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Limit Cycles of Differential Equations

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Limit Cycles of Differential Equations

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Foreword

This book contains two sets of revised and augmented notes prepared for the Advanced Course on Limit Cycles and Differential Equations given at the Centre de Recerca Matemàtica in June 2006, as part of its year-long research programme on Hilbert's 16th problem. The common goal of the two sets of notes is to help young mathematicians enter a very active area of research lying on the borderline between dynamical systems, analysis and applications.

The first part of the book, by Colin Christopher, considers some of the topics which surround the Poincar center-focus problem for polynomial systems, a subject closely tied with the integrability of polynomial systems. The second part, by Chengzhi Li, is devoted to the introduction of some basic concepts and methods in the study of Abelian integrals and applications to the weak Hilbert's 16th problem.

Besides our indebtedness to the Centre de Recerca Matemàtica, thanks are due to Jaume Llibre and Armengol Gasull, the course co-ordinators, for giving us this challenging but rewarding opportunity and for providing such a pleasant environment during the programme.

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Part I

Around the Center-Focus Problem

Colin Christopher

Preface

My aim in these notes is to consider some of the topics which surround the Poincaré center-focus problem for polynomial systems. That is, given a polynomial system

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y),$$

with a critical point whose linearization gives a center, under what conditions can we conclude that the point is a center for the nonlinear system?

Clearly, the subject is closely tied with what mechanisms underlie the local integrability of polynomial systems, since the existence of a center implies the existence of a local analytic first integral.

Because these systems are defined algebraically, we expect these mechanisms to be algebraic too, in some sense. This indeed seems to be the case, but the situation is far from being understood except for a growing number of explicit examples.

The choice of topics covered in these notes is very much a personal one, being in the main problems that I have been involved in myself or found interesting. Unfortunately, this has meant that there is much that is missing from this presentation which I felt less competent to comment on. In particular, very little is said on the many detailed analyses of particular systems, nor on the more far-reaching work on holomorphic foliations.

The first part of the notes considers the two main mechanisms known to produce centers in polynomial systems, namely Darboux integrability and algebraic symmetries. The second part considers several topics loosely associated with the idea of monodromy. Though diverse, they share a common theme of teasing out concrete global information from trying to extend the known local behavior, surely one of the most beguiling aspects of the center-focus problem.

Chapter 1

Centers and Limit Cycles

In this chapter I want to give a general background to the center-focus problem, and then to show why the problem is interesting: both in what it tells us about the distinctive algebraic features of polynomial vector fields, and also in the simple concrete estimates it gives of the number of limit cycles which can exist in these vector fields.

1.1 Outline of the Center-Focus Problem

Let X be a polynomial vector field

$$X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}, \tag{1.1}$$

where P and Q are real polynomials of degree at most d. We will identify this vector field with the pair of first-order differential equations

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y). \tag{1.2}$$

We are interested in the situation where this vector field has a critical point which we can choose, without loss of generality, to be at the origin.

The associated linearized system at the origin is given by calculating the Jacobian matrix $J_{(0,0)}$ where

$$J_{(x,y)} = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix}.$$

Then

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = J_{(0,0)} \begin{pmatrix} x \\ y \end{pmatrix} + O(2),$$

where $O(2)$ represents terms of degree 2 or higher in x and y.

If the determinant of $J_{(0,0)}$ is non-zero (the critical point is *non-degenerate*), then the Hartman–Grossman theorem tells us that in a sufficiently small neighborhood of the origin, the system is topologically equivalent to its linear part (i.e. we can ignore the terms of higher order) as long as the eigenvalues of $J_{(0,0)}$ are not pure imaginary. That is, as long as the linear parts do not give a center. This result also holds when P and Q are just continuously differentiable.

The *center-focus problem* asks for the criteria which determine whether a critical point whose linear parts give a center, really is a center.

If the critical point is either a center or focus, we shall use the more general term *monodromic* to cover both cases. The following proposition is straightforward from the Hartman–Grossman theorem. A focus whose linearization gives a center is called a *weak focus*.

Proposition 1.1. *Suppose that the polynomial system* (1.2) *has a non-degenerate critical point at the origin. If the critical point is monodromic, then we can bring the vector field to the form*

$$\dot{x} = -y + \lambda x + p(x,y), \qquad \dot{y} = x + \lambda y + q(x,y), \tag{1.3}$$

by a linear transformation, where p and q are polynomials without constant or linear terms. The case when $\lambda = 0$ corresponds to a weak focus or a center.

From now on we take our polynomial system in the form (1.3) with p and q polynomials of degree at most n.

Example 1.1. The linear parts of the system

$$\dot{x} = -y + x^3, \qquad \dot{y} = x + y^3, \tag{1.4}$$

about the origin give a center, but for the nonlinear system we have

$$\frac{d}{dt}(x^2 + y^2) = 2(x^4 + y^4), \tag{1.5}$$

and so trajectories travel away from the origin, and the system has therefore an unstable focus there.

For differentiable systems, the behavior at the origin can be hard to determine as the following well-known example shows.

Example 1.2. The C^∞ system

$$\dot{x} = -y + xf(x,y), \qquad \dot{y} = x + yf(x,y), \tag{1.6}$$

with

$$f(x,y) = \sin\left(\frac{1}{x^2+y^2}\right)e^{-1/(x^2+y^2)},$$

has an infinite number of limit cycles, $x^2 + y^2 = 1/n\pi$, for $n \in \mathbb{Z}_+$ accumulating at the origin.

However for polynomial (or analytic) systems, this situation does not occur. A critical point whose linear parts give a center is either asymptotically stable, asymptotically unstable or it is a center. This can be most easily seen by computing the return map at the origin.

That is, we choose a one-sided analytic transversal at the origin with a local analytic parameter c, and represent the return map by an expansion

$$c \mapsto h(c) = c + \sum_{i=1}^{\infty} \alpha_i \, c^i. \tag{1.7}$$

By expressing (1.3) in polar coordinates,

$$\dot{r} = \lambda r + O(r^2), \qquad \dot{\theta} = 1 + O(r), \tag{1.8}$$

and invoking standard theorems on analytic dependence on parameters for solutions of the system (1.8), we see that the map (1.7) is analytic in c and also in the parameters of the system, so the expansion (1.7) is valid.

The stability of the origin is clearly given by the sign of the first non-zero α_i, and if all the α_i are zero, then the origin is a center.

However, we can say more. The terms α_{2k} are just analytic functions (with zero constant term) of the previous α_i, so the only interesting functions are the ones of the form α_{2i+1}. If α_{2k+1} is the first non-zero one of these, then at most k limit cycles can bifurcate from the origin. We call this a *weak focus of order k*. Provided we have sufficient choice in the coefficients α_i, we can also obtain that many limit cycles in a simultaneous bifurcation from the critical point.

We call the functions α_{2i+1} the *Lyapunov quantities* of the critical point, and denote them $L(i)$. If all the $L(i)$ vanish, then the critical point clearly is a center. When $\lambda = 0$, the $L(i)$ turn out to be polynomials in the parameters of the system. By the Hilbert basis theorem, the vanishing of all the $L(i)$ must be equivalent to the vanishing of the first N of them, for some integer N. Thus the set of points where we have a center must be an algebraic set, which we call the *center variety.*

It would appear that one part of the center-focus problem is therefore quite easy, as the calculation of the $L(i)$ is computationally straightforward and has been implemented by many authors. However, this is deceptive in two ways. First, because the actual calculation of the common zeros of the first N Lyapunov quantities is computationally intensive (and in general intractable for even quite simple systems), and second because the Hilbert Basis Theorem gives us no explicit value for N. That is, we have no idea in practice when to stop calculating values of $L(i)$.

In order to remedy the second problem, we need to know what mechanisms in polynomial systems force the origin to be a center, then we can show that a particular set of parameter values do indeed give a center. Here lies the main interest in the center-focus problem. This is because these mechanisms should reflect something of the algebraic nature of the systems in which they arise. And indeed, this seems to be the case, at least for the families of systems whose centers have been classified to date. In contrast a generic finite dimensional family of analytic systems of the form (1.3) will have only trivial centers, because the existence of a center has an arbitrarily high codimension generically.

It is conjectured that there are only two main mechanisms which underlie the existence of a center. One is the existence of enough invariant algebraic solutions that an integrating factor can be constructed from them; we consider this case in the next chapter. That is, we seek a first integral or an integrating factor of the form

$$e^{g/h} \prod f_i^{l_i},$$

where f_i, f and g are polynomials, and $f_i = 0$ and $h = 0$ define invariant algebraic curves in system (1.3). We call such a function a *Darboux* function, and the center a *Darboux center.*

The second mechanism is the existence of an algebraic symmetry, that is a map $(x, y) \mapsto (X(x, y), Y(x, y))$, where X and Y are algebraic functions of x and y, which keeps the system fixed but "reverses" time. Any critical points which lie on the set of fixed points of this transformation will be forced to be centers. We consider this case in more detail in Chapter 4. Both these mechanisms clearly have important global consequences for the systems which exhibit them.

We shall see below that the calculation of the Lyapunov quantities can be made purely algebraic, and their vanishing corresponds to the algebraic fact of the existence of a formal power series ϕ such that $X(\phi) = 0$. Seen in this light, the center-focus problem becomes an algebraic question of showing that the existence of a formal local first integral implies the global existence of algebraic solutions or symmetries. It is this fascinating, and unobvious, connection between the local and global properties of polynomial systems that underlies part of the fascination of the center-focus problem.

The other spur to understand what underlies centers in more detail is that they seem a natural "organizing center" for the dynamics of polynomial systems. Due to their algebraic structure, they are also much easier to analyze by perturbation methods. Indeed, many of the strongest conclusions about Hilbert's 16th problem on the number of limit cycles of (1.2) have come exactly from analyzing bifurcations from centers. We give examples of this at the end of this chapter.

With this background in hand, the main conjecture for the center focus problem was first formally stated by Żołądek. I have replaced the original *rationally reversible* by the more general algebraic symmetries as it seems we really do need these in more complex examples.

Conjecture 1.2 (Żołądek). Suppose (1.2) has a center; then the center is either Darboux, or arises from an algebraic symmetry.

1.2 Calculating the Conditions for a Center

In practice, the computation of the Lyapunov quantities from the return map $h(c)$ is not the most efficient way to proceed. Instead we use a method which turns out to be equivalent. It is clear that to find a center, we only need to calculate the Lyapunov quantities $L(k)$ modulo the previous $L(i)$, $i < k$. In particular, $L(0)$ is a multiple of λ and so we can assume that $\lambda = 0$ when we calculate the $L(k)$ for $k > 0$.

We seek a function $V = x^2 + y^2 + \cdots$ such that for our vector field

$$X = (-y + p(x, y))\frac{\partial}{\partial x} + (x + q(x, y))\frac{\partial}{\partial y},$$

we have

$$X(V) = \eta_4(x^2 + y^2)^2 + \eta_6(x^2 + y^2)^3 + \cdots, \tag{1.9}$$

for some polynomials η_{2k}. The calculation is purely formal, and the choice of V can be made uniquely if, for example, we specify that $V(x, 0) - x^2$ is an odd function. It turns out that the polynomials η_{2k+2} for $k > 0$ are equivalent to $L(k)/\pi$ modulo the previous $L(i)$ with $i < k$.

This can be seen by taking the polar coordinates as for the return map and then taking

$$\rho = \sqrt{V(r\cos\theta, r\sin\theta)} = r + O(r^2),$$

to give

$$\dot{\rho} = \eta_{2k+2}\rho^{2k+1}/2 + O(\rho^{2k+2})$$

where η_{2k} is the first non-vanishing of the η_{2i}. Calculation of the return map in the new coordinate system gives the result.

Though this is a purely algebraic way to calculate the Lyapunov quantities, it turns out that if the origin is a center, then the expression for V converges to an analytic function. Thus the existence of a formal first integral V and an analytic one are equivalent in this case. This justifies our assertion that the center-focus problem is a purely algebraic phenomena.

If the linear parts of the system are not quite in the form of (1.3), then rather than transform the system to (1.3), we can replace the terms $x^2 + y^2$ in expansion of V by the equivalent positive definite quadratic form which is annihilated by the linear parts of X.

We note that if we have a center at the origin with first integral V as above, we can always choose coordinates X and Y such that $V = X^2 + Y^2$. The system is thus orbitally equivalent to the linear center:

$$\dot{X} = -Y, \qquad \dot{Y} = X.$$

That is, the system can be brought to this form after multiplying by some analytic function $h(X, Y)$ with $h(0, 0) \neq 0$.

Thus, from the analytic point of view all centers are equivalent. It is only as we restrict our attention to algebraic phenomena that we see the richness of the various center types.

1.3 Bifurcation of Limit Cycles from Centers

As mentioned above, for a generic family of analytic vector fields, the existence of a center has infinite codimension, and therefore centers will not appear. But in polynomial systems, the set of parameters which give centers form significant strata in the set of all polynomial vector fields. The strata therefore are likely organizing centers for the behavior of the systems in their neighborhood in parameter space.

In this last section, we give a nice application of how the knowledge of a strata of the center variety in a family of systems can give good estimates of the number of limit cycles in the whole family.

If we count free parameters in the expression for the return map $h(c)$ in (1.7), we would expect that in general the codimension of the center variety should be one more than the number of limit cycles that can bifurcate from the center as we move away from the center variety. Though this is not true in general, it does seem to hold in many cases. Furthermore, as we show below, it is often sufficient just to look at the linearization of the Lyapunov quantities to determine this.

Suppose that the coefficients of (1.3) depend polynomially on a finite set of parameters Λ, which includes the parameter λ. We choose a transversal at the origin and calculate the return map $h(c)$ as before. The limit cycles of the system are locally given by the roots of the expression

$$P(c) = h(c) - c = \alpha_1\, c + \sum_{i=2}^{\infty} \alpha_k\, c^k,$$

where the α_i are analytic functions of Λ.

We are interested in a fixed point of the parameter space, K, which we can without loss of generality choose to be the origin (λ must be zero at a bifurcation point, and the other parameters can be translated appropriately).

More detailed calculations show that $\alpha_1 = e^{2\pi\lambda} - 1 = 2\pi\lambda\,(1 + O(\lambda))$ and that

$$\alpha_k = \beta_k + \sum_{i=1}^{k-1} \beta_i w_{ik}, \quad (k > 1)$$

where the β_i are polynomials in the coefficients of p and q. The w_{ik} are analytic functions of Λ. We set $\beta_1 = 2\pi\lambda$. Furthermore, β_{2k} always lies in the ideal generated by the previous β_i $(1 \le i \le 2k-1)$ in the polynomial ring generated by the coefficients in Λ. This means that in the calculations below the β_{2i} turn out to be almost redundant. The β_{2i+1} are of course just the Lyapunov quantities $L(i)$.

Suppose now that at the origin of K, we have $L(i) = 0$ for all i, then the critical point is a center. Let $\mathbb{R}[\Lambda]$ denote the coordinate ring generated by the parameters $\Lambda = \{\lambda_0, \ldots, \lambda_r\}$, with $\lambda_0 = \lambda$, and I the ideal generated in this ring by the Lyapunov quantities. As above, the Hilbert basis theorem shows that there is some number N for which the first N of the $L(i)$ generate I.

Since all the β_{2k}'s lie in the ideal generated by the $L(i)$ with $i < k$, we can write

$$P(c) = \sum_{i=0}^{N} b_{2i+1} c^{2i+1} (1 + \Psi_{2i+1}(c, \lambda_0, \ldots, \lambda_r)), \tag{1.10}$$

where the functions Ψ_{2i+1} are analytic in their arguments and $\Psi(0,0) = 0$. A standard argument from [4] shows that at most N limit cycles can bifurcate.

To find the cyclicity of the whole of the center variety, not only is it necessary to know about the zeros of the $L(i)$, but also the ideal that they generate. It is no surprise therefore that few examples are known of center bifurcations [4, 60].

However, if we work about a specific point on the center variety, we can simplify these calculations greatly. Instead of taking the polynomial ring generated by the $L(i)$, we can take the ideal generate by the $L(i)$ in $\mathbb{R}\{\{\Lambda\}\}$, the power series ring of Λ about $0 \in K$ instead. This also has a finite basis, by the equivalent Noetherian properties of power series rings.

What makes this latter approach so useful is that in many cases this ideal will be generated by just the linear terms of the $L(i)$. In which case we have the following theorem.

Theorem 1.3. *Suppose that $s \in K$ is a point on the center variety and that the first k of the $L(i)$ have independent linear parts (with respect to the expansion of $L(i)$ about s); then s lies on a component of the center variety of codimension at least k, and there are bifurcations which produce $k-1$ limit cycles locally from the center corresponding to the parameter value s.*

If, furthermore, we know that s lies on a component of the center variety of codimension k, then s is a smooth point of the variety, and the cyclicity of the center for the parameter value s is exactly $k-1$.

In the latter case, $k-1$ is also the cyclicity of a generic point on this component of the center variety.

Proof. The first statement is obvious. As above we can without loss of generality choose s to be the origin. Since the theorem is local about the origin of K, we can perform a change of coordinates so that the first k of the $L(i)$ are given by λ_i.

Now since we can choose the λ_i independently, we can take $\lambda_i = m_i \epsilon^{2(k-i)}$ for some fixed values m_i $(0 \le i \le k-1)$, and $m_k = 1$. The return map will therefore be an analytic function of ϵ and c. From (1.10) above, we see that

$$P(c)/c = \sum_{i=0}^{k} m_i c^{2i} \epsilon^{2(k-i)} + \Phi(c, \epsilon).$$

Here Φ contains only terms of order greater than $2k$ in c and ϵ. For appropriate choices of the m_i, the linear factors of $\sum_{i=0}^{r} m_i c^{2i} \epsilon^{2(k-i)}$ can be chosen to be distinct and real, and none tangent to $\epsilon = 0$; whence $P(c)/c$ has an ordinary $2k$-fold point at the origin as an analytic function of c and ϵ. Now it is well known that in this case each of the linear factors $c - v_i\epsilon$ of the terms of degree $2k$ can be extended to an analytic solution branch $c = v_i\epsilon + O(\epsilon^2)$ of $P(c)/c = 0$. This gives $2k$ distinct zeros for small ϵ, and the second statement follows.

The third statement follows from noticing that the first k of the $L(i)$ must form a defining set of equations for the component of the center variety. Any $L(i)$ for $i > k$ must therefore lie in the ideal of the $L(i)$ if we work over $\mathbb{R}\{\{\Lambda\}\}$. The result follows from Bautin's argument mentioned above [4].

The last statement follows from the fact that the points where the center variety is not smooth, or where the linear terms of the first k Lyapunov quantities are dependent, form a closed subset of the component of the center variety we are on. $\square$

Armed with this result, we can do two things. One is to try to find complete components of the center variety by comparing the dimension of a known algebraic subset of the center variety with its codimension calculated above. Another is to try to find some family of centers of high codimension to see how many limit cycles we can produce. We give two examples of the latter.

Theorem 1.4. *There exists a class of cubic systems with* 11 *limit cycles bifurcating from a critical point. There exists a class of quartic systems with* 15 *limit cycles bifurcating from a critical point.*

Proof. We first consider the family of cubic systems C_{31} in Żołądek's most recent classification [62]. These systems have a Darboux first integral of the form

$$\phi = \frac{(xy^2 + x + 1)^5}{x^3(xy^5 + 5xy^3/2 + 5y^3/2 + 15xy/8 + 15y/4 + a)^2}. \tag{1.11}$$

There is a critical point at

$$x = \frac{6(8a^2 + 25)}{(32a^2 - 75)}, \qquad y = \frac{70a}{(32a^2 - 75)}.$$

If we translate this point to the origin and put $a = 2$ we find we have the system,

$$\begin{aligned} \dot{x} &= 10(342 + 53x)(289x - 2112y + 159x^2 - 848xy + 636y^2), \\ \dot{y} &= 605788x - 988380y + 432745xy - 755568y^2 + 89888xy^2 - 168540y^3, \end{aligned}$$

whose linear parts give a center.

We consider the general perturbation of this system in the class of cubic vector fields. That is, we take a parameter for each quadratic and cubic term and also a parameter to represent λ above, when the system is brought to the normal form (1.3).

Routine computations now show that the linear parts of $L(0), \ldots, L(11)$ are independent in the parameters and therefore 11 limit cycles can bifurcate from this center.

For the quartic result, we look at a system whose first integral is given by

$$\phi = \frac{(x^5+5x^3+y)^6}{(x^6+6x^4+6/5xy+3x^2+a)^5}. \tag{1.12}$$

The form is inspired by Żołądek's system C45 in [61]. We take $a=-8$ which gives a center at $x=2$, $y=-50$, which we move to the origin. This gives a system

$$\begin{aligned} \dot{x} &= -510x - 6y - 405x^2 - 3xy - 120x^3 - 15x^4, \\ \dot{y} &= 49950x + 510y + 22500x^2 - 1335xy - 15y^2 \\ &\quad +2850x^3 - 630x^2y - 300x^4 - 105x^3y. \end{aligned} \tag{1.13}$$

This time we take a general quartic bifurcation and find that the linear parts of $L(0)$ to $L(15)$ are independent. Hence we can produce 15 limit cycles from this center by bifurcation. □

Remark 1.5. The results in this section can be generalized to take second-order terms in the Lyapunov quantities. If we do so, we find that the quartic system above can actually generate 17 limit cycles.

We give one final result, which uses centers given by both Darboux first integrals and symmetries.

Theorem 1.6. *There exists a quartic system with* 22 *limit cycles. The cycles appear in two nests of 6 cycles and one nest of* 10.

Proof. We work with the cubic center $C_{4,5}$, which was the one considered in Żołądek in [63]. This is of the form

$$\dot{x} = 2x^3 + 2xy + 5x + 2a, \dot{y} = -2x^3a + 12x^2y - 6x^2 - 4ax + 8y^2 + 4y, \tag{1.14}$$

with first integral

$$\phi = \frac{(x^4+4x^2+4y)^5}{(x^5+5x^3+5xy+5x/2+a)^4}. \tag{1.15}$$

When $a=3$, the system has a center at the point $(-3/2, -11/4)$. We translate the system by $(x,y) \mapsto (x-1, y+3)$, which brings the critical point to $(-5/2, 1/4)$. Now we perform a singular transformation $(x,y) \mapsto (x, y^2)$. After multiplying the resulting equation through by y we get the quartic system

$$\begin{aligned} \dot{x} &= y(2x^3 + 6x^2 + 2xy^2 + 5x + 2y^2 + 7), \\ \dot{y} &= -3x^3 + 6x^2y^2 - 30x^2 + 12xy^2 - 57x + 4y^4 - 16y^2. \end{aligned} \tag{1.16}$$

This system has a center at the origin, and we calculate that the linear parts of the Lyapunov quantities $L(0)$ to $L(11)$ are independent.

Now, suppose we add perturbation terms to the system (1.14) in such a way that applying the same transform as that given above we still obtain a system of degree 4. Clearly any perturbation of this form does not affect the center of (1.16) which is given by symmetry.

Furthermore, we can calculate that the new perturbation terms have the linear parts of $L(0)$ to $L(7)$ independent and so can produce 6 limit cycles, which will be doubled by the singular transformation. Thus we have 22 limit cycles in all. □

Notes

The calculation of center conditions for quadratic and homogeneous cubic systems is well known and we do not repeat them here. Recent accounts can be found in [56, 57]. A similar result for cubic systems is well beyond the computational capabilities of even the most powerful computers. Apart from these "standard" results, there are a very large number of finite families of polynomial systems for which center conditions have been calculated. We do not try to summarize them here. A common e-resource for known center conditions (especially the many families of cubic systems that have been discovered) put in a common format and classified according to type would be a real bonus to further research in this area.

The calculation of Lyapunov quantities is also well-trodden ground. Algorithms have been implemented in various ways by many authors. Again, we do not attempt to survey them here. Generally, speaking the generation of Lyapunov quantities is usually the straightforward part. The real computational difficulties arise as we try to find their common zeros. A book by Romanovski and Shafer explaining these techniques and results is currently in preparation [52].

We have not considered at all the equivalent of the center-focus problem for degenerate centers. That is, degenerate critical points with neighborhoods consisting of closed trajectories. The decision problem for whether a general family of monodromic critical points is a center or not for certain parameter values has been shown to be non-algebraic by Il'yashenko (see the account in [1]). There have also been attempts to apply holonomy techniques in the analytic case to show that such points can have more complicated mechanisms which govern the production of a center [5].

More details of the calculations for the center bifurcations can be found in [19], from which the examples in the last section were drawn.

Although our interest is at the moment in real centers, there are good reasons for working over the complex numbers. We can take the existence of a local analytic first integral as the definition of a center in this case. In the case of a real saddle which has a local first integral, we will also use the term *integrable saddle*. We can bring a complex saddle with 1:-1 eigenvalues to the form

$$\dot{x} = x + p(x, y), \qquad \dot{y} = -y + q(x, y), \tag{1.17}$$

which is the complex analog of a weak focus or center. One can calculate Lyapunov quantities for (1.17) exactly as before; these are better known as *saddle quantities* in this case. It seems that the various classes of complex centers arising in quadratic and symmetric cubic systems intersect much more naturally with the real integrable saddles than with the real centers [25].

There is a very close connection between the study of planar polynomial systems and the theory of holomorphic foliations of codimension 1. We will will only mention one nice application of the center-focus problem to holomorphic foliations here. Suppose ω is an integrable polynomial 1-form in $\mathbb{C}^n$ of degree 2: that is, $d\omega \wedge \omega = 0$. Since ω is integral, about any non-singular point in $\mathbb{C}^n$ we have a local analytic first integral. Restricting to a general 2-plane, we get a quadratic system whose critical points must be integrable. Cerveau and Lins Neto [13] have shown that from the knowledge of the classification of centers of quadratic systems it is possible to classify all the possible forms ω can take.

Chapter 2

Darboux Integrability

In this chapter, we consider one of the two main mechanisms which seem to underlie the existence of centers in polynomial vector fields. We only hint at the historical side, which is covered in detail by Schlomiuk [57].

2.1 Invariant Algebraic Curves

We consider the system (1.2). For the statements of the following definitions and propositions it is often more convenient to work with the associated vector field (1.1).

Definition 2.1. Let $f \in \mathbb{C}[x,y]$. If the algebraic curve $f=0$ is invariant by a vector field X of degree d, then $X(f)/f$ is a polynomial of degree at most $d-1$. In this case we say that $f=0$ is an *invariant algebraic curve* of X and $L_f = X(f)/f$ is its *cofactor*.

Note that, if the vector field X has several invariant algebraic curves of different degrees, the cofactors will all lie in $\mathbb{C}_{d-1}[x,y]$, the vector space of polynomials of degree at most $d-1$. This allows us to reduce the problem of Darboux integrability to one of linear algebra. The proof of the next proposition is clear.

Proposition 2.2. *Let $f \in \mathbb{C}[x,y]$ and $f = f_1^{n_1} \cdots f_r^{n_r}$ be its factorization in irreducible factors. Then, for a vector field X, $f=0$ is an invariant algebraic curve with cofactor L_f if, and only if $f_i = 0$ is an invariant algebraic curve for each $i = 1, \ldots, r$ with cofactor L_{f_i}. Moreover $L_f = n_1 L_{f_1} + \cdots + n_r L_{f_r}$.*

Definition 2.3. Let $f, g \in \mathbb{C}[x,y]$; we say that $e = \exp(g/f)$ is an *exponential factor* of the vector field X of degree d, if $X(e)/e$ is a polynomial of degree at most $d-1$. This polynomial is called the *cofactor* of the exponential factor e, which we denote by L_e. The quotient g/f is an *exponential coefficient* of X.

Exponential factors represent the coalescence of two or more invariant algebraic curves and so appear natural in families of vector fields with invariant algebraic curves.

Proposition 2.4. *If* $e = \exp(g/f)$ *is an exponential factor for the vector field* X*, then* f *is an invariant algebraic curve and* g *satisfies the equation*

$$X(g) = gL_f + fL_e \ , \tag{2.1}$$

where L_f *is the cofactor of* f*.*

2.2 The Darboux Method

We are interested in the role of invariant algebraic curves in constructing first integrals and integrating factors of *Darboux* type: that is, functions which are expressible as products of invariant algebraic curves and exponential factors. We recall the following definitions.

Definition 2.5. Let P/Q be a rational function in x and y, with P and Q coprime, then its *degree* is the maximum of the degrees of P and Q.

Definition 2.6. A (multi-valued) function is said to be *Darboux* if it is of the form

$$e^{g/h} \prod_{i=1}^{r} f_i^{\,l_i}, \tag{2.2}$$

where the f_i, g and h are polynomials, and the l_i are complex numbers.

We shall see in the next chapter that the set of such functions is precisely the set of exponentials of integrals of closed rational 1-forms in x and y.

Definition 2.7. Let U be an open subset of $\mathbb{C}^2$. We say that a non-constant function $H : U \to \mathbb{C}$ is a *first integral* of a vector field X on U if, and only if, $X|_U(H) = 0$. When H is the restriction of a rational (resp. Darboux) function to U, then we say that H is a *rational (resp. Darboux) first integral.*

Definition 2.8. We say that a non-zero function $R : U \to \mathbb{C}$ is an *integrating factor* of a vector field X on U if, and only if, $X(R) = -\mathrm{div}X \cdot R$ on U, where div denotes the divergence of the vector field.

If we know an integrating factor we can compute by quadrature a first integral of the system up to a constant. Reciprocally, if H is a first integral of the vector field (1.2), then there is a unique integrating factor R satisfying

$$R\,a = \frac{\partial H}{\partial y} \quad \text{and} \quad R\,b = -\frac{\partial H}{\partial x} \ . \tag{2.3}$$

Such R is called the *integrating factor associated to* H.

A theorem of Singer [58] shows that if H is a Liouvillian function, then the integrating factor is Darboux. In an earlier work, Prelle and Singer [50] show that if H is an elementary function, then the integrating factor is the N-th root of a rational function. We shall demonstrate Singer's theorem in the next chapter.

The idea behind the Darboux method is to use the invariant algebraic curves of the system to find an integrating factor of the form (2.2). This, in turn, is purely a matter of linear algebra from Proposition 2.2 as all the cofactors lie in $\mathbb{C}_{d-1}[x, y]$. A simple introduction to these things can be found in [20].

For example, we can find a Darboux first integral (2.2) if we can find constants l_i and m_i such that

$$\sum_{i=1}^{r} l_i L_{f_i} + \sum_{j=1}^{s} m_j L_{e_j} = 0,$$

where the L_{f_i} and L_{e_j} represent the cofactors of f_i and $\exp(g_j/h_j)$ respectively. In particular, this will always happen if there are more than $d(d+1)/2$ such curves or exponential factors.

Proposition 2.9. *Let X be a vector field. If X admits p distinct invariant algebraic curves $f_i = 0$, for $i = 1, \ldots, p$, and q independent exponential factors e_j, for $j = 1, \ldots, q$. Then the following statements hold.*

(a) *There are $\lambda_i, \rho_j \in \mathbb{C}$, not all zero, such that $\sum_{i=1}^{p} \lambda_i L_{f_i} + \sum_{j=1}^{q} \rho_j L_{e_j} = 0$ if and only if the (multi-valued) function $f_1^{\lambda_1} \cdots f_p^{\lambda_p} e_1^{\rho_1} \cdots e_q^{\rho_q}$ is a first integral of the vector field X.*

(b) *There are $\lambda_i, \rho_j \in \mathbb{C}$, not all zero, such that $\sum_{i=1}^{p} \lambda_i L_{f_i} + \sum_{j=1}^{q} \rho_j L_{e_j} = -\mathrm{div}(X)$ if and only if the function $f_1^{\lambda_1} \cdots f_p^{\lambda_p} e_1^{\rho_1} \cdots e_q^{\rho_q}$ is an integrating factor of X.*

Thus, the problem of finding first integrals or integrating factors is reduced to a question of linear algebra on the set of cofactors. In order to reduce the dimension of this space we introduce the following concepts from [15].

Proposition 2.10. *Let p be a critical point of the vector field X. Then if f is an invariant algebraic curve of X which does not vanish at p, its cofactor L_f must vanish at p. Furthermore, if $e = \exp(g/f)$ is an exponential factor of X, then L_e must vanish at p too.*

Proof. This follows directly from the equations $X(f) = L_f f$ and $X(g) = L_f g + L_e f$. □

Definition 2.11. Let X be a vector field of degree d, and $S \subset \mathbb{C}^2$ a finite set of points (possibly empty). The *restricted cofactor space with respect to S*, Σ_S, is defined by

$$\Sigma_S = \cap_{p \in S} m_p \cap \mathbb{C}_{d-1}[x, y] \ ,$$

where m_p is the maximal ideal of $\mathbb{C}[x, y]$ corresponding to the point p.

If S consists of s points, then we say that they are *independent* with respect to $\mathbb{C}_{d-1}[x, y]$ if

$$\sigma := \dim \Sigma_S = \dim \mathbb{C}_{d-1}[x, y] - s = \frac{1}{2}(d+1)(d+2) - s \ .$$

Theorem 2.12. *Let X be a vector field of degree d. Assume that X has p distinct invariant algebraic curves $f_i = 0$, $i = 1, \ldots, p$ and q exponential factors $e_i = \exp(g_i/h_i)$, $i = 1, \ldots, q$, where each h_i is equal to f_k for some k. Suppose, furthermore, that there are s critical points $p_1, \ldots, p_r$ which are independent with respect to $\mathbb{C}_{d-1}[x, y]$, and $f_j(p_k) \neq 0$ for $j = 1, \ldots, p$ and $k = 1, \ldots, r$. Then the following statements hold.*

(a) *If $p + q \geq \sigma + 2$, then X has a rational first integral.*

(b) *If $p + q \geq \sigma + 1$, then X has a Darboux first integral.*

(c) *If $p + q \geq \sigma$, and $\operatorname{div}(X)$ vanishes at the p_i, then X has either a Darboux first integral or a Darboux integrating factor.*

Proof. Statements (b) and (c) follow from counting dimensions and applying Proposition 2.9. One has just to observe that all possible cofactors are contained in Σ_S by Proposition 2.10.

When $p + q \geq \sigma + 2$, we apply (b) to obtain two independent Darboux first integrals, say H_1 and H_2. We can see easily that the integrating factor R_i associated to $\log H_i$ is a rational function. Since the quotient of two integrating factors is a first integral, the statement (a) follows from the independence of H_1 and H_2. □

Definition 2.13. If (1.2) has a center given by a Darboux first integral or integrating factor, we call it a *Darboux* center.

Unfortunately, given the degree of the system, there is no bound on the degree of the invariant algebraic curves. In fact systems are known with curves of arbitrary degree. This causes problems in trying to find all Darboux centers.

The following conjecture would at least show that the systems (1.2) with invariant algebraic curves form an algebraic subset in the set of parameters.

Conjecture 2.14. There is a number $N(d)$ such that if (1.2) has an invariant algebraic curve, then it has an invariant algebraic curve of degree at most $N(d)$.

A similar conjecture would also be useful for applications to the center-focus problem.

Conjecture 2.15. There is a number $N(d)$ such that if (1.2) has an invariant algebraic curve of degree greater than $N(d)$, then it has a Darboux first integral or integrating factor.

Of course, in both cases we would prefer to have some concrete way of determining $N(d)$.

A quadratic system with an invariant algebraic curve of degree 12 but not Darboux integrable is given in [23].

2.3 Multiple Curves and Exponential Factors

If we are interested in families of Darboux centers, then we need to be able to understand how the family will change at the points where one or more curves of the system coalesce. In general, this will give rise to exponential factors, as the following simple example makes clear.

Example 2.1. Consider the vector field

$$X = x\frac{\partial}{\partial x} + ((1+\ell)y + x)\frac{\partial}{\partial y}$$

with invariant algebraic curves $x = 0$ and $x+\ell y = 0$. As ℓ tends to zero, then these two curves coalesce. However we can recover an exponential factor, by taking the limit of the Darboux function $((x+\ell y)/x)^{1/\ell}$ which tends to $\exp(y/x)$.

In general, we would hope that an exponential factor $\exp(f/g)$ corresponds to the coalescence of two invariant algebraic curves $f = 0$ and $f + \ell g = 0$ as ℓ tends to zero. That is, we consider $\exp(f/g)$ as

$$\exp(f/g) = \lim_{\ell\to 0}((f+\ell g)/f)^{1/\ell}.$$

Although a general explanation of this phenomena is not yet known, we give here a summary of the case when $f = 0$ is given by an irreducible polynomial f. The proofs of these results and precise definitions can be found in [21].

Suppose $X = X_\lambda$ depends on a parameter λ. If X_λ has m invariant algebraic curves $f_{\lambda,i} = 0$, $i = 1, \ldots, m$, which converge to the curve $f = 0$ as λ tends to zero, then we say that the curve $f = 0$ of X_0 has multiplicity m.

It turns out that a multiple curve of multiplicity m will have associated to it $m-1$ exponential factors of the form $\exp(g_i/f^i)$ for g_i a polynomial of degree at most $i\deg(f)$.

How do we detect such multiple curves without knowing a priori a family which they lie in? It turns out that there are several equivalent definitions of multiplicity of an algebraic curve and a couple of these are very computational in form.

First, we can associate to a multiple curve a *generalized invariant algebraic curve* of the form

$$F = f_0 + \varepsilon f_1 + \cdots + \varepsilon^{k-1} f_{k-1}, \qquad f_0 = f,$$

where each of the polynomials f_i have degree at most $\deg(f)$, and ε is an algebraic quantity with $\varepsilon^k = 0$. That is, we have a curve with some "infinitesimal" information attached. This generalized invariant algebraic curve satisfies

$$X(F) = FL_F, \qquad L_F = L_0 + \varepsilon L_1 + \cdots + \varepsilon^{k-1} L_{k-1}.$$

Conversely, it is easy to see that if we have such a generalized curve satisfying the above equation, then the coefficients of ε^i of

$$\log(F) = \log(f_0) + \varepsilon\frac{f_1}{f_0} + \varepsilon^2\frac{f_2 - f_1^2/2}{f_0^2} + \cdots$$

give $m-1$ exponential factors.

A more computational approach to finding this multiplicity is given by computing the extactic:

$$\mathcal{E}_n \det\begin{pmatrix} v_1 & v_2 & \cdots & v_l \\ X(v_1) & X(v_2) & \cdots & X(v_l) \\ \vdots & \vdots & \cdots & \vdots \\ X^{l-1}(v_1) & X^{l-1}(v_2) & \cdots & X^{l-1}(v_l) \end{pmatrix}, \tag{2.4}$$

where $n = \deg(f)$, and $v_1, v_2, \ldots, v_l$ is a basis of $\mathbb{C}_n[x,y]$, the $\mathbb{C}$-vector space of polynomials in $\mathbb{C}[x,y]$ of degree at most n, and we take $l = (n+1)(n+2)/2$, $X^0(v_i) = v_i$ and $X^j(v_i) = X^{j-1}(X(v_i))$. It turns out that the multiplicity can also be given by the maximum power of f appearing in $\mathcal{E}_n$.

Finally, computations of multiplicity can also be obtained from the holonomy group, which we examine in Chapter 9. Here we also need to impose some restrictions on the critical points which can appear on $f = 0$. Further details for all these equivalent definitions can be found in [21].

Notes

Darboux functions are ubiquitous in the area of polynomial systems. For example, when the system is non-integrable, but has sufficient limit cycles, they can be used to give geometric non-existence results for limit cycles [20]. Another area of application is for showing not only integrability but linearizability [41]. Here we look for substitutions of the form

$$X = x\,m(x,y), \qquad Y = y\,n(x,y),$$

where m and n are Darboux functions, in order to bring the system to a linear form.

Movasati [45] has shown that given integers d_i, $i = 1, \ldots, r$ with $\sum d_i = d+1$, the set of centers which have a Darboux first integral

$$\prod_{i=1}^{r} f_i^{l_i}, \qquad \deg(f_i) = d_i$$

form a complete component of the center variety.

I'd like to mention one problem here which is quite intriguing though of minor importance. Given a system (1.2) with a Darboux integrating factor, what can be said about its first integral. In the case where the invariant algebraic curves $f_i = 0$ are in generic position, and we have no exponential factors, we can also find a Darboux first integral.

This has been shown algebraically in [22], however it also has a nice geometric interpretation. Let D be the Darboux integrating factor, then

$$\phi = \int D(P\,dy - Q\,dx)$$

defines a multi-valued integral outside the set Z of zeros of the curves $\{f_i = 0\}$. The effect of passing around a non-trivial loop γ in the complement of Z takes ϕ to $h_\gamma(\phi) = a_\gamma \phi + b_\gamma$ for some constants a_γ and b_γ. If the set Z together with the line at infinity has only nodal singularities (which will be true in the generic case when the $f_i = 0$ are smooth and intersect transversally with each other and infinity), it is well known that the complement of Z has an abelian fundamental group. This means that the maps h_γ must commute. It is then straightforward to show that either $a_\gamma = 1$ for all γ or the addition of a constant to ϕ makes all the b_γ vanish. In the first case, $\phi - \sum c_i \log(f_i)$ is single-valued for some constants c_i given by the b_γ, and in the second, $\phi / \prod f_i^{m_i}$ is single-valued for some constants m_i given by the a_γ. Application of growth estimates shows that these functions are therefore rational and hence gives either $\exp(\phi)$ or ϕ as a Darboux first integral. In the case when D contains exponential terms, then it is necessary to use the extended monodromy group of Żołądek [65].

In general, it seems that the first integrals have the property that they are a sum of a Darboux function plus a sum of several one-dimensional integrals of the form

$$\int^{h(x,y)} e^{s(u)} \prod r_i(u)_i^{\lambda},$$

where h, s and the r_i are rational functions. Żołądek calls these *Darboux–Schwartz–Christoffel* integrals, and conjectures that all first integrals of Darboux centers can be obtained in this form. Interesting examples can be found in [65].

Chapter 3

Liouvillian Integrability

In this chapter we want to prove that Darboux integrability corresponds to the notion of Liouvillian integrability, or "solution by quadratures".

3.1 Differential Fields and Liouvillian Extensions

Let (K, Δ) be a *differential field*. That is, a field K equipped with a set of commuting operators $\delta : K \to K$, $\delta \in \Delta$, called *derivations* satisfying

$$\delta(x + y) = \delta x + \delta y, \qquad \delta(xy) = (\delta x)y + x(\delta y).$$

We shall assume that all fields have characteristic zero.

A *differential field extension* $(K', \Delta') \supset (K, \Delta)$ is a field extension $K' \supset K$ for which the restriction of each $\delta' \in \Delta'$ to K is given by some $\delta \in \Delta$. Because of this compatibility condition we can use Δ to represent both sets of derivations without confusion.

From now on we will drop the explicit references to the derivations Δ of the differential fields unless necessary.

Example 3.1. If K' is an algebraic extension of a differential field K, then the derivations of K extend to K' in an unique way. This does not hold true if the characteristic of the field is different from zero.

Example 3.2. If K is a differential field, we can form the extension field $K' = K(t)$, for some t transcendental over K. To say that this is a differential extension is to say that there are elements $a_\delta \in K(t)$ such that $\delta t = a_\delta$ for all $\delta \in \Delta$. The commutativity of the derivations means that $\delta_1 a_{\delta_2} = \delta_2 a_{\delta_1}$ for all δ_1, $\delta_2 \in \Delta$.

Definition 3.1. A differential field extension $K \supset k$ is called *Liouvillian* if it can be written as a tower of differential extensions

$$k = K_0 \subset K_1 \subset \cdots \subset K_n = K,$$

where at each step we have one of the following conditions:

(i) K_{i+1} is a finite algebraic extension of K_i.

(ii) $K_{i+1} = K_i(t)$ for some t with $\delta t/t \in K_i$ for all $\delta \in \Delta$.

(iii) $K_{i+1} = K_i(t)$ for some t with $\delta t \in K_i$ for all $\delta \in \Delta$.

That is, our new field contains functions which can be got by successive (i) solutions of algebraic equations, (ii) exponentials of integrals and (iii) integrals.

In our applications we will always take the extensions starting from the field $K_0 = \mathbb{C}(x, y)$ with the standard derivations $\Delta = \{\partial/\partial x, \partial/\partial y\}$. We shall write $\partial f/\partial x$ for $(\partial/\partial x)f$.

We say that (1.2) has a Liouvillian first integral if there exists a function ϕ in some Liouvillian extension of $\mathbb{C}(x, y)$ such that $X(\phi) = 0$, where X is the associated vector field in (1.1), and at least one of $\partial\phi/\partial x$ and $\partial\phi/\partial y$ is not zero.

Theorem 3.2 (Singer [58])**.** *The system* (1.2) *has a Liouvillian first integral, if and only if it has a Darboux integrating factor.*

In fact, Singer's original proof [58] shows slightly more than this. He proves that if a polynomial system has a trajectory whose phase curve is given by the zeros of a Liouvillian function, then the trajectory is either algebraic, or we have a Liouvillian first integral. This demonstrates something of the importance of invariant algebraic curves in the analysis of polynomial systems.

The "if" statement in the theorem is clear: if D is our Darboux integrating factor, then $\delta D/D$ is a rational function and we can therefore add D to our field by one extension of type (ii). The converse will be proved in the next section.

3.2 Proof of Singer's Theorem

Proposition 3.3. *If the system* (1.2) *has a Liouvillian first integral, then it has an integrating factor of the form*

$$\exp\Big(\int U\,dx + V\,dy\Big), \qquad \frac{\partial U}{\partial y} = \frac{\partial V}{\partial x},$$

where U and V are rational functions.

Proof. From the hypothesis of the theorem, there exists a ϕ in some Liouvillian extension field K of $\mathbb{C}(x, y)$ with $X(\phi) = 0$. Thus,

$$h\frac{\partial\phi}{\partial x} = Q, \qquad h\frac{\partial\phi}{\partial y} = -P,$$

for some $h \in K$. Thus, putting

$$A = \frac{1}{h}\frac{\partial h}{\partial x}, \qquad B = \frac{1}{h}\frac{\partial h}{\partial y}$$

we have elements A and B in K such that

$$PA + QB = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}, \qquad \frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}. \tag{3.1}$$

We want to show that if the above equation can be satisfied for A and B in K_{i+1} it can be satisfied in K_i. The conclusion follows directly from the case $i = 0$, putting $U = A$ and $V = B$.

We consider each type of extension in turn. Without loss of generality, we can assume that the extensions in (ii) and (iii) are transcendental, else we consider them under (i).

(i) Let $\tilde{K}_{i+1}$ be the normal closure of K_{i+1}. Let Σ be the set of automorphisms of K_{i+1} fixing K_i. We also write N for $|\Sigma|$. Then,

$$\sum_{\sigma\in\Sigma} \sigma(PA + QB) = N(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}),$$

and

$$\frac{\partial \sigma(A)}{\partial y} = \frac{\partial \sigma(B)}{\partial x},$$

for all σ in Σ. Hence, taking

$$\bar{A} = \frac{1}{N}\sum_{\sigma\in\Sigma} \sigma(A), \qquad \bar{B} = \frac{1}{N}\sum_{\sigma\in\Sigma} \sigma(B),$$

we have

$$P\bar{A} + Q\bar{B} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}, \qquad \frac{\partial \bar{A}}{\partial y} = \frac{\partial \bar{B}}{\partial x},$$

where $\bar{A}$ and $\bar{B}$ must lie in K_i.

(ii) We have $A = a(t)$ and $B = a(t)$ for some rational functions a and b with coefficients in K_i. Since t is transcendental, we can expand $a(t)$ and $b(t)$ formally as Laurent series in t. Let a_0 and b_0 be the coefficients of t^0 in the expansions of $a(t)$ and $b(t)$ respectively, then the coefficient of t^0 in (3.1) can be seen to give

$$Pa_0 + Qb_0 = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}, \qquad \frac{\partial a_0}{\partial y} = \frac{\partial b_0}{\partial x}.$$

(iii) We take $A = a(t)$ and $B = b(t)$ as above, but now expand formally as a Laurent series in $1/t$. Let r be the highest power of t appearing in the expansion of either $a(t)$ or $b(t)$, and let a_r and b_r be the respective coefficients of t^r in these expansions. Equating powers of t^r in (3.1), we get

$$Pa_r + Qb_r = 0, \qquad \frac{\partial a_r}{\partial y} = \frac{\partial b_r}{\partial x},$$

for $r \neq 0$, and

$$Pa_0 + Qb_0 = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}, \qquad \frac{\partial a_0}{\partial y} = \frac{\partial b_0}{\partial x},$$

for $r = 0$. In the latter case the inductive step is complete, and in the former we can find an element $h \in K_i$ such that

$$P = -b_r h, \qquad Q = a_r h.$$

Then

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = P\frac{1}{h}\frac{\partial h}{\partial x} + Q\frac{1}{h}\frac{\partial h}{\partial y},$$

and taking

$$A = \frac{1}{h}\frac{\partial h}{\partial x}, \qquad B = \frac{1}{h}\frac{\partial h}{\partial y},$$

we complete the inductive step in this case also.

Thus we can repeat the inductive step until we can find solutions to (3.1) in $K_0 = \mathbb{C}(x, y)$ and we are finished. □

To finish the proof of Theorem 3.2 we need the following proposition.

Proposition 3.4. *If the system* (1.2) *has an integrating factor of the form*

$$\exp\Big(\int U\,dx + V\,dy\Big), \qquad \frac{\partial U}{\partial y} = \frac{\partial V}{\partial x},$$

where U and V are rational functions of x and y, then there exists an integrating factor of the form

$$exp(g/f)\prod {f_i}^{l_i},$$

where g, f and the f_i are polynomials in x and y.

Proof. Let K be a normal algebraic extension of $\mathbb{C}(y)$ which is a splitting field for the numerators and denominators of U and V considered as polynomials in x over $\mathbb{C}(y)$. We can thus rewrite U and V in their partial fraction expansions

$$U = \sum_{i=1}^{r}\sum_{j=1}^{n_i}\frac{\alpha_{i,j}}{(x-\beta_i)^j} + \sum_{i=0}^{N}\gamma_i x^i, \qquad V = \sum_{i=1}^{\bar{r}}\sum_{j=1}^{\bar{n}_i}\frac{\bar{\alpha}_{i,j}}{(x-\beta_i)^j} + \sum_{i=0}^{\bar{N}}\bar{\gamma}_i x^i,$$

where the $\alpha_{i,j}$, $\bar{\alpha}_{i,j}$, β_i, γ_i and $\bar{\gamma}_i$ are elements of K. By taking $\alpha_{i,j}$, $\bar{\alpha}_{i,j}$, γ_i and $\bar{\gamma}_i$ to be zero outside their defined values, we can neglect the explicit mention of the summation limits without confusion.

We now apply the condition $U_y = V_x$ to the above expressions. Gathering terms and using the uniqueness of the partial fraction expansion, we see that (using f' to denote df/dx)

$$\gamma_i' = \bar{\gamma}_{i+1}(i+1), \qquad \alpha_{i,j+1}' + j\beta_i'\alpha_{i,j} + j\bar{\alpha}_{i,j} = 0. \tag{3.2}$$

In particular we have $\alpha'_{i,1} = 0$.

We now write down a function and show that it is indeed the integral $\int U\,dx + V\,dy$. Let ϕ be given by

$$\phi = \sum \alpha_{i,1} \log(x - \beta_i) + \sum \frac{\alpha_{i,j}}{(x-\beta_i)^{j-1}}\Big(\frac{-1}{j-1}\Big) + \sum \frac{\gamma_i x^{i+1}}{i+1} + \int \bar{\gamma}_0\,dy,$$

where the last term represents any primitive of $\bar{\gamma}$. It is easy to verify that $\partial\phi/\partial x = U$ and $\partial\phi/\partial y = V$ using (3.2).

We now let Σ represent the group of automorphisms of K over $\mathbb{C}(y)$, and let $N = |\Sigma|$. As in the previous proof, we define

$$\bar{\phi} = \frac{1}{N}\sum_{\sigma\in\Sigma}\sigma(\phi),$$

where we note that

$$\sigma(\alpha_{i,1}\log(x-\beta_i)) = \alpha_{i,1}\log(x - \sigma(\beta_i)),$$

and

$$\sigma\Big(\int \bar{\gamma}_0\,dy\Big) = \int \sigma(\bar{\gamma}_0)\,dy,$$

is only defined up to an arbitrary constant.

It is clear that we still have $\partial\bar{\phi}/\partial x = U$ and $\partial\bar{\phi}/\partial y = V$. Furthermore we have

$$\bar{\phi} = \sum l_i \log(R_i(x,y)) + R(x,y) + \int S(y)\,dy,$$

where R_i, R and S are rational functions. We can evaluate the integral in the last term via the partial fraction expansion of S as

$$\int S(y)\,dy = \sum \alpha_i \log(S_i(y)) + S_0(y),$$

where the S_i are polynomials in y. Taking exponentials, the integrating factor obtained is of the form desired. $\square$

3.3 Riccati equations

We mention briefly here another possible mechanism which guarantees that a critical point is a center. However, although we shall give non-trivial examples of this mechanism for the integrability of a saddle, we do not know of any non-trivial example of such a case for real centers. These systems exhibit both properties of symmetric systems in that there is a reduction to a simpler system, and Darboux systems, in that there is a first integral in the form of a Darboux first integral, but where the factors are solutions of a second-order differential equation.

We first give an example of a $1 : -\lambda$ resonant saddle which is integrable via the solutions of a Riccati equation.

Example 3.3. The system

$$\dot{x} = x(1-x), \qquad \dot{y} = -\lambda y + d_1 x^2 + d_2 xy + y^2, \tag{3.3}$$

with $\lambda > 0$ but not an integer has a first integral

$$\phi = x^\lambda \frac{(y-\alpha x)F_1(x) + x(1-x)F_1'(x)}{(\lambda - y + (\alpha-\lambda)x)F_2(x) - x(1-x)F_2'(x)}, \tag{3.4}$$

where $F_1(x) = F(a,b;c;x)$ and $F_2(x) = F(a-c+1, b-c+1; 2-c; x)$ where $F(a,b;c;x)$ is the Gauss hypergeometric function

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n$$

with

$$(a)_n = \begin{cases} a(a+1)\dots(a+n-1) & n \geq 1 \\ 1 & n = 0 \end{cases}$$

and similarly for $(b)_n$ and $(c)_n$. Here we define α as a root of $\alpha^2 - (\lambda - d_2)\alpha + d_1 = 0$, $c = \lambda + 1$ and a and b are the roots of $A^2 - (1 + 2\alpha + d_2)A + \alpha(\lambda+1) = 0$.

It is easy to see that a transformation $x \mapsto X^n$ will give non-trivial examples of integrable $1 : -1$ saddles. The functions in the numerator and denominator of (3.4) satisfy a similar equation to the polynomials defining invariant algebraic curves. In particular, they have polynomial cofactors.

The following example appears quite naturally in the class of cubic systems with a $1 : -1$ saddle (see [51]).

Example 3.4. Consider the system

$$\dot{x} = x - 9bx^3 - ay^3, \qquad \dot{y} = -y + bxy - 6b^2x^2y. \tag{3.5}$$

We perform a transformation

$$X = y^3(1-3bx)^{-2}, \qquad Y = x - aX/4,$$

which brings the system to the form

$$\dot{X} = -3X - 6abX^2, \qquad \dot{Y} = Y + \frac{39}{16}a^2bX^2 + \frac{9}{2}abXY + 3bY^2,$$

which can be brought to the form of Example 3.3 by a simple scaling of the X and Y axes.

It would be interesting to consider extending the work in the previous chapter to include solutions of second-order linear differential equations to see if the above types of integrals are the only ones.

Notes

There are several simplified proofs of Singer's theorem in the literature [12, 48, 64]. We have followed loosely the one in Pereira [48]. Singer's original proof is in [58]. The proof can be carried over very simply to the case of holomorphic foliations of codimension 1, and has been done so by several of the authors cited.

If we restrict items (ii) and (iii) in Theorem 3.2 to just the addition of exponentials and logarithms,

(ii)′ $K_{i+1} = K_i(t)$ for some t with $\delta t/t = g$ for some $g \in K_i$ and all $\delta \in \Delta$,

(iii)′ $K_{i+1} = K_i(t)$ for some t with $\delta t = \delta g/g$ for some $g \in K_i$ and all $\delta \in \Delta$,

we obtain the concept of an *elementary* first integral. This corresponds to integrals in "closed form" (without quadratures). In an earlier paper, Prelle and Singer [50] showed that all elementary first integrals have integrating factors which are fractional powers of a rational function in x and y. The proof is similar to the Liouvillian case. There have been attempts to implement these results as symbolic integration routines. One indication of how this can be done is given in [40]. Liouvillian integrability is discussed in [3]. There are also some partial results known about Liouvillian integrability for higher order differential equations (see [2] for example).

The role of Riccati equations for the center problem was discussed by Żołądek in [64], who also gives the example analyzed in Example 3.3.

Of course it would be interesting to examine other types of extensions. Casale [12] gives results for extensions involving the solution of Riccati equations. The question is closely tied with the existence of Godbillon–Vey sequences of finite length. Those of length 1 correspond to Darboux integrable systems, and those of length 2 to Riccati systems. It is not clear whether higher length sequences play a role in planar systems or not. For some suggestive results in this direction see [14].

Chapter 4

Symmetry

In this chapter we consider the second mechanism which gives rise to centers in polynomial systems: the existence of an algebraic symmetry.

After some brief preliminary comments, we shall show that such symmetries can be used to obtain a complete classification of centers in polynomial Liénard systems.

4.1 Algebraic Symmetries

Let $(x,y) \mapsto (X(x,y), Y(x,y))$ be an analytic transformation which is a local involution in the neighborhood of the origin. After a linear transformation we can assume that

$$X = x + r(x,y), \qquad Y = -y + s(x,y).$$

Choosing new coordinates

$$\xi = \frac{x + X(x,y)}{2} = x + O(2), \qquad \zeta = \frac{y - Y(x,y)}{2} = y + O(2),$$

the involution is brought to the form $(\xi, \zeta) \mapsto (\xi, -\zeta)$ and so we have a symmetry with respect to the line $\zeta = 0$.

If the critical point at the origin is monodromic, then it is clear that the symmetry condition implies that we have a center. However, we can see this in another way which will also apply to the case of complex centers.

In the new coordinates, (1.2) becomes

$$\dot{\xi} = \zeta \tilde{P}(\xi, \zeta^2) h(\xi, \zeta), \qquad \dot{\zeta} = \tilde{Q}(\xi, \zeta^2) h(\xi, \zeta), \tag{4.1}$$

with $h(0,0) \neq 0$. We can remove the factor h without loss of generality, as we are only interested in the orbital behavior of the system. Doing this, and taking $Z = \zeta^2$, we obtain (after again ignoring a common factor ζ) the *reduced* system

$$\dot{\xi} = \tilde{P}(\xi, Z), \qquad \dot{Z} = 2\tilde{Q}(\xi, Z). \tag{4.2}$$

We call the map $(\xi, \zeta) \mapsto (\xi, \zeta^2)$ a *reducing transformation.* Now, (4.2) no longer has a singular point at the origin, and hence there is a local first integral $\phi(\xi, Z)$ in the neighborhood of the origin. The pull back of this first integral via the reducing transformation gives a first integral $\phi(\xi, \zeta^2)$ of (4.1), and hence (4.1) has a center at the origin.

We saw in Chapter 1 that every center is orbitally equivalent to the linear center by an analytic change of coordinates, and hence every center has an analytic symmetry. So the existence of an analytic symmetry does not carry with it much information about the real "cause" of the center.

To understand the mechanisms behind a center-focus problem, we will only be interested in functions $X(x, y)$ and $Y(x, y)$ which are algebraic over $\mathbb{C}(x, y)$. This gives us a nice algebraic and global mechanism for a center.

Example 4.1. The Kukles' system

$$\dot{x} = y, \qquad \dot{y} = -x + a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2 + a_7y^3,$$

has a symmetry in the x-axis for $a_2 = a_5 = a_7 = 0$, and a symmetry in the y-axis for $a_1 = a_3 = a_5 = a_7 = 0$. From the symmetry in the x-axis we obtain a reducing transformation $Z = y^2$, and a reduced system

$$\dot{x} = 1, \qquad \dot{Z} = 2(-x + a_1x^2 + a_3Z + a_4x^3 + a_6xZ),$$

having removed a common factor y from the system.

Those centers which are given by algebraic symmetries seem to form components of the center variety of relatively large dimension compared with the Darboux centers. In the case of systems of low degree, however, the reduced systems tend to be sufficiently simple that they can be algebraically integrated, and so are Darboux as well as symmetric.

We want to show that algebraic symmetries comprise all the centers for Polynomial Liénard systems. In order to do this, we need a nice characterization of analytic conditions for a center, given by Cherkas.

4.2 Centers for analytic Liénard equations

In this section we describe the analytic conditions for a center for the system

$$\dot{x} = y, \qquad \dot{y} = -g(x) - yf(x), \tag{4.3}$$

where f and g are real polynomials in x. This was first obtained by Cherkas in [16].

The system (4.3) arises from the second-order nonlinear equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{4.4}$$

which generalizes the van der Pol Oscillator, and is ubiquitous in the study of polynomial systems.

Any critical point of (4.3) lies on the x-axis. We can assume, after a translation of the x-axis, that the critical point we are interested in is at the origin. The condition that this critical point should be non-degenerate and a focus or center implies that $g(0) = 0$ and $g'(0) > 0$. It also implies that $f(0)^2 < 4g'(0)$, however we do not need this condition here.

We now wish to transform (4.3) into a more amenable form. Denote

$$F(x) = \int_0^x f(\xi)\, d\xi, \qquad G(x) = \int_0^x g(\xi)\, d\xi.$$

Under the Liénard transformation $y \mapsto y + F(x)$, the system is brought to the form

$$\dot{x} = y - F(x), \qquad \dot{y} = -g(x). \tag{4.5}$$

We can simplify (4.5) further by a transformation which effectively removes g. Let u be the positive root of $2G$. From the conditions on g given above it is clear that this root is well defined and analytic in a neighborhood of $x = 0$, Thus

$$u = (2G(x))^{1/2}\,\mathrm{sgn}(x) = (g'(0))^{1/2}x + O(x^2) \tag{4.6}$$

defines an invertible analytic transformation in a neighborhood of $x = 0$.

Let $x(u)$ denote the inverse of this function. The transformation (4.6) takes the system (4.5) to the system

$$\dot{u} = \frac{g(x(u))}{u}\left(y - F(x(u))\right), \qquad \dot{y} = -g(x(u)).$$

Since $g(x(u))/u = (g'(0))^{1/2} + O(u)$ is analytic and non-zero in a neighborhood of the origin, we can rescale (4.6) by multiplying the right-hand side by $u/g(x(u))$ which gives

$$\dot{u} = y - F(x(u)), \qquad \dot{y} = -u. \tag{4.7}$$

This system has exactly the same direction field as (4.6) in a neighborhood of the origin, and hence the local qualitative behavior of the system, in particular the existence of a center, is not altered by this scaling.

We write the power series for $F(x(u))$ as $\sum_1^\infty a_i u^i$. It turns out that the origin of (4.7) is a center if and only if all the a_{2i+1} vanish. To see this we introduce the function $F^*(u) = \sum_1^\infty a_{2i} u^{2i}$, analytic in a neighborhood of the origin, and consider the system

$$\dot{u} = y - F^*(u), \qquad \dot{y} = -u. \tag{4.8}$$

Since the flow of (4.8) is monodromic, it is clear that it must have a center at the origin due to symmetry in the u-axis. However, the system (4.7) is rotated with respect to (4.8) in a neighborhood of the origin unless all the terms a_{2i+1} vanish.

Thus (4.7) cannot have a center at the origin unless all the a_{2i+1} vanish. On the other hand if all the a_{2i+1} do vanish, then the system is a center by symmetry.

We can express this necessary and sufficient condition in a more geometrical form:

Theorem 4.1. *The system* (4.3) *has a center at the origin if and only if* $F(x) = \Phi(G(x))$, *for some analytic function* Φ, *with* $\Phi(0) = 0$.

Proof. The argument above shows that there is a center if and only if $F(x(u)) = \phi(u^2)$ for some analytic function ϕ, $\phi(0) = 0$. But $u^2 = 2G(x)$, so set $\Phi(w) = \phi(2w)$. □

Now consider the function $z(x)$ defined in a neighborhood of the origin by $z(x) = x(-u(x))$. We can also describe $z(x)$ as the unique analytic function which satisfies

$$G(x) = G(z), \qquad (z(0) = 0,\ z'(0) < 0).$$

That this equation defines a unique analytic function $z(x)$ is clear from the conditions on g since

$$G(x) - G(z) = (x - z)\Big(\tfrac{1}{2}g'(0)(x + z) + o(x, z)\Big) = 0$$

has two analytic branches at the origin $z = x$ and $z = -x + o(x)$. The conditions on $z'(0)$ then selects the second of these. Now $2G(x(u)) = u^2 = 2G(x(-u))$ whence $G(x) = G(x(-u(x)))$. Furthermore, as $x(-u(x)) = -x + O(x^2)$, this solution must be $x(-u(x))$.

We know that the origin is a center if and only if the function $F(x(u))$ is even. That is, $F(x(u)) - F(x(-u))$ vanishes identically. But this is equivalent to saying that $F(x(u(x))) - F(x(-u(x))) = F(x) - F(z) = 0$. Thus, we have the following characterization of centers.

Theorem 4.2. *The system* (4.3) *has a center at the origin if and only if there exists a function* $z(x)$ *satisfying*

$$F(x) = F(z), \qquad G(x) = G(z), \qquad (z(0) = 0,\ z'(0) < 0). \tag{4.9}$$

This result also works when f and g are only analytic functions in a neighborhood of the origin. However, if f and g are also polynomials, then the solution $z(x)$ must correspond to a common factor between the functions $F(x) - F(z)$ and $G(x) - G(z)$ other than $(x - z)$. Thus, the following corollary is clear:

Corollary 4.3. *If the system* (4.3) *with* f *and* g *polynomials has a center at the origin, then it is necessary that the resultant of*

$$\frac{F(x) - F(z)}{x - z} \quad \textit{and} \quad \frac{G(x) - G(z)}{x - z}$$

with respect to x *or* z *vanishes. This condition is sufficient if the common factor of the two polynomials vanishes at* $x = z = 0$.

4.3 Centers for polynomial Liénard equations

Corollary 4.3 gives algebraic conditions for a center, but does not indicate how systems satisfying these conditions arise. We now want to show that the centers are in fact given by algebraic symmetries in the x variable.

Consider the subfield of $\mathbb{R}(x)$ generated by the polynomials F and G. Call this field $\mathcal{F}$. The field $\mathcal{F}$ shares an important property with F and G:

Lemma 4.4. *Suppose there exists an analytic function $z(x)$ with $z(0) = 0$, $z'(0) < 0$ such that both $F(z(x)) = F(x)$ and $G(z(x)) = G(x)$ in a neighborhood of $x = 0$, Then for all elements H of the field $\mathcal{F}$ generated by F and G we have $H(z(x)) = H(x)$ considered as meromorphic functions of x about $x = 0$.*

Proof. Note first that $H(z(x)) = 0$ if and only if $H(x) = 0$. Thus we need only verify that addition, multiplication and inversion of non-zero elements of $\mathcal{F}$ preserve this property, which is clearly the case. □

Recall that Lüroth's Theorem states that if k is a field, any subfield of $k(x)$ which strictly contains k is isomorphic to $k(x)$. That is to say, the subfield is just $k(r)$ for some $r \in k(x)$. But $\mathcal{F}$ is a subfield of $\mathbb{R}(x)$ strictly containing $\mathbb{R}$, and so we must have $\mathcal{F} = \mathbb{R}(r)$ for some rational function $r \in \mathbb{R}(x)$.

Let us write r as A/B with A, $B \in \mathbb{R}[x]$. The field generated over $\mathbb{R}$ by r can also be generated by

$$\frac{\alpha r + \beta}{\gamma r + \delta} = \frac{\alpha A + \beta B}{\gamma A + \delta B}$$

for any constants α, β, γ and δ, such that $\alpha\delta - \beta\gamma \neq 0$. By choosing these constants to ensure that the degree of the denominator is less than the degree of the numerator, we can assume without loss of generality that B has degree less than A. We can also assume that all common factors between A and B are canceled and that B is monic.

Now F and G are in $\mathcal{F}$, and so

$$F = \frac{F_1(A, B)}{F_2(A, B)}, \qquad G = \frac{G_1(A, B)}{G_2(A, B)},$$

where the F_i and G_i are homogeneous polynomials which we choose to have no common factors as polynomials in A and B. We now show that:

Lemma 4.5. $B = 1$.

Proof. We first factor the expressions for F_1 and F_2 over $\mathbb{C}[A, B]$ to obtain

$$F_1(A, B) = \prod_{i=1}^{r}(\lambda_1 A + \mu_i B), \quad F_2(A, B) = \prod_{i=r+1}^{r+s}(\lambda_1 A + \mu_i B),$$

for some complex constants λ_i and μ_i.

Now note that if $\lambda_1 A+\mu_1 B$ and $\lambda_2 A+\mu_2 B$ have a common factor as polynomials in x, then they are multiples of each other, since A and B have no common factors in x (over $\mathbb{R}$ and therefore over $\mathbb{C}$ too). However we chose F_1 and F_2 to have no common factors as polynomials in A and B, hence they must have no common factors as polynomials in x.

Thus the denominator of F as a rational function of x after cancellation with the numerator is just $F_2(A(x), B(x))$, and so

$$\prod_{i=r+1}^{r+s} (\lambda_1 A + \mu_i B) \in \mathbb{R}.$$

Since the degree of A is larger than the degree of B, this can only happen when $\lambda_i = 0$ for all $i = r+1, \ldots, s$ and hence B must be a constant polynomial, and therefore equal to 1. Similar considerations show that $G_2(A(x), B(x))$ is also a constant. □

Thus we have shown that both F and G are polynomials of some polynomial $A \in \mathcal{F}$. The final step follows.

Theorem 4.6. *The system* (4.3) *with $g(0) = 0$ and $g'(0) > 0$ has a non-degenerate center at the origin if and only if $F(x)$ and $G(x)$ are both polynomials of a polynomial $A(x)$ with $A'(0) = 0$ and $A''(0) \neq 0$.*

Proof. By Theorem 4.2, if there is a center at the origin of (4.3), then there is a function $z(x)$ with $z(0) = 0$ and $z'(0) < 0$ such that $F(z(x)) = F(x)$ and $G(z(x)) = G(x)$. By Lemma 4.4, the polynomial generator of $\mathcal{F}$, A also satisfies $A(z(x)) = A(x)$, and hence its linear term must vanish. Now $G(x)$ is a polynomial in A with $G''(0) > 0$, which means that the quadratic term of A cannot vanish.

Conversely, assume F and G are polynomials of a polynomial A with a non-zero quadratic term but no linear term. From the conditions on A, we can find an analytic function satisfying $A(z(x)) = A(x)$ with $z(0) = 0$, $z'(0) < 0$. Clearly F and G must then satisfy condition (4.9) of Theorem 4.2, and the origin is therefore a center. □

Corollary 4.7. *The system* (4.3) *has a non-degenerate center at the point $x = p$ if and only $g(p) = 0$, $g'(p) > 0$ and F and G are both polynomials of a polynomial A which satisfies $A'(p) = 0$ with $A''(p) \neq 0$.*

Proof. If we shift the x-axis to bring $x = p$ to the origin, then it is clear that the new F and G calculated will differ from the original ones only by a constant. The rest follows quite easily from Theorem 4.6. □

Theorem 4.8. *If the system* (4.3) *has a non-degenerate center at the origin, then the transformation $x \mapsto z(x)$, given from the conditions of Theorem 4.2, takes the direction field of* (4.3) *into itself, reversing the directions. Thus the origin has a generalized symmetry.*

Alternatively, the system (4.3) *can be seen to have a center via the reducing transformation* $w \mapsto h(x)$ *for some polynomial* $h(x) = x^2 + O(x^3)$ *from the system*

$$\dot{w} = y, \quad \dot{y} = -m(w) - l(w)y, \tag{4.10}$$

after a scaling. Here l *and* m *are polynomials* $m(0) > 0$*, and the transformation* $v \mapsto h(x)$ *takes a non-critical point at the origin of* (4.10) *and "unfolds" it into the center of* (4.3).

Proof. The first assertion is a direct calculation from condition (6) of Theorem 4.2. The generalized symmetry condition means that trajectories lying in $x \geq 0$ can be mapped onto trajectories in $x \leq 0$, with the points on $x = 0$ being fixed. If we know that the flow encircles the origin, then trajectories sufficiently close to the origin must be closed. Thus if the critical point is known to be of focal type, this generalized symmetry is enough to imply the existence of a center.

For the second part, we take the polynomial A of Theorem 4.6, and consider

$$h(x) = 2\frac{A(x) - A(0)}{A''(0)} = x^2 + O(x^3).$$

Clearly F and G are also polynomials of this polynomial, so that $F = L(h(x))$ and $G = M(h(x))$ for some polynomials L and M. From the condition on $g'(0)$, we see that $M'(0) > 0$. Take $l = L'$ and $m = M'$, then system (4.10) transforms (after scaling by $h'(x)$) to

$$\dot{x} = y, \quad \dot{y} = -h'(x)m(h(x)) - h'(x)l(h(x))y = -g(x) - f(x)y.$$

The origin of (4.10) is not a critical point, but locally the trajectories are of the form

$$w = \alpha - \frac{1}{2m(\alpha)}y^2 + O(y^3)$$

for small values of α, where the $O(y^3)$ term is analytic in α as well as y. The transformation takes these trajectories to the curves

$$x^2 + O(x^3) = \alpha - \frac{1}{2m(\alpha)}y^2 + O(y^3),$$

for α sufficiently small. These trajectories are thus closed curves approximating the ellipses $x^2 + y^2/(2m(0)) = \alpha$, and the origin is a center. $\square$

Notes

The role of symmetries in proving the existence of centers is of course standard, but a systematic investigation of algebraic symmetries was carried out by Żołądek[61], who coined the name "rationally reversible" for this phenomenon. We have preferred to consider the more general concept of algebraic symmetries, as these seem to be needed in some of the more complex center examples [26].

The classification of centers in Liénard systems can be found in [18].

Chapter 5

Cherkas' Systems

In this chapter we give an extended example of a non-trivial classification of centers which involves both Darboux and symmetry mechanisms for producing a center. Further details can be found in [26], which we follow closely.

We first need a result concerning algebraic solutions to transcendental equations due to Rosenlicht [53].

Theorem 5.1. *Let $(k,')$ be a differential field of characteristic zero with differential extension field $(K,')$ with the same field of constants and such that k is algebraically closed in K, i.e. all elements in K which are algebraic over k also lie in k. We also assume that K is a finite algebraic extension of $k(t)$, where t is transcendental over k and such that $t' \in k$. Suppose that*

$$\sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v' \in k,$$

where $c_1, \ldots c_n$ are constants of k which are linearly independent over $\mathbb{Q}$ and $u_1, \ldots u_n$ and v are in K. Then $u_1, \ldots, u_n \in k$ and $v = ct + d$, with c a constant of k and $d \in k$.

In our applications, we shall take $k = \mathbb{C}$ to ensure that k is algebraically closed. In particular, all elements of k are constants. The application we need of this result could probably be obtained by an application of complex variables, but we use the above result to emphasize the algebraic nature of the computations.

We consider the system

$$\dot{x} = y, \qquad \dot{y} = P_0(x) + P_1(x)y + P_2(x)y^2. \tag{5.1}$$

We prove the following theorem for this system.

Theorem 5.2. *A system of the form* (5.1) *with* $P_0(0) = 0$ *and* $P_0'(0) < 0$*, which has a center at the origin, satisfies one of the following (possibly overlapping) conditions.*

(i) *The system is algebraically reducible via the map* $(x, y) \mapsto (x, y^2)$ *and thus it has a symmetry in the* x*-axis.*

(ii) *The system is algebraically reversible at the origin. In fact, it is algebraically reducible via a map* $(x, y) \mapsto (r(x), y)$ *for some polynomial* $r(x)$ *over* $\mathbb{R}$.

(iii) *There is a local first integral of Darboux type.*

Proof. We break the proof into several steps.

Step 1. We perform the change of variables used by Cherkas,

$$y = Y \exp\,(\int_0^x P_2(\xi)\,d\xi)$$

to arrive, after renaming the variable Y as y, at the system

$$\dot{x} = y, \qquad \dot{y} = g(x) + f(x)y.$$

Here

$$g(x) = P_0(x)\exp(-2\int_0^x P_2(\xi)\,d\xi), \qquad f(x) = P_1(x)\exp(-\int_0^x P_2(\xi)\,d\xi). \quad (5.2)$$

We note that the transformation of Cherkas changed the system into one which is polynomial in y but with coefficients in a Liouvillian differential field extension of $(\mathbb{C}(x), d/dx)$ generated by adjoining the exponentials of integrals in (5.2). On the other hand the above transformation reduced to the first degree, the polynomial in y in the right side of the second differential equation, making it possible to apply the result of Cherkas (Theorem 4.2).

Step 2. We now apply Cherkas' Theorem.

It is easy to verify that the conditions on $P_0(x)$ given above imply the hypotheses of Theorem 4.2 on $g(x)$. Therefore the conclusion of Theorem 4.2 tells us that (5.1) has a center at the origin if and only if there is a real analytic function $z(x)$ in the neighborhood of the origin, with $z(0) = 0$ and $z'(0) = -1$ which simultaneously satisfies

$$\int_0^x f(\xi)\,d\xi = \int_0^z f(\xi)\,d\xi, \qquad \int_0^x g(\xi)\,d\xi = \int_0^z g(\xi)\,d\xi, \quad (5.3)$$

or equivalently,

$$f(x)\,dx = f(z)\,dz, \qquad g(x)\,dx = g(z)\,dz. \quad (5.4)$$

We first dismiss the trivial case where $f(x)$ vanishes identically, as this implies that P_1 is identically zero. The origin is, in this case, a center by symmetry in the x axis. Alternatively, the system can be algebraically reduced to the system

$$\dot{\bar{x}} = 1, \qquad \dot{\bar{y}} = 2P_0(\bar{x}) + 2P_2(\bar{x})\bar{y}$$

by the map $(x, y) \mapsto (\bar{x}, \bar{y}) = (x, y^2)$. This means that the first integral defined at the origin of this system in $(\bar{x}, \bar{y})$ can be pulled back to a first integral in a neighborhood of the origin of (5.1), giving a center.

Thus we shall assume from now on that f and g do not vanish identically. We shall also exclude the case where P_2 vanishes identically in (5.1) as this has been covered in the previous chapter. In fact this case is just a subcase of Case 1 below.

From (5.4) we obtain

$$g(x)/f(x) = g(z)/f(z), \tag{5.5}$$

as local meromorphic functions in x around the origin (in the right side of (5.5) z is actually $z(x)$) and hence $z(x)$ satisfies the equation

$$\frac{P_0(x)}{P_1(x)} \exp\left(-\int_0^x P_2(\xi)\, d\xi\right) = \frac{P_0(z)}{P_1(z)} \exp\left(-\int_0^z P_2(\xi)\, d\xi\right). \tag{5.6}$$

As a real local analytic function, $z(x)$ may be considered as an element of $\mathbb{R}\{\{x\}\}$ which determines an element of $\mathbb{C}\{\{x\}\}$ and hence a local complex analytic function which we also denote by $z(x)$. We now divide our investigation into two cases: the first case is when $z(x)$ is algebraic over $\mathbb{C}(x)$, and the second when $z(x)$ is transcendental over $\mathbb{C}(x)$.

Step 3. $\mathbf{z(x)}$ is algebraic over $\mathbb{C}(x)$. In this case we can apply the results of Theorem 5.1. First note that $\int P_2(x)\, dx$ is a non-constant polynomial, and that the equation (5.6) gives

$$\frac{P_0(x)/P_1(x)}{P_0(z)/P_1(z)} \exp\left(-\int_0^x P_2(\xi)\, d\xi + \int_0^z P_2(\xi)\, d\xi\right) = 1. \tag{5.7}$$

Thus we have

$$\Psi(x) = R_1(x, z(x))e^{R_2(x,z(x))} = 1, \tag{5.8}$$

where

$$\begin{aligned} R_1(x, y) &= \frac{P_0(x)/P_1(x)}{P_0(y)/P_1(y)} \in \mathbb{R}(x, y), \\ R_2(x, y) &= \int_0^y P_2(\xi)\, d\xi - \int_0^x P_2(\xi)\, d\xi \in \mathbb{R}[x, y]. \end{aligned}$$

$\overline{R_1}(x) = R_1(x, z(x))$ and $\overline{R_2}(x) = R_2(x, z(x))$ therefore lie in the algebraic differential field extension $(\mathbb{C}(x)[z(x)], d/dx)$ of $(\mathbb{C}(x), d/dx)$ generated by $z(x)$.

Below we denote the derivations in the two fields by $'$. Considering now the expression for Ψ'/Ψ from (5.8) we obtain:

$$(\overline{R_1})'/\overline{R_1} + (\overline{R_2})' = 0. \tag{5.9}$$

Thus, by Theorem 5.1 where we take $k = \mathbb{C}$ and $K = \mathbb{C}(x)[z]$, we get that $R_1(x, z(x))$ is a constant and $R_2(x, z(x))$ is of the form $cx + d$, for constants c and d. Substituting back into (5.9), we see that c must vanish. Furthermore, it is clear that at $x = 0$, $R_2(x, z(x))$ vanishes, and hence $d = 0$. Lastly, from (5.8), $R_1(x, z(x)) = 1$.

Thus, we have arrived at the equations

$$P_0(x)/P_1(x) = P_0(z)/P_1(z), \qquad \int_0^x P_2(\xi)\, d\xi = \int_0^z P_2(\xi)\, d\xi. \tag{5.10}$$

Consider the subfield F of $\mathbb{R}(x)$ generated by all rational functions $S(x)$ such that $S(x) = S(z(x))$. By Lüroth's theorem, the field is isomorphic to $\mathbb{R}(r/s)$, for some function $r(x)/s(x)$ with $r(x), s(x) \in \mathbb{R}[x, y]$. Without loss of generality, we can choose the degree of r to be greater than the degree of s, with r and s coprime. Hence, we can write

$$\int_0^x P_2(\xi)\, d\xi = \phi(r(x)/s(x)), \tag{5.11}$$

for some rational function ϕ over $\mathbb{R}$, in one variable.

Now, working over $\mathbb{C}$, the right-hand side of (5.11) can be written as

$$\prod_{i=1}^{q}(\alpha_i r + \beta_i s) \Big/ \prod_{i=1}^{q}(\gamma_i r + \delta_i s),$$

As in Section 4.3, if $(\alpha_i r + \beta_i s)$ shares a common factor with $(\gamma_j r + \delta_j s)$, these two polynomials must differ by a constant, whence we can assume that the fraction above allows no further cancellations. Since the left-hand side of (5.11) is a polynomial, $\prod(\gamma_i r + \delta_i s)$ must be a constant, and hence the denominator has no dependence on r, and s must be a constant.

Without loss of generality, we can take $s(x) = 1$ and $r(0) = 0$. Furthermore, from (5.11), we now see that ϕ must be a polynomial.

From (5.10), we must also have

$$P_0(x)/P_1(x) = \psi_0(r/s)/\psi_1(r/s) = \psi_0(r)/\psi_1(r)$$

for some polynomials in one variable ψ_0 and ψ_1 over $\mathbb{R}$ with $(\psi_0, \psi_1) = 1$. From the equality above we get

$$P_0(x)/\psi_0(r(x)) = P_1(x)/\psi_1(r(x)) = K(x),$$

with $K(x)$ a rational function in x over $\mathbb{R}$. Thus,

$$P_0(x) = K(x)\psi_0(r(x)), \qquad P_1(x) = K(x)\psi_1(r(x)), \tag{5.12}$$

which implies that $K(x)$ is a polynomial over $\mathbb{R}$.

Using the expression for P_1 in (5.12) and replacing it into the second part of (5.2) we have:

$$f(x) = K(x)\psi_1(r(x))\exp(-\int_0^x P_2(\xi)\,d\xi).$$

Substituting the above in the first part of (5.4) we obtain:

$$K(x)\psi_1(r(x))\exp(-\int_0^x P_2(\xi)\,d\xi)\,dx = K(z)\psi_1(r(z))\exp(-\int_0^z P_2(\xi)\,d\xi)\,dz.$$

But $r(x) = r(z(x))$ and, using the second part of (5.10), we obtain:

$$K(x)\,dx = K(z)\,dz.$$

However, from $r(x) = r(z)$, we also have $r'(x)\,dx = r'(z)\,dz$, and hence,

$$K(x)/r'(x) = K(z)/r'(z).$$

So $K(z)/r'(z)$ is in the field F. This implies that $K(x) = r'(x)\chi(r(x))$ for some rational function χ over $\mathbb{R}$, which in view of the preceding equality must be a polynomial by a comparison of the degrees of r and r'. Since $r(x) = r(z)$ with $z'(0) = -1$, we must have $r'(0) = 0$. However, from the expression of $P_0(x)$ in (5.12) and using the expression $K(x) = r'(x)\chi(r(x))$ we get that $P_0'(0) = r''(0)\chi(0)\psi(0)$ and hence $r''(0) \neq 0$. Without loss of generality, we can choose r so that $r''(0) = 1$.

Putting together this information, there exist polynomials A_0, A_1 and A_2 such that

$$P_0(x) = A_0(r(x))r'(x), \qquad P_1(x) = A_1(r(x))r'(x), \qquad P_2(r(x)) = A_2(r(x))r'(x),$$

with $A_i = \chi\psi_i$ for $i = 0, 1$ and $A_2 = \phi'$. The system is then algebraically reducible. Indeed the map $(x, y) \mapsto (\bar{x}, \bar{y}) = (r(x), y)$ reduces (5.1) to the system

$$\dot{\bar{x}} = \bar{y}, \qquad \dot{\bar{y}} = A_0(\bar{x}) + A_1(\bar{x})\bar{y} + A_2(\bar{x})\bar{y}^2.$$

This system is non-singular at the origin since the conditions on P_0 imply that $A_0(0) < 0$.

Alternatively, the center can be seen to be given by a reversing transformation

$$(x, y, t) \mapsto (\bar{x}, y, -t),$$

where $r(x) = r(\bar{x})$, $\bar{x}(0) = 0$ and $\bar{x}'(0) < 0$. We have seen in the previous chapter that the case when P_2 vanishes identically follows the same pattern.

Step 4. $\mathbf{z(x)}$ is transcendental over $\mathbb{C}(x)$. In this case, we first consider some consequences of (5.5). Differentiating (5.5) we obtain

$$(g/f)'(x)\,dx = (g/f)'(z)\,dz$$

where here and also in the equality below, $'$ in the right side just means the differentiation with respect to z. This gives:

$$\left[\frac{1}{f}\left(\frac{g}{f}\right)'\right](x) = \left[\frac{1}{f}\left(\frac{g}{f}\right)'\right](z),$$

which gives

$$\left[\frac{P_2}{P_1}\left(\frac{P_0}{P_1}\right) - \frac{1}{P_1}\left(\frac{P_0}{P_1}\right)'\right](x) = \left[\frac{P_2}{P_1}\left(\frac{P_0}{P_1}\right) - \frac{1}{P_1}\left(\frac{P_0}{P_1}\right)'\right](z).$$

Since this is an algebraic equation between z and x and since both x and z are transcendental over $\mathbb{C}$, then both sides of the above equality must be a constant c. In particular, the fraction P_0/P_1 must in fact be a polynomial.

Hence we consider the equality in $\mathbb{R}(x)$:

$$P_2P_0P_1 + P_0P_1' - P_1P_0' = cP_1^3. \tag{5.13}$$

For $k \in \mathbb{C}$ we define

$$C_k = y + k(P_0/P_1).$$

We seek invariant algebraic curves in the family of curves $C_k = 0$. Recall that a curve $C_k = 0$ is an invariant algebraic curve of the differential system (3.1) if and only if $DC_k/C_k \in C[x, y]$, where D is the operator

$$y\frac{\partial}{\partial x} + (P_0 + P_1y + P_2y^2)\frac{\partial}{\partial y}.$$

We determine the condition on the constant k such that this be satisfied.

We first compute

$$DC_k = k(P_0/P_1)'y + P_0 + P_1y + P_2y^2.$$

Using (5.13) we have that

$$(P_0/P_1)' = (P_0'P_1 - P_0P_1')/P_1^2 = P_2P_0/P_1 - cP_1,$$

and hence

$$DC_k = P_2y^2 + k(P_0P_2/P_1 - cP_1)y + P_0 + P_1y. \tag{5.14}$$

We search for polynomials $A_k(x)$ and $B_k(x)$ such that

$$DC_k = C_k(A_ky + B_k) = (y + kP_0/P_1)(A_ky + B_k). \tag{5.15}$$

Clearly, we have from conditions (5.14) and (5.15) that

$$A_k = P_2, \qquad B_k = P_1/k,$$

and

$$-ckP_1 + P_1 = P_1/k,$$

yielding

$$ck^2 - k + 1 = 0.$$

Depending on the value of c this equation has one or two distinct solutions. If $c = 1/4$ it has only one solution $k = 2$, and if $c \neq 1/4$ it has two distinct solutions k_1 and k_2. Thus

$$DC_k = C_k(P_2 y + P_1/k)$$

where $k = 2$ in case $c = 1/4$ and $k = k_j$, $j = 1, 2$ in case $c \neq 1/4$.

Before considering each one of these two cases we observe that the system (5.1) admits the expression

$$C = \exp\left(\int_0^x P_2(x)\,dx\right)$$

as an exponential factor, i.e., $DC/C \in \mathbb{C}[x, y]$. Indeed, we have

$$DC = C\,(P_2 y).$$

We now consider the two possible cases. First let us suppose $c \neq 1/4$. In this case we construct a Darboux first integral from these three functions C, C_{k_1} and C_{k_2} of the form

$$\Big(y + k_1(P_0/P_1)\Big)^{r_1}\Big(y + k_2(P_0/P_1)\Big)^{r_2}\exp\left(r_3\int_0^x P_2(x)\,dx\right).$$

It is immediately verified that if we take $r_1 = 1$, $r_2 = -k_2/k_1$ and $r_3 = -1 + k_2/k_1$, then we have the linear combination of their corresponding cofactors

$$r_1(P_2 y + P_1/k_1) + r_2(P_2 y + P_1/k_2) + r_3 P_2 y = 0.$$

In the case $c = 1/4$ we only have one invariant algebraic curve, i.e. $C_2 = 0$. We recall that we also have the exponential factor C. We now consider the expression

$$\tilde{C} = \exp(P_0/(P_1 y + 2P_0)),$$

then $\tilde{C}$ is another Darboux exponential factor. Indeed calculations yield

$$D\tilde{C} = \tilde{C}(-P_1/4).$$

In this case we construct a Darboux first integral by using the curve $C_2 = y + 2(P_0/P1)(x)$ and the two exponential factors C and $\tilde{C}$ defined above. This first integral is of the form

$$(C_2)^{r_1} C^{r_2} (\tilde{C})^{r_3}.$$

It is easy to see that if we take $r_1 = 2$, $r_2 = -1$ and $r_3 = 1$ we obtain the following linear combination of their corresponding cofactors:

$$r_1(-P_1/4) + r_2(P_2 y) + r_3(P_2 y + P_1/2) = 0.$$

These integrals are well defined and holomorphic in a neighborhood of the origin, and hence, by Poincaré's result the origin is a center. $\square$

Notes

Further details can be found in the paper [26] where the more general case

$$\dot{x} = yP(x), \qquad \dot{y} = P_0(x) + P_1(x)y + P_2(x)y^2 + P_2(x)y^3,$$

with $P(x)$ is a polynomial with $P(0) \neq 0$, is also treated.

We note that in this latter case, the reducing transformations are no longer polynomial, but only algebraic.

There is a method of also dealing with generalized Kukles' systems

$$\dot{x} = yP_4(x), \qquad \dot{y} = P_0(x) + P_1(x)y + P_2(x)y^2 + P_3(x)y^3,$$

where all the P_i are polynomials. The reduction stage is more delicate, but the results are essentially the same: either there is a Darboux first integral, or there is an algebraic symmetry. Unfortunately, there does not seem to be any method for tackling more complex systems in this way.

Chapter 6

Monodromy

In this chapter we begin the second part of these notes, looking at some ideas based on the concept of *monodromy*. Very roughly, this is the study of how objects depending on a parameter, and which are locally constant in some sense, change as the parameter moves around a non-trivial path. This idea is particulary appropriate for the center-focus problem, as the essence of this problem is about trying to make global extensions of local information. For example, we might naïvely hope to be able to extend the local first integral at the origin to a global first integral. This is not possible in general, but even if we could do so, the first integral would certainly ramify as a global object. Our desire would then be to read off some important information about the system from this global ramification.

Over the next five chapters we present several topics very loosely connected with this idea. In this chapter we give two basic and classical examples of monodromy, and then discuss an extended example related to the Model problem of Briskin, Françoise and Yomdin. This is a problem which has close connections with the center-focus problem.

6.1 Some Basic Examples

Our first example is the monodromy of an algebraic function of one variable, $x(c)$ say. The special case where $x(c)$ satisfies $f(x) = c$, for some polynomial f, will be used in the final section and in Chapter 8.

Example 6.1. Suppose, we have a polynomial $F(x, c)$ in $\mathbb{C}[x, c]$. Let $\Delta(c)$ denote the discriminant of F as a polynomial in x over $\mathbb{C}(c)$, and n the degree of F with respect to x. In a neighborhood of each value of c where $\Delta(c) \neq 0$, we can solve $F(x, c) = 0$ to obtain n distinct roots $x_i(c)$, algebraic over c. As these roots do not coalesce unless $\Delta(c) = 0$, we have a locally constant picture as c varies in $\mathbb{C} - S$, where S is the set of c where $\Delta(c) = 0$. That is, although the precise value of the x_i will change, of course, there is a unique way of associating the x_i's at each value of c on a path in $\mathbb{C} - S$. However, moving c around a loop which includes a point in S, we find that the x_i will swap amongst themselves in general.

If we fix a base point c_0, and consider the effect on the roots $x_i(c_0)$ as we move around non-trivial loops in $\mathbb{C}-S$ based at c_0, we get a map from $\pi_1(\mathbb{C}-S, c_0)$ to the symmetric group on n elements. This is called the monodromy group of the algebraic function given by $F(x, c) = 0$. We will usually drop the reference to the base point c_0.

It can be shown (for example in [30]) that the monodromy group is equivalent to the Galois group of $\mathbb{C}(x_1(c), \ldots, x_n(c))$ over $\mathbb{C}(c)$.

Our second example examines what happens to the topology of the level curves of a polynomial as a parameter varies. It will be used and generalized in the next two chapters

Example 6.2. Consider the level curves of the function x^2+y^2. That is, we consider $x^2 + y^2 = c$ as c varies. If we draw this as a two-sheeted covering of $\mathbb{C}$, then we have branch points at $\pm\sqrt{c}$ and can consider a cut lying between these two points. As c makes a loop around $c = 0$ these two branch points swap over.

Now consider two curves. One δ is the loop which surrounds the two branch points, and the other δ' is a loop which begins at ∞ on one sheet, passes through the cut and ends at ∞ on the other sheet. We let (δ, δ') denote the intersection number of δ and δ'.

As c makes the tour around $c = 0$, δ is taken to itself, but δ' tends to $\delta' + (\delta', \delta)\delta$. This is a simple example of the Picard–Lefshetz formula. The transformation on the surface itself (a cylinder) is called a *Dehn twist*.

6.2 The Model Problem

We now discuss a longer example where monodromy is used to tackle a problem which is closely related to the center-focus problem.

We consider the Abel equation

$$\frac{dy}{dx} = p(x)y^2 + q(x)y^3, \qquad a \le x \le b \tag{6.1}$$

where p and q are polynomials and a is a fixed constant, We denote the solution of (6.1) by $y(x, c)$, where $y(a, c) = c$. Standard existence theorems ensure that $y(x, c)$ is well defined and analytic in both its arguments for c sufficiently small. If $y(b, c) = c$, then we call $y(x, c)$ a periodic solution. Likewise if $y(b, c) \equiv c$ for all c close to 0 we say that the system has a *center* between a and b. The numbers a and b are not important; by a simple transformation we can always choose $a = 0$ and $b = 1$.

It is clear that this has strong connections with the center-focus problem. We note that one strong motivation for studying Abel equations is that the family of systems,

$$\dot{x} = -y + M(x, y), \qquad \dot{y} = x + N(x, y), \tag{6.2}$$

where M and N are homogeneous polynomials of the same degree n, can be brought to the form (6.1) with p and q trigonometric polynomials. Setting $a = 0$

and $b = 2\pi$, the definitions of periodic solution and center for (6.1) coincide with their usual definitions in the planar system (6.2).

The analogue of the center focus problem in this case is to see what conditions the existence of a center in (6.1) imposes on the defining equations. We shall always denote the antiderivative of the polynomials p and q as P and Q. That is:

$$P(x) = \int_a^x p(\xi)\,d\xi, \qquad Q(x) = \int_a^x q(\xi)\,d\xi.$$

The question of whether there is a center or not depends on computing the expansion of $y(1, c) = \sum \alpha_i c^i$. This can be done as in the case of a center, but involves similar difficulties to the center-focus problem. In the light of this, Briskin, Françoise and Yomdin [7, 8, 9] have proposed a simplified center problem called the *model problem.*

Here we ask for conditions that the center lies in a parametric family of centers of the form

$$\frac{dy}{dx} = p(x)y^2 + \epsilon q(x)y^3, \qquad 0 \le x \le 1, \tag{6.3}$$

and has a center for all ϵ sufficiently small.

A study of the return map calculations described above shows that this condition is equivalent to

$$P(1) = P(0) = 0, \qquad \int_0^1 P(x)^n q(x)\,dx = 0, \qquad n = 0, 1, \ldots. \tag{6.4}$$

The problem is then to find out which P and Q can satisfy these equations.

One simple condition, which guarantees that we have a center, is that P and Q are both polynomials of a polynomial A with $A(0) = A(1)$. This is equivalent to the symmetry condition discussed in Chapter 4. We say that the center satisfies the *composition condition* in this case.

In fact, from the form of the return map, this problem turns out to be the same as saying that the center at $\epsilon = 0$ is not destroyed to first order by perturbing ϵ.

We will use some simple ideas using monodromy to show that, if we assume that both $p(0)$ and $p(1)$ are non-zero, then this symmetry condition comprises all cases of centers for the model problem. That is, all centers of this form satisfy the *composition condition.*

6.3 Applying Monodromy to the Model Problem

Let us assume that (6.4) holds with the additional assumption that $p(0)$ and $p(1)$ are non-zero. Without loss of generality we can take P to be monic. Furthermore, if (6.4) holds for some polynomial $P(x)$, then it must also hold for any polynomial of the form $P(x)+k$, with k a constant; whence we can also assume that $P(0)=0$. The $n=0$ equation of (6.4) implies that $Q(0)=Q(1)$. In fact the condition $P(0)=P(1)$ is also deducible from the other conditions.

It is clear that the polynomial $P-c$ can have no roots in the interval $[0,1]$ for all c such that

$$|c| > K := \sup_{x\in[0,1]} |P(x)|.$$

Thus, if $|c|>K$, $(P(x)-c)^{-1}$ has a well-defined expansion, and

$$I(c) = \int_0^1 \frac{q\,dx}{P(x)-c} = \sum_{i\geq 0} \frac{1}{c^{i+1}} \int_0^1 P(x)^i q(x)\,dx.$$

The hypothesis (6.4) is therefore equivalent to the condition

$$I(c) \equiv 0, \qquad |c| \gg 0.$$

Note that $I(c)$ is related to the Melnikov function which describes the bifurcation of periodic solutions of (6.3) for small ϵ.

We shall work over $\mathbb{C}$ from now on. Let $S=\{c_0,c_1,\ldots,c_n\}$ be the critical values of $P(x)$; that is the values of $P(x)$ when $P'(x)=0$. In addition we shall also include in S the values $c_0=P(0)=0$ and $c_1=P(1)$, even if they are not critical. For all values of $c\in\mathbb{C}-S$ the polynomial $P(x)-c$ has distinct roots none of which are 0 or 1.

Let $\alpha_i(c)$ be the roots of the polynomial equation $P(x)=c$. Clearly these functions are well defined and non-ramified on the universal cover of $\mathbb{C}-S$, which we denote $\tilde{\mathbb{C}}$, with projection $\pi:\tilde{\mathbb{C}}\to\mathbb{C}-S$. We also lift c and $I(c)$, ($|c|>K$) to $\tilde{\mathbb{C}}$. We shall show that $I(c)$ has a well-defined expression over $\tilde{\mathbb{C}}$ which, as it vanishes identically in some domain, must vanish throughout $\tilde{\mathbb{C}}$. The proof that all centers satisfy the *composition condition* will follow from an examination of the monodromy of this expression.

Clearly, on $\tilde{\mathbb{C}}$,

$$P(x)-c = \prod (x-\alpha_i).$$

We therefore obtain the following partial fraction expansion over $\tilde{\mathbb{C}}$:

$$\frac{q(x)}{P(x)-c} = r(x,c) + \sum_i \frac{m(\alpha_i(c))}{x-\alpha_i(c)},$$

where r is a polynomial in x and c and $m(x)=q(x)/p(x)$.

Using this expression, we fix a point $c'\in\tilde{\mathbb{C}}$ with $|\pi(c')|>K$ and find that

$$I(c) = R(c) + \sum_i m(\alpha_i)\ln_i(1-\frac{1}{\alpha_i}), \tag{6.5}$$

in a neighborhood of c'. Here R is a polynomial in c, and the $\ln_i$'s are specific branches of the logarithm, chosen individually for each α_i. It is clear that each term of this expression can be analytically continued to $\tilde{\mathbb{C}}$ as the only points of indeterminacy are those where the algebraic functions $\alpha_i(c)$ are ramified or where one of the $\alpha_i(c)$ attains the value 0 or 1. Both of these possibilities have been taken care of by removing the set S from consideration.

It is an interesting fact that the remainder of the proof does not depend on the specific choice of these logarithmic branches.

From standard results on algebraic functions, we know that for c sufficiently close to 0, we can describe all the solutions of $P(x) - c$ as Puisseaux expansions in c. We label the distinct roots (except $x = 1$) of $P(x) = 0$ as x_i, $i = 0, \ldots, r-1$. Since $P(0) = 0$, we can choose $x_0 = 0$ without loss of generality. If we also have $P(1) = 0$, then we shall label this root x_r. The contribution of this term in the expressions derived below should be taken to be zero if $P(1) \neq 0$. We write l_i for the multiplicity of x_i as a root of $P(x) = 0$.

About the point $x = x_i$ we have

$$c = k_i(x - x_i)^{l_i}(1 + U_i(x - x_i)), \qquad U_i(0) = 0,$$

for some constant $k_i \neq 0$ and some polynomial U_i. Close to $c = 0$, therefore, the solutions of $P(x) = c$ can be written as

$$x_{i,j} - x_i = \zeta_i^j k_i^{-1/l_i} c^{1/l_i} \left(1 + h_i(\zeta_i^j c^{1/l_i})\right), \quad i = 0, \ldots, r, \quad j = 0, \ldots, l_i - 1$$

where h_i is analytic with $h_i(0) = 0$, and ζ_i is a primitive l_i-th root of unity. We fix the choice of k_i^{-1/l_i} for each i. Clearly $x_{i,j}(0) = x_j$.

If we now analytically continue the expression (6.5) to a neighborhood of $c = 0$, then the solutions α_i of $P(x) = c$ can be relabeled as the corresponding $x_{ij}(c)$. In the same way we write $\ln_{i,j}$ for the corresponding branch of the logarithm given in (6.5). Thus,

$$I(c) = R(c) + \sum_{i=0}^{r} \sum_{j=0}^{l_i - 1} m(x_{i,j}) \ln_{i,j}(1 - \frac{1}{x_{i,j}}). \tag{6.6}$$

For ease of explanation, we shall evaluate the second index of $x_{i,j}$ modulo l_i. Clearly, as $c \in \mathbb{C} - S$ moves around a sufficiently small circle about 0, γ say, the $x_{i,j}$, $j = 0, \ldots, l_i - 1$ move around x_i swapping roles as follows:

$$x_{i,j} \mapsto x_{i,j+1}, \qquad (j = 0, \ldots l_i - 1).$$

Alternatively, we can consider the effect of the corresponding deck transformation of $\tilde{\mathbb{C}}$ on the right-hand side of (6.6). We denote this transformation by σ. Thus

$$\sigma^k \ln_{i,j}(1 - \frac{1}{x_{i,j}}) = \ln_{i,j}(1 - \frac{1}{x_{i,j+k}}).$$

If γ is sufficiently small, the paths of the $x_{i,j}$, $i \neq 0, r$ will not encircle the values 0 or 1. Thus, applying σ l_i to $\ln_{i,j}(1 - 1/x_{i,j})$ will return it to its original value.

However, when $i = 0, r$, the roots $x_{0,j}$ cycle around 0 and 1 and so

$$\begin{aligned} \sigma^{l_0} \ln_{0,j}(1 - \frac{1}{x_{0,j}}) &= \ln_{0,j}(1 - \frac{1}{x_{0,j}}) - 2\pi i, \\ \sigma^{l_r} \ln_{r,j}(1 - \frac{1}{x_{r,j}}) &= \ln_{r,j}(1 - \frac{1}{x_{r,j}}) + 2\pi i. \end{aligned}$$

Under the assumption that $p(0)$ and $p(1)$ are non-zero, we must have $l_0 = 1$ and $l_r \leq 1$. Letting N be the lowest common multiple of the l_i we obtain

$$\sigma^N I(c) - I(c) = -2\pi N im(x_{0,0}) + 2\pi N im(x_{r,0}).$$

Now $\sigma^N I(c) \equiv 0 \equiv I(c)$ and so $m(x_{0,0}) = m(x_{r,0})$. Since $q(x_{0,0})$ cannot vanish, the term $x_{r,0}$ must also exist and so we deduce $l_0 = 1$ and $P(1) = 0$.

Finally, we consider

$$M(c) = Q(x_{0,0}) - Q(x_{r,0}).$$

It is easy to see that

$$M'(c) = m(x_{0,0}) - m(x_{r,0}) = 0, \qquad M(0) = 0,$$

and so $M(c)$ is identically zero. We therefore have two polynomials P and Q for which their values on $x_{0,0}(c)$ and $x_{r,0}(c)$ are the same. The set of all rational functions for which this is true forms a subfield $\mathcal{F}$ of $\mathbb{C}(x)$, Following an identical line of reasoning to that used in Chapter 4 we see that both P and Q must be polynomials of a polynomial A in $\mathcal{F}$. Furthermore, since $A(x_{0,0}) = A(x_{r,0})$, we must have $A(0) = A(1)$. This proves that the composition condition is a necessary and sufficient condition for a center in this case ($p(0)$, $p(1) \neq 0$).

Notes

It was originally conjectured that the *composition condition* held for all systems (6.1) with centers. However, Pakovich [47] has shown that this is not the case (although there is still a non-trivial decomposition of P in his example). The results on the model problem presented here first appeared in [17].

Much work has been carried out on Abel equations by the team of Briskin, Françoise and Yomdin since the works [7, 8, 9]. In particular newer works take the ideas gained from the polynomial Abel equations back to the trigonometry equations, and in this way hope to obtain information about the center-focus problem for the homogenous systems (6.2).

Ideas of monodromy are ubiquitous in mathematics occurring even in places where they would seem unlikely (for example they play a key role in arithmetic algebraic geometry).

Żołądek's book [66] contains much more detail on many topics associated to monodromy in differential equations, and further afield.

Chapter 7

The Tangential Center-Focus Problem

As is well known, the second part of Hilbert's 16th problem is concerned with bounding the number of limit cycles in a polynomial system (1.2) of degree n in terms of n. This is a very hard problem, but Arnold has suggested a "Weak Hilbert's 16th problem" which seems far more tractable: to find a bound on the number of limit cycles which can bifurcate from a first-order perturbation of a Hamiltonian system,

$$\dot{x} = -\frac{\partial H}{\partial y} + \epsilon P, \qquad \dot{y} = \frac{\partial H}{\partial x} + \epsilon Q, \tag{7.1}$$

where the Hamiltonian, H, is a polynomial of degree $n+1$ and the perturbation terms, P and Q are polynomials of degree m.

Over time, the term "Tangential Hilbert's 16th problem" seems to have become the more popular (and descriptive) phrase for this problem. Although this is still a very difficult problem, finiteness results are known in this case, and several exact results have been obtained for Hamiltonians of low degree.

Recall that, to first order, the limit cycles appearing in the perturbed system (7.1) are given by the zeros of the abelian integral

$$I_c = \oint_{\gamma_c} P\, dy - Q\, dx, \tag{7.2}$$

where γ_c is a family of closed loops in the level curves $H = c$ of the Hamiltonian.

We want to suggest here an analogous "tangential center-focus problem", which seems to have a similar property of being much easier to tackle, whilst retaining something of the structure of the original problem.

Suppose that the Hamiltonian H has a Morse point at the origin. That is, $\partial H/\partial x = \partial H/\partial y = 0$ at (0,0) and the matrix of second derivative of H is sign-definite. Then (taking $-H$ if necessary) we can write $H = H(0,0) + X^2 + Y^2$ for some suitable choice of local coordinates, and the system (7.1) clearly has a center at the origin for $\epsilon = 0$. Without loss of generality we can take $H(0,0) = 0$. The curves $X^2 + Y^2 = c$, for c close to zero, give a family of closed curves tending to the origin as c tends to zero. We call such curves *vanishing cycles*.

We call the origin a *tangential center* if

$$\oint_{\gamma_c} P\,dy - Q\,dx \equiv 0, \tag{7.3}$$

for γ_c the vanishing cycle $X^2 + Y^2 = c$ given above, and c sufficiently small.

The *tangential center-focus problem* then asks for the conditions on P and Q which give a tangential center.

We shall sketch a proof in this chapter that for a generic Hamiltonian the answer for this problem is quite simple: P and Q must satisfy the equation

$$P\,dy - Q\,dx = dA + B\,dH, \tag{7.4}$$

for some polynomials A and B in x and y. In this case we say that the 1-form $P\,dy - Q\,dx$ is *relatively exact*.

The proof is due to Il'yashenko [36] who used the result to count the dimension of the space of non-trivial perturbations of the Hamiltonian system (7.1) and hence give a lower bound on the maximum number of limit cycles which can be produced by such perturbations.

Françoise [31] has shown that if the Hamiltonian has the property that every tangential center must come from a relatively exact perturbation, then it is possible to calculate the higher order perturbation terms of (7.1) as abelian integrals. This is known not to be true in general. Françoise calls this property "condition $(*)$".

The interesting question is how degenerate a Hamiltonian needs to become before condition $(*)$ does not hold. We shall show in the next chapter that, in the case of hyperelliptic Hamiltonians, the failure of condition $(*)$ implies the existence of a non-trivial symmetry.

7.1 Preliminaries

Suppose we have a tangential center (7.3) and want to investigate its global consequences. Since (7.2) is analytic in c, we can extend the condition (7.3) up to the boundary of the period annulus around the origin. However, if we want to extend further, we can only do this by trying to work over the complex numbers.

If we do so, the level curves $H(x, y) = c$ become Riemann surfaces, and the curves γ_c closed curves on this surface. The equations (7.2) and (7.3) carry across in an obvious way. By Cauchy's theorem we only need to consider the curves γ_c up to homotopy.

If we exclude a number of *critical values*, $S = \{c_1, \dots, c_r\}$, the map $(x, y) \mapsto H(x, y)$ defines a fibration over $\mathbb{C} - S$, with fibre $H(x, y) = c$ of constant topological type. If we consider the γ_c only up to homotopy, then there is a unique way to transport the γ_c as c varies.

If we move around a non-trivial loop in $\mathbb{C}-S$, the curve γ_c will not in general return to itself, but some other element in the homology group of the Riemann surface.

Thus we get a map

$$\sigma : \pi_1(\mathbb{C} - S) \to Aut(H_1(\phi_c, \mathbb{Z})), \tag{7.5}$$

where $H_1(\phi_c, \mathbb{Z})$ denotes the first homology group of the Riemann surface $\phi_c = \{H(x,y) = c\}$ with $\mathbb{Z}$ coefficients.

Our aim will be to show that for a generic Hamiltonian we can generate the whole homology group $H_1(\phi_c, \mathbb{Z})$ from the one cycle γ_c. By this we mean the following.

Definition 7.1. We say that γ_c generates the homology $H_1(\phi_c, \mathbb{Z})$ if there exist loops ℓ_i, $i = 1, \ldots, l_k$, in $\pi_1(\mathbb{C} - S)$ such that

$$\sum_{i=1}^{k} \mathbb{Z}\,\sigma(\ell_i)\gamma_c = H_1(\phi_c, \mathbb{Z}).$$

Similarly, we say that γ_c generates $H_1(\phi_c, \mathbb{Q})$ if we can find ℓ_i as above such that

$$\sum_{i=1}^{k} \mathbb{Q}\,\sigma(\ell_i)\gamma_c = H_1(\phi_c, \mathbb{Q}).$$

We first consider a simple case. Take the Hamiltonian

$$H = y^2 + x^2 + x^3.$$

This has critical points at $(0,0)$ and $(-2/3, 0)$. The critical values where the topology of $H = c$ changes are therefore $c_1 = 0$ and $c_2 = 4/27$.

Take γ_c as the vanishing cycle at the origin for $0 < c \ll 1$ close to 0. We now move c along a path which makes a positive loop around $c = c_2$ and then $c = c_1$. At both the critical values there is a Dehn twist. Let γ_c' be the vanishing cycle at $c = c_2$, then the twist at $c = c_2$ takes γ_c to $\gamma_c + \gamma_c'$. Then the twist at $c = c_1$ takes $\gamma_c + \gamma_c'$ to γ_c'. Since γ_c and γ_c' form a basis for $H_1(\phi_c, \mathbb{Z})$, we have shown that γ_c generates $H_1(\phi_c, \mathbb{Z})$.

7.2 Generic Hamiltonians

We want to generalize the construction in the previous section. We follow the original paper of Il'yashenko [36].

Consider the space B of Hamiltonians of degree $n+1$ with only Morse singularities, with distinct critical values, and whose highest order terms have no multiple factors. Equivalently, we require the equations $\partial H/\partial x = 0$ and $\partial H/\partial y = 0$

to have n^2 distinct solutions and the values of H on these solutions to also be distinct. These conditions are generic, and hence the complement of B in the space of all polynomials of degree $n+1$, $\mathbb{C}^{(n+2)(n+3)/2}$ is a proper algebraic subset. In particular B is path connected.

Now it is clear by continuity that if we are in the space B, the property that a vanishing cycle γ_c generates the homology is preserved under perturbation, so that if it holds for any Hamiltonian in B it holds in all of them.

Let $S = \{c_1, \ldots, c_k\}$ be the critical values as above, and denote by $\tilde{C}$ the universal cover of $C = \mathbb{C} - S$.

Near to each critical value c_i we have a vanishing cycle δ_i (unique up to homotopy). We fix a point $\tilde{c}_0 \in \tilde{C}$ with image $c_0 \in C$ and paths l_i in C from a sufficiently small neighborhood of each of the c_i to c_0 in C. We denote the result of transporting δ_i along l_i by $\delta_i(\tilde{c}_0)$ and can extend this to a cycle $\delta_i(\tilde{c})$ for all $\tilde{c} \in \tilde{C}$.

We say that the set of vanishing cycles is good if the cycles $\delta_i = \delta_i(\tilde{c}_0)$ generate the homology $H_1(\phi_{c_0}, \mathbb{Z})$ and we can connect any two cycles, d_i and d_j by a chain of cycles

$$\delta_i = \delta_{n_1}, \delta_{n_2}, \ldots, \delta_{n_k} = \delta_j,$$

with $(\delta_{n_r}, \delta_{n_{r+1}}) = \pm 1$ for each r. Here, (δ', δ) denotes the intersection number of the curves δ and δ' on the Riemann surface ϕ_c.

Clearly, if a Hamiltonian $H \in B$ has a good set of cycles for $\tilde{c} = \tilde{c}_0$, they will also be good for every value of $\tilde{c}$. Furthermore they will transport in an obvious way to a good set of cycles for any Hamiltonian in B.

If we have a good set of cycles, then it is now easy to show that γ_c generates the homology $H_1(\phi_{c_0}, \mathbb{Z})$. Suppose we have two vanishing cycles δ_i and δ_j with $(\delta_i, \delta_j) = \pm 1$. For each i, let $\bar{l}_i$ be the path from c_0 to itself which moves along l_i^{-1}, around c_i in the positive sense and back along l_i to c_0. Then $\sigma(\bar{l}_j)$ takes δ_i to

$$\delta_i + (\delta_j, \delta_i)\delta_j,$$

and $\sigma(\bar{l}_i)$ takes this to

$$\delta_i + (\delta_j, \delta_i)\delta_j + (\delta_i, \delta_i + (\delta_j, \delta_i)\delta_j)\delta_i = \pm\delta_j,$$

where equality is taken in $H_1(\phi_{c_0}, \mathbb{Z})$. Thus, since any two vanishing cycles can be connected by a chain of cycles with intersection numbers ± 1, we can generate the homology $H_1(\phi_{c_0}, \mathbb{Z})$ from any vanishing cycle.

To finish, we sketch the proof that we can find a good set of cycles. The idea is to consider a Hamiltonian

$$\tilde{H} = \prod_{i=1}^{n+1} r_i,$$

given by the product of $n+1$ real linear factors, such that the lines $r_i = 0$ are in general position. This Hamiltonian does not lie in the space B as the critical value $c = 0$ is very degenerate. However, we do not need that all the critical values are distinct to define a good system of cycles. Since this system lies in the complement of B it can be moved into B by a perturbation which will maintain the good system of cycles.

Consider the Riemann surface for a general value of c. This is non-singular in the finite plane with $n+1$ points of intersection with infinity and genus $n(n-1)/2$. Thus it has Betti number

$$2\frac{n(n-1)}{2} + (n+1) - 1 = n^2.$$

Now we consider the level curves of $\tilde{H} = c$. For $c = 0$ we have $n(n+1)/2$ points corresponding to the intersections of the $r_i = 0$. The other $n(n-1)/2$ vanishing cycles come from the centers which lie in the bounded connected components of $H > 0$ and $H < 0$. This gives us a full set of generators for the homology.

All critical values lie on the real line. We take c_0 to be a point sufficiently close to the origin and take paths from neighborhoods of the other critical points to c_0 passing $c = 0$ by making a half turn in the positive direction if necessary. It can be readily verified that this will give us a good system of cycles.

Note that even if we have a good system of cycles for $\tilde{H}$, it is not possible to conclude that the vanishing cycles generate the homology, as we require that the critical values be distinct in order to apply the argument above.

7.3 Relative exactness

We have shown that we can generate the homology $H_1(\phi_c, \mathbb{Z})$ from the vanishing cycle γ_c. Thus we are left to show that the condition (7.3) implies that the 1-form $P\,dy - Q\,dx$ is relatively exact. However, we have shown that γ_c generates $H_1(\phi_c, \mathbb{Z})$, and since (7.2) is analytic in c we must have

$$\oint_\gamma P\,dy - Q\,dx = 0, \tag{7.6}$$

for any closed curve γ in ϕ_c. We say that $P\,dy - Q\,dx$ is *topologically exact* on ϕ_c.

We want to show that if a polynomial 1-form is topologically exact on ϕ_c for all c in a continuum, then the 1-form must be relatively exact. We follow the paper of Bonnet [6].

Proposition 7.2. *If ϕ_c is a smooth irreducible curve, then any 1-form ω which is topologically exact on ϕ_c can be written in the form*

$$\omega = dR + (H - c)\eta,$$

for some polynomial R and some polynomial 1-form η.

Proof. The condition (7.6) means that we can integrate the 1-form ω on ϕ_c and get a single-valued function $\bar{R}$. The function R is analytic on ϕ_c, and by considering the growth of R at infinity, it must be a rational function and therefore realizable as the restriction of a rational function f/g in $\mathbb{C}^2$ whose denominator does not vanish on ϕ_c. This latter condition implies that $A(H-c)+Bg=1$ for some polynomials A and B by the Nullstellensatz, and so the $f/g = Bf$ on ϕ_c. We take $R = Bf$ whose restriction to ϕ is $\bar{R}$. Then $\omega - dR$ vanishes on ϕ_c and hence the result. □

Proposition 7.3. *Suppose that there exist polynomials R_c and polynomial 1-forms η_c such that*

$$\omega = dR_c + (H-c)\eta_c, \tag{7.7}$$

for an uncountable number of parameter values of c; then there exists a polynomial $P(H)$ such that

$$P(H)\omega = dA + B\,dH,$$

for some polynomials A and B.

Proof. Consider the space of 1-forms η_c from (7.7). Since the space of polynomial 1-forms is countable, but the number of η_c are uncountable, then there must exist λ_i and c_i, $i = 1,\ldots,r$, such that

$$\sum_{i=1}^{r} \lambda_i \eta_{c_i} = 0.$$

Hence,

$$\sum_{i=1}^{r} \frac{\lambda_i}{H-c_i}\,\omega = \sum_{i=1}^{r} \frac{\lambda_i dR_{c_i}}{H-c_i}.$$

On multiplying by the product of the $H - c_i$ and integrating by parts we obtain the desired result. □

Theorem 7.4. *Suppose that for all c, the curve $\phi_c = \{H = c\}$ is connected and contains only finitely many singular points of H. If a 1-form is relatively exact on ϕ_c for an uncountable number of values of c, then the 1-form is relatively exact.*

Proof. From Proposition 7.3 we only need to show that if $(H-c)\omega$ is relatively exact, then so is ω. Thus, suppose that

$$(H-c)\omega = dA + B\,dH,$$

for some polynomials A and B. On $H = c$ we have $dA = 0$ thus A is a constant on $H = c$ since $H = c$ is connected. Thus $A = k + (H-c)\bar{A}$ for some constant k and some polynomial $\bar{A}$, and we can write

$$(H-c)\omega = (H-c)d\bar{A} + (\bar{A}+B)\,dH.$$

Finally, since there are only a finite number of points on $H = c$ where dH vanishes, $(H-c)$ must divide $\bar{A}+B$ and we are done. □

The hypothesis for this theorem is satisfied if we assume that the Hamiltonian H lies in B and therefore we have shown that for a generic Hamiltonian all tangential centers arise from relatively exact perturbation terms.

Notes

In the paper of Bonnet [6], a more detailed result is shown. Polynomial $P(H)$ in Proposition 7.3 can be reduced to

$$P(H) = \prod_{i=1}^{r} (H - c_i)^{m_i},$$

where $m_i = 1$ if $H = c_i$ contains infinitely many singular points of H; m_i is arbitrary if $H = c_i$ is not connected; and $m_i = 0$ otherwise.

That more complex things can happen for more general Hamiltonians is clear in the case of the Hamiltonian $\tilde{H}$ mentioned in Section 7.2. In fact, taking

$$P\,dy - Q\,dx = \sum \alpha_i K_i dr_i, \qquad K_i = \tilde{H}/r_i,$$

for any constants α_i, it is clear that on $\tilde{H} = c$ we have

$$P\,dy - Q\,dx = \sum \alpha_i c dr_i / r_i,$$

and hence (7.3) holds for each of the real centers lying in the bounded regions of $H > 0$ and $H < 0$. However the form cannot be relatively exact as the integrals around the vanishing cycles for $c = 0$ will not vanish in general. Perturbations from such Hamiltonians have been studied by Movasati [45] and Uribe [59].

Such 1-forms, which generalize the relatively exact 1-forms, seem to be the tangential analog of the Darboux centers. We shall show in the next section that the symmetric centers also have a prominent place in the tangential center-focus problem. Thus although the tangential center-focus problem is a considerable simplification of the original center-focus problem, much of the complexity of the full center-focus problem is retained, suggesting that further investigation into this subject would be very valuable.

Chapter 8

Monodromy of Hyperelliptic Abelian Integrals

We want to show that in the case of Hamiltonians of the form

$$H(x,y) = y^2 + f(x),$$

where $f(x)$ is a polynomial of degree n, the existence of a tangential center implies that either $P\,dx - Q\,dy$ is relatively exact, or the polynomial $f(x)$ is *composite.* That is, it can be expressed as a polynomial of a polynomial, $f(x) = a(b(x))$, in a non-trivial way.

The factorization is related to the existence of a symmetry in the Hamiltonian. That is, a factorization

$$H(x,y) = \tilde{H}(r(x,y), s(x,y)), \tag{8.1}$$

where $r(x,y)$ and $s(x,y)$ are polynomials whose Jacobian vanishes at some point in $\mathbb{C}^2$, and hence on some algebraic curve C.

Conversely, suppose we are given a factorization of a Hamiltonian H as in (8.1). Let $\pi : \mathbb{C}^2 \to \mathbb{C}^2$ be the projection $(x,y) \mapsto (r(x,y), s(x,y))$. Suppose further that we have a family of vanishing cycles γ_c whose image under π is homotopic to zero. Let $\tilde{w}$ be a polynomial 1-form on $\mathbb{C}^2$, and $w = \pi^*\tilde{w}$, then

$$\oint_{\gamma_c} w = \oint_{\pi(\gamma_c)} \tilde{w} \equiv 0,$$

thus the 1-form w gives a tangential center. However, in general this 1-form is not relatively exact, since any family of vanishing cycles which are not sent to a curve homotopic to zero can have non-trivial integrals given by the integral of $\tilde{w}$.

In this section we want to show the following result.

Theorem 8.1. *If the system (7.1), with $H = y^2 + f(x)$, has a tangential center with associated vanishing cycle γ_c, then one of the following must hold.*

(i) *γ_c generates the homology $H_1(\phi_c, \mathbb{Q})$.*

(ii) *f is decomposible.*

(iii) *f is a Chebyshev polynomial of prime degree.*

Remark 8.2. Note that the theorem doesn't say anything about the perturbation terms, it is purely topological. However, in generic cases of (i) we can conclude that the perturbation terms would have to be relatively exact. In case (ii) we would expect that the 1-form given by the perturbation terms would be the pull back of a 1-form on the factorized space. However we could not show this at present. In case (iii) it is possible by analyzing the monodromy in more detail to show that in fact we can reduce to Case (iii). However, in this case we do have a non-trivial splitting of $H_1(\phi_c, \mathbb{Q})$ into invariant subspaces over an extension of $\mathbb{Q}$. It is not clear whether splittings can occur for any other Hamiltonians not in Case(ii).

8.1 Some Group Theory

We recall some definitions from group theory.

Definition 8.3. (1) Let G be a group acting on a finite set S; then we say that the action is *imprimitive* if there exists a non-trivial decomposition of S, $S = \bigcup S_i$, such that for each element of g and each i, g sends S_i into S_j for some j. The action is called *primitive* if it is not imprimitive.

(2) The action is *transitive* if, given any pair of elements of S, s_1 and s_2, there is an element $g \in G$ which sends s_1 to s_2.

(3) The action is *2-transitive* if, given any two pairs of elements of S, (s_1, s_2) and (s_3, s_4), there is an element $g \in G$ which sends s_1 to s_3 and s_2 to s_4.

(4) The action is *regular* if, given two elements s_1 and s_2 of S, there is a unique element g of G which sends s_1 to s_2.

Clearly, if G is transitive and imprimitive, then all the sets S_i must be of the same size.

Recall that the affine group $\mathrm{Aff}(\mathbb{Z}_p)$ is the group of all affine transformations of $\mathbb{Z}_p$ to itself. That is, it is the group of all maps from $\mathbb{Z}_p$ to itself of the form $x \mapsto ax + b$ for a, $b \in \mathbb{Z}_p$ with multiplication given by composition.

Theorem 8.4 (Burnside–Schur). *Every primitive finite permutation group containing a regular cyclic subgroup is either 2-transitive or permutationally isomorphic to a subgroup of the affine group Aff(p), where p is a prime.*

Proof. See [29] or [28]. □

8.2 Monodromy groups of polynomials

Let $f(x)$ be a polynomial of degree $n > 0$, and consider the solutions, $x_i(c)$, of the equation $f(x) = c$. Let S be the set of critical points $c \in \mathbb{C}$ for which $f(x) = c$ and $f'(x) = 0$ has a common solution. Clearly there are at most $n(n-1)$ of these points. As c takes values in $\mathbb{C} - S$ the functions $x_i(c)$ are well defined. The group $G = \pi_1(\mathbb{C} - S)$ acts on the $x_i(c)$. The action is always transitive if we consider large values of c.

Definition 8.5. Let G be as above, then the action of G on the set of x_i is called the *monodromy* group of the polynomial f, denoted $\mathrm{Mon}(f)$.

As mentioned in Chapter 6, we have the following theorem [30].

Theorem 8.6. *The monodromy group is isomorphic to the Galois group of $f(x) - c$ considered as a polynomial over $\mathbb{C}(c)$.*

Definition 8.7. We say that a polynomial $f(x)$ is *decomposable* if and only if there exist two polynomials g and h, both of degree greater than 1, such that $f(x) = g(h(x))$.

Proposition 8.8. *Let f be a polynomial as above and let G be its monodromy group. Then:*

(i) *the action of G is imprimitive if and only if the polynomial f is decomposable;*

(ii) *the action is 2-transitive if and only if the divided differences polynomial $H(x, y) = (f(x) - f(y))/(x - y)$ is irreducible.*

We do not prove this proposition as we do not need the results below, however we note that quantities of the form $(f(x) - f(y))/(x - y)$ were also the ones appearing in Chapter 3.

Definition 8.9. The unique polynomial $T_n(x)$ which satisfies $T_n(\cos(\theta)) = \cos(n\theta)$ is called the *Chebyshev* polynomial of degree n. Equivalently $T_n((z + z^{-1})/2) = T_n((z^n + z^{-n})/2)$.

From the definition, the Chebyshev polynomial T_n has $n-1$ distinct turning points when $T_n = \pm 1$. Hence

$$T_n'^2 \mid (T_n(x)^2 - 1).$$

Conversely, any polynomial with this property is equivalent to a Chebyshev polynomial after composing with a linear function.

Theorem 8.10. *Let $f(x)$ be a polynomial of degree n and $G = Mon(f)$, then one of the following holds.*

(i) *The action of G on the x_i is 2-transitive.*

(ii) *The action of G on the x_i is imprimitive.*

(iii) f *is equivalent to a Chebyshev polynomial* T_p *where* p *is prime.*

(iv) f *is equivalent to* x^p *where* p *is prime.*

Remark 8.11. In particular, the question of whether f is a composite polynomial or not, can be solved very simply by considering whether or not the divided differences polynomial factorizes or not, having excluded the two exceptional cases above. "Equivalence" refers to pre- and post- composition by linear functions.

Proof. When c is large the x_i can be expanded as

$$x_i = \omega^r c^{1/n} + O(c^{(1/n)-1}),$$

where ω is an n-th root of unity. Thus, taking a sufficiently large loop in $\mathbb{C} - C$, we obtain an element of G which is an n-cycle. This element generates a subgroup $\mathbb{Z}_n$ of G which acts regularly on the roots of $f = c$.

Thus we can apply the Burnside–Schur Theorem above to show that the group must be 2-transitive, imprimitive, or a subgroup of $\mathrm{Aff}(\mathbb{Z}_p)$. In the latter case we note that every element of $\mathrm{Aff}(\mathbb{Z}_p)$ fixes at most one element of $\mathbb{Z}_p$. This means that for every critical value of f there is at most one x_i that remains fixed as we turn around this value.

Now, suppose f has r distinct critical values, $c_1, \ldots, c_r$, and f has r_i distinct turning points associated to the critical value c_i. Let the multiplicities of the roots of f' at these turning points be $m_{i,1}, \ldots, m_{i,r_i}$. Since a root of multiplicity $m_{i,j}$ gives a cycle of order $m + 1$, then for all i we must have

$$n - 1 \leq \sum_{j=1}^{r_i} (m_{i,j} + 1) \leq n, \tag{8.2}$$

since at most one of the x_i remains fixed turning around each critical value. Summing these equations over i we obtain

$$r(n-1) \leq \sum_{i=1}^{r} \sum_{j=1}^{r_i} (m_{i,j} + 1) \leq rn. \tag{8.3}$$

But the number of turning points of f counted with multplicity is just the sum of the $m_{i,j}$, and hence

$$r(n-1) \leq (n-1) + \sum_{i=1}^{r} r_i \leq rn. \tag{8.4}$$

Since the sum of the r_i is at most $n - 1$ we must have $r \leq 2$.

If $r = 1$, then (8.4) shows that $r_1 = 1$, and therefore $f(x)$ must have a root of multiplicity n. This is just Case (iv).

If $r = 2$ we need $n - 1 \leq r_1 + r_2 \leq n + 1$. But since r_i can be no more than $n/2$ this means that both r_i lie between $(n-1)/2$ and $n/2$. This implies that every turning point must have multiplicity 1 and the polynomial must be Chebyshev. $\square$

8.3 Proof of the theorem

We consider the level curves of the Hamiltonian $H = y^2 - f(x) = c$ as a two-sheeted covering of the complex plane $\mathbb{C}$ given by projection onto the x-axis. The sheets ramify at the roots of $f(x) = c$. Taking S to be the set of critical points as above, we let c vary in $\mathbb{C} - S$, and follow the effect on the homology group $H_1(\phi_c, \mathbb{Z})$. We wish to relate this group to the monodromy group of the polynomial $f(x)$. As x tends to infinity along the positive real axis, we can distinguish the two sheets as "upper" and "lower" depending on whether $y = \pm x^{n/2}$. We let t denote the deck transformation which takes y to $-y$ fixing x.

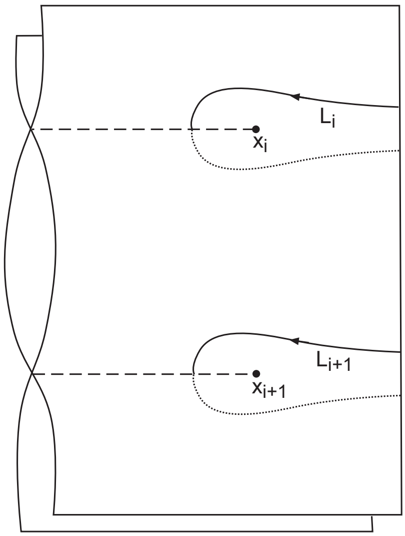

Figure 8.1: The loops L_i.

Let $H_1^c(\phi_c, \mathbb{Z})$ represent the closed homology group of ϕ_c over $\mathbb{Z}$. This can be obtained from $H_1(\phi_c, \mathbb{Z})$ by adding curves starting and finishing at infinity. Let $x_i(c)$ be the roots of $f(x) = c$. Generically, the x_i will have distinct imaginary parts, and so any closed path in $\mathbb{C} - S$ can be deformed so that only two of the x_i's have the same imaginary part at the same time. In other words, we can decompose every element of $\mathrm{Mon}(f)$ as a number of swaps of x_i's with neighboring real values.

Suppose that the x_i are initially numbered in order of decreasing imaginary part for a value of c close to zero. We let L_i represent the path from infinity (from the direction of the positive real axis) on the upper sheet, turning around x_i in the positive direction and returning to infinity on the lower sheet. Clearly $t(L_i) + L_i$ is homotopic to zero, and so the L_i generate $H_1^c(\phi_c, \mathbb{Z})$. Furthermore, the elements $L_i - L_{i+1}$ generate $H_1(\phi_c, \mathbb{Z})$.

The effect of a swap of x_i and x_{i+1} is to take L_{i+1} to L_i and L_i to $2L_i - L_{i+1}$. This is a little too complex to analyze in general, except for very specific systems. Instead we shall work for the moment over $\mathbb{Z}_2$. That is, we consider the images of the L_i in $H_1^c(\phi_c, \mathbb{Z}_2)$ and $H_1^c(\phi_c, \mathbb{Z}_2)$.

Working modulo 2 means that a swap of x_i and x_{i+1} takes L_{i+1} to L_i and L_i to L_{i+1}. That is, the action of $\mathrm{Mon}(f)$ on the $L_i \pmod 2$ is exactly the same as the action on the x_i.

We now apply the results of Theorem 8.10 in order to prove Theorem 8.1. According to Theorem 8.10 we only need to consider four cases. The last two of these are easy. In the case (iii) holds, we trivially have case (iii) of Theorem 8.1, and in case (iv) the Hamiltonian does not have a Morse point, and hence no tangential center. We shall show that the cases (i) and (ii) of Theorem 8.10 correspond to cases (i) and (ii) of Theorem 8.1, and the proof is complete.

Case (i). If the monodromy group of f is 2-transitive, then we can find a transformation which takes any two x_i's to any other two. Since, working modulo 2, the action on the loops L_i is the same as the action on the x_i, we can find an element of the monodromy group which takes $L_i - L_{i+1}$ to $L_j - L_{j+1}$ modulo 2 for all i and j.

Now, the vanishing cycle γ_c occurs at the coalescence of two of these x_i's and so must correspond to one of the $L_k - L_{k+1}$ for some k. Thus, there exist paths ℓ_i in $\mathbb{C} - C$ such that

$$\sigma(\ell_i)\gamma_c = L_i - L_{i+1} \pmod 2,$$

for all i.

Now let $N = 2\lfloor (n-1)/2 \rfloor$. Then $L_i - L_{i+1}$ form a basis of $H_1(\phi_c, \mathbb{Z})$. From the discussion above, we have

$$\begin{pmatrix} \sigma(\ell_1)\gamma_c \\ \sigma(\ell_2)\gamma_c \\ \vdots \\ \sigma(\ell_N)\gamma_c \end{pmatrix} = A \begin{pmatrix} L_1 - L_2 \\ L_2 - L_3 \\ \vdots \\ L_N - L_{N+1} \end{pmatrix},$$

where the matrix A reduces to the identity matrix if we reduce modulo 2. In particular, A is invertible, and we can express the basis of $H_1(\phi_c, \mathbb{Z})$ as sums of the $\sigma(\ell_i)\gamma_c$ with coefficients in $\mathbb{Q}$. That is, γ_c generates $H_1(\phi_c, \mathbb{Q})$. This gives us Case (i) of Theorem 8.1.

Case (ii). In this case $\mathrm{Mon}(f)$ is imprimitive (but nevertheless transitive) on the set of roots, $S = \{x_1, \ldots, x_n\}$. Let S_1 be one of the subsets in the decomposition $S = \bigcup S_i$, and let $s = x_k$ be an element of S_1. We denote G_s and H the subgroups of $G = \mathrm{Mon}(f)$ which leave s and S_1 fixed respectively. Then Case (iii) implies that

$$G_s \subsetneq H \subsetneq G. \tag{8.5}$$

As stated in Section 8.2, the group $G = \mathrm{Mon}(f)$ is just the Galois group of $\mathbb{C}(x_1(c), \ldots, x_n(c))$ over $\mathbb{C}(c)$. We consider the corresponding fixed fields of the groups in (8.5) under the Galois correspondence, to obtain

$$\mathbb{C}(x_k(c)) \supsetneq K \supsetneq \mathbb{C}(c), \tag{8.6}$$

where K is the fixed field of H. From Lüroth's theorem, we must have $K = \mathbb{C}(r(x_k))$, for some rational function $r(x_k) \in \mathbb{C}(c_k)$. Then (8.6) implies that $c = s(r(x_k))$ for some rational function s. Thus $f(x) = s(r(x))$, and a similar argument to the one given in Lemma 4.5 shows that s and r can in fact be chosen to be polynomials.

This completes the proof of Theorem 8.1. □

Notes

The monodromy group of a polynomial is an object of some interest in the inverse Galois problem (see, for example, the work of Müller [46], from whom I first learnt about the Burnside–Schur results).

Complete details of the above result together with an analysis of the Chebyshev case and other related results can be found in [24].

Chapter 9

Holonomy and the Lotka–Volterra System

In this section we give another idea related to monodromy. This is the holonomy of the foliation $P\,dy - Q\,dx = 0$ associated to the system (1.2) in the neighborhood of an invariant curve. This object, roughly speaking, is the nonlinear analog to the monodromy of the solutions of a linear differential equation as they turn around a singular point. Alternatively, it can be thought of as a kind of Poincaré return map for foliations.

We shall define the holonomy in the next section and give a very basic theorem which will guarantee the integrability of a critical point. Recall this means the following:

Definition 9.1. The origin of the analytic system

$$\dot{x} = P(x,y) = x + \tilde{P}(x,y), \qquad \dot{y} = Q(x,y) = -\lambda y + \tilde{Q}(x,y), \tag{9.1}$$

where $\tilde{P}$ and $\tilde{Q}$ contain terms of order 2 or higher, is *integrable* if there exists an analytic change of coordinates $(X,Y) = (x+o(x,y), y+o(x,y))$ in the neighborhood of the origin transforming the system into

$$\dot{X} = Xh(X,Y), \qquad \dot{Y} = -\lambda Y h(X,Y), \tag{9.2}$$

where $h = 1 + O(X,Y)$. Alternatively, the origin of (9.1) is integrable if and only if there exist holomorphic functions $X = x + o(x,y)$ and $Y = y + o(x,y)$ such that $X^\lambda Y$ is a first integral of (9.1).

When $\lambda = p/q$ we call (9.1) a *p:-q saddle*. If (9.1) it is not linearizable, we call it a *resonant saddle*. The problem of finding whether a $p:-q$ saddle is integrable or not is in exact analogy to the center-focus problem. Indeed, if we take $x = X^p$ and $y = Y^q$ we obtain a new system (after scaling time by a constant factor p).

$$\dot{x} = X + \tilde{P}(X^p, Y^q), \qquad \dot{Y} = -Y + \tilde{Q}(X^p, Y^q)q/p, \tag{9.3}$$

and the origin of (9.1) is integrable if and only if the origin of (9.3) is integrable (i.e., it is complex center). We can therefore apply the same techniques of computing Lyapunov quantities etc., in order to detect whether the system (9.1) is integrable or not.

The holonomy around the separatrix of a saddle turns out to be linearizable if and only if the saddle is integrable. Furthermore, if the separatrix is known, then we can relate this holonomy to the holonomy of the other critical points on the separatrix. Under favorable conditions, we can show that the original holonomy must be linearizable and hence give a potentially new method of finding integrable critical points.

We apply our results to the origin of the Lotka–Volterra system

$$\dot{x} = x(1 + ax + by), \qquad \dot{y} = y(-\lambda + cx + dy), \tag{9.4}$$

and show that this technique seems to explain many of the cases of integrability when $\lambda = p/q$ for $p+q \leq 12$, apart from a small number of (apparently exceptional) Darboux cases.

Further details can be found in [25] from whence the material is drawn.

9.1 The monodromy group of a separatrix

In this section we consider a saddle point with a separatrix given by either an invariant line or a non-singular conic and give sufficient conditions for the integrability of a saddle point by looking at the monodromy group of the separatrix. We apply this to the Lotka–Volterra equations, to obtain four classes of explicit conditions which give integrable critical points.

The surprising thing is that, even though these conditions on the monodromy groups are elementary, they comprise all the known cases of integrability for the Lotka–Volterra equations, except for the case where the system has an invariant straight line and two exceptional Darboux integrable cases [11, 44].

Consider the foliation on $\mathbb{CP}^2$ generated by the 1-form associated to the vector field. Let Γ be an invariant line or conic for the 1-form, and $Q_1, \ldots, Q_n$ be the singular points of the foliation which lie on Γ. For (9.4) we have three such lines: the two axes and the line at infinity. Clearly $\Gamma' = \Gamma - \{Q_1, \ldots, Q_n\}$ is isomorphic to an n-punctured sphere.

Choose a family of analytic transversals, Σ_x, through each point x in Γ', and fix a base point, P, in Γ', and an analytic parameterization z of Σ_P with $z = 0$ corresponding to the point P. For each path γ in $\pi(\Gamma', P)$, we can define a map from a neighborhood of P in Σ_P to Σ_P by lifting the path γ to the leaf of the foliation though $s \in \Sigma_P$ via the transversals Σ_x, $x \in \gamma$. Using the parameter z, this map can be identified with the germ of a diffeomorphism from $\mathbb{C}$ to itself, fixing $z = 0$. We call the set of all such diffeomorphisms $\mathrm{Diff}(\mathbb{C}, 0)$.

Clearly the map $M : \pi(\Gamma', P) \to \mathrm{Diff}(\mathbb{C}, 0)$ is in fact a group homomorphism. We denote the image of the path γ by M_γ. The *monodromy group* is the image of M. The *monodromy of one singular point* Q_i is M_γ where γ is a loop turning around Q_i exactly once in the positive direction and not containing any other singular point in its interior.

The map M_γ depends only on the homotopy type of γ in Γ'. If we use a different base point P_1, then the two monodromy groups are conjugate. Likewise a different choice of transversals and their parameterizations has the effect of conjugating the group. Thus the following notions for the monodromy of a singular point are intrinsic:

- the monodromy of the singular point is the identity;
- the monodromy of the singular point is linearizable.

Theorem 9.2. *Consider a polynomial system with a saddle point at the origin*

$$\begin{aligned} \dot{x} &= x(1+P(x,y)) = x(1+O(x,y)), \\ \dot{y} &= -\lambda y + Q(x,y) = -\lambda y + o(x,y), \end{aligned} \tag{9.5}$$

where $\lambda > 0$*. If all singular points of the system on the* y*-axis except the origin are integrable and if all of them but one have identity monodromy maps corresponding to the invariant* y*-axis, then the origin is also integrable.*

Proof. We consider the completion of the line $x = 0$ as the Riemann sphere S^1. Let $Q_1, \ldots, Q_n$ be the singular points of the system on that leaf.

Let Q_i be a point of saddle or node type. It is known that Q_i is integrable if and only if the corresponding monodromy map is linearizable (this is proved in [43] and [49] for a saddle. For a node it can easily be proved by considering the analytic normal form at the node).

Take a base point $y_0 \in S^1 - \{Q_1, \ldots, Q_n\}$ and loops γ_i from y_0 winding once around the singular points Q_i in the positive sense; then γ_1 is homotopic to $\gamma_n^{-1} \circ \cdots \circ \gamma_2^{-1}$, with appropriate re-labelling of the Q_i. As a result M_{γ_1} is conjugate to $M_{\gamma_n}^{-1} \circ \cdots \circ M_{\gamma_2}^{-1}$. Since all of them are the identity except one which is linearizable, then the map M_{γ_1} is linearizable. $\square$

9.2 Integrable points in Lokta–Volterra systems

We apply these results to the Lotka–Volterra family (9.4). This family is invariant under

$$(x, y, t, \lambda, a, b, c, d) \mapsto (-\lambda y, -\lambda x, -\frac{t}{\lambda}, \frac{1}{\lambda}, d, c, b, a) \tag{9.6}$$

and corresponding cases under this invariance are called *dual.*

Lemma 9.3. *A node is linearizable if and only it it has two analytic separatrices.*

Proof. A node with eigenvalues λ_1, λ_2 whose quotient is in $\mathbb{R}^+$ can always be brought to normal form by an analytic change of coordinates. When $\frac{\lambda_2}{\lambda_1} \notin \mathbb{N} \cup 1/\mathbb{N}$, then the normal form is linear and the two axes are analytic separatrices. When $\frac{\lambda_2}{\lambda_1} = n \in \mathbb{N}$ the normal form is

$$\begin{array}{rcl} \dot{x} & = & \lambda_1 x, \\ \dot{y} & = & \lambda_2 y + \alpha x^n. \end{array} \tag{9.7}$$

If $\alpha = 0$, then the system is linear as before and all integral curves through the origin are analytic, while if $\alpha \neq 0$ the curve $x = 0$ is the unique analytic integral curve through the origin. Similarly for $\frac{\lambda_2}{\lambda_1} \in 1/\mathbb{N}$. $\square$

Theorem 9.4. *We consider the Lotka–Volterra system* (9.4) *with* $\lambda > 0$*. Then the origin is integrable if one of the following conditions is satisfied.*

(A_n). $\lambda + \frac{c}{a} = n$ *with* $n \in \mathbb{N}$, $2 \leq n < \lambda + 1$.

(B_n). $\frac{b}{d} + \frac{1}{\lambda} = n$ *with* $n \in \mathbb{N}$, $2 \leq n < \frac{1}{\lambda} + 1$.

(C_n). $\frac{c}{a} + n = 0$ *with* $n \in \mathbb{N} \cup \{0\}$ *and* $n < \lambda$ *and* $\lambda \neq n + \frac{1}{m}$ *with* $m \in \mathbb{N}$. *If* $\lambda = n + \frac{1}{m}$, *then an additional condition is necessary for integrability.*

(D_n). $\frac{b}{d} + n = 0$ *with* $n \in \mathbb{N} \cup \{0\}$ *and* $n < \frac{1}{\lambda}$ *and* $\frac{1}{\lambda} \neq n + \frac{1}{m}$ *with* $m \in \mathbb{N}$. *If* $\frac{1}{\lambda} = n + \frac{1}{m}$, *then an additional condition is necessary for integrability.*

$(E_{n,m})$. $\lambda + \frac{c}{a} = n$ *and* $1 - \frac{b}{d} = \frac{1}{m}$ *with* $n, m \in \mathbb{N}$, $n > 1$ *and* $0 < \frac{(c-a)(d-b)}{ad-bc} \notin \mathbb{N}$.

$(F_{n,m})$. $\frac{1}{\lambda} + \frac{b}{d} = n$ *and* $1 - \frac{c}{a} = \frac{1}{m}$ *with* $n, m \in \mathbb{N}$, $n > 1$ *and* $0 < \frac{(c-a)(d-b)}{ad-bc} \notin \mathbb{N}$.

$(G_{n,m})$. $\lambda + \frac{c}{a} = n$, $1 - \frac{b}{d} > 0$ *and* $\frac{ad-bc}{(c-a)(d-b)} = m$ *with* $m, n \in \mathbb{N} - \{1\}$.

$(H_{n,m})$. $\frac{1}{\lambda} + \frac{b}{d} = n$, $1 - \frac{c}{a} > 0$ *and* $\frac{ad-bc}{(c-a)(d-b)} = m$ *with* $n, m \in \mathbb{N} - \{1\}$.

(Note that some strata with different names may be identical for some values of λ *and of the indices. This can for instance happen with* $(E_{n,m})$ *and* $(G_{n,m'})$*.)*

Proof. To apply the previous theorem and corollary we need to calculate the Jacobian matrix and the eigenvalues at all singular points along the axes and along infinity. On each separatrix there are three critical points: the one at the origin with ratio of eigenvalues $-\lambda$, one in the finite plane, and one where the axes cross the line at infinity. The Jacobians for the finite critical points $P_1 = (-\frac{1}{a}, 0)$ (resp. $P_2 = (0, \frac{\lambda}{d})$) on the x-axis (resp. y-axis) are

$$\begin{pmatrix} -1 & -\frac{b}{a} \\ 0 & -\lambda - \frac{c}{a} \end{pmatrix} \quad \textit{resp.} \quad \begin{pmatrix} 1 + \lambda\frac{b}{d} & 0 \\ \lambda\frac{c}{d} & \lambda \end{pmatrix}, \tag{9.8}$$

showing that the monodromy of the finite critical points on the x-axis (resp. y-axis) is the identity if $\lambda + \frac{c}{a} = n$ (resp. $\frac{b}{d} + \frac{1}{\lambda} = n$) with $n \in \mathbb{N}$, $n \geq 2$.

We now study the singular points at infinity. For that purpose we first consider the chart $(u, z) = (y/x, 1/x)$ to calculate the Jacobian matrix at the intersection of the line at infinity with the x-axis, which we denote $P_x = (0, 0)$. We can also calculate the Jacobian at the other critical point $P_\infty = (\frac{a-c}{d-b}, 0)$ on the line at infinity. After multiplication by z, the system becomes:

$$\begin{aligned} \dot{u} &= (c-a)u + (d-b)u^2 - (1+\lambda)uz, \\ \dot{z} &= -az - buz - z^2, \end{aligned} \tag{9.9}$$

yielding the following Jacobian matrices for P_x and P_∞:

$$\begin{pmatrix} c-a & * \\ 0 & -a \end{pmatrix} \quad resp. \quad \begin{pmatrix} -(c-a) & * \\ 0 & \frac{ad-bc}{b-d} \end{pmatrix}. \tag{9.10}$$

Similarly the chart $(v, w) = (x/y, 1/y)$ is used to study the infinite singular point P_y along the y-axis. Its Jacobian matrix is given by

$$\begin{pmatrix} b-d & * \\ 0 & -d \end{pmatrix}. \tag{9.11}$$

We can represent the ratios of eigenvalues on the diagram below, where the arrows represent the direction of the eigenvalue which is the numerator of the eigenvalue ratio.

Note that the sum of the eigenvalue ratios along any line is equal to 1. This follows from the index formula of Lins Neto [38].

We now prove the cases (A)–(H) given above. We may remove the indices when they are not necessary.

Case $(A_n)/(B_n)$: In Case (A_n), the condition implies that the monodromy of P_1 corresponding to the invariant x-axis is the identity and the critical point P_x is a node. It is always linearizable since there are two analytic separatrices. Case (B_n) is the dual of Case (A_n).

Case $(C_n)/(D_n)$: Case (C_n) is similar to Case (A), but now the monodromy at P_x is the identity corresponding to the invariant x-axis, and P_1 is a node. It is linearizable if $\lambda \neq n + \frac{1}{m}$ with $m \in \mathbb{N}$, otherwise (the case of a resonant node) the obstruction to linearizability consists of only one condition. Case (D_n) is the dual of Case (C_n).

Case $(E_{n,m})/(F_{n,m})$: Case $(E_{n,m})$ requires a double application of Theorem 9.2. The conditions imply that the monodromy of P_1 corresponding to the invariant x-axis is the identity. Thus the monodromy at the origin is conjugate to the inverse of the monodromy of P_x corresponding to the invariant x-axis. Now, this monodromy is linearizable if and only if P_x is integrable. This is the case if and only if the monodromy of P_x corresponding to the other separatrix (in this case, the line at infinity) is linearizable. Now, the conditions given in Case $(E_{n,m})$ guarantee that the monodromy of P_y corresponding to the line at infinity is the identity. (P_y is a node with ratio of eigenvalues $m \in N$.) Hence the origin is integrable if and only if the monodromy of P_∞ is linearizable corresponding to the line at infinity. Now the final condition in Case $(E_{n,m})$ guarantees that P_∞ is a

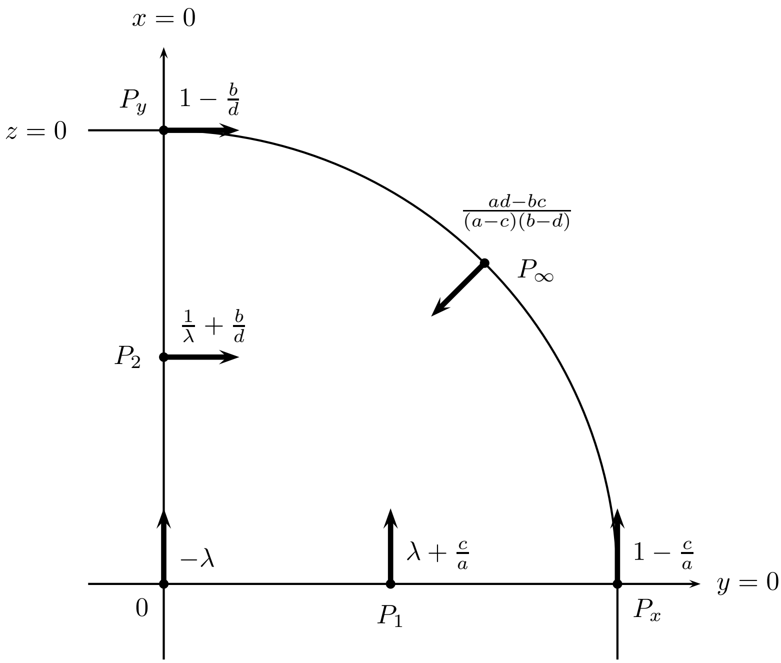

Figure 9.1: Ratio of eigenvalues for the Lotka–Volterra system

non-resonant node, and therefore linearizable. Case $(F_{n,m})$ is the dual of Case $(E_{n,m})$.

CASE $(G_{n,m})/(H_{n,m})$: Case $(G_{n,m})$ is the same as Case (E) except that now, the monodromy of P_∞ corresponding to the line at infinity is the identity and the point P_y is a node (necessarily linearizable). Case $(H_{n,m})$ is the dual of Case $(G_{n,m})$. □

We have the following conjecture.

Conjecture 9.5. The Lotka–Volterra system (9.4) with $\lambda \in \mathbb{Q}^+$ is integrable if and only if either

1. the system has a third invariant line, i.e.,

$$\lambda ab + (1-\lambda)ad - cd = 0; \tag{9.12}$$

2. one of the conditions of Theorem 9.4 is satisfied;

3. or there is an invariant algebraic curve, $f = 0$.

The first and third items above will give Darboux centers. In fact, we know from the lists given in [11, 44] that there are essentially only two cases (with $\lambda = 8/7$ and $13/7$ and their duals) where this last condition holds, which are not contained in the previous two conditions. These (after scaling) are the systems

$$\begin{aligned} \dot{x} &= x(1 - 2x + y), \\ \dot{y} &= y(-\tfrac{8}{7} + 4x + y), \end{aligned} \tag{9.13}$$

with invariant cubic

$$F(x, y) = 1372xy(3x - y) - 1764xy - 63y - 72 = 0, \tag{9.14}$$

and

$$\begin{aligned} \dot{x} &= x(1 - 2x + y), \\ \dot{y} &= y(-\tfrac{13}{7} + 4x + y), \end{aligned} \tag{9.15}$$

with the invariant quartic

$$F(x, y) = 343x^2y(3x - y) - 588x^2y + 21xy + 18x - 9 = 0, \tag{9.16}$$

together with their duals.

Theorem 9.6. *The conjecture is proved for $\lambda = \frac{p}{q}$ with $p + q \leq 12$ and all $\lambda = \frac{n}{2}$ and $\lambda = \frac{2}{n}$ for $n \in \mathbb{N}$.*

Proof. The proof consists in calculating the Lyapunov quantities for the origin of (9.4) and finding their common roots. As many as three quantities are needed to give complete conditions. The conditions for $\lambda = 1/n$ and $2/n$ for $n \in \mathbb{N}$ were given in [32] and [34] respectively. In these cases we can prove by a counting argument that the list of conditions is necessary and sufficient: it is easy to prove that the first two saddle quantities cannot vanish elsewhere than the known sufficient conditions. Independent calculations of some of these cases have been done in [39]. □

Notes

It is not clear whether the cases proved to be a center by these monodromy arguments comprise new types of centers or whether they can be related to the other known mechanisms. A more detailed examination of the Lotka–Volterra system can be found in [25].

It would be also interesting to know if there can be non-trivial real centers given by monodromy arguments.

There are many applications of holonomy techniques to the whole area of integrability of polynomial vector fields. In particular, there is a nice dictionary which in generic cases matches Liouvillian solutions in a neighborhood of an invariant algebraic curve to the solvability of its monodromy group.

The general analytic classification of resonant saddles and their holonomy has been done by Martinet and Ramis [42], however we do not need to use this work here, as we are only interested in integrable saddles and these have only linear holonomy maps. The classification of saddles for irrational λ is much more complex and is not known in general.

Chapter 10

Other Approaches

In this final chapter I want to mention briefly three other approaches to the general center-focus problem. In the first, we try to identify whole components of the center variety by finding their intersections with specific subsets of parameter space and then showing that the type of center is "rigid". In the second approach, we try to see the consequences of a center on its bounding graphic. Monodromy-type arguments play an implicit role in both of these approaches. The last section describes an experimental approach to the center-focus via intensive computations using modular arithmetic and an application of the Weil conjectures. It makes a fitting conclusion to our range of monodromy techniques, since the arithmetic analog of monodromy was an essential ingredient in Deligne's proof of the Weil conjectures [27].

10.1 Finding components of the center variety

In order to be able to use results from algebraic geometry, we shall consider the center-focus problem for systems (1.2) with complex coefficients. We shall also compactify the space of parameters to $\mathbb{CP}(N)$ for some N. This can be done, for example, by taking the space of coefficients of P and Q in (1.2) modulo the action of $\mathbb{C}^*$ obtained by multiplying all coefficients by a constant (which of course does not affect the existence of a center).

Now, the closure of the center variety therefore becomes an algebraic subset of $\mathbb{CP}(n)$, which we denote Σ. Suppose Σ_1 is an irreducible component of Σ of dimension r, and let H be a subspace of $\mathbb{CP}(n)$ of dimension s; then if $r + s > n$ the two spaces must intersect in a non-empty space of dimension $r + s - n$.

In particular, if we restrict our attention to H, we can see a trace of all components of the center variety whose dimension is $n - s$ or greater. However, it might be possible that more than one component intersects H in the same subset, and so we cannot distinguish between the different components of Σ by just looking at H.

In order to be able to improve this result we need to be able to show that certain types of centers are "rigid". That is, if we know that at a certain parameter value we have this type of center, then all centers lying close to this value will also have a center of this type.

The most general result in this direction comes from the work of Movasati [45]. In his paper, he shows that for a polynomial system (1.2) of degree d, the subset $\Sigma(d_1, \ldots, d_r)$ of the center variety which is composed of Darboux centers given by the curves $f_i = 0$ of degree d_i, $i = 1, \ldots, r$ with $d + 1 = \sum d_i$ and their limits is a full component of the center variety.

Thus any point in H which has a center which is a generic point of $\Sigma(d_1, \ldots, d_r)$ cannot also be in the intersection of H with any other component of the center variety. Unfortunately, in [45], no method is given of determining whether a point is generic or not. This would be an interesting topic to understand much better.

The proof is based on a reduction to exactly the case in Chapter 7 where the Hamiltonian has $d+1$ invariant lines in general position. The author then examines the tangent space of the center variety at that point using similar arguments to the ones given there, but including higher degree perturbations. The conclusion is that the tangent cone of Σ at this point is exactly spanned by the spaces $\Sigma(d_1, \ldots, d_r)$ as the d_i run over all subsets of integers summing to $d+1$. This is enough to show that the $\Sigma(d_1, \ldots, d_r)$ are complete components of Σ.

We note that a similar idea has been suggested in [10] for Abel systems (6.1). Here the authors consider the limit of families of centers of the form (6.3) as ϵ tends to infinity. Alternatively, after a rescaling, we could consider the systems

$$\frac{dy}{dx} = \epsilon p(x) y^2 + q(x) y^3, \qquad 0 \le x \le 1, \tag{10.1}$$

as ϵ tends to zero. The situation is slightly more subtle than the one we have described above, as the limiting case always has a center. Thus the authors take the "tangential" part of the center conditions in order to define a "center at infinity". They then show that these centers must be solutions to the moment problem (6.4), which give centers of (6.3) for all ϵ. Every component of the variety of centers at infinity of dimension n must therefore extend to a component of the center variety of (6.3) of dimension $n + 1$. Some more work is needed to show that this is the unique component which intersects in this way with the centers at infinity. Conversely, every component of the center variety of dimension $n + 1$ must correspond to a component of the variety of centers at infinity of dimension n. Thus, the only centers which do not satisfy (6.4) exist only for discrete parameter values.

10.2 Extending Centers

A second approach is to analyze what happens at the boundary of a period annulus. Our hope is that the effect of a separatrix cycle having an identity return map has strong global consequences for the system. We could even hope that the local first integral of the center could be extended, in some ramified way, past these boundaries to obtain further global consequences of a center.

In general this seems a very difficult problem, but in one case this problem is quite easy. This is when the boundary of the center is a homoclinic loop attached to a saddle.

The nature of the return map in the neighborhood of a saddle in this case is well known [37, 54, 55]. In particular, the asymptotics are governed by two sets of interleaving terms. One set comes from the loop and the others are essentially governed by the Lyapunov quantities of the saddle. In order for the return map for a homoclinic loop to be the identity (or even analytic), we need to have all the Lyapunov quantities vanish, and therefore we can conclude that the saddle is integrable.

Thus in this case, the local integrability of the center has a global effect on a neighboring critical point. Furthermore, since the saddle is integrable, the local first integral of the center can be extended beyond the boundary of the homoclinic loop and hopefully could give further information about the system.

Example 10.1. We give an application of this idea to the problem of the center for the Abel equations (6.1). We assume that $q(x)$ does not vanish at $x = a$ or $x = b$, and has at most one root in (a, b) with $q' < 0$ at this root, and show that the system must then satisfy the composition condition of Chapter 6. We take $a = 0$ and $b = 1$ as before.

In order to prove this result, we note that the transformation $z = 1/y$ brings (6.1) to the form

$$\dot{x} = z, \qquad \dot{z} = -q - zp. \tag{10.2}$$

If $y(x, c)$ is the solution of (6.1), the solution of (10.2) is given by $z(x, c) = 1/y(x, 1/c)$ with $z(0, c) = c$ for c sufficiently large. Thus, if there is a center for (6.1), then there is a band of trajectories of (10.2) which passes between $x = 0$ and $x = 1$ with $z(1, c) = z(0, c)$ for c sufficiently large. We shall first show that the boundary of this band of trajectories must consist of a non-degenerate saddle and two of its separatrices.

Consider the path of the trajectory $z(x, c)$ as c decreases. Elementary considerations show that such a trajectory which crosses the line $z = 0$ in $(0, 1)$ cannot pass from $x = 0$ to $x = 1$. On the other hand, since all the critical points of the system lie on the line $z = 0$, we can certainly decrease c until $z(x, c)$ impinges on $z = 0$ at some point. Clearly there are only two possibilities: either we can decrease c to zero, with $z(x, 0) \geq 0$ in $(0, 1)$ and $z(0, 0) = z(1, 0) = 0$ or the trajectory $z(x, c)$ tends to a critical point on $z = 0$ as c tends to $c_0 > 0$, which must be a saddle by the conditions on $q'(x)$. In the former case, since the direction of the trajectories across $z = 0$ is given by $-q$, we would have a root with non-negative derivative between 0 and 1, contradicting the hypothesis. In the latter case, the boundary of the band of solutions $z(x, c_0)$ must comprise the saddle and two of its trajectories as stated above. Let $(p, 0)$ denote the position of this saddle.

Now suppose for some value k in $(0,1)$ we have $z(k,c) = c$ for all c sufficiently large. Clearly this cannot be at $k = p$ since the boundary must touch the z axis at $x = p$, but crosses the line $x = 0$ at $c_0 > 0$. Thus both of the regions $[0,k]$ and $[k,1]$ satisfy the hypothesis, but they cannot both contain saddles. Thus for any $k \in (0,1)$ the system cannot have a center between 0 and k or k and 1.

We now consider the return map near the trajectory $z(x, c_0)$. As the return map is analytic, the saddle can contribute no non-analytic terms, and hence it is integrable. We can now adapt the arguments of Chapter 4 to show that P and Q must be polynomials of a polynomial $A = (x-p)^2 + O((x-p)^3)$. Once again from the conditions on q, A' has only one root at $x = p$. Thus the transformation $u = \sqrt{A}\,\mathrm{sgn}(x-p)$ is well defined and analytic in $[0,1]$, and P and Q are now polynomials of u^2.

Let $P = f(u^2)$ and $Q = g(u^2)$; then the transformation brings (6.1) to the form

$$\frac{dy}{du} = 2uf'(u^2)y^2 + 2ug'(u^2)y^3. \tag{10.3}$$

Clearly this equation is symmetric with respect to the transformation $u \to -u$. Therefore, there is a center between any two points $u = -\ell$ and $u = \ell$. Under this transformation the points $x = 0$ and $x = 1$ correspond to the values u_0 and u_1 with $u_0 < 0 < u_1$. If $-u_0 < u_1$, then it is clear that $u(k) = -u_0$ for some k in $(p,1)$. But then we have a center of (6.1) with $a = k$ corresponding to the center of (10.3) between $-u_0$ and $u(k)$, which cannot happen; thus $-u_0 \geq u_1$. In a similar manner $-u_0 \leq u_1$ and so $-u_0 = u_1$. Hence, $A(0) = u_0^2 = u_1^2 = A(1)$ and we have proved the existence of a symmetry (the composition condition of Chapter 6) in this case. □

When the center bounds on a separatrix cycle with two saddles, then it can be shown that the two saddles have equivalent holonomies. That is, there is a map taking the neighborhood of one critical point into the other. If the holonomies are both linearizable, then we can find a local Darboux first integral. If they are not linearizable, then the maps taking one saddle to the other are more restricted and we might hope that they could be extended to a symmetry of the separatrix cycle.

Conjecture 10.1. Suppose a center is bounded by a separatrix cycle consisting of two curves $\ell_1 = 0$ and $\ell_2 = 0$ which intersect in two saddles p_1 and p_2. Then there exists a neighborhood U of the separatrix cycle with either a local Darboux first integral $\ell_1^\lambda \ell_2$, or a symmetry of the system $f : U \to U$ swapping p_1 and p_2.

The conjecture is the simplest case of a "center-focus problem" for period annuli. If true, it would be a satisfying first step in trying to see how to extract the right sort of global information from the existence of a center.

10.3 An Experimental Approach

The final approach we want to discuss is based around some recent work of H.C. Graf v. Bothmer [33]. It allows one to obtain information about the dimensions of specific components of the center variety which are inaccessible with more conventional computer algebra.

We calculate the Lyapunov quantities as before, but now modulo a prime number p. It turns out [33] that an algorithm for computing the Lyapunov quantities can be found which works modulo p up to $L((p-3)/2)$ (after which the denominators may vanish).

Let Σ be the center variety as before (considered as an affine space again), and let $\Sigma(\mathbb{Z}_p)$ denote the points in Σ with coefficients in $\mathbb{Z}_p$. Let n_p denote the fraction of $\mathbb{Z}_p$ points in Σ compared with $\mathbb{Z}_p^N$, where N is the number of parameters. That is,

$$n_p = |\Sigma(\mathbb{Z}_p)|/p^N.$$

One consequence of the Weil conjectures [3, 35] is that

$$n_p = r\Big(\frac{1}{p}\Big)^c + O\Big(\Big(\frac{1}{p}\Big)^{c+1}\Big), \tag{10.4}$$

where c is the lowest codimension amongst the components of Σ and r is the number of components of that codimension.

This number can be estimated by computing a number of random points in $\mathbb{Z}_p^N$ and calculating the empirical fraction of these which lie in $\Sigma(\mathbb{Z}_p)$. We denote this number by $\tilde{n}_p$. This quantity $\tilde{n}_p$ allows us to make estimates of r and c via (10.4).

However, we would like also to be able to say something about the components of the center variety with higher codimensions. This can be done as follows. Suppose we want to calculate the number of components r' of Σ which have codimension c'. We need to exclude all points on components of Σ which have codimension less than c'. However this can be done in a nice way: given a point on $\Sigma(\mathbb{Z}_p)$ we can calculate the tangent space to $\Sigma(\mathbb{Z}_p)$ at that point. If the codimension of this tangent space is less than c' we reject this point.

The reason this works is that the codimension of the tangent space at a point on a component of Σ is always less than or equal to the codimension of the component itself. For a generic point however the codimension will be the same. Thus if we reject all points whose tangent spaces have codimension less than c', we are rejecting precisely all points on components with codimension less than c' and some non-generic points of the other components.

We calculate the fraction of points satisfying the codimension criteria above and estimate r' and c' from (10.4) as before.

This method has been shown to give accurate results for quadratic systems (where the complete classification is known) and also for cubic systems up to codimension 7, tying in with known irreducible components of the center variety [45, 62].

Higher codimension cases will require much more computational power, but the rate of growth of the complexity is much more favorable than using more standard symbolic routines.

Notes

It is clear that the general solution to the center-focus problem seems very far away, in spite of a growing number of techniques and known cases.

A more realistic goal over the next few years would be to classify all centers in cubic systems. All three of the approaches suggested above could be used in this task, but some new ideas will probably be needed to bridge the still wide computational gap between what has been achieved to date, and the full complexity of cubic systems.

Perhaps this is a case where a coordinated effort by the many researchers who have an interest in this area could yield some very tangible results in the not-too-distant future.

Part II

Abelian Integrals and Applications to the Weak Hilbert's 16th Problem

Chengzhi Li

Dedicated to Professor Zhifen Zhang on the occasion of her 80th birthday

Preface

The second part of Hilbert's 16th problem, asking for the maximum $H(n)$ of the numbers of limit cycles and their relative positions for all planar polynomial differential systems of degree n, is still open even for the quadratic case ($n = 2$).

A weak form of this problem, proposed by Arnold, asking for the maximum $Z(m, n)$ of the numbers of isolated zeros of Abelian integrals of all polynomial 1-forms of degree n over algebraic ovals of degree m, is also extremely hard to grasp. The number $\tilde{Z}(n) = Z(n + 1, n)$ can be chosen as a lower bound of $H(n)$; so far only $\tilde{Z}(2) = 2$ has been proved.

These lecture notes are devoted to the introduction of some basic concepts and methods in the study of Abelian integrals and applications to the weak Hilbert's 16th problem. In Chapter 1 we briefly introduce Hilbert's 16th problem and its weak form. In Chapter 2 we explain the relation between the study of Abelian integrals and the study of limit cycles. In Chapter 3 we use several methods to study the number of zeros of the Abelian integrals associated with perturbations of the Bogdanov–Takens system. At last, in Chapter 4 we introduce a proof of $\tilde{Z}(2) = 2$, the method of the proof is unified for all regions of the parameter space.

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Chapter 1

Hilbert's 16th Problem and Its Weak Form

1.1 Hilbert's 16th Problem

Consider the planar differential systems

$$\dot{x} = P_n(x, y), \quad \dot{y} = Q_n(x, y), \tag{1.1}$$

where P_n and Q_n are real polynomials in x, y and the maximum degree of P and Q is n. The second half of the famous Hilbert's 16th problem, proposed in 1900, can be stated as follows (see [70]):

For a given integer n, what is the maximum number of limit cycles of system (1.1) *for all possible P_n and Q_n ? And how about the possible relative positions of the limit cycles ?*

Usually, the maximum of the number of limit cycles is denoted by $H(n)$, and is called the Hilbert number. Recall that a *limit cycle* of system (1.1) is an isolated closed orbit. It is the ω- (forward) or α- (backward) limit set of nearby orbits. In many applications the number and positions of limit cycles are important to understand the dynamical behavior of the system. Note that the problem is trivial for $n = 1$: a linear system may have periodic orbits but have no limit cycle, so we assume $n \geq 2$.

This problem is still open even for the case $n = 2$, and there is no answer if $H(2)$ is finite or not. In [158] S. Smale said: "Except for the Riemann hypothesis it seems to be the most elusive of Hilbert's problems" (see also [157]).

Below we list some results, among a lot of works on this problem.

1.1.1 The finiteness problem

- In 1923 H. Dulac [36] claimed the *individual finiteness* of limit cycles, i.e., for a *given* system (1.1) the number of limit cycles is finite. A gap in his arguments was found in the early 1980s.
- In 1985 R. Bamon [6] proved this individual finiteness property for the quadratic case ($n = 2$).
- In the early 1990s Yu. Ilyasenko and J. Ecale published, independently in two long papers [86] and [50], new proofs of the individual finiteness theorem, filling up the gap in Dulac's paper. This "is the most spectacular and the most general fact established so far in connection with the Hilbert 16th problem" (see S. Yakovenko [169]); and "these two papers have yet to be thoroughly digested by the mathematical community" (see S. Smale [158]). Naturally, the next step is to prove the *uniform* finiteness , i.e., $H(n) < \infty$.
- In 1988 R. Roussarie [141] proposed a program to prove the uniform finiteness by reducing this problem, via the compactification of the systems and of the parameter space, to the problem of proving the finite cyclicity of limit periodic sets (see also J.-P. Françoise & C.C. Pugh [53]). F. Dumortier, R. Roussarie & C. Rousseau in [47, 48] started this program for the quadratic case and listed 121 graphics as all limit periodic sets which are necessary in this study. A series of papers, among them [37, 38, 39, 144, 145, 190], continue this program, and about 85 of the 121 graphics have been studied. The remaining graphics are more degenerate and the study of them is more difficult.

For a detailed introduction to the finiteness problem we refer to a recent article by D. Schlomiuk (the first chapter of [153]).

1.1.2 Configuration of limit cycles

There are many papers dealing with the related positions of limit cycles for system (1.1). A general result was obtained by J. Llibre and G. Rodríguez [116] in 2004. Let us briefly introduce their result.

A *configuration of limit cycles* is a finite set $C = \{C_1, \ldots, C_m\}$ of disjoint simple closed curves of the plane such that $C_i \cap C_j = \emptyset$ for all $i \neq j$.

Given a configuration of limit cycles $C = \{C_1, \ldots, C_m\}$ the curve C_i is *primary* if there is no curve C_j of C contained in the bounded region limited by C_i. Two configurations of limit cycles C and C' are (topologically) *equivalent* if there is a homeomorphism in $\mathbb{R}^2$, mapping C to C'.

A system (1.1) *realizes* the configuration of limit cycles C if the set of all its limit cycles is equivalent to C.

Theorem 1.1 ([116]). *Let C be a configuration of limit cycles, and let r be its number of primary curves. Then $C = \{C_1, \ldots, C_m\}$ is realizable as algebraic limit cycles by a polynomial system (1.1) of degree $n \leq 2(m+r)-1$.*

This theorem can be seen as a partial answer to the position question of the second part of Hilbert's 16th problem. The remaining question is : For a *fixed* integer n what kinds of configurations of limit cycles of systems (1.1) are possible ? In the next subsection there is some information about this question in the quadratic case.

1.1.3 Some results on quadratic systems

- In 1952 H.H. Bautin [7] proved a fundamental fact for quadratic systems: at most three limit cycles can bifurcate from a weak focus or center of system (1.1) for $n = 2$. A *weak focus* means a focus at which the linear part of the system has a center. See [51] for a computation of the focal values in Bautin's formula.

- In 1955 I.G. Petrovskii & E.M. Landis [135] attempted to prove that "$H(2) = 3$".

- In 1959 C. Tung [161] found some important properties of quadratic systems: a closed orbit is convex; there is a unique singularity in the interior of it; two closed orbits are similarly (resp. oppositely) oriented if their interiors have (resp. do not have) common points. Hence, the distribution of limit cycles of quadratic systems have only one or two nests.

- In the 1960s Y. Ye classified the quadratic systems into 3 classes: any quadratic system with limit cycle(s) can be transformed into the form $\dot{x} = -y + \delta x + lx^2 + mxy + ny^2, \dot{y} = x(1 + ax + by)$. It belongs to class I if $a = b = 0$, to class II if $b = 0, a \neq 0$ and to class III if $b \neq 0$. It was proved in [18, 170] that at most one limit cycle exists for systems in class I, see also [171].

- In 1979 S. Shi [155], L. Chen & M. Wang [17] found counter-examples to the result of Petrovskii–Landis. In both examples the four limit cycles are located in two nests, with at least three in one nest and at least one in another (called (3,1)-distribution of limit cycles). Nowadays most mathematicians in this field believe that $H(2) = 4$. If this were true, then how about the (4,0)- and (2,2)- distribution of limit cycles ?

- In 1986 C. Li [94] proved that there is no limit cycle surrounding a weak focus of third order for any quadratic system, which gives no possibility to construct (4,0)-distribution of limit cycles by perturbing a quadratic system with a weak focus of third order, because there is no limit cycle surrounding the focus before perturbation.

- Around 2000, in a series of papers (see [176, 177]), P. Zhang proved that there is at most one limit cycle surrounding a weak focus of second order for any quadratic system, and if a quadratic system has two nests of limit cycles, then at least one nest contains a unique limit cycle, hence (2,2)-distribution of limit cycles for a quadratic system is impossible.
- There is a series of papers towards a systematic study of the global geometry of quadratic differential systems. Among them, R. Roussarie & D. Schlomiuk [143] and D. Schlomiuk [152] give a general framework of study of the class of all quadratic systems; D. Schlomiuk and N. Vulpe [154] studies the geometry of quadratic systems in the neighborhood of infinity; in the paper [152] D. Schlomiuk gives a short history of invariant theory and motivation for using invariants in the global theory; J. Llibre & D. Schlomiuk [117] determines the global geometry of quadratic systems with a weak focus of third order; J.C. Artés, J. Llibre & D. Schlomiuk [5] makes a global study of the closure of systems having a weak focus of second order within quadratic systems. The global geometry of this class reveals interesting bifurcation phenomena; for example, all phase portraits with limit cycles obtained in this class can be produced by perturbations of symmetric (reversible) quadratic systems with a center. The study of perturbations of centers is an important part of the study of the weak Hilbert's 16th problem, which is the main topic of this discussion.

1.1.4 Some results on cubic and higher degree systems

- In 1954 K.C. Sibirskii [156] proved that at most five limit cycles can appear by a Hopf bifurcation for cubic systems without quadratic terms. In 1987 J. Li & Q. Huang [104] constructed an example showing $H(3) \geq 11$. The 11 limit cycles form "compound eyes": a big limit cycle surrounds two smaller limit cycles, each of them surrounds two nests, with at least two limit cycles in each nest. In 2005 P. Yu & M. Han [174] gave an example for $H(3) \geq 12$, with (6,6)-distribution of limit cycles.
- In 1995 H. Żołądek [192] proved that surrounding a focus of a cubic system there may exist 11 limit cycles. Recently C. Christopher [27] confirmed this result, and established a quartic system with 17 limit cycles bifurcating from a non-degenerate center, and another quartic system with at least 22 limit cycles globally. Hence $H(4) \geq 22$.
- In 1954 N.F. Otrokov [128] proved that $H(n) \geq \frac{1}{2}(n^2 + 5n - 14)$ for $n \geq 6$ even, and $H(n) \geq \frac{1}{2}(n^2 + 5n - 26)$ for $n \geq 7$ odd. In his study, all the limit cycles are located in a small neighborhood of one singular point. In 1995 C. Christopher & N.G. Lloyd [28] proved that $H(n) \geq k\,n^2 \ln n$ for some constant k. In this result, the limit cycles surround many singular points. In 2003 J. Li improved this result in a survey paper [103], and proved that $H(n) \geq \frac{1}{4}(n+1)^2(1.442695 \ln(n+1) - \frac{1}{6}) + n - \frac{2}{3}$.

1.1.5 Some results on Liénard equations

- For the generalized Liénard equation $\ddot{x} + f(x)\,\dot{x} + g(x) = 0$ (or equivalently, the planar systems $\dot{x} = y - F(x)$, $\dot{y} = -g(x)$, where $F(x) = \int_0^x f(x)\,dx$), Z. Zhang [178, 179] proved a theorem in 1958 that if $f(x)/g(x)$ is monotone, then the limit cycle (if it exists) is unique. In particular, if $g(x) = x$ and F is a cubic polynomial, then $f(x)/x$ is monotone, so the corresponding Liénard equation has no more than one limit cycle. This theorem and different forms of its generalization were used widely, for example in [18, 170, 176, 177]. Note that if a quadratic system is transformed to a Liénard equation, the functions F and g, in general, are no longer polynomials.
- Concerning the number of limit cycles for a polynomial Liénard equation (i.e., F and g are polynomials), there is the so-called *Lins–De Melo–Pugh conjecture* in [107]: if $g(x) = x$ and deg$F = 2n + 1$ or $2n + 2$ ($n \geq 1$) then the maximal number of limit cycles is n. See also [157]. This conjecture was proved only for the case deg$F$=3 in the same paper [107]. Note that this result can be proved by the theorem of [178] as mentioned above. Recently a counterexample to this conjecture was found in [45] for the case deg$F$=7 with four limit cycles.
- Concerning the number of small amplitude limit cycles in Liénard systems there is a series of works by N.G. Lloyd, C. Christopher and S. Lynch, see for example [27, 28, 29, 118, 119], and by Y. Liu and J. Li, see [109, 110, 111]. Related to this topic, there are a lot of works by J. Llibre, A. Gasull and the research group in Barcelona, see for example [31, 56, 59, 61, 69, 114].

For more details about limit cycles and Hilbert's 16th problem, we refer to the survey papers [11, 23, 33, 87, 103, 113, 152], and the books [1, 24, 35, 71, 88, 90, 105, 120, 138, 142, 150, 171, 172, 180].

1.2 Weak Hilbert's 16th Problem

Now we turn to a weak version of the problem. Let $H = H(x, y)$ be a polynomial in x, y of degree $m \geq 2$, and the level curves $\gamma_h \subset \{(x, y) : H(x, y) = h\}$ form a continuous family of ovals $\{\gamma_h\}$ for $h_1 < h < h_2$. Consider a polynomial 1-form $\omega = f(x, y)dy - g(x, y)dx$, where max(deg($f$),deg($g$)) $= n \geq 2$. V.I. Arnold in [2, 3] proposed the following problem:

For fixed integers m and n find the maximum $Z(m, n)$ of the numbers of isolated zeros of the Abelian integrals

$$I(h) = \oint_{\gamma_h} \omega. \tag{1.2}$$

Recall that an *Abelian integral* is the integral of a rational 1-form along an algebraic oval. Note that in the above problem one must consider all possible H

with all possible families of ovals $\{\gamma_h\}$, and arbitrary f and g. So it does not matter if we put $-$ or $+$ before g in ω. Remark also that the function $I(h)$ may be multivalued since it is possible that several ovals lie on the same level curve $H^{-1}(h)$.

At a first look, this problem has no relation with Hilbert's 16th problem at all. We will explain in the next section how these two problems are related to each other. Roughly speaking, the function $I(h)$, given by the Abelian integral (1.2), is the first approximation in ε of the "displacement function" of the Poincaré map on a segment transversal to γ_h (at least locally) for the system

$$\dot{x} = -\frac{\partial H(x,y)}{\partial y} + \varepsilon f(x,y), \quad \dot{y} = \frac{\partial H(x,y)}{\partial x} + \varepsilon g(x,y), \tag{1.3}$$

where H, f and g are the same as above when defining the Abelian integral $I(h)$. Hence the number of isolated zeros of $I(h)$ (taking into account the multiplicities) gives an upper bound of the number of limit cycles of system (1.3) with small ε.

It is clear that if one takes $m = n + 1$, then system (1.3) is a special form of system (1.1), close to Hamiltonian for small ε. In this sense the second problem usually is called the *weak (or tangential, infinitesimal)* Hilbert's 16th problem, and the number $\tilde{Z}(n) = Z(n+1, n)$ can be chosen as a lower bound of the Hilbert number $H(n)$.

A. Varchenko and A. Khovanskii proved that for given m and n the number $Z(m,n)$ is uniformly bounded with respect to the choice of the polynomial H, the family of ovals $\{\gamma_h\}$ and the 1-form ω.

Theorem 1.2 ([92, 164]). *$Z(m,n) < \infty$.*

This result certainly is important. However, it is a purely existential statement, giving no information on the number $Z(m,n)$. To find an explicit expression for $Z(m,n)$ in general, even to find an explicit bound to $Z(m,n)$, is extremely hard. There are many works dealing with restricted versions of the problem (restriction on H or ω), some of them will be briefly introduced in these notes. It is natural to think about a possibility to find $\tilde{Z}(n) = Z(n+1, n)$ for lower n, and this was done by several authors over a period of about 10 years and only for $n = 2$. We will first introduce this result in the next subsection, then give more detailed information about a unified proof in Chapter 4.

1.2.1 The study of $\tilde{Z}(2) = 2$

We consider all cubic polynomials $H(x,y)$ with a continuous family of ovals $\{\gamma_h\}$ in $H^{-1}(h)$ for $h_c < h < h_s$, where h_c and h_s correspond to the critical values of the corresponding quadratic Hamiltonian system X_H (i.e., (1.3) for $\varepsilon = 0$) at a center and a saddle loop respectively (the discussion below and Figure 1 show that this is the case for generic quadratic Hamiltonian systems). The family of the ovals forms an *annulus*. We first give a definition of the *cyclicity of the annulus.*

Definition 1.3. For $0 < \varepsilon \ll 1$ let N_ε be the maximum number of limit cycles which bifurcate from the compact region $\cup_{h\in[h_c+\varepsilon,h_s-\varepsilon]}\gamma_h$ of X_H by quadratic perturbations. The *cyclicity of the period annulus* of X_H under quadratic perturbations is $\sup_{0<\varepsilon\ll 1} N_\varepsilon$.

Recall that the quadratic systems with at least one center are always integrable. They can be classified into the following five classes : Hamiltonian (Q_3^H), reversible (Q_3^R), generalized Lotka–Volterra (Q_3^{LV}), co-dimension 4 (Q_4) and the Hamiltonian triangle ([79] by using the terminology from [191], see also [151]).

Definition 1.4. ([79]) A quadratic integrable system is said to be *generic* if it belongs to one of the first four classes and does not belong to other integrable classes. Otherwise, it is called *degenerate*.

It was shown by I.D. Iliev in [79] that if $X_H \in Q_3^H$ is generic, then the number $\tilde{Z}(2)$ gives the cyclicity of the period annulus of X_H. E. Horozov and I.D. Iliev proved in [74] that any cubic Hamiltonian, with at least one period annulus contained in its level curves, can be transformed into the following normal form,

$$H(x,y) = \frac{1}{2}(x^2+y^2) - \frac{1}{3}x^3 + axy^2 + \frac{1}{3}by^3,$$

where a, b are parameters lying in the region

$$\bar{G} = \left\{(a,b) : -\frac{1}{2} \le a \le 1,\ 0 \le b \le (1-a)\sqrt{1+2a}\right\},$$

and moreover, their respective vector fields X_H are generic if $(a,b) \in G = \bar{G} \setminus \partial\bar{G}$ and degenerate if $X_H \in \partial\bar{G}$. Figure 1 shows all possible phase portraits of X_H for

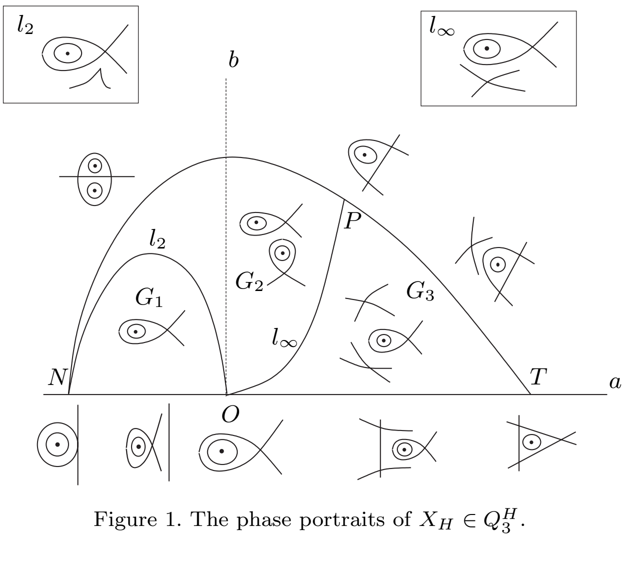

Figure 1. The phase portraits of $X_H \in Q_3^H$.

different ranges of a and b, where G is divided into three regions G_1, G_2 and G_3 by two curves l_2 and l_∞ (l_2 and l_∞ also belong to G). Along l_2 two singularities of X_H coincide, and when (a,b) tends to l_∞ a finite singularity of X_H coalesces with an infinite singularity. Hence, besides the two critical situations along l_2 and l_∞, X_H has one, two or three saddle points if $(a,b) \in G_1, G_2$ or G_3 respectively. X_H has two period annuli if $(a,b) \in G_2$ and the arc PN on $\partial\bar{G}$, and one period annulus in the other cases.

Theorem 1.5. $\tilde{Z}(2) = 2$.

E. Horozov and I.D. Iliev [74] proved that the least upper bound of the number of zeroes of related Abelian integrals is 2 for $(a,b) \in G_3$, then L. Gavrilov [64] obtained the same conclusion for $(a,b) \in G_1 \cup G_2$ (the method is also valid for $(a,b) \in G_3$). Since a basic assumption in [74] and [64] is that $H(x,y)$ has four distinct critical values (in the complex plane), the cases $(a,b) \in l_2 \cup l_\infty$ must be considered separately. Papers [125] and [182] independently gave different proofs for $(a,b) \in l_\infty$, and [102] proved the same conclusion for the last case $(a,b) \in l_2$. A unified proof appears recently in [15].

Remark 1.6. If X_H is degenerate (i.e., $(a,b) \in \partial\bar{G}$), then it is not difficult to show that $I(h)$ has at most one zero. But this gives no information about the cyclicity of the period annulus, higher approximations must be considered. Iliev in [79] gives formulas (called second- or third- order Melnikov function, which will be discussed in Chapter 2) to determine the cyclicity for all degenerate cases. The cyclicity of the period annulus (or annuli) is 3 for the Hamiltonian triangle case ([78]), and is 2 for all other seven cases (see [65], [77], [188], [189], [25] and a recent paper [97]).

Remark 1.7. X_H has two period annuli when $(a,b) \in G_2$. To prove $\tilde{Z}(2)$ still is 2 in this case, implying that only (1,1)-distribution of limit cycles is possible if there are two nests of limit cycles after perturbation, [64] uses a result in [175] while [15] gives a direct proof. There is a similar study in [25] for the period annuli bounded by the elliptic-segment loops.

Remark 1.8. By a result of R. Roussarie [140] and P. Mardesic [121] the conclusion of Theorem 1.5 can be extended to the the case $h \in [h_c, h_s]$ if $(a,b) \in G_1 \cup G_2 \cup G_3$, i.e., the period annulus terminates at a homoclinic loop of a hyperbolic saddle. This means that the perturbed system has at most two limit cycles, including that bifurcating from the saddle loop. A similar conclusion holds if (a,b) belongs to the open segment NT on $\partial\bar{G}$. But the problem of the number of limit cycles bifurcating from heteroclinic loop(s) or from infinity is still open; only some partial results appear in [66, 99].

1.2.2 Perturbations of elliptic and hyperelliptic Hamiltonians

Now we restrict the function H to the following form:

$$H(x,y) = \frac{y^2}{2} + P_m(x), \tag{1.4}$$

where P_m is a polynomial in x of degree m. The level curves of H are rational for $m = 1, 2$, elliptic for $m = 3, 4$ and hyperelliptic for $m \geq 5$. We assume $m \geq 2$ since the level curves have no oval if $m = 1$.

We first give a general lemma.

Lemma 1.9. *Suppose that for the function H defined in* (1.4) *there is a family of ovals $\gamma_h \subset H^{-1}(h)$, and ω is an arbitrary polynomial 1-form of degree n; then*

$$\oint_{\gamma_h} \omega = \begin{cases} \oint_{\gamma_h} p_1(x)\, y\, dx, & n = 2, \\ \oint_{\gamma_h} p_k(x, h)\, y\, dx, & n \geq 3, \end{cases}$$

where p_1 is a linear function in x, and $p_k(x, h)$ is a polynomial in x and h of degree $k = \frac{m(n-1)}{2}$ if n is odd and $k = \frac{m(n-2)}{2} + 1$ if n is even.

Proof. For any integers $i, j \geq 0$ it is easy to see that

$$\oint_{\gamma_h} x^i\, y^j\, dy = \begin{cases} 0, & i = 0, \\ -\frac{i}{j+1} \oint_{\gamma_h} x^{i-1}\, y^{j+1}\, dx, & i \geq 1. \end{cases}$$

Hence, without loss of generality we only consider $\omega = f(x, y)dx$, where f is a polynomial in x and y of degree n. On the other hand, we have

$$\oint_{\gamma_h} x^i\, y^j\, dx = \begin{cases} 0, & j = 2l, \\ \oint_{\gamma_h} x^i\, [2(h - P_m(x))]^l y\, dx, & j = 2l + 1. \end{cases}$$

The statements of the lemma immediately follow. $\square$

(i) *The case $m = 2$.*

In this case we may choose $H = \frac{x^2+y^2}{2}$ (to put the center of X_H at the origin). The ovals are circles $\{x^2 + y^2 = h^2\}$. Suppose that the 1-form ω is of degree n, then by using the polar coordinates one finds that

$$\oint_{\gamma_{h^2}} \omega = h^2 Q_{n-1}(h),$$

where $Q_{n-1}(h)$ is a polynomial in h of degree $(n - 1)$, but depends only on h^2 by symmetry. $I(h)$ has at most $[(n - 1)/2]$ zeros except the trivial zero at $h = 0$, which corresponds to the singularity at the origin.

(ii) *The elliptic Hamiltonian of degree* 3.

In this case if we suppose that the level curves of H contains a continuous family of ovals, then the two singularities of the corresponding vector field X_H must be a center and a saddle, which is chosen (without loss of generality) at $(-1, 0)$ and $(1, 0)$ respectively, and the elliptic Hamiltonian reads as

$$H(x, y) = \frac{y^2}{2} - \frac{x^3}{3} + x. \tag{1.5}$$

In this case the continuous family of ovals is given by

$$\{\gamma_h\} = \{(x,y) : \ H(x,y) = h, \ -2/3 \le h \le 2/3\}, \tag{1.6}$$

see Figure 2. By Lemma 1.9 the Abelian integral $I(h)$ can be expressed in the form

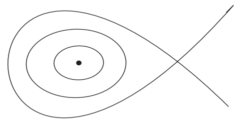

Figure 2. The family of ovals in case $m = 3$.

$\oint_{\gamma_h} p_k(x,h)ydx$, where p_k is a polynomial in x and h. An important observation is that along γ_h,

$$0 \equiv dH = H_x dx + H_y dy = (1-x^2)dx + ydy,$$

which implies $(1-x^2)ydx + y^2dy \equiv 0$, hence $I_2(h) \equiv I_0(h)$, where we define $I_j(h) = \oint_{\gamma_h} x^j ydx$. Similarly, we have

$$\oint_{\gamma_h} x^k(x^2-1)ydx = \oint_{\gamma_h} x^k y^2 dy = \oint_{\gamma_h} x^k(2h + 2x^3/3 - 2x)dy.$$

Using integration by parts on the right-hand side we find the following induction formula,

$$(2k+9)I_{k+2}(h) - 3(2k+3)I_k(h) + 6khI_{k-1}(h) = 0,$$

where $k \ge 1$. Hence, it is not hard to prove the following result.

Lemma 1.10 ([130]). *Suppose that $I(h)$ is the Abelian integral of the polynomial 1-form ω of degree at most n over the ovals γ_h defined in* (1.6), *then*

$$I(h) = Q_0(h)I_0(h) + Q_1(h)I_1(h),$$

where Q_0 and Q_1 are polynomials, $\deg Q_0 \le [\frac{n-1}{2}]$, $\deg Q_1 \le [\frac{n}{2}] - 1$, *and as usual $[\xi]$ means the integer part of ξ.*

If we denote $[\frac{n-1}{2}] = n_0$ and $[\frac{n}{2}] - 1 = n_1$, then $n_0 + n_1 = n - 2$, and any $I(h)$, defined in Lemma 1.10, can be expressed as a linear combination of the n independent functions

$$I_0(h), hI_0(h), h^2I_0(h), \ldots, h^{n_0}I_0(h); \ I_1(h), hI_1(h), h^2I_1(h), \ldots, h^{n_1}I_1(h).$$

Hence, it is possible to find a special $I(h)$, having $n-1$ zeros for $h \in (-\frac{2}{3}, \frac{2}{3})$. On the other hand, one may expect that all such $I(h)$ have at most $n-1$ zeros, counting their multiplicities. This result was proved by Petrov in [132]. Before stating his result we recall a definition.

Definition 1.11. ([123]) A $(k+1)$-tuple of smooth functions $(J_0, \dots, J_k)$ defined on some interval (h_0, h_1), is a *Chebychev system*, if for any $\ell \le k$, a nontrivial linear combination of the $\ell + 1$ functions $(J_0, \dots, J_\ell)$ has at most ℓ zeros in (h_0, h_1) counting their multiplicities.

The simplest example is the set of monomials $(1, x, x^2, \dots, x^k)$, which is a Chebychev system on any interval.

Theorem 1.12 ([131, 132])**.** *The space of functions $\{I(h)\}$, defined in Lemma* 1.10*, has the Chebyshev property on $h \in (-2/3, 2/3)$. This means that any nontrivial $I(h)$ has at most $n-1$ zeros, and there exists a* 1*-form ω, such that $I(h)$ has exactly $n-1$ zeros.*

In fact, Petrov made an analytic extension of $I(h)$ from $(-2/3, 2/3)$ to a domain D in the complex plane, and proved by using the Argument Principle that the space of extended functions has the Chebyshev property in D. We will introduce this proof in Chapter 3.

Remark 1.13. One motivation for studying the perturbations of the elliptic Hamiltonians comes from the so-called Bogdanov–Takens bifurcation, see [8, 160]. If a C^∞ planar system has a nilpotent linear part, a truncated normal form up to degree 2 looks like

$$\dot{x} = y, \quad \dot{y} = ax^2 + bxy. \tag{1.7}$$

If $ab \neq 0$, then the problem has codimension 2, and by a scaling (a, b) can be changed to $(1, \pm 1)$. A universal unfolding (in C^∞ function class) could be (see, for example, [24])

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \mu_1 + \mu_2 y + x^2 + xyF(x, \mu) + y^2 G(x, y, \mu), \end{aligned} \tag{1.8}$$

where $\mu = (\mu_1, \mu_2)$ are small parameters, $F, G \in C^\infty$, and $F(0,0) = \pm 1 = \mathrm{sgn}(ab)$. There is no bifurcation for $\mu_1 > 0$ and the saddle-node bifurcation happens for $\mu_1 = 0$. The most interesting phenomenon appears for $\mu_1 < 0$, in this case by a change of coordinates and parameters

$$\mu_1 = -\varepsilon^4,\ \mu_2 = \alpha\varepsilon^2,\ x = \varepsilon^2 \bar{x},\ y = \varepsilon^3 \bar{y},\ t = \bar{t}/\varepsilon,$$

where $\varepsilon > 0$ small, system (1.8) (changing $(\bar{x}, \bar{y}, \bar{t})$ back to (x, y, t)) becomes

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -1 + x^2 + \varepsilon(\alpha \pm x)y + O(\varepsilon^2), \end{aligned} \tag{1.9}$$

which is exactly the perturbation of X_H with H in the form (1.5), and the corresponding Abelian integral is

$$I(h) = \alpha I_0(h) \pm I_1(h). \tag{1.10}$$

As a typical example, we will introduce several methods to study the number of zeros of this Abelian integral in Chapter 3.

If $b = 0$ in (1.7), then the problem has higher codimension; the study of the bifurcations for codimensions 3 and 4 (not only the study of zeros of the corresponding Abelian integrals, but also the number of limit cycles and the bifurcation diagrams) was given in [49] and [100], respectively.

(iii) *The elliptic Hamiltonian of degree* 4.

In this case we may take the function H in the form

$$H(x, y) = \frac{y^2}{2} + a\frac{x^4}{4} + b\frac{x^3}{3} + c\frac{x^2}{2}, \tag{1.11}$$

where $a \neq 0$. There are five types of continuous families of ovals on the level curves of H, shown in Figure 3 depending on the values of the parameters (a, b, c), called the truncated pendulum case, the saddle loop case, the global center case, the cuspidal loop case, and the figure-eight loop (Duffing oscillator) case, respectively.

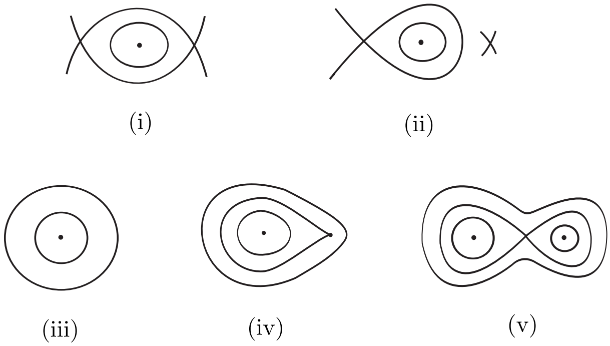

Figure 3. The families of ovals for the case $m = 4$.

The first two cases correspond to $a < 0$ while the last three casescorrespond to $a > 0$. Note that in the figure-eight loop case the corresponding Abelian integral is a multi-valued function, since an oval in the left annulus and an oval in the right annulus (surrounded by the figure-eight loop) may correspond to a same value of h.

In [133] and [134] G.S. Petrov considered the figure-eight loop case ($a > 0$, $b < 0$ and $H(x, 0)$ has only three real different critical values), and the case $(a, b, c) = (1, 1, 1)$, a special case of the global center, respectively. In the first case he obtained the result concerning the two annuli surrounded by the figure-eight loop. Recently C. Liu in [108] studied the region outside the figure-eight loop, and considered the total number of zeros for the ovals in the two annuli surrounded by the figure-eight loop. We state their results in the following theorems.

Theorem 1.14 ([133]). *Let H be as in* (1.11) *with the figure-eight loop. Then the space of the elliptic integral $I(h)$ of a* 1*-form of degree n over cycles vanishing at one of the two singularities of X_H surrounded by the figure-eight loop has the Chebyshev property on the corresponding interval of h. This means that the number of zeros of nontrivial $I(h)$ is less than the dimension of the space. This dimension is $n + [(n-1)/2]$.*

The conclusion of [134] for $(a, b, c) = (1, 1, 1)$ (a global center case) is similar, the dimension of the space of Abelian integrals in this case is $2[(n-1)/2] + 1$.

Theorem 1.15 ([108]). *Let H be as in* (1.11) *with the figure-eight loop and ω be a polynomial* 1*-form of degree n. Then the following statements hold.*

(A) *The total number of zeros of $I(h)$ (taking into account their multiplicity) for the ovals in the two annuli surrounded by the figure-eight loop does not exceed $2n - 1$ for n even, or $2n + 1$ for n odd.*

(B) *The number of zeros (taking into account their multiplicity) of $I(h)$ for the ovals outside the figure-eight loop does not exceed $2n+1$ for n even, or $2n+3$ for n odd.*

There is a series of papers dealing with the exact number of zeros of the Abelian integrals over all types of ovals in Figure 3, but which only consider 1-forms of degree 3 as follows.

$$\omega = (\alpha + \beta x + \gamma x^2)\, y\, dx. \tag{1.12}$$

Comparing with the general 1-form of degree 3, one term, $y^3 dx$, is omitted. Let us first explain why the 1-form (1.12) is interesting. Consider a cubic Liénard equation with a small quadratic damping:

$$\ddot{x} + \varepsilon\, p_2(x)\, \dot{x} + p_3(x) = 0,$$

where ε is a small parameter and p_k is a polynomial in x of degree k. This equation is equivalent to the planar system

$$\dot{x} = y, \quad \dot{y} = -p_3(x) - \varepsilon p_2(x)\, y. \tag{1.13}$$

The study of the number of limit cycles of system (1.13) naturally leads to the study of the Abelian integral of 1-form (1.12) over the ovals of (1.11).

Theorem 1.16. *Let $I(h)$ be an Abelian integral of the polynomial* 1*-form* (1.12) *over the ovals contained in the level curves of the elliptic Hamiltonian of degree* 4 (1.11)*. Then the maximal number of zeros of $I(h)$ (taking into account their multiplicity) is*

(A) 1 *in the truncated pendulum case* ([73])*;*

(B) 2 *in the saddle loop case* ([40])*;*

(C) 4 *in the global center case, and the four zeros of the elliptic integral can be simple or multiple exhibiting a complete unfolding of a zero of multiplicity four* ([41])*;*

(D) 4 *in the cuspidal loop case, moreover, if restricting to the level curves "inside" and "outside" the cuspidal loop, we found the sharp upper bound to be, respectively,* 2 *and* 3 ([42])*;*

(E) 5 *in the figure-eight loop case, and there are three kinds of zeros for the elliptic integrals, depending on the integral over compact level curves inside the left loop, inside the right loop, or outside the figure-eight loop. We denote their respective number by n_1, n_2, n_3 respectively, then $n_1 + n_2 \leq 2$, $n_3 \leq 4$ and $n_1 + n_2 + n_3 \leq 5$ (see* [43] *for a precise description).*

Remark 1.17. The results in Theorem 1.16 are valid for $b \neq 0$. If $b = 0$, then the Hamiltonians (1.11) are symmetric (called also reversible); this may happen for the cases (i), (iii) and (v) of Figure 3. If $0 < b \ll 1$, the parameter b breaks this symmetry in a generic way, one has to add it into the parameter space of perturbations, and for a description of the bifurcation diagram of the unfolding in a full neighborhood of the origin in the parameter space, see [83, 98].

(iv) The hyperelliptic case.

In this case the polynomial $P(x)$ in (1.4) has degree at least 5. To find the exact number of zeros of the Abelian integrals for small n or to give an explicit upper bound of the number of zeros in general, such as introduced above for $m = 3, 4$, is extremely hard. The only general result was announced by D. Novikov and S. Yakovenko as follows:

Theorem 1.18 ([126])**.** *For any real polynomial $P(x) \in \mathbb{R}[x]$ of degree m and any polynomial 1-form ω of degree n, the number of real ovals $\gamma \subset \{y^2 + P(x) = h\}$ yielding an isolated zero of the integral $I(h) = \oint_\gamma \omega$, is bounded by a primitive recursive (in fact, elementary) function $B(m; n)$ of two integer variables m and n, provided that all critical values of P are real.*

The authors of [126] explained that "the function $B(m; n)$ grows no faster than a certain tower function (iterated exponent) of height 5 or perhaps 6. In any case, this bound is too excessive to believe that it might be realistic: this is the main reason why we never tried to write it explicitly".

There are some other works ([89, 127], for example) which give an explicit upper bound of the number $Z(m, n)$ by certain tower functions, not restricted to the hyperelliptic case, but with other restrictions on the Hamiltonians.

Before closing this chapter we introduce a result concerning a lower bound for $\tilde{n} = Z(n + 1, n)$.

Theorem 1.19 ([84]). *If $H \in \mathbb{R}[x, y]$ is a Morse polynomial of degree $n+1$ transversal to infinity, then for any $N = \frac{1}{2}(n+1)(n-2)$ real ovals $\{\gamma_h \subset H^{-1}(h)\}$ on $\mathbb{R}^2$ one can construct a form $\omega = P(x, y)dx + Q(x, y)dy$, $P, Q \in \mathbb{R}[x, y]$, deg $P, Q \leq n$, such that the perturbation $\{dH + \varepsilon\omega = 0\}$ produces at least N limit cycles which converge to the specified ovals as $\varepsilon \to 0$.*

Note that a *Morse function* means all its critical points are non-degenerate, and all critical values are different (see [4], for example); a polynomial $f \in \mathbb{C}[x, y]$ of degree $n + 1 \geq 2$ is called *transversal to infinity*, if one of the two equivalent conditions holds:

(1) Its principal homogeneous part factors out as the product of $n + 1$ pairwise different linear forms.

(2) Its principal homogeneous part has an isolated critical point of multiplicity n^2 at the origin.

Chapter 2

Abelian Integrals and Limit Cycles

In this chapter we will explain the relation between the number of zeros of the Abelian integrals and the number of limit cycles of the corresponding planar polynomial differential systems.

2.1 Poincaré–Pontryagin Theorem

Now we consider a polynomial $H(x, y)$ of degree m as in the previous chapter, the corresponding Hamiltonian vector field X_H:

$$\frac{dx}{dt} = -\frac{\partial H(x, y)}{\partial y}, \quad \frac{dy}{dt} = \frac{\partial H(x, y)}{\partial x}, \tag{2.1}$$

and a perturbed system

$$\frac{dx}{dt} = -\frac{\partial H(x, y)}{\partial y} + \varepsilon f(x, y), \quad \frac{dy}{dt} = \frac{\partial H(x, y)}{\partial x} + \varepsilon g(x, y), \tag{2.2}$$

where f and g are polynomials in x, y of degrees at most n, and ε is a small parameter.

Suppose that there is a family of ovals, $\gamma_h \subset H^{-1}(h)$, continuously depending on a parameter $h \in (a, b)$. Then we may define the Abelian integral as before

$$I(h) = \oint_{\gamma_h} f(x, y)dy - g(x, y)dx. \tag{2.3}$$

It is clear that all γ_h, filling up an annulus for $h \in (a, b)$, are periodic orbits of the Hamiltonian system (2.1).

Consider the question: How many orbits keep being unbroken and become the periodic orbits of the perturbed system (2.2) for small ε ? Note that if the number of such orbits is finite, then they are limit cycles of (2.2).

This question can be proposed in the converse way: Is it possible to find a value $h \in (a, b)$, and some periodic orbits Γ_ε of the perturbed systems (2.2), such that Γ_ε tends to γ_h (in the sense of Hausdorff distance) as $\varepsilon \to 0$? And how many such Γ_ε for a same h ?

To answer this question, we take a segment σ, transversal to each oval γ_h. We choose the values of the function H itself to parameterize σ, and denote by $\gamma(h, \varepsilon)$ a piece of the orbit of the perturbed system (2.2) between the starting point h on σ and the next intersection point $P(h, \varepsilon)$ with σ, see Figure 4. The

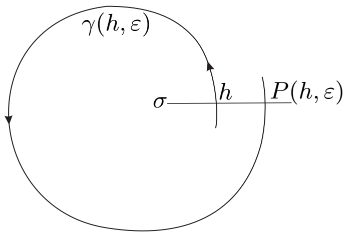

Figure 4. Construction of displacement function.

"next intersection" is possible for sufficiently small ε, since $\gamma(h, \varepsilon)$ is close to γ_h. As usual, the difference $d(h, \varepsilon) = P(h, \varepsilon) - h$ is called the *displacement function.*

Theorem 2.1 (Poincaré–Pontryagin [136, 137])**.** *We have that*

$$d(h, \varepsilon) = \varepsilon\,(I(h) + \varepsilon\phi(h, \varepsilon)), \quad \text{as } \varepsilon \to 0, \tag{2.4}$$

where $\phi(h, \varepsilon)$ is analytic and uniformly bounded for (h, ε) in a compact region near $(h, 0)$, $h \in (a, b)$.

Proof. By the construction above, the displacement function is given by the difference of the function H between the endpoints of $\gamma(h, \varepsilon)$, that is

$$d(h, \varepsilon) = \int_{\gamma(h,\varepsilon)} dH = \int_{\gamma(h,\varepsilon)} \left(\frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt} \right)\bigg|_{(2.2)} dt.$$

Substituting (2.2) into the right-hand side, we find

$$d(h, \varepsilon) = \varepsilon \int_{\gamma(h,\varepsilon)} \left(\frac{\partial H}{\partial x} f + \frac{\partial H}{\partial y} g \right)\bigg|_{(2.2)} dt.$$

Note that $\gamma(h, \varepsilon)$ converges to γ_h uniformly as $\varepsilon \to 0$ since γ_h is compact, and $H_x dt = dy$, $H_y dt = -dx$ along γ_h by (2.1) , we immediately obtain (2.4), where $I(h)$ is given by (2.3). □

Remark 2.2. Note that the number of zeros of the displacement function is independent of the choice of the transversal segment σ.

From Theorem 2.1 we obtain the following result giving an answer to the above question. We use X_H and $X_{H,\varepsilon}$ to denote the Hamiltonian system (2.1) and its perturbation (2.2) respectively, and first give a definition for convenience.

Definition 2.3. If there exist an $h^* \in (a,b)$ and an $\varepsilon^* > 0$ such that $X_{H,\varepsilon}$ has a limit cycle Γ_ε for $0 < |\varepsilon| < \varepsilon^*$, and Γ_ε tends to γ_{h^*} as $\varepsilon \to 0$, then we will say that Γ_ε *bifurcates from* γ_{h^*}. We say that a limit cycle Γ of $X_{H,\varepsilon}$ *bifurcates from the annulus* $\cup_{h\in(a,b)}\gamma_h$ of X_H, if there is a $h \in (a,b)$ such that Γ bifurcates from γ_h.

Theorem 2.4. *We suppose that $I(h)$ is not identically zero for $h \in (a,b)$, then the following statements hold.*

(A) *If $X_{H,\varepsilon}$ has a limit cycle bifurcating from γ_{h^*}, then $I(h^*) = 0$.*

(B) *If there exists an $h^* \in (a,b)$ such that $I(h^*) = 0$ and $I'(h^*) \neq 0$, then $X_{H,\varepsilon}$ has a unique limit cycle bifurcating from γ_{h^*}, moreover, this limit cycle is hyperbolic.*

(C) *If there exists an $h^* \in (a,b)$ such that $I(h^*) = I'(h^*) = \cdots = I^{(k-1)}(h^*) = 0$, and $I^{(k)}(h^*) \neq 0$, then $X_{H,\varepsilon}$ has at most k limit cycles bifurcating from the same γ_{h^*}, taking into account the multiplicities of the limit cycles.*

(D) *The total number (counting the multiplicities) of the limit cycles of $X_{H,\varepsilon}$, bifurcating from the annulus $\cup_{h\in(a,b)}\gamma_h$ of X_H, is bounded by the maximum number of isolated zeros (taking into account their multiplicities) of the Abelian integral $I(h)$ for $h \in (a,b)$.*

Proof. (A) Suppose that a limit cycle Γ_ε of $X_{H,\varepsilon}$ bifurcates from γ_{h^*}. By Theorem 2.1, there exist an $\varepsilon^* > 0$ and $h_\varepsilon \to h^*$ as $\varepsilon \to 0$, such that

$$d(h_\varepsilon, \varepsilon) = \varepsilon(I(h_\varepsilon) + \varepsilon\phi(h_\varepsilon, \varepsilon)) \equiv 0, \quad 0 < |\varepsilon| < \varepsilon^*.$$

Dividing by ε on both sides, and taking the limit as $\varepsilon \to 0$, we obtain $I(h^*) = 0$.

(B) Suppose that there exists an $h^* \in (a,b)$ such that $I(h^*) = 0$ and $I'(h^*) \neq 0$. Since we consider limit cycles for small ε and $\varepsilon \neq 0$, instead of the displacement function $d(h,\varepsilon)$ we may study the zeros of $\tilde{d}(h,\varepsilon) = d(h,\varepsilon)/\varepsilon$. By Theorem 2.1 we have

$$\tilde{d}(h,\varepsilon) = I(h) + \varepsilon\phi(h,\varepsilon),$$

where ϕ is analytic and uniformly bounded in a compact region near $(h^*, 0)$. Since $\tilde{d}(h^*, 0) = I(h^*) = 0$ and $\tilde{d}'_h(h^*, 0) = I'(h^*) \neq 0$, by the Implicit Function Theorem, we find an $\varepsilon^* > 0$, an $\eta^* > 0$ and a unique function $h = h(\varepsilon)$ defined in $U^* = \{(h,\varepsilon) : |h - h^*| \leq \eta^*, |\varepsilon| \leq \varepsilon^*\}$, such that $h(0) = h^*$ and $\tilde{d}(h(\varepsilon), \varepsilon) \equiv 0$ for $(h,\varepsilon) \in U^*$. Hence, the unique $h(\varepsilon)$ gives a unique limit cycle Γ_ε of system (2.2) for each small ε. We need to prove that the bifurcated limit cycle Γ_ε is hyperbolic (for small ε). This fact is easy to understand because it comes from a simple zero of $I(h)$ at h^*. We give a precise proof below. We write $I(h)$ in (2.3) as $I_1(h) + I_2(h)$, where

$$I_1(h) = \oint_{\gamma_h} f(x,y)dy, \quad I_2(h) = -\oint_{\gamma_h} g(x,y)dx.$$

In the first integral we treat x as a function of y and h, and along γ_h we have $H_x x_h = 1$ and $dy = H_x dt$. This gives

$$I_1'(h) = \oint_{\gamma_h} f_x x_h dy = \oint_{\gamma_h} f_x dt.$$

Similarly we obtain

$$I_2'(h) = -\oint_{\gamma_h} g_y y_h dx = \oint_{\gamma_h} g_y dt.$$

Hence $\oint_{\gamma_{h^*}} (f_x + g_y)dt = I'(h^*) \neq 0$, which implies

$$\oint_{\Gamma_\varepsilon} \text{trace}(2.2)dt = \varepsilon \oint_{\Gamma_\varepsilon} (f_x + g_y)dt \neq 0 \quad 0 < |\varepsilon| \ll 1,$$

since $\Gamma_\varepsilon \to \gamma_{h^*}$ as $\varepsilon \to 0$. The hyperbolicity of Γ_ε follows.

(C) Assume that there exists an $h^* \in (a,b)$ such that $I(h^*) = I'(h^*) = \cdots = I^{(k-1)}(h^*) = 0$, and $I^{(k)}(h^*) \neq 0$. We need to show that there exist a $\delta > 0$ and an $\eta > 0$, such that for any $(h, \varepsilon) \in U = \{|h - h^*| < \eta, |\varepsilon| < \delta\}$, the displacement function $d(h, \varepsilon)$ has at most k zeros in h, taking into account their multiplicities. Suppose the contrary, then for any integer j there exist $\varepsilon_j > 0$ and $\eta_j > 0$, $\varepsilon_j \to 0$ and $\eta_j \to 0$ as $j \to \infty$, such that for any ε_j the function $d(h, \varepsilon_j)/\varepsilon_j$ has at least $k+1$ zeros for $|h - h^*| < \eta_j$. By using the Rolle Theorem we find an h_j such that $|h_j - h^*| < \eta_j$ and

$$I^{(k)}(h_j) + \varepsilon_j \frac{\partial^k}{\partial^k h}\phi(h_j, \varepsilon_j) = 0,$$

which implies $I^{(k)}(h^*) = 0$ by taking the limit as $j \to \infty$, leading to a contradiction.

(D) This statement is a consequence of the first three statements. In fact, for any small $\sigma > 0$ we may consider the number of limit cycles bifurcating from $h \in [a + \sigma, b - \sigma]$ for small ε. By Theorem 1.2 this number is uniformly bounded. We take the maximum of this number as $\sigma \to 0$, then we get the *cyclicity of the period annulus*, see Definition 1.3. □

Example. Consider the van der Pol equation

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0,$$

which is equivalent to the system

$$\dot{x} = y, \quad \dot{y} = -x + \varepsilon(1 - x^2)y. \tag{2.5}$$

When $\varepsilon = 0$, equation (2.5) is a Hamiltonian system with a family of ovals

$$\gamma_h = \{(x,y) : H(x,y) \equiv x^2 + y^2 = h^2,\ h > 0\}.$$

By using the polar coordinates $x = h\cos\theta, y = h\sin\theta$, and noting that the orientation of γ_h is clockwise, from formula (2.3) we have

$$I(h) = -\oint_{\gamma_h} (1-x^2)ydx = \int_0^{2\pi} (1-h^2\cos^2\theta)h^2(-\sin^2\theta)d\theta = \pi h^2\left(\frac{h^2}{4}-1\right).$$

The zero $h = 0$ corresponds to the singularity of the system and $h = 2$ is the only positive zero of $I(h)$. On the other hand, it is easy to find $I'(2) = 4\pi$. Using Theorem 2.4 we conclude that for small ε system (2.5) has a unique limit cycle which is hyperbolic and tends to the circle of radius 2 as $\varepsilon \to 0$.

Remark 2.5. If both the Hamiltonian and the perturbation are given without parameters (except ε), as in the above example, then Theorem 2.4 works well to get a definite result. But in many cases the perturbations are given in a function space (with parameters), and the Hamiltonian may also depend on some parameters, this causes some problem in using Theorem 2.4 and we explain it below.

Since the zeros of $I(h)$ also depend on the parameters, appearing in perturbations, they may tend to the endpoints of (a, b), corresponding to critical values of H. At these special values the Implicit Function Theorem can not, in general, be applied to the displacement function, so it is difficult to give a uniform estimate of the number of zeros for $h \in [a, b]$. It is well known that if one of the endpoints, say a, corresponds to the center of X_H, then $I(h)$ can be extended to the value a analytically (see Lemma 20 of [12] for example), and nontrivial $I(h)$ has at most a finite number of zeros near a uniformly with respect to parameters, hence the statement (C) of Theorem 2.4 can be extended to $[a, b)$. On the other hand, if an endpoint, say b, corresponds to a polycycle (homoclinic or heteroclinic orbit) of X_H, the better conclusions are the following. Statement (C) can be extended to $[a, b]$ if b corresponds to a homoclinic loop, see Roussarie [140] as we mentioned in Remark 1.8; and in general it surely could not be extended to $[a, b]$ if b corresponds to a heteroclinic loop, as it has been shown by a counter-example with two-saddle loop in the recent papers [46, 13], also see [72]. At last, if $b = \infty$ (the annulus tends to infinity), we could make a conclusion about the number of limit cycles only in *any compact region* of the annulus.

If $H = H_\nu$ depends on some parameter ν, then for some special values, say ν^*, H_{ν^*} may be *degenerate*, for example has some symmetries, see the cases $X_H \in Q_3^H \cap \{Q_3^R \cup Q_3^{LV}\}$ (i.e., $(a, b) \in \partial\bar{G}$ in Figure 2), or the cases (i), (iii) and (v) in Figure 3 for some special values of the parameters of H. As Iliev explained in [79] the degeneracy causes a lower bound for the number of zeros of $I(h)$ than the expected one, and the function $I(h)$ (even in the case that $I(h)$ is not identically zero) can never yield the maximum number of zeros of the displacement function $d(h, \varepsilon)$ for the whole class of perturbations. In this case a higher order (in ε) of approximation for $d(h, \varepsilon)$ is needed, as in the study of the cyclicity of the period annulus for $X_H \in Q_3^H \cap \{Q_3^R \cup Q_3^{LV}\}$ (see Remark 1.6); or the parameters which break down the symmetry of H and the perturbation parameters should be considered together in higher dimensional space to give a "principal part" of the displacement function, as in the study of perturbations of symmetric elliptic Hamiltonians of degree 4 in [98].

Remark 2.6. Theorems 2.1 and 2.4 were proved for the polynomial Hamiltonian systems (2.1) and their polynomial perturbation (2.2), but there are essentially the same proofs for analytic vector fields.

2.2 Higher Order Approximations

It is shown in the last section that the Abelian integral $I(h)$, related to X_H, gives the first order approximation of the displacement function of the perturbed system $X_{H,\varepsilon}$, hence the number of isolated zeros of $I(h)$ gives an upper bound of the number of limit cycles of $X_{H,\varepsilon}$, if $I(h)$ is not identically zero. In this section, we continue the discussion of the problem if $I(h) \equiv 0$ for $h \in (a, b)$.

It is very natural to express the displacement function in the form

$$d(h,\varepsilon) = \varepsilon\, I_1(h) + \varepsilon^2\, I_2(h) + \cdots + \varepsilon^j\, I_j(h) + O(\varepsilon^{j+1}), \tag{2.6}$$

where $I_1(h) \equiv I(h)$, ε small. The question is that if $I_1(h) \equiv 0$, then how to compute the second order approximation $I_2(h)$ and so on ?

The following algorithm to compute $I_{k+1}(h)$, if $I_j(h) \equiv 0$ for $j = 1, 2, \ldots k$, was given by J.-P. Françoise in [52], see also [168].

Denote $dH = H_x dx + H_y dy$, $\omega = f dy - g dx$, where H, f and g are polynomials in x and y, deg(H)=$n+1$, max(deg(f),deg(g))=n. Then equations (2.1) and (2.2) can be written in Pfaffian forms $dH = 0$ and $dH - \varepsilon\omega = 0$ respectively. As before, we use γ_h to denote the family of ovals contained in the level curves $H^{-1}(h)$, σ a segment transversal to γ_h and parameterized by H, and $\gamma(h,\varepsilon)$ a piece of the orbit of $dH - \varepsilon\omega = 0$ between the starting point h on σ and the next intersection point $P(h,\varepsilon)$ with σ. By using these notations the theorem of Poincaré–Pontryagin can be shown in brief as follows.

The integration of $dH - \varepsilon\omega = 0$ over $\gamma(h,\varepsilon)$ gives

$$d(h,\varepsilon) = \int_{\gamma(h,\varepsilon)} dH = \varepsilon \int_{\gamma(h,\varepsilon)} \omega = \varepsilon \int_{\gamma_h} \omega + O(\varepsilon^2).$$

Following [52], we say that the polynomial H satisfies the condition $(*)$ if and only if for all polynomial 1-forms ω:

$$\int_{\gamma_h} \omega = 0 \;\Leftrightarrow\; \text{there are polynomials } q \text{ and } R \text{ such that } \omega = q dH + dR. \tag{$*$}$$

Theorem 2.7 ([52]). *Assume that H satisfies the condition $(*)$ and $I_j(h) \equiv 0$ (in the formula (2.6)) for $j = 1, 2, \ldots, k$. Then there are $q_1, \ldots, q_k$; $R_1, \ldots, R_k$ such that $\omega = q_1\, dH + dR_1, q_1\omega = q_2\, dH + dR_2, \ldots, q_{k-1}\omega = q_k\, dH + dR_k$ and*

$$I_{k+1}(h) = \int_{\gamma_h} q_k\, \omega. \tag{2.7}$$

Proof. The proof can be done by induction.

(1) Assume that $I_1(h) \equiv 0$. By the condition $(*)$ we find two polynomials q_1 and R_1 such that $\omega = q_1\, dH + dR_1$, which implies equality

$$(1 + \varepsilon q_1)(dH - \varepsilon\, \omega) = d\,(H - \varepsilon R_1) - \varepsilon^2 q_1\, \omega.$$

After integrating the above equality over $\gamma(h, \varepsilon)$, along which $dH - \varepsilon\omega = 0$, we obtain the displacement function

$$d(h, \varepsilon) = \int_{\gamma(h,\varepsilon)} dH = \varepsilon^2 \int_{\gamma(h,\varepsilon)} q_1\omega + \varepsilon \int_{\gamma(h,\varepsilon)} dR_1.$$

Treating $dR_1 = \frac{\partial R_1}{\partial x}dx + \frac{\partial R_1}{\partial y}dy$ as a polynomial 1-form and using $I_1(h) \equiv 0$, we get $\int_{\gamma(h,\varepsilon)} dR_1 = O(\varepsilon^2)$. Thus we obtain

$$d(h, \varepsilon) = \varepsilon^2 \int_{\gamma(h,\varepsilon)} q_1\omega + O(\varepsilon^3) = \varepsilon^2 \int_{\gamma_h} q_1\omega + O(\varepsilon^3),$$

i.e., $I_2(h) = \int_{\gamma_h} q_1\omega$.

(2) Suppose that $q_{j-1}\, \omega = q_j\, dH + dR_j$ (denote $g_0 = 1$) and $I_j(h) \equiv 0$ for $j = 1, \ldots, k$. Then the former gives the equality

$$(1 + \varepsilon q_1 + \cdots + \varepsilon^k q_k)(dH - \varepsilon\, \omega) = d\,(H - \varepsilon R_1 - \cdots - \varepsilon^k R_k) - \varepsilon^{k+1} q_k\, \omega,$$

and the latter gives $\int_{\gamma(h,\varepsilon)} d\,(\varepsilon R_1 + \cdots + \varepsilon^k R_k) = O(\varepsilon^{k+2})$. Hence

$$d(h, \varepsilon) = \varepsilon^{k+1} \int_{\gamma(h,\varepsilon)} q_k\omega + O(\varepsilon^{k+2}) = \varepsilon^{k+1} \int_{\gamma_h} q_k\omega + O(\varepsilon^{k+2}),$$

i.e., $I_{k+1}(h) = \int_{\gamma_h} q_k\omega$. □

Two important questions arise: (1) Is there an integer K such that the above procedure stops at order K (i.e., if $I_j(h) \equiv 0$ for $j = 1, \ldots, K$, then all $I_j(h) \equiv 0$ for $j > K$) ? (2) What kind of functions H satisfy the condition $(*)$?

The answer to the first question is positive, but a new problem is that there is no efficient method to find such integer K. Usually, the condition $I_j(h) \equiv 0$ for $j = 1, \ldots, l$ gives restrictions on the parameters which appear on the 1-form ω, and one needs to check if $dH - \varepsilon\, \omega$ is integrable at this stage. If the answer is yes, then $K = l$.

Concerning the second question, the following result by Gavrilov shows that for "generic" polynomial Hamiltonian H, the condition $(*)$ holds. Before stating the result, we need to give some definitions.

Definition 2.8. ([63]) A polynomial $f \in \mathbb{R}[x, y]$ is called *weighted homogeneous of weighted degree* j *and type* (w_x, w_y) if there are $w_x, w_y \in \mathbb{N}^+$ and $j \in \mathbb{N}$, such that

$$f(z^{w_x} x, z^{w_y} y) = z^j f(x, y), \quad \forall\, z \in \mathbb{R}.$$

A polynomial $f \in \mathbb{R}[x,y]$ is called *semiweighted homogeneous of weighted degree* k *and type* (w_x, w_y) if it can be written as $f = \sum_{j=0}^{k} f_j$, where f_j are weighted homogeneous polynomials of weighted degree j and type (w_x, w_y), and the polynomial $f_k(x,y)$ has an isolated critical point at the origin.

Theorem 2.9 (Proposition 3.2 of [63]). *Let $\gamma_h \subset H^{-1}(h) \subset \mathbb{R}^2$ be a continuous family of ovals surrounding a single critical point of H. If $H \in \mathbb{R}[x,y]$ is a semiweighted homogeneous Morse polynomial with distinct critical values, then the space of all real polynomial 1-forms satisfies the condition $(*)$.*

The above result was obtained by Ilyashenko [84] when H is a polynomial of degree m with $(m-1)^2$ distinct critical points. Some further discussions concerning Theorem 2.7 can be found in [68].

Example. It was shown in [63] that if $P_m(x)$ is a polynomial of degree m with $m-1$ distinct critical values, then the Hamiltonian function $H(x,y) = \frac{1}{2}y^2 + P_m(x)$ satisfies the condition $(*)$. This result was found earlier in [52] for the case $H = \frac{1}{2}(x^2 + y^2)$.

The result in Theorem 2.7 was generalized by I.D. Iliev in [80] to the case $\omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots$. He considers polynomial perturbations of Hamiltonian systems with elliptic or hyperelliptic Hamiltonians and gives a formula for the second variation of the displacement function in terms of the coefficients of the perturbations. We briefly introduce this result below.

Let $H(x,y) = \frac{y^2}{2} - U(x)$, where $U(x)$ is a polynomial of degree n, and $n \geq 2$. Consider the perturbations

$$\begin{aligned} \dot{x} &= H_y + \varepsilon\, f(x,y,\varepsilon), \\ \dot{y} &= -H_x + \varepsilon\, g(x,y,\varepsilon), \end{aligned} \tag{2.8}$$

where $f(x,y,\varepsilon)$ and $g(x,y,\varepsilon)$ are polynomials in x, y and depend analytically on a small parameter ε. System (2.8) can be written in a Pfaffian form

$$dH - \varepsilon\,\omega = 0, \quad \omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots, \tag{2.9}$$

where $\omega_0 = -f(x,y,0)dy + g(x,y,0)dx, \omega_1 = -f_\varepsilon(x,y,0)dy + g_\varepsilon(x,y,0)dx, \ldots$.

Suppose that the continuous family of ovals $\gamma_h \subset H^{-1}(h)$ for $h \in (a,b)$, which form an annulus D. If we take a transversal segment to $\{\gamma_h\}$ and parameterize it using the level value h, then, by Theorem 2.1, the displacement function for small ε has the form

$$d(h,\varepsilon) = \varepsilon\, M_1(h) + \varepsilon^2\, M_2(h) + \cdots,$$

where $M_1(h) = \oint_{\gamma_h} \omega_0$.

Theorem 2.10 ([80]). *Under the above assumptions the following statements hold.*

(A) *If $M_1(h) \equiv 0$, then there exist in D a continuous function $q_0(x, y)$ and a locally Lipschitz continuous function $Q_0(x, y)$ such that the form ω_0 can be expressed as $\omega_0 = q_0\, dH + dQ_0$, and*

$$M_2(h) = \oint_{\gamma_h} (q_0\omega_0 + \omega_1).$$

(B) *If $M_k(h) \equiv 0, 1 \le k \le m$, then define the 1-forms $\Omega_0, \Omega_1, \ldots, \Omega_m$ successively as follows:*

$$\Omega_0 = \omega_0, \quad \Omega_k = \omega_k + \Sigma_{i+j=k-1}\, q_i\, \omega_j, \ 1 \le k \le m,$$

where, for $0 \le k \le m-1$, the 1-form Ω_k can be expressed as $\Omega_k = q_k\, dH + dQ_k$ with q_k, Q_k as in statement (A), *and*

$$M_{m+1}(h) = \oint_{\gamma_h} \Omega_m.$$

(C) *The function $M_2(h)$, $h \in (a, b)$ can be explicitly expressed as*

$$\begin{aligned} M_2(h) \quad &= \oint_{\gamma_h} [G_{1h}(x,y)P_2(x,h) - G_1(x,y)P_{2h}(x,h)]dx \\ &\quad - \oint_{\gamma_h} \frac{F(x,y)}{y}[f_x(x,y,0) + g_y(x,y,0)]dx \\ &\quad + \oint_{\gamma_h} g_\varepsilon(x,y,0)dx - f_\varepsilon(x,y,0)dy, \end{aligned} \tag{2.10}$$

where

$$F(x,y) = \int_0^y f(x,s,0)ds - \int_0^x g(s,0,0)ds, \ G(x,y) = g(x,y,0) + F_x(x,y),$$

and $G_1(x, y)$ and $G_2(x, y)$ are the odd and even parts of $G(x, y)$ with respect to y:

$$G(x,y) = G_1(x,y) + G_2(x,y), \ G_1(x,y) = y\, p_1(x,y^2), \ G_2(x,y) = p_2(x,y^2).$$

Finally, $P_2(x, h)$ is the polynomial

$$P_2(x,h) = \int_0^x p_2(s, 2h + 2U(s))ds.$$

Note that $G_{1h}(x, y) = G_{1y}(x, y)/y$ on the oval $\gamma(h)$. If the divergence $f_x + g_y$ is either an odd or an even function of y, then formula (2.10) can be written in the more compact form:

$$M_2(h) = -\oint_{\gamma_h} \frac{F(x,y)}{y}[f_x + g_y]|_{\varepsilon=0}\, dx + \oint_{\gamma_h} (g_\varepsilon\, dx - f_\varepsilon\, dy)|_{\varepsilon=0}.$$

Remark 2.11. In 1995, before the general result in Theorem 2.7, B. Li and Z. Zhang [93] deduced the second order Melnikov function $M_2(h)$ for the codimension 2 Bogdanov–Takens bifurcation problem, see also [181]. We will introduce this result in Section 3.3, together with a later result of [81] on $M_k(h)$ for arbitrary k.

Remark 2.12. When the condition $(*)$ is not satisfied, it is also possible to use the Françoise recursion formula (Theorem 2.7), but the functions q_k and R_k may not be polynomials, see [77] for example, where the fractional function and logarithm function appear in q_k and R_k. Of course, in this case the study of "generalized Abelian integrals" is more difficult, see the next section.

Remark 2.13. By generalizing the Françoise procedure, A. Gasull and J. Torregrosa constructed a new algorithm of the computation of the Lyapunov constants for some degenerate critical points in [59], and studied the relation between the degenerate Hopf bifurcation and the method of Abelian integrals near the singularity in [60].

Remark 2.14. In the paper [162] M. Viano, J. Llibre and H. Giacomini gave a different recursive procedure for calculation of higher order Melnikov functions.

2.3 The Integrable and Non-Hamiltonian Case

To attack Hilbert's 16th problem, we need to consider the cyclicity of period annulus (or annuli) under polynomial perturbations not only from the polynomial Hamiltonian systems, as we explained in the last two sections, but also from polynomial integrable and non-Hamiltonian systems. To see it clearly, let us list all integrable quadratic systems with at least one center. By using the terminology from [191], Iliev [79] classified them into the following five classes using complex notation:

$$
\begin{array}{lll}
(1) & \dot z = -iz - z^2 + 2|z|^2 + (b+ic)\overline{z}^2, & \text{Hamiltonian } (Q_3^H) \\
(2) & \dot z = -iz + az^2 + 2|z|^2 + b\overline{z}^2, & \text{reversible } (Q_3^R) \\
(3) & \dot z = -iz + 4z^2 + 2|z|^2 + (b+ic)\overline{z}^2, & |b+ic| = 2,\ \text{codimension } 4\ (Q_4) \\
(4) & \dot z = -iz + z^2 + (b+ic)\overline{z}^2, & \text{generalized Lotka} - \text{Volterra } (Q_3^{LV}) \\
(5) & \dot z = -iz + \overline{z}^2, & \text{Hamiltonian triangle}
\end{array}
$$

where the parameters a, b and c are real, and $z = x + i\,y$.

An integrable quadratic system is called *generic*, if it belongs to one of the first four classes and does not belong to other classes of the classification given above. Otherwise, it is *degenerate*.

The discussions about Abelian integrals so far were only valid for the generic Hamiltonian class. As an example of integrable and non-Hamiltonian case, we consider the reversible class. Taking $z = x + iy$, and the change $t \to -t$, we obtain from (2) that

$$
\begin{aligned}
\dot x &= -y - (a+b+2)x^2 + (a+b-2)y^2, \\
\dot y &= x - 2(a-b)xy.
\end{aligned}
$$

If $c = a - b \neq 0$, then making the scaling $(x, y) \mapsto (x/c, y/c)$ and changing the parameters $(-\frac{a+b+2}{a-b}, \frac{a+b-2}{a-b}) \mapsto (a, b)$, we obtain

$$\begin{aligned} \dot{x} &= -y + ax^2 + by^2, \\ \dot{y} &= x(1 - 2y). \end{aligned} \tag{2.11}$$

Using the following coordinates and time scaling,

$$x = \frac{1}{2}\,\bar{x}, \quad y = -\frac{1}{2}\,(\bar{y} - 1), \quad t = 2\,\bar{t},$$

and then writing (x, y, t) instead of $(\bar{x}, \bar{y}, \bar{t})$, we obtain

$$\begin{aligned} \dot{x} &= ax^2 + by^2 - 2(b-1)y + (b-2), \\ \dot{y} &= -2xy. \end{aligned} \tag{2.12}$$

System (2.12) has an invariant straight line $\{y = 0\}$, and has a center at $(0, 1)$. The singularity $(0, -\frac{2-b}{b})$ is also a center if $0 < b < 2$, and is a saddle if $b < 0$ or $b > 2$. If $a(2 - b) > 0$, then the system has two saddles at $(\pm\sqrt{\frac{2-b}{a}}, 0)$.

If $a(a + 1)(a + 2) \neq 0$, then the first integral of system (2.12) is given by

$$F = |y|^a (x^2 + Ly^2 + My + N) = h, \tag{2.13}$$

where

$$L = \frac{b}{a+2}, \quad M = \frac{2(1-b)}{a+1}, \quad N = \frac{b-2}{a}. \tag{2.14}$$

Note that if $a \neq 1$, then system (2.12) is not Hamiltonian, and the integrating factor is $\mu = |y|^{a-1}$. In the period annulus surrounding a center of system (2.12), we denote the ovals by

$$\Gamma_h = \{(x, y) \in \mathbb{R}^2 \,:\, F(x, y) = h,\, h_c < h < h_s\},$$

where h_c is the critical value of H at a center, and h_s is the value of H for which the period annulus ends at a separatrix polycycle or at infinity. We can suppose that $h_c < h_s$, otherwise we can change the sign of H to ensure it.

We consider quadratic perturbations of (2.12):

$$\begin{aligned} \dot{x} &= ax^2 + by^2 - 2(b-1)y + (b-2) + \epsilon f(x, y), \\ \dot{y} &= -2xy + \epsilon g(x, y), \end{aligned}$$

where ϵ is a small parameter, and f and g are quadratic polynomials in x and y.

If $a \neq 1$, $a + b \neq 0$, $(a, b) \neq (-4, 2)$ and $(a, b) \neq (-2/3, 0)$, then the reversible system (2.12) is generic. If, in addition, $a(a+1)(a+2) \neq 0$, then the cyclicity of the period annulus of (2.12) under quadratic perturbations is equal to the maximal number of isolated zeros in (h_c, h_s), counting multiplicities, of the following integral (see [79])

$$M(h) = \int_{\Gamma_h} |y|^{a-2} (\alpha + \beta y + \gamma y^2)\, x\, dy, \tag{2.15}$$

where α, β and γ are real constants, and the orientation of the integral is given by the vector field.

It is clear now that for most values of a, say a is not an integer, both the first integral F in (2.13) and the integrand function in (2.15) are no longer polynomials. Hence, the integral $M(h)$ is not an Abelian integral in the strict meaning. Usually it is called *generalized* Abelian integral or *pseudo-Abelian* integral, or simply Abelian integral as before.

Finally, let us give a general setting for the integrable and non-Hamiltonian case. Suppose that the unperturbed system has a first integral $F(x,y)$ with an integrating factor $\mu(x,y) = 1/R(x,y)$; then the perturbed system can be written in the form

$$\begin{cases} \dot{x} = -\dfrac{\partial F(x,y)}{\partial y} R(x,y) + \varepsilon f(x,y)\,, \\ \dot{y} = \dfrac{\partial F(x,y)}{\partial x} R(x,y) + \varepsilon g(x,y)\,, \end{cases} \tag{2.16}$$

and associated to it we define the (generalized) Abelian integral

$$I(h) = \int_{\gamma_h} \frac{f(x,y)dy - g(x,y)dx}{R(x,y)}\,, \tag{2.17}$$

where $\{\gamma_h\}$ are the family of ovals contained in the level curves $\{F(x,y) = h\}$. By the same mechanisms as in the last sections, the integral $I(h)$ gives the first approximation of the displacement function.

Note that some traditional methods, such as the derivation of the Picard–Fuchs equation or Picard–Lefschetz formula etc, fail for this generalized form of Abelian integrals. For this reason, the study of perturbations from the reversible class Q_3^R is very difficult. On the other hand, this study, comparing with the quadratic perturbations from other quadratic integrable classes, is the most interesting one. We list some results concerning the quadratic perturbations from the reversible system (2.11): [22] for the isochronous centers; [159] for the bifurcation curve of the unbounded heteroclinic loop; [44, 129, 173, 82] for $a = -3$ with different values of b; [16] for $a \sim 2$ with different values of b. There are some other works dealing with the perturbations from integrable and non-Hamiltonian systems, among them [5, 32, 58, 95, 96, 106, 115, 187].

2.4 The Study of the Period Function

We use the same notation as before to denote a continuous family of ovals $\gamma_h \subset H^{-1}(h)$, where H is the Hamiltonian function (or the first integral) of a planar Hamiltonian (or integrable) system. Each γ_h is a periodic orbit of the system, so we have a *period function* $T(h)$, parameterized by the same $h \in (a,b)$. If the period function is a constant for all h, then the period annulus is *isochronous.* If the isochronous period annulus surrounds a center, then the center is called isochronous. If the period function is strictly increasing or decreasing, then we say that the period function is *monotone*. Otherwise, the period function has *critical points.*

The study of the period function and the study of the Abelian integral have some relations, at least from the following two points of view.

First, the study of the number of critical points of the period function by perturbing an isochronous center inside a certain class of integrable systems, is comparable to the study of the number of zeros of an Abelian integral by perturbing an integrable system inside a certain class of systems. We will introduce a work by E. Freire, A. Gasull and A. Guillamon [54] on this aspect.

Secondly, the study of the period function is useful for the study of the Abelian integral. For example, the Abelian integral $I_0(h) = \oint_{\gamma_h} y\, dx$ gives the area of the region surrounded by γ_h. Here we suppose the orientation of γ_h is clockwise, $\gamma_h \subset H^{-1}(h)$, and the related Hamiltonian system is $X_H = H_y\partial/\partial x - H_x\partial/\partial y$. Hence, $I_0(h) > 0$ for $h > a$ and $I_0'(h) > 0$ gives the period of γ_h. In fact,

$$I_0'(h) = \oint_{\gamma_h} \frac{\partial y}{\partial h}\, dx = \oint_{\gamma_h} (H_y)^{-1}\, dx = \oint_{\gamma_h} dt = T(h).$$

If the period function is monotone, then $I_0''(h) \neq 0$. In some studies of Abelian integrals this information is needed to define a function by a ratio of two Abelian integrals with second order derivative, see for example [15, 16, 25, 40, 41, 43, 82], and section 4.2 below. If $I_0''(h)$ has some zeros, i.e., the period function has critical points, then the use of the ratio becomes complicated, see [42]. For this reason, we will also briefly introduce some results on period functions.

Before stating the result of [54], we give the characterizations of isochronous centers: A center point p of a planar smooth vector field X in an annulus D is an isochronous center if and only if one of the following assertions holds:

- (i) There exists a smooth change of coordinates in a neighborhood of p that linearizes X (a classical result of Poincaré).
- (ii) There exists a transversal vector field U, commuting with X, i.e., $[X, U] = DUX - DXU = 0$, see [146, 167].
- (iii) There exists a transversal vector field U and a scale function μ such that $[X, U] = \mu X$ and
$$\int_0^{T_r} \mu(x(t), y(t))\, dt = 0,$$
where $\gamma = \{\varphi(t) = (x(t), y(t)), t \in [0, T_r]\}$ is any period orbit of X in D, and T_r is its period. In this case U is called a *normalizer* of X, see [55].

Theorem 2.15 ([54])**.** *Suppose that a vector field X has an isochronous center of period T_0 in D. Consider a vector field U transversal to X such that $[X, U] = 0$. Let $\gamma(t) := \{\varphi(t; \psi(h)), t \in [0, T_0]\}$ be the set of periodic orbits of X in D parameterized by the time flow of U. Consider the family of vector fields $X_\varepsilon = X + \varepsilon Y$ having also a center; write Y as $Y = aX + bU$ and denote by $\gamma_\varepsilon(h)$ a generic closed orbit of X_ε passing through $\psi(h)$. The following statements hold:*

(i) *The period function associated to* $\gamma_\varepsilon(h)$ *is*

$$T_{\gamma_\varepsilon}(h) = T_0 + \varepsilon T_1(h) + O(\varepsilon^2),$$

where

$$T_1(h) = -\int_0^{T_0} a(\varphi(t;\psi(h)))\,dt.$$

(ii) *The derivative of* T_1 *with respect to* h *is:*

$$T_1'(h) = -\int_0^{T_0} \nabla a(x)\cdot U(x)|_{\{x=\varphi(t;\psi(h))\}}dt.$$

(iii) *If* h^* *is a simple zero of* $T_1'(h)$*, then for small* ε *there is exactly one critical period of* X_ε *close to* h^* *which tends to* h^* *as* $\varepsilon \to 0$*.*

Example ([54]). Consider the system

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + \varepsilon\, G'(x). \end{aligned} \tag{2.18}$$

Then, for ε sufficiently small the zeros of

$$I(s) = \int_0^{2\pi} \left.\frac{x(xG''(x) - G'(x))}{x^2+y^2}\right|_{x=s\cos t, y=s\sin t} dt$$

give rise to critical periods of (2.18).

Moreover, if $G'(x)$ is a polynomial of degree n vanishing at zero, then the maximum number of simple zeros of $I(s)$ is $[(n-3)/2]$, i.e., at most $[(n-3)/2]$ critical periods bifurcate from the closed orbits of $(2.18)_{\varepsilon=0}$ in any fixed compact set in the annulus region.

The above result can be proved by Theorem 2.15. In fact, let $X = (-y, x)$, $U = (x, y)$ and $Y = (0, G'(x))$. Then

$$a = \frac{x\,G'(x)}{x^2+y^2}, \quad b = \frac{y\,G'(x)}{x^2+y^2},$$

and

$$\nabla a \cdot U = \frac{x(xG''(x) - G'(x))}{x^2+y^2}.$$

Taking $\psi(h) = (e^h, 0)$, $\varphi(t;\psi(h)) = (e^h\cos t, e^h\sin t)$ and renaming e^h by s, the result follows by applying Theorem 2.15.

Now we list some results concerning the period function.

- A survey article about isochronous centers can be found in [14].

- Every center of a polynomial Hamiltonian system of degree 4 (that is, with its homogeneous part of degree 4 not identically zero) is non-isochronous ([91]).
- Suppose $H(x, y) = F(x) + G(y)$ and the origin is a non-degenerate center of X_H. More concretely, if $T(h)$ denotes the period of the periodic orbit contained in $H(x, y) = h$, then [30] solved the inverse problem of characterizing all systems with a given function $T(h)$, characterized the limiting behavior of T at infinity when the origin is a global center and applied this result to prove, among other results, that there are no nonlinear polynomial isochronous centers in this family.
- An analytic vector field has a finite number of critical periods in any compact region inside an annulus ([21]).
- For elliptic Hamiltonian $H = \frac{y^2}{2} + P(x)$, where P is a polynomial, the period function of the corresponding system X_H is monotone if $\deg(P) = 3$ and has at most one critical point if $\deg(P) = 4$. In the latter case only the global center (case (iii) of Figure 3) has a critical period ([26, 62]).
- If the Hamiltonian system with Hamiltonian $H = \frac{y^2}{2} + V(x)$, where V is a smooth function, has a non-degenerate relative minimum at $x = 0$, then the period function is monotone if $V/(V')^2$ is convex ([19]). This condition was generalized in [30, 9].

The following results show the behavior of the period function for quadratic integrable systems with period annulus (or annuli). The notation is introduced at the beginning of the previous section.

- At most two *local critical periods* bifurcate from quadratic centers. Here local means when the perturbation parameter ε tends to zero, the level curves with bifurcating critical periods shrinks to the center point ([22]).
- It was conjectured in [19] that all the centers encountered in the family of second order differential equations $\ddot{x} = V(x, \dot{x})$, being V a quadratic polynomial, should have a monotone period function, and some cases were solved in that paper. The remaining cases were completely solved in [57]. Note that this equation can be written as the planar system $\dot{x} = y, \dot{y} = -x + ax^2 + bxy + cy^2$.
- The period function for the quadratic Hamiltonian systems ($X \in Q_3^H$) is monotone ([34]).
- The period function for the quadratic codimension 4 systems ($X \in Q_4$) is monotone ([184]).
- The period function for the *classical* quadratic Lotka–Volterra systems is monotone ([139, 148, 163]). But in general for $X \in Q_3^{LV}$ the problem is still open. Some partial results about the monotonicity of the period function in this class, especially the monotone property of the period function near the saddle loop, were obtained in [165].

- The behavior of the period function for $X \in Q_3^R$ (the family of quadratic reversible systems) is more complicated. The first example in this family with non-monotone period function was given in [20]. Then C. Chicone conjectured that the reversible centers have at most two critical periods (see Math. Review 94h:58072). As it was shown in the previous section, Q_3^R is a family of two parameters. The papers [185, 186] analyze some 1-parameter families inside Q_3^R (including the example of [20]) and in them it is proved that at most one critical period may happen. In [166] several two-dimensional regions in the parameter plane were determined for which the corresponding center has monotone period function. In the recent paper [124] the behavior of the period function for $X \in Q_3^R$ is determined near the saddle loop. Some regions in the parameter plane were determined, for which the corresponding system has one or two critical periods near the saddle loop, and the local bifurcation diagram was given.

Chapter 3

Estimate of the Number of Zeros of Abelian Integrals

To study the weak Hilbert's 16th problem by using Abelian integrals, it is crucial to estimate the number of zeros of the Abelian integral. In this chapter, we introduce several methods to study the number of zeros of the Abelian integral $I(h)$ given in (1.10), which is related to the codimension 2 Bogdanov–Takens bifurcation problem, as we explained in subsection 1.2.2.

3.1 The Method Based on the Picard–Fuchs Equation

Recall that we consider the elliptic Hamiltonian of degree 3 in the form

$$H(x, y) = \frac{y^2}{2} - \frac{x^3}{3} + x, \tag{3.1}$$

with the continuous family of ovals

$$\{\gamma_h\} = \{(x, y) : \ H(x, y) = h, \ -2/3 \le h \le 2/3\}, \tag{3.2}$$

shown in Figure 2. We denote the corresponding Hamiltonian system by X_H. When $h \to -2/3$ the oval γ_h shrinks to the center of X_H at $(-1, 0)$ while when $h \to 2/3$ the oval γ_h terminates at the homoclinic loop with saddle $(1, 0)$. The perturbation of X_H has the form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -1 + x^2 + \varepsilon(\alpha + x)y, \end{aligned} \tag{3.3}$$

where α is a constant and ε is a small parameter. The corresponding Abelian integral is

$$I(h) = \alpha I_0(h) + I_1(h), \quad I_j(h) = \oint_{\gamma_h} x^j y \, dx, \ j = 0, 1, 2, \ldots \tag{3.4}$$

Since the orientation of γ_h is clockwise, by (3.3), by using the Green's formula it is easy to know that $I_0(h)$ is the area of the region surrounded by γ_h, hence $I_0(h) > 0$ for $h > -2/3$. Let $(\xi_h, 0)$ and $(\eta_h, 0)$ $(\xi_h < -1 < \eta_h < 1)$ be the two intersection points of γ_h with the x-axis, then by using (3.1) it is easy to find

$$I_j(h) = 2\int_{\xi_h}^{\eta_h} x^j y(x,h)\,dx, \quad I_j'(h) = 2\int_{\xi_h}^{\eta_h} \frac{x^j}{y(x,h)}\,dx, \tag{3.5}$$

where $y(x,h) \geq 0$ is determined from $H(x,y) = h$. Hence, $\lim_{h\to -2/3} \frac{I_1(h)}{I_0(h)} = -1$, and we may define the function

$$P(h) = \begin{cases} \dfrac{I_1(h)}{I_0(h)}, & h \in (-2/3, 2/3]; \\ -1, & h = -2/3. \end{cases} \tag{3.6}$$

Then (3.3) can be written as

$$I(h) = I_0(h)(\alpha + P(h)). \tag{3.7}$$

We will prove that $P'(h) > 0$ for $h \in (-2/3, 2/3)$, implying at most one zero of $I(h)$.

Lemma 3.1. *$I_0(h)$ and $I_1(h)$ satisfy the Picard–Fuchs equation*

$$(9h^2 - 4)\frac{d}{dh}\begin{pmatrix} I_0 \\ I_1 \end{pmatrix} = \begin{pmatrix} \frac{15h}{2} & 7 \\ 5 & \frac{21h}{2} \end{pmatrix}\begin{pmatrix} I_0 \\ I_1 \end{pmatrix}. \tag{3.8}$$

Proof. By using (3.5) and the fact that $y^2 = 2h + \frac{2}{3}x^3 - 2x$ along γ_h, we find

$$I_j(h) = \int_{\gamma_h} \frac{x^j y^2}{y}\,dx = 2hI_j'(h) - 2I_{j+1}'(h) + \frac{2}{3}I_{j+3}'(h). \tag{3.9}$$

On the other hand, using the formula of integration by parts and the fact that $ydy = (-1 + x^2)dx$ along γ_h, we get

$$I_j(h) = \frac{1}{j+1}(I_{j+1}'(h) - I_{j+3}'(h)). \tag{3.10}$$

Removing $I_{j+3}'(h)$ from (3.9) and (3.10), we obtain

$$(2j+5)I_j(h) = 6hI_j'(h) - 4I_{j+1}'(h).$$

Taking $j = 0, 1$, we have

$$\begin{aligned} 5I_0(h) &= 6hI_0'(h) - 4I_1'(h), \\ 7I_1(h) &= 6hI_1'(h) - 4I_2'(h). \end{aligned} \tag{3.11}$$

Note that along γ_h holds $y^2 dy = (-1 + x^2)ydx$, which implies $I_2(h) \equiv I_0(h)$. Using $I_0'(h)$ instead of $I_2'(h)$ in (3.11), and solving $I_0'(h)$ and $I_1'(h)$ from this equation, one obtains (3.8). $\square$

Theorem 3.2. *The function $P(h)$ defined in* (3.6) *is strictly increasing for $h \in (-2/3, 2/3)$.*

Proof. By definition (3.6) we have

$$P'(h) = \frac{I_1'(h)}{I_0(h)} - \frac{I_0'(h)}{I_0(h)} P(h).$$

Substituting (3.8) into the above equality, we obtain the Riccati equation

$$(9h^2 - 4)P' = -7P^2 + 3hP + 5,$$

which is equivalent to the system

$$\begin{aligned} \tfrac{dP}{dt} &= -7P^2 + 3hP + 5, \\ \tfrac{dh}{dt} &= 9h^2 - 4. \end{aligned} \tag{3.12}$$

This system has the invariant lines $\{h = \pm 2/3\}$, and all four singularities of the system are located on these two lines: a saddle at $S_-(-2/3, -1)$ and a node at $N_-(2/3, -5/7)$ on the lower half plane while a saddle at $S_+(2/3, 1)$ and a node at $N_+(-2/3, 5/7)$ on the upper half plane. Definition (3.6) shows that the graph of the function $P = P(h)$ is the stable manifold of the saddle S_-, and it must go to

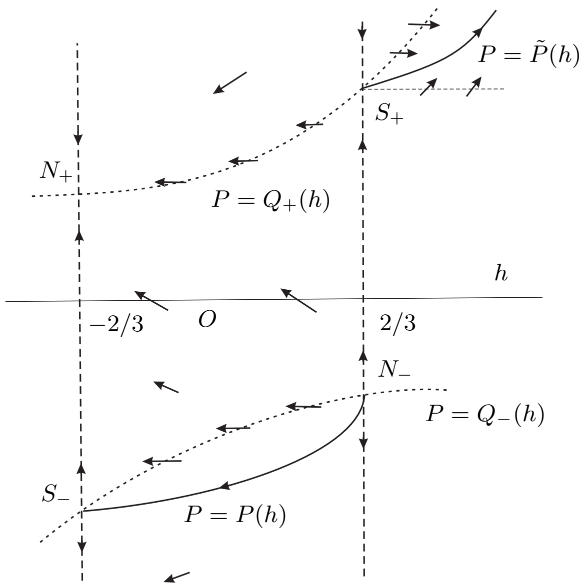

Figure 5. The behavior of the vector field (3.12) and of the function $P(h)$.

the unstable node N_- as h increases, since the vector field is upwards on the line $\{(h, P) : P = 0\}$, see Figure 5 (we need only the lower part of this figure in this proof, but we need the upper part in a proof of a lemma in section 3.3). On the other hand, from the first equation of (3.12) one finds that the horizontal isocline is given by the hyperbola with two branches $P = Q_{\pm}(h)$, where $Q_{\pm}(h) = (3h \pm \sqrt{9h^2 + 140})/14$. The two branches divide the strip $\{(h, P) : -2/3 < h < 2/3\}$ into three regions, and the vector field (3.12) is downwards on the top and lower regions, and upwards in the middle region. A direct calculation shows that the slope of the curve $P = P(h)$ at the point S_- is $1/8$ while the slope of $P = Q_-(h)$ at the same point is $1/4$. Hence the curve $P = P(h)$ is located below $P = Q_-(h)$ near this point, therefore it remains below $P = Q_-(h)$ for all $h \in (-2/3, 2/3)$, see Figure 5. This implies $P'(h) > 0$, since $dP/dt < 0$ below the branch $P = Q_-(h)$ and $dh/dt < 0$ for all $h \in (-2/3, 2/3)$. □

Remark 3.3. Since equation (3.12) has only regular singular points at $h = \pm 2/3$, it is of *Fuchsian type* (see a detailed proof in Lemma 3.8 of section 3.3). A Fuchsian equation is said to be of *Picard–Fuchs type*, provided that it possesses a fundamental set of solutions which are Abelian integrals, see [68] for example.

3.2 A Direct Method

To prove the monotonicity of the function $P = P(h)$, defined as a ratio of two Abelian integrals, one may hope to find some direct ways. For example to construct an auxiliary function defined directly from the Hamiltonian and the integrand functions, some property of the auxiliary function will give the monotonicity of the ratio of the two Abelian integrals. The following result from [101] shows this method under certain restriction on the form of the Hamiltonian.

We first consider the Hamiltonians with the form

$$H(x, y) = \Phi(x) + \Psi(y), \tag{3.13}$$

where $\Phi \in C^2[\alpha, A]$, $\Psi \in C^2[\beta, B]$. Denote by $\{\gamma_h\}$ the continuous family of ovals contained in the level curves $\{(x, y) : H(x, y) = h, h_1 < h < h_2\}$. Assume that there exist an $a \in (\alpha, A)$ and a $b \in (\beta, B)$, such that the following hypothesis is satisfied:

(H_1) (i) $\Phi'(x)(x - a) > 0$ (or < 0) for $x \in (\alpha, A)\backslash\{a\}$,
(ii) $\Psi'(y)(y - b) > 0$ (or < 0) for $y \in (\beta, B)\backslash\{b\}$.

This hypothesis implies that the point (a, b) is the center of X_H. Suppose that for each $h \in (h_1, h_2)$, γ_h cuts the line $\{y = b\}$ at the points $(\alpha(h), b)$ and $(A(h), b)$ and cuts the line $\{x = a\}$ at the points $(a, \beta(h))$ and $(a, B(h))$ respectively, where $\alpha \le \alpha(h) \le a \le A(h) \le A$, $\beta \le \beta(h) \le b \le B(h) \le B$; then for each $x \in (\alpha(h), a)$, there exists a one-to-one mapping $x \mapsto \tilde{x} \in (a, A(h))$, such that $\Phi(x) = \Phi(\tilde{x})$; and for each $y \in (\beta(h), b)$, there exists a one-to-one mapping $y \mapsto \tilde{y} \in (b, B(h))$, such that $\Psi(y) = \Psi(\tilde{y})$.

Now we consider the ratio of the two Abelian integrals

$$F(h) = \frac{I_2(h)}{I_1(h)}, \quad I_k(h) = \oint_{\gamma_h} f_k(x)g(y)dx,$$

where $f_k(x) \in C^1(\alpha, A)$, $k = 1, 2$, $g(y) \in C^2(\beta, B)$.

The hypothesis (H_1) implies that $\Phi(x)\Phi'(\tilde{x}) < 0$ and $\Psi(y)\Psi'(\tilde{y}) < 0$ for $x \in (\alpha(h), a)$ and $y \in (\beta(h), b)$. We define two auxiliary functions:

$$\xi(x) = \frac{f_2(x)\Phi'(\tilde{x}) - f_2(\tilde{x})\Phi'(x)}{f_1(x)\Phi'(\tilde{x}) - f_1(\tilde{x})\Phi'(x)}, \quad \eta(y) = \frac{(g(\tilde{y}) - g(y))\Psi'(\tilde{y})\Psi'(y)}{g'(\tilde{y})\Psi'(y) - g'(y)\Psi'(\tilde{y})},$$

where $\tilde{x} = \tilde{x}(x)$ and $\tilde{y} = \tilde{y}(y)$ are defined above.

In order to guarantee that the denominators of $\xi(x)$ and $\eta(y)$ are different from zero, we need two more hypotheses:

(H_2) $f_1(x)f_1(\tilde{x}) > 0$ for $x \in (\alpha, a)$;

(H_3) $g'(y)g'(\tilde{y}) > 0$ for $y \in (\beta, b)$.

Remark that the hypothesis (H_2) implies $I_1(h) \neq 0$. If $f_2(x)f_2(\tilde{x}) > 0$ for $x \in (\alpha, a)$, we may use the ratio $I_1(h)/I_2(h)$ instead of $I_2(h)/I_1(h)$.

Theorem 3.4 ([101]). *Assume that $H(x, y)$ has the form* (3.13)*, and the hypotheses (H_1), (H_2) and (H_3) are satisfied, then $\xi'(x)\eta'(y) > 0$ (resp. < 0) for $x \in (\alpha, a)$ and $y \in (\beta, b)$ implies $F'(h) > 0$ (resp. < 0) for $h \in (h_1, h_2)$.*

Proof. We just give a main idea of the proof. Consider

$$\begin{aligned}
&I_2'(h)I_1(h) - I_1'(h)I_2(h) \\
&= \int_{\alpha(h)}^{A(h)} f_2(x)\left(\frac{g'(\tilde{y}(x))}{\Psi'(\tilde{y}(x))} - \frac{g'(y(x))}{\Psi'(y(x))}\right)dx \int_{\alpha(h)}^{A(h)} f_1(x)(g(\tilde{y}(x)) - g(y(x)))dx \\
&\quad - \int_{\alpha(h)}^{A(h)} f_1(x)\left(\frac{g'(\tilde{y}(x))}{\Psi'(\tilde{y}(x))} - \frac{g'(y(x))}{\Psi'(y(x))}\right)dx \int_{\alpha(h)}^{A(h)} f_2(x)(g(\tilde{y}(x)) - g(y(x)))dx \\
&= \frac{1}{2}\int_{\alpha(h)}^{A(h)}\int_{\alpha(h)}^{A(h)} [G(x_1, x_2) + G(x_2, x_1)]dx_1dx_2,
\end{aligned}$$

where $y = y(x)$ and $y = \tilde{y}(x)$ are the two branches of γ_h, and

$$\begin{aligned}
G(x_1, x_2) &= (g(\tilde{y}(x_2) - g(y(x_2)))\left(\frac{g'(\tilde{y}(x_1))}{\Psi'(\tilde{y}(x_1))} - \frac{g'(y(x_1))}{\Psi'(y(x_1))}\right) \\
&\quad \cdot(f_2(x_1)f_1(x_2) - f_2(x_2)f_1(x_1)).
\end{aligned}$$

By using the definition of $\tilde{x}(x)$ and $\tilde{y}(y)$, and transforming the integration limits of the double integral from $[\alpha(h), A(h)] \times [\alpha(h), A(h)]$ to $[\alpha(h), a] \times [\alpha(h), a]$, we can get the result. $\square$

Now let us use this theorem for the same problem discussed in the last section. Recall that we need to prove the monotonicity of the ratio of the two Abelian integrals $P(h) = I_2(h)/I_1(h)$, see (3.4). The Hamiltonian is given in (3.1). Hence in this case we have $\Phi(x) = x - x^3/3$ and $\Psi(y) = y^2/2$; $(a, b) = (-1, 0)$; $f_1(x) \equiv 1$ and $f_2(x) = x$; $g(y) = y$. Hence all the conditions in $(H_1) - (H_3)$ are satisfied. And it is obvious that $\tilde{y}(y) = -y$, which implies $\eta(y) = y^2$ and $\eta'(y) = 2y < 0$, since $y < 0 < \tilde{y}$. On the other hand, it is easy to compute

$$\xi(x) = \frac{(1 + x\tilde{x})}{x + \tilde{x}}.$$

Note that (H_1) implies $d\tilde{x}/dx = \Phi'(x)/\Phi'(\tilde{x}) < 0$, hence

$$\xi'(x) = \frac{1}{(x + \tilde{x})^2}\left((\tilde{x}^2 - 1) + (x^2 - 1)\frac{d\tilde{x}}{dx}\right) < 0,$$

since $x < -1 < \tilde{x} < 1$. By Theorem 3.4, we obtain $P'(h) > 0$.

Next, we extend the above result to the Hamiltonian of the form

$$H(x, y) = \phi(x)y^2 + \Phi(x), \tag{3.14}$$

where $\phi(x), \Phi(x) \in C^1$, and $\phi(x)$ has a fixed sign. Without loss of generality, we may assume $\phi(x) > 0$. Consider the ratio of the Abelian integrals

$$K(h) = \frac{\int_{\gamma_h} f_2(x)ydx}{\int_{\gamma_h} f_1(x)ydx},$$

where γ_h is the same as before, and $f_1(x)$ and $f_2(x)$ are continuous.

If $\Phi(x)$ satisfies the hypothesis (H_1)(i) and $f_1(x)$ satisfies (H_2), then we can define $\tilde{x} = \tilde{x}(x)$ by $\Phi(x) = \Phi(\tilde{x})$ for $x < a < \tilde{x}$ as before, and define

$$\zeta(x) = \frac{f_2(x)\sqrt{\phi(\tilde{x})}\Phi'(\tilde{x}) - f_2(\tilde{x})\sqrt{\phi(x)}\Phi'(x)}{f_1(x)\sqrt{\phi(\tilde{x})}\Phi'(\tilde{x}) - f_1(\tilde{x})\sqrt{\phi(x)}\Phi'(x)},$$

where $\tilde{x} = \tilde{x}(x)$ for $\alpha < x < a$.

Theorem 3.5 ([101])**.** *Suppose that $H(x, y)$ has the form* (3.14)*, and the hypotheses* (H_1)(i) *and* (H_2) *are satisfied, then the increasing (resp. decreasing) of $\zeta(x)$ for $\alpha < x < a$ implies the decreasing (resp. increasing) of $K(h)$ for $h_1 < h < h_2$.*

Remark 3.6. The monotonicity of the two Abelian integrals can give the uniqueness of limit cycles in a codimension 2 bifurcation problem, as shown above. It is also often used in higher codimension problems, for example see [40, 41, 42, 43, 82, 15] etc, and we will use it again in Chapter 4.

3.3 The Method Based on the Argument Principle

In this section we introduce a method to study the number of zeros of Abelian integrals which use the Argument Principle. G.S. Petrov used this method to study the perturbations of elliptic Hamiltonian of degree 3 and degree 4 in a series of papers [130]–[134], hence in some literature it is called Petrov's Method. We will use this method to study the same Hamiltonian (3.1), but the Abelian integral $I(h)$ is obtained by polynomial perturbations of arbitrary degree n. A part of the proof below is from [183], we will explain it in a remark at the end of this section.

The main result in this section is the following theorem.

Theorem 3.7 ([131, 132])**.** *Any nontrivial $I(h)$, the Abelian integral of the polynomial 1-form of degree at most n over the oval* (3.2)*, has at most $n-1$ zeros for $h \in (-2/3, 2/3)$.*

We will prove that $I(h)$ can be extended to the following domain D as a single-valued analytic function.

$$D = \mathbb{C}\backslash\{h \in \mathbb{R}, h \geq 2/3\}.$$

We still use $I(h)$ for the extended complex function in D. In order to apply the Argument Principle to $I(h)$, we define $G = G_{R,\varepsilon} \subset D$ (a simply connected region) with $\partial G = C_{R,\varepsilon}$ a simple closed curve,

$$C_{R,\varepsilon} = \{C_R\} \cup \{C_\varepsilon\} \cup \{L_\pm\},$$

where $C_R = \{h \in \mathbb{C}, |h| = R \gg 1\}$, $C_\varepsilon = \{h \in \mathbb{C}, |h - 2/3| = \varepsilon \ll 1\}$, and $L_\pm$ are

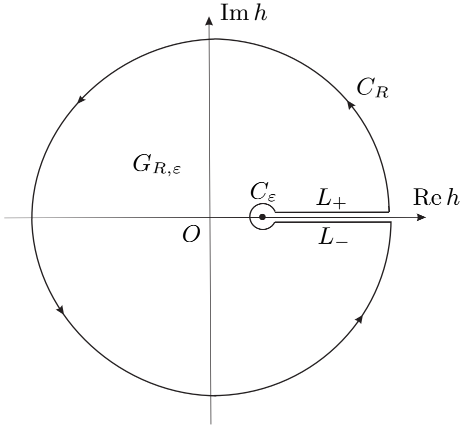

Figure 6. The domain $G_{R,\varepsilon}$ and its boundary.

the upper and lower banks of the cut $\{2/3 \leq h \leq R\}$, see Figure 6.

We first state some lemmas, then prove Theorem 3.7 by using these lemmas, and finally give proofs of the lemmas.

Lemma 3.8. *$I(h)$ can be extended to D as a single-valued analytic function. Moreover, $I_0(h) \sim h^{5/6}$, $I_1(h) \sim h^{7/6}$ for h in a neighborhood of the infinity.*

Lemma 3.9 ([183])**.** *$Im\, I_0(h) \neq 0$, $Im\, I_1(h) \neq 0$ for $h \in L_+ \cup L_-$.*

Lemma 3.10 ([183])**.** *$I_0(h) \neq 0$ for $h \in G \backslash \{-2/3\}$.*

Lemma 3.11. *$Im\, (I_1(h)/I_0(h)) \neq 0$ for $h \in L_+ \cup L_-$.*

Proof of Theorem 3.7. By Lemma 1.10 (in subsection 1.2.2) the Abelian integral $I(h)$ can be expressed in the form

$$I(h) = Q_0(h)I_0(h) + Q_1(h)I_1(h), \tag{3.15}$$

where Q_0 and Q_1 are polynomials, $\deg Q_0 \leq [\frac{n-1}{2}] = n_0$, $\deg Q_1 \leq [\frac{n}{2}] - 1 = n_1$, and $I_j(h)$ are defined in (3.4). Note that $n_0 + n_1 = n - 2$. Here we may suppose that $Q_0(h)$ and $Q_1(h)$ have no common factors.

By Lemma 3.10, $h = -2/3$ is the unique zero of $I_0(h)$, and the limit of $I_1(h)/I_0(h)$ is -1 as $h \to -2/3$ (see (3.6)), hence, instead of $I(h)$ in the above form, we consider the number of zeros of the function

$$F(h) = Q_0 + (I_1/I_0)\, Q_1.$$

Note that the trivial zero of $I(h)$ at $h = -2/3$ may be removed for $F(h)$. Now we use the Argument Principle for $F(h)$ to $G_{R,\varepsilon}$, with R and $1/\varepsilon$ positive and big enough. We will prove that the rotation number of F when h turns around the boundary of $G_{R,\varepsilon}$ is at most $n - 1$.

By virtue of Lemma 3.8, when the variable h moves counterclockwise along C_R the function $F(h)$ makes at most $\max(n_0, n_1 + 1/3)$ rotations around zero. By Lemma 3.11 the number of zeros of $\mathrm{Im}(F)$ for $h \in L_+ \cup L_-$ is at most $2n_1$. Since each complete turn of $F(h)$ forces at least two zeros of $\mathrm{Im}(F(h))$ we get that the number of complete turns on these two banks is at most $n_1 + 1$ (we add less than one half turn on each bank). Finally, when h moves along C_ε clockwise, the number of complete turns of F goes to 0, when $\varepsilon \to 0$, since I_1/I_0 tends to a constant when $h \to 2/3$, and $Q_0^2(2/3) + Q_1^2(2/3) \neq 0$ ($Q_0(h)$ and $Q_1(h)$ have no common factors). Summing up the above estimate, we get that the total number of rotation is at most $\max(n_0 + n_1 + 1,\, 2n_1 + 4/3) = n - 1$. Note that the rotation number must be an integer. □

Proof of Lemma 3.8. We know that $I_0(h)$ and $I_1(h)$ are real analytic on $h \in (-2/3, 2/3)$. To extend them to a complex domain, we first deduce some differential equations satisfied by them. Rewrite (3.11) as

$$\begin{pmatrix} I_0 \\ I_1 \end{pmatrix} = \begin{pmatrix} \frac{6h}{5} & -\frac{4}{5} \\ -\frac{4}{7} & \frac{6h}{7} \end{pmatrix} \begin{pmatrix} I_0 \\ I_1 \end{pmatrix}'. \tag{3.16}$$

Taking derivatives with respect to h on both sides, we get

$$\begin{pmatrix} -\frac{1}{5} & 0 \\ 0 & \frac{1}{7} \end{pmatrix} \begin{pmatrix} I_0 \\ I_1 \end{pmatrix}' = \begin{pmatrix} \frac{6h}{5} & -\frac{4}{5} \\ -\frac{4}{7} & \frac{6h}{7} \end{pmatrix} \begin{pmatrix} I_0 \\ I_1 \end{pmatrix}''. \tag{3.17}$$

Substituting (3.17) into (3.16) we obtain

$$I_0 = \frac{4}{5}(4-9h^2)\,I_0'', \quad I_1 = \frac{4}{7}(9h^2-4)\,I_1''. \tag{3.18}$$

It is clear now that if we try to extend h to the complex plane $\mathbb{C}$, then the only singularities of these two equations are located at $h = \pm 2/3$, and the equations are of Fuchsian type. It is easy to check that near the point $h = -2/3$ both equations have solutions in power series $\sum_{k=0}^{\infty} a_k(h+2/3)^k$, which are convergent and $a_k \in \mathbb{C}$. This means that the singularity $h = -2/3$ can be removed, and $I_0(h)$ and $I_1(h)$ are holomorphic at this point (we omit the detailed computation). This fact can also be obtained directly from Lemma 20 of [12]. Near the singularity $h = 2/3$, corresponding to the saddle loop Γ of X_H, if we restrict to $h \in (-2/3, 2/3)$ then the solutions of the above equations have the form (see [140, 121]):

$$\begin{aligned} I_0(h) &= a_0 + b_0\,(2/3-h)\,\ln(2/3-h) + o((2/3-h)\,\ln(2/3-h)),\\ I_1(h) &= a_1 + b_1\,(2/3-h)\,\ln(2/3-h) + o((2/3-h)\,\ln(2/3-h)), \end{aligned} \tag{3.19}$$

where $a_0 = \oint_\Gamma y dx \neq 0$ and $a_1 = \oint_\Gamma xy dx \neq 0$. A computation shows that in our case $b_0 = b_1 = 1$ and $a_0 b_1 - a_1 b_0 \neq 0$. It is clear that $I_0(h)$ and $I_1(h)$ can not be extended to $\mathbb{C}$ as a single-valued analytic function, unless we cut a ray, starting at the point $h = 2/3$, from $\mathbb{C}$. The first part of the statement of Lemma 3.8 is proved. Now we still use the same $I_0(h)$ and $I_1(h)$ for the extended functions on D.

To study the behavior of these two functions near infinity, we let $t = 1/h$ and $K_j(t) = I_j(h)$, then $I_j'' = t^4\ddot{K}_j + 2t^3\dot{K}_j$, where $' = d/dh$, $\cdot = d/dt$, and the first equation in (3.18) becomes

$$t^2\,\ddot{K} + 2t\,\dot{K} + (5/36 + O(t^2))\,K = 0.$$

The index ρ satisfies the equation $\rho(\rho-1)+2\rho+5/36 = 0$, which gives two indices $-1/6$ and $-5/6$. Hence $I_0(h) \sim h^{1/6}$ or $h^{5/6}$ as $h \to \infty$. Similarly, we find from the second equation of (3.18) that $I_1(h) \sim h^{5/6}$ or $h^{7/6}$ as $h \to \infty$. Comparing the coefficients of the leading terms on both sides of (3.16) as $h \to \infty$, we find that the only possibility is $I_0(h) \sim h^{5/6}$ and $I_1(h) \sim h^{7/6}$ as $h \to \infty$. □

Proof of Lemma 3.9. When $h \in L_+ \cup L_-$, equation (3.8) is real analytic, hence ($\mathrm{Im}I_1$, $\mathrm{Im}I_0$) is a solution of it, and $\tilde{P}$ $=\mathrm{Im}I_1/\mathrm{Im}I_0$ satisfies the same equation (3.12). By (3.19) we find $\mathrm{Im}I_1/\mathrm{Im}I_0 \to 1$ as $h \in L_+ \cup L_-$ and $h \to 2/3+0$, which implies that the graph of $\tilde{P}$ $=\mathrm{Im}I_1/\mathrm{Im}I_0$ is the unstable manifold of the saddle point $S_+(2/3, 1)$ of system (3.12), see Figure 5 in section 3.1. By the same analysis as in the proof of Theorem 3.2 we conclude that when $h > 2/3$ this manifold must stay between the horizontal isocline $P = Q_+(h)$ (the upper branch of the hyperbola) and the ray $\{(h, P) : P = 1, h \geq 2/3\}$, along which the vector field (3.12) is upwards. Hence we obtain that $1 < \mathrm{Im}I_1/\mathrm{Im}I_0 < \infty$ for $h \in L_+ \cup L_-$. □

Proof of Lemma 3.10. We use the same procedure as in the proof of Theorem 3.7 to consider the rotation number of $I_0(h)$ when h turns around the boundary of $G_{R,\varepsilon}$. By using Lemmas 3.8 and 3.9, one can easily obtain that the total rotation number is not bigger than $5/6+1$. Since this number is an integer, $I_0(h)$ has at most one zero in G. As we already have $I_0(-2/3)=0$, hence $I_0(h)$ has no other zeros in G. □

Proof of Lemma 3.11. We suppose the contrary: there exists an $h^* \in L_+ \cup L_-$ such that $\mathrm{Im}(I_1(h^*)/I_0(h^*))=0$, which implies

$$\mathrm{Re}(I_0(h^*))\,\mathrm{Im}(I_1(h^*)) - \mathrm{Re}(I_1(h^*))\,\mathrm{Im}(I_0(h^*)) = 0.$$

This means the two vectors $(\mathrm{Im}(I_0(h^*)), \mathrm{Im}(I_1(h^*)))$ and $(\mathrm{Re}(I_0(h^*)), \mathrm{Re}(I_1(h^*)))$ would be proportional. Note that for $h \in L_+ \cup L_-$, these two vector functions are solutions of the real linear differential equation (3.16), hence, being proportional at one point, they are proportional on the entire banks of the cut, i.e., $I_1(h)/I_0(h)$ is real for $h \in L_+ \cup L_-$. From (3.19) we find that for $h \in L_+ \cup L_-$ and h near $2/3$,

$$\mathrm{Im}\left(I_1(h)/I_0(h)\right) \sim c(a_0b_1 - a_1b_0)(2/3-h),$$

where c is a non-zero real number. This gives a contradiction. □

Remark 3.12. By using the result in [140], P. Mardesic [121] generalized the conclusion about the number of zeros of an Abelian integral in this section to the conclusion about the number of limit cycles. That is the maximal number of limit cycles, including the ones bifurcated from the saddle loop, is also $n-1$, if the polynomial perturbation is of degree at most n.

Remark 3.13. We follow the basic idea of Petrov's proof, but with some changes. For example, there is a claim without proof in Lemma 6 of [132] that the function $\mathrm{Im}I_1(h)$ does not have zeros on the open cut, but this fact is not obvious. So we use some proofs from [183].

Note that Theorem 3.7 gives the maximal number of zeros of the *first* variation of the displacement function (the first order Melnikov function $M_1(h)$). As discussed in Section 2.2 (Theorem 2.10), the displacement function has the form:

$$d(h,\varepsilon) = \varepsilon M_1(h) + \varepsilon^2 M_2(h) + \cdots . \tag{3.20}$$

The following result gives an estimate of arbitrary order of the Melnikov function for the same Hamiltonian (3.1) under polynomial perturbations of arbitrary degree n.

Theorem 3.14. *The following statements hold.*

(A) *If $M_1(h) \equiv 0$, then $M_2(h)$ has at most $2(n-1)$ zeros for n even and at most $2n-3$ zeros for n odd, counting the multiplicities* ([93]).

(B) *If $M_k(h)$ is the first Melnikov function in* (3.20) *which does not vanish identically, then $M_k(h)$ has no more than $k(n-1)$ zeros, counting the multiplicities. Moreover, if $k \geq 3$ (or $k \geq 2$ and n is odd) then $M_k(h)$ has no more than $k(n-1)-1$ zeros, counting the multiplicities* ([81]).

Remark 3.15. In [93] B. Li and Z. Zhang gave results not only for the upper bound of the number of zeros of the second order Melnikov function, but also extend them to the maximal number of limit cycles by using the technique from [121] (if $M_2(h)$ does not vanish identically). In [81] I.D. Iliev also mentioned this extension of the result for $k \geq 3$.

In [67] L. Gavrilov and I.D. Iliev studied two-dimensional Fuchsian systems in a general setting: under certain conditions the function space of the Abelian integrals obey the Chebyshev property, and there is no need to use the Argument Principle each time. We briefly introduce their result below.

As in (3.15) we consider functions

$$I(h) = p_1(h)I_1(h) + p_2(h)I_2(h), \quad h \in (a, b), \tag{3.21}$$

where $p_1(h)$ and $p_2(h)$ are polynomials, $I_1(h)$ and $I_2(h)$ are complete Abelian integrals over $\gamma_h \subset H^{-1}(h)$, defined as before, and the vector function $\mathbf{I}(h) = (I_1(h), I_2(h))^{\mathsf{T}}$ satisfies a two-dimensional first-order Fuchsian system

$$\mathbf{I}(h) = \mathbf{A}(h)\,\mathbf{I}'(h), \quad ' = d/dh, \tag{3.22}$$

with a first-degree polynomial matrix $\mathbf{A}(h)$, as in (3.16).

The main assumptions on (3.22) are the following:

(H1) The matrix $\mathbf{A}'$ is constant having real distinct eigenvalues.

(H2) The equation $\det\mathbf{A}(h) = 0$ has real distinct roots h_0, h_1 and the identity trace $\mathbf{A}(h) \equiv (\det\mathbf{A}(h))'$ holds.

(H3) The function $\mathbf{I}(h)$ is analytic in a neighborhood of h_0.

The conditions that $\mathbf{A}'$ is a constant matrix and $\det\mathbf{A}(h) = 0$ has distinct roots imply that the singular points of the system

$$\mathbf{I}'(h) = \mathbf{A}^{-1}(h)\,\mathbf{I}(h)$$

(including ∞) are regular, i.e., it is of Fuchs type. The condition trace $\mathbf{A}(h) \equiv (\det\mathbf{A}(h))'$ implies that the characteristic exponents of (3.22) at h_0 and h_1 are $\{0, 1\}$. In the formulation here it is assumed for definiteness that $h_0 < h_1$. A similar result holds if $h_0 > h_1$. Clearly if $h_0 < h_1$ and the function $\mathbf{I}(h)$ is analytic in a neighborhood of $h = h_0$, then it also possesses an analytic continuation in the complex domain $\mathbb{C}\backslash[h_1, \infty)$, as we proved for the special case of Lemma 3.8.

We reformulate the *Chebyshev property* as follows.

Definition 3.16 ([67]). The real vector space of functions V is said to be *Chebyshev* in the complex domain $G \subset \mathbb{C}$ provided that every function $I \in V\backslash\{0\}$ has at most $\dim V - 1$ zeros in G. V is said to be *Chebyshev with accuracy* k *in* G if any function $I \in V\backslash\{0\}$ has at most $k + \dim V - 1$ zeros in G.

Definition 3.17 ([67]). Let $I(h), h \in \mathbb{C}$ be a function, locally analytic in a neighborhood of ∞, and $s \in \mathbb{R}$. Write $I(h) \lesssim h^s$, provided that for every sector S centered at ∞ there exists a non-zero constant C_S such that $|I(h)| \leq C_s|h|^s$ for all sufficiently big $|h|$, $h \in S$.

For systems (3.22) satisfying (H1) and (H2), the characteristic exponents at infinity are $-\lambda$ and $-\mu$ where $\lambda' = 1/\lambda$ and $\mu' = 1/\mu$ are the eigenvalues of the constant matrix $\mathbf{A}'$. According to (H2), $\lambda + \mu = 2$. Denote $\lambda^* = 2$ if λ is integer and $\lambda^* = \max\left(|\lambda - 1|, 1 - |\lambda - 1|\right)$ otherwise.

Take $s \geq \lambda^*$ and consider the real vector space of functions

$$V_s = \{I(h) = P(h)I_1(h) + Q(h)I_2(h) : P, Q \in \mathbb{R}[h], I(h) \lesssim h^s\},$$

where $\mathbf{I}(h) = (I_1(h), I_2(h))^{\mathsf{T}}$ is a non-trivial solution of (3.22), holomorphic in a neighborhood of $h = h_0$. As $\lambda, \mu \notin \{0, 1, 2\}$ the vector function $\mathbf{I}(h)$ is uniquely determined, up to multiplication by a constant, and $I_1(h_0) = I_2(h_0) = 0$. Clearly, V_s is invariant under linear transformations in (3.22) and affine changes of the argument h. The restriction $s \geq \lambda^*$ is taken to guarantee that V_s is not empty. Recall that $h_0 < h_1$ are the roots of $\det(\mathbf{A}(h)) = 0$.

Theorem 3.18 ([67]). *Assume that conditions* (H1)–(H3) *hold. If* $\lambda \notin \mathbb{Z}$, *then* V_s *is a Chebyshev vector space with accuracy* $1 + [\lambda^*]$ *in the complex domain* $G = \mathbb{C}\backslash[h_1, \infty)$. *If* $\lambda \in \mathbb{Z}$, *then* V_s *coincides with the space of real polynomials of degree at most* $[s]$ *which vanish at* h_0 *and* h_1.

As an application, we use this theorem for the case discussed in the first part of the present section. From (3.16) we have that the matrix in (3.22) is

$$\mathbf{A}(h) = \begin{pmatrix} \frac{6h}{5} & -\frac{4}{5} \\ -\frac{4}{7} & \frac{6h}{7} \end{pmatrix}.$$

It is easy to check that all conditions (H1)–(H3) hold, and $\lambda = 5/6$. Hence, by Theorem 3.18, an upper bound of the number of zeros of $I(h) = Q_0(h)I_0(h) + Q_1(h)I_1(h)$ is $(n-1)+1 = n$, since the dimension of the function space in this case is n (see Lemma 1.10 and the explanation below it). If we remove the trivial zero at h_0 (in the proof of Theorem 3.7 the zero at $h_0 = -2/3$ is removed), then the upper bound is $n - 1$.

3.4 The Averaging Method

In this section we briefly introduce an application of the averaging method to the study of the weak Hilbert's 16th problem in [112] and [10]. Although this study is equivalent to the study by using Abelian integrals, in some cases one of them is more convenient than the other. We first state some general theorems on the averaging method, then use this method to study the quadratic perturbations of a quadratic reversible and non-Hamiltonian system.

The setting of averaging theory is in general in an arbitrary dimensional space. Since we will use it in an autonomous planar system, we will state it in one-dimensional form. The following theorem gives a first order averaging; for a proof, see [147].

Theorem 3.19. *Consider the two initial value problems*

$$\dot{x} = \varepsilon f(t,x) + \varepsilon^2 h(t,x,\varepsilon), \quad x(0) = x_0, \tag{3.23}$$

and

$$\dot{y} = \varepsilon f^0(y), \quad y(0) = x_0, \tag{3.24}$$

where $x, y, x_0 \in D$, D *is an open subset of* $\mathbb{R}$, $t \in [0,\infty), \varepsilon \in (0,\varepsilon_0]$, f *and* h *are periodic of period* T *in* t, *and*

$$f^0(y) = \frac{1}{T}\int_0^T f(t,y)dt. \tag{3.25}$$

Suppose that

(i) $f, \partial f/\partial x, \partial^2 f/\partial x^2$ *and* $\partial h/\partial x$ *are defined, continuous and bounded by a constant independent of* ε *in* $[0,\infty) \times D$ *and* $\varepsilon \in (0,\varepsilon_0]$;

(ii) T *is independent of* ε;

(iii) $y(t)$ *belongs to* D *on the time-scale* $1/\varepsilon$.

Then the following statements hold.

(a) *On the time-scale* $1/\varepsilon$ *we have that*

$$x(t) - y(t) = O(\varepsilon), \quad \text{as } \varepsilon \to 0.$$

(b) *If* p *is an equilibrium of the averaging system* (3.24) *such that*

$$\partial f^0/\partial y|_{y=p} \neq 0, \tag{3.26}$$

then there exists a T*-periodic solution* $\phi(t,\varepsilon)$ *of equation* (3.23) *which is close to* p *such that* $\phi(t,\varepsilon) \to p$ *as* $\varepsilon \to 0$.

(c) *If* (3.26) *is negative, then the corresponding periodic solution* $\phi(t,\varepsilon)$ *in the space* (t,x) *is asymptotically stable for* ε *sufficient small. If* (3.26) *is positive, then it is unstable.*

The next theorem provides a second order averaging; see [147] and [112] for a proof.

Theorem 3.20. *Consider the two initial value problems*

$$\dot{x} = \varepsilon f(t,x) + \varepsilon^2 g(t,x) + \varepsilon^3 h(t,x,\varepsilon), \quad x(0) = x_0, \tag{3.27}$$

and

$$\dot{y} = \varepsilon f^0(y) + \varepsilon^2 f^{10}(y) + \varepsilon^2 g^0(y), \quad y(0) = x_0, \tag{3.28}$$

with f, g:$[0,\infty) \times D \to G : [0,\infty) \times D \times (0, \varepsilon_0] \to \mathbb{R}$, D *an open subset of* $\mathbb{R}$, f, g *and* h *periodic of period* T *in* t, *and*

$$f^1(t,x) = \frac{\partial f}{\partial x} y^1(t,x) - \frac{\partial y^1}{\partial x} f^0(x),$$

where

$$y^1(t,x) = \int_0^t (f(s,x) - f^0(x))ds + z(x),$$

with $z(x)$ *a* C^1 *function such that the averaging of* y^1 *is zero. Besides,* f^0, f^{10} *and* g^0 *denote the averaging functions of* f, f^1 *and* g, *respectively, defined as in* (3.25). *Suppose that*

(i) $\partial f / \partial x$ *is Lipschitz in* x *and all these functions are continuous on their domain of definition;*

(ii) $|h(t,x,\varepsilon)|$ *is bounded by a constant uniformly in* $[0, L/\varepsilon) \times D \times (0, \varepsilon_0]$;

(iii) T *is independent of* ε;

(iv) $y(t)$ *belongs to* D *on the time-scale* $1/\varepsilon$.

Then

(a) *On the time-scale* $1/\varepsilon$ *we have that*

$$x(t) = y(t) + \varepsilon y^1(t, y(t)) + O(\varepsilon^2), \quad \text{as } \varepsilon \to 0.$$

If, in addition, $f^0(y) \equiv 0$, *then the following statements hold.*

(b) *If* p *is an equilibrium of the averaging system* (3.28) *such that*

$$\left.\frac{\partial}{\partial y}(f^{10}(y) + g^0(y))\right|_{y=p} \neq 0, \tag{3.29}$$

then there exists a T*-periodic solution* $\phi(t,\varepsilon)$ *of equation* (3.27) *which is close to* p *such that* $\phi(t,\varepsilon) \to p$ *as* $\varepsilon \to 0$.

(c) *If* (3.29) *is negative, then the corresponding periodic solution* $\phi(t,\varepsilon)$ *in the space* (t,x) *is asymptotically stable for* ε *sufficient small. If* (3.29) *is positive, then it is unstable.*

Now we consider a perturbation of a planar integrable system of the form

$$X_\varepsilon : \quad \begin{aligned} \dot{x} &= P(x,y) + \varepsilon\, p(x,y), \\ \dot{y} &= Q(x,y) + \varepsilon\, q(x,y), \end{aligned} \tag{3.30}$$

where $P, Q, p, q \in C^1(\mathbb{R}^2, \mathbb{R})$. We suppose that X_0 has an integrating factor $\mu(x,y)\,(\neq 0)$, a first integral H and a continuous family of ovals $\{\gamma_h\}$:

$$\gamma_h \subset \{(x,y) : H(x,y) = h, h_1 < h < h_2\}. \tag{3.31}$$

To study the number of limit cycles of system (3.30) for sufficiently small ε by using the above theorems, a natural question is how do we transform this system to the form (3.23) or (3.27). The following result gives an answer.

Theorem 3.21 ([10]). *Assume that $xQ(x,y) - yP(x,y) \neq 0$ for all (x,y) in the period annulus formed by the ovals $\{\gamma_h\}$. Then there is a continuous function $\rho : (\sqrt{h_1}, \sqrt{h_2}) \times [0, 2\pi) \to [0, \infty)$ such that*

$$H(\rho(R,\varphi)\cos\varphi,\ \rho(R,\varphi)\sin\varphi) = R^2, \tag{3.32}$$

for all $R \in (\sqrt{h_1}, \sqrt{h_2})$ and all $\varphi \in [0, 2\pi)$, and the differential equation which describes the dependence between the square root of energy, $R = \sqrt{h}$, and the angle φ for system (3.30) *is*

$$\frac{dR}{d\varphi} = \varepsilon\, \frac{\mu\,(x^2+y^2)(Qp - Pq)}{2R(Qx - Py)} \left(1 - \varepsilon\, \frac{qx - py}{Qx - Py}\right) + O(\varepsilon^3), \tag{3.33}$$

where $x = \rho(R,\varphi)\cos\varphi$ and $y = \rho(R,\varphi)\sin\varphi$.

Example ([10]). Consider

$$\begin{aligned} \dot{x} &= -y + x^2 + \varepsilon\, p(x,y), \\ \dot{y} &= x + xy + \varepsilon\, q(x,y), \end{aligned} \tag{3.34}$$

where $p(x,y) = a_1x - a_3x^2 + (2a_2 + a_5)xy + a_6y^2$ and $q(x,y) = a_1y + a_2x^2 + a_4xy - a_2y^2$.

Note that when $\varepsilon = 0$ system (3.34) is reversible and non-Hamiltonian (the center is isochronous), which has the first integral

$$H(x,y) = \frac{x^2+y^2}{(1+y)^2},$$

and the integrating factor is $2(1+y)^{-3}$. We use Theorem 3.21 and taking the transformation $x = \rho\cos\varphi, y = \rho\sin\varphi$, where

$$\rho = \rho(R,\varphi) = \frac{R}{1 - R\cos\varphi}, \quad 0 < R < 1,\ \varphi \in [0, 2\pi),$$

then system (3.34) becomes

$$\frac{dR}{d\varphi} = \varepsilon\,\frac{a_1R + a(\varphi)R^2 + b(\varphi)R^3}{1 - R\sin\varphi} + O(\varepsilon^2),$$

where

$$\begin{aligned} a(\varphi) &= (-2a_1 + 3a_2 + a_5)\sin\varphi + (a_4 + a_6)\cos\varphi \\ &\quad -(4a_2 + a_5)\sin^3\varphi - (a_3 + a_4 + a_6)\cos^3\varphi, \\ b(\varphi) &= a_1 + a_2 - (a_1 + 2a_2)\cos^2\varphi - a_4\sin\varphi\cos\varphi. \end{aligned}$$

By integration one has the averaging function

$$\begin{aligned} f^0(R) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{a_1R + a(\varphi)R^2 + b(\varphi)R^3}{1 - R\sin\varphi}\,d\varphi \\ &= \frac{1}{2(R\sqrt{1-R^2})}[2a_2R^4 + (6a_2 + a_5 - 2a_1)R^2\sqrt{1-R^2} - (10a_2 + 2a_5)R^2 \\ &\quad -(2a_5 + 8a_2)\sqrt{1-R^2} + 8a_2 + 2a_5]. \end{aligned}$$

Note that $R \in (0,1)$. By the substitution $\xi = \sqrt{1-R^2}$ it is not hard to find that $f^0(R)$ has at most two zeros for $R \in (0,1)$. Hence the system (3.34) has at most two limit cycles for ε sufficiently small.

Before closing this chapter, we remark that the methods introduced in this chapter mainly remain in the real domain (except for the use of the Argument Principle). There is also a method based on complexification of the Abelian integrals. The basic construction, see [4] for example, is the following. Consider the Hamiltonian function $H \in \mathbb{C}[x,y]$ as a map $\mathbb{C}^2 \to \mathbb{C}$, and define the Abelian integral as an integral of the complex polynomial 1-form over some cycle on the level set $H^{-1}(h)$ (a 1-dimensional complex manifold). This permits us to construct the monodromy map, and it is possible to apply the Picard–Lefschetz formula and to use the tools of algebraic topology. We list some works below in which this complex method was successfully applied. In [85] Yu. Ilyashenko studied the codimension two Bogdanov–Takens bifurcation (the same problem introduced in this chapter). In [122] P. Mardesic gave an explicit bound for the multiplicity of zeros of Abelian integrals under certain generic assumptions. In [64] L. Gavrilov studied the quadratic perturbation of quadratic Hamiltonian systems (the weak Hilbert's 16th problem for $n = 2$). In [126] D. Novikov and S. Yakovenko gave an explicit upper bound of the polynomial perturbations of the hyperelliptic Hamiltonian under some generic condition. For a more detailed introduction of this complex theory we refer to an early work of Yu. Ilyaskenko [84], the lecture notes [169] and the forthcoming book by Yu. Ilyashenko and S. Yakovenko [90].

Chapter 4

A Unified Proof of the Weak Hilbert's 16th Problem for n=2

4.1 Preliminaries and the Centroid Curve

As we explained in Subsection 1.2.1, any cubic generic Hamiltonian, with at least one period annulus contained in its level curves, can be transformed into the normal form

$$H(x,y) = \frac{1}{2}(x^2+y^2) - \frac{1}{3}x^3 + axy^2 + \frac{1}{3}by^3, \tag{4.1}$$

where a, b are parameters lying in the open region

$$G = \left\{(a,b) : -\frac{1}{2} < a < 1,\ 0 < b < (1-a)\sqrt{1+2a}\right\}. \tag{4.2}$$

Figure 1 (in Subsection 1.2.1) shows all five possible phase portraits of X_H in the generic cases. Here X_H is the Hamiltonian vector field corresponding to H, i.e.,

$$X_H = H_y\frac{\partial}{\partial x} - H_x\frac{\partial}{\partial y}. \tag{4.3}$$

The vector field X_H has a center at the origin in the (x,y)-plane, and the continuous family of ovals, surrounding the center, is

$$\{\gamma_h\} \subset \{(x,y) : H(x,y) = h,\ 0 < h < 1/6\}. \tag{4.4}$$

The oval γ_h shrinks to the center as $h \to 0$, and the oval γ_h terminates at the saddle loop of the saddle point $(1,0)$ when $h \to 1/6$.

We consider any quadratic perturbation of X_H, i.e.,

$$X_\epsilon = X_H + \epsilon Y_\epsilon, \tag{4.5}$$

where

$$Y_\epsilon = f(x,y,\epsilon)\frac{\partial}{\partial x} + g(x,y,\epsilon)\frac{\partial}{\partial y},$$

with f and g polynomials in x and y of degree 2, and their coefficients depend analytically on the parameter ϵ.

In this chapter we will study the Abelian integral

$$I(h) = \oint_{\gamma_h} f(x,y,0)dy - g(x,y,0)dx, \tag{4.6}$$

and prove the following theorem (see subsection 1.2.1 and [15]).

Theorem 4.1. *For any cubic polynomial H with $(a,b) \in G$ and any quadratic polynomials f and g, the least upper bound of the number of zeros of the Abelian integrals* (4.6) *is* 2*.*

Since the orientation of the integral over γ_h is clockwise, and f and g are polynomials in x and y of degree 2, we may rewrite the integral (4.6) into the form

$$I(h) = -\iint_{\mathrm{Int}(\gamma_h)} \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)\bigg|_{\varepsilon=0} dxdy = \iint_{\mathrm{Int}(\gamma_h)} (\alpha + \beta x + \gamma y)\, dx\, dy, \tag{4.7}$$

where α, β and γ are arbitrary constants. Following the notation of [74], we define

$$\begin{aligned} M(h) &= \textstyle\iint_{\mathrm{Int}(\gamma_h)} dx\, dy, \qquad & X(h) &= \textstyle\iint_{\mathrm{Int}(\gamma_h)} x\, dx\, dy, \\ Y(h) &= \textstyle\iint_{\mathrm{Int}(\gamma_h)} y\, dx\, dy, \qquad & K(h) &= \textstyle\iint_{\mathrm{Int}(\gamma_h)} xy\, dx\, dy. \end{aligned}$$

Since $M(h)$ is the area of $\mathrm{Int}(\gamma_h)$ and $M'(h)$ is the period of γ_h, we have that $M(h) > 0$ and $M'(h) > 0$ for $h \in (0, 1/6)$. Hence (4.7) can be written as

$$I(h) = \alpha M(h) + \beta X(h) + \gamma Y(h) = M(h)\left[\alpha + \beta p(h) + \gamma q(h)\right], \tag{4.8}$$

where

$$p(h) = \frac{X(h)}{M(h)}\ , \quad q(h) = \frac{Y(h)}{M(h)}. \tag{4.9}$$

The following results are easily obtained from the definitions of $I(h)$, $p(h)$ and $q(h)$. The second claim of statement (3) was first proved in Theorem 2.4 of [74], see also Lemma 4.1 of [102].

Lemma 4.2. *For any $(a,b) \in G$ we have:*

(1) $I(0) \equiv 0$ *for any constants* α, β *and* γ.

(2) $p(0) = \lim_{h\to 0+0} p(h) = 0$ *and* $q(0) = \lim_{h\to 0+0} q(h) = 0$.

(3) $0 < p(h) < 1$ *and* $q(h) < 0$ *for* $h \in (0, 1/6]$.

(4) $p, q \in C^\infty\,[0, 1/6) \bigcup C^0\,[0, 1/6]$.

Note that in the (x,y)-plane the point $(p(h), q(h))$ is the coordinate of the center of mass of $\mathrm{Int}(\gamma_h)$ with uniform density, so following [74] we give the definition below.

Definition 4.3. In the (p,q)-plane for $(a,b) \in G$ the curve

$$\Sigma_{a,b} = \{(p,q) : p = p(h), q = q(h),\ 0 \le h \le 1/6\} \tag{4.10}$$

is called a *centroid curve.*

The geometric meaning of this curve is that for any fixed $(a,b) \in G$, the parameter h gives a point on $\Sigma_{a,b}$, which is the center of mass of $\mathrm{Int}(\gamma_h)$ if we identify the (p,q)-plane with the (x,y)-plane. Hence, it is natural that $(p(h), q(h)) \to (0,0)$ as $h \to 0$; and $(p(1/6), q(1/6))$ is the center of mass of the region surrounded by the saddle loop $\gamma_{1/6}$.

It is obvious from (4.8) that for any constants α, β and γ the number of zeros of $I(h)$ for $h > 0$ (counting the multiplicities) equals the number of intersection points (counting the multiplicities) of the curve $\Sigma_{a,b}$ with the straight line

$$L_{\alpha\beta\gamma}: \quad \alpha + \beta p + \gamma q = 0 \tag{4.11}$$

in the (p,q)-plane, where $\beta^2 + \gamma^2 \neq 0$.

Definition 4.4. A plane curve is called *sectorial*, if it is smooth, and when running it, the tangential vector rotates through an angle less than π.

If X_H has only one period annulus, then Theorem 4.1 follows from the following result.

Theorem 4.5. *For any $(a,b) \in G$ the curve $\Sigma_{a,b}$ is sectorial, and is strictly convex with non-zero curvature.*

For $(a,b) \in G_2$, X_H has two period annuli, hence there are two centroid curves $\Sigma^i_{a,b}$ for $i = 1,2$. To finish the proof of Theorem 4.1, we also need the following result, which was first proved in [76] based on some other results, and we will give a direct proof in the last section of this chapter.

Theorem 4.6. *For any $(a,b) \in G_2$ both centroid curves are strictly convex with non-zero curvature, and any straight line cuts $\Sigma^1_{a,b} \cup \Sigma^2_{a,b}$ at most at two points, counting the multiplicities.*

4.2 Basic Lemmas and the Geometric Proof of the Result

Computation shows that it is impossible to deduce a third order Picard–Fuchs equation satisfied by $M(h), X(h)$ and $Y(h)$. In fact, it is necessary to add one more function, for example $K(h) = \iint_{\mathrm{Int}(\gamma_h)} xy\,dx\,dy$, then one may deduce a Picard–Fuchs equation of order 4. Thus, it is very difficult to study the global behavior of the curve $\Sigma_{a,b}$, by using this Picard–Fuchs equation directly, except for some of its local properties for h near 0 and near 1/6, shown in the following result.

Lemma 4.7. *For any $(a,b) \in G$ the curvature of $\Sigma_{a,b}$ near its two endpoints is non-zero.*

This result is equivalent to saying that for a generic quadratic Hamiltonian system the order of the Hopf bifurcation and of the homoclinic bifurcation is at most 2, and it basically follows from [7] and [75], respectively.

Taking the derivative on $I(h)$ twice, we get

$$I''(h) = \alpha M''(h) + \beta X''(h) + \gamma Y''(h) = M''(h)\Big[\alpha + \beta\nu(h) + \gamma\omega(h)\Big], \qquad (4.12)$$

where

$$\nu(h) = \frac{X''(h)}{M''(h)}, \quad \omega(h) = \frac{Y''(h)}{M''(h)}. \qquad (4.13)$$

Note that $M'(h)$ is the period function of γ_h and it is monotone for quadratic Hamiltonian vector fields (see [34]), hence $M''(h) \neq 0$. By our choice of h, we have $M''(h) > 0$. We define the curve in the (ν,ω)-plane

$$\Omega_{a,b} = \Big\{(\nu,\omega)(h) : 0 \le h \le 1/6\Big\}. \qquad (4.14)$$

Hence the number of zeros of $I''(h)$ equals the number of intersection points (counting multiplicities) of the curve $\Omega_{a,b}$ with the straight line

$$L'_{\alpha\beta\gamma}: \quad \alpha + \beta\nu + \gamma\omega = 0 \qquad (4.15)$$

in the (ν,ω)-plane.

Lemma 4.8. *For any $(a,b) \in G$ the following statements hold, which imply the regularity of the curve $\Omega_{a,b}$.*

(1) $\big[\omega'(h)\big]^2 + \big[\nu'(h)\big]^2 \neq 0$ *for* $h \in (0,1/6)$, *and*

(2) $(\nu,\omega)(h_1) \neq (\nu,\omega)(h_2)$ *for* $h_1 \neq h_2$ *and* $h_1, h_2 \in [0,1/6]$.

To prove Theorem 4.5, we suppose the contrary: for some $(a,b) \in G$ the curve $\Sigma_{a,b}$ has zero curvature at some points, and we denote by $(p,q)(h^*)$ the nearest such point to the endpoint $(p,q)(0)$. By Lemma 4.7, $h^* \in (0,1/6)$. Now we denote the arc of $\Sigma_{a,b}$ from $h=0$ to $h=h^*$ by $\Sigma^*_{a,b}$. We will prove the following property of $\Sigma^*_{a,b}$.

Lemma 4.9. *For any $(a,b) \in G$ the following statements hold.*

(1) *Along* $\Sigma^*_{a,b}$ *for* $h \in (0,h^*)$ *we have*

$$\frac{d^2q}{dp^2} > 0, \quad k_0(a,b) < \frac{dq}{dp} < k_1(a,b), \qquad (4.16)$$

where

$$k_0(a,b) = \lim_{h\to 0}\frac{dq}{dp} = \frac{b}{a-1} < 0, \;\; k_1(a,b) = \lim_{h\to 1/6}\frac{dq}{dp} = \frac{-q(1/6)}{1-p(1/6)} > 0. \qquad (4.17)$$

(2) *The curve* $\Sigma^*_{a,b}$ *is smooth and* $p'(h) > 0$ *for* $h \in [0, h^*]$.

If such h^* *does not exist, then the above statements hold along* $\Sigma_{a,b}$ *for* $h \in (0, 1/6)$.

One of the crucial steps in this proof is to identify the (p, q)-plane with the (ν, ω)-plane, hence the two straight lines $L_{\alpha\beta\gamma}$ and $L'_{\alpha\beta\gamma}$ are identified. We denote the set of tangent lines to $\Sigma_{a,b}$ (resp. $\Omega_{a,b}$) by $T_{\Sigma_{a,b}}$ (resp. $T_{\Omega_{a,b}}$), that is

$$T_{\Sigma_{a,b}} = \{\xi_h : \text{the tangent line to } \Sigma_{a,b} \text{ at } (p, q)(h), h \in [0, 1/6]\}\,, \tag{4.18}$$

$$T_{\Omega_{a,b}} = \{\eta_h : \text{the tangent line to } \Omega_{a,b} \text{ at } (\nu, \omega)(h), h \in [0, 1/6]\}\,. \tag{4.19}$$

We will prove

Lemma 4.10. *For any* $(a, b) \in G$ *we have :*

(1) $\xi_t \cap \Omega_{a,b} \neq \emptyset$ *for any* $t \in (0, 1/6)$.

(2) $\xi_0 \cap \Omega_{a,b} = \{(\nu, \omega)(0)\}$, $\xi_{1/6} \bigcap \Omega_{a,b} = \{(\nu, \omega)(1/6)\}$, *and the crossing is transversal.*

(3) $\{(\nu, \omega)(0) \cup (\nu, \omega)(1/6)\} \bigcap \xi_h = \emptyset$ *for any* $\xi_h \in T_{\Sigma^*_{a,b}}$ *and any* $h \in (0, h^*]$.

(4) $\Sigma^*_{a,b}$ *and* $\Omega_{a,b}$ *have no common tangent line.*

Now, assuming Lemmas 4.7–4.10 we may give a proof of the basic Theorem 4.5.

Proof of Theorem 4.5. If $\Sigma_{a,b}$ is not globally convex with non-zero curvature for all $(a, b) \in G$, then there exists a $(a^*, b^*) \in G$ and an $h^* \in [0, 1/6]$ such that the curvature of Σ_{a^*,b^*} at h^* is zero. By Lemma 4.7, $h^* \in (0, 1/6)$ and we may choose it in such a way that it is the nearest one to the endpoint $(p, q)(0)$ with zero curvature. Lemma 4.9 implies the sectorial and convexity property (with non-zero curvature) of the curve $\Sigma^*_{a^*,b^*}$ (i.e., a piece of the curve Σ_{a^*,b^*} with h restricted to $[0, h^*)$). We move $\xi_t \in T_{\Sigma^*_{a^*,b^*}}$ with the tangent point $(p, q)(t)$ along $\Sigma^*_{a^*,b^*}$ as t increases from 0 to h^*, and consider the number of intersection points of $\xi_t \cap \Omega_{a^*,b^*}$. By Lemma 4.10 (1) and (2) and Lemma 4.7, $\xi_t \cap \Omega_{a^*,b^*}$ consists of exactly one point if $0 < t \ll 1$, counting its multiplicity. As we have supposed that $\Sigma^*_{a^*,b^*}$ has zero curvature at $h = h^* \in (0, 1/6)$, by taking $L_{\alpha\beta\gamma} = \xi_{h^*}$ the function $I(h)$ has at least a triple zero at $h = h^*$ plus a zero at $h = 0$ (Lemma 4.2 (1)), this implies that $\xi_{h^*} \cap \Omega_{a^*,b^*}$ consists of at least two points. By Lemma 4.10 (2),(3) and Lemma 4.9, the two endpoints of Ω_{a^*,b^*} stay on different sides of ξ_h for all $h \in (0, h^*]$. Hence, the increase in the number of intersection points of $\xi_t \cap \Omega_{a^*,b^*}$, as t increases from 0 to h^*, causes the existence of an $h' \in (0, h^*)$ such that $\xi_{h'} \in T_{\Sigma^*_{a^*,b^*}}$ is also tangent to Ω_{a^*,b^*}. This contradicts Lemma 4.10 (4). $\square$

Before proving the lemmas, we give a more geometric explanation of the above proof. By Lemma 4.2 for any α, β and γ we have $I(0) = 0$. If we choose $L_{\alpha\beta\gamma}$ such that it is tangent to $\Sigma_{a,b}$ at a point $(p, q)(t)$, then the graph of $I(h)$

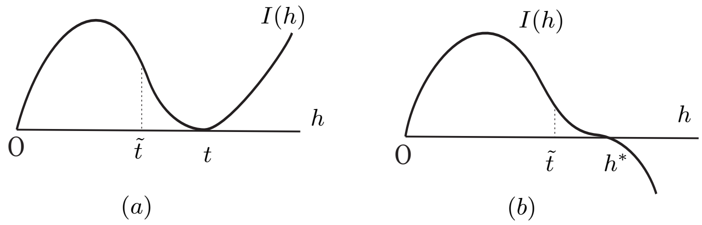

Figure 7. The behavior of curves $I(h)$.

has at least one inflection point at some value $\tilde{t} \in (0,t)$, see Figure 7 (a). In other words, in the identified (p,q)- and (ν,ω)-plane, the tangent line ξ_t to $\Sigma_{a,b}$ at $(p,q)(t)$ must cross the curve $\Omega_{a,b}$ at a point $(\nu,\omega)(\tilde{t})$ with $0 < \tilde{t} < t$, see Figure 8.

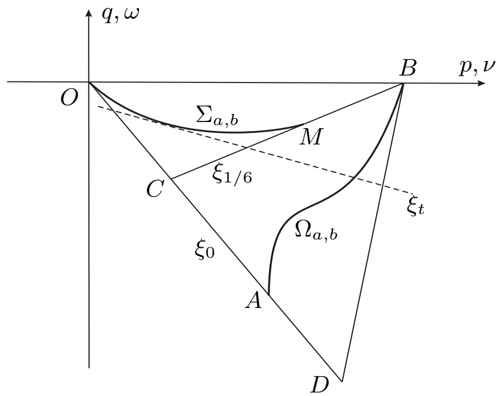

Figure 8. The relative positions of $\Sigma_{a,b}$, $\Omega_{a,b}$ and ξ_t.

If h^* is the (first) zero-curvature point on $\Sigma_{a,b}$, then h^* is at least a triple zero of $I(h)$, hence the graph of $I(h)$ has at least two inflection points at h^* and at some value $\tilde{t} \in (0,h^*)$, see Figure 7 (b). This means that the tangent line $\xi_t = L_{\alpha\beta\gamma}$ of $\Sigma_{a,b}$ crosses the curve $\Omega_{a,b}$ at least at two points. Lemma 4.10 tells us that the tangent line ξ_0 (to $\Sigma_{a,b}$ at the endpoint $(p,q)(0) = (0,0)$) crosses $\Omega_{a,b}$ at its (left) endpoint A while the tangent line $\xi_{1/6}$ (to $\Sigma_{a,b}$ at the endpoint $M = (p,q)(1/6)$) crosses $\Omega_{a,b}$ at its (right) endpoint B, and when t moves from 0 to $1/6$, the two points A and B always stay (respectively) below and above the tangent line ξ_t. Hence the increasing of the number of intersection points of ξ_t with $\Omega_{a,b}$ must be through a position that ξ_t is also tangent to $\Omega_{a,b}$. But Lemma 4.10 (4) excludes this possibility, hence the transversality of ξ_t with Ω for $0 \le t \ll 1$ remains true for all $t \in [0,1/6]$, and this implies the non-zero curvature along $\Sigma_{a,b}$ for all $(a,b) \in G$.

4.3 The Picard–Fuchs Equation and the Riccati Equation

When $(a,b) \in G \backslash l_\infty$, by using a standard method, such as in Section 3.1, one can obtain the following Picard–Fuchs equation of order 4, satisfied by $X(h)$, $Y(h)$, $M(h)$ and $K(h)$ (see Lemma 3.3 of [74]):

$$\begin{aligned}
&-6bhM' + bX' - (a+1)Y' - 2a(a+1)K' + 4bM = 0,\\
&(6\lambda h + a + 1)Y' + (4a(a+1)^2 - \lambda)K' + b(a+1)M - 6\lambda Y = 0,\\
&b\lambda(6h-1)X' + a(\lambda - 2a(a+1))Y' + ((4a^2+3a+1)\lambda\\
&-8a^3(a+1)^2)K' - 6b\lambda X + b(\lambda - 2a^2(a+1))M = 0,\\
&((1-6h)\lambda^2 - (8a^3+12a^2+5a+1)\lambda + 16a^3(a+1)^3)K' + a(4a(a+1)^2 - \lambda)\\
&\cdot(Y' + bM) + \lambda(-(a+1)bX + 8\lambda K + (4a^2(a+1) - \lambda)Y) = 0,
\end{aligned} \tag{4.20}$$

where $' = \frac{d}{dh}$, and $\lambda = 4a^3 - b^2$. Note that $\lambda = 0$ corresponds to $(a,b) \in l_\infty$, and in this case the last three equations in (4.20) are not independent, and a Picard–Fuchs equation of order 3, similar to (4.20), can be obtained ($K(h)$ and $K'(h)$ do not appear). Hence the results, parallel to Lemma 4.11 and equation (4.25) in this section, can be obtained for $\lambda = 0$. In fact, they are limits as $\lambda \to 0$ of the corresponding results here. Hence the discussions are valid for all $(a,b) \in G$.

For simplicity we use the notation

$$\begin{aligned}
&\lambda_1(a,b) = (1-a)^2(2a+1) - b^2, \quad \lambda_2(a,b) = (3a+1)^2 + b^2,\\
&\lambda_3(a,b) = (3a-1)^2 + 5b^2 + 4.
\end{aligned} \tag{4.21}$$

Note that $\lambda_1(a,b) > 0$ for $(a,b) \in G$. The following result follows from (4.20) (see the proof of Lemma 2 of [40] or Lemma 3.3 of [74]).

Lemma 4.11. *For $0 < h \ll 1$ we have*

$$\begin{aligned}
p(h) &= \tfrac{1}{2}(1-a)h + \tfrac{1}{72}[-5(11a+1)b^2 - (a-1)(63a^2+18a+55)]h^2 + O(h^3),\\
q(h) &= -\tfrac{b}{2}h + \tfrac{b}{72}(-55b^2 - 183a^2 + 42a + 5)h^2 + O(h^3).
\end{aligned}$$

Lemma 4.12. *For $(a,b) \in G$ we have:*

(1) $\lim_{h\to 0+0} \dfrac{dq}{dp} = \dfrac{b}{a-1} < 0.$

(2) $\lim_{h\to 0+0} \dfrac{d^2q}{dp^2} = \dfrac{20}{3}\dfrac{b\lambda_1(a,b)}{(1-a)^3} > 0.$

(3) $\lim_{h\to 1/6-0} \dfrac{dq}{dp} = -\dfrac{q(1/6)}{1-p(1/6)} > 0.$

Proof. Statements (1) and (2) are easily deduced from Lemma 4.11. Statement (3) can be proved by using the expansions of $M(h)$, $X(h)$ and $Y(h)$ in h near 1/6 as follows,

$$c_1 + c_2(h - 1/6)\ln(1/6 - h) + c_3(h - 1/6) + o(h - 1/6),$$

see [140] or (1.8) of [74]. □

Taking derivatives with respect to h in the first three equations of (4.20), and removing M', we can express X'', K'' through M'', Y'' as follows,

$$\begin{aligned} X'' &= d_1(h)M'' + d_2(h)Y'', \\ K'' &= d_3(h)M'' + d_4(h)Y'', \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} d_1(h) &= \frac{6\lambda_1(a,b)h}{L(h)}, \\ d_2(h) &= \frac{[12(3a^2+2a+1)b^2 - 24a^3(3a+1)(a-1)]h + (a-1)\lambda_2(a,b)}{bL(h)}, \\ d_3(h) &= \frac{-6b(a+1)(6h-1)h}{L(h)}, \\ d_4(h) &= \frac{6[12(4a^3-b^2)h + b^2 - 6a^3 - 3a^2 + 1]h}{L(h)}, \\ L(h) &= 12(a^3 - 6a^2 - 3a - b^2)h + \lambda_2(a,b). \end{aligned} \tag{4.23}$$

Note that $L(0) > 0$, $L(1/6) = \lambda_1(a,b) > 0$ (see (4.21)), hence the linear function $L(h) \neq 0$ for all $h \in [0, 1/6]$.

Taking derivatives in (4.20) with respect to h once more, and using (4.22) we get

$$T(h)\frac{d}{dh}\begin{pmatrix} M'' \\ Y'' \end{pmatrix} = \begin{pmatrix} e_1(h) & e_2(h) \\ e_3(h) & e_4(h) \end{pmatrix}\begin{pmatrix} M'' \\ Y'' \end{pmatrix}, \tag{4.24}$$

where

$$\begin{aligned} T(h) &= -6bh(6h-1)L(h)\bar{T}(h), \\ \bar{T}(h) &= 36(4a^3-b^2)^2h^2 - 6[b^4 + 2(6a^2+3a+1)b^2 + 8a^3(3a+1)]h + \lambda_2(a,b), \end{aligned}$$

and

$$e_i(h) = \sum_{k=0}^{4} e_{ik}h^k,$$

with e_{ik} polynomials in a and b. We omit their expressions here; our readers can find them in [15].

From the definition of $\omega(h)$ (see (4.13)), we have

$$\omega'(h) = \frac{(Y''(h))'}{M''(h)} - \frac{(M''(h))'}{M''(h)}\omega(h).$$

Combining this fact with (4.24), we obtain a 2-dimensional system of equations

$$\dot{h} = T(h), \qquad \dot{\omega} = \phi(h, \omega), \tag{4.25}$$

where $\phi(h, \omega) = -e_2(h)\omega^2 + (e_4(h) - e_1(h))\omega + e_3(h)$, and the dot denotes the derivative with respect to an arbitrary variable s. Note that system (4.25) is equivalent to a Riccati equation with dependent variable ω and independent h.

Remark 4.13. We note that $T(h) \neq 0$ for $h \in (0, 1/6)$, hence system (4.25) has no singularities for $h \in (0, 1/6)$. In fact, we have shown that $L(h)$ has no zeros for $h \in (0, 1/6)$. If $(a, b) \in G_1$, then $\bar{T}(h)$ has no real roots. If $(a, b) \in G_2 \cup G_3 \cup l_2 \cup l_\infty$, then the roots of $\bar{T}(h)$ correspond to other singularities of X_H, besides the center $O(0, 0)$ and the saddle $S(1, 0)$. By the monotonic property of the level curves of the Hamiltonian vector field and the relative positions of the singularities, we immediately obtain that the roots of $\bar{T}(h)$ must be greater than $1/6$.

By Remark 4.13 and direct computations we obtain the following result.

Lemma 4.14. *For $h \in [0, 1/6]$ system (4.25) has 4 singularities: two improper nodes at $(0, 0)$ and $(1/6, 0)$ and two hyperbolic saddles at $(0, \omega_0)$ and $(1/6, \omega_1)$, where*

$$\omega_0 = \frac{-6b}{\lambda_3(a, b)} < 0, \quad \omega_1 = \frac{-6b(2a + 1)}{5b^2 - 82a^3 - 93a^2 - 36a - 5}. \tag{4.26}$$

When $5b^2 - 82a^3 - 93a^2 - 36a - 5 \to 0$, the singularity $(1/6, \omega_1)$ goes to infinity.

We recall that an *improper node* is a node such that all the orbits arrive to or exit from it in one direction.

Let

$$C_\omega = \{(h, \omega) : 0 \leq h \leq 1/6,\ \omega = \omega(h) \text{ is defined in (4.13)}\}. \tag{4.27}$$

The following lemma can be proved in the same way as the proof of Lemma 3.1 in [102], except statement (2) which is a consequence of Lemma 4.21 below. Statement (1) of the next lemma shows that C_ω is the unstable manifold from the saddle $(0, \omega_0)$ to the improper node $(1/6, 0)$ of system (4.25).

Lemma 4.15. (1) $\lim_{h\to 0+0} \omega(h) = \omega_0$*,* $\lim_{h\to 1/6-0} \omega(h) = 0$.

(2) $\omega(h) < 0$ *for* $h \in (0, 1/6)$.

(3) $\lim_{h\to 0+0} \nu(h) = \nu_0$, $\lim_{h\to 1/6-0} \nu(h) = 1$, *where*

$$\nu_0 = \frac{6(1 - a)}{\lambda_3(a, b)} > 0. \tag{4.28}$$

(4) *We have*

$$\lim_{h\to 0+0} \omega'(h) = \frac{5}{2}\frac{bf_1(a, b)}{(\lambda_3(a, b))^2}, \quad \lim_{h\to 0+0} \nu'(h) = \frac{5}{2}\frac{f_2(a, b)}{(\lambda_3(a, b))^2},$$

$$\lim_{h\to 0+0}[\omega''(h)\nu'(h)-\omega'(h)\nu''(h)] = \frac{175}{6}\frac{b\lambda_1(a,b)\lambda_2(a,b)f_3(a,b)}{(\lambda_3(a,b))^3},$$

where

$$f_1(a,b) = 7b^4 + (42a^2+60a-70)b^2 - (189a^4 - 180a^3 + 174a^2 + 12a - 67),$$
$$f_2(a,b) = (7a-67)b^4 + (162a^3 - 270a^2 - 58a + 70)b^2 + (a-1)(3a+1) \cdot(9a^3 - 27a^2 - 21a + 7),$$
$$f_3(a,b) = 55b^4 + (126a^2 - 204a - 106)b^2 - 81a^4 + 324a^3 + 162a^2 - 204a + 55.$$

(5) *We have*

$$\lim_{h\to 1/6-0}\frac{\nu'(h)}{\omega'(h)} = -\frac{(2a+1)(3a+1)}{b}. \qquad \square$$

From (4.22) and definition (4.13) we obtain the expression of $\nu(h)$ as a function of h and $\omega(h)$ as follows

$$\nu(h) = d_1(h) + d_2(h)\,\omega(h), \tag{4.29}$$

where $d_i(h) = d_i(h;a,b)$, $i=1,2$ are given in (4.23).

We consider the following transformation from the (h,ω)-plane to the (ν,ω)-plane:

$$\nu = d_1(h) + d_2(h)\omega, \qquad \omega = \omega. \tag{4.30}$$

It is easy to see that (4.30) maps the straight line $\{(h,\omega) : h = h_0\}$ ($h_0 \in [0,1/6]$) in the (h,ω)-plane to a straight line in the (ν,ω)-plane. In particular, it maps $\{(h,\omega) : h = 0\}$ to L_0, and maps $\{(h,\omega) : h = 1/6\}$ to L_3, where

$$L_0 = \left\{(\nu,\omega) : \nu = \frac{a-1}{b}\omega\right\},\ L_3 = \left\{(\nu,\omega) : \nu = -\frac{(2a+1)(3a+1)}{b}\omega + 1\right\}. \tag{4.31}$$

We note that if $a=0$, then L_0 is parallel to L_3, and if $a \neq 0$, then $L_0 \cap L_3 = \{(\hat{\nu},\hat{\omega})\}$, where

$$\hat{\nu} = \frac{a-1}{6a(a+1)},\ \ \hat{\omega} = \frac{b}{6a(a+1)}. \tag{4.32}$$

Let

$$D_{a,b} = \{(h,\omega)\in\mathbb{R}^2 : 0 \le h \le 1/6;\ -\infty < \omega < \infty \text{ if } a=0,$$
$$-\infty < \omega < \hat{\omega} \text{ if } a>0,\ \hat{\omega} < \omega < \infty \text{ if } a<0\}.$$

Correspondingly, let $D'_{a,b}$ be the region in the (ν,ω)–plane, which is the strip $-\frac{1}{b}\omega \le \nu \le -\frac{1}{b}\omega + 1$ if $a=0$; and is the corresponding sector region, limited by the two straight lines L_0 and L_3 with vertex at $(\hat{\nu},\hat{\omega})$ if $a\neq 0$, see Figure 9. Note that the vertex $(\hat{\nu},\hat{\omega})$ is not included in $D'_{a,b}$. The Jacobian of transformation (4.30),

$$\frac{D(\nu,\omega)}{D(h,\omega)} = d_1'(h) + d_2'(h)\omega = -\frac{6\lambda_1(a,b)\lambda_2(a,b)[6a(a+1)\omega - b]}{bL^2(h)},$$

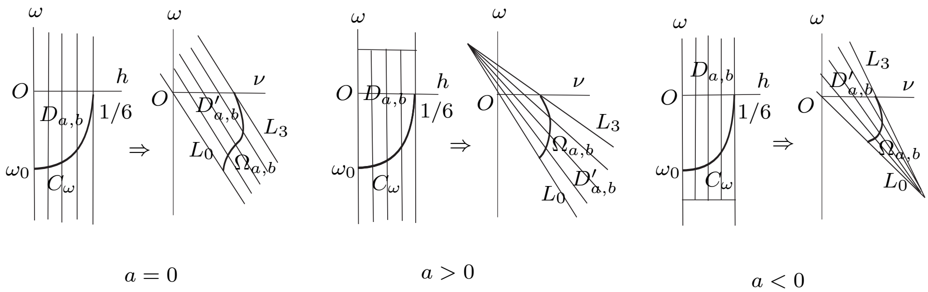

Figure 9. From $D_{a,b}$ to $D'_{a,b}$ through the transformation (4.30).

is non-zero if $a = 0$, and is zero only for $\omega = \hat{\omega}$ if $a \neq 0$. Hence, we immediately have the following result.

Lemma 4.16. *For any* $(a, b) \in G$ *the transformation* (4.30) *from* $D_{a,b}$ *to* $D'_{a,b}$ *is a smooth diffeomorphism. Hence, system* (4.25) *in* $D_{a,b}$ *becomes the smooth system*

$$\dot{\nu} = \varphi_1(\nu, \omega), \qquad \dot{\omega} = \varphi_2(\nu, \omega), \tag{4.33}$$

in $D'_{a,b}$.

From Remark 4.13, Lemmas 4.14 and 4.16 we obtain the following result.

Lemma 4.17. *For* $(a.b) \in G$ *we have*

(1) *Any orbit of system* (4.25), *especially* C_ω, *is transversal to all lines* $\{h = h_0, h_0 \in [0, 1/6)\}$ *in* $D_{a,b}$.

(2) *Any orbit of system* (4.33), *especially* $\Omega_{a,b}$, *is transversal to all straight lines between* L_0 *and* L_3 *in* $D'_{a,b}$, *the lines are parallel if* $a = 0$, *or are in the sectorial region with vertex* $(\hat{\nu}, \hat{\omega})$ *if* $a \neq 0$, *see Figure* 9.

We denote by $L^*_{\alpha\beta\gamma}$ the part of the straight line $L'_{\alpha\beta\gamma}$ in the (ν, ω)-plane, which is contained in $D'_{a,b}$. Let $C_U = \{(h, \omega) : 0 \leq h \leq 1/6, \omega = U(h)\}$ where

$$U(h) = U(h; a, b, \alpha, \beta, \gamma) = \frac{Z(h)}{N(h)} \equiv \frac{z_1 h + z_0}{n_1 h + n_0}, \tag{4.34}$$

with

$$\begin{aligned}
z_1 &= 6b[2(b^2 - a^3 + 6a^2 + 3a)\alpha - \lambda_1(a, b)\beta], \\
z_0 &= -b\alpha\lambda_2(a, b), \\
n_1 &= 12b(a^3 - 6a^2 - 3a - b^2)\gamma + [12(3a^2 + 2a + 1)b^2 - 24a^3(3a + 1)(a - 1)]\beta, \\
n_0 &= \lambda_2(a, b)[(a - 1)\beta + b\gamma].
\end{aligned}$$

Lemma 4.18. *For any* $(a, b) \in G$ *and any constants* α, β *and* γ, $L^*_{\alpha\beta\gamma}$ *is tangent to an orbit of system* (4.33) *of order* k *(in particular to* $\Omega_{a,b}$, *at a point* $(\nu, \omega)(h_0)$ *for* $h_0 \in (0, 1/6)$*), if and only if* C_U *is tangent to the corresponding orbit of system* (4.25) *of order* k *(in particular to* C_ω, *at* $(h_0, \omega(h_0))$*).*

Proof. Under the transformation (4.30) the line $L'_{\alpha\beta\gamma}$ becomes

$$\alpha + \beta\nu + \gamma\omega = \frac{N(h)\omega - Z(h)}{bL(h)} = 0, \tag{4.35}$$

where $L(h) \neq 0$ for $h \in [0, 1/6]$ is given in (4.23), and the linear functions $N(h)$ and $Z(h)$ are defined in (4.34). If $N(h_0) \neq 0$, then for h near h_0 we can rewrite the above equality as

$$\alpha + \beta\nu + \gamma\omega = \frac{N(h)}{bL(h)}[\omega - U(h)] = 0.$$

This means that the transformation (4.30) maps the straight line $L^*_{\alpha\beta\gamma}$ to the curve C_U, and the lemma is proved for h near h_0 by Lemma 4.16. Next, we show that we can skip all zero points of $N(h)$ for $h \in [0, 1/6]$. In fact, if $N(h_0) = 0$ but $Z(h_0) \neq 0$, then equation (4.35) is not satisfied and we do not need to consider it. If $N(h_0) = Z(h_0) = 0$, then the resultant of $N(h)$ and $Z(h)$ must be zero. By a direct computation and using $(a, b) \in G$, we obtain

$$\beta\,[6a(a+1)\alpha + (a-1)\beta + b\gamma] = 0.$$

If $\beta = 0$, then $N(h) = bL(h)\gamma$ and $Z(h) = -bL(h)\alpha$, which contradicts the non-zero property of $L(h)$ for $h \in [0, 1/6]$. If $6a(a+1)\alpha + (a-1)\beta + b\gamma = 0$, then $L^*_{\alpha\beta\gamma} \in D'_{a,b}$ is parallel to L_0 and L_3 when $a = 0$, or passes through the vertex $(\hat{\nu}, \hat{\omega})$ of the sector when $a \neq 0$, see (4.31) and (4.32). By Lemma 4.17, there is no orbit of system (4.33) tangent to it, and the assumption of the lemma is not satisfied. $\square$

Lemma 4.19. *For any $(a, b) \in G$ and any constants α, β, and γ, there exist at most four points on $L^*_{\alpha\beta\gamma}$, counting their multiplicities, such that at each of these points the vector field (4.33) is tangent to $L^*_{\alpha\beta\gamma}$. In particular, if one of the endpoints of $L^*_{\alpha\beta\gamma}$ is $(\nu, \omega)(0)$ or $(\nu, \omega)(1/6)$, then the endpoint is included in these tangent points.*

Proof. By Lemma 4.18 we only need to consider the number of tangent points on C_U (corresponding to $L^*_{\alpha\beta\gamma}$) with respect to the vector field (4.25) in the (h, ω)-plane. By using (4.25) and (4.34) we obtain

$$\dot{\omega} - U'(h)\dot{h}\big|_{\omega = U(h)} = \phi(h, U(h)) - U'(h)T(h) = \frac{b^2 L^2(h)\, F(h)}{N^2(h)}, \tag{4.36}$$

where $F(h) = F(h; a, b, \alpha, \beta, \gamma)$ is a polynomial in all its arguments, and of degree 4 in h. Besides, $F(h)$ has the factor h or $(h - 1/6)$ if $L^*_{\alpha\beta\gamma}$ has the endpoint $(\nu, \omega)(0)$ or $(\nu, \omega)(1/6)$, respectively. Note that we may suppose that $N(h) \neq 0$ for $h \in [0, 1/6]$, see the proof of Lemma 4.18. $\square$

By using the variation argument and the fact that the function $F(h)$ in (4.36) is a polynomial in h of degree 4, we can prove the following results.

Lemma 4.20. *For any $(a,b) \in G$ if $\Omega_{a,b}$ has an inflection point, then the tangent line to $\Omega_{a,b}$ at this point does not pass through the point C, where $\{C\} = \xi_0 \cap \xi_{1/6}$ (see Figure 8).*

Lemma 4.21. *For any $(a,b) \in G$:*

(1) *The curve $\Omega_{a,b}$ is located in the region $D'_{a,b}$, i.e., between the lines ξ_0 and $\eta_{1/6}$, and on the right-hand side of the line $\xi_{1/6}$, except the endpoint $\{B\} = (\nu,\omega)(1/6) = (1,0)$.*

(2) *The curve $\Sigma_{a,b}$ is located inside the closed triangle with vertices O, C and B, denoted by $\Delta_{a,b}$, see Figure 8.*

4.4 Outline of the Proofs of the Basic Lemmas

Proof of Lemma 4.8. From (4.29) we see that the transformation (4.30) maps C_ω to $\Omega_{a,b}$, and by Lemmas 4.14 and 4.15, C_ω, satisfying $\dot{h} \neq 0$ for $h \in (0, \frac{1}{6})$, is a regular curve. Hence, by Lemma 4.16, to prove the regularity of $\Omega_{a,b}$ it is enough to show that C_ω stays in $D_{a,b}$, i.e., C_ω does not meet the straight line $\{\omega = \hat{\omega}\}$ in the (h,ω)-plane for $a \neq 0$. This fact can be proved basically by using the 2-dimensional system (4.25) and the equality (4.36), which gives the number of contact points of system (4.25) with the curve $\omega = U(h)$. $\square$

Proof of Lemma 4.9. (1) Lemma 4.21 shows that for any $(a,b) \in G$ the centroid curve $\Sigma_{a,b}$ is located inside the triangle region $\Delta_{a,b}$ (see Figure 8) and the curve $\Omega_{a,b}$ is located on the right-hand side of the straight line BC (i.e., $\xi_{1/6}$). From Lemma 4.12 we see that $d^2q/dp^2 > 0$ and dq/dp at $(p,q)(h) \in \Sigma_{a,b}$ is increasing from $k_0(a,b)$ as h increases from 0, until h^* or the first value h_∞ that $\lim_{h \to h_\infty} dq/dp = \infty$. We claim that the latter case is impossible. In fact, $dq/dp < k_1(a,b)$ for any point on $\Sigma_{a,b}$. If this is not true, then we would find a point on $\Sigma_{a,b}$, such that the tangent line at this point is parallel to $\xi_{1/6}$, hence it has no intersection with the curve $\Omega_{a,b}$, giving a contradiction, since any tangent line to $\Sigma_{a,b}$ must cross $\Omega_{a,b}$.

(2) By using the fact that any tangent line to $\Omega_{a,b}$ does not pass through the point $D(\hat{\nu},\hat{\omega})$ (see Figure 8 and (4.32)) one can prove that $p(h) < \nu(h)$ for $h \in (0,h^*)$, implying $p'(h) > 0$ for $h \in (0,h^*)$. In fact, Lemma 4.11 implies $p(0) = 0$ and $p'(0) > 0$ for $(a,b) \in G$. We suppose that $h_0 = \inf_{h \in (0,h^*)}\{h : p'(h) = 0\}$, then $h_0 > 0$, $p'(h_0) = 0$ and $p'(h) > 0$ for $0 < h_0 - h \ll 1$. By definitions (4.9) and (4.13), $X'(h_0)M(h_0) - M'(h_0)X(h_0) = 0$, and for $0 < h_0 - h \ll 1$ we have

$$\begin{aligned} M^2(h)p'(h) &= (X'(h)M(h) - M'(h)X(h)) - (X'(h_0)M(h_0) - M'(h_0)X(h_0)) \\ &= (X''(\theta)M(\theta) - M''(\theta)X(\theta))(h - h_0) \\ &= M(\theta)M''(\theta)[\nu(\theta) - p(\theta)](h - h_0) < 0, \quad \theta \in (h,h_0), \end{aligned}$$

since for all $h \in (0, h^*)$ we have $M(h) > 0$ (the area of γ_h), $M''(h) > 0$ (the derivative of the period function which is positive by [34]), and, by assumption, $\nu(\theta) - p(\theta) > 0$. This gives a contradiction. □

Proof of Lemma 4.10. Statement (1) was mentioned several times; it can be seen clearly from Figure 7 (a) that for any double zero t ($t > 0$) of $I(h)$ there exists at least one inflection point of $I(h)$ at some point $\tilde{t} \in (0, t)$. Statement (2) follows from statements (1) and (3) of Lemma 4.12, and the transversality can be obtained by direct computation. To prove statement (3), we note that $\Sigma^*_{a,b}$ is convex, hence $(\nu, \omega)(0) \cap \xi_h = \emptyset$ for $h \in (0, h^*]$ is obviously true. If $(\nu, \omega)(\frac{1}{6}) \cap \xi_h \neq \emptyset$ for some $h \in (0, h^*]$, then, by the facts that $\Sigma_{a,b}$ is located inside the triangle region OCB (see Figure 8) and the curve $\Omega_{a,b}$ is located on the right-hand side of the straight line $\xi_{1/6}$, we conclude that the point $B = (\nu, \omega)(1/6)$ is the only intersection point of $\Omega_{a,b} \cap \xi_h$, and this contradicts the fact that ξ_h must intersect $\Omega_{a,b}$ at some point $(\nu, \omega)(t)$ with $t \in (0, h)$.

Finally, we prove statement (4): there is no common tangent line for the two curves $\Sigma_{a,b}$ and $\Omega_{a,b}$. The idea is to consider the motion of the tangent line η_h,with tangent point moving on $\Omega_{a,b}$, as h decreases from $\frac{1}{6}$. Since $\Omega_{a,b}$ is on the right-hand side of the line BC (see Figure 8), there are only two possibilities for η_h also tangent to $\Sigma_{a,b}$: either η_h passes over the point C upwards, and enters the triangle region OBC, or η_h passes the points $B(1,0)$, $O(0,0)$, and over $\Sigma_{a,b}$ downwards, and gets a tangent position. The latter case contradicts statement (2) of Lemma 4.17. In the former case, by using a deformation argument we would find a $(\bar{a}, \bar{b}) \in G$, such that $\Omega_{\bar{a},\bar{b}}$ has an inflection point, and its tangent line at this point passes through the point C, contradicting Lemma 4.20. □

4.5 Proof of Theorem 4.6

For $(a, b) \in G_2$, the Hamiltonian vector field X_H has two centers C, C', two saddles S, S', two saddle loops γ, γ', and the corresponding period annuli $D(\gamma)$, $D(\gamma')$. Hence we have two centroid curves $\Sigma \subset D(\gamma)$ and $\Sigma' \subset D(\gamma')$. Since the convexity does not change under affine transformations, we can move C or C' (resp. S or S') to $(0, 0)$ (resp. (1,0)) and obtain the normal form (4.1) by an affine transformation, so from Theorem 4.5 we conclude that both Σ and Σ' are strictly convex. Note that X_H is a quadratic system; the four singularities form a quadrilateral with C and C' as a pair of opposite vertices and S and S' as another opposite pair (see, for example [171]). If we exchange C to C' and S to S' by doing an affine transformation, then we must reverse the direction of one coordinate axis (or with a rotation π), hence Σ and Σ' must be one convex and one concave.

Now we denote by L_c (resp. L_s) the straight line passing through C and C' (resp. S and S'); by O the intersection point of L_c and L_s; by Δ (resp. Δ') the interior of the triangle with vertices C, S and O (resp. C', S' and O). Next we denote by t_c the straight half-line which is tangent to Σ at C and points to the direction of the convexity; by t_s the straight half-line from S to another endpoint Z of Σ (the centroid point of $D(\gamma)$); by M the intersection point of t_c and t_s. By Lemma 4.10(2), t_s is tangent to Σ at Z (note that the point $(\nu,\omega)(\frac{1}{6})$ corresponds to a saddle), and by Lemma 4.9, M is located on the same side of the convexity of Σ. We similarly define the straight half-lines t'_c, t'_s and let $\{M'\} = t'_c \cap t'_s$, see Figure 10.

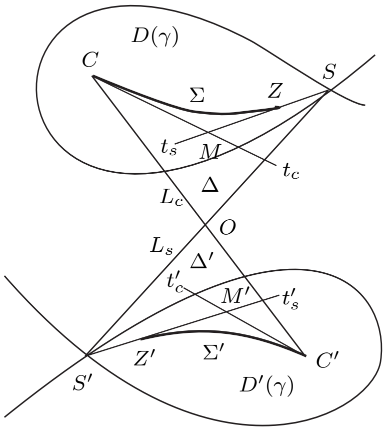

Figure 10. The relative positions of the two centroid curves.

As has been pointed out by Horozov and Iliev [76] to finish the proof of Theorem 4.6, it is enough to show that for any $(a,b) \in G_2$, $M \in \Delta$ and $M' \in \Delta'$. This follows from the claims: (1) t_s and t'_s are located on different sides of L_s; and (2) t_c and t'_c are located on different sides of L_c.

Claim (1) follows from the simple fact that for a quadratic system on any straight line there are at most two points at which the vector field is tangent to this line ([161, 33]). Now L_s passes through the two singular points S and S', so it must stay outside $D(\gamma)$ and $D'(\gamma)$. Otherwise, one orbit inside $D(\gamma)$ or $D'(\gamma)$ would be tangent to it. On the other hand, the point Z (resp. Z') is inside $D(\gamma)$ (resp. $D'(\gamma)$). Claim (2) can be verified by means of a direct calculation. For the normal form (4.1), $C(0,0)$ is a center and the slope of t_c is $k_0 = b/(a-1)$. Note that the equations of l_2 and l_∞ are given by $b = \sqrt{-4a(2a+1)}, -1/2 < a < 0$ and $b = 2\sqrt{a^3}, 0 < a < 1/2$ respectively. We find that the condition for $(a,b) \in G_2$ is : $\xi_1 \equiv b^2 - 4a^3 > 0$, $\xi_2 \equiv b^2 + 4a(2a+1) > 0$ and $\xi_3 \equiv (1-a)^2(2a+1) - b^2 > 0$, where $0 < b < 1$, and $|a| < \frac{1}{2}$. The other center is $C'(x', y')$ with

$$x' = \frac{4a^2 + b^2 + b\sqrt{\xi_2}}{2\xi_1}, \quad y' = \frac{-(2ax'+1)}{b}.$$

Hence, the slope of L_c is $k' = y'/x'$, and we have

$$k_0 - k' = \frac{\eta_1 + \eta_2\sqrt{\xi_2}}{(1-a)(4a^2 + b^2 + b\sqrt{\xi_2})},$$

where $\eta_1 = b(2 - 6a^2 - b^2)$ and $\eta_2 = 2a - 2a^2 - b^2$. From $\xi_3 > 0$ we have $-b^2 > 3a^2 - 2a^3 - 1$, which implies $\eta_1 > b(1-2a)(a+1)^2 > 0$ because $|a| < \frac{1}{2}$ for $(a, b) \in G_2$. Hence, if $\eta_2 \geq 0$, then we have $k_0 - k' > 0$. If $\eta_2 < 0$, then a computation gives

$$\eta_1^2 - \eta_2^2\xi_2 = 4\xi_1\xi_3 > 0,$$

and we obtain the same conclusion. Since we may change t_c' by t_c through an affine transformation, and reverse one coordinate axis as explained before, t_c' must stay on the other side of L_c.

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