Foundations and Trends® in
Accounting
1:1 (2006)

Earnings, Earnings Growth and Value

James Ohlson and Zhan Gao

now
the essence of knowledge

Earnings, Earnings Growth and Value

Earnings, Earnings Growth and Value

James Ohlson

Zhan Gao

William P. Carey School of Business
Arizona State University
Tempe, AZ 85287-3606
USA

the essence of knowledge

Boston – Delft

Foundations and Trends® in Accounting

Published, sold and distributed by:
now Publishers Inc.
PO Box 1024
Hanover, MA 02339
USA
Tel. +1-781-985-4510
www.nowpublishers.com
sales@nowpublishers.com

Outside North America:
now Publishers Inc.
PO Box 179
2600 AD Delft
The Netherlands
Tel. +31-6-51115274

A Cataloging-in-Publication record is available from the Library of Congress

The preferred citation for this publication is J. Ohlson and Z. Gao, Earnings, Earnings Growth and Value, Foundation and Trends® in Accounting, vol. 1, no. 1, pp. 1–70, 2006

Printed on acid-free paper

ISBN: 1-933019-88-3

Foundations and Trends® in
Accounting
Volume 1 Issue 1, 2006

Editorial Scope

Foundations and Trends® in Accounting will publish survey and tutorial articles in the following topics:

- Auditing
- Corporate Governance
- Cost Management
- Disclosure
- Event Studies/Market Efficiency Studies
- Executive Compensation
- Financial Reporting
- Financial Statement Analysis and Equity Valuation
- Management Control
- Performance Measurement
- Taxation

Information for Librarians

Foundations and Trends® in Accounting, 2006, Volume 1, 4 issues. ISSN paper version 1554-0642. ISSN online version 1554-0650. Also available as a combined paper and online subscription.

Foundations and Trends® in
Accounting
Vol. 1, No. 1 (2006) 1–70

DOI: 10.1561/1400000001

the essence of knowledge

Earnings, Earnings Growth and Value

James Ohlson and Zhan Gao

William P. Carey School of Business, Arizona State University, Tempe, AZ 85287-3606, USA

Abstract

A recent paper by Ohlson and Juettner-Nauroth (2005) develops a model in which a firm's expected earnings and their growth determine its value. At least on its surface, the model appeals because it embeds the core principle used in investment practice and, further, generalizes the Constant Growth model (Gordon and Williams) without restricting the firm's dividend policy. This text reviews the valuation model and its properties. It also extends previous results by analyzing a number of issues not adequately covered in the original paper. These topics include the precise nature of dividend policy irrelevancy, how the model relates to other well-known valuation models, the role of accounting principles, and how it can be developed on the basis of an underlying information dynamics. A central result shows why the model should be accorded "benchmark" status.

Contents

1

Introduction

Equity valuation in practice relies on an easy-to-state principle: As a first-cut, the price to forward-earnings ratio should relate positively to the subsequent growth in expected earnings. The claim is readily appreciated if one simply reviews financial media, such as Barron's, or summaries of financial analysts' reports. Still, in spite of the principle's centrality in investment practice, textbooks of equity valuation often allocate most of their space to what appears to be competing valuation methodologies, namely, the Free Cash Flows model and the Residual Income Valuation (RIV) model. But textbooks do leave some space for the first-cut investment practice principle. In deference to the apparent need for a model that embodies the first-cut investment principle they provide the so-called Constant Growth model (often attributed to Gordon or Williams), which assumes, of course, a constant dividend to earnings payout ratio and a constant growth for the two variables. The setup guarantees, in a crude way, that the growth in expected earnings relates positively to the price to forward-earnings ratio. However, the model's earnings-construct fails the smell-test because it introduces earnings via an arbitrary rescaling of dividends. Such a model runs at stark cross-purposes with the Miller and Modigliani concept of dividend policy irrelevancy, not to speak of empirical realities. Due to these limitations, a more appealing model could potentially be beneficial to investment practice and research. "Is there a better way to model earnings and dividends that captures the principle of equity valuation?" becomes the obvious question.

A recent paper by Ohlson and Juettner-Nauroth (2005) develops a model of earnings and dividends leading up to the core principle that growth in earnings explains the price to forward-earnings ratio. We will refer to this model as the OJ model. The OJ model takes into account two growth measures of earnings – the near term and the long term – to explain the price to forward-earnings ratio. Further, the model allows for a broad set of dividend policies: The model does not rely on a dividend payout parameter, and it permits, for example, zero expected dividends for any number of future periods. The paper shows that the Constant Growth model obtains as a special case. On the surface at least, the OJ model would seem to be a worthwhile generalization of the Constant Growth model.

This paper revisits the OJ model. We start from basics and derive the valuation formula which shows how value depends on earnings and their growth. An extensive examination of the formula's properties follows. The remainder of the work addresses the many subtle issues which the original paper either treats too crudely, incompletely, or not at all. Each of the topics enhances an understanding of how the model deals with various aspects of accounting and economics. We also provide a message that concerns the uniqueness of the model. Broadly speaking, we will argue that no model other than the OJ model can parsimoniously explain the price to forward-earnings ratio in terms of growth in earnings (given that value also equals the present value of expected dividends). Thus the OJ model extends the model of value that disregards the issue of growth, i.e., the so-called "permanent earnings" model in which next period's expected earnings capitalized, by itself, determines value. In other words, the analysis here speaks to the question: "How do we move from a model of next-period earnings capitalization to a simple model that admits growth in earnings without putting a burden on the dividend policy?"

Among the topics not (adequately) covered by the original paper which we develop here are the following: dividend policy irrelevancy (DPI) and its central role in the model; properties of the primitive variable "x_t" and reasons why it makes sense to label it earnings; how one extends the model to incorporate an underlying information dynamic in the spirit of Ohlson (1995); accounting rules and their influence on the model; the ways in which the model can be extended to reflect operating vs. financial activities much like Feltham and Ohlson (1995).

Aside from the original OJ paper, and its companion Ohlson (2005), the analysis draws on Christensen and Feltham (2003), Fairfield (1994), Feltham and Ohlson (1995), Ohlson (1995, 1999a, 1999b, 2005), Ohlson et al. (2006), Ohlson and Zhang (1999), Olsson (2005), Ozair (2003), Penman (2005, 2006), Ryan (1986), Sougiannis and Yaekura (2001), and Yee (2005, 2006).

Finally, we should note here that this paper will not discuss many empirical papers that have looked at, or used, the OJ model and similar valuation formulas (e.g., Botosan and Plumlee, 2005, Begley and Feltham, 2002, Cheng, 2005, Cheng et al., 2006, Daske, 2006, Easton, 2004, Easton, 2006, Easton and Monahan, 2005, Easton et al., 2002, Francis et al., 2004, Gebhardt et al., 2001, Gode and Mohanram, 2003, Hutton, 2000, Ohlson, 2001, Thomas and Zhang, 2006). Our sole interest pertains to the conceptual underpinnings and implications of the model. These aspects, we believe, are of sufficient interest although there will always be numerous questions related to how the model holds up in empirical and practical applications.

2

The OJ Model: An Overview

Before dealing with the logical exercises, triplets of assumption-derivation-conclusion, it helps to have a general understanding of the OJ model's properties. In this section, we summarize these properties to provide a better feel for what the model accomplishes and how it aligns with common-sense aspects of equity valuation in practice and textbooks. An overview of the model also facilitates an appreciation of how it can potentially address research questions about cost of equity capital, analyst earnings forecasts, market efficiency, etc.

- The OJ valuation formula, which thus identifies a firm's equity value, depends on four variables: (i) next year's (FY1) expected earnings (forward earnings); (ii) short-term growth in expected earnings, FY2 vs. FY1; (iii) long-term, or the asymptotic, growth in expected earnings; and (iv) the discount factor, or the cost of equity capital. The availability of analysts' earnings forecasts makes it easy to apply the model for any cost of capital specification.
- The OJ formula always conforms to the idea that value should be equivalent to the present value of future expected

dividends. Yet the model does not depend on specific dividend policies.

- Both measures of the growth in expected earnings have a positive influence on the price to forward-earnings ratio.
- The price to forward-earnings ratio can be relatively large (40, say) and exceed the inverse of the cost of equity capital.
- The near term growth in expected earnings might well exceed the cost of equity capital.
- The underlying accounting in the model must be conservative in the sense that, on average, market values exceed book values.
- In a "reverse engineering" of the OJ formula, one can express the cost of equity capital as a function of forward-earnings yield and the two growth measures of expected earnings. Thus one infers a firm's cost of equity capital from price and analysts' forecasts. Because any measurement of a firm's cost of equity capital reflects many aspects of the valuation environment, the model generates a number of researchable questions:

  (a) Does a relatively large inferred cost of equity capital correlate positively with measures of risk such as beta, the variance of the market return, leverage, etc?
  (b) Does a relatively large inferred cost of equity capital correlate positively with analysts' earnings forecasts being too optimistic?
  (c) Does a relatively large inferred cost of equity capital correlate positively with near term downward revisions in analysts' forecasts of FY1 earnings?
  (d) Does a relatively large inferred cost of equity capital correlate positively with a firm being relatively overvalued?

- Well-known valuation models turn out to be special cases of the OJ model. With added structure one derives: (i) the Market-to-Book model based on constant growth in residual earnings, and (ii) the Free Cash Flow model when free cash flows grow at a constant rate.

- The model bears on what information explains the unexpected market return. Aside from unexpected earnings, two variables pick up information about subsequent expected earnings and their subsequent growth.
- Standard assumptions distinguishing between operating vs. financial activities fit into the framework.

3

Basics of the OJ Model

3.1 Broad setup

The following notation will be used throughout:

$p_0 =$ Price (or value) of equity at date zero (today),
$x_t =$ Expected earnings for period t given today's information,
$d_t =$ Expected dividends at date t given today's information.
$R = 1 + r =$ the discount factor, i.e., $r =$ the cost of equity capital.

We view these variables as being on a per-share basis. To keep matters simple, we assume that there is only one share outstanding at all points in time (for sure); thus (p, x, d) also represent total dollar values. In this spirit we also assume that the firm has only one owner at all points in time so that d_t can be negative as well as positive. In other words, we view dividends as being dividends net of capital contributions on a market value basis. These assumptions can be relaxed, but we stick to them to avoid complications related to potential transfers of wealth across different classes of future and existing shareholders due to expected capital transactions.

Some settings invoke the Clean Surplus Relation (CSR); this requires additional notation:

b_t = Expected book value at date t, given today's information,
$x_t^a = x_t - r \cdot b_{t-1}$ = Expected residual, or abnormal, earnings for period t, given today's information.

Throughout the analysis, value, or price, equals the present value of expected dividends, or PVED for short:

$$p_0 = \sum_{t=1}^{\infty} R^{-t} d_t, \tag{PVED}$$

It is understood that a firm's risk and risk-free rate influence the discount factor R. A well-known result states that the expected market return equals r, i.e., $E_t[\Delta\tilde{p}_{t+1} + \tilde{d}_{t+1}]/p_t = r$, assuming R is fixed across dates. In a standard neo-classical framework the expected return should reflect risk as well as the time value of money. That said, the PVED modeling here leaves out the nature of risk and how it affects the discount factor. We treat R as an unexplained and exogenous constant.[1] Because of the lack of economics concerning the discount factor, one can usefully think of it as simply being the internal rate of return that equates PVED to an observed price. Taking this perspective allows us to think of R in concrete terms, yet we can be agnostic about the influence of risk on the value of a firm's equity.

Next, putting accounting or economics aside, consider the following algebraic *zero-sum equality*:

$$\begin{aligned} 0 &= y_0 + R^{-1}(y_1 - Ry_0) + R^{-2}(y_2 - Ry_1) + \cdots \\ &= y_0 + \sum_{t=1}^{\infty} R^{-t}(y_t - Ry_{t-1}). \end{aligned} \tag{3.1}$$

Expression (3.1) holds for *any* sequence $\{y_t\}_{t=0}^{\infty}$ as long as it satisfies the transversality condition $\lim_{t\to\infty} R^{-t} y_t = 0$. Though (3.1) seems trivial,

[1] The assumption of an inter-temporally constant R is obviously one of (analytical) convenience since the term-structure of interest rates is not only not flat, it also changes stochastically from one period to the next. Students of fixed income securities are all too aware of this simple empirical fact. Our approach to the discount rate can only be justified from the perspective that it reflects the state-of-the-art when it comes to equity valuation.

the equality will usefully speed up and streamline derivations leading up to the OJ model.

Adding the zero-sum series (3.1) to PVED one obtains:

$$p_0 = y_0 + \sum_{t=1}^{\infty} R^{-t} z'_t, \tag{3.2}$$

where

$$z'_t = y_t + d_t - R y_{t-1}.$$

We emphasize that z'_t should be viewed as a function of y_t and y_{t-1} and thus the above relation holds as long as the transversality condition is met.

Expression (3.2) has two parts, y_0 and the PV of z'_t. The former provides the starting point in valuation, and the PV-term acts as its complement. Investment practice suggests that capitalized forward earnings ought to be the starting point. In this case

$$y_0 = x_1/r.$$

Proceeding in a logical fashion,

$$y_t = x_{t+1}/r, \quad \text{for } t = 1, 2, \dots .$$

Given this specification it follows that

$$z'_t = \frac{1}{r}(\Delta x_{t+1} - r(x_t - d_t)), \quad t = 1, 2, \dots .$$

It will be convenient to define

$$z_t \equiv r \cdot z'_t = \Delta x_{t+1} - r(x_t - d_t), \quad t = 1, 2, \dots$$

so that

$$p_0 = \frac{1}{r} \cdot x_1 + \frac{1}{r} \sum_{t=1}^{\infty} R^{-t} z_t. \tag{3.3}$$

Expression (3.3) equates value to capitalized forward earnings, x_1/r, plus an adjustment for subsequent superior, or abnormal, (dollar) growth in expected earnings. The word superior is appropriate because $z_t = 0$ is the benchmark when the (dollar) earnings growth is neutral. Now the valuation reduces to the simplest possible formula,

$p_0 = x_1/r$. A savings account illustrates the benchmark $z_t = 0$. No superior growth is feasible, as the earnings dynamic $x_{t+1} = R \cdot x_t - r \cdot d_t = x_t + r(x_t - d_t)$ makes clear. Regardless of the dividend d_t, a savings account satisfies $z_t = 0$ because d_t reduces x_{t+1} with the "right" amount. In other words, start-of-period dividends lead to foregone earnings, and the definition of z_t ensures the appropriate adjustment. The idea is a general one: The earnings increment Δx_{t+1} must be adjusted by the term $r(x_t - d_t)$, which identifies the earnings due to earnings retained (or reinvested) in the firm. With a full payout the benchmark is zero growth in earnings; with zero payout earnings need to grow at least at the rate r to be labeled superior.

Expression (3.3) is of interest in its own right, and the literature refers to it as the Abnormal Earnings Growth model or AEG model. Just as RIV explains the market value minus book value premium $(p_0 - b_0)$ in terms of superior growth in expected book value (or residual earnings), AEG explains the market value minus capitalized forward earnings premium $(p_0 - x_1/r)$ in terms of superior growth in subsequent expected earnings.[2] Both formulae help us to understand value because they focus on the premia for two natural value anchors. In a comparison of the two, AEG aligns with investment practice better than RIV does because it focuses on earnings rather than book value. That said, the AEG expression lacks real content without additional assumptions since it can be viewed as an identity (or tautology) with its negative connotation. The point underscores that it takes more than the zero-sum expression (3.1) and usage of the word "earnings" to achieve sharper insights about how value relates to earnings and their growth.

Before continuing with assumptions/derivations, an appreciation of the meaning of superior earnings growth helps. Intuition suggests that the occurrence of superior earnings growth originates from expectations that the firm undertakes positive net present value projects. This way of looking at superior earnings growth makes sense, provided that the expected net benefits from future investments cannot be capitalized

[2] Finance textbooks, like Brealey and Myers (1984), often refer to the premium $p_0 - x_1/r$ as "PVGO" or the "present value of future growth opportunities". (That said, finance textbooks do not articulate the analytics of AEG.)

and put on the balance sheet today (which is how GAAP works). An expectation of positive NPV projects is, however, not necessary for superior earnings growth to exist. Conservative balance sheet values, today and in the future, also suffice for superior earnings growth. Conservatism pushes earnings recognition into the future as long as the firm is expected to grow. Thus one can visualize that more conservative accounting in a growth setting reduces x_1/r while at the same time it increases z_t such that p_0 remains the same (which makes sense since PVED has not been affected.)

3.2 Adding structure to AEG

Researchers who develop a model with a present value evaluation tend to consider what happens if the variable in question grows at a constant rate. The OJ model maintains this time-honored practice. Hence we assume that

$$z_{t+1} = \gamma \cdot z_t, \quad t = 1, 2, \ldots \tag{3.4}$$

where $\gamma(< R)$ identifies the growth parameter. Since (3.4) implies that $\{R^{-t} z_t\}_t$ satisfies a geometric sequence, one obtains

$$PV \;\; of \;\; z = \frac{z_1}{R - \gamma}.$$

The above, strikingly simple, assumptions and derivations result in the OJ model.

Proposition 3.1 Assume PVED and

$$z_{t+1} = \gamma \cdot z_t, \quad t = 1, 2, \ldots,$$

where $\gamma < R$ and

$$z_t \equiv \Delta x_{t+1} - r(x_t - d_t).$$

Then

$$p_0 = \frac{x_1}{r} + \frac{1}{r} \cdot \frac{z_1}{(R - \gamma)} = \frac{x_1}{r} \left[\frac{g_2 - (\gamma - 1)}{r - (\gamma - 1)} \right], \tag{3.5}$$

where

$$g_2 \equiv (\Delta x_2 + r \cdot d_1)/x_1.$$

Expression (3.5) has two variations, depending on whether the term that augments x_1/r is additive or multiplicative. The latter approach appeals more than the former because, consistent with investment practice, it introduces a measure of percentage growth in near-term earnings, g_2. This measure of growth corrects in the numerator for foregone period 2 earnings due to date 1 dividends. Hence, $r \cdot d_1$ must be added to Δx_2.

The dynamic (3.4) supporting the OJ formula has two degrees of freedoms: (i) the initialization z_1 and (ii) the growth parameter γ. Unless indicated otherwise, we maintain $z_1 \geq 0$ and $\gamma \geq 1$, which represent the normal cases.

As mentioned earlier, although the OJ formula (3.5) indeed equals PVED, the model requires no payout parameter. This contrasts with the Constant Growth model, which introduces a sequence of earnings via a payout parameter and an explicit sequence of dividends. The point should be underscored. Proposition 3.1 depends neither on a payout parameter nor on an explicit sequence of dividends, except for d_1. Even so, d_1 is irrelevant in the sense that $x_2 + r \cdot d_1$ (or $\Delta x_2 + r \cdot d_1$) does not depend on d_1. (Section 4 provides a full treatment of the issue.)

The reader may have noted that Proposition 3.1 leaves out whether the transversality condition $\lim_{t\to\infty} R^{-t} x_{t+1} = 0$ has been met. It is not a problem, as a subsequent section explains why. Here we simply note that an assumption $d_t = K \cdot x_t$ (used in the Constant Growth model) guarantees the transversality condition. It also guarantees that p_0 does not depend on K, an issue that relates closely to the fact that $\lim_{t\to\infty} R^{-t} x_{t+1} = 0$. For this class of settings, at least, dividend policy irrelevancy (DPI) applies. Proving this is reasonably straightforward, but we skip it because the DPI concept is much deeper and more general. (Section 4 develops DPI.)

Another question pertains to the necessity of the dynamic (3.4). Given PVED, are there starting points other than (3.4) which lead to the OJ formula? Routine analysis verifies that the answer is no. One concludes that the dynamic (3.4) fully describes the OJ model if one takes PVED for granted. To appreciate the model it therefore helps to take a closer look at the dynamic. We make two points here, both of which will be relevant in some of the subsequent sections.

First, if CSR holds, then the dynamic (3.4) corresponds to

$$\Delta x^a_{t+1} = \gamma \cdot \Delta x^a_t, \quad t = 2, 3, \ldots .$$

Second, as a *special* case of this setting, one obtains

$$x^a_t = \gamma \cdot x^a_{t-1}, \quad t = 2, 3, \ldots .$$

The second equation is a special case of the first; i.e., it is sufficient but not necessary. To validate that it is not necessary, note that $x^a_t = \gamma \cdot x^a_{t-1} + \beta$ implies $\Delta x^a_{t+1} = \gamma \cdot \Delta x^a_t$ even if $\beta \neq 0$.

It is not obvious how one should interpret γ as a practical matter. It would be nice if γ provides a measure of long-term growth in expected earnings. After all, the quantity g_2 clearly provides a measure of the near-term growth in expected earnings (after an adjustment for the foregone earnings due to expected dividends). Though the analysis involves some subtleties, γ can be related to the asymptotic growth in earnings in a qualified sense.[3]

Proposition 3.2 Assume

$$z_{t+1} = \gamma \cdot z_t, \quad t = 1, 2, \ldots$$

where $\gamma < R$, and

$$z_t \equiv \Delta x_{t+1} - r(x_t - d_t), z_1 > 0.$$

Assume further that $d_t/x_t = k \geq (R - \gamma)/r$ for all $t \geq T$, some T. Then

$$\lim_{t \to \infty} x_{t+1}/x_t = \gamma.$$

Proof. Refer to Appendix 1. □

The minimum payout rate never exceeds 1 given $1 \leq \gamma < R$. As a numerical example, if $R = 1.09$ and $\gamma = 1.035$, then k must be at least $(1.09 - 1.035)/0.09 \approx 61\%$.[4]

To appreciate the nature of this asymptotic result it is also worthwhile to keep in mind that γ also determines the asymptotic growth in dividends.

[3] Olsson (2005) derives a similar result in a different context.

[4] A very small payout means that the growth rate x_{t+1}/x_t will be close to R. The conclusion makes intuitive sense if one considers a savings account.

Corollary 3.3 Given the assumptions of Proposition 3.2,

$$\lim_{t\to\infty} d_{t+1}/d_t = \gamma.$$

The restriction on the dividend payout ratio, to be sure, only serves the role of allowing us to interpret γ. It is not necessary for the OJ model per se; expression (3.5) does not depend on the restriction on the dividend payout ratio stated in Proposition 3.2. If the dividend payout ratio is low enough, namely $k < (R-\gamma)/r$, then $\lim_{t\to\infty} x_{t+1}/x_t = \lim_{t\to\infty} d_{t+1}/d_t = R - r\cdot k$. Nevertheless, even for this class of dividend policies, it is still true that $\lim_{t\to\infty} R^{-t}x_{t+1} = 0$, which satisfies the transversality condition in Proposition 3.1. (An examination of the proof of Proposition 3.2 validates the claim.)

The two propositions lead to the question of how the OJ model relates to the Constant Growth model. It is immediate that if one restricts (x_1, x_2) such that $x_2/x_1 = \gamma$, then the OJ model reduces to the constant growth valuation formula, $p_0 = (d_1/x_1)\cdot x_1/(R-\gamma)$. But this analysis actually makes a more important point: The last expression was derived *without* assuming constant growth in dividends. Nor does the analysis depend on the payout being constant or on x_{t+1}/x_t being constant for $t \geq 2$. One interprets the condition $x_2/x_1 = \gamma$ as the direct consequence of a very specific year 1 dividend policy. To appreciate this point, note that one can think of the sum of $x_2 + r\cdot d_1$ as being independent of d_1 (which, as noted before, is precisely how a savings account works). Then one can always pick a d_1 such that the resulting x_2 satisfies $x_2/x_1 = \gamma$. Subsequent years' dividend policies are irrelevant. This identification of the constant growth valuation formula differs from its standard derivation, which depends on the more restrictive assumption $d_{t+1}/d_t =$ constant for *all* t.

We now return to the dynamic (3.4) more generally. To get a feel for how it evolves over time, we illustrate three cases numerically. All of these assume $R = 1.1$, $\gamma = 1.06$, and $x_2 = 1.12, x_1 = 1.0$. Near-term earnings growth rate thus equals 12%, but the cases differ in their g_2-measure due to the dividends at date 1, i.e., d_1. Further, to derive an earnings sequence we have to make assumptions about dividends more

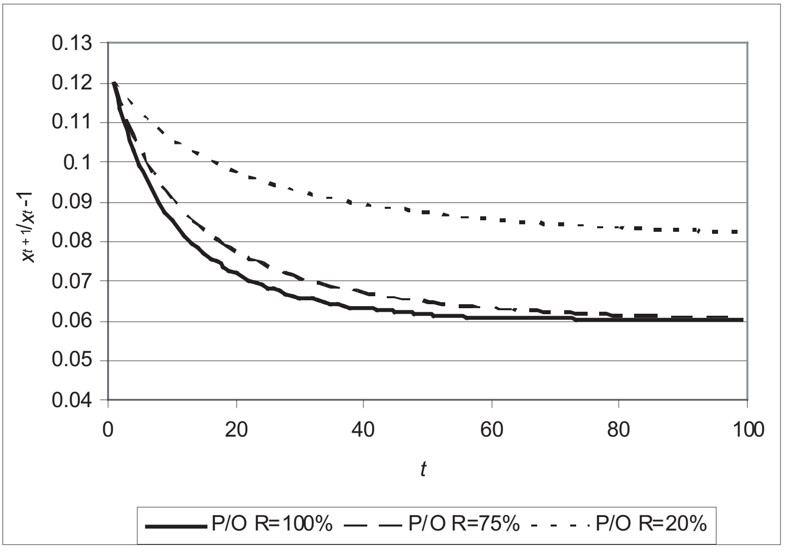

Fig. 3.1 Earnings growth under different payout ratios

generally. For ease of numerical analysis we consider three different constant payout ratios: 100%, 75%, and 20%. These specifications allow us to generate the sequences $\{x_t\}_{t=1}^{\infty}$ and $\{x_{t+1}/x_t\}_{t=1}^{\infty}$. Figure 3.1 plots the latter sequence for the three cases. All three plots show a gradual attenuation of the growth rate and the long-term steady states are approached in a smooth manner. Consistent with Proposition 3.2, the asymptotic growth equals 6% only in the first two cases. For the case of $d_t/x_t = 20\%$, the dividend payout ratio is too small to make the growth rate converge to 6%. Instead, the growth rate converges to $1.1 - 0.1 \times 0.2 = 1.08$, or 8%.

The above concludes our basic derivations related to the OJ model. The next section establishes various properties of the valuation formula.

3.3 Properties of the OJ valuation formula

This subsection takes a very close look at the OJ formula (3.5) and its properties. Thus, consider

$$p_0 = \frac{x_1}{r}\left[\frac{g_2 - (\gamma - 1)}{r - (\gamma - 1)}\right].$$

Looking at the RHS, we view price as a function of x_1 holding g_2 constant although g_2 depends on x_1. One readily sees that p_0 increases as either x_1, g_2, or γ increases, and p_0 decreases as r increases. These conclusions are necessary to have a sensible model. More to the point, the OJ model subsumes the basic principle of equity valuation: *The price to forward-earnings ratio*, p_0/x_1, *increases as either of the two growth measures*, g_2 *and* γ, *increase.*

We further note that $p_0/x_1 = 1/r$ if and only if $g_2 = r$, which corresponds to $z_1 = 0$.[5] In other words, the price to forward-earnings ratio builds in a premium only if there is an expectation of superior growth in subsequent expected earnings. The argument is pleasing, and it brings to the fore the meaning of superior growth in earnings. The proper growth measures must adjust for the growth due to reinvestment of earnings. Hence g_2 corrects for the foregone period 2 earnings due to date 1 dividends, consistent with our discussion of the AEG formula. There is no need for a similar adjustment in the denominator because today's price is an ex-dividend price. It becomes clear that g_2 is dividend policy independent.

The centrality of near-term earnings growth in practical investment analysis motivates a statement of the linear equation that explains the price to forward-earnings ratio as a function of g_2:

$$p_0/x_1 = k_1 + k_2 \cdot g_2,$$

where

$$\begin{aligned} k_1 &= -(\gamma - 1)/(r(R - \gamma)) \leq 0, \\ k_2 &= 1/(r(R - \gamma)) > 0 \end{aligned}$$

Note that as γ increases, the slope increases and the negative intercept becomes even more negative, i.e., p_0 becomes more sensitive to short-term growth as long-term growth increases. Such a relation would seem to make sense since it captures the idea that g_2 is more important when

[5] It may be useful to make a few comments about the initialization $z_1 < 0$, which the text has not considered. The equity is now worth less than capitalized earnings ($p_0 < x_1/r$), consistent with future abnormal earnings being below normal ($\Delta x_{t+1} - r(x_t - d_t) < 0$). To make sense of this setting, one puts $\gamma < 1$, which means $\lim_{t\to\infty} z_t = 0$. In other words, the below-normal expected earnings growth (adjusted for dividends) will be gradually eliminated as t increases.

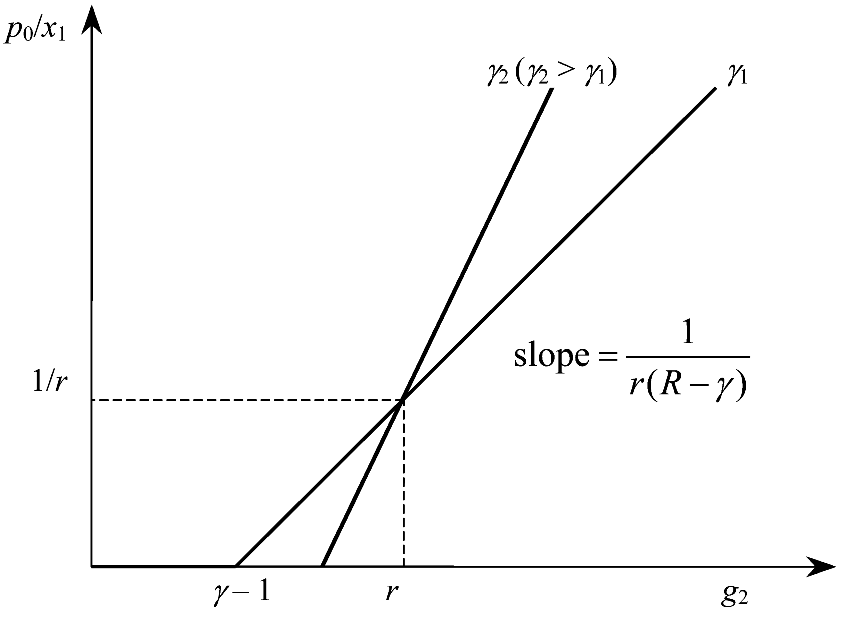

Fig. 3.2 Linear relation between P/E ratio and g_2, given γ

log-term growth is also important. Figure 3.2 illustrates how $p_0/x_1 = k_1(\gamma) + k_2(\gamma) \cdot g_2$ depends on two different γs.

Another way of looking at the OJ formula views p_0 as a function of the two expected earnings quantities for FY1 and FY2, $x_1, x_2 + r \cdot d_1$, in addition to γ and r. (As always, we require a correction of x_2 for foregone earnings due to date 1 dividends.) Manipulations of the valuation formula lead to [6]

$$p_0 = w \cdot f_1 + (1 - w) \cdot f_2,$$

where

$$w \equiv -\gamma/(R - \gamma), \text{and}$$
$$f_1 \equiv x_1/r,$$
$$f_2 \equiv (x_2 + rd_1)/rR.$$

[6] This observation is due to Yee (2005).

On the RHS, each of the two variables, $f_j, j = 1, 2$, can be thought of as an indicator of value. To appreciate this point, consider a savings account. In such case, the two value indicators equal each other as well as the date zero intrinsic value: $p_0 = x_1/r = (p_1 + d_1)/R = (x_2 + r \cdot d_1)/rR$. It follows that putting weights on the fs is irrelevant, an unsurprising conclusion since a savings account rules out earnings growth beyond the effect due to retained earnings. With the weights being irrelevant, the same is true for γ. But unless $z_1 = 0$, the two indicators will generally differ and thus their relative weights influence value. We then note that the weight on f_1 is *negative*, which means that *value decreases as forward-earnings increases.* At first glance the observation may seem counter-intuitive, but it does make sense if one keeps in mind that f_2 has been held constant. With f_2 constant, it follows that g_2 increases as f_1 decreases. Given the latter, there is no surprise since g_2 has a positive influence on equity value.

Rewriting the last expression clarifies the significance of near-term growth:

$$p_0 = f_1 + (1 - w)(f_2 - f_1),$$

where $1 - w = R/(R - \gamma) > 1$. The term $f_2 - f_1$ measures growth in dollars rather than as a percentage, $(f_2 - f_1)/f_1$. Hence dollar growth adds to value with an elasticity of $R/(R - \gamma)$. This elasticity is large in the sense that it is no less than $R/r(> 1/r)$ given that $\gamma = 1$. Of course, the elasticity increases as γ increases, provided that $f_2 > f_1$ (which corresponds to $g_2 > r$).

Rather than anchor value around f_1, consider the alternative f_2:

$$p_0 = f_2 - w \cdot (f_2 - f_1),$$

where $w = -\gamma/(R - \gamma) < 0$. One concludes that

$$p_0 > f_2 \text{ as well as} f_1,$$

provided that $f_2 > f_1$ (or $g_2 > r$). In other words, due to growth, value exceeds the first-cut estimates of value, x_1/r, and $(x_2 + r \cdot d_1)/rR$.

No long-term growth in expected earnings, or $\gamma = 1$, implies that

$$p_0 = (\Delta x_2 + r \cdot d_1)/r^2.$$

In this special case, the next period's expected earnings do *not* affect value, *given* the *change* in expected earnings (adjusted for the foregone earnings due to dividends). Hence, $\gamma = 1$ reduces the information required from $(x_1, \Delta x_2 + r \cdot d_1)$ to simply $\Delta x_2 + r \cdot d_1$ to value the equity. It goes almost without saying that this specification is irregular and of modest practical/empirical interest except, perhaps, as a very crude approximation of a firm's value.

Any application of the OJ formula requires a specification of γ, presumably something other than $\gamma = 1$. What is the appropriate way of thinking about its numeric specification? Proposition 3.2 and Corollary 3.3 help to address the question. The parameter γ equals the (very) long-term growth in expected dividends and earnings (given an adequate payout policy). This observation suggests that one puts γ equal to the long-term growth in GNP, say 3.5%. This way of thinking about γ further suggests that γ should be the same for all firms. To assume that two firms have the same growth in expected earnings in the *very* long term, no matter how different they may be right now, appeals in its simplicity.

Treating γ as a "universal constant" valid for all firms has the disadvantage of eliminating a degree of freedom in a cross-section. Only two degrees of freedom remain – the near-term measure of the growth in earnings, g_2, and R – to explain the price to forward-earnings ratio. In many investment or research settings this approach may be too narrow, and there may be good reasons why one would want γ to represent more of an average growth rate for the "foreseeable future". This perspective is consistent with the idea that γ represents not only the asymptotic growth in the sense of Proposition 3.2, but also the *rate of change* in growth going from g_2 to the asymptotic growth. Granted, the additional degree of freedom leads to greater subjectivity as to how to apply the model. But this problem is inescapable in any model that uses parameters that characterize expectations about the future.

To get a feel for the OJ formula as a practical tool, consider GE at the end of 2005 (December 12, 2005 to be precise). On that date analysts' consensus estimates of EPS 06 and 07 were 1.98 and 2.10 respectively. DPS for 2006 were about 1.00 (for an approximate 50% payout). We put γ equal to 1.035 and, somewhat arbitrarily, $r = 7\%$ to estimate GE's value on the basis of the formula. Specifically,

$$g_2 = \frac{2.1 + 0.07 \times 1.0}{1.98} - 1 = 9.6\%,$$

i.e., a forecasted 9.6% growth for near-term, dividend-adjusted, expected earnings. Hence,

$$p_0 = \frac{1.98}{0.07} \times \frac{0.096 - 0.035}{0.07 - 0.035} = 49.27.$$

This value estimator can be compared with actual value at that date, 36.06 (at the end of the trading day). Such a difference is not trivial. It reflects the sensitivity of p_0 to the choice of the discount factor (and the specification of γ). If one changes r to 8% from 7%, then one obtains a value much closer to 36.06 (in fact, $p_0 = 36.31$). In this regard the OJ model has the same problem as other valuation models.

All valuation models, including the OJ model, must deal with the fact that the discount factor is never a known constant. Rather than relying on educated guesses or sensitivity analysis investment practice often circumvents the issue via the use of so-called reverse engineering. That is, one solves for r by equating the right-hand side of any valuation formula to the actual, observed, price. For the OJ model one obtains a square-root formula:

$$r = A + \sqrt{A^2 + \frac{x_1}{p_0}\left(\frac{\Delta x_2}{x_1} - (\gamma - 1)\right)},$$

where

$$A \equiv (\gamma - 1 + d_1/p_0)/2.$$

For the special case when $\gamma = 1$ the above formula reduces to

$$r = \sqrt{PEG^{-1}},$$

where $PEG \equiv (p_0/x_1)/g_2$. The formula is of some interest because PEG is often used in practice as a first-cut relative-value indicator (the less, the better). Still, as previously noted, $\gamma = 1$ is a peculiar case because of the irrelevance of forward earnings given the earnings change $\Delta x_2 + r \cdot d_1$.[7]

[7] As yet another special case, if the earnings growth is constant in the sense that $\Delta x_2/x_1 = \gamma - 1$, then one obtains the textbook relation $r = (\gamma - 1) + d_1/p_0$.

We apply the square-root formula for IBM at the end of 2005: The closing price of IBM on 16 December 2005 was USD83.37. On the same date analysts' consensus estimate of EPS in 2006 was 5.66. The near-term expected earnings growth (without dividends adjustment) at that date was approximately 9%. DPS for 2006 was about 0.91 (i.e., 16% payout). Again, we put γ equal to 1.035. These quantities yield the estimated cost of equity capital

$$A = (0.035 + 0.91/83.37)/2 = 0.02$$

$$r = 0.02 + \sqrt{(0.02)^2 + \frac{5.66}{83.37} \cdot (0.09 - 0.035)} = 0.09.$$

"Reverse engineering" works as a general scheme, and one does not have to target the discount factor. It can be applied no less to γ, for example. In this case r must be determined using some independent scheme (like the CAPM). One then infers γ from the formula $\gamma = R - [(g_2/r - 1)/(p_0/x_1 - 1/r)]$.[8]

3.4 A special case of the OJ model: The market-to-book model

Financial statement analysis textbooks, such as Penman (2006), often present the Market-to-Book Valuation Formula, or M/B model. The accounting in this model relies on CSR, a condition which is generally unnecessary in the OJ model. The valuation highlights that the firm's upcoming expected return on equity, $roe_1 = x_1/b_0$, explains the M/B ratio:

$$p_0/b_0 = \frac{roe_1 - (\gamma - 1)}{r - (\gamma - 1)}. \quad (3.6)$$

To derive the formula (3.6), the standard development follows two steps. First, given CSR, PVED is equivalent to RIV, i.e.,

[8] As yet another approach to reverse engineering, consider the following. Let $\hat{r}$ and $\hat{\gamma}$ represent reasonable but tentative guesstimates of the two parameters in question (e.g., averages, somehow determined, over a population). Thereafter one can minimize $(r - \hat{r})^2 + (\gamma - \hat{\gamma})^2$ over all (r, γ)-pairs consistent with the OJ formula (given p_0, x_1, x_2, d_1). (Of course, the metric is merely suggestive; as an alternative one may use, say, $\log(r/\hat{r}) + \log(\gamma/\hat{\gamma})$.)

$PVED = b_0 + \sum_{t=1}^{\infty} R^{-t} x_t^a$. Second, the dynamic $x_{t+1}^a = \gamma \cdot x_t^a, t \geq 1$, generates $\{x_t^a\}$. Hence the PV of the x_t^a-sequence equals $x_1^a/(R-\gamma)$; simple manipulation of $b_0 + x_1^a/(R-\gamma)$ then leads to the M/B valuation formula (3.6).

At first glance the M/B model, with its emphasis on *roe* and a firm's book value, may seem like a competing valuation scheme which has a different flavor compared to the OJ model's singular emphasis on earnings. Such is not the case, however. The M/B model derives from special assumptions embedded in the OJ model.

An earlier section noted that $x_{t+1}^a = \gamma \cdot x_t^a$ implies $\Delta x_{t+1}^a = \gamma \cdot \Delta x_t^a$, which represents the OJ model's dynamic if CSR applies. It follows that the OJ formula combined with CSR and the more restrictive dynamic $x_{t+1}^a = \gamma \cdot x_t^a$ reduces to the M/B formula.

To work through the details, first note that, as a matter of the definition of x_1^a,

$$x_1/r = b_0 + x_1^a/r.$$

The OJ formula can now be expressed as

$$p_0 = b_0 + x_1^a/r + \Delta x_2^a/(r(R-\gamma)).$$

Second, $x_2^a = \gamma \cdot x_1^a$ implies that $\Delta x_2^a = (\gamma - 1) \cdot x_1^a$. Substituting Δx_2^a into the last equation results in

$$\begin{aligned} p_0 &= b_0 + \frac{x_1^a}{r} + \frac{(\gamma-1)x_1^a}{r(R-\gamma)} \\ &= b_0 + \frac{x_1^a}{R-\gamma} \\ &= b_0 \cdot \frac{roe_1 - (\gamma-1)}{r-(\gamma-1)}. \end{aligned}$$

Summarizing, we have the following:[9]

Proposition 3.4 Assume PVED, CSR, and the dynamic

$$x_{t+1}^a = \gamma x_t^a, \quad t = 1, 2, \ldots,$$

[9] To be sure, there will be OJ formulas that do *not* reduce to the M/B model.

where $\gamma < R$. Then the OJ model reduces to the M/B model:

$$p_0 = \frac{x_1}{r} + \frac{\Delta x_2^a}{r(R-\gamma)} = b_0 \cdot \frac{roe_1 - (\gamma - 1)}{r - (\gamma - 1)}.$$

Though the M/B model anchors value to book value, one can shift the perspective away from the market-to-book ratio p_0/b_0 to the price to forward-earnings ratio p_0/x_1, and let p_0/x_1 (like p_0/b_0) be a function of roe_1. Specifically, the M/B model can be restated as

$$p_0/x_1 = k_1 + k_2/roe_1,$$

where

$$k_1 = 1/(R-\gamma),$$
$$k_2 = (1-\gamma)/(R-\gamma).$$

To evaluate roe_1's influence on p_0/x_1, we consider two different specifications of γ and x_t^a:

(i) If $\gamma \geq 1$ then assume $x_1^a \geq 0$ (or $roe_1 \geq r$). The last restriction indicates that it makes no accounting/economic sense to let x_t^a be negative for all future periods, which is what would happen if $x_1^a < 0$ given $\gamma \geq 1$. Thus, in this setting, the dollar amount of residual earnings grows without bound over time. One can think of this as being due to conservative accounting combined with growth in the business.

(ii) If $\gamma < 1$ then assume $x_1^a < 0$ (or $roe_1 < r$). Here the firm is unprofitable relative to the benchmark r in the upcoming year, but as time passes the profitability is expected to improve and it approaches the benchmark asymptotically, i.e., $x_t^a < x_{t+1}^a < \cdots \to 0$ as $t \to \infty$.[10]

Setting (i) implies $k_2 < 0$. Thus p_0/x_1 is bounded below by $1/r$ and the ratio p_0/x_1 increases as roe_1 increases (where $roe_1 > r$). Setting (ii) implies the converse, $k_2 > 0$. Again p_0/x_1 is bounded below by $1/r$ but the ratio now decreases as roe_1 increases (where $roe_1 < r$).

[10] Because the model has only two degrees of freedom, there is no way to parameterize $x_t^a < 0$, for $t = 1, 2, \ldots, T$ but $x_t^a > 0$ for $t > T$.

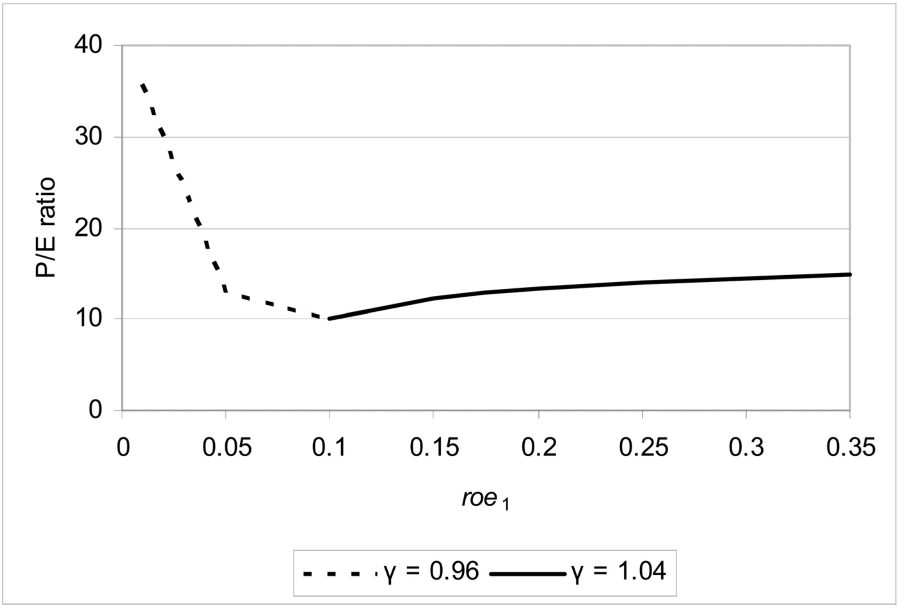

Fig. 3.3 P/E ratio as function of roe_1 and γ

Figure 3.3 combines the two cases in one graph, which is U-shaped. It illustrates an (as of yet) untested empirical proposition. Firms with relatively high *or* low roe_1 should have relatively high p_0/x_1 ratio.

A final point related to the M/B model pertains to how one infers a firm's cost of equity capital through reverse engineering. Solving for r one obtains the simple formula

$$r = \frac{p_0 - b_0}{p_0} \cdot (\gamma - 1) + \frac{x_1}{p_0}.$$

The sign of $\frac{p_0 - b_0}{p_0} \cdot (\gamma - 1)$ is always positive for both settings (i) and (ii). To prove this note that $(p_0 - b_0)$ is positive if and only if $x_1^a \geq 0$, and the sign of x_t^a is the same as $\gamma - 1$ by assumption.

One obtains three conclusions with empirical content. First, as alluded to previously and shown in Figure 3.3, r always exceeds forward-earnings yield (x_1/p_0). The hypothesis is not completely satisfactory since the real world allows occasionally for the discount setting $p_0 < x_1/r$. (The more general OJ model does, in fact, allow for a discount when $\gamma < 1$ and $z_1 < 0$.) Second, for a given M/B ratio, r increases as x_1/p_0 increase. This observation has obvious appeal. Third, the market-to-book ratio also influences r, but it does so in a fairly complicated manner. Fixing the earnings yield, for a profitable firm r increases as the market-to-book ratio increases. But for an unprofitable firm the converse is the case. Of course, whether the model does an adequate job in explaining empirical regularities concerning a firm's r (or risk) becomes a tricky matter insofar as the model is relatively rigid in its assumptions and conceivable settings.

Though the OJ model in its full generality adds an extra degree of freedom – by allowing $\beta \neq 0$ in the expression $x^a_{t+1} = \gamma \cdot x^a_t + \beta$ – there is of course no guarantee that this helps if one wants to explain the price to forward-earnings ratio. It is easy to see that at least under some circumstances such ought to be the case. The most obvious one occurs if it is desirable to force x^a_t to switch signs at some future date. If x^a_1 is negative and γ is slightly greater than 1, then a (sufficiently large) β ensures that $x^a_t (t \geq 2)$ will be positive. Another case of interest occurs if one wants x^a_t to decay rapidly in the near term because x^a_1 is perceived to be unsustainably large. A negative β now makes x^a_2 much smaller than x^a_1. Yet the model ensures that β does not influence the asymptotic growth as long as γ exceeds 1. In sum, while it may be useful to view p_0/x_1 as a function of roe_1, it is an open question whether this works for a meaningfully high proportion of the kinds of firms that are usually subject to empirical examination.[11]

3.5 Another special case of the OJ model: Free cash flows and their growth

Virtually all textbooks, in finance and financial statement analysis, develop a model focusing on firms' free cash flows as the PV-attribute.

[11] One can generalize the OJ dynamic in the same way that we generalize $x^a_{t+1} = \gamma \cdot x^a_t$. Consider the dynamic $z^a_{t+1} = \gamma \cdot z^a_t + \mu$, where μ is not necessarily equal to zero ($\mu = 0$ would take us back to the OJ dynamic). From an analytical point of view this generalization poses no problems, and one readily derives the valuation function. However, the usefulness of this more general approach is not clear to us at this point. The question arises whether the dynamic $\Delta z^a_{t+1} = \gamma \cdot \Delta z^a_t$ captures some interpretable aspects of the forecasting of a firm's performance. Our best guess is no.

The basic idea is that a firm's value splits into two separate parts. One part comprises the financial assets, net of financial obligations (i.e., interest-bearing debt). One values these assets/liabilities by referring to their unambiguous market values. Forecasting plays no role, akin to a savings account. The other part pertains to the present value of the net benefits expected from operating activities (as opposed to financial activities). The expected free cash flows determine these net benefits. This approach relies on the assumption that the net financial assets can be valued without ambiguity because there are no frictions – such as positive probability of bankruptcy and related costs, taxes, or agency costs – that would interfere with their valuation. All financial activities are zero NPV activities. Operating activities, on the other hand, may involve positive NPV projects (in expectation). Such projects face no financial constraints because borrowing/lending is always available at the rate r.

To formalize the model, we start out by stating the expression for value:

$$p_0 = fa_0 + \sum_{t=1}^{\infty} R^{-t} c_t \tag{3.7}$$

where

fa_0 = financial assets, net of debt, on date 0,
c_t = expected free cash flows from operations, period t.

One defines the free cash flow implicitly from the (T-account) relation that updates the fa-balance:

$$fa_t = fa_{t-1} + fx_t + c_t - d_t, \text{ for } t = 1,2,\ldots, \tag{A 1}$$

where

fx_t = expected financial income, or interest income, on financial assets, in period t.

We will refer to this equation as Assumption (A 1).

Because financial activities must be zero NPV, we add a second assumption [12]

$$fx_t = r \cdot fa_{t-1}, \quad \text{for } t = 1, 2, \ldots. \tag{A 2}$$

Combined with PVED, (A 1) and (A 2) imply the valuation formula (3.7). Next assume that the free cash flows grow at a constant rate:

$$c_{t+1} = \gamma \cdot c_t, \quad \text{for } t = 1, 2, \ldots. \tag{A 3}$$

One obtains the well-known valuation formula

$$p_0 = fa_0 + c_1/(R - \gamma).$$

This setting, it turns out again, is a special case of the OJ formula. To appreciate this point, consider the setting in Proposition 3.1 with the added requirement that the firm uses cash accounting. It follows that

$$x_t = fx_t + c_t,$$

since fx_t is essentially equivalent to cash. Combining this relation with CSR implies $b_t = fa_t$ and $x_t^a = x_t - fx_t = c_t$. Moreover, $\Delta x_2^a = c_2 - c_1 = -(1 - \gamma)c_1$ and thus

$$\begin{aligned} p_0 &= \frac{x_1}{r} + \frac{\Delta x_2^a}{r(R - \gamma)} \\ &= \frac{fx_1 + c_1}{r} + \frac{-(1 - \gamma)c_1}{r(R - \gamma)} \\ &= fa_0 + \frac{c_1}{R - \gamma} \end{aligned}$$

[12] The reader will note that, generally, the discount factor in expression (3.7) does not have to be same as in A 2. In other words, as the textbooks underscore, the weighted average cost of capital, or the discount factor related to operating activities, differs from the (after-tax) borrowing/lending rate. To deal with this matter on a sound theoretical basis one must refer to the foundations of neo-classical valuation; i.e., the no-arbitrage precept in friction-free markets and the resulting existence of non-negative state-contingent prices. Such a framework can be employed in this section, but we refrain from doing so because it would radically escalate the level of analytical abstraction. In the background is the unsolved issue of how one achieves readily derived, concrete and pragmatically useful conclusions. Thus we stick to our current framework in which the reader can think of operating and financial activities as belonging to the same risk-class from the investor's perspective.

Summarizing, we have the following result

Proposition 3.5 Assume PVED and

$$fa_t = fa_{t-1} + fx_t + c_t - d_t, \tag{A 1}$$

$$fx_t = r \cdot fa_{t-1}, \tag{A 2}$$

$$c_{t+1} = \gamma \cdot c_t. \tag{A 3}$$

Assume further that $x_t = fx_t + c_t, b_t = fa_t$, i.e., there is cash accounting. Then the OJ model reduces to the Free Cash Flows discounting model:

$$p_0 = \frac{x_1}{r} + \frac{\Delta x_2^a}{r(R-\gamma)} = fa_0 + c_1/(R-\gamma).$$

One can think of this equivalence in a slightly different way. The free cash flow approach is actually equivalent to the M/B model, given that the accounting is one of cash accounting: $b_t = fa_t$, as noted, and $x_t^a = c_t$. Of course, since the M/B model is a special case of the OJ model, it follows that the free cash flow model (PVED, (A 1), (A 2), and (A 3)) must also be a special case of the OJ model.

All of the above analysis is straightforward, and it may look somewhat pedantic. Nevertheless, it serves the useful purpose of highlighting that parsimonious valuation approaches will end up being special cases of the OJ model. *In no sense do these models compete with the OJ model since they reflect sharper conclusions due to additional assumptions about either the underlying dynamic or the accounting.*

The analysis of the free cash flows model raises a poignant question. In what ways do accounting rules influence the OJ formula? It makes sense to address to what extent one can rely on a broad set of accounting rules and yet maintain the formula because the structure of the underlying dynamic has not changed. We analyze this matter in Section 8.

4

The OJ Model and Dividend Policy Irrelevancy

We noted early on that the OJ model embeds a dividend policy irrelevancy property, or DPI for short. A case in point pertains to the fixed payout setting $d_t = K \cdot x_t$, where PVED accordingly does not depend on K. But DPI is far more general and the model permits much more complicated policies. Aside from a weak regularity condition, DPI requires no particular restrictions on the payout policy ratio d_t/x_t (such as $\lim_{t\to\infty} d_t/x_t = K$ or even $0 < d_t/x_t \leq 1$ for all t sufficiently large).

This section analyzes this DPI property in full. It is shown to be robust, and we do not have to introduce unappealing assumptions to maintain DPI as part of the OJ model. We also take the opportunity to address an issue that was touched upon, but which we left dangling, when developing Proposition 3.1: Does $R^{-t}y_t \equiv R^{-t}(x_{t+1}/r)$, in fact, converge to zero as t goes to infinity? The answer is affirmative under mild assumptions on the dividend policy.

Before proceeding with the analysis, it is worthwhile to rearticulate DPI's sharp content. DPI means that one can determine value (p_0) *without* having any *particular* information about the d-sequence. Nor can one infer any properties of the sequence, besides the PVED solution itself. Dividends may, or may not, equal zero for the next 10 years, for example. The OJ formula, and the underlying dynamic, has no bearing upon this issue except for the fact that d_1 shows up in the formula. d_1 can be made equal to any value as long as one makes the appropriate correction of x_2. Still, p_0 does in fact equal PVED, so the irrelevance of the sequence may at first glance seem puzzling. It is less so as long as one keeps firmly in mind that any change in the dividend policy merely reshuffles the expected dividends across dates without changing their present value.

To appreciate the analytical concept of DPI and its irrelevance it helps to first consider a savings account. The OJ model still holds but with the additional restriction $z_1 = z_2 = \ldots = 0$. The earnings dynamic is now specified by

$$x_{t+1} = R \cdot x_t - r \cdot d_t, \quad \text{for } t = 1, 2, \ldots.$$

Next, we also need a dividend policy concept. To determine a date $t+1$ dividend, we specify an equation analogous to the one for earnings, namely,

$$d_{t+1} = c_1 \cdot x_t + c_2 \cdot d_t, \quad \text{for } t = 1, 2, \ldots,$$

where c_1 and c_2 are two dividend policy parameters. Given these two equations, the two dynamic equations generate a specific sequence $d_2, d_3, \ldots$, for any initialization x_1 and d_1. Thus one can evaluate PVED as a function of (x_1, d_1) and R, c_1, c_2.

To ensure a finite PVED, we need two convergence conditions: (i) $c_1 > 0$ and (ii) $|c_2| < R$. These two conditions correspond to a standard regularity condition that the maximum root (modulus) of the implied transition matrix $\begin{bmatrix} R & -r \\ c_1 & c_2 \end{bmatrix}$ is strictly less than R. This condition effectively resolves the problem we encountered in the development of the OJ model and Proposition 3.1, namely, the convergence of $R^{-t}x_t$ to zero as $t \to \infty$. One sees that PVED will be finite if, and only if, $\lim_{t\to\infty} R^{-t}x_t = 0$ (in which case $\lim_{t\to\infty} R^{-t}d_t = 0$).

With the convergence being guaranteed, it follows that $p_0 = x_1/r$ for *all* values of the two dividend policy parameters (c_1, c_2) satisfying the regularity condition. Hence, DPI holds; we can infer value, since it equals x_1/r, *without* referring to the elements in the sequence of dividends.

It goes almost without saying that one can come up with more general dividend policies without affecting the DPI conclusion. For example, one can add a condition $d_t = 0$ for even years so that $d_{t+1} = c_1 \cdot x_t + c_2 \cdot d_t = c_1 \cdot x_t$ applies only if t (on the RHS) is even. A simple spreadsheet analysis will show that the value conclusion $p_0 = x_1/r$ remains and so does DPI. In fact, for a savings account, no matter what the dividend policy $d_{t+1} = f_t(x_t, x_{t-1}, \cdots, x_1; d_t, d_{t-1}, \cdots, d_1)$ is, it is not difficult to show that $\lim_{t\to\infty} R^{-t}x_t = 0$ suffices for DPI. It also turns out to be necessary. As we move on to the OJ model we generalize this regularity condition by requiring that no variable can grow at the rate R or more.

To see how the OJ model enfolds DPI we next consider a problem that allows a more complete characterization of DPI. This setting elaborates on DPI's many subtle aspects, and it goes beyond a savings account by looking at a 3×3 dynamics. Thus the vector (x_{1t}, x_{2t}, d_t) identifies the state variables, and x_{1t} and x_{2t} need no "labels". Following this generic setup to analyze DPI we can turn our attention to the OJ model.

Lemma 4.1 Consider the 3×3 dynamics

$$\begin{bmatrix} x_{1t+1} \\ x_{2t+1} \\ d_{t+1} \end{bmatrix} = \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ 0 & \omega_{22} & 0 \\ \omega_{31} & \omega_{32} & \omega_{33} \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \\ d_t \end{bmatrix}, \quad \text{for } t = 1, 2, \ldots,$$

with the regularity condition that the maximal root of the matrix $[\omega_{ij}]$ is strictly less than R. Then PVED does not depend on the dividend policy parameters ω_3 if and only if $\omega_{11} = R$. Moreover, $\omega_{11} = R$ implies $PVED = PVED(x_{11}, x_{21}, d_1)$ is independent of d_1 and conversely.

Lemma 4.1 is a special case of a problem which allows for arbitrary $n \times n$ dynamics. Appendix 2 states and proves this general DPI result.

Lemma 4.1 brings out the essence of DPI because it characterizes the necessary and sufficient condition of the inter-period behavior of x_1: On the margin the x_1 variable must grow exactly at the rate of the discount-factor, R, i.e., the condition is represented by $\partial x_{1t+1}/\partial x_{1t} = \omega_{11} = R$. One can interpret this as a "no-arbitrage" condition on the x_1 variable in the sense that the choice of a dividend policy cannot manipulate the x_1 variable to take a path that increases (or decrease) today's value. The x_1 variable thereby becomes the most important variable as one compares and assesses the economic significance of the three variables, (x_1, x_2, d), in the setup. x_2 has its own evolution regardless of x_1 and d; d is relevant only because it influences the behavior of x_1 via ω_{13}.[1] Hence the burden to attain DPI is placed on x_1. These aspects of the 3×3 setting with DPI suggest that x_1 has much in common with the construct of earnings as it is generally understood. If we cannot center the analysis around dividends, then we need something else and this suggests that an earnings construct should play the central role. We expand on this theme in the next section.

Another aspect of DPI in the lemma concerns the forecasting of the upcoming period's expected dividend. The policy parameters $(\omega_{31}, \omega_{32}, \omega_{33})$ are of no valuation relevance, an unsurprising condition that is not only necessary but also sufficient. One sees again that dividends are of no interest *except that they influence the forecasting of the x_1 variable* (through the parameter ω_{13}). In this way the two conditions of Lemma 4.1 suggest that DPI is not only built into the OJ model but it also suffices for the OJ model if a 3-dimensional space underpins the valuation framework.

The dynamics of Lemma 4.1 fit into the modeling that supports Proposition 3.1. To identify the OJ dynamics as a special case of the last lemma, consider the following. Let (x_t, z_t) correspond to (x_{1t}, x_{2t}) and put $\omega_{11} = R$, $\omega_{12} = 1$, $\omega_{13} = r$ so that, (i) $x_{t+1} = Rx_t - rd_t + z_t$ or $z_t = \Delta x_{t+1} - r(x_t - d_t)$, and (ii) z_t grows at the constant rate $\gamma = \omega_{22}$. These observations reveal that the sequence of expected dividends is indeed part of Proposition 3.1, but they need not be explicated: The dividend policy, which $(\omega_{31}, \omega_{32}, \omega_{33})$ determines, does *not* affect the present value of expected dividends.

[1] It can be shown that the regularity conditions of the proposition imply $\omega_{13} \neq 0$.

Proposition 4.2 Given the assumptions of Lemma 4.1 and $\omega_{11} = R$, $\omega_{12} = 1$, $\omega_{13} = -r$, $\omega_{22} = \gamma$, one obtains the OJ dynamic

$$z_{t+1} = \gamma \cdot z_t,$$

where

$$z_t = \Delta x_{t+1} - r(x_t - d_t),$$

and

$$\lim_{t\to\infty} R^{-t} x_t = 0.$$

Proposition 4.2 uses the regularity condition stated in Lemma 4.1 for the conclusion $\lim_{t\to\infty} R^{-t} x_t = 0$. The condition "maximum root of the matrix $[\omega_{i,j}] < R$" rules out that any of the three variables (x_{1t}, x_{2t}, d_t) can grow at a rate R or more. Hence, $\lim_{t\to\infty} R^{-t} x_t = 0$ for the set of permissible dividend policies.

As the discussion of the savings account indicated, the linear dividend policy equation is an assumption of convenience. Though linearity in the third equation simplifies the analysis, it is by no means crucial. The reader who is interested can combine the dynamic $z_{t+1} = \gamma \cdot z_t (x_1, x_2, d_1, z_1$ being exogenous) with a nonlinear dividend equation such as $d_t = 0.3 x_t \sin^2(x_t)$, and then verify via a spreadsheet that a direct evaluation of PVED in fact equals the value per the OJ valuation formula.[2]

As a final point, the analysis here indicates that to achieve useful insights about accounting data and value requires DPI. Without DPI the value function becomes much more complicated because the policy parameters would have a direct influence on any formula that determines value. Such a possibility would almost surely exclude any practical or intuitive results, quite aside from the mathematical mess it would cause.

[2] The function $0.3x_t \sin^2(x_t)$ works such that $0 < d_t/x_t < 1$ for almost every x_t, but the limit of d_t/x_t does not exist due to the oscillating property of $\sin^2(x_t)$.

5

The Labeling of x_t as Expected Earnings

5.1 The analytical properties of x_t

The OJ model claims to focus on expected *earnings* and their subsequent growth as representing value. But the reader may ask: Are there, in fact, any good reasons for the labeling of x_t as expected earnings, and, if so, what properties inherent in the OJ model can provide such reasons? The second part of the question is important because it means that the labeling issue can be resolved only if we introduce the OJ model itself into the analysis. Vague reference to investment practice will not do. How to approach the issue is of course a tricky matter. The word "earnings" evokes all sorts of meanings depending on the context; nevertheless, we will make the case that there are good reasons for the label of x_t as earnings.

This subsection proceeds as follows. First, we state the underlying dynamics of the OJ model in terms of its three primitives, (x_t, z_t, d_t). Second, on the basis of these underpinnings we establish a number of analytical properties of x_t from a time-series perspective. In no way does this analysis depend on PVED, DPI, or the OJ *valuation* formula. Reintroduction of PVED takes place in the next subsection; now it turns out that one can derive the OJ formula and DPI by assuming PVED in conjunction with all the properties of earnings which this section articulates.

The previous section identified the 3×3 dynamics which support the OJ model:

$$\begin{bmatrix} x_{t+1} \\ z_{t+1} \\ d_{t+1} \end{bmatrix} = \begin{bmatrix} R & 1 & -r \\ 0 & \gamma & 0 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x_t \\ z_t \\ d_t \end{bmatrix}, \quad t = 1, 2, \ldots .$$

The third equation, which forecasts the next period's dividends given (x_t, z_t, d_t), can be generalized. We use the current linear modeling of the dividend policy only because of its analytical convenience. (To be sure, the analysis could easily handle the case of a fixed payout, $d_t = K \cdot x_t$: let $c_1 = K \cdot R,\ c_2 = K,\ c_2 = -K \cdot r$). Leaving the third equation aside, no other dynamic equations can describe the underlying dynamics in the OJ model. Further note that if $z_1 = 0$, then the mathematics reduces to the savings account setting. Thus $z_1 = 0$ constitutes the benchmark permanent earnings model, and one can read x_t as expected earnings instead of interest income. For this specialized setting, at least, the label "(expected) earnings" would seem to be the only one acceptable.

A standard aspect of any (linear) dynamic modeling stipulates that there can be no explicit or implicit *contemporaneous* dependence among the (three) state variables. Any setting with dependence would undermine the foundations of the dynamics. Accordingly, the analysis here does not deviate from this independence-condition. This modeling leads to the following observation: x_t's independence of d_t mirrors standard accounting (including GAAP) for earnings, i.e, earnings do not depend on contemporaneous dividends. (In contrast, a firm's book value does depend on contemporaneous dividends.) On this dimension the model setup appeals.

Looking at the dynamics from a time series perspective one can deduce additional properties of x_t that makes the label "earnings" plausible. We pursue the issue by evaluating how marginal changes in x_t and d_t influence *subsequent* x and keeping z_t constant, consistent with the requirements of the dynamics.

Specifically, one readily shows that:

(i) $\partial x_{t+1}/\partial d_t = -r$. The distribution of dividends foregoes, on the margin, subsequent (expected) earnings at the rate r.
(ii) $\partial x_{t+1}/\partial x_t = R$. On the margin, (expected) earnings beget more (expected) earnings at the accretion rate R.
(iii) $\partial(x_{t+2} + r \cdot d_{t+1} + x_{t+1})/\partial d_t = -(R^2 - 1)$. This property generalizes (i) for two future periods.
(iv) $\partial(x_{t+2} + r \cdot d_{t+1} + x_{t+1})/\partial x_t = R^2 + R$. This property generalizes (ii) for two future periods.

The first two properties are as straightforward as they are sensible if one wants to label x_t as (expected) earning. The remaining two are somewhat more complicated. Nevertheless, they capture the idea that (expected) earnings, corrected for foregone earnings due to dividends, satisfy intertemporal aggregation properties.[1] In (iii) the effect of increments in dividends reduces subsequent earnings in a systematic fashion regardless of horizon, and in (iv) earnings beget more subsequent earnings in a systematic fashion, also reflecting that one can pick any horizon.

Properties (i) through (iv) do not bear on the extent to which the accounting depends on (expected) accruals. Some form of cash accounting is not ruled out. This aspect must be acknowledged. That said, the properties make it clear that earnings differ from revenues because there are no reasons suggesting that revenues ought to satisfy (i) through (iv) properties. The same can be said for any sub-category of earnings, such as some construct of operating earnings. Perhaps one can argue that the two properties (i) and (iii) can hold if x_t pertains to earnings before certain non-cash charges such as write-offs. But these are only two of the four properties, and would then have to confront the issue of how such a before no-cash charges earnings construct can reconcile with (ii) and (iv). Developing such a model would seem to pose considerable challenges, and, at the least, it would require variables that go beyond (x, z, d).

[1] The two properties, (iii) and (iv), generalize for a T-horizon perspective. Define $AE_T \equiv \sum_{t=1}^{T} x_t + \sum_{t=1}^{T} d_t(R^{T-t} - 1)$. Then (iii) and (iv) generalized equal $\partial AE_T/\partial d_1 = -(R^T - 1)$ and $\partial AE_T/\partial x_1 = R^T + R^{T-1} + \cdots + R$.

None of the derived properties comes as a surprise. They reduce to the idea that the OJ model preserves the earnings properties of a savings account *on the margin*. While this embededness does not necessarily settle the labeling issue, at least it eliminates what otherwise might lead to a hard-to-deflect question about the OJ dynamic: How can one justify a model of expected earnings that does not satisfy the properties (i) through (iv)?

Up to this point, none of the conclusions has depended on PVED. The next subsection reintroduces PVED. It is then shown that the four properties of earnings restrict the environment such that the OJ valuation formula holds.

5.2 The OJ model derived from the four properties of earnings

To appreciate the main result in this subsection, it helps to consider a simpler problem first. Consider the following 2×2 dynamics:

$$\begin{aligned} x_{t+1} &= \omega_{11} \cdot x_t + \omega_{12} \cdot d_t \\ d_{t+1} &= \omega_{21} \cdot x_t + \omega_{22} \cdot d_t. \end{aligned}$$

Note that the dynamics impose no particular restrictions on the four parameters, except for the standard regularity condition such that (x_t, d_t) does not grow in excess of R as $t \to \infty$. Suppose further PVED holds. One can now ask: Under what conditions will $p_t = x_{t+1}/r$? A sufficient condition is obviously implied by the savings account dynamic $\omega_{11} = R$ and $\omega_{12} = -r$, and where the two remaining parameters (ω_{21}, ω_{22}) are irrelevant. These conditions are also necessary, as Section 4 made clear. In other words, if one assumes the general 2×2 dynamics for (x_t, d_t), then the earnings properties (i) and (ii) (stated in section 5.1) identify the ideal earnings construct – permanent earnings (in expectation).

Given this idea that restrictions on earnings properties result in the valuation function, we can next ask: What does the valuation look like if one replaces x_t with two variables, x_{1t}, x_{2t} and, to maintain consistency, replaces the above 2×2 transition matrix with a 3×3 transition matrix? Expressed somewhat differently, the question aims at finding the model that "comes after" the one based on ideal earnings. It goes without saying that such a model needs to subsume a concept of growth in earnings that goes beyond the growth due to retained earnings.

The degrees of freedom in the dynamics have increased from $4(= 2 \times 2)$ to $9(= 3 \times 3)$, so the earnings properties (i) and (ii) alone imposed on x_{1t+1} do not suffice to achieve any useful insights. To further restrict the ω_{ij} parameters obviously requires additional assumptions on the candidate earnings variable, x_1. It turns out – as the reader surely must have already conjectured – that the earnings properties (iii) and (iv) added to (i) and (ii) lead to the OJ valuation formula. To summarize, we have the following result.

Proposition 5.1 Consider the 3×3 linear dynamics

$$\begin{bmatrix} x_{1t+1} \\ x_{2t+1} \\ d_{t+1} \end{bmatrix} = \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \\ d_t \end{bmatrix}, \quad t = 1, 2, \ldots,$$

and where the standard regularity condition is implied. Assume further that the dynamics satisfy the four properties:

(i) $\partial x_{1t+1}/\partial d_t = -r$;
(ii) $\partial x_{1t+1}/\partial x_{1t} = R$;
(iii) $\partial(x_{1t+2} + x_{1t+1} + r \cdot d_{t+1})/\partial d_t = -(R^2 - 1)$;
(iv) $\partial(x_{1t+2} + x_{1t+1} + r \cdot d_{t+1})/\partial x_{1t} = R^2 + R$.

Then $\omega_{11} = R$, $\omega_{13} = -r$, $\omega_{21} = \omega_{23} = 0$; $\omega_{12} = 1$ without loss of generality unless $\omega_{12} = 0$. Further, if PVED and $\omega_{22} < R$ are assumed, then the OJ formula holds:

$$p_0 = \frac{x_{11}}{r} + \frac{x_{21}}{r(R - \omega_{22})}.$$

Proof. Refer to Appendix 1. □

Starting with nine degrees of freedom, the four assumptions on earnings thus restrict four parameters, ensure the irrelevance of three parameters, and allow us to put one parameter (ω_{12}) equal to one without loss of generality. Restrictions $\omega_{11} = R$ and $\omega_{13} = -r$ are trite. More complicated are $\omega_{21} = \omega_{23} = 0$, which depend on properties (iii) and (iv). With respect to ω_{12}, if it equals zero then the model reduces to a savings account that makes the second equation irrelevant. If ω_{12} differs from zero, then, because $\omega_{21} = \omega_{23} = 0$, one can put $\omega_{12} = 1$ without loss of generality (an issue of scaling). Of course, the built-in DPI makes ω_{31}, ω_{32}, ω_{33} irrelevant.

In the preceding proposition the variable x_{2t} ends up being equivalent to z_t since one can infer that $x_{2t} = \Delta x_{1t+1} - r(x_{1t} - d_t)$, given the implied restrictions on the ω_{ij}-parameters. Thus the proposition admits a conclusion that short-term and long-term expected earnings growth explain the price to forward-earnings ratio. Proving it depends explicitly on the four properties of expected earnings. Proposition 3.1, in contrast, reaches the same conclusion without making any references to the underlying properties of earnings. Hence, Proposition 3.1 paints a coarser picture as to what makes the OJ model work. The more general setup in Proposition 5.1 enhances the appeal of the OJ model.

A final result deals with how one can think of the OJ model's x-variable as being equivalent to an ideal earnings construct perturbed by an additive error. The ideal earnings construct is determined by permanent earnings (in expectation), i.e., the dynamic of a savings account: The error corresponds to the rescaled z-variable (or x_{2t} in the last proposition). Previous analysis thus leads to the following essentially immediate result.

Proposition 5.2 Assume PVED. Further, assume that ${x_t}^*$ satisfies the dynamic

$$x^*_{t+1} = R \cdot x^*_t - r \cdot d_t, \tag{5.1}$$

given any sequence d_1, $d_2, \ldots$ that implies $\lim_{t\to\infty} R^{-t} x^*_t = 0$. Define

$$x_t = x^*_t - err_t, \quad \text{for } t = 1, 2, \ldots.$$

Then each of the following two statements implies the other:

(i) $err_{t+1} = \gamma \cdot err_t$, for $t = 1, 2, \ldots$ and $err_1 \geq 0$;
(ii) $x_{t+2} + r \cdot d_{t+1} - R \cdot x_{t+1} = (R - \gamma) \cdot err_{t+1}$, for all t.

The result brings out how closely the OJ model relates to ideal earnings and the way in which it provides the most obvious extension of the ideal earnings model. With respect to the error, what assumption can be simpler than constant growth? Having made this assumption on the error leads to the unique identification of the error. Conversely, the OJ model implies a constant growth in "what is missing" in ideal earnings.

In light of the traditional (textbook) Constant Growth model, the last proposition is not without irony. An assumption of constant growth is indeed a powerful and usable assumption in the analysis of equity value. But to impose it on dividends fatally undermines the very foundation on which the analysis now must be based. It removes the idea of DPI as a valid construct, and, unsurprisingly, it becomes all but impossible to introduce earnings in any meaningful sense. In sharp contrast, the analysis here shows that there is indeed ample room for a constant growth assumption provided that one starts from an ideal earnings construct (permanent earnings in expectation) that embeds DPI. The next step, which introduces into ideal earnings an error that grows at a constant rate, is more or less immediate if one wants to maintain analytical simplicity.

6

Capitalized Expected Earnings as an Estimate of Terminal Value

Equity valuation in practice sometimes uses a horizon approach. It exploits the notion that the forecasted earnings provide a better indicator of value if it pertains to a period farther out in the future than the next year. Implementation of the approach includes two parts. First, evaluate the expected present value of expected dividends up to the horizon. Second, estimate the so-called terminal value by capitalizing the expected earnings at the horizon date. This way of looking at value raises the following question: Can the x-variable in the OJ model sensibly serve the role of estimating the *terminal* value? The analysis of this question ties in with the previous section. Both bear on whether one can reasonably attach the label "expected earnings" to the x-variable.

The analysis starts from the relation

$$p_0 = \sum_{t=1}^{T} R^{-t} d_t + R^{-T} p_T,$$

where T represents the horizon date. This expression, of course, readily derives from PVED. With this horizon expression in place one can analyze the nature of the valuation-error if one uses x_{T+1}/r as an estimate of p_T, i.e., as an estimate of the so-called terminal value.

To tackle this problem, define

$$TrErr_T = p_0 - \left[\sum_{t=1}^{T} R^{-t} d_t + R^{-T}(x_{T+1}/r)\right],$$

where one reads *TrErr* as truncation error. In general, since $p_T \neq x_{T+1}/r$, $TrErr_T$ differs from zero.

For the asymptotic horizon, when T approaches infinity, $TrErr_T$ goes to zero because, under the usual regularity condition of the OJ model, $R^{-T}x_T$ goes to zero. This conclusion is straightforward. Its lack of sharpness, however, makes it relatively uninteresting. A much trickier question addresses whether the absolute magnitude of $TrErr_T$ decreases monotonically. The desired conclusion reflects the general idea that while it is certainly harder to forecast earnings farther out in the future, at the same time there should be an offsetting benefit because the discounted measurement error in the *expected* earnings declines for the longer horizon. If it does not, then the earnings construct lacks a proper foundation, and it would make no sense to apply the horizon concept. It turns out that the x-variable in the OJ model meets the desired requirement.

Proposition 6.1 Assume PVED and the dynamic $z_{t+1} = \gamma \cdot z_t$, for $t = 1, 2, \ldots$, where

$$z_t \equiv \Delta x_{t+1} - r(x_t - d_t).$$

Then

$$|TrErr_{T+1}| < |TrErr_T| \quad \text{for all } T,$$

and $TrErr_T$ goes to zero as T goes to infinity for any dividend policy.

How best to implement the horizon approach as a practical matter goes beyond the OJ model. The insight of the proposition bears on the nature of the x-variable and why we can argue it meets characteristics commonly associated with expected earnings.

The idea of a horizon approach also leads to a straightforward extension of the OJ model. Specifically, one can relax the assumption on the z_t-dynamic so that it satisfies $z_{t+1} = \gamma \cdot z_t$ for $t \geq T$ and where the starting date T needs not equal 1. The more general assumption implies the valuation formula

$$p_0 = PVED_T + R^{-T} p_T^*.$$

Here, p_T^* stands for the estimate of the terminal value:

$$p_T^* = \frac{x_{T+1}}{r} + \frac{1}{r} \cdot \frac{z_{T+1}}{(R-\gamma)} = \frac{x_{T+1}}{r} \left[\frac{g_{T+2} - (\gamma - 1)}{r - (\gamma - 1)} \right],$$

where

$$g_{T+2} \equiv (\Delta x_{T+2} + r \cdot d_{T+1})/x_{T+1}.$$

This mode analysis can be generalized to embed the spirit of Proposition 3.4 by assuming $x_{t+1}^a = \gamma \cdot x_t^a$ for $t \geq T$ for some T which may exceed 1.

The horizon approach generalization comes at a cost. It requires additional input. Not only z_{T+1} and x_{T+1} are exogenous; so is the sequence of expected dividends up to the horizon date.

7

The OJ Model and Cost of Equity Capital

All equity valuation models, including the OJ model, require a cost of equity capital parameter.[1] Its presence has been ubiquitous, appearing in all valuation formulae and representations of the underlying dynamic. This section takes a closer look at the parameter r and how it should be interpreted in various expressions. One reason for doing this is that it helps to understand the OJ model. There is also a second reason, which we emphasize here: It elaborates on the cost of capital concept itself. Some of these aspects have not generally been appreciated in the literature.

Valuation models introduce the cost of equity capital parameter as the discounting factor needed to let PVED determine value. One can also think of r as the market's required rate of return, conforming to the fact that the expected market return equals to r, underscored by textbooks (and noted in Section 3). This discounting factor attribute of r is perhaps all too easy to taken for granted. It is, however, incomplete

[1] As indicated earlier, we do not distinguish between the cost of equity capital and the weighted-average cost of capital. To do so would cause considerable difficulties if we started from the first principle of valuation. Hence, we view operating and financial activities as belonging to the same risk-class.

because the r in the PVED formula should ultimately depend on the firm's opportunities and plans; i.e., the pricing that takes place in the equity market must be consistent with the firm's expected transactions and their economic consequences. Introducing r in PVED before having considered the underlying dynamic therefore puts the horse before the cart. Hence we need to consider r's presence in the dynamic $x_{t+1} = R \cdot x_t - r \cdot d_t + z_t$, where $z_{t+1} = \gamma \cdot z_t$.

The cost of equity capital, as the phrase itself suggests, refers to the idea that investors have expectations about the subsequent payoff when they supply capital to the firm. The OJ dynamic formalizes the relation through $\partial x_{t+1}/\partial(-d_t) = r$, where one reads $-d$ as capital contribution. The property thus describes the economics when the firm transacts with its owners. The effect of dividends is on the next-period's expected *earnings*, and the effect can be identified even though earnings by themselves do not provide sufficient information to determine value. That is, the effect on next-period earnings of a dollar contributed is r even though p_t differs from x_{t+1}/r. The analysis shows, not without subtlety, that the earnings variable only has to capture the marginal effect of the capital contribution.

The cost of equity capital parameter also influences the behavior of expected earnings: In expectation, earnings beget more earnings such that, on the margin, earnings grow at the cost of capital rate, i.e., $\partial x_{t+1}/\partial x_t = R$. This observation extends the previous one, $\partial x_{t+1}/\partial(-d_t) = r$. Now it reflects that the supply of capital leads to a multi-period stream of expected benefits. These can unfold as either earnings or dividends. Hence the cost of capital must influence the time-series behavior of earnings. Risk can therefore be viewed as an attribute inherent in the expected earnings behavior; this attribute then carries over into the market's "required rate of return", i.e., the determinant of r in PVED.

One can develop the points in the last two paragraphs from a more integrated perspective. Consider the (expected) earnings dynamic written as $x_{t+1} = x_t + r \cdot (x_t - d_t) + z_t$. It shows that any investments made by the firm that are "financed by retained earnings" earn a rate specified by the discount factor. But the extent of such "financing" makes no difference because of its zero NPV characteristic. DPI, of course, ensures such indifference. As before, this analysis refers to what happens on the margin, not on the average. The setup is fully consistent with the idea that a firm may plan to undertake positive NPV investments. The variable z_t handles potential positive NPV investments, a point made early on when we developed the model's dynamic foundation.[2]

[2] To be sure, $z_t \geq 0$ does not necessarily imply positive NPV investments since z_t also depends on the accounting rules.

8

Accounting Rules and the OJ Formula

Expected earnings and their growth depend, at least in principle, on how the firm plans to keep its books.[1] This observation raises questions about how the OJ model relates to accounting rules. Does the model allow for flexibility in the accounting? Specifically, can the valuation formula remain valid across two distinct (sets of) rules? If so, how does the extent of conservative accounting affect the input required for the valuation formula (3.5)? The first two questions concern how one models alternative accounting measures that are consistent with the OJ framework. Putting the analytical machinery into place then sets the stage for the question about accounting conservatism. We summarize the points to be made in this section as follows.

First, we identify admissible changes in the accounting rules such that the forward earnings and their near-term growth change, yet the LHS of the OJ formula (3.5) – i.e., the price – remains the same. Second, if one makes the accounting more conservative, in the sense of lower expected book values for all future dates, then forward earnings decrease while there is an effective increase in the near-term growth in

[1] This section depends on ideas found in Yee (2006).

expected earnings. No change in price is consistent with the idea that a so-called cosmetic change in the accounting rules does not alter expectations about the firm's underlying economic realities. Third, under weak assumptions, changing the accounting to make it more or less conservative does not change the long-term growth of earnings as measured by γ. This invariance aspect reflects that, while the accounting may affect both numerator and denominator of the earnings growth measure, x_{t+1}/x_t, these effects may cancel each other as $t \to \infty$. There is no need to reconfigure Proposition 3.2 and γ is indeed a "fixed parameter" in the OJ model.

To motivate the analysis, one can think of a firm that contemplates a change in its depreciation method to make it less conservative. In such a case, current and carrying values of expected future plant, property and equipment increase. Expected book values also increase. Moreover, if one assumes generic growth and CSR, then the entire sequence of expected earnings also increases. The CSR structure of basic accounting guarantees this effect on earnings. But can the OJ formula remain intact? The answer is "yes" given the appropriate modeling. The strategy is to ensure that the underlying dynamic $\Delta x^a_{t+1} = \gamma \cdot \Delta x^a_t$ remains the same though the accounting has been altered. Such invariance in the dynamics is necessary and sufficient for the OJ formula to result in the same value (LHS). (The point is obvious given the analysis in Section 3.1.) We proceed as follows.

Let $(x_t,\, b_t)$ represent the accounting under current rules. Consider next the following change in current and future book values:

$$\hat{b}_t(K) \equiv \gamma^t K + b_t, \quad \text{for } t = 0, 1, \ldots,$$

where $K > 0$ means the accounting is less conservative (in expectation). Thus the term $\gamma^t K$ represents the total increase in the book value at date t due to the change in depreciation method, or $\hat{b}_t - b_t$. The specific structure appeals because it embodies the idea that the additional amount in PPE should grow as the firm grows. (That is, if the difference today is $K = 100$ million, then a growth rate of say 4% means that the difference has grown to $1.04^5 \times 100 = 122$ million five years later). Given CSR it follows that expected earnings also change:

$$\hat{x}_t(K) \equiv \gamma^{t-1} K(\gamma - 1) + x_t.$$

This simple modeling yields the following invariance property.

Lemma 8.1 Assume CSR and consider

$$\hat{b}_t(K) \equiv \gamma^t K + b_t,$$
$$\hat{x}_t(K) \equiv \gamma^{t-1}(\gamma - 1)K + x_t, \quad t = 1, 2, \ldots.$$

Then $x^a_{t+1} = \gamma \cdot x^a_t$ implies

$$\hat{x}^a_{t+1}(K) = \gamma \cdot \hat{x}^a_t(K),$$

for any K and conversely.

Notice that the initialization $\Delta\hat{x}^a_2(K)$ depends on K. Whatever this initialization might be (assuming it is positive) the dynamic structure of the sequence does not depend on K. It follows that the OJ model holds for every K. Less apparent, but still readily verifiable, the LHS of the OJ formula does not depend on K. That is, the effects of K on $\hat{x}_1(K)$ and $\Delta\hat{x}^a_2(K)$ cancel each other.

Proposition 8.2 The assumptions of Lemma 8.1 imply

$$\hat{x}_1(K) = K(\gamma - 1) + x_1,$$

and $\hat{x}(K)$ depends on K. But

$$\hat{p}_0(K) \equiv \frac{\hat{x}_1(K)}{r} + \frac{\Delta\hat{x}^a_2(K)}{r(R - \gamma)}$$

does not depend on K.

Moreover,

$$\hat{x}_1(K) > x_1 (= \hat{x}_1(0))$$

if and only if $\hat{g}_2(K) < g_2(= \hat{g}_2(0))$, where $\hat{g}_2(K) = (\Delta\hat{x}_2(K) + r \cdot d_1)/\hat{x}_1(K)$.

Proof. Refer to Appendix 1. □

The proposition makes sense in the way it articulates the accounting-dependence of forward earnings and their growth. Conservative accounting and its extent have a clear influence on the triplet

(b_0, x_1, g_2) given generic growth, i.e., the parameter γ exceeds 1. Shifting the accounting so it becomes less conservative increases forward earnings but it also has a concomitant reducing effect on the near-term growth in earnings. In the context of Proposition 3.4 it becomes apparent how conservative accounting increases the market-to-book ratio with an offsetting increased expected return on equity. (While the latter has a long history of being present in research and textbooks, the empirical evaluation of the extent to which conservative accounting influences the triplet (b_0, x_1, g_2) as opposed to (b_0, roe_1) has on the whole not been done. It should be quite feasible to do so because the market-to-book ratio can serve as a construct to measure the degree of conservatism.)

Proposition 8.2 admits a straightforward generalization by allowing for the so-called "canceling error" concept. To be precise, instead of assuming $\hat{b}(K) \equiv \gamma^t K + b_t$, consider the more general structure $\hat{b}_t(K_1, K_2) \equiv \gamma^t K_1 + K_2 + b_t$, where K_1 and K_2 are constants. The conclusions of Lemma 8.1 still apply if one replaces $\hat{x}^a_{t+1}(K) = \gamma \cdot \hat{x}^a_t(K)$ with $\Delta\hat{x}^a_{t+1}(K_1, K_2) = \gamma \cdot \Delta\hat{x}^a_t(K_1, K_2)$, and thus Proposition 8.2 also applies. But, to be sure, this canceling error concept cannot be applied for the M/B model.

9

Information Dynamics that Sustain the OJ Model

Expectations, and thus valuation, in financial markets depend on the underlying information.[1] This truism has been neglected so far. The development of the OJ model rests on an assumption of how expectations evolve subsequent to date $t + 2$ by taking forward earnings and their near-term growth as exogenous quantities. Nothing has been said about the information that influences these expectations. A more general approach is feasible, as will be shown in this section. One can start from a vector of information variables which evolve stochastically such that any realization of the vector yields forward earnings, their growth, and the function that converts observed current information to value. Using the appropriate set of assumptions on the information-vector and its stochastic process then leads to the OJ model.

An information-based approach allows us to develop broader insights which bear on how the market return across two adjacent dates, $(\Delta\tilde{p}_{t+1} + \tilde{d}_{t+1})/p_t$, depends on "new" information. Moreover, the modeling of the information dynamics shows how the OJ model forces accounting to be conservative. As in previous sections, the analysis embeds DPI although PVED determines price at all dates.

[1] This section summarizes Ozair (2003).

Consider the following information dynamics (ID, for short):

$$\begin{bmatrix} \tilde{x}^a_{t+1} \\ \tilde{v}_{1t+1} \\ \tilde{v}_{2t+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \gamma & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^a_t \\ v_{1t} \\ v_{2t} \end{bmatrix} + \begin{bmatrix} \widetilde{\varepsilon}_{1t+1} \\ \widetilde{\varepsilon}_{2t+1} \\ \widetilde{\varepsilon}_{3t+1}, \end{bmatrix} \tag{ID}$$

where the $\widetilde{\varepsilon}_{t+1}$ are unpredicted disturbance terms with zero means. We model ε_{2t} such that $v_{1t} > 0$ for all t with probability one. This poses no problem because we allow $\text{var}_t\ (\widetilde{\varepsilon}_{2t+1})$ to depend on date t information. More generally, the disturbance terms satisfy no particular probability distribution; these can change in a random fashion from one date to the next. The disturbance terms $(\varepsilon_{1t+1}, \varepsilon_{2t+1}, \varepsilon_{3t+1})$ resolve the uncertainty as time passes from date t to $t + 1$. The two variables $(\widetilde{v}_1, \widetilde{v}_2)$ reflect "other (observable) information" that goes beyond the basic accounting data, (b, x, d).

To keep matters as simple as possible, the accounting satisfies CSR. Referring to the Residual Income Valuation approach, the vector $(b_t, x^a_t, v_{1t}, v_{2t})$ provides sufficient information for value. This way of looking at the information dynamics is convenient.

ID builds in the dynamics of expectations consistent with the previous sections. Specifically, given any realization (x^a_t, v_{1t}, v_{2t}), one readily shows that ID implies

$$E_t[\Delta\widetilde{x}^a_{t+2}] = \gamma \cdot v_{1t}.$$

Further,

$$E_t[\Delta\widetilde{x}^a_{\tau+1}] = \gamma \cdot E_t[\Delta\widetilde{x}^a_{\tau}], \quad \text{for } \tau \geq t + 2,$$

because of the second equation in the ID. With respect to forward earnings, the first equation in ID yields in the forecast

$$E_t[\widetilde{x}_{t+1}] = R \cdot x_t - r \cdot d_t + v_{1t} + v_{2t}.$$

As an immediate consequence one obtains:

Proposition 9.1 Assume PVED and ID, combined with any regular dividend policy.

Then,

(i) The OJ model holds;

(ii) $p_t = b_t + \beta_1 \cdot x_t^a + \beta_2 \cdot v_{1t} + \beta_3 \cdot v_{2t}$,

where $\boldsymbol{\beta} = [1/r, R/(r(R-\gamma)), 1/r]$, or

$$p_t = \alpha_1 \cdot x_t + \alpha_2 \cdot d_t + \alpha_3 \cdot v_{1t} + \alpha_4 \cdot v_{2t},$$

where $\boldsymbol{\alpha} = [R/r, -1, R/(r(R-\gamma)), 1/r]$.

The proposition sets the stage for a statement of how the period $(t, t+1)$ excess return, $\widetilde{r}^e_{t+1} \equiv (\widetilde{p}_{t+1} + \widetilde{d}_{t+1})/p_t - R$, depends on the period's uncertainty resolution, $(\varepsilon_{1t+1}, \varepsilon_{2t+1}, \varepsilon_{3t+1})$.

Corollary 9.2 Given the assumptions of Proposition 9.1,

$$\widetilde{r}^e_{t+1} = \sum_{k=1}^{3} \mu_k (\widetilde{\varepsilon}_{k,t+1}/p_t), \tag{9.1}$$

where

$$\boldsymbol{\mu} = (R/r, R/(r(R-\gamma)), 1/r).$$

The coefficients appeal intuitively. First, unexpected earnings has a coefficient of R/r, consistent with contemporaneous earnings having a multiplier of R/r on value; see part (ii) of the above proposition. Second, the model picks up the information, ε_{2t+1}, that changes perceptions about subsequent, near-term, growth in expected earnings. The coefficient μ_2 equals $R/(r(R-\gamma))$ which essentially corresponds to the coefficient in the OJ formula, $1/(r(R-\gamma))$. (The difference is due to the need to convert the information into forward value.) Third, the information ε_{3t+1} modifies expectation about the next period's expected earnings that goes beyond realized earnings. Hence, the coefficient related to ε_{3t+1} (or ε_{3t+1}/p_t to be precise) equals $1/r$. That is, the coefficient must be the same as the one for $E_t[\tilde{x}_{t+1}]$ in the basic OJ formula, given that the information about earnings growth has been picked up by the second term in expression (9.1). Fourth, to be sure, expression (9.1) does not include a term related to unexpected dividends. DPI causes this irrelevance, of course.

The existence of other information, $(\widetilde{v}_1, \widetilde{v}_2)$, makes the model flexible as to how accounting data relate to value. One cannot sign $p_t - b_t$

or $p_t - \left(\frac{R}{r}x_t - d_t\right)$, which means that the two premia can be either positive or negative. From this perspective ID is realistic. The model also builds in realism from the perspective that the accounting must be conservative on the average, or, equivalently, in expectation. The statement obviously applies to earnings because ID implies that $p_t > E_t[\widetilde{x}_{t+1}]/r$ such that $E_t[\widetilde{p}_{t+\tau}] > E_t[(R/r)\widetilde{x}_{t+\tau} - \widetilde{d}_{t+\tau}]$ for all $\tau \geq 2$. A weaker conclusion holds for book value.

Proposition 9.3 Assume PVED and the information dynamic [ID]. Then

$$\lim_{\tau\to\infty} E_t[\widetilde{p}_{t+\tau} - \widetilde{b}_{t+\tau}] > K > 0.$$

Proof. Refer to Appendix 1. □

As the proof indicates, the conclusion follows because $p_t - b_t = \sum_{\tau=1}^{\infty} R^{-\tau} E_t[\widetilde{x}^a_{t+\tau}]$ and $\lim_{s\to\infty} E_t[E_{t+\tau}[\widetilde{x}^a_{t+\tau+s}]] > 0$ for all $\tau \geq 1$. Loosely speaking, on average we can expect future expected abnormal earnings to be positive.

Proposition 9.3 is of obvious interest in view of a well-known empirical regularity: Most firms have market values exceeding their book values of equity. That is, the accounting is conservative from a balance sheet perspective.

10

Operating Versus Financial Activities

Valuation research and practice frequently use the idea that a firm's activities can be split into operating and financial activities. Subsection 3.5 is a case in point. Here we retain this framework, but we also extend it to identify implications that bear on the OJ model. We show how the model works when one shifts the focus from the bottom-line, earnings, to the bottom-line before financial expenses/revenues, namely, operating earnings. The valuation of operating activities will thus depend on expected operating earnings and their subsequent growth. Such a framework introduces no substantive complications, and it yields unsurprising conclusions. In the OJ formula one replaces earnings with operating earnings and dividends with cash flows. However, a subtle concept must be dealt with: DPI must be extended to Cash Flows Irrelevancy (CFI). The analysis accordingly admits an opportunity to revisit central aspects of equity valuation, with specific emphasis on the role of (operating) earnings and cash flows.

Two straightforward premises motivate the operating versus financial activities approach. First, as a matter of definition, all financial activities have zero NPV and one infers their values from the balance sheet. Like a savings account, the valuation of such an asset/liability eliminates the need to make any forecasts. Second, operating activities may have positive (expected) NPV on average, and their value today must reflect the present value of the expected net cash flows. The latter presents no problems as long as the operating activities can be conceptualized without any reference to specific borrowing/lending activities, i.e., there can be no synergy between the two kinds of activities. The accounting carrying value of (net) operating assets in the balance sheet has no particular relation to their economic value because the latter partially depends on positive NPV investments that are expected to be undertaken in the future. Intangible assets, in the abstract, can thereby exist but they pertain solely to operating activities.

To develop the results, we introduce symbols to denote accounting measures for operating vs. financial activities:

$ox_t =$ operating earnings, period t
$fx_t =$ financial earnings, period t
$oa_t =$ operating assets, net of operating liabilities, date t
$fa_t =$ financial assets, net of financial liabilities, date t.

The first assumption pertains to the accounting beyond CSR:

$$ox_t = \Delta oa_t + c_t, \qquad \text{(A 4)}$$
$$fx_t = \Delta fa_t - c_t + d_t,$$

Adding the two equations results in CSR: $x_t = \Delta b_t + d_t$. (The setup is arguably more definitional than an assumption.)

The second assumption pertains to the zero NPV property of financial activates (same as in Subsection 3.5):

$$fx_t = r \cdot fa_{t-1}. \qquad \text{(A 5)}$$

(A 2) reflects financial activities only, so that, generally, $ox_t \neq r \cdot oa_{t-1}$. One then obtains the counterpart of Proposition 3.1.

Proposition 10.1 Consider the assumptions in Proposition 3.1, combined with (A 4) and (A 5).

Then,

$$p_0 - fa_0 = \frac{ox_1}{r} + \frac{\Delta ox_2^a}{r(R-\gamma)} = \frac{ox_1}{r}\left[\frac{\hat{g}_2 - (\gamma - 1)}{r - (\gamma - 1)}\right],$$

where

$$\hat{g}_2 \equiv (\Delta ox_2 + r \cdot c_1)/ox_1.$$

With the above result in place it follows that one can state modified versions of Proposition 3.2 and on, in a spirit no different from one restatement of Proposition 3.1. The matter simply involves replacing $(x_t,\ d_t,\ b_t)$ with $(ox_t,\ c_t,\ oa_t)$ and thus shifts the focus to the valuation of operating activities, rather than the valuation of equity. This refocus of the analysis would seem to be unproblematic from a strict logical perspective. From a broader point of view one has to ask whether the language and motivation attached to the reconsidered exercises need to change. Presumably, one has to do more than change the language by substituting dividends for cash flows, and earnings for operating earnings. Two related issues come to mind. For starters, CFI can hardly be thought of as being the property of a set of policies available to a firm, i.e., the spirit of DPI cannot be retained. Readers may be comfortable with a relation in which $d_t = K \cdot x_t$ for a set of values of the dividend-policy parameter K (like $0 < K \leq 1$); at the same time readers may doubt that $c_t = \hat{K} \cdot ox_t$ makes any economic or accounting sense if one insists that $\hat{K}$ remains a policy parameter no different from K. The meaning of such an interpretation, if maintained, would seem to require considerable elaboration.

There is a second, more subtle, issue. While it poses no problem to say that x_t does not depend on d_t – after all, that is how standard accounting works – to claim that the same kind of independence applies to ox_t as it relates to c_t is a different matter. If one thinks about c_t as being the net of various components (wages, capital expenditures etc.), then the idea that ox_t can be independent of c_t breaks down. A similar multidimensionality does not, of course, arise for dividends: Nothing can be gained by looking at various types of "dividends" (a firm buying and selling its own equity shares, for example) when the firm is a single-person proprietorship. In a similar vein, to say that cash accounting corresponds to putting $ox_t = c_t$ poses no problems and such a setting can be of interest, as Proposition 3.5 suggests. In sharp contrast, to put $x_t = d_t$ and claim that such a model represents "cash accounting for the equity" would seem to be of no interest, as well as awkward.

With the above caveats in mind, one can still meaningfully refer to CFI. There is no need to refer to a policy concept. Instead, CFI means that, given the assumptions A1 and A2, one can infer the value of operating activities without knowing the elements in the sequence of expected cash flows. That is, the OJ valuation formula for operating activities and related input on RHS do not per se imply any particular sequence of cash flows. An exception occurs only if one imposes an additional prior restriction like cash accounting for operating activities (i.e., $ox_t = c_t$).

To illustrate CFI we generalize the information dynamics in Proposition 9.1 to separate operating from financial activities. The setup is fully consistent with Proposition 10.1, and it includes the equation that updates the financial assets, $fa_t = R \cdot fa_{t-1} + c_t - d_t$. Appendix 3 explicates the various dynamic equations and their valuation implications.

Appendix 3 serves an additional purpose. It allows us to show why it makes sense to distinguish between net earnings as opposed to comprehensive earnings in a valuation context. These two concepts of earnings arise naturally for the information dynamics specified because, consistent with GAAP, one can treat unpredictable, windfall, gains/losses of financial assets as part of other comprehensive income (OCI). Such unpredictable gains/losses have the same effect on value as dividends: Just as a dollar of dividend reduces value by a dollar so does a dollar of windfall loss resulting from the holding of financial assets. Such gain/loss items must be distinguished from (i) the *expected* earnings due to the holding of financial assets and (ii) realized operating earnings. The latter two income components add in the income statement without loss of information, which contrasts with the windfall gains/losses.

Appendix

Appendix 1 Proofs of select propositions in the text

Proposition 3.2 Note that for $t \geq T$, the dynamic of z and $d_t/x_t = k$ implies the following bi-variate dynamic system

$$\begin{bmatrix} x_{t+1} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} R - r \cdot k & 1 \\ 0 & \gamma \end{bmatrix} \begin{bmatrix} x_t \\ z_t \end{bmatrix}.$$

The limit growth of x is governed by the dominant eigenvalue of the transition matrix $\begin{bmatrix} R - r \cdot k & 1 \\ 0 & \gamma \end{bmatrix}$. When $k \geq (R - \gamma)/r$, it is easy to verify γ is the dominant eigenvalue. Thus, $\lim_{t\to\infty} x_{t+1}/x_t = \gamma$.

Proposition 5.1 We prove the first half of the proposition only; the second half is immediate given the first half and PVED.

First note that the dynamic of x_1 implies

$$\partial x_{1t+1}/\partial x_{1t} = \omega_{11}.$$

Then property (i) holds if and only if $\omega_{11} = R$. Similarly, property (ii) implies $\omega_{13} = -r$. Further define

$$AE_2 \equiv x_{1t+2} + x_{1t+1} + r \cdot d_{t+1}.$$

Noting $x_{1t+2} = R \cdot x_{1t+1} + \omega_{12} \cdot x_{2t+1} - r \cdot d_{t+1}, AE_2$ can be rewritten as

$$AE_2 = (R+1)x_{1t+1} + \omega_{12} \cdot x_{2t+1}.$$

For property (iii), take partial derivative on AE_2 with respect to d_t

$$\partial AE_2/\partial d_t = -(R^2 - 1) + \omega_{12} \cdot \omega_{23}.$$

Thus, property (iii) holds if and only if $\omega_{12} \cdot \omega_{23} = 0$. Because $\omega_{12} = 0$ corresponds to the case of a savings account, one concludes $\omega_{23} = 0$ (when $\omega_{12} \neq 0$). Similarly, property (iv) implies $\omega_{21} = 0$ because $\partial x_{2t+1}/\partial x_{1t} = \omega_{21}$. Finally, since $\omega_{21} = \omega_{23} = 0$, one can always rescale x_2 such that $\omega_{12} = 1$.

Proposition 8.2 The assumptions in Lemma 8.1 imply

$$\hat{x}_1(K) = K(\gamma - 1) + x_1,$$

and

$$\begin{aligned}\Delta\hat{x}_2^a(K) &\equiv \Delta\hat{x}_2(K) - r \cdot \Delta\hat{b}_1(K) \\ &= (\gamma - R)(\gamma - 1)K + \Delta x_2^a.\end{aligned}$$

It immediately follows that

$$\hat{p}_0(K) \equiv \frac{x_1(K)}{r} + \frac{\Delta x_2^a(K)}{r(R-\gamma)} = \frac{x_1}{r} + \frac{\Delta x_2^a}{r(R-\gamma)},$$

which does not depend on K.

Now prove the remaining part. Because $\hat{p}_0$ is shown invariant to K,

$$\frac{\hat{x}_1(K)}{r} + \frac{\Delta\hat{x}_2^a(K)}{r(R-\gamma)} = \frac{\hat{x}_1(0)}{r} + \frac{\Delta\hat{x}_2^a(0)}{r(R-\gamma)}.$$

Hence, $\hat{x}_1(K) > \hat{x}_1(0)(> 0)$ if and only if $\Delta\hat{x}_2^a(K) < \Delta\hat{x}_2^a(0)$. With CSR, the latter inequality is equivalent to

$$(\hat{g}_2(K) - r) \cdot \hat{x}_1(K) < (\hat{g}_2(0) - r) \cdot \hat{x}_1(0).$$

Again, $\hat{x}(K) > \hat{x}_1(0)(> 0)$ if and only if $\hat{g}_2(K) < \hat{g}_2(0)$ for any K.

Proposition 9.3 First note that RIV implies $E_t[p_{t+\tau} - b_{t+\tau}] = \sum_{s=1}^{\infty} R^{-s} E_t[\widetilde{x}^a_{t+\tau+s}]$. Thus, to prove the conclusion, it suffices to show that there exists $T > 0$ such that for all $\tau > T$, $E_t[\widetilde{x}^a_{t+\tau}] > 0$. Iterating $E_t[\widetilde{x}^a_{t+\tau}]$ backwards yields

$$E_t[\widetilde{x}^a_{t+\tau}] = x^a_t + v_{2t} + \frac{v_{1t}}{\gamma - 1}(\gamma^\tau - 1).$$

Because $\gamma > 1$ and $v_{1t} > 0$, there exists T such that for all $\tau > T$, $E_t[\widetilde{x}^a_{t+\tau}] > 0$.

Appendix 2 The generalization of Lemma 4.1

Consider the following two settings:

(1) The $n \times n$ dynamics

$$\mathbf{z}_{t+1} = \left[\frac{\mathbf{A}|\mathbf{b}}{\mathbf{c}}\right] \mathbf{z}_t, \quad \text{for } t = 1, 2, \ldots,$$

where

$$\mathbf{z}_t \equiv [x_{1t}, \cdots, x_{n-1,t}, d_t]^T \text{and } \mathbf{z}_1 \text{ is an arbitrary initialization,}$$

$$\mathbf{A} \equiv \begin{bmatrix} \omega_{11} & \cdots & \omega_{n-1,1} \\ \vdots & & \vdots \\ \omega_{n-1,1} & \cdots & \omega_{n-1,n-1} \end{bmatrix},$$

$$\mathbf{b} \equiv [\omega_{1n}, \ldots, \omega_{n-1,n}]^T,$$

$$\mathbf{c} \equiv [\omega_{n1}, \ldots, \omega_{nn}], \text{ the dividend policy parameters.}$$

Moreover, the maximal root of $\left[\frac{\mathbf{A}|\mathbf{b}}{\mathbf{c}}\right]$ is strictly less than R.

(2) PVED.

(1) and (2) lead to

$$p_0 = \boldsymbol{\alpha}\mathbf{z}_1 \equiv \sum_{k=1}^{n-1} \alpha_k x_{k1} + \alpha_n d_1, \qquad \text{(B 1)}$$

where $\boldsymbol{\alpha} \equiv [\alpha_1, \ldots, \alpha_n]$ is a function of $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$ (and R).

Proposition. Assume (1) and (2). Then any of the three statements implies the remaining two:

(i) $\boldsymbol{\alpha}$ does not depend on $\mathbf{c}$, i.e., DPI holds;
(ii) $\mathbf{A}$ has a root R;
(iii) $\alpha_n = 0$.

Proof. It proceeds in two parts: The first part shows the equivalency between statement (i) and (ii); the second part establishes the equivalency between (ii) and (iii). The equivalency between (i) and (iii) then follows immediately. □

1. DPI $\Leftrightarrow \alpha_n = 0$.

(a) Suppose $\alpha_n = 0$. For any $\mathbf{z}_t$, the equivalency of PVED, $R \cdot p_{t-1} = p_t + d_t$, and (B 1), implies that

$$R\boldsymbol{\alpha}\mathbf{z}_t = \boldsymbol{\alpha}\mathbf{H}\mathbf{z}_t + d_t. \tag{B 2}$$

Because $\alpha_n = 0$, (B 2) can be rewritten as

$$R\sum_{k=1}^{n-1}\alpha_k x_k = \sum_{k=1}^{n-1}\beta_k x_k + \beta_n d_t.$$

Here $\boldsymbol{\beta} \equiv [\beta_1, \ldots, \beta_n]$ depends, at the most, on $\mathbf{A}$, $\mathbf{b}$, and $\boldsymbol{\alpha}$, but *not* on $\mathbf{c}$ (it can be easily verified). Finally, because $\alpha_1, \ldots, \alpha_{n-1}$ are the unique linear solution to the equation for *all* $\mathbf{z}_t, \alpha_1, \ldots, \alpha_{n-1}$ do not depend on $\mathbf{c}$, either. Thus DPI holds if $\alpha_n = 0$.

(b) Suppose DPI holds. Now let p_0 be the price associated with the dividend policy $\mathbf{c}$, with $\boldsymbol{\alpha}$ defined in (B 1). Because DPI holds, for a different dividend policy $\mathbf{c} + [0, \ldots, 0, \Delta c](\Delta c \neq 0), p_0$ is still the price, with the same $\boldsymbol{\alpha}$. Given any ($\mathbf{x}_0$, d_0), (B 1) leads to

$$\begin{aligned} p_0 &= (\boldsymbol{\alpha}_{n-1}\mathbf{A} + \alpha_n \mathbf{c}_{n-1})\mathbf{x}_0 + (\boldsymbol{\alpha}_{n-1}\mathbf{b} + \alpha_n(c_n + \Delta c))d_0 \\ &= p_0 + \alpha_n \cdot \Delta c \cdot d_0. \end{aligned}$$

It means $\alpha_n \cdot d_0 = 0$. Since it holds for any d_0, then $\alpha_n = 0$.

2. $\alpha_n = 0 \Leftrightarrow \mathbf{A}$ has a root of R.

(a) Suppose $\alpha_n = 0$. From (B 1), it is easy to see that α_n is the column-n, row-n entry of the matrix $\left(R\mathbf{I}_n - \left[\frac{\mathbf{A}|\mathbf{b}}{\mathbf{c}}\right]\right)^{-1}$. The explicit expression for α_n is

$$\alpha_n = \frac{\det(R\mathbf{I}_{n-1} - \mathbf{A})}{\det\left(R\mathbf{I}_n - \left[\frac{\mathbf{A}|\mathbf{b}}{\mathbf{c}}\right]\right)} \tag{B 3}$$

Because $\alpha_n = 0$, $\det(R\mathbf{I}_{n-1} - \mathbf{A}) = 0$, which is equivalent to $\mathbf{A}$ having a root of R.

(b) Suppose $\mathbf{A}$ has a root R. $\alpha_n = 0$ is immediate from (B 3).

Appendix 3 Information dynamics for operating and financial activities

The model presumes CSR and distinguishes between operating and financing activities:

$$\begin{aligned} b_t &= oa_t + fa_t, \\ x_t &= ox_t + fx_t, \end{aligned} \tag{B 4}$$

where x_t thus corresponds to comprehensive earnings. Free cash flows, c_t, equals

$$c_t = ox_t - \Delta oa_t, \tag{B 5}$$

or, equivalently,

$$c_t = \Delta fa_t - fx_t + d_t.$$

The information dynamics related to operating activities satisfy

$$\begin{bmatrix} o\widetilde{x}^a_{t+1} \\ \widetilde{v}_{1t+1} \\ \widetilde{v}_{2t+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \gamma & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} ox^a_t \\ v_{1t} \\ v_{2t} \end{bmatrix} + \begin{bmatrix} \widetilde{\varepsilon}_{1t+1} \\ \widetilde{\varepsilon}_{2t+1} \\ \widetilde{\varepsilon}_{3t+1} \end{bmatrix}. \tag{B 6}$$

To be sure, the first equation in (B 6) can be rewritten as

$$o\widetilde{x}_{t+1} = R \cdot ox_t - r \cdot c_t + v_{1t} + v_{2t} + \widetilde{\varepsilon}_{1t+1}$$

and one can further specify the dynamic of (free) cash flows

$$\widetilde{c}_{t+1} = \theta_1 \cdot ox_t + \theta_2 \cdot c_t + \theta_3 \cdot v_{1t} + \theta_4 \cdot v_{2t} + \widetilde{\varepsilon}_{4t+1}.$$

This equation serves no analytical purpose, however, because CFI applies. There is no need to specify a dividend policy either. That is, the model builds in both DPI and CFI.

The dynamic related to financial activities is

$$fx_{t+1} = r \cdot fa_t + \widetilde{\varepsilon}_{5t+1}.$$

With all assumptions above, PVED implies the following valuation function

$$p_t = AI_t + OI_t \tag{B 7}$$

where

$$AI_t \equiv fa_t + oa_t + \frac{ox_t^a}{r} = \text{“accounting information”},$$

$$OI_t = \frac{R \cdot v_{1t}}{r(R-\gamma)} + \frac{v_{2t}}{r} = \text{“other information”}.$$

Next, consider the concept of net earnings as opposed to x_t, which here is comprehensive earnings. Specifically, define

$$ne_t \equiv ox_t + r \cdot fa_{t-1}$$

so that

$$\varepsilon_{5t} = x_t - ne_t = \text{other comprehensive earnings.}$$

Consistent with GAAP, “windfall” gains and losses on holding financial assets bypass the income statement and show up as a direct debit or credit to shareholders’ equity. Thus note that

$$AI_t = (R/r) \cdot ne_t - (d_t - \varepsilon_{5t}). \tag{B 8}$$

As the model meets the assumptions of Proposition 9.1 in the text, it follows that the OJ formula holds not only for aggregated activities, but for operating activities alone, adjusted for financial assets. Specifically,

$$p_t = \frac{E_t[\widetilde{x}_{t+1}]}{r} \cdot [\frac{g_{t+2} - (\gamma - 1)}{r - (\gamma - 1)}],$$

where

$$g_{t+2} \equiv \frac{E_t[\Delta \widetilde{x}_{t+2} + r \cdot \widetilde{d}_{t+1}]}{E_t[\widetilde{x}_{t+1}]},$$

and for operating activities one obtains

$$p_t = fa_t + \frac{E_t[o\widetilde{x}_{t+1}]}{r} \cdot \left[\frac{h_{t+2} - (\gamma - 1)}{r - (\gamma - 1)}\right], \tag{B 9}$$

where

$$h_{t+2} \equiv \frac{E_t[\Delta o\widetilde{x}_{t+2} + r \cdot \widetilde{c}_{t+1}]}{E_t[o\widetilde{x}_{t+1}]}.$$

At last, one can explain market return in excess of expected return over the period $(t, t+1)$, $\widetilde{r}^e_{t+1} \equiv (\widetilde{p}_{t+1} + \widetilde{d}_{t+1})/p_t - R$, by the expression

$$\widetilde{r}^e_{t+1} = \frac{\widetilde{\varepsilon}_{1t+1}}{r} + \frac{R \cdot \widetilde{\varepsilon}_{2t+1}}{r(R-\gamma)} + \frac{\widetilde{\varepsilon}_{3t+1}}{r} + \widetilde{\varepsilon}_{5t+1}. \tag{B 10}$$

Acknowledgement

The authors wish to thank Jing Liu, Stephen Penman, and Ken Yee for valuable comments on earlier versions of this text.

References

Begley, J. and G. Feltham (2002), 'The relation between market value, earning forecasts, and reported earnings'. *Contemporary Accounting Research* **19**(1), 1–48.

Botosan, C. and M. Plumlee (2005), 'Assessing alternative proxies for the expected risk premium'. *The Accounting Review* **80**(1), 21–53.

Brealey, J. and S. Myers (1984), *Principles of Corporate Finance*. New York: McGraw-Hill, second edition.

Cheng, C. S. A., D. Collins, and H. H. Huang (2006), 'Shareholder rights, financial disclosure and the cost of equity capital'. *Review of Quantitative Finance and Accounting*. Forthcoming.

Cheng, Q. (2005), 'What determines residual income?'. *The Accounting Review* **80**(1), 85–112.

Christensen, P. and G. Feltham (2003), Economics of accounting, *Volume I – Information in markets*. Amsterdam: Kluwer Academic Publishers.

Daske, H. (2006), 'Economic benefits of adopting IFRS or USGAAP – have the expected cost of equity capital really decreased?'. *Journal of Business Finance & Accounting* **33**(3–4), 329–373.

Easton, P. (2004), 'PE ratio, PEG ratios, and estimating the implied expected rate of return on equity capita'. *The Accounting Review* **79**(1), 79–95.

Easton, P. (2006), 'Use of forecasts of earnings to estimate and compare cost of capital across regimes'. *Journal of Business Finance & Accounting* **33**(3–4), 374–394.

Easton, P. and S. Monahan (2005), 'An evaluation of accounting-based measures of expected return'. *The Accounting Review* **80**(2), 501–538.

Easton, P., P. Taylor, and T. Sougiannis (2002), 'Using forecasts of earnings to simultaneously estimate growth and the rate of return on equity investment'. *Journal of Accounting Research* **40**(3), 657–676.

Fairfield, P. M. (1994), 'P/E, P/B and the preset value of future dividends'. *Financial Analysts Journal* **50**(4), 23–31.

Feltham, G. and J. A. Ohlson (1995), 'Valuation and clean surplus accounting for operating and financial activities'. *Contemporary Accounting Research* **11**(2), 689–731.

Francis, J., R. LaFond, P. Olsson, and K. Schipper (2004), 'Costs of equity and earnings attributes'. *The Accounting Review* **79**(4), 967–1010.

Gebhardt, W., C. Lee, and B. Swaminathan (2001), 'Toward an implied cost of capital'. *Journal of Accounting Research* **39**(1), 135–176.

Gode, D. and P. Mohanram (2003), 'Inferring the cost of capital using the Ohlson-Juettner model'. *Review of Accounting Studies* **8**(4), 399–431.

Hutton, A. (2000), 'A valuation framework for quantitative research'. Harvard Business School. Case, N9-1-100-088.

Ohlson, J. A. (1995), 'Earnings, book values, and dividends in equity valuation'. *Contemporary Accounting Research* **11**(2), 661–687.

Ohlson, J. A. (1999a), 'Discussion of an analysis of historical and future-oriented information in accounting-based security valuation models'. *Contemporary Accounting Research* **16**(2), 381–384.

Ohlson, J. A. (1999b), 'On transitory earnings'. *Review of Accounting Studies* **4**(3–4), 145–162.

Ohlson, J. A. (2001), 'Earnings, book values and dividends in equity valuation: An empirical perspective'. *Contemporary Accounting Research* **18**(1), 107–120.

Ohlson, J. A. (2005), 'On accounting-based valuation formulae'. *Review of Accounting Studies* **10**(2–3), 323–347.

Ohlson, J. A. and B. Juettner-Nauroth (2005), 'Expected EPS and EPS growth as determinants of value'. *Review of Accounting Studies* **10**(2–3), 349–365.

Ohlson, J. A., A. Ostaszewski, and Z. Gao (2006), 'Dividend policy irrelevancy and the construct of earnings'. Arizona State University. Working paper.

Ohlson, J. A. and X. J. Zhang (1999), 'Accrual accounting and equity valuation'. *Journal of Accounting Research* **36**, 85–111. Supplement.

Olsson, R. (2005), 'Implications of constant growth of abnormal earnings in perpetuity for equity premia, discount rates, dividends, book values and key financial ratios: An extension of Claus and Thomas'. University of Umea. Working paper.

Ozair, M. (2003), 'Growth in expected earnings and equity valuation'. Tel Aviv University. Working Paper.

Penman, S. (2005), 'Discussion of "On accounting-based valuation formulae" and "Expected EPS and EPS growth as determinants of value"'. *Review of Accounting Studies* **10**(2–3), 367–378.

Penman, S. (2006), *Financial Statement Analysis and Security Valuation*. New York: McGraw-Hill, third edition.

Ryan, S. (1986), 'Structural Models of the Price to Earnings Relation: Measurement Errors in Accounting Earnings'. Doctoral Dissertation, Graduate School of Business. Stanford University.

Sougiannis, T. and T. Yaekura (2001), 'The accuracy and bias of equity values inferred from analysts' earnings forecasts'. *Journal of Accounting, Auditing and Finance* **16**, 331–362.

Thomas, J. and H. Zhang (2006), 'Another look at P/E ratios'. Yale School of Management. Working Paper.

Yee, K. (2005), 'Aggregation, dividend irrelevancy, and earnings-value relations'. *Contemporary Accounting Research* **22**(2), 453–480.

Yee, K. (2006), 'Capitalization of costs and expected earnings growth'. Columbia University. Working Paper.