $p \in W \setminus W'$. Moreover one has $\mathrm{MT}(\mathcal{V}_{p'}) \subset \mathrm{MT}(\mathcal{V}_p)$ for all $p' \in W'$ and $p \in W \setminus W'$.*

J.C. Rohde, *Cyclic Coverings, Calabi-Yau Manifolds and Complex Multiplication*, Lecture Notes in Mathematics 1975, DOI: 10.1007/978-3-642-00639-5_4,

Proof. (see [43], Subsection 1.2) □

Remark 3.1.2. There exist the following versions of the previous proposition:

If one replaces W by a connected complex algebraic manifold in the previous proposition, the submanifolds $W' \subset W$ of the previous proposition are algebraic, too (see also [43], Subsection 1.2).

Now let F be a totally real number field, W be a complex connected algebraic manifold, $\mathcal{A} \to W$ be a family of abelian varieties and $\mathcal{V}$ be its polarized variation of F-Hodge structures of weight 1 over W. Then there is a countable union $W' \subset W$ of subvarieties such that all $\mathrm{MT}(\mathcal{V}_p)$ coincide (up to conjugation by integral matrices) for all closed $p \in W \setminus W'$ (see [42], Subsection 1.2).

The previous remark motivates the definition of the generic Mumford-Tate group $\mathrm{MT}_F(\mathcal{V})$ of a polarized variation $\mathcal{V}$ of F-Hodge structures of weight 1 of a family of abelian varieties over a connected complex algebraic manifold W. Moreover the preceding proposition motivates the definition of the generic Mumford-Tate group $\mathrm{MT}(\mathcal{V})$ of a polarized variation $\mathcal{V}$ of $\mathbb{Q}$-Hodge structures of weight k on a connected complex manifold. The generic Mumford-Tate group is given by $\mathrm{MT}_F(\mathcal{V}) = \mathrm{MT}_F(\mathcal{V}_p)$ resp., $\mathrm{MT}(\mathcal{V}) = \mathrm{MT}(\mathcal{V}_p)$ for all closed $p \in W \setminus W'$.

Since the image of the embedding $\mathrm{SL}(\mathcal{V}_{F,p}) \hookrightarrow \mathrm{GL}(\mathcal{V}_{F,p})$ is independent with respect to the chosen coordinates on $\mathcal{V}_{F,p}$, Lemma 1.3.17 allows us to define the generic Hodge group $Hg_F(\mathcal{V}) := (\mathrm{MT}_F(\mathcal{V}) \cap \mathrm{SL}_F(\mathcal{V}))^0$ such that $\mathrm{Hg}_F(\mathcal{V}) = \mathrm{Hg}_F(\mathcal{V}_p)$ for all (closed) $p \in W \setminus W'$.

Definition 3.1.3. Let $\mathbb{Q} \subseteq K \subseteq \mathbb{R}$ be a field and $\mathcal{V} = (\mathcal{V}_K, \mathcal{F}^\bullet, Q)$ be a polarized variation of K Hodge structures on a connected complex manifold D. Then $\mathrm{Mon}^0_K(\mathcal{V})_p$ denotes the connected component of identity of the Zariski closure of the monodromy group in $\mathrm{GL}((\mathcal{V}_K)_p)$ for some $p \in D$. For simplicity we write $\mathrm{Mon}^0(\mathcal{V})_p$ instead of $\mathrm{Mon}^0_{\mathbb{Q}}(\mathcal{V})_p$.

Theorem 3.1.4. *Keep the assumptions and notations of Proposition* 3.1.1*. One has that* $\mathrm{Mon}^0_F(\mathcal{V})_p$ *is a subgroup of* $\mathrm{MT}^{\mathrm{der}}_F(\mathcal{V}_p)$ *for all* $p \in W \setminus W'$*. Moreover for a variation of* $\mathbb{Q}$ *Hodge structures one has that* $\mathrm{Mon}^0(\mathcal{V})_p$ *is a normal subgroup of* $\mathrm{MT}^{\mathrm{der}}(\mathcal{V}_p)$ *and*

$$\mathrm{Mon}^0(\mathcal{V})_p = \mathrm{MT}^{\mathrm{der}}(\mathcal{V}_p)$$

for all $p \in W \setminus W'$*, if* $\mathcal{V}_{\mathbb{Q}}$ *has a CM point.*

Proof. (see [43], Theorem 1.4 for the statement about the variations of $\mathbb{Q}$ Hodge structures and [42], Properties 7.14 for the statement about the variations of F Hodge structures) □

Corollary 3.1.5. *Keep the assumptions of Proposition* 3.1.1*. Then the group* $\mathrm{Mon}^0(\mathcal{V})$ *is semisimple.*

Proof. By Theorem 3.1.4, the Lie subalgebra $Lie(\mathrm{Mon}^0_{\mathbb{Q}}(\mathcal{V})_{\mathbb{R}})$ of $Lie(\mathrm{MT}^{\mathrm{der}}_{\mathbb{Q}}(\mathcal{V})_{\mathbb{R}})$ is an ideal. Recall that $\mathrm{MT}^{\mathrm{der}}_{\mathbb{Q}}(\mathcal{V})_{\mathbb{R}}$ is semisimple. Hence the algebra $Lie(\mathrm{Mon}^0_{\mathbb{Q}}(\mathcal{V})_{\mathbb{R}})$ consists of the direct sum of simple subalgebras of $Lie(\mathrm{MT}^{\mathrm{der}}_{\mathbb{Q}}(\mathcal{V})_{\mathbb{R}})$. Thus $\mathrm{Mon}^0_{\mathbb{Q}}(\mathcal{V})_{\mathbb{R}}$ and $\mathrm{Mon}^0(\mathcal{V})$ are semisimple. □

3.2 Families of covers of the projective line

Let S be some $\mathbb{C}$-scheme. Recall that the covers $c_1 : V_1 \to \mathbb{P}^1_S$ and $c_2 : V_2 \to \mathbb{P}^1_S$ are equivalent, if there is a S-isomorphism $j : V_1 \to V_2$ such that $c_1 = c_2 \circ j$.

In this section we construct a family of cyclic covers of $\mathbb{P}^1$ such that all equivalence classes of covers with a fixed number of branch points with fixed branch indices are represented by some of its fibers. For us it is sufficient to start with a space, which is not a moduli scheme, but whose closed points "hit" all equivalence classes of covers of $\mathbb{P}^1$ with Galois group $G = (\mathbb{Z}/m, +)$ and a fixed number of branch points with fixed branch indices.

We start with the space

$$(\mathbb{P}^1)^{n+3} \supset \mathcal{P}_n := (\mathbb{P}^1)^{n+3} \setminus \{z_i = z_j | i \neq j\},$$

which parametrizes the injective maps $\phi : N \to \mathbb{P}^1$, where $N := \{s_1, \ldots, s_{n+3}\}$. Thus a point $q \in \mathcal{P}_n$ corresponds to an injective map $\phi_q : N \to \mathbb{P}^1$.[1] One can consider $\mathcal{P}_n$ as configuration space of $n+3$ ordered points, too.

We endow the points $s_k \in N$ with some local monodromy data $\alpha_k = e^{2\pi i \mu_k}$, where

$$\mu_k \in \mathbb{Q}, \quad 0 < \mu_k < 1 \quad \text{and} \quad \sum_{k=1}^{n+3} \mu_k \in \mathbb{N}.$$

Now we construct a family of covers of $\mathbb{P}^1$ by these local monodromy data:

Construction 3.2.1. Let m be the smallest integer such that $m\mu_k \in \mathbb{N}$ for $k = 1, \ldots, n+3$, and $D_k \subset \mathbb{P}_{\mathcal{P}_n} := \mathbb{P}^1 \times \mathcal{P}_n$ be the prime divisor given by

$$D_k = \{(a_k, a_1, \ldots, a_k, \ldots, a_{n+3})\}.$$

[1] The set N is some arbitrary finite set, where the set S of the preceding chapter is a concrete set $S \subset \mathbb{P}^1$ given by $S = \phi_q(N)$ for some $q \in \mathcal{P}_n$.

Let D be the divisor

$$D := \sum_{k=1}^{n+3} m\mu_k D_k \sim mD_0 \quad \text{with} \quad D_0 := (\sum_{k=1}^{n+3} \mu_k) \cdot (\{0\} \times \mathcal{P}_n).$$

By the sheaf $\mathcal{L} := \mathcal{O}_{\mathbb{P}_{\mathcal{P}_n}}(D_0)$ and the divisor D, we obtain an irreducible cyclic cover $\mathcal{C}$ of degree m onto $\mathbb{P}_{\mathcal{P}_n}$ as in [20], §3 (where irreducible means that the covering variety is irreducible). By $\pi : \mathcal{C} \to \mathbb{P}^1 \times \mathcal{P}_n \stackrel{pr_2}{\to} \mathcal{P}_n$, this cyclic cover yields a family of irreducible cyclic covers of degree m onto $\mathbb{P}^1$.

Suppose that r divides m. By taking the quotient of the subgroup of order r of the Galois group of the cyclic cover $\mathcal{C} \to \mathbb{P}^1 \times \mathcal{P}_n$, one gets a family $\pi_r : \mathcal{C}_r \to \mathcal{P}_n$ of cyclic covers of degree $\frac{m}{r}$ onto $\mathbb{P}^1$. Let $\phi_r : \mathcal{C} \to \mathcal{C}_r$ denote the quotient map. One has

$$\pi = \pi_r \circ \phi_r.$$

Remark 3.2.2. Without loss of generality one may assume that $q := (a_1, \ldots, a_{n+3}) \in \mathcal{P}_n$ is contained in $\mathbb{A}^{n+3}$, too. Thus the fiber $\mathcal{C}_q$ is given by the equation

$$y^m = (x - a_1)^{d_1} \cdot \ldots \cdot (x - a_{n+3})^{d_{n+3}}$$

with $d_k = m\mu_k$. By Remark 2.1.5, the local monodromy datum α_k describes the lifting of a path γ_k around $a_k \in \mathbb{P}^1$.[2] One checks easily that each equivalence class of cyclic covers of degree m with $n + 3$ branch points and fixed branch indexes $d_1, \ldots, d_{n+3}$ is represented by some fibers of $\mathcal{C}$. Moreover for $C = \mathcal{C}_q$ the quotient C_r of Remark 2.2.10 is given by the fiber $(\mathcal{C}_r)_q$.

A family of smooth algebraic curves over $\mathbb{C}$ determines a proper submersion $\tau : X \to Y$ in the category of differentiable manifolds ([61], Proposition 9.5). By the Ehresmann theorem, we obtain that over any contractible submanifold W of Y the family is diffeomorphic to $X_0 \times W$, where X_0 is the fiber of some point $0 \in W$. This fact has some consequences for the monodromy representation of the variation of integral Hodge structures.

Recall that $R^1\tau_*(\mathbb{Z})$ is the sheaf associated to the presheaf given by

$$V \to H^1(\tau^{-1}(V), \mathbb{Z}|_{\pi^{-1}(V)})$$

for all open subsets $V \subset \mathcal{P}_n$. Moreover we have

$$H^1(X_0, \mathbb{Z}) = H^1(X_W, \mathbb{Z}) = (R^1\tau_*(\mathbb{Z}))(W)$$

for some contractible $W \subset \mathcal{P}_n$ with $0 \in W$, which implies that $R^1\tau_*(\mathbb{Z})$ is a local system (see [61], 9.2.1).

[2] This circumstance explains the term "local monodromy datum".

By using these facts, one can easily ensure that the monodromy group of the VHS of a family of curves can be calculated over any arbitrary field $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$:

Lemma 3.2.3. *Let K be a field with $char(K) = 0$. Moreover let $\tau : X \to Y$ be a holomorphic family of curves. Then we obtain*

$$R^1\tau_*(K) = R^1\tau_*(\mathbb{Z}) \otimes_{\mathbb{Z}} K.$$

Proof. The sheaf $R^1\tau_*(K)$ is given by the sheafification of the presheaf

$$V \to H^1(\tau^{-1}(V), K|_{\tau^{-1}(V)}).$$

Hence by the description of the cohomology by Čech complexes, this presheaf is given by

$$V \to H^1(\tau^{-1}(V), \mathbb{Z}|_{\tau^{-1}(V)}) \otimes_{\mathbb{Z}} K.$$

By the fact that a local section of $\mathbb{Z}$ or K on a connected component of V resp., $\tau^{-1}(V)$ is constant, one does not need to differ between the locally constant sheaves given by $\mathbb{Z}$ resp., K on X or Y for the computation of $R^1\tau_*(K)$. This yields the desired identification. $\square$

By the fact that the integral cohomology of a curve does not have torsion, one concludes:

Corollary 3.2.4. *Keep the assumptions of Lemma* 3.2.3*. Then the monodromy representations of $R^1\tau_*(\mathbb{Z})$ and $R^1\tau_*(K)$ coincide.*

Remark 3.2.5. Recall that we have an eigenspace decomposition of

$$H^1(\mathcal{C}_0, \mathbb{C}) = H^1(\mathcal{C}_0, \mathbb{Z}) \otimes \mathbb{C}$$

with respect to the Galois group action. By $H^1(\mathcal{C}_0, \mathbb{C}) = (R^1\pi_*(\mathbb{C}))(W)$ for some contractible $W \subset \mathcal{P}_n$ with $0 \in W$, we obtain an eigenspace decomposition of $(R^1\pi_*(\mathbb{C}))(W)$. Since we have this decomposition over all contractible $W \subset \mathcal{P}_n$, we can glue these eigenspaces, which yields a decomposition of the whole sheaf $R^1\pi_*(\mathbb{C})$ into eigenspaces with respect to the Galois group action.

Recall that we have an identification between the characters of the Galois group of some fiber and the elements $j \in \mathbb{Z}/(m)$. This identification allows a compatible identification between the characters of the Galois group of the family and the elements $j \in \mathbb{Z}/(m)$. Let $\mathcal{L}_j$ denote the eigenspace of $R^1\pi_*(\mathbb{C})$ with respect to the character j.

Remark 3.2.6. Let $0 \in \mathcal{P}_n$. We have a monodromy action $\rho_{\mathcal{C}}$ by diffeomorphisms on the fiber $\mathcal{C}_0$, which is induced by the gluing diffeomorphisms of the locally constant family of manifolds given by $\mathcal{C}$. Since these gluing diffeomorphisms induce the gluing homomorphisms of $R^1\pi_*(\mathbb{Z})$ in the obvious natural way, the monodromy representation ρ of $R^1\pi_*(\mathbb{Z})$ is given by

$$\rho(\gamma)(\eta) = (\rho_{\mathcal{C}}(\gamma))_*(\eta) \quad (\forall \ \ \eta \in H^1(\mathcal{C}_0, \mathbb{Z})).$$

Remark 3.2.7. Since each gluing diffeomorphism of the locally constant family of manifolds corresponding to $\mathcal{C}$ respects intersection form, Remark 3.2.6 implies that the monodromy group of $R^1\pi_*(\mathbb{C})$ respects the polarization of the Hodge structures. Assume that $H^1_j(\mathcal{C}_q, \mathbb{C}) = (\mathcal{L}_j)_q$ satisfies that $H^{1,0}_j(\mathcal{C}_q) = n_1$ and $H^{0,1}_j(\mathcal{C}_q)_2 = n_2$. This means that the polarized variation of integral Hodge structure endows $(\mathcal{L}_j)_q$ with an Hermitian form with signature (n_1, n_2). Hence the monodromy group of this eigenspace is contained in $\mathrm{U}(n_1, n_2)$. In this sense we say that $\mathcal{L}_j$ is of type (n_1, n_2).

3.3 The homology and the monodromy representation

In this section we study the monodromy representation of $\pi_1(\mathcal{P}_n)$ on the dual of $R^1\pi_*(\mathbb{C})$ given by the complex homology. This will yield corresponding statements for the monodromy representation of $R^1\pi_*(\mathbb{C})$.

In the case of the configuration space $\mathcal{P}_n$ of $n+3$ points, we make a difference between these different points. One says that the points are "colored" by different "colors". Moreover one can identify its fundamental group with the subgroup of the braid group on $n+3$ strands in $\mathbb{P}^1$, which is given by the braids leaving the strands invariant (see [24], Chapter **I**. 3.). This subgroup of the braid group is called the colored braid group. An element of this group is for example given by the Dehn twist T_{k_1,k_2} with $1 \leq k_1 < k_2 \leq n+3$. The Dehn twist T_{k_1,k_2} is given by leaving a_{k_2} "run" counterclockwise around a_{k_1}.

Now we consider a fiber $C = \mathcal{C}_q$ of $\mathcal{C}$. Recall that C is a cyclic cover of $\mathbb{P}^1$ described in Chapter 2. Let ψ denote the generator of the Galois group as in Section 2.2. We keep the notation of Chapter 2.

Consider the eigenspace $\mathbb{L}_j$, which can be extended from a local system on $\mathbb{P}^1 \setminus S$ to a local system on $\mathbb{P}^1 \setminus S_j$ with $S_j = \{a_1, \ldots, a_{n_j+3}\}$. For simplicity one may without loss of generality assume that $a_{n_j+3} = \infty$ and $a_k \in \mathbb{R}$ such that $a_k < a_{k+1}$ for all $k = 1, \ldots, n_{j+2}$. Here we assume that δ_k is the oriented path from a_k to a_{k+1} given by the straight line.

Construction 3.3.1. Let ζ be a path on $\mathbb{P}^1$. Assume without loss of generality that $\zeta((0,1))$ is contained in a simply connected open subset U of $\mathbb{P}^1 \setminus S$. Otherwise we decompose ζ into such paths. It has m liftings $\zeta^{(0)}, \ldots, \zeta^{(m-1)}$ to C such that $\psi(\zeta^{(\ell)}) = \zeta^{([\ell-1]_m)}$. By the tensorproduct of $\mathbb{C}$ with the free abelian group generated by the paths on C, one obtains the vector space of $\mathbb{C}$-valued paths on C. Now let $c \in \mathbb{C}$ and take the linear combination of $\mathbb{C}$-valued paths on C given by

$$\hat{\zeta} = c\zeta^{(0)} + \ldots + ce^{2\pi i \frac{jr}{m}}\zeta^{(r)} + \ldots + ce^{2\pi i \frac{j(m-1)}{m}}\zeta^{(m-1)}.$$

By the assumptions, one verifies easily that $\psi(\hat{\zeta}) = e^{2\pi i \frac{j}{m}}\hat{\zeta}$. Moreover by the local sections given by $c, \ldots, ce^{2\pi i \frac{jr}{m}}, \ldots, ce^{2\pi i \frac{j(m-1)}{m}}$ on the corresponding sheets over U containing the different $\zeta^{(\ell)}((0,1))$, one obtains a corresponding section $\tilde{c} \in \mathbb{L}_j(U)$. In this sense we have a $\mathbb{L}_j$-valued path $\tilde{c} \cdot \zeta$ on $\mathbb{P}^1$.

Remark 3.3.2. Consider the (oriented) path δ_k from the branch point a_k to the branch point a_{k+1}. Let e_k be a non-zero local section of $\mathbb{L}_j$ defined over an open set containing $\delta_k((0,1))$. The $\mathbb{L}_j$-valued path $e_k \cdot \delta_k$ yields an element $[e_k \cdot \delta_k]$ of the homology group of $H_1(C, \mathbb{C})$, which is represented by the corresponding linear combination of paths in C lying over δ_k. It has the character j with respect to the Galois group representation. Let $H_1(C, \mathbb{C})_j$ denote the corresponding eigenspace.

Definition 3.3.3. A partition of S_j into some disjoint sets $S^{(1)} \cup \ldots \cup S^{(\ell)} = S_j$ is stable with respect to the local monodromy data μ_k of $\mathbb{L}_j$, if

$$\sum_{a_k \in S^{(1)}} \mu_k \notin \mathbb{N}, \ldots, \sum_{a_k \in S^{(\ell)}} \mu_k \notin \mathbb{N}.$$

Theorem 3.3.4. *Assume that* $S_j = \{a_i : i = 1, \ldots, n_j + 3\}$ *has the stable partition* $\{a_1, \ldots, a_{\ell+1}\}, \{a_{\ell+2}, \ldots, a_{n_j+3}\}$ *for some* $1 \leq \ell \leq n_j + 1$*. Then the eigenspace* $H_1(C, \mathbb{C})_j$ *of the complex homology group of* C *has a basis given by*

$$\mathcal{B} = \{[e_k\delta_k] : k = 1, \ldots, \ell\} \cup \{[e_k\delta_k] : k = \ell + 2, \ldots, n_j + 2\}.$$

Proof. By [36], Lemma 4.5, one has that $\{[e_k\delta_k] : k = 1, \ldots, n_j + 1\}$ is a basis of $H_1(C, \mathbb{C})_j$. Hence $\{[e_k\delta_k] : k = 1, \ldots, n_j + 2\}$ is not linearly independent.

One can compute a non-trivial linear combination, which yields 0, in the following way: Choose a non-zero section of $\mathbb{L}_j$ over

$$U = \mathbb{P}^1 \setminus (\bigcup_{k=1}^{n_j+2} \delta_k).$$

This yields a linear combination of the sheets over U, on which ψ acts by j. By the boundary operator ∂, one gets the desired non-trivial linear combination of $\mathbb{L}_j$-valued paths, which is equal to 0. Now let α_k denote the local monodromy datum of $\mathbb{L}_j$ around $a_k \in S_j$ for all $k = 1, \ldots, n_j + 3$. By the local monodromy data, one can easily compute this linear combination. This computation yields that $\{\delta_1, \ldots, \delta_\ell\} \cup \{\delta_{\ell+2}, \ldots, \delta_{n_j+2}\}$ is linearly independent, if and only if $\{a_1, \ldots, a_{\ell+1}\}, \{a_{\ell+2}, \ldots, a_{n_j+3}\}$ is a stable partition. $\square$

Let α_k denote the local monodromy datum of $\mathbb{L}_j$ around $a_k \in S_j$ for all $k = 1, \ldots, n_j + 3$. One has up to a certain normalization of the basis vectors $[e_1\delta_1], \ldots [e_1\delta_{n_j+1}]$ the following description of the monodromy representation:

The Dehn twist $T_{k,k+1}$ leaves obviously δ_ℓ invariant for all $|k - \ell| > 1$. Moreover by following a path representing $T_{k,k+1}$, one sees that the monodromy action of $T_{k,k+1}$ on $H_1(C, \mathbb{C})_j$ (induced by push-forward) is given by

$$\begin{aligned} [e_{k-1}\delta_{k-1}] &\to [e_{k-1}\delta_{k-1}] + \alpha_k(1-\alpha_{k+1})[e_k\delta_k], \\ [e_k\delta_k] &\to \alpha_k\alpha_{k+1}[e_k\delta_k] \\ \text{and} \quad [e_{k+1}\delta_{k+1}] &\to [e_{k+1}\delta_{k+1}] + (1-\alpha_k)[e_k\delta_k]. \end{aligned}$$

Hence the monodromy representation is given by:

Proposition 3.3.5. *The monodromy representation of $T_{\ell,\ell+1}$ on $H_1(C, \mathbb{C})_j$ is given with respect to the basis $\{[e_k\delta_k] | k = 1, \ldots n_j + 1\}$ of $H_1(C, \mathbb{C})_j$ by the matrix with the entries:*

$$M_{\ell,\ell+1}(a,b) = \begin{cases} 1 & : \ a = b \quad \textit{and} \quad a \neq \ell \\ \alpha_\ell\alpha_{\ell+1} & : \ a = b = \ell \\ \alpha_\ell(1-\alpha_{\ell+1}) & : \ a = \ell \quad \textit{and} \quad b = \ell - 1 \\ 1 - \alpha_\ell & : \ a = \ell \quad \textit{and} \quad b = \ell + 1 \\ 0 & : \ \textit{elsewhere} \end{cases}$$

Remark 3.3.6. The monodromy representation of Proposition 3.3.5 corresponds to an eigenspace in the local system given by the direct image of the complex homology. By integration over $\mathbb{C}$-valued paths, this eigenspace is the dual local system of $\mathcal{L}_{m-j}$. By the polarization, $\mathcal{L}_j$ is the dual of $\mathcal{L}_{m-j}$, too. Hence Proposition 3.3.5 yields the monodromy representation of $\mathcal{L}_j$.

Chapter 4
The Galois group decomposition of the Hodge structure

In this chapter we make some general observations about the VHS of $\mathcal{C} \to \mathcal{P}_n$ and its generic Hodge group. Moreover we will give an upper bound for the generic Hodge group and a sufficient criterion for dense sets of complex multiplication fibers.

Let ξ be a primitive m-th. root of unity and $r < m$ be a divisor of m. Recall that a fiber C of one of our families $\pi : \mathcal{C} \to \mathcal{P}_n$ is given by

$$y^m = (x - a_1)^{d_1} \cdot \ldots \cdot (x - a_{n+3})^{d_{n+3}}.$$

By the Galois group action we have a decomposition of $H^1(C, \mathbb{Q})$ into subspaces $N^1(C_r, \mathbb{Q})$ such that the Galois group action endows $N^1(C_r, \mathbb{Q})$ with the structure of a $\mathbb{Q}(\xi^r)$-vector space as we see in Section 4.1. In Section 4.2 we see that this decomposition is also a decomposition into sub-Hodge structures, which are closely related to the quotients C_r of C. By the centralizer of the Galois group action, we obtain an upper bound for the generic Hodge group in Section 4.3. The real valued points of the centralizer are given by the direct product of the unitary groups with respect to the Hermitian forms on the eigenspaces $\mathcal{L}_j$ with $j \leq \frac{m}{2}$. By using this description of the centralizer, we define pure $(1, n)$ variations of Hodge structures and show that a family $\mathcal{C}$ with a pure $(1, n) - VHS$ has a dense set of CM fibers in Section 4.4.

4.1 The Galois group representation on the first cohomology

Let $\pi : C \to \mathbb{P}^1$ be a cyclic cover of degree m. The elements of $\mathrm{Gal}(\pi)$ act as $\mathbb{Z}$-module automorphisms on $H^1(C, \mathbb{Z})$. This induces a faithful representation

$$\rho^1 : \mathrm{Gal}(\pi) \to \mathrm{GL}(H^1(C, \mathbb{Q})). \tag{4.1}$$

J.C. Rohde, *Cyclic Coverings, Calabi-Yau Manifolds and Complex Multiplication*, Lecture Notes in Mathematics 1975, DOI: 10.1007/978-3-642-00639-5_5,

By the Galois group representation of a cyclic cover of degree m, we have the following eigenspace decomposition:

$$H^1(C,\mathbb{Q}) \otimes \mathbb{Q}(\xi) = H^1(C,\mathbb{Q}(\xi)) = \bigoplus_{i=1}^{m-1} H^1_j(C,\mathbb{Q}(\xi))$$

Recall that $\pi : C \to \mathbb{P}^1$ is given by some fibers of a family $\pi : \mathcal{C} \to \mathcal{P}_n$. The monodromy representation of $R^1\pi_*(\mathbb{C})$ has a decomposition into subrepresentations on the different eigenspaces. In general there is no $\mathbb{Q}(\xi)$ structure on $H^1(C,\mathbb{Q})$, which turns $H^1(C,\mathbb{Q})$ into a $\mathbb{Q}(\xi)$-vector space. But in this section we will see that $H^1(C,\mathbb{Q})$ has a direct sum decomposition into sub-vector spaces with different $\mathbb{Q}(\xi^r)$ structures, where $r|m$. Moreover we will see that the monodromy representation respects the different $\mathbb{Q}(\xi^r)$ structures, which we will study.

Let ψ denote a generator of $\mathrm{Gal}(\pi)$ as in Chapter 2. The characteristic polynomial of $\rho^1(\psi)$ decomposes into the product of the minimal polynomials of the different ξ^r, where $r|m$ and ξ is a m-th. primitive root of unity. By [33], Satz 12.3.1., we have a decomposition of $H^1(C,\mathbb{Q})$ into subvector spaces $N^1(C_r,\mathbb{Q})$ such that the $\mathbb{Q}$-vector space automorphism $\rho^1(\psi)|_{N^1(C_r,\mathbb{Q})}$ is (up to conjugation) given by a matrix

$$\begin{pmatrix} M & & 0 \\ & \ddots & \\ 0 & & M \end{pmatrix},$$

where M is the $k \times k$ matrix given by

$$M = \begin{pmatrix} 0 & 0 & \dots & 0 & -p_0 \\ 1 & 0 & \dots & 0 & -p_1 \\ 0 & 1 & \ddots & 0 & -p_2 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & -p_{k-1} \end{pmatrix},$$

where $x^k + p_{k-1}x^{k-1} + \ldots + p_1x + p_0$ is the minimal polynomial of ξ^r.[1] We call a $\mathbb{Q}$-vector space with such an automorphism of the form $\mathrm{diag}(M,\ldots,M)$ a $\mathbb{Q}(\xi^r)$-structure. By $\xi^r \cdot v := g(v)$, this defines a scalar multiplication of $\mathbb{Q}(\xi^r)$, which turns $N^1(C_r,\mathbb{Q})$ into a $\mathbb{Q}(\xi^r)$-vector space. We obtain:

[1] In the next section we will see that there is a correspondence between the covers C_r and the subvector spaces $N^1(C_r,\mathbb{Q})$, which justifies this notation.

Proposition 4.1.1. *The direct sum decomposition*

$$H^1(C,\mathbb{Q}) = \bigoplus_{r|m} N^1(C_r,\mathbb{Q})$$

is a direct sum of $\mathbb{Q}(\xi^r)$*-structures.*

Next we consider the trace map

$$\mathrm{tr} : H^1_j(C,\mathbb{Q}(\xi)) \to H^1(C,\mathbb{Q}) \text{ given by } v \to \sum_{\gamma\in \mathrm{Gal}(\mathbb{Q}(\xi);\mathbb{Q})} \gamma v,$$

which will be one of our main tools in this chapter. By the Galois group action, the vector space $N^1(C_r,\mathbb{Q}(\xi^r))$ decomposes into eigenspaces $H^1_j(C,\mathbb{Q}(\xi^r)))$ such that

$$H^1_j(C,\mathbb{Q}(\xi)) = H^1_j(C,\mathbb{Q}(\xi^r)) \otimes_{\mathbb{Q}(\xi^r)} \mathbb{Q}(\xi).$$

Lemma 4.1.2. *Let* $r|m$ *and* $r = \gcd(j,m)$*. Then* $\mathrm{tr}|_{H^1_j(C,\mathbb{Q}(\xi^r))}$ *is a monomorphism.*

Proof. Let $f \in H^1_j(C,\mathbb{Q}(\xi^r)) \setminus \{0\}$. We need some Galois theory. By the fact that $\mathbb{Q}(\xi^r)$ is a Galois extension of $\mathbb{Q}$, the group $\Gamma_r := \mathrm{Aut}(\mathbb{Q}(\xi);\mathbb{Q}(\xi^r))$ is a normal subgroup of $(\mathbb{Z}/(m))^* \cong \Gamma := \mathrm{Gal}(\mathbb{Q}(\xi);\mathbb{Q})$, which is the kernel of the epimorphism $\Gamma \to \mathrm{Gal}(\mathbb{Q}(\xi^r);\mathbb{Q})$ given by $\gamma \to \gamma|_{\mathbb{Q}(\xi^r)}$ for all $\gamma \in \mathrm{Gal}(\mathbb{Q}(\xi);\mathbb{Q})$. Hence we obtain that

$$\mathrm{tr}(f) = \sum_{\gamma\in \mathrm{Gal}(\mathbb{Q}(\xi);\mathbb{Q})} \gamma f = \sum_{[\gamma]\in\Gamma/\Gamma_r} [\gamma] \sum_{\gamma\in\Gamma_r} \gamma f = \sum_{\gamma\in \mathrm{Gal}(\mathbb{Q}(\xi^r);\mathbb{Q})} \gamma|\Gamma_r| f.$$

Since ψ acts by an integral matrix, one has $\gamma\circ\psi = \psi\circ\gamma$ for all $\gamma\in\Gamma$. This implies that

$$\gamma(\xi^r)\gamma(f) = \gamma(\xi^r f) = (\gamma\circ\psi)(f) = \psi(\gamma f). \tag{4.2}$$

Thus $\gamma(f) \in H^1_{j_0 j}(C,\mathbb{Q}(\xi))$, where $j_0 \in (\mathbb{Z}/(m))^*$ corresponds to γ. By the fact that we have a direct sum of eigenspaces, we conclude that

$$\mathrm{tr}(f) = \sum_{\gamma\in \mathrm{Gal}(\mathbb{Q}(\xi^r);\mathbb{Q})} \gamma|\Gamma_r| f \neq 0.$$

□

Now we consider the restriction of the trace map to

$$R := \bigoplus_{r|m} H^1_r(C,\mathbb{Q}(\xi^r)).$$

Proposition 4.1.3. *The trace map* $\mathrm{tr}|_R : R \to H^1(C,\mathbb{Q})$ *is an isomorphism of* $\mathbb{Q}$*-vector spaces.*

Proof. Let

$$v := \sum_{r|m} v_r \in R$$

with $v_r \in H^1_r(C,\mathbb{Q}(\xi^r))$. By the proof of the preceding lemma, we know that

$$\mathrm{tr}(v_r) = \sum_{\gamma\in \mathrm{Gal}(\mathbb{Q}(\xi^r);\mathbb{Q})} \gamma|\Gamma_r|v_r \in \bigoplus_{j\in(\mathbb{Z}/(\frac{m}{r}))^*} H^1_j(C,\mathbb{Q}(\xi)).$$

These ξ^{jr} with $j \in (\mathbb{Z}/(\frac{m}{r}))^*$ are exactly the $\frac{m}{r}$-th. primitive roots of unity. Thus they are the elements with order $\frac{m}{r}$ in the multiplicative group generated by ξ. Hence by the fact that we have a direct sum of eigenspaces, we conclude that $\mathrm{tr}(v) = 0$ implies that $\mathrm{tr}(v_r) = 0$ for all r with $r|m$. By the preceding lemma, this implies that $v_r = 0$ for all r with $r|m$. Hence $v = 0$. Thus the map $\mathrm{tr}|_R$ is injective, and we have only to verify that $\dim_{\mathbb{Q}}(R) = \dim_{\mathbb{Q}}(H^1(C,\mathbb{Q}))$:

$$\dim_{\mathbb{Q}} R = \sum_{r|m} \dim_{\mathbb{Q}(\xi)}(H^1_r(C,\mathbb{Q}(\xi))) \cdot [\mathbb{Q}(\xi^r);\mathbb{Q}]$$

$$= \sum_{r|m} \dim_{\mathbb{Q}(\xi)}(H^1_r(C,\mathbb{Q}(\xi))) \cdot \sharp\{\text{primitive } \frac{m}{r}\text{-th. roots of unity}\}$$

$$= \sum_{j=1}^{m-1} \dim_{\mathbb{Q}(\xi)}(H^1_j(C,\mathbb{Q}(\xi))) = \dim_{\mathbb{Q}(\xi)}(H^1(C,\mathbb{Q}(\xi))) = \dim_{\mathbb{Q}}(H^1(C,\mathbb{Q}))$$

□

Remark 4.1.4. We know that the monodromy representation fixes $H^1(C,\mathbb{Q})$ and each $H^1_j(C,\mathbb{Q}(\xi))$. By the fact that

$$N^1(C_r,\mathbb{Q}) = N^1(C_r,\mathbb{Q}(\xi)) \cap H^1(C,\mathbb{Q}),$$

we conclude that the monodromy representation fixes $N^1(C_r,\mathbb{Q}))$, too.

Proposition 4.1.5. *The monodromy representation* ρ *on* $N^1(C_r,\mathbb{Q})$ *is given by*

$$\rho(\omega) = \begin{pmatrix} \gamma_1 M_\omega & & \\ & \ddots & \\ & & \gamma_k M_\omega \end{pmatrix},$$

where M_ω *denotes the image of* ω *in the monodromy of* $H^1_r(C,\mathbb{Q}(\xi^r))$*, and*

$$\{\gamma_1,\ldots,\gamma_k\} = \mathrm{Gal}(\mathbb{Q}(\xi^r);\mathbb{Q}).$$

Proof. Since $\rho(\gamma)$ fixes the eigenspaces, it acts by $\mathrm{diag}(M_1, \ldots, M_k)$, where each M_ℓ with $1 \leq \ell \leq k$ describes the action of $\rho(\omega)$ on $\gamma_\ell H^1_r(C, \mathbb{Q}(\xi^r))$. Let $j_\gamma \in (\mathbb{Z}/(\frac{m}{r}))^*$ and γ correspond. The description of the $M_1, \ldots, M_k$ follows from the facts that each $\rho(\omega)$ commutes with each $\gamma \in \mathrm{Gal}(\mathbb{Q}(\xi^r); \mathbb{Q})$, and that $\gamma H^1_r(C, \mathbb{Q}(\xi^r)) = H^1_{rj_\gamma}(C, \mathbb{Q}(\xi^r))$ (see (4.2)) for all $\gamma \in \mathrm{Gal}(\mathbb{Q}(\xi^r); \mathbb{Q})$. □

Now let N_ω denote the restriction of $\rho(\omega)$ on $N^1(C_r, \mathbb{Q})$ and $v \in N^1(C_r, \mathbb{Q})$ given by $v = \mathrm{tr}(w)$ for some $w \in H^1_r(C, \mathbb{Q}(\xi^r))$. By the preceding proposition, we have:

$$N_\omega(v) = N_\omega([\mathbb{Q}(\xi); \mathbb{Q}(\xi^r)] \sum_{\gamma \in \mathrm{Gal}(\mathbb{Q}(\xi^r);\mathbb{Q})} \gamma w) = [\mathbb{Q}(\xi); \mathbb{Q}(\xi^r)] \sum_{i=1}^{k} \gamma_i M_\omega(\gamma_i(w))$$

$$= [\mathbb{Q}(\xi); \mathbb{Q}(\xi^r)] \sum_{i=1}^{k} \gamma_i(M_\omega(w)) = tr(M_\omega(w))$$

The trace map $H^1_r(C, \mathbb{Q}(\xi^r)) \to N^1(C_r, \mathbb{Q})$ is an isomorphism of $\mathbb{Q}(\xi^r)$-vector spaces with respect to the $\mathbb{Q}(\xi^r)$ structure on $N^1(C_r, \mathbb{Q})$. Thus one has:

Proposition 4.1.6. *The monodromy representation on $N^1(C_r, \mathbb{Q})$ is a representation on a $\mathbb{Q}(\xi^r)$-vector space given by the $\mathbb{Q}(\xi^r)$ structure, which coincides up to the trace map with the monodromy representation on $H^1_r(C, \mathbb{Q}(\xi^r))$.*

We will need a decomposition of $H^1(C, \mathbb{R})$ into a direct sum of certain sub-vector spaces fixed by the Galois group representation. This decomposition is defined over

$$\mathbb{Q}(\xi^j)^+ = \mathbb{Q}(\xi^j) \cap \mathbb{R}$$

and given by the sub-vector spaces

$$\Re\mathbb{V}(j) := (H^1_j(C, \mathbb{Q}(\xi)) \oplus H^1_{m-j}(C, \mathbb{Q}(\xi))) \quad \cap \quad H^1(C, \mathbb{Q}(\xi^j)^+).$$

Since the monodromy representation fixes

$$H^1_j(C, \mathbb{Q}(\xi)), \quad H^1_{m-j}(C, \mathbb{Q}(\xi)) \quad \text{and} \quad H^1(C, \mathbb{Q}(\xi^j)^+),$$

it fixes $\Re\mathbb{V}(j)$, too.

Remark 4.1.7. One has that $\mathrm{tr} : H^1_j(C, \mathbb{Q}(\xi^j)) \to N^1(C_j, \mathbb{Q})$ coincides with the composition

$$H^1_j(C, \mathbb{Q}(\xi^j)) \xrightarrow{\mathrm{tr}} \Re\mathbb{V}(j) \xrightarrow{\mathrm{tr}} N^1(C_j, \mathbb{Q}).$$

Hence the latter trace map $\Re\mathbb{V}(j) \xrightarrow{\mathrm{tr}} N^1(C_j, \mathbb{Q})$ induces a $\mathbb{Q}(\xi^j)^+$-structure on $N^1(C_j, \mathbb{Q})$, which is compatible with the $\mathbb{Q}(\xi^j)$-structure via

$\mathbb{Q}(\xi^j)^+ \hookrightarrow \mathbb{Q}(\xi^j)$. Thus by the preceding results about the monodromy representation on $N^1(C_j,\mathbb{Q})$, the monodromy representation on $N^1(C_j,\mathbb{Q})$ is a $\mathbb{Q}(\xi^j)^+$-vector space representation with respect to the $\mathbb{Q}(\xi^j)^+$-structure.

Remark 4.1.8. In the case of $H^1_{\frac{m}{2}}(C,\mathbb{Q}(\xi^{\frac{m}{2}}))$ one gets that $\mathbb{Q}(\xi^{\frac{m}{2}}) = \mathbb{Q}(-1) = \mathbb{Q}$. In other terms: The monodromy group on $H^1_{\frac{m}{2}}(C,\mathbb{Q}(\xi^{\frac{m}{2}}))$ is the monodromy group on the rational vector space $N^1(C_{\frac{m}{2}},\mathbb{Q})$.

4.2 Quotients of covers and Hodge group decomposition

In this section we consider our quotient families $\pi_r : \mathcal{C}_r \to \mathcal{P}_n$ of covers, and their Hodge groups. Moreover we will explain the notation $N^1(C_r,\mathbb{Q})$ and show that the decomposition of $H^1(C,\mathbb{Q})$ into these $\mathbb{Q}(\xi^r)$ structures is a decomposition into rational sub-Hodge structures. Recall that $\mathcal{C}_r$ is given by a quotient of the subgroup of order r of the Galois group of $\mathcal{C}$ (see Construction 3.2.1).

Let C and C_r denote a fiber of $\mathcal{C}$ and the corresponding fiber of $\mathcal{C}_r$ over the same point. The natural cover $\phi_r : C \to C_r$ induces an embedding of Hodge structures, which gives a direct sum decomposition of $H^1(C,\mathbb{Q})$ into two rational sub-Hodge structures (see [61], 7.3.2 and [61], Lemme 7.26).

The Hodge structure on $H^1(C_r,\mathbb{Q})$ is the sub-Hodge structure of $H^1(C,\mathbb{Q})$ fixed by $\mathrm{Gal}(\phi_r)$. Hence the eigenspaces of $H^1(C_r,\mathbb{C})$ with respect to the Galois group π_r can be identified with the eigenspaces of $H^1(C,\mathbb{C})$, on which $\mathrm{Gal}(\phi_r)$ acts trivial. Thus one obtains

$$H^1(C_r,\mathbb{C}) = \bigoplus_{j=1}^{\frac{m}{r}-1} H^1_{jr}(C,\mathbb{C}) \hookrightarrow \bigoplus_{j=1}^{m-1} H^1_j(C,\mathbb{C}) = H^1(C,\mathbb{C}).$$

Recall that every eigenspace $\mathcal{L}_j$ of $R^1\pi_*(\mathbb{C})$ is a local system. We consider the eigenspace $(\mathcal{L}_j)_{\mathcal{C}_r}$ of $R^1(\pi_r)_*(\mathbb{C})$ with the character j and the eigenspace $\mathcal{L}_{jr}$ of $R^1\pi_*(\mathbb{C})$. Proposition 2.2.11 tells us that the local monodromy data of $(\mathbb{L}_j)_{C_r}$ and $\mathbb{L}_{rj}$ coincide. By Proposition 3.3.5, these monodromy data determine the dual monodromy representations of the eigenspaces of the dual VHS given by the homology. Thus we obtain:

Proposition 4.2.1. *The local systems $(\mathcal{L}_j)_{\mathcal{C}_r}$ and $\mathcal{L}_{jr}$ coincide.*

The following statements will explain the notation “$N^1(C_r,\mathbb{Q})$”. One has that

$$N^1(C_r,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{j\in(\mathbb{Z}/\frac{m}{r})^*} H^1_{jr}(C,\mathbb{C}).$$

Since each $H^1_{jr}(C,\mathbb{C}) \subset N^1(C_r,\mathbb{C})$ has a decomposition into

$$H_j^{1,0}(C_r) \oplus H_j^{0,1}(C_r), \quad \text{where} \quad \overline{H_j^1(C_r,\mathbb{C})} = H^1_{m-j}(C_r,\mathbb{C}) \subset N^1(C_r,\mathbb{C}),$$

each $N^1(C_r,\mathbb{Q})$ is a rational sub-Hodge structure of $H^1(C,\mathbb{Q})$. Moreover each $N^1(C_r,\mathbb{Q})$ is the maximal sub-Hodge structure of $H^1(C_r,\mathbb{Q})$, which is orthogonal (with respect to the polarization) to each sub-Hodge structure of $H^1(C_r,\mathbb{Q})$ given by a quotient $H^1(C_{r'},\mathbb{Q})$ with $r < r' < m$, $r|r'$ and $r'|m$. By using Lemma 2.4.1, we have the result:

Proposition 4.2.2. *We have a decomposition*

$$H^1(C,\mathbb{Q}) = \bigoplus_{r|m} N^1(C_r,\mathbb{Q})$$

into rational Hodge structures and a natural embedding

$$\mathrm{Hg}(C) \hookrightarrow \prod_{r|m} \mathrm{Hg}(N^1(C_r,\mathbb{Q}))$$

such that the natural projections

$$\mathrm{Hg}(C) \to \mathrm{Hg}(N^1(C_r,\mathbb{Q}))$$

are surjective for all r.

Remark 4.2.3. Note that the preceding section yields a corresponding statement about the Zariski closures of the monodromy group of $R^1\pi_*(\mathbb{Q})$ and the restricted representations monodromy representations on the different $N^1(C_r,\mathbb{Q})$. These two facts will play a very important role.

4.3 Upper bounds for the Mumford-Tate groups of the direct summands

The different $N^1(C_r,\mathbb{Q})$ on the fibers induce a decomposition of $R^1\pi_*(\mathbb{Q})$ into a direct sum of local systems $\mathcal{N}^1(\mathcal{C}_r,\mathbb{Q})$. Now we consider the induced variations $\mathcal{V}_r$ of rational Hodge structures on the local systems $\mathcal{N}^1(\mathcal{C}_r,\mathbb{Q})$. Let Q_r denote the alternating form on $N^1(C_r,\mathbb{Q})$ obtained by the restriction of the intersection form Q of the curve C. One has that each element of $\rho(\pi_1(\mathcal{P}_n))$ commutes with the Galois group. The same holds true for the image of the homomorphism

$$h : \mathbb{S} \to \mathrm{GSp}(H^1(C,\mathbb{R}),Q)$$

corresponding to the Hodge structure of an arbitrary fiber. Since the Galois group respects the intersection form, its representation on $N^1(C_r, \mathbb{Q})$ is contained in $\mathrm{Sp}(N^1(C_r, \mathbb{Q}), Q_r)$. Let $C_r(\psi)$ denote the centralizer of the Galois group in $\mathrm{Sp}(N^1(C_r, \mathbb{Q}), Q_r)$ and $GC_r(\psi)$ denote the centralizer of the Galois group in $\mathrm{GSp}(N^1(C_r, \mathbb{Q}), Q_r)$. One concludes:

Proposition 4.3.1. *The centralizer $GC_r(\psi)$ contains the generic Mumford-Tate group $\mathrm{MT}(\mathcal{V}_r)$. Moreover the centralizer $C_r(\psi)$ contains the generic Hodge group $\mathrm{Hg}(\mathcal{V}_r)$ and $\mathrm{Mon}^0(\mathcal{V}_r)$.*

We write

$$C(\psi) := \prod_{r|m} C_r(\psi).$$

Remark 4.3.2. If $r \neq \frac{m}{2}$, the preceding proposition yields some information. In the case $r = \frac{m}{2}$ the elements of the Galois group act as the multiplication with 1 or -1 on $N^1(C_{\frac{m}{2}}, \mathbb{Q})$. Since id resp., $-\mathrm{id}$ is contained in the center of $\mathrm{Sp}(N^1(C_{\frac{m}{2}}, \mathbb{Q}), Q_{\frac{m}{2}})$, this proposition does not give any new information in this case.

Now let us assume that $r \neq \frac{m}{2}$. We describe $C_r(\psi)$ by its $\mathbb{R}$-valued points. Let ξ^j be a $\frac{m}{r}$-th. primitive root of unity such that $H^1_j(C, \mathbb{C}) \subset N^1(C_r, \mathbb{C})$, $v \in H^1_j(C, \mathbb{C})$ and $M \in C_r(\psi)(\mathbb{R})$. Then one gets

$$\psi M(v) = M(\psi v) = M(\xi^j v) = \xi^j M(v).$$

Thus M leaves each $H^1_j(C, \mathbb{C})$ invariant.

For our description of $C_r(\psi)$ we introduce the trace map

$$\mathrm{tr} : \mathrm{GL}(H^1_j(C, \mathbb{C})) \to \mathrm{GL}(\Re\mathbb{V}(j)_{\mathbb{R}})$$

given by

$$\mathrm{GL}(H^1_j(C, \mathbb{C})) \ni N \to N \times \bar{N} \in \mathrm{GL}(H^1_j(C, \mathbb{C})) \times \mathrm{GL}(H^1_{m-j}(C, \mathbb{C})), \quad (4.3)$$

where $\bar{N}$ denotes the matrix, which satisfies that $\bar{N}\bar{v} = \overline{Nv}$ for all $v \in H^1_j(C, \mathbb{C})$. Recall that we have a fixed complex structure. Thus one checks easily that $N \times \bar{N}$ leaves $\Re\mathbb{V}(j)_{\mathbb{R}}$ invariant. Hence we consider it as a real matrix.

For the Hermitian form $H(\cdot, \cdot) := iE(\cdot, \bar{\cdot})$ and $v, w \in H^1_j(C, \mathbb{C})$ one obtains

$$H(v, w) = iE(v, \bar{w}) = iE(Mv, M\bar{w}) = iE(Mv, \overline{Mw}) = H(Mv, Mw).$$

Thus the matrix $M|_{\Re\mathbb{V}(j)_{\mathbb{R}}}$ is contained in $\mathrm{tr}(\mathrm{U}(H^1_j(C, \mathbb{C}), H|_{H^1_{m-j}(C,\mathbb{C})}))$.

Assume conversely that $M \in \mathrm{GL}(N^1(C_r,\mathbb{C}))$ satisfies that

$$M|_{\Re\mathbb{V}(j)_{\mathbb{R}}} \in \mathrm{tr}(\mathrm{U}(H^1_j(C,\mathbb{C}), H|_{H^1_j(C,\mathbb{C})}))$$

for each $\frac{m}{r}$-th. primitive root of unity ξ^j. Since M fixes all $H^1_j(C,\mathbb{C}) \subset N^1(C_r,\mathbb{C})$, it commutes with the Galois group representation on $N^1(C_r,\mathbb{R})$. Now let $N \in \mathrm{GL}(H^1_j(C,\mathbb{C}))$ be the matrix with $\mathrm{tr}(N) = M|_{\Re\mathbb{V}(j)_{\mathbb{R}}}$. One has that

$$iE(v,\bar{w}) = iE(Nv,\overline{Nw}) \Leftrightarrow E(v,\bar{w}) = E(Nv,\overline{Nw})$$

for all $v,w \in H^1_j(C,\mathbb{C})$. By the fact that E is an alternating form, one gets

$$E(\bar{v},w) = E(\overline{Nv},Nw),$$

too. The elements of $\Re\mathbb{V}(j)_{\mathbb{C}}$ are given by $v_1+\bar{v}_2$ and $w_1+\bar{w}_2$ with

$$v_1,v_2,w_1,w_2 \in H^1_j(C,\mathbb{C}).$$

Thus one concludes that

$$\begin{aligned} E(v_1+\bar{v}_2,w_1+\bar{w}_2) &= E(v_1,\bar{w}_2)+E(\bar{v}_2,w_1) = E(Nv_1,\overline{Nw_2})+E(\overline{Nv_2},Nw_1) \\ &= E(Mv_1,M\bar{w}_2)+E(M\bar{v}_2,Mw_1) = E(M(v_1+\bar{v}_2),M(w_1+\bar{w}_2)). \end{aligned}$$

Hence M is contained in the symplectic group. Altogether we conclude:

Theorem 4.3.3. *If $r \neq \frac{m}{2}$, the group Lie $C_r(\psi)(\mathbb{R})$ is isomorphic to the direct product of the Lie groups given by the $\mathbb{R}$-valued points of the unitary groups over $\Re\mathbb{V}(j)_{\mathbb{R}} \subset N^1(C_r,\mathbb{R})$ induced by the trace maps and the unitary groups $\mathrm{U}(H^1_j(C,\mathbb{C}),H|_{H^1_j(C,\mathbb{C})})$.*

Recall the definition of the type (a,b) of an eigenspace $\mathcal{L}_j$ in Remark 3.2.5. If there is an eigenspace of $N^1(C_r,\mathbb{C})$ of type (a,b) with $a>0$ and $b>0$, we call $N^1(C_r,\mathbb{Q})$ general. Otherwise we call it special. Now assume that $N^1(C_r,\mathbb{Q})$ is special. In this case $h(\mathbb{S})$ is contained in the center of $GC_r(\psi)_{\mathbb{R}}$, and $h(S^1)$ is contained in the center of $C_r(\psi)_{\mathbb{R}}$. Thus one concludes:

Remark 4.3.4. Assume that $N^1(C_r,\mathbb{Q})$ is special. Then the center $Z(GC_r(\psi))$ of $GC_r(\psi)$ contains $\mathrm{MT}(\mathcal{V}_r)$. Moreover the center $Z(C_r(\psi))$ of $C_r(\psi)$ contains $\mathrm{Hg}(\mathcal{V}_r)$.

Remark 4.3.5. One has that $C_r(\psi)_{\mathbb{R}}$ consists of $\mathrm{U}(s)^t$ for some $s,t \in \mathbb{N}_0$, if $N^1(C_r,\mathbb{Q})$ is special. Thus in this case the monodromy group is a discrete sub-group of the compact group $\mathrm{U}(s)^t$. Hence it is finite and $\mathrm{Mon}^0(\mathcal{V}_r)$ is trivial in this case.

4.4 A criterion for complex multiplication

In this short section we find a sufficient condition for the existence of a dense set of CM fibers of a family of cyclic covers. By technical reasons, we do not consider the family $\mathcal{C} \to \mathcal{P}_n$, but a family over the space $\mathcal{M}_n$, which can be considered as the quotient

$$\mathcal{M}_n = \mathcal{P}_n/\mathrm{PGL}_2(\mathbb{C}).$$

One has an embedding $\iota_{a,b,c} : \mathcal{M}_n \to \mathcal{P}_n$, too. Its image is the subspace of $\mathcal{P}_n$, which parametrizes the maps $\phi : N \to \mathbb{P}^1$ satisfying $\phi(a) = 0$, $\phi(b) = 1$ and $\phi(c) = \infty$ for some fixed $a, b, c \in N$ (compare to [18], 3.7).

Remark 4.4.1. One can move 3 arbitrary branch points of a fiber of $\mathcal{C} \to \mathcal{P}_n$ to 0, 1 and ∞. Hence one has that all fibers of the geometric points of $\mathcal{P}_n$ occur as fibers of the restricted family $\mathcal{C}_{\mathcal{M}_n} \to \mathcal{M}_n$, too. Hence the generic Hodge groups and the generic Mumford-Tate groups of the both families coincide.

4.4.2. Each curve C with $g(C) > 1$ has at most $84(g-1)$ automorphisms (see [26], **IV**. Exercise 2.5). Thus C can have only finitely many cyclic covering maps onto $\mathbb{P}^1$ with different Galois groups. Moreover, there is an automorphism α of $\mathbb{P}^1$, if the Galois groups of the covers of $\mathcal{C}_{p_1}$ and $\mathcal{C}_{p_2}$ can be conjugate by an isomorphism ι such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{C}_{p_1} \arrow[r, "\iota"] \arrow[d] & \mathcal{C}_{p_2} \arrow[d] \\ \mathbb{P}^1 \arrow[r, "\alpha"] & \mathbb{P}^1 \end{tikzcd}$$

Thus C occurs only as finitely many fibers of $\mathcal{C}_{\mathcal{M}_n}$, if $g(C) \geq 2$.

Recall that we have defined the type of an eigenspace $\mathcal{L}_j$ in Remark 3.2.5.

Definition 4.4.3. A family $\pi : \mathcal{C} \to \mathcal{P}_n$ of cyclic covers has a pure $(1, n) - VHS$, if it has its VHS only one eigenspace $\mathcal{L}_j$ of type $(1, n)$ such that $\mathcal{L}_{m-j}$ is of type $(n, 1)$, and all other eigenspaces are of type $(a, 0)$ or of type $(0, b)$ for some $a, b \in \mathbb{N}_0$.

Theorem 4.4.4. *Let $\mathcal{C}_{\mathcal{M}_n} \to \mathcal{M}_n$ be a family of cyclic covers of $\mathbb{P}^1$ and C be a fiber with $g(C) \geq 2$ as before. Assume that $\mathcal{C}$ has a pure $(1, n) - VHS$. Then the family $\mathcal{C}_{\mathcal{M}_n} \to \mathcal{M}_n$ has a dense set of complex multiplication fibers.*

Proof. We have to show that over an arbitrary open simply connected subset W of $\mathcal{M}_n(\mathbb{C})$ there are infinitely many CM points of the VHS of $\mathcal{C}_{\mathcal{M}_n}$. Let $q_0 \in W$ and $\mathcal{L}_j$ be the eigenspace of type $(1, n)$. We have a trivialization

$$R^1\pi_*(\mathbb{C})|_W = H^1(\mathcal{C}_{q_0}, \mathbb{C}) \times W \quad \text{such that} \quad \mathcal{L}_j|_W \cong H^1_j(\mathcal{C}_{q_0}, \mathbb{C}) \times W.$$

Let $q \in W$ and $\varpi^{(j)}_q \in H^{1,0}_j(\mathcal{C}_q)\setminus\{0\}$. By the holomorphic VHS of the family, one obtains a holomorphic "fractional period" map

$$p : W \to \mathbb{P}(H^1_j(\mathcal{C}_{q_0}, \mathbb{C})) \quad \text{via} \quad q \to [\varpi^{(j)}_q].$$

By the assumptions, the integral Hodge structure depends uniquely on the class $[\varpi^{(j)}_q] \in \mathbb{P}(H^1_j(\mathcal{C}_{q_0}, \mathbb{C}))$. Since for each fiber there are only finitely many isomorphic fibers (see 4.4.2) and two curves have isomorphic polarized integral Hodge structures, if and only if they are isomorphic, the fibers of p have the dimension 0. Thus [49], Chapter **VII**. Proposition 4 and the fact that

$$\dim W = \dim \mathbb{P}(H^1_j(\mathcal{C}_{q_0}, \mathbb{C}))$$

tell us that p is open.

The natural embedding $C(\psi) \hookrightarrow \mathrm{GL}(H^1(\mathcal{C}_{q_0}, \mathbb{C}))$ induces a holomorphic variation of Hodge structures over the bounded symmetric domain associated with $C(\psi)(\mathbb{R})/K$ (see Theorem 1.4.11). This VHS depends uniquely on the fractional VHS on the eigenspace $H^1_j(\mathcal{C}_{q_0}, \mathbb{C})$ of type $(1, n)$. Hence this VHS yields a holomorphic injection

$$\varphi : C(\psi)(\mathbb{R})/K \to \mathbb{P}(H^1_j(\mathcal{C}_{q_0}, \mathbb{C})).$$

Recall that that homogeneous space $C(\psi)(\mathbb{R})/K$ parametrizes the integral Hodge structures of type $(1,0), (0,1)$ on $H^1(\mathcal{C}_{q_0}, \mathbb{C})$, whose Hodge group is contained in $C(\psi)$. Hence altogether the map $\varphi^{-1} \circ p$, which assigns to each fiber $\mathcal{C}_q$ its integral Hodge structure, is open. Since the set of CM points on $C(\psi)(\mathbb{R})/K$ is dense (see Theorem 1.7.2), this yields the desired statement. $\square$

Chapter 5
The computation of the Hodge group

Recall that $\mathcal{P}_n$ is the configuration space of $n+3$ points and $\mathcal{M}_n = \mathcal{P}_n/\mathrm{PGL}_2(\mathbb{C})$. In this chapter we try to compute the derived group $\mathrm{Hg}^{\mathrm{der}}(\mathcal{V})$ of the generic Hodge group of a family $\mathcal{C} \to \mathcal{P}_n$ by using $\mathrm{Mon}^0(\mathcal{V})$. We will get many information and for infinitely many examples we will obtain

$$\mathrm{MT}^{\mathrm{der}}(\mathcal{V}) = \mathrm{Hg}^{\mathrm{der}}(\mathcal{V}) = \mathrm{Mon}^0(\mathcal{V}) = C^{\mathrm{der}}(\psi).$$

Our motivation is to try to prove that the criterion of Theorem 4.4.4 given by the existence of a pure $(1,n)-VHS$ is also necessary under some additional assumptions. Finally we will see that a family $\mathcal{C} \to \mathcal{M}_1$ induces an open period map

$$p : \mathcal{M}_1(\mathbb{C}) \to \mathrm{MT}^{\mathrm{ad}}(\mathcal{V})/K,$$

if and only if it has a pure $(1,1)-VHS$.

In Section 5.1 we show that for all eigenspaces $\mathcal{L}_j$ of type (p,q) with $p,q>0$ the group $\mathrm{Mon}^0_{\mathbb{R}}(\Re V(j))$ is given by the unitary group of the Hermitian form on $\mathcal{L}_j$ with respect to the polarization, if $j \neq \frac{m}{2}$ or $\mathcal{L}_j$ is of type $(1,1)$. We make some general observations about $\mathrm{Mon}^0(\mathcal{V}_r)$ in Section 5.2. Since $\mathrm{Mon}^0(\mathcal{V}_r) \subset C_r^{\mathrm{der}}(\psi)$, the latter group is an upper bound of $\mathrm{Mon}^0(\mathcal{V}_r)$. For $\mathrm{Mon}^0(\mathcal{V}_r)$ we give a sufficient criterion of the reaching of this upper bound in Section 5.3. In Section 5.4 we consider the exceptional cases, which do not satisfy this sufficient criterion. We see that $Mon^0_{\mathbb{R}}(\mathcal{V}_r)$ is a proper subgroup of $C_r^{\mathrm{der}}(\psi)_{\mathbb{R}}$ in some of these cases. For completeness we show that $\mathrm{Hg}(\mathcal{V}) \cong \mathrm{Sp}_{\mathbb{Q}}(2g)$ in the case of an universal family of hyperelliptic curves of genus g in Section 5.5. In Section 5.6 we collect the previous results and consider $\mathrm{Mon}^0(\mathcal{V})$. W! e finish this section with the proof of the result that a family $\mathcal{C} \to \mathcal{M}_1$ induces an open period map

$$p : \mathcal{M}_1(\mathbb{C}) \to \mathrm{MT}^{\mathrm{ad}}(\mathcal{V})/K,$$

if and only if it has a pure $(1,1)-VHS$.

J.C. Rohde, *Cyclic Coverings, Calabi-Yau Manifolds and Complex Multiplication*, Lecture Notes in Mathematics 1975, DOI: 10.1007/978-3-642-00639-5_6,

5.1 The monodromy group of an eigenspace

Let $j \in \{1, \dots, m-1\}$. Then we have an eigenspace $\mathcal{L}_j$ in the variation of Hodge structures of a family $\mathcal{C} \to \mathcal{P}_n$ of cyclic degree m covers of $\mathbb{P}^1$. There are $p, q \in \mathbb{N}_0$ such that the eigenspace $H^1_j(C, \mathbb{C})$ of an arbitrary fiber C is of type (p, q), where (p, q) is the signature of the restricted polarization of the latter eigenspace. The type of $\mathcal{L}_j$ is given by the type of $H^1_j(C, \mathbb{C})$. The embedding $\mathbb{R} \hookrightarrow \mathbb{C}$ allows to consider $H^1_j(C, \mathbb{C})$ as $\mathbb{R}$-vector space. Let $\mathrm{Mon}^0(\mathcal{L}_j)$ denote the identity component of the Zariski closure of the monodromy group of $\mathcal{L}_j$ in $\mathrm{GL}_{\mathbb{R}}(H^1_j(C, \mathbb{C}))$.

We show in this section:

Theorem 5.1.1. *Let $\mathcal{L}_j$ be of type (p, q) with $p, q \geq 1$. Moreover assume that $j \neq \frac{m}{2}$ or $p = q = 1$. Then*

$$\mathrm{Mon}^0(\mathcal{L}_j) = \mathrm{SU}(p, q).$$

If $p = 0$ or $q = 0$, the statement of the preceding theorem does not hold true in general as one can conclude by Remark 4.3.5.

We give a proof of Theorem 5.1.1 by induction over the integer given by $p + q$.

By the following lemma, we start the proof of Theorem 5.1.1:

Lemma 5.1.2. *If $\mathcal{L}_j$ is of type $(1, 1)$, its monodromy group contains infinitely many elements.*

Proof. There are two cases: In the first case there are some local monodromy data α_1 and α_2 of the eigenspace $\mathbb{L}_j$ in $(\pi_q)_*(\mathbb{C}_C)|_{\mathbb{P}^1 \setminus S_j}$ for the fiber $C := \mathcal{C}_q$ of some arbitrary $q \in \mathcal{P}_n$ such that $\alpha_1 \alpha_2 = 1$. In this case the Dehn twist $T_{1,2}$ yields a unipotent triangular matrix (follows by Proposition 3.3.5) and we are done.

Otherwise each Dehn twist $T_{k,\ell}$ provides a semisimple matrix, where its eigenvalues are given by 1 and a m-th. root of unity. Note that the matrices induced by the Dehn twists $T_{1,2}$ and $T_{2,3}$ do not commute. In the considered case $\{a_1, a_2\}, \{a_3, a_4\}$ is a stable partition. Hence one can choose the basis $\mathcal{B} = \{[e_1\gamma_1], [e_3\gamma_3]\}$ of $H_1^j(C, \mathbb{C})$. By the fact that these two cycles do not intersect each other, this basis is orthogonal with respect to the Hermitian form induced by the intersection form. Hence by normalization, this basis is orthonormal with respect to the Hermitian form such that the Hermitian form is without loss of generality given by $\mathrm{diag}(1, -1)$ with respect to $\mathcal{B}$. The matrix induced by $T_{1,2}$ is given by $\mathrm{diag}(\xi, 1)$ with respect to $\mathcal{B}$, where ξ is a m-th. root of unity. Since the matrix A of $T_{2,3}$ with respect to $\mathcal{B}$ does not commute with $\mathrm{diag}(\xi, 1)$, it is not a diagonal matrix. Now we compute the commutator

$$K = A \cdot \mathrm{diag}(\xi, 1) \cdot A^{-1} \cdot \mathrm{diag}(\bar{\xi}, 1).$$

Since the monodromy representation respects the Hermitian form on the eigenspace, one can replace A by a non-diagonal matrix in $\mathrm{SU}(1,1)$ and the matrix $\mathrm{diag}(\xi, 1)$ by $\mathrm{diag}(e, \bar{e}) \in \mathrm{SU}(1,1)$, where $e^2 = \xi$, for the computation of K. By [54], page 59, one has a description of the matrices in $\mathrm{SU}(1,1)(\mathbb{R})$ such that

$$K = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \mathrm{diag}(e, \bar{e}) \begin{pmatrix} \bar{a} & -b \\ -\bar{b} & a \end{pmatrix} \mathrm{diag}(\bar{e}, e) = \begin{pmatrix} a\bar{a} - e^{-2}b\bar{b} & ab - e^2 ab \\ \bar{a}\bar{b} - e^{-2}\bar{a}\bar{b} & a\bar{a} - e^2 b\bar{b} \end{pmatrix}.$$

Hence

$$\begin{aligned} tr(K) - 2 &= 2a\bar{a} - 2\Re(e^2)b\bar{b} - 2 = 2a\bar{a} - 2\Re(e^2)b\bar{b} - a\bar{a} + b\bar{b} - 1 \\ &\geq (a\bar{a} - |\Re(e^2)|b\bar{b}) + (b\bar{b} - |\Re(e^2)|b\bar{b}) - 1 \geq a\bar{a} - |\Re(e^2)|b\bar{b} - 1 \geq 0. \end{aligned}$$

If the eigenvalues of K would be roots of unity (if it is not unipotent), one would have $|tr(K)| < 2$. Hence by the fact that $tr(K) \geq 2$, one concludes that K is unipotent or has eigenvalues v with $|v| \neq 1$. In both cases K has infinite order. $\square$

For the proof of Theorem 5.1.1 we need to recall some facts about complex simple Lie algebras. The complex simple Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ will be very important:

Remark 5.1.3. The Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ is given by

$$\mathfrak{sl}_n(\mathbb{C}) = \{M \in M_{n\times n}(\mathbb{C}) : \mathrm{tr}(M) = 0\}.$$

The Cartan subalgebra of $\mathfrak{sl}_n(\mathbb{C})$ is given by

$$\mathfrak{h} = \{\mathrm{diag}(a_1, \ldots, a_n) : \sum_{i=1}^{n} a_i = 0\}.$$

Each root space is given by the matrices $(a_{i,j})$, which have exactly one entry $a_{i_0,j_0} \neq 0$ for a fixed pair (i_0, j_0) with $i_0 \neq j_0$.

We want to show a statement about unitary groups, and not about special linear groups. The fact, which makes $\mathfrak{sl}_n(\mathbb{C})$ interesting for us, is given by the following remark:

Remark 5.1.4. We can obviously embed $\mathfrak{su}_{p,q}(\mathbb{R})$ into $\mathfrak{sl}_{p+q}(\mathbb{C})$, since $\mathrm{SU}(p,q)(\mathbb{R})$ is a Lie subgroup of $\mathrm{SL}_{p+q}(\mathbb{C})$. Moreover $i\mathfrak{su}_{p,q}(\mathbb{R})$ is a subvector space of $\mathfrak{sl}_{p+q}(\mathbb{C})$ (considered as real vector space). One has that

$$\mathfrak{su}_{p,q}(\mathbb{C}) = \mathfrak{su}_{p,q}(\mathbb{R}) \oplus i\mathfrak{su}_{p,q}(\mathbb{R}) = \mathfrak{sl}_{p+q}(\mathbb{C}).$$

(see [21], page 433).

Moreover we need to compare the monodromy group of $\mathcal{L}_j$ with the monodromy groups of some of its restrictions over certain subspaces of $\mathcal{P}_n$.

Remark 5.1.5. Consider some embedding $\iota_{a,b,c} : \mathcal{M}_n \hookrightarrow \mathcal{P}_n$. By the holomorphic diffeomorphism

$$\mathrm{PGL}_2(\mathbb{C}) \times \iota_{a,b,c}(\mathcal{M}_n)(\mathbb{C}) \ni M \times q \to M(q) \in \mathcal{P}_n(\mathbb{C}),$$

we have that

$$\mathrm{PGL}_2(\mathbb{C}) \times \mathcal{M}_n \cong \mathcal{P}_n \text{ and } \pi_1(\mathrm{PGL}_2(\mathbb{C})) \times \pi_1(\mathcal{M}_n) \cong \pi_1(\mathcal{P}_n),$$

where $\pi_1(\mathrm{PGL}_2(\mathbb{C})) \cong \mathbb{Z}/(2)$ (compare [18], 3.7 and [18], 3.15).

For technical reasons, we need to introduce an additional subspace of $\mathcal{P}_n$:

$$\mathcal{P}_n^{(a_k)} = \{q \in \mathcal{P}_n | \phi_q(a_k) = \infty\}$$

Let G_T denote the group of triangular matrices given by

$$G_T = \{\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \in M_{2\times 2}(\mathbb{C}) | a \neq 0\}.$$

We have obviously an embedding $\iota_{a,b,c} : \mathcal{M}_n \hookrightarrow \mathcal{P}_n^{(a_{n+3})}$ such that we get a holomorphic diffeomorphism

$$G_T \times \iota_{a,b,c}(\mathcal{M}_n)(\mathbb{C}) \ni M \times q \to M(q) \in \mathcal{P}_n^{(a_{n+3})}(\mathbb{C}).$$

Hence we have that

$$G_T \times \mathcal{M}_n \cong \mathcal{P}_n^{(a_{n+3})} \text{ and } \pi_1(G_T) \times \pi_1(\mathcal{M}_n) \cong \pi_1(\mathcal{P}_n^{(a_{n+3})}),$$

where $\pi_1(G_T) \cong \mathbb{Z}/(2)$.

The space $\mathcal{P}_n^{(a_{n+3})}$ has a natural interpretation as configuration space of $n+2$ points on $\mathbb{R}^2$. Its fundamental group is the colored braid group on $n+2$ strands in $\mathbb{R}^2$.

Lemma 5.1.6. *The fundamental group of the configuration space of $n+2$ points on $\mathbb{R}^2$ is generated by the Dehn twists T_{k_1,k_2} with $1 \leq k_1 < k_2 \leq n+2$.*

Proof. (see [24], Chapter **I**. 4) □

5.1.7. By the preceding results, the monodromy groups of $\mathcal{L}_j$, $(\mathcal{L}_j)_{\mathcal{M}_n}$ and $(\mathcal{L}_j)_{\mathcal{P}_n^{(a_{n+3})}}$ are commensurable. Therefore their $\mathbb{R}$-Zariski closures have the same connected component of identity. Thus we do not need to distinguish between them and we will call all of them simply $\mathrm{Mon}^0(\mathcal{L}_j)$.

Again assume that $\mathcal{L}_j$ is of type $(1,1)$. By Lemma 5.1.6, the monodromy group $\rho_j(\pi_1(\mathcal{P}_1^{(a_4)}))$ of $(\mathcal{L}_j)_{\mathcal{P}_1^{(a_4)}}$ is generated by the matrices $\rho_j(T_{k,\ell})$ for $k,\ell \in \{1,2,3\}$. For each Dehn twist T one can choose a suitable numbering of the branch points such that $T = T_{1,2}$. Hence by Proposition 3.3.5, one concludes that the generators of the monodromy group are contained in the group given by

$$\{M \in GL_2(\mathbb{C}) | \det(M)^m = 1\}.$$

Since $\mathrm{Mon}^0(\mathcal{L}_j)$ is contained in $\mathrm{U}(1,1)$, one concludes that $\mathrm{Mon}^0(\mathcal{L}_j) \subseteq \mathrm{SU}(1,1)$. Thus the complexification of the Lie algebra of $\mathrm{Mon}^0(\mathcal{L}_j)$ is contained in $\mathfrak{sl}_2(\mathbb{C})$. Note that the real Zariski closure $\mathrm{Mon}^0(\Re\mathbb{V}(j)_{\mathbb{R}})$ is isomorphic to $\mathrm{Mon}^0(\mathcal{L}_j)$ and $\mathrm{Mon}^0(\Re\mathbb{V}(j)_{\mathbb{R}})$ is a quotient of the semisimple group $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r)$. Thus by the kernel, which is semisimple, we have an exact sequence of algebraic groups. This yields an exact sequence of semisimple Lie algebras such that $\mathrm{Mon}^0(\mathcal{L}_j)$ must be semisimple. One has that $\mathrm{Mon}^0_{\mathbb{C}}(\mathcal{L}_j) \subseteq \mathrm{SU}_{\mathbb{C}}(1,1)$. Since $\mathfrak{su}_{1,1}(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C})$ is the smallest semisimple non-trivial complex Lie algebra (see [21], §14.1, Step 3) and $\mathrm{Mon}^0(\mathcal{L}_j)$ is infinite by Lemma 5.1.2, one concludes:

Proposition 5.1.8. *If* $\mathcal{L}_j$ *is of type* $(1,1)$*, then* $\mathrm{Mon}^0(\mathcal{L}_j) = \mathrm{SU}(1,1)$.

Recall that we want to give a proof of Theorem 5.1.1 by induction. The following construction explains our method to compare the monodromy groups of eigenspaces of different type, which we will need for the induction:

Construction 5.1.9 (Collision of points). Let $\mathbb{L}_j$ be an eigenspace in the cohomology of a fiber $C = \mathcal{C}_q$ with the local monodromy data α_k on a_k. Now let

$$b := \{a_{n_j+2}, a_{n_j+3}\} \text{ and } P = \{\{a_1\}, \ldots, \{a_{n_j+1}\}, b\}$$

be a stable partition of $N = \{a_1, \ldots, a_{n_j+3}\}$. Let $\phi_P : P \to \mathbb{P}^1$ be some embedding and the local system $\mathbb{L}(P)_j$ on $\mathbb{P}^1 \setminus \phi_P(P)$ have the local monodromy data

$$\alpha_b = \alpha_{a_{n_j+2}} \alpha_{a_{n_j+3}} \text{ and otherwise } \alpha_{\{a_k\}} = \alpha_{a_k}.$$

By Construction 3.2.1, these monodromy data allow the construction of a family of cyclic covers

$$\pi(P) : \mathcal{C}(P) \to \mathcal{P}_{n_j-1}.$$

The higher direct image sheaf $R^1\pi(P)_*(\mathbb{C})$ has an eigenspace with respect to the character given by 1, which we denote by $\mathcal{L}(P)_j$.[1] By the description of the respective monodromy representations in Proposition 3.3.5, we can

[1] This definition may seem to be a little bit odd. But it is motivated by some reasons, which should become clearer by Remark 5.1.10.

identify the monodromy group of $(\mathcal{L}(P)_j)_{\mathcal{P}^{(b)}_{n_j-1}}$ with the subgroup of the monodromy group of $(\mathcal{L}_j)_{\mathcal{P}^{(a_{n_3})}_n}$ generated by the Dehn twists $T_{a_{k_1},a_{k_2}}$ with $k_1, k_2 \leq n_j + 1$.

Remark 5.1.10. The local system $\mathcal{L}(P)_j$ is in general not the j-th. eigenspace of a family of irreducible covers of degree m obtained by a collision of two branch points of a family of irreducible covers of degree m. The problem is given by the irreducibility of the resulting family obtained by collision. Take for example the family $\mathcal{C} \to \mathcal{P}_2$ with generic fibers given by

$$y^4 = (x - a_1)(x - a_2)(x - a_3)^2 \cdot \ldots \cdot (x - a_5)^2.$$

By the collision of a_1 and a_2, one does not obtain an irreducible family of degree 4 covers. But the resulting local system $\mathcal{L}(P)_1$ is the eigenspace with respect to the character 1 on the higher direct image sheaf of the family $\mathcal{C}(P) \to \mathcal{P}_1$ with generic fibers given by

$$y^2 = (x - a_1) \cdot \ldots \cdot (x - a_4).$$

Now let $\mathcal{L}_j$ be of type (p, q) with $p, q > 0$. By the collision of two points and Proposition 2.3.4, one gets an eigenspace of type $(p, q-1)$ or of type $(p-1, q)$, if there is a suitable corresponding stable partition. A little bit later we will see that this construction yields an induction step such that the statement of Theorem 5.1.1 for local systems of type $(p, q-1)$ (if $p, q-1 \geq 1$) and of type $(p-1, q)$ (if $p-1, q \geq 1$) implies the statement of Theorem 5.1.1 for local systems of type (p, q).

For the application of the step of induction we will need a pair of stable partitions such that the resulting two eigenspaces satisfy the assumptions of Theorem 5.1.1. Moreover one can assume that for each fiber S_j contains at least 5 different points. Otherwise $\mathcal{L}_j$ is of type $(1, 1)$ or unitary. By the following technical lemma, we start to show that there exists a suitable pair of stable partitions, if the assumptions of Theorem 5.1.1 are satisfied and if S_j contains at least 5 points:

Lemma 5.1.11. *Assume that $j \neq \frac{m}{2}$. Then there is an $a_k \in S_j$ with $\mu_k \neq \frac{1}{2}$.*

Proof. Assume that all $a_k \in S_j$ satisfy $\mu_k = \frac{1}{2}$ and $j \neq \frac{m}{2}$. One has that $\mathcal{C}_r$ (with $r = \gcd(m, j)$) is a family of irreducible cyclic covers of $\mathbb{P}^1$ of degree $\frac{m}{r} > 2$ given by $\mu_1, \ldots, \mu_{n+3}$ in the sense of Construction 3.2.1. By the assumption that all $a_k \in S_j$ satisfy $\mu_k = \frac{1}{2}$, each branch point has the same branch index $\frac{m}{2r}$, which divides the degree $\frac{m}{r}$. Since we assume that $j \neq \frac{m}{2}$, one concludes that the branch indices given by $\frac{m}{2r}$ are not 1. Thus $\mathcal{C}_r$ is not a family of irreducible cyclic covers. Contradiction! □

Next we show that a $\mu_k \neq \frac{1}{2}$ yields two stable partitions:

Lemma 5.1.12. *Assume that S_j contains at least 5 different points such that there is an $a_k \in S_j$ with $\mu_k \neq \frac{1}{2}$. Then there are some pairwise different $\mu_h, \mu_i, \mu_s, \mu_t \in S_j$ such that*

$$\mu_h + \mu_i \neq 1, \quad \textit{and} \quad \mu_s + \mu_t \neq 1.$$

Proof. Assume that each pair $h, i' \in \{1, \ldots, n+3\}$ with $h \neq i'$ satisfies $\mu_h + \mu_{i'} = 1$. This implies that $\mu_h = \mu_{i'} = \frac{1}{2}$ for each pair h, i'. But this contradicts the assumptions of this lemma. Hence by the assumptions, there must be a pair (h, i') such that $\mu_h + \mu_{i'} \neq 1$.

Now consider $S'_j := S_j \setminus \{a_h, a_{i'}\}$. Let us assume that each pair $a_{s'}, a_{t'} \in S'_j$ with $s' \neq t'$ satisfies $\mu_{s'} + \mu_{t'} = 1$. Since $|S'_j| \geq 3$, one concludes that $\mu_{s'} = \mu_{t'} = \frac{1}{2}$. Since $\mu_h = \frac{1}{2}$ or $\mu_{i'} = \frac{1}{2}$ would contradict the assumptions in this case, one concludes that $\mu_h, \mu_{i'} \neq \frac{1}{2}$. Hence put $i := s', s := i', t := t'$, and we are done in this case.

If there are $a_{s'}, a_{t'} \in S'_j$ with $s' \neq t'$ and $\mu_{s'} + \mu_{t'} \neq 1$, we put $i := i'$, $s := s', t := t'$, and we are done. $\square$

By Lemma 5.1.11 and Lemma 5.1.12, one concludes immediately:

Corollary 5.1.13. *Assume that S_j contains at least 5 different points and $j \neq \frac{m}{2}$. Then there are some pairwise different $\mu_h, \mu_i, \mu_s, \mu_t \in S_j$ such that*

$$\mu_h + \mu_i \neq 1, \quad \textit{and} \quad \mu_s + \mu_t \neq 1.$$

Remark 5.1.14. The condition that

$$\mu_h + \mu_i \neq 1, \quad \text{and} \quad \mu_s + \mu_t \neq 1$$

implies that

$$(\mu_h \neq \frac{1}{2} \quad \text{or} \quad \mu_i \neq \frac{1}{2}) \quad \text{and} \quad (\mu_s \neq \frac{1}{2} \quad \text{or} \quad \mu_t \neq \frac{1}{2}).$$

Therefore the resulting eigenspace obtained by the collision of a_h and a_i resp., a_s and a_t satisfies that there is a local monodromy datum $\mu_k \neq \frac{1}{2}$. Hence the resulting eigenspace is not a middle part $\mathcal{L}_{\frac{m}{2}}$ of the VHS of the family obtained by the respective collision of two points. It remains to ensure that the resulting eigenspaces are not of type $(a, 0)$ resp., $(0, b)$ in order to satisfy the assumptions of Theorem 5.1.1 in this case.

5.1.15. Assume that $\mathcal{L}_j$ is of type $(1, n)$ with $n > 1$. By Proposition 2.3.4, one calculates that

$$\sum_{i=1}^{n+3} \mu_i = 2$$

in this case. One can choose the indices such that

$$\mu_1 \leq \ldots \leq \mu_{n+3}.$$

Hence one has

$$\mu_1 + \mu_3 \leq \mu_2 + \mu_4 \leq \mu_3 + \mu_5.$$

By the fact that

$$(\mu_2 + \mu_4) + (\mu_3 + \mu_5) < 2 \quad \text{and} \quad \mu_2 + \mu_4 \leq \frac{1}{2}((\mu_2 + \mu_4) + (\mu_3 + \mu_5)),$$

one has

$$\mu_1 + \mu_3 \leq \mu_2 + \mu_4 < 1.$$

Since the local systems with respect to the corresponding stable partitions of the collision of a_1 and a_3 resp., the collision of a_2 and a_4 are of type $(1, n-1)$ as one can calculate by Proposition 2.3.4, one can apply the induction hypothesis for these partitions.

Now let $\mathcal{L}_j$ be of type $(n, 1)$. Then the monodromy representation of $\mathcal{L}_j$ is the complex conjugate of the monodromy representation of $\mathcal{L}_{m-j}$, which is of type $(1, n)$ in this case. Hence first the induction step yields the statement for all $\mathcal{L}_j$ of type $(1, n)$. Then we have the statement for all $\mathcal{L}_j$ of type $(n, 1)$, too.

Assume that $\mathcal{L}_j$ is of type (p, q) with $p, q \geq 2$ and satisfies the assumptions of Theorem 5.1.1. By Corollary 5.1.13, one has a pair of stable partitions. Remark 5.1.14 and the fact that $p, q \geq 2$ imply that the corresponding eigenspaces satisfy the assumptions of Theorem 5.1.1, too.

Now we must only prove and explain the step of induction:

One has without loss of generality the stable partitions

$$P_1 = \{\{a_1\}, \ldots, \{a_{n+1}\}, \{a_{n+2}, a_{n+3}\}\}, \quad \text{and} \quad P_2 = \{\{a_1, a_2\}, \{a_3\}, \ldots, \{a_{n+3}\}\}.$$

Here we assume without loss of generality that $a_k \in \mathbb{R}$ and $a_k < a_{k+1}$ such that δ_k is the oriented path from a_k to a_{k+1} given by the straight line.

Let $q \in \mathcal{P}_n$. We consider the monodromy representation with respect to the basis $\mathcal{B}$ of $(\mathcal{L}_j)_q$ given by

$$\mathcal{B} = \{[e_1\delta_1], \ldots, [e_n\delta_n], [e_{n+2}\delta_{n+2}]\}.$$

One has obviously that $\mathrm{Mon}^0(\mathcal{L}_j(P_1))$ leaves $\langle [e_1\delta_1], \ldots, [e_n\delta_n] \rangle$ invariant and fixes all vectors in $\langle [e_{n+2}\delta_{n+2}] \rangle$. Now let U_1 be a small open neighborhood of the identity in $\mathrm{Mon}^0(\mathcal{L}_j(P_1))(\mathbb{R})$ such that the "inverse"

$$\log : U_1 \to Lie(\mathrm{Mon}^0(\mathcal{L}_j(P_1)))$$

of the exponential map is defined on U_1. By Remark 5.1.4 and the induction hypothesis, $\log(U_1)$ generates a Lie algebra, whose complexification L_1 is with respect to $\mathcal{B}$ given by the matrices

$$\begin{pmatrix} a_{1,1} & \dots & a_{1,n} & 0 \\ \vdots & & \vdots & \vdots \\ a_{n,1} & \dots & a_{n,n} & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix}, \quad \text{where} \quad N := \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix}$$

is an arbitrary $n \times n$ matrix with $\mathrm{tr}(N) = 0$. Note that $\mathrm{Mon}^0(\mathcal{L}_j(P_2))$ fixes all vectors in $\langle [e_1\delta_1] \rangle$ and leaves $\langle [e_3\delta_3], \dots, [e_{n+2}\delta_{n_2}] \rangle$ invariant. Hence in a similar way $\log(U_2)$ $(e \in U_2 \subset \mathrm{Mon}^0(\mathcal{L}_j(P_2))(\mathbb{R}))$ generates a Lie algebra. Its complexification L_2 is given by the matrices

$$\begin{pmatrix} 0 & v \\ 0 & N \end{pmatrix},$$

where N is again an arbitrary $n \times n$ matrix with $\mathrm{tr}(N) = 0$ and

$$v = (v_1, \dots, v_n)$$

is a vector depending on N. It is easy to see that L_1 and L_2 generate $\mathfrak{sl}_{n+1}(\mathbb{C})$.

Since $\mathrm{Mon}^0(\mathcal{L}_j)$ is contained in $\mathrm{SU}(p,q)$ and $\mathfrak{su}_{p,q} \otimes \mathbb{C} \cong \mathfrak{sl}_{n+1}(\mathbb{C})$, the group $\mathrm{Mon}^0(\mathcal{L}_j)$ is isomorphic to $\mathrm{SU}(p,q)$.

5.2 The Hodge group of a general direct summand

The VHS of a family $\mathcal{C} \to \mathcal{P}_n$ has a decomposition into rational subvariations $\mathcal{V}_r$ of Hodge structures, which where introduced in Section 4.3. Recall that $\mathcal{V}_r$ is general, if its monodromy group is infinite. Otherwise we call it special. Let $r \neq \frac{m}{2}$, $\mathcal{V}_r$ be general and $\mathcal{L}_j \subset \mathcal{V}_r$ in this section. Moreover recall that $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r)_q$ denotes the connected component of identity of the Zariski closure of the monodromy group in $\mathrm{GL}(((\mathcal{V}_r)_{\mathbb{R}})_q)$ for some $q \in \mathcal{P}_n$. Since $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r)_{q_1}$ and $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r)_{q_2}$ are conjugated, we write $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r)$ instead of $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r)_q$ for simplicity.

Remark 5.2.1. The group $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r)$ does not need to be equal to $\mathrm{Mon}^0(\mathcal{V}_r) \times_{\mathbb{Q}} \mathbb{R}$. It satisfies only $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r) \subseteq \mathrm{Mon}^0(\mathcal{V}_r) \times_{\mathbb{Q}} \mathbb{R}$. Hence $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r)$ yields a lower bound for $\mathrm{Mon}^0(\mathcal{V}_r)$. Thus one obtains

$$C_r^{\mathrm{der}}(\psi) = \mathrm{Hg}^{\mathrm{der}}(\mathcal{V}) = \mathrm{Mon}^0(\mathcal{V}_r),$$

if $C_r^{\mathrm{der}}(\psi)_{\mathbb{R}} = \mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r)$.

By the preceding section, we know that $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r) \to \mathrm{Mon}^0(\mathcal{L}_j)$ can be considered as the projection onto some $\mathrm{SU}(a,b)$, if $\mathcal{L}_j$ is of type (a,b) with $a,b > 0$. Otherwise one can use induction with the corresponding stable partitions again. We only consider the start of induction:

Assume that $S_j = 4$. hence one has without loss of generality $\mathcal{C}_r \to \mathcal{P}_1$. By our assumptions, there is an eigenspace $\mathcal{L}_{j_2}$ in $\mathcal{N}^1(\mathcal{C}_r, \mathbb{C})$ of type $(1,1)$, whose monodromy group is infinite. Since the monodromy group of $\mathcal{L}_j$ is conjugated to the monodromy of $\mathcal{L}_{j_2}$ by some $\gamma \in Gal(\mathbb{Q}(\xi^r); \mathbb{Q})$, it is infinite, too. One concludes similarly to the preceding section that $\mathrm{Mon}^0(\mathcal{L}_j) = \mathrm{SU}(2)$ (since $\mathfrak{su}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C})$ by [21], page 433, too). The rest of the proof is an induction analogue to the induction of the preceding section.

By the preceding considerations, one has:

Proposition 5.2.2. *Assume that $\mathcal{V}_r$ is general. Then the image of the natural projection $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r) \to \mathrm{GL}(\Re\mathbb{V}(j)_{\mathbb{R}})$ is given by the special unitary group induced by the trace map and the special unitary group $\mathrm{SU}(H^1_j(C,\mathbb{C}), H|_{H^1_j(C,\mathbb{C})})$ described in Section 4.3.*

Moreover we know that $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r)$ is contained in $C_r^{\mathrm{der}}(\psi)_{\mathbb{R}}$, which is given by a direct product of certain groups $\mathrm{SU}(a,b)$. Either $\mathrm{Mon}^0(\mathcal{V}_r) = C_r^{\mathrm{der}}(\psi)$ or it is given by a proper subgroup. We want to examine the conditions of the case $\mathrm{Mon}^0(\mathcal{V}_r) \neq C_r^{\mathrm{der}}(\psi)$. This will yield information and some criteria for the structure of $\mathrm{Mon}^0(\mathcal{V}_r)$.

First let us make a simple, but very useful observation:

Remark 5.2.3. Let $G_1, \ldots, G_t$ be connected simple Lie groups and $N \subset G_1 \times \ldots \times G_t =: G$ be a normal connected subgroup. One has that $Lie(G)$ is a direct sum of the simple ideals $Lie(G_1), \ldots, Lie(G_t)$, which implies that each ideal is a sum of certain $Lie(G_i)$ (see [27], **II**. Corollary 6.3). Since the normal connected subgroups of G and the ideals of $Lie(G)$ correspond (follows by [21], Proposition 8.41 and [21], Exercise 9.2), one obtains that

$$N = G_1 \times \ldots \times G_{t_0} \times \{e\} \times \ldots \times \{e\}$$

for some $t_0 \leq t$ with respect to a suitable numbering.

The decomposition of the rational Hodge structure $N^1(C_r, \mathbb{Q})$ into the $\mathbb{Q}(\xi^r)^+$-Hodge structures $\Re\mathbb{V}(j)$ yields a decomposition of the variation $\mathcal{V}_r$ of rational Hodge structures into the variations $\Re\mathcal{V}(j)$ of $\mathbb{Q}(\xi^r)^+$-Hodge structures.

By technical reasons, we consider the semisimple adjoint group $\mathrm{Mon}^{\mathrm{ad}}_{\mathbb{R}}(\mathcal{V}_r)$ instead of $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r)$ first. By Remark 5.2.3, one concludes that $\mathrm{Mon}^{\mathrm{ad}}_{\mathbb{R}}(\mathcal{V}_r)$ is isomorphic to the direct product of $\mathrm{Mon}^{\mathrm{ad}}(\Re\mathcal{V}(j)_{\mathbb{R}})$ and the kernel K_j of the natural projection $\mathrm{Mon}^{\mathrm{ad}}_{\mathbb{R}}(\mathcal{V}_r) \to \mathrm{Mon}^{\mathrm{ad}}(\Re\mathcal{V}(j)_{\mathbb{R}})$. Moreover one has:

Lemma 5.2.4. *Let $G_1, \ldots, G_t$ be simple adjoint Lie groups and G be a semisimple subgroup of $G_1 \times \ldots \times G_t$ such that each natural projection*

$$G \hookrightarrow G_1 \times \ldots \times G_t \xrightarrow{pr_j} G_j$$

is surjective. One has $G \neq G_1 \times \ldots \times G_t$, if and only if there are some $j_1, j_2 \in \{1, \ldots, t\}$ with $j_1 \neq j_2$ such that G contains a simple subgroup G' isomorphically mapped onto G_{j_1} and G_{j_2} by the natural projections.

Proof. The "if" part is easy to see. The "only if" part follows by induction. □

Note that we have a natural embedding

$$\mathrm{Mon}^{\mathrm{ad}}_{\mathbb{R}}(\mathcal{V}_r) \hookrightarrow \prod_{j \in \mathbb{Z}/\frac{m}{r}, j \leq \frac{m}{2}} \mathrm{Mon}^{\mathrm{ad}}(\Re\mathcal{V}(j)_{\mathbb{R}}).$$

Thus the preceding lemma and our assumption that $\mathrm{Mon}^0(\mathcal{V}_r) \neq C_r^{\mathrm{der}}(\psi)$ imply that there is a direct simple factor of $\mathrm{Mon}^{\mathrm{ad}}_{\mathbb{R}}(\mathcal{V}_r)$, which isomorphically mapped onto $\mathrm{Mon}^{\mathrm{ad}}(\Re\mathcal{V}(j_1)_{\mathbb{R}})$ and $\mathrm{Mon}^{\mathrm{ad}}(\Re\mathcal{V}(j_2)_{\mathbb{R}})$ for some j_1 and j_2 with $j_2 \neq j_1$ and $m - j_1$. By Remark 5.2.3, $\mathrm{Mon}^{\mathrm{ad}}_{\mathbb{R}}(\mathcal{V}_r)$ is a direct product of the kernel of the both projections and this direct simple factor.

Thus the natural projections onto $\mathrm{Mon}^{\mathrm{ad}}(\Re\mathcal{V}(j_1)_{\mathbb{R}})$ and $\mathrm{Mon}^{\mathrm{ad}}(\Re\mathcal{V}(j_2)_{\mathbb{R}})$ yield an isomorphism

$$\alpha^{\mathrm{ad}} : \mathrm{Mon}^{\mathrm{ad}}(\Re\mathcal{V}(j_1)_{\mathbb{R}}) \to \mathrm{Mon}^{\mathrm{ad}}(\Re\mathcal{V}(j_2)_{\mathbb{R}}).$$

Moreover one concludes that the image $\mathrm{Mon}^{\mathrm{ad}}(\Re\mathcal{V}(j_1, j_2)_{\mathbb{R}})$ of the projection

$$\mathrm{Mon}^{\mathrm{ad}}_{\mathbb{R}}(\mathcal{V}_r) \to \mathrm{Mon}^{\mathrm{ad}}(\Re\mathcal{V}(j_1)_{\mathbb{R}}) \times \mathrm{Mon}^{ad}(\Re\mathcal{V}(j_2)_{\mathbb{R}})$$

is given by the graph of α^{ad}.

5.2.5. For the image $\mathrm{Mon}^0(\Re\mathcal{V}(j_1, j_2)_{\mathbb{R}})$ of the projection

$$\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r) \to \mathrm{Mon}^0(\Re\mathcal{V}(j_1)_{\mathbb{R}}) \times \mathrm{Mon}^0(\Re\mathcal{V}(j_2)_{\mathbb{R}})$$

this implies that the natural projections

$$p_1 : \mathrm{Mon}^0(\Re\mathcal{V}(j_1, j_2)_{\mathbb{R}}) \to \mathrm{Mon}^0(\Re\mathcal{V}(j_1)_{\mathbb{R}})$$

and

$$p_2 : \mathrm{Mon}^0(\Re\mathcal{V}(j_1, j_2)_{\mathbb{R}}) \to \mathrm{Mon}^0(\Re\mathcal{V}(j_2)_{\mathbb{R}})$$

are isogenies. Since

$$\mathrm{Mon}^0(\Re\mathcal{V}(j_1)_{\mathbb{R}})(\mathbb{C}) = \mathrm{Mon}^0(\Re\mathcal{V}(j_2)_{\mathbb{R}})(\mathbb{C}) = \mathrm{SL}_{a+b}(\mathbb{C}),$$

where (a, b) is the type of $\mathcal{L}_{j_1}$, and the Lie group $\mathrm{SL}_{a+b}(\mathbb{C})$ is simply connected (see [21], Proposition 23.1), the induced isogenies of Lie groups of $\mathbb{C}$-valued points are isomorphisms. Hence the isogenies p_1 and p_2 are isomorphisms.

Hence our assumption implies the existence of an isomorphism

$$\alpha : \mathrm{Mon}^0(\Re\mathcal{V}(j_1)_{\mathbb{R}}) \to \mathrm{Mon}^0(\Re\mathcal{V}(j_2)_{\mathbb{R}}),$$

which satisfies that $\mathrm{Mon}^0(\Re\mathcal{V}(j_1, j_2)_{\mathbb{R}})$ is given by $\mathrm{Graph}(\alpha)$.

5.3 A criterion for the reaching of the upper bound

In this section we give a necessary criterion for the existence of an isomorphism α. This yields a sufficient condition that $\mathrm{Mon}^0(\mathcal{V}_r)$ reaches the upper bound $C_r^{\mathrm{der}}(\psi)$. In addition we will see that $\mathrm{Mon}^0(\mathcal{V}) = \mathrm{Mon}^0(\mathcal{V}_1)$ reaches the upper bound, if the degree m of the covers given by the fibers of $\mathcal{C} \to \mathcal{P}_n$ is a prime number > 2.[2]

We say that a Dehn twist T is semisimple (with respect to $\mathcal{V}_r$), if the monodromy representation ρ_j of one (and hence of all) $\mathcal{L}_j \subset \mathcal{V}_r$ yields a semisimple matrix $\rho_j(T)$. By the trace map (see (4.3)), we can identify $\mathrm{Mon}^{\mathrm{ad}}(\Re\mathcal{V}(j)_{\mathbb{R}})$ and $\mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_j)$. Thus $\mathrm{Mon}^0(\Re\mathcal{V}(j_1, j_2)_{\mathbb{R}})$ is equal to $\mathrm{Graph}(\alpha)$, if and only if one has a corresponding isomorphism α^{ad} : $\mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_1) \to \mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_2)$ such that $\mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{j_1} \oplus \mathcal{L}_{j_2})$ is given by $\mathrm{Graph}(\alpha^{\mathrm{ad}})$. By an abuse of notation, we will write α instead of α^{ad} from now on.

First let us formulate a sufficient criterion for the non-existence of α in the case $\mathcal{C} \to \mathcal{P}_1$:

Proposition 5.3.1. *Let $\mathcal{V}_r$ be general and $\mathcal{L}_{j_1}, \mathcal{L}_{j_2} \subset \mathcal{V}_r$ be of type (a, b), where $a + b = 2$. Moreover let z_i denote the non-trivial eigenvalue of $\rho_{j_i}(T)$ with respect to a semisimple Dehn twist T for $i = 1, 2$. Then there is not any isomorphism $\alpha : \mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_1) \to \mathrm{Mon}^{ad}(\mathcal{L}_2)$ such that $P\rho_{j_2} = \alpha \circ P\rho_{j_1}$, if there is a semisimple Dehn twist T such that the non-trivial eigenvalue z_2 of $\rho_{j_2}(T)$ is not contained in $\{z_1, \bar{z}_1\}$.*

Proof. Assume that $\mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_1)$ and $\mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_2)$ are isomorphic and T satisfies the assumptions of this proposition. Thus $\rho_{j_1}(T)$ generates a finite commutative subgroup FT of $\mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{j_1})$. Our assumption that $a+b = 2$ implies that $\mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{j_1}) \cong \mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{j_2})$ is isomorphic to $\mathrm{PU}(1, 1)$ or $\mathrm{PU}(2)$. Note that the elements of $FT(\mathbb{R})$ are up to conjugation given by classes of diagonal matrices. The elements of $FT(\mathbb{R})$ commute exactly with the $\mathbb{R}$-rational elements of the maximal torus G of $\mathrm{PU}(1, 1)$ resp., $\mathrm{PU}(2)$ which is (up to conjugation)

[2] For $m = 2$ we will later see that $\mathrm{Mon}^0(\mathcal{V})$ reaches the upper bound as well.

given by the classes of diagonal matrices in PU(1,1) resp., PU(2). One checks easily that $G(\mathbb{R})$ is isomorphic to $S^1(\mathbb{R})$. Hence one can identify $FT(\mathbb{R})$ with some $\langle \xi^s \rangle \subset S^1(\mathbb{R})$. Now let $1 \neq \zeta \in \langle \xi^s \rangle$ satisfy the property that there is a closed interval on $S^1(\mathbb{R})$ with end points 1 and ζ, which does not contain any other element of $\langle \xi^s \rangle$. Hence there is a closed interval I on G with ending points $[\mathrm{diag}(1,1)]$ and $[\mathrm{diag}(\zeta,1)] \in FT$, which does not contain any other element of FT.

Now assume such an isomorphism α exists. Note that we have an identification $\alpha(G)(\mathbb{R}) = S^1(\mathbb{R})$, too. But our assumptions imply that

$$\alpha(\mathrm{diag}(\zeta,1)) \notin \{\mathrm{diag}(\zeta,1), \mathrm{diag}(\bar{\zeta},1)\}.$$

Hence by our identification $\alpha(G)(\mathbb{R}) = S^1(\mathbb{R})$, one obtains that

$$\alpha(\zeta) \notin \{\zeta, \bar{\zeta}\}.$$

Thus $\alpha(I) \subset \alpha(G)(\mathbb{R})$ is not a connected interval, which does not contain any other element of $\langle \xi^s \rangle$ except of 1 and $\alpha(\zeta)$. But α must be a homeomorphism on the $\mathbb{R}$-valued points. Contradiction! □

By the preceding proposition, we can use certain semisimple Dehn twists for the study of the generic Hodge group. Hence we make some observations about the orders and the existence of semisimple Dehn twists:

Lemma 5.3.2. *Let $j \neq \frac{m}{2}$ and $v | \frac{m}{r}$, where*

$$1 \neq v, \quad r := \gcd(m,j) \quad \textit{and} \quad 1,2 \neq \frac{m}{rv}.$$

Then there exists a Dehn twist $T \in \pi_1(\mathcal{P}_n)$ such that $\rho_j(T) \in \mathrm{Mon}(\mathcal{L}_j)$ is semisimple and $|\langle \rho_j(T) \rangle|$ does not divide v.

Proof. One can replace $\mathcal{C}$ by $\mathcal{C}_r$ and choose a suitable collection of local monodromy data for $\mathcal{C}$ such that $j = 1$. By an isomorphism $\langle \xi \rangle \cong \mathbb{Z}/(m)$, the non-trivial eigenvalues of the semisimple Dehn twists T_{k_1,k_2} correspond to some elements $[b_{k_1,k_2}] \in \mathbb{Z}/(m)$, where $b_{k_1,k_2} := d_{k_1} + d_{k_2}$ and d_{k_1} and d_{k_2} denote the branch indices of a_{k_1} and a_{k_2}.

Assume that each semisimple Dehn twist satisfies that its order divides some v with $v|m$. This implies that $\frac{m}{v} | b_{k_1,k_2}$ for all b_{k_1,k_2}. Hence for all $k = 1, \ldots, n+3$ one has that $\frac{m}{v}$ divides

$$2d_k = (d_k + d_{k_1}) + (d_k + d_{k_2}) - (d_{k_1} + d_{k_2}) = b_{k,k_1} + b_{k,k_2} - b_{k_1,k_2}.$$

Since there does not exist any integer $N \neq 1$, which divides each d_k, one has that $\frac{m}{v}$ divides 2. This implies that $\frac{m}{v} = 1$ or $\frac{m}{v} = 2$. □

For the formulation of our criterion in the higher dimensional case we need the following lemma:

Lemma 5.3.3. *Let $q \in \mathcal{P}_n$. Assume that we have a stable partition*

$$P := \{\{a_1\}, \{a_2\}, \{a_3\}, \{a_4, \ldots, a_{n_j+3}\}\}$$

with respect to the local monodromy data of $(\mathcal{L}_j)_q$ such that we can define the eigenspace $\mathcal{L}_j(P)$ over $\mathcal{P}_1$ with $b = \{a_4, \ldots, a_{n_j+3}\}$ as in Construction 5.1.9. *Then the monodromy group $\rho_j(P)(\pi_1(\mathcal{P}_1))$ of $\mathcal{L}_j(P)$ has a subgroup of finite index generated by $\rho_j(T_{1,2})$ and $\rho_j(T_{2,3})$.*

Proof. The stability of the partition ensures that $\alpha_b = \alpha_{a_4} \ldots \alpha_{n_j+3} \neq 1$. It is a well-known fact that $\pi_1(\mathcal{M}_1(\mathbb{C}))$ is generated by the two loops around 0 and 1, where we identify $\mathbb{A}^1 \setminus \{0,1\} = \mathcal{M}_1$. By the embedding $\mathcal{M}_1 \to \mathcal{P}_1$ given by

$$a_1 = 0, \quad a_3 = 1, \quad a_4 = \infty,$$

we can identify the generators of $\pi_1(\mathcal{M}_1(\mathbb{C}))$ with the Dehn twists $T_{1,2}$ and $T_{2,3}$. The statement follows from the fact that the monodromy group of $\mathcal{L}_j(P)|_{\mathcal{M}_1}$ has finite index in the monodromy group of $\mathcal{L}_j(P)$. $\square$

Proposition 5.3.4. *Let $\mathcal{L}_{j_1}, \mathcal{L}_{j_2} \subset (\mathcal{V}_1)_{\mathbb{C}}$ with $j_1 \neq j_2$ and $j_1 \neq m - j_2$. Assume that we have a stable partition*

$$P := \{\{a_1\}, \{a_2\}, \{a_3\}, \{a_4, \ldots, a_{n+3}\}\}$$

such that the monodromy group of $\mathcal{L}_{j_1}(P)$ or $\mathcal{L}_{j_2}(P)$ is infinite. Let $\mathrm{Mon}^0(\mathcal{L}_{j_1}(P))$ *and* $\mathrm{Mon}^0(\mathcal{L}_{j_2}(P))$ *be not isomorphic or $T_{k,\ell}$ be a semisimple Dehn twist with $k, \ell \in \{1,2,3\}$ such that the non-trivial eigenvalue z_2 of $\rho_{j_2}(T_{k,\ell})$ is not contained in $\{z_1, \bar{z}_1\}$, where z_1 denotes the non-trivial eigenvalue of $\rho_{j_1}(T_{k,\ell})$. Then*

$$\mathrm{Mon}^0(\Re\mathcal{V}(j_1,j_2)_{\mathbb{R}}) = \mathrm{Mon}^0(\Re\mathcal{V}(j_1)_{\mathbb{R}}) \times \mathrm{Mon}^0(\Re\mathcal{V}(j_2)_{\mathbb{R}}).$$

Proof. By Lemma 5.3.3 and the fact that the monodromy group of $\mathcal{L}_j|_{\mathcal{M}_n}$ has finite index in the monodromy group of $\mathcal{L}_j$, one concludes that the group generated by $\rho_{j_1}(T_{1,2})$ and $\rho_{j_1}(T_{2,3})$ resp., $\rho_{j_2}(T_{1,2})$ and $\rho_{j_2}(T_{2,3})$ has finite index in the monodromy representation of $\mathcal{L}_{j_1}(P)$ resp., $\mathcal{L}_{j_2}(P)$. Therefore an isomorphism

$$\alpha : \mathrm{Mon}^0(\Re\mathcal{V}(j_1)_{\mathbb{R}}) \to \mathrm{Mon}^0(\Re\mathcal{V}(j_2)_{\mathbb{R}})$$

yields an isomorphism

$$\alpha(P) : \mathrm{Mon}^0(\mathcal{L}_{j_1}(P)) \to \mathrm{Mon}^0(\mathcal{L}_{j_2}(P)).$$

Thus one only needs to apply Proposition 5.3.1. □

Now let us first define the condition for the reaching of the upper bound and then write down the obvious theorem:

Definition 5.3.5. Assume that one has for each $\mathcal{L}_{j_1}, \mathcal{L}_{j_2} \subset \mathcal{V}_r$ with $j_1 \neq j_2, m - j_2$ and $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{L}_{j_1}) \cong \mathrm{Mon}^0_{\mathbb{R}}(\mathcal{L}_{j_1})$ a stable partition

$$P := \{\{a_1\}, \{a_2\}, \{a_3\}, \{a_4, \ldots, a_{n_j+3}\}\}$$

(with respect to a suitable enumeration) such that the monodromy group of $\mathcal{L}_{j_1}(P)$ or $\mathcal{L}_{j_2}(P)$ is infinite and one of the following conditions is satisfied:

1. $\mathrm{Mon}^0(\mathcal{L}_{j_1}(P))$ and $\mathrm{Mon}^0(\mathcal{L}_{j_2}(P))$ are not isomorphic.
2. There is a semisimple Dehn twist $T_{k,\ell}$ with $k, \ell \in \{1, 2, 3\}$ such that the non-trivial eigenvalue z_2 of $\rho_{j_2}(T_{k,\ell})$ is not contained in $\{z_1, \bar{z}_1\}$, where z_1 denotes the non-trivial eigenvalue of $\rho_{j_1}(T_{k,\ell})$.

We call $\mathcal{V}_r$ very general in this case.

A direct summand $\mathcal{V}_r$ is exceptional, if it is general, but not very general.

By Proposition 5.3.4, one concludes:

Theorem 5.3.6. *If $\mathcal{V}_r$ is very general,* $\mathrm{Mon}^0(\mathcal{V}_r)$ *reaches the upper bound* $C^{\mathrm{der}}(\psi)$.

Theorem 5.3.7. *If the degree m of the covers given by the fibers of $\mathcal{C} \to \mathcal{P}_n$ is a prime number $m > 2$,* $\mathrm{Mon}^0(\mathcal{V}) = \mathrm{Mon}(\mathcal{V}_1)$ *reaches the upper bound.*

Proof. By the preceding theorem, we have only to show that $\mathrm{Mon}^0(\mathcal{V}) = \mathrm{Mon}(\mathcal{V}_1)$ is very general. Note that Lemma 5.3.2 implies that there is a semisimple Dehn twist for $m > 2$.

Assume that we are in the case of a family $\mathcal{C} \to \mathcal{P}_1$, and that $j_1 \neq j_2, m - j_2$. Since $\mathbb{Z}/(m)$ is a field in our case, one has that each semisimple Dehn twist satisfies that the non-trivial eigenvalue of $\rho_{j_2}(T)$ is not contained in $\{z_1, \bar{z}_1\}$, where z_1 denotes the non-trivial eigenvalue of $\rho_{j_2}(T)$. Thus in this case the statement follows from Proposition 5.3.1.

Otherwise we have to find a stable partition P as in Proposition 5.3.4. One has without loss of generality the semisimple Dehn twist $T_{1,2}$. Moreover assume without loss of generality that $d_1 + d_2 = m - 1$. One has two cases: Either there is some a_3 such that

$$P = \{\{a_1\}, \{a_2\}, \{a_3\}, \{a_4, \ldots, a_{n+3}\}\}$$

is the desired stable partition or one has that

$$d_3 = \ldots = d_{n+3} = 1.$$

Since in the case $m = 3$ there is nothing to show, one can otherwise assume that $m > 3$ and take the stable partition

$$P = \{\{a_3\}, \{a_4\}, \{a_5\}, \{a_1, a_2, a_6, \ldots, a_{n+3}\}\}.$$

□

5.4 The exceptional cases

At this time the author does not see a possibility to calculate the monodromy group of the VHS of an arbitrary family $\mathcal{C} \to \mathcal{P}_n$. Therefore we consider mainly a family $\mathcal{C} \to \mathcal{P}_1$.

5.4.1. Let ρ_{j_1} and ρ_{j_2} denote the monodromy representations of $\mathcal{L}_{j_1}, \mathcal{L}_{j_2} \subset \mathcal{V}_r$. Proposition 3.3.5 yields a description of $\rho_{j_1}(T)$ and $\rho_{j_2}(T)$ for some Dehn twist T. By this description, the entries of the matrices $\rho_{j_1}(T)$ and $\rho_{j_2}(T)$ differ by some $\gamma \in \mathrm{Gal}(\mathbb{Q}(\xi^r);\mathbb{Q})$. By its action on $\langle \xi^r \rangle \cong \mathbb{Z}/(\frac{m}{r})$, each γ can be identified with some $[v] \in (\mathbb{Z}/(\frac{m}{r}))^*$ such that $[\frac{j_1}{r}v]_{\frac{m}{r}} = [\frac{j_2}{r}]_{\frac{m}{r}}$. One has a subgroup $H_1(\gamma)$ of $\langle \xi^r \rangle$ consisting of roots of unity fixed by γ and a subgroup $H_2(\gamma)$ of $\langle \xi^r \rangle$ consisting of roots of unity, on which γ acts by complex conjugation. Since $j_1 \neq j_2, m - j_2$, one has that γ is neither given by the complex conjugation nor by the identity. Thus $H_1(\gamma)$ resp., $H_2(\gamma)$ is given by $\{1\}$ or some proper subgroup of $\langle \xi^r \rangle$ generated by $\xi^{rt_1(\gamma)}$ resp., $\xi^{rt_2(\gamma)}$, where $1 \neq t_1(\gamma)$ and $1 \neq t_2(\gamma)$ divide $\frac{m}{r}$.

For the rest of this section we consider only families $\mathcal{C} \to \mathcal{P}_1$ of degree m with an exceptional part $\mathcal{V}_r$. Assume without loss of generality that $\mathcal{V}_1$ is exceptional and $j_1 = 1$. Let γ correspond to v. For simplicity we write t_1 and t_2 instead of $t_1(\gamma)$ and $t_2(\gamma)$, and H_1 and H_2 instead of $H_1(\gamma)$ and $H_2(\gamma)$.

Lemma 5.4.2. *Let $\mathcal{C} \to \mathcal{P}_1$ be a family of degree m covers such that $\mathcal{V}_1$ is exceptional. Then one is without loss of generality in one of the following cases:*

1. *(Complex case) $t_1 | d_1 + d_2$, $t_1 | d_2 + d_3$ and $t_2 | d_1 + d_3$, where t_1 does not divide $d_1 + d_3$.*
2. *(Separated case) $t_1 = 2$ and 2 divides $d_1 + d_2$, $d_2 + d_3$ and $d_1 + d_3$.*

Proof. If $\mathcal{V}_1$ is exceptional, then $d_1 + d_2$, $d_2 + d_3$ and $d_1 + d_3$ are divided by t_1 or t_2. Assume that t_1 (resp., t_2) divides $d_1 + d_2$, $d_2 + d_3$ and $d_1 + d_3$. Hence one has $t_1 = 2$ (resp., $t_2 = 2$) as in the proof of Lemma 5.3.2. Otherwise one has only to choose a suitable enumeration such that one is in the complex case.

□

Remark 5.4.3. It can occur that one is in the complex case and the separated case with respect to the same eigenspaces (up to complex conjugation). Consider the family $\mathcal{C} \to \mathcal{P}_1$ of degree 12 covers given by

$$d_1 = 5, \quad d_2 = 1, \quad d_3 = 11, \quad d_4 = 7.$$

Let $v = 5$. Then one has $t_1 = 3$ and $t_2 = 2$ such that $3|d_1 + d_2$, $3|d_2 + d_3$ and $2|d_1 + d_3$. Now let $v = 7$. In this case one has $t_1 = 2$ and 2 divides $d_1 + d_2$, $d_2 + d_3$ and $d_1 + d_3$. By 5.4.10, we will see that there is an isomorphism $\alpha : \mathrm{Mon}^0(\Re\mathcal{V}(1))_{\mathbb{R}} \to \mathrm{Mon}^0(\Re\mathcal{V}(5))_{\mathbb{R}}$.

On the other hand consider the family $\mathcal{C} \to \mathcal{P}_1$ of degree 12 covers given by

$$d_1 = 11, \quad d_2 = 1, \quad d_3 = 11, \quad d_4 = 1.$$

Again by the same arguments, we are in the complex case and the separated case at the same time. But in this case the existence of a suitable isomorphism

$$\alpha : \mathrm{Mon}^0_{\mathbb{R}}(\Re\mathcal{V}(1)) \to \mathrm{Mon}^0_{\mathbb{R}}(\Re\mathcal{V}(5))$$

is not known to the author at this time.

5.4.4. Assume that the direct summand $\mathcal{V}_1$ is separated with respect to $[v]_m \in (\mathbb{Z}/(m))^*$ for a family $\mathcal{C} \to \mathcal{P}_1$ of degree m covers. One has $[v2] = [2]$ in each separated case. This implies that $[2][v-1] = [0]$. Therefore one has $[v] = [\frac{m}{2} + 1] \in (\mathbb{Z}/(m))^*$ in each separated case. Hence $v \in (\mathbb{Z}/(m))^*$ is an involution. The fact that $[v] = [\frac{m}{2} + 1] \in (\mathbb{Z}/(m))^*$ implies that $\frac{m}{2} + 1$ is odd. Hence 4 divides m. In the separated case $r_1 = 2$ divides each $d_k + d_\ell$. Thus $\mathcal{V}_1$ is separated, if and only if $4|m$ and each d_k is odd.

Therefore there are infinitely many cases of families $\mathcal{C} \to \mathcal{P}_1$ such that $\mathcal{V}_1$ is separated. At this time the author can not give an isomorphism

$$\alpha : \mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_1) \to \mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{\frac{m}{2}+1})$$

for each separated example.

By the preceding point we have classified and described all examples $\mathcal{C} \to \mathcal{P}_1$ such that $\mathcal{V}_1$ is separated. Hence we consider only the case of a family $\mathcal{C} \to \mathcal{P}_1$ such that $\mathcal{V}_1$ is complex for the rest of this section.

Lemma 5.4.5. *Assume that $\mathcal{V}_1$ is complex. Then one has:*

$$\ell := \mathrm{lcm}(t_1, t_2) = \begin{cases} m & : \quad m \quad \textit{is odd} \\ \frac{m}{2} & : \quad m \quad \textit{is even} \end{cases}$$

Proof. If m is odd, $H_1 \cap H_2 = \{1\} = \{\xi^m\}$. If m is even, $H_1 \cap H_2 = \{1, -1\} = \langle \xi^{\frac{m}{2}} \rangle$. □

Lemma 5.4.6. *Assume that $\mathcal{V}_1$ is complex. Then one has that $t_1 t_2 = m$ or $t_1 t_2 = \frac{m}{2}$. Moreover one has that $t_1 t_2 = m$, if m is odd, and $t_1 t_2 = \frac{m}{2}$, if $2|m$, but 4 does not divide m.*

Proof. If m is odd, one has $\ell = \mathrm{lcm}(t_1, t_2) = m$. Hence one obtains $t'_1 t'_2 g = m$ for $g := \gcd(t_1, t_2)$ and $t_i = g t'_i$. Hence $|H_1| = t'_2$ and $|H_2| = t'_1$. If $g > 2$, there is a semisimple Dehn twist, whose order does not divide $t'_1 t'_2$ (follows from Lemma 5.3.2). But this can not occur by our assumption that $\mathcal{V}_1$ is complex. Hence $g = 1$, since $g = 2$ is not possible for m odd.

If m is even, one has $\ell = \mathrm{lcm}(t_1, t_2) = \frac{m}{2}$. Hence one has $t'_1 t'_2 g = \frac{m}{2}$ for $g := \gcd(t_1, t_2)$ and $t_i = g t'_i$. If $g > 2$, there is a semisimple Dehn twist, whose order does not divide $t'_1 t'_2$. Hence one has $g = 1$ or $g = 2$. Thus $t_1 t_2 = m$ or $t_1 t_2 = \frac{m}{2}$.

Now assume that $2|m$, but 4 does not divide m. Then one has that $\frac{m}{2} = \mathrm{lcm}(t_1, t_2)$ is odd. Hence one can not have that $g = 2$ in this case. Thus $g = 1$ and $t_1 t_2 = \frac{m}{2}$. □

Example 5.4.7. In the case $4|m$ both $t_1 t_2 = m$ and $t_1 t_2 = \frac{m}{2}$ can occur. Let $m = 24$ and take $v = 5$ for the corresponding automorphism of $\mathbb{Q}(\xi)$. In this case one has $t_1 = 6$ and $t_2 = 4$ such that $t_1 t_2 = 24 = m$.

Now let $m = 24$ and take $v = 7$. In this case one has $t_1 = 4$ and $t_2 = 3$ such that $t_1 t_2 = 12 = \frac{m}{2}$.

Proposition 5.4.8. *Assume $\gamma \in Gal(\mathbb{Q}(\xi); \mathbb{Q})$ yields an example of a complex case. Then γ is an involution.*

Proof. Let $[v] \in \mathbb{Z}/(m)^*$ correspond to γ. One has that $t_1 t_2 = m$ or $t_1 t_2 = \frac{m}{2}$. Since one has that $[vt_1]_m = [t_1]_m$ and $[vt_2]_m = -[t_2]_m$, one gets that

$$(v-1)t_1 \in (m) \quad \text{and} \quad (v+1)t_2 \in (m).$$

This implies that $t_2|(v-1)$ and $t_1|(v+1)$ or (if $t_1 t_2 = \frac{m}{2}$) that $2t_2|(v-1)$ and $2t_1|(v+1)$. Hence in each case one obtains that

$$v^2 - 1 = (v-1)(v+1) \in (m).$$

□

Theorem 5.4.9. *Let $\mathcal{C} \to \mathcal{P}_1$ be a family of degree m covers. Then $\mathcal{V}_1$ is complex, if and only if the fibers of $\mathcal{C}$ have the branch indices $d_1, \ldots, d_4$ with $2m = d_1 + \ldots + d_4$ such that*

$$[vd_2]_m = [d_1 + d_2 + d_3]_m, \quad [vd_1]_m = [-d_3]_m, \quad [vd_3]_m = [-d_1]_m$$

or

$$[vd_2]_m = [d_1 + d_2 + d_3 + \frac{m}{2}]_m, \quad [vd_1]_m = [-d_3 + \frac{m}{2}]_m, \quad [vd_3]_m = [-d_1 + \frac{m}{2}]_m$$

for some v with $[v^2]_m = [1]_m$ and $[v]_m \notin \{[1]_m, [m-1]_m\}$.

Proof. The condition $2m = d_1 + \ldots + d_4$ ensures that $\mathcal{V}_1$ is not special.

By an abuse of notation, each integer z denotes the residue class $[z]_m$ in this proof. Assume that $\mathcal{V}_1$ is complex. Hence by Lemma 5.4.2, one has that

$$\begin{aligned}
2vd_2 &= v((d_1+d_2)-(d_1+d_3)+(d_2+d_3)) = (d_1+d_2)+(d_1+d_3)+(d_2+d_3)\\
&= 2(d_1+d_2+d_3),\\
2vd_1 &= v((d_1+d_2)+(d_1+d_3)-(d_2+d_3)) = (d_1+d_2)-(d_1+d_3)-(d_2+d_3)) = -2d_3,\\
2vd_3 &= v(-(d_1+d_2)+(d_1+d_3)+(d_2+d_3)) = -(d_1+d_2)-(d_1+d_3)+(d_2+d_3)) = -2d_1.
\end{aligned}$$

Hence one has two cases:

$$vd_2 = d_1+d_2+d_3 \quad \text{or} \quad vd_2 = d_1+d_2+d_3+\frac{m}{2}$$

In the first case (resp., the second case) the fact that $v(d_1+d_2) = d_1+d_2$ implies that $vd_1 = -d_3$ (resp., $vd_1 = -d_3+\frac{m}{2}$). Moreover in the first case (resp., the second case) the fact that $v(d_2+d_3) = d_2+d_3$ implies that $vd_3 = -d_1$ (resp., $vd_3 = -d_1+\frac{m}{2}$). Hence we have obtained the claimed equations.

Assume conversely that the family $\mathcal{C} \to \mathcal{P}_1$ satisfies one of the two systems of equations of this theorem. Then one can easily calculate that $\mathcal{V}_1$ is complex. □

5.4.10. Let $\mathcal{C} \to \mathcal{P}_1$ be a family of degree m covers. Assume that d_1, d_2, d_3 satisfy the first system of equations of Theorem 5.4.9 with respect to some v with $[v^2] = [1]_m$, which satisfies that $[v]_m \notin \{[1]_m, [m-1]_m\}$. Moreover let $j \in (\mathbb{Z}/(m))^*$ such that $\mathcal{L}_j \subset \mathcal{V}_1$ with monodromy representation ρ_j. Now we calculate that $\mathrm{Mon}^0_{\mathbb{Q}(\xi)^+}(\mathcal{V}_1)$ does not reach the upper bound $C_1^{\mathrm{der}}(g)_{\mathbb{Q}(\xi)^+}$ in this case.

Let $a_1 = 0$, $a_3 = 1$ and $a_4 = \infty$. The fundamental group of the corresponding copy of $\mathcal{M}_1$ is generated by $T_{1,2}$ and $T_{2,3}$. One obtains that

$$\rho_j(T_{1,2}) = \begin{pmatrix} \xi^{jd_1+jd_2} & 1-\xi^{jd_1} \\ 0 & 1 \end{pmatrix}, \quad \rho_j(T_{2,3}) = \begin{pmatrix} 1 & 0 \\ \xi^{jd_2} - \xi^{jd_2+jd_3} & \xi^{jd_2+jd_3} \end{pmatrix}.$$

Let $\gamma_v \in \mathrm{Gal}(\mathbb{Q}(\xi);\mathbb{Q})$ denote the automorphism corresponding to $[v]$. The monodromy representation of $\mathcal{L}_{jv}$ is given by

$$\rho_{jv}(T_{1,2}) = \begin{pmatrix} \xi^{jd_1+jd_2} & 1-\xi^{-jd_3} \\ 0 & 1 \end{pmatrix}, \; \rho_{jv}(T_{2,3}) = \begin{pmatrix} 1 & 0 \\ \xi^{jd_1+jd_2+jd_3} - \xi^{jd_2+jd_3} & \xi^{jd_2+jd_3} \end{pmatrix}.$$

One calculates easily that

$$\frac{1-\xi^{jd_1}}{1-\xi^{-jd_3}} \cdot \frac{\xi^{jd_2}-\xi^{jd_2+jd_3}}{\xi^{jd_1+jd_2+jd_3}-\xi^{jd_2+jd_3}} = \frac{\xi^{jd_2}-\xi^{jd_2+jd_3}-\xi^{jd_1+jd_2}+\xi^{jd_1+jd_2+jd_3}}{\xi^{jd_1+jd_2+jd_3}-\xi^{jd_2+jd_3}-\xi^{jd_1+jd_2}+\xi^{jd_2}} = 1.$$

Hence there is a $z \in \mathbb{Q}(\xi)$ such that $\gamma_v|_{<\rho_j(T_{1,2}),\rho_j(T_{2,3})>}$ coincides with $\alpha|_{<\rho_j(T_{1,2}),\rho_j(T_{2,3})>}$, where α is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & zb \\ z^{-1}c & d \end{pmatrix}.$$

Thus by Lemma 5.2.4, the group $\mathrm{Mon}^0_{\mathbb{Q}(\xi)^+}(\mathcal{V}_1)$ does not attain its upper bound in this case. In addition one calculates easily that α is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} \sqrt{z} & 0 \\ 0 & \sqrt{z^{-1}} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \sqrt{z^{-1}} & 0 \\ 0 & \sqrt{z} \end{pmatrix}.$$

Thus the monodromy representations of $\mathcal{L}_j$ and $\mathcal{L}_{jv}$ coincide up to conjugation such that $\mathcal{L}_j$ and $\mathcal{L}_{jv}$ are isomorphic for each $j \in (\mathbb{Z}/(m))^*$.

Corollary 5.4.11. *There are infinitely many families $\mathcal{C} \to \mathcal{P}_1$ such that $\mathcal{V}_1$ is complex and $\mathrm{Mon}^0_{\mathbb{Q}(\xi)^+}(\mathcal{V}_1)$ does not reach its upper bound.*

Proof. Let $p, q \in \mathbb{N}$ such that $\gcd(p,q) = 1$ with $p, q \notin \{1,2\}$ and $m := pq$. Hence $\mathbb{Z}/(m) = \mathbb{Z}/(p) \times \mathbb{Z}/(q)$. Let $v < m$ correspond to $(1,-1) \in \mathbb{Z}/(p) \times \mathbb{Z}/(q)$. Thus we get $[v^2] = [1]_m$ and $[v]_m \notin \{[1]_m, [m-1]_m\}$. One has that

$$d_1 = v, \quad d_2 = 1, \quad d_3 = m-1$$

satisfies the first system of equations of Theorem 5.4.9, which guarantees by 5.4.10 that $\mathrm{Mon}^0_{\mathbb{Q}(\xi)^+}(\mathcal{V}_1)$ does not reach its upper bound. Since there are infinitely many possible choices for $p, q \in \mathbb{N}$ such that $\gcd(p,q) = 1$ with $p, q \notin \{1,2\}$, one obtains infinitely many families $\mathcal{C} \to \mathcal{P}_1$ such that $\mathrm{Mon}^0_{\mathbb{Q}(\xi)^+}(\mathcal{V}_1)$ does not reach its upper bound. □

5.5 The Hodge group of a universal family of hyperelliptic curves

If the middle part $\mathcal{V}_{\frac{m}{2}}$ is of type $(1,1)$, one obtains $\mathrm{Mon}^0(\mathcal{V}_{\frac{m}{2}}) = \mathrm{Sp}_{\mathbb{Q}}(2)$, since $\mathrm{Sp}_{\mathbb{R}}(2) \cong \mathrm{SU}(1,1)$, and $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_{\frac{m}{2}}) = \mathrm{SU}(1,1)$ as one has by Theorem 5.1.1.

By using [63] Theorem 10.1 and Remark 10.2, one can conclude that the Hodge group $\mathrm{Hg}(\mathcal{V}_{\frac{m}{2}})$ of an arbitrary middle part $\mathcal{V}_{\frac{m}{2}}$ coincides with $\mathrm{Sp}(\mathcal{V}_{\frac{m}{2}}, Q_{\mathcal{V}_{\frac{m}{2}}})$. For completeness we give an elementary proof. We use the the fact that

$$\mathrm{Mon}^0(\mathcal{V}_{\frac{m}{2}}) \subseteq \mathrm{Hg}(\mathcal{V}_{\frac{m}{2}}) \subseteq \mathrm{Sp}(\mathcal{V}_{\frac{m}{2}}, Q_{\mathcal{V}_{\frac{m}{2}}})$$

and show by explicit calculations that the dimensions of the Lie algebras of $\mathrm{Mon}^0(\mathcal{V}_{\frac{m}{2}})$ and $\mathrm{Sp}(\mathcal{V}_{\frac{m}{2}}, Q_{\mathcal{V}_{\frac{m}{2}}})$ coincide.

By Proposition 3.3.5, each Dehn twist $T_{\ell,\ell+1}$ yields a unipotent subgroup of $\mathrm{Mon}^0(\mathcal{V}_{\frac{m}{2}})$ isomorphic to $\mathbb{G}_a$. Its corresponding subvector space of the Lie algebra is generated by

$$A_{\ell,\ell+1}(a,b) = \begin{cases} -1 & : \quad a=\ell \quad \text{and} \quad b=\ell-1 \\ 1 & : \quad a=\ell \quad \text{and} \quad b=\ell+1 \\ 0 & : \quad \text{elsewhere} \end{cases} .$$

Now we consider the middle part of type $(2,2)$. Hence we are in the case of the genus 2 curves. For $\ell = 1,\ldots,4$ the matrices $A_{\ell,\ell+1}$ generate a 4 dimensional vector space. Moreover by $[A_{i,i+1}, A_{i+1,i+2}]$ for $i = 1,2,3$, we get the 3 additional linearly independent matrices

$$\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

By

$$[A_{2,3}, [A_{3,4}, A_{4,5}]] \quad \text{resp.,} \quad [[A_{1,2}, A_{2,3}], A_{3,4}],$$

we obtain the two further linearly independent matrices

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus the Lie algebra has at least dimension 9. Moreover one checks easily that

$$[[A_{1,2}, A_{2,3}], [A_{3,4}, A_{4,5}]] = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

is a tenth linearly independent matrix. Thus the well-known fact that $Sp_{\mathbb{Q}}(4)$ has dimension 10 implies:

Proposition 5.5.1. *If $\mathcal{V}_{\frac{m}{2}}$ is of type $(2,2)$, then* $\mathrm{Mon}^0(\mathcal{V}_{\frac{m}{2}}) \cong \mathrm{Sp}(\mathcal{V}_{\frac{m}{2}}, Q_{\mathcal{V}_{\frac{m}{2}}})$.

Note that the quotient of $\mathrm{Sp}_4(\mathbb{R})$ by its maximal compact subgroup is Siegel's upper half plane $\mathfrak{h}_2$, which has dimension 3. Since $\mathcal{M}_3$ has dimension 3, one concludes for the restricted family $\mathcal{C}_{\mathcal{M}_3} \to \mathcal{M}_3$ of genus 2 curves:

Corollary 5.5.2. *The family $\mathcal{C}_{\mathcal{M}_3} \to \mathcal{M}_3$ of genus 2 curves has a dense set of CM fibers.*

Proof. One has (similarly to the proof of Theorem 4.4.4) that the holomorphic period map $p : \mathcal{M}_3 \to \mathfrak{h}_2$ has fibers of dimension 0. Since

$\dim(\mathfrak{h}_2) = \dim(\mathcal{M}_3) = 3$, one concludes that p is open. Hence the statement follows from the fact that $\mathfrak{h}_2$ has a dense set of CM points. $\square$

We will use Proposition 5.5.1 and the calculations, which yield this proposition, to show the following theorem by induction:

Theorem 5.5.3. *If $\mathcal{V}_{\frac{m}{2}}$ is of type (g, g), then* $\mathrm{Mon}^0(\mathcal{V}_{\frac{m}{2}}) \cong \mathrm{Sp}(\mathcal{V}_{\frac{m}{2}}, Q_{\mathcal{V}_{\frac{m}{2}}})$.

Corollary 5.5.4.

$$\mathrm{Hg}(\mathcal{V}_{\frac{m}{2}}) = \mathrm{Sp}(\mathcal{V}_{\frac{m}{2}}, Q_{\mathcal{V}_{\frac{m}{2}}}) \quad \textit{and} \quad \mathrm{MT}(\mathcal{V}_{\frac{m}{2}}) = \mathrm{GSp}(\mathcal{V}_{\frac{m}{2}}, Q_{\mathcal{V}_{\frac{m}{2}}})$$

It is a well-known fact that $\dim(\mathrm{Sp}_{\mathbb{Q}}(2g)) = 2g^2 + g$.[3] Hence one gets

$$\dim(\mathrm{Sp}_{\mathbb{Q}}(2g+1)) = 2(g+1)^2 + g + 1 = (2g^2 + g) + (4g + 3).$$

We will show by induction that for each $g \in \mathbb{N}$ the matrices $A_{\ell,\ell+1}$ generate a Lie algebra, which has at least the same dimension as $\mathfrak{sp}_{2g}(\mathbb{Q})$. This yields Theorem 5.5.3. Since we have shown the statement for $g = 1, 2$, we will only give the induction step:

Recall that we have defined $\mathbb{L}_j$-valued paths $[e_k\delta_k]$ in Section 3.3. We consider a middle part of type $(g+1, g+1)$ with respect to the basis $\mathcal{B} = \{[e_1\delta_1], \ldots, [e_{2g+2}\delta_{2g+2}]\}$. The Dehn twists $T_{\ell,\ell+1}$ for $\ell = 1, \ldots, 2g$ yield the monodromy group G_1 of a middle part of type (g, g). Therefore by the induction hypothesis, they yield a group isomorphic to $\mathrm{Sp}_{2g}(\mathbb{Q})$.

Remark 5.5.5. One has the obvious embedding of $G_1 \hookrightarrow \mathrm{GL}(N^1(C_{\frac{m}{2}}, \mathbb{Q}))$ with respect to the basis $\mathcal{B}_1 := \{[e_1\delta_1], \ldots, [e_{2g}\delta_{2g}], [e_{2g+2}\delta_{2g+2}], [e_{2g+3}\delta_{2g+3}]\}$ such that

$$G_1 \ni A \to \begin{pmatrix} A & & \\ & 1 & 0 \\ & 0 & 1 \end{pmatrix} \in \mathrm{GL}(N^1(C_{\frac{m}{2}}, \mathbb{Q})).$$

Moreover this embedding of G_1 into $\mathrm{GL}(N^1(C_{\frac{m}{2}}, \mathbb{Q}))$ is given by

$$G_1 \ni A \to \begin{pmatrix} A & v & \\ & 1 & 0 \\ & 0 & 1 \end{pmatrix} \in \mathrm{GL}(N^1(C_{\frac{m}{2}}, \mathbb{Q})),$$

with respect to the basis $\mathcal{B}$, where $v^t = (v_1, \ldots, v_{2g})$ is a vector depending on A.

Since we consider the embedding with respect to the latter basis, we want to understand v, which is possible, if we understand the base change between the bases of the preceding remark.

[3] Otherwise one has a description of $\mathfrak{sp}_{2g}(\mathbb{C})$ in [21], page 239. By this description, one can easily determine its dimension.

Lemma 5.5.6. *Let $C \to \mathbb{P}^1$ be a hyperelliptic curve of genus $g+1$. One has (up to a suitable normalization)*

$$\sum_{i=0}^{g+1} [e_{2i+1}\delta_{2i+1}] = 0.$$

Proof. Let $\zeta \in H_2(C, \mathbb{C})$ be a nontrivial linear combination of the closures of the sheets of $\mathbb{P}^1 \setminus S$, on which ψ acts via push-forward by the character $1 \in \mathbb{Z}/(2)$. One has that $\partial\zeta$ represents a linear combination of $[e_1\delta_1], \ldots, [e_{2g+1}\delta_{2g+3}] \in H_1(C, \mathbb{C})_1$, which is equal to zero. Recall that over $\delta_1 \cup \ldots \cup \delta_{2g+3}$ the gluing of these sheets depends on the local monodromy data determined by the branch indices of the branch points a_k. Since each a_k has the local monodromy datum -1, this linear combination is (up to a suitable normalization of $[e_1\delta_1], \ldots, [e_{2g+1}\delta_{2g+3}]$) given by

$$\sum_{i=0}^{g+1} [e_{2i+1}\delta_{2i+1}] = 0.$$

$\square$

5.5.7. By the preceding lemma, the matrices of base change between the bases $\mathcal{B}$ and $\mathcal{B}_1$ are given by

$$M_{\mathcal{B}}^{\mathcal{B}_1}(\mathrm{id}) = \begin{pmatrix} 1 & & & & -1 \\ & \ddots & & & \vdots \\ & & 1 & & -1 \\ & & & 1 & 0 \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix} \quad \text{and} \quad M_{\mathcal{B}_1}^{\mathcal{B}}(\mathrm{id}) = \begin{pmatrix} 1 & & & -1 & \\ & \ddots & & \vdots & \\ & & 1 & -1 & \\ & & & 1 \; 0 & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}$$

such that

$$\begin{pmatrix} A & v & \\ & 1 & 0 \\ & 0 & 1 \end{pmatrix} = M_{\mathcal{B}}^{\mathcal{B}_1}(\mathrm{id}) \cdot \begin{pmatrix} A & & \\ & 1 & 0 \\ & 0 & 1 \end{pmatrix} \cdot M_{\mathcal{B}_1}^{\mathcal{B}}(\mathrm{id}).$$

Thus one calculates easily that $v_1 = 0$, if $a_{1,1} = 1$ and $a_{1,j} = 0$ for $2 \leq j \leq 2g$ and $A = (a_{i,j})$. The exponential map exp is a diffeomorphism on a neighborhood of 0. Hence by the definition

$$\exp(m) = 1 + m + \frac{m^2}{2} + \frac{m^3}{6} + \ldots,$$

one concludes that each $(m_{i,j}) \in Lie(G_1)$ satisfies that $m_{1,2g+1} = 0$, if $m_{1,j} = 0$ for all $j = 1, \ldots, 2g$, which will play a very important role later. Otherwise exp would yield a matrix with $a_{1,1} = 1$, $a_{1,j} = 0$ for $2 \leq j \leq 2g$

and $v_1 \neq 0$ as one can calculate by the fact that each $(m_{i,j}) \in Lie(G_1)$ satisfies that $m_{i,j} = 0$ for $i > 2g$.

Lemma 5.5.8. *Let $i_0 \leq 2g$ and $j_0 < 2g$ be integers such that $i_0 - j_0 > 0$. In the Lie algebra $Lie(G_1)$ one finds an element $(x_{i,j}^{(i_0,j_0)})$ with $x_{i_0,j_0}^{(i_0,j_0)} \neq 0$ and $x_{i,j}^{(i_0,j_0)} = 0$, if $i > i_0$ or $j < j_0$ or $i = 1$.*

Proof. Let $k_0 := i_0 - j_0 > 0$. We show the statement by induction over k_0. Each pair (i_0, j_0) with $i_0 - j_0 = k_0 = 1$ is given by $(i_0, i_0 - 1)$. By A_{i_0,i_0+1}, such an element is given for each $(i_0, i_0 - 1)$.

Now let (i_0, j_0) be a pair with $k_0 := i_0 - j_0 > 1$ and assume that the statement is satisfied for $k_0 - 1, \ldots, 1 > 0$. Hence one has $(x_{i,j}^{(i_0,j_0+1)}), A_{j_0+1,j_0+2} \in Lie(G_1)$. By

$$(x_{i,j}^{(i_0,j_0)}) := [(x_{i,j}^{(i_0,j_0+1)}), A_{j_0+1,j_0+2}],$$

one obtains the desired element of $Lie(G_1)$, since one has the entry

$$x_{i_0,j_0}^{(i_0,j_0)} = x_{i_0,j_0+1}^{(i_0,j_0+1)} \cdot (A_{j_0+1,j_0+2})_{j_0+1,j_0} \neq 0.$$

□

Moreover the Dehn twists $T_{2n-1,2n}, \ldots, T_{2g+2,2g+3}$ generate a group G_2 isomorphic to the monodromy group of a middle part of type $(2, 2)$, which has dimension 10. One can easily compare the matrices of $Lie(G_2)$ with the above explicitly given matrices of a middle part of type $(2, 2)$: "The restriction of the matrices of $Lie(G_2)$ to the lower right corner looks like the matrices of the Lie algebra of the monodromy group of a middle part of type $(2, 2)$."

Since the vectors

$$A_{2g-1,2g}, \quad A_{2g,2g+1} \quad \text{and} \quad [A_{2g-1,2g}, A_{2g,2g+1}]$$

are contained in $Lie(G_1) \cap Lie(G_2)$, both Lie algebras yield together a $2g^2 + g + 7$-dimensional vector space of matrices $(x_{i,j})$, whose entries $x_{i,j}$ vanish for $j < 2g - 3$ and $i > 2g$. Hence by using

$$[A_{2g+1,2g+2}, (x_{i,j}^{(2g,j_0)})] \quad \text{and} \quad [[A_{2g+1,2g+2}, A_{2g+2,2g+3}], x_{i,j}^{(2g,j_0)}]$$

for $j_0 < 2g - 3$, one has $4g - 6$ additional linearly independent vectors. Thus we have altogether $(2g^2 + g) + (4g + 1)$ linearly independent vectors. Hence 2 remaining linearly independent vectors are to find. Since $x_{i,j}^{(i_0,j_0)} = 0$ for $i = 1$, in the constructed vector space of matrices $(m_{i,j})$ the coordinate $m_{1,2g+1}$ depends uniquely on the vectors in $Lie(G_1)$ such that $m_{1,2g+1} = 0$, if $m_{1,j} = 0$ for all $j = 1, \ldots, 2g$ as we have seen in 5.5.7. Let

$$Lie(G_1) \ni (y_{i,j}) = [A_{1,2}, [A_{2,3}, [\ldots [A_{2g-1,2g}, A_{2g,2g+1}] \ldots]].$$

One checks easily that

$$y_{1,2g+1} \neq 0.$$

Now the matrix

$$(y'_{i,j}) = [(y_{i,j}), [A_{2g+1,2g+2}, A_{2g+2,2g+3}]]$$

satisfies $y'_{1,2g+1} \neq 0$, $y'_{i,j} = 0$ for $i, j \leq 2g$ and $y'_{1,2g+2} = 0$. Thus we have found a new vector not contained in the vector space, which we have constructed by $Lie(G_1)$, $Lie(G_2)$ and some Lie brackets at the present.

Note that all matrices $(x_{i,j})$, which we have found, satisfy $x_{1,2g+2} = 0$. But

$$(z_{i,j}) := [(y_{i,j}), A_{2g+1,2g+2}]$$

satisfies $z_{1,2g+2} \neq 0$. Therefore we are done.

5.6 The complete generic Hodge group

By this section, we finish our calculation (of the derived group) of the generic Hodge group and obtain the final result:

Theorem 5.6.1. *One has*

$$\mathrm{Mon}^0(\mathcal{V}) = \prod_{r|m} \mathrm{Mon}^0(\mathcal{V}_r)$$

in the following cases:

1. *The degree m of the covers given by the fibers of $\mathcal{C} \to \mathcal{P}_n$ is odd.*
2. *$\mathcal{P}_n = \mathcal{P}_1$ and 6 does not divide m.*

Corollary 5.6.2. *Assume that $\mathcal{C} \to \mathcal{P}_n$ satisfies one of the following conditions:*

1. *The degree m of the covers given by the fibers of $\mathcal{C} \to \mathcal{P}_n$ is odd.*
2. *$\mathcal{P}_n = \mathcal{P}_1$ and 6 does not divide m.*

Then one has

$$\mathrm{MT}^{\mathrm{der}}(\mathcal{V}) = \mathrm{Hg}^{\mathrm{der}}(\mathcal{V}) \supseteq \prod_{r|m} \mathrm{Mon}^0(\mathcal{V}_r).$$

By Theorem 2.4.4, one has a CM-fiber, if the fibers of $\mathcal{C} \to \mathcal{P}_n$ have $n+1$ branch points with the same branch index d. Thus by the fact that this implies the equality of $\mathrm{Mon}^0(\mathcal{V})$ and $\mathrm{MT}^{\mathrm{der}}(\mathcal{V})$ (see Theorem 3.1.4), one concludes:

Corollary 5.6.3. *Let the fibers of $\mathcal{C} \to \mathcal{P}_n$ have $n+1$ branch points with the same branch index d and $\mathcal{C} \to \mathcal{P}_n$ satisfy one of the following conditions:*

1. The degree m of the covers given by the fibers of $\mathcal{C} \to \mathcal{P}_n$ is odd.
2. $\mathcal{P}_n = \mathcal{P}_1$ and 6 does not divide m.

Then
$$\mathrm{MT}^{\mathrm{der}}(\mathcal{V}) = \mathrm{Hg}^{\mathrm{der}}(\mathcal{V}) = \prod_{r|m} \mathrm{Mon}^0(\mathcal{V}_r).$$

Since $C_r^{\mathrm{der}}(g)$ is an upper bound for $\mathrm{Hg}^{\mathrm{der}}(\mathcal{V}_r)$, one concludes finally:

Corollary 5.6.4. *Assume that $\mathcal{C} \to \mathcal{P}_n$ satisfies one of the following conditions:*

1. The degree m of the covers given by the fibers of $\mathcal{C} \to \mathcal{P}_n$ is odd.
2. $\mathcal{P}_n = \mathcal{P}_1$ and 6 does not divide m.

If all $\mathcal{V}_r$ except of the middle part are very general or special, one has

$$\mathrm{MT}^{\mathrm{der}}(\mathcal{V}) = \mathrm{Hg}^{\mathrm{der}}(\mathcal{V}) = \mathrm{Mon}^0(\mathcal{V}) = \prod_{r|m} \mathrm{Mon}^0(\mathcal{V}_r).$$

Recall that we search for families $\mathcal{C} \to \mathcal{P}_n$ with dense set of complex multiplication fibers. One obtains dense set of complex multiplication fibers, if one has an open (multivalued) period map

$$p : \mathcal{M}_n(\mathbb{C}) \to \mathrm{MT}^{\mathrm{der}}(\mathcal{V})(\mathbb{R})/K$$

given by the VHS. Hence for our applications we need to know $\mathrm{MT}^{\mathrm{der}}(\mathcal{V})$ and the dimension of $\mathrm{MT}^{\mathrm{der}}(\mathcal{V})(\mathbb{R})/K$, but not $\mathrm{MT}(\mathcal{V})$ itself. Let us first prove Theorem 5.6.1. After this proof we will see that the (multivalued) period map of a family $\mathcal{C} \to \mathcal{M}_1$ onto $\mathrm{MT}^{\mathrm{der}}(\mathcal{V})(\mathbb{R})/K$ is open, if and only if one has a $(1,1) - VHS$.

For the proof of Theorem 5.6.1 we use the same methods as before. One has that $\mathrm{Mon}^{\mathrm{ad}}(\mathcal{V})$ is the direct product of the kernel of the natural projection

$$p_1 : \mathrm{Mon}^{\mathrm{ad}}(\mathcal{V}) \to \mathrm{Mon}^{\mathrm{ad}}(\mathcal{V}_{r_1})$$

and an adjoint semisimple group G_{r_1} isomorphic to $\mathrm{Mon}^{\mathrm{ad}}(\mathcal{V}_{r_1})$. Moreover one has that
$$\mathrm{Mon}^{\mathrm{ad}}(\mathcal{V}) = \prod_{r|m} \mathrm{Mon}^{\mathrm{ad}}(\mathcal{V}_r),$$

if and only if each G_{r_1} is contained in the kernels of the natural projections onto all $\mathrm{Mon}^{\mathrm{ad}}(\mathcal{V}_{r_2})$ with $r_1 \neq r_2$.

We give a proof of Theorem 5.6.1 by contradiction. Thus we assume that

$$\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}) \neq \prod_{r|m} \mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_r). \text{ This implies } \mathrm{Mon}^{\mathrm{ad}}_{\mathbb{R}}(\mathcal{V}) \neq \prod_{r|m} \mathrm{Mon}^{\mathrm{ad}}_{\mathbb{R}}(\mathcal{V}_r).$$

Hence some G_{r_1} is not contained in the kernel of the projection onto $\mathrm{Mon}^{\mathrm{ad}}(\mathcal{V}_{r_2})$ for some $r_2 \neq r_1$. Since all simple direct factors of G_{r_1} resp., G_{r_2} project isomorphically onto some $\mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{j_1})$ resp., $\mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{j_2})$, one gets an isomorphism

$$\alpha : \mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{j_1}) \to \mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{j_2}),$$

which respects the respective projective monodromy representations. But by the following proposition, the isomorphism α can not exist, if the assumptions of Theorem 5.6.1 are satisfied. This yields the proof of Theorem 5.6.1.

Proposition 5.6.5. *Assume that* $r_1 := \gcd(m, j_1) \neq r_2 := \gcd(m, j_2)$. *Moreover assume that one of the following cases holds true:*

1. *m is odd.*
2. *$\mathcal{P}_n = \mathcal{P}_1$ and 6 does not divide m.*

Then an isomorphism

$$\alpha : \mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{j_1}) \to \mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{j_2}),$$

which respects the respective projective monodromy representations, can not exist.

Proof. Assume without loss of generality that $r_1 < r_2$. This implies $\frac{m}{r_1} > \frac{m}{r_2}$. There are two cases: Either $2r_1 \neq r_2$ or $2r_1 = r_2$.

If m is odd, one has $\frac{r_1}{2} \neq g := \gcd(\frac{m}{r_1}, \frac{m}{r_2})$. Hence by Lemma 5.3.2, one finds a Dehn twist T such that $P\rho_{j_1}(T)$ is semisimple and the order of $P\rho_{j_1}(T)$ does not divide g. One has that $P\rho_{j_2}(T)$ is either unipotent or semisimple. If $P\rho_{j_2}(T)$ is semisimple, its order divides $\frac{m}{r_2}$. But the order of $P\rho_{j_1}(T)$ does not divide $\frac{m}{r_2}$. If $P\rho_{j_2}(T)$ is unipotent, its order is infinite. But $P\rho_{j_1}(T)$ has finite order. However $P\rho_{j_1}(T)$ and $P\rho_{j_2}(T)$ do not have the same order. Hence such an isomorphism $\alpha : \mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{j_1}) \to \mathrm{Mon}^{\mathrm{ad}}(\mathcal{L}_{j_2})$, which respects the respective projective monodromy representations, can not exist in this case.

Now assume that we are in the case of a family $\mathcal{C} \to \mathcal{P}_1$, where 6 does not divide m. There is a Dehn twist T such that $P\rho_{j_1}(T)$ is semisimple. If $P\rho_{j_1}(T)$ and $P\rho_{j_2}(T)$ do not have the same order, one can argue as above. Otherwise all semisimple Dehn twists have the same order. Hence one must have $2r_1 = r_2$. The nontrivial eigenvalue of $\rho_{j_2}(T)$ is given by the square of the nontrivial eigenvalue ξ of $\rho_{j_1}(T)$. Note that the corresponding maximal tori are isomorphic to S^1, where $S^1_{\mathbb{C}} \cong \mathbb{G}_{m,\mathbb{C}}$. Thus its character group is isomorphic to $\mathbb{Z}$. Hence the induced map of the corresponding maximal tori can be an isomorphism, only if one has $\xi^2 = \xi^{-1} = \bar{\xi}$. In this case ξ would be a primitive cubic root of unity, which implies that 3 divides m. Since we have that $2r_1 = r_2$, 6 would divide m. But by the assumptions, this is not possible. $\square$

Remark 5.6.6. If $2r_1 = r_2$, there are many additional cases, in which α can not exist. These obvious cases are given, if for a Dehn twist T the order of the semisimple matrix $\rho_{r_1}(T)$ does not divide $\frac{m}{2r_1}$, if $\rho_{r_1}(T)$ is semisimple and $\rho_{j_2}(T)$ is unipotent or if $\mathcal{L}_{j_1}$ and $\mathcal{L}_{j_1}$ are of type (a_1, b_1) and (a_2, b_2) such that

$$(a_1, b_1) \neq (a_2, b_2) \quad \text{and} \quad (a_1, b_1) \neq (b_2, a_2).$$

But in the case of the family $\mathcal{C} \to \mathcal{P}_1$ of degree 6 covers given by the local monodromy data

$$d_1 = d_2 = 1, \quad d_3 = d_4 = 5$$

nothing of them holds true with respect to $\mathcal{L}_1$ and $\mathcal{L}_4$. In this case the situation is not clear.

Now let us finish this chapter and state the final result about the period map:

Theorem 5.6.7. *In the case of a family $\mathcal{C} \to \mathcal{M}_1$ the period map*

$$p : \mathcal{M}_1(\mathbb{C}) \to \mathrm{MT}^{\mathrm{der}}(\mathcal{V})(\mathbb{R})/K$$

is open, if and only if one has a pure $(1,1) - VHS$.

Proof. As we have seen in the proof Theorem 4.4.4, the period map is open, if one has a pure $(1,1) - VHS$.

For the other direction assume that the period map is open and there are up to complex conjugation at least two different eigenspaces, which are not unitary.

Lemma 5.6.8. *Assume that we have a family $\mathcal{C}_{\mathcal{M}_1} \to \mathcal{M}_1$. Only if all $\mathcal{V}_r$ except for exactly one $\mathcal{V}_{r_0}$ are special, the period map*

$$p : \mathcal{M}_1(\mathbb{C}) \to \mathrm{MT}^{\mathrm{der}}(\mathcal{V})(\mathbb{R})/K$$

can be open.

Proof. Assume that r_1 and r_2 divide m such that $r_1 \neq r_2$ and $\mathcal{V}_{r_1}$ and $\mathcal{V}_{r_2}$ are not special. If $2r_1 \neq r_2$ or if there is a Dehn twist, whose finite order with respect to $\mathcal{V}_{r_1}$ does not divide $\frac{m}{r_2} = \frac{m}{2r_1}$, the same arguments as in the proof of Proposition 5.6.5 imply that

$$\dim(\mathrm{MT}^{\mathrm{der}}(\mathcal{V})(\mathbb{R})/K) > 1 = \dim(\mathcal{M}_1).$$

Therefore the period map can not be open.

Otherwise assume without loss of generality that $r_1 = 1$ and all semisimple Dehn twists have an order dividing $\frac{m}{2}$. This implies that all d_k are odd and the degree m is even. Hence $\mathrm{Mon}^0(\mathcal{V}_{\frac{m}{2}})$ is isomorphic to $\mathrm{Sp}_{\mathbb{Q}}(2)$, where its monodromy representation sends all Dehn twists to unipotent matrices. Thus $\dim(\mathrm{MT}^{\mathrm{der}}(\mathcal{V})(\mathbb{R})/K) > 1$. □

By Lemma 5.6.8, these two eigenspaces, which are not unitary, must be contained in the same $\mathcal{V}_{r_0}$, which must be exceptional. Hence assume without loss of generality that $\mathcal{V}_{r_0} = \mathcal{V}_1$.

In the separated case, the fact that all d_k are odd (compare to 5.4.4) implies that $\mathrm{Mon}^0_{\mathbb{R}}(\mathcal{V}_{\frac{m}{2}}) = \mathrm{Sp}_{\mathbb{R}}(2)$. Hence by Lemma 5.6.8, we have a contradiction.

In the complex case Lemma 5.4.2 implies without loss of generality that

$$t_1|d_1 + d_2, \quad t_1|d_2 + d_3, \quad t_1|d_1 + d_4, \quad t_1|d_3 + d_4.$$

This implies that t_1 divides each d_k or that t_1 does not divide any d_k. Thus t_1 does not divide any d_k. Hence $\mathcal{C}_{\frac{m}{t_1}}$ is a family of covers with 4 branch points, where $\rho_{\frac{m}{t_1}}(T_{1,2})$ and $\rho_{\frac{m}{t_1}}(T_{2,3})$ are unitary. Hence $\mathcal{V}_{\frac{m}{t_1}}$ has an infinite monodromy group resp., it is not special. Thus by Lemma 5.6.8, we have a contradiction. □

Chapter 6
Examples of families with dense sets of complex multiplication fibers

In this chapter we classify all families $\mathcal{C} \to \mathcal{P}_n$ of covers with a pure $(1,n) - VHS$. Due to Theorem 4.4.4, all these families have a dense set of CM fibers. We say that a pure $(1,n) - VHS$ is primitive, if the $(1,n)$ eigenspace $\mathcal{L}_j$ satisfies that $j \in (\mathbb{Z}/(m))^*$. Otherwise the pure $(1,n) - VHS$ is derived.

In Section 6.1 we give an integral condition for the branch indices d_k of the family $\mathcal{C}$ with the fibers given by

$$y^m = (x - a_1)^{d_1} \cdot \ldots \cdot (x - a_n)^{d_n}.$$

This integral condition is stronger than the similar integral condition INT of P. Deligne and G. D. Mostow [18]. Thus we call this strong integral condition $SINT$. We show that this condition is necessary for the existence of a primitive pure $(1,n) - VHS$. By using this condition, we compute all examples of families $\mathcal{C} \to \mathcal{P}_1$ of covers with a primitive pure $(1,1) - VHS$ in Section 6.2, which will be listed in Section 6.3. By using the list of examples satisfying INT for $n > 1$ in [18], we give in Section 6.3 the complete lists of families with a primitive pure $(1,n) - VHS$. In Section 6.3 we give also the complete list of examples with a derived pure $(1,n) - VHS$, which will be verified in Section 6.4.

6.1 The necessary condition *SINT*

By Theorem 4.4.4, one has a sufficient criterion for a dense set of CM fibers of a family $\mathcal{C}_{\mathcal{M}_n} \to \mathcal{M}_n$. This criterion is satisfied, if $\mathcal{C}$ has a pure $(1,n) - VHS$ (i.e. its VHS contains one eigenspace of type $(1,n)$, a complex conjugate eigenspace of type $(n,1)$ and otherwise only eigenspaces of the type $(a,0)$ and $(0,b)$ for some $a, b \in \mathbb{N}_0$).

Remark 6.1.1. Assume that the family $\mathcal{C} \to \mathcal{P}_n$ of cyclic covers of degree m has a pure $(1,n)$-VHS and that $\mathcal{L}_{j_0}$ is the eigenspace of type $(1,n)$.

J.C. Rohde, *Cyclic Coverings, Calabi-Yau Manifolds and Complex Multiplication*, Lecture Notes in Mathematics 1975, DOI: 10.1007/978-3-642-00639-5_7,

Let $j_0 \notin (\mathbb{Z}/(m))^*$. Then we have $1 < r_0 := gcd(j_0, m)$. By Section 4.2, the family $\mathcal{C}_{r_0}$ has a pure $(1,n)$-VHS, too.

Definition 6.1.2. A pure $(1,n) - VHS$ is primitive, if $j_0 \in (\mathbb{Z}/(m))^*$. Otherwise it is a derived pure $(1,n) - VHS$ with the associated primitive pure $(1,n) - VHS$ induced by $\mathcal{C}_{r_0}$, where $\mathcal{C}_{r_0}$ is given by the preceding remark.

Hence first we search for families with a primitive pure $(1,n)-VHS$. Later we will look for families with a derived pure $(1,n)-VHS$. It is helpful to have a necessary condition to find the families with a primitive pure $(1,n)-VHS$. In [18] P. Deligne and G. D. Mostow have formulated the following integral condition INT:

Definition 6.1.3. A local system on $\mathbb{P}^1 \setminus S$ of monodromy $(\alpha_s)_{s \in S}$ with $\alpha_s = exp(2\pi i \mu_s)$ and $\mu_s \in \mathbb{Q}$ for all $s \in S$ satisfies the condition INT, if:

1. $0 < \mu_s < 1$ for all $s \in S$.
2. We have for all $s, t \in S$: $(1-\mu_s-\mu_t)^{-1}$ is an integer, if $s \neq t$ and $\mu_s+\mu_t < 1$.
3. $\sum \mu_s = 2$.

Remark 6.1.4. By P. Deligne and G. D. Mostow [18], the monodromy of the fractional period map of an eigenspace $\mathcal{L}_j$ of type $(1,n)$ is discrete in the unitary group $U(1,n)$ and has finite covolume, if one has that $H_1(\mathcal{C}_q, \mathbb{C})_j$ satisfies INT for some $q \in \mathcal{P}_n$. This sufficient condition can be replaced by a weaker condition ΣINT as G. D. Mostow [44] has shown. Later G. D. Mostow [45] has shown that ΣINT is necessary for the discreteness, if $n > 3$.

One can identify the local monodromy data, which yield the family $\mathcal{C} \to \mathcal{P}_n$ by Construction 3.2.1, with the local monodromy data of the eigenspace $\mathbb{L}_1$ of some fiber $\mathcal{C}_q$ for an arbitrary $q \in \mathcal{P}_n$. Hence one can formulate the condition INT for the local monodromy data of the family. For the latter data we give a corresponding stronger integral condition $SINT$:

Definition 6.1.5. A family $\mathcal{C} \to \mathcal{P}_n$ of cyclic covers of $\mathbb{P}^1$ given by the local monodromy data given by $\mu_k \in \mathbb{Q}$ around $s_k \in N$ satisfies $SINT$, if we have:

1. $\mu_{k_1} + \mu_{k_2} = 1$ or $(1 - \mu_{k_1} - \mu_{k_2})^{-1} \in \mathbb{Z}$ for all $s_{k_1}, s_{k_2} \in N$ with $s_{k_1} \neq s_{k_2}$.
2. $\sum \mu_s = 2$.

Remark 6.1.6. The reader checks easily that for a family $\mathcal{C} \to \mathcal{P}_1$ the conditions INT and $SINT$ are equivalent. Moreover by the list on [18], page 86, each family $\mathcal{C} \to \mathcal{P}_n$ with $n \geq 2$, which satisfies INT, satisfies $SINT$, too.

At the present the author can not explain this fact. We use $SINT$ instead of INT, since this is a stronger and hence a more helpful condition.

By the following theorem, we have our helpful necessary condition for families $\mathcal{C}$, which have a primitive pure $(1,n) - VHS$:

Theorem 6.1.7. *If the family $\mathcal{C} \to \mathcal{P}_n$ has a primitive pure $(1,n) - VHS$, its local monodromy data can be given rational numbers satisfying $SINT$.*

For the proof of Theorem 6.1.7 we first reduce the situation to the case of a family $\mathcal{C} \to \mathcal{P}_1$ of covers with only 4 branch points. That means we will consider a pair of branch points of a fiber of $\mathcal{C} \to \mathcal{P}_n$, where $\mathcal{C}$ has a primitive pure $(1,n) - VHS$, as a pair of branch points with the same branch indices of a fiber of a family $\mathcal{C}(P) \to \mathcal{P}_1$, which has a primitive pure $(1,1) - VHS$. The following lemma will make it possible in almost all cases:

Lemma 6.1.8. *Assume that $\mathcal{C}$ is given by local monodromy data on at least 5 points, where one does not have $\mu_3 = \ldots = \mu_{n+3} = \frac{1}{2}$. Then there exists a stable partition P with $\{a_1\}, \{a_2\} \in P$ such that $|P| = 4$.*[1]

Proof. One can without loss of generality assume that $\mu_1 + \mu_2 \leq 1$. Otherwise we take the local monodromy data of $\mathcal{L}_{m-1}$.

Now assume that such a stable partition P with $\{a_1\}, \{a_2\} \in P$ does not exist. Hence one must have $\mu_1 + \mu_2 + \mu_k = 1$ for all $3 \leq k \leq n+3$. Otherwise one obtains the stable partition

$$P = \{\{a_1\}, \{a_2\}, \{a_k\}, \{a_3, \ldots, a_{k-1}, a_{k+1}, \ldots, a_{n+3}\}\}$$

Thus one must have

$$\mu := \mu_3 = \ldots = \mu_{n_j+3}.$$

Since

$$P = \{\{a_1\}, \{a_2\}, \{a_3, a_4\}, \{a_5, \ldots, a_{n_j+3}\}\}$$

is not a stable partition by our assumption, too, one has

$$2\mu = \mu_3 + \mu_4 = 1. \text{ Hence } \mu = \frac{1}{2}.$$

□

6.1.9. The family of irreducible cyclic covers of $\mathbb{P}^1$ given by the local monodromy data

$$\mu_1 = \mu_2 = \frac{1}{4}, \quad \mu_3 = \mu_4 = \mu_5 = \frac{1}{2}$$

has a primitive pure $(1,2) - VHS$. Moreover it is easy to calculate that this family satisfies $SINT$.

This is the only example of a family $\mathcal{C} \to \mathcal{P}_n$ with a primitive pure $(1,n) - VHS$ for $n > 1$, which does not satisfy the assumptions of Lemma 6.1.8: It is very easy to see that this is the only degree 4 example with a primitive pure $(1,n) - VHS$ for $n > 1$, which contradicts the assumptions of Lemma 6.1.8. If $m > 4$, $\mathcal{L}_3$ must be unitary. But in this case the condition that

$$n + 3 > 4 \text{ and } [3\mu_3]_1 = \ldots = [3\mu_{n+3}]_1 = \frac{1}{2}$$

[1] Since the assumptions of this lemma are sufficient, we do not restrict to the interesting case of a family with a primitive pure $(1,n) - VHS$.

and Proposition 2.3.4 imply that

$$h_3^{1,0}(C) \geq (\sum_{k\geq 3}[3\mu_k]_1) - 1 = (\sum_{k\geq 3}\frac{1}{2}) - 1 > 0$$

and

$$h_3^{0,1}(C) \geq \sum_{k\geq 3}(1-[3\mu_k]_1) - 1 = (\sum_{k\geq 3}\frac{1}{2}) - 1 > 0.$$

Thus $\mathcal{L}_3$ is not unitary.

6.1.10. Assume that $\mathcal{C} \to \mathcal{P}_n$ has a primitive pure $(1,n) - VHS$. Hence $\mathcal{L}_1$ is without loss of generality the eigenspace of type $(1,n)$. For our application of Lemma 6.1.8 we must check that the collision of Lemma 6.1.8 resp., its corresponding stable partition yields a family $\mathcal{C}(P) \to \mathcal{P}_1$, which has a primitive pure $(1,1) - VHS$. The family $\mathcal{C}(P)$ is given by $N = P$ with the local monodromy data

$$\alpha_{\{a_k,\ldots,a_\ell\}} = \alpha_k \cdot \ldots \cdot \alpha_\ell \quad (\forall \quad \{a_k,\ldots,a_\ell\} \in P)$$

as in Construction 3.2.1. The fibers of $\mathcal{C}(P)$ have degree m', where m' divides m. For $j = 1,\ldots,m'-1$ and $q \in \mathcal{M}_1$, the eigenspace $\mathbb{L}_j(P)$ in the Hodge structure of $\mathcal{C}(P)_q$ with the character j is given by the local monodromy data

$$[j\mu_1]_1, \quad [j\mu_2]_1, \quad [j\mu_3 + \ldots + j\mu_k]_1, \quad [j\mu_{k+1} + \ldots + j\mu_{n+3}]_1.$$

If the eigenspace $\mathcal{L}_j$ in the VHS of $\mathcal{C}$ is of type $(0,a)$, Proposition 2.3.4 implies that its local monodromy data satisfy

$$[j\mu_1]_1 + \ldots + [j\mu_{n+3}]_1 = 1.$$

Hence one has that

$$[j\mu_1]_1 + [j\mu_2]_1 + [j\mu_3 + \ldots + j\mu_k]_1 + [j\mu_{k+1} + \ldots + j\mu_{n+3}]_1 = 1,$$

too. Thus by Proposition 2.3.4, $\mathbb{L}_j(P)$ is of type $(0,a')$.

If $\mathcal{L}_j$ is of type $(a,0)$, $\mathcal{L}_{m-j}$ is of type $(0,a)$. The dual eigenspace $\mathbb{L}_j(P)^\vee$ of $\mathbb{L}_j(P)$ is given by

$$[(m-j)\mu_1]_1, \quad [(m-j)\mu_2]_1, \quad [(m-j)\mu_3 + \ldots + (m-j)\mu_k]_1,$$
$$[(m-j)\mu_{k+1} + \ldots + (m-j)\mu_{n+3}]_1.$$

The same arguments as above tell us that $\mathbb{L}_j(P)^\vee$ is of type $(0,a')$. Thus $\mathbb{L}_j(P)$ is of type $(a',0)$.

The restricted family $\mathcal{C}_{\mathcal{M}_1}(P) \to \mathcal{M}_1$ of cyclic covers with 4 different branch points has a non-trivial variation of Hodge structures. This follows from the fact that each fiber of $\mathcal{C}_{\mathcal{M}_n} \to \mathcal{M}_n$ is isomorphic to only finitely many other fibers (compare to 4.4.2). Therefore the eigenspaces $\mathbb{L}_1(P)$ and $\mathbb{L}_{m'-1}(P)$ are of type $(1,1)$. In addition one concludes that $m' = 2$ or $m' = m$.

Now we are without loss of generality in the case of a family $\mathcal{C} \to \mathcal{P}_1$ with a primitive pure $(1,1) - VHS$. For the proof of Theorem 6.1.7 we need the following lemma:

Lemma 6.1.11. *Let C and C' be curves and $\gamma : \mathrm{Jac}(C) \to \mathrm{Jac}(C')$ be an isomorphism of principally polarized abelian varieties. Then there exists a unique isomorphism $f : C \to C'$ such that*

$$\pm\gamma \circ \alpha_p = \alpha_{f(p)} \circ f$$

for each $p \in C$, where α_p and $\alpha_{f(p)}$ denote the respective Abel-Jacobi maps.

Proof. By [39], Theorem 12.1, for each $p \in C$ and $p' \in C'$ there is a unique isomorphism $f : C \to C'$ and a unique $c \in \mathrm{Jac}(C')$ such that

$$\pm\gamma \circ \alpha_p + c = \alpha_{p'} \circ f.$$

Since $(\gamma \circ \alpha_p)(p) = 0 \in \mathrm{Jac}(C')$ and $(\alpha_{p'} \circ f)(p) = [f(p) - p'] \in \mathrm{Jac}(C')$, one has $c = 0$ for $p' = f(p)$. $\square$

By the next proposition, we will apply Lemma 6.1.11 for our proof of Theorem 6.1.7:

Proposition 6.1.12. *Let $q_1, q_2 \in \mathcal{P}_n$ and $\mathcal{C} \to \mathcal{P}_n$ be a family of cyclic covers. Assume there is an isomorphism between the polarized integral Hodge structures of the fibers $\mathcal{C}_{q_1}$ and $\mathcal{C}_{q_2}$, which respects the eigenspace decompositions of $H^1(\mathcal{C}_{q_1}, \mathbb{C})$ and $H^1(\mathcal{C}_{q_1}, \mathbb{C})$. Then there is an isomorphism $\iota : \mathcal{C}_{p_1} \to \mathcal{C}_{p_2}$ and an isomorphism $\alpha : \mathbb{P}^1 \to \mathbb{P}^1$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathcal{C}_{p_1} \arrow[r, "\iota"] \arrow[d] & \mathcal{C}_{p_2} \arrow[d] \\ \mathbb{P}^1 \arrow[r, "\alpha"] & \mathbb{P}^1 \end{tikzcd}$$

Proof. Let γ be an isomorphism of polarized Hodge structures respecting the eigenspace decompositions of $H^1(\mathcal{C}_{q_1}\mathbb{C})$ and $H^1(\mathcal{C}_{q_1}\mathbb{C})$. Then there exists a suitable pair (ψ_1, ψ_2) of generators of the Galois groups of $\mathcal{C}_{q_1}$ and $\mathcal{C}_{q_1}$ such that

$$\gamma \circ (\psi_1)_* = (\psi_2)_* \circ \gamma.$$

For simplicity we write ψ instead of ψ_1 and ψ_2.

By the exponential exact sequence, an isomorphism $\gamma : H^1(\mathcal{C}_{q_1}, \mathbb{Z}) \to H^1(\mathcal{C}_{q_2}, \mathbb{Z})$ of polarized Hodge structures commuting with the action of ψ

on these integral Hodge structures induces an isomorphism $\gamma' : \mathrm{Jac}(\mathcal{C}_{q_1}) \to \mathrm{Jac}(\mathcal{C}_{q_2})$ commuting with ψ_*. In other terms one has

$$\gamma' \circ \psi_* = \psi_* \circ \gamma'$$

for the Jacobians.

By Lemma 6.1.11, one obtains a unique isomorphism $\mathcal{C}_{p_1} \xrightarrow{\iota} \mathcal{C}_{p_2}$ such that

$$\iota \circ \psi = \psi \circ \iota.$$

Thus one obtains the desired automorphism α. □

6.1.13. Now assume that $\mathcal{C} \to \mathcal{P}_n$ has a primitive pure $(1,n) - VHS$. Moreover one can without loss of generality assume that $\mathcal{L}_1$ is the eigenspace of type $(1,n)$. Choose $s_1, s_2 \in N$.

If $\mu_1 + \mu_2 = 1$, there remains nothing to prove for these two points with respect to Theorem 6.1.7.

Otherwise we let the branch points collide as in Lemma 6.1.8, if we are not in the only exceptional case, which satisfies $SINT$ as we have seen in 6.1.9. Thus we can restrict to the case $\mathcal{C} \to \mathcal{P}_1$. Assume that all 4 branch points of a fiber of $\mathcal{C} \to \mathcal{P}_1$ have pairwise different branch indices. By Proposition 6.1.12, there will not be an isomorphism α between different fibers, which respects the action of the Galois group. Hence the fractional period map according to $\mathcal{L}_1|_{\mathcal{M}_1}$ is injective. Now choose the embedding $\mathcal{M}_1 \hookrightarrow \mathbb{P}^1$ corresponding to

$$p_1 = 0, \quad p_3 = 1, \quad p_4 = \infty.$$

By [36], Section 4, one can identify the fractional period map concerning $\mathcal{L}_1$ with some multivalued map, which is called Schwarz map. The Schwarz map is the composition of the multivalued map studied by P. Deligne and G. D. Mostow in [18], which is defined by some integrals, with the natural map $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n_{\mathbb{C}}$. By [18], 9.6 and the preceding description of the fractional period map, there exists a sufficiently small neighborhood U of $0 \in \mathbb{P}^1 \setminus \mathcal{M}_1$ such that the fractional period map concerning $\mathcal{L}_1$ is (up to a biholomorphic map) given by $x \to x^{1-\mu_1-\mu_2}$ on $U \setminus \{0\}$. Hence the injectivity of the period map implies that $(1-\mu_1-\mu_2)^{-1} \in \mathbb{Z}$. This yields $SINT$.

6.1.14. Now we have the problem that we can not directly apply Proposition 6.1.12 as before, if we assume that there are 4 branch points, where exactly two of them have the same branch index: Let p_1 and p_4 have the same branch index and p_3 run around p_2, where

$$p_1 = 0, \quad p_2 = 1, \quad p_4 = \infty.$$

The automorphism $x \to x^{-1}$ interchanges 0 and ∞ and leaves a basis of neighborhoods of $1 \in \mathbb{P}^1 \setminus \mathcal{M}_1$ invariant. We have obviously the same problem, if we let p_1 run around p_4. But for all other pairs $k_1, k_2 \in \{1,2,3,4\}$ with $k_1 \neq k_2$ and the coordinates

$$k_1 = 0, \quad k_3 = 1, \quad k_4 = \infty,$$

Proposition 6.1.12 implies that the multivalued period map is injective on $U \setminus \{0\}$, where U is a sufficiently small neighborhood of $0 \in \mathbb{P}^1 \setminus \mathcal{M}_1$. Thus k_1 and k_2 satisfy the integral condition

$$1 - \mu_{k_1} - \mu_{k_2} = 0 \text{ or } (1 - \mu_{k_1} - \mu_{k_2})^{-1} \in \mathbb{Z}.$$

Hence one must ensure that the remaining pairs satisfy this latter condition, in order to show that $SINT$ is satisfied:

Let us change the enumeration and assume that $\mu_1 = \mu_2$. By Proposition 6.1.12, we can have

$$1 - \mu_1 - \mu_2 = \frac{1}{\ell} \text{ or } 1 - \mu_1 - \mu_2 = \frac{2}{\ell} \text{ or } 1 - \mu_1 - \mu_2 = 0$$

for some odd $\ell \in \mathbb{Z}$. Note that $1-\mu_1-\mu_2 = -(1-\mu_3-\mu_4)$, if $\mu_1+\ldots+\mu_4 = 2$. Hence we only have to exclude the second case

$$(1 - \mu_1 - \mu_2) = \frac{2}{\ell}.$$

First assume that m is odd. In this case $m - 2d_1$ is odd and the second case can not occur. Hence assume that m is even and let $m = 2^s r$, where $r = k \cdot \ell$ is odd. If the second case holds true, one has

$$\frac{2^s k\ell - 2d_1}{2^s k\ell} = \frac{2}{\ell} \Leftrightarrow 2^{s-1}k\ell - d_1 = 2^s k \Leftrightarrow d_1 = 2^{s-1}k(\ell - 2).$$

If $s \geq 2$, one has that $d_1 = d_2$ is even. Since $\mathcal{C}_2$ must have a trivial VHS, one has without loss of generality that $d_3 = 2^{s-1}k\ell$. Since we have

$$2m = d_1 + \ldots + d_4,$$

which is even, where d_1, d_2, d_3 are even, too, d_4 must be even. But in this case the cover is not irreducible. Hence we must have $s = 1$. Since $\mathcal{C}_2$ must have a trivial VHS, one has without loss of generality $d_3 = k\ell$. Since we have

$$2m = d_1 + \ldots + d_4,$$

which is even, where d_1, d_2, d_3 are odd, one must have that d_4 is odd, too. But in this case $\mathcal{C}_{k\ell}$ is the family of elliptic curves and we do not have a primitive pure $(1,1) - VHS$. Therefore the second case is excluded.

If we have 4 branch points and more than exactly two of them have the same branch index, one can have the additional simple cases

$$\mu_1 = \mu_2 = \mu_3 \text{ or } \mu_1 = \mu_2, \quad \mu_3 = \mu_4.$$

For these very simple cases one can directly calculate all occurring examples of families $\mathcal{C} \to \mathcal{P}_1$ with a primitive pure $(1,1) - VHS$. Then one can verify by their local monodromy data that Theorem 6.1.7 holds true in these cases as we will do now.

Remark 6.1.15. One must without loss of generality have

$$d_1 < d_4 \text{ resp.}, \; d_1 = d_2 = d_3 < d_4 \text{ or } d_1 = d_2 < d_3 = d_4$$

in the simple cases, if $m > 2$. Otherwise we would obtain

$$d_1 = d_2 = d_3 = d_4 = \frac{m}{2},$$

which implies that $\mathcal{C}$ is not irreducible, if $m > 2$.

Lemma 6.1.16. *Assume that the family $\mathcal{C} \to \mathcal{P}_1$ with the branch indices $d_1 = d_2 = d_3 \neq d_4$ has a primitive pure $(1,1) - VHS$. Then the degree m is odd and satisfies $m \leq 9$. Moreover one has without loss of generality that $d_1 = d_2 = d_3 = 1$.*

Proof. By the assumptions we have that $2m = 3d_1 + d_4$. Hence $g = \gcd(m, d_1) = 1$ divides d_4, too, which implies by the irreducibility of the fibers of $\mathcal{C}$ that $g = 1$. Thus if m is even, we have that $d_1 = d_2 = d_3$ and d_4 are odd. But in this case $\mathcal{C}_{\frac{m}{2}}$ would be the family of elliptic curves such that $\mathcal{L}_{\frac{m}{2}}$ is of type $(1,1)$. Contradiction! Hence m must be odd.

It remains to show that $m \leq 9$. Since $\gcd(m, d_1) = 1$, the fibers are without loss of generality given by

$$y^m = x(x-1)(x-\lambda)$$

such that $\mathcal{L}_{[\frac{m}{2}]}$ is of type $(1,1)$ as one can calculate by Proposition 2.3.4. By Proposition 2.3.4, one can calculate the type of $\mathcal{L}_{\frac{m-3}{2}}$ by its local monodromy data, too. For this local system one gets that

$$3[\frac{m-3}{2m}]_1 + [\frac{(m-3)(m-3)}{2m}]_1$$

$$= 3\frac{m-3}{2m} + (m-3)\frac{m-3}{2m} - [\frac{(m-3)(m-3)}{2m}] = \frac{m-3}{2} - [\frac{(m-3)(m-3)}{2m}].$$

Now let us assume that $9 < m$. Since m must be odd, we obtain

$$3[\frac{m-3}{2m}]_1 + [\frac{(m-3)(m-3)}{2m}]_1 = \frac{m-3}{2} - [\frac{m-6}{2} + \frac{9}{2m}] = \frac{m-3}{2} - \frac{m-7}{2} = 2.$$

This result and Proposition 2.3.4 imply that $\mathcal{L}_{\frac{m-3}{2}}$ is of type $(1,1)$ in this case, too. Hence we do not have a pure $(1,1)-VHS$, if $9 < m$. $\square$

Remark 6.1.17. In the case of the preceding lemma one obtains all examples of families $\mathcal{C} \to \mathcal{P}_1$ with a primitive pure $(1,1)-VHS$ by $m = 5, 7, 9$, which satisfy $SINT$ as one can calculate easily, too.

Remark 6.1.18. If we are in the second simple case $d_1 = d_2 \neq d_3 = d_4$, one obtains

$$d_1 + d_3 = d_2 + d_4 = m.$$

By the fact that $d_1 \neq d_3$, one concludes that $\mu_1, \mu_3 \neq \frac{1}{2}$. Hence the local monodromy data of $\mathcal{L}_2$ satisfy $[2\mu_i]_1 \neq 0$ for all $i = 1, \ldots, 4$. Moreover one has

$$[2\mu_1]_1 + [2\mu_3]_1 = [2\mu_2]_1 + [2\mu_4]_1 = 1.$$

Hence $\mathcal{L}_2$ is of type $(1,1)$ and $\mathcal{C}$ can have a primitive $(1,1)-VHS$, only if $m = 3$. Thus the only possible case is given by

$$\mu_1 = \mu_2 = \frac{1}{3} \text{ and } \mu_3 = \mu_4 = \frac{2}{3},$$

which satisfies $SINT$ as one can easily verify.

6.2 The application of $SINT$ for the more complicated cases

In the preceding section we have seen that $SINT$ is a necessary condition for families $\mathcal{C} \to \mathcal{P}_n$ with a primitive pure $(1,n)-VHS$. In addition we have given all examples of families $\mathcal{C} \to \mathcal{P}_1$ with a primitive pure $(1,1)-VHS$, which do not satisfy that at most two branch points have the same branch index. Here we calculate all examples of families $\mathcal{C} \to \mathcal{P}_1$ with a primitive pure $(1,1)-VHS$, which satisfy that at most two branch points have the same branch index.

By technical reasons, we will sometimes assume $m \geq 4$. Note that the only possible case of a family $\mathcal{C} \to \mathcal{P}_1$ of degree 3 covers with a pure $(1,1)-VHS$ is given by Remark 6.1.18, where the only possible case of degree 2 covers is given by the elliptic curves. Thus this assumption does not provide any restriction for the more complicated cases.

Note that in the case of a family $\mathcal{C} \to \mathcal{P}_1$ the condition $SINT$ is equivalent to INT.

Remark 6.2.1. By [18], 14.3, one can describe all families of covers $\mathcal{C} \to \mathcal{P}_1$, whose local monodromy data satisfy INT, such that there is not any pair $k_1, k_2 \in \{1,2,3,4\}$ with $k_1 \neq k_2$ satisfying $\mu_{k_1} + \mu_{k_2} = 1$, in the following way: Let $(p,q,r) \in \mathbb{N}^3$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ and $1 < p \leq q \leq r < \infty$. Then in the case of 4 branch points these solutions of INT for covers can be given by:

$$\mu_1 = \frac{1}{2}(1 - \frac{1}{p} - \frac{1}{q} + \frac{1}{r}), \quad \mu_2 = \frac{1}{2}(1 - \frac{1}{p} + \frac{1}{q} - \frac{1}{r}),$$
$$\mu_3 = \frac{1}{2}(1 + \frac{1}{p} - \frac{1}{q} - \frac{1}{r}), \quad \mu_4 = \frac{1}{2}(1 + \frac{1}{p} + \frac{1}{q} + \frac{1}{r})$$

We have that

$$\mu_1 + \mu_2 = 1 - \frac{1}{p}, \quad \mu_1 + \mu_3 = 1 - \frac{1}{q}, \quad \mu_2 + \mu_3 = 1 - \frac{1}{r}.$$

Thus p ,q, and r divide the degree m of the cover. This fact and the equations, which use p, q, and r for the definition of the different μ_i, imply that we have

$$m = \mathrm{lcm}(p,q,r) \quad \text{or} \quad m = 2 \cdot \mathrm{lcm}(p,q,r).$$

If we are in the case of a family with a primitive pure $(1,1) - VHS$ such that all local monodromy data satisfy $\mu_{k_1} + \mu_{k_2} \neq 1$ and at most two branch points have the same index, we are in the case of Remark 6.2.1 with the additional condition $p < r$. Hence let us first consider this case. Later we will consider families with at most two branch points with the same branch index and some $\mu_{k_1} + \mu_{k_2} = 1$, which is the last remaining subcase.

Now let $\ell := \mathrm{lcm}(p,q,r)$.

Lemma 6.2.2. *Let $\mathcal{C} \to \mathcal{P}_1$ be given by p,q,r as in Remark* 6.2.1*, where $p < r$, and have a primitive pure* $(1,1) - VHS$*. Then one has*

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{r}.$$

Proof. Since $p|\ell$ resp., $p|m$, we have the family $\mathcal{C}_p$, which must have a trivial VHS. This implies that there is a d_{i_0} with $\frac{m}{p}|d_{i_0}$, which implies that $\frac{\ell}{p}|d_{i_0}$. Since

$$d_{i_0} = \ell \pm \frac{\ell}{p} \pm \frac{\ell}{q} \pm \frac{\ell}{r} \quad \text{or} \quad 2d_{i_0} = \ell \pm \frac{\ell}{p} \pm \frac{\ell}{q} \pm \frac{\ell}{r},$$

one concludes that $\frac{\ell}{p}|(\frac{\ell}{q} \pm \frac{\ell}{r})$. From the fact that $\frac{\ell}{p} \geq \frac{\ell}{q}$ and $\frac{\ell}{p} > \frac{\ell}{r}$, one obtains

$$\frac{\ell}{p} = \frac{\ell}{q} + \frac{\ell}{r}. \quad \text{Hence} \quad \frac{1}{p} = \frac{1}{q} + \frac{1}{r}.$$

□

Lemma 6.2.3. *Let $\mathcal{C} \to \mathcal{P}_1$ be a family with a primitive pure $(1,1) - VHS$, which is given by p, q, r as in Remark 6.2.1, where $p < r$. Then the family $\mathcal{C}$ and the eigenspace $\mathbb{L}_1$ are given by the local monodromy data*

$$\mu_1 = \frac{1}{2} - \frac{1}{q}, \quad \mu_2 = \frac{1}{2} - \frac{1}{r}, \quad \mu_3 = \frac{1}{2}, \quad \mu_4 = \frac{1}{2} + \frac{1}{p}.$$

Proof. By Lemma 6.2.2, we have

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{r}.$$

This equation and Remark 6.2.1, this imply that $\mathcal{C}$ and $\mathbb{L}_1$ have the local monodromy data

$$\mu_1 = \frac{1}{2} - \frac{1}{q}, \quad \mu_2 = \frac{1}{2} - \frac{1}{r}, \quad \mu_3 = \frac{1}{2}, \quad \mu_4 = \frac{1}{2} + \frac{1}{p}.$$

$\square$

Remark 6.2.4. Let $\mathcal{C} \to \mathcal{P}_1$ is a family of covers of degree $m \geq 4$ with a primitive pure $(1,1) - VHS$ satisfying the assumptions of Lemma 6.2.3. Moreover assume that $3 \notin (\mathbb{Z}/(m))^*$. Hence the assumption that $\mathcal{C} \to \mathcal{P}_1$ has primitive pure $(1,1) - VHS$ implies that the family $\mathcal{C}_3$ must have a trivial VHS. Thus all fibers of $\mathcal{C}_3$ must be isomorphic. Hence they are ramified over at most 3 points. By Lemma 6.2.3, one concludes that

$$0 = [\frac{1}{2} - \frac{3}{q}]_1, \quad 0 = [\frac{1}{2} - \frac{3}{r}]_1 \ \text{ or } \ 0 = [\frac{1}{2} + \frac{3}{p}]_1.$$

Since $\mu_4 = \frac{1}{2} + \frac{1}{p} < 1$, one concludes that $2 < p \leq q \leq r$. Thus one has

$$p = 6, \quad q = 6 \ \text{ or } \ r = 6.$$

Hence one can determine all examples of families $\mathcal{C} \to \mathcal{P}_1$ with a primitive pure $(1,1) - VHS$ in this case as we will do now:

6.2.5. Keep the assumptions of Remark 6.2.4. In the case $p = 6$ one has that $[3\mu_4]_1 = 0$. One can have $q = 7, 8, 9, 10, 11, 12$, where $q = 12$ implies that

$$\frac{1}{6} = \frac{1}{p}, \quad \frac{1}{q} = \frac{1}{r} = \frac{1}{12},$$

which leads to a family with a primitive pure $(1,1) - VHS$. Now we verify that $q = 7, 8, 9, 10, 11$ do not lead to a family with a primitive pure $(1,1) - VHS$: One must have that $\mathbb{L}_5$ is unitary. It has the local monodromy data

$$\mu_3 = \frac{1}{2} \ \text{ and } \ \mu_4 = [\frac{1}{2} + \frac{5}{6}]_1 = \frac{1}{3}.$$

Hence one must have that

$$\frac{1}{6} \geq \mu_1 = [\frac{1}{2} - \frac{5}{q}]_1,$$

which is satisfied for $q = 10, 11$, but not for $q = 7, 8, 9$. For $q = 10$ we have that

$$\frac{1}{r} = \frac{1}{p} - \frac{1}{q} = \frac{1}{15}.$$

This leads to a family given by the local monodromy data

$$\mu_1 = \frac{4}{10}, \quad \mu_2 = \frac{13}{30}, \quad \mu_3 = \frac{1}{2}, \quad \mu_4 = \frac{2}{3}.$$

One calculates easily that the eigenspace $\mathcal{L}_7$ in the VHS of this family is given by

$$\mu_1 = \frac{4}{5}, \quad \mu_2 = \frac{1}{30}, \quad \mu_3 = \frac{1}{2}, \quad \mu_4 = \frac{2}{3}.$$

Hence this family has not a pure $(1,1) - VHS$.

For $q = 11$ we have that

$$\frac{1}{p} - \frac{1}{q} = \frac{5}{66}.$$

Hence the equation of Lemma 6.2.2 can not be satisfied in this case.

6.2.6. Keep the assumptions of Remark 6.2.4. Moreover assume that $q = 6$. In this case we can have $p = 3, 4, 5$, where $p = 3$ implies

$$\frac{1}{p} = \frac{1}{3}, \quad \frac{1}{q} = \frac{1}{r} = \frac{1}{6},$$

which yields an example of a family with a primitive $(1,1) - VHS$. For $p = 4$ resp., $p = 5$ Lemma 6.2.2 and Lemma 6.2.3 yield a family of covers given by the local monodromy data

$$\mu_1 = \frac{1}{3}, \quad \mu_2 = \frac{5}{12} \quad \mu_3 = \frac{1}{2}, \quad \mu_4 = \frac{3}{4}$$

resp.,

$$\mu_1 = \frac{1}{3}, \quad \mu_2 = \frac{14}{30} \quad \mu_3 = \frac{1}{2}, \quad \mu_4 = \frac{7}{10}.$$

Hence one can easily verify that $\mathcal{L}_5$ is an eigenspace of type $(1,1)$ in both cases. Thus $p = 4, 5$ do not lead to a primitive pure $(1,1) - VHS$.

6.2.7. Keep the assumptions of Remark 6.2.4. Moreover assume that $r = 6$. In this case Lemma 6.2.2 implies that

$$\frac{1}{p} \geq \frac{2}{r} = \frac{1}{3}.$$

Hence one has $p = 2$ or $p = 3$, where $p = 2$ would imply that $\mu_4 = 1$, and $p = 3$ yields the same example of a family with a primitive pure $(1,1) - VHS$ as in 6.2.6.

Now we have considered the subcase given by $3 \notin (\mathbb{Z}/(m))^*$. We start the consideration of the subcase given by $3 \in (\mathbb{Z}/(m))^*$ by the following lemma:

Lemma 6.2.8. *Let $\mathcal{C} \to \mathcal{P}_1$ be a family with a primitive pure $(1,1) - VHS$, which satisfies that each $\mu_{k_1} + \mu_{k_2} \neq 1$. Then one has $m > 4$.*

Proof. We know that one must have $m \geq 4$ in the considered case. Thus we must only exclude $m = 4$. Since for a family $\mathcal{C}$ of degree 4 covers with a primitive pure $(1,1) - VHS$ the family $\mathcal{C}_2$ must have a trivial VHS, one has without loss of generally $d_1 = 2$. By the assumption that each $\mu_{k_1} + \mu_{k_2} \neq 1$, one concludes that d_1, d_2, d_3 are not equal to 2. Hence d_1, d_2, d_3 are odd. But this contradicts our assumptions, which imply that we have the even sum

$$2m = d_1 + \ldots + d_4.$$

□

Remark 6.2.9. Keep the assumption of Lemma 6.2.3. If $m > 4$, the eigenspace $\mathbb{L}_3$ is not of type $(1,1)$. Assume that 3 is a unit in $\mathbb{Z}/(m)$. Thus Lemma 6.2.8 implies for the local monodromy data of $\mathbb{L}_3$ that $\sum \mu_i = 3$ or $\sum \mu_i = 1$. Recall that $\mu_3 = \frac{1}{2}$. Hence $\sum \mu_i = 3$ implies that

$$\mu_1, \mu_2, \mu_4 > \frac{1}{2}.$$

By Lemma 6.2.3, one concludes that

$$\mu_1 = \frac{3}{2} - \frac{3}{q} \quad \text{and} \quad \mu_2 = \frac{3}{2} - \frac{3}{r}.$$

By Lemma 6.2.2, this implies that

$$\mu_4 = 3 - \mu_1 - \mu_2 - \mu_3 = 3 - \frac{3}{2} + \frac{3}{q} - \frac{3}{2} + \frac{3}{r} - \frac{1}{2} = -\frac{1}{2} + \frac{3}{p} + \frac{3}{q} = -\frac{1}{2} + \frac{3}{p}$$

in this case. This implies that $p, q, r < 6$.

In the case $\sum \mu_i = 1$ one gets $\mu_1, \mu_2, \mu_4 < \frac{1}{2}$. By Lemma 6.2.2 and Lemma 6.2.3, this implies that

$$\mu_1 = \frac{1}{2} - \frac{3}{q}, \quad \mu_2 = \frac{1}{2} - \frac{3}{r} \quad \text{and} \quad \mu_4 = -\frac{1}{2} + \frac{3}{p}$$

such that $p < 6$ and $q, r > 6$.

Remark 6.2.10. The case $p, q, r < 6$ does not yield any example of a family with a primitive pure $(1,1) - VHS$, since no triple $(p,q,r) \in \mathbb{N}^3$ with $2 \leq p \leq q \leq r < 6$ satisfies both

$$\frac{1}{p} - \frac{1}{q} = \frac{1}{r} \text{ and } \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$$

as one can check by calculation for each example.

6.2.11. Assume that we are in the case $p < 6$ and $q, r > 6$. Since $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, one has $\frac{1}{p} \leq 2\frac{1}{q}$ such that $6 < q \leq 2p$ and $3 < p < 6$. Hence one has two cases: $p = 4$ or $p = 5$. Thus by using that $\frac{1}{p} - \frac{1}{q} = \frac{1}{r}$, one calculates that only the examples given by

$$p = 4, \quad q = r = 8 \text{ and } p = 5, \quad q = r = 10$$

have a primitive pure $(1,1) - VHS$ in this case.

Now we consider the last remaining case of a family $\mathcal{C} \to \mathcal{P}_1$ with a primitive pure $(1,1) - VHS$. In this case there are at most 2 branch indices equal and one has some $\mu_{k_1} + \mu_{k_2} = 1$.

Lemma 6.2.12. *Let $\mathcal{C} \to \mathcal{P}_1$ a family of cyclic covers. If there are $k_1, k_2 \in \{1,2,3,4\}$ such that $d_{k_1} + d_{k_2} = m$ with $d_1 \leq d_2 \leq d_3 \leq d_4$, then one has*

$$d_1 + d_4 = m \quad \textit{and} \quad d_2 + d_3 = m.$$

Proof. (quite easy to see) □

Remark 6.2.13. By the preceding lemma, we have that $d_1 + d_4 = d_2 + d_3 = m$, if there are $k_1, k_2 \in \{1,2,3,4\}$ such that $d_{k_1} + d_{k_2} = m$ with $d_1 \leq d_2 \leq d_3 \leq d_4$. Hence if $d_1 + d_3 = m$ resp., $d_3 = d_4$, one gets $d_1 = d_2$, too. But this contradicts the assumption that at most 2 branch indexes are equal. Hence by $SINT$, one gets

$$\mu_1 + \mu_2 = 1 - \frac{1}{p} < 1, \quad \mu_1 + \mu_3 = 1 - \frac{1}{q} < 1, \quad \mu_2 + \mu_3 = 1$$

with $p, q \in \mathbb{N}$ and $p \leq q$. Hence one obtains similarly to Remark 6.2.1 with $\frac{1}{p} + \frac{1}{q} < 1$:

$$\mu_1 = \frac{1}{2}(1 - \frac{1}{p} - \frac{1}{q}), \quad \mu_2 = \frac{1}{2}(1 - \frac{1}{p} + \frac{1}{q}),$$
$$\mu_3 = \frac{1}{2}(1 + \frac{1}{p} - \frac{1}{q}), \quad \mu_4 = \frac{1}{2}(1 + \frac{1}{p} + \frac{1}{q})$$

Lemma 6.2.14. *Assume that the local monodromy data of Remark* 6.2.13 *yield a family of degree* $m \geq 4$ *with a primitive pure* $(1,1) - VHS$*. Then one has* $p = q$ *and* m *is even.*

Proof. In the case of Remark 6.2.13 the eigenspace $\mathbb{L}_2$ is given by the local monodromy data

$$\mu_1 = 1 - \frac{1}{p} - \frac{1}{q}, \quad \mu_2 = [1 - \frac{1}{p} + \frac{1}{q}]_1, \quad \mu_3 = [\frac{1}{p} - \frac{1}{q}]_1, \quad \mu_4 = \frac{1}{p} + \frac{1}{q}.$$

Thus in this case $\mathcal{L}_2$ is of type $(1,1)$, if and only if $p < q$. Hence one can obtain a primitive pure $(1,1) - VHS$, only if $p = q$. Now $p = q$ implies that $\mu_2 = \mu_3 = 0$ for the local monodromy data of $\mathbb{L}_2$. Hence the family of covers has an even degree. □

Proposition 6.2.15. *Assume that the local monodromy data of Remark* 6.2.13 *yield a family of degree* $m \geq 4$ *with a primitive pure* $(1,1) - VHS$*. Then* $p = q \leq 6$*.*

Proof. By the preceding lemma, the assumptions imply that $p = q$. Hence by Remark 6.2.13, we have:

$$\mu_1 = \frac{p-2}{2p}, \quad \mu_2 = \mu_3 = \frac{1}{2}, \quad \mu_4 = \frac{p+2}{2p} \tag{6.1}$$

If $p > 6$, then $\mathbb{L}_3$ has the local monodromy given by

$$\mu_1 = \frac{p-6}{2p}, \quad \mu_2 = \mu_3 = \frac{1}{2}, \quad \mu_4 = \frac{p+6}{2p}.$$

Hence Proposition 2.3.4 implies that $\mathcal{L}_3$ is of type $(1,1)$ in this case. □

Lemma 6.2.16. *Assume that the local monodromy data of Remark* 6.2.13 *yield a family of degree* $m \geq 4$ *with a primitive pure* $(1,1) - VHS$*. Then* p *must be even.*

Proof. Assume that p is odd. Since $gcd(p-2, 2p) = 1$ in this case, one gets a family of degree $2p$ with branch indices

$$d_1 = p - 2, \quad d_2 = d_3 = p, \quad d_4 = p + 2.$$

Thus all branch indices are odd, and $\mathcal{C}_p$ is a family of elliptic curves such that $\mathcal{L}_p$ is of type $(1,1)$. Contradiction! □

Remark 6.2.17. Keep the assumptions of the preceding lemma. Since one must have $\mu_1 > 0$, the preceding proposition and (6.1) imply that

$$3 \leq p \leq 6.$$

Since $p = q$ must be even, one can only have $p = 4$ and $p = 6$.

1. For $p = 4$ one obtains the example of a family with a primitive pure $(1,1) - VHS$ given by
$$\mu_1 = \frac{1}{4}, \quad \mu_2 = \mu_3 = \frac{1}{2}, \quad \mu_4 = \frac{3}{4}.$$
2. If $p = 6$ one has the example of a family with a primitive pure $(1,1) - VHS$ given by
$$\mu_1 = \frac{1}{3}, \quad \mu_2 = \mu_3 = \frac{1}{2}, \quad \mu_4 = \frac{2}{3}.$$

6.3 The complete lists of examples

In this section we give the complete lists of examples of families $\mathcal{C} \to \mathcal{P}_n$ with primitive pure $(1,n)$-variations of Hodge structures and derived pure $(1,n)$-variations of Hodge structures.

By our preceding calculations, we get the following complete list of families of covers $\mathcal{C} \to \mathcal{P}_1$ with a primitive pure $(1,1) - VHS$, where "ref" denotes the number of the preceding remark, lemma, proposition or point yielding the respective example:

number	degree	branch points with branch index	genus	ref
1	2	1 1 1 1	1	(known)
2	3	1 2 2 1	2	6.1.18
3	4	1 2 2 3	2	6.2.17, (1)
4	5	1 3 3 3	4	6.1.17
5	6	1 4 4 3	3	6.2.6, 6.2.7
6	6	2 3 3 4	2	6.2.17, (2)
7	7	2 4 4 4	6	6.1.17
8	8	2 5 5 4	5	6.2.11
9	9	3 5 5 5	7	6.1.17
10	10	3 6 6 5	6	6.2.11
11	12	4 7 7 6	7	6.2.5

We will later see that each derived pure $(1,n) - VHS$ is in fact a derived pure $(1,1) - VHS$. In the next section we will verify that we get the following complete list of families of covers $\mathcal{C} \to \mathcal{P}_1$ with a derived pure $(1,1) - VHS$, where N_{r_0} means the number of $\mathcal{C}_{r_0}$ in the preceding list, which has the corresponding primitive pure $(1,1) - VHS$:

degree	branch points with branch index	genus	r_0	N_{r_0}
4	1 1 1 1	3	2	1
6	1 1 1 3	4	3	1
6	1 2 2 1	4	2	2

Note that any family $\mathcal{C} \to \mathcal{P}_n$ with a primitive pure $(1,n) - VHS$ satisfies $SINT$, which implies INT. Hence by consulting the list of [18] on page 86, which contains all examples satisfying INT for $n \geq 2$, and the computation of the types of the eigenspaces of the corresponding covers), we have the following complete list of families of covers with a primitive pure $(1,n)-VHS$ for $n > 1$:

degree	branch points with branch index	genus
3	2 1 1 1 1	3
4	2 2 2 1 1	3
5	2 2 2 2 2	6
6	3 3 3 2 2	4
3	1 1 1 1 1 1	4

In [11] R. Coleman formulated the following conjecture:

Conjecture 6.3.1. *Fix an integer $g \geq 4$. Then there are only finitely many complex algebraic curves C of genus g such that* $\mathrm{Jac}(C)$ *is of CM type.*

Remark 6.3.2. J. de Jong and R. Noot [29] resp., E. Viehweg and K. Zuo [58] have given counterexamples of families with infinitely many CM fibers for $g = 4, 6$. In our lists here we have counterexamples for $g = 5, 7$.

J. de Jong and R. Noot resp., E. Viehweg and K. Zuo needed to find a fiber with CM for the proofs that their examples of families have infinitely many CM fibers. In the proof of Theorem 4.4.4, which implies that the examples of this section have dense set of complex multiplication fibers, we did not need to find one CM fiber first.

By the fact that our examples $\mathcal{C} \to \mathcal{M}_n$ with a dense set of CM fibers satisfy that $n+1$ branch points have the same branch index, Theorem 2.4.4 yields the CM-type of one CM fiber and hence by Lemma 1.7.3, the CM-type of a dense set of CM fibers.

6.4 The derived variations of Hodge structures

In this section we determine the families of cyclic covers with a derived pure $(1,n) - VHS$ and verify that the list of examples in the preceding section is complete.

Remark 6.4.1. Assume that the family $\mathcal{C}$ of degree dm covers has a derived pure $(1,n) - VHS$ induced by $\mathcal{C}_d$. Let

$$d = p_1^{n_1} \cdot \ldots \cdot p_t^{n_t}$$

be the decomposition of d into its prime factors. Then there exists a family of covers of degree p_1m with a derived pure $(1,n)-VHS$. Hence there are two cases to consider first: d is a prime number and divides m, or d is a prime number and does not divide m.

Lemma 6.4.2. *Let p be a prime number. Assume that d is a prime number such that $\gcd(d,p) = 1$. Then a family $\mathcal{C}$ of covers of degree $p \cdot d$ with a derived pure $(1,n)-VHS$ induced by $\mathcal{C}_d$ can not exist, if all Dehn twists yield semisimple matrices with respect to the monodromy representation of $\mathcal{L}_d$.*

Proof. Since $\mathcal{C}_p$ must have a trivial VHS, there exists a d_2 such that d divides d_2. Moreover there is a d_1 such that d does not divide d_1. Hence $\gcd(d, d_1 + d_2) = 1$. by the fact that $\mathcal{C}_d$ has the property that its local monodromy data satisfy $\mu_1 + \mu_2 \neq 1$, one concludes that $\gcd(p, d_1 + d_2) = 1$, too. Hence $[d_1 + d_2]_{dp}$ is a unit in $\mathbb{Z}/(dp)$. Thus there exists a $d_0 \in (\mathbb{Z}/(dp))^*$ such that $d_0[d_1 + d_2]_{dp} = 1$. One obtains that the sum of the integers of $\{1, \ldots, p-1\}$ representing $[d_0d_1]_{dp}$ and $[d_0d_2]_{dp}$ is given by $dm + 1$. By Proposition 2.3.4, one concludes that $\mathcal{L}_{d_0}$ is not of type $(0, n+1)$. Moreover the fact that the local monodromy data of $\mathcal{L}_{d_0}$ satisfy

$$\mu_1 + \mu_2 = \frac{dp+1}{dp}, \quad \mu_3 \leq \frac{dp-1}{dp}, \quad \mu_4 + \ldots + \mu_{n+3} < n$$

tells us that

$$\mu_1 + \ldots + \mu_{n+3} < n + 2.$$

Hence one concludes by Proposition 2.3.4 that $\mathcal{L}_{d_0}$ is not of type $(n+1, 0)$, too. $\square$

Lemma 6.4.3. *Let $m = 2^tp$, where $p \neq 2$ is a prime number and $t \geq 1$. Assume that d is a prime number such that $\gcd(d,m) = 1$. Then a family $\mathcal{C}$ of degree $m \cdot d$ covers with a derived pure $(1,n) - VHS$, which is induced by $\mathcal{C}_d$, can not exist.*

Proof. Since $\mathcal{C}_2$ must have a trivial VHS, one has $d_1 = \ldots = d_n = 2^{t-1}dp$. By the fact that $\mathcal{C}_p$ must have a trivial VHS, we obtain that 2^td divides n different branch indices. Since there must be at least two different branch indices, which are not divided by d, d_{n+2} and d_{n+3} are not divided by d. By the fact that $d_1 = \ldots = d_n = 2^{t-1}dp$ is not divided by 2^td, one must have $n = 1$ and that 2^td divides d_2. Moreover the facts that

$$d_1 + \ldots + d_4 \in (m) = (2^tpd) \ \text{ and } \ 2|d_2$$

imply without loss of generality that 2 does not divide d_3. We have two cases: Either $p|d_3$ or this does not hold true. In the first case one has that 2, p and d do not divide $d_2 + d_3$. Hence $d_2 + d_3$ is a unit, and again we use the argument that there is a $d_0 \in (\mathbb{Z}/(dm))^*$ such that $[d_0d_2 + d_0d_3] = 1$.

In the other case d_3 yields a unit of $\mathbb{Z}/(dm)$. Hence we have without loss of generality $d_3 = 1$. Thus $g := \gcd(dm, d_1 + d_3) \in \{1, 2\}$. If $g = 1$, we are done again. Otherwise we must have $t = 1$, if $g = 2$. Hence

$$[(pd - 2)(d_1 + d_3)]_{dm} = pd + pd - 2 = dm - 2$$

such that $\mathcal{L}_{pd-2}$ is neither of type $(0, n+1)$ nor of type $(n+1, 0)$, since the fact that $2^t d$ divides d_2 implies that $[(pd-2)d_2]_{dm} \neq [1]_{dm}$. □

Lemma 6.4.4. *Let p be a prime number and $m = p^t$ with $t \geq 2$. Assume that d is a prime number such that $\gcd(d, p) = 1$. Then there can not be a family $\mathcal{C}$ of degree $m \cdot d$ covers with a derived pure $(1, n) - VHS$, which is induced by $\mathcal{C}_d$.*

Proof. Since $\mathcal{C}_p$ must have a trivial VHS, one concludes without loss of generality that dp^{t-1} divides $d_1, \ldots, d_n$. Since d and p divide

$$dp^t = d_1 + \ldots + d_{n+3},$$

too, p resp., d does not divide at least two different elements of $\{d_{n+1}, d_{n+2}, d_{n+3}\}$. Hence there is an element of $\{d_{n+1}, d_{n+2}, d_{n+3}\}$, which is not divided by both d and p. Without loss of generality d_{n+1} is a unit in $\mathbb{Z}/(2^t d)$. Hence one has without loss of generality $[d_1 + d_{n+1}]_{dm} = [1]_{dm}$. □

There are only few remaining examples, which do not satisfy the assumptions of the preceding lemmas. One of these examples is considered in the following lemma:

Lemma 6.4.5. *Let $d \neq 3$ be a prime number. There can not be a family of covers of degree $3d$ with a derived pure $(1, 2) - VHS$ induced by $\mathcal{C}_d$ given by the local monodromy data*

$$\mu_1 = \ldots = \mu_4 = \frac{1}{3}, \quad \mu_5 = \frac{2}{3}.$$

Proof. Let $\gcd(d, 3) = 1$ and $\mathcal{C}$ be a family of degree $3d$ with a derived pure $(1, 2) - VHS$. Since $\mathcal{C}_3$ should have a trivial VHS, one has with a new enumeration $d|d_1$ and $d|d_2$. Moreover one has without loss of generality that d_3 and d_4 are not divided by d. Hence d divides neither $d_1 + d_3$ nor $d_2 + d_4$. Moreover the local monodromy data of $\mathcal{C}_d$ tell us that 3 does not divide $d_1 + d_3$ or $d_2 + d_4$. Hence without loss of generality $d_1 + d_3$ is a unit in $\mathbb{Z}/(3d)$ such there is a $d_0 \in (\mathbb{Z}/(3d))^*$ with the property that $[d_0 d_1 + d_0 d_3]_{3d} = 1$, which implies that $\mathbb{L}_{d_0}$ is of type $(1, 2)$ or of type $(2, 1)$. □

The reader checks easily that all examples of families with a primitive pure $(1, n) - VHS$ satisfy with two exceptions the assumptions of one of the preceding lemmas. These two exceptions yield examples of families with a derived pure $(1, n) - VHS$ as we will see now.

6.4.6. Now we consider the case of the elliptic curves. Let d be a prime number with $\gcd(d,2) = 1$ and $\mathcal{C}$ be a family of degree $d \cdot 2$ covers with a derived pure $(1,1) - VHS$ induced by $\mathcal{C}_d$. Thus $d_1, \ldots, d_4$ must be odd. Without loss of generality we have $d_4 = d$, since $\mathcal{C}_2$ must have a trivial VHS. Since $d_3 = d$ would imply that $\mathbb{L}_1$ is of type $(1,1)$, one has that $d_1, d_2, d_3 \in (\mathbb{Z}/(2d))^*$. We have two cases. Either $d_1 = d_2$ or this does not hold true. In the first case we put $d_1 = d_2 = d - 2$. One has

$$2d < d_1 + d_2 + d_4 < 2 \cdot 2d$$

such that $\mathbb{L}_1$ is of type $(1,1)$, if $4 < d$. Thus one can have $d = 3$. In this case one has a family of degree 6 covers, where $d_4 = 3$. Hence one must have

$$\mu_1 = \mu_2 = \mu_3 = \frac{1}{6}, \quad \mu_4 = \frac{1}{2}.$$

In the second case, one puts $d_3 = d - 2$. This implies that $d_3 + d_4 = 2d - 2$. Since $d_1 \neq d_2$, one can not have $d_1 = d_2 = 1$ such that $\mathbb{L}_1$ is of type $(1,1)$ in this case.

6.4.7. Now we consider the case number 2 in the list of examples with a primitive pure $(1,1) - VHS$. Let d be a prime number with $\gcd(d,3) = 1$ and $\mathcal{C}$ be a family of degree $d \cdot 3$ covers with a derived pure $(1,1) - VHS$ induced by $\mathcal{C}_d$. Assume without loss of generality that d divides d_1 and $d_1 + \ldots + d_4 = 3d$. We have 2 cases: Either d divides d_2, d_3 or d_4, or d does not divide d_2, d_3 and d_4. In the first case one has without loss of generality that d divides d_2. Since d divides d_1 and $d_1 + \ldots + d_4 = 3d$, one concludes that $d_1 = d_2 = d$. This implies that $\mathcal{L}_2$ is of type $(1,1)$ such that $d = 2$. In addition one concludes that

$$d_1 = d_2 = 2, \quad d_3 = d_4 = 1.$$

In the second case one has that 3 does not divide $d(d_1 + d_i)$ for exactly one $k \in \{2,3,4\}$, which follows by the branch indices in the case number 2. Hence 3 does not divide $d_1 + d_k$. Moreover d does not divide $d_1 + d_k$, too. Hence $d_1 + d_k \in (\mathbb{Z}/(3d))^*$.

Proposition 6.4.8. *Let d be a prime number, which divides m and $\mathcal{C}$ be a family of covers of degree md. Assume a Dehn twist yields a semisimple matrix of maximal order m with respect to the monodromy representation of $\mathcal{L}_d$. Then $\mathcal{C}$ can not have a derived pure $(1,n) - VHS$ induced by $\mathcal{C}_d$.*

Proof. Assume without loss of generality that $\rho_d(T_{1,2})$ yields a matrix of order m. In this case $[d(d_1 + d_2)] \in \mathbb{Z}/(dm)$ has the order m. Hence the fact that d divides m implies that $d_1 + d_2 \in (\mathbb{Z}/(dm))^*$. □

Remark 6.4.9. One can easily check that the assumptions of the preceding proposition are satisfied for all examples of families with a primitive pure

$(1, n) - VHS$ except of the case of elliptic curves. In this case we have in fact an example of a family of degree 4 covers with a derived pure $(1, 1) - VHS$. Without loss of generality we have

$$d_1 + \ldots + d_4 = 4.$$

Hence the only possibility is given by

$$d_1 = \ldots = d_4 = 1.$$

6.4.10. In the case of the elliptic curves we have families of degree 6 and degree 4 covers with a derived pure $(1, 1) - VHS$. Hence one must check that there is not a family of degree 8, 12 or 18 covers with derived pure $(1, 1) - VHS$ in this case.

First we check that there is not a family $\mathcal{C}$ of degree 8 covers with a derived pure $(1, 1) - VHS$. Otherwise one has such a family $\mathcal{C}$ of degree 8 covers such that $\mathcal{C}_2$ is the family of degree 4 covers with a derived pure $(1, 1) - VHS$. This implies that each d_k satisfies $[d_k]_4 = [1]_4$ or each d_k satisfies $[d_k]_4 = [3]_4$. Moreover one has without loss of generality that $d_1 + \ldots + d_4 = 8$. Hence it is not possible that each d_k satisfies $[d_k]_4 = [3]_4$. Thus the only possibility is (up to the numbering) given by

$$d_1 = d_2 = d_3 = 1, \quad d_4 = 5.$$

But in this case $\mathcal{L}_3$ is of type $(1, 1)$. Thus there can not exist a family of degree 8 covers with a derived pure $(1, 1) - VHS$.

There can not be a family of degree 12 covers with a derived pure $(1, 1) - VHS$ induced by $\mathcal{C}_6$. Otherwise one has that $\mathcal{C}_3$ the example of degree 4 covers with a derived pure $(1, 1) - VHS$. Thus one concludes that

$$[d_1]_4 = \ldots = [d_4]_4 = [1]_4 \;\text{ or }\; [d_1]_4 = \ldots = [d_4]_4 = [3]_4.$$

Since one has without loss of generality that $d_1 + \ldots + d_4 = 12$, the only possibilities are given by

$$d_1 = d_2 = 5, \quad d_3 = d_4 = 1 \;\text{ and }\; d_1 = 9, \quad d_2 = d_3 = d_4 = 1.$$

In the first case $\mathcal{L}_2$ is of type $(1, 1)$ and in the second case $\mathcal{L}_5$ is of type $(1, 1)$.

There can not be a family of degree 18 covers with a derived $(1, 1) - VHS$ induced by $\mathcal{C}_9$. Otherwise one has that $\mathcal{C}_3$ is the example of degree 6 covers with a derived pure $(1, 1) - VHS$ induced by the elliptic curves. Thus one concludes that

$$[d_1]_6 = \ldots = [d_3]_6 = [1]_6 \;\text{ and }\; [d_4]_6 = [3]_6$$

or

$$[d_1]_6 = \ldots = [d_3]_6 = [5]_6 \text{ and } [d_4]_6 = [3]_6.$$

Since one has without loss of generality that $d_1 + \ldots + d_4 = 18$, the only possibilities are given by:

$$d_1 = 13, \quad d_2 = d_3 = 1, \quad d_4 = 3$$

$$d_1 = d_2 = 7, \quad d_3 = 1, \quad d_4 = 3$$

$$d_1 = 7, \quad d_2 = d_3 = 1, \quad d_4 = 9$$

$$d_1 = d_2 = d_3 = 1, \quad d_4 = 15$$

$$d_1 = d_2 = d_3 = 5, \quad d_4 = 3.$$

One has that $\mathcal{L}_5$ is of type $(1,1)$ in case 1, $\mathcal{L}_2$ is of type $(1,1)$ in case 2, $\mathcal{L}_5$ is of type $(1,1)$ in case 3, $\mathcal{L}_7$ is of type $(1,1)$ in case 4 and $\mathcal{L}_2$ is of type $(1,1)$ in case 5.

6.4.11. It remains to show that there can not exist a degree 12 cover with a derived $(1,1) - VHS$ induced by the degree 3 cover given by

$$d_1 = d_2 = 1, \quad d_3 = d_4 = 2.$$

Otherwise one has such a family $\mathcal{C}$ of degree 12 covers such that $\mathcal{C}_2$ is the family of degree 6 covers with a derived pure $(1,1) - VHS$ by the degree 3 example above. Thus one concludes that

$$[d_1]_6 = [d_2]_6 = [2]_6 \text{ and } [d_3]_6 = [d_4]_6 = [1]_6$$

or

$$[d_1]_6 = [d_2]_6 = [4]_6 \text{ and } [d_3]_6 = [d_4]_6 = [5]_6$$

Since one has without loss of generality that $d_1 + \ldots + d_4 = 12$, the only possibilities are given by

$$d_1 = 8, \quad d_2 = 2, \quad d_3 = d_4 = 1 \text{ and } d_1 = d_2 = 2, \quad d_3 = 7, \quad d_4 = 1.$$

One has that $\mathcal{L}_5$ is of type $(1,1)$ in the first case and one has that $\mathcal{L}_3$ is of type $(1,1)$ in the second case.

Chapter 7
The construction of Calabi-Yau manifolds with complex multiplication

In this chapter we explain the basic construction methods of Calabi-yau manifolds with complex multiplication and give a first new example. We call a family of Calabi-Yau n-manifolds, which contains a dense set of fibers X such that the Hodge group of the Hodge structure on $H^k(X, \mathbb{Q})$ is a torus for all k, a $CMCY$ family of n-manifolds.

In Section 7.1 we explain the technical facts, which we will need for the construction of $CMCY$ families. By using the mirror construction of C. Borcea [9] and C. Voisin [60], we give a method to construct an infinite tower of $CMCY$ families in Section 7.2. In Section 7.3 we discuss the construction method of E. Viehweg and K. Zuo [58]. By using this method given by a tower of cyclic covers, E. Viehweg and K. Zuo [58] have constructed an example of a $CMCY$ family of 3-manifolds. We finish this chapter with the example

$$\mathbb{P}^3 \supset V(y_2^4 + y_1^4 + x_1(x_1 - x_0)(x_1 - \lambda x_0)x_0) \to \lambda \in \mathcal{M}_1$$

of a family of $K3$ surfaces with a dense set of CM fibers. This example is obtained from the Viehweg-Zuo tower, which starts with the family

$$\mathbb{P}^2 \supset V(y_1^4 + x_1(x_1 - x_0)(x_1 - \lambda x_0)x_0) \to \lambda \in \mathcal{M}_1$$

of curves. This family has a dense set of CM fibers by the previous chapter. By some of its involutions the family of $K3$ surfaces above is suitable for the construction of a Borcea-Voisin tower.

7.1 The basic construction and complex multiplication

Now we have finished our considerations on Hodge structures of cyclic covers of $\mathbb{P}^1$. We start with the second part, which is devoted to the construction of families of Calabi-Yau manifolds with dense set of complex multiplication fibers.

J.C. Rohde, *Cyclic Coverings, Calabi-Yau Manifolds and Complex Multiplication*, 143
Lecture Notes in Mathematics 1975, DOI: 10.1007/978-3-642-00639-5_8,

In the works of C. Borcea [8], [9], of E. Viehweg and K. Zuo [58] and of C. Voisin [60] the methods to obtain higher dimensional Calabi-Yau manifolds contain one common basic construction. In this section we describe this construction and explain how it yields complex multiplication. For this construction we use Kummer coverings. Let $A - B$ be a principal divisor with $(f) = A - B$ for some $f \in \mathbb{C}(X)$. The Kummer covering given by $\mathbb{C}(X)(\sqrt[m]{\frac{A}{B}})$ is nothing but the normalization of X in $\mathbb{C}(X)(\sqrt[m]{f})$.

Let V_1 and V_2 be irreducible complex algebraic manifolds and $\mathcal{A}$ resp., $\mathcal{B}$ be a bundle of irreducible algebraic manifolds with universal fiber A resp., B over V_1 resp., V_2. Moreover let $\mathcal{Z}$ resp., Σ be a cyclic Galois cover of $\mathcal{A}$ resp., a cyclic Galois cover of $\mathcal{B}$ of degree m over V_1 resp., V_2 ramified over a smooth divisor. We assume that the irreducible components of these ramification divisors intersect each fiber of $\mathcal{Z}$ resp., Σ transversally in smooth subvarieties of codimension 1. Thus we assume that $\mathcal{Z}$ and Σ are given by Kummer coverings of the kind

$$\mathbb{C}(W) = \mathbb{C}(X)(\sqrt[m]{\frac{D_1 + \ldots + D_k}{D_0^m}}),$$

where $D_1, \ldots, D_k$ are (reduced) smooth prime divisors, which do not intersect each other.

Example 7.1.1. By a cyclic degree 2 cover $S \to R$ of surfaces (or in general algebraic varieties), one has an involution on S. Let us assume that the surface S is a smooth $K3$ surface. Moreover assume that there exists an involution ι on S, which acts via pull-back by -1 on $\Gamma(\omega_S)$. It has the property that it fixes at most a divisor D, whose support consists of smooth curves, which do not intersect each other (see [60], 1.1). Moreover by [60], 1.1, to give an involution ι on S, which acts by -1 on $\Gamma(\omega_S)$, is the same as to give a cyclic degree 2 cover $S \to R$ of smooth surfaces. In this case R is rational, if and only if $D \neq 0$.

We consider the following commutative diagram, which yields the basic construction:

$$\begin{tikzcd}
\mathcal{Z} \times \Sigma \arrow[r, "\gamma"] & \mathcal{Y}' \arrow[r, "\alpha"] & \mathcal{A} \times \mathcal{B} \arrow[r] & V_1 \times V_2 \\
\widetilde{\mathcal{Z} \times \Sigma} \arrow[u, "\beta"] \arrow[r, "\tilde{\gamma}"] & \tilde{\mathcal{Y}} \arrow[u, "\delta"] \arrow[r, "\tilde{\alpha}"] & \hat{\Pi} \arrow[u, "\zeta"] &
\end{tikzcd} \tag{7.1}$$

First we explain the upper line of this diagram: The cyclic covers $\mathcal{Z}$ and Σ can locally be described by equations of the type

$$y^m = \prod_{i=1,\ldots,k} f_i(x_1, \ldots, x_j)$$

over any open affine set $\mathbb{A}$ of $\mathcal{A}$ resp., $\mathcal{B}$, where f_i is the (reduced) equation of D_i in $\mathbb{A}$. The Galois transformations are given by

$$(y, x_1, \ldots, x_j) \xrightarrow{g_k} (e^{2\pi\sqrt{-1}\frac{k}{m}} y, x_1, \ldots, x_j)$$

for some $k \in \mathbb{Z}/(m)$. Hence we have a natural identification between $\mathbb{Z}/(m)$ and the Galois groups given by $[k]_m \to g_k$. By the description of the covers above in terms of Kummer coverings, this identification is independent of the chosen open affine subset. Now γ is the quotient by

$$G := \langle (1,1) \rangle \quad \subset \quad G' := \mathrm{Gal}(\mathcal{Z}; \mathcal{A}) \times \mathrm{Gal}(\Sigma; \mathcal{B}),$$

and α is the quotient by G'/G. The morphism ζ is given by the blowing up of the fiber product of the supports of the branch divisors of $\mathcal{Z}$ and Σ. Moreover δ is the blowing up along the singular points of $\mathcal{Y}'$, which is given by the intersection locus of the ramification divisors, and β is the blowing up with respect to the corresponding inverse image ideal sheaf. Hence $\tilde{\alpha}$ and $\tilde{\gamma}$ are the unique cyclic covers obtained by the universal property of the blowing up (compare to [26], **II**. Corollary 7.15). By the construction of α, one can easily check that $\tilde{\alpha}$ is not ramified over the exceptional divisor. Hence the branch locus of $\tilde{\alpha}$ is smooth. This implies that $\tilde{\mathcal{Y}}$ is smooth, too. The ramification locus of $\tilde{\gamma}$ is given by the smooth exceptional divisor of β, since G leaves the generators of the inverse image ideal sheaf invariant as one can see by the following remark:

Remark 7.1.2. Now we describe $\widetilde{\mathcal{Z} \times \Sigma}$. A neighborhood of the preimage point $p \in \mathcal{Z} \times \Sigma$ of a singular point can be identified with an open neighborhood of $0 \in \mathbb{C}^2 \times \mathbb{B}$, where $\mathbb{B}$ is a ball of suitable dimension and the Galois group acts via $(x_1, x_2) \to (e^{\frac{2\pi i}{m}} x_1, e^{\frac{2\pi i}{m}} x_2)$ with respect to the coordinates on $\mathbb{C}^2$. Due to [6], **III**. Proposition 5.3, each singular point of $\mathcal{Y}'$ has an analytic neighborhood isomorphic to $V(x^m = y^{m-1}z) \times \mathbb{B}$. Hence locally we have the product of a cover of surfaces with $\mathbb{B}$. One should have $\mathbb{B}$ in mind. But for the description of $\widetilde{\mathcal{Z} \times \Sigma}$, it is sufficient to consider only covers of surfaces. The inverse image ideal sheaf with respect to this cover is generated by $\{x_1^{m-i} x_2^i : i = 0, 1, \ldots, m\}$. By the Veronese embedding for relative projective manifolds, one can easily identify the blowing up with respect to this ideal with the blowing up with respect to the ideal generated by $\{x_1, x_2\}$. But this is the blowing up of the origin resp., the preimage point of the singular point. Hence $\widetilde{\mathcal{Z} \times \Sigma}$ is given by the blowing up of the reduced preimage $\gamma^{-1}(S)$, where S is the singular locus of $\mathcal{Y}'$.

Now we have described the basic construction. Next we see that this construction yields complex multiplication. We use following fact:

Proposition 7.1.3. *For all $\tilde{a} \in \mathcal{A}$, and $\tilde{b} \in \mathcal{B}$, we have the following tensor product of rational Hodge structures on the fibers:*

$$H^n(\mathcal{Z}_{\tilde{a}} \times \Sigma_{\tilde{b}}, \mathbb{Q}) = \bigoplus_{a+b=n} H^a(\mathcal{Z}_{\tilde{a}}, \mathbb{Q}) \otimes H^b(\Sigma_{\tilde{b}}, \mathbb{Q})$$

such that

$$H^{r,s}(\mathcal{Z}_{\tilde{a}} \times \Sigma_{\tilde{b}}) = \bigoplus_{p+p'=r, q+q'=s} H^{p,q}(\mathcal{Z}_{\tilde{a}}) \otimes H^{p',q'}(\Sigma_{\tilde{b}})$$

Proof. (follows from [61], Théorème 11.38) □

We want to construct higher dimensional varieties with complex multiplication. The first main tool is:

Proposition 7.1.4. *Let h_1 and h_2 be rational polarized Hodge structures. Then*

$$h_3 = h_1 \otimes h_2$$

is of CM type, if and only if h_1 and h_2 are of CM type.

Proof. (see [8], Proposition 1.2) □

By the fact that $\mathcal{Y}'$ is not smooth, but the blowing up $\tilde{\mathcal{Y}}$ is smooth, $\tilde{\mathcal{Y}}$ will be our candidate for a family of Calabi-Yau manifolds with dense set of complex multiplication fibers. Hence we must consider the behavior of the Hodge structures under blowing up:

Lemma 7.1.5. *Let X be an algebraic manifold of dimension n and $\tilde{X}$ be the blowing up X with respect to some submanifold $Z \supset X$ of codimension 2. Then $\mathrm{Hg}(H^k(\tilde{X}, \mathbb{Z}))$ is commutative, if and only if $\mathrm{Hg}(H^k(X, \mathbb{Z}))$ and $\mathrm{Hg}(H^{k-2}(Z, \mathbb{Z}))$ are commutative, too.*

Proof. By [61], Théorème 7.31, we have an isomorphism

$$H^k(X, \mathbb{Z}) \oplus H^{k-2}(Z, \mathbb{Z}) \cong H^k(\tilde{X}, \mathbb{Z})$$

of Hodge structures, where $H^{k-2}(Z, \mathbb{Z})$ is shifted by $(1,1)$ in bi-degree. Since

$$\mathrm{Hg}(H^k(\tilde{X}, \mathbb{Z})) = \mathrm{Hg}(H^k(X, \mathbb{Z}) \oplus H^{k-2}(Z, \mathbb{Z})) \subset \mathrm{Hg}(H^k(X, \mathbb{Z})) \times \mathrm{Hg}(H^{k-2}(Z, \mathbb{Z}))$$

such that the natural projections

$$\mathrm{Hg}(H^k(\tilde{X}, \mathbb{Z})) \to \mathrm{Hg}(H^k(X, \mathbb{Z})) \text{ and } \mathrm{Hg}(H^k(\tilde{X}, \mathbb{Z})) \to \mathrm{Hg}(H^{k-2}(Z, \mathbb{Z}))$$

are surjective (see Lemma 2.4.1), we obtain the result. □

Corollary 7.1.6. *Let X be a smooth surface and $\tilde{X}$ be the blowing up of some point $p \in X$. Then X has complex multiplication, if and only if $\tilde{X}$ has complex multiplication, too. Moreover we obtain that*

$$\mathrm{Hg}(H^2(\tilde{X}, \mathbb{Z})) \cong \mathrm{Hg}(H^2(X, \mathbb{Z})).$$

Now we want to consider the behavior of the fibers. Hence for simplicity we assume now that $V_1 = V_2 = \mathrm{Spec}(\mathbb{C})$ in diagram (7.1). By the fact that $\tilde{\mathcal{Y}}$ has the Hodge structure given by the Hodge sub-structure of $\widetilde{\mathcal{Z} \times \Sigma}$ invariant under the Galois group, one concludes:

Theorem 7.1.7. *If for all k the groups $\mathrm{Hg}(H^k(\mathcal{Z}, \mathbb{Q}))$, $\mathrm{Hg}(H^k(\Sigma, \mathbb{Q}))$ and $\mathrm{Hg}(H^k(Z_i, \mathbb{Q}))$ are commutative,*[1] *then $\mathrm{Hg}(H^k(\tilde{\mathcal{Y}}, \mathbb{Q}))$ is commutative for all k, too.*

Remark 7.1.8. At first sight the condition that for all k the groups $\mathrm{Hg}(H^k(\mathcal{Z}, \mathbb{Q}))$, $\mathrm{Hg}(H^k(\Sigma, \mathbb{Q}))$ and $\mathrm{Hg}(H^k(Z_i, \mathbb{Q}))$ have to be commutative may seem to be a little bit restrictive. But by the Hodge diamond of a Calabi-Yau n-manifold with $n \leq 3$ or the Hodge diamond of a Calabi-Yau n-manifold given by a projective hypersurface, one sees that the condition that all its Hodge groups are commutative is equivalent to the condition that it has complex multiplication. Moreover we will need this condition for an inductive construction of families of Calabi-Yau manifolds with dense set of complex multiplication fibers in arbitrary high dimension in the next section.

7.2 The Borcea-Voisin tower

Recall that we want to construct families of Calabi-Yau manifolds with a dense set of CM fibers. Hence let us now define Calabi-Yau manifolds:

Definition 7.2.1. A Calabi-Yau manifold X of dimension n is a compact Kähler manifold of dimension n such that $\Gamma(\Omega^i_X) = 0$ for all $i = 1, \ldots, n-1$ and $\omega_X \cong \mathcal{O}_X$.

By the construction of the preceding section, which we will use, we need more and we get more than only complex multiplication. Hence let us define, which we will get:

Definition 7.2.2. A $CMCY$ family $\mathcal{X} \to \mathcal{B}$ of n-manifolds is a (smooth) family of Calabi-Yau manifolds of dimension n, which has a dense set of fibers $\mathcal{X}_b$ satisfying the property that $\mathrm{Hg}(H^k(\mathcal{X}_b, \mathbb{Q}))$ is commutative for all k.

In this section the degree m of all cyclic covers, which will occur, is equal to 2. We apply the construction of a Calabi-Yau manifold with an involution by two Calabi-Yau manifolds with involutions by C. Borcea [9]. This yields an

[1] One needs in fact the condition that each $\mathrm{Hg}(H^k(Z_i, \mathbb{Q}))$ is commutative. The argument is similar to the argument in the proof of Proposition 10.3.2.

iterative construction of $CMCY$ families with involutions in arbitrary high dimension by $CMCY$ families in lower dimension.[2]

Construction 7.2.3. Let $\mathcal{Z}_1 \to \mathcal{M}$ be a $CMCY$ family of n-manifolds covering the A bundle $\mathcal{A}$ with ramification locus R_1, which satisfies the assumptions for $\mathcal{Z}$ in diagram (7.1). Moreover let Σ_i be a $CMCY$ family $\Sigma_i \to \mathcal{M}^{(i)}$ of n_i-manifolds covering the B_i bundle $\mathcal{B}_i$ over $\mathcal{M}^{(i)}$ with ramification locus $R^{(i)}$, which satisfies the assumptions for Σ in diagram (7.1), for all $1 < i \in \mathbb{N}$.

Let us assume that there is a dense subset of points $m^{(i)} \in \mathcal{M}^{(i)}$ resp., $p \in \mathcal{M}$, which have the property that each $\mathrm{Hg}(H^k((\Sigma_i)_{m^{(i)}}, \mathbb{Q}))$ and each $\mathrm{Hg}(H^k(R^{(i)}_{m^{(i)}}, \mathbb{Q}))$ resp., each $\mathrm{Hg}(H^k((\mathcal{Z}_1)_p, \mathbb{Q}))$ and each $\mathrm{Hg}(H^k((R_1)_p, \mathbb{Q}))$ is commutative.

We define an iterative tower of covers

$$\mathcal{Z}_i \to V^{(i)} := \mathcal{M} \times \mathcal{M}^{(2)} \times \ldots \times \mathcal{M}^{(i)}$$

given by

$$\mathcal{Z}_i = \tilde{\mathcal{Y}}_i,$$

where $\tilde{\mathcal{Y}}_i$ is obtained from $\tilde{\mathcal{Y}}$ in the diagram (7.1) with $V_1 = V^{(i-1)}$, $V_2 = \mathcal{M}^{(i)}$, $\Sigma = \Sigma_i$ and $\mathcal{Z} = \mathcal{Z}_{i-1}$ for all $i \in \mathbb{N}$. Let us call such a construction Borcea-Voisin tower.

The assumption that we have ramification in codimension 1 on the fibers of a family of Calabi-Yau manifolds leads to the important property that the corresponding involutions act by -1 on the global sections of their canonical sheaves, as we see by the following Lemma:

Lemma 7.2.4. *Let C be a Calabi-Yau manifold and ι be an involution on it. Assume that the points fixed by ι are given by a non-trivial smooth divisor D. Then ι acts by -1 on $H^0(C, \omega_C)$.*

Proof. By our assumptions, the induced natural cyclic cover $\gamma : C \to C/\iota$ is ramified over a smooth non-trivial divisor D such that C/ι is smooth. Hence one has a cyclic cover of manifolds and one can apply the Hurwitz formula (compare [6], **I**. 16). Since C has a trivial canonical divisor, the Hurwitz formula implies that $\mathcal{O}_C(-D) \cong \gamma^*(\omega_{C/\iota})$. This implies that $\omega_{C/\iota}$ does not contain any global section. Since $\omega_{C/\iota}$ yields the eigenspace for the character 1 of $\gamma_*(\omega_C)$ (see [20], §3), the character of the action of ι on $H^0(C, \omega_C)$ is not given by 1. Thus it is given by -1. □

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[2] The construction of C. Borcea is repeated in Proposition 7.2.5. By C. Voisin [60], the same construction was used to construct Calabi-Yau 3-manifolds by $K3$-surfaces with involutions and elliptic curves. This is the reason that our construction here is called "Borcea-Voisin tower". Here this construction is introduced as a systematic method to construct Calabi-Yau manifolds with complex multiplication in an arbitrary dimension which has never been done by C. Borcea or C. Voisin in this way.

Proposition 7.2.5. *Assume that $\gamma_1 : C_1 \to M_1$ and $\gamma_2 : C_2 \to M_2$ are cyclic covers of degree 2 with the involutions ι_1 and ι_2 and ramification divisors $D_1 \subset C_1$ and $D_2 \subset C_2$, which are not trivial and consist of disjoint smooth hypersurfaces. Moreover assume that C_1 and C_2 are Calabi-Yau manifolds of dimension n_1 and n_2. Let $\widetilde{C_1 \times C_2}$ denote the blowing up of $C_1 \times C_2$ with respect to $D_1 \times D_2$. Then by the involution on $\widetilde{C_1 \times C_2}$ given by (ι_1, ι_2), one obtains a cyclic cover $\gamma : \widetilde{C_1 \times C_2} \to C$ such that C is a Calabi-Yau manifold.*

Proof. We assume that each C_i is a Calabi-Yau manifold such that $h^{t,0}(C_i) = 0$ for all $t = 1, \ldots, n_i - 1$. By the assumption that one has the ramification divisors D_1 and D_2 and Lemma 7.2.4, the corresponding involution of each γ_i acts by -1 on each ω_{C_i}. Thus one concludes that $h^{j,0}(C) = 0$ for all $j = 1, \ldots, (n_1 + n_2) - 1$.

The canonical divisor $K_{\widetilde{C_1 \times C_2}}$ of $\widetilde{C_1 \times C_2}$ is given by the exceptional divisor E of the blowing up $\widetilde{C_1 \times C_2} \to C_1 \times C_2$. Moreover the ramification divisor R of γ coincides with E. Hence by the Hurwitz formula ([6], **I**.16), we have

$$\mathcal{O}_{\widetilde{C_1 \times C_2}}(R) \cong \mathcal{O}_{\widetilde{C_1 \times C_2}}(K_{\widetilde{C_1 \times C_2}}) = \omega_{\widetilde{C_1 \times C_2}} \cong \gamma^*(\omega_C) \otimes \mathcal{O}_{\widetilde{C_1 \times C_2}}(R).$$

Thus one concludes that $\gamma^*(\omega_C) \cong \mathcal{O}$.

Since ι_1 and ι_2 act by the character -1, the involution (ι_1, ι_2) on $\widetilde{C_1 \times C_2}$ leaves the global sections of $\omega_{\widetilde{C_1 \times C_2}}$ invariant. Now recall that $\gamma_*(\omega_{\widetilde{C_1 \times C_2}})$ consists of a direct sum of invertible sheaves, which are the eigenspaces with respect to the characters of the Galois group action. By [20], §3, the eigenspace for the character 1 is given by ω_C. Thus ω_C has a non-trivial global section. Hence the canonical divisor of C satisfies (up to linear equivalence) $K_C \geq 0$. Thus by the fact that $\gamma^*(\omega_C) \cong \mathcal{O}$, we have the desired result $K_C \sim 0$. □

Altogether one has the following result:

Theorem 7.2.6. *Each family $\mathcal{Z}_i \to \mathcal{M} \times \mathcal{M}^{(2)} \times \ldots \times \mathcal{M}^{(i)}$ obtained by the Borcea-Voisin tower is a CMCY family of $n + n_2 + \ldots + n_i$-manifolds.*

Proof. The statement that each $(\mathcal{Z}_i)_p$ is a Calabi-Yau manifold follows fiberwise by induction. By the assumptions, we have the result for $n = 1$. First by induction, one can show that the ramification loci are given by smooth divisors. By using this fact and the induction hypothesis, one can apply Lemma 7.2.4 such that each involution acts by the character -1 on each $\Gamma(\omega)$. Hence the assumptions of Proposition 7.2.5 are satisfied, which provides the induction step.

Next we show the statement about the commutativity of all Hodge groups over a dense subset of the basis. Due to the situation described in diagram (7.1) the connected components of the ramification locus $(R_{i+1})_{p \times m^{(i+1)}}$ of

$(\mathcal{Z}_{i+1})_{p\times m^{(i+1)}}$ over $p \times m^{(i+1)} \in V^{(i)} \times \mathcal{M}^{(i+1)}$ are given by the connected components of $(\mathcal{Z}_i)_p \times R^{(i+1)}_{m^{(i+1)}}$ and by the connected components of $(R_i)_p \times (\Sigma_{i+1})_j$, where $(R_i)_p$ is the ramification locus of $(\mathcal{Z}_i)_p$.

Hence it is sufficient to use an inductive argument and to show the following Claim: □

Claim 7.2.7. *Assume that for all k the Hodge group* $\mathrm{Hg}(H^k((\mathcal{Z}_i)_p, \mathbb{Z}))$ *is commutative and each connected component Z of the ramification locus $(R_i)_p$ satisfies that each* $\mathrm{Hg}(H^k(Z, \mathbb{Z}))$ *is commutative. In addition we assume that for all k the Hodge group* $\mathrm{Hg}(H^k(\Sigma_{i+1})_{m^{(i+1)}}, \mathbb{Z}))$ *is commutative and each connected component Z_{i+1} of $R^{(i+1)}_{m^{(i+1)}}$ satisfies that each* $\mathrm{Hg}(H^k(Z_{i+1}, \mathbb{Z}))$ *is commutative. Then for all k each connected component $\tilde{Z}$ of $(R_{i+1})_{p\times m^{(i+1)}}$ satisfies that each* $\mathrm{Hg}(H^k(\tilde{Z}, \mathbb{Z}))$ *is commutative and for all k* $\mathrm{Hg}(H^k((\mathcal{Z}_{i+1})_{p\times m^{(i+1)}}, \mathbb{Z}))$ *is commutative.*

Proof. By the assumptions of this claim and the description of R_{i+1} above, one obtains obviously that the connected components $\tilde{Z}$ of $(R_{i+1})_{p\times m^{(i+1)}}$ satisfy that each $\mathrm{Hg}(H^k(\tilde{Z}, \mathbb{Z}))$ is commutative. Then one must simply use Theorem 7.1.7 and one obtains that each $\mathrm{Hg}(H^k(\mathcal{Z}_{i+1})_{p\times m^{(i+1)}}, \mathbb{Z}))$ is commutative, too. □

7.3 The Viehweg-Zuo tower

By the Borcea-Voisin tower, one can construct $CMCY$ families of manifolds in arbitrary high dimension. But one needs $CMCY$ families of manifolds (in low dimension) with a suitable involution, which can be used to be $\mathcal{Z}_1$ or some Σ_i. One way to obtain some suitable $CMCY$ families of n-manifolds (in low dimension) is given by the Viehweg-Zuo tower, which we introduce now.

E. Viehweg and K. Zuo [58] have constructed a tower of projective algebraic manifolds starting with a family $\mathcal{F}_1$ of cyclic covers of $\mathbb{P}^1$ given by

$$\mathbb{P}^2 \supset V(y_1^5 + x_1(x_1 - x_0)(x_1 - \alpha x_0)(x_1 - \beta x_0)x_0) \to (\alpha, \beta) \in \mathcal{M}_2,$$

which has a dense set of CM fibers. This is one example of a family of cyclic covers, which has a primitive pure $(1, 2) - VHS$ as one can easily verify by using Proposition 2.3.4. Since each of these covers given by the fibers of the family can be embedded into $\mathbb{P}^2$, the fibers of $\mathcal{F}_1$ are the branch loci of the fibers of a family $\mathcal{F}_2$ of cyclic covers of $\mathbb{P}^2$ of degree 5. Moreover the fibers of $\mathcal{F}_2$, which can be embedded into $\mathbb{P}^3$, are the branch loci of the fibers of a family $\mathcal{F}_3$ of cyclic covers of $\mathbb{P}^3$, which can be embedded into $\mathbb{P}^4$. The family $\mathcal{F}_3$ is given by

$$\mathbb{P}^4 \supset V(y_3^5 + y_2^5 + y_1^5 + x_1(x_1 - x_0)(x_1 - \alpha x_0)(x_1 - \beta x_0)x_0) \to (\alpha, \beta) \in \mathcal{M}_2.$$

The fibers of $\mathcal{F}_3$ are Calabi-Yau 3-manifolds. By an inductive argument, this latter family has a dense set of CM points on the basis given by the dense set of the CM points of the family of curves we have started with (see [58]). Since only the Hodge group of the Hodge structure on $H^3(X,\mathbb{Q})$ of a projective hypersurface $X \subset \mathbb{P}^4$ can be non-trivial, the family $\mathcal{F}_3$ is a $CMCY$ family of 3-manifolds.

Example 7.3.1. By Theorem 2.4.4, the fibers of $\mathcal{F}_1$ isomorphic to

$$V(y_1^5+x_1^5+x_0^5), \quad V(y_1^5+x_1(x_1^4+x_0^4)), \quad V(y_1^5+x_1(x_1^3+x_0^3)x_0) \subset \mathbb{P}^2$$

have CM. Thus the fibers of $\mathcal{F}_3$ isomorphic to

$$V(y_3^5+y_2^5+y_1^5+x_1^5+x_0^5),\ V(y_3^5+y_2^5+y_1^5+x_1(x_1^4+x_0^4)),\ V(y_3^5+y_2^5+y_1^5+x_1(x_1^3+x_0^3)x_0) \subset \mathbb{P}^4$$

have CM, too.

Example 7.3.2. We consider the $CMCY$ family $\mathcal{F}_3$

$$\mathbb{P}^4 \supset V(y_3^5+y_2^5+y_1^5+x_1(x_1-x_0)(x_1-\alpha x_0)(x_1-\beta x_0)x_0) \to (\alpha,\beta) \in \mathcal{M}_2$$

constructed by E. Viehweg and K. Zuo. On each fiber $(\mathcal{F}_3)_p$ the involution ι given by

$$\iota(y_3 : y_2 : y_1 : x_1 : x_0) = (y_2 : y_3 : y_1 : x_1 : x_0)$$

leaves the smooth divisor $\mathcal{D}_p$ given by the equation $y_3 = y_2$ invariant. Moreover one has that $\mathcal{D}_p \cong (\mathcal{F}_2)_p$. Therefore there is a dense set of points $p \in \mathcal{M}_2$, which have the property that for all k the Hodge groups of $H^k(\mathcal{D}_p,\mathbb{Q})$ and $H^k((\mathcal{F}_3)_p,\mathbb{Q})$ are commutative. Hence one can use $\mathcal{F}_3$ to be $\mathcal{Z}_1$ or some Σ_i for the construction of a Borcea-Voisin tower of $CMCY$ families of n-manifolds.

Example 7.3.3. Let $\mathbb{F}_d$ denote the Fermat curve of degree $d > 2$. The curve $\mathbb{F}_d$ has complex multiplication (see [22] and [32]). By the construction of E. Viehweg and K. Zuo in [58], one concludes that the Calabi-Yau manifold H_d given by

$$V(\sum_{i=0}^{d-1} x_i^d) \subset \mathbb{P}^{d-1}$$

has complex multiplication. Since H_d is a projective hypersurface, this implies that H_d has only commutative Hodge groups. We have the involution ι_a given by

$$(x_{d-1} : \ldots : x_2 : x_1 : x_0) \to (x_{d-1} : \ldots ; x_2 : x_0 : x_1)$$

on H_d. If d is even, one has the additional involution ι_b given by

$$(x_{d-1} : \ldots : x_1 : x_0) \to (x_{d-1} : \ldots : x_1 : -x_0).$$

The involution ι_a resp., ι_b (if it is given on H_d) fixes the points of a smooth divisor on H_d, which is isomorphic to

$$V(\sum_{i=0}^{d-2} x_i^d) \subset \mathbb{P}^{d-2}.$$

Therefore by the same arguments as in Example 7.3.2, one can use H_d to be $\mathcal{Z}_1$ or some Σ_i with $\mathcal{M} = \mathrm{Spec}(\mathbb{C})$, resp., $\mathcal{M}^{(i)} = \mathrm{Spec}(\mathbb{C})$ for the construction of a Borcea-Voisin tower of $CMCY$ families of n-manifolds.

We want to start the construction of a Viehweg-Zuo tower (of projective hypersurfaces as in [58] or the construction of a modified version) with a family of cyclic covers $\mathcal{C} \to \mathcal{M}_n$ of $\mathbb{P}^1$ with a dense set of CM fibers. For the smoothness of the higher dimensional fibers of the resulting families, we will use the assumption that the fibers of $\mathcal{C}$ are given by

$$V(y^m + x(x-1)(x-a_1)\ldots(x-a_n)) \subset \mathbb{A}^2, \tag{7.2}$$

where m divides $n+3$ such that all branch indices coincide.

By our preceding results, we have only the following examples of families of cyclic covers of $\mathbb{P}^1$ with a dense set of CM fibers, which satisfy this assumption:

degree m	number of ramification points of the fibers
2	4
2	6
3	6
4	4
5	5

Remark 7.3.4. The case with $m = 2$ and 4 ramification points is the case of elliptic curves, which has been considered by C. Borcea in [8]. The case with $m = 5$ yields the example by E. Viehweg and K. Zuo in [58].

The case with $m = 3$ is one of the examples of a family of covers of $\mathbb{P}^1$ with a dense set of CM fibers by J. de Jong and R. Noot [29]. We must a bit work to give a suitable modified construction of a Viehweg-Zuo tower for this example. The next chapter is devoted to this modified construction of a Viehweg-Zuo tower.

In the case of the family $\mathcal{C} \to \mathcal{M}_3$ of genus 2 curves the author does not see a possibility for the construction of a Viehweg-Zuo tower.[3]

[3] One natural choice for an embedding of the fibers of the family of genus 2 curves is given by the weighted projective space $\mathbb{P}(3,1,1)$. But the canonical divisor of the desingularization of $\mathbb{P}(3,1,1)$ does not allow a natural construction of a Viehweg-Zuo tower as in the case of $\mathbb{P}(2,1,1)$, which we will see in the next chapter for the degree 3 case.

The case with $m = 4$ yields the Shimura- and Teichmüller curve of M. Möller [41], which provides the example of the next section.

7.4 A new example

Here we see that the Shimura- and Teichmüller curve of M. Möller yields an example of a Viehweg-Zuo tower. Moreover we will see that the resulting $CMCY$ family of 2-manifolds is endowed with some involutions, which make it suitable for the construction of a Borcea-Voisin tower. In addition we give some explicit CM fibers and try to decide, which involutions provide isomorphic quotients resp., isomorphic $CMCY$ families by the construction of a Borcea-Voisin tower.

Proposition 7.4.1. *The family $\mathcal{C}_2 \to \mathcal{M}_1$ given by*

$$\mathbb{P}^3 \supset V(y_2^4 + y_1^4 + x_1(x_1 - x_0)(x_1 - \lambda x_0)x_0) \to \lambda \in \mathcal{M}_1$$

is a $CMCY$ family of 2-manifolds.

Proof. It is well-known that a hypersurface of $\mathbb{P}^3$ of degree 4 is a $K3$-surface.

By [58], Notation 2.2, and Corollary 8.5, we have that λ_0 is a CM-point of $\mathcal{C}_2$, if λ_0 is a CM-point of the family $\mathcal{C}_1 \to \mathcal{M}_1$ given by

$$\mathbb{P}^2 \supset V(y_1^4 + x_1(x_1 - x_0)(x_1 - \lambda x_0)x_0) \to \lambda \in \mathcal{M}_1.$$

Note that $\mathcal{C}_1$ has in fact a dense set of CM fibers, since it has a derived pure $(1,1)-VHS$ as we have seen. Since only the Hodge group of the Hodge structure on $H^2(X, \mathbb{Q})$ can be non-trivial for a $K3$-surface X (follows by definition resp., by the Hodge diamond of a $K3$-surface), the family $\mathcal{C}_2$ is a $CMCY$ family of 2-manifolds. □

Now we give some examples of CM fibers of $\mathcal{C}_2$:

Remark 7.4.2. Consider the family $\mathcal{E} \to \mathcal{M}_1$ of elliptic curves given by

$$\mathbb{P}^2 \supset V(y^2 x_0 + x_1(x_1 - x_0)(x_1 - \lambda x_0)) \to \lambda \in \mathcal{M}_1.$$

Note that $\mathcal{C}_1$ has a derived pure $(1,1) - VHS$, where $\mathcal{E}$ has the associated primitive pure $(1,1) - VHS$. Thus the Hodge structure decomposition of Proposition 4.2.2 tells us that the fiber $(\mathcal{C}_1)_\lambda$ has CM, if the fiber $\mathcal{E}_\lambda$ has CM. In the proof of Proposition 7.4.1 we have seen that $(\mathcal{C}_2)_\lambda$ has CM, if $(\mathcal{C}_1)_\lambda$ has CM. Thus by the CM fibers of $\mathcal{E}$, we can determine CM fibers of $\mathcal{C}_2$.

Example 7.4.3. By Remark 7.4.2, the well-known CM curves with j invariant 0 and 1728 yield CM fibers of $\mathcal{C}_2$ isomorphic to

$$V(y_2^4 + y_1^4 + (x_1^3 - x_0^3)x_0), \quad V(y_2^4 + y_1^4 + x_1^4 + x_0^4) \subset \mathbb{P}^3.$$

Theorem 2.4.4 yields the same examples.

Example 7.4.4. By [26], **IV**. Proposition 4.18, one concludes that an elliptic curve has complex multiplication, if it has a non-trivial isogeny with itself. The elliptic curve with j invariant 8000 resp., -3375 is given by

$$y^2x_0 = x_1(x_1-x_0)(x_1-(1+\sqrt{2})^2x_0) \text{ resp., } y^2x_0 = x_1(x_1-x_0)(x_1-\frac{1}{4}(3+i\sqrt{7})^2x_0)$$

and has an isogeny of degree 2 with itself. Moreover the elliptic curve with j invariant 1728 has an isogeny of degree 2 with itself. This follows from the solution of [26], **IV**. Exercise 4.5, which we will partially sketch. Thus the $K3$ surfaces given by

$$y_2^4+y_1^4+x_1(x_1-x_0)(x_1-(1+\sqrt{2})^2x_0)x_0 \text{ and } y_2^4+y_1^4+x_1(x_1-x_0)(x_1-\frac{1}{4}(3+i\sqrt{7})^2x_0)x_0$$

have complex multiplication.

We sketch how we obtain the given examples: First note that each degree 2 cover $u : \mathbb{P}^1 \to \mathbb{P}^1$ is up to a changement of coordinates given by $x \to x^2$. This follows from the fact that u has two ramification points by the Hurwitz formula. Without loss of generality the elliptic curve E is endowed with a degree two cover $i : E \to \mathbb{P}^1$ such that there exists a λ such i is ramified over $0, 1, \lambda, \infty$ resp., E is locally given by

$$V(y^2 - x(x-1)(x-\lambda)) \subset \mathbb{A}^2. \tag{7.3}$$

Since an isogeny $f : E \to E$ is a morphism of abelian varieties, one concludes that for each $(x, y) = f(P) \in E$ one has $f(-P) = -(x, y) = (x, -y)$. Hence one concludes that there exist the degree 2 covers $u_f : \mathbb{P}^1 \to \mathbb{P}^1$ and $h_f : E \to \mathbb{P}^1$ such that

$$i \circ f = u_f \circ h_f.$$

It is a very easy exercise to check that u_f can be given by $x \to x^2$ in this case for some suitable λ, which yields E. Thus one concludes that h_f is ramified over

$$1, -1, \sqrt{\lambda}, -\sqrt{\lambda},$$

which follows from considering the ramification indices. By a changement of coordinates, E is given by

$$0, 1, \frac{(\sqrt{\lambda}+1)^2}{(\sqrt{\lambda}-1)^2}, \infty,$$

too. Note that λ and $1-\lambda$ yield the same elliptic curve. We substitute $t = \sqrt{\lambda}$ and resolve the equations

$$t^2 = \frac{(t+1)^2}{(t-1)^2} \quad \text{and} \quad t^2 = 1 - \frac{(t+1)^2}{(t-1)^2}$$

by using the computer algebra program MATHEMATICA in the case of the ground field $\mathbb{C}$. This yields the stated elliptic curves E with an isogeny $f : E \to E$ of degree 2. It remains to prove the completeness of the given examples, which is a well-known fact.

Example 7.4.5. Elliptic curves with CM has been well studied by number theorists. In [55], Appendix **C**, §3 there is a list of 13 isomorphy classes of elliptic curves with complex multiplication containing all classes represented by the preceding 4 examples. Two examples of the list, which have the j invariants 54000 and 16581375, are given by the equations

$$y^2 = x^3 - 15x + 22, \quad y^2 = x^3 - 595x + 5586.$$

The equations allow an explicit determination of involutions on these examples. The given equations for the 7 remaining isomorphy classes of elliptic curves do not allow an immediate description of involutions.

As we will see, the family $\mathcal{C}_2$ has some involutions, which make it suitable for the construction of a Borcea-Voisin tower. The following lemma is obvious:

Lemma 7.4.6. *Over the basis $\mathcal{M}_1$ the family $\mathcal{C}_2$ has three involutions given by*

$$\iota_1(y_2 : y_1 : x_1 : x_0) = (-y_2 : y_1 : x_1 : x_0), \quad \iota_2(y_2 : y_1 : x_1 : x_0) = (y_2 : -y_1 : x_1 : x_0),$$

$$\iota_3(y_2 : y_1 : x_1 : x_0) = (-y_2 : -y_1 : x_1 : x_0),$$

which constitute with the identity map a subgroup of the $\mathcal{M}_1$-automorphism group of $\mathcal{C}_2$ isomorphic to the Kleinsche Vierergruppe.

Remark 7.4.7. Over $\mathcal{M}_1$ there are at least the 4 following additional involutions on $\mathcal{C}_2$:

$$\iota_4(y_2 : y_1 : x_1 : x_0) = (y_1 : y_2 : x_1 : x_0), \quad \iota_5(y_2 : y_1 : x_1 : x_0) = (iy_1 : -iy_2 : x_1 : x_0),$$

$$\iota_6(y_2 : y_1 : x_1 : x_0) = (-y_1 : -y_2 : x_1 : x_0), \; \iota_7(y_2 : y_1 : x_1 : x_0) = (-iy_1 : iy_2 : x_1 : x_0)$$

Theorem 7.4.8. *By the involutions ι_1 and ι_4, the family $\mathcal{C}_2$ can be used to be $\mathcal{Z}_1$ or some Σ_i for the construction of a Borcea-Voisin tower of CMCY families of n-manifolds.*

Proof. The divisor of the fiber $(\mathcal{C}_2)_\lambda$, which is fixed by ι_1 resp., ι_4 is given by $y_2 = 0$ resp., $y_2 = y_1$. Hence both divisors are smooth and isomorphic to the fiber $(\mathcal{C}_1)_\lambda$ given by

$$\mathbb{P}^2 \supset V(y_1^4 + x_1(x_1 - x_0)(x_1 - \lambda x_0)x_0) \to \lambda \in \mathcal{M}_1.$$

We use the same arguments as in the proof of Proposition 7.4.1: If $(\mathcal{C}_1)_\lambda$ has complex multiplication, then $(\mathcal{C}_2)_\lambda$ and the divisor fixed by ι_1 resp., ι_4 have complex multiplication, too. Hence by the fact that $\mathcal{C}_1$ has a dense set of complex multiplication fibers, $\mathcal{C}_2$ and ι_1 resp., $\mathcal{C}_2$ and ι_4 satisfy the assumptions of Construction 7.2.3. □

Remark 7.4.9. By the fact that

$$\iota_2 = \iota_4 \circ \iota_1 \circ \iota_4,$$

the involution ι_2 is suitable for the construction of a Borcea-Voisin tower, too. But according to the construction of C. Voisin [60], this implies that ι_2 yields a $CMCY$ family of 3-manifolds over $\mathcal{M}_1 \times \mathcal{M}_1$, which is isomorphic to the corresponding family obtained by ι_1.

Let α denote the $\mathcal{M}_1$-automorphism of $\mathcal{C}_2$ given by

$$(y_2 : y_1 : x_1 : x_0) \to (iy_2 : y_1 : x_1 : x_0).$$

One calculates easily that

$$\iota_5 = \alpha \circ \iota_4 \circ \alpha^{-1}, \quad \iota_6 = \alpha^2 \circ \iota_4 \circ \alpha^{-2}, \quad \iota_7 = \alpha^{-1} \circ \iota_4 \circ \alpha.$$

Hence one has that $\mathcal{C}_2/\iota_4, \ldots, \mathcal{C}_2/\iota_7$ resp., the resulting $CMCY$ families of 3-manifolds obtained by the method of C. Voisin [60] are isomorphic as $\mathcal{M}_1$-schemes resp., as $\mathcal{M}_1 \times \mathcal{M}_1$-schemes.

Since

$$\iota_3 = \iota_1 \iota_2,$$

the involution ι_3 acts by id on each $\Gamma(\omega_{(\mathcal{C}_2)_\lambda})$ such that it can not be used for the construction of a Borcea-Voisin tower.

Remark 7.4.10. By Example 7.4.3, Example 7.4.4 and Example 7.4.5, one has 6 explicitly given elliptic curves with CM and explicitly given involutions, which yields 6 $K3$ surfaces with CM. By using the method of C. Voisin [60], these examples yield 36 explicitly given fibers with CM for each of our resulting $CMCY$ family of 3-manifolds.

Remark 7.4.11. The author does not see a way to conjugate ι_1 into ι_4. Moreover we will see that the fibers of the resulting $CMCY$ families of 3-manifolds constructed with ι_1 and ι_4 according to C. Voisin [60] have the same Hodge numbers. This means that the question for isomorphisms between these two families remains open.

Chapter 8
The degree 3 case

In this chapter we give a modified construction of a Viehweg-Zuo tower, which yields a $CMCY$ family of 2-manifolds suitable for the construction of a Borcea-Voisin tower.

Let R^1 the desingularization of the weighted projective space $\mathbb{P}(2,1,1)$, which is obtained from blowing up the singular point. We start with the family $\mathcal{C}$ of curves given by

$$R^1 \supset V(y^3 - x_1(x_1 - x_0)(x_1 - a_1x_0)(x_1 - a_2x_0)(x_1 - a_3x_0)x_0) \to (a_1, a_2, a_3) \in \mathcal{M}_3.$$

This family has a dense set of CM fibers. Since the degree of these covers of $\mathbb{P}^1$ does not coincide with the sum of their branch indices, it is not possible to work with usual projective spaces. Thus we work with weighted projective spaces $\mathbb{P}(2, \ldots, 2, 1, 1)$ resp., their desingularizations to obtain Calabi-Yau hypersurfaces and a tower of cyclic coverings similar to the construction of E. Viehweg and K. Zuo [58]. For this construction we have to recall some facts and to make some preparations in Section 8.1. In Section 8.2 we give our modified version of the construction of Viehweg and Zuo, which yields a $CMCY$ family of Calabi-Yau 2-manifolds. Let R^2 be the desingularization of the weighted projective space $\mathbb{P}(2,2,1,1)$, which is obtained from blowing up the singular locus. The $CMCY$ family of Calabi-Yau 2-manifolds is given by

$$R^2 \supset \tilde{V}(y_2^3 + y_1^3 - x_1(x_1 - x_0)(x_1 - a_1x_0)(x_1 - a_2x_0)(x_1 - a_3x_0)x_0) \to (a_1, a_2, a_3) \in \mathcal{M}_3.$$

We indicate some involutions of this family, which make it suitable for the construction of a Borcea-Voisin tower, in Section 8.3.

J.C. Rohde, *Cyclic Coverings, Calabi-Yau Manifolds and Complex Multiplication*, 157
Lecture Notes in Mathematics 1975, DOI: 10.1007/978-3-642-00639-5_9,

8.1 Prelude

Recall that the usual projective space $\mathbb{P}^n$ is given by $\mathrm{Proj}(\mathbb{C}[z_n, \ldots, z_1, z_0])$, where each z_j (with $j = 0, \ldots, n$) has the weight 1. Our weighted projective space Q^n is given by $\mathrm{Proj}(\mathbb{C}[y_n, \ldots, y_1, x_1, x_0])$, where each y_j (with $j = 1, \ldots, n$) has the weight 2, and x_0 and x_1 have the weight 1.

First we investigate and describe the projective space Q^n. The following well-known Lemma will be very useful here:

Lemma 8.1.1. *(Veronese embedding) Let R be a graded ring. Then we have*

$$\mathrm{Proj}(R) \cong \mathrm{Proj}(R^{[d]}).$$

Proposition 8.1.2. *The weighted projective space Q^n is isomorphic to the irreducible singular hypersurface in $\mathbb{P}^{n+2}$ given by the equation $z_1 z_3 = z_2^2$. The singular locus of Q^n is given by $V(z_1, z_2, z_3)$.*

Proof. By the Veronese embedding, we have

$$Q^n \cong \mathrm{Proj}(k[x_0^2, x_0 x_1, x_1^2, y_1, \ldots, y_n]).$$

Therefore we obtain a closed embedding of Q^n into $\mathbb{P}^{n+2}$ given by

$$x_0^2 \to z_1, \quad x_0 x_1 \to z_2, \quad x_1^2 \to z_3, \quad y_1 \to z_4, \quad \ldots, \quad y_n \to z_{n+3}.$$

We have that $Q^n \backslash V(x_0^2)$ is isomorphic to $\mathbb{A}^{n+1}$. Hence $\dim(Q^n) = n+1$, which implies that its projective cone, which is contained in $\mathbb{A}^{n+3}$, has the dimension $n+2$. By [26], **I**. Proposition 1.13, each irreducible component of dimension $n+2$ of this cone is given by an ideal generated by one irreducible polynomial. The corresponding polynomial of the unique irreducible component of Q^n is

$$f(z_1, z_2, z_3) = z_1 z_3 - z_2^2,$$

since each point $p \in Q^n \subset \mathbb{P}^{n+2}$ satisfies $f(p) = 0$ and f is irreducible. The last statement about the singular locus follows from calculating the partial derivatives of f. □

Let $a_1, \ldots, a_{2m} \in \mathbb{C}$, and $m \in \mathbb{N} \setminus \{1\}$. Then $C_{(n)} \subset Q^n$ is the subvariety, which is given by the homogeneous polynomial

$$y_n^m + \ldots + y_1^m + (x_1 - a_1 x_0) \ldots (x_1 - a_{2m} x_0).$$

It is a very easy exercise to check that this polynomial is irreducible.

Proposition 8.1.3. *There exists a homogeneous polynomial $G \in \mathbb{C}[z_1, z_2, z_3]$ of degree m such that $C_{(n)} \subset \mathbb{P}^{n+2}$ is given by the ideal generated by h and f, where*

$$h = z_{n+3}^m + \dots z_4^m + G.$$

Proof. We can obviously choose a polynomial G such that

$$G(x_0^2, x_0x_1, x_0^2) = (x_1 - a_1x_0)\dots(x_1 - a_{2m}x_0).$$

Now let $h = z_{n+3}^m + \dots z_4^m + G$, and

$$\phi : \mathbb{C}[z_1, \dots, z_{n+3}] \to \mathbb{C}[x_0^2, x_0x_1, x_1^2, y_1, \dots, y_n]$$

be the homomorphism associated to the closed embedding $Q^n \hookrightarrow \mathbb{P}^{n+2}$, which has the kernel (f). We obtain

$$\phi(h) = y_n^m + \dots + y_1^m + (x_1 - a_1x_0)\dots(x_1 - a_{2m}x_0).$$

Hence $C_{(n)} \subset \mathbb{P}^{n+2}$ is given by the prime ideal

$$\phi^{-1}(\mathcal{I}(C_{(n)})) = (h, f).$$

□

Proposition 8.1.4. *The singular locus of $C_{(n)}$ is given by $C_{(n)} \cap V(z_1, z_2, z_3)$.*

Proof. On $Q^n \backslash V(x_0) \cong \mathrm{Spec}(\mathbb{C}[x_1, y_1, \dots, y_n])$ the hypersurface $C_{(n)}$ is given by the equation

$$0 = y_n^m + \dots + y_1^m + (x_1 - a_1)\dots(x_1 - a_{2m}).$$

By the partial derivatives of the polynomial on the right hand, one can easily check that there are no singularities of $C_{(n)}$ in this affine subset. The same arguments give the same statement for $Q^n \setminus V(x_1)$. Hence all singularities of $C_{(n)}$ are contained in $V = V(z_1, z_2, z_3)$. For all $P \in C_{(n)} \cap V$, the Jacobian matrix of $C_{(n)}$ at P does not have the maximal rank 2, where this is obtained by explicit calculation of the partial derivatives of f and h. □

8.1.5. The variety Q^n has a natural interpretation as degree 2 cover of the variety given by $\{z_2 = 0\}$ ramified over $\{z_1 = z_2 = 0\}$ and $\{z_2 = z_3 = 0\}$. Hence by blowing up $V = V(z_1, z_2, z_3)$, the proper transform $R^n := \tilde{Q}_V^n$ is the natural degree 2 cover of the proper transform of $\{z_2 = 0\}$ ramified over the disjoint proper transforms of $\{z_1 = z_2 = 0\}$ and $\{z_2 = z_3 = 0\}$. Thus R^n is non-singular.

Note that the general construction of the blowing up yields a natural embedding of an open subset of R^n into $\mathbb{A}^{n+2} \times \mathbb{P}^2$. Hence the Jacobian matrix at each point of R^n has the maximal rank 3 with respect to this local embedding. The Jacobian matrix of the proper transform $\tilde{C}_n$ of $C_{(n)}$ is given by adding the line of the partial derivatives of h to the Jacobian matrix of R^n. Without loss of generality we are on the open subset $\{y_1 = 1\}$. On the exceptional divisor E the polynomial G vanishes. Thus all points of $\tilde{C}_n \cap E$ satisfy

$$y_n^m + \ldots + y_2^n + 1 = 0.$$

Hence for each $p \in \tilde{C}_n \cap E$ there is a partial derivative $\partial h/\partial y_i(P) \neq 0$. Since all partial derivatives of the equations defining R^n with respect to y_i vanish, the Jacobian matrix of $\tilde{C}_n$ has the maximal rank 4 at each point on the exceptional divisor. Thus $\tilde{C}_n$ is smooth.

Remark 8.1.6. Note that Q^1 has a natural interpretation as projective closure of the affine cone of a rational curve of degree 2 in $\mathbb{P}^2$. By [26], **V**. Example 2.11.4, one has that R^1, which is the blowing up of the unique singular point given by the vertex of the cone, is a rational ruled surface isomorphic to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} + \mathcal{O}_{\mathbb{P}^1}(2))$, where the exceptional divisor has the self-intersection number -2.

By [26], **II**. Proposition 8.20, one has for $n \geq 1$:

$$\omega_{Q^n \setminus V(z_1,z_2,z_3)} = \omega_{\mathbb{P}^{n+2} \setminus V(z_1,z_2,z_3)} \otimes \mathcal{I}(Q^n \setminus \ V(z_1, z_2, z_3)) \otimes \mathcal{O}_{Q^n \setminus V(z_1,z_2,z_3)}$$

$$= \mathcal{O}_{Q^n \setminus V(z_1,z_2,z_3)}(-(n+1)V(z_4))$$

By [4], Theorem 2.7 and the fact that the self-intersection number of the exceptional divisor is -2, the pull-back of the canonical divisor of Q^1 with respect to the blowing up morphism is the canonical divisor of R^1. Note that the canonical divisor of Q^1 yields the canonical divisor of $Q^1 \setminus \{s\}$, where s denotes the singular point. Thus:

Corollary 8.1.7. *The canonical divisor of R^1 is given by $-2V(z_4)$.*

The following lemma describes the construction of this section. One has the following commutative diagram of closed embeddings:

$$\begin{tikzcd}
C_{(0)} \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & C_{(n)} \arrow[r] \arrow[d] & C_{(n+1)} \arrow[r] \arrow[d] & \cdots \arrow[d] \\
Q^0 \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & Q^n \arrow[r] \arrow[d] & Q^{n+1} \arrow[r] \arrow[d] & \cdots \arrow[d] \\
\mathbb{P}^2 \arrow[r] & \cdots \arrow[r] & \mathbb{P}^{n+2} \arrow[r] & \mathbb{P}^{n+3} \arrow[r] & \cdots
\end{tikzcd}$$

The ideal sheaf of each blowing up $\tilde{C}_n \to C_{(n)}$ and $R^n \to Q^n$ is generated by z_1, z_2, z_3. Moreover this ideal sheaf is obviously the inverse image ideal sheaf of the ideal sheaf generated by z_1, z_2, z_3 with respect to all embeddings. Hence we obtain by [26], II. Corollary 7.15 for $V := V(z_1, z_2, z_3)$:

Lemma 8.1.8. *We have the commutative diagram*

$$\begin{tikzcd}
\tilde{C}_0 \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \tilde{C}_n \arrow[r] \arrow[d] & \tilde{C}_{n+1} \arrow[r] \arrow[d] & \cdots \arrow[d] \\
R^0 \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & R^n \arrow[r] \arrow[d] & R^{n+1} \arrow[r] \arrow[d] & \cdots \arrow[d] \\
\tilde{\mathbb{P}}^2_V \arrow[r] & \cdots \arrow[r] & \tilde{\mathbb{P}}^{n+2}_V \arrow[r] & \mathbb{P}^{n+3}_V \arrow[r] & \cdots
\end{tikzcd}$$

of closed embeddings.

Remark 8.1.9. Note that $C_{(0)} = \tilde{C}_0$, $C_{(1)} = \tilde{C}_1$ and $Q^0 = R^0$.

Theorem 8.1.10. *The canonical divisor of R^n is given by $-(n+1)\tilde{V}(z_4)$ for $n \geq 1$.*

Proof. By Corollary 8.1.7, we have the statement for $n = 1$.

We use induction for higher n. Let E_n denote exceptional divisor of the blowing up $R^n \to Q^n$. The open subset $R^n \backslash E_n$ is isomorphic to $Q^n \backslash V(z_1, z_2, z_3)$. We know that $-(n+1)\tilde{V}(z_4)$ is the canonical divisor of $Q^n \backslash V(z_1, z_2, z_3)$. Hence we conclude that

$$K_{R^{n+1}} = -(n+2)\tilde{V}(z_4) + zE_{n+1}$$

for some $z \in \mathbb{Z}$. We have that $R^n \sim \tilde{V}(z_4)$ in $Cl(R^{n+1})$. By the induction hypothesis, we have

$$\mathcal{O}_{R^n}(-(n+1)\tilde{V}(z_4)) \cong \omega_{R^n} \cong \mathcal{O}_{R^{n+1}}(\tilde{V}(z_4)) \otimes \omega_{R^{n+1}} \otimes \mathcal{O}_{R^n}$$

such that $z = 0$ and $-(n+2)\tilde{V}(z_4)$ is the canonical divisor of R^{n+1}. $\square$

Since we want to construct a family of Calabi-Yau manifolds, we note:

Theorem 8.1.11. *The hypersurface $\tilde{C}_{m-1} \subset R^{m-1}$ is a Calabi-Yau manifold.*

Proof. By Theorem 8.1.10, $-m\tilde{V}(z_4)$ is the canonical divisor of R^{m-1}. Hence [26], **II**. Proposition 8.20 and $\tilde{C}_{m-1} \sim m\tilde{V}(z_4)$ imply that

$$\omega_{\tilde{C}_{m-1}} = \mathcal{O}_{\tilde{C}_{m-1}}.$$

By the fact that $h^{q,0}$ is a birational invariant of non-singular projective varieties (see [26], page 190), and R^{m-1} is birationally equivalent to $\mathbb{P}^m$, we obtain that $h^{q,0}(R^{m-1}) = 0$ for all $1 \leq q \leq m$. By Hodge symmetry and Serre duality, we obtain that $h^q(R^{m-1}, \mathcal{O}) = 0$ for all $1 \leq q \leq m$ and $h^q(R^{m-1}, \omega) = 0$ for all $0 \leq q \leq m-1$. Since the canonical divisor of R^{m-1} is linearly equivalent to $-\tilde{C}_{m-1}$, we obtain the exact sequence

$$0 \to \omega_{R^{m-1}} \to \mathcal{O}_{R^{m-1}} \to \mathcal{O}_{\tilde{C}_{m-1}} \to 0.$$

This implies that $h^i(\tilde{C}_{m-1}, \mathcal{O}) = 0$ for $1 \leq i < m-1 = \dim(\tilde{C}_{m-1})$. Hence $\tilde{C}_{m-1}$ is a Calabi-Yau manifold. □

8.1.12. The projection $\mathbb{P}^{n+2} \setminus \{(1:0:\ldots:0)\} \to \mathbb{P}^{n+1}$ given by

$$(z_{n+3} : \ldots : z_1) \to (z_{n+2} : \ldots : z_1)$$

induces a cyclic cover $C_{(n+1)} \to Q^n$ of degree m ramified over $C_{(n)}$. The Galois group is generated by

$$(z_{n+3} : z_{n+2} : \ldots : z_1) \to (\xi z_{n+3} : z_{n+2} : \ldots : z_1),$$

where ξ is a primitive m-th. root of unity.

Recall the commutative diagram of Lemma 8.1.8. Let $\mathbb{A}^4$ be given by $\{z_4 = 1\} \subset \mathbb{P}^4$ and $\mathbb{A}^3$ be given by $\{z_4 = 1\} \subset \mathbb{P}^3$. Then the projection above yields a morphism

$$f : \mathbb{A}^4 \times \mathbb{P}^2 \to \mathbb{A}^3 \times \mathbb{P}^2. \tag{8.1}$$

Since the blowing up yields natural embeddings of open subsets of $\tilde{C}_2$ and R^1 into the varieties of (8.1), f induces a rational map $\tilde{C}_2 \to R^1$. Now this rational map $\tilde{C}_2 \to R^1$ is again a cyclic cover of degree m with the Galois group as above (on the open locus of definition). On the complements of the exceptional divisors it coincides with the cyclic cover $C_{(2)} \to Q^1$ above. Hence by gluing, one has a cyclic cover $\tilde{C}_2 \to R^1$ ramified over $C_{(1)}$.

8.2 A modified version of the method of Viehweg and Zuo

The following construction is a modified version of the construction in [58], Section 5. Here we show that $\tilde{C}_2$ has CM, if $C_{(1)}$ has CM. In the next section we will use the construction of the preceding section to define a family of $K3$-surfaces. In this section we give the argument that this family of $K3$-surfaces is a $CMCY$ family of 2-manifolds.

For our application, it is sufficient to consider the situation fiberwise and to work with $\mathbb{P}^1$-bundles over $\mathbb{P}^1$ resp., with rational ruled surfaces. Let π_n :

$\mathbb{P}_n \to \mathbb{P}^1$ denote the rational ruled surface given by $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ and σ denote a non-trivial global section of $\mathcal{O}_{\mathbb{P}^1}(6)$, which has the six different zero points represented by a point $q \in \mathcal{M}_3$. The sections E_σ, E_0 and E_∞ of $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(6))$ are induced by

$$\mathrm{id} \oplus \sigma : \mathcal{O} \to \mathcal{O} \oplus \mathcal{O}(6), \quad \mathrm{id} \oplus 0 : \mathcal{O} \to \mathcal{O} \oplus \mathcal{O}(6)$$
$$\text{and} \quad 0 \oplus \mathrm{id} : \mathcal{O}(6) \to \mathcal{O} \oplus \mathcal{O}(6)$$

resp., by the corresponding surjections onto the cokernels of these embeddings as described in [26], **II**. Proposition 7.12.

Remark 8.2.1. The divisors E_σ and E_0 intersect each other transversally over the 6 zero points of σ. Recall that $\mathrm{Pic}(\mathbb{P}_6)$ has a basis given by a fiber and an arbitrary section. Hence by the fact that E_σ and E_0 do not intersect E_∞, one concludes that they are linearly equivalent with self-intersection number 6. Since E_∞ is a section, it intersects each fiber transversally. Thus one has that $E_\infty \sim E_0 - (E_0.E_0)F$, where F denotes a fiber. Therefore one concludes

$$E_\infty.E_\infty = E_\infty.(E_0 - (E_0.E_0)F) = -(E_0.E_0) = -6.$$

Next we establish a morphism $\mu : \mathbb{P}_2 \to \mathbb{P}_6$ over $\mathbb{P}^1$. By [26], **II**. Proposition 7.12., this is the same as to give a surjection $\pi_2^*(\mathcal{O} \oplus \mathcal{O}(6)) \to \mathcal{L}$, where $\mathcal{L}$ is an invertible sheaf on $\mathbb{P}_2$. By the composition

$$\pi_2^*(\mathcal{O} \oplus \mathcal{O}(6)) = \pi_2^*(\mathcal{O}) \oplus \pi_2^*\mathcal{O}(6) \hookrightarrow \bigoplus_{i=0}^{3} \pi_2^*\mathcal{O}(2i) = Sym^3(\pi_2^*(\mathcal{O} \oplus \mathcal{O}(6))) \to \mathcal{O}_{\mathbb{P}_2}(3),$$

where the last morphism is induced by the natural surjection $\pi_2^*(\mathcal{O} \oplus \mathcal{O}(2)) \to \mathcal{O}_{\mathbb{P}_2}(1)$ (see [26], **II**. Proposition 7.11), we obtain a morphism μ^* of sheaves. This morphism μ^* is not a surjection onto $\mathcal{O}_{\mathbb{P}_2}(3)$, but onto its image $\mathcal{L} \subset \mathcal{O}_{\mathbb{P}_2}(3)$. Locally over $\mathbb{A}^1 \subset \mathbb{P}^1$ all rational ruled surfaces are given by $\mathrm{Proj}(\mathbb{C}[x])[y_1, y_2]$, where x has the weight 0. Hence we have locally that $\pi_2^*(\mathcal{O} \oplus \mathcal{O}(6)) = \mathcal{O}e_1 \oplus \mathcal{O}e_2$. Over $\mathbb{A}^1$ the morphism μ^* is given by

$$e_1 \to y_1^3, e_2 \to y_2^3$$

such that the sheaf $\mathcal{L} = im(\mu^*) \subset \mathcal{O}_{\mathbb{P}_2}(3)$ is invertible. Thus the morphism $\mu : \mathbb{P}_2 \to \mathbb{P}_6$ corresponding to μ^* is locally given by the ring homomorphism

$$(\mathbb{C}[x])[y_1, y_2] \to (\mathbb{C}[x])[y_1, y_2] \quad \text{via} \quad y_1 \to y_1^3 \quad \text{and} \quad y_2 \to y_2^3.$$

Construction 8.2.2. One has a commutative diagram

$$
\begin{tikzcd}
\mathcal{Y}' \arrow[r, "\tau'"] & \mathbb{P}_2' \arrow[r, "\mu'"] & \mathbb{P}^1 \times \mathbb{P}^1 \\
\hat{\mathcal{Y}} \arrow[u, "\delta"] \arrow[r, "\hat{\tau}"] \arrow[d, "\rho"'] & \hat{\mathbb{P}}_2 \arrow[u, "\delta_2"'] \arrow[r, "\hat{\mu}"] \arrow[d, "\rho_2"] & \hat{\mathbb{P}}_6 \arrow[u, "\delta_6"'] \arrow[d, "\rho_6"] \\
\mathcal{Y} \arrow[r, "\tau", "\sqrt[3]{\frac{\mu^* E_\sigma}{3\cdot(\mu^* E_0)_{red}}}"'] \arrow[d, "\pi"'] & \mathbb{P}_2 \arrow[r, "\mu", "\sqrt[3]{\frac{E_\infty + 6\cdot F}{E_0}}"'] \arrow[d, "\pi_2"] & \mathbb{P}_6 \arrow[d, "\pi_6"] \\
\mathbb{P}^1 \arrow[r, "\mathrm{id}"] & \mathbb{P}^1 \arrow[r, "\mathrm{id}"] & \mathbb{P}^1
\end{tikzcd}
$$

of morphisms between normal varieties with:

(a) δ, δ_2, δ_6, ρ, ρ_2 and ρ_6 are birational.
(b) π is a family of curves, π_2 and π_d are $\mathbb{P}^1$-bundles.
(c) All the horizontal arrows (except for the ones in the bottom line) are Kummer coverings of degree 3.

Proof. One must only explain δ_6 and ρ_6. Recall that E_σ is a section of $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(6))$, which intersects E_0 transversally in exactly 6 points. The morphism ρ_6 is the blowing up of the six intersection points of $E_0 \cap E_\sigma$. The preimage of the six points given by $q \in \mathcal{M}_3$ with respect to $\pi_6 \circ \rho_6$ consists of the exceptional divisor $\hat{D}_1$ and the proper transform $\tilde{D}_2$ of the preimage of these six points with respect to ρ_6 given by 6 rational curves with self-intersection number -1. The morphism δ_6 is obtained by blowing down $\tilde{D}_2$. $\square$

Remark 8.2.3. The section σ has the zero divisor given by some $q \in \mathcal{P}_3$. Hence one obtains $\mu^*(E_\sigma) \cong \mathcal{C}_q$, where $\mathcal{C} \to \mathcal{P}_3$ denotes the family of cyclic covers of $\mathbb{P}^1$ with a pure $(1,3)-VHS$ of degree 3. Since τ is the unique cyclic degree 3 covering of $\mathbb{P}_2 \cong R^1$ ramified over $\mu^*(E_\sigma) \cong \mathcal{C}_q$, the surface $\mathcal{Y}$ is isomorphic to some $K3$-surface $\tilde{C}_2$ of the preceding section.

Recall that $\mathbb{F}_n$ denotes the Fermat curve of degree n.

Proposition 8.2.4. *The surface $\mathcal{Y}$ is birationally equivalent to $\mathcal{C}_q \times \mathbb{F}_3/\langle(1,1)\rangle$.*[1]

Proof. Let $\tilde{E}_\bullet$ denote the proper transform of the section $E_\bullet$ with respect to ρ_6. Then $\hat{\mu}$ is the Kummer covering given by

[1] Similarly to [58], Construction 5.2, we show that $\mathcal{Y}'$ is birationally equivalent to $\mathcal{C}_q \times \mathbb{F}_3/\langle(1,1)\rangle$.

$$\sqrt[3]{\frac{\tilde{E}_\infty + 6 \cdot F}{\tilde{E}_0 + \hat{D}_1}},$$

where $\hat{D}_1$ denotes the exceptional divisor of ρ_6. Thus the morphism μ' is the Kummer covering

$$\sqrt[3]{\frac{(\delta_6)_*\tilde{E}_\infty + 6 \cdot (\delta_6)_*F}{(\delta_6)_*\tilde{E}_0 + (\delta_6)_*\hat{D}_1}} = \sqrt[3]{\frac{\mathbb{P}^1 \times \{\infty\} + 6 \cdot (P \times \mathbb{P}^1)}{\mathbb{P}^1 \times \{0\} + \Delta \times \mathbb{P}^1}},$$

where Δ is the divisor of the 6 different points in $\mathbb{P}^1$ given by $q \in \mathcal{M}_3$ and $P \in \mathbb{P}^1$ is the point with the fiber F. Since $E_0 + E_\sigma$ is a normal crossing divisor, $\tilde{E}_\sigma$ neither meets $\tilde{E}_0$ nor $\tilde{D}_2$, where $\tilde{D}_2$ is the proper transform of $\pi_6^*(\Delta)$. Therefore $(\delta_6)_*\tilde{E}_\sigma$ neither meets

$$(\delta_6)_*\tilde{E}_0 = \mathbb{P}^1 \times \{0\} \ \text{ nor } \ (\delta_6)_*\tilde{E}_\infty = \mathbb{P}^1 \times \{\infty\}.$$

Hence one can choose coordinates in $\mathbb{P}^1$ such that $(\delta_6)_*\tilde{E}_\sigma = \mathbb{P}^1 \times \{1\}$.

By the definition of τ, we obtain that $\hat{\tau}$ is given by

$$\sqrt[3]{\frac{\rho_2^*\mu^*(E_\sigma)}{\rho_2^*\mu^*(E_0)}} = \sqrt[3]{\frac{\hat{\mu}^*(\tilde{E}_\sigma)}{\hat{\mu}^*(\tilde{E}_0)}},$$

and τ' is given by

$$\sqrt[3]{\frac{\mu'^*(\mathbb{P}^1 \times \{1\})}{\mu'^*(\mathbb{P}^1 \times \{0\})}}.$$

By the fact that the last function is the third root of the pullback of a function on $\mathbb{P}^1 \times \mathbb{P}^1$ with respect to μ', it is possible to reverse the order of the field extensions corresponding to τ' and μ' such that the resulting varieties obtained by Kummer coverings are birationally equivalent. Hence we have the composition of $\beta : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ given by

$$\sqrt[3]{\frac{\mathbb{P}^1 \times \{1\}}{\mathbb{P}^1 \times \{0\}}}$$

with

$$\sqrt[3]{\frac{\beta^*(\mathbb{P}^1 \times \{\infty\}) + 6 \cdot (P \times \mathbb{P}^1)}{\beta^*(\mathbb{P}^1 \times \{0\}) + (\Delta \times \mathbb{P}^1)}},$$

which yields the covering variety isomorphic to $\mathbb{F}_3 \times \mathcal{C}_q / \langle (1,1) \rangle$. $\square$

Hence $\tilde{C}_2 \cong \mathcal{Y}$ is birationally equivalent to the algebraic manifold $\hat{\mathcal{Y}}$ in the diagram (7.1) with $\mathcal{Z} = C_{(1)}$ and $\Sigma = \mathbb{F}_3$. Therefore by Corollary 7.1.6, we obtain:

Corollary 8.2.5. *If the curve $\mu^*(E_\sigma)$ has complex multiplication, then the $K3$-surface $\mathcal{Y}$ has complex multiplication, too.*

8.3 The resulting family and its involutions

8.3.1. Let us summarize the things we have done. By using the Veronese embedding, the weighted projective space $Q^2 = \mathbb{P}_{\mathbb{C}}(2,2,1,1)$ is given by $V(z_1 z_3 = z_2^2) \subset \mathbb{P}^4$. Moreover there exists a homogeneous polynomial $G_{(a_1,a_2,a_3)} \in \mathbb{C}[z_1,z_2,z_3]$ of degree 3 such that

$$G_{(a_1,a_2,a_3)}(x^2,x,1) = x(x-1)(x-a_1)(x-a_2)(x-a_3)$$

for each $(a_1,a_2,a_3) \in \mathcal{M}_3$. Let $W \hookrightarrow Q^2 \times \mathcal{M}_3 \stackrel{pr_2}{\to} \mathcal{M}_3$ be the family with the fibers given by $W_q = V(z_1 z_3 - z_2^2,\ \ z_5^3 + z_4^3 + G_q)$ for all $q \in \mathcal{M}_3$. Moreover let $\mathcal{W} \hookrightarrow R^2 \times \mathcal{M}_3 \to \mathcal{M}_3$ be the smooth family obtained by the proper transform of W with respect to the blowing up of $V(z_1,z_2,z_3) \times \mathcal{M}_3$. Since the family $\mathcal{C} \to \mathcal{M}_3$ given by

$$R^1 \supset V(y^3 - x_1(x_1-x_0)(x_1-a_1x_0)(x_1-a_2x_0)(x_1-a_3x_0)x_0) \to (a_1,a_2,a_3) \in \mathcal{M}_3$$

has dense set of complex multiplication fibers, Corollary 8.2.5 implies that $\mathcal{W}$ is a $CMCY$ family of 2-manifolds.

Next we will find and study involutions on $\mathcal{W}$ over $\mathcal{M}_3$ satisfying the assumptions for the construction of a Borcea-Voisin tower.

Remark 8.3.2. We have the involutions on W over $\mathcal{M}_3$ given by

$$\begin{aligned}
\gamma^{(1)}(z_5 : z_4 : z_3 : z_2 : z_1) &= (z_4 : z_5 : z_3 : z_2 : z_1),\\
\gamma^{(2)}(z_5 : z_4 : z_3 : z_2 : z_1) &= (\xi z_4 : \xi^2 z_5 : z_3 : z_2 : z_1),\\
\gamma^{(3)}(z_5 : z_4 : z_3 : z_2 : z_1) &= (\xi^2 z_4 : \xi z_5 : z_3 : z_2 : z_1),
\end{aligned}$$

where ξ is a fixed primitive cubic root of unity. For simplicity we write γ instead of $\gamma^{(1)}$, too. Since the ideal sheaf of $V(z_1,z_2,z_3) \cap W$ coincides with its inverse image ideal sheaf with respect to $\gamma^{(i)}$ (for all $i = 1,2,3$), each $\gamma^{(i)}$ induces an involution on $\mathcal{W}$ over the basis $\mathcal{M}_3$ denoted by $\gamma^{(i)}$, too.

Remark 8.3.3. We have the $\mathcal{M}_3$-automorphism κ of W given by

$$\kappa(z_5 : z_4 : z_3 : z_2 : z_1) = (\xi z_5 : z_4 : z_3 : z_2 : z_1) \text{ with}$$

$$\kappa^{-1}(z_5 : z_4 : z_3 : z_2 : z_1) = (\xi^2 z_5 : z_4 : z_3 : z_2 : z_1)$$

such that by the same argument as in Remark 8.3.2, we obtain an automorphism of $\mathcal{W}$ over $\mathcal{M}_3$ denoted by κ, too. On W and hence on $\mathcal{W}$ one has

$$\gamma^{(2)} = \kappa \circ \gamma \circ \kappa^{-1} \text{ and } \gamma^{(3)} = \kappa^{-1} \circ \gamma \circ \kappa.$$

Hence these involutions act by the same character on the global differential forms of the fibers of $\mathcal{W}$, and all quotients $\mathcal{W}/\gamma^{(i)}$ are isomorphic. Therefore it is sufficient to consider the quotient by γ.

Proposition 8.3.4. *On each fiber of $\mathcal{W}$ the involution γ fixes exactly the points on the divisor given by $V(z_4 = z_5)$ and one exceptional line over one singular point of the corresponding fiber of W.*

Proof. Let $q \in \mathcal{M}_3$ and let S denote the singular locus of W_q. On $W_q \setminus S$ the points fixed by γ are given by the divisor $V(z_4 = z_5)$. Now let us consider the exceptional divisors of the blowing up, which turns W into the family $\mathcal{W}$ of smooth $K3$-surfaces. There are exactly 3 points of S given by $z_1 = z_2 = z_3 = 0$ and $z_4^3 + z_5^3 = 0$. The involution γ fixes $(1 : -1 : 0 : 0 : 0)$ and interchanges the other two singular points. Since the generators of the ideal of the blowing up are invariant under γ, one concludes that each point on the exceptional line over $(1 : -1 : 0 : 0 : 0)$ is fixed by γ. □

Since the divisor on W_q given by $V(z_4 = z_5)$ is isomorphic to $\mathcal{C}_q$ and the projective line providing the fixed exceptional divisor has CM, one has by Corollary 8.2.5:

Theorem 8.3.5. *By the involution γ, the family $\mathcal{W}$ can be used to be some $\mathcal{Z}_1$ or Σ_i in the construction of a Borcea-Voisin tower.*

Remark 8.3.6. By Example 7.4.3, Example 7.4.4 and Example 7.4.5, one has 6 explicitly given elliptic curves with CM and explicitly given involutions. Theorem 2.4.4 yields the $K3$ surfaces isomorphic to

$$V(y_2^3+y_1^3+x_1^6+x_0^6),\ V(y_2^3+y_1^3+x_1(x_1^5+x_0^5)),\ V(y_2^3+y_1^3+x_1(x_1^4+x_0^4)x_0) \subset R^2$$

with complex multiplication. Thus by using the method of C. Voisin [60], one obtains 18 explicitly given fibers with CM for the resulting $CMCY$ family of 3-manifolds.

Chapter 9
Other examples and variations

In this chapter we consider the automorphism groups of our examples of $CMCY$ families. We want to find some new examples of $CMCY$ families of n-manifolds as quotients by cyclic subgroups of these automorphism groups. By using [20], Lemma $3.16, d$), one can easily determine the character of the action of these cyclic groups on the global sections of the canonical sheaves of the fibers. In this chapter we state this character with respect to the pull-back action.

In Section 9.1 we see that the $CMCY$ family $\mathcal{W}$ of 2-manifolds given by

$$R^2 \supset \tilde{V}(y_2^3+y_1^3 = x_1(x_1-x_0)(x_1-a_1x_0)(x_1-a_2x_0)(x_1-a_3x_0)x_0) \to (a_1,a_2,a_3) \in \mathcal{M}_3$$

has a degree 3 quotient, which is birationally equivalent to a $CMCY$ family of 2-manifolds. This quotient is also suitable for the construction of a Borcea-Voisin tower. By using degree 3 automorphisms of $\mathcal{W} \to \mathcal{M}_3$ and the Fermat curve $\mathbb{F}_3$ of degree 3, we construct the $CMCY$ families $\mathcal{Q} \to \mathcal{M}_3$ and $\mathcal{R} \to \mathcal{M}_3$ of 3-manifolds in Section 9.2.

In Section 9.3 we consider a subgroup of the automorphism group of the $CMCY$ family $\mathcal{C}_2 \to \mathcal{M}_1$ given by

$$\mathbb{P}^3 \supset V(y_2^4 + y_1^4 + x_1(x_1 - x_0)(x_1 - \lambda x_0)x_0) \to \lambda \in \mathcal{M}_1.$$

We find some degree 2 quotients of this family, which are birationally equivalent to $CMCY$ families of 2-manifolds. In Section 9.4 we see that these families have involutions suitable for the construction of Borcea-Voisin towers. We consider a larger subgroup of the automorphism group of $\mathcal{C}_2$ in Section 9.5.

In Section 9.6 we study the automorphism group of the $CMCY$ family of 3-manifolds

$$\mathbb{P}^4 \supset V(y_3^5 + y_2^5 + y_1^5 + x_1(x_1 - x_0)(x_1 - ax_0)(x_1 - bx_0)x_0) \to (a,b) \in \mathcal{M}_2$$

constructed by E. Viehweg and K. Zuo [58].

J.C. Rohde, *Cyclic Coverings, Calabi-Yau Manifolds and Complex Multiplication*, Lecture Notes in Mathematics 1975, DOI: 10.1007/978-3-642-00639-5_10,

9.1 The degree 3 case

Let ξ denote a fixed primitive cubic root of unity. In 8.3.1 we have constructed the $CMCY$ family $\mathcal{W} \to \mathcal{M}_3$ given by

$$R^2 := \tilde{\mathbb{P}}_{\mathbb{C}}(2,2,1,1) \supset \tilde{V}(y_2^3+y_1^3+x_1(x_1-1)(x_1-a_1x_0)(x_1-a_2x_0)(x_1-a_3x_0)x_0)$$
$$\to (a_1,a_2,a_3) \in \mathcal{M}_3.$$

First we introduce an $\mathcal{M}_3$-automorphism group $\mathbb{G}_3$ of the family $\mathcal{W}$. The elements $g \in \mathbb{G}_3$ can be uniquely written as a product $g = abc$ with $a \in \langle\alpha\rangle$, $b \in \langle\beta\rangle$ and $c \in \langle\gamma\rangle$, where:

$$\alpha(z_5 : z_4 : z_3 : z_2 : z_1) = (\xi z_5 : z_4 : z_3 : z_2 : z_1),$$
$$\beta(z_5 : z_4 : z_3 : z_2 : z_1) = (z_5 : \xi z_4 : z_3 : z_2 : z_1),$$
$$\gamma(z_5 : z_4 : z_3 : z_2 : z_1) = (z_4 : z_5 : z_3 : z_2 : z_1)$$

The group $\mathbb{G}_3$ contains exactly 18 elements. The action of $\mathbb{G}_3$ on the global sections of the canonical sheaves of the fibers induces a surjection of $\mathbb{G}_3$ onto the multiplicative group of the 6-th. roots of unity. Its kernel is the cyclic group of order 3 generated by $\alpha\beta^{-1}$.

Remark 9.1.1. Since $\alpha\beta^{-1}$ is an $\mathcal{M}_3$-automorphism, one obtains the quotient family $\mathcal{W}/\langle\alpha\beta^{-1}\rangle \to \mathcal{M}_3$. One checks easily that $\alpha\beta^{-1}$ leaves exactly the sections given by $z_5 = z_4 = 0$ invariant. Let $q \in \mathcal{M}_3$. The fiber $(\mathcal{W}/\langle\alpha\beta^{-1}\rangle)_q$ of $\mathcal{W}/\langle\alpha\beta^{-1}\rangle$ has quotient singularities of the type $A_{3,2}$ (see [6], **III.** Proposition 5.3). We blow up the sections of fixed points on $\mathcal{W}$ and call the resulting exceptional divisor E_1. On each connected component of E_1 one has two disjoint sections of fixed points again. But on a fiber the quotient map sends any fixed point onto a singularity of the type $A_{3,1}$.[1] Hence let us blow up these latter sections of fixed points with exceptional divisor E_2. The canonical divisor of the resulting fibers $\tilde{\mathcal{W}}_q$ is given by

$$K_{\tilde{\mathcal{W}}_q} = (\tilde{E}_1)_q + 2(E_2)_q,$$

where quotient map φ induced by $\alpha\beta^{-1}$ has ramification on E_2. Thus by the Hurwitz formula, one calculates that $\varphi^*(\omega_q) = \mathcal{O}((\tilde{E}_1)_q)$. Note that the irreducible components of the exceptional curve $(E_1)_q$ have selfintersection-number -1. Since $(E_2)_q$ is the exceptional divisor of the blowing up of two points of each irreducible component of $(E_1)_q$, each irreducible component of $(\tilde{E}_1)_q$ has selfintersection-number -3. By the fact that the quotient map

[1] For this description consider the corresponding action of the cyclic group on an analytic open neighborhood of a fixed point.

$\varphi : \tilde{\mathcal{W}}_q \to (\tilde{\mathcal{W}}/\langle \alpha\beta^{-1}\rangle)_q$ is not ramified over $\varphi((\tilde{E}_1)_q)$, the irreducible components of $\varphi((\tilde{E}_1)_q)$ have selfintersection-number -1.

From now on let $\mathcal{X} := \tilde{\mathcal{W}}/\langle \alpha\beta^{-1}\rangle$.

Proposition 9.1.2. *One can blow down $\varphi(\tilde{E}_1)$ such that the blowing down morphism $\phi : \mathcal{X} \to \mathcal{Y}$ yields a CMCY family $\mathcal{Y} \to \mathcal{M}_3$ of 2-manifolds.*

Proof. By the construction of the projective family, one has an invertible relatively very ample sheaf $\mathcal{A} := \mathcal{O}_{\mathcal{X}}(D)$ on $\mathcal{X}$. Let P denote some connected component of $\varphi(\tilde{E}_1)$. Note that $\varphi(\tilde{E}_1)$ consists of different copies of $\mathbb{P}^1_{\mathrm{Spec}(R)}$ with $\mathrm{Spec}(R) = \mathcal{P}_n$ such that each invertible sheaf on P is uniquely determined by its degree. Thus the intersection number $\mu_P := D_q.P_q$ is independent of $q \in \mathcal{P}_n$. As in the proof of the Castelnuovo Theorem in [26], **V.** Theorem 5.7 the invertible sheaf

$$\mathcal{L} := \mathcal{A}(\sum_{P \subset \varphi(\tilde{E}_1)} \mu_P P)$$

yields the blowing down morphism on the fibers. Since this $\mathcal{P}_n$-morphism is globally defined, one obtains a global blowing down morphism f such that the resulting family $\mathcal{Y} = f(\mathcal{X})$ is smooth.

By the fact that $\alpha\beta$ acts by the character 1 on $\Gamma(\omega_{\mathcal{W}_q})$, one concludes easily that $\mathcal{Y} \to \mathcal{M}_3$ is a family of $K3$ surfaces. Since $\mathcal{W}$ has a dense set of CM fibers, one concludes that $\mathcal{X} = \tilde{\mathcal{W}}/\langle \alpha\beta\rangle$ and $\mathcal{Y}$ have dense sets of CM fibers, too. $\square$

By the blowing down of $\varphi(\tilde{E}_1)$, we get the following situation:

$$\begin{tikzcd}
\tilde{E}_1 \cup E_2 \arrow[r, two heads, "\varphi"] \arrow[d, hook] & \varphi(\tilde{E}_1 \cup E_2) \arrow[r, two heads, "\phi"] \arrow[d, hook] & \phi \circ \varphi(E_2) \arrow[d, hook] \\
\tilde{\mathcal{W}} \arrow[r, "\varphi", "\mathrm{mod}\langle \alpha\beta^2\rangle"'] & \mathcal{X} \arrow[r, "\phi", "\mathrm{Bl}(\varphi(\tilde{\mathrm{E}}_1))"'] & \mathcal{Y}
\end{tikzcd}$$

Proposition 9.1.3. *The $\mathcal{M}_3$-automorphism γ of $\mathcal{W}$ yields an involution on $\mathcal{Y}$, which makes it suitable for the construction of a Borcea-Voisin tower.*

Proof. One has the following commutative diagram:

$$\begin{tikzcd}
\tilde{\mathcal{W}} \arrow[r, "\alpha\beta^{-1}"] \arrow[d, "\gamma"'] & \tilde{\mathcal{W}} \arrow[d, "\gamma"] \\
\tilde{\mathcal{W}} \arrow[r, "\alpha^{-1}\beta"] & \tilde{\mathcal{W}}
\end{tikzcd}$$

Thus γ yields an involution on $\mathcal{X} = \tilde{\mathcal{W}}/\langle\alpha\beta^{-1}\rangle$. By the fact that $\gamma(E_1) = E_1$, it induces an involution on the complement of the sections of $\mathcal{Y}$ obtained by blowing down $\varphi(\tilde{E}_1)$. Since these sections have codimension 2, the involution extends to a holomorphic involution on $\mathcal{Y}$ (by Hartog's Extension Theorem [61], Théorème 1.25). By the fact that γ acts by -1 on $\Gamma(\omega_{\mathcal{W}_q})$, the same holds true for $\mathcal{X}_q$ and $\mathcal{Y}_q$.

Let $\mathcal{C} \to \mathcal{M}_3$ denote the family of degree 3 covers with a pure $(1,3)-VHS$. We have seen that $\mathcal{W}_q$ has CM, if $\mathcal{C}_q$ has CM. Therefore $H^k(\mathcal{Y}_q, \mathbb{Q})$ has a commutative Hodge group for all k, if $\mathcal{C}_q$ has CM. Thus the following point describes the ramification divisor of γ_q on $\mathcal{Y}_q$ and ensures that there is a dense set of CM fibers $\mathcal{Y}_q$ such that the ramification divisor of γ_q has CM, too. □

9.1.4. Now we describe the divisor of points of $\mathcal{Y}_q$ fixed by γ for some $q \in \mathcal{M}_3$. Each point of $\mathcal{Y}_q \setminus (\phi \circ \varphi(E_2))$ can be given by the image $[p]$ of a point $p \in \mathcal{W}_q$ with respect to the quotient map according to $\langle\alpha\beta^{-1}\rangle$. One has that a point $[p] \in \mathcal{Y}_q \setminus (\phi \circ \varphi(E_2))$ is fixed by γ, if and only if $\gamma(p) \in \langle\alpha\beta^2\rangle \cdot p$. These points $p \in \mathcal{W}_q$ are exactly given by $\langle\alpha\beta^2\rangle \cdot V(y_2 = y_1)$ and the exceptional divisor of $\mathcal{W}_q \to W_q$.

By the fact that $\langle\alpha\beta^2\rangle \cdot V(y_2 = y_1)$ interchanges all 3 irreducible components of $\langle\alpha\beta^2\rangle \cdot V(y_2 = y_1)$ and all 3 irreducible components of the exceptional divisor of $\mathcal{W} \to W$, one obtains a divisor of fixed points on $\mathcal{Y}_q$ given by $\mathcal{C}_q$ and one copy of $\mathbb{P}^1$. Since γ is given by $(y_2 : y_1) \to (y_1 : y_2)$ on E_1 and $\alpha\beta^2$ is given by $(y_2 : y_1) \to (y_2 : \xi y_1)$ on E_1, γ interchanges each two irreducible components of E_2, which intersect the same irreducible component of $\tilde{E}_1$. Thus the ramification divisor of $\mathcal{Y} \to \mathcal{Y}/\gamma$ given by a family of rational curves and $\mathcal{C}$, where $\mathcal{C}$ denotes the example of a family of degree 3 covers with a pure $(1,3) - VHS$.

9.2 Calabi-Yau 3-manifolds obtained by quotients of degree 3

We have seen that the family $\mathcal{W}$ of $K3$-surfaces given by

$$R^2 := \tilde{\mathbb{P}}_{\mathbb{C}}(2,2,1,1) \supset \tilde{V}(y_2^3+y_1^3+x_1(x_1-1)(x_1-a_1x_0)(x_1-a_2x_0)(x_1-a_3x_0)x_0)$$

$$\to (a_1, a_2, a_3) \in \mathcal{M}_3$$

has a dense set of fibers $\mathcal{W}_q$ such that $H^k(\mathcal{W}_q, \mathbb{Q})$ has a commutative Hodge group for all k.

Recall that the canonical divisor of $R^1 \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$ is given by $-2\tilde{V}(z_4)$. Now we consider the up to isomorphisms unique cyclic cover of degree 3 given by $\mathcal{W}_q \to R^1$ ramified over $\mathcal{C}_q$, whose Galois group is generated by α.

Moreover consider the cyclic degree 3 cover $\mathbb{F}_3 \to \mathbb{P}^1$, where $\mathbb{F}_3 = V(x^3 + y^3 + z^3) \subset \mathbb{P}^2$ denotes the Fermat curve of degree 3 and $\alpha_{\mathbb{F}_3}$ given by

$$(x : y : z) = (x : y : \xi z),$$

is a generator of the Galois group, which acts by the character ξ on $\Gamma(\omega_{\mathbb{F}_3})$.

Let X be a singular variety of dimension n such that each irreducible component of its singular locus S has at least the codimension 2. Then we call X a singular Calabi-Yau n-manifold, if $h^0(X \setminus S, \Omega^k_{X\setminus S}) = 0$ for all $k = 1, \ldots, n-1$ and $\omega_{X\setminus S} \cong \mathcal{O}_{X\setminus S}$. With the notation of diagram (7.1) one gets:

Proposition 9.2.1. *The quotient of $\mathcal{W} \times \mathbb{F}_3$ by $\langle(1,2)\rangle$ yields a family of singular Calabi-Yau 3-manifolds with a dense set of CM fibers.*

Proof. Note that the VHS of the family $\mathcal{W} \times \mathbb{F}_3/\langle(1,2)\rangle$ is the sub-VHS fixed by $\langle(1,2)\rangle$.[2] Since $\mathbb{F}_3$ has complex multiplication, a CM fiber of $\mathcal{W}$ yields a corresponding CM fiber of $\mathcal{W} \times \mathbb{F}_3/\langle(1,2)\rangle$.

Let φ denote the quotient map

$$\varphi : \mathcal{W} \times \mathbb{F}_3 \to \mathcal{W} \times \mathbb{F}_3/\langle(1,2)\rangle$$

and S denote the singular locus of $\mathcal{W} \times \mathbb{F}_3/\langle(1,2)\rangle$. Over each point, which lies not in the singular locus given by 3 copies of $\mathcal{C}$, one does not have ramification. Hence by the Hurwitz formula, $\varphi^*(\omega_{(\mathcal{W}\times\mathbb{F}_3/\langle(1,2)\rangle)\setminus S})$ is given by the structure sheaf. Since $\langle(1,2)\rangle$ acts on $\Gamma(\omega_{\mathcal{W}\times\mathbb{F}_3})$ by the character 1, the sheaf $\omega_{(\mathcal{W}\times\mathbb{F}_3/\langle(1,2)\rangle)\setminus S}$ has global sections. Hence

$$\omega_{(\mathcal{W}\times\mathbb{F}_3/\langle(1,2)\rangle)\setminus S} = \mathcal{O}_{(\mathcal{W}\times\mathbb{F}_3/\langle(1,2)\rangle)\setminus S}.$$

In addition the reader checks easily that $\langle(1,2)\rangle$ does not act by the character 1 on a non-trivial sub-vector space of $H^{1,0}(\mathcal{W} \times \mathbb{F}_3)$ or $H^{2,0}(\mathcal{W} \times \mathbb{F}_3)$. Thus $\mathcal{W} \times \mathbb{F}_3/\langle(1,2)\rangle$ is a family of singular Calabi-Yau 3-manifolds. $\square$

Now consider a fiber $\mathcal{W}_q$, which is a family of curves given by

$$\mathcal{W}_q \to R^2 \to \mathbb{P}^1.$$

Thus

$$\mathcal{W}_q \times \mathbb{F}_3 \to R^2 \to \mathbb{P}^1$$

is a family of surfaces. The singular locus given by 3 copies of $\mathcal{C}_q$ does not consist of sections.

[2] For a short introduction to such orbifolds and their Hodge theory see [12], Appendix A.3.

Here we do not blow up sections of $\mathcal{W}_q \times \mathbb{F}_3 \to \mathbb{P}^1$. Hence here one can not formulate a relative version of the Castelnuovo Theorem as in Proposition 9.1.2. Thus we use complex analytic methods:

9.2.2. Let X be a non-compact complex analytic surface. Note that self-intersection numbers are also defined for compact complex curves on non-compact complex analytic surfaces X. On X one can blow down a compact rational curve with self-intersection number -1 such that one obtains a smooth complex analytic surface. Moreover for the blowing up $\phi : \tilde{X} \to X$ of the point $p \in X$ with exceptional divisor E, one has

$$\omega_{\tilde{X}} = \phi^*(\omega_{\tilde{X}})(E).$$

We have also the adjunction formula for X and $\tilde{X}$ such that $E^2 = -1$. Moreover one has for each compact curve C on X

$$\phi^*(C)^2 = C^2.$$

(see [6], **I. - III.**)

Hence we can repeat the procedure of the previous section for small open analytic subsets and glue. We will locally blow down a divisor to a codimension 2 submanifold Z. Note that we have for the complement of Z gluing morphisms, since the blowing down morphism φ yields an obvious isomorphism between the complements of Z and $\varphi^{-1}(Z)$. Hence the gluing of our local blowing down morphisms follows from Hartog's extension theorem and the uniqueness of this extension, which follows from the continuity of holomorphic maps:

Theorem 9.2.3 (Hartog). *Assume that U is an open subset of $\mathbb{C}^N$ and f is a holomorphic function on $U \setminus \{u_1 = u_2 = 0\}$. Then f extends to a holomorphic function on U.*

Proof. (see [61], Théorème 1.25) □

9.2.4. Now consider a fiber $(\mathcal{W} \times \mathbb{F}_3/\langle(1,2)\rangle)_q$ of $\mathcal{W} \times \mathbb{F}_3/\langle(1,2)\rangle$ and its singularities in the complex analytic setting. For the construction of the blowing up of a complex submanifold we refer to [61], 3.3.3. As in [61], 3.3.3 described, one constructs the blowing up over open sets first. The global blowing up is given by gluing the local blowing ups. Here we consider the situation on sufficiently small complex open submanifolds.

The $\mathcal{M}_3$-automorphism α acts on y_2 by ξ. On each fiber $\mathcal{W}_q$ the curve $\mathcal{C}_q$ defines the ramification locus of $\mathcal{W}_q \to R^2$, which is fixed by α. A local parameter $p_{\mathcal{C}_q}$ on $\mathcal{C}_q$ yields a local parameter on $\mathcal{W}_q$ fixed by α. By z, we denote a local parameter for the neighborhoods of the ramification points of $\mathbb{F}_3$. On a small open subset, which intersects the ramification locus of

$$\varphi_q : (\mathcal{W} \times \mathbb{F}_3)_q \to (\mathcal{W} \times \mathbb{F}_3/\langle(1,2)\rangle)_q,$$

one has the three local parameters given by y_2, $p_{\mathcal{C}_q}$ and z.

By the action of $\langle(1,2)\rangle$ on the local parameters, the singular loci of the family $\mathcal{W} \times \mathbb{F}_3/\langle(1,2)\rangle$ are locally given by the product of the 4-ball $\mathbb{B}_4$ with a surface, which has a singularity of the type $A_{3,2}$ (with the notation in [6], **III**. Section 5). Let us blow up the family of fixed curves on $\mathcal{W} \times \mathbb{F}_3$ with respect to $\langle(1,2)\rangle$ and let E_1 denote the exceptional divisor. On each connected component of E_1 one has two disjoint families of fixed curves with respect to the action of $\langle(1,2)\rangle$ again. Again this follows from the consideration of the action of $\langle(1,2)\rangle$ on local parameters of a small open subset. On a fiber the quotient map sends any neighborhood of a point on these latter curves onto the product of the 1-ball $\mathbb{B}_1$ with a surface with a singularity of the type $A_{3,1}$. Hence let us blow up these latter two families of curves with exceptional divisor E_2. The canonical divisor of the resulting fibers $(\widetilde{\mathcal{W} \times \mathbb{F}_3})_q$ is given by

$$K_{(\widetilde{\mathcal{W}\times\mathbb{F}_3})_q} = (\tilde{E}_1)_q + 2(E_2)_q,$$

where quotient map φ by $\langle(1,2)\rangle$ is ramified over E_2. Thus by the Hurwitz formula, one calculates that

$$\varphi^*(\omega_q) = \mathcal{O}((\tilde{E}_1)_q). \tag{9.1}$$

By 9.2.2 and our blowing up construction, one concludes that $\tilde{E}_1$ is locally given the product of a rational -3 curve with $\mathbb{B}_4$. Thus the divisor $D = \varphi(\tilde{E}_1)$ is covered by open analytic subsets U on $\widetilde{\mathcal{W} \times \mathbb{F}_3}/\langle(1,2)\rangle$ such that U is of the type $X \times \mathbb{B}_4$, where X is a complex analytic surface containing a -1 curve E' and D is given by $E' \times \mathbb{B}_4$. By 9.2.2, we can blow down E'. This yields a local blowing down of D. We have explained that we can glue these blowing down maps. Thus we obtain a family

$$\mathcal{R} \to \mathcal{M}_3.$$

By (9.1), one concludes easily that the fibers have a trivial canonical bundle. Moreover one sees quite easily that the fibers are Calabi-Yau 3-manifolds.

At present it is not clear to the author that the family $\mathcal{R} \to \mathcal{M}_3$ is algebraic, since we have used analytic methods. Note that this construction is a relative version of a construction by S. Cynk and K. Hulek [13], which yields a result written down in Proposition 10.4.3. Thus the fibers are algebraic.

Note we have blown up copies of the family $\mathcal{C}$ of degree 3 covers. Moreover note that $\mathbb{F}_3$ has CM and that for all $q \in \mathcal{M}_3$ the fiber $\mathcal{W}_q$ has CM, if the fiber $\mathcal{C}_q$ has CM. Since $\mathcal{C}$ has a dense set of CM fibers, we conclude:

Proposition 9.2.5. *The family $\mathcal{R} \to \mathcal{M}_3$ is a (holomorphic) CMCY family of 3-manifolds.*

Let us construct an other example: $\alpha\beta$ acts by the character ξ^2 on $\Gamma(\omega_{\mathcal{W}_q})$ for all $q \in \mathcal{M}_3$. Moreover we have a Galois cover $\mathbb{F}_3 \to \mathbb{P}^1$ of degree 3 with a generator $\alpha_{\mathbb{F}_3}$ given by

$$(x : y : z) = (x : y : \xi z),$$

which acts by the character ξ on $\gamma(\omega_{\mathbb{F}_3})$. Hence $\alpha_2 := (\alpha\beta, \alpha_{\mathbb{F}_3})$ leaves $\Gamma(\omega_{\mathcal{W}_q \times \mathbb{F}_3})$ invariant.

The automorphism α_2 fixes a finite number of points on $\mathcal{W}_q \times \mathbb{F}_3$ given by

$$\{z_5 = z_4 = 0\} \times \{z = 0\},$$

and α_2 fixes in addition the points on the curves given by the fiber product of $\{z = 0\}$ with the exceptional divisor of the blowing up $\mathcal{W}_q \to W_q$. The latter statement about the exceptional divisor of $\mathcal{W}_q \to W_q$ follows from the fact that $\alpha\beta$ fixes the generators of the corresponding ideal sheaf of the blowing up and the singular points of W_q given by

$$(1 : -1 : 0 : 0 : 0), \quad (1 : -\xi : 0 : 0 : 0) \text{ and } (1 : -\xi^2 : 0 : 0 : 0).$$

9.2.6. Now we determine the action of $\alpha\beta$ on the local parameters, whose zero-loci are given by the exceptional divisor $E_{\mathcal{W}_q}$ of $\mathcal{W}_q \to W_q$. The action of $\alpha\beta$ on $W_q \subset \mathbb{P}^4$ is given by

$$(z_5 : z_4 : z_3 : z_2 : z_1) \to (\xi z_5 : \xi z_4 : z_3 : z_2 : z_1) \text{ resp.,}$$
$$(z_5 : z_4 : z_3 : z_2 : z_1) \to (z_5 : z_4 : \xi^{-1} z_3 : \xi^{-1} z_2 : \xi^{-1} z_1).$$

By using the explicit equations for W_q in 8.3.1, one can very easily calculate that $\alpha\beta$ acts by ξ^{-1} on these local parameters.[3]

Hence the singularities of $\mathcal{W}_q \times \mathbb{F}_3/\langle\alpha_2\rangle$, which result by the exceptional divisor of $\mathcal{W}_q \to W_q$, are locally given by the product of $\mathbb{B}_1$ with a singularity of the type $A_{3,2}$.

Now we construct a desingularization of $\mathcal{W} \times \mathbb{F}_3/\langle\alpha_2\rangle$, which is a $CMCY$ family of 3-manifolds. Let $E_{\mathcal{W}}$ denote the exceptional divisor of $\mathcal{W} \to W$. We start with the blowing up of the family of rational curves given by the fiber-product of $E_{\mathcal{W}}$ with the points on $\mathbb{F}_3$ fixed by $\alpha_{\mathbb{F}_3}$. This yields the exceptional divisor E_C consisting of 9 rational ruled surfaces. By the same arguments as in 9.2.4, each connected component of E_C contains two families of rational

[3] The singular locus of W_q is contained in $W_q \cap \{z_5 = 1\}$. Thus one can calculate the desingularization with the usual equations $z_i t_j = z_j t_i$ for $i, j = 1, 2, 3$. On $\{t_i = 1\}$ the zero locus of the local parameter z_i yields the exceptional divisor. The local parameter fixed by $\alpha\beta$ can be given by t_1/t_i or t_3/t_i.

curves of fixed points. The blowing up $\widetilde{\mathcal{W} \times \mathbb{F}_3}$ of these latter families has a quotient

$$\widetilde{\mathcal{W} \times \mathbb{F}_3}/\alpha_2$$

with quotient map given by φ such that on the complement of the isolated sections fixed by φ

$$\varphi^*\omega_q = \mathcal{O}((\tilde{E}_C)_q).$$

9.2.7. Recall that R^1 is a rational ruled surface, where the exceptional divisor E_{R^1} of the blowing up $R^1 \to Q^1$ is a section of $R^1 \to \mathbb{P}^1$ (see Remark 8.1.6). A fiber $\mathcal{W}_q$ can be considered as a family

$$\mathcal{W}_q \xrightarrow{f} R^1 \to \mathbb{P}^1$$

of curves, where f is constructed in 8.1.12. By 8.3.1 and the projection $R^2 \to R^1$, the morphism f extends to a morphism $f : \mathcal{W} \to R^1 \times \mathcal{M}_3$ such that the exceptional divisor $E_{\mathcal{W}}$ of the blowing up $\mathcal{W} \to W$ is send to the exceptional divisor $E_{R^1 \times \mathcal{M}_3} = E_{R^1} \times \mathcal{M}_3$ of the blowing up $R^1 \times \mathcal{M}_3 \to Q^1 \times \mathcal{M}_3$. The following commutative diagram describes the situation:

$$\begin{tikzcd}
E_{\mathcal{W}} \arrow[r] \arrow[d, "f"'] & \mathcal{W} \arrow[r] \arrow[d, "f"'] & W \arrow[d, "f"'] \\
E_{R^1} \times \mathcal{M}_3 \arrow[r] \arrow[d] & R^1 \times \mathcal{M}_3 \arrow[r] \arrow[d] & Q^1 \times \mathcal{M}_3 \\
\mathbb{P}^1 \times \mathcal{M}_3 \arrow[r, "\mathrm{id}"] & \mathbb{P}^1 \times \mathcal{M}_3 &
\end{tikzcd}$$

9.2.8. Thus

$$g : \mathcal{W} \xrightarrow{f} R^1 \times \mathcal{M}_3 \to \mathbb{P}^1 \times \mathcal{M}_3$$

is a family of curves, which has 3 distinguished sections given by the exceptional divisor $E_{\mathcal{W}}$ of $\mathcal{W} \to W$. Moreover by the description of $f : \mathcal{W}_q \to R^1$ as degree 3 cover, one can easily see that the fibers of g are given by the Fermat curve of degree 3 or consist of 3 smooth rational curves intersecting each other in exactly one point, which does not lie on $(E_{\mathcal{W}})_q$. Over $\mathbb{P}^1 \setminus \{\infty\} \times \mathcal{M}_3$ and $\mathbb{P}^1 \setminus \{0\} \times \mathcal{M}_3$ one can embed the restricted family into some copy of $\mathbb{P}^2_{\mathbb{A}^1 \times \mathcal{M}_3}$.

Therefore we obtain the family

$$\mathcal{W} \times \mathbb{F}_3 \to \mathbb{P}^1 \times \mathcal{M}_3$$

of surfaces, which has sections given by the fiberproduct of the exceptional divisors of $\mathcal{W} \to W$ with the points fixed by $\alpha_{\mathbb{F}_3}$, which do not meet any singular point of a fiber. In addition α_2 is a $\mathbb{P}^1 \times \mathcal{M}_3$-automorphism of this family. Hence by the same arguments as in the proof of Proposition 9.1.2, we can blow down $\varphi(\tilde{E}_C)$ over $(\mathbb{P}^1 \setminus \{\infty\}) \times \mathcal{M}_3$ and $(\mathbb{P}^1 \setminus \{0\}) \times \mathcal{M}_3$. By gluing, we obtain the family $\hat{\mathcal{Q}}$. Note that the singular fibers of $\mathcal{W} \times \mathbb{F}_3 \to \mathbb{P}^1 \times \mathcal{M}_3$ are given by 3 copies of $\mathbb{P}^1 \times \mathbb{F}_3$. Hence by the restriction of the sheaf, which yields the blowing down morphism, to the corresponding copies of $\widetilde{\mathbb{P}^1 \times \mathbb{F}_3}/\langle\alpha_2\rangle$, one obtains smooth blowing down morphisms on these copies.

Construction 9.2.9. But $\hat{\mathcal{Q}}$ has 18 sections of singular points given by the 18 isolated sections fixed by α_2 on $\widetilde{\mathcal{W} \times \mathbb{F}_3}$. Recall that these sections are given by

$$\{z_5 = z_4 = 0\} \times \{z = 0\}.$$

Let $\mathcal{Q} \to \hat{\mathcal{Q}}$ denote the blowing up of the singular sections of $\hat{\mathcal{Q}}$ and

$$\widetilde{\widetilde{\mathcal{W} \times \mathbb{F}_3}} \to \widetilde{\mathcal{W} \times \mathbb{F}_3}$$

denote the blowing up of these 18 sections. By the same arguments as in Remark 7.1.2, we obtain the following commutative diagram:

$$\begin{tikzcd}
\widetilde{\widetilde{\mathcal{W}}} \arrow[r, "\tilde{\varphi}"] \arrow[d] & \mathcal{Q} \arrow[d] \\
\widetilde{\mathcal{W}} \arrow[r, "\varphi"] & \hat{\mathcal{Q}}
\end{tikzcd}$$

Note that $\tilde{\varphi}$ is a cyclic cover on the complement of $\tilde{E}_C$. Thus by the Hurwitz formula and the fact that α_2 acts by the character 1 on $\Gamma(\omega_{\mathcal{W}_q \times \mathbb{F}_3})$ for each $q \in \mathcal{M}_3$, one concludes that $\mathcal{Q}$ is a family of Calabi-Yau 3-manifolds.

Proposition 9.2.10. *The family $\mathcal{Q} \to \mathcal{M}_3$ is a CMCY family of 3-manifolds.*

Proof. Note that on each fiber we blow up some points and several copies of $\mathbb{P}^1$, which have CM. Hence by Theorem 7.1.7, we must only apply the facts that $\mathbb{F}_3$ has CM and $\mathcal{W}$ has a dense set of fibers $\mathcal{W}_q$ such that $\mathrm{Hg}(H^k(\mathcal{W}_q, \mathbb{Q}))$ is commutative for all k. $\square$

9.3 The degree 4 case

Consider the $CMCY$ family $\mathcal{C}_2 \to \mathcal{M}_1$ of 2-manifolds given by

$$\mathbb{P}^3 \supset V(y_2^4 + y_1^4 + x_1(x_1 - x_0)(x_1 - \lambda x_0)x_0) \to \lambda \in \mathcal{M}_1$$

which we have constructed in Section 7.4. In this section we construct quotients of $\mathcal{C}_2$ by cyclic subgroups of its group of $\mathcal{M}_1$-automorphisms, which will be suitable to obtain new $CMCY$ families of 2-manifolds. In the next section we will see that these new examples are endowed with involutions, which make them suitable for the construction of the Borcea-Voisin tower. Hence by the Hurwitz formula and some other obvious reasons, one has:

Claim 9.3.1. *Let C be a $K3$ surface and α be an involution on C, which admits a finite set S of fixed points on C. Then the quotient $\tilde{C}/\alpha$, where $\tilde{C}$ denotes the blowing up of C with respect to the subvariety given by S, is a $K3$ surface, too. Moreover $\tilde{C}/\alpha$ has complex multiplication, if C has complex multiplication.*

Now we introduce a group $\mathbb{G}_4$ of $\mathcal{M}_1$-automorphisms of the $CMCY$ family $\mathcal{C}_2 \to \mathcal{M}_1$. The elements $g \in \mathbb{G}_4$ can be uniquely written as a product $g = abc$ with $a \in \langle \alpha \rangle$, $b \in \langle \beta \rangle$, and $c \in \langle \iota_4 \rangle$, where:

$$\alpha(y_2 : y_1 : x_1 : x_0) = (iy_2 : y_1 : x_1 : x_0), \;\; \beta(y_2 : y_1 : x_1 : x_0) = (y_2 : iy_1 : x_1 : x_0),$$

$$\iota_4(y_2 : y_1 : x_1 : x_0) = (y_1 : y_2 : x_1 : x_0)$$

Therefore the group $\mathbb{G}_4$ contains exactly 32 elements. The action of $\mathbb{G}_4$ on the global sections of the canonical sheaves of the fibers induces a surjection of $\mathbb{G}_4$ onto the multiplicative group of the 4-th. roots of unity.

Its kernel $\mathbb{K}_4$ is a normal subgroup of order 8. It contains the following automorphisms of order 4:

$$\delta(y_2 : y_1 : x_1 : x_0) = (-y_1 : y_2 : x_1 : x_0), \;\; \epsilon(y_2 : y_1 : x_1 : x_0) = (iy_2 : -iy_1 : x_1 : x_0),$$

$$\eta(y_2 : y_1 : x_1 : x_0) = (iy_1 : iy_2 : x_1 : x_0)$$

One has that

$$\iota_3 = \delta^2 = \epsilon^2 = \eta^2 = (\alpha\beta)^2.$$

Moreover one checks easily that $\mathbb{K}_4$ is isomorphic to the quaternion group and has the generators δ, ϵ and η. Thus one has

$$\mathbb{K}_4/\langle \iota_3 \rangle = (\mathbb{Z}/2)^2. \tag{9.2}$$

One can easily calculate that

$$\alpha\langle\delta\rangle\alpha^{-1} = \langle\eta\rangle.$$

By the fact that $\mathbb{K}_4$ has 2 residue classes with respect to $\langle\delta\rangle$ resp., $\langle\epsilon\rangle$ resp., $\langle\eta\rangle$, one concludes that $\langle\delta\rangle$ resp., $\langle\epsilon\rangle$ resp., $\langle\eta\rangle$ is a normal subgroup of $\mathbb{K}_4$. Since $[\alpha]_{\mathbb{K}_4}$ generates $\mathbb{G}_4/\mathbb{K}_4$ and

$$\alpha\langle\epsilon\rangle\alpha^{-1} = \langle\epsilon\rangle,$$

$\langle\epsilon\rangle$ is a normal subgroup of $\mathbb{G}_4$.

9.3.2. Recall that ι_3 denotes the involution given by

$$\iota_3(y_2 : y_1 : x_1 : x_0) = (-y_2 : -y_1 : x_1 : x_0).$$

Let $\mathcal{C}_{\langle\iota_3\rangle}$ be the $CMCY$ family of 2-manifolds given by the quotient $\tilde{\mathcal{C}}_2/\langle\iota_3\rangle$, where $\tilde{\mathcal{C}}_2$ denotes the blowing up of $\mathcal{C}_2$ with respect to the 8 sections fixed by ι_3. Four sections fixed by ι_3 are given by $(1 : \zeta : 0 : 0)$, where ζ runs through the primitive 8-th. roots of unity. The other 4 sections are given by

$$(0:0:0:1), \quad (0:0:1:1), \quad (0:0:\lambda:1) \quad \text{and} \quad (0:0:1:0).$$

Since the generators α, β and ι_4 of $\mathbb{G}_4$ leave the ideal sheaf corresponding to these 8 sections invariant, all automorphisms of $\mathbb{G}_4$ induce automorphisms on $\tilde{\mathcal{C}}_2$. Note that ι_3 commutes with each $\tau \in \mathbb{G}_4$. For each $\tau \in \mathbb{G}_4$ one finds open affine subsets invariant under $\langle\tau, \iota_3\rangle$. On these affine sets the global sections of the structure sheaf invariant under $\langle\tau, \iota_3\rangle$ are contained in $\mathcal{O}^{\langle\iota_3\rangle}$, where τ leaves $\mathcal{O}^{\langle\iota_3\rangle}$ invariant. Therefore τ induces an automorphism on $\mathcal{C}_{\langle\iota_3\rangle}$. One checks easily that δ, η and ϵ yield involutions on $\mathcal{C}_{\langle\iota_3\rangle}$ leaving only finitely many sections fixed. Thus by using Claim 9.3.1, these involutions yield the $CMCY$ families of 2-manifolds

$$\mathcal{C}_{\langle\delta\rangle} \cong \mathcal{C}_{\alpha\langle\delta\rangle\alpha^{-1}} = \mathcal{C}_{\langle\eta\rangle} \quad \text{and} \quad \mathcal{C}_{\langle\epsilon\rangle}.$$

9.4 Involutions on the quotients of the degree 4 example

In Section 7.4 we introduced several $\mathcal{M}_1$-involutions $\iota_1, \ldots, \iota_7$ of $\mathcal{C}_2$. We have seen that ι_3 acts by the character 1 on the global sections of the canonical sheaves of the fibers. Moreover $\iota_1, \iota_2, \iota_4, \ldots, \iota_7$ act by the character -1 on the global sections of the canonical sheaves of the fibers. Here we show that each for each $i = 1, 2, 4, \ldots, 7$ the involution ι_i induces $\mathcal{M}_1$-involutions on the quotient families of 9.3.2, which make them suitable for the construction of a Borcea-Voisin tower.

Remark 9.4.1. One can use Example 7.4.3, Example 7.4.4 and Example 7.4.5 and determine some explicitly given CM fibers of the new quotient families. By using the method of C. Voisin [60], these new $K3$ surfaces with complex multiplication and our explicit examples of elliptic curves with complex multiplication yield new Calabi-Yau 3-manifolds with complex multiplication

We fix some new notation. Let C_2 be an arbitrary fiber of $\mathcal{C}_2$, $p \in C_2$, where p is not fixed by ι_3, and F_i denote the curve of fixed points on C_2 with respect to ι_i for all $i = 1, 2, 4, \ldots, 7$.

9.4.2. The involutions ι_1 and ι_2 induce the same involution on $\mathcal{C}_{\langle\iota_3\rangle}$. One has that $\iota_1([p]_{\langle\iota_3\rangle}) = [p]_{\langle\iota_3\rangle}$, if and only if $p \in F_1 \cup F_2$. The involution ι_3 induces an involution on the curve F_1 and on the curve F_2. Each of the covers induced by these involutions has 4 ramification points. Hence by the Hurwitz formula, ι_1 induces an involution on $\mathcal{C}_{\langle\iota_3\rangle}$, which has a divisor of fixed points containing two families of elliptic curves. By [60], 1.1, the ramification divisor of our involution on a fiber of $\mathcal{C}_{\langle\iota_3\rangle}$ has at most one irreducible component of genus $g > 0$ or consists of two elliptic curves. Thus it consists of two elliptic curves. It is quite easy to check that by this involution ι_1, the family $\mathcal{C}_{\langle\iota_3\rangle}$ is suitable for the construction of a Borcea-Voisin tower.

9.4.3. The involutions ι_4 and ι_6 induce the same involution on $\mathcal{C}_{\langle\iota_3\rangle}$. One has that $\iota_4([p]_{\langle\iota_3\rangle}) = [p]_{\langle\iota_3\rangle}$, if and only if $p \in F_4 \cup F_6$. The involution ι_3 induces an involution on the curve F_4 and on the curve F_6. Each of the covers induced by these involutions have 4 ramification points. Hence by the same arguments as in 9.4.2, the involution ι_4 induces an involution on $\mathcal{C}_{\langle\iota_3\rangle}$, which has a divisor of fixed points consisting of two families of elliptic curves. It is quite easy to check that by this involution ι_1, the family $\mathcal{C}_{\langle\iota_3\rangle}$ is suitable for the construction of a Borcea-Voisin tower.

Since $\alpha\iota_4\alpha^{-1} = \iota_5$ and $\alpha\iota_6\alpha^{-1} = \iota_7$, the involutions ι_5 and ι_7 induce up isomorphisms the same involution as ι_4 and ι_6 on $\mathcal{C}_{\langle\iota_3\rangle}$.

Recall the $\mathcal{M}_1$-automorphisms

$$\delta(y_2 : y_1 : x_1 : x_0) = (-y_1 : y_2 : x_1 : x_0),\ \epsilon(y_2 : y_1 : x_1 : x_0) = (iy_2 : -iy_1 : x_1 : x_0)$$

of $\mathcal{C}_2$ of order 4.

Remark 9.4.4. Now we consider the quotient families $\mathcal{C}_{\langle\delta\rangle}$ and $\mathcal{C}_{\langle\epsilon\rangle}$ in 9.3.2. Moreover one has that δ and ϵ act as involutions on the 4 sections given by $(1 : \zeta : 0 : 0)$, where ζ runs through the primitive 8-th. roots of unity, and leave the sections given by

$$(0 : 0 : 0 : 1), \quad (0 : 0 : 1 : 1), \quad (0 : 0 : \lambda : 1), \quad (0 : 0 : 1 : 0)$$

invariant.

One can easily verify there does not exist a point $p \in C_2$ on the complement of these eight sections such that $\delta(p) = \iota_3(p)$ or $\epsilon(p) = \iota_3(p)$.

Therefore either p is contained in one of the 8 sections fixed by ι_3 or $\langle\delta\rangle \cdot p$ and $\langle\epsilon\rangle \cdot p$ contain 4 different elements. For our notation we will assume that p is not fixed by ι_3 as above.

9.4.5. The involutions ι_1 and ι_2 commute with ϵ. Thus the same holds true with respect to the involutions on $\mathcal{C}_{\langle\iota_3\rangle}$ induced by ι_1, ι_2 and ϵ. Hence one concludes that ι_1 and ι_2 induce an involution an $\mathcal{C}_{\langle\epsilon\rangle}$. Since ι_1 and ι_2 induce the same involution on $\mathcal{C}_{\langle\iota_3\rangle}$, the involutions ι_1 and ι_2 induce the same involution on $\mathcal{C}_{\langle\epsilon\rangle}$.

A point $[p]$ on the fiber $C_{\langle\epsilon\rangle}$ of $\mathcal{C}_{\langle\epsilon\rangle}$ is fixed by ι_1, if $\iota_1(p) = \epsilon^i(p)$ for $i = 0, \ldots, 3$. This is exactly satisfied on F_1 and F_2 for $i = 0$ or $i = 2$. The automorphism ϵ yields a quotient of F_1 resp., F_2 of degree 4 fully ramified over 4 points. Hence by the Hurwitz formula, $F_1/\langle\epsilon\rangle$ and $F_2/\langle\epsilon\rangle$ are rational curves.

By the definitions of ι_1 and ϵ, one checks easily that their actions coincide on the exceptional divisor on $\tilde{\mathcal{C}}_2$ over the four sections given by $V(y_2, y_1)$. Moreover by the definitions of ι_1 and ϵ, one checks easily that for each primitive 8-th. root ζ of unity

$$\iota_1(1 : \zeta : 0 : 0) = \epsilon(1 : \zeta : 0 : 0) = (1 : -\zeta : 0 : 0).$$

Both $\mathcal{M}_1$-automorphisms fix the local parameters x_1 and x_0.

Thus altogether the involution ι_1 induces an involution on $\mathcal{C}_{\langle\epsilon\rangle}$, which has a divisor of fixed points consisting of 8 disjoint families of rational curves. It is quite easy to check that $\mathcal{C}_{\langle\epsilon\rangle}$ is suitable for the construction of a Borcea-Voisin tower by this involution.

9.4.6. The involutions $\iota_4, \ldots, \iota_7$ do not commute with ϵ. But one has $\epsilon\iota_i = \iota_i\epsilon^3$ for all $i = 4, \ldots, 7$. Hence ι_i $(i = 4, \ldots, 7)$ induces an involution on $\mathcal{C}_{\langle\iota_3\rangle}$. Since $\iota_5 = \epsilon\iota_4$, $\iota_6 = \epsilon^2\iota_4$ and $\iota_7 = \epsilon^3\iota_4$, these involutions induce the same involution on $\mathcal{C}_{\langle\epsilon\rangle}$.

A point $[p] \in C_{\langle\epsilon\rangle}$ is invariant under ι_4, if $\iota_4(p) = \epsilon^i(p)$ for $i = 0, \ldots, 3$. One has that $\iota_4(p) = (p)$ on F_4, $\iota_4(p) = \epsilon^1(p)$ on F_7, $\iota_4(p) = \epsilon^2(p)$ on F_6 and $\iota_4(p) = \epsilon^3(p)$ on F_5. Note that $\epsilon(F_4) = F_6$, $\epsilon(F_6) = F_4$, $\epsilon^2(F_4) = F_4$ and $\epsilon^2(F_6) = F_6$. Moreover one has $\epsilon(F_5) = F_7$, $\epsilon(F_7) = F_7$, $\epsilon^2(F_5) = F_5$ and $\epsilon^2(F_5) = F_5$. The automorphism $\epsilon^2 = \iota_3$ yields a quotient of F_4, F_5, F_6 resp., F_7 of degree 2 ramified over 4 points, where F_4 and F_6 resp., F_5 and F_7 are mapped onto the same quotient by ϵ. Hence by the Hurwitz formula, the quotient consists of two families of elliptic curves.

By [60], 1.1, the ramification divisor of our involution on $C_{\langle\epsilon\rangle}$ has at most one irreducible component of genus $g > 0$ or consists of two elliptic curves. Thus ι_4 induces an involution on $\mathcal{C}_{\langle\epsilon\rangle}$, which has a divisor of fixed points consisting of 2 families of elliptic curves. It is quite easy to check that this involution makes $\mathcal{C}_{\langle\epsilon\rangle}$ suitable for the construction of a Borcea-Voisin tower.

9.4.7. The involutions ι_4 and ι_6 do not commute with δ. But one has $\delta\iota_4 = \iota_4\delta^3$ and $\delta\iota_6 = \iota_6\delta^3$. Moreover one has

$$\iota_1 = \delta \circ \iota_4, \quad \iota_6 = \delta^2 \circ \iota_4, \quad \text{and} \quad \iota_2 = \delta^3 \circ \iota_4.$$

Hence ι_1, ι_2, ι_4 and ι_6 induce the same involution on $\mathcal{C}_{<\delta>}$.

A point $[p] \in C_{\langle\delta\rangle}$ is invariant under ι_4, if $\iota_4(p) = \delta^i(p)$. This occurs, if and only if

$$p \in F_1 \cup F_2 \cup F_4 \cup F_6.$$

Note that $\delta(F_4) = F_6$ and $\delta(F_1) = F_2$. Moreover δ yields a degree 4 quotient of $F_4 \cup F_6$, and a degree 4 quotient of $F_1 \cup F_2$. Thus the divisor of fixed points contains two families of elliptic curves.

By the same arguments as in 9.4.6, the involution ι_4 induces an involution on $\mathcal{C}_{\langle\delta\rangle}$, which has a divisor of fixed points consisting of 2 families of elliptic curves and makes $\mathcal{C}_{\langle\delta\rangle}$ suitable for the construction of a Borcea-Voisin tower.

9.4.8. The involution ι_5 commutes with δ. One has that $p = \iota_5(p)$, if $p \in F_5$ and $\delta^2(p) = \iota_5(p)$, if $p \in F_7$. Note that δ acts as degree 4 automorphism on F_5 resp., F_7. Each of the corresponding quotient maps is fully ramified over 4 points. By the same arguments as in 9.4.5, the $\mathcal{M}_1$-automorphisms ι_5 and δ act in the same way on the exceptional divisor of $\tilde{\mathcal{C}}_2$. Thus ι_5 induces an involution on $\mathcal{C}_{\langle\delta\rangle}$, which fixes a divisor consisting of 8 families of rational curves. Moreover it is quite easy to check that this involution makes $\mathcal{C}_{\langle\delta\rangle}$ suitable for the construction of a Borcea-Voisin tower.

9.4.9. Since $\alpha\iota_1\alpha^{-1} = \iota_1$ and $\alpha\delta\alpha^{-1} = \eta$, one concludes that the involution induced by ι_1 on $\mathcal{C}_{\langle\eta\rangle}$ coincides up to an isomorphism with the involution induced by ι_1 on $\mathcal{C}_{\langle\delta\rangle}$.

Since $\alpha\iota_5\alpha^{-1} = \iota_6$ and $\alpha\delta\alpha^{-1} = \eta$, one concludes that the involution induced by ι_6 on $\mathcal{C}_{\langle\eta\rangle}$ coincides up to an isomorphism with the involution induced by ι_5 on $\mathcal{C}_{\langle\delta\rangle}$.

9.5 The extended automorphism group of the degree 4 example

The group $\mathbb{G}_4$ of $\mathcal{M}_1$-automorphisms of $\mathcal{C}_2$ does not contain all $\mathcal{M}_1$-automorphisms of $\mathcal{C}_2$. In this section we give an additional group $\mathbb{E}_4$ of $\mathcal{M}_1$-automorphisms such that $\mathbb{G}_4$ and $\mathbb{E}_4$ generate an extended $\mathcal{M}_1$-automorphism group $\bar{\mathbb{G}}_4$. Moreover we will make some remarks about $\bar{\mathbb{G}}_4$ and $\mathbb{E}_4$.

We obtain due to [28], Proposition 9 and the notations of [28], Section 2:

Proposition 9.5.1. *The family $\mathcal{C}_2$ has a group $\mathbb{E}_4$ of $\mathcal{M}_1$-automorphisms consisting of 16 different automorphisms given by $(\alpha\beta)^\nu$ with $\nu = 0, \ldots, 3$ and:*

$$\alpha_\zeta(y_2 : y_1 : x_1 : x_0) = (\zeta y_2 : \zeta y_1 : x_1 - \lambda x_0 : x_1 - x_0), \quad \zeta^4 = (1-\lambda)^2$$

$$\beta_\varsigma(y_2 : y_1 : x_1 : x_0) = (\varsigma y_2 : \varsigma y_1 : x_1 - x_0 : \frac{1}{\lambda}x_1 - x_0), \quad \varsigma^4 = (1 - \frac{1}{\lambda})^2$$

$$\gamma_\kappa(y_2 : y_1 : x_1 : x_0) = (\kappa y_2 : \kappa y_1 : \lambda x_0 : x_1), \quad \kappa^4 = \lambda^2$$

The involutions of $\mathbb{E}_4$ are given by $(\alpha\beta)^\nu$, α_ζ, β_ς and γ_κ for $\nu = 2$, $\zeta^2 = 1-\lambda$, $\varsigma^2 = 1 - \frac{1}{\lambda}$ and $\kappa^2 = \lambda$. The group $\mathbb{E}_4$ has a subgroup isomorphic to the *quaternion group given by* $(\alpha\beta)^\nu$, α_ζ, β_ς *and* γ_κ *for* $\nu = 0, 2$, $\zeta^2 = -1+\lambda$, $\varsigma^2 = -1 + \frac{1}{\lambda}$ *and* $\kappa^2 = -\lambda$.

One can ask for the character of the action of the involutions of $\mathbb{E}_4$ on $\Gamma(\omega_{(\mathcal{C}_2)_q})$ for each $q \in \mathcal{M}_1$ and the possibilities to use these involutions for the construction of Borcea-Voisin towers. For example one has:

Example 9.5.2. One checks easily that $\gamma_{\sqrt{\lambda}}$ resp., $\gamma_{-\sqrt{\lambda}}$ fixes the family curves on $\mathcal{C}_2$ given by

$$x_1 = \sqrt{\lambda}x_0 \text{ resp., } x_1 = -\sqrt{\lambda}x_0.$$

This family of curves is isomorphic to the constant family with universal fiber given by the Fermat curve $\mathbb{F}_4$ of degree 4, which has the genus 3. Thus it acts by the character -1 on $\Gamma(\omega_{(\mathcal{C}_2)_q})$ for each $q \in \mathcal{M}_1$. Since $\mathbb{F}_4$ has complex multiplication, $\gamma_{\sqrt{\lambda}}$ and $\gamma_{-\sqrt{\lambda}}$ make $\mathcal{C}_2$ suitable for the construction of a Borcea-Voisin tower.

The following claim implies that $\gamma_{\sqrt{\lambda}}$ and $\gamma_{-\sqrt{\lambda}}$ yield isomorphic families by the Borcea-Voisin tower:

Claim 9.5.3. *One can conjugate* $\gamma_{\sqrt{\lambda}}$ *and* $\gamma_{-\sqrt{\lambda}}$ *in* $\mathbb{E}_4$.

Proof. There exists some g of order 4 contained in the quaternion subgroup of $\mathbb{E}_4$ such that

$$\gamma_{\sqrt{\lambda}} = (\alpha\beta)g \text{ and } \gamma_{-\sqrt{\lambda}} = (\alpha\beta)^3 g = (\alpha\beta)(\alpha\beta)^2 g = (\alpha\beta)g^{-1}.$$

It is a well-known fact that there is a g_2 contained in the quaternion group such that

$$g^{-1} = g_2 \circ g \circ g_2^{-1}.$$

Since $(\alpha\beta)$ is contained in the center of $\mathbb{E}_4$, one obtains the result. □

Finally the question for isomorphy between $\mathcal{C}_2/\iota_1$ and $\mathcal{C}_2/\iota_4$ resp., the corresponding $CMCY$ families of 3-manifolds constructed by the method of C. Voisin [60] remains open, since we have:

Remark 9.5.4. By the description of $\mathbb{E}_4$ in Proposition 9.5.1, one checks easily that the generators α, β, ι_4 of $\mathbb{G}_4$ commute with each element of $\mathbb{E}_4$. Hence each element of $\bar{\mathbb{G}}_4$, which is the group generated by $\mathbb{G}_4$ and $\mathbb{E}_4$, can be written as $\kappa\tau$ with $\kappa \in \mathbb{E}_4$ and $\tau \in \mathbb{G}_4$. Thus for each $\sigma \in \mathbb{G}_4$ one obtains

$$(\kappa\tau)^{-1}\sigma(\kappa\tau) = \tau^{-1}\sigma\tau. \tag{9.3}$$

Hence the fact that ι_1 and ι_4 are not conjugate in $\mathbb{G}_4$ implies that ι_1 and ι_4 are not conjugate in $\bar{\mathbb{G}}_4$.

Moreover (9.3) implies that $\gamma_{\sqrt{\lambda}}$ is not conjugate to ι_1 or ι_4 in $\bar{\mathbb{G}}_4$.

Remark 9.5.5. One may search for additional involutions in $\bar{\mathbb{G}}_4$ and try to determine the character of the actions of all involutions on $\Gamma(\omega_{(\mathcal{C}_2)_q})$ for each $q \in \mathcal{M}_1$. In addition one can try to determine the involutions, which are suitable for the construction of a Borcea-Voisin tower and try to repeat the construction of the preceding section for arbitrary induced involutions on suitable quotients by cyclic subgroups of $\bar{\mathbb{G}}_4$.

9.6 The automorphism group of the degree 5 example by Viehweg and Zuo

We consider the $CMCY$ family $\mathcal{F}_3$

$$\mathbb{P}^4 \supset V(y_3^5 + y_2^5 + y_1^5 + x_1(x_1 - x_0)(x_1 - a_1x_0)(x_1 - a_2x_0)x_0) \to (a_1, a_2) \in \mathcal{M}_2$$

of 3-manifolds constructed by E. Viehweg and K. Zuo. Let ξ denote a fixed primitive 5-th. root of unity. We introduce an $\mathcal{M}_2$-automorphism group $\mathbb{G}_5$ of the family $\mathcal{F}_3 \to \mathcal{M}_2$. The elements $g \in \mathbb{G}_5$ can be uniquely written as a product $g = abcd$ with $a \in \langle\alpha\rangle$, $b \in \langle\beta\rangle$, $c \in \langle\gamma\rangle$ and $d \in S_3$, where:

$$\alpha(y_3 : y_2 : y_1 : x_1 : x_0) = (\xi y_3 : y_2 : y_1 : x_1 : x_0),$$
$$\beta(y_3 : y_2 : y_1 : x_1 : x_0) = (y_3 : \xi y_2 : y_1 : x_1 : x_0),$$
$$\gamma(y_3 : y_2 : y_1 : x_1 : x_0) = (y_3 : y_2 : \xi y_1 : x_1 : x_0),$$
$$d(y_3 : y_2 : y_1 : x_1 : x_0) = (y_{d(3)} : y_{d(2)} : y_{d(1)} : x_1 : x_0)$$

Therefore the group $\mathbb{G}_5$ contains exactly $5 \cdot 5 \cdot 5 \cdot 6 = 750$ elements. The action of $\mathbb{G}_5$ on the global sections of the canonical sheaves of the fibers induces a surjection of $\mathbb{G}_5$ onto the multiplicative group of the 10-th. roots of unity.[4]

Its kernel $\mathbb{K}_5$ is a normal subgroup of order 75. It contains the subgroup $\langle\alpha\beta^{-1}, \beta\gamma^{-1}\rangle$ of automorphisms of order 5. This group has 25 elements. Moreover it contains the cyclic group given by the permutations of A_3 of order 3. Therefore all elements of $\mathbb{K}_5$ are determined.

9.6.1. Let us consider all cyclic groups $\langle g\rangle \subset \mathbb{K}_5$ with $g = abc \neq e$ as above. If $a = e$ or $b = e$ or $c = e$, the group $\langle g\rangle$ is given by $\langle\alpha\beta^{-1}\rangle$, $\langle\beta\gamma^{-1}\rangle$ or $\langle\alpha\gamma^{-1}\rangle$. These groups are conjugate by $(1,2), (1,3), (2,3) \in S_3$.

Now consider the cyclic group $\langle g\rangle \subset \mathbb{K}_5$ with $g = abc$ and $a, b, c \neq e$. One has that $\langle g\rangle$ contains an element $\alpha\beta^b\gamma^{4-b}$ with $b \in \{1,2,3\}$. Hence by $e \in S_3$ or $(2,3) \in S_3$, it is conjugate to $\langle\alpha\beta\gamma^3\rangle$ or $\langle\alpha\beta^2\gamma^2\rangle$. By the cycle $(1,3) \in S_3$, these both groups are conjugate. By the fact that $\langle\alpha\beta\gamma^3\rangle$ leaves only finitely

[4] Note that S_3 is generated by the involutions given by the cycles $(1,2)$ and $(2,3)$, which act by the character -1 on the global sections of the canonical sheaves of the fibers.

many points invariant on each fiber, but $\langle \alpha\beta^{-1} \rangle$ leaves a curve invariant on each fiber, both groups can not be conjugate.

Therefore we have two conjugacy classes of cyclic subgroups $\langle g \rangle \subset \mathbb{K}_5$ with $g = abc \neq e$ represented by $\langle \alpha\beta^{-1} \rangle$ and $\langle \alpha\beta\gamma^3 \rangle$.

Claim 9.6.2. *Any automorphism $\tau \in \mathbb{K}_5$, which is not given by*

$$\tau(y_3 : y_2 : y_1 : x_1 : x_0) = (\xi^s y_3 : \xi^t y_2 : \xi^{5-s-t} y_1 : x_1 : x_0)$$

for some $s, t \in \mathbb{Z}$, satisfies $\tau^3 = \mathrm{id}$.

Proof. If τ satisfies the assumptions of the Claim, then τ or τ^{-1} is given by

$$(y_3 : y_2 : y_1 : x_1 : x_0) \to (\xi^s y_1 : \xi^t y_3 : \xi^{5-s-t} y_2 : x_1 : x_0) \tag{9.4}$$

for some $s, t \in \mathbb{Z}$. Hence we assume without loss of generality that τ is given by (9.4) and verify the statement by calculation:

$$\tau^3(y_3 : y_2 : y_1 : x_1 : x_0) = \tau^2(\xi^s y_1 : \xi^t y_3 : \xi^{-s-t} y_2 : x_1 : x_0)$$

$$= \tau(\xi^{-t} y_2 : \xi^{s+t} y_1 : \xi^{-s} y_3 : x_1 : x_0) = (y_3 : y_2 : y_1 : x_1 : x_0)$$

$\square$

For each τ as in (9.4) one can easily calculate that $\alpha^{-s}\beta^{-s-t} \circ \tau \circ \alpha^s \beta^{s+t}$ is given by

$$(y_3 : y_2 : y_1 : x_1 : x_0) \to (y_1 : y_3 : y_2 : x_1 : x_0).$$

Therefore all cyclic subgroups of $\mathbb{K}_5$ are up to conjugation determined. Hence:

Proposition 9.6.3. *The family $\mathcal{F}_3$ has up to isomorphisms the following quotient families of Calabi-Yau orbifolds with dense sets of CM fibers:*

$$\mathcal{F}_3/\langle \alpha\beta^4 \rangle, \quad \mathcal{F}_3/\langle \alpha\beta\gamma^3 \rangle, \quad \mathcal{F}_3/\langle (1,2,3) \rangle$$

Proof. The existence of dense sets of CM fibers follows, since the VHS of a quotient family of $\mathcal{F}_3$ is a sub-VHS of $\mathcal{F}_3$. $\square$

Chapter 10
Examples of *CMCY* families of 3-manifolds and their invariants

In this chapter we collect all examples of $CMCY$-families from the previous chapters, determine the *length* of their Yukawa couplings and compute the Hodge numbers of their fibers. In Section 10.4 we will also give an outlook to the possible construction methods of Calabi-Yau manifolds by using other Calabi-Yau manifolds in lower dimensions and cyclic automorphism groups. We recall the definition of the *length* of the Yukawa coupling and its computation methods in Section 10.1. Since there are equations for the Hodge numbers of the Calabi-Yau 3-manifolds obtained from the Borcea-Voisin method, we only need to list them in Section 10.2. In Section 10.3 we need to calculate a little bit to get the Hodge numbers of Calabi-yau 3-manifolds obtained from $K3$ surfaces with a degree 3 automorphism and Fermat curve of degree 3.

10.1 The *length* of the Yukawa coupling

First let us construct the Yukawa coupling. A little bit later in this short section we will give a motivation to consider it and describe how to calculate its *length* for our examples of $CMCY$ families of 3-manifolds. For this section we refer [57], [58] and [59].

Construction 10.1.1. Assume that U is a quasi projective variety and $\mathcal{V}$ is a complex polarized variation of Hodge structures of weight n on U. It is a well-known fact that there exists a suitable finite cover of U such that the pullback of $\mathcal{V}$ has local unipotent monodromy. We replace U by this finite cover. There exists a smooth projective compactification Y of U such that $S := Y \setminus U$ is a normal crossing divisor. Then one can construct the Deligne extension $\mathcal{H}$ of $\mathcal{V} \otimes_{\mathbb{C}} \mathcal{O}_U$ (i.e., the unique extension such that the Gauß-Manin connection yields the structure of a logarithmic Higgs bundle (F, θ) on the associated graded bundle and the real components of eigenvalues of the residues are contained in $[0, 1)$). The graduation gives a decomposition of

J.C. Rohde, *Cyclic Coverings, Calabi-Yau Manifolds and Complex Multiplication*, Lecture Notes in Mathematics 1975, DOI: 10.1007/978-3-642-00639-5_11,

F into locally free sheaves $E^{p,n-p}$ and the Gauß-Manin connection induces an $\mathcal{O}_Y$-linear morphism

$$E^{p,n-p} \to E^{n-1,n-p+1} \otimes \Omega^1_Y(\log S),$$

called Higgs field. The Yukawa coupling θ_i (for $i \leq n$) is defined by the composition

$$\theta_i : E^{n,0} \xrightarrow{\theta_{n,0}} E^{n-1,1} \otimes \Omega^1_Y(\log S) \xrightarrow{\theta_{n-1,1}} E^{n-2,2} \otimes \mathrm{Sym}^2\Omega^1_Y(\log S) \xrightarrow{\theta_{n-2,2}} \dots$$
$$\xrightarrow{\theta_{n-i+1,i-1}} E^{n-i,i} \otimes \mathrm{Sym}^i\Omega^1_Y(\log S).$$

Definition 10.1.2. Let $f : V \to U$ be a family with fibers of dimension n as in Construction 10.1.1. The *length* $\zeta(f)$ of the Yukawa coupling is given by

$$\zeta(f) := \min\{i \geq 1; \theta_i = 0\} - 1.$$

We say that the Yukawa coupling has *maximal length*, if $\zeta(f) = n$.

The family $f : V \to U$ is rigid, if there does not exist a non-trivial deformation of f over a nonsingular quasi-projective curve T.

The following proposition yields our motivation to consider the *length* of the Yukawa coupling:

Proposition 10.1.3. *If the Yukawa coupling has* maximal length*, the family is rigid.*

Proof. (see [57], Section 8) □

The statements of the following lemma, which allow the computation of *length* of the Yukawa couplings of our examples of $CMCY$ families of 3-manifolds by their construction, are well-known:

Lemma 10.1.4. *For two variations of Hodge structures* $\mathbb{V}$ *and* $\mathbb{W}$ *on a holomorphic manifold one has*

$$\zeta(\mathbb{V} \otimes \mathbb{W}) = \zeta(\mathbb{V}) + \zeta(\mathbb{W}) \quad \text{and} \quad \zeta(\mathbb{V} \oplus \mathbb{W}) = \max\{\zeta(\mathbb{V}), \zeta(\mathbb{W})\}.$$

10.2 Examples obtained by degree 2 quotients

Let $\mathcal{Z}_1 \to \mathcal{M}$ be one of the examples of a $CMCY$ family of 2-manifolds, which we have constructed in the preceding chapters, with a suitable involution ι such that it satisfies the assumptions for $\mathcal{Z}_1$ in the construction of a Borcea-Voisin tower. Here we list all examples of $CMCY$ families $\mathcal{Z}_2$ of 3-manifolds obtained by the Borcea-Voisin tower starting with such a family

$\mathcal{Z}_1$ and Σ_2 given by the family $\mathcal{E} \to \mathcal{M}_1$ of elliptic curves endowed with its natural involution. By the definition of Calabi-Yau manifolds, Serre duality and Hodge symmetry, all Hodge numbers of the fibers of the resulting $CMCY$ family $\mathcal{Z}_2$ of 3-manifolds are determined by $h^{1,1}$ and $h^{2,1}$.

Claim 10.2.1. *Keep the assumptions above. Let $(\mathcal{Z}_1)_p \to (\mathcal{Z}_1)_p/\iota$ be ramified over N curves with genus $g_1, \ldots, g_N$ for all $p \in \mathcal{M}$. Then the fibers of $\mathcal{Z}_2$ have the Hodge numbers*

$$h^{1,1} = 11 + 5N - N' \text{ and } h^{2,1} = 11 + 5N' - N, \quad \text{where} \quad N' = \sum g_i.$$

Proof. (see [60], Corollaire 1.8) □

Hence for our examples of $CMCY$ families of 3-manifolds obtained by using the Borcea-Voisin tower and $CMCY$ families of 2-manifolds with suitable involutions, we have the following table:

family $\mathcal{Z}_1$	basis $\mathcal{M}$	involution ι	N	N'	$h^{1,1}$	$h^{2,1}$	ζ	reference
$\mathcal{C}_2$	$\mathcal{M}_1$	ι_1	1	3	13	25	2	7.4.8
$\mathcal{C}_2$	$\mathcal{M}_1$	ι_4	1	3	13	25	2	7.4.8
$\mathcal{C}_2$	$\mathcal{M}_1$	$\gamma_{\sqrt{\lambda}}, \gamma_{\sqrt{-\lambda}}$	1	3	13	25	2	9.5.2
$\mathcal{C}_{\langle\iota_3\rangle}$	$\mathcal{M}_1$	ι_1	2	2	19	19	2	9.4.2
$\mathcal{C}_{\langle\iota_3\rangle}$	$\mathcal{M}_1$	ι_4	2	2	19	19	2	9.4.3
$\mathcal{C}_{\langle\epsilon\rangle}$	$\mathcal{M}_1$	ι_1	8	0	51	3	2	9.4.5
$\mathcal{C}_{\langle\epsilon\rangle}$	$\mathcal{M}_1$	ι_4	2	2	19	19	2	9.4.6
$\mathcal{C}_{\langle\delta\rangle}$	$\mathcal{M}_1$	$\iota_1 = \iota_4$	2	2	19	19	2	9.4.7
$\mathcal{C}_{\langle\delta\rangle}$	$\mathcal{M}_1$	ι_5	8	0	51	3	2	9.4.8
$\mathcal{W}$	$\mathcal{M}_3$	γ	2	4	17	29	2	8.3.4, 8.3.5
$\mathcal{Y}$	$\mathcal{M}_3$	γ	2	4	17	29	2	9.1.3, 9.1.4

10.3 Examples obtained by degree 3 quotients

In this section we determine the Hodge numbers of the $CMCY$ family $\mathcal{Q}$ of 3-manifolds obtained by Proposition 9.2.10 and the Hodge numbers of the $CMCY$ family $\mathcal{R}$ of 3-manifolds obtained by Proposition 9.2.5.

Remark 10.3.1. In the case of the $CMCY$ families $\mathcal{Q}$ and $\mathcal{R}$ one has $\zeta = 1$ for the *length* of the Yukawa coupling as one concludes by their constructions and using Lemma 10.1.4.

Let X be a complex manifold and γ an automorphism of X of order m. Then $H^k(X, \mathbb{C})_\ell$ denotes the eigenspace of $H^k(X, \mathbb{C})$, on which γ acts via pullback by the character $e^{2\pi i \frac{\ell}{m}}$. For the computation of the Hodge numbers of the fibers of $\mathcal{Q}$ and $\mathcal{R}$ we will need the following proposition:

Proposition 10.3.2. *Let X be a Kähler manifold of dimension 3. Moreover let φ be an automorphism of X fixing a finite set of some isolated points Z_0 and a finite set Z_1 of disjoint curves such that $\varphi^m = \mathrm{id}$ for some $m \in \mathbb{N}$. Then one has the following eigenspaces:*

$$H^2(\tilde{X}_{Z_1 \cup Z_0}, \mathbb{Z})_0 \cong H^2(X, \mathbb{Z})_0 \oplus H^0(Z_1, \mathbb{Z}) \oplus H^0(Z_0, \mathbb{Z}),$$
$$H^3(\tilde{X}_{Z_1 \cup Z_0}, \mathbb{Z})_0 \cong H^3(X, \mathbb{Z})_0 \oplus H^1(Z_1, \mathbb{Z})$$

Proof. Let Y be a Kähler manifold and Z be a submanifold of codimension r. Then the Hodge structure of the blowing up $\tilde{Y}_Z$ along Z is given by

$$H^k(Y, Z) \oplus \bigoplus_{i=0}^{r-2} H^{k-2i-2}(Z, \mathbb{Z}) \cong H^k(\tilde{Y}_Z, \mathbb{Z}),$$

where $H^{k-2i-2}(Z, \mathbb{Z})$ shifted by $(i+1, i+1)$ in bi-degree (see [61], Théorème 7.31).

Thus one has:

$$H^2(\tilde{X}_{Z_1 \cup Z_0}, \mathbb{Z}) \cong H^2(X, \mathbb{Z}) \oplus H^0(Z_1, \mathbb{Z}) \oplus H^0(Z_0, \mathbb{Z}),$$
$$H^3(\tilde{X}_{Z_1 \cup Z_0}, \mathbb{Z}) \cong H^3(X, \mathbb{Z}) \oplus H^1(Z_1, \mathbb{Z})$$

Hence it remains to show that $H^0(Z_1, \mathbb{Z})$, $H^0(Z_0, \mathbb{Z})$ and $H^1(Z_1, \mathbb{Z})$ are invariant as sub-Hodge structures by φ. Therefore one considers the proof of [61], Théorème 7.31. These sub-Hodge structures are given by the image of $j_* \circ (\pi|_{Z_1 \cup Z_0})^*(H^0(Z_1 \cup Z_0, \mathbb{Z}))$ and $j_* \circ (\pi|_{Z_1 \cup Z_0})^*(H^1(Z_1 \cup Z_0, \mathbb{Z}))$, where j denotes the embedding of the exceptional divisor E of the blowing up morphism $\pi : \tilde{X}_{Z_1 \cup Z_0} \to X$.[1] One has the following commutative diagram:

$$\begin{tikzcd}
\tilde{X}_{Z_1 \cup Z_0} \arrow[r, "\varphi"] & \tilde{X}_{Z_1 \cup Z_0} \\
E \arrow[u, "j"] \arrow[r, "\varphi"] \arrow[d, "\pi|_E"] & E \arrow[u, "j"] \arrow[d, "\pi|_E"] \\
Z_1 \cup Z_0 \arrow[r, "\varphi"] & Z_1 \cup Z_0
\end{tikzcd}$$

[1] In general one has $\bigoplus_{i=0}^{r-2} j_* \circ h^i \circ (\pi|_{Z_1 \cup Z_0})^*$ instead of $j_* \circ (\pi|_{Z_1 \cup Z_0})^*$ for $i = 0, \ldots, r-2$ in [61], Théorème 7.31, where h denotes the cup-product with $c_1(\mathcal{O}_E(1))$ and the sheaf $\mathcal{O}_E(1)$ of the projective bundle E is described in [61], Subsection 3.3.2. But here the weight of the Hodge structures is to small for $i > 0$.

Since φ acts as the identity on $Z_1 \cup Z_0$, the same holds true for the Hodge structures on $Z_1 \cup Z_0$. Hence by the commutative diagram, the same holds true for the sub-Hodge structures on $\tilde{X}$ given by $j_* \circ (\pi|_{Z_1 \cup Z_0})^*$. $\square$

Proposition 10.3.3. *For all $q \in \mathcal{M}_3$ the action of the cyclic group $\langle \alpha\beta \rangle$ on $\mathcal{W}$ yields an eigenspace decomposition of $H^{1,1}(\mathcal{W}_q)$ of the dimensions*

$$h^{1,1}(\mathcal{W}_q)_0 = 14, \quad h^{1,1}(\mathcal{W}_q)_1 = 3, \quad h^{1,1}(\mathcal{W}_q)_2 = 3.$$

Proof. Let $\tilde{\mathcal{W}} \to \mathcal{W}$ be the blowing up of the six sections fixed by $\alpha\beta$. By the same arguments as in the proof of the preceding proposition, each fiber $\tilde{\mathcal{W}}_q$ has the Hodge numbers

$$h^{2,0} = 1, \quad h^{1,1} = 26, \quad h^{0,2} = 1.$$

Let $M := \tilde{\mathcal{W}}_q / \langle \alpha\beta \rangle$. Now we consider the quotient morphism $\varphi : \tilde{\mathcal{W}}_q \to M$. By the Hurwitz formula, one concludes that

$$\varphi^*(K_M) = -2E - E^{(2)},$$

where E is the exceptional divisor of $\mathcal{W}_q \to W_q$ given by three -2 curves and $E^{(2)}$ is the exceptional divisor of $\tilde{\mathcal{W}}_q \to \mathcal{W}_q$. From [61], Proposition 21.14, we have that $3 \cdot K_M^2 = (\varphi^*(K_M))^2$. Since

$$(\varphi^*(K_M))^2 = (-2E - E^{(2)})^2 = 4 \cdot (-6) - 6 = -30$$

and $c_1(M)^2 = K_M^2$ (see [26], Appendix A, Example 4.1.2), one obtains

$$c_1(M)^2 = K_M^2 = -10.$$

By the Noether formula (compare to [26], Appendix A, Example 4.1.2 and [61], Remarque 23.6), one has

$$\chi(\mathcal{O}_M) = \frac{1}{12}(c_1(M)^2 + c_2(M)) \quad \text{with} \quad c_2(M) - 2 = b_2(M)$$

in our case. From the fact that $\chi(\mathcal{O}_M) = 1$, one calculates that

$$h^{1,1}(\tilde{\mathcal{W}}_q)_0 = b_2(M) = 20.$$

By the fact that the blowing up morphism $\tilde{\mathcal{W}}_q \to \mathcal{W}_q$ has an exceptional divisor consisting of 6 rational curves, we conclude similar to Proposition 10.3.2 that

$$h^{1,1}(\mathcal{W}_q)_0 = h^{1,1}(\tilde{\mathcal{W}}_q)_0 - 6 = 20 - 6 = 14.$$

Since the $K3$ surface $\mathcal{W}_q$ has the Hodge number

$$h^{1,1}(\mathcal{W}_q) = 20 \text{ and } h^{1,1}(\mathcal{W}_q)_1 = h^{1,1}(\mathcal{W}_q)_2,$$

one concludes that

$$h^{1,1}(\mathcal{W}_q)_1 = h^{1,1}(\mathcal{W}_q)_2 = 3.$$

□

Proposition 10.3.4. *For all $q \in \mathcal{M}_3$ the action of the cyclic group $\langle\alpha\rangle$ on $\mathcal{W}$ yields an eigenspace decomposition of $H^{1,1}(\mathcal{W}_q)$ of the dimensions*

$$h^{1,1}(\mathcal{W}_q)_0 = 2, \quad h^{1,1}(\mathcal{W}_q)_1 = 9, \quad h^{1,1}(\mathcal{W}_q)_2 = 9.$$

Proof. Recall that

$$\mathcal{W}_q/\langle\alpha\rangle = R^1.$$

Since $-2V(z_4)$ is the canonical divisor of R^1 (see Corollary 8.1.7), one obtains

$$c_1(R^1)^2 = K^2_{R^1} = 8.$$

By the Noether formula, one has

$$\chi(\mathcal{O}_{R^1}) = \frac{1}{12}(c_1(R^1)^2 + c_2(R^1)) \text{ with } c_2(R^1) - 2 = b_2(R^1).$$

From the fact that $\chi(\mathcal{O}_{R^1}) = 1$, one calculates that

$$h^{1,1}(\tilde{\mathcal{W}}_q)_0 = b_2(R^1) = 2.$$

Since the $K3$ surface $\mathcal{W}_q$ has the Hodge number

$$h^{1,1}(\mathcal{W}_q) = 20 \text{ and } h^{1,1}(\mathcal{W}_q)_1 = h^{1,1}(\mathcal{W}_q)_2,$$

one concludes that

$$h^{1,1}(\mathcal{W}_q)_1 = h^{1,1}(\mathcal{W}_q)_2 = 9.$$

□

Proposition 10.3.5. *For all $q \in \mathcal{M}_3$ one has*

$$h^{1,1}(\mathcal{Q}_q) = 51.$$

Proof. Since

$$h^{0,0}(\mathcal{W}_q)_0 = h^{0,0}(\mathbb{F}_3)_0 = h^{1,1}(\mathbb{F}_3)_0 = 1, \quad b_1(\mathcal{W}_q) = 0$$

and Proposition 10.3.3 tells us that

$$h^{1,1}(\mathcal{W}_q)_0 = 14,$$

one concludes that $h^{1,1}(\mathcal{W}_q \times \mathbb{F}_3)_0 = 15$. Note that α_2 fixes $6 \cdot 3 = 18$ points. Moreover we have an additional exceptional divisor consisting of $3 \cdot 3 \cdot 3 = 27$ rational ruled surfaces. In the construction of $\mathcal{Q}$ we blow down 9 of these families of ruled surfaces. Hence by Proposition 10.3.2,

$$h^{1,1}(\mathcal{Q}_q) = 15 + 18 + 27 - 9 = 51.$$

□

Proposition 10.3.6. *For all $q \in \mathcal{M}_3$ one has*

$$h^{1,1}(\mathcal{R}_q) = 9.$$

Proof. Since

$$h^{0,0}(\mathcal{W}_q)_0 = h^{0,0}(\mathbb{F}_3)_0 = h^{1,1}(\mathbb{F}_3)_0 = 1, \quad b_1(\mathcal{W}_q) = 0$$

and Proposition 10.3.4 tells us that

$$h^{1,1}(\mathcal{W}_q)_0 = 2,$$

one concludes that $h^{1,1}(\mathcal{W}_q \times \mathbb{F}_3)_0 = 3$. Note that α_2 fixes 3 copies of the genus 4 curve $\mathcal{C}_q$. Each of these copies yields 3 blowing ups of a copy of $\mathcal{C}_q$ and one blowing down to a copy of $\mathcal{C}_q$. Hence by Proposition 10.3.2,

$$h^{1,1}(\mathcal{R}_q) = 3 + 9 - 3 = 9.$$

□

Proposition 10.3.7. *For all $q \in \mathcal{M}_3$ one has*

$$h^{1,2}(\mathcal{Q}_q) = h^{2,1}(\mathcal{Q}_q) = 3.$$

Proof. Recall that $\alpha\beta$ acts by the character $e^{2\pi i \frac{2}{3}}$ on the global sections of $\omega_{\mathcal{W}_q}$ for all $q \in \mathcal{P}_n$ and $\alpha_{\mathbb{F}_3}$ acts by the character $e^{2\pi i \frac{1}{3}}$ on the global sections of $\omega_{\mathbb{F}_3}$. Hence one obtains

$$h^{1,0}(\mathbb{F}_3)_1 = h^{0,1}(\mathbb{F}_3)_2 = h^{2,0}(\mathcal{W}_q)_2 = h^{0,2}(\mathcal{W}_q)_1 = 1$$

and

$$h^{1,0}(\mathbb{F}_3)_2 = h^{0,1}(\mathbb{F}_3)_1 = h^{2,0}(\mathcal{W}_q)_1 = h^{0,2}(\mathcal{W}_q)_2 = 0.$$

Note that $b_1(\mathcal{W}_q) = b_3(\mathcal{W}_q) = 0$, $h^{1,1}(\mathcal{W}_q)_0 = 14$ and $h^{1,1}(\mathcal{W}_q)_1 = h^{1,1}(\mathcal{W}_q)_2 = 3$. Recall that

$$H^3(\mathcal{W}_q \times \mathbb{F}_3, \mathbb{C})_0 = \bigoplus_{t=0}^{2} H^2(\mathcal{W}_q, \mathbb{C})_t \otimes H^1(\mathbb{F}_3, \mathbb{C})_{[3-t]_3}.$$

Hence one concludes that

$$H^3(\mathcal{W}_q \times \mathbb{F}_3, \mathbb{C})_0 = (H^{2,0}(\mathcal{W}_q)_2 \oplus H^{1,1}(\mathcal{W}_q)_2) \otimes H^{1,0}(\mathbb{F}_3)_1$$
$$\oplus (H^{1,1}(\mathcal{W}_q)_1 \oplus H^{0,2}(\mathcal{W}_q)_1) \otimes H^{0,1}(\mathbb{F}_3)_2.$$

This implies that

$$H^{2,1}(\mathcal{W}_q \times \mathbb{F}_3)_0 = H^{1,1}(\mathcal{W}_q)_2 \otimes H^{1,0}(\mathbb{F}_3)_1 \text{ such that } h^{2,1}(\mathcal{W}_q \times \mathbb{F}_3)_0 = 3.$$

Hence by Proposition 10.3.2 and the fact that $b_1(\mathbb{P}^1) = 0$, one obtains the statement. □

Proposition 10.3.8. *For all $q \in \mathcal{M}_3$ one has*

$$h^{1,2}(\mathcal{R}_q) = h^{2,1}(\mathcal{R}_q) = 33.$$

Proof. The automorphism α acts by the character $e^{2\pi i \frac{1}{3}}$ on the global sections of $\omega_{\mathcal{W}_q}$ for all $q \in \mathcal{P}_n$ and $\alpha^2_{\mathbb{F}_3}$ acts by the character $e^{2\pi i \frac{2}{3}}$ on the global sections of $\omega_{\mathbb{F}_3}$. Hence one obtains

$$h^{1,0}(\mathbb{F}_3)_1 = h^{0,1}(\mathbb{F}_3)_2 = h^{2,0}(\mathcal{W}_q)_2 = h^{0,2}(\mathcal{W}_q)_1 = 0$$

and

$$h^{1,0}(\mathbb{F}_3)_2 = h^{0,1}(\mathbb{F}_3)_1 = h^{2,0}(\mathcal{W}_q)_1 = h^{0,2}(\mathcal{W}_q)_2 = 1.$$

Note that $b_1(\mathcal{W}_q) = b_3(\mathcal{W}_q) = 0$, $h^{1,1}(\mathcal{W}_q)_0 = 2$ and $h^{1,1}(\mathcal{W}_q)_1 = h^{1,1}(\mathcal{W}_q)_2 = 9$. Recall that

$$H^3(\mathcal{W}_q \times \mathbb{F}_3, \mathbb{C})_0 = \bigoplus_{t=0}^{2} H^2(\mathcal{W}_q, \mathbb{C})_t \otimes H^1(\mathbb{F}_3, \mathbb{C})_{[3-t]_3}.$$

Hence one concludes that

$$H^3(\mathcal{W}_q \times \mathbb{F}_3, \mathbb{C})_0 = (H^{2,0}(\mathcal{W}_q)_1 \oplus H^{1,1}(\mathcal{W}_q)_1) \otimes H^{1,0}(\mathbb{F}_3)_2$$
$$\oplus (H^{1,1}(\mathcal{W}_q)_2 \oplus H^{0,2}(\mathcal{W}_q)_2) \otimes H^{0,1}(\mathbb{F}_3)_1.$$

This implies that

$$H^{2,1}(\mathcal{W}_q \times \mathbb{F}_3)_0 = H^{1,1}(\mathcal{W}_q)_2 \otimes H^{1,0}(\mathbb{F}_3)_1 \text{ such that } h^{2,1}(\mathcal{W}_q \times \mathbb{F}_3)_0 = 9.$$

Hence by Proposition 10.3.2 and the fact that we have 6 copies of $\mathcal{C}_q$ with $H^{1,0}(\mathcal{C}_q) = 4$, one obtains the statement. □

Next we show that $\mathcal{Q}$ is a maximal family of Calabi-Yau manifolds. First let us define maximality. For this definition recall:

Proposition 10.3.9. *Each Calabi-Yau manifold X has a local universal deformation $\mathcal{X} \to B$, where*

$$\dim(B) = h^{2,1}(X).$$

Proof. (see [61], 10.3.2) □

Definition 10.3.10. A family $\mathcal{F} \to Y$ of Calabi-Yau manifolds is maximal in $0 \in Y$, if the universal property of the local universal deformation $\mathcal{X} \to B$ of $\mathcal{F}_0$ yields a surjection of a neighborhood of 0 onto B. The family $\mathcal{F} \to Y$ is maximal, if it is maximal in all $0 \in Y$.

Remark 10.3.11. If the family $\mathcal{F} \to Y$ of Calabi-Yau manifolds is maximal in some $0 \in Y$, its restriction to the complement of a closed analytic subvariety of Y is maximal.

Remark 10.3.12. Since $\mathcal{W}_q$ is birationally equivalent to $\mathbb{F}_3 \times \mathcal{C}_q/\langle(1,1)\rangle$ (see Proposition 8.2.4), one has

$$H^{2,0}(\mathcal{W}_q) = H^{1,0}(\mathbb{F}_3)_1 \otimes H^{1,0}(\mathcal{C}_q)_2,$$

where $\mathcal{C}$ denotes the family of degree 3 covers with a pure $(1,3)-VHS$. Thus by our former notation with respect to the push forward action, the VHS of $\mathcal{W}$ depends uniquely on the fractional VHS of the eigenspace $\mathcal{L}_1$ of the VHS of $\mathcal{C}$.

In Section 9.2 we have seen that $\mathcal{Q}$ is birationally equivalent to a quotient of $\mathcal{W} \times \mathbb{F}_3$. It differs by some blowing up morphism with respect to some families of rational curves and some isolated sections. Thus by similar arguments, the VHS of $\mathcal{Q}$ depends on the VHS of $\mathcal{W}$. Hence the VHS of $\mathcal{Q}$ depends uniquely on the fractional VHS of $\mathcal{L}_1$. Thus the period map of $\mathcal{Q}$ can be considered as a multivalued map to the ball $\mathbb{B}_3$.

The preceding remark tells us the period map of the family $\mathcal{Q} \to \mathcal{M}_3$ is locally injective. Hence by the Torelli theorem for Calabi-Yau manifolds, one concludes:

Theorem 10.3.13. *The family $\mathcal{Q} \to \mathcal{M}_3$ is maximal.*

10.4 Outlook onto quotients by cyclic groups of high order

Recall that we used $K3$ surfaces S and elliptic curves E with cyclic degree m covers $S \to R$ and $E \to \mathbb{P}^1$ to construct Calabi-Yau 3-manifolds by a quotient, where $m = 2, 3$. In this chapter we give an outlook on the possibilities to use of cyclic groups of higher order for the construction of Calabi-Yau 3-manifolds by an elliptic curve and a $K3$-surface.

First the following Lemma shows that there are only finitely many elliptic curves with an action of a cyclic group with order $m > 2$, which could be suitable:

Lemma 10.4.1. *Let E be an elliptic curve, and $f : E \to \mathbb{P}^1$ be a cyclic cover. Then one obtains*

$$m := deg(f) = 2, \quad 3, \quad 4 \quad or \quad 6.$$

For each $m > 2$ there is at most only one elliptic curve having a cyclic cover $f : E \to \mathbb{P}^1$ of degree m.[2]

Proof. We use Proposition 2.3.4 and Corollary 2.3.5. Let $f : E \to \mathbb{P}^1$ be be a cyclic cover of degree $m > 2$. Moreover if f has n branch points, then $\mathbb{L}_1$ is of type (p, q) with $p+q = n-2$. Thus there must be at least 2 branch points. If there are only 2 branch points, we are in the case of the cover $\mathbb{P}^1 \to \mathbb{P}^1$ given by $x \to x^m$. Since $\mathbb{L}_1$ is of type (p, q) with $p + q = n - 2$, the curve C can be an elliptic curve for $m > 2$, only if $n = 3$.

For $n = 3$ and $m > 2$ we have that $\mathbb{L}_1$ is of type (p, q) with $p + q = 1$. Without loss of generality we assume that $p = 0$ and $q = 1$. Hence by Proposition 2.3.4, one concludes that

$$\mu_1 + \mu_2 + \mu_3 = 1.$$

If $m = 3$, one has only the case of the Fermat curve of degree 3 given by

$$\mu_1 = \mu_2 = \mu_3 = \frac{1}{3}.$$

If $m > 3$, $\mathbb{L}_2$ must be of type $(0, 0)$, which implies without loss of generality that $\mu_1 = \frac{1}{2}$. Hence for $m = 4$ we have only the case of the cover given by

$$\mu_1 = \frac{1}{2}, \quad \mu_2 = \mu_3 = \frac{1}{4}.$$

---

[2] The well-educated reader knows the automorphism group of the abelian variety given by one elliptic curve. But the quotient map by a cyclic subgroup of this automorphism group is fully ramified at the zero-point. There could be cyclic covers, which are not fully ramified over all branch points. Hence for the proof of this lemma, it is not sufficient to know the automorphism group of this abelian variety.

If $m > 4$, $\mathbb{L}_2$ and $\mathbb{L}_3$ must be of type $(0,0)$, which implies without loss of generality that $\mu_1 = \frac{1}{2}$ and $\mu_2 = \frac{1}{3}$. Hence we obtain the only additional case given by the degree 6 cover with the local monodromy data

$$\mu_1 = \frac{1}{2}, \quad \mu_2 = \frac{1}{3}, \quad \mu_3 = \frac{1}{6}.$$

□

Let S be a $K3$-surface, E be an elliptic curve and the cyclic groups $\langle \gamma_S \rangle$ and $\langle \gamma_E \rangle$ of order $m > 1$ acting on S and E with the ramification loci F_S and F_E such that γ_S and γ_E act by -1 on the global sections of the respective canonical sheaves. The aim is the construction of a Calabi-Yau 3-manifold by a desingularization of $S \times E/\langle(\gamma_S, \gamma_E)\rangle$. The following proposition tells us that there are singularities on $S \times E/\langle(\gamma_S, \gamma_E)\rangle$, if $m > 2$. Thus one has to find a suitable desingularization in these cases.

Proposition 10.4.2. *Let $m > 2$. Then γ_S must have ramification.*

Proof. If γ_S does not have ramification, one concludes by the Hurwitz formula $\varphi_S^*\omega = \mathcal{O}$. Thus the quotient has a canonical sheaf ω with $\omega^{\otimes m} = \mathcal{O}$ for $m > 2$. Moreover it has the Betti number $b_1 = 0$. In addition it must be a minimal model, since a rational -1 curve would lie in the support of the canonical divisor K and forbid any torsion of K. But by the Enriques-Kodaira classification (compare to [6], **VI.**), such a minimal model does not exist. □

In the cases of Calabi-Yau manifolds with degree 3 and 4 automorphisms S. Cynk and K. Hulek [13] have given general methods to obtain Calabi-Yau manifolds in higher dimension. These methods are written down in the following two propositions. Note that in the examples of the constructions of the $CMCY$ families $\mathcal{Q}$ and $\mathcal{R}$ of 3-manifolds we have already used the general method of S. Cynk and K. Hulek for the degree 3 case. In the degree 6 case no method is known to the author.

Proposition 10.4.3. *Let X_1 and X_2 be Calabi-Yau manifolds and ξ be a primitive cubic root of unity. Assume that for $i = 1, 2$ the Calabi-Yau manifold X_i has an automorphism η_i of order 3 such that η_i acts via pullback by the character ξ^i on $H^0(X_i, \omega_{X_i})$. Moreover assume that the fixed point set on X_1 is a smooth divisor and that the fixed point set on X_2 consists of a disjoint union of a smooth divisor and a smooth submanifold of codimension 2. Assume that η_1 is locally given by $(\xi, 1, \ldots, 1)$ near the divisor of fixed points and η_2 is locally given by $(\xi^2, 1, \ldots, 1)$ near the divisor of fixed points and $(\xi, \xi, 1, \ldots, 1)$ near the submanifold of fixed points of codimension 2.*

Then $X_1 \times X_2/\langle(\eta_1, \eta_2)\rangle$ has a resolution X of singularities, which is a Calabi-Yau manifold. The Calabi-Yau manifold X admits an action of $\mathbb{Z}/(3)$, which satisfies the same assumptions as for X_2.

Proof. (see [13], Proposition 3.1) □

Proposition 10.4.4. *Let X_1 and X_2 be Calabi-Yau manifolds. Assume that X_1 has an automorphism η_1 of order 4 such that η_1 acts via pullback by the character i on $H^0(X_1, \omega_{X_1})$ and X_2 has a automorphism η_2 of order 4 such that η_2 acts via pullback by the character $-i$ on $H^0(X_2, \omega_{X_2})$. Moreover assume that the fixed point set on X_1 is a smooth divisor and that the fixed point set on X_2 consists of a disjoint union of smooth submanifolds of codimension one, two or three. Assume that η_1 is locally given by $(i, 1, \ldots, 1)$ near the divisor of fixed points and η_2 is locally given by $(-i, 1, \ldots, 1)$, $(-1, i, 1, \ldots, 1)$ or $(i, i, i, 1, \ldots, 1)$ near the respective submanifolds of fixed points.*

Then $X_1 \times X_2 / \langle(\eta_1, \eta_2)\rangle$ has a resolution X of singularities, which is a Calabi-Yau manifold. The Calabi-Yau manifold X admits an action of $\mathbb{Z}/(4)$, which satisfies the same assumptions as for X_2.

Proof. (see [13], Proposition 4.1) □

Chapter 11
Maximal families of *CMCY* type

In this chapter we use the classification of involutions on $K3$ surfaces S by V. V. Nikulin [51], which act by -1 on $H^0(S, \omega_S)$. If the divisor of fixed points consists at most of rational curves, the Borcea-Voisin construction yields a maximal holomorphic $CMCY$ family of 3-manifolds.

After we have recalled some basic facts in Section 11.1, we define a Shimura datum by using involutions on the integral lattice in Section 11.2. Each of the points of a dense open subset of the bounded symmetric domain obtained from this Shimura datum represents a marked $K3$ surface with involution. By using this fact, we obtain our examples of maximal holomorphic $CMCY$ families of 3-manifolds in Section 11.3. For each $n \in \mathbb{N}$ with $n \leq 11$ there is a holomorphic maximal $CMCY$ family over a basis of dimension n.

11.1 Facts about involutions and quotients of $K3$-surfaces

In this section we collect some known facts about $K3$ surfaces and their involutions, which we will need in the sequel.

11.1.1. The integral cohomology $H^2(S, \mathbb{Z})$ is a lattice of rank 22. We have the cup-product $(\cdot, \cdot)$ on $H^2(S, \mathbb{Z})$, which yields a symmetric bilinear form. Let $L := (H^2(S, \mathbb{Z}), (\cdot, \cdot))$. One has the orthogonal direct sum decomposition

$$L \cong (-E_8) \oplus (-E_8) \oplus H \oplus H \oplus H,$$

where $-E_8$ consists of $\mathbb{Z}^8$ endowed with a negative definite integral bilinear form given by the matrix

J.C. Rohde, *Cyclic Coverings, Calabi-Yau Manifolds and Complex Multiplication*, Lecture Notes in Mathematics 1975, DOI: 10.1007/978-3-642-00639-5_12,

$$\begin{pmatrix} -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

and H denotes the hyperbolic plane, i. e. $H = (\mathbb{Z}^2, \langle \cdot, \cdot \rangle)$, where $\langle \cdot, \cdot \rangle$ is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(see [6], **VIII**. Proposition 3.3 and the notation in [6], **VIII**. Section 1 and also [6], **I**. Examples 2.7 for details).

Remark 11.1.2. Let S be a $K3$-surface and $L = H^2(S, \mathbb{Z})$, where L is endowed with an involution ι. Assume that ι corresponds to an involution on S, which acts by the character -1 on $\Gamma(\omega_S)$. Then the involution induces a degree 2 cover $\gamma : S \to R$ onto a smooth surface R. Moreover the divisor of fixed points, which yields the ramification divisor of γ, consists of a disjoint union of smooth curves or it is the zero-divisor. The involution ι yields integral sub-Hodge structures $H^2(S, \mathbb{Z})_0$ and $H^2(S, \mathbb{Z})_1$ of $H^2(S, \mathbb{Z})$ such that ι acts by $(-1)^i$ on $H^2(S, \mathbb{Z})_i$. Since ι acts by -1 on $\Gamma(\omega_S)$ and

$$H^2(R, \mathbb{Q}) = H^2(S, \mathbb{Q})_0,$$

one has that

$$H^{2,0}(S), H^{0,2}(S) \subset H^2(S, \mathbb{C})_1.$$

Moreover the intersection form has the signature $(2, r)$ on $H^2(S, \mathbb{Q})_1$ (compare to [60], §1 and [60], 2.1).

Remark 11.1.3. Let

$$D = \{[\omega] \in \mathbb{P}(H^2(S, \mathbb{C})_1) | (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0\}.$$

By the Torelli theorem, each marked $K3$ surface $(S', \phi_{S'})$ endowed with an involution, which yields the the same involution ι on his cohomology lattice, yields a unique one dimensional vector space $H^{2,0}(S') \subset H^2(S, \mathbb{C})_1$ corresponding to some $p \in D$.

11.2 The associated Shimura datum

The Hodge structure of a $K3$ surface S with a cyclic degree 2 cover onto a rational surface resp., Enriques surface R has a decomposition into two rational Hodge structures $H^2(S,\mathbb{Q})_1$ and $H^2(S,\mathbb{Q})_0$. We consider $H^2(S,\mathbb{Q})_1$, since the variation of Hodge structures given by $H^2(S,\mathbb{Q})_0$ is trivial.

The Hodge decomposition of $H^2(S,\mathbb{C})$ is orthogonal with respect to the Hermitian form $(\cdot,\bar{\cdot})$. Therefore the corresponding embedding

$$h : S^1 \to \mathrm{SL}(H^2(S,\mathbb{R})_1)$$

factors trough the special orthogonal group $\mathrm{SO}(H^2(S,\mathbb{R})_1)$ with respect to the symmetric form given by the cup product pairing, where $\mathrm{SO}(H^2(S,\mathbb{R})_1)$ is isomorphic to $\mathrm{SO}(2,r)_{\mathbb{R}}$. Let $\omega \in \omega_S \setminus \{0\}$,

$$\Re\omega := \frac{1}{2}(\omega+\bar{\omega}), \quad \Im\omega := \frac{i}{2}(\omega-\bar{\omega})$$

and $\{v_1,\dots v_r\}$ be a basis of $H^{1,1}(X,\mathbb{R})_1$. One has the basis

$$\{\Re\omega, \Im\omega, v_1,\dots,v_r\}$$

of $H^1(X,\mathbb{R})_1$ such that the intersection form is without loss of generality given by the matrix $\mathrm{diag}(1,1,-1,\dots,-1)$ with respect to this basis. The subgroup, whose elements are invariant under

$$g \to h(i)gh(i^{-1}),$$

is given by $\mathrm{S}(\mathrm{O}(2)\times\mathrm{O}(r))$, where

$$h(i) = h(i^{-1}) = \mathrm{diag}(-1,-1,1,\dots,1).$$

Since $h^2(i) = h(-1) = \mathrm{diag}(1,\dots,1)$, the action of i is an involution. This implies that one has a decomposition of $\mathfrak{so}_{2,r}(\mathbb{R})$ into 2 eigenspaces with respect to the eigenvalues 1 and -1. Hence $h(\sqrt{i})$ yields a complex structure on the eigenspace with eigenvalue -1. The eigenspace for the eigenvalue 1 is given by the Lie algebra of $\mathrm{S}(\mathrm{O}(2)\times\mathrm{O}(r))$. Thus we have a decomposition

$$\mathfrak{so}_{2,r}(\mathbb{C}) = \mathfrak{h}_+ \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_-$$

such that S^1 acts by the characters $z/\bar{z}$, 1 and $\bar{z}/z$ on the respective complex sub-vector spaces.

We continue our consideration of the involution ι given by

$$\iota(g) = h(i)gh^{-1}(i).$$

The matrices $M_1 \in \mathrm{SO}(2,r)(\mathbb{C})$ with $\bar{M}_1 = \iota(M_1)$ satisfy that

$$\bar{M}_1 = \mathrm{diag}(-1,-1,1,\ldots,1) \cdot M_1 \cdot \mathrm{diag}(-1,-1,1,\ldots,1)$$

$$= \mathrm{diag}(1,1,-1,\ldots,-1) \cdot M_1 \cdot \mathrm{diag}(1,1,-1,\ldots,-1).$$

Since $\mathrm{SO}(2,r)(\mathbb{C})$ is given by the matrices M satisfying

$$M^t \cdot \mathrm{diag}(1,1,-1,\ldots,-1) \cdot M = \mathrm{diag}(1,1,-1,\ldots,-1)$$

$$\Leftrightarrow M^{-1} = \mathrm{diag}(1,1,-1,\ldots,-1) \cdot M^t \cdot \mathrm{diag}(1,1,-1,\ldots,-1),$$

each matrix M_1 satisfies

$$M_1^{-1} = \bar{M}_1^t.$$

Thus M_1 is contained in the compact group $\mathrm{SU}(2+r)$, and one concludes:

Proposition 11.2.1. *Our morphism*

$$h : S^1 \to \mathrm{SO}(H^2(S,\mathbb{Q})_1)_{\mathbb{R}}$$

yields a Shimura datum.[1]

Remark 11.2.2. Note that the simple Lie group $\mathrm{SO}(2,r)(\mathbb{R})$ consists of two connected components (see [21], Exercise 7.2). Since the Lie group $\mathrm{SO}(2+r)(\mathbb{C}) \cong \mathrm{SO}(H^2(S,\mathbb{R})_1)(\mathbb{C})$ is connected (see [27], **IX**. Lemma 4.2), the algebraic group $\mathrm{SO}(H^2(S,\mathbb{R})_1)$ is connected, too. Recall that all Cartan involutions of the simple algebraic group $\mathrm{SO}(H^2(S,\mathbb{R})_1)$ are conjugate. The action of S^1 on $H^2(S,\mathbb{R})_1$ is given by its action on $\langle \Re\omega, \Im\omega \rangle$ and S^1 fixes all vectors of $H^{1,1}(S,\mathbb{R})_1$. This implies that all morphisms

$$h : S^1 \to \mathrm{SO}(H^2(S,\mathbb{R})_1),$$

which yields the Hodge structure of a $K3$ surface, satisfy that their images $h(S^1)$ are conjugate. The definition of the Hodge structure on $H^2(S,\mathbb{R})_1$ implies that the $\mathbb{R}$-valued points of the kernel of h are given by $\{1,-1\} \in S^1(\mathbb{R})$. Let $\iota_{S^1} : S^1 \to S^1$ be the involution given by $x \to x^{-1}$. For each morphism h_1 in the conjugacy class of h, there exists exactly one other morphism h_2

[1] This Shimura datum is not a Shimura datum in the sense of Definition 1.3.22. Let $\mathrm{GO}(H^2(S,\mathbb{Q})_1)$ denote the general orthogonal group, which preserves the symmetric bilinear form up to a scalar. The Hodge structure defines a corresponding homomorphism

$$h : \mathbb{S} \to \mathrm{GO}(H^2(S,\mathbb{Q})_1)_{\mathbb{R}},$$

which is a Shimura datum in the sense of Definition 1.3.22 and whose restriction to S^1 is the morphism of the proposition. By arguments analogous to the arguments in Remark 1.4.13, we can consider this restricted morphism as Shimura datum, too.

with $h_1(S^1) = h_2(S^1)$ and kernel given by $\{1, -1\} \in S^1(\mathbb{R})$, which is given by $h_2 = h_1 \circ \iota_{S^1}$. The conjugation by $\mathrm{diag}(-1, 1, -1, 1, \ldots, 1)$ yields an inner automorphism φ of $\mathrm{SO}(H^2(S, \mathbb{R})_1)$ such that $h_2 = \varphi \circ h_1$. Thus each Hodge structure of a $K3$ surface obtained by some $p \in D$ is obtained by some element of the conjugacy class of our morphism $h : S^1 \to \mathrm{SO}(H^2(S, \mathbb{R})_1)$. Moreover note that the holomorphic VHS over the bounded symmetric domain associated with $\mathrm{SO}(H^2(S, \mathbb{R})_1)(\mathbb{R})^+/K$, which is induced by the natural embedding $\mathrm{SO}(H^2(S, \mathbb{Q})_1) \to \mathrm{GL}(H^2(S, \mathbb{Q})_1)$, is uniquely determined by the variation of the subbundle of rank 1 given by $H^{2,0}$. Since

$$r = \dim(D) = \dim(\mathrm{SO}(H^2(S, \mathbb{R})_1)(\mathbb{R})/K),$$

this VHS yields a biholomorphic map from the bounded symmetric domain associated with $\mathrm{SO}(H^2(S, \mathbb{R})_1)(\mathbb{R})^+/K)$ onto D^+.

The preceding remark and Theorem 1.7.2 imply:

Theorem 11.2.3. *There is a dense set of CM points on D with respect to the VHS on D obtained by Remark* 11.2.2.

11.3 The examples

First we construct a holomorphic family of marked $K3$-surfaces with a global involution over its basis:

Construction 11.3.1. There exists a universal family $u : \mathcal{X} \to B$ of marked analytic $K3$-surfaces, whose basis is not Hausdorff (see [6], **VIII**. Section 12). Let ϕ denote the global marking of the family $\mathcal{X} \to B$. We consider an involution ι on a marked $K3$ surface (S, ϕ), which acts by -1 on $H^{2,0}(S)$. This involution yields an involutive isometry ι on the lattice L. Thus the involution ι endows $\mathcal{X} \to B$ with a new marking $\iota \circ \phi$. By the universal property of the universal family, this new marking yields an involution of the family:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\iota_{\mathcal{X}}"] \arrow[d, "u"'] & \mathcal{X} \arrow[d, "u"] \\ B \arrow[r, "\iota_B"] & B \end{tikzcd}$$

Let $\Delta : B \to B \times B$ denote the diagonal embedding. We define

$$B_\iota = \mathrm{Graph}(\iota_B) \cap \Delta(B) \subset B \times B.$$

Note that each point $b \in B_\iota$ has an analytic neighborhood $U \subset B$ such that $\mathcal{X}_U \to U$ is given by the Kuranishi family and yields an injective period map for U. Thus on $U \times U$ the diagonal $\Delta(U)$ and $\mathrm{Graph}(\iota_B|_U)$ are closed analytic submanifolds. Hence B_ι has the structure of an analytic variety, which is not necessarily Hausdorff, and can have singularities. The composition $\Delta \circ u$ allows to consider $\Delta(B)$ as basis of the universal family of the marked $K3$ surfaces. By the restricted family $\mathcal{X}_{B_\iota} \to B_\iota$, we obtain a holomorphic family with a global involution over the basis B_ι. For simplicity we write $\mathcal{X}_\iota \to B_\iota$ instead of $\mathcal{X}_{B_\iota} \to B_\iota$.

Remark 11.3.2. The fibers of $\mathcal{X}_\iota \to B_\iota$ have by the involution ι a cyclic covering onto a projective surface (compare to [60], 2.1). Thus the fibers of $\mathcal{X}_\iota \to B_\iota$ are algebraic.

Proposition 11.3.3. *Assume that for all $b \in B_\iota$ the involution $\iota_{\mathcal{X}_b}$ on $\mathcal{X}_b$ has a locus of fixed points consisting of rational curves. Then the holomorphic family $\mathcal{X}_\iota \to B_\iota$ is due to its global involution suitable for the construction of a holomorphic Borcea-Voisin tower.*

Proof. Let $b_0 \in B_\iota$ and $U \subset B_\iota$ be a small open neighborhood of b_0. The eigenspace decomposition with respect to ι yields a variation of Hodge structures on the eigenspace with respect to -1. The corresponding period map yields an open injection of U into D. By the fact that D has a dense set of CM points, the family $\mathcal{X}_\iota \to B_\iota$ has a dense set of CM fibers. Since the locus of fixed points with respect to $\iota_{\mathcal{X}_b}$ consists of rational curves, this locus of fixed points has complex multiplication, too. Hence $\mathcal{X}_\iota \to B_\iota$ can be used for the construction of a holomorphic Borcea-Voisin tower. □

Assume that $\mathcal{X}_\iota \to B_\iota$ satisfies the assumptions of Proposition 11.3.3. Then let $\mathfrak{X}_\iota \to B_\iota \times \mathcal{M}_1$ denote the family obtained by the holomorphic Borcea-Voisin tower from $\mathcal{X}_\iota \to B_\iota$ and $\mathcal{E} \to \mathcal{M}_1$ denote the family of elliptic curves.

Definition 11.3.4. A family $\mathcal{F} \to Y$ of Calabi-Yau manifolds is maximal in $0 \in Y$, if the universal property of the local universal deformation $\mathcal{X} \to B$ of $\mathcal{F}_0$ yields a surjection of a neighborhood of 0 onto B. The family $\mathcal{F} \to Y$ is maximal, if it is maximal in all $0 \in Y$.

Theorem 11.3.5. *The family $\mathfrak{X}_\iota$ is maximal.*

Proof. By the following lemma, we start to prove Theorem 11.3.5:

Lemma 11.3.6.

$$H^3((\mathfrak{X}_\iota)_{p\times q}) = H^2((\mathcal{X}_\iota)_p, \mathbb{Q})_1 \otimes H^1(\mathcal{E}_q, \mathbb{Q})$$

Proof. Due to Proposition 10.3.2 and the fact that the exceptional divisors consist of some rational curves, one only needs to determine $H^3((\mathcal{X}_\iota)_p \times \mathcal{E}_q, \mathbb{Q})_0$. Since $b_1((\mathcal{X}_\iota)_p) = b_3((\mathcal{X}_\iota)_p) = 0$ and $H^1(\mathcal{E}_q, \mathbb{Q}) = H^1(\mathcal{E}_q, \mathbb{Q})_1$, we are done. □

By using the preceding lemma, we prove the following proposition.

Proposition 11.3.7. *One has that* $\dim(B_\iota \times \mathbb{B}_1)$ *and* $h^{2,1}((\mathfrak{X}_\iota)_{p\times q})$ *coincide.*

Proof. By Proposition 11.3.6,

$$H^3((\mathfrak{X}_\iota)_{p\times q}) = H^2((\mathcal{X}_\iota)_p, \mathbb{Q})_1 \otimes H^{1,0}(\mathcal{E}_q, \mathbb{Q}) \oplus H^2((\mathcal{X}_\iota)_p, \mathbb{Q})_1 \otimes H^{0,1}(\mathcal{E}_q, \mathbb{Q}).$$

Therefore

$$\begin{aligned} h^{2,1}((\mathfrak{X}_\iota)_{p\times q}) &= h^{1,1}((\mathcal{X}_\iota)_p, \mathbb{Q})_1 \cdot h^{1,0}(\mathcal{E}_q, \mathbb{Q}) + h^{2,0}((\mathcal{X}_\iota)_p, \mathbb{Q})_1 \cdot h^{0,1}(\mathcal{E}_q, \mathbb{Q}) \\ &= h^{1,1}((\mathcal{X}_\iota)_p, \mathbb{Q})_1 + h^{2,0}((\mathcal{X}_\iota)_p, \mathbb{Q})_1 = h^{1,1}((\mathcal{X}_\iota)_p, \mathbb{Q})_1 + 1. \end{aligned}$$

Recall that D^+ is the bounded symmetric domain obtained by $\mathrm{SO}(2,r)^+(\mathbb{R})$, where $r = h^{1,1}((\mathcal{X}_\iota)_p, \mathbb{R})_1$. By [27], **IX**. Table **II**, the domain D has the complex dimension r.[2] Since the period map $p : B_\iota \to D$ of $\mathcal{X}_\iota \to B_\iota$ is locally bijective, one concludes

$$h^{1,1}((\mathcal{X}_\iota)_p, \mathbb{Q})_1 = r = \dim(D) = \dim(B_\iota),$$

which yields the result. □

By the following proposition, we finish the proof of Theorem 11.3.5: □

Proposition 11.3.8. *The period map yields a multivalued map from* $\mathcal{M}_1 \times B_\iota$ *to the period domain, which is locally injective.*

Proof. Let $\mathbb{B}$ be a small open subset of $\mathcal{M}_1 \times B_\iota$ and let $x_1, x_2 \in \mathbb{B}$. Note that the period map p on $\mathcal{M}_1 \times B_\iota$ yields different image points $p(x_1)$ and $p(x_2)$, if the classes of $H^{3,0}((\mathfrak{X}_\iota)_{x_1})$ and $H^{3,0}((\mathfrak{X}_\iota)_{x_2})$ in $\mathbb{P}(H^3((\mathfrak{X}_\iota)_{x_1}, \mathbb{C}))$ do not coincide. The respective period maps on B_ι and $\mathcal{M}_1$ are locally injective and depend only on $\omega_{\mathcal{E}_q}$ and $\omega_{(\mathcal{X}_\iota)_p}$. Since

$$H^{3,0}((\mathfrak{X}_\iota)_{p\times q}) \subset H^3((\mathfrak{X}_\iota)_{p\times q}) = H^2((\mathcal{X}_\iota)_p, \mathbb{Q})_1 \otimes H^1(\mathcal{E}_q, \mathbb{Q})$$

is given by $H^{2,0}((\mathcal{X}_\iota)_p) \otimes H^{1,0}(\mathcal{E}_q)$, the period map concerning $\mathfrak{X}_\iota$ is locally injective, too. □

It remains to classify the involutions ι on L, which provide our families $\mathcal{X}_\iota \to B_\iota$ with a global involution.

Remark 11.3.9. The involutions on L, which yields involutions on certain $K3$ surfaces, are characterized by the triples of the following integers (compare to [51]):

- The integer t is the rank of the sublattice $\mathrm{Pic}(S)_0$ of the Picard lattice of an arbitrary fiber S of $\mathcal{X}_\iota$, which is invariant under the global involution.

[2] By [27], **IX**. Table **II**, the domain D has the dimension $2r$ as real manifold.

- By the intersection pairing, one obtains a homomorphism $\mathrm{Pic}(S)_0 \to \mathrm{Pic}(S)_0^\vee$. The integer a is given by

$$(\mathbb{Z}/(2))^a \cong \mathrm{Pic}(S)_0^\vee/\mathrm{Pic}(S)_0.$$

- By the morphism $\mathrm{Pic}(S)_0 \to \mathrm{Pic}(S)_0^\vee$, the intersection form on $\mathrm{Pic}(S)_0$ yields a quadratic form q on $\mathrm{Pic}(S)_0^\vee$ with values in $\mathbb{Q}$. The integer δ is 0, if q has only values in $\mathbb{Z}$ and 1 otherwise.

For a fixed triple (t, a, δ) we write $\mathcal{X}_{(t,a,\delta)} \to B_{(t,a,\delta)}$ instead of $\mathcal{X}_\iota \to B_\iota$ and $\mathfrak{X}_{(t,a,\delta)}$ instead of $\mathfrak{X}_\iota$.

Remark 11.3.10. The ramification locus of the fibers with respect to the involution on $\mathcal{X} \to \mathcal{B}_{t,a,\delta}$ is given by two elliptic curves, if $(t, a, \delta) = (10, 8, 0)$, is empty, if $(t, a, \delta) = (10, 10, 0)$, and otherwise given by $C_{N'} + E_1 + \ldots + E_{N-1}$, where $C_{N'}$ is a curve of genus

$$N' = \frac{1}{2}(22 - t - a), \quad \text{and} \quad N = \frac{1}{2}(t - a) + 1.$$

(compare to [51]).

Therefore the triples

$$(t, a, \delta) = (10, 10, 0) \quad \text{and} \quad (t, a, \delta) \quad \text{with} \quad t + a = 22$$

yield the examples of families $\mathcal{X}_{(t,a,\delta)} \to B_{(t,a,\delta)}$ with global involutions over the basis, whose locus of fixed points consists at most of families of rational curves. Hence by Proposition 11.3.3, these triples yield maximal holomorphic $CMCY$ families of 3-manifolds.

11.3.11. By [51], Figure 2, one gets the following complete list of holomorphic maximal $CMCY$ families $\mathfrak{X}_{(t,a,\delta)} \to B_{(t,a,\delta)} \times \mathcal{M}_1$ of 3-manifolds obtained by this method. By Claim 10.2.1, we obtain the Hodge numbers $h^{1,1}$ and $h^{2,1}$ of the fibers of $\mathfrak{X}_{(t,a,\delta)}$.

t	a	δ	N	$h^{1,1}$	$h^{2,1}$
10	10	0	0	11	11
11	11	1	1	16	10
12	10	1	2	21	9
13	9	1	3	26	8
14	8	1	4	31	7
15	7	1	5	36	6
16	6	1	6	41	5
17	5	1	7	46	4
18	4	1	8	51	3
18	4	0	8	51	3
19	3	1	9	56	2
20	2	1	10	61	1

Remark 11.3.12. C. Borcea [8] has constructed Calabi-Yau manifolds of dimension 3 with CM by using 3 elliptic curves with involutions. This construction yields a $CMCY$ family of 3-manifolds over $\mathcal{M}_1 \times \mathcal{M}_1 \times \mathcal{M}_1$. The fibers have the Hodge numbers $h^{1,1} = 51$ and $h^{2,1} = 3$. By similar arguments as in Theorem 11.3.5, this family is maximal. The associated period domain is given by $\mathbb{B}_1 \times \mathbb{B}_1 \times \mathbb{B}_1$.

As we have seen in Section 10.3, the family $\mathcal{Q} \to \mathcal{M}_3$ is a maximal $CMCY$ family of 3-manifolds, whose fibers have the same Hodge numbers $h^{1,1} = 51$ and $h^{2,1} = 3$. The associated period domain is given by $\mathbb{B}_3$.

Moreover by Theorem 11.3.5 and the preceding point, we have two additional holomorphic maximal $CMCY$ families of 3-manifolds, whose fibers have the same Hodge numbers $h^{1,1} = 51$ and $h^{2,1} = 3$. The associated period domain is given by $\mathbb{B}_1 \times D$, where D denotes the bounded domain given by $\mathrm{SO}(2,2)(\mathbb{R})/K$.

Hence there exist 4 maximal $CMCY$ families of 3-manifolds, whose fibers have the Hodge numbers $h^{1,1} = 51$ and $h^{2,1} = 3$. One can easily check that the example of [8] has a Yukawa coupling of *length* 3, where the Yukawa coupling of the family $\mathcal{Q} \to \mathcal{M}_3$ constructed in Section 9.2 has the *length* 1. Hence there are not any open sets of the respective bases, which allow a local identification of these two families.

By using the involutions on elliptic curves, one gets a local identification between $\mathcal{E} \times \mathcal{E}/\langle(\iota_{\mathcal{E}}, \iota_{\mathcal{E}})\rangle \to \mathcal{M}_1 \times \mathcal{M}_1$, which yields the example of [8], with one of our examples $\mathcal{X}_{(t,a,\delta)} \to B_{(t,a,\delta)}$ with $t = 18$ and $a = 4$. This implies a local identification between the resulting $CMCY$ families of 3-manifolds obtained by the Borcea-Voisin tower.

Remark 11.3.13. By Example 7.4.5, there are 13 explicit examples of elliptic curves with CM. Thus for the $CMCY$ family of C. Borcea [8], which we have discussed in the preceding remark, one obtains up to birational equivalence 455 different examples of CM fibers. For 6 of these 13 elliptic curves, we have an explicitly given involution. Thus we can at least describe the 56 Calabi-Yau 3-manifolds, which are obtained by some of the latter 6 elliptic curves, by local equations.

Remark 11.3.14. It would be interesting to consider the following question: Is the maximal $CMCY$ family $\mathfrak{X}_{(10,10,0)}$ its own mirror family?

Let S denote a $K3$ surface with an involution, which acts by -1 on $\Gamma(\omega_S)$. In [60] the triples (t, a, δ), which yield our families $\mathcal{X}_{(t,a,\delta)} \to B_{(t,a,\delta)}$ satisfying the assumptions of Proposition 11.3.3, do not satisfy the assumptions of the technical Lemma [60], Lemme 2.5. This Lemma guarantees the existence of a hyperbolic plane $H \subset H^2(S,\mathbb{Z})_1$, which is needed for the mirror construction in [60]. Hence these triples (t, a, δ) do not satisfy the assumptions of the Mirror Theorem [60], Théorème 2.17. But by [12], Lemma 4.4.4, there is a hyperbolic plane $H \subset H^2(S,\mathbb{Z})_1$ for these triples. Moreover by [6], **VIII**. 19, one has a description of the corresponding involution on the cohomology lattice, which yields the existence of a hyperbolic plane $H \subset H^2(S,\mathbb{Z})_1$.

In her construction of a Calabi-Yau 3-manifold ([60], Lemme 1.3) C. Voisin assumes that the involution on the $K3$ surface is not given by the triple $(10, 10, 0)$, since it is easy to see that the resulting 3-manifold is not simply connected in this case. But by Proposition 7.2.5 the resulting 3-manifold satisfies our definition of a Calabi-Yau manifold (Definition 7.2.1) in this case, too.

The mirror of a fiber of $\mathfrak{X}_{(10,10,0)}$ must have the same Hodge numbers $h^{1,1} = h^{2,1} = 11$. By Claim 10.2.1, this implies for an involution on a $K3$ surface:

$$5N - N' = 5N' - N = 0$$

Hence one calculates easily that $N = N' = 0$. Thus by V. V. Nikulins [51] classification of involutions on $K3$ surfaces, the Voisin-Borcea Mirror (in the notation of [12]) of a fiber of $\mathfrak{X}_{(10,10,0)}$ should be obtained by the triple $(10, 10, 0)$, too. Hence the author has the impression that one can consider the maximal $CMCY$ family $\mathfrak{X}_{(10,10,0)}$ of 3-manifolds as its own mirror family, but one must check the details.

Appendix A
Examples of Calabi-Yau 3-manifolds with complex multiplication

Introduction

The previous examples of Calabi-Yau manifolds with CM occur as fibers of a family over a Shimura variety, which has a dense set of complex multiplication fibers. Here we give some examples, which are not necessarily fibers of a non-trivial family with a dense set of complex multiplication fibers.

The first two sections give two different classes of examples by using involutions on $K3$ surfaces. In each of the both Sections we will use a modified version of the construction of Viehweg and Zuo to obtain $K3$ surfaces, which are suitable for the construction of a Borcea-Voisin tower.

In the third section we will prove that a $K3$ surface with a degree 3 automorphism has complex multiplication. By using methods, which has been introduced in Section 9.1 and Section 9.2, we will use this automorphism and the Fermat curve of degree 3 for the construction of a Calabi-Yau 3-manifold with complex multiplication.

A.1 Construction by degree 2 coverings of a ruled surface

We start by finding curves with complex multiplication. The following proposition yields some examples:

Proposition A.1.1. *Let $0 < d_1, d < m$, and ξ_k denote a primitive k-th. root of unity for all $k \in \mathbb{N}$. Then the curve C, which is locally given by*

$$y^m = x^{d_1} \prod_{i=1}^{n-2} (x - \xi_{n-2}^i)^d,$$

is covered by the Fermat curve $\mathbb{F}_{(n-2)m}$ *locally given by*

$$y^{(n-2)m} + x^{(n-2)m} + 1 = 0$$

and has complex multiplication.

Proof. (see Theorem 2.4.4) □

Example A.1.2. By the preceding proposition, the curves locally given by

$$y^4 = x_1^8 + x_0^8, \quad y^4 = x_1(x_1^7 + x_0^7), \quad y^4 = x_1(x_1^6 + x_0^6)x_0$$

have complex multiplication. These curves are degree 4 covers of the projective line and have the genus 9 as one can easily calculate by the Hurwitz formula.

The curves of the preceding example have a natural interpretation as cyclic covers of $\mathbb{P}^1$ of degree 4. One can identify these covers with the set of their 8 branch points in $\mathbb{P}^1$. Thus let $\mathcal{P}_8$ denote the configuration space of 8 different points in $\mathbb{P}^1$. We use a modified version of the construction in [58], Section 5 to construct $K3$ surfaces with complex multiplication by Example $A.1.2$ in a first step. This method is nearly the same method as in Section 8.2.

For our application, it is sufficient to work with $\mathbb{P}^1$-bundles over $\mathbb{P}^1$ resp., with rational ruled surfaces. Let $\pi_n : \mathbb{P}_n \to \mathbb{P}^1$ denote the rational ruled surface given by $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ and σ denote a non-trivial global section of $\mathcal{O}_{\mathbb{P}^1}(8)$, which has the 8 different zero points represented by a point $q \in \mathcal{P}_8$. The sections E_σ, E_0 and E_∞ of $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(8))$ are induced by

$$\mathrm{id} \oplus \sigma : \mathcal{O} \to \mathcal{O} \oplus \mathcal{O}(8), \quad \mathrm{id} \oplus 0 : \mathcal{O} \to \mathcal{O} \oplus \mathcal{O}(8)$$
$$\text{and} \quad 0 \oplus \mathrm{id} : \mathcal{O}(8) \to \mathcal{O} \oplus \mathcal{O}(8)$$

resp., by the corresponding surjections onto the cokernels of these embeddings as described in [26], **II**. Proposition 7.12.

Remark A.1.3. The divisors E_σ and E_0 intersect each other transversally over the 8 zero points of σ. Recall that $\mathrm{Pic}(\mathbb{P}_8)$ has a basis given by a fiber and an arbitrary section. Hence by the fact that E_σ and E_0 do not intersect E_∞, one concludes that they are linearly equivalent with self-intersection number 8. Since E_∞ is a section, it intersects each fiber transversally. Thus one has that $E_\infty \sim E_0 - (E_0.E_0)F$, where F denotes a fiber. Therefore one concludes

$$E_\infty.E_\infty = E_\infty.(E_0 - (E_0.E_0)F) = -(E_0.E_0) = -8.$$

Next we establish a morphism $\mu : \mathbb{P}_2 \to \mathbb{P}_8$ over $\mathbb{P}^1$. By [26], **II**. Proposition 7.12., this is the same as to give a surjection $\pi_2^*(\mathcal{O} \oplus \mathcal{O}(8)) \to \mathcal{L}$, where $\mathcal{L}$ is an invertible sheaf on $\mathbb{P}_2$. By the composition

$$\pi_2^*(\mathcal{O}\oplus\mathcal{O}(8)) = \pi_2^*(\mathcal{O})\oplus\pi_2^*\mathcal{O}(8) \hookrightarrow \bigoplus_{i=0}^{4} \pi_2^*\mathcal{O}(2i) = Sym^4(\pi_2^*(\mathcal{O}\oplus\mathcal{O}(8))) \to \mathcal{O}_{\mathbb{P}_2}(4),$$

where the last morphism is induced by the natural surjection $\pi_2^*(\mathcal{O}\oplus\mathcal{O}(2)) \to \mathcal{O}_{\mathbb{P}_2}(1)$ (see [26], **II**. Proposition 7.11), we obtain a morphism μ^* of sheaves. This morphism μ^* is not a surjection onto $\mathcal{O}_{\mathbb{P}_2}(4)$, but onto its image $\mathcal{L} \subset \mathcal{O}_{\mathbb{P}_2}(4)$. Over $\mathbb{A}^1 \subset \mathbb{P}^1$ all rational ruled surfaces are locally given by $\mathrm{Proj}(\mathbb{C}[x])[y_1, y_2]$, where x has the weight 0. Hence we have locally that $\pi_2^*(\mathcal{O} \oplus \mathcal{O}(8)) = \mathcal{O}e_1 \oplus \mathcal{O}e_2$. Over $\mathbb{A}^1$ the morphism μ^* is given by

$$e_1 \to y_1^4, e_2 \to y_2^4$$

such that the sheaf $\mathcal{L} = im(\mu^*) \subset \mathcal{O}_{\mathbb{P}_2}(4)$ is invertible. Thus the morphism $\mu : \mathbb{P}_2 \to \mathbb{P}_8$ corresponding to μ^* is locally given by the ring homomorphism

$$(\mathbb{C}[x])[y_1, y_2] \to (\mathbb{C}[x])[y_1, y_2] \ \text{ via } \ y_1 \to y_1^8 \ \text{ and } \ y_2 \to y_2^8.$$

Construction A.1.4. One has a commutative diagram

$$\begin{tikzcd}
\mathcal{Y}' \arrow[r, "\tau'"] & \mathbb{P}_2' \arrow[r, "\mu'"] & \mathbb{P}^1 \times \mathbb{P}^1 \\
\hat{\mathcal{Y}} \arrow[u, "\delta"] \arrow[r, "\hat{\tau}"] \arrow[d, "\rho"'] & \hat{\mathbb{P}}_2 \arrow[u, "\delta_2"'] \arrow[r, "\hat{\mu}"] \arrow[d, "\rho_2"] & \hat{\mathbb{P}}_8 \arrow[u, "\delta_8"'] \arrow[d, "\rho_8"] \\
\mathcal{Y} \arrow[r, "\tau", "\sqrt[2]{\frac{\mu^* E_\sigma}{3\cdot(\mu^* E_0)_{red}}}"'] \arrow[d, "\pi"'] & \mathbb{P}_2 \arrow[r, "\mu", "\sqrt[4]{\frac{E_\infty+8\cdot F}{E_0}}"'] \arrow[d, "\pi_2"] & \mathbb{P}_8 \arrow[d, "\pi_8"] \\
\mathbb{P}^1 \arrow[r, "\mathrm{id}"] & \mathbb{P}^1 \arrow[r, "\mathrm{id}"] & \mathbb{P}^1
\end{tikzcd}$$

of morphisms between normal varieties with:

(a) δ, δ_2, δ_8, ρ, ρ_2 and ρ_8 are birational.
(b) π is a family of curves, π_2 and π_8 are $\mathbb{P}^1$-bundles.

Proof. One must only explain δ_8 and ρ_8. Recall that E_σ is a section of $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(8))$, which intersects E_0 transversally in exactly 8 points. The morphism ρ_8 is the blowing up of the 8 intersection points of $E_0 \cap E_\sigma$. The preimage of the 8 points given by $q \in \mathcal{P}_8$ with respect to $\pi_8 \circ \rho_8$ consists of the exceptional divisor $\hat{D}_1$ and the proper transform $\tilde{D}_2$ of the preimage of these 8 points with respect to ρ_8 given by 8 rational curves with self-intersection number -1. The morphism δ_8 is obtained by blowing down $\tilde{D}_2$. $\square$

Remark A.1.5. The section σ has the zero divisor given by some $q \in \mathcal{P}_8$. Hence one obtains $\mu^*(E_\sigma) \cong C$, where $C \to \mathbb{P}^1$ is a cyclic cover of degree 4 as in Example A.1.2 ramified over the 8 points given by σ. The surface $\mathcal{Y}$ is a cyclic degree 2 cover of $\mathbb{P}_2$ ramified over C. Thus it has an involution. It is given by the invertible sheaf

$$\mathcal{L} = \omega_{\mathbb{P}_2}^{-1}$$

and the divisor

$$B = \mu^*(E_\sigma), \quad \text{where} \quad \mathcal{O}(B) \cong \mathcal{L}^2,$$

with the notation of [6] **I**. Section 17. Thus [6] **I**. Lemma 17.1 implies that $\mathcal{Y}$ is a $K3$ surface.

By Lemma 10.4.1, there is only one elliptic curve with a cyclic degree 4 cover onto $\mathbb{P}^1$. Let $\mathbb{E}$ denote this curve, which is locally given by

$$y^4 = x(x-1)^2.$$

One can easily see that $\mathbb{E}$ has the j invariant 1728. Thus $\mathbb{E}$ has complex multiplication.

We fix some notation. Let $n \in \mathbb{N}$, let ξ be a fixed primitive n-th. root of unity and let C_1 and C_2 be curves locally given by

$$y^n = f_1(x) \quad \text{and} \quad y^n = f_2(x),$$

where $f_1, f_2 \in \mathbb{C}[x]$. By $(x,y) \to (x, \xi y)$, one can define an automorphism γ_i on C_i for $i = 1, 2$. The surface $C_1 \times C_2/\langle(1,1)\rangle$ is the quotient of $C_1 \times C_2$ by $\langle(\gamma_1, \gamma_2)\rangle$.

Proposition A.1.6. *The surface $\mathcal{Y}$ is birationally equivalent to $C \times \mathbb{E}/\langle(1,1)\rangle$.*[1]

Proof. Let $\tilde{E}_\bullet$ denote the proper transform of the section $E_\bullet$ with respect to ρ_8. Then $\hat{\mu}$ is the Kummer covering given by

$$\sqrt[4]{\frac{\tilde{E}_\infty + 8 \cdot F}{\tilde{E}_0 + \hat{D}_1}},$$

where $\hat{D}_1$ denotes the exceptional divisor of ρ_8. Thus the morphism μ' is the Kummer covering

$$\sqrt[4]{\frac{(\delta_8)_*\tilde{E}_\infty + 8 \cdot (\delta_8)_* F}{(\delta_8)_*\tilde{E}_0 + (\delta_8)_*\hat{D}_1}} = \sqrt[4]{\frac{\mathbb{P}^1 \times \{\infty\} + 8 \cdot (P \times \mathbb{P}^1)}{\mathbb{P}^1 \times \{0\} + \Delta \times \mathbb{P}^1}},$$

[1] Similarly to [58], Construction 5.2, we show that $\mathcal{Y}'$ is birationally equivalent to $C \times \mathbb{E}/\langle(1,1)\rangle$.

where Δ is the divisor of the 8 different points in $\mathbb{P}^1$ given by $q \in \mathcal{P}_8$ and $P \in \mathbb{P}^1$ is the point with the fiber F. Since $E_0 + E_\sigma$ is a normal crossing divisor, $\tilde{E}_\sigma$ neither meets $\tilde{E}_0$ nor $\tilde{D}_2$, where $\tilde{D}_2$ is the proper transform of $\pi_8^*(\Delta)$. Therefore $(\delta_8)_*\tilde{E}_\sigma$ neither meets

$$(\delta_8)_*\tilde{E}_0 = \mathbb{P}^1 \times \{0\} \quad \text{nor} \quad (\delta_8)_*\tilde{E}_\infty = \mathbb{P}^1 \times \{\infty\}.$$

Hence one can choose coordinates in $\mathbb{P}^1$ such that $(\delta_8)_*\tilde{E}_\sigma = \mathbb{P}^1 \times \{1\}$.

By the definition of τ, we obtain that $\hat{\tau}$ is given by

$$\sqrt[2]{\frac{\rho_2^*\mu^*(E_\sigma)}{\rho_2^*\mu^*(E_0)}} = \sqrt[2]{\frac{\hat{\mu}^*(\tilde{E}_\sigma)}{\hat{\mu}^*(\tilde{E}_0)}},$$

and τ' is given by

$$\sqrt[2]{\frac{\mu'^*(\mathbb{P}^1 \times \{1\})}{\mu'^*(\mathbb{P}^1 \times \{0\})}}.$$

By the fact that the last function is the root of the pullback of a function on $\mathbb{P}^1 \times \mathbb{P}^1$ with respect to μ', it is possible to reverse the order of the field extensions corresponding to τ' and μ' such that the resulting varieties obtained by Kummer coverings are birationally equivalent. Hence we have the composition of $\beta : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ given by

$$\sqrt[2]{\frac{\mathbb{P}^1 \times \{1\}}{\mathbb{P}^1 \times \{0\}}}$$

with

$$\sqrt[4]{\frac{\beta^*(\mathbb{P}^1 \times \{\infty\}) + 8 \cdot (P \times \mathbb{P}^1)}{\beta^*(\mathbb{P}^1 \times \{0\}) + (\Delta \times \mathbb{P}^1)}},$$

which yields the covering variety isomorphic to $\mathbb{E} \times C/\langle(1,1)\rangle$. □

As in Section 8.2 we conclude:

Corollary A.1.7. *If the curve C has complex multiplication, the $K3$-surface $\mathcal{Y}$ has complex multiplication, too.*

By the the preceding corollary, our Example $A.1.2$ yields 3 different $K3$ surfaces with complex multiplication locally given by

$$y_2^2 + y_1^4 + x_1^8 + x_0^8, \quad y_2^2 + y_1^4 + x_1(x_1^7 + x_0^7), \quad y_2^2 + y_1^4 + x_1(x_1^6 + x_0^6)x_0.$$

Proposition A.1.8. *For $i = 1, 2$ assume that C_i is a Calabi-Yau i-manifold with complex multiplication endowed with the involution ι_i such that ι_i acts by -1 on $\Gamma(\omega_{C_i})$. By blowing up the singular locus of $C_1 \times C_2/\langle(\iota_1, \iota_2)\rangle$, one obtains a Calabi-Yau 3-manifold with complex multiplication.*

Proof. It is well-known that an involution on a Calabi-Yau 2-manifold resp., a $K3$ surface, which acts by -1 on $\Gamma(\omega)$, has a smooth divisor of fixed points or it has not any fixed point. Thus the proof follows from the same methods as in Section 7.2. □

Now we need some elliptic curves with complex multiplication:

Example A.1.9. Elliptic curves with CM has been well studied by number theorists. Some examples of elliptic curves with complex multiplication are given by the following list:

equation	j invariant
$y_1^2 x_0 = x_1^3 - x_0^3$	0
$y_1^2 x_0 = x_1(x_1 - x_0)(x_1 - 2x_0)$	1728
$y^2 x_0 = x_1(x_1 - x_0)(x_1 - (1+\sqrt{2})^2 x_0)$	8000
$y^2 x_0 = x_1(x_1 - x_0)(x_1 - \frac{1}{4}(3+i\sqrt{7})^2 x_0)$	−3375
$y^2 x_0 = x_1^3 - 15x_1 x_0^2 + 22x_0^3$	54000
$y^2 x_0 = x^3 - 595x_1 x_0^2 + 5586x_0^3$	16581375

Note that the equations allow an explicit definition of an involution on these elliptic curves. (see Section 7.4)

A.1.10. By combining our 3 examples of $K3$ surfaces and the 6 elliptic curves and using Proposition $A.1.8$, we have 18 examples of Calabi-Yau 3-manifolds with complex multiplication. By [60], one has equations to determine the Hodge numbers of these examples. Let C_2 be a $K3$ surface satisfying the assumptions of Proposition $A.1.8$, let N be the number of curves in the ramification locus of the quotient map $C_2 \to C_2/\iota_2$ and let N' be given by

$$N' = g_1 + \ldots + g_N,$$

where g_i denotes the genus of the i-th. curve in the ramification locus. Then one has for the Calabi-Yau 3-manifold, which results by Proposition $A.1.8$:

$$h^{1,1} = 11 + 5N - N'$$

$$h^{2,1} = 11 + 5N' - N$$

In our case the ramification locus of $C_2 \to C_2/\iota_2$ is given by one genus 9 curve. Thus in our case the Hodge numbers are given by

$$h^{1,1} = 7 \text{ and } h^{2,1} = 55.$$

A.2 Construction by degree 2 coverings of $\mathbb{P}^2$

Example A.2.1. By Proposition $A.1.1$, the projective curves given by

$$y^6 = x_1^6 + x_0^6, \quad y^6 = x_1(x_1^5 + x_0^5), \quad y^6 = x_1(x_1^4 + x_0^4)x_0$$

have complex multiplication. These curves have the genus 10 as one can easily calculate by the Hurwitz formula.

Let $\mathcal{P}_6$ denote the configuration space of 6 different points in $\mathbb{P}^1$. Again we use a modified version of the construction in [58], Section 5. Let σ denote a non-trivial global section of $\mathcal{O}_{\mathbb{P}^1}(6)$, which has the 6 different zero points represented by a point $q \in \mathcal{P}_6$.

Here the sections E_σ, E_0 and E_∞ of $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(6))$ are induced by

$$\mathrm{id} \oplus \sigma : \mathcal{O} \to \mathcal{O} \oplus \mathcal{O}(6), \quad \mathrm{id} \oplus 0 : \mathcal{O} \to \mathcal{O} \oplus \mathcal{O}(6)$$
$$\text{and} \quad 0 \oplus \mathrm{id} : \mathcal{O}(6) \to \mathcal{O} \oplus \mathcal{O}(6)$$

resp., by the corresponding surjections onto the cokernels of these embeddings as described in [26], **II**. Proposition 7.12.

One concludes similarly to the preceding section that

$$E_\infty.E_\infty = E_\infty.(E_0 - (E_0.E_0)F) = -(E_0.E_0) = -6.$$

By the composition

$$\pi_1^*(\mathcal{O} \oplus \mathcal{O}(6)) = \pi_1^*(\mathcal{O}) \oplus \pi_1^*\mathcal{O}(6) \hookrightarrow \bigoplus_{i=0}^{6} \pi_1^*\mathcal{O}(i) = Sym^6(\pi_1^*(\mathcal{O} \oplus \mathcal{O}(6))) \to \mathcal{O}_{\mathbb{P}_1}(6),$$

where the last morphism is induced by the natural surjection $\pi_2^*(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathcal{O}_{\mathbb{P}_1}(1)$ (see [26], **II**. Proposition 7.11), we obtain a morphism μ^* of sheaves as in the preceding section. The morphism $\mu : \mathbb{P}_1 \to \mathbb{P}_6$ corresponding to μ^* is locally given by the ring homomorphism

$$(\mathbb{C}[x])[y_1, y_2] \to (\mathbb{C}[x])[y_1, y_2] \quad \text{via} \quad y_1 \to y_1^6 \quad \text{and} \quad y_2 \to y_2^6.$$

Construction A.2.2. One has a commutative diagram

$$\begin{tikzcd}
\mathcal{Y}' \arrow[r, "\tau'"] & \mathbb{P}_1' \arrow[r, "\mu'"] & \mathbb{P}^1 \times \mathbb{P}^1 \\
\hat{\mathcal{Y}} \arrow[u, "\delta"] \arrow[r, "\hat{\tau}"] \arrow[d, "\rho"'] & \hat{\mathbb{P}}_1 \arrow[u, "\delta_1"'] \arrow[r, "\hat{\mu}"] \arrow[d, "\rho_1"] & \hat{\mathbb{P}}_6 \arrow[u, "\delta_6"'] \arrow[d, "\rho_6"] \\
\mathcal{Y} \arrow[r, "\tau", "\sqrt[2]{\frac{\mu^* E_\sigma}{3\cdot(\mu^* E_0)_{red}}}"'] \arrow[d, "\pi"'] & \mathbb{P}_1 \arrow[r, "\mu", "\sqrt[6]{\frac{E_\infty + 6\cdot F}{E_0}}"'] \arrow[d, "\pi_1"] & \mathbb{P}_6 \arrow[d, "\pi_6"] \\
\mathbb{P}^1 \arrow[r, "\mathrm{id}"] & \mathbb{P}^1 \arrow[r, "\mathrm{id}"] & \mathbb{P}^1
\end{tikzcd}$$

of morphisms between normal varieties with:

(a) δ, δ_1, δ_1, ρ, ρ_1 and ρ_6 are birational.
(b) π is a family of curves, π_1 and π_6 are $\mathbb{P}^1$-bundles.

Proof. One must only explain δ_6 and ρ_6. These morphisms are given by blowing up morphisms similar to the preceding section. □

Remark A.2.3. The section σ has the zero divisor given by some $q \in \mathcal{P}_6$. Hence one obtains $\mu^*(E_\sigma) \cong C$, where $C \to \mathbb{P}^1$ is a cyclic cover of degree 6 as in Example A.2.1 ramified over the 6 points given by σ. The surface $\mathcal{Y}$ is a cyclic degree 2 cover of $\mathbb{P}_1$ ramified over C. Thus it is birationally equivalent to the $K3$ surface given the degree 2 cover of $\mathbb{P}^2$ ramified over C.

Let C' denote the projective smooth curve locally given by

$$y^6 = x(x-1).$$

By Proposition A.1.1, it has complex multiplication.

Proposition A.2.4. *The surface $\mathcal{Y}$ is birationally equivalent to $C \times C'/\langle(1,1)\rangle$.*

Proof. Let $\tilde{E}_\bullet$ denote the proper transform of the section $E_\bullet$ with respect to ρ_6. Then $\hat{\mu}$ is the Kummer covering given by

$$\sqrt[6]{\frac{\tilde{E}_\infty + 6 \cdot F}{\tilde{E}_0 + \hat{D}_1}},$$

where $\hat{D}_1$ denotes the exceptional divisor of ρ_6. Thus the morphism μ' is the Kummer covering

$$\sqrt[6]{\frac{(\delta_6)_*\tilde{E}_\infty + 6 \cdot (\delta_6)_* F}{(\delta_6)_*\tilde{E}_0 + (\delta_6)_*\hat{D}_1}} = \sqrt[6]{\frac{\mathbb{P}^1 \times \{\infty\} + 6 \cdot (P \times \mathbb{P}^1)}{\mathbb{P}^1 \times \{0\} + \Delta \times \mathbb{P}^1}},$$

where Δ is the divisor of the 6 different points in $\mathbb{P}^1$ given by $q \in \mathcal{P}_6$ and $P \in \mathbb{P}^1$ is the point with the fiber F. Since $E_0 + E_\sigma$ is a normal crossing divisor, $\tilde{E}_\sigma$ neither meets $\tilde{E}_0$ nor $\tilde{D}_2$, where $\tilde{D}_2$ is the proper transform of $\pi_6^*(\Delta)$. Therefore $(\delta_6)_*\tilde{E}_\sigma$ neither meets

$$(\delta_6)_*\tilde{E}_0 = \mathbb{P}^1 \times \{0\} \text{ nor } (\delta_6)_*\tilde{E}_\infty = \mathbb{P}^1 \times \{\infty\}.$$

Hence one can choose coordinates in $\mathbb{P}^1$ such that $(\delta_6)_*\tilde{E}_\sigma = \mathbb{P}^1 \times \{1\}$.

By the definition of τ, we obtain that $\hat{\tau}$ is given by

$$\sqrt[2]{\frac{\rho_1^*\mu^*(E_\sigma)}{\rho_1^*\mu^*(E_0)}} = \sqrt[2]{\frac{\hat{\mu}^*(\tilde{E}_\sigma)}{\hat{\mu}^*(\tilde{E}_0)}},$$

and τ' is given by

$$\sqrt[2]{\frac{\mu'^*(\mathbb{P}^1 \times \{1\})}{\mu'^*(\mathbb{P}^1 \times \{0\})}}.$$

By the fact that the last function is the root of the pullback of a function on $\mathbb{P}^1 \times \mathbb{P}^1$ with respect to μ', it is possible to reverse the order of the field extensions corresponding to τ' and μ' such that the resulting varieties obtained by Kummer coverings are birationally equivalent. Hence we have the composition of $\beta : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ given by

$$\sqrt[2]{\frac{\mathbb{P}^1 \times \{1\}}{\mathbb{P}^1 \times \{0\}}}$$

with

$$\sqrt[6]{\frac{\beta^*(\mathbb{P}^1 \times \{\infty\}) + 6 \cdot (P \times \mathbb{P}^1)}{\beta^*(\mathbb{P}^1 \times \{0\}) + (\Delta \times \mathbb{P}^1)}},$$

which yields the covering variety isomorphic to $C' \times C/\langle(1,1)\rangle$. □

Hence $\mathcal{Y}$ is birationally equivalent to $C' \times C/\langle(1,1)\rangle$. As in Section 8.2 we conclude:

Corollary A.2.5. *If the curve C has complex multiplication, the $K3$-surface $\mathcal{Y}$ has complex multiplication, too.*

A.2.6. By the preceding corollary, our Example $A.2.1$ yields 3 different $K3$ surfaces with complex multiplication as degree 2 covers of $\mathbb{P}^2$, which are locally given by

$$y_2^2 + y_1^6 = x_1^6 + x_0^6, \quad y_2^2 + y_1^6 = x_1(x_1^5 + x_0^5), \quad y_2^2 + y_1^6 = x_1(x_1^4 + x_0^4)x_0.$$

By an elliptic curve with complex multiplication, these $K3$ surfaces yield Calabi-Yau 3-manifolds with complex multiplication. We obtain 18 Calabi-Yau 3-manifolds with complex multiplication by using Example $A.1.9$. By the same methods as in $A.1.10$, one calculates easily that the resulting Calabi-Yau 3-manifolds have the Hodge numbers

$$h^{1,1} = 6 \text{ and } h^{2,1} = 60.$$

A.3 Construction by a degree 3 quotient

Consider the $K3$ surface

$$S = V((y_2^3 - y_1^3)y_1 + (x_1^3 - x_0^3)x_0) \subset \mathbb{P}^3.$$

By using the partial derivatives of the defining equation, one can easily verify that S is smooth. First we prove that this surface has complex multiplication. In a second step we consider an automorphism of degree 3 on this surface, which allows the construction of a Calabi-Yau 3-manifold with complex multiplication.

Proposition A.3.1. *The $K3$ surface S has complex multiplication.*

Proof. Consider the isomorphic curves

$$C_1 = V(z_1^4 - (y_2^3 - y_1^3)y_1) \subset \mathbb{P}^2,$$

$$C_2 = V(z_2^4 - (x_1^3 - x_0^3)x_0) \subset \mathbb{P}^2.$$

Since the elliptic curve with j invariant 0 given by

$$V(y^2x_0 + x_1^3 + x_0^3) \subset \mathbb{P}^2$$

has complex multiplication, one concludes as in Remark 7.4.2 that C_1 and C_2 have complex multiplication, too. The $K3$ surface S is birationally equivalent to

$$T = C_1 \times C_2/\langle(1,1)\rangle.$$

This follows from the rational map $C_1 \times C_2 \to S$ given by

$$((z_1 : y_2 : y_1), (z_2 : x_1 : x_0)) \to (\frac{z_2}{z_1}y_2 : \frac{z_2}{z_1}y_1 : x_1 : x_0).$$[2]

There exists a suitable sequence of blowing ups turning $C_1 \times C_2$ into $\widetilde{C_1 \times C_2}$ such that

$$\tilde{T} = \widetilde{C_1 \times C_2}/\langle(1,1)\rangle$$

is smooth. Since we only blow up points, $\widetilde{C_1 \times C_2}$ has CM, too (see Corollary 7.1.6). Thus the quotient has CM. Since $\tilde{T}$ is birationally equivalent to S, there exists a sequence of blowing ups of smooth points and blowing downs to smooth points, which turns $\tilde{T}$ into S. By Corollary 7.1.6, the fact that $\tilde{T}$ has CM implies that S has CM. □

A.3.2. Let ξ denote $e^{\frac{2\pi i}{3}}$. The $K3$ surface S has an automorphism γ of degree 3 given by

$$(y_2 : y_1 : x_1 : x_0) \to (\xi y_2 : y_1 : \xi x_1 : x_0).$$

On $\{x_0 = 1\}$ we have the 4 fixed points given by

$$(0 : \sqrt[4]{-1} : 0 : 1).$$

[2] In [9], Section 5 one finds a similar rational map.

Now assume $x_0 = 0$. By the equation of S, this yields

$$(y_2^3 - y_1^3)y_1 = 0.$$

Thus in addition the line given by $y_1 = x_0 = 0$ is fixed.

Proposition A.3.3. *The automorphism γ acts via pullback by ξ^2 on $\Gamma(\omega_S)$.*

Proof. By the multiplication of i with z_1 and z_2, one defines an action of the group of the 4-th. roots of unity on the curves C_1 and C_2 given by

$$V(z_1^4 - (y_2^3 - y_1^3)y_1) \subset \mathbb{P}^2 \quad \text{and} \quad C_2 = V(z_2^4 - (x_1^3 - x_0^3)x_0) \subset \mathbb{P}^2.$$

The -1 eigenspace in $\Gamma(\omega_{C_1})$ and $\Gamma(\omega_{C_2})$ with respect to the action of i comes from the cohomology of the elliptic curve E_0 given by

$$y^2 x_0 = x_1^3 - x_0^3$$

(see Section 4.2). Note that the action of $\langle(1,1)\rangle$ on $\omega_{C_1\times C_2}$ fixes exactly the tensor product of the -1 eigenspaces in $\Gamma(\omega_{C_1})$ and $\Gamma(\omega_{C_2})$. Thus one concludes that $\Gamma(\omega_S)$ is given by the tensor product of the -1 eigenspaces in $\Gamma(\omega_{C_1})$ and $\Gamma(\omega_{C_2})$.

The automorphism $\gamma_{\mathbb{F}_3} : E_0 \to E_0$ given by $x_1 \to \xi x_1$ is the generator of the Galois group of the degree 3 cover, which allows an identification of E_0 with the Fermat curve $\mathbb{F}_3$ of degree 3. It acts via pullback by ξ on $\Gamma(\omega_{\mathbb{F}_3})$. Thus the corresponding automorphisms $\varphi_{C_1} : C_1 \to C_1$ and $\varphi_{C_2} : C_2 \to C_2$ act by ξ on the -1 eigenspace with respect to $H^0(\omega_{C_1})$ and $H^0(\omega_{C_2})$. Note that $(\varphi_{C_1}, \varphi_{C_2})$ yields an automorphism of $C_1 \times C_2/\langle(1,1)\rangle$. By the birational map to S, this automorphism corresponds to γ and one verifies easily that γ acts via pullback by ξ^2 on $\Gamma(\omega_S)$. $\square$

A.3.4. Consider the blowing up $\tilde{\mathbb{P}}^3$ of $\mathbb{P}^3$ with respect to $\{y_2 = x_1 = 0\}$. Let $\tilde{S}$ denote the proper transform of the blowing up of S with respect to the latter blowing up, which has the exceptional divisor E consisting of four -1 curves over the 4 points given by $(0 : \sqrt[4]{-1} : 0 : 1)$. Consider the projection

$$p : S\backslash\{y_2 = x_1 = 0\} \hookrightarrow \mathbb{P}^3\backslash\{y_2 = x_1 = 0\} \to \mathbb{P}^1 \text{ given by } (y_2 : y_1 : x_1 : x_0) \to (y_2 : x_1).$$

Over $\{x_0 = 1\}$ one has an embedding of an open subset of $\tilde{\mathbb{P}}^3$ into $\mathbb{P}^1 \times \mathbb{A}^3$, which yields an open embedding e of an open subset U of $\tilde{S}$ into $\mathbb{P}^1 \times \mathbb{A}^3$. Note that $\mathbb{P}^1 \times \mathbb{A}^3$ is endowed with a natural projection $pr_1 : \mathbb{P}^1 \times \mathbb{A}^3 \to \mathbb{P}^1$. Over $U \setminus \{y_2 = x_1 = 0\}$ one has

$$p = pr_1 \circ e.$$

Thus by gluing, p extends to a morphism $\tilde{S} \to \mathbb{P}^1$, which is a family of projective curves of degree 4. This family has a section $D = \{y_1 = x_0 = 0\}$.

One checks easily the singular loci of the fibers do not meet D (since $y_2 \neq 0$ or $x_1 \neq 0$). By $\tilde{S} \times \mathbb{F}_3 \to \mathbb{P}^1$, we have a family of surfaces. Let denote the generator of the Galois group of $\mathbb{F}_3 \to \mathbb{P}^1$, which acts via pullback by ξ on $\omega_{\mathbb{F}_3}$. The quotient map onto $\tilde{S} \times \mathbb{F}_3/\langle(\gamma, \gamma_{\mathbb{F}_3})\rangle$ yields three quotient singularities of type $A_{3,2}$ with the notation of [6], **III**. Section 5. As in Section 9.2 described one must blow up the three corresponding sections obtained from D and in a second step one blows up the fixed locus of the exceptional divisor over D. Now we blow down the image of the proper transform of the exceptional divisor over D and obtain the orbifold X_1. Note that the exceptional divisor E of the blowing up $\tilde{S} \to S$ and the 3 points on $\mathbb{F}_3$ fixed by $\gamma_{\mathbb{F}_3}$ yield a singular locus consisting of 12 curves.

On $S \times \mathbb{F}_3$ we blow up the 12 points given by the product of $\{(0 : \sqrt[4]{-1} : 0 : 1)\}$ with the three points fixed by $\gamma_{\mathbb{F}_3}$. Since $(\gamma, \gamma_{\mathbb{F}_3})$ acts by ξ on all local parameters of each of these points, the exceptional divisor over these points is contained in the ramification locus of the quotient map onto

$$X_2 = \widetilde{S \times \mathbb{F}_3}/\langle(\gamma, \gamma_{\mathbb{F}_3})\rangle.$$

X_2 is a orbifold with three $A_{3,2}$ singularities obtained from D. By gluing the complements of the singular loci of X_1 and X_2, one obtains a Calabi-Yau 3-manifold X. By the same arguments as in Section 9.2, the Calabi-Yau manifold X has obviously complex multiplication.

Thus the Calabi-Yau manifold X is obtained by the method of S. Cynk and K. Hulek [13], which we have written down in Proposition 10.4.3.

A.3.5. For the computation of the Hodge numbers we use the same methods as in Section 10.3. During Section 10.3 these methods are explained in-depth. The automorphism γ of S acts on $\tilde{S}$, too. The quotient map φ onto $M = \tilde{S}/\gamma$ is ramified over E and $D = \{y_1 = x_0 = 0\}$. Since D is a rational curve on a $K3$ surface, the adjunction formula implies that $D.D = -2$. By the Hurwitz formula, one has

$$\varphi^* K_M \sim -2D - E.$$

Since

$$3 \cdot K_M^2 = (\varphi^* K_M)^2,$$

one concludes that

$$c_1(M)^2 = K_M^2 = -4.$$

Thus the Noether formula

$$\chi(\mathcal{O}_M) = \frac{1}{12}(c_1(M)^2 + c_2(M)) \quad \text{and} \quad c_2(M) - 2 = b_2(M)$$

tell us that $b_2(M) = 14$. Since we have blown up 4 points, one obtains $h_0^{1,1}(S) = 10$. Thus

$$h_1^{1,1}(S) = h_2^{1,1}(S) = 5.$$

By the fact that one has an exceptional divisor consisting of 12 copies of $\mathbb{P}^2$ and 6 rational ruled surfaces and $b_1(S) = 0$, one obtains as in Section 10.3:

$$h^{1,1}(X) = h_0^{0,0}(\mathbb{F}_3) \cdot h_0^{1,1}(S) + h_0^{0,0}(S) \cdot h_0^{1,1}(\mathbb{F}_3) + 18 = 10 + 1 + 18 = 29$$

$$h^{2,1}(X) = h_1^{1,0}(\mathbb{F}_3) \cdot h_2^{1,1}(S) = 5$$

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