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real r&d options

EDITED BY
Dean A. Paxson

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Chapter 1

Introduction to real R&D options

DEAN A. PAXSON

Real R&D options are usually embedded in projects or processes, where management has the on-going capacity to alter the R&D investment timing, amounts and/or ultimate project, or downsize or abandon the R&D. In contrast to real property and real resource options, it is difficult to accurately predict 'discoveries' or estimate future unit sales of R&D products, and there is no established forward unit price market.

There are some excellent classics on real R&D options, starting some 20 years ago, which were not originally considered real options, but where the R&D discovery, volume of sales, as well as the unit prices of the development were considered uncertain. Over the last five years, there has been a growth of real R&D option models, applied to a host of industries, especially biotechnology, e-commerce, internet and telecommunications, as well as the exploration phase of natural resources.

This book contains seven articles (most of which have been revised) presented at the symposium on real R&D options held at Manchester Business School on 12 July 2000, and published in the April 2001 issue of the *R&D Management* journal (Chapters 2, 3, 4, 5, 8, 9, 13). In addition there are six new papers (Chapters 6, 7, 10, 11, 12, 14), which were discussed in the MBS autumn 2001 doctoral finance seminar by the participants, José Azevedo-Pereira, João Duque, Sydney Howell, I-Doun Kou, Jongwoo Lee, David Newton, Helena Pinto and Siqin Xu.

There are nine relatively new models and four new applications or refinements of previous models, although what should be considered new and refinement is debatable. After all, we are building on Thales, Jevons, Samuelson and others. The new models are: analytical values for real options when there are information costs and implementation costs uncertainty; analytical solutions for exit/entry decisions when the future cash flows are mean-reverting, or alternatively finite and fat-tailed; finite options with possibly endogenous learning and exogenous and experiential shocks; analytical approximations for real American sequential option values; pre-emption options for temporary first mover advantages; analytical real option values given externalities and government subsidies, also for mean-reverting stochastic processes. The new applications are R&D expenditures modeled as forward start premiums for 'new product development' (NPD) options; valuing the exploration options in natural resources; viewing technological innovation models under incomplete information costs; and calculating the follower's and leader's real value functions, with a time-varying market share. In addition, there is a biotechnology case study at the end (Chapter 15), utilizing some of these models in financial analysis and planning, and then a review of some of the classical real R&D option articles (Chapter 16).

1.1 CHALLENGES IN VALUING REAL R&D OPTIONS

There are several major challenges in valuing real R&D options, including:

(1) Modeling the duration, dimension and diffusion processes of the eventual R&D payoff values.
(2) It is not always realistic to assume that the eventual project or product will be a perpetuity [as in some land developments and (practically) some natural resources].
(3) Identifying the time-varying volatilities of the processes and of the underlying eventual values.
(4) Including the possibility of success or failure of the venture, which may also be time-varying, in a real option model.
(5) Identifying the stages of R&D management flexibility and actions.
(6) Dealing with the usual environment where R&D is budgeted and the expenditure consists of salaries and experiments occurring continuously in time rather than instantaneously at a point of time.
(7) In some R&D projects, the real options might become proprietary, where patents or orphan drug status might be available. In other industries, which are wholly or partly competitive, the first mover's advantages are not necessarily pre-emptive, so that the advantages of deferral are partly dependent on competitive (follower's) actions.
(8) R&D data is not always public, or even available within research enterprises, and often not suitable as input for economic models.
(9) While the traditional literature on real options in R&D often focuses on the upside, there are no doubt put options written by R&D enterprises (such as product guarantees, reimbursement obligations, requirements for further testing, product liabilities), as well as suboptimal exercise of real options, which may destroy value.
(10) Finally, the link between basic research and the eventual discovery or project value is seldom well identified; indeed, the eventual project

value may not be imagined at the research stage, and other aspects such as developing technical and intellectual competency, reducing information costs of the eventual project and the real option, and increasing the implementation capacity of the firm may be important products of R&D.

1.2 COMPLEXITY OF R&D MODELS

The authors have been organized (loosely) on the basis of the complexity of the R&D and project stages of their models, and on how they deal with many of the challenges above (see Table 1.1). The following four chapters assume the eventual project is (or can be valued like) a perpetuity and there is an instantaneous R&D expenditure. The next two chapters assume that the underlying project distributions are not lognormal, but instead resemble Student distributions, or (possibly) mixed jump and geometric Brownian motion (gBm). The next two chapters assume limited stages in expenditures and a finite life (or sale) for the project. The next three chapters assume that there are first adopters/first movers who are influenced by attitudes towards risk and growth and the action of other players. The last two chapters contrast public and private objectives, in the context of determining optimal fiscal policies for R&D expenditures and disinvestments.

1.2.1 Underlying R&D project value

All authors assume that the ‘underlying’ eventual project value volatility is constant or deterministic over time, and that there is a constant (sometimes risk-adjusted) project drift. Usually the eventual project cash flows or values are modeled as gBm.

The exception to these general model rules are: Tsekrekos (Chapter 3) and Jou and Lee (Chapter 13), who allow for a Poisson hazard rate for implementation jumps and technology shifts; Biekpe, Klumpes and Tippett (BKT; Chapter 5), Rhys and Tippett (Chapter 6) and Jou and Lee (Chapter 14), who deal with mean-reverting cash flows; and Martzoukas (Chapter 7), who considers mixed diffusion processes, incorporating jumps. Most of these authors have the luxury (and innovation) of closed-form solutions, and usually derive the critical prices which justify irreversible investments (and entry/exit decisions).

1.2.2 Sequential decisions

Six chapters allow for a (slightly more) realistic environment, where R&D expenditures are not instantaneous, and decisions to enter, adopt an innovation, continue expenditures, or exit are made sequentially. In some cases, the volatility of the expenditures and/or the volatility of the outcome are not necessarily constant. Lint and Pennings (Chapter 4) assume gBm in two stages, R&D and NPD, and allow for a volatility of R&D outcome in excess of that for NPD. Lee and Paxson (Chapter 8) also consider stages, but where R&D and investment cost volatility is different from the value volatility, although correlation and both volatilities are time-invariant. Cortazar, Schwartz and Casassus (CSC; Chapter 9) also examine various stages, and in addition allow for an exploration volatility, in excess of eventual project volatility, to be dependent on the exploration expenditures (a process of reducing volatility through a type of learning). Bellalah (Chapter 10) examines adoption strategies over initial and subsequent technological advances. Tsekrekos (Chapter 11) and Paxson and Pinto (Chapter 12) consider an entry stage for the leader, which is partly dependent on the entry timing for the follower. None of these authors allow for multiple stages, or stochastic volatility.

1.2.3 Real R&D option model solutions

There is a variety of real R&D option model solutions proposed by these authors. Many articles model real R&D expenditure and its underlying project value as a perpetual American call (entry) or put (exit) option. Lint and Pennings (Chapter 4) use a forward start option, for an American perpetuity. Two chapters (8 and 9) propose compound options for the stages of R&D and investment, an American sequential exchange compound option with approximated values, and then a complex mixed European and American compound option requiring a numerical solution. There is an analytical solution for various jump processes. Two chapters (5 and 6) provide new real option model solutions based on mean-reverting stochastic processes, and then Student distributions. There are analytical solutions for the follower and leader value functions in a duopoly. The Genzyme case study (Chapter 15) shows the possible use of some of these analytical real option models in relating internal R&D project valuation to external financial market valuation.

1.2.4 Real R&D option 'Greeks'

Most authors have provided the partial derivatives of each real option model with respect to the underlying project or discovery value, and in some cases with respect to other parameters. The Black–Scholes first derivative of the call option value with respect to the underlying asset value (V) is 0 at $V = 0$ and approaches 1 as V approaches infinity (or some large number). The derivative of a European option with respect to volatility is always positive. These are also two commonsense derivatives for real options, a departure from which requires explanation. Where the underlying R&D project value is lognormally distributed, and the analytical solution is relatively simple, most real option model 'Greeks' herein are consistent with these tests. The Lint and Pennings forward start option, the jump models, and the sequential compound call options all have deltas that are similar to, but not always the same as, the cumulative standardized normal distribution function ranging from 0 to 1. Also, all of these real option values increase with increased project value volatility, keeping all other parameters constant.

However, when the underlying is a cash flow, or a price (then continuing in perpetuity, or multiplied by the quantity of production, for instance), some of the deltas with respect to the state variable are a multiple of that variable. In addition, some mean-reverting and Student distribution deltas appear different from ordinary call option deltas; and the option vegas are complex. For CSC, the numerical delta (value of the real option value with respect to the copper price) increases from 0 to over 14, although the gamma and vega are similar to those for ordinary options. Where there is partial pre-emption, some competitive environment deltas are not necessarily positive, since after the follower's entry trigger point, the leader's market share, or profitability, declines.

Several authors have provided other partial derivatives particular to their model. Bellalah (Chapters 2 and 10) shows the sensitivity of real option values to information costs. Tsekrekos (Chapter 3) provides the analytical partial derivative of real option value with respect to the implementation and hazard rate parameters, respectively. Martzoukos (Chapter 7) shows the partial derivative of the endogenous control option with respect to the size and volatility of the control (jump) variable. Jou and Lee (Chapters 13 and 14) show the sensitivity of optimal capital stock for the firm and for the industry to changes in several parameters.

Besides the commonsense aspect of viewing these partial derivatives, the real option Greeks and sensitivity analyses may ultimately be useful for issuers of (and investors in) related derivatives, indirectly through equity in such R&D projects, more or less directly through 'tracking stock' with embedded options, or eventually in synthetic real R&D options, conceivably exchange traded.

1.2.5 Empirical contributions

One inherent problem in real R&D model development and publication is the typical confidentiality of R&D results and empirical parameters. For commercial reasons, most R&D enterprises require hypothetical (or disguised) empirical illustrations. While most authors herein have not emphasized empirical studies, each chapter illustrates the option values, given ranges for the input parameters required. Bellalah (Chapter 2) requires data or estimates for information costs for both projects and options. Tsekrekos (Chapter 3) requires estimates of implementation costs and the hazard rate for implementation uncertainty. Both the pattern of future cash flows (long-term mean reversion level and speed of reversion) and the cash flow volatilities (related to the cash flow level, or its square root) are required in the BKT model. Rhys and Tippett (Chapter 6) require not only the eventual project cash flow and reversion speed, but also a volatility estimate related to the cash flow level. Martzoukas (Chapter 7) requires jump frequency, jump size and volatility estimates for both exogenous and endogenous systems. Bellalah (Chapter 10) requires data or estimates for information costs for both projects and options, as well as migration inclinations of technological adopters. Tsekrekos (Chapter 11) requires an estimate of the proportion of project profitability temporarily (before followers) and permanently (even after followers) retained by the leader, as well as other parameters. Paxson and Pinto (Chapter 12) require rates of new customer arrivals, and old customer departures from the market, data which is often available. Jou and Lee (Chapters 13 and 14) require a large number of estimated inputs, especially concerning externalities (production function for an individual firm that is affected by aggregate industry investment) and eventually the mean-reverting parameters.

Some authors provide 'empirical' data (even though disguised) for their models, which one might take as (more or less) representative. Lint and Pennings (Chapter 4) show the real option value of the NPD option to launch for a unit investment and range of NPD values and volatilities. Lee and Paxson (Chapter 8) use project revenues and costs, R&D effectiveness estimates, R&D and investment costs, and associated volatilities and correlations. Cortazar, Schwartz and Casassus (Chapter 9) provide sample 'mine profiles' and occurrence probabilities, along with specific exploration and investment expenditures.

1.3 SUMMARY OF CHAPTERS

Chapter 2. Mondher Bellalah focuses on the possibility that R&D investments may be conducted in an environment of information uncertainty, regarding the quality, quantity and persistence of future projects and options on projects. In the context of such incomplete information, both analytical solutions and illustrated numerical results are provided.

Chapter 3. Andrianos Tsekrekos supposes that even with research success, development and production timing and value may be exposed to implementation uncertainty. Since implementation uncertainty may affect both the level and timing of project profitability, the option deferral value of R&D and other irreversible investments will be dependent on the resolution of such uncertainty.

Chapter 4. Onno Lint and Enrico Pennings consider the new product development process as a series of real options with reducing uncertainty over time. In electronics R&D and new products, particular projects may be viewed in terms of a matrix of volatility versus R&D and new product value. For low volatility projects, high NPV projects should be adopted immediately and low NPV projects abandoned. For high volatility NPD, high NPV projects should be adopted, but low NPV projects have primarily real option value and commencement of the NPD should be deferred.

Chapter 5. Nicholas Biekpe, Paul Klumpes and Mark Tippett derive the critical level of future cash flows for triggering commencement of serious research units, or product development and marketing, when cash flows are expected to be mean-reverting. Analytical expressions (using the complementary error function) are proposed for optimal (dis)investment decisions, even when the option to enter or exit a business is significant.

Chapter 6. Huw Rhys and Mark Tippett derive an explicit formula for the value of the option to invest in a capital project when the difference between the benefits and costs of the investment decision are generated by a general class of Student distributions. These processes encapsulate the 'fat tail' property of some of the characteristic R&D project payoffs. Their analytic solution is based on the assumption that the option to invest has a finite life.

Chapter 7. Spiros Martzoukos values real investment options in the presence of endogenous and exogenous learning. *Endogenous* learning is captured through optimally activated controls arising because of costly managerial actions such as R&D intended to enhance value and reveal information. *Exogenous* learning is captured through random information arrival of rare events (jumps resulting from technological and other shocks) that follow a Poisson process and have a size drawn from a mixed distribution. *Experiential* shocks are captured by a dynamic volatility similar to that observed in the financial options markets. An optimization problem is solved by considering the trade-off between the benefits and costs of R&D actions.

Chapter 8. Jongwoo Lee and Dean Paxson model stages of R&D expense and then the ultimate e-commerce and internet project values as real sequential (compound) exchange options. In the sequential investment phases, future revenues and investment costs are stochastic, and the investments can be initiated or abandoned at any time. Approximate analytical solutions (based on the confined exponential distribution) are provided, where the input parameters are consistent with market volatilities.

Chapter 9. Gonzalo Cortazar, Eduardo Schwartz and Jaime Casassus consider several real options from natural resource exploration to development and then to mine operations. There are joint price and geological–technical uncertainties, which are collapsed into a one-factor model for tractability. The investment schedules for exploration and development are flexible, and the technological uncertainty is reduced by exploration expenditures. An implicit finite difference numerical approach is used to compute the value of the operational, development and exploration options, for different levels of eventual mine values.

Chapter 10. Mondher Bellalah examines how frictions such as costly information affect equilibrium in capital and real markets. Investment in technological innovations, with stochastic arrival times and profitability, may require gathering information before deciding on the appropriate technology. Important characteristics of real-world technology adopters are considered in the context of varying project and option information costs to derive firm policies.

Chapter 11. Andrianos Tsekrekos examines the strategic exercise of real options, when the first mover's full advantage is temporary, in a competitive environment. Otherwise identical firms will invest at different critical prices (and thus at different times) depending on the scale of the advantage, and the economic characteristic of the underlying R&D project value.

Chapter 12. Dean Paxson and Helena Pinto model a leader and follower function (in a duopoly) where the market (and thus the market share) evolves according to new arrival (birth) and departure (death, or churn) processes. This is characteristic of new product markets, where the total market is not well defined, and the behavior of adopters and leavers (or adopters of the followers' innovations) are similar to well-studied stochastic processes in other fields.

Chapter 13. Jyh-bang Jou and Tan Lee ask whether governments should subsidize R&D directly (through scientific funding institutions) or indirectly (through tax credits and other incentives). Suppose R&D capital exhibits both irreversibility and externality through the learning-by-doing effect. Given private risk aversion, private enterprise will not invest sufficiently in R&D, so subsidy is justified, along with taxation of disinvestments.

Chapter 14. Jyh-bang Jou and Tan Lee now assume that the return to R&D capital is driven by a technological factor that follows a mean-reverting process. The optimal paths for R&D capital under both a decentralized and a centralized economy are derived and then compared, using the confluent hypergeometric function. In theory, an equal rate of investment tax credits should be given to both costlessly reversible investments and irreversible R&D, regardless of whether the ultimate R&D project values are a geometric Brownian motion or a mean-reverting process.

1.4 FUTURE REAL R&D OPTION MODELS

These authors (and the classical authors) provide what eventually will be regarded as elementary models for realistic R&D environments. As partly noted in Section 1.1, R&D is typically expended continuously over time, sometimes with greater emphasis, depending on the urgency of outcomes, availability of personnel and facilities, likelihood of successful results and, of course, R&D fashion.

Empirical aspects are eventually critical for the implementation of real R&D option models in aiding decisions regarding the amount, timing, direction and value of R&D. The empirical problems that have to be addressed in practice are: (a) the direct link between R&D and underlying discovery value, and hence the effectiveness of specific R&D expenditures over time; (b) the distribution of that underlying value, and the time-varying parameters; (c) the correlation or dependency relationships between cost and value, and among different projects (often with non-normal distributions); (d) identification of analogous securities (to R&D discovery value and processes) for estimating volatility and drifts; and (e) estimates of the information cost reduction and/or implementation improvement contributed by R&D. Each of these challenges will probably be the topic of lots of R&D and many articles in the future.

Table 1.1 *Real R&D options*

Chapter	Author	Application	Diffusion variables (new)	Type of real option	Stages	Duration	Diffusion: Process	Diffusion: Success	Solution	Data required
2	Bellalah	INFORMATION COST	$\lambda_{\text{(OPTION INFO)}}$, $\lambda_{\text{(PROJECT INFO)}}$	PA	N/A	PA	gBm		CF	INFO COST ESTIMATES
3	Tsekrekos	IMPLEMENTATION	a(IMPLE), h(HAZARD)	PA	N/A	PA	MJD	$h(a)$	CF	IMPLE & HAZARD RATE
4	Lint & Pennings	R&D $\Rightarrow$ NPD	$\sigma_{\text{(R\&D)}} = \beta\sigma_{\text{(NPD)}}$	Forward Start	R&D (t), NPD (T_{L})	PA	gBm		CF	σ(STAGES), T(L)
5	BKT	MEAN-REVERTING CASH FLOWS	β, μ(REVERSIONS)	OU(PA)	N/A	PA	OU		SERIES	OU(β, μ, σ)
6	Rhys & Tippett	STUDENT'S DISTRIBUTION	β(REVERT), $\sigma^2 = (k^2 + 2rx^2)$	STUDENT_finite	N/A	Finite	$\sigma = \phi[+\rho(x)]$		CF	(β, σ), x(CF), r
7	Martzoukos	ENDO & EXO LEARNING	σ, δ, σ_i, $\gamma_{i\text{(JUMP)}}$	Euro_Jumps	N/A	Finite	MJD		CF	σ_j, $\gamma_{j\text{(EXO)}}$, λ, σ_i, $\gamma_{i\text{(ENDO)}}$
8	Lee & Paxson	E-COM SEQUENTIAL	$\sigma_{\text{(E-COM)}}$, $\sigma_{\text{(INVEST)}}$, ρ	Amer_CO Sequential	SEQ_AMER	Finite	gBm		CF(Bivariate)	VALUE & INVEST σ, ρ
9	CSC	EXPLORATION	$\sigma_{\text{(TECH)}}$, $\sigma_{\text{(PRICE)}}$, E, D, I, MINE	Euro_Compound	FOUR	Finite	gBm(P)	EXPLORE(α^i)	NUMERICAL	G(TECH), MINE PROFILE
10	Bellalah	INNOVATION ADOPTION	$\lambda_{\text{(OPTION INFO)}}$, $\lambda_{\text{(PROJECT INFO)}}$	PA	INNO STAGES	PA	gBm		CF	INFO COST ESTIMATES
11	Tsekrekos	FIRST-MOVER ADVANTAGE	a(LEADER), σ, x	PA	LEADER/ FOLLOWER	PA	gBm		CF	a(LEADER), σ, x
12	Paxson & Pinto	VARYING MARKET SHARE	a, $(\lambda/\nu), \sigma, x$	PA	LEADER/ FOLLOWER	PA	gBm		CF	a, $(\lambda/\nu), \sigma, x$
13	Jou & Lee	EXTERNALITIES & SUBSIDY	INDUSTRY CAP, EXTERNALITIES	PA	N/A	PA	gBm	TECH_HAZARD	CF	12 Parameters
14	Jou & Lee	MEAN-REVERTING CASH FLOWS	Z(TECH_SHIFT), η, σ	PA	N/A	PA	OU		CF	11 Parameters

gBm	geometric Brownian motion
PA	perpetual American option
CO	compound option
OU	Ornstein–Uhlenbech mean-reverting
MJD	mixed jump diffusion
CF	closed-form solution or approximation
SERIES	series sum
N/A	not applicable (e.g. no stages)
R	R&D stage
D	development stage
L	project launch stage

Chapter 2

On irreversibility, sunk costs and investment under incomplete information

MONDHER BELLALAH

SUMMARY

This chapter presents a framework for the valuation of investment opportunities by accounting for the effects of incomplete information regarding the firm and its cash flows. We present some simple models of irreversible investment to illustrate the option-like characteristics of investment opportunities under incomplete information. We show how optimal investment rules can be obtained using real option theory under shadow costs of incomplete information. Simulations are provided to illustrate our main results.

2.1 INTRODUCTION

Several models in financial economics are proposed to deal with the ability to delay an irreversible investment expenditure. These models undermine the theoretical foundation of standard neoclassical investment models and invalidate the net present value criteria in investment choice under uncertainty.

Pindyck (1991) reviews some of the results of basic models of irreversible investment and uses the theory of option pricing to illustrate the option-like characteristics of investment opportunities. Roberts and Weitzman (1981) developed a model of sequential investment that puts the stress on the role of information gathering during the investment process. In this model, information gathering adds a shadow value to the early stages of the investment. This latter result applies whenever information gathering, rather than waiting, yields information. A simple example is given in Pindyck (1991) to illustrate this result. We believe that it is possible to extend the analysis in Pindyck (1991) by accounting for information costs. Our definition of information costs refers to the models in Merton (1987) and Bellalah (1999a). Merton (1987) introduced a modified capital asset pricing model where each investor can participate only in markets contained in an exogenous, investor-specific subset of all asset markets.

Since the acquisition of information and its dissemination are central activities in finance and in the investment process, Merton's (1987) simple model of capital market equilibrium with incomplete information might provide some insights into the behavior of security prices. This model can be applied in the investment decision and in the derivation of equilibrium option prices. Bellalah (2000) applies Merton's (1987) model to study the cost of capital and extends the standard Modigliani–Miller analysis to account for shadow costs of incomplete information.

It is well known that an investment opportunity is like a call option. This option gives the holder the right, for some specified amount of time, to pay an exercise price and in return receive the underlying asset. Using a similar context as that of Black and Scholes (1973) in a Merton (1987) economy, it is possible to present option valuation formulas for the investment opportunity in the presence of information costs as in Bellalah (1990, 1999a,b) and Bellalah and Jacquillat (1995). Bellalah (2000) applies a similar analysis to the investment decision in the study of strategic investments.

In this chapter, we present some models of basic investment by accounting for the option-like characteristics of investment opportunities in an incomplete information context. In this context, we obtain optimal investment rules using option pricing theory. Section 2.2 presents a justification for the foundations of information costs in investment decisions. These costs are based on the shadow costs of incomplete information in the spirit of Merton's model. We provide a basic continuous-time model of irreversible investment in the presence of information costs. The model shows when the firm should invest in a project in the presence of incomplete information. Section 2.3 extends the basic model so that the price of the firm's output is random and the firm can stop production whenever the price falls below variable cost. We derive the value of the project and the value of the firm's option to invest in the project as well as the optimal investment rule in the presence of information costs. Simulations are provided to illustrate our main results. Section 2.4 compares some of our results with respect to some standard models. Section 2.5 concludes.

2.2 THE PROBLEM OF INVESTMENT TIMING AND THE ANALOGY TO FINANCIAL OPTIONS WITH INCOMPLETE INFORMATION

Before introducing the model, we justify the main assumptions in this chapter regarding information costs.

2.2.1 The role of information in investment decisions

The introduction of information costs on the option 'market' and the underlying asset markets can be better understood by reviewing the main results in Merton's (1987) model. In Merton's model, the expected returns increase with systematic risk, firm-specific risk, and relative market value. The expected returns decrease with relative size of the firm's investor base, referred to in Merton's model as the 'degree of investor recognition'.

Merton's model may be stated as follows:

$$R_V - r = \beta_V[R_m - r] + \lambda_V - \beta_V\lambda_m \tag{2.1}$$

where:

R_V = the equilibrium expected return on an asset V,
R_m = the equilibrium expected return on the market portfolio,
r = the riskless rate of interest,
$\beta_V = \text{cov}(R_V/R_m)/\text{var}(R_m)$,
λ_V = the equilibrium aggregate 'shadow cost' for the asset V (of the same dimension as the expected rate of return on this asset V),
λ_m = the weighted average shadow cost of incomplete information over all assets.

The model is based on the assumption that there are several factors in addition to incomplete information that may explain this behavior for individuals and firms. Hence, the presence of prudent-investing laws and traditions and other regulatory constraints can rule out investment in a particular firm by some investors. Using this assumption, Merton shows that the expected returns depend on other factors in addition to market risk. The main intuition behind this result is that the absence of a firm-specific risk component in the capital asset pricing model (CAPM) comes about because such risk can be eliminated (through diversification) and is not priced. Merton's (1987) model is supported by several authors, including Amihud and Mendelson (1989), Kadlec and McConnell (1994), Kang and Stulz (1997), Coval and Moskowitz (1999) and Stulz (1999).

Merton's model is an extension of the CAPM to a context of incomplete information. The model gives a general method for discounting future cash flows under uncertainty. In this model, assets with higher idiosyncratic risk are rationally priced to earn a higher expected return. It appears in this model that taking into account the effect of incomplete information on the equilibrium price of an asset or an investment opportunity is similar to applying an additional discount rate to its future cash flows. In fact, the expected return on the asset is given by the appropriate discount rate that must be applied to its future cash flows. Then, relying on Merton's model to derive a valuation formula should lead to more accurate theoretical prices since information costs are included for assets and options. Merton's model is used to price contingent claims in Bellalah (1990), Bellalah and Jacquillat (1995) and Bellalah (1999a,b) and capital budgeting decisions in Bellalah (2000, 2001).

For a project, information costs correspond to the costs of collecting information about the investment opportunity. These costs are specific to each asset and to each investment opportunity. Empirically, they can be estimated using the estimation procedure in Merton (1987) or in Kadlec and McConnell (1994). They can also be estimated implicitly using market option prices. In this case, a procedure is applied to estimate information costs from option data in the same way as we estimate implied volatilities.

2.2.2 Investment timing and the pricing of assets

The investment opportunity is analogous to a call option on a common stock since it gives the right to make an investment expenditure at the strike price and to receive the project. The firm's option to invest refers to the possibility of paying a sunk cost I and receiving a project, which is worth V. Unlike standard options, this call is perpetual and has no expiration date. This result is used in McDonald and Siegel (1986) and Pindyck (1991). The decision regarding the timing of the investment is equivalent to the choice of the exercise time of this option.

The dynamics of the project's value can be described by the following equation:

$$\mathrm{d}V/V = \alpha\,\mathrm{d}t + \sigma\,\mathrm{d}z \tag{2.2}$$

where α and σ refer to the instantaneous rate of return and the standard deviation of the project, and $\mathrm{d}z$ is a geometric Brownian motion. This equation shows that the current project value is known, whereas its future values are lognormally distributed. Now, let X denote the price of an asset or a dynamic portfolio of assets perfectly correlated with V. The dynamics of X are given by:

$$\mathrm{d}X/X = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}z \tag{2.3}$$

where μ stands for the expected return from owning a completed project.

Let $\delta = \mu - \alpha$. If V were the price of a share, δ would be the dividend rate on the stock. In this context, δ represents an opportunity cost of delaying investment. If δ is zero, then there is no opportunity cost to keeping the option alive. Hence, the value of δ must be positive. Let $C(V)$ denote the value of the firm's option to invest, which corresponds also to an investment timing option. Using Merton's model, Bellalah and Jacquillat (1995) and Bellalah (1990, 1999a,b) obtain option prices in the context of incomplete information. The derivation is reproduced in the appendix (Section 2.6). Consider the return on the following portfolio: hold an option which is worth $C(V)$ and go short C_V units of the project where the subscript V refers to the partial derivative with respect to V. The value of this portfolio is:

$$P = C - C_V V \tag{2.4}$$

Over a short interval, the change in the value of V induces changes in the value of C_V and in the portfolio's value. The short position requires a payment of $\delta V C_V$ dollars per time period, where δV refers to a dividend stream. If this is not verified, no rational investor will enter into the long side of the transaction. Since the short position includes C_V units of the project, it requires the paying out of an amount $\delta V C_V$. The total return for this portfolio over a short interval of time $\mathrm{d}t$ is:

$$\mathrm{d}C - C_V\,\mathrm{d}V - \delta V C_V\,\mathrm{d}t \tag{2.5}$$

To avoid riskless arbitrage, the value of this portfolio must be the riskless rate. However, since there are information costs embedded in the option and in its underlying assets, the return must be equal to $(r + \lambda_V)$ for the project and $(r + \lambda_c)$ for the option, where λ_V and λ_c refer respectively to the information costs on the project and the option. These parameters represent sunk costs, which are necessary before entering into a project. They are incurred during the phase of gathering information about the project and the opportunity to invest. Since the project may not have the same value for all the firms, this information cost can be specific to each firm. Therefore, the costs of gathering information and data about the project and the investment opportunity are present in the discounting procedure. It is important to note the presence of a shadow cost of incomplete information for each asset. In this context, we have:

$$\mathrm{d}C - C_V\,\mathrm{d}V - \delta V C_V\,\mathrm{d}t = (r + \lambda_c)C\,\mathrm{d}t + (r + \lambda_V)VC_V\,\mathrm{d}t \tag{2.6}$$

Assuming a hedged position is constructed and 'continuously' rebalanced, and since $\mathrm{d}C$ is a continuous and differentiable function, it is possible to use a Taylor series expansion to expand $\mathrm{d}C(V)$. When limiting arguments are used and second-order terms ignored, we get:

$$\mathrm{d}C = \tfrac{1}{2}C_{VV}(\mathrm{d}V)^2 + C_V\,\mathrm{d}V$$

This is just an extension of simple results to get Itô's lemma. The application of this lemma gives: $\mathrm{d}C = \frac{1}{2}C_{VV}\sigma^2V^2\,\mathrm{d}t + C_V\,\mathrm{d}V$. Since $\alpha = \mu - \delta$, the value of $\mathrm{d}C$ is:

$$\mathrm{d}C = \tfrac{1}{2}C_{VV}\sigma^2V^2\,\mathrm{d}t + (\mu - \delta)C_V V\,\mathrm{d}t + C_V V\sigma\,\mathrm{d}z \tag{2.7}$$

If we substitute equation (2.7) into equation (2.6), we get after simplification:

$$\tfrac{1}{2}C_{VV}\sigma^2V^2 + (r + \lambda_V - \delta)VC_V - (r + \lambda_c)C = 0 \tag{2.8}$$

Since there are two assets C and V, there are two information costs: one regarding C and the other regarding V. Note that this is a modified Black–Scholes (1973) equation, in which the interest rate is adjusted by the effect of incomplete information regarding the two assets.

This equation for the value of the investment timing option $C(V)$ must satisfy the following conditions:

$$C(0) = 0 \tag{2.9}$$

$$C(V^*) = V^* - I \tag{2.10}$$

$$C_V(V) = 1 \tag{2.11}$$

The value V^* is the project value at which it is optimal to invest. At that time, the firm receives the difference $V^* - I$. The last condition is the 'smooth pasting' condition. The solution to the differential equation under the above conditions gives the value of the investment timing option $C(V)$. The solution under the first condition is:

$$C(V) = aV^\beta \tag{2.12}$$

where a is a constant and:

$$\beta = \tfrac{1}{2} - (r - \delta + \lambda_V)/\sigma^2 + \left\{\left[(r - \delta + \lambda_V)/\sigma^2 - \tfrac{1}{2}\right]^2 + 2(r + \lambda_c)/\sigma^2\right\}^{0.5}$$

The delta in this case is $\Delta = a\beta V^{\beta-1}$, if $V < V^*$. The value of the constant a and the critical value V^* are determined using the other two boundary conditions. Substituting equation (2.12) into equations (2.10) and (2.11) gives:

$$V^* = \beta I/(\beta - 1) \quad \text{and} \quad a = (V^* - I)/(V^{*\beta})$$

This solution gives the value of the investment opportunity or the investment timing option contingent on a value V^* where V^* corresponds to an optimal timing of the investment. This value maximizes the firm's market value. These equations also give the optimal investment rule in the presence of information costs. The opportunity cost is $C(V)$. The firm must invest only when V is greater than V^*. When V is less than V^*, then $V < I + C(V)$. Hence, the value of the project is less than its full cost, i.e. the direct cost I plus the opportunity cost of terminating the option. The value of $C(V)$ and the critical value V^* increase with the volatility parameter. They also increase with interest rates and information costs. Since the present value of an investment I made at an instant T is $I\ \exp[-(r+\lambda_c)T]$, the present value of the project received for this expenditure is $V\exp[-(\delta+\lambda_V)T]$. In this case, an increase in r and information costs reduces the present value of the cost of investing, but not the payoff.

Table 2.1 shows the optimal investment rule using equation (2.12). For $\sigma = 20\%$, $\beta = 2.1583$, the critical value of V at which it is optimal to invest is $V^* = 186.332$. For $\sigma = 30\%$, $\beta = 1.6667$, $V^* = 250$. For $\sigma = 40\%$, $\beta = 1.4354$, $V^* = 329.666$. In all cases, $V^* > I$. Note that for all values of $V < V^*$, $V < I + C(V)$, since the value of the project is less than its full cost. This corresponds to the cost I plus the opportunity cost of terminating the investment option. The critical value V^* is an increasing function of the volatility parameter for fixed levels of information costs. This confirms the analysis in Pindyck (1991). However, when δ increases, the critical value decreases and the expected rate of growth of V falls. In this case, it becomes costlier to wait rather than invest now. An increase in the interest rate and in the information costs leads also to an increase in the option value and the critical value. In fact, an increase in the discounting rate given by the interest rate and the information cost reduces the present value of the investment expenditure but does not reduce the payoff. The results can depend as before on the interaction between the different values that determine the sign of the quantity $(r - \delta + \lambda_V)$ and its magnitude with respect to σ^2.

Table 2.1 *Simulation of the effect of volatility on the value of the investment opportunity or the investment timing option C(V) given by equation (2.12) as a function of the project value V in the presence of information costs* $r = 4\%$, $\delta = 6\%$, $I = 100$, $\lambda_c = 1\%$, $\lambda_V = 2\%$

$[V]$	$C(V)$		
	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$
12	0.232	0.951	1.976
24	1.035	3.019	5.343
36	2.484	5.934	9.562
48	4.622	9.585	14.451
60	7.481	13.903	19.907
72	11.089	18.840	25.862
84	15.466	24.359	32.267
96	20.632	30.431	39.084
108	26.604	37.031	46.283
120	33.396	44.139	53.839

In this table, r is the interest rate, δ is the opportunity cost of delaying a project or a constant payout rate, I is the cost of investment or investment expenditure, σ is the volatility, λ_c (λ_V) is the information cost related to $C(V)$ (V).

2.3 THE INVESTMENT DECISION AND THE PROJECT'S VALUE WITH INCOMPLETE INFORMATION

It is well known that option pricing models can be used to value projects and to search for the optimal investment rule. While in equation (2.2) the value of the project is a random walk, it is more realistic to assume that the price of the output of the project is a random walk. We assume that the output price P follows the dynamics:

$$dP/P = \alpha\,dt + \sigma\,dz \tag{2.13}$$

and that $\delta = \mu - \alpha$. When the output is a storable commodity, then δ refers to the net marginal convenience yield from the storage activity. This refers to the difference between the flow of benefits and the storage costs offered by a marginal stored unit. Following the analysis in Pindyck (1991), let us assume that: marginal and average production costs are equal to a constant co; the project can be shut down when P falls below co; it can be restarted when P is above the cost co; and the project yields one unit of output per period. The sunk cost is I and the project is infinitely lived. What is the value of the project $V(P)$? What is the value of the firm's option to invest in this project?

The value of the project can be studied by viewing the project as a set of options, where for each option the firm pays co and receives P. The value of the option to invest can be determined with respect to the critical price, P^*, above which the firm will invest.

2.3.1 The project's value with incomplete information

Using the same analogy with respect to option pricing theory, it is possible to construct a portfolio which comprises a long position in the project, $V(P)$, and a short position in V_P units of P. This project leads to an instantaneous cash flow $j(P - co)dt - \delta V_P P\,dt$. The value $j = 1$ holds when the firm is producing, i.e. $P > co$, otherwise $j = 0$. In this context, the total return on the portfolio is:

$$dV - V_P\,dP + j(P - co)dt - \delta V_P P\,dt$$

In the presence of information costs regarding the project and the output in a risk-neutral world, this return must be:

$$(r + \lambda_V)V\,dt - (r + \lambda_P)V_P P\,dt$$

By applying Itô's lemma for dV and substituting for dP, we obtain the following differential equation for the value of the project V:

$$\tfrac{1}{2}V_{PP}\sigma^2 P^2 + (r + \lambda_P - \delta)PV_P - (r + \lambda_V)V + j(P - co) = 0 \tag{2.14}$$

This equation for the value of the project $V(P)$ must satisfy the following conditions:

$$V(0) = 0 \tag{2.15}$$

$$V(co^-) = V(co^+) \tag{2.16}$$

$$V_P(co^-) = V_P(co^+) \tag{2.17}$$

$$\lim(P \to \infty)\ V = (P/\delta) - (co/r) \tag{2.18}$$

The first condition shows that when P is zero, the project has no value. The following two conditions show that the project's value is a function which is continuous and smooth in the output price. The last condition shows that for large values of the price P, the project's value tends toward the difference between two perpetuities: a flow of revenue P discounted at δ and a flow of cost co discounted at r.

As in Pindyck, the solution has two parts according to the position of P with respect to co. Using the above equation, the first and the last conditions, the project value $V(P)$ in the presence of information costs is given by:

$$\begin{aligned} V(P) &= A_1 P^{\beta_1} && \text{if } P < co \\ V(P) &= A_2 P^{\beta_2} + (P/\delta) - (co/r) && \text{if } P \geq co \end{aligned} \tag{2.19}$$

where:

$$\beta_1 = \tfrac{1}{2} - [(r - \delta + \lambda_P)/\sigma^2] + \left\{\left[(r - \delta + \lambda_P)/\sigma^2 - \tfrac{1}{2}\right]^2 + 2(r + \lambda_V)/\sigma^2\right\}^{0.5}$$

and

$$\beta_2 = \tfrac{1}{2} - [(r - \delta + \lambda_P)/\sigma^2] - \left\{\left[(r - \delta + \lambda_P)/\sigma^2 - \tfrac{1}{2}\right]^2 + 2(r + \lambda_V)/\sigma^2\right\}^{0.5}$$

The values of the constants A_1 and A_2 can be found using the second and third conditions, or:

$$A_1 = \left\{[r - \beta_2(r - \delta)]/r\delta(\beta_1 - \beta_2)\right\} co^{(1-\beta_1)}$$

and

$$A_2 = \left\{[r - \beta_1(r - \delta)]/r\delta(\beta_1 - \beta_2)\right\} co^{(1-\beta_2)}$$

The delta (with respect to P) is $\Delta(P) = A_1 \beta_1 P^{\beta_1 - 1}$, if $P < co$.

It is important to note that this formula exhibits two information costs: the first λ_V concerns V and the second λ_P is linked to P. When $P < co$, the project is inactive and the value of the firm's options for the future are given by $A_1 P^{\beta_1}$, if P increases. When $P > co$, the project is active and the present value of the firm's future flow of profits is $(P/\delta) - (co/r)$. If the price falls, the firm has the option to stop production. The value of these options is $A_2 P^{\beta_2}$.

Table 2.2 simulates the project's value in the presence of information costs using the solution (2.19). The solution shows that the project is inactive when the output price is less than the marginal or average production cost. In this case, the solution gives the value of the options to produce in the future. The table gives the value of the project, the options to produce and to suspend production for different levels of the volatility parameter in the presence of information costs. For $\sigma = 20\%$, the parameters defining the two parts of the solution for different levels of the volatilities are $A_1 = 0.7142$, $A_2 = 2258.77$, $\beta_1 = 2$, $\beta_2 = -1.5$. For $\sigma = 40\%$, $A_1 = 5.0380$, $A_2 = 439.3214$, $\beta_1 = 1.4077$,

Table 2.2 *Simulation of the effect of volatility on the value of the project V(P) as a function of the output price P using equation (2.19) in the presence of information costs.* $r = 4\%$, $\delta = 4\%$, $co = 10$, $\lambda_V = 2\%$, $\lambda_P = 1\%$

$[P]$	$V(P)$		
	$\sigma = 0.20$	$\sigma = 0.40$	$\sigma = 0.50$
1	0.7142	5.0380	7.6799
2	2.8571	13.3671	18.8007
3	6.4285	23.6555	31.7409
4	11.4285	35.4663	46.0249
5	17.8571	48.5560	61.3994
6	25.7142	62.7641	77.7028
7	35.0000	77.9752	94.8216
8	45.7142	94.1012	112.6707
9	57.8571	111.0723	131.1840
10	71.4285	128.8313	150.3081
11	86.9131	147.4528	170.0774
12	104.3375	166.9060	190.4613
13	123.1900	187.0255	211.3447
14	143.1201	207.6887	232.6409
15	163.8807	228.8022	254.2833
16	185.2932	250.2938	276.2194
17	207.2254	272.1062	298.4076
18	229.5776	294.1937	320.8138
19	252.2735	316.5192	343.4106
20	275.2538	339.0521	366.1748

In this table, r is the interest rate, δ is an opportunity cost of delaying a project, co is the marginal or average production cost, σ is the volatility, λ_P (λ_V) is the information cost related to the output price P (the value of the project V).

$\beta_2 = -0.5327$. For $\sigma = 50\%$, $A_1 = 7.6799$, $A_2 = 353.6775$, $\beta_1 = 1.2916$, $\beta_2 = -0.3716$. It is notable that the higher the volatility, the greater the expected future flow of profit and the higher the value of the project.

Table 2.3 gives the values of the project and the options for different levels of information costs. For $\lambda_V = 3\%$, the parameters defining the two parts of the solution are $A_1 = 0.0644$, $A_2 = 644.6651$, $\beta_1 = 2.3333$, $\beta_2 = -0.6667$. For $\lambda_V = 4\%$, $A_1 = 0.0407$, $A_2 = 742.2238$, $\beta_1 = 2.4056$, $\beta_2 = -0.7389$. For $\lambda_V = 5\%$, $A_1 = 0.0244$, $A_2 = 850.5883$, $\beta_1 = 2.4748$, $\beta_2 = -0.8081$. Note that the higher the information costs relative to P, the lower the project's value. The effect of information costs seems to be important.

2.3.2 The investment decision and the option's value with incomplete information

Since we know the project's value, it is possible to determine the value of the firm's option to invest. This option depends on the output price P and its critical

Table 2.3 *Simulation of the value of the project V(P) in the presence of information costs using equation (2.19). $r = 4\,\%$, $\delta = 8\,\%$, $co = 10$, $\sigma = 30\,\%$, $\lambda_P = 1\,\%$*

$[P]$	$V(P)$		
	$\lambda_V = 0.03$	$\lambda_V = 0.04$	$\lambda_V = 0.05$
1	0.0644	0.0407	0.0244
2	0.3248	0.2160	0.1360
3	0.8367	0.5729	0.3711
4	1.6373	1.1446	0.7564
5	2.7559	1.9579	1.3141
6	4.2171	3.0359	2.0634
7	6.0426	4.3988	3.0218
8	8.2517	6.0652	4.2052
9	10.8617	8.0520	5.6283
10	13.8888	10.3785	7.3050
11	17.8383	13.6679	9.9969
12	22.9928	18.3105	14.1791
13	29.1018	24.0152	19.5271
14	35.9810	30.5722	25.8054
15	43.4920	37.8245	32.8387
16	51.5283	45.6519	40.4936
17	60.0067	53.9612	48.6669
18	68.8611	62.6783	57.2772
19	78.0381	71.7441	66.2596
20	87.4845	81.1106	75.5616

In this table, δ is the opportunity cost of delaying a project, co is the marginal or average production cost, λ_P (λ_V) is the information cost related to the output price P (the value of the project V).

level P^*. At this level, the firm exercises the option by paying an amount I in exchange for the project. Using the same steps as before, it is possible to construct a hedging portfolio to show that the firm's option to invest, $C(P)$, obeys the following partial differential equation:

$$\tfrac{1}{2}C_{PP}\sigma^2 P^2 + (r + \lambda_P - \delta)PC_P - (r + \lambda_c)C = 0 \tag{2.20}$$

Since there is a shadow cost of incomplete information for each asset, this formula shows an information cost for P, λ_P, and a cost for the option C, λ_c. The firm's option to invest must satisfy the following conditions:

$$C(0) = 0 \tag{2.21}$$

$$C(P^*) = V(P^*) - I \tag{2.22}$$

$$C_P(P^*) = V_P(P^*) \tag{2.23}$$

These conditions are similar to those of the preceding section. The main difference is that the payoff is a function of the output price P. The solution to equation (2.20) under the first condition (2.21) is:

$$\begin{aligned} C(P) &= aP^{\beta_1} && \text{if } P < P^* \\ C(P) &= V(P) - I && \text{if } P > P^* \end{aligned} \tag{2.24}$$

where:

$$\begin{aligned}\beta_1 = \tfrac{1}{2} &- [(r - \delta + \lambda_P)/\sigma^2] \\ &+ \left\{\left[(r - \delta + \lambda_P)/\sigma^2 - \tfrac{1}{2}\right]^2 + 2(r + \lambda_c)/\sigma^2\right\}^{0.5}\end{aligned}$$

The delta of equation (2.24) with respect to price is: $\Delta(P) = a\beta_1 P^{\beta_1 - 1}$, if $P < P^*$; that is the option with respect to price increases by a multiple of the price.

The solution for $V(P)$ is similar to equation (2.19). The two conditions (2.22) and (2.23) are used to search for the critical price P^* and the constant a. The value of a is:

$$a = (\beta_2 A_2/\beta_1)(P^*)^{(\beta_2 - \beta_1)} + (1/\beta_1\delta)(P^*)^{(1-\beta_1)} \tag{2.25}$$

The critical price P^* is the solution to the following equation:

$$[A_2(\beta_1 - \beta_2)/\beta_1](P^*)^{\beta_2} + (\beta_1 - 1)/(\beta_1\delta)P^* - (co/r) - I = 0 \tag{2.26}$$

where:

$$\begin{aligned}\beta_2 = \tfrac{1}{2} &- [(r - \delta + \lambda_P)/\sigma^2] \\ &- \left\{\left[(r - \delta + \lambda_P)/\sigma^2 - \tfrac{1}{2}\right]^2 + 2(r + \lambda_c)/\sigma^2\right\}^{0.5}\end{aligned}$$

Using an iterative procedure, the numerical solution to equation (2.26) gives the optimal investment rule. Note, as for the previous model, that an increase in σ produces a higher $V(P)$ for any price P. This result comes from the fact that a project is a set of call options and an increase in price volatility leads to higher call values.

This model shows the effect of uncertainty over future prices on the value of the project and on the timing of the investment decision. It includes the sunk costs regarding gathering information and analyzing data. The model can have practical implications if the produced commodity is a traded asset, like copper, coffee, oil, etc. In this case, the values of σ, δ and information costs can be inferred from futures, spot and options data as in Bellalah (1999a,b).

Table 2.4 gives the value of the firm's option to invest as a function of the price P using equations (2.24)–(2.26). The optimal investment rule in the presence of information costs is calculated for different levels of volatility. The table gives, for different levels of P, the opportunity cost of investing $C(P)$ and the project value. For $\sigma = 20\%$, the parameters defining the solution for different levels of the volatility parameter are $A_1 = 0.7142$, $A_2 = 2258.769$, $\beta_1 = 2$,

Table 2.4 *Simulation of the effect of volatility on the value of the firm's option to invest C(P) as a function of the price P using equations (2.24)–(2.26). The table gives the opportunity cost of investing C(P) and the value of the project V(P).* $r = 4\%$, $\delta = 4\%$, $I = 100$, $co = 10$, $\lambda_c = 2\%$, $\lambda_P = 1\%$

$[P]$	$C(P)$			$V(P)$		
	$\sigma = 0.20$	$\sigma = 0.30$	$\sigma = 0.40$	$\sigma = 0.20$	$\sigma = 0.30$	$\sigma = 0.40$
1	0.4691	1.8643	3.9632	0.7142	2.5339	5.0380
2	1.8764	5.6804	10.5154	2.8571	7.7204	13.3671
3	4.2220	10.8996	18.6090	6.4285	14.8141	23.6555
4	7.5058	17.3073	27.9002	11.4285	23.5230	35.4663
5	11.7278	24.7739	38.1973	17.8571	33.6712	48.5560
6	16.8881	33.2096	49.3744	25.7142	45.1365	62.7641
7	22.9866	42.5470	61.3404	35.0000	57.8273	77.9752
8	30.0233	52.7327	74.0262	45.7142	71.6712	94.1012
9	37.9983	63.7234	87.3768	57.8571	86.6090	111.0723
10	46.9115	75.4824	101.3472	71.4285	102.5911	128.8313
11	56.7629	87.9785	115.8998	86.9131	119.7923	147.4528
12	67.5526	101.1847	131.0027	104.3375	138.1913	166.9060
13	79.2805	115.0771	146.6280	123.1900	157.5257	187.0255
14	91.9466	129.6343	162.7516	143.1201	177.6052	207.6887
15	105.5510	144.8372	179.3520	163.8807	198.2884	228.8022
16	120.0935	160.6687	196.4102	185.2932	219.4679	250.2938
17	135.5744	177.1129	207.2254	241.0607	213.9089	272.1062
18	151.9934	194.1555	231.8327	229.5776	263.0015	294.1937
19	169.3507	211.7831	250.1672	252.2735	285.2383	316.5192
20	187.6462	229.9834	268.8997	275.2538	307.7289	339.0521

$\beta_2 = -1.5$, $a = 0.4691$. In this case, the critical price P^* at which the firm should exercise the option to invest an amount $I = 100$ to purchase the project is $P^* = 25.5516$. For $\sigma = 30\%$, $A_1 = 2.5339$, $A_2 = 692.866$, $\beta_1 = 1.6073$, $\beta_2 = -0.8295$, $a = 1.8643$, $P^* = 30.5528$. For $\sigma = 40\%$, $A_1 = 5.0380$, $A_2 = 439.3214$, $\beta_1 = 1.4078$, $\beta_2 = -0.5328$, $a = 3.9632$, $P^* = 35.9260$. The higher the volatility, the higher the critical price. This price is higher than the cost co, and $V(P^*) > I$. This indicates the project must show a large positive NPV before the firm decides to invest. An increase in the volatility parameter leads to a higher $V(P)$ for any value of P. In fact, since a project is viewed as a set of call options on future production, a higher volatility implies a higher option value. For any increase in the value of P, the opportunity cost of investing $C(P)$ increases the same in relative terms as the project's value $V(P)$, until $P > co$, when $C(P)$ increases by more than $V(P)$ in relative terms.

Table 2.5 shows the value of the firm's option to invest as a function of the price P using equations (2.24)–(2.26). The optimal investment rule is calculated for different levels of option information costs. The table shows, for different levels of P, the values of $C(P)$ and $V(P)$. For $\lambda_c = 3\%$, the parameters defining

Table 2.5 *Simulation of the effect of information costs on the value of the firm's option to invest as a function of the price P using equations (2.24)–(2.26). The table gives the opportunity cost of investing C(P) and the value of the project V(P).* $r = 4\%$, $\sigma = 20\%$, $\delta = 4\%$, $I = 100$, $co = 10$, $\lambda_P = 1\%$

$[P]$	$C(P)$			$V(P)$		
	$\lambda_c = 0.03$	$\lambda_c = 0.04$	$\lambda_c = 0.05$	$\lambda_c = 0.03$	$\lambda_c = 0.04$	$\lambda_c = 0.05$
1	0.2981	0.1967	0.1337	0.4825	0.3364	0.2406
2	1.3111	0.9459	0.6990	2.1233	1.6179	1.2576
3	3.1208	2.3703	1.8392	5.0512	4.0541	3.3091
4	5.7720	4.5484	3.6537	9.3422	7.7795	6.5739
5	9.2996	7.5408	6.2225	15.0519	12.8976	11.1956
6	13.7313	11.3975	9.6137	22.2249	19.4939	17.2972
7	19.0902	16.1615	13.8875	30.8984	27.6422	24.9868
8	25.3960	21.8709	19.0983	41.1046	37.4073	34.3622
9	32.6664	28.5599	25.2956	52.8722	48.8479	45.5125
10	40.9173	36.2596	32.5254	66.2266	62.0173	58.5205
11	50.1628	44.9988	40.8306	81.6570	77.4121	73.8924
12	60.4162	54.8042	50.2515	99.1333	94.9483	91.4927
13	71.6896	65.7006	60.8263	118.0977	114.0246	110.6788
14	83.9942	77.7116	75.5912	138.1727	134.2385	131.0249
15	97.3407	90.8593	85.5808	159.0949	155.3118	152.2395
16	111.7390	105.1648	99.8282	180.6757	177.0474	174.1178
17	127.1985	120.6481	115.3650	202.7767	199.3019	196.5120
18	143.7280	137.3282	132.2218	225.2949	221.9691	219.3136
19	161.3360	155.2236	150.4281	248.1518	244.9689	242.4413
20	180.0305	174.3518	170.0124	271.2867	268.2400	265.8330

the solution are $A_1 = 0.4825$, $A_2 = 2874.0292$, $\beta_1 = 2.1375$, $\beta_2 = -1.6375$, $a = 0.2981$, $P^* = 24.2464$. For $\lambda_c = 4\%$, $A_1 = 0.3364$, $A_2 = 3614.7458$, $\beta_1 = 2.2656$, $\beta_2 = -1.7656$, $a = 0.1967$, $P^* = 23.2856$. For $\lambda_c = 5\%$, $A_1 = 0.2406$, $A_2 = 4501.0051$, $\beta_1 = 2.386$, $\beta_2 = -1.886$, $a = 0.1337$, $P^* = 22.5434$. The higher the information cost, the lower the critical price at which the firm should invest. Note that the higher the information cost, the lower the values of options on future production and the smaller the values of the project.

2.3.3 Alternative models

The way the price of the commodity is represented may correspond to only some specific assets. It is possible to use other processes like the mean-reverting process in the description of the price dynamics. For example, suppose that the price follows a mean-reverting process:

$$\mathrm{d}P/P = \kappa(P' - P)\mathrm{d}t + \sigma\,\mathrm{d}z \tag{2.27}$$

In this context, the price P tends to revert back to its 'normal' level P'. The term P' may be the long-run marginal cost. Using the same arguments as before, the value $V(P)$ must satisfy the following equation:

$$\begin{aligned}&\tfrac{1}{2}V_{PP}\sigma^2P^2 + [(r + \lambda_P - \mu - \kappa)P + \kappa P']PV_P \\ &\qquad - (r + \lambda_V)V + j(P - co) = 0\end{aligned} \tag{2.28}$$

This equation for the value of the project $V(P)$ must satisfy the conditions (2.15) to (2.18). The value of the investment option $C(P)$ must obey the following equation:

$$\tfrac{1}{2}C_{PP}\sigma^2P^2 + [(r + \lambda_P - \mu - \kappa)P + \kappa P']PC_P - (r + \lambda_V)V = 0 \tag{2.29}$$

This equation must be solved under boundary conditions (2.21) to (2.23). Equations (2.28) and (2.29) can be solved by numerical methods.

2.4 COMPARISONS TO OTHER MODELS

The models presented show how to value a project and an investment opportunity as a set of options in the presence of information costs. These costs concern the gathering of information about the project, some of which may be research and development costs. These are sunk costs which are different from those used in the standard literature.

In Myers and Majd (1990), the sunk costs are related to the decision to exit or abandon a project for different reasons, including severance pay for workers and land reclamation for the case of a mine. In the Brennan and Schwartz (1985) model, the decision to invest contains the sunk cost of land reclamation.

However, these models do not account for information costs. The Brennan and Schwartz (1985) model shows how sunk costs of opening and closing a mine can explain the 'hysteresis' observed in extraction industries. However, their sunk costs are different from those presented here. If we recognize the fact that information costs are related to the decision to open and close a mine, then it is straightforward to derive their models in our context by accounting for information costs.

In a different context, Majd and Pindyck (1987) study sequential investment programs considered as compound options. Their analysis concerns a contingent plan for making sequential and irreversible expenditures. In their model, the firm invests continuously until the project is completed. Hence, each dollar spent buys an option to spend the following dollar. In that model, investment can be stopped and later restarted costlessly. Their analysis can be extended by accounting for the information costs concerning the study of the decision to suspend or to start the investment as in the above models.

The model proposed in Roberts and Weitzman (1981) stresses the role of information gathering in sequential investment. In this type of investment, early stages offer information about the net payoffs and costs of later stages. In fact, the engineering prototype production and testing stages provide information about the final costs and revenues. In the same way, the research and development and testing stages of the development of a new product contribute to the value of the final product. The shadow costs in that model are not very different from ours. However, their model assumes that prices and costs do not evolve stochastically. Hence, the analysis shows that the process of information gathering might add a shadow value to the early stages of the investment. Their results apply whenever information gathering, rather than waiting, yields information. In practice, gathering information can reduce uncertainty but it does not eliminate it. Therefore, it is possible to 'pay' informational sunk costs even when waiting for new information. This is one of the main ideas in our formulation.

2.5 CONCLUSION

This chapter develops some simple models for the analysis of the investment decision under uncertainty, irreversibility and sunk costs like shadow costs of incomplete information. We focus our analysis on investment in capital goods and R&D, but the results apply to a broad variety of problems showing irreversibility. We first provide a justification for the use of information costs in investment decisions. These costs refer to the shadow costs of incomplete information in the spirit of Merton's model. We develop a continuous-time model of irreversible investment in the presence of information costs. Then, we extend the model so that the price of the firm's output is random and the firm can stop production whenever the price falls below variable cost. This allows us to derive the value of the project and the value of the firm's option to invest in the project as well as the optimal investment rule in the presence of information costs. We provide some analytical solutions and equations that can be solved by numerical methods. Several simulations are run in our illustrations of the effects of various parameters on project valuation and real option prices. The results are roughly similar to those obtained in standard models, with an important difference. The behavior of the option to invest and the project value depend on the possible interaction between the different values that determine the sign of the algebraic sum of the interest rate, the opportunity cost of delaying the project, the values of the information costs and the variance parameter. The models can provide some insights into the importance of shadow costs or information costs in the study of the irreversibility and the ranges of opportunity costs implied in this context. This analysis can be applied to capital goods, R&D, labor markets, natural resources and the environment, and to a broad variety of investment problems in the presence of irreversibility and information costs. Our analysis can be applied to the valuation of all well-known real options in the presence of information costs. It can also be applied to the valuation of political risks.

2.6 APPENDIX

An alternative derivation of the formula is as follows.

The relation between an option's beta and its underlying security's beta is:

$$\beta_c = V(C_V/C)\beta_V \tag{A2.1}$$

where β_c is the option's beta and β_V is the stock's beta.

According to Merton's model, the expected return on a security should be:

$$R_V - r = \beta_V[R_{\mathrm{m}} - r] + \lambda_V - \beta_V\lambda_{\mathrm{m}} \tag{A2.2}$$

where R_V is the expected return on the asset V over a short interval of time. Equation (A2.2) may also be written as:

$$E(\mathrm{d}V/V) = [r + \beta_V(R_{\mathrm{m}} - r) + \lambda_V - \beta_V\lambda_{\mathrm{m}}]\mathrm{d}t \tag{A2.3}$$

Using Merton's model, the expected return on a call option should be:

$$E(\mathrm{d}C/C) = [r + \beta_c(R_{\mathrm{m}} - r) + \lambda_c - \beta_c\lambda_{\mathrm{m}}]\mathrm{d}t \tag{A2.4}$$

Note that an information cost λ_c appears in the expression of the option's expected return. Multiplying equation (A2.3) and equation (A2.4) by V and C yields:

$$E(\mathrm{d}V) = [rV + V\beta_V(R_\mathrm{m} - r) + V\lambda_V - V\beta_V\lambda_\mathrm{m}]\mathrm{d}t \tag{A2.5}$$

$$E(\mathrm{d}C) = [rC + C\beta_c(R_\mathrm{m} - r) + C\lambda_c - C\beta_c\lambda_\mathrm{m}]\mathrm{d}t \tag{A2.6}$$

When substituting for the option's elasticity from equation (A2.1), the equation for $E(\mathrm{d}C)$ becomes after transformation:

$$E(\mathrm{d}C) = [rC + VC_V\beta_V(R_\mathrm{m} - r) + C\lambda_c - VC_V\beta_V\lambda_\mathrm{m}]\mathrm{d}t \tag{A2.7}$$

Taking expectations of both sides and replacing $\mathrm{d}V$, we get:

$$E(\mathrm{d}C) = \tfrac{1}{2}C_{VV}\sigma^2V^2\,\mathrm{d}t + C_V E(\mathrm{d}V) + C_t\,\mathrm{d}t \tag{A2.8}$$

Replacing the expected value of $\mathrm{d}V$ gives:

$$\begin{aligned} E(\mathrm{d}C) &= \tfrac{1}{2}C_{VV}\sigma^2V^2\,\mathrm{d}t \\ &\quad + C_V[rV + V\beta_V(R_\mathrm{m} - r) + V\lambda_V - V\beta_V\lambda_\mathrm{m}]\,\mathrm{d}t + C_t\,\mathrm{d}t \end{aligned} \tag{A2.9}$$

Combining and rearranging yields the following differential equation:

$$\tfrac{1}{2}C_{VV}\sigma^2V^2 + (r + \lambda_V)VC_V - (r + \lambda_c)C + C_t = 0 \tag{A2.10}$$

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Chapter 3

Investment under economic and implementation uncertainty

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SUMMARY

Some investment decisions are exposed to uncertainty over their implementation phase apart from the underlying economic uncertainty. We provide a general way of introducing implementation uncertainty, which includes prior research as a special case. The generality of our treatment stems from the fact that implementation uncertainty is allowed to affect both the level and the timing of project profitability. In a case explicitly addressed, implementation uncertainty might even cause earlier investment if the probability of uncertainty resolution exceeds the opportunity cost of delaying investment. Investment will be earlier, the higher the effect of uncertainty resolution on project profitability.

3.1 INTRODUCTION

When a firm is contemplating entry into a new market or investment in a research project, its decision must be made in an uncertain environment and in most cases it entails costs, which are at least partly irreversible. Uncertainty arises from the stochastic nature of the economic value of the investment. Since the return to a new product design or production process is derived from product market profitability, the value of the investment is affected by fluctuations in expected cash flows or market demand. On the other hand, R&D or investment expenditures may be sunk costs either because of the specificity of their nature in a particular firm/industry or because of what is termed the 'lemons' effect (see Akerlof, 1970).

A growing line of research known as 'real options', by exploiting the analogy between real and financial investment decisions, has stressed the fact that uncertainty and irreversibility give rise to option values which must be taken into account when making optimal investment/entry decisions. This insight, applied to the analysis of natural resource extraction by Brennan and Schwartz (1985), improves upon traditional investment appraisal approaches (decision trees or NPV-based criteria) by allowing the value of delay and the importance of flexibility to be incorporated into the assessment.[1] Since then, a substantial number of papers have explored this idea. McDonald and Siegel (1985, 1986) and Dixit (1989) price option values associated with entry and exit from a productive capacity. Pindyck (1988) examines irreversible investment decisions where capacity utilization is a choice variable. Trigeorgis (1993) deals with the interaction of real option opportunities, while Dixit and Pindyck (1994) survey the literature as a whole.

However, most of the real option models seem to abstract from the complexities and the uncertainty surrounding the implementation phase of a project. The characteristics of the implementation stage of an investment will have an impact both on the timing and the level of profitability realized. For example, large-scale projects have substantial time lags between the decision to invest and the realization of cash returns, thus the length of the time lag has to be taken into consideration[2] (notable examples would be the aircraft and mining industries). In product markets, the level of profitability of a new product will depend on distribution channels and the accessibility to selling points. The ability to service the whole potential market is of great concern in some commodity markets.[3] In technology-intensive industries, the uncertainty concerning the discovery of innovation will have an impact on the profitability of the project. In industries where marketing considerations are important, the relative time of adoption of a new product by consumers will affect the timing and level of profitability realized.[4]

Notable exceptions in the real options literature are Majd and Pindyck (1987) and Weeds (1999). The former look at option values and sequential investment decisions when projects have a 'time-to-build' element. The rate of construction in their model is deterministic and cash flows accrue to the investor only when the project is completed. Weeds, on the other hand, deals with the technological uncertainty of research projects by allowing cash flows to be realized only after a random event (i.e. discovery). However, uncertainty over the implementation phase in these models only affects the timing and not the level of profitability realized from the project.

In this chapter we provide a general framework for incorporating uncertainty over the implementation phase of investment, which allows both the timing and the level of profitability realized to be affected. Implementation uncertainty is introduced as an exogenous parameter a, which summarizes the effect of this uncertainty on the cash flows realized. In addition, this implementation uncertainty is allowed to be resolved randomly according to a Poisson arrival. Thus, the optimal investment timing of our firm will also be directly influenced by the implementation uncertainty, through the effect on the level of cash flows.

Our findings imply that uncertainty over the implementation phase of a project might cause earlier or later optimal investment compared to the corresponding certainty case. The relative magnitudes of the *probability of uncertainty resolution* and the *opportunity cost of delay* will determine which is the case. Moreover, the value of the project is also affected by the *direction* (favorable or not) of the resolution of implementation uncertainty. However, in a case explicitly addressed in this chapter, favorable resolution does not necessarily imply higher project value.

The exposition is based on the model by Dixit (1989), modified to allow for implementation uncertainty. Specifically, as implementation uncertainty is eliminated, the model collapses to his model. More importantly, our framework is general enough to encompass the work of Majd and Pindyck (1987) and Weeds (1999) as special cases, and has the ability to generate a range of other possible outcomes.

The structure of the chapter is as follows. Section 3.2 describes our framework for incorporating implementation uncertainty and presents the basic setting. Section 3.3 examines the optimal investment strategy of our firm. The value of the project and the critical value that triggers investment are derived in closed form and comparative statics are presented. Section 3.4 concludes.

3.2 THE MODEL

A single risk-neutral firm is contemplating investment in a new project, facing no actual or potential competitors in the area. The decision to invest is assumed to be irreversible and the profitability of the project, as summarized by the state variable x, evolves stochastically over time. The market profit flow, x, evolves exogenously according to a geometric Brownian motion with drift given by the following expression:

$$\mathrm{d}x_t = \mu x_t \,\mathrm{d}t + \sigma x_t \,\mathrm{d}W_t \tag{3.1}$$

where $\mu \in [0, r)$ is the drift parameter, measuring the expected growth rate[5] of x, $\sigma > 0$ is the instantaneous standard deviation or volatility parameter, and $\mathrm{d}W$ is the increment of a standard Wiener process, $\mathrm{d}W_t \sim N(0, \mathrm{d}t)$.

Note that geometric Brownian motion is a Markov process with continuous sample paths. The probability distribution for the value of the process at any future date depends only on its own current value, i.e. it is unaffected either by past values of the process or by any other current information. Thus, to make a best estimate of the future value of the process, all that is needed is the current level of x, along with the parameter values μ and σ.

When the firm invests, it pays a sunk cost $K > 0$. To introduce implementation uncertainty, we assume that upon investment the firm receives ax with $a \in [0, 1]$. The implementation parameter a collectively summarizes any exogenous effect of the implementation phase on the project value, i.e. time-to-build effects, distribution difficulties, technological uncertainty, inability to service the whole market, etc. Obviously, letting $a = 1$ abstracts from any complications arising in the implementation phase of the project. If $a < 1$, we allow the implementation uncertainty to be resolved randomly according to a Poisson arrival with parameter (or hazard rate) h, i.e.

$$\mathrm{d}q = \begin{cases} \xi(a) & \text{w.p.} \quad h\,\mathrm{d}t \\ 0 & \text{w.p} \quad 1 - h\,\mathrm{d}t \end{cases} \tag{3.2}$$

where ξ, the effect of the uncertainty resolution (i.e. the Poisson jump) on the value of the project, depends on the parameter a, and 'w.p.' = with probability.

The flexibility that our framework provides could be displayed with reference to Figure 3.1. In this figure, let T stand for the random time at which our firm optimally invests and $\tau > T$ stand for the expected time at which implementation uncertainty is resolved, conditional on investment having occurred. Panel (a) corresponds to the case where implementation uncertainty only affects the timing of the project cash flows, i.e. the cases treated by Majd and Pindyck (1987) and Weeds (1999). In Weeds (1999) the firm receives no cash flows ($a = 0$) even after investment, until implementation uncertainty is resolved (in the form of a Poisson jump, $a = 1$). Similarly, in Majd and Pindyck (1987) cash flows accrue to the project only when implementation (in their case building the project) is completed.[6] Panels (b) and (c) represent intermediate cases where implementation uncertainty is allowed to affect both the level and the timing of the project's profitability. Implementation uncertainty can be resolved favorably (panel b) or unfavorably (panel c) for our firm. These cases demonstrate the generality of our framework.

In the next section we derive the optimal investment behavior for our firm under the 'regime' of panel (b). Similar reasoning would provide the optimal solution for any possible uncertainty resolution case.

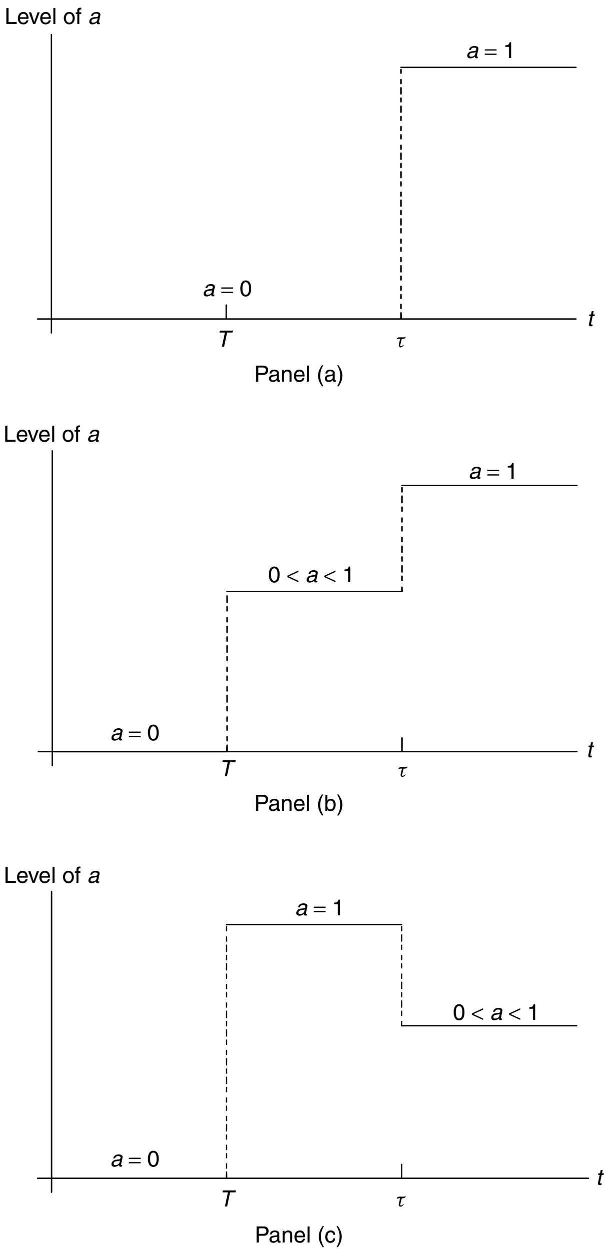

Figure 3.1 *Alternative formulations depending on different evolution for the exogenous implementation parameter $a \in [0, 1]$. T is the optimal investment/entry time and τ is the random time at which implementation uncertainty is resolved. In panel (a), which corresponds to the cases treated by Majd and Pindyck (1987) and Weeds (1999), our firm receives no cash flows ($a = 0$) until implementation uncertainty is resolved. Our specification allows the firm to earn cash flows from the time of entry with the possibility of favorable (panel b) or unfavorable (panel c) uncertainty resolution*

3.3 OPTIMAL INVESTMENT TIMING

The investment decision that our firm faces could be formulated as the following optimal stopping time problem:

$$\begin{aligned} V^* = \max_T E \Big\{ \mathrm{e}^{-rT} \Big[\int_T^\infty \mathrm{e}^{-r(t-T)} a x_t \,\mathrm{d}t \\ + \int_\tau^\infty \mathrm{e}^{-r(t-T)} (1-a) x_t \,\mathrm{d}t - K \Big] \Big\} \end{aligned} \tag{3.3}$$

where E denotes the risk-neutral expectation, T is the unknown future stopping time at which the investment is made, and τ is the random time of the Poisson jump conditional on investment having occurred. Equation (3.3) simply says that upon investment at T the firm gets ax_t perpetually and after a random time $\tau \geq T$ (when implementation uncertainty is resolved) the remaining fraction $(1-a)x_t$. The stopping time T is optimally chosen so as to maximize this equation.

Note that the only decision variable for our firm is the stopping time T at which the cost K is sunk. The resolution of implementation uncertainty is random and our firm has no control over it. Our solution methodology draws on the *Hamilton–Bellman–Jacobi principle of optimality*[7] (hereafter HBJ). Namely, the value of the project before and after investment is examined and the optimal investment timing is derived from boundary conditions so as to maximize the value of the firm.

3.3.1 Value of the project before investment

Let $V_0(x)$ denote the value of the project in the *continuation region* (values of x for which it is not yet optimal to invest). Prior to investment, the firm only holds the opportunity to invest in the project and get the flow ax_t perpetually upon entry. It has no cash flows but may experience a capital gain or loss on the value of its option. Hence, in this region the HBJ equation for the value of the investment opportunity $V_0(x)$ is given by:

$$rV_0(x)\mathrm{d}t = E[\mathrm{d}V_0(x)] \tag{3.4}$$

Expanding $\mathrm{d}V_0(x)$ using Itô's lemma, we can write:

$$\mathrm{d}V_0(x) = V_0'(x)\mathrm{d}x + \tfrac{1}{2}V_0''(x)(\mathrm{d}x)^2$$

Substituting from equation (3.1) and noting that $E[\mathrm{d}W_t] = 0$, we can write:

$$E[\mathrm{d}V_0(x)] = \left[\mu x V_0'(x) + \tfrac{1}{2}\sigma^2 x^2 V_0''(x)\right]\mathrm{d}t$$

Thus the HBJ equation (3.4) gives rise to the following second-order differential equation:

$$\tfrac{1}{2}\sigma^2 x^2 V_0''(x) + \mu x V_0'(x) - r V_0(x) = 0 \tag{3.5}$$

From equation (3.1) it can be seen that if x ever goes to zero[8] it then stays there. Therefore the option to invest in the project should be worthless and $V_0(x)$ must satisfy the following boundary condition:

$$\lim_{x \to 0_+} V_0(x) = 0 \tag{3.6}$$

Solving the differential equation (3.5) subject to (3.6), the following solution for the value of the project is obtained:

$$V_0(x) = B(a)x^{\lambda} \tag{3.7}$$

where $B(a) \geq 0$ is a constant whose value is determined as part of the solution and λ is the positive root of the *fundamental quadratic*:[9]

$$Q = \tfrac{1}{2}\sigma^2 \lambda(\lambda - 1) + \mu\lambda - r = 0 \tag{3.8}$$

3.3.2 Value of the project after investment

Now consider the value of the project in the *stopping region*, i.e. values of x for which it is optimal to undertake the investment at once. Note that in this region, the value of the project is exposed not only to economic uncertainty – the evolution of the state variable in equation (3.1), but also to implementation uncertainty – the independent Poisson jump in equation (3.2). Moreover, since investment is irreversible, the value of the project in the stopping region, $V_1(x)$ is given by the expected value alone with no option value terms. Thus the HBJ equation in this region is given by:

$$rV_1(x)\mathrm{d}t = E[\mathrm{d}V_1(x)] + ax\,\mathrm{d}t \tag{3.9}$$

where the last term in equation (3.9) recognizes the fact that in this region the firm receives a perpetual cash flow of magnitude ax every instant. Expanding $\mathrm{d}V_1(x)$ using Itô's lemma yields:

$$\mathrm{d}V_1(x) = V_1'(x)\mathrm{d}x + \tfrac{1}{2}V_1''(x)(\mathrm{d}x)^2 + V_1'(x)\mathrm{d}q$$

Substituting from (3.1), (3.2) and noting that $E[\mathrm{d}W_t] = 0$ gives:

$$E[\mathrm{d}V_1(x)] = \left\{rxV_1'(x) + \tfrac{1}{2}\sigma^2x^2V_1''(x) + hE\left[V_1(x + z(a)) - V_1(x)\right]\right\} \mathrm{d}t$$

where the last term in the square brackets captures the effect of the Poisson jump on the value of the project. It is easy to see that for the case where $0 < a < 1$ after investment at T [panel (b) in Figure 3.1], this term will be $E[V_1(x + \xi(a)) - V_1(x)] = (1 - a)x/(r - \mu)$ and the previous equation would now be:

$$E[\mathrm{d}V_1(x)] = \left[\mu xV_1'(x) + \frac{1}{2}\sigma^2x^2V_1''(x) + h\frac{(1-a)x}{r-\mu}\right] \mathrm{d}t$$

which simply states that with conditional probability $h\,\mathrm{d}t$, implementation uncertainty is resolved and the remaining fraction of profitability $(1 - a)x$ is realized perpetually. Substituting in equation (3.9) and rearranging gives rise to the following second-order differential equation:

$$\tfrac{1}{2}\sigma^2x^2V_1''(x) + \mu xV_1'(x) + H(a, h)x - rV_1(x) = 0 \tag{3.10}$$

where $H(a, h) \equiv [a(r - \mu - h) + h]/(r - \mu)$. Keeping in mind that the firm has no option value components in this region, a simple substitution would verify that the solution for the value of the active firm is given by:

$$V_1(x) = \frac{x}{r-\mu}H(a, h) \tag{3.11}$$

3.3.3 Value of the project and the investment trigger

The boundary between the continuation and the stopping region is given by a critical value of the stochastic process or trigger point such that continued delay (immediate investment) is optimal for values of x below (above) this level. Let $\bar{x}$ denote this critical value of the state variable. The optimal stopping time T is then defined as the first time that the state variable enters the interval $[\bar{x}, \infty)$, i.e.

$$T = \inf\{t > 0 : x \geq \bar{x}\}$$

At the boundary between regions, the critical value $\bar{x}$ must satisfy the following conditions by arbitrage:

$$V_0(\bar{x}) = V_1(\bar{x}) - K \tag{3.12}$$

$$V_0'(x) = V_1'(x) \tag{3.13}$$

Condition (3.12), also known as the *value-matching condition*, simply states that when the critical value $\bar{x}$ is reached, the firm exercises its option to invest in the project by paying the sunk cost K, to get the value of the active project. Condition (3.13), also known as the *smooth-pasting condition*, says that at the critical value, the value function of the continuation and stopping regions must meet smoothly to ensure optimality.[10] Substituting expressions for the value functions from equations (3.7) and (3.11), these two conditions uniquely determine the critical value $\bar{x}$ and the unknown coefficient $B(a)$. The result can be summarized in the following proposition.

Proposition 3.1 *Under the setting described in the last two sections, the value of the project prior to investment is*:

$$V(x) = \left(\frac{\bar{x}}{r-\mu}H(a,h) - K\right)\left(\frac{x}{\bar{x}}\right)^{\lambda} \tag{3.14}$$

where $H(a,h)$ and λ are as previously defined. The firm's optimal strategy consists of entering when x_t first crosses $\bar{x}$ from below, where $\bar{x}$ is given by:

$$\bar{x} = \frac{\lambda}{\lambda-1}(r-\mu)K\frac{1}{H(a,h)} \tag{3.15}$$

Proofs of this and subsequent propositions are provided in the appendix (Section 3.5). The project value and optimal investment trigger in this model are essentially modified versions of the value and optimal exercise strategy of an infinite American call option (see Merton, 1973).

Figure 3.2 shows the value of the project and critical investment threshold for different values of a and how they converge to the implementation certainty case ($a = 1$).[11] Let $\phi \equiv \bar{x}/x_{\mathrm{m}}$ denote the ratio of the investment trigger of our setting to the trigger that would emerge by abstracting from implementation phase uncertainty (denoted by x_{m}). If $\phi > 1$ ($\phi < 1$), our firm optimally invests later (earlier) in the face of implementation uncertainty. Compared to the case without implementation uncertainty, the optimal policy depends on the extra term H, which is a function of a (the parameter that determines the magnitude of the jump), h (the parameter that determines the timing of the Poisson arrival), and $\delta \equiv r - \mu$ (the opportunity cost of delaying investment and instead keeping the option to invest alive).[12] A simple substitution would verify that ϕ is equal to the reciprocal of the $H(a,h)$ term, i.e.

$$\phi = \frac{1}{H(a,h)} = \frac{\delta}{a\delta + (1-a)h} \tag{3.16}$$

From equation (3.16), the next result follows naturally.

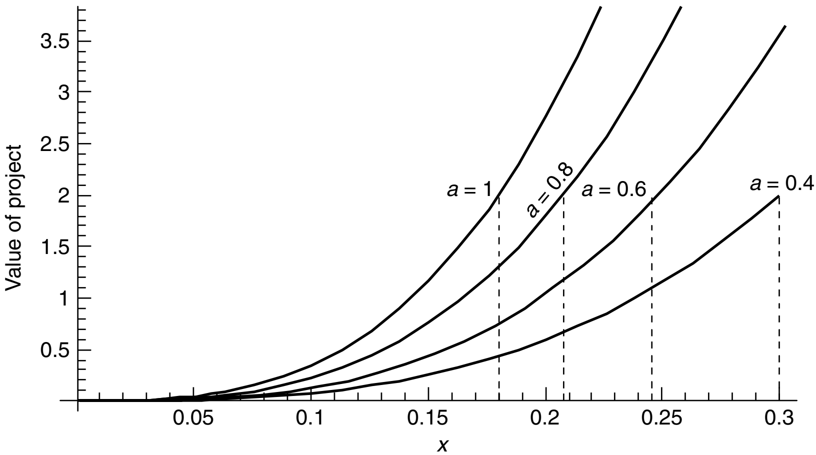

Figure 3.2 *The value of the project as a function of the state variable x, for different values of the implementation parameter $a \in [0, 1]$. The dashed lines indicate the optimal investment threshold $\bar{x}$ for each case, which are (rounded to four decimals): 0.1800, 0.2077, 0.2455 and 0.3000 for $a = 1$ to $a = 0.4$, respectively. The parameters used are: $r = 0.03$, $\mu = 0$, $\sigma = 0.1$, $K = 4$ and $h = 0.01$*

Corollary 3.1 *If the hazard rate $h < \delta$, ϕ exceeds unity. If $h > \delta$, ϕ is less than one. In the special case where $h = \delta$, implementation uncertainty considerations are not important as far as the investment policy is concerned ($\phi = 1$).*

The above corollary simply states that the investment timing in our setting might lead or lag the timing when implementation uncertainty is not present, depending on the magnitude of the Poisson density and the profitability return shortfall.

Figure 3.3 plots the critical value $\bar{x}$ as a function of a under the two different 'regimes' of Corollary 3.1. We also include the implementation certainty critical value x_{m}, which is between 'regimes' as Corollary 3.1 states. As we move away from implementation certainty ($a = 1$), our firm optimally enters later (earlier) depending on the magnitude of the conditional probability of the profitability jump against the 'dividend-like' shortfall in the state variable. If our firm knows that it will take a considerable amount of time to fully exploit the profitability of the market (low h), option values are preserved in the face of implementation uncertainty and our firm will optimally delay incurring the sunk cost K until further into the future. However, if implementation uncertainty is short-lived (high h), our firm will be eager to undertake the project earlier and obtain the cash flows that actual investment guarantees.

3.3.4 Comparative statics and numerical example

Our analytical solution in Proposition 3.1 lends itself easily to the examination of comparative statics over key parameters of the specification. Differentiating

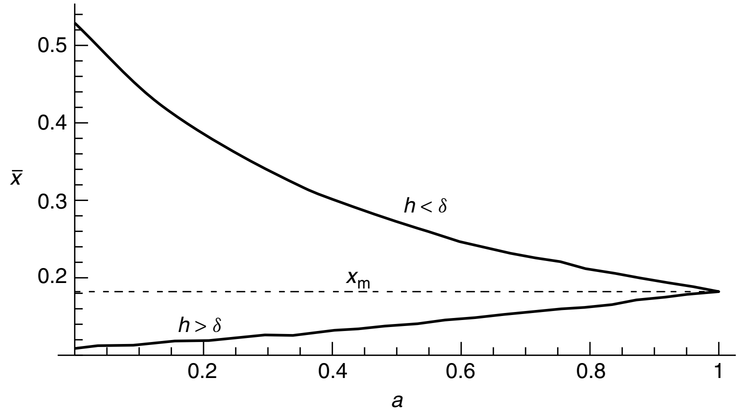

Figure 3.3 *The critical investment threshold $\bar{x}$ as a function of the implementation parameter $a \in [0, 1]$, under two different 'regimes': $h > \delta$ ($h = 0.05$, $\delta = 0.03$) and $h < \delta$ ($h = 0.01$, $\delta = 0.03$). The dashed line corresponds to the critical investment threshold with implementation certainty, x_m (i.e. $h = \delta$). The parameters used are: $r = 0.03$, $\mu = 0$, $\sigma = 0.1$ and $K = 4$*

equation (3.14) with respect to the implementation parameter a yields:

$$\frac{\partial V}{\partial a} = [\lambda(\lambda - 1)K]^{1-\lambda} \left[\frac{x}{r - \mu} H(a, h) \right]^{\lambda} (\delta - h) \tag{3.17}$$

The sign of this partial derivative depends on the sign of the last term, $\delta - h$. Thus, when the opportunity cost of delay exceeds the intensity of the uncertainty resolution, the greater the effect of the implementation uncertainty on profitability (higher a), the higher the value of the project. However, if the conditional probability of uncertainty resolution is high enough, a higher a would lead to a project value decrease. This is illustrated in the numerical example summarized in Table 3.1.

Project values and optimal investment thresholds (in parentheses) are reported for a set of varying parameters.[13] As the implementation parameter a increases (from 0.75 to 0.95), the value of the project increases (from 0.0136 to 0.0194) when $\delta = 0.02 > h = 0.01$, but decreases (from 0.0536 to 0.0249) when the inequality is reversed (δ is 0.02 and h increases to 0.05).

Figure 3.4 plots the value of the project as a function of a under both cases. By examining Figures 3.3 and 3.4, our model suggests that when $h > \delta$, our firm optimally enters earlier than in the implementation certainty case and the value of the project is negatively related to a. On the other hand, in cases where the opportunity cost of delay exceeds the probability of uncertainty resolution, our firm optimally delays incurring the sunk cost K until further into the future and the value of the project will increase with a. Figure 3.4 and Table 3.1 show that changes in a affect project value in a non-linear fashion, since a, δ and h

Table 3.1 *Numerical example of the project value $V(x, a)$ for a range of the key parameters. The optimal investment threshold $\bar{x}$ is reported in parentheses*

	$a = 0.5$								
	$\sigma = 0.05$			$\sigma = 0.10$			$\sigma = 0.25$		
	$\delta = 0.02$	$\delta = 0.03$	$\delta = 0.07$	$\delta = 0.02$	$\delta = 0.03$	$\delta = 0.07$	$\delta = 0.02$	$\delta = 0.03$	$\delta = 0.07$
$h = 0.01$	0.0002	0.0000	0.0000	0.0099	0.0008	0.0000	0.1466	0.0413	0.0012
	(0.1369)	(0.2207)	(0.5600)	(0.1750)	(0.2700)	(0.6396)	(0.3472)	(0.4800)	(0.9446)
$h = 0.05$	0.0086	0.0001	0.0000	0.0867	0.0065	0.0000	0.4979	0.1251	0.0029
	(0.0587)	(0.1103)	(0.3733)	(0.0750)	(0.1350)	(0.4264)	(0.1488)	(0.2400)	(0.6297)
$h = 0.10$	0.0994	0.0012	0.0000	0.3450	0.0279	0.0000	1.0841	0.2720	0.0059
	(0.0342)	(0.0697)	(0.2635)	(0.0437)	(0.0831)	(0.3010)	(0.0868)	(0.1477)	(0.4445)
	$a = 0.75$								
$h = 0.01$	0.0003	0.0000	0.0000	0.0136	0.0014	0.0000	0.1756	0.0552	0.0021
	(0.1208)	(0.1839)	(0.4308)	(0.1544)	(0.2250)	(0.4920)	(0.3064)	(0.4000)	(0.7266)
$h = 0.05$	0.0037	0.0000	0.0000	0.0536	0.0047	0.0000	0.3796	0.1057	0.0033
	(0.0708)	(0.1226)	(0.3500)	(0.0905)	(0.1500)	(0.3998)	(0.1796)	(0.2667)	(0.5903)
$h = 0.10$	0.0244	0.0003	0.0000	0.1559	0.0135	0.0000	0.6928	0.1845	0.0051
	(0.0467)	(0.0865)	(0.2835)	(0.0597)	(0.1059)	(0.3239)	(0.1184)	(0.1882)	(0.4783)
	$a = 0.95$								
$h = 0.01$	0.0006	0.0000	0.0000	0.0194	0.0025	0.0000	0.2140	0.0748	0.0036
	(0.1053)	(0.1522)	(0.3343)	(0.1346)	(0.1862)	(0.3819)	(0.2671)	(0.3310)	(0.5639)
$h = 0.05$	0.0010	0.0000	0.0000	0.0249	0.0030	0.0000	0.2464	0.0832	0.0038
	(0.0955)	(0.1424)	(0.3246)	(0.1221)	(0.1742)	(0.3708)	(0.2423)	(0.3097)	(0.5476)
$h = 0.10$	0.0016	0.0000	0.0000	0.0330	0.0038	0.0000	0.2888	0.0942	0.0041
	(0.0855)	(0.1318)	(0.3133)	(0.1094)	(0.1612)	(0.3578)	(0.2170)	(0.2866)	(0.5284)
	$a = 1$								
$h = \delta$	0.0007	0.0000	0.0000	0.0207	0.0027	0.0000	0.2220	0.0789	0.0039
	(0.1027)	(0.1471)	(0.3200)	(0.1312)	(0.1800)	(0.3655)	(0.2604)	(0.3200)	(0.5397)

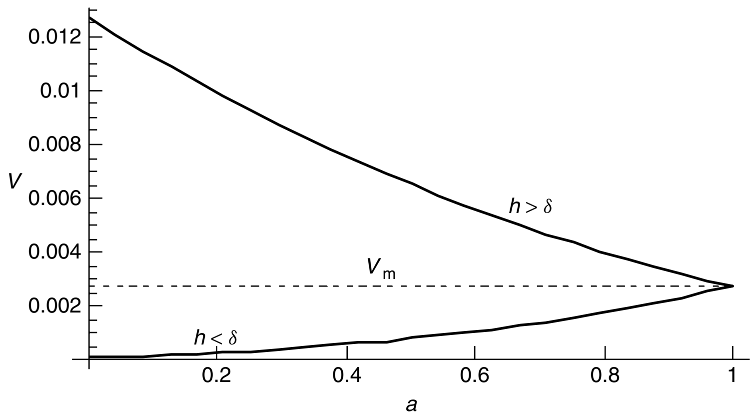

Figure 3.4 *The value of the project $V(x, a)$ as a function of the implementation parameter $a \in [0, 1]$, under two different 'regimes': $h > \delta$ ($h = 0.05$, $\delta = 0.03$) and $h < \delta$ ($h = 0.01$, $\delta = 0.03$). The dashed line corresponds to the value of the project under implementation certainty, V_m (i.e. $h = \delta$). The parameters used are: $r = 0.03$, $\mu = 0$, $\sigma = 0.1$, $K = 4$ and $x = 0.02$*

are included in the *non-linear* part of equation (3.17) through the $H(a, h)$ term. The effect of a changing a seems to induce larger decreases in value ($h > \delta$) when a is low but larger increases ($h < \delta$) when it is high.

Differentiating now with respect to the hazard rate of the Poisson arrival gives:

$$\frac{\partial V}{\partial h} = (1-a)\left[\frac{\lambda}{\lambda-1}K\right]^{1-\lambda}\left[\frac{x}{(r-\mu)^2}\right]^{\lambda} H(a, h)(r-\mu) \geq 0 \qquad (3.18)$$

which implies that an increase in the probability of the uncertainty resolution has a positive impact on the value of the project.

Figure 3.5 plots the value function against the hazard rate [equation (3.18)] and the numerical example confirms the non-linear positive dependence evident in the graph. The positive dependence of project value on h is more pronounced the lower the level of volatility ($\sigma = 0.05$) and a. The intuition is that lower market volatility means a larger fraction of the project's upside potential is represented by the Poisson arrival, thus an increase in intensity (h) has a more significant impact on the firm's option. On the other hand, bear in mind that $h\,\mathrm{d}t$ is the conditional probability that implementation uncertainty is resolved, and the remaining $(1-a)x$ is realized. Thus, the larger the part of market profitability subject to implementation uncertainty (i.e. higher a), the higher the impact of an increased intensity on project value.

All other comparative statics yield intuitive results, namely $\partial V/\partial\sigma > 0$, $\partial V/\partial\mu > 0$, $\partial V/\partial r < 0$. An increase in σ and μ (decrease in r) causes an

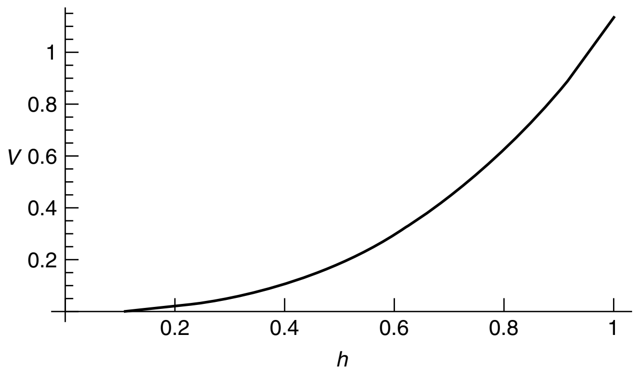

Figure 3.5 *The value of the project $V(x, a)$ as a function of the intensity of the Poisson arrival $h \in [0, 1]$. The parameters used are: $r = 0.03$, $\mu = 0$, $\sigma = 0.1$, $K = 4$, $x = 0.02$ and $a = 0.8$*

increase in the value of the project. The effect of a volatility increase is more evident for high values of a and low values of the 'return shortfall' δ.

3.4 CONCLUDING REMARKS

Part of the uncertainty inherent in project investments arises from the unknown future economic value of the project. However, in some industries there is also significant uncertainty over the implementation phase of a project. In product markets, the sales of a new product at launch will be influenced by the uncertainty over distribution channels and access to customer selling points. In large-scale projects (aircraft, mining), uncertainty over the construction and completion of the project is significant. It is natural to expect that in such cases, the level and the timing of the project's profitability will be affected by the resolution of implementation uncertainty.

All previous models of investment under uncertainty, with the exception of Weeds (1999) and Majd and Pindyck (1987), abstract from complexities over the implementation phase of a project. We provide a general framework for incorporating implementation uncertainty, which includes the two previously mentioned models as special cases, by allowing both the level and the timing of project profitability to be exposed to such uncertainty. Intuitively, the project value is found to depend on the direction (favorable or not) of uncertainty resolution, but in a non-linear fashion.

In a case explicitly addressed, implementation uncertainty does not necessarily guarantee further delay in the investment decision. If implementation uncertainty is resolved favorably, the optimal investment timing is found to depend on the magnitude of the opportunity cost of delay compared to the probability of uncertainty resolution. Naturally, our model implies investment delay if the probability of uncertainty resolution is low (lower than the opportunity cost of delay). However, if the opportunity cost of delaying investment is relatively low compared to the probability of resolution, a firm will optimally invest earlier, so as to benefit from the possible resolution of implementation uncertainty after investment. The higher the possible gain of uncertainty resolution, the earlier is optimal investment.

3.5 APPENDIX

3.5.1 Proof of Proposition 3.1

The general solution to the differential equation (3.5) is:

$$V(x) = B_1(a)x^{\lambda_1} + B_2(a)x^{\lambda_2} \tag{A3.1}$$

where $B_1(a)$ and $B_2(a)$ are constants (depending on the parameter a) to be determined from the boundary conditions and λ_1 and λ_2 are the negative and positive roots of the fundamental quadratic in equation (3.8) (see for example Shimko, 1992, chapter 2). As $x \to 0$, the right-hand side of equation (A3.1) explodes if $B_1(a) \neq 0$. Since this contravenes the boundary condition in equation (3.6), i.e. $\lim_{x \to 0_+} V(x, a) = 0$, we conclude that $B_1(a) = 0$.

Substituting $V(x) = B_2(a)x^{\lambda_2}$ in boundary condition (3.12), i.e.

$$V(\bar{x}) = \frac{\bar{x}}{r - \mu} H(a, h) - K \tag{A3.2}$$

and dropping the subscript on the positive root, λ_2 for simplicity, we obtain the expression given in equation (3.14). The optimal investment trigger, $\bar{x}$, is found as the root of the smooth-pasting condition in equation (3.13). It is easy to verify that the second-order condition corresponding to this optimality condition is $\partial^2 V(x, a)/\partial x^2 > 0$. **QED**

3.5.2 Proof of Corollary 3.1

From the definition of ϕ in equation (3.16), it is straightforward to derive that:

$$\phi = \frac{\delta}{a\delta + (1-a)h} > 1 \Rightarrow \delta > a\delta + (1-a)h \Rightarrow \delta > h \tag{A3.3}$$

$$\phi = \frac{\delta}{a\delta + (1-a)h} < 1 \Rightarrow \delta < a\delta + (1-a)h \Rightarrow \delta < h \tag{A3.4}$$

In the special case that $h = \delta$, the model collapses to the implementation certainty case ($a = 1$). **QED**

ACKNOWLEDGMENTS

I am grateful to Mark Shackleton, Rafal Wojakowski, Stephen Taylor, Dean Paxson, and an anonymous referee for helpful comments and discussions. This work has benefited from the comments of participants at the Math-Finance seminar (Lancaster University) and the Real R&D Options Symposium (Manchester Business School). Of course, all errors remain my own responsibility.

NOTES

1. The reconciliation of traditional investment appraisal techniques and real option valuation paradigms is an area of active research. For some recent results, see Teisberg (1995) and Kasanen and Trigeorgis (1995).
2. The importance of sequential investment and time to build have been demonstrated by Kydland and Prescott (1982) in the context of a general equilibrium model.
3. Alfred S. Eichner (1970) deals with this inability to service the whole potential market at initial stages in his case study of the sugar refining industry in the US.
4. See Kotler (1988) and his adoption categorization theory.
5. The restriction that $\mu < r$, commonly found in real option models, is necessary to ensure that there is a positive opportunity cost to holding the option, so that it would not be held indefinitely. In more technical terms, the integral over time of discounted expected values of x_t would be unbounded if this restriction did not apply. Since the model is concerned with the effects of implementation uncertainty, not expected drifts, the conclusions from the analysis are unaffected by this assumption.
6. Even though both models conform to the same case of our framework, there is a qualitative difference, which should be stressed. The implementation time lag $\tau - T$ is random (subject to a Poisson jump) in Weeds, while in Majd and Pindyck it is a deterministic function of time depending on construction rates.
7. For a rigoros exposition of the technique and its intuition see Bellman (1957) or Bellman and Dreyfus (1962). Alternatively, Dixit and Pindyck (1994, chapter 4) present applications of the technique in a real options context.
8. Technically this is a zero probability event if x_0, the initial value of the state variable, is positive.
9. The positive root λ is a function of the drift and volatility parameters, μ and σ, and the riskless rate r, and has the following analytic expression:

$$\lambda = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

10. Alternatively, if a kink arose in $V(x)$ at $\bar{x}$, a deviation from the supposedly optimal policy would raise the firm's expected payoff. See Dumas (1991) for a clear presentation of the smooth-pasting and high-contact conditions or Dixit and Pindyck (1994, chapter 4, Appendix C), for a less technical explanation.

11. This and subsequent figures employ the parameter values used by Dixit (1989) as a base case, namely $\mu = 0$, $\sigma = 0.1$, $r = 0.03$, $K = 4$ and $h = 0.01$.
12. Drawing upon the analogy of the investment opportunity with financial options, δ is sometimes referred to as a *dividend* or *convenience yield*. It collectively accounts for any shortfall in the return of the state variable that makes exercise in finite time optimal. For an exposition of this case see the seminal paper of McDonald and Siegel (1985) or the survey work of Dixit and Pindyck (1994).
13. In Table 3.1, $K = 4$ and the state variable x is set at a level of 0.02.

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Chapter 4

An options approach to new product development: experience from Philips Electronics

ONNO LINT AND ENRICO PENNINGS

SUMMARY

This chapter considers the product development process as a series of (real) options with reducing uncertainty over time. Criteria are developed to decide on speeding up or delaying the development process. The chapter demonstrates how, in the R&D phase, any particular project may be assigned within a 2×2 matrix of uncertainty versus R&D option value. A similar matrix can be established for the product launch phase. The matrices support portfolio management throughout the different phases of development and enable management to decide on an appropriate point at which to abandon individual projects. The approach originates from applying real options insights to the product development process at Philips Electronics. The chapter is illustrated with some actual R&D projects.

4.1 INTRODUCTION

Management is often confronted with the dilemma of whether or not to invest in a particular stage of the new product development (NPD) program, given a high level of market and technology uncertainty. Rapid product obsolescence and the emergence of new markets require a fast resource allocation process in NPD, while at the same time market and technology uncertainty demand flexibility in the program (see Sanchez, 1995 and Wind and Mahajan, 1988). The trade-off between accelerating time to market and making pre-launch improvements in product performance is a topical concern. On the one hand, an accelerated market introduction may lead to a substantial gain of future market share (see Urban et al., 1986; Liebermann and Montgomery, 1988; or Brown and Lattin, 1994). Millson et al. (1992) and Urban and Hauser (1993) give surveys on the acceleration of the NPD process by skipping phases. On the other hand, as argued by Griffin and Page (1993), reducing time to market is only advisable when this does not limit the probability of success of the final product to be introduced to the market.

Given that financial risk assessment is important in the NPD process, standard discounted cash flow methods such as the NPV method dominate the project evaluation process (Newton et al., 1996). However, only when the NPD process is treated as a fixed series of investments can the application of the NPV method be justified. With an NPV approach, differing management forecasts of project values are weighted in order to get a single forecast of the average project value, thereby neglecting the extra information in the entire data set (i.e. the NPV rule assumes a fixed scenario without any contingencies).

In this chapter, an option approach to the NPD process is used to develop a framework that incorporates market and technology uncertainties in all go/no-go decisions. In particular, the product launch phase is treated as an American perpetual call option, which is a timing option with no limitation to the length of the exercise time. In order to create this option, a firm has to successfully fulfill the R&D stage. The value of the product launch option during the R&D stage is the discounted expected value of the opportunity to launch the product after R&D completion. The value of this option is derived and explicit decision criteria are developed that enhance decision making on abandoning, mothballing, delaying or speeding up R&D projects during the NPD trajectory. The calculation of this so-called forward start American perpetual call option adds to the literature on forward start options (see Hull, 2000).

The traditional process for new product development is the sequential approach. Several refinements to this approach have been proposed, mainly because of the lack of speed and flexibility in the sequential approach. Takeuchi and Nonaka (1986) propose a holistic approach. They acknowledge that the NPD process involves different stages, but stress that these stages interact with each other. Their approach to NPD builds upon the iterative communication between the functional specialists and the parallel processing of tasks. Since the process does not delay when one functional department is lagging behind, this NPD process is flexible and effective. The holistic approach improves the sequential approach, but lacks criteria on how much integration is necessary and this may hamper its use in practice (Gupta et al., 1986). Also, the approach does not explicitly capture market and technology uncertainty, nor does it give guidelines for the optimal time to abandon a project or introduce the technology to the market. Since development already starts when research is still in its embryonic stage, projects are liable to continue once research is finished. Quality function deployment (QFD) provides another approach that aims at simultaneous development across functions (Hauser and Clausing, 1988; Griffin, 1992; Griffin and Hauser, 1992). QFD uses a customer's perceptions of several physical product characteristics and requirements of a new product in order to arrive at the final new product. However, the proposed framework is less suited for the whole NPD process, but is particularly useful in the design phase of the NPD process.

A classic redesign of the NPD process in such a way that management can better deal with market and technology uncertainty is the phase review process (Cooper, 1990). Phase review processes divide NPD into a predetermined set of stages with checkpoints (gates). Each gate is characterized by a set of deliverables or inputs, a set of criteria and an output. The inputs of the functional area at each stage are the deliverables from the functional area at the preceding stage. The criteria are the hurdles that the project must pass at the gate to have it opened to the next stage. At these stages, management can perform different kinds of assessment such as market, technical, financial and legal. By dividing the NPD process into different stages at which a go/no-go decision is made, this approach deals effectively with market and technology uncertainty surrounding the new product. However, the stage–gate approach does not consider option characteristics within NPD.

By treating NPD as an incremental process, the option approach gives explicit decision rules for the trade-off between validating the project or market preemption. We set up a framework for NPD that can be regarded as (i) an extension of stagewise NPD processes, (ii) an interface between marketing, finance and R&D, and (iii) an amalgamation of qualitative and quantitative NPD assessment criteria. Our main contribution is to derive economic criteria for the go/no-go decision before and after the R&D stage – including the decision to launch a new product – based upon the flexibility to opt out at each decision node. We derive pre- and post-R&D option portfolios that enable a thoughtful comparison between feasible projects. This way, we provide a justified assessment of idiosyncratic new product initiatives at all stages of the NPD process in an uncertain environment. When economic criteria for assessing a project's flexibility in terms of the value to opt out are lacking, decision rules are invalid and a balanced dynamic portfolio of feasible business initiatives cannot be created.

The essential parameter is the uncertainty both during and after the R&D stage. When uncertainty is high, there is a high probability that a high (low) project value turns out to be low (high). Since management has the flexibility to opt out of the NPD process at all stages, downward risk is limited, while upward potential is not. Therefore, a higher uncertainty is beneficial during the initial stages of the NPD process. However, with a high uncertainty at the final stages, the option approach induces firms to postpone market introduction, as there is a high risk of failure. Thus, our option approach is consistent with the observation made by Crawford (1992) that delaying market introduction, while gathering more information, is valuable in the case of high uncertainty.

The following studies are related to the framework presented in this chapter. Baldwin and Clark (1998) analyze the option value of modularity in product design. Their approach involves simultaneous development of complementary projects, while our approach assumes that projects are mutually exclusive. Cohen et al. (1996) examine the performance and time-to-market trade-off. The authors analyze in a deterministic setting the optimal time to market and the product performance target of a single new product. We focus on uncertainty during the NPD process, the decision to skip the validation stage in the NPD process or entirely refrain from market introduction. Huchzermeier and Loch (2001) use a real option approach to evaluate flexibility in R&D and extensively address the distinction between financial uncertainty and stochastic variability in operations (budget, schedule, technical product performance and market requirements). The authors focus on the value of a single R&D project, while we concentrate on portfolios of R&D options and product launch options. Finally, Childs and Triantis (1999) propose a framework for dynamic R&D investment policies. They consider a multi-period model for the R&D process and provide a numerical solution procedure to derive the optimal investment strategy and to value the resulting R&D program. In particular, they build a three-dimensional lattice to model two projects and demonstrate how R&D characteristics such as learning, interaction between projects, and competition can be incorporated. The authors concentrate on managerial flexibility during the R&D stage, while we also take into account managerial flexibility subsequent to completion of the R&D stage.

The chapter is organized as follows. Section 4.2 discusses investment under uncertainty and the specific conditions and assumptions that are fundamental to the option approach to the NPD process. Section 4.3 presents our option framework for NPD that helps to solve timing issues of sequential investments in NPD. Subsequently, Section 4.4 analyzes the value of managerial flexibility at the R&D and launching stages, thereby creating explicit rules for the decision to conduct R&D. In Section 4.5, we introduce the option portfolios which enable management to discriminate between different but equally feasible alternatives. Finally, Section 4.6 concludes the chapter.

4.2 ASSUMPTIONS

The real option approach (McDonald and Siegel, 1986; Dixit and Pindyck, 1994; Trigeorgis, 1996) states that firms should only make an irreversible investment when the value of the investment opportunity, V, exceeds some critical value, V^*, which reflects the required fixed investment sum, I, and the value of waiting to invest. The critical value can be written formally as $g(\sigma_2, \delta)I$, with $g(\sigma_2, \delta) \geq 1$ defined in the appendix (Section 4.7). The parameter σ_2 denotes the standard deviation per unit time of the growth in V after the R&D stage and δ reflects the value that is lost by waiting. Let T_L denote the time when R&D is finished and product launching could take place. Product launch prior to this lower bound of market introduction entails substantial technology risk and may have severe consequences for future market share.

The standard deviation is a measure of the uncertainty surrounding the project value. As uncertainty increases, the probability that V ex ante appears to be lower than the value expected at the moment of market introduction increases. At the same time, as uncertainty increases, the chances of attaining a higher project value by waiting increase. Therefore management will require a higher V^* in an environment with higher uncertainty. When there is no uncertainty at all, market introduction will take place when the (deterministic) value of the investment opportunity exceeds the investment sum. This investment rule is just the traditional net present value (NPV) rule. The NPV rule is encompassed in this microeconomic framework since $g(\sigma_2, \delta) \rightarrow 1$ in the case of $\sigma_2 \rightarrow 0$.

We define the following four specific assumptions with respect to the option approach to the NPD process.

(A1) The capital and marketing expenditures that are required for market introduction are significantly larger than the cost of R&D. Also, these expenditures are known ex ante and are assumed to occur instantaneously at a specific time.

When capital and marketing expenditures are relatively small, but R&D investment is large, the investment decision is not about creating the launching opportunity, but about conducting R&D (and subsequent product launch) or not. For empirical examples supporting assumption A1, see (Urban and Hauser, 1993, chapter 3, p. 60).

(A2) The product launch investments, consisting of the capital and marketing expenditures that are required for successful market introduction, are assumed to be irreversible.

The expenditures cannot be undone or in some way be recovered. Without this (reasonable) assumption, there is no final go/no-go decision as a project can be stopped after product launch of the new product without major financial consequences.

(A3) The length of the R&D stage is fixed and R&D are considered as one stage.

When the R&D stage is complete, management holds the option to introduce the newly developed product to the market. In order to calculate the value of the

R&D, the option value must be discounted to get the present value. For reasons of tractability, we assume that the length of the R&D stage is fixed. Note that although the length of the R&D stage is fixed, the outcome of the project value is stochastic. The length of the R&D stage is given by $T_{\mathrm{L}} - t_{\mathrm{RD}}$, where t_{RD} is the moment at which R&D starts. Treating T_{L} as an optimization parameter requires a separate analysis. We refer to Granot and Zuckerman (1991) for an interesting multi-period model in which the stopping time of the R&D process is introduced as a decision variable to be determined endogenously.

(A4) The project value follows a geometric Brownian motion[1] with drift μ and standard deviation σ_1 for $t \leq T_{\mathrm{L}}$ and a geometric Brownian motion with zero drift and standard deviation σ_2 for $t > T_{\mathrm{L}}$. So, $\mathrm{d}V = \mu V\,\mathrm{d}t + \sigma_1 V\,\mathrm{d}z$ for $t < T_{\mathrm{L}}$ and $\mathrm{d}V = \sigma_2 V\,\mathrm{d}z$ for $t \geq T_{\mathrm{L}}$.

Once the R&D stage is successfully completed, the uncertainty surrounding technology is resolved. Also, part of the market uncertainty is resolved. Since product specifications are set and scans of targeted markets are conducted, product–market combinations become clear. Therefore, σ_1 will drop[2] to a lower value σ_2 after T_{L}. In our model, σ_2 represents the market uncertainty to which the project value is exposed, while σ_1 represents both market and technology uncertainty to which the project value is exposed.

4.3 AN IMPROVED INVESTMENT APPROACH TO THE NPD PROCESS

Our option approach to the NPD process takes financial elements into account within the NPD process to capture the flexibility value and extend the R&D–marketing interface to a cross-functional interaction between R&D, marketing and finance. At each stage of the NPD process, management has the possibility, but not the obligation, to step into the next stage. In the last part of the product development cycle this turns into the possibility of launching the new product. From the possibility of stopping the NPD process at each stage, including the possibility of refraining from product launch when market and technology conditions turn out to be unfavorable, downward risk is limited while upward potential is maximized. Flexibility, though it does not offer benefits in an environment known with certainty, is advantageous under uncertainty. At each stage this flexibility stems from different options. First, there exists the option to conduct R&D without the obligation of product launching. Second, when R&D is complete, management has the option of validation or market introduction.

Our framework is based upon the stage–gate approach and is graphically presented in Figure 4.1. The framework consists of four stages: (i) idea generation and initial screen, (ii) research and development, (iii) validation, and (iv) product launch. Between all stages, option assessment of each individual

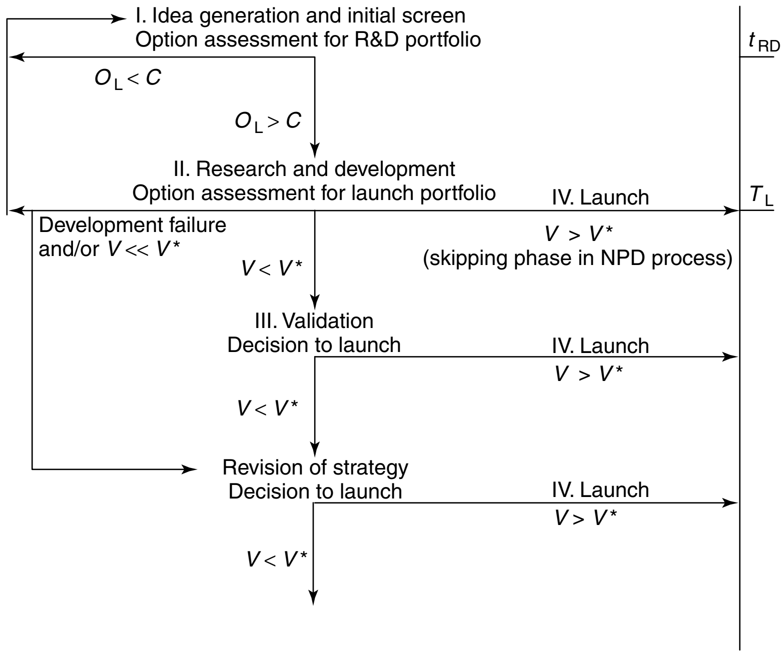

Figure 4.1 *Framework*

project will take place. We explain the specific contribution of the option approach at each of these stages.

4.3.1 Idea generation and initial screen

At the initial screen, the product idea is evaluated and a go/no-go decision whether to embark on large-scale R&D is made. All involved functional departments – marketing, finance, R&D and engineering – will contribute to and evaluate the input parameters for option decision making. This way, the potential value of the project (V) as well as the aggregate market and technology uncertainty (σ_1) surrounding the project value can be analyzed. With managerial judgement, the fraction of market uncertainty (σ_2) within the aggregate uncertainty must be captured. Also, the capital and marketing expenditures that are required for market introduction, as well as the length of the R&D stages, are determined. At the same time a concept business plan is written with the vision, scope, strategic fit, unique selling points, entry barriers, challenges, competencies and a suitable marketing mix.

The interaction between the qualitative strategic assessment and the quantitative option assessment will lead to realistic business plans, as well as to realistic input parameters as inaccurate financial, marketing or R&D assessments will be detected and changed to meet reality. Finally, the expenditures that are required for successful R&D are determined. When all information is gathered and the tasks are complete, the value of the option to launch (O_L) can be calculated and compared to the costs of the option (C). These costs denote the R&D effort that is required for successful completion of the research stage. When $C > O_L$, the product should move back to the initial screen. Consequently, the vision, scope and unique selling points of the new product have to be redefined in order to attain a project value that sufficiently increases the option value.

4.3.2 Research and development

During this stage, R&D is completed and business plans with detailed financial, marketing and manufacturing plans are worked out. The option portfolio will be created on an on-going basis, thereby tracking the dynamics in the value during the R&D stage that result from new information about the market and technology. A discussion of the R&D options portfolio is presented in Section 4.5. At the end of this stage all technology uncertainty is resolved and all involved functional departments will recalculate the project value.

In case of successful development, the option approach provides a decision tool for assessing the timing of new product development. A company can launch immediately when first, the R&D stage is successfully completed and second, the project value has crossed the critical value of product launch. When project uncertainty has decreased sufficiently, so that $V > V^*$, the option approach provides a clear solution to the dilemma of early market entry or not. With product launch just after the R&D stage, one phase in the NPD process, the validation stage, is skipped. Skipping a phase in the NPD process offers a substantial contribution to accelerate NPD, and therefore shorten the time to market.

When R&D has been completed, but the uncertainty surrounding the new product is still very high and expected future sales do not sufficiently exceed the required capital and marketing expenditures, the required additional resolution of uncertainty can be achieved in the validation stage.

4.3.3 Validation

The validation stage reviews the R&D stage of the projects that are not suitable for early market introduction. In the case that $V < V^*$, additional information about the value of the project will be pulled from the market. The company may test market the new product, trial sell the new product by means of a phased roll-out (Pennings and Lint, 2000) or use simulated test markets in order to get more insight regarding the market. The purpose of these procedures is to attain better forecasts of volume and profitability. If development fails on technological grounds, product definitions have to be re-examined and a renewed option assessment must be conducted.

The outcome of the validation can be that the project value is higher than predicted just after the R&D stage, resulting in either support for market entry or postponing product launch once more. In the latter case, the product strategy can be revised – for example by market introduction in targeted markets only (regional, professional, etc.). The procedure of strategy reformulation can in principle be repeated ad infinitum.[3]

4.3.4 Product launch

Once the option approach supports market introduction, the product has a high probability of success. The product that enters a market characterized by high uncertainty is expected to yield high benefits, since otherwise the option approach will indicate not to go to market. This way, the option approach prevents a company from commercializing failures arising from high market and technology uncertainties.

4.4 THE OPTION VALUE AT EACH STAGE

By combining microeconomic theory with an NPD perspective, we derive and apply a model for assessing the value of the options in the NPD process as described. When the R&D stage is complete and the possibility of market introduction is thus created, the value of the timing option $[F(V(T_L))]$ can be calculated as:

$$E\left[\max_{T \geq T_L}(\mathrm{e}^{-\rho(T-t_{RD})}(V(T)-I))\right] \tag{4.1}$$

where $E[\cdot]$ denotes the mathematical expectation operator, ρ is the appropriate discount factor and T is the ex ante optimal time to market. The option value reflects the expected net value of the launching opportunity. This value is always non-negative since the product will not be launched into the market as long as the investments required for a proper product launch (I) exceed the value of market introduction (V). When it is expected that V will never surmount I, the optimal time to market (T) will tend to infinity and, by discounting, the option value will tend to zero.

Samuelson (1965) and Dixit and Pindyck (1994) argue that product launching can be regarded as an American perpetual call option. The option is American since it can be exercised any moment after completion of the R&D stage, while it is perpetual since there are in principle no limitations to the length of the exercise time. When $V(T_L) > V^*$, the investment threshold has been crossed at

T_L and there is no value in waiting to invest. Economic theory suggests that the option should be exercised immediately and market introduction should take place. Hence, the net value of the market introduction opportunity, $F(V(T_L))$, equals $V(T_L) - I$. When it appears that the investment threshold has not been crossed after the R&D stage, continuing the NPD process is valuable. Dixit and Pindyck (1994) show that the timing option can be calculated as:

$$F(V(T_L)) = AV^{\beta}(T_L) \tag{4.2}$$

with $A > 0$ and $\beta > 1$ as defined in the appendix (Section 4.7).[4] Although uncertainty is advantageous during the R&D stage, it limits market introduction afterwards. When T_L is passed, project uncertainty has to be resolved in order to lower the threshold and make an improved go/no-go decision. Management must proactively handle the relevant information content and resolve uncertainty by increasing research efforts and team capabilities, by enhanced customer orientation, by product tests, (pre)test marketing, by setting up appropriate distribution channels and by managerial decision making (see Lint and Pennings, 1999). When the threshold is lower, crossing the threshold becomes more likely.

In order to create the timing option, a firm has to successfully fulfill the R&D stage. So, the value of the product launch option is the discounted expected value of the opportunity to go to market after the R&D stage at any moment, and can be written as:

$$O_L(t_{RD}) = \exp(-\rho(T_L - t_{RD}))E_{t_{RD}}[F(V(T_L))] \tag{4.3}$$

In the appendix, the value of the product launch option[5] is derived as:

$$\begin{aligned} O_L(t_{RD}) = {} & AV^{\beta}(t_{RD})\exp(\beta w_{\mu} + \tfrac{1}{2}\beta^2 w_{\sigma}^2)\Phi(\kappa_1) \\ & + V(t_{RD})\exp(\mu(T_L - t_{RD}))\Phi(\kappa_2) - I\Phi(\kappa_3) \end{aligned} \tag{4.4}$$

where κ_1, κ_2 and κ_3 are parameters that are defined in the appendix.[6] In Figure 4.2 we illustrate the value of the product launch option for different project values and uncertainty during the R&D stage. In the figure we assume that market uncertainty is proportional to technology uncertainty ($\sigma_2 = \sigma_1/2$).

Projects for which the product launch option value exceeds the R&D cost enter the portfolio of research projects, which we examine in the next section. In addition to the project value (V) and R&D stage volatility (σ_1), the portfolio explicitly takes into account the cost of investment (I) and the cost of the R&D stage (C). This way, the portfolio enables a transparent comparison of the project value, the complementary investment cost for product launch and the required R&D budget across projects differing in size.

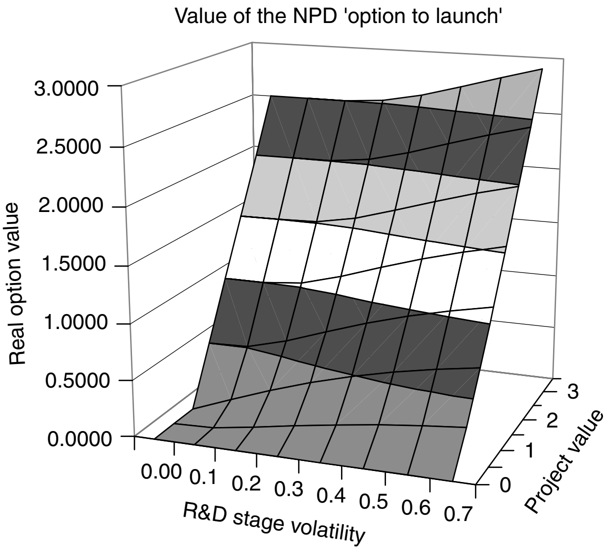

Figure 4.2 *The value of the product launch option for* $0.001 \leq \sigma_1 \leq 0.7$ *and* $0 \leq V \leq 3.5$. *Other parameter values:* $\delta = 0.04$, $\mu = 0$, $\rho = 0.04$, $\sigma_2 = \sigma_1/2$, $I = 1$ *and* $T_L - t_{RD} = 2$

4.5 MANAGEMENT OF R&D INVESTMENTS: THE OPTIONS PORTFOLIOS

We develop two portfolios. The first one consists of projects on which management has to decide whether R&D will be abandoned or continued. The other one consists of projects after the R&D stage on which management has to decide whether to introduce these directly to the market, to delay market introduction or to abandon market introduction.

4.5.1 The R&D options portfolio

Considering the first portfolio, projects can be classified at the R&D stage in four categories, depending on two variables. The first variable is the level of joint market and technology uncertainty (σ_1). We will distinguish between projects with a low σ_1 and projects with a high σ_1. The second variable is the expected value at present of the difference between the expected project value and the investment threshold and the R&D costs: $\text{PV}[E(V(T_\text{L})) - V^*] - C$.[7] We divide the sample of projects into projects with a positive value and a negative value. The four categories derived this way are illustrated in Figure 4.3 and can be described as follows.

(A) These projects are so valuable that their market introduction is expected as soon as R&D is complete. These projects have a low market and technology uncertainty. This implies that the investment threshold (V^*) is close to

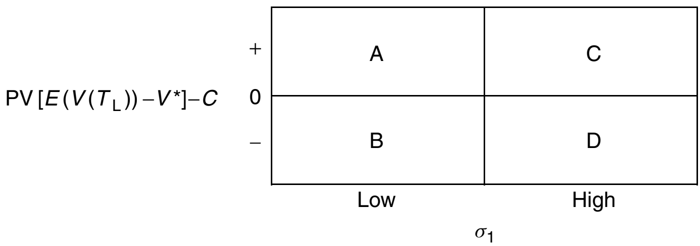

Figure 4.3 *The R&D options portfolio*

the cost of investment (I). Also, $E(V(T_L)) - V^*$ is positive. Because of the low uncertainty surrounding the project outcomes, the traditional NPV rule can be applied.

(B) These projects are liable to be market failures. Since these projects have a low market and technology uncertainty, the threshold is close to the cost of investment and the traditional NPV method can be applied. Because of the negative NPV, these projects will be abandoned.

(C) These projects are exposed to a high market and technology uncertainty. The threshold can be relatively low in the case of low market uncertainty or relatively high in the case of high market uncertainty. Nevertheless, it is ex ante expected that the threshold will be crossed at the first moment of potential product launching. This means that there are two kinds of projects in this cell. On the one hand, there are projects which are surrounded by a relatively large market uncertainty, but with expected sales that compensate the uncertainty to a great extent. On the other hand, cell C includes projects with a relatively low market uncertainty, but relatively high technology uncertainty. Development of the required technology is a bottleneck, but market demand forecasts are accurate. The option value during the R&D stage will be relatively high for both projects in this cell, and will likely exceed the R&D costs. Regardless of the kind of project, implicitly built into the model by the dividend yield δ, the option approach may support rapid completion of R&D for high-yield projects in order to create possible first mover and pioneering advantages.

(D) Like projects in cell C, these projects are surrounded by a relatively high market and technology uncertainty. Again, market uncertainty can be relatively high or relatively low. In each case, however, the investment threshold will ex ante exceed the expected project value, so there is a high probability that market introduction will be postponed after the R&D stage and validation is recommended. Since these projects can be stopped at the end of the R&D stage, the relatively high market and technology uncertainty during this stage will provide an option value that may exceed the R&D costs. At the end of the R&D stage, management can determine whether the investment threshold has been crossed and market introduction can take place. When market uncertainty can be decreased, marketing theory advocates embarkment upon a validation stage.

The portfolio consists of all projects in the R&D stage. At each moment, if new information that affects the project values arrives, the portfolio can be revised (Lint and Pennings, 1998). Consequently, a dynamic portfolio is built and management can fine-tune the allocation of R&D resources.

We made a first step at Philips Electronics Corporate Research to build a portfolio along the lines outlined above. Typical A projects, with a high expected payoff and relatively low uncertainty, appear in the field of lighting. The option approach does not support B projects. As a consequence, they are abandoned and are not part of the actual option portfolio. The Philips' R&D pipeline is well fuelled with C projects. Typical C projects are multimedia applications, such as optical tape recording and speech recognition systems. D projects appear to involve new technologies with an application base that is dominated by mature product–market combinations. A representative example is polymer light emitting diodes (LEDs), which could replace existing liquid crystal displays (LCDs) in existing products.

4.5.2 The product launch options portfolio

The second portfolio deals with projects at the NPD stage. We add projects that have just departed from the R&D stage to existing projects in the validation stage. Like the R&D options portfolio, the product launch portfolio can be represented by a 2 × 2 matrix (see Figure 4.4).

(AA) These projects are characterized by a low market uncertainty and hence the traditional NPV rule applies. The investment hurdle of these projects has been crossed or, alternatively, these projects have a positive NPV. Consequently, these options should be exercised immediately. Therefore, this cell is likely to consist only of projects that recently completed the R&D stage.

(BB) Like the investment projects in cell AA, these projects are subject to a low market uncertainty, but in contrast to the projects in cell AA, the investment hurdle has not been crossed. These projects typically have a negative NPV and

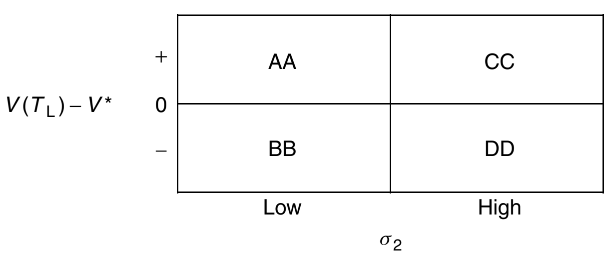

Figure 4.4 *The product launch options portfolio*

will be abandoned. Like cell AA, this cell is likely to contain projects that just left the R&D stage.

(CC) Projects in cell CC can be identified as projects with a high market uncertainty surrounding the project value. Hence, the investment hurdle is relatively high, but despite this high hurdle the project value exceeds the investment hurdle. Because of this high project value, the options in cell CC can be exercised immediately.

(DD) Like the projects in cell CC, the value of the projects in cell DD is subject to a high market uncertainty, but contrary to the projects in cell CC, it has not crossed the investment hurdle yet. Since market uncertainty is high, the investment hurdle in this cell exceeds the project value and immediate exercise of these options is not optimal. It might be true that the project value exceeds the irreversible costs and thus that the NPV rule – neglecting the high market uncertainty – would support immediate investment. However, the option approach shows that postponement of market introduction and waiting until the project value has crossed the investment hurdle is optimal for these projects. This cell may include a lot of projects, since projects can stay within this cell for an unknown period.

4.6 DISCUSSION

Although the option approach to NPD enhances the traditional phase review process, the approach simultaneously may introduce new pitfalls. A general limitation of the phase review approach is that the framework is only suitable when clearly defined milestones exist at which investment decisions have to be taken. The approach is not capable of valuing NPD processes with an integrated design, manufacturing and roll-out, which is current practice in the software industry (see Cusumano and Selby, 1996; Iansiti, 1998). In this industry, market uncertainty already resolves at the R&D stage.

Assumption A1 may hold for several new products (for example consumer goods and high-tech products), but industrial chemicals require high R&D costs relative to the investment needed for successful product launch (see Mansfield and Wagner, 1975). Moreover, the assumption will be violated in the case of uncertain product launch investments. This, however, can easily be overcome by implementing stochastic product launch investments. Assumption A2 may also not always hold. For example, when a company works with advertising agencies on a 'no cure no pay' basis, the advertising costs can be recovered when market introduction appears to be unsuccessful. Additionally, when machines and equipment can be used for manufacturing of alternative products, the capital expenditures may be recovered. The assumption that the length of the R&D stage is fixed (A3) may not always hold as R&D breakthroughs or failures may occur randomly over the R&D stage. Moreover, research and development can be treated as sequential, so that research offers a European compound option on the product launch option (see Lee and Paxson, 2001, for treatment of real American sequential exchange options). Finally, a major drawback is the exclusion of explicit actions by competitors in the model. Though competitive action is implicitly built into the model by the dividend yield δ, explicit modeling is an interesting but complex avenue for further research (see Lambrecht and Perraudin, 1997 or Kulatilaka and Perotti, 1998).

In conclusion, using the same decision tool for different projects in different stages of NPD, a more sophisticated comparison between all projects will be possible. This way, a transparent and consistent portfolio of projects can be built. With the option approach, the portfolio can be updated each time new information arrives with substantial impact.

4.7 APPENDIX

When the possibility of product launch has been created, the value of the timing option can be calculated analogous to Dixit and Pindyck (1994, pp. 140–144):

$$F(V(T_{\mathrm{L}})) = \begin{cases} AV^{\beta}(T_{\mathrm{L}}) & V(T_{\mathrm{L}}) < V^* \\ V(T_{\mathrm{L}}) - I & V(T_{\mathrm{L}}) \geq V^* \end{cases} \tag{A4.1}$$

with:

$$\beta = \frac{1}{2} - \frac{\rho - \delta}{\sigma_2^2} + \sqrt{\left(\frac{1}{2} - \frac{\rho - \delta}{\sigma_2^2}\right)^2 + \frac{2\rho}{\sigma_2^2}}, \quad A = \frac{(\beta - 1)^{\beta - 1}}{\beta^{\beta} I^{\beta - 1}},$$

$$V^* = g(\sigma_2, \delta)I, \quad g(\sigma_2, \delta) = \frac{\beta}{\beta - 1}$$

Under assumption A4, $\rho = \delta$, so:

$$\beta = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\rho}{\sigma_2^2}}$$

Now, we can determine the value of the product launch option (O_{L}) as the discounted expected value of the timing option as follows. Note that:

$$O_{\mathrm{L}}(t_{\mathrm{RD}}) = \exp(-\rho(T_{\mathrm{L}} - t_{\mathrm{RD}}))E_{t_{\mathrm{RD}}}[F(V(T_{\mathrm{L}}))] \tag{A4.2}$$

By applying Itô's lemma, it holds that the process for $\ln(V)$ for $t < T_{\mathrm{L}}$ is:

$$\mathrm{d}\ln(V) = \left(\mu - \tfrac{1}{2}\sigma_1^2\right)\mathrm{d}t + \sigma_1\,\mathrm{d}z \tag{A4.3}$$

Therefore, it is true that:

$$\ln(V(T_L)) - \ln(V(t_{RD})) \sim N\left(\left(\mu - \tfrac{1}{2}\sigma_1^2\right)(T_L - t_{RD}), \sigma_1^2(T_L - t_{RD})\right) \tag{A4.4}$$

Let $w_\mu = (\mu - \frac{1}{2}\sigma_1^2)(T_L - t_{RD})$ and $w_\sigma = \sigma_1\sqrt{T_L - t_{RD}}$. Now we can write $V(T_L)$ as:

$$V(T_L) = V(t_{RD})\exp(w_\mu + w_\sigma x) \tag{A4.5}$$

where x is distributed according to a standard normal distribution.

Now:

$$\begin{aligned} O_L(t_{RD}) = \exp(-\rho(T_L - t_{RD}))\Bigg[&\int_{x<x^*} AV^\beta(t_{RD})\exp(\beta w_\mu + \beta w_\sigma x)\varphi(x)\mathrm{d}x \\ &+ \int_{x\geq x^*}\{V(t_{RD})\exp(w_\mu + w_\sigma x) - I\}\varphi(x)\mathrm{d}x\Bigg] \end{aligned} \tag{A4.6}$$

where $\varphi(\cdot)$ denotes the density function of a standard normal distribution and

$$x^* = [\ln(V^*) - \ln(V(t_{RD})) - w_\mu]/w_\sigma \tag{A4.7}$$

Equation (A4.6) can be written as:

$$\begin{aligned} O_L(t_{RD}) = {}& AV^\beta(t_{RD})\exp\left(\beta w_\mu + \tfrac{1}{2}\beta^2 w_\sigma^2\right)\Phi(\kappa_1) \\ &+ V(t_{RD})\exp(\mu(T_L - t_{RD}))\Phi(\kappa_2) - I\Phi(\kappa_3) \end{aligned} \tag{A4.8}$$

where $\Phi(\cdot)$ denotes the cumulative probability distribution function of a standard normal variable and where $\kappa_1 = x^* - \beta w_\sigma$, $\kappa_2 = -x^* + w_\sigma$ and $\kappa_3 = -x^*$.

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NOTES

1. The described decision rules are based on specific assumptions about the diffusion process of V. More specifically, it is assumed that V follows a geometric

Brownian motion after time T_L. Consequently, V is distributed according to a lognormal distribution at any point in time.
2. Thus, we assume that technology uncertainty is constant during the R&D stage and drops to zero at R&D completion. Alternatively, technology uncertainty may be modeled as a decreasing function of time. Since only aggregate uncertainty during R&D matters for the calculation of the product launch option value, such adjustments can be incorporated.
3. In practice, the cost of keeping the option alive will induce management to put an end to it at some point in time.
4. The synthetic timing option can be (delta) hedged by an investment banker taking a position in an asset which is perfectly correlated with V. The position in the correlated asset equals the ratio of V and the correlated asset multiplied by the option delta, which is $A\beta V^{\beta-1}(T_L)$ if $V(T_L) < V^*$ and 1 otherwise.
5. The delta of the product launch option can be calculated as $A\beta V^{\beta-1}(t_{RD}) \exp(\beta w_\mu + \frac{1}{2}\beta^2 w_\sigma^2)\{\beta\Phi(\kappa_1) - \varphi(\kappa_1)/w_\sigma\} + \exp(\mu(T_L - t_{RD}))\{\Phi(\kappa_2) + \varphi(\kappa_2)/w_\sigma\} - I\varphi(\kappa_3)/Vw_\sigma$.
6. It should be noted that the effect of market uncertainty on the value of the product launch option (the so-called vega of the option) is difficult to calculate, as market uncertainty affects both σ_1 and σ_2.
7. This value can be determined analogous to the calculations in Figure 4.2.

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Chapter 5

Analytic solutions for the value of the option to (dis)invest

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SUMMARY

It is well known that costly reversibility complicates capital investment analysis to the point where closed-form expressions for the value of a firm's investment opportunities seldom exist. In such circumstances numerical evaluation is normally taken as the most practical (and often the only) way of determining investment value. However, we demonstrate that power series expansions can often be used to obtain *analytic* expressions for the value of a firm's investment opportunities. We use them in a research and development (R&D) setting to determine investment value when cash flows are generated by two well-known stochastic processes. The first is based on the Cox et al. (1985) 'square root' process; the second on the Uhlenbeck and Ornstein (1930) mean-reverting random walk. The criteria which lead to optimal investment decisions when the options to abandon or take up investment opportunities have the non-trivial values implied by these processes are also briefly examined.

5.1 INTRODUCTION

One of the most significant developments in capital investment analysis since the seminal works of Irving Fisher (1907, 1930) has come with the realization that some of the central tenets of neoclassical investment theory do not hold up in practice. It is now generally conceded, for example, that irreversibility and the possibility of delay are critical characteristics of any investment decision. This means that firms are viewed as having the right but not the obligation to incur future capital investment expenditures. However, when firms do elect to exercise these options and thus take irreversible investment decisions, they give up the possibility of waiting for new information, and this could affect the desirability and/or timing of the investment expenditures they actually end up making. Furthermore, this lost option value is an opportunity cost, which must be included in the assessment of a capital project's viability. In other words, the irreversibility of investment decisions means that firms ought to invest to the point where the expected present value of project cash flows just exceeds the purchase and installation costs by an amount equal to the value of keeping the investment option alive (Dixit and Pindyck, 1994; Trigeorgis, 1996). The accumulated weight of both the empirical and analytical evidence is that the inclusion of these (formerly ignored) option values can lead to significantly different investment evaluation criteria to those based on the simple net present value rules of traditional neoclassical theory (Burgstahler and Dichev, 1997).

One of the difficulties with these more realistic investment appraisal techniques, however, is that they may lead to intractable expressions for the value of the firm's investment opportunities. In such cases custom dictates that numerical evaluation is the most practical (and often the only) way of determining optimal investment expenditures (Trigeorgis, 1996, chapter 10). However, we demonstrate that under fairly mild regularity conditions, power series expansions can often be used to obtain *analytic* expressions for the value of a firm's investment opportunities and that these permit a somewhat deeper understanding of the criteria leading to the optimal investment decisions made by firms. In Section 5.2 we begin our analysis by considering the two crucial propositions which underpin much of the analytical and applied work which has been published in the irreversibility literature. We then illustrate the application of these propositions for two particular stochastic processes. The first (Section 5.3) is based on the purely non-negative project cash flows implied by the Cox et al. (1985) 'square root' process. The second (Section 5.4), based on the Uhlenbeck and Ornstein (1930) process, takes the more realistic focus of allowing the project cash flows to have potentially negative values. In Section 5.5 we then briefly examine the criteria which lead to optimal investment decisions when the options to abandon or take up investment opportunities have the non-trivial values implied by these processes. Finally, Section 5.6 contains our summary and conclusions.

5.2 FUNDAMENTAL PROPOSITIONS

Our analysis is based on the assumption that capital projects are valued by discounting their expected future cash flows.[1] We further assume that a project's cash flows are made up of two components. The first of these, which we would normally expect to be positive, consists of the 'current' operating cash flows arising out of the physical investment and research and development decisions taken in the past. The second, generally negative component, arises out of the project's 'current' physical investment and research and development activities. The sum of these two cash flow streams is defined as the project's 'current' net cash flow. The operative decision for an 'active' firm is whether the project's current operating cash flow is of sufficient magnitude to justify continuing with its investment and research and development activities. An 'inactive' firm must decide whether it ought to invoke them.

We now show that whichever state the firm is in (active or inactive), capital project values will satisfy a fundamental valuation equation known as the Feynman–Kac 'killing' equation or formula (Dixit and Pindyck, 1994, p. 123). It is often the case that expected present values are more easily computed by determining solutions to this equation rather than by evaluating the present value integral itself. It also permits a more general treatment of the issues arising out of irreversibility problems. We can thus begin by summarizing the Feynman–Kac formula in terms of the following proposition.

Proposition 5.1 *Suppose a capital project's instantaneous net cash flows are generated by the process*:

$$\mathrm{d}C(t) = b(C(t))\mathrm{d}t + \sqrt{a(C(t))}\,\mathrm{d}z(t) \tag{5.1}$$

where $E[\mathrm{d}C(t)]/\mathrm{d}t = b(C(t))$ *is the expected instantaneous net cash flow* (*per unit time*) *at time* t, $\mathrm{d}z(t)$ *is a standard Gauss–Wiener process and* $\mathrm{Var}[\mathrm{d}C(t)]/\mathrm{d}t = a(C(t))$ *is the variance* (*per unit time*) *of the instantaneous net cash flows at time* t. *Let* i *be the discount rate so that*:

$$V_{\mathrm{p}}(C) = E\left[\int_0^{\infty} \mathrm{e}^{-it} C(t)\mathrm{d}t\right] \tag{5.2}$$

is the expected present value of the net cash flows over the time interval $[0, \infty)$. *Then* $V_{\mathrm{p}}(C)$ *is a 'particular solution' of the fundamental valuation equation:*

$$\tfrac{1}{2}a(C)V''(C) + b(C)V'(C) - iV(C) + C = 0 \tag{5.3}$$

Proof: A modified version of the proof is contained in Karlin and Taylor (1981, pp. 203–204).

Here, however, it is important to distinguish between a particular solution of the fundamental valuation equation defined in this proposition and the general solution of the same equation. A particular solution of a given differential equation is *any* solution which satisfies the equation. Now, there are potentially an infinite number of (different) particular solutions which satisfy the fundamental valuation equation. The functional relationship describing the (complete) set of all such particular solutions is called the general solution of the given differential equation. Hence, every particular solution can be obtained by imposing a 'particular' set of restrictions on the functional form which characterizes the general solution of the given differential equation.[2] A more precise statement of these ideas is contained in the following proposition.

Proposition 5.2 *Every solution of the fundamental valuation equation (defined in Proposition 5.1) takes the form:*

$$V(C) = V_{\mathrm{p}}(C) + a_1 V_1(C) + a_2 V_2(C) \tag{5.4}$$

where $V_{\mathrm{p}}(C)$ is an arbitrarily chosen 'particular' solution of the fundamental valuation equation, a_1 and a_2 are constants and $V_1(C)$ and $V_2(C)$ are linearly independent 'complementary' functions satisfying the auxiliary equation:

$$\tfrac{1}{2}a(C)V''(C) + b(C)V'(C) - iV(C) = 0 \tag{5.5}$$

Proof: Boyce and DiPrima (1997, p. 163).

Now suppose a firm has a given capital project in place. Then we can choose the particular solution, $V_{\mathrm{p}}(C)$, so that it represents the expected present value of the net cash flows from maintaining the capital project intact. It then follows that if the firm follows optimal investment policies, the complementary functions, $V_1(C)$ and $V_2(C)$, must capture the value associated with the embedded option to abandon (or terminate) its commitment to the physical investment and research and development policies associated with the capital project, at some future point in time (Dixit, 1989, p. 626). That is, a linear combination of the complementary functions returns the value of the option to abandon its continuing investment in the project.[3]

A particular difficulty here, however, is that it is only on rare occasions that the auxiliary equation submits to closed-form solution. When it is not possible to obtain a closed-form solution, convention has it that numerical procedures ought to be used to approximate the true solution. By a numerical procedure for solving the auxiliary equation we mean a method for constructing approximate values, $V^{\mathrm{a}}(C_0)$, $V^{\mathrm{a}}(C_1)$, $V^{\mathrm{a}}(C_2), \ldots, V^{a}(C_n)$, of the true solution, $V(C_0), V(C_1), V(C_2), \ldots, V(C_n)$, at the net cash flow 'mesh' points $C_0, C_1, C_2, \ldots, C_n$. There are, however, some significant limitations associated with the numerical evaluation of differential equations of the kind arising in the irreversibility literature (Boyce and DiPrima, 1997, pp. 445–453). Probably best known amongst these is that numerical techniques will often return systematically biased approximations of the true solution, especially when the true solution is either continuously increasing or decreasing (i.e. monotonic), as will often be the case with irreversible investment decisions (Boyce and DiPrima, 1997, pp. 423–429). Normally this problem is addressed by applying a combination of sophisticated numerical methods (e.g. multistep Milne or Runge–Kutta predictor–corrector) to a more ‘compact’ set of mesh points (Boyce and DiPrima, 1997, pp. 429–434). However, it may be shown that this procedure is often characterized by problems of ‘overfitting’; for as the distance between the mesh points declines the approximation error at first decreases, but then a point will be reached where the error progressively increases. Boyce and DiPrima (1997, p. 447) note that no method is available for identifying where the ‘turning point’ in the error occurs, and since the error beyond this point grows at an exponential rate, the potential for large and significant errors in the approximating solution is very high. For a more detailed account of this and some other problems which arise with the numerical solution of differential equations, the reader is referred to Boyce and DiPrima (1997, pp. 445–453).

There is, however, an alternative procedure which can be used when the auxiliary equation assumes the linear form defined in Proposition 5.2. For then some mild regularity conditions show that it will be possible to represent the general solution of the auxiliary equation in terms of an infinite series expansion of the form (Boyce and DiPrima, 1997, pp. 262–275):[4]

$$V(C) = \sum_{j=0}^{\infty} a_j C^{j+r} \tag{5.6}$$

where the a_j are coefficients and r, which is known as the ‘exponent of singularity’, is determined by solving what is known as the ‘indicial equation’ (Boyce and DiPrima, 1997, p. 264). We now demonstrate how the indicial equation is obtained and the exponent of singularity computed from it by assuming that a capital project’s net cash flows are generated by the Cox et al. (1985) elastic (mean-reverting) random walk or ‘square root’ process. Here Dixit (1989, p. 634) notes that the net cash flows from capital projects often show a tendency to revert ‘toward some predictable long-run equilibrium level ... even though they may fluctuate in response to various random short-run influences.’ Unfortunately, these processes also lead to auxiliary equations which do not, in general, possess closed-form solutions (Dixit, 1989, p. 634) and, as a consequence, relatively little work has been published on irreversibility problems where project cash flows are generated by mean-reversion processes of this type.

5.3 COX, INGERSOLL AND ROSS (CIR) 'SQUARE ROOT' CASH FLOWS

We thus consider a capital project whose instantaneous net cash flows evolve in accordance with the stochastic differential equation:

$$\mathrm{d}C(t) = \beta(\mu - C(t))\mathrm{d}t + \sigma\sqrt{C(t)}\,\mathrm{d}z(t) \tag{5.7}$$

This process characterizes the net cash flows as an elastic (or mean-reverting) random walk with instantaneous changes in the net cash flow being described by a normal distribution with a mean (per unit time) of $b(C) = \beta(\mu - C)$. Hence, if the net cash flow, C, exceeds the long-run expected cash flow of μ, then it will be drawn back towards μ with a force which is proportional to the difference between μ and C. Here $\beta > 0$, the speed of adjustment coefficient, is the constant of proportionality which measures the intensity with which the net cash flow is drawn back towards its long-run mean. Note also that σ is an 'intensity parameter' defined on $\mathrm{d}z(t)$, a standard Gauss–Wiener process, in which case the variance (per unit time) of instantaneous changes in the cash flow amounts to $a(C) = \sigma^2 C$. Hence, the uncertainty surrounding future net cash flows increases with the magnitude of the current net cash flow.

We can now determine the expected present value of the net cash flows generated by the CIR process by first noting that integration by parts implies:

$$\begin{aligned} V_\mathrm{p}(C) &= E\left\{\int_0^\infty \mathrm{e}^{-it}C(t)\mathrm{d}t\right\} \\ &= E\left\{\left[-\frac{1}{i}\mathrm{e}^{-it}C(t) + \frac{1}{i}\int \mathrm{e}^{-it}\mathrm{d}C(t)\right]_0^\infty\right\} \end{aligned} \tag{5.8}$$

However, since $E[\mathrm{d}C(t)] = b(C)\mathrm{d}t = \beta(\mu - C(t))\mathrm{d}t$ will be the expected instantaneous change in the net cash flow, it follows that:

$$\begin{aligned} E\left\{\frac{1}{i}\int \mathrm{e}^{-it}\mathrm{d}C(t)\right\} &= \frac{\beta}{i}E\left\{\int \mathrm{e}^{-it}(\mu - C(t))\mathrm{d}t\right\} \\ &= \frac{\beta\mu}{i}\int \mathrm{e}^{-it}\mathrm{d}t - \frac{\beta}{i}E\left\{\int \mathrm{e}^{-it}C(t)\mathrm{d}t\right\} \end{aligned} \tag{5.9}$$

Hence, the expected present value of the net cash flows may be restated as:

$$V_\mathrm{p}(C)\left(1 + \frac{\beta}{i}\right) = E\left\{\left[-\frac{1}{i}\mathrm{e}^{-it}C(t) + \frac{\beta\mu}{i}\int \mathrm{e}^{-it}\,\mathrm{d}t\right]_0^\infty\right\} \tag{5.10}$$

Thus, if we impose the transversality condition $\lim_{t\to\infty}\mathrm{e}^{-it}E[C(t)] = 0$ (project cash flows grow more slowly than the discount rate) and evaluate the right-hand side of the above equation on this basis, it follows that the expected present value of the future net cash flows will be:

$$V_{\mathrm{p}}(C) = \frac{\mu}{i} + \left(\frac{C-\mu}{\beta+i}\right) \tag{5.11}$$

The intuition behind this result becomes clearer if we recall that $E[C(t)] = \mu$ is the instantaneous net cash flow expected over the 'long run' and so, if expectations are realized, the above result implies $E[V_{\mathrm{p}}(C)] = \mu/i$, as one would expect.[5]

We can now substitute $b(C) = \beta(\mu - C)$ and $a(C) = \sigma^2 C$ into the fundamental valuation equation of Proposition 5.1 to give:

$$\tfrac{1}{2}\sigma^2 C V''(C) - \beta(C-\mu)V'(C) - iV(C) + C = 0 \tag{5.12}$$

where, it will be recalled, $V(C)$ is the combined expected present value of the project's net cash flows and the value of the option to abandon (or activate) the investment in the capital project which generates these cash flows. Direct substitution shows

$$V_{\mathrm{p}}(C) = E\left\{\int_0^\infty \mathrm{e}^{-it} C(t)\mathrm{d}t\right\} = \frac{\mu}{i} + \left(\frac{C-\mu}{\beta+i}\right)$$

to be a particular solution of this equation. Hence, as expected from the discussion surrounding Propositions 5.1 and 5.2, the expected present value of the net cash flows from maintaining the capital project in place is a particular solution of the fundamental valuation equation. It thus follows that a linear combination of the complementary functions, $V_1(C)$ and $V_2(C)$, gives the value of the option to abandon (or activate) the investment in the capital project. Recall that whilst, in principle, these functions are retrieved as solutions of the auxiliary equation defined by Proposition 5.2, it is only rarely that it will be possible to solve this equation in closed form. However, our previous discussion shows that it will normally be possible to represent any given solution of the auxiliary equation in terms of an infinite series expansion. We now demonstrate how this is achieved for the CIR mean-reversion process considered here.

For the present example, Proposition 5.2 shows that the auxiliary equation takes the form:

$$\tfrac{1}{2}\sigma^2 C V''(C) - \beta(C-\mu)V'(C) - iV(C) = 0 \tag{5.13}$$

Unfortunately, there is no general closed-form solution for this equation. However, in the appendix (Section 5.7) we show that it does possess two linearly independent series solutions. The first has an exponent of singularity of $r = 0$ and takes the following form:

$$V_1(C) = 1 + \sum_{j=1}^{\infty} \left\{ \frac{i(\beta + i)\ldots((j-1)\beta + i)}{\beta\mu(\sigma^2 + 2\beta\mu)\ldots((j/2)(j-1)\sigma^2 + j\beta\mu)} \right\} C^j \quad (5.14)$$

Note that since β, μ, σ^2 and i are all positive, each term in this series expansion will also be positive. This, in turn, implies that $V_1(C)$ is a strictly positive and increasing function of C or that $V_1'(C) > 0$. In other words, if C increases in value then $V_1(C)$ also increases in value, whilst if C decreases in value then $V_1(C)$ also decreases in value. Finally, in the appendix we show that the series expansion for $V_1(C)$ converges (to a finite sum) for all positive values of C.[6]

We also demonstrate in the appendix that the second series solution for the auxiliary equation has the exponent of singularity $r = 1 - 2\beta\mu/\sigma^2$ and takes the form:[7]

$$V_2(C) = \frac{1 + \sum_{j=1}^{\infty} \left\{ \frac{\left[\beta\left(1 - \frac{2\beta\mu}{\sigma^2}\right) + i\right]\left[\beta\left(2 - \frac{2\beta\mu}{\sigma^2}\right) + i\right] \ldots \left[\beta\left(j - \frac{2\beta\mu}{\sigma^2}\right) + i\right]}{(\sigma^2 - \beta\mu)(3\sigma^2 - 2\beta\mu)\ldots((j/2)(j+1)\sigma^2 - j\beta\mu)} \right\} C^j}{C^{\left(\frac{2\beta\mu}{\sigma^2} - 1\right)} \quad (5.15)$$

We can, however, simplify this solution considerably if we assume that the discount rate can be represented in the parametric form $i = \beta(2\beta\mu/\sigma^2 - q) > 0$, for some integral value of q.[8] It then follows that $\beta(j - 2\beta\mu/\sigma^2) + i = -\beta(q - j)$ when $j < q$ and $\beta(q - 2\beta\mu/\sigma^2) + i = 0$ when $j = q$. Substitution then shows that the series expansion for $V_2(C)$ consists of q terms only. We can then use these results to show that the series expansion for $V_2(C)$ reduces to the simpler form:

$$V_2(C) = \frac{1 + \sum_{j=1}^{q-1} \left\{ \frac{\beta^j (q-1)(q-2)\ldots(q-j)}{(\beta\mu - \sigma^2)(2\beta\mu - 3\sigma^2) \ldots (j\beta\mu - (j/2)(j+1)\sigma^2)} \right\} C^j}{C^{\left(\frac{2\beta\mu}{\sigma^2} - 1\right)} \quad (5.16)$$

For positive net cash flows, convergence of the series expansion for $V_2(C)$ is guaranteed by the fact that it consists of a finite number of terms. Note also that the exponent of the last term in the series expansion for $V_2(C)$ is:

$$C^{\left(1 - \frac{2\beta\mu}{\sigma^2}\right)} C^{q-1} = C^{\left(q - \frac{2\beta\mu}{\sigma^2}\right)}$$

Furthermore, the parametric representation of the discount rate implies:

$$0 > q - \frac{2\beta\mu}{\sigma^2} > (q-1) - \frac{2\beta\mu}{\sigma^2} > \cdots > 1 - \frac{2\beta\mu}{\sigma^2}$$

or that the exponent associated with C for each term in the series expansion for $V_2(C)$ is negative. Hence, it is of crucial importance to note that if $2\beta\mu > \sigma^2$, an assumption which is satisfied by the parametric representation of the discount rate used here, then $C > 0$ or that the net cash flow from operations will always be positive (Feller, 1951; Cox et al., 1985, p. 391).[9] In other words, the origin (where $C = 0$) is inaccessible and so we need not worry about the fact that $V_2(C)$ is unbounded at this point. Furthermore, since $i = \beta(2\beta\mu/\sigma^2 - q) > 0$ then $2\beta\mu - j\sigma^2 > 2\beta\mu - q\sigma^2 > 0$ for all $j < q$ and so both the numerator and denominator of each term in the series expansion for $V_2(C)$ are positive. This in turn implies that $V_2(C)$ is a strictly positive function. Finally, differentiation shows $V_2'(C) < 0$ or that $V_2(C)$ is a strictly decreasing function of the project cash flows. Hence, when C increases, $V_2(C)$ declines in value; when C decreases, then $V_2(C)$ increases in value.

Now consider a firm in the active state; that is, a firm which currently owns and operates a capital project which generates an instantaneous net cash flow of $C(t)$ at time t. If this net cash flow is 'large', then the value of the option to abandon the investment in the capital project will be 'small'. However, if the net cash flow is 'small', then the value of the option to abandon the investment in the capital project will be 'large'. The only way we can insure that these requirements are simultaneously satisfied is when $a_1 = 0$ and $a_2 > 0$. It then follows that the combined expected present value of the project's net cash flows and the option to terminate the investment opportunity will be $V_A(C) = V_p(C) + a_2V_2(C)$, or:

$$V_A(C) = \frac{\mu}{\beta\left(\frac{2\beta\mu}{\sigma^2} - q\right)} + \frac{(C-\mu)}{\beta\left(\frac{2\beta\mu}{\sigma^2} - (q-1)\right)} + \frac{a_2\left[1 + \sum_{j=1}^{q-1}\left\{\frac{\beta^j(q-1)(q-2)\ldots(q-j)}{(\beta\mu-\sigma^2)(2\beta\mu-3\sigma^2)\ldots(j\beta\mu-(j/2)(j+1)\sigma^2)}\right\}C^j\right]}{C^{\left(\frac{2\beta\mu}{\sigma^2}-1\right)}} \tag{5.17}$$

where $V_A(C)$ is defined as the value of an active firm and we have used the fact that:

$$V_p(C) = \frac{\mu}{i} + \left(\frac{C-\mu}{\beta+i}\right) = \frac{\mu}{\beta\left(\frac{2\beta\mu}{\sigma^2} - q\right)} + \frac{(C-\mu)}{\beta\left(\frac{2\beta\mu}{\sigma^2} - (q-1)\right)}$$

due to the parametric representation of the discount rate, $i = \beta(2\beta\mu/\sigma^2 - q)$.

For a firm in the inactive state, the option is whether or not to activate the potential investment in the research and development and physical investment policies implied by the capital project. Hence, if there is a large net cash flow, the value of the option to invest must also have a large value. If, on the other hand, there is only a small net cash flow, then the option to invest must also have a small value. The only way we can insure that these requirements will be simultaneously satisfied is when $a_1 > 0$ and $a_2 = 0$. It then follows that the value of the option to delay incurring the research and development and physical investment expenditures implied by the project will be $V_N(C) = a_1 V_1(C)$, or:

$$V_N(C) = a_1 \left[1 + \sum_{j=1}^{\infty} \left\{ \frac{\beta^{j-1}\left(\frac{2\beta\mu}{\sigma^2} - q\right)\left(\frac{2\beta\mu}{\sigma^2} - (q-1)\right) \dots \left(\frac{2\beta\mu}{\sigma^2} - (q-j+1)\right)}{\mu(\sigma^2 + 2\beta\mu)\dots((j/2)(j-1)\sigma^2 + j\beta\mu)} \right\} C^j \right] \tag{5.18}$$

where $V_N(C)$ is defined as the value of an inactive firm and we have again substituted the parametric expression for $i = \beta(2\beta\mu/\sigma^2 - q)$ into the previous expression for $V_1(C)$. Now we can use the expressions for $V_A(C)$ and $V_N(C)$ to determine the instantaneous net cash flow that induces an inactive firm to exercise its option to invest in the capital project and also the net cash flow which induces an active firm to abandon its investment in the capital project. Before doing so, however, we demonstrate how the techniques used to obtain series expansions for the value of capital projects with CIR cash flows can also be extended to determine the value of capital projects with the potential to return *negative* cash flows.[10]

5.4 ORNSTEIN–UHLENBECK (OU) CASH FLOWS

A significant drawback with the CIR square root cash flow process and many other processes used to model irreversibility problems is that they are based on the assumption that project net cash flows can never be negative. Yet we know the potential negative cash flow characteristic to be an important feature of most capital projects in practice. Fortunately, it is a relatively simple matter to adapt the CIR process to accommodate negative cash flows. We thus consider the Uhlenbeck and Ornstein (1930) process, which may be summarized as follows:

$$dC(t) = \beta(\mu - C(t))dt + \sigma\, dz(t) \tag{5.19}$$

Here, as with the CIR process, $C(t)$ represents the capital project's instantaneous net cash flow, μ is the expected cash flow over the long run, $\beta > 0$ is the speed of adjustment coefficient and σ is an intensity parameter defined on $dz(t)$, a standard Gauss–Wiener process. Hence, the OU process also characterizes project net cash flows as an elastic (or mean-reverting) random walk with the same mean as the CIR process. However, the variance of instantaneous changes in the net cash flow no longer depends on the current cash flow itself and this means that the OU process accommodates the possibility of negative project cash flows.

We can now follow procedures similar to those invoked in the previous section to show that the OU and CIR processes return the same *expression* for the expected present value of the project's net cash flows, namely:[11]

$$V_{\mathrm{p}}(C) = E\left\{\int_0^\infty \mathrm{e}^{-it} C(t)\mathrm{d}t\right\} = \frac{\mu}{i} + \left(\frac{C-\mu}{\beta+i}\right) \tag{5.20}$$

Furthermore, we can also substitute $b(C) = \beta(\mu - C)$ and $a(C) = \sigma^2$ into Proposition 5.1 to obtain the fundamental valuation equation for OU cash flows, namely:

$$\tfrac{1}{2}\sigma^2 V''(C) - \beta(C-\mu)V'(C) - iV(C) + C = 0 \tag{5.21}$$

Here it will be recalled that $V(C)$ is the total of the expected present value of the project's net cash flows and the value of the option (to abandon or take up the investment) in the capital project which generates these cash flows. Now, the reader will verify that:

$$V_{\mathrm{p}}(C) = \frac{\mu}{i} + \left(\frac{C-\mu}{\beta+i}\right) \tag{5.22}$$

is a particular solution of the OU fundamental valuation equation. This, taken in conjunction with Proposition 5.2, implies that the value of the option (to abandon or take up the investment) in the capital project is determined as the solution of the auxiliary equation for OU cash flows, namely:

$$\tfrac{1}{2}\sigma^2 V''(C) - \beta(C-\mu)V'(C) - iV(C) = 0 \tag{5.23}$$

Again, there is no general closed-form solution for this equation. However, applying similar procedures to those used for the CIR cash flows in the previous section shows that:

$$V_1(C) = 1 + \sum_{j=1}^{\infty}\left\{\frac{2^j i(2\beta+i)(4\beta+i)\ldots(2(j-1)\beta+i)}{(2j)!\sigma^{2j}}\right\}(C-\mu)^{2j} \tag{5.24}$$

and

$$V_2(C) = (C-\mu) + \sum_{j=1}^{\infty} \left\{ \frac{2^j(\beta+i)(3\beta+i)\ldots((2j-1)\beta+i)}{(2j+1)!\sigma^{2j}} \right\} \times (C-\mu)^{2j+1} \tag{5.25}$$

are series expansions for the two complementary functions which are solutions of the OU auxiliary equation. Now β, i and σ^2 are all positive and so the bracketed coefficients for both complementary functions are strictly positive. It thus follows that $V_1(C)$ is a strictly positive and symmetric (or even) function with a minimum value of unity at $C = \mu$. However, $V_2(C)$ is an odd function, being positive when $C > \mu$ and negative when $C < \mu$. It may also be shown that both series expansions are convergent for all values of the project's net cash flow. Hence, the option value associated with abandoning or taking up a capital project with OU cash flows can be expressed in terms of (a linear combination of) these two series expansions.

It facilitates the further discussion of these complementary functions, however, if we substitute the 'low order' mean-reversion assumption $i = 2\beta$ into each of the above series expansions.[12] It then follows that the series expansion for the first complementary function reduces to:

$$V_1(C) = 1 + \frac{\sqrt{2\beta}}{\sigma}(C-\mu) \sum_{j=1}^{\infty} \frac{1}{(1)(3)\ldots(2j-1)} \times \left[\frac{\sqrt{2\beta}}{\sigma}(C-\mu)\right]^{2j-1} \tag{5.26}$$

However, using Spiegel (1974, exercise 118, p. 257) shows that this is the series expansion for the following closed-form expression:

$$V_1(C) = 1 + \sqrt{\frac{\beta\pi}{\sigma^2}}(C-\mu)\exp\left[\frac{\beta(C-\mu)^2}{\sigma^2}\right]\operatorname{erf}\left[\frac{\sqrt{\beta}}{\sigma}(C-\mu)\right] \tag{5.27}$$

where $\exp(\cdot)$ is the exponential operator and $\operatorname{erf}(x) = (2/\sqrt{\pi}) \int_0^x \exp(-z^2)\mathrm{d}z$ is the error function of mathematical physics (Crank, 1975, p. 14). Furthermore, substituting $i = 2\beta$ into the second complementary function shows that its series representation becomes:

$$V_2(C) = (C-\mu)\left[1 + \sum_{j=1}^{\infty} \left(\frac{\beta}{\sigma^2}\right)^j \frac{(C-\mu)^{2j}}{j!}\right] \tag{5.28}$$

However, Kiselev et al. (1967, p. 128) show this to be the series expansion for the following closed-form expression:[13]

$$V_2(C) = (C - \mu)\exp\left[\frac{\beta(C-\mu)^2}{\sigma^2}\right] \tag{5.29}$$

Hence, the low order mean-reversion assumption employed here enables us to state the two complementary functions in closed form. Recall, however, that the importance of these results stems from the fact that Proposition 5.2 shows that every solution of the OU fundamental valuation equation can be expressed as a linear sum of the particular solution and the two complementary functions. We now use this proposition and these results to determine the value of investment options available to firms with OU cash flows.

Consider then a firm in the active state, that is a firm which currently owns and operates a capital project with research and development and physical investment policies which results in an instantaneous net cash flow, C, generated by an OU process. Now, if the project's net cash flows are positive and large, then the value of the option to abandon the investment in the capital project will be small. Furthermore, if there is a large negative project net cash flow, then the value of the option to abandon the investment project will be large. Now suppose, as with the CIR cash flow example in the previous section, we define $V_A(C)$ to be the value of the firm in the active state. Then inspection of the two complementary functions shows we can insure that these two requirements will be simultaneously satisfied when $V_A(C) = V_p(C) + a_1[V_1(C) - V_2(C)]$, or:

$$V_A(C) = \frac{\mu}{2\beta} + \left(\frac{C-\mu}{3\beta}\right) + a_1\left\{1 - \sqrt{\frac{\beta\pi}{\sigma^2}}(C-\mu)\exp\left[\frac{\beta(C-\mu)^2}{\sigma^2}\right] \times\left[1 - \operatorname{erf}\left(\frac{\sqrt{\beta}}{\sigma}(C-\mu)\right)\right]\right\} \tag{5.30}$$

where $a_1 > 0$. Note that in this solution we have substituted $i = 2\beta$ into the expression for the expected value of the future project cash flows:

$$V_p(C) = \frac{\mu}{i} + \left(\frac{C-\mu}{\beta + i}\right) = \frac{\mu}{2\beta} + \left(\frac{C-\mu}{3\beta}\right) \tag{5.31}$$

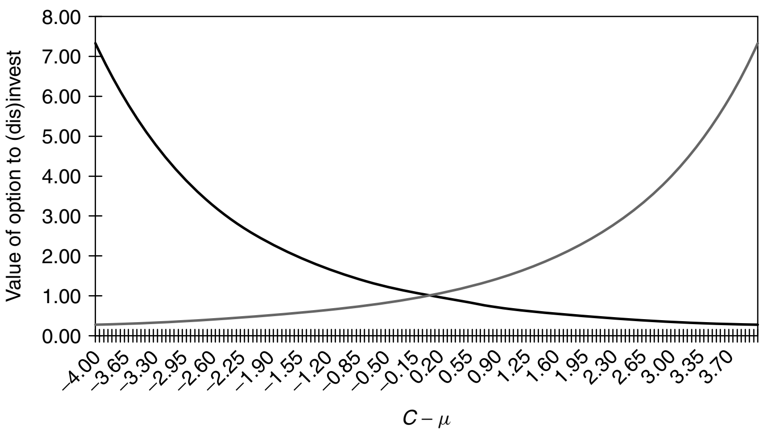
on which the investment options are based.[14]

For a firm in the inactive state, the option is whether or not to invest in the capital project with research and development and physical investment policies which results in an instantaneous net cash flow generated by an OU process. Here it will be recalled that if the project's net cash flows are large and positive, the value of the option to invest must also have a large value. If, on the other hand, the project's net cash flows are negative and large, then the option to invest must have a small value. Now suppose, as previously, we define $V_N(C)$ to be the value of the firm in the inactive state. Then we can insure that these two requirements will be simultaneously satisfied when $V_N(C) = a_2[V_1(C) + V_2(C)]$, or:[15]

$$V_N(C) = a_2 \left\{ 1 + \sqrt{\frac{\beta\pi}{\sigma^2}}(C-\mu)\exp\left[\frac{\beta(C-\mu)^2}{\sigma^2}\right] \times \left[1 + \operatorname{erf}\left(\frac{\sqrt{\beta}}{\sigma}(C-\mu)\right)\right]\right\} \tag{5.32}$$

where $a_2 > 0$ is also a constant. In Figure 5.1 we graph the option values to (dis)invest for OU project cash flows when the speed of adjustment coefficient is $\beta = 0.05$, there is a unit variance, $\sigma^2 = 1$, and $a_1 = a_2 = 1$. Note that the value of the option to invest uniformly increases with the project's net cash flows whilst the option to disinvest decreases. This is, of course, precisely what one would expect.

The magnitude of the values contained in these graphs also gives one a good 'feel' for the significant errors, which may arise when embedded option values are ignored in project evaluation. We now turn to the issue of how we



The graph with a negative slope is:

$$1 - \sqrt{\frac{\beta\pi}{\sigma^2}}\,(C-\mu)\exp\left[\frac{\beta(C-\mu)^2}{\sigma^2}\right]\left[1 - \operatorname{erf}\left(\frac{\sqrt{\beta}}{\sigma}(C-\mu)\right)\right]$$

which is the normalized value of the option to disinvest, for an active firm. The graph with a positive slope is:

$$1 + \sqrt{\frac{\beta\pi}{\sigma^2}}\,(C-\mu)\exp\left[\frac{\beta(C-\mu)^2}{\sigma^2}\right]\left[1 + \operatorname{erf}\left(\frac{\sqrt{\beta}}{\sigma}(C-\mu)\right)\right]$$

which is the normalized value of the option to invest, for an inactive firm.

Figure 5.1 *Value of option to (dis)invest for Ornstein–Uhlenbeck cash flows:* $\beta = 0.05$, $\sigma^2 = 1$, $a_1 = a_2 = 1$

might determine the instantaneous net cash flows that induce an inactive firm to exercise its option to activate its potential investment in the capital project and also the cash flows which will induce an active firm to abandon (or terminate) its commitment to the research and development and physical investment policies implied by a capital project with OU cash flows.[16]

5.5 DETERMINATION OF ENTRY AND EXIT OU PROJECT CASH FLOWS

Now, consider an inactive firm which has an investment opportunity whose 'immediate' purchase and installation costs amount to I. Then we use the OU analyses with $i = 2\beta$ of the previous section to illustrate how the instantaneous cash flow, C_{h}, which will induce the firm to avail itself of the investment opportunity is determined. The operative rule is that the value of the investment opportunity must exceed the purchase and installation costs by an amount equal to the value of keeping the investment option alive. Now, the value of the investment opportunity to an active firm is composed of two parts: the expected present value of the project's future net cash flows and the value of the option to 'kill' or abandon the investment opportunity at some future point in time. Their combined value is given by the equation for $V_{\mathrm{A}}(C)$ defined in the previous section. The value of the investment opportunity to an inactive firm, however, is composed entirely of the value of the option to activate the investment opportunity at some future point in time, $V_{\mathrm{N}}(C)$, also defined in the previous section. Hence, if we add the purchase and installation costs, I, to $V_{\mathrm{N}}(C)$ and set their sum equal to $V_{\mathrm{A}}(C)$ then we can determine the instantaneous cash flow, C_{h}, which will induce the firm to avail itself of the investment opportunity. The operative rule will thus be to determine the C_{h} for which we have $V_{\mathrm{A}}(C_{\mathrm{h}}) = V_{\mathrm{N}}(C_{\mathrm{h}}) + I$, or:

$$\begin{aligned}
&\frac{\mu}{2\beta} + \frac{\sigma}{3\beta\sqrt{\beta}}\, x_{\mathrm{h}} + a_1 \left\{1 - \sqrt{\pi}\, x_{\mathrm{h}} \exp(x_{\mathrm{h}}^2)[1 - \operatorname{erf}(x_{\mathrm{h}})]\right\} \\
&\quad = I + a_2 \left\{1 + \sqrt{\pi}\, x_{\mathrm{h}} \exp(x_{\mathrm{h}}^2)[1 + \operatorname{erf}(x_{\mathrm{h}})]\right\} \qquad (5.33)
\end{aligned}$$

where $x_{\mathrm{h}} = (\sqrt{\beta}/\sigma)(C_{\mathrm{h}} - \mu)$, in terms of the equations for the OU process determined in the previous section.

Now consider an active firm which is considering whether or not to abandon the investment project. We assume that if it does so, it incurs immediate decommissioning costs amounting to E. If the firm does elect to abandon the investment opportunity it also loses the expected present value of the future net cash flows and the value of its option to kill or abandon the investment opportunity at some future point in time. Their combined value is given by the equation for $V_{\mathrm{A}}(C)$ defined in the previous section. In return it gains the value of the option to reactivate the investment opportunity at some future point in time, $V_N(C)$, also defined in the previous section. Hence, if we add the decommissioning costs, E, to $V_A(C)$ and set their sum equal to $V_N(C)$ then we can determine the instantaneous net cash flow, C_u, which will induce the firm to abandon the investment opportunity. The operative rule will thus be to determine the C_u for which we have $V_A(C_u) + E = V_N(C_u)$, or:

$$E + \frac{\mu}{2\beta} + \frac{\sigma}{3\beta\sqrt{\beta}} x_u + a_1 \left\{1 - \sqrt{\pi}\, x_u \exp(x_u^2)[1 - \mathrm{erf}(x_u)]\right\}$$
$$= a_2 \left\{1 + \sqrt{\pi}\, x_u \exp(x_u^2)[1 + \mathrm{erf}(x_u)]\right\} \quad (5.34)$$

where $x_u = (\sqrt{\beta}/\sigma)(C_u - \mu)$. Note that this analysis returns four unknowns: C_h, C_u, a_1 and a_2. There are, however, presently only two equations from which to determine these four variables.

The remaining two equations are provided, however, by the Samuelson 'smooth-pasting' conditions. The smooth-pasting conditions rule out the possibility of arbitrage profits at the net cash flows which will just induce the firm to exercise its option to activate the capital project (C_h), or abandon its commitment to it (C_u). In other words, if we are to rule out arbitrage opportunities at these cash flows, the following smooth-pasting conditions will have to apply:[17]

$$\left.\frac{\mathrm{d}V_A(C)}{\mathrm{d}C}\right|_{C_h} = \left.\frac{\mathrm{d}V_N(C)}{\mathrm{d}C}\right|_{C_h} \quad \text{and} \quad \left.\frac{\mathrm{d}V_A(C)}{\mathrm{d}C}\right|_{C_u} = \left.\frac{\mathrm{d}V_N(C)}{\mathrm{d}C}\right|_{C_u}$$

Differentiating the affected equations shows that these conditions are satisfied at the two roots for which the following equation holds:

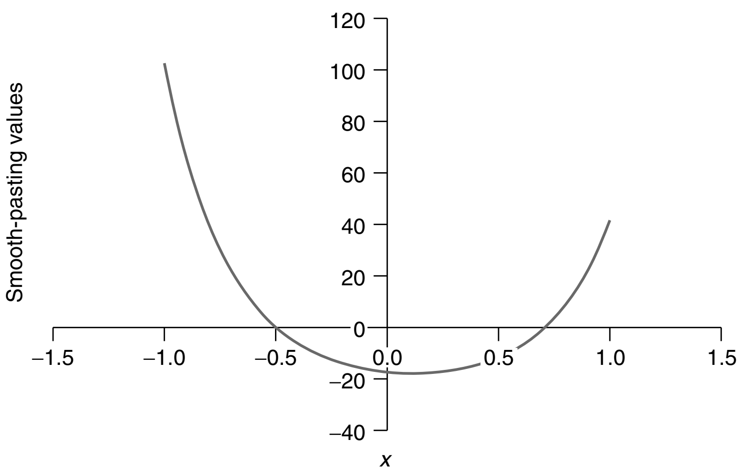
$$\left(\frac{-\sigma}{3\beta\sqrt{\beta}} + \frac{H}{2}\right) + Gx + Hx^2 = 0 \quad (5.35)$$

where $x = (\sqrt{\beta}/\sigma)(C_h - \mu)$; $(\sqrt{\beta}/\sigma)(C_u - \mu)$ are the two roots of this equation, $G = -2(a_1 - a_2)$ and $H = 2\sqrt{\pi}\{a_1 \exp(x^2)[1 - \mathrm{erf}(x)] + a_2 \exp(x^2)[1 + \mathrm{erf}(x)]\}$. This means that we now have four equations through which to determine the four unknowns: C_h, C_u, a_1 and a_2.

We demonstrate the implementation of the above procedures by considering a capital project whose purchase and installation costs amount to $I = 11.6924$. Decommissioning costs of $E = 3.4566$ will have to be paid in the event that the firm abandons its investment in the research and development and physical investment activities implied by the capital project. Finally, we assume that the project's net cash flows are generated by an OU process with an expected cash flow over the long run of $\mu = 0$, a mean-reversion coefficient of $\beta = 0.05$, an instantaneous variance of $\sigma^2 = 1$ and a cost of capital amounting to $i = 2\beta = 0.10$. Substitution then shows that $C = x/\sqrt{0.05}$ and this, taken in conjunction with the optimality conditions derived above, leads to the following 'entry' and 'exit' net cash flows and coefficients defining the option values:

$$C_{\mathrm{h}} = 3.1743, \quad C_{\mathrm{u}} = -2.2220, \quad a_1 = 4.6020, \quad a_2 = 2.4512$$

In other words, the firm would implement the physical investment and research and development activities associated with this capital project once its instantaneous net cash flow rises above $C_{\mathrm{h}} = 3.1743$. It would, however, abandon them once its instantaneous net cash flow falls below $C_{\mathrm{u}} = -2.2220$. Figure 5.2 contains the graph of the equation summarizing the smooth-pasting conditions, for this example. Note that there are two roots for this equation, the lower root of $x = \sqrt{0.05}\,C = -0.4969$ corresponding to the 'exit' net cash flow of $C_{\mathrm{u}} = -2.2220$ and the upper root $x = \sqrt{0.05}\,C = 0.7098$ corresponding to the 'entry' net cash flow of $C_{\mathrm{h}} = 3.1743$. The reader will find a good illustration of the numerical procedures which can be used for determining these roots in Dixit and Pindyck (1994, chapter 6), whilst Carnahan et al. (1969,



This is the graph of the smooth-pasting condition for the Ornstein–Uhlenbeck process:

$$\left(\frac{-\sigma}{3\beta\sqrt{\beta}} + \frac{H}{2}\right) + Gx + Hx^2 = 0$$

where $x = \sqrt{\beta}\ (C - \mu)/\sigma$, $G = -2(a_1 - a_2)$ and $H = 2\sqrt{\pi}\ \{a_1 \cdot \exp\ (x^2)[1 - \mathrm{erf}(x)] + a_2 \cdot \exp(x^2) \times [1 + \mathrm{erf}(x)]\}$. The two roots of this equation define the entry (C_h) and exit (C_u) cash flows.

Figure 5.2 *Determination of entry and exit cash flows for an Ornstein–Uhlenbeck process:* $\beta = 0.05$, $\sigma^2 = 1$, $a_1 = 4.6020$, $a_2 = 2.4512$, $\mu = 0$

pp. 308–320) contains a useful discussion of the underlying analytical theory and convergence properties of the non-linear iteration techniques on which they are based.

Table 5.1 contains a more detailed summary of the particular version of the OU process illustrated here. Thus, when the discount rate is $i = 20\%$, the cash flow which will induce an inactive firm to invoke the physical investment and research and development activities implied by the OU cash flows will be $C_\mathrm{h} = 4.3646$. For an active firm, however, the cash flows will have to fall below $C_\mathrm{u} = -2.2742$ before the firm would exercise its option to abandon the physical investment and research and development activities implied by this process. Finally, the coefficients defining the option values at this discount rate are $a_1 = 0.9003$ and $a_2 = 0.0909$.

Table 5.1 *Ornstein–Uhlenbeck process: entry (C_h) and exit (C_u) residual income triggers and constants (a_1 and a_2) associated with the option to implement or terminate production and investment plans; $\mu = 0$, $i = 2\beta$, $\sigma = 1$*

i	C_h	C_u	a_1	a_2
0.01	2.3287	−2.2450	248.1824	244.2726
0.02	2.4116	−2.2414	82.2459	78.5432
0.03	2.4974	−2.2380	42.0623	38.5658
0.04	2.5859	−2.2347	25.7116	22.4191
0.05	2.6771	−2.2316	17.3363	14.2451
0.06	2.7711	−2.2287	12.4395	9.5460
0.07	2.8678	−2.2262	9.3181	6.6183
0.08	2.9673	−2.2242	7.2033	4.6922
0.09	3.0694	−2.2223	5.7036	3.3757
0.10	3.1743	−2.2220	4.6020	2.4512
0.11	3.2818	−2.2221	3.7697	1.7893
0.12	3.3920	−2.2230	3.1262	1.3089
0.13	3.5048	−2.2250	2.6189	0.9570
0.14	3.6202	−2.2280	2.2125	0.6980
0.15	3.7382	−2.2323	1.8824	0.5068
0.16	3.8587	−2.2379	1.6109	0.3658
0.17	3.9816	−2.2448	1.3853	0.2621
0.18	4.1069	−2.2531	1.1962	0.1861
0.19	4.2346	−2.2629	1.0363	0.1308
0.20	4.3646	−2.2742	0.9003	0.0909
0.21	4.4969	−2.2870	0.7838	0.0624
0.22	4.6313	−2.3013	0.6835	0.0422
0.23	4.7679	−2.3171	0.5968	0.0281
0.24	4.9064	−2.3343	0.5216	0.0184
0.25	5.0470	−2.3530	0.4561	0.0119
0.26	5.1895	−2.3731	0.3989	0.0075
0.27	5.3338	−2.3944	0.3489	0.0047
0.28	5.4799	−2.4169	0.3052	0.0028
0.29	5.6277	−2.4407	0.2668	0.0017
0.30	5.7772	−2.4655	0.2331	0.0010

5.6 SUMMARY AND CONCLUSIONS

The realization that investment opportunities also carry embedded options has revolutionized the theory of capital investment analysis. Unfortunately, it has also complicated project evaluation to the point where optimal investment decisions invariably depend on a series of intractable valuation expressions. In such cases, custom dictates that numerical evaluation is the most practical (and often the only) way of determining optimal investment expenditures. However, we demonstrate that it is often possible to use power series expansions to obtain analytic expressions for the value of a firm's investment opportunities. These will normally permit a somewhat deeper investigation of the criteria leading to the optimal investment decisions made by firms. Furthermore, they can often identify the specialized circumstances under which it will be possible to obtain closed-form expressions for the value of a firm's investment opportunities. Here, we demonstrate how these procedures are applied when a firm's investment opportunities have net cash flows which are generated by the CIR 'square root process' and also by the OU process. The latter process has the advantage of allowing project cash flows to assume negative values.

Other stochastic processes are accommodated by our analysis in a fairly straightforward manner. Here, a particularly interesting candidate is the 'scaled' t distribution of Praetz (1972) and Blattberg and Gonedes (1974). This process is based on the assumption that expected changes in operating cash flows are always towards a long-run mean, are potentially negative and have a variance which depends on the difference between the 'current' and long-run operating cash flows. Hence, the uncertainty associated with future cash flows depends on the current level of the operating cash flow – something that intuition suggests ought to be the case. It thus combines the most attractive features from the CIR and OU processes. Determining the value of the embedded investment options for this stochastic process would be a very useful addition to the literature. However, Kendall et al. (1987, p. 216) note that the Pearson Type IV distribution, of which the scaled t is a special case, is 'difficult to handle in practice.' Furthermore, the valuation equations based on it lead to complementary functions involving intricate convergence issues. Hence, it seems sensible that we leave consideration of the scaled t operating cash flow processes until another occasion.

5.7 APPENDIX

5.7.1 Series solution of the CIR auxiliary equation

The solution $V(C) = \sum_{j=0}^{\infty} a_j C^{j+r}$ implies that $V'(C) = \sum_{j=0}^{\infty}(j+r)a_j C^{j+r-1}$ and $V''(C) = \sum_{j=0}^{\infty}(j+r)(j+r-1)a_j C^{j+r-2}$. Substitute these expressions into the CIR auxiliary equation to give:

$$\sum_{j=0}^{\infty}\left\{\tfrac{1}{2}\sigma^2(j+r)(j+r-1)a_jC^{j+r-1}\right.$$
$$\left.+\beta\mu(j+r)a_jC^{j+r-1}-\beta(j+r)a_jC^{j+r}-ia_jC^{j+r}\right\}=0 \qquad \text{(A5.1)}$$

Noting that the last two summations in this equation may be restated as $\sum_{j=0}^{\infty}(j+r)a_jC^{j+r}=\sum_{j=1}^{\infty}(j+r-1)a_{j-1}C^{j+r-1}$ and $\sum_{j=0}^{\infty}a_jC^{j+r}=\sum_{j=1}^{\infty}a_{j-1}C^{j+r-1}$ and combining terms gives:

$$a_0\left\{\tfrac{1}{2}\sigma^2r(r-1)+\beta\mu r\right\}C^{r-1}+\sum_{j=1}^{\infty}\left\{\tfrac{1}{2}\sigma^2(j+r)(j+r-1)a_j\right.$$
$$\left.+\beta\mu(j+r)a_j-\beta(j+r-1)a_{j-1}-ia_{j-1}\right\}C^{j+r-1}=0 \qquad \text{(A5.2)}$$

Now the coefficient associated with the first term of this expansion, when set equal to zero, is known as the indicial equation (Boyce and DiPrima, 1997, pp. 262–266). Hence, for the CIR process the indicial equation is $\frac{1}{2}\sigma^2r(r-1)+\beta\mu r=0$. Furthermore, the roots of the indicial equation are known as the 'exponents of singularity' for the differential equation (Boyce and DiPrima, 1997, p. 264). Hence, for the square root process, $r=0$ and $r=1-2\beta\mu/\sigma^2$ are the exponents of singularity. Note that if we let $r=0$ then the series expansion becomes:

$$\sum_{j=1}^{\infty}\left\{\tfrac{1}{2}\sigma^2j(j-1)a_j+j\beta\mu a_j-\beta(j-1)a_{j-1}-ia_{j-1}\right\}C^{j-1}=0 \qquad \text{(A5.3)}$$

or the following recurrence relationship holds:

$$a_j=\left\{\frac{2((j-1)\beta+i)}{j(j-1)\sigma^2+2j\beta\mu}\right\}a_{j-1} \qquad \text{(A5.4)}$$

Letting j vary over all integral values then implies the following solution to the auxiliary equation:

$$V_1(C)=1+\sum_{j=1}^{\infty}\left\{\frac{i(\beta+i)\ldots((j-1)\beta+i)}{\beta\mu(\sigma^2+2\beta\mu)\ldots((j/2)(j-1)\sigma^2+j\beta\mu)}\right\}C^j \qquad \text{(A5.5)}$$

Note that the limit of the ratio of the $(j+1)$st and jth terms of this expansion is:

$$\lim_{j\to\infty}\left[\frac{2(j\beta+i)}{j(j+1)\sigma^2+2(j+1)\beta\mu}\right]C=0$$

Hence, by the ratio test (Spiegel, 1974, p. 226; Boyce and DiPrima, 1997, pp. 226–227), the above series converges for all C.

For the second exponent of singularity, $r = 1 - 2\beta\mu/\sigma^2$, similar analysis returns the recurrence relationship:

$$a_j = \left\{ \frac{2\left[\beta\left(j - \frac{2\beta\mu}{\sigma^2}\right) + i\right]}{j(j+1)\sigma^2 - 2j\beta\mu} \right\} a_{j-1} \tag{A5.6}$$

Hence, if we let j vary over all integral values we obtain the following solution to the auxiliary equation:

$$V_2(C) = \frac{1 + \sum_{j=1}^{\infty} \left\{ \frac{\left[\beta\left(1 - \frac{2\beta\mu}{\sigma^2}\right) + i\right]\left[\beta\left(2 - \frac{2\beta\mu}{\sigma^2}\right) + i\right] \ldots \left[\beta\left(j - \frac{2\beta\mu}{\sigma^2}\right) + i\right]}{(\sigma^2 - \beta\mu)(3\sigma^2 - 2\beta\mu)\ldots((j/2)(j+1)\sigma^2 - j\beta\mu)} \right\} C^j}{C^{\left(\frac{2\beta\mu}{\sigma^2} - 1\right)} \tag{A5.7}$$

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NOTES

1. Rubinstein (1974, 1976) summarizes some mild regularity conditions which can be used to justify this assumption.
2. As a simple example, consider the differential equation $y''(x) + y(x) + x = 0$. Both $y_p(x) = -x$ and $y_p(x) = \cos(x) - x$ are particular solutions of this equation. However, the general solution takes the form $y(x) = a_1 \cos(x) + a_2 \sin(x) - x$, where a_1 and a_2 are constants. Hence, the first particular solution is obtained by letting $a_1 = a_2 = 0$, whilst the second particular solution is obtained by letting $a_1 = 1$ and $a_2 = 0$. Hence, by letting a_1 and a_2 vary over all real numbers, we can define the infinite number of particular solutions which exist for this differential equation.
3. For an inactive firm, $V_1(C)$ and $V_2(C)$ capture the value of the option to activate the capital project at some future point in time.

4. These mild regularity conditions are that $2C[b(C)/a(C)]$ and $2C^2[i/a(C)]$, where $b(C)$ and $a(C)$ are defined in Proposition 5.1, must both possess convergent Taylor series expansions over some finite domain (Boyce and DiPrima, 1997, pp. 229, 272–273).
5. It may also be shown when $C \neq \mu$, that the expected present value of the future 'abnormal' cash flows amounts to $(C - \mu)/(\beta + i)$.
6. Convergence for this series can, however, be extremely slow in the sense that a large number of terms are required before the remaining terms are insignificantly small. However, modern computing power means that slow convergence is unlikely to be of any practical significance.
7. The two series solutions given here are derived on the assumption that the absolute difference between the two exponents of singularity is of non-integral value. When this assumption is not satisfied, the solutions take a slightly different form. See Boyce and DiPrima (1997, pp. 272–273) for further details.
8. This assumption simplifies the analysis without detracting from its generality.
9. The proof follows from substituting $a = \sigma^2/2$, $\beta = -b$ and $c = \beta\mu$ into equation (1.1) of Feller (1951, p. 173) and using lemma 6 (p. 179) of the same article.
10. Cox et al. (1985, pp. 396–397) determine the value of a (European) investment option with CIR cash flows but which expires on a fixed and known date.
11. Note, however, that the actual value for the discount rate, i, will in general be different between the two processes, reflecting the differing risks associated with the cash flows they imply.
12. The discount rate, i, is typically small and so this assumption implies that the speed of adjustment coefficient, β, will be of even lower order. We should also emphasize that this assumption does not affect the generality of the results which we are about to report. We invoke this assumption purely because of the tractability and pedagogic convenience which arises out of the closed-form solutions it implies.
13. In subsequent analysis we normalize this complementary function so that it becomes:

$$V_2(C) = \sqrt{\frac{\beta\pi}{\sigma^2}}(C - \mu)\exp\left[\frac{\beta(C - \mu)^2}{\sigma^2}\right]$$

This renders it compatible with the expression for $V_1(C)$.
14. The substitution $x = (\sqrt{\beta}/\sigma)(C - \mu)$ shows:

$$\sqrt{\frac{\beta\pi}{\sigma^2}}(C - \mu)\exp\left[\frac{\beta(C - \mu)^2}{\sigma^2}\right]\left[1 - \mathrm{erf}\left(\frac{\sqrt{\beta}}{\sigma}(C - \mu)\right)\right]$$

$$= 2x\ \exp(x^2)\int_x^{\infty}\exp(-z^2)\mathrm{d}z = \frac{\displaystyle\int_x^{\infty}\exp(-z^2)\mathrm{d}z}{\dfrac{1}{2x}\ \exp(-x^2)}$$

It then follows:

$$\lim_{x\to\infty}\left[\frac{\displaystyle\int_x^\infty \exp(-z^2)\mathrm{d}z}{\dfrac{1}{2x}\exp(-x^2)}\right] = \lim_{x\to\infty}\left[\frac{-\exp(-x^2)}{\dfrac{-1}{2x^2}\exp(-x^2)-\exp(-x^2)}\right]$$

$$= \lim_{x\to\infty}\left[\frac{2x^2}{2x^2+1}\right] = 1$$

by virtue of L'Hôpital's rule (Spiegel, 1974, p. 62). Furthermore, Cauchy's generalized theorem of the mean (Spiegel, 1974, p. 71) shows for non-negative x:

$$0 \le \frac{\displaystyle\int_x^\infty \exp(-z^2)\mathrm{d}z}{\dfrac{1}{2x}\exp(-x^2)} \le 1$$

Together, these results imply that the embedded option value, as given by the term $a_1[V_1(C) - V_2(C)]$, can never be negative – as one would expect.

15. Analysis similar to that contained in the previous note shows $V_\mathrm{N}(C)$ to be a strictly positive function. Furthermore, Boyce and DiPrima (1997, exercise 12, p. 144) note that if $V_1(C)$ and $V_2(C)$ are two complementary functions for the auxiliary equation $\frac{1}{2}a(C)V''(C) + b(C)V'(C) - iV(C) = 0$, then $V_1(C) - V_2(C)$ and $V_1(C) + V_2(C)$ can also be taken as the complementary functions if it turns out that this is more convenient. This is, in fact, what we have done in this section in deriving the value of the embedded options for OU cash flows.
16. Cox and Ross (1976, pp. 162–163) and Egginton et al. (1989, pp. 265–266) determine the value of a (European) investment option with OU cash flows but which expires on a fixed and known date.
17. Dixit and Pindyck (1994, pp. 130–132) contains an intuitive and very readable account of the arbitrage ideas which underscore the smooth-pasting conditions.

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Chapter 6

Student's distribution and the value of real options

HUW RHYS AND MARK TIPPETT

SUMMARY

In this chapter we derive an explicit formula for the value of the option to invest in a capital project when the difference between the benefits and costs of the investment decision are generated by a general class of Student distributions. These processes encapsulate the 'fat tail' property that characterizes the empirical distributions of some of the asset pricing literature. Furthermore, since many real-life investment opportunities are not infinitely lived but expire and become worthless at a known point in time, our analysis is based on the assumption that the option to invest has a finite life.

6.1 INTRODUCTION

In this chapter we derive an explicit formula for the value of the option to invest in a capital project, but where the option to do so has a finite life. The seminal and most often quoted papers in this area are those of Margrabe (1978) and McDonald and Siegel (1986), both of which solve the valuation problem by assuming that the benefits and costs associated with the capital project are generated by a geometric Brownian motion. Virtually all the subsequent work in the area, such as that by Myers and Majd (1990), Olsen and Stensland (1992), Quigg (1993), Carr (1988, 1995) and Schroder (1999), has continued with the Brownian motion assumption. There is growing evidence, however, that asset prices evolve in terms of distributions which possess 'fat tails' when compared to the normal distribution on which the Brownian motion is based (Praetz, 1972; Blattberg and Gonedes, 1974; Theodossiou, 1998; Barndorff-Nielsen and Shephard, 2001). Hence, our purpose here is to present a closed-form solution for the value of the option to invest in a capital project when the difference between the benefits and costs of the investment decision are generated by a general class of Student distributions. These processes exhibit the 'fat tail' properties that characterize the (probable) distributions of some R&D projects, which either fail after large expenditures in development and clinical trials, or succeed as blockbuster products. Section 6.2 develops the valuation formula and analyzes some of its more important properties. Section 6.3 contains our summary conclusions.

6.2 THE VALUE OF THE INVESTMENT OPTION

Our focus is on determining the value of the option to invest in a capital project whose *net* present value at time t is $x(t)$. We emphasize that $x(t)$ is a net present value; it represents the discounted expected value of the capital project's net cash flows (composed of the operating cash flow less any investment outlays incurred in each period) given the information that is available about it at time t.[1] As such, $x(t)$ may assume both positive and negative values. Furthermore, the 'fat tail' property alluded to earlier will mean that the variance of instantaneous increments, $\mathrm{d}x(t)$, in the capital project's net present value will have to depend on the level, $x(t)$, of the variable itself. Our analysis encapsulates both these requirements by assuming that instantaneous increments in the capital project's net present value evolve in terms of the following elastic (or mean-reverting) random walk:

$$\mathrm{d}x(t) = -\beta x(t)\mathrm{d}t + \sqrt{k^2 + 2rx^2(t)}\,\mathrm{d}z(t) \tag{6.1}$$

where $\beta \geq 0$ is the 'speed of adjustment' coefficient, $k^2 > 0$ is the variance of instantaneous increments in the capital project's net present value when there are no economic rents (that is, when $x(t) = 0$), $r > 0$ is the intertemporally constant risk-free rate of interest and $\mathrm{d}z(t)$ is a 'white noise' process with unit variance.[2] Hence, under these assumptions the expected instantaneous increment (per unit time) in the net present value variable will be $E_t[\mathrm{d}x(t)]/\mathrm{d}t = -\beta x(t)$, where $E_t(\cdot)$ is the expectations operator taken at time t. This means that we can expect any economic rents to be eliminated with a force which is proportional to the existing rents; the constant of proportionality being defined by the speed of adjustment coefficient, β. Furthermore, the variance (per unit time) of instantaneous variations in the net present value variable will be $\mathrm{Var}_t[\mathrm{d}x(t)]/\mathrm{d}t = k^2 + 2rx^2(t)$, where $\mathrm{Var}_t(\cdot)$ is the variance operator (also taken at time t). This expression reflects the belief that the uncertainty associated with variations in asset values increases as the value of the affected asset becomes larger [Fama, 1965; Blattberg and Gonedes, 1974; Cox et al., 1985; Theodossiou, 1998; Barndorff-Nielsen and Shephard, 2001].[3]

Here we need to emphasize, however, that assuming *increments*, $\mathrm{d}x(t)$, in the net present value of the capital project are generated by some form of Gaussian process (as is the case above) does not necessarily imply that the (*level* of the) net present value variable, $x(t)$, itself will also be Gaussian. In fact, the exact relationship between the distributions describing how the increments to a stochastic variable evolve and that which describes the levels of the variable itself are summarized by the Fokker–Planck equation (Cox and Miller, 1965, pp. 213–225). In the present context Merton (1975, pp. 389–391) and Karlin and Taylor (1981, pp. 219–221) amongst others show that the solution to the Fokker–Planck equation takes the form:

$$g(x) = \frac{c}{a(x)} \exp\left[\int^{x} \frac{2b(y)}{a(y)}\,\mathrm{d}y\right] \tag{6.2}$$

where x is the instantaneous level of the stochastic variable, $g(x) \geq 0$ is its probability density, $b(x) = E_t[\mathrm{d}x(t)]/\mathrm{d}t$ is the expected instantaneous increment (per unit time) in the stochastic variable, $a(x) = \mathrm{Var}_t[\mathrm{d}x(t)]/\mathrm{d}t$ is the variance (per unit time) of the instantaneous increment in the stochastic variable and c is a normalizing constant which ensures that there is a unit area under the probability density. Substituting the expressions for $b(x)$ and $a(x)$ into the Fokker–Planck equation shows that the probability density for (the level of) the net present value variable, $x(t)$, is given by:

$$g(x) = c\left(1 + \frac{x^2}{\nu_1}\right)^{-(1+\nu_2)} \tag{6.3}$$

where $\nu_1 = k^2/2r > 0$, $\nu_2 = \beta/2r \geq 0$, $c = \Gamma(\nu_2 + 1)/[\sqrt{\pi\nu_1}\Gamma((2\nu_2 + 1)/2)]$ and $\Gamma(\cdot)$ is the gamma function. Furthermore, substitution shows $z = \sqrt{(2\nu_2 + 1)/\nu_1}x$ is distributed as a Student's 't' variate with $2\nu_2 + 1$ degrees of freedom. This identifies $g(x)$ as the 'scaled' Student distribution first introduced into the financial economics literature by Praetz (1972) and Blattberg and Gonedes (1974). It is well known that this distribution has 'fat' tails relative to the normal distribution, reflecting the fact that in a research and development context, capital projects either tend to pay off as blockbuster products or be dismal failures. However, relatively little is known about the properties of options written on securities that evolve in terms of fat-tailed distributions like these, and so it is to this issue we now turn.

Using the standard arbitrage arguments demonstrated in the appendix (Section 6.4) (Smith, 1976, pp. 20–23) we can show that the value of an option, $V(x, t)$, written on the net present value of the underlying capital project, $x(t)$, will have to satisfy the fundamental valuation equation:

$$\frac{1}{2}(k^2 + 2rx^2)\frac{\partial^2 V}{\partial x^2} + rx\frac{\partial V}{\partial x} + \frac{\partial V}{\partial t} - rV(x, t) = 0 \tag{6.4}$$

For simplicity we assume that the option will only be exercised if the net present value of the underlying capital project is positive. This means that the option value will also have to satisfy the following boundary condition:

$$V(x, T) = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \tag{6.5}$$

where $T \geq t$ is the date on which a decision about whether to invest in the capital project (or not) must be taken. The more general case, which requires that the option will only be exercised if the project's net present value exceeds a prespecified and non-trivial (positive) hurdle value at maturity, is treated in the appendix.

Progress towards solving this boundary value problem can be made by making the substitution $V(x, t) = \exp[-r(T - t)]F(\xi, \eta)$, based on the transformed co-ordinate system $\xi = (1/\sqrt{2r}) \log(x + \sqrt{x^2 + k^2/2r})$ and $\eta = (T - t)/2$. We can demonstrate the significance of these transformations by recalling that the martingale equivalent pricing property for 'viable' arbitrage economies 'causes every security to earn (in expected value) at the riskless rate ....' (Harrison and Kreps, 1979, p. 383). In other words, the continuous trading property of an arbitrage economy means that there is 'a redistribution of probability mass', the effect of which is to change the underlying distribution (of the security's return) so that it has an expected return equal to that on the riskless asset. All the 'higher' moments, however, remain unchanged. This means that the arbitrage process will be based on a security whose value evolves in accordance with the following stochastic differential equation:[4]

$$\mathrm{d}x(t) = rx(t)\mathrm{d}t + \sqrt{k^2 + 2rx^2(t)}\,\mathrm{d}z(t) \tag{6.6}$$

Thus, in the arbitrage economy depicted here, the net present value variable will have increments with an instantaneous mean (per unit time) of $E_t[\mathrm{d}x(t)]/\mathrm{d}t = rx(t)$. The instantaneous variance of increments in the net present value variable, however, remains unchanged at $\mathrm{Var}_t[\mathrm{d}x(t)]/\mathrm{d}t = k^2 + 2rx^2(t)$. Furthermore, using this result in conjunction with Itô's lemma shows that instantaneous increments in ξ evolve in terms of a white noise process with unit variance parameter, or:

$$\begin{aligned} \mathrm{d}\xi &= \frac{\partial \xi}{\partial x}\mathrm{d}x + \frac{1}{2}(\mathrm{d}x)^2 \frac{\partial^2 \xi}{\partial x^2} = \frac{rx\,\mathrm{d}t + \sqrt{k^2 + 2rx^2}\,\mathrm{d}z}{\sqrt{k^2 + 2rx^2}} \\ &\quad - \frac{(k^2 + 2rx^2)rx\,\mathrm{d}t}{[\sqrt{k^2 + 2rx^2}]^3} = \mathrm{d}z(t) \end{aligned} \tag{6.7}$$

This means that ξ itself is a Wiener–Lévy process and, as such, will be normally distributed with a mean of zero and a variance of t. Now, given the well-known relationship between the normal distribution and the diffusion equation of mathematical physics (Apostol, 1969, p. 292), it ought to come as no surprise that under the (ξ, η) transformations, the fundamental valuation equation assumes the canonical form:[5]

$$\frac{\partial^2 F}{\partial \xi^2} = \frac{\partial F}{\partial \eta} \tag{6.8}$$

with the boundary condition:

$$F(y, T) = h(y) = \begin{cases} \left[\dfrac{2r\exp(\sqrt{2r}y) - k^2\exp(-\sqrt{2r}y)}{4r}\right] \\ \quad \text{if } y \geq \dfrac{1}{\sqrt{2r}} \log\left(\dfrac{k}{\sqrt{2r}}\right) \\ 0 \quad \text{if } y < \dfrac{1}{\sqrt{2r}} \log\left(\dfrac{k}{\sqrt{2r}}\right) \end{cases} \tag{6.9}$$

Now, it is well known that the general solution to this equation takes the form (Weinberger, 1965, pp. 327–328):

$$F(\xi, \eta) = \frac{1}{\sqrt{4\pi\eta}} \int_{-\infty}^{\infty} h(y) \exp\left[\frac{-(\xi - y)^2}{4\eta}\right] \mathrm{d}y \tag{6.10}$$

Thus, if we substitute the boundary condition, $h(y)$, into this equation and simplify, we end up with the following unique solution to the fundamental valuation equation:

$$V(x, t) = \frac{1}{2}\left(x + \sqrt{x^2 + \frac{k^2}{2r}}\right) N(d_1) - \frac{1}{2}\left(\sqrt{x^2 + \frac{k^2}{2r}} - x\right) N(d_2) \tag{6.11}$$

where:[6]

$$d_1 = \frac{\log\left(\dfrac{\sqrt{2r}}{k}x + \sqrt{\dfrac{2rx^2}{k^2} + 1}\right) + 2r(T - t)}{\sqrt{2r(T - t)}}$$

$$d_2 = \frac{\log\left(\dfrac{\sqrt{2r}}{k}x + \sqrt{\dfrac{2rx^2}{k^2} + 1}\right) - 2r(T - t)}{\sqrt{2r(T - t)}}$$

and $N(d) = (1/\sqrt{2\pi}) \int_{-\infty}^{d} \exp(-z^2/2)\mathrm{d}z$ is the accumulated area under the standard normal distribution.[7]

Table 6.1 *Value of the option to implement the capital project whose net present value is x(t), when there is one year to run until maturity*

	$k = 1$			$k = 3$			$k = 5$		
$x(t)$	$r = 0.03$	$r = 0.06$	$r = 0.09$	$r = 0.03$	$r = 0.06$	$r = 0.09$	$r = 0.03$	$r = 0.06$	$r = 0.09$
−4.00	0.0000	0.0001	0.0003	0.1308	0.1339	0.1366	0.5958	0.5905	0.5851
−3.75	0.0001	0.0002	0.0004	0.1546	0.1571	0.1592	0.6492	0.6427	0.6361
−3.50	0.0002	0.0004	0.0007	0.1821	0.1838	0.1851	0.7065	0.6986	0.6908
−3.25	0.0004	0.0007	0.0011	0.2137	0.2145	0.2149	0.7676	0.7584	0.7494
−3.00	0.0008	0.0013	0.0018	0.2498	0.2495	0.2490	0.8328	0.8223	0.8121
−2.75	0.0015	0.0022	0.0030	0.2909	0.2894	0.2878	0.9022	0.8905	0.8790
−2.50	0.0029	0.0039	0.0049	0.3373	0.3346	0.3319	0.9759	0.9630	0.9504
−2.25	0.0055	0.0067	0.0079	0.3895	0.3856	0.3816	1.0541	1.0401	1.0264
−2.00	0.0100	0.0115	0.0128	0.4481	0.4428	0.4376	1.1369	1.1218	1.1071
−1.75	0.0179	0.0194	0.0208	0.5133	0.5068	0.5003	1.2243	1.2083	1.1926
−1.50	0.0308	0.0322	0.0334	0.5855	0.5778	0.5702	1.3166	1.2997	1.2832
−1.25	0.0515	0.0524	0.0531	0.6653	0.6564	0.6478	1.4138	1.3961	1.3788
−1.00	0.0833	0.0832	0.0830	0.7527	0.7430	0.7333	1.5159	1.4976	1.4797
−0.75	0.1298	0.1285	0.1272	0.8483	0.8377	0.8273	1.6230	1.6042	1.5858
−0.50	0.1952	0.1926	0.1901	0.9520	0.9409	0.9299	1.7352	1.7161	1.6973
−0.25	0.2828	0.2792	0.2758	1.0642	1.0527	1.0414	1.8525	1.8332	1.8142
0.00	0.3950	0.3911	0.3873	1.1850	1.1733	1.1619	1.9749	1.9555	1.9365
0.25	0.5328	0.5292	0.5258	1.3142	1.3027	1.2914	2.1025	2.0832	2.0642
0.50	0.6952	0.6926	0.6901	1.4520	1.4409	1.4299	2.2352	2.2161	2.1973
0.75	0.8798	0.8785	0.8772	1.5983	1.5877	1.5773	2.3730	2.3542	2.3358
1.00	1.0833	1.0832	1.0830	1.7527	1.7430	1.7333	2.5159	2.4976	2.4797
1.25	1.3015	1.3024	1.3031	1.9153	1.9064	1.8978	2.6638	2.6461	2.6288
1.50	1.5308	1.5322	1.5334	2.0855	2.0778	2.0702	2.8166	2.7997	2.7832
1.75	1.7679	1.7694	1.7708	2.2633	2.2568	2.2503	2.9743	2.9583	2.9426
2.00	2.0100	2.0115	2.0128	2.4481	2.4428	2.4376	3.1369	3.1218	3.1071
2.25	2.2555	2.2567	2.2579	2.6395	2.6356	2.6316	3.3041	3.2901	3.2764
2.50	2.5029	2.5039	2.5049	2.8373	2.8346	2.8319	3.4759	3.4630	3.4504
2.75	2.7515	2.7522	2.7530	3.0409	3.0394	3.0378	3.6522	3.6405	3.6290
3.00	3.0008	3.0013	3.0018	3.2498	3.2495	3.2490	3.8328	3.8223	3.8121
3.25	3.2504	3.2507	3.2511	3.4637	3.4645	3.4649	4.0176	4.0084	3.9994
3.50	3.5002	3.5004	3.5007	3.6821	3.6838	3.6851	4.2065	4.1986	4.1908
3.75	3.7501	3.7502	3.7504	3.9046	3.9071	3.9092	4.3992	4.3927	4.3861
4.00	4.0000	4.0001	4.0003	4.1308	4.1339	4.1366	4.5958	4.5905	4.5851

Table 6.1 contains a summary of the option values implied by this pricing formula for three values of the risk-free rate of interest ($r = 3\%$, 6% and 9%) and three values of the 'zero rent' variance parameter ($k = 1$, 3 and 5) when the option has a year to run until maturity ($T - t = 1$). Thus, when the risk-free rate of interest is $r = 6\%$, the zero rent volatility parameter is $k = 3$ and the net present value of the capital project is currently $x = -£2$, then the option to invest in the capital project will have a value of 44.28p.

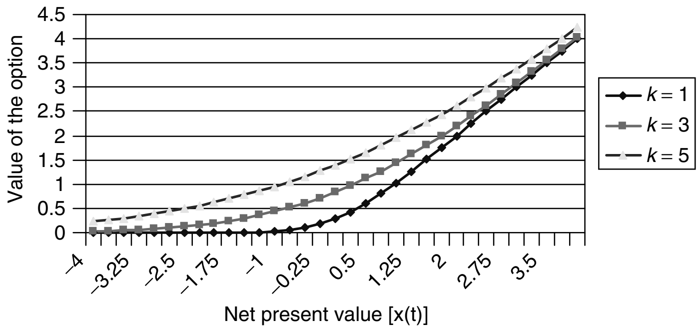

Figure 6.1 *Value of the option on the capital project ($T = 6$ months, $r = 6\%$)*

Figure 6.1 provides an equivalent illustration for the case when all parameters are the same as in Table 6.1 except that the option now has only six months ($T - t = 1/2$) to run until maturity. Note that as k grows (there is more uncertainty associated with the capital project's future net present value), the value of the option also grows. Furthermore, the option's value always exceeds what could be obtained from exercising it before its maturity date; that is, $V(x, t) > x$ for all values of x and $t < T$ (Merton, 1973, pp. 143–144). This means that the holder of the option would be foolish to exercise the option prematurely, since there is a greater financial advantage to be had from selling the option itself rather than implementing the capital project on which it is written.

The sensitivity of the option formula, $V(x, t)$, to small changes in its determining variables sheds further light on the impact that Student distributions can have on option values. We begin with the most commonly computed sensitivity measure; namely, the option delta, which is the derivative of the option value, $V(x, t)$, with respect to the net present value, x, of the capital project, or:

$$\Delta = \frac{\partial V}{\partial x} = \frac{1}{2}\left(1 + \sqrt{\frac{x^2}{\frac{k^2}{2r} + x^2}}\right) N(d_1) + \frac{1}{2}\left(1 - \sqrt{\frac{x^2}{\frac{k^2}{2r} + x^2}}\right) N(d_2) \tag{6.12}$$

Numerical values of this derivative are summarized in Table 6.2 and Figure 6.2. The table contains the option delta for three values of the risk-free rate of interest ($r = 3\%, 6\%$ and 9%) and three values of the zero rent variance parameter ($k = 1, 3$ and 5) when the option has a year to run until maturity. Figure 6.2 provides an equivalent illustration for the case when all parameters are the same as in Table 6.2, except that the option now has only six months to run until maturity. These show that the option delta will be near to zero for deep out-of-the-money

Table 6.2 *Value of delta ($\Delta = \partial V/\partial x$) when the option to implement the capital project whose net present value is $x(t)$ has one year to run until maturity*

	$k=1$			$k=3$			$k=5$		
$x(t)$	$r=0.03$	$r=0.06$	$r=0.09$	$r=0.03$	$r=0.06$	$r=0.09$	$r=0.03$	$r=0.06$	$r=0.09$
−4.00	0.0001	0.0003	0.0005	0.0886	0.0862	0.0840	0.2065	0.2014	0.1966
−3.75	0.0003	0.0005	0.0008	0.1025	0.0996	0.0968	0.2212	0.2161	0.2112
−3.50	0.0006	0.0009	0.0013	0.1180	0.1145	0.1113	0.2366	0.2314	0.2265
−3.25	0.0011	0.0016	0.0022	0.1351	0.1311	0.1274	0.2525	0.2474	0.2425
−3.00	0.0021	0.0029	0.0035	0.1540	0.1496	0.1455	0.2691	0.2640	0.2592
−2.75	0.0040	0.0050	0.0058	0.1746	0.1699	0.1654	0.2861	0.2813	0.2766
−2.50	0.0075	0.0086	0.0094	0.1971	0.1920	0.1873	0.3038	0.2991	0.2946
−2.25	0.0135	0.0145	0.0154	0.2212	0.2161	0.2112	0.3219	0.3175	0.3132
−2.00	0.0237	0.0244	0.0248	0.2472	0.2420	0.2371	0.3404	0.3364	0.3324
−1.75	0.0401	0.0400	0.0398	0.2747	0.2697	0.2649	0.3594	0.3558	0.3521
−1.50	0.0654	0.0640	0.0627	0.3038	0.2991	0.2946	0.3788	0.3755	0.3723
−1.25	0.1025	0.0996	0.0968	0.3342	0.3300	0.3260	0.3985	0.3957	0.3930
−1.00	0.1540	0.1496	0.1455	0.3658	0.3623	0.3588	0.4184	0.4162	0.4139
−0.75	0.2212	0.2161	0.2112	0.3985	0.3957	0.3930	0.4386	0.4369	0.4352
−0.50	0.3038	0.2991	0.2946	0.4319	0.4300	0.4281	0.4590	0.4578	0.4567
−0.25	0.3985	0.3957	0.3930	0.4658	0.4648	0.4639	0.4795	0.4789	0.4783
0.00	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
0.25	0.6015	0.6043	0.6070	0.5342	0.5352	0.5361	0.5205	0.5211	0.5217
0.50	0.6962	0.7009	0.7054	0.5681	0.5700	0.5719	0.5410	0.5422	0.5433
0.75	0.7788	0.7839	0.7888	0.6015	0.6043	0.6070	0.5614	0.5631	0.5648
1.00	0.8460	0.8504	0.8545	0.6342	0.6377	0.6412	0.5816	0.5838	0.5861
1.25	0.8975	0.9004	0.9032	0.6658	0.6700	0.6740	0.6015	0.6043	0.6070
1.50	0.9346	0.9360	0.9373	0.6962	0.7009	0.7054	0.6212	0.6245	0.6277
1.75	0.9599	0.9600	0.9602	0.7253	0.7303	0.7351	0.6406	0.6442	0.6479
2.00	0.9763	0.9756	0.9752	0.7528	0.7580	0.7629	0.6596	0.6636	0.6676
2.25	0.9865	0.9855	0.9846	0.7788	0.7839	0.7888	0.6781	0.6825	0.6868
2.50	0.9925	0.9914	0.9906	0.8029	0.8080	0.8127	0.6962	0.7009	0.7054
2.75	0.9960	0.9950	0.9942	0.8254	0.8301	0.8346	0.7139	0.7187	0.7234
3.00	0.9979	0.9971	0.9965	0.8460	0.8504	0.8545	0.7309	0.7360	0.7408
3.25	0.9989	0.9984	0.9978	0.8649	0.8689	0.8726	0.7475	0.7526	0.7575
3.50	0.9994	0.9991	0.9987	0.8820	0.8855	0.8887	0.7634	0.7686	0.7735
3.75	0.9997	0.9995	0.9992	0.8975	0.9004	0.9032	0.7788	0.7839	0.7888
4.00	0.9999	0.9997	0.9995	0.9114	0.9138	0.9160	0.7935	0.7986	0.8034

options (when the capital project has a large negative net present value) and near to unity for deep in-the-money options (when the capital project has a large positive net present value). Note also that the delta changes more quickly when the capital project's net present value is close to zero, although increasing values of the zero rent variance parameter k have a moderating influence on the rate at which delta grows. Furthermore, comparing Table 6.2 with Figure 6.2 shows that for out-of-the-money options, delta converges towards zero as the option's maturity date approaches; for in-the-money options, however, delta converges towards unity as the maturity date approaches.

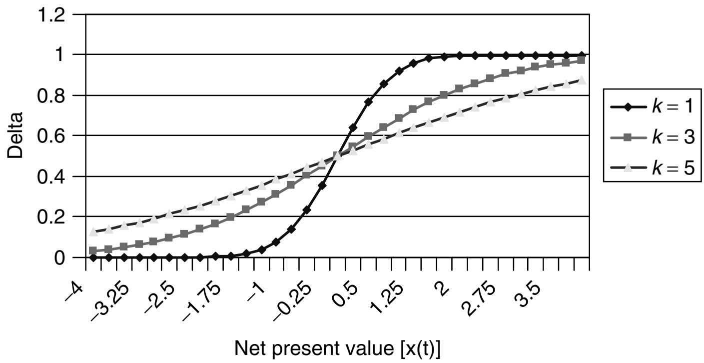

Figure 6.2 *Value of delta for the option on the capital project ($T = 6$ months, $r = 6\%$)*

A second sensitivity measure is the option's theta which is obtained by differentiating the option valuation formula, $V(x, t)$, with respect to calendar time, or:

$$\Theta = \frac{\partial V}{\partial t} = -\frac{1}{2}\left(x + \sqrt{\frac{k^2}{2r} + x^2}\right) N'(d_1)\sqrt{\frac{2r}{T - t}}$$

$$= -\frac{1}{2}\left(\sqrt{\frac{k^2}{2r} + x^2} - x\right) N'(d_2)\sqrt{\frac{2r}{T - t}} \tag{6.13}$$

where $N'(d) = (1/\sqrt{2\pi})\exp(-d^2/2)$ is the derivative of (the accumulated area under) the standard normal distribution. Numerical values of this derivative are summarized in Table 6.3 and Figure 6.3. Table 6.3 contains the option theta for the three values of the risk-free rate ($r = 3\%, 6\%$ and 9%) and the three values of the zero rent variance parameter ($k = 1, 3$ and 5) previously used, given that the option has a year to run until maturity. Figure 6.3 provides an equivalent illustration for the case when all parameters are the same as in Table 6.3, except that the option now has only six months to run until maturity. These show that as the capital project's net present value, x, grows, the option theta at first decreases, reaching a minimum when the capital project's net present value is zero. The option theta then increases in a symmetrical fashion towards zero. Comparing Table 6.3 with Figure 6.3 shows that for at-the-money options (where the capital project's net present value is not far removed from zero) theta is decreasing as the option's maturity date approaches. For (moderately or deep) in- or out-of-the-money options, however, the option theta is either increasing or decreasing as the time to maturity declines, depending on the exact values of the risk-free rate of interest, r, and the zero rent variance parameter, k.

The final sensitivity measure we consider is the option's rho, which is obtained by differentiating the valuation formula with respect to the risk-free rate of

Table 6.3 *Value of theta ($\Theta = \partial V/\partial t$) when the option to implement the capital project whose net present value is x(t) has one year to run until maturity*

	$k=1$			$k=3$			$k=5$		
$x(t)$	$r=0.03$	$r=0.06$	$r=0.09$	$r=0.03$	$r=0.06$	$r=0.09$	$r=0.03$	$r=0.06$	$r=0.09$
−4.00	−0.0004	−0.0009	−0.0017	−0.2460	−0.2452	−0.2439	−0.7057	−0.6874	−0.6696
−3.75	−0.0007	−0.0015	−0.0025	−0.2721	−0.2698	−0.2669	−0.7329	−0.7133	−0.6943
−3.50	−0.0013	−0.0024	−0.0037	−0.2993	−0.2953	−0.2909	−0.7594	−0.7386	−0.7184
−3.25	−0.0023	−0.0038	−0.0054	−0.3273	−0.3215	−0.3157	−0.7849	−0.7631	−0.7417
−3.00	−0.0041	−0.0060	−0.0080	−0.3557	−0.3483	−0.3409	−0.8095	−0.7865	−0.7642
−2.75	−0.0070	−0.0094	−0.0117	−0.3841	−0.3752	−0.3664	−0.8328	−0.8089	−0.7857
−2.50	−0.0118	−0.0146	−0.0170	−0.4123	−0.4019	−0.3918	−0.8547	−0.8299	−0.8059
−2.25	−0.0192	−0.0221	−0.0246	−0.4397	−0.4280	−0.4166	−0.8750	−0.8495	−0.8247
−2.00	−0.0302	−0.0329	−0.0351	−0.4659	−0.4530	−0.4404	−0.8937	−0.8675	−0.8421
−1.75	−0.0456	−0.0476	−0.0491	−0.4904	−0.4765	−0.4629	−0.9105	−0.8837	−0.8577
−1.50	−0.0659	−0.0667	−0.0671	−0.5128	−0.4980	−0.4835	−0.9254	−0.8981	−0.8716
−1.25	−0.0907	−0.0899	−0.0890	−0.5326	−0.5170	−0.5019	−0.9381	−0.9104	−0.8836
−1.00	−0.1186	−0.1161	−0.1136	−0.5494	−0.5332	−0.5175	−0.9487	−0.9207	−0.8935
−0.75	−0.1466	−0.1427	−0.1389	−0.5629	−0.5463	−0.5301	−0.9571	−0.9288	−0.9013
−0.50	−0.1709	−0.1660	−0.1612	−0.5727	−0.5558	−0.5394	−0.9631	−0.9346	−0.9070
−0.25	−0.1876	−0.1821	−0.1767	−0.5787	−0.5616	−0.5450	−0.9667	−0.9381	−0.9104
0.00	−0.1936	−0.1879	−0.1823	−0.5807	−0.5636	−0.5469	−0.9679	−0.9393	−0.9115
0.25	−0.1876	−0.1821	−0.1767	−0.5787	−0.5616	−0.5450	−0.9667	−0.9381	−0.9104
0.50	−0.1709	−0.1660	−0.1612	−0.5727	−0.5558	−0.5394	−0.9631	−0.9346	−0.9070
0.75	−0.1466	−0.1427	−0.1389	−0.5629	−0.5463	−0.5301	−0.9571	−0.9288	−0.9013
1.00	−0.1186	−0.1161	−0.1136	−0.5494	−0.5332	−0.5175	−0.9487	−0.9207	−0.8935
1.25	−0.0907	−0.0899	−0.0890	−0.5326	−0.5170	−0.5019	−0.9381	−0.9104	−0.8836
1.50	−0.0659	−0.0667	−0.0671	−0.5128	−0.4980	−0.4835	−0.9254	−0.8981	−0.8716
1.75	−0.0456	−0.0476	−0.0491	−0.4904	−0.4765	−0.4629	−0.9105	−0.8837	−0.8577
2.00	−0.0302	−0.0329	−0.0351	−0.4659	−0.4530	−0.4404	−0.8937	−0.8675	−0.8421
2.25	−0.0192	−0.0221	−0.0246	−0.4397	−0.4280	−0.4166	−0.8750	−0.8495	−0.8247
2.50	−0.0118	−0.0146	−0.0170	−0.4123	−0.4019	−0.3918	−0.8547	−0.8299	−0.8059
2.75	−0.0070	−0.0094	−0.0117	−0.3841	−0.3752	−0.3664	−0.8328	−0.8089	−0.7857
3.00	−0.0041	−0.0060	−0.0080	−0.3557	−0.3483	−0.3409	−0.8095	−0.7865	−0.7642
3.25	−0.0023	−0.0038	−0.0054	−0.3273	−0.3215	−0.3157	−0.7849	−0.7631	−0.7417
3.50	−0.0013	−0.0024	−0.0037	−0.2993	−0.2953	−0.2909	−0.7594	−0.7386	−0.7184
3.75	−0.0007	−0.0015	−0.0025	−0.2721	−0.2698	−0.2669	−0.7329	−0.7133	−0.6943
4.00	−0.0004	−0.0009	0.0017	−0.2460	−0.2452	−0.2439	−0.7057	−0.6874	−0.6696

interest, or:

$$\rho = \frac{\partial V}{\partial r} = -\frac{k^2}{8r^2}\frac{1}{\sqrt{\frac{k^2}{2r}+x^2}}[N(d_1) - N(d_2)] + \frac{1}{2}\left(\sqrt{\frac{k^2}{2r}+x^2}+x\right)N'(d_1)\sqrt{\frac{2(T-t)}{r}} \tag{6.14}$$

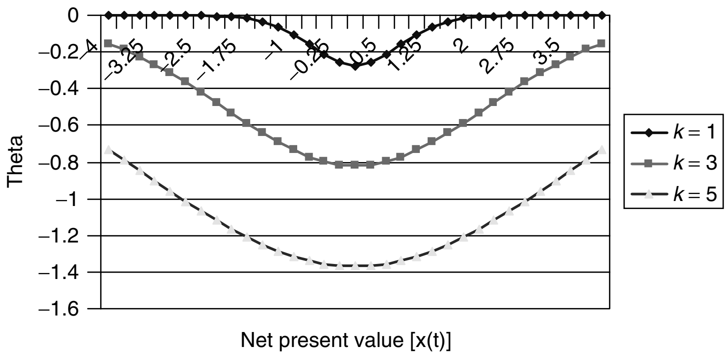

Figure 6.3 *Value of theta for the option on the capital project ($T = 6$ months, $r = 6\%$)*

Numerical values of this derivative are summarized in Table 6.4 and Figure 6.4. Table 6.4 contains the option rho for the three values of the risk-free rate ($r = 3\%, 6\%$ and 9%) and the three values of the zero rent variance parameter ($k = 1, 3$ and 5) previously used, given that the option has a year to run until maturity. Figure 6.4 provides an equivalent illustration for the case when all parameters are the same as in Table 6.4, except that the option now has only six months to run until maturity. These show that as the capital project's net present value, x, grows, the option rho at first increases towards a maximum and then declines towards a minimum which is reached when the capital project has a net present value of zero.[8] Rho then grows again in symmetrical fashion towards another maximum before asymptotically declining towards zero. Comparing Table 6.4 with Figure 6.4 shows that for at-the-money options, the option rho is increasing as the option's maturity date approaches. For in- or out-of-the-money options, however, the option rho is either increasing or decreasing as the time to maturity approaches, depending on the exact values of the risk-free rate of interest, r, and the zero rent variance parameter, k.

There is an important relationship between the elasticities implied by each of the determining variables considered above. Here it will be recalled that the elasticity of the option price with respect to its determining variables is the proportionate change in the option price divided by the proportionate change in the determining variable. This means that the elasticity of the option price with respect to the net present value of the capital project will be $(x/V)(\partial V/\partial x)$. Hence, if the proportionate change in the option price exceeds the proportionate change in the net present value of the capital project, then the elasticity of the option value with respect to this variable will exceed unity; that is, $(x/V)(\partial V/\partial x) > 1$. Likewise, if the proportionate change in the option price is less than the proportionate change in the net present value of the capital project, then the elasticity of the option value with respect to this variable will be less than unity; that is, $(x/V)(\partial V/\partial x) < 1$. Now, suppose also that we let T

Table 6.4 *Value of rho ($\rho = \partial V/\partial r$) when the option to implement the capital project whose net present value is x(t) has one year to run until maturity*

	$k = 1$			$k = 3$			$k = 5$		
$x(t)$	$r = 0.03$	$r = 0.06$	$r = 0.09$	$r = 0.03$	$r = 0.06$	$r = 0.09$	$r = 0.03$	$r = 0.06$	$r = 0.09$
−4.00	0.0022	0.0039	0.0056	0.1112	0.0969	0.0843	−0.1761	−0.1780	−0.1795
−3.75	0.0037	0.0059	0.0077	0.0882	0.0755	0.0642	−0.2198	−0.2197	−0.2193
−3.50	0.0062	0.0086	0.0106	0.0606	0.0498	0.0402	−0.2636	−0.2617	−0.2596
−3.25	0.0099	0.0125	0.0144	0.0287	0.0199	0.0122	−0.3073	−0.3037	−0.3000
−3.00	0.0153	0.0177	0.0191	−0.0073	−0.0137	−0.0195	−0.3503	−0.3452	−0.3401
−2.75	0.0227	0.0242	0.0249	−0.0466	−0.0508	−0.0545	−0.3922	−0.3857	−0.3793
−2.50	0.0318	0.0319	0.0314	−0.0884	−0.0904	−0.0921	−0.4324	−0.4248	−0.4173
−2.25	0.0416	0.0398	0.0378	−0.1319	−0.1318	−0.1316	−0.4705	−0.4619	−0.4536
−2.00	0.0500	0.0462	0.0427	−0.1757	−0.1738	−0.1719	−0.5061	−0.4967	−0.4876
−1.75	0.0534	0.0483	0.0437	−0.2186	−0.2153	−0.2120	−0.5386	−0.5286	−0.5189
−1.50	0.0478	0.0424	0.0376	−0.2594	−0.2549	−0.2504	−0.5677	−0.5573	−0.5470
−1.25	0.0294	0.0252	0.0214	−0.2967	−0.2912	−0.2859	−0.5931	−0.5822	−0.5716
−1.00	−0.0024	−0.0046	−0.0065	−0.3292	−0.3231	−0.3172	−0.6143	−0.6031	−0.5922
−0.75	−0.0440	−0.0439	−0.0439	−0.3558	−0.3493	−0.3429	−0.6311	−0.6197	−0.6086
−0.50	−0.0865	−0.0850	−0.0835	−0.3756	−0.3688	−0.3622	−0.6432	−0.6317	−0.6205
−0.25	−0.1186	−0.1164	−0.1143	−0.3877	−0.3808	−0.3741	−0.6506	−0.6390	−0.6277
0.00	−0.1306	−0.1283	−0.1260	−0.3918	−0.3849	−0.3781	−0.6530	−0.6415	−0.6301
0.25	−0.1186	−0.1164	−0.1143	−0.3877	−0.3808	−0.3741	−0.6506	−0.6390	−0.6277
0.50	−0.0865	−0.0850	−0.0835	−0.3756	−0.3688	−0.3622	−0.6432	−0.6317	−0.6205
0.75	−0.0440	−0.0439	−0.0439	−0.3558	−0.3493	−0.3429	−0.6311	−0.6197	−0.6086
1.00	−0.0024	−0.0046	−0.0065	−0.3292	−0.3231	−0.3172	−0.6143	−0.6031	−0.5922
1.25	0.0294	0.0252	0.0214	−0.2967	−0.2912	−0.2859	−0.5931	−0.5822	−0.5716
1.50	0.0478	0.0424	0.0376	−0.2594	−0.2549	−0.2504	−0.5677	−0.5573	−0.5470
1.75	0.0534	0.0483	0.0437	−0.2186	−0.2153	−0.2120	−0.5386	−0.5286	−0.5189
2.00	0.0500	0.0462	0.0427	−0.1757	−0.1738	−0.1719	−0.5061	−0.4967	−0.4876
2.25	0.0416	0.0398	0.0378	−0.1319	−0.1318	−0.1316	−0.4705	−0.4619	−0.4536
2.50	0.0318	0.0319	0.0314	−0.0884	−0.0904	−0.0921	−0.4324	−0.4248	−0.4173
2.75	0.0227	0.0242	0.0249	−0.0466	−0.0508	−0.0545	−0.3922	−0.3857	−0.3793
3.00	0.0153	0.0177	0.0191	−0.0073	−0.0137	−0.0195	−0.3503	−0.3452	−0.3401
3.25	0.0099	0.0125	0.0144	0.0287	0.0199	0.0122	−0.3073	−0.3037	−0.3000
3.50	0.0062	0.0086	0.0106	0.0606	0.0498	0.0402	−0.2636	−0.2617	−0.2596
3.75	0.0037	0.0059	0.0077	0.0882	0.0755	0.0642	−0.2198	−0.2197	−0.2193
4.00	0.0022	0.0039	0.0056	0.1112	0.0969	0.0843	−0.1761	−0.1780	−0.1795

be the calendar date on which the option matures. It then follows that $\tau = T - t$ will be the time until the option's maturity – in which case $(\tau/V)(\partial V/\partial\tau)$ will be the elasticity of the option price with respect to the time the option has remaining until its maturity. Furthermore, the elasticity of the option price with respect to the risk-free rate of interest will be $(r/V)(\partial V/\partial r)$. Differentiating through equation (6.11) and a little algebra shows that these three elasticity measures are related by the formula:

$$\frac{x}{V}\frac{\partial V}{\partial x} + 2\left(\frac{\tau}{V}\frac{\partial V}{\partial \tau} - \frac{r}{V}\frac{\partial V}{\partial r}\right) = 1 \qquad (6.15)$$

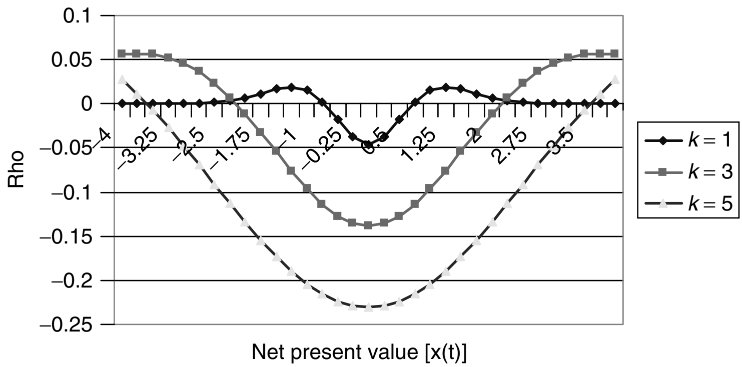

Figure 6.4 *Value of rho for the option on the capital project ($T = 6$ months, $r = 6\%$)*

This provides a useful device for checking whether the numerical values obtained for the important sensitivity measures (Δ, Θ and ρ) have been correctly computed. Needless to say, it is a relationship that is satisfied by the values reported for the sensitivity measures in the above tables.

6.3 SUMMARY CONCLUSIONS

Our purpose here has been to present a closed-form solution for the value of the option to invest in a capital project when the option to do so has a finite life. In doing so, we assume that the capital project's net present value evolves in terms of the Student distributions which exhibit the 'fat tail' properties characterizing at least some of the empirical distributions of R&D. However, most analytical work conducted in this area assumes that the option to undertake an investment project has an infinite life (McDonald and Siegel, 1986). Unfortunately, many real-life investment opportunities are not infinitely lived but expire and become worthless at a known point in time. Probably the best example of this is a finitely lived patent, which gives the holder the option to invest at any time before a given expiration date. Our analysis provides an explicit closed-form solution for the valuation of finite-lived derivative securities of this kind.

Closed-form solutions, such as the one derived here, are, however, notoriously difficult to come by. Yet, despite the difficulties associated with obtaining closed-form solutions, there is one variation to our analysis that is worthy of further investigation. This stems from the fact that many asset prices appear to evolve in terms of distributions which exhibit not only 'fat' tails but also significant skewness (Theodossiou, 1998; Barndorff-Nielsen and Shephard, 2001). Here it is important to note that the symmetric Student distributions on which our analysis is based are a specific instance of a more general class of Student distributions known as the Pearson Type IV. These distributions, which Kendall and Stuart (1977,

p. 163) emphasize are 'very difficult to handle ... in practice', encapsulate not only the 'fat' tail property which characterizes many asset prices, but also allow for skewness in the increments of asset values. It would be a useful exercise to explore whether, for these more general Student distributions, it is possible to obtain closed-form formulae for the value of the option to invest in a capital project.

6.4 APPENDIX

Applying Itô's lemma to the value of the option, $V(x, t)$, implies:

$$\mathrm{d}V = \frac{\partial V}{\partial x}\,\mathrm{d}x + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(\mathrm{d}x)^2 + \frac{\partial V}{\partial t}\,\mathrm{d}t \tag{A6.1}$$

or

$$\mathrm{d}V = \left[-\beta x\frac{\partial V}{\partial x} + \frac{1}{2}(k^2 + 2rx^2)\frac{\partial^2 V}{\partial x^2} + \frac{\partial V}{\partial t}\right]\mathrm{d}t + \sqrt{k^2 + 2rx^2}\,\frac{\partial V}{\partial x}\,\mathrm{d}z \tag{A6.2}$$

This means that the instantaneous rate of return on the option will be:

$$\frac{\mathrm{d}V}{V} = \theta\,\mathrm{d}t + \gamma\,\mathrm{d}z \tag{A6.3}$$

where:

$$\theta = \frac{-\beta x\dfrac{\partial V}{\partial x} + \dfrac{1}{2}(k^2 + 2rx^2)\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial V}{\partial t}}{V} \quad \text{and} \quad \gamma = \frac{\sqrt{k^2 + 2rx^2}\,\dfrac{\partial V}{\partial x}}{V}$$

Now, suppose we invest W_1 in the underlying capital project, W_2 in options and W_3 in the risk-free asset, but in such a way that the investment is self-financing, or $W_1 + W_2 + W_3 = 0$. Then differentiation shows:

$$W_1\frac{\mathrm{d}W_1}{W_1} + W_2\frac{\mathrm{d}W_2}{W_2} + W_3\frac{\mathrm{d}W_3}{W_3} = 0 = W_1\frac{\mathrm{d}x}{x} + W_2\frac{\mathrm{d}V}{V} - (W_1 + W_2)r\,\mathrm{d}t \tag{A6.4}$$

Substituting the stochastic differential equations for $\mathrm{d}x$ and $\mathrm{d}V$ into the right-hand side of this expression will then give:

$$\left(-\beta\,\mathrm{d}t + \sqrt{\frac{k^2}{x^2} + 2r}\,\mathrm{d}z\right)W_1 + W_2(\theta\,\mathrm{d}t + \gamma\,\mathrm{d}z) - (W_1 + W_2)r\,\mathrm{d}t = 0 \tag{A6.5}$$

Now, suppose we pursue an investment policy which eliminates all uncertainty, so that:

$$\left(\sqrt{\frac{k^2}{x^2}+2r}\right)W_1+\gamma W_2=0=\left(\sqrt{\frac{k^2}{x^2}+2r}\right)W_1+\frac{\sqrt{k^2+2rx^2}\,\dfrac{\partial V}{\partial x}}{V}W_2 \tag{A6.6}$$

or

$$-\frac{x}{V}\frac{\partial V}{\partial x}W_2=W_1$$

Since all uncertainty has been eliminated, arbitrage will dictate that the non-stochastic component of the investment policy will also have to be zero, or:

$$\begin{aligned}-(\beta+r)W_1+(\theta-r)W_2=0&=\frac{(\beta+r)x}{V}\frac{\partial V}{\partial x}W_2\\&+\left[\frac{-\beta x\dfrac{\partial V}{\partial x}+\dfrac{1}{2}(k^2+2rx^2)\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial V}{\partial t}}{V}-r\right]W_2\end{aligned} \tag{A6.7}$$

Simplifying the right-hand side of this equation gives:

$$\frac{1}{2}(k^2+2rx^2)\frac{\partial^2 V}{\partial x^2}+rx\frac{\partial V}{\partial x}+\frac{\partial V}{\partial t}-rV(x,t)=0 \tag{A6.8}$$

which is the fundamental valuation equation contained in the text.

In the text we develop a pricing formula for a call option written on the net present value variable, with an exercise price of zero. Here, we solve the fundamental valuation equation under the more general boundary condition:

$$V(x,T)=\begin{cases}x-E & \text{if } x\geq E\\ 0 & \text{if } x<E\end{cases} \tag{A6.9}$$

where E is a (non-trivial) exercise price. We again make the substitution $V(x,t)=\exp[-r(T-t)]F(\xi,\eta)$, based on the co-ordinate system $\xi=(1/\sqrt{2r})\log(x+\sqrt{x^2+k^2/2r})$ and $\eta=(T-t)/2$. The fundamental valuation equation will then assume the canonical form:

$$\frac{\partial^2 F}{\partial \xi^2}=\frac{\partial F}{\partial \eta} \tag{A6.10}$$

subject to the boundary condition:

$$F(y,T)=h(y)=\begin{cases}\dfrac{2r\exp(\sqrt{2r}\,y)-k^2\exp(-\sqrt{2r}\,y)}{4r}-E \\ \qquad \text{if } y\geq \dfrac{1}{\sqrt{2r}}\log\left[E+\sqrt{E^2+\dfrac{k^2}{2r}}\right] \\ 0 \quad \text{if } y<\dfrac{1}{\sqrt{2r}}\log\left[E+\sqrt{E^2+\dfrac{k^2}{2r}}\right]\end{cases} \tag{A6.11}$$

The general solution to this equation will now take the form (Weinberger, 1965, pp. 327–328):

$$F(\xi,\eta)=\frac{1}{\sqrt{4\pi\eta}}\int_{-\infty}^{\infty}h(y)\exp\left[\frac{-(\xi-y)^2}{4\eta}\right]\mathrm{d}y \tag{A6.12}$$

Evaluating this integral shows the more general option pricing formula to be:

$$V(x,t)=\frac{1}{2}\left(x+\sqrt{x^2+\frac{k^2}{2r}}\right)N(d_1)-\frac{1}{2}\left(\sqrt{x^2+\frac{k^2}{2r}}-x\right)N(d_2) \\ -E\,\mathrm{e}^{-rt}N(d_3) \tag{A6.13}$$

where:[9]

$$d_1=\frac{\log\left(x+\sqrt{x^2+\frac{k^2}{2r}}\right)-\log\left(E+\sqrt{E^2+\frac{k^2}{2r}}\right)+2r(T-t)}{\sqrt{2r(T-t)}}$$

$$d_2=\frac{\log\left(x+\sqrt{x^2+\frac{k^2}{2r}}\right)-\log\left(E+\sqrt{E^2+\frac{k^2}{2r}}\right)-2r(T-t)}{\sqrt{2r(T-t)}}$$

$$d_3=\frac{\log\left(x+\sqrt{x^2+\frac{k^2}{2r}}\right)-\log\left(E+\sqrt{E^2+\frac{k^2}{2r}}\right)}{\sqrt{2r(T-t)}}$$

and $N(d)=(1\sqrt{2\pi})\int_{-\infty}^{d}\exp(-z^2/2)\mathrm{d}z$ is the accumulated area under the standard normal distribution.

As a second application of the above analysis, consider the case of a perpetual call option, for which $t \to \infty$ with an exercise price of $E = 0$. The value, $V(x)$, of this option will have to satisfy the following ordinary differential equation:

$$\frac{1}{2}(k^2 + 2rx^2)\frac{\mathrm{d}^2 V}{\mathrm{d}x^2} + rx\frac{\mathrm{d}V}{\mathrm{d}x} - rV(x,t) = 0 \tag{A6.14}$$

Now, direct substitution shows that $V_1(x) = x$ is a solution of this equation and so, by reduction of order, it is readily shown that a second linearly independent solution is (Boyce and DiPrima, 1969, pp. 103–106):

$$V_2(x) = x\int_x^{\infty} \frac{\mathrm{d}y}{y^2\sqrt{\alpha^2 + y^2}} = -\frac{x}{\alpha^2} + \frac{\sqrt{\alpha^2 + x^2}}{\alpha^2} \tag{A6.15}$$

where $\alpha^2 = k^2/2r$. It thus follows that the general solution of the ordinary differential equation (A6.14) is:

$$V(x) = c_1 V_1(x) + c_2 V_2(x) = c_1 x + c_2\left[-\frac{x}{\alpha^2} + \frac{\sqrt{\alpha^2 + x^2}}{\alpha^2}\right] \tag{A6.16}$$

where c_1 and c_2 are constants determined by boundary conditions which require that $V(x) > x$ for all x, $V(x) \to 0$ as $x \to -\infty$ and $V(x) \to x$ as $x \to \infty$. Setting $c_1 = c_2/(\alpha^2 - 1)$ and $c_2 = \alpha^2/2$ returns the solution satisfying these boundary conditions, namely:

$$V(x) = \frac{1}{2}x + \frac{1}{2}\sqrt{x^2 + \frac{k^2}{2r}} \tag{A6.17}$$

ACKNOWLEDGMENTS

We are indebted to Sydney Howell and Dean Paxson for constructive criticisms and suggestions. However, since we have not always followed the counsel of our critics, all remaining errors and omissions are the sole responsibility of the authors.

NOTES

1. It warrants emphasizing that our model does *not* assume there is a one-off investment outlay incurred immediately before installation of the capital project. We assume, instead, that operating cash flows and investment outlays accumulate over the entire term of the investment project. This means that the net cash flows from the project (by which we mean the operating cash flow less any investment

outlays incurred in each period) might be negative, and so the present value of these cash flows can also be negative.

2. A ‘white noise’ process (Hoel et al., 1972, p. 142) is the derivative of a Wiener–Lévy process (Hoel et al., 1972, p. 123). Furthermore, the coefficient associated with $x^2(t)$ in the stochastic term of this formulation – namely $2r$ – is chosen for computational and analytical convenience. By allowing $\mathrm{d}z(t)$ to be a white noise process with variance parameter σ^2 not, in general, equal to unity, we can reduce the impact of this assumption.
3. For some alternative empirical evidence (and models) on this issue see the articles by Heston (1993) and Hull and White (1988).
4. A simple application of Itô’s lemma shows that the solution to this equation is:

$$x(t) = \frac{2r\ \exp[\sqrt{2r}\,z(t)] - k^2\ \exp[-\sqrt{2r}\,z(t)]}{4r}$$

where $z(t)$ is a Wiener–Lévy process with unit variance parameter.

5. The Black and Scholes (1973) option valuation formula is likewise founded on a co-ordinate system which transforms the assumed geometric Brownian motion on which asset prices are based into a white noise process. The particular transformation used by Black and Scholes (1973, p. 643) is $\xi = \log(x) + (r - \frac{1}{2}\sigma^2)(T - t)$ where $\mathrm{d}x/x = r\,\mathrm{d}t + \sigma\,\mathrm{d}z(t)$ is a geometric Brownian motion based on a white noise process, $\mathrm{d}z(t)$, with variance parameter σ^2 and an instantaneous drift term equal to the risk-free rate of interest, r. A simple application of Itô’s lemma then shows:

$$\mathrm{d}\xi = \frac{\mathrm{d}x}{x} - \frac{1}{2}\left(\frac{\mathrm{d}x}{x}\right)^2 - \left(r - \frac{1}{2}\sigma^2\right)\mathrm{d}t = \sigma\,\mathrm{d}z(t)$$

in which case it follows that instantaneous increments in ξ are generated by a white noise process with variance parameter σ^2. Hence, if there is a transformation which reduces the underlying stochastic process on which asset prices are based into a white noise process, then it will always be possible to recast and solve the valuation problem in terms of the diffusion equation of mathematical physics (Crank, 1975).

6. More parsimonious, but perhaps less familiar, expressions for these variables are:

$$d_1 = \frac{\sinh^{-1}\left(\frac{\sqrt{2r}}{k}x\right) + 2r(T - t)}{\sqrt{2r(T - t)}} \quad \text{and}$$

$$d_2 = \frac{\sinh^{-1}\left(\frac{\sqrt{2r}}{k}x\right) - 2r(T - t)}{\sqrt{2r(T - t)}}$$

7. In the appendix there is a brief treatment of the option valuation formula for the more usual (restricted) scenario which assumes the capital project has an infinite life (McDonald and Siegel, 1986).

8. Cox and Rubinstein (1985, p. 226) show that for the Black and Scholes (1973) option pricing formula, ρ is uniformly positive. The negative numbers for ρ which arise for the option pricing formula (6.11) considered here are due to the fact that the variance of instantaneous changes in the net present value variable, x, depend on the return on the risk-free asset, r. In the Black and Scholes (1973) formula, however, the instantaneous variance of changes in the share's price does not depend on the return on the risk-free asset.
9. More parsimonious, but perhaps less familiar, expressions for these variables are:

$$d_1 = \frac{\sinh^{-1}\left(\frac{\sqrt{2r}}{k}x\right) - \sinh^{-1}\left(\frac{\sqrt{2r}}{k}E\right) + 2r(T-t)}{\sqrt{2r(T-t)}},$$

$$d_2 = \frac{\sinh^{-1}\left(\frac{\sqrt{2r}}{k}x\right) - \sinh^{-1}\left(\frac{\sqrt{2r}}{k}E\right) - 2r(T-t)}{\sqrt{2r(T-t)}},$$

$$d_3 = \frac{\sinh^{-1}\left(\frac{\sqrt{2r}}{k}x\right) - \sinh^{-1}\left(\frac{\sqrt{2r}}{k}E\right)}{\sqrt{2r(T-t)}}$$

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Chapter 7

Real R&D options with endogenous and exogenous learning

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SUMMARY

We demonstrate the valuation of real (investment) options in the presence of endogenous and exogenous learning. *Endogenous* learning is captured through optimally activated controls arising because of costly managerial actions (R&D, marketing research, advertisement, etc.) intended to enhance value and reveal information. The realization of the controls are jumps with a random size. The decision-maker solves an optimization problem by considering the trade-off between the benefits of the R&D actions and their cost. *Exogenous* learning is captured through random information arrival of rare events (jumps resulting from technological, competitive, regulatory or political risk shocks, etc.) that follow a Poisson process and have a size drawn from a mixed distribution. In addition, *experiential* learning is captured by a dynamic volatility similar to that observed in the financial options markets.

7.1 INTRODUCTION

Very little has appeared in the academic literature to capture the ability of managers to intervene in order to add value and/or learn more in the context of the contemporary theory of investments under uncertainty. It has long been known that even pure learning but optional actions should have a positive effect on the value of investment opportunities (see Roberts and Weitzman, 1981). This is clearly observed in practice, as a recent article in *Business Week* demonstrates the high value of oil exploration rights (Exploiting Uncertainty, *Business Week*, 7 June 1999, pp. 118–124).

Investment theory under uncertainty is now synonymous with the term real options and the pricing tools are those of contingent claims (option) pricing theory, in its contemporary form initiated by Black and Scholes (1973) and Merton (1973a). The same methodology has also been used for evaluation of many real-life investment decisions under uncertainty, following the seminal paper of McDonald and Siegel (1986) (see also Dixit and Pindyck, 1994 and Trigeorgis, 1996). Much of the theory of real (investment) options captures the value of waiting to invest, and the flexibility to switch among modes of operation without explicit consideration of managerial actions to enhance value and acquire more information. Notable exceptions are Epstein et al. (1999) and Childs et al. (2001), who use a filtering approach, and Martzoukos (2000a) who considers (European and American) real options with embedded costly and optional control/learning (i.e. R&D) actions. He captures these actions using costly and optimally activated proportional jumps defined with a continuous distribution. In the real options literature it is often assumed that assets underlying the options are *observed* variables, and that they follow a continuous-time geometric Brownian motion. Often it is more realistic to assume that they represent subjective management estimates (incomplete information). Since these are simply estimates, the management would *act* in order to improve information, and/or add value (R&D and market research or advertisement). Acts like these would come at a cost, but they would affect the estimates of the uncertain variables, and outcome is uncertain. This method of active learning is in contrast with Hendricks (1992), who proposes a deterministically decreasing instantaneous variance as a tool to capture passive learning over time (see also Kolstad, 1992), and also with Majd and Pindyck (1989), who use an auxiliary state variable that captures continuous learning as a linear function of the time that an operation – which can be switched on and off – has been active, thus making stochastic the time that any particular level of learning is reached. Note also the use of a scenario-type approach called technical uncertainty, often related to the technological uncertainty of construction, project completion, etc. (see Pindyck, 1993 and Cortazar et al., 2001).

The empirical return distributions of most financial assets can exhibit significant skewness and fat tails. This may be due in part to a higher likelihood of rare events (jumps), which naturally affects the value of financial and real options. In the absence of controls, Trigeorgis (1991) values real options in the presence of multiple (but single-class) rare events induced by competitive entries, and Martzoukos and Trigeorgis (2002) study real options in the presence of multiple classes (sources) of rare events. A rare event class is characterized by the frequency and distribution of jump arrivals, and by the distributional characteristics of the jump size. The relevant parameter values may differ across different types of events (jumps). Examples of different sources of rare events might be technological innovations (positive jumps), competitive entry or arrival of substitute products (negative jumps), a court ruling or a new regulatory or environmental decision (positive or negative jumps), and expropriation or political risk (negative jump). For example, Grenadier and Weiss (1997) present a real options application with technological innovations, while Pennings and Lint (1997) and Mauer and Ott (1995) present applications with a single source of rare events representing R&D and cost-reducing innovations, respectively. Brennan and Schwartz (1982a,b) discuss the impact of regulation uncertainty, and Teisberg (1993, 1994) presents a real options application. Wagner (1997) discusses the importance of political risk, while Clark (1997) presents a real options application with a single source of (catastrophic) rare events.

In this chapter we are interested in the European real option under incomplete information in the presence of endogenously generated actions of control (R&D) and exogenous information arrival. The uncertainty about the evolution of the value of the asset underlying an investment opportunity (or the value of a relevant state variable) is described by either a geometric Brownian motion or a jump-diffusion process. We capture management intervention through optimal activation of proportional jumps of random size (random controls). In the case of control, the arrival of the jump is not random, but is optimally activated by management that wishes to *maximize* the value of the real option. For the equivalence between contingent claims valuation and dynamic optimization, see Dixit and Pindyck (1994). We consider control actions that directly aim at enhancing real option value (impact controls) by enhancing the value of the underlying asset or decreasing the cost, or actions that will enhance value by improving information (pure learning controls). Examples of the first type are research in high-tech or chemical firms to enhance product value/attributes, but also pre-launch advertisement, etc. Examples of the pure learning actions could be oil exploration activities, soil structural experiments before large infrastructure projects are committed, and marketing research.

We proceed as follows. In Section 7.2 we elaborate on the model with endogenous costly control/learning. Then in Section 7.3 the model with exogenous learning is demonstrated. In Section 7.4 the two are synthesized and the model with both endogenous and exogenous (plus experiential) learning is presented. Numerical results are presented and discussed in Section 7.5, and Section 7.6 concludes.

7.2 THE MODEL WITH ENDOGENOUS COSTLY LEARNING

We consider a real (investment) option on a real asset S with a deterministic exercise price (capital cost) X. The decision-maker has monopoly power over this investment option. For option pricing we resort to the risk-neutral approach (which in its present form is due to Cox and Ross, 1976 and Harrison and Pliska, 1981). Since in the case of real options the underlying asset is often not traded, for contingent-claims valuation we do not need to invoke the replication and continuous trading arguments in Black and Scholes (1973). We assume instead that an intertemporal capital asset pricing model holds (see Merton, 1973b) and we draw on Constantinides (1978) (see also Cox et al., 1985). We assume that markets are complete and all uncertainties are spanned by traded assets. In the real probability measure, S follows the process:

$$\frac{\mathrm{d}S}{S} = g\,\mathrm{d}t + \sigma\,\mathrm{d}Z^R + \sum_{i=1}^{I} k_i\,\mathrm{d}q_i \tag{7.1a}$$

and in the risk-neutral measure, S follows:

$$\frac{\mathrm{d}S}{S} = (r - \delta)\mathrm{d}t + \sigma\,\mathrm{d}Z + \sum_{i=1}^{I} k_i\,\mathrm{d}q_i \tag{7.1b}$$

Thus, we have the deterministic drift component g and two random components. The first random component is the increment $\mathrm{d}Z$ of the standard Wiener process, with σ a constant or deterministic function of time. The second component is a controlled jump of (random) size k, and the counter increment $\mathrm{d}q$ is zero before the jump is activated, and one afterwards. This differs from the jump-diffusion assumption of random (Poisson distributed) jump arrival, since here $\mathrm{d}q$ is not a random but a control variable. The riskless rate of interest is denoted by r. The dividend yield δ of the underlying asset or state variable can be interpreted as a shortfall between the required return and the actual growth rate (McDonald and Siegel, 1984; see also Brennan, 1991 for a convenience yield interpretation).

We also consider that the realization of the control will be random, so we must make specific assumptions about the distribution of $1 + k$ and assume that the mean $E[k]$ and its variance are known (with $E[\cdot]$ denoting the expectations operator). We can assume many plausible distributional assumptions for the outcome of the control, but it will be more convenient to assume that $1 + k$ follows a lognormal distribution. These specific distributional assumptions make available analytic solutions isomorphic to the familiar Black and Scholes model. The realization of the control's outcome is independent of the increment of the standard Wiener process, and is incurred at a cost c. We also make the assumption that jump k carries a diversifiable risk. In the case of a jump that is induced for *pure learning* (in contrast to *impact* control), the expected size k of the jump is zero, $E[k] = 0$. If the outcome is observed instantaneously, the expected value of $S(t^+)$ at time t, conditional on the occurrence of the jump, equals its value $S(t^-)$ before the jump: $E[S(t^+)|\mathrm{d}q = 1, S(t^-)] = S(t^-)$.

We consider the real investment (call) option C to acquire S by paying a capital cost X. Similarly we can solve for a put option to sell S in order to receive X. In general, the objective is to find the optimal control/learning policy that maximizes the value of the real claim. Each control action 'i' has a prespecified timing $t(i)$, so the decision-maker for each available control 'i', has the option to activate it at time $t(i)$ by paying cost c_i. Many such controls can exist at different times $t(i)$, and several controls could coincide in time and be mutually exclusive. In this chapter and in order to focus on an analytic solution, we consider only the European option, and controls that are available at time $t = 0$.

In general the problem is stated as follows:

$$\underset{i,\,t(i)}{\text{maximize}}[C(t, S, c_i)] \tag{7.2}$$

subject to:

$$\frac{\mathrm{d}S}{S} = (r - \delta)\mathrm{d}t + \sigma\,\mathrm{d}Z + \sum_{i=1}^{I}(k_i\,\mathrm{d}q_i)$$

$$\ln(1 + k_i)\text{ normally distributed with mean: }\gamma_i - 0.5\sigma_i^2\text{ and variance: }\sigma_i^2$$

$$E[k_i] = \mathrm{e}^{\gamma_i} - 1 \tag{7.3}$$

and the terminal condition at maturity T that defines the claim as a call (or alternatively as a put) option. Below we give the option value *conditional* on activation of i controls.

Proposition 7.1 *Given the above assumptions, the European call option C conditional on activation of i random controls at $t = 0$ equals*:

$$C_{\text{cond}}(S, X, T, \sigma, \delta, r, \gamma_i, \sigma_i) = \mathrm{e}^{-rT}E[(S_T - X)^+|i\text{ controls}] \tag{7.4}$$

The discounted risk-neutral expectation, derived along the lines of the Black–Scholes model but conditional on control activation equals:

$$\begin{aligned}&\mathrm{e}^{-rT}E[(S_T - X)^+|i\text{ controls}]\\&\quad = S\exp\left[-\delta T + \sum_{i=1}^{I}(\gamma_i)\right]N(d_1) - X\,\mathrm{e}^{-rT}N(d_2)\end{aligned}$$

where:

$$d_1 \equiv \frac{\ln(S/X) + (r - \delta)T + \sum_{i=1}^{I}(\gamma_i) + 0.5\sigma^2 T + \sum_{i=1}^{I}(0.5\sigma_i^2)}{\left[\sigma^2 T + \sum_{i=1}^{I}(\sigma_i^2)\right]^{1/2}}$$

and

$$d_2 \equiv d_1 - \left[\sigma^2 T + \sum_{i=1}^{I}(\sigma_i^2)\right]^{1/2}$$

with $N(d)$ again denoting the cumulative standard normal density evaluated at d. Note that if controls were costless, the conditional value would be the same even if activation time differs from $t = 0$, due to the proportionality of the impact.

The sensitivities (the *Greeks*) of the European call option in respect to asset S, the control's mean γ_i and the control's volatility σ_i equal:

$$\frac{\partial C}{\partial S} = N(d_1) \exp\left[-\delta T + \sum_{i=1}^{I}(\gamma_i)\right] > 0 \tag{7.5}$$

$$\frac{\partial C}{\partial \gamma_i} = N(d_1) S \exp\left[-\delta T + \sum_{i=1}^{I}(\gamma_i)\right] > 0 \tag{7.6}$$

and

$$\frac{\partial C}{\partial \sigma_i} = \frac{X \exp(-rT - d_2^2/2)}{\sqrt{2\pi\left[\sigma^2 T + \sum_{i=1}^{I}(\sigma_i^2)\right]}}\sigma_i > 0 \tag{7.7}$$

Now we wish to find the control that maximizes option value. First assume the existence of a single control at $t = 0$.

Proposition 7.2 *For every single control i affecting a European call option, the optimal value equals*:

$$\max[C_{\text{cond}}(S, X, T, \sigma, \delta, r, \gamma_i, \sigma_i) - c_i, C(S, X, T, \sigma, \delta, r)] \tag{7.8}$$

If more controls can be activated, the optimal combination will be chosen. For example, if controls are mutually exclusive, only the best one will be considered among the alternatives $[C_{\text{cond}}(S, X, T, \sigma, \delta, r, \gamma_1, \sigma_1) - c_1, \ldots, C_{\text{cond}}(S, X, T, \sigma, \delta, r, \gamma_I, \sigma_I) - c_I]$. Similarly, we proceed if combinations of controls are permissible. The problem above has analytic solutions for European call and put options when controls can be exercised at $t = 0$. When this is not the case, numerical solutions like in Martzoukos (2000a) are needed.

It is known in general that the probability that a call option will be exercised (that at T the value of asset S will be above X) is $N(d_2)$ under risk-neutrality. In the real probability measure, this probability equals $N(d_2)$ if in d_2 we replace $(r-\delta)$ with the real expected growth rate g. In the case of one control action i we can calculate the risk-neutral probability $P(S_T > X) = N(d_2)$, with:

$$d_1 \equiv \frac{\ln(S/X) + (r-\delta)T + \gamma_i + 0.5\sigma^2 T + 0.5\sigma_i^2}{(\sigma^2 T + \sigma_i^2)^{1/2}}$$

and

$$d_2 \equiv d_1 - (\sigma^2 T + \sigma_i^2)^{1/2}$$

whereas for the real probability, $(r-\delta)$ is again replaced by g.

Thus, $E[S_T \mid \text{given activation of control } i] = S_0\, e^{(r-\delta)T+\gamma_i}$ and $S_0\, e^{gT+\gamma_i}$ in the risk-neutral and the real measure, respectively. If we want to isolate the impact of a control, we can see the expected value $E[S \mid \text{given activation of control } i] = S_0\, e^{\gamma_i}$, and the probability that S will be above any value X:

$$P(S > X \mid \text{ given activation of control } i) = N(d_2) \tag{7.9}$$

with:

$$d_1 \equiv \frac{\ln(S/X) + \gamma_i + 0.5\sigma_i^2}{(\sigma_i^2)^{1/2}}$$

and

$$d_2 \equiv d_1 - (\sigma_i^2)^{1/2}$$

The control impact is rather unlikely to be present in any observed time series. Equation (7.9) can be very useful in creating confidence intervals in trying to estimate these parameters subjectively. In the presence of differing opinions (i.e. when there are several experts), Lindley and Singpurwalla (1986) recommend a Bayesian method to combine subjective experts' opinions (see also Cooke, 1991).

The value of a European put option P is similarly shown to be $P_{\text{cond}}(S, X, T, \sigma, \delta, r, \gamma_i, \sigma_i) = e^{-rT} E[\max(X - S_T, 0) \mid i \text{ controls}]$, etc., and the optimal unconditional value equals $\max\,[P_{\text{cond}}(S, X, T, \sigma, \delta, r, \gamma_i, \sigma_i) - c,\ P(S, X, T, \sigma, \delta, r)]$. Like in Propositions 7.1 and 7.2, we can derive the *Greeks* and the probability of exercise of the European put option.

7.3 THE MODEL WITH EXOGENOUS LEARNING

Stochastic processes with discontinuous (Poisson distributed) events have been studied by Kushner (1967, 1990) and Davis (1976) and, in the context of option pricing, by Merton (1976). Later applications in option pricing include Ball and Torous (1985), Amin (1993), and Bates (1991). The finance literature has mostly focused on the case of a single source of discontinuity (information arrival). Notable exceptions are Jones (1984), who studied hedging of financial (European) options under two classes of jumps, and Abraham and Taylor (1997), who considered a jump with anticipated and a jump with random arrival to price European financial options.

Here we assume the existence of multiple ($j = 1, \ldots, J$) sources of jumps, such that the underlying asset S follows a continuous-time stochastic process of the form:

$$\frac{\mathrm{d}S}{S} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}Z^R + \sum_{j=1}^{J}(k_j\,\mathrm{d}q_j) \tag{7.10a}$$

Summation is over the j classes of rare events, μ is the drift and σ the instantaneous standard deviation (excluding the impact of the jumps), $\mathrm{d}Z$ is an increment to a standard Wiener process, λ_j is the frequency of the Poisson arrival of a jump of type j, $\mathrm{d}q_j$ is a jump counter that takes a value ($\mathrm{d}q_j = 1$) with probability $\lambda_j\,\mathrm{d}t$ or a value ($\mathrm{d}q_j = 0$) with probability $(1 - \lambda_j)\mathrm{d}t$, and k_j is the jump size of each event class. Due to the impact of the jumps, the actual trend of this process equals $\mu + \sum_{j=1}^{J}(\lambda_j\bar{k}_j)$ with a term $\lambda\bar{k} \equiv \lambda E[k]$ for each event class, and with $E[\cdot]$ denoting the expectations operator. The overall frequency of rare events equals $\sum_{j=1}^{J}\lambda_j$, with jump size drawn from a mixed distribution j with probability equal to $\lambda_j/\sum_{j=1}^{J}\lambda_j$.

Under risk-neutral valuation the state variable S follows the process:

$$\frac{\mathrm{d}S}{S} = \left[r - \delta - \sum_{j=1}^{J}(\lambda_j\bar{k}_j)\right]\mathrm{d}t + \sigma\,\mathrm{d}Z + \sum_{j=1}^{J}(k_j\,\mathrm{d}q_j) \tag{7.10b}$$

Following Merton (1976), we assume the jump risk to be diversifiable, and that an intertemporal capital asset pricing model holds, as for example in Merton (1973b). Again we do not need to invoke the replication and continuous trading arguments of Black and Scholes (1973). The parameter δ represents any form of a 'dividend yield'. The presence of jumps affects the underlying stochastic process in two ways. First, volatility is higher due to the randomness of the jump size and of the jump arrival. Second, jumps can affect the actual drift by adding a component equal to $\lambda\bar{k} \equiv \lambda E[k]$ for each jump class. The latter effect can be avoided if the deterministic drift component includes a '*compensation*' term $-\lambda\bar{k}$ for each event class. This term is present in the jump-diffusion model of Merton (1976), where the underlying asset is traded to ensure that the total expected return of the asset equals the required (risk-neutral) return r. Jump classes that are not due to discontinuities in the price process of a traded asset would not require such a compensation term, in which case the jump assumption affects not only volatility but also the actual drift. This compensation term for example is missing from the jump-diffusion model in Dixit and Pindyck (1994, p. 171), where the authors assume that the jumps affect both the drift and the volatility. Finally, the stochastic differential equation can alternatively be expressed in stochastic integral form as:

$$\ln[S(T)] - \ln[S(0)] = \int_0^T \left[r - \delta - \sum_{j=1}^{J} (\lambda_j \bar{k}_j) - 0.5\sigma^2 \right] \mathrm{d}t + \int_0^T \sigma \, \mathrm{d}Z(t) + \sum_{j=1}^{J} \sum_{q=1}^{n_j} [\ln(1 + k_{j,q})] \tag{7.11}$$

where the nested summation is over the realizations for all j classes of jumps of size $k_{j,q}$, with $n = (n_1, \ldots, n_J)$ a J-element vector with each element being the number of realized j-type jump occurrences. For each *event class*, we assume that the distribution of the jump size, $1 + k_j$, is lognormal: $\ln(1 + k_j) \sim N(\gamma_j - 0.5\sigma_j^2, \sigma_j^2)$, with $N(.,.)$ denoting the normal density function with mean $\gamma_j - 0.5\sigma_j^2$ and variance σ_j^2, and $E[k_j] \equiv \bar{k}_j = \mathrm{e}^{\gamma_j} - 1$. This multi-class assumption is an extension of Merton (1976) and like in his case this equation in general is hard to solve. Following the approach in Merton (see also Jones, 1984) we can value a European call option on asset S with time to maturity T and exercise price X, assuming independence between the different event classes and the Wiener process $\mathrm{d}Z$.

Proposition 7.3 *The solution to the European call option with multiple sources of jumps is given by the iterated integral* (*see Martzoukos and Trigeorgis, 2002*):

$$C(S, X, T, \sigma, \delta, r, \lambda_j, \gamma_j, \sigma_j) = \mathrm{e}^{-rT} \sum_{n_1=0}^{\infty} \ldots \sum_{n_J=0}^{\infty} \{P(n_1, \ldots, n_J) E[(S_T - X)^+ | (n_1, \ldots, n_J) \text{ jumps}]\} \tag{7.12}$$

where $P(n_1, \ldots, n_J)$ denotes the joint probabilities of any random realization of $n = (n_1, \ldots, n_J)$ jumps. Because of the independence assumption, these joint probabilities simplify to the J-term product:

$$P(n_1, \ldots, n_J) = \prod_{j=1}^{J} [\mathrm{e}^{-\lambda_j T} (\lambda_j T)^{n_j} / n_j!]$$

The discounted risk-neutral expectation, conditional on $n = (n_1, \ldots, n_J)$ equals:

$$\begin{aligned}
&\mathrm{e}^{-rT} E[(S_T - X)^+ | (n_1, \ldots, n_J) \text{ jumps}] \\
&\quad = S \exp\left\{\left[-\delta - \sum_{j=1}^{J}(\lambda_j \bar{k}_j)\right] T + \sum_{j=1}^{J}(n_j \gamma_j)\right\} N(d_{1n}) - X\mathrm{e}^{-rT} N(d_{2n})
\end{aligned}$$

where:

$$d_{1n} \equiv \frac{\begin{array}{c}\ln(S/X) + \left[r - \delta - \sum_{j=1}^{J}(\lambda_j \bar{k}_j)\right] T + \sum_{j=1}^{J}(n_j \gamma_j) \\ + 0.5\sigma^2 T + 0.5 \sum_{j=1}^{J}(n_j \sigma_j^2)\end{array}}{\left[\sigma^2 T + \sum_{j=1}^{J}(n_j \sigma_j^2)\right]^{1/2}}$$

and

$$d_{2n} \equiv d_{1n} - \left[\sigma^2 T + \sum_{j=1}^{J}(n_j \sigma_j^2)\right]^{1/2}$$

$N(d)$ again denotes the cumulative standard normal density evaluated at d. The value of a European put option with multiple types of jumps is similarly shown.

7.4 THE MODEL WITH ENDOGENOUS AND EXOGENOUS LEARNING

Now we combine the previous results and consider both controlled actions and rare events. There are $(i = 1, \ldots, I)$ controls and $(j = 1, \ldots, J)$ classes of rare events. In addition we can add experiential learning by assuming that the instantaneous variance $\sigma^2 = \sigma^2(t)$ is a (potentially decreasing) function of time.

Proposition 7.4 *The solution to the European call option with multiple sources of jumps conditional on activation of controlled actions is given by the iterated integral*:

$$\begin{aligned}
&C_{\text{cond}}(S, X, T, \sigma, \delta, r, \gamma_i, \sigma_i, \lambda_j, \gamma_j, \sigma_j) = \mathrm{e}^{-rT} \sum_{n_1=0}^{\infty} \cdots \sum_{n_J=0}^{\infty} \\
&\quad \times \{P(n_1, \ldots, n_J) E[(S_T - X)^+ \mid (n_1, \ldots, n_J) \text{ jumps}, i \text{ controls}]\} \qquad (7.13)
\end{aligned}$$

where $P(n_1, \ldots, n_J)$ denotes the joint probabilities of any random realization of $n = (n_1, \ldots, n_J)$ jumps. Because of the independence assumption, these joint probabilities simplify to the J-term product:

$$P(n_1, \ldots, n_J) = \prod_{j=1}^{J} [\mathrm{e}^{-\lambda_j T} (\lambda_j T)^{n_j} / n_j!]$$

The discounted risk-neutral expectation, finally conditional on both random information arrival $n = (n_1, \ldots, n_J)$ and control activation, is:

$$\begin{aligned} &\mathrm{e}^{-rT} E[(S_T - X)^+ | (n_1, \ldots, n_J) \text{ jumps}, i \text{ controls}] \\ &\quad = S \exp\left\{ \left[-\delta - \sum_{j=1}^{J} (\lambda_j \bar{k}_j) \right] T + \sum_{i=1}^{I} (n_i \gamma_i) + \sum_{j=1}^{J} (n_j \gamma_j) \right\} N(d_{1n}) \\ &\qquad - X \mathrm{e}^{-rT} N(d_{2n}) \end{aligned}$$

where:

$$\begin{aligned} d_{1n} \equiv {} & \frac{\ln(S/X) + \left[r - \delta - \sum_{j=1}^{J} (\lambda_j \bar{k}_j) \right] T + \sum_{i=1}^{I} (n_i \gamma_i) + \sum_{j=1}^{J} (n_j \gamma_j)}{\left[\int_{t=0}^{T} \sigma^2(t) \mathrm{d}t + \sum_{i=1}^{I} (n_i \sigma_i^2) + \sum_{j=1}^{J} (n_j \sigma_j^2) \right]^{1/2}} \\ & + 0.5 \frac{\int_{t=0}^{T} \sigma^2(t) \mathrm{d}t + \sum_{i=1}^{I} (n_i \sigma_i^2) + \sum_{j=1}^{J} (n_j \sigma_j^2)}{\left[\int_{t=0}^{T} \sigma^2(t) \mathrm{d}t + \sum_{i=1}^{I} (n_i \sigma_i^2) + \sum_{j=1}^{J} (n_j \sigma_j^2) \right]^{1/2}} \end{aligned}$$

and

$$d_{2n} \equiv d_{1n} - \left[\int_{t=0}^{T} \sigma^2(t) \mathrm{d}t + \sum_{i=1}^{I} (n_i \sigma_i^2) + \sum_{j=1}^{J} (n_j \sigma_j^2) \right]^{1/2}$$

The sensitivities to $S, \gamma_i, \sigma_i, \gamma_j, \sigma_j$, etc. are derived exactly like for the model in the first proposition, but probability is weighted for all possible realizations of random jumps and all jump classes $\sum_{n_1=0}^{\infty} \cdots \sum_{n_J=0}^{\infty} \{P(n_1, \ldots, n_J)[Greek| (n_1, \ldots, n_J) \text{ jumps}, i \text{ controls}]\}$.

Proposition 7.5 *For every single control i affecting a European call option, the optimal value in the presence of rare events equals*:

$$\max[C_{\text{cond}}(S, X, T, \sigma, \delta, r, \gamma_i, \sigma_i, \lambda_j, \gamma_j, \sigma_j) - c_i, C(S, X, T, \sigma, \delta, r, \lambda_j, \gamma_j, \sigma_j)] \tag{7.14}$$

Again, in the presence of mutually exclusive controls or combinations of controls, the optimal one is chosen from the set:

$$[C_{\text{cond}}(S, X, T, \sigma, \delta, r, \gamma_1, \sigma_1, \lambda_j, \gamma_j, \sigma_j), \ldots, C_{\text{cond}}(S, X, T, \sigma, \delta, r, \gamma_I, \sigma_I, \lambda_j, \gamma_j, \sigma_j)]$$

Similar results can also be derived for the put option.

7.5 NUMERICAL RESULTS AND DISCUSSION

Let us see some numerical results for a real (call) option. The endogenous costly learning is presented in Table 7.1. The case with $\gamma_i = 0$ is the pure learning case. Control is not expected to change the value of the underlying asset, only to reveal information. Think for example of the case where S is the product of two variables, a stochastic price and a (potentially) constant but unknown quantity. If the learning action is not activated, the real option will be valued and exercised, calculating S from the (subjective) estimate of this quantity. If the learning action is activated, the true quantity (or a better estimate) will be revealed (see discussions on issues of path dependency in Martzoukos and Trigeorgis, 2001). The cases with $\gamma_i \neq 0$ are for a normal control where the effort is to change the value of the underlying asset, albeit with uncertain outcome ($\gamma_i > 0$ seems more

Table 7.1 *Real option values with activation of a control*

		$S = 75.00$	$S = 100.00$	$S = 125.00$
$\gamma i = ---$	$\sigma_i = ---$	**0.005**	**3.608**	**22.666**
$\gamma_i = 0.10$	$\sigma_i = 0.10$	0.500	11.413	34.574
	$\sigma_i = 0.30$	4.411	17.311	37.182
$\gamma_i = 0.00$	$\sigma_i = 0.10$	0.086	5.101	22.971
	$\sigma_i = 0.30$	2.426	11.368	27.116
$\gamma_i = -0.10$	$\sigma_i = 0.10$	0.010	1.716	13.297
	$\sigma_i = 0.30$	1.243	7.053	18.939

Call option parameters are: exercise price $X = 100.00$, time to maturity $T = 1$, standard deviation $\sigma = 0.10$, dividend yield $\delta = 0.10$, riskless rate $r = 0.10$. Control has been activated at $t = 0$, and option value is before considering the control's cost. Figures in bold demonstrate option value in the absence of control.

relevant for a call option and $\gamma_i < 0$ for a put option). We observe the results conditional on control activation with the use of equation (7.4). The control's cost should be subtracted from the figures in order to compare with the base case without control activation (i.e. 3.608 when $S = 100.00$) and derive the optimal decision through equation (7.8). See the top figure in the middle column. The control should be activated only if its cost is less than $11.413 - 3.608$. It is obvious that the relative impact of control learning is much higher for out-of-the-money options. Martzoukos and Trigeorgis (2001) demonstrate that the absolute impact of controls is the highest for options that are about at-the-money, and they also discuss the optimal control exercise boundary.

In Table 7.2 we see numerical results for an option in the presence of rare events of information arrival. The four rows after the base case are for an increasing degree of expected directional impact in the case of a rare event arrival. Notice the significant difference with the case $\gamma_j = 0.00$ (which could be an erroneous but tempting simplification in order to use a single class of jumps with twice the frequency). The second column provides results when the compensation terms are not present in the drift of the stochastic equation (7.10b). In that case the jump means affect the stochastic process not only through the volatility (of the arrival time and the jump size), but also through the drift. When the compensation terms are present like in Merton (1976), jumps affect the option value only through the volatility since the expected directional impact is removed (compensated for). This is obvious by comparing the figures of the last two rows.

The random controls (endogenous learning) and the rare events are combined in Tables 7.3 (for pure learning actions) and 7.4 (for impact control). The insights gained earlier do not change. The numerical results are conditional on

Table 7.2 *Real option values with exogenous learning*

Rare event parameters	*compns* = YES	*compns* = NO
$\gamma_j = ---\ \ (\sigma_j = ---)$	**3.608**	**3.608**
$\gamma_1 = \gamma_2 = 0.00$	4.951	4.951
$\gamma_1 = -\gamma_2 = 0.10$	5.949	6.192
$\gamma_1 = -\gamma_2 = 0.30$	11.232	13.626
$\gamma_1 = \gamma_2 = 0.30$	12.689	39.261
$\gamma_1 = \gamma_2 = -0.30$	11.126	1.352

Call option parameters are: exercise price $X = 100.00$, underlying asset $S = 100.00$, time to maturity $T = 1$, standard deviation $\sigma = 0.10$, dividend yield $\delta = 0.10$, riskless rate $r = 0.10$. Jump parameters are: two jump classes with frequency 0.50 per year each, and $\sigma_1 = \sigma_2 = 0.10$. If *compns* = YES for both jumps the compensation term is included in the drift of the process, otherwise it is not. Figures in bold demonstrate option value in the absence of rare events.

Table 7.3 *Real option values with endogenous (pure learning) and exogenous learning*

Rare event parameters	Control $\sigma_i = 0.10$, $\gamma_i = 0.00$		Control $\sigma_i = 0.30$, $\gamma_i = 0.00$	
	compns = YES	*compns* = NO	*compns* = YES	*compns* = NO
$\gamma_j = ---\ (\sigma_j = ---)$	**3.608**	**3.608**	**3.608**	**3.608**
$\gamma_1 = \gamma_2 = 0.00$	6.164	6.164	11.906	11.906
$\gamma_1 = -\gamma_2 = 0.10$	7.028	7.274	12.409	12.669
$\gamma_1 = -\gamma_2 = 0.30$	11.917	14.354	15.832	18.366
$\gamma_1 = \gamma_2 = 0.30$	13.172	39.892	16.522	43.486
$\gamma_1 = \gamma_2 = -0.30$	11.618	1.961	15.329	5.194

Call option parameters are: exercise price $X = 100.00$, underlying asset $S = 100.00$, time to maturity $T = 1$, standard deviation $\sigma = 0.10$, dividend yield $\delta = 0.10$, riskless rate $r = 0.10$. Jump parameters are: two jump classes with frequency 0.50 per year each, and $\sigma_1 = \sigma_2 = 0.10$. If *compns* =YES for both jumps the compensation term is included in the drift of the process, otherwise it is not. Control parameters: one (pure learning) control activated at $t = 0$. Option value is before considering the control's cost. Figures in bold demonstrate option value in the absence of either rare events or controls.

Table 7.4 *Real option values with endogenous (impact control) and exogenous learning*

Rare event parameters	Control $\sigma_i = 0.10$, $\gamma_i = 0.10$		Control $\sigma_i = 0.30$, $\gamma_i = 0.10$	
	compns = YES	*compns* = NO	*compns* = YES	*compns* = NO
$\gamma_j = ---\ (\sigma_j = ---)$	**3.608**	**3.608**	**3.608**	**3.608**
$\gamma_1 = \gamma_2 = 0.00$	12.345	12.345	17.852	17.852
$\gamma_1 = -\gamma_2 = 0.10$	13.134	13.506	18.357	18.699
$\gamma_1 = -\gamma_2 = 0.30$	17.810	21.120	21.813	25.015
$\gamma_1 = \gamma_2 = 0.30$	18.603	52.126	22.291	55.078
$\gamma_1 = \gamma_2 = -0.30$	18.080	4.524	21.563	8.187

Call option parameters are: exercise price $X = 100.00$, underlying asset $S = 100.00$, time to maturity $T = 1$, standard deviation $\sigma = 0.10$, dividend yield $\delta = 0.10$, riskless rate $r = 0.10$. Jump parameters are: two jump classes with frequency 0.50 per year each, and $\sigma_1 = \sigma_2 = 0.10$. If *compns* =YES for both jumps the compensation term is included in the drift of the process, otherwise it is not. Control parameters: one control activated at $t = 0$. Option value is before considering the control's cost. Figures in bold demonstrate option value in the absence of either rare events or controls.

control activation and are given through the use of equation (7.13). Accounting for the control's cost through the use of equation (7.14) would provide the optimal activation decision. It would be easy to also add experiential learning by considering volatility of the Brownian motion component of the stochastic process to be a function of time. New industries or products would have at first a higher volatility. In that case, as we have already shown in equation (7.13), we replace $\sigma^2 \mathrm{d}t$ with $\int_{t=0}^{T} \sigma^2(t)\mathrm{d}t$. In the simple case where the variance $\sigma^2(t)$ is a linear function of time, the integral is replaced by $\bar{\sigma}^2\,\mathrm{d}t$, where $\bar{\sigma}^2$ is the arithmetic average of $\sigma^2(t = 0)$ and $\sigma^2(t = T)$. For example, if we have a decreasing σ from 0.30 to 0.10:

$$\bar{\sigma}^2 = [\sigma^2(t = 0) + \sigma^2(t = T)]/2 = (0.30^2 + 0.10^2)/2$$
$$= 0.05 \Rightarrow \bar{\sigma} = 0.22361$$

and the option value (for the parameter values of Table 7.2) in the absence of rare events or controls equals 8.055.

In this chapter, the exercise price of the investment option is assumed constant. The results are practically unaffected under the assumption of a stochastic exercise price that follows a geometric Brownian motion, like in McDonald and Siegel (1986). The volatility parameter σ^2 of the economic uncertainty will be replaced by a function of the two volatilities and the covariance between the two. Again, if these parameters are functions of time, this function will be replaced by its integral in respect to time like in the previous section. These results hold due to the proportionality of the jump even when S follows jump-diffusion, and of course for such assumptions they also hold in the presence of activated random controls. In the same McDonald and Siegel paper, in addition to the geometric Brownian motion assumption, a catastrophic risk (with a Poisson distribution of arrival time) is added. Before the real option is exercised, there is a chance that a rare (catastrophic) event driving the value of the underlying asset to zero will occur, and the call option will expire worthless. An example is clinical trials about the suitability (safety) of a new drug. The intensity of this process provides a simple adjustment to the risk-neutral process parameters, which can also easily be accommodated for in our framework.

The simplicity of the analytic solution makes the model a useful tool for both application and pedagogical purposes. In the analysis presented here we assumed the existence of a European option with one stochastic variable and controls available at time zero. Many realistic situations though would require more uncertainties and/or numerical solutions. For an extension to analytic and numerical (lattice-based) results with more state variables, see Martzoukos (2000a), and for an explicit treatment of controls with path dependency (including the path dependency induced due to the optimal timing of a single control), see Martzoukos and Trigeorgis (2001). Beyond lattice-based numerical methods, the random controls methodology can also be incorporated in a numerical framework for the direct solution of partial differential equations. Note finally that in the traditional real options literature we assume that the firm has monopoly power over the investment opportunity. There is a growing literature where this assumption is relaxed, and strategic game-theoretic interactions are considered. Notable examples directly relevant to R&D decisions are Smit and Ankum (1993), Joaquin and Butler (2000), Lambrecht (2000), Jou and Lee (2001), and in a random controls framework Martzoukos and Zacharias (2001).

7.6 CONCLUSIONS

In this chapter we elaborated on the analytic solution of the continuous random controls and then we presented a more general model that included both endogenously generated (costly) learning and exogenously generated information through the arrival of rare events. Many industries with heavy R&D expenditures, exploration, experimentation or clinical testing activities, and similar actions like marketing research and advertisement, face too many uncertainties in their capital-intensive investment process to ignore efforts for information acquisition, which of course come at a cost. The model presented allows managers to estimate how much to spend in order to enhance the value of their investment opportunities. This value enhancement they pursue either directly through impact control-type actions or indirectly through pure learning (information acquisition) actions. Modeling management's decision process with contingent claims (real options) tools is required; neglecting such actions would provide erroneous results for both the investment value and the optimal investment timing decision.

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Chapter 8

Approximate valuation of real R&D American sequential exchange options

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SUMMARY

We model the stages of R&D expense and then the ultimate discovery value (given an estimated development cost for the discovery) using real sequential (compound) exchange option models. This is applied to e-commerce R&D, so the timing is relatively short-term, with initial R&D, a second phase of R&D, and a final development phase, when the project values are realized. Proxies from the financial markets are used for expected project value and cost volatilities (and correlation). An approximate American sequential exchange option is compared to somewhat easier, although less realistic, exchange option values. Through comparing the real option worth to the initial R&D cost of separate projects, this approach serves as an R&D fund allocation criterion, in order to improve overall R&D portfolio value/costs.

8.1 INTRODUCTION

We address overall R&D budget and allocation problems through evaluating R&D worth using real American sequential exchange option values (RASE). The analysis is based on the R&D specifications and timing of required initial expenditures R&D0, a second phase of required R&D1 expenditures, and a final development phase, when the project values are realized. The essential aspect of this characterized R&D program is that managers have the timing choice for both the R&D1 expenditures, and then the timing of development costs (K) (and development versus sale) of the development package (P). The R&D program constitutes call options on further call options. If all costs are considered 'sunk costs', R&D0 expense at t_0 is an irrecoverable premium for a call option to pay R&D1 at t_1, which is itself a premium for an option to pay K at t_2, to receive then the project values. Without management flexibility not to pay R&D1 or K, perhaps such an R&D program should be valued using present values. With management discretion, real option models are appropriate since future expenditures can be canceled. The first stage decisions are based on the difference between perceived value (including future options) and cost at or before exercise dates (perhaps best pictured as R&D budget review dates). The transitions between the stages are sequential exchange options.

This model is suitable for any R&D program, where there are required interim expenditures for program continuance, such as: (a) a telecommunications company contemplating providing intermediate services and looking to maintain or increase line usage, or a mobile operator initially bidding for a 3G license, which requires R&D at a first stage, and then implementation expenditures; (b) an e-commerce software or an internet service provider, which aims to add advertising, and then content in sequences, each requiring R&D and marketing expenditures; and (c) a wholesale or retail merchant studying e-commerce as a supplement to (or eventual replacement of) mail catalog or physical store sales, where there are interim customer survey or marketing expenditures.

Hypothetical inputs have been provided for a particular e-commerce R&D project. The timing of the investment is assumed to be in one year for the R&D1 phase and two years for the development costs. The expenditure in one year t_1 is a proportion of K. The expenditure is stochastic as is the developed value, and there is an allowance for a correlation between the two stochastic processes.

Four real option valuation methods are considered, two 'exchange' options and two 'sequential exchange' options. The simplest European exchange model is from Margrabe (1978), which assumes that development costs and the e-commerce development are stochastic, and costs (R&D1 and K) are exchanged for the underlying asset at the development time. Then the Margrabe model has been adjusted to reflect possible early exercise, that is assuming the combined amount of R&D1 and K can be spent any time before t_2. Analytic approximations for American exchange option models are in Carr (1988), Bjerksund and Stensland (1993),[1] Lee and Paxson (2000b). We use the latter as the second model.[2]

The sequential exchange option model is a more realistic characterization of R&D options than standard American or European exchange option models, if R&D projects take the form of stages of research and/or sequential investment opportunities. We use as the third model the Carr (1988) model, which integrates both elements of the Geske (1979) compound option and the Margrabe (1978) exchange option to value European sequential exchange options. This model may be interpreted as a combination of a time-to-build option (growth option) and an option to exchange (operating option). Carr (1988) also provides an approximation for an American sequential exchange option. We use as the fourth model the Lee and Paxson (2000b) approximation of American sequential exchange option value, where the asset is expected to have a significant current income.

Then we apply the last model to an overall R&D budget and to each individual R&D project comprising the aggregate budget. Each project is expected to produce a different pattern of future cash flows. The conclusion is that given the assumptions, the aggregate initial budget for R&D is not quite justified (since the total real option value is \$0.12 million less than the t_0 R&D0 cost), but this varies widely among research projects. One of the smallest currently budgeted R&D projects has a 165% ratio of RASE/cost, and yet it has a high ratio of the critical development value (which justifies spending R&D1) to currently estimated project value. The largest current R&D project is worth far less than the cost. Just selecting two R&D projects would increase the ratio of real option worth to cost to nearly 1.5, so project selection would increase the total R&D value over the R&D0 cost by \$1.28 million.

Section 8.2 surveys some of the studies and strategies on sequential R&D options. Section 8.3 reviews the stochastic processes for R&D costs and benefits, and presents the closed-form solutions for sequential exchange options. Section 8.4 describes the empirical basis for the parameter inputs for these real R&D options. Section 8.5 shows the calculation of real R&D American option values. Section 8.6 discusses some of the unsolved problems herein and the likely uses of this e-commerce R&D options approach.

8.2 STUDIES ON R&D SEQUENTIAL OPTIONS

There is a rich literature on modeling R&D sequential options, although these have not always been recognized as real options. Roberts and Weitzman (1981) modeled the terminal R&D benefits as a geometric Brownian motion process, where the concern was an optimal stopping problem. Since costs were deterministic, R&D provided information learned in stages. Weitzman et al. (1981) used similar assumptions, except that costs were stochastic, and process volatility decreased over time.

Carr (1988), building on Margrabe (1978) and Geske (1979), provided the valuation of sequential exchange options. This has been applied to R&D real options indirectly in Taudes (1997), Childs et al. (1998), and Bar-Ilan and Strange (1998). The Carr (1988) European sequential exchange model assumes that the process is a compound option, with stochastic exercise prices (R&D1 at t_1 and K at t_2). The solution utilizes the bivariate normal distribution, with a correlation equal to the square root of (t_1/t_2). Schwartz and Moon

(2000a) provided a numerical solution (successive over-relaxation iterations) for multiple sequential exchange options. Lee and Paxson (2000b) suggested some methods for (increasingly accurate) approximate value of American exchange, and American sequential exchange options.

Note that simple real R&D sequential option models are not meant to be prescriptive, and are not a recommendation that the structure of sequential choices is necessarily the best way to organize R&D, or to achieve R&D effectiveness. Where it is possible to have frequent experimentation and quantification of R&D performance, as in some user testing of software, the additional information is likely to be helpful in deciding the usefulness of current R&D, exercise of options for further research along the same lines, and the value of the underlying innovation.

Other authors have suggested that sequential R&D organization is often inefficient and ineffective, especially for development (or close to production) R&D as in some software companies. Iansiti (1998) outlined an R&D model divided into two stages, concept development after which the concept or specification is frozen, and implementation. Alternative flexible development (technology integration) models have the implementation stage beginning before the concept stage is completed, so that there is feedback to the specifications before 'concept freeze'. He noted that Microsoft now has the policy of letting the specification change during the project development, driven by daily integration and experimentation efforts. Cusumano and Selby (1995) described in detail this 'synch-and-stabilize' process, where code written during the day is 'frozen', then immediately tested internally, and frequently externally. Implementation stages (and marketing stages) appear to be determined partly by the 'bug rate', although another policy is apparently to be ready to 'ship a new product' at any time, in response to competition.

Perhaps this type of R&D strategy could be modeled as multiple sequential options, except that the sequences would be daily (or longer practical review intervals). Such a strategy is close to manufacturing flexibility, where quality control and customer responsiveness are important. This would not be difficult to model, if the sequential options were more or less identical, as in capped floating-rate mortgages. However, where the nature of the options and the underlying asset (software products) change daily, along with the R&D investment required (in terms of person-years, or person-years weighted by salaries), modeling will be a challenge.

8.3 STOCHASTIC PROCESSES FOR R&D, DEVELOPMENT COSTS AND VALUE

In so far as developed e-commerce systems are securitized through the vehicles of traded shares, one might assume that developed values (P) follow a geometric Brownian motion:

$$\mathrm{d}P = (\mu_P - \delta_P)P\,\mathrm{d}t + \sigma_P P\,\mathrm{d}z_P \tag{8.1}$$

where μ_P is the equilibrium expected rate of return perfectly correlated with the developed e-commerce system, δ_P is the income rate (or dividend rate) of P, and σ_P is the volatility of the e-commerce development. Then suppose that the R&D1 and development costs (K) follow a similar diffusion process:

$$\mathrm{d}K = \mu_K K\,\mathrm{d}t + \sigma_K K\,\mathrm{d}z_K \tag{8.2}$$

where μ_K is the drift term (the expected cost escalation, which we assume is nil), σ_K is the volatility of the e-commerce development cost, and the correlation between the Wiener processes is ρ. If the developed project values and R&D1 and development costs (equal to exercise price) are lognormally distributed and occur when the e-commerce value is developed, e-commerce value options can be priced as European-style options to exchange R&D1 and development costs for the e-commerce values (especially if technical progress is restricted so that the development cannot occur until a specified date). The solution for a European exchange option (in this case of development costs for the developed system) was provided in Margrabe (1978):

$$\begin{aligned} &V(P, K, \mathrm{R\&D1}, \sigma_P, \sigma_K, \rho, r, \delta_P, t_2) \\ &\quad = P\,\mathrm{e}^{-\delta_P t_2} N(d_1) - (K + \mathrm{R\&D1}) N(d_2) \end{aligned} \tag{8.3}$$

where $P =$ the value of developed system, $K =$ the development cost, $\mathrm{R\&D1} =$ second phase R&D expenditure, $t_2 =$ time of the development, $\sigma = \sqrt{\sigma_P^2 - 2\rho\sigma_P\sigma_K + \sigma_K^2}$, $N(\cdot) =$ cumulative standard normal distribution function and:

$$d_1 = \frac{\ln(P/(K + \mathrm{R\&D1})) + (-\delta_P + 0.5\sigma^2)t_2}{\sigma\sqrt{t_2}}, \quad d_2 = d_1 - \sigma\sqrt{t_2}$$

The European option model assumes that R&D1 and K cannot occur until t_2. However, if the development has substantial income (which is not always the case for e-commerce companies), it may be preferable to start development earlier. Modifying Ho et al. (1994), Lee and Paxson (2000a) suggested a two-point confined exponential extrapolation method for an American option by extending the Geske and Johnson (1984) compound option approach. Applying the symmetry property, Lee and Paxson (2000b) gave the following analytic approximation for the value of an American exchange option:

$$V_{\mathrm{A}}(P, K, \mathrm{R\&D1}, \sigma_P, \sigma_K, \rho, r, \delta_P, t_2) = \left[V_\infty - \frac{(V_\infty - V_2)^2}{V_\infty - V_1} \right] \tag{8.4}$$

V_1 is the European exchange option price expiring at t_2, and V_2 is the price of a twice-exercisable exchange option, which can be exercised either at maturity (t_2) or at the midpoint of its time to maturity ($t_2/2$), given by:

$$V_2 = P\,\mathrm{e}^{-\delta_P t_2} B(-a_1, d_1; -\rho') - (K + \mathrm{R\&D1}) B(-a_2, d_2; -\rho') \\ + P\,\mathrm{e}^{-0.5\delta_P t_2} N(a_1) - (K + \mathrm{R\&D1}) N(a_2) \quad (8.5)$$

where:

$$a_1 = \frac{\ln(P/P^*) + (-\delta_P + 0.5\sigma^2)0.5t_2}{\sigma\sqrt{0.5t_2}}, \quad a_2 = a_1 - \sigma\sqrt{0.5t_2},$$
$$\rho' = \sqrt{0.5t_2/t_2}$$

and $B(a, d; \rho')$ is the bivariate cumulative standard normal distribution function with correlation coefficient ρ'.

The critical price, P^*, is obtained by solving the value matching condition:

$$P^* - (K + \mathrm{R\&D1}) = P^*\,\mathrm{e}^{-\delta_P(t_2-0.5t_2)} N(d_1') - (K + \mathrm{R\&D1}) N(d_2') \quad (8.6)$$

where:

$$d_1' = \frac{\ln(P^*/(K + \mathrm{R\&D1})) + (-\delta_P + 0.5\sigma^2)(t_2 - 0.5t_2)}{\sigma\sqrt{t_2 - 0.5t_2}},$$
$$d_2' = d_1' - \sigma\sqrt{t_2 - 0.5t_2}$$

The value of the perpetual American exchange option, V_∞, is given[3] by:

$$V_\infty = \frac{K + \mathrm{R\&D1}}{\theta - 1}\left(\frac{P}{P^*_\infty}\right)^\theta \quad (8.7)$$

Here, P^*_∞ denotes the optimal exercise price at which the perpetual American exchange option should be exercised:

$$P^*_\infty = \frac{\theta}{\theta - 1}(K + \mathrm{R\&D1}) \quad (8.8)$$

where:

$$\theta = \frac{-(-\delta_P - 0.5\sigma^2) + \sqrt{(-\delta_P - 0.5\sigma^2)^2}}{\sigma^2}$$

Although these American exchange options cover the flexibility inherent in investment opportunities, in contrast to the European counterpart, these standard exchange options do not model the sequential nature inherent in many R&D investment projects. Suppose we simplify the R&D1 process and assume that it is identical to the development cost stochastic process, except that it occurs (entirely and instantaneously) at time t_1 (that is $t_1 - t_0$ years after today), and is proportional to the development cost. These assumptions enable this entire process to be valued as a sequential European exchange option, see Carr (1988) and Lee and Paxson (2000b). The value of such a European sequential exchange option is:

$$W(P, K, \text{R\&D1}, \sigma_P, \sigma_K, \rho, r, \delta_P, t_1, t_2) \\ = P\,\mathrm{e}^{-\delta_P t_2} B(a_1', d_1''; \rho'') - KB(a_2', d_2''; \rho'') - QKN(a_2') \tag{8.9}$$

where $Q =$ fraction of K required for R&D1, $R = P/K$, $t_1 =$ time of the R&D1 expenditure:

$$a_1' = \frac{\ln(R/R^*) + (-\delta_P + 0.5\sigma^2)t_1}{\sigma\sqrt{t_1}}, \quad a_2' = a_1' - \sigma\sqrt{t_1}$$

$$d_1'' = \frac{\ln(R) + (-\delta_P + 0.5\sigma^2)t_2}{\sigma\sqrt{t_2}}, \quad d_2'' = d_1'' - \sigma\sqrt{t_2}, \quad \rho'' = \sqrt{t_1/t_2}$$

The critical price ratio R^* above which the simple exchange option should be acquired at t_1 is found by solving the following equation for R:

$$V_1(P, K, 0, \sigma_P, \sigma_K, \rho, r, \delta_P, t_2 - t_1) = Q \tag{8.10}$$

where $V_1 =$ the Margrabe exchange option, adjusted for t_1.

The European sequential option model assumes that R&D1 cannot occur until t_1 and K is only exercised at t_2 (in the Carr formula). Like American exchange options, when the asset to be received in the exchange has a sufficiently large dividend yield, there is always a probability that the compound option and the underlying American exchange option should be exercised prior to expiration. So consider a 'pseudo-American sequential exchange option' model, where the first compound option can be exercised (investment made) at a fixed time t_1, then evaluate an underlying American exchange option, which can be exercised at any time before t_2 (during $t_2 - t_1$). This is the sum of the Carr European compound exchange option and the early exercise premium for the underlying exchange option between $t_2 - t_1$.[4] The value of such an American sequential exchange option is given in Lee and Paxson (2000b) as:

$$W_A(P, K, \text{R\&D1}, \sigma_P, \sigma_K, \rho, r, \delta_P, t_1, t_2) \\ = W(P, K, \text{R\&D1}, \sigma_P, \sigma_K, \rho, r, \delta_P, t_1, t_2) \\ + V_{A1}(P, K, 0, \sigma_P, \sigma_K, \rho, r, \delta_P, t_2 - t_1) \\ - V_1(P, K, 0, \sigma_P, \sigma_K, \rho, r, \delta_P, t_2 - t_1) \tag{8.11}$$

The first part of the right-hand side of equation (8.11) is a European compound option with R&D1 given by Carr (1988), and the last two parts are the early exercise premium inherent in an underlying American exchange option with time interval $t_2 - t_1$(R&D1 = 0).

8.4 EMPIRICAL MODELING OF REAL R&D OPTIONS

This section discusses the empirical inputs required to analyze the real option value for a specific R&D program overall and on a project-by-project basis. Implicit in all of these option calculations is that e-commerce system returns and development cost 'returns' are multivariate normally distributed, so that the normal standard distribution and bivariate distributions represent the underlying distributions.

8.4.1 Inputs

First of all the expected revenues, operating costs, R&D expenditures and development costs are estimated for each R&D project, and aggregated for purposes of the overall R&D budget benefits/costs. Table 8.1 (based on disguised information) is an example of the e-commerce R&D estimates for project I. It is assumed that small revenues and (larger) operating costs are currently in process, that the evaluation concerns R&D0 in 1998, that net operating profits

Table 8.1 *Hypothetical e-commerce R&D estimates: project I*

	$ Millions						
Year	1998	1999	2000	2001	2002	2003	Terminal value
Project I							
Revenues	2.00	18.00	68.00	194.50	445.00	611.50	
Op. costs	18.13	42.05	92.08	207.35	421.95	537.95	
Op. profit	−16.13	−24.05	−24.08	−12.85	23.05	73.55	490.33
INVESTMENT			90.00	115.00	115.00	150.00	1000.00
R&D1		25.00	37.50	50.00			
R&D0	5.71						
Net present values							
E-commerce value = P	172.21						
E-commerce invest = K	823.77						
E-commerce R&D (99+) = R&D1	82.97						
Discount rate	0.15						
'Moneyness'	0.21						

from e-commerce are treated as perpetuity after year 2003, and capitalized at the company's discount rate for similar risky projects. The e-commerce present value is calculated as of 1998, the R&D1 (the present value of all subsequent R&D expenses after 1998) as of 1999, and the development costs as of 2000. The actual R&D0 spend is $5 710 000. The 'moneyness' is the ratio of the e-commerce current present value to the present value at time t_2 of the e-commerce investment costs. This corresponds to the 'moneyness' of a financial option (as defined by some authors), which is the ratio of the current stock price to the exercise price.

Table 8.1 shows the project I proposal is to spend $5.7 million at t_0, for the option to spend on R&D1 the equivalent of $83 million at t_1 in order to continue the option to spend a development cost equivalent to $824 million at t_2, in order to obtain a project value with a current PV of $172 million. Since this proposal clearly has a large negative net present value, it might be considered worthless and beyond consideration, but valued as a sequential option it has some positive value.

Holland et al. (1992), Holland and Lockett (1996, 1997), and Hagel and Armstrong (1997) show that the R&D1 and K are e-commerce system-specific, and so require separate R&D cost and e-commerce value projections for each type of research. The value volatility and correlation for each R&D project is built up as a composite volatility and correlation of separate e-commerce value and supplier securities.[5] The volatility of each R&D1 stage for each project is assumed to be constant and the same as for K. The cost volatilities are assumed to be the same for all projects.

Conveniently, there are quoted shares and traded options for a wide variety of e-commerce developments, including more or less pure R&D, software development, internet facilities, and e-commerce applications. The underlying assets of R&D on e-commerce are (ultimately) developed networks and systems, which are often securitized in the US and Europe, and traded on NASDAQ or on some of the new European exchanges. We use e-commerce value historical volatilities from a time series (November 1997 through December 1998) selection of securities including Amazon.com,[6] Cybercash, Harbinger, Open Market, Sterling Commerce and Wave Systems Corporation. The historical time series of e-commerce service companies and software suppliers are used to calculate the correlation between value and cost.

The e-commerce cost elements could be considered (in part) securitized, in so far as software companies are also traded on NASDAQ, including Microsoft and SAP, which have e-commerce services bundled with other software. For those securities with traded options, the implied volatility of e-commerce values and costs is also used as (weighted) input to derive the value and cost volatilities for each R&D project.

8.5 REAL R&D AMERICAN SEQUENTIAL EXCHANGE OPTION EXAMPLES

We calculate first the European and American exchange option values of the overall R&D budget, which assumes that the R&D1 and K are fixed at t_2 (for the Margrabe European formula and for the Lee and Paxson American exchange formulae).

8.5.1 E-commerce R&D: RASE options

Table 8.2 shows the valuation of the overall e-commerce hypothetical projects, supposing that the eventual cash flows are as specified, the R&D1 occurs after one year, and the development occurs in the second year. The correlation between the costs and the e-commerce values is assumed to be around 0.29, and the volatilities are assumed to be around 97% (value) and 30% (cost). Valued as simple European or even American exchange options, this aggregate R&D program has a value of some \$22 million, even though the net present value is negative. However, assuming that R&D1 expenditure is required at t_1 in order to keep the program alive, the sequential exchange option value is only \$9.7 million, or almost \$2 million below the R&D0 budget.

Because eventually an e-commerce development is expected to yield some income, and because the timing of R&D1 and development costs can (more or less) be initiated any time over a planning horizon, the more realistic real option value is an American sequential exchange option. As an alternative to a numerical solution, we calculate an approximate ‘pseudo-American’ sequential exchange option value.

In order to approximate the value of an American sequential exchange option value, one has to numerically find two critical points: one critical price ratio from European sequential exchange options at t_1 and one critical price from an American exchange option at $t_2/2$. One point is the critical price ratio, R^*, the ratio of e-commerce value/costs which would justify paying the R&D1 costs at t_1, which triggers the manager to start up the project. The other point is the critical price, P^*, from the American exchange option, which triggers the manager to exercise or exchange the project after it has been started at t_1 (this critical price is also calculated as P^*/K). Note the R^* ratio is 64% and the critical price P^* is around \$2736 million (or a ratio of 166%), which is far from the actual P/K ratio [see equation (8.6) for P^* and equation (8.10) for R^*].

Given the inputs and assumptions, and considering the adjustment (favoring early exercise) for the expected development income, the approximate American sequential exchange option value is around \$9.72 million, which is slightly greater than the Carr European sequential exchange option value of around

Table 8.2 *E-commerce R&D: real American option*

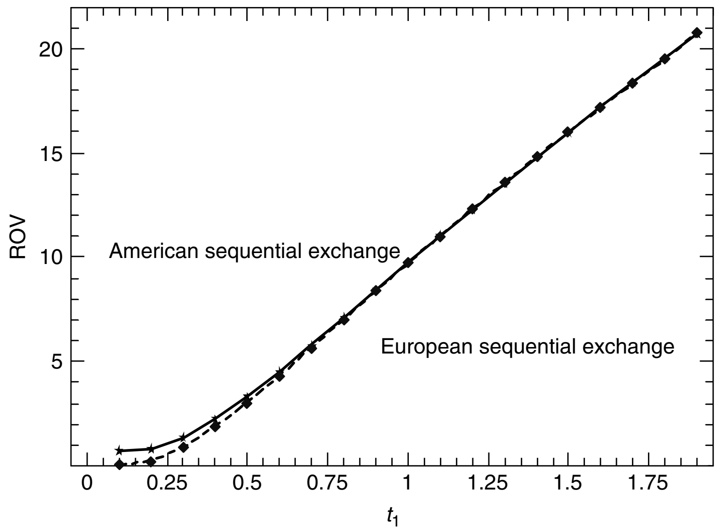
Net income rate of e-commerce development	0.15	
R&D1 phase time	1	
Investment phase time	2	
Cost & value correlation	0.29	
Value volatility	0.97	
Cost volatility	0.30	
Interest rate	0.06	
Exchange volatility	0.92	
Development value = P	344	\$ millions
Development cost = K	1648	\$ millions
R&D1 phase cost	166	\$ millions
(Eq 8.3) V = Margrabe real European exchange option	22.20	\$ millions
(Eq 8.4) V_A = Lee & Paxson real American exchange option	22.87	\$ millions
(Eq 8.9) W = Carr European sequential exchange option	9.69	\$ millions
Lee & Paxson real American sequential exchange option		
(Eq 8.10) R^* = critical price ratio	0.64	
(Eq 8.6) P^* = critical price	2736	\$ millions
(Eq 8.4) V_{A1} = Lee & Paxson American exchange with $t_2 - t_1$(R&D1 = 0)	7.34	\$ millions
(Eq 8.3) V_1 = European exchange with $t_2 - t_1$(R&D1 = 0)	7.31	\$ millions
(Eq 8.5) V_2 = twice exercisable exchange with $t_2 - t_1$(R&D1 = 0)	7.33	\$ millions
(Eq 8.7) V_∞ = perpetual exchange (R&D1 = 0)	80.25	\$ millions
(Eq 8.11) W_A = approximate American sequential exchange option	9.72	\$ millions
option value % e-commerce value	2.82%	

The first seven inputs are the e-commerce income rate, R&D1 and K timing estimates, the value and cost volatilities and correlation from e-commerce securities, and the riskless rate.
The next three inputs are from the R&D e-commerce project estimates.
The Margrabe European and the American exchange option values (Lee and Paxson) assume R&D1 and K are stochastic and at t_2.
The Lee and Paxson American sequential exchange option values assume R&D1 at t_1, and K is exercised at any time after t_1.
R^* is the price ratio P/K at which it will pay to spend R&D1 at t_1.
P^* is the critical price above which the option should be exercised after t_1.
The approximate American sequential exchange option value is $W + V_{A1} - V_1$, where V_{A1}, the confined exponential approximation for the American exchange option with time interval $t_2 - t_1$, is calculated as $V_\infty - (V_\infty - V_2)^2/(V_\infty - V_1)$.

\$9.69 million. Since the budgeted R&D0 effective cost is over \$11 million, cancellation of the R&D activity and/or selectivity among R&D projects is still warranted.

Figure 8.1 shows that the American sequential exchange option value is more valuable than the Carr European sequential exchange option value as time t_1 decreases, since the early exercise premium inherent in the underlying American exchange option becomes large.

However, it should be noted that this outcome is highly sensitive to the underlying assumptions. Also, only the overall R&D budget has thus far been considered. The differences between the American and European sequential



The inputs are the same as in Table 8.2 with different time t_1 between 0.1 and 1.9.
The approximate American sequential exchange option value is the European sequential exchange option (W) + the American exchange option with time interval $t_2 - t_1$ (R&D1 = 0) – the European exchange option with time interval $t_2 - t_1$ (R&D1 = 0).

Figure 8.1 *Real American versus European sequential exchange option values (as a function of t_1)*

exchange options are not going to be significant for way out-of-the-money options like all of these R&D projects.

8.5.2 Allocation among competing R&D projects

The R&D project allocation focuses on the problem of allocating the R&D budget among 'competing' projects. The expected R&D project cash flows must be specified, and any R&D program dependencies identified (we assume the four R&D programs are independent). Then particular projects are associated with specific e-commerce securities and software suppliers correlations and (in many cases) historical volatilities. For convenience, the cost volatility is assumed to be the same for all projects, and the timeframe for all R&D1 and all K is assumed to be t_1 and t_2. Then the European and approximate American sequential exchange option values are calculated for each project.

As summarized in Table 8.3, there is a range of RASE option values, expressed as a percentage of e-commerce value. The project cost and value correlations range from −0.03 to 0.42, and the value volatilities range from 87% to 107%. The highest RASE (as $\%P$) is project III at 7.2%, due to the high value volatility and negative correlation (remember these are way out-of-the-money options, characteristic of many e-commerce and internet enterprises).

Table 8.3 *Real American sequential exchange option value: comparison of R&D projects*

R&D project	I	II	III	IV	
E-commerce income rate	0.15	0.15	0.15	0.15	
R&D1 phase time	1	1	1	1	
Investment phase time	2	2	2	2	
Cost & value correlation	0.05	0.07	-0.03	0.42	
Value volatility	0.88	0.87	1.07	1.06	
Cost volatility	0.30	0.30	0.30	0.30	
Interest rate	0.06	0.06	0.06	0.06	
Exchange volatility	0.91	0.90	1.12	0.97	
Development value $= P$	172	86	43	43	\$ millions
Development cost $= K$	824	412	206	206	\$ millions
R&D1 phase cost	83	23	23	37	\$ millions
(Eq 8.9) $W =$ Carr European sequential exchange option	4.56	2.80	3.09	1.10	\$ millions
Lee & Paxson real American sequential exchange option					
(Eq 8.10) $R^* =$ critical price ratio	0.64	0.51	0.56	0.80	
(Eq 8.6) $P^* =$ critical price	1358	672	395	353	\$ millions
(Eq 8.4) $V_{A1} =$ Lee & Paxson American exchange with $t_2 - t_1$(R&D1 $= 0$)	3.48	1.57	2.42	1.19	\$ millions
(Eq 8.3) $V_1 =$ European exchange with $t_2 - t_1$(R&D1 $= 0$)	3.46	1.57	2.40	1.18	\$ millions
(Eq 8.5) $V_2 =$ twice exercisable exchange with $t_2 - t_1$(R&D1 $= 0$)	3.47	1.57	2.41	1.19	\$ millions
(Eq 8.7) $V_\infty =$ perpetual exchange (R&D1 $= 0$)	39.32	20.06	14.20	10.07	\$ millions
(Eq 8.11) $W_A =$ approximate American sequential exchange option	4.58	2.80	3.11	1.10	\$ millions
Option value % e-commerce value	2.66%	3.25%	7.22%	2.56%	\$ millions

The first seven inputs are the e-commerce income rate, R&D1 and K timing estimates, the value and cost volatilities and correlation from e-commerce securities, and the riskless rate.
The next three inputs are from the R&D e-commerce project estimates.
The Carr European exchange option values (W) assume R&D1 and K are stochastic and at t_2.
The Lee and Paxson American sequential exchange option values assume R&D1 at t_1, and K is exercised at any time after t_1.
P^* is the critical price above which the option should be exercised after t_1.
The approximate American sequential exchange option value is $W + V_{A1} - V_1$, where V_{A1}, the confined exponential approximation for the American exchange option with time interval $t_2 - t_1$, is calculated as $V_\infty - (V_\infty - V_2)^2/(V_\infty - V_1)$.

Table 8.4 compares the RASE option worth to the R&D0 cost. The R&D project III is associated with high volatility and low correlation, but its real option worth/cost ratio is 0.93. The favored project is IV and then project II, whereas project I should be canceled. Note that the RASE option values of four projects is \$11.59 million, higher than the overall RASE option value of \$9.72 million, which is consistent with the value of a portfolio of options exceeding the option on a portfolio.[7] Thus the aggregate R&D budget is (almost)

Table 8.4 *Allocating R&D budget among separate projects ($ millions)*

	I Project	II Project	III Project	IV Project	Aggregate
Real option worth	4.58	2.80	3.11	1.10	11.59
R&D0 cost	5.71	2.00	3.33	0.67	11.71
Real option worth/cost	0.80	1.40	0.93	1.65	0.99
Critical price = P^*	1358.32	671.57	394.67	353.02	
Ratio P^*/P	7.90	7.81	9.19	8.21	

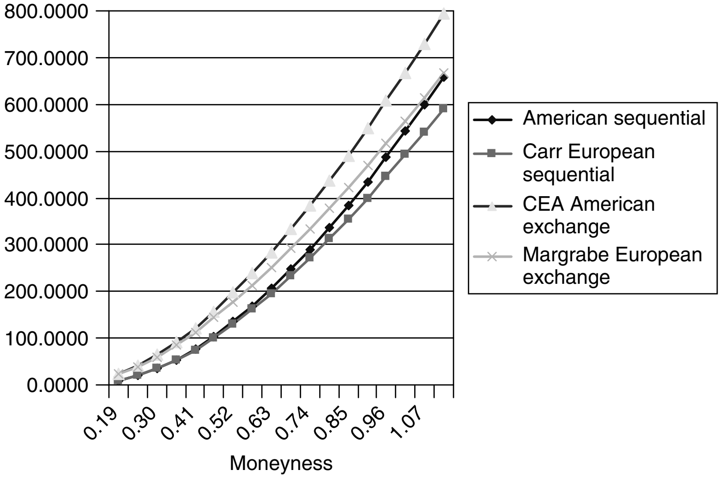
P^* is the critical price for e-commerce value above which the exchange option should be exercised after t_1.

justified considering the separate RASE option valuation, especially if project selectivity is feasible. Selecting the two best R&D projects with a positive worth/cost ratio would increase the ratio of real option worth to cost to nearly 1.5. Project selection would increase the total R&D value over the R&D0 cost by $1.35 million. Focus on developed e-commerce value volatility, development cost volatility, and the correlation of separate R&D projects by research management is critical in assessing both appropriate budget allocation and the appropriate level of the total R&D budget.

8.5.3 Real option value sensitivities

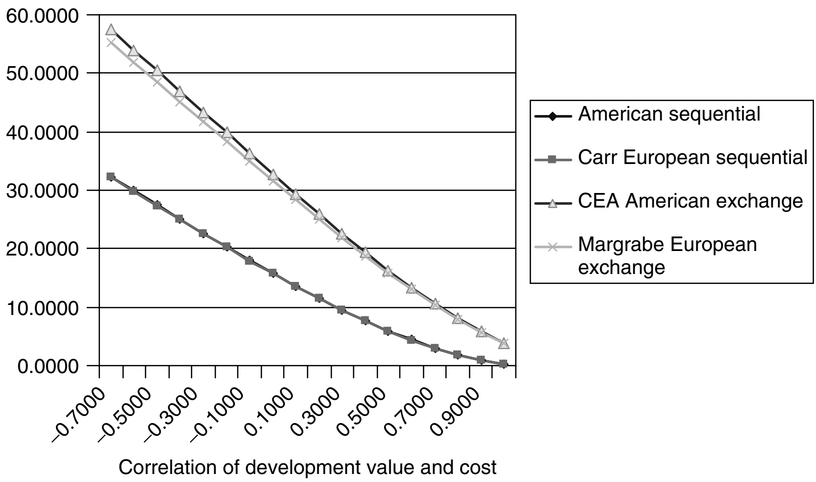
The real option value would be increased if the expected operating cash flows are increased, or if the R&D1 and development costs are reduced, or if the time to development is extended, or if the expected volatilities are increased or the correlation reduced. Figures 8.2 and 8.3 show graphically the sensitivity of the RASE option values to changes in some separate factors. With other factors held constant, the option values (both European and American) increase and the early exercise premium of the RASE becomes larger as development value increases (Figure 8.2). As P approaches $K +$ R&D1, the American is worth much more than the European exchange option, which itself is not much greater than the American sequential exchange option. For almost in-the-money options, having to pay the R&D1 is not much of a penalty. For way out-of-the-money options, the sequential exchange option values are worth much less than the exchange options, because of the requirement to spend the R&D1 cost. Comparison of these option values over a range of moneyness might provide some guidance on the appropriate structuring and strategy for R&D management.

As expected, Figure 8.3 shows that a lower cost and value correlation makes all exchange and sequential exchange options more valuable. For example, a decrease in the correlation (from 0.29 to zero) brings about a 62% increase in RASE option value (from $9.72 million to $15.75 million). Note that the spread between exchange and sequential exchange options increases with a decrease in correlation.



The inputs are the same as in Table 8.2 with different development value P between the basic value of \$344 million and \$2044 million.

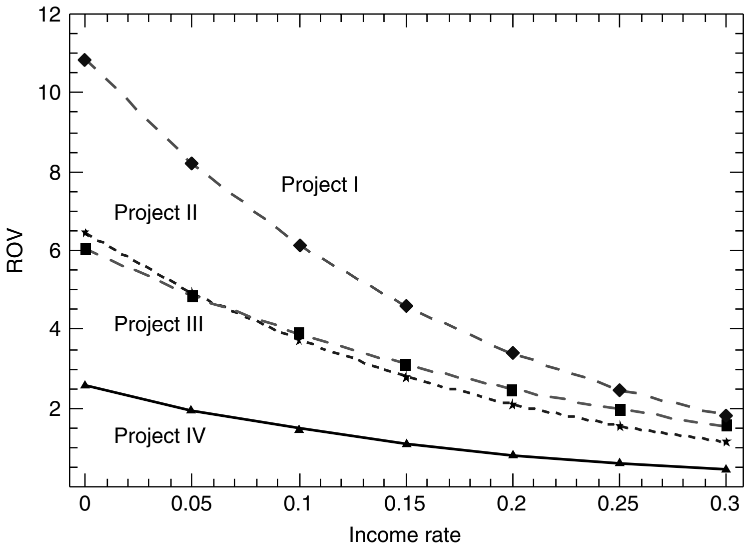
Figure 8.2 *American versus European exchange and sequential exchange options*



The inputs are the same as in Table 8.2 with different cost and value correlation between –0.7 and +1.

Figure 8.3 *Sensitivity of real American and European exchange and sequential exchange options to correlation*

Also, American sequential exchange options are highly sensitive to changes in e-commerce income. Assuming all of the inputs for the American option values for the separate R&D projects remain constant, we 'simulate' the RASE assuming that only income rate changes. American approximate value decreases



The inputs are the same as in Table 8.3 with different income rates between 0% and 30%. The approximate American sequential exchange option value is the European sequential exchange option (W) + the American exchange option with time interval $t_2 - t_1$ (R&D1 = 0) – the European exchange option with time interval $t_2 - t_1$ (R&D1 = 0). Assumes change in income rate does not affect project present value.

Figure 8.4 *Sensitivity of real American sequential exchange option value to income rates*

as income rates are increased. Figure 8.4 shows that the RASE value declines sharply as income rate is increased from 0% only to 5% and decreases more than 50% as income rates are increased from 0% to 30%, and that this decline varies among the four projects.

8.6 FURTHER RESEARCH AND CONCLUSION

There are four alternative real exchange and sequential exchange option models considered herein. The European and American sequential exchange option values are similar (since the options examined in this chapter are extremely out-of-the-money). Comparisons should be made for all of these methods to numerical solutions for different ranges of moneyness and cost of carry.

Some very broad assumptions as to e-commerce values and development costs have been made herein to assess the possible benefits of e-commerce R&D and real option values. The multivariate normal assumption presumes that both value and cost 'return' proxies are normally distributed and stationary. Other remaining theoretical problems include using multivariate normal distributions that can model several investment stages over time; volatility and correlation matrices for these sequential investments; allowing for variable e-commerce income and development cost escalation assumptions over time; and using mixed diffusion-jump processes for both value and cost.[8] Determining the optimal exercise time for R&D1 and development stage expenditures is an important consideration. These RASE models do not allow for the abandonment option (selling R&D information to date while canceling R&D1 and *K* expenditures) or for the more realistic R&D1 and *K* continuous expenditures over time.[9]

The primary uses of these RASE models are: (i) determining the appropriate R&D strategy and budget; (ii) allocating the overall budget among competing research proposals; and (iii) determining the optimal timing (and likely occurrence) of the subsequent stage R&D phase, if the timing is flexible (that is any time before a given date). Following from (i) are some corporate finance issues, such as issuing equity if the R&D enterprise market capitalization (MC) exceeds the RASE adjusted for other enterprise values, cutting R&D and repurchasing shares in the opposite case, and buying other enterprises with large relative RASE/MC.

ACKNOWLEDGMENTS

Many thanks to Sydney Howell, Kjeld Jensen, Peter Oliver, Pedro Santa-Clara, Philip Ka-Jung Sung, Mark Tippett, Paul Warren and four referees for helpful comments on earlier drafts of this chapter and related papers.

NOTES

1. Lee and Paxson (2000b) show that the confined exponential approximation (CEA) is around three times more accurate (using a PSOR scheme for 3TLD finite difference numerical method as a benchmark) than the Carr model, and comparable to the Bjerksund and Stensland (1993) model. If the CEA model is refined using tighter upper bounds, it is around three times more accurate than the Bjerksund and Stensland (1993) model.
2. Carr (1995) also proposes an analytic approximation for an American exchange option by generalizing the solution of Geske and Johnson (1984) using three-point Richardson extrapolation. However, this approach is still not inexpensive and is not considered in this chapter.
3. McDonald and Siegel (1986) gave an explicit analytic solution for the value of an infinitely lived American exchange option, where project value and investment cost both follow geometric Brownian motions. Gerber and Shiu (1996) derived a closed-form solution for a perpetual American exchange option applying a martingale approach.
4. It is usual to assume that the compound option is European-style since the option holder would not want to exercise the compound option before the beginning of the underlying option (see Lee and Paxson, 2000b).
5. Since the composite weightings would indirectly reveal the type and scope of research considered herein, the exact weightings are not disclosed.

6. Schwartz and Moon (2000b) showed that the implied volatility of options on Amazon.com is not consistent with their real option valuation of the underlying security.
7. This would be true if the volatilities and correlations were the same on all of the projects.
8. Suggested by a referee.
9. Noted by another referee.

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Chapter 9

Optimal exploration investments under price and geological–technical uncertainty: a real options model

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SUMMARY

This chapter develops a real options model for valuing natural resource exploration investments (e.g. oil or copper) when there is joint price and geological–technical uncertainty. After a successful several-stage exploration phase, there is a development investment and an extraction phase. All phases are optimized contingent on price and geological–technical uncertainty.

Several real options are considered. There are flexible investment schedules for all exploration stages and a timing option for the development investment. Once the mine is developed, there are closure, opening and abandonment options for the extraction phase. Our model maintains a relatively simple valuation structure by collapsing price and geological–technical uncertainty into a one-factor model.

We apply the model to a copper exploration prospect and find that a significant fraction of total project value is due to the operational, development and exploration options available to project managers.

9.1 INTRODUCTION

We present a real options model for valuing natural resource exploration investments (e.g. oil or copper) when there is joint price and geological–technical uncertainty. Price risk refers to output market value, while geological–technical risk applies to reserves, development investments and cost structure. Continuous-time Brownian motions are used to model both uncertainty processes assuming a futures market for output prices[1], and a declining geological–technical risk level as exploration investments are undertaken. In case of finding an economically feasible mine, there may be a development investment phase, to be followed by an extraction phase. All phases are optimized contingent on price and geological–technical uncertainty.

Several real options are considered. The exploration investment schedule is flexible and may be stopped and/or resumed at any moment depending on cash flow expectations, which in turn depend on current commodity price and geological–technical expectations. The model allows for several exploration phases, each one with its own investment schedule and probabilities of success. In the event of an exploration success there is a timing option for the development investment, and closure, opening and abandonment options for the extraction phase.

The model has the virtue of maintaining a relatively simple structure by collapsing price and geological–technical uncertainty into a one-factor model. The model can be applied to value oil or other natural resource investments and has been used by a major copper company to value real exploration prospects. We solve the model using implicit finite-difference numerical methods and present results for a specific case.

Our model follows the rapidly increasing literature on the real options approach for the valuation of investments under uncertainty. Among them, Majd and Pindyck (1989) include the effect of the learning curve by considering that accumulated production reduces unit costs, Trigeorgis (1993) combines real options and their interactions with financial flexibility, McDonald and Siegel (1986) and Majd and Pindyck (1987) optimize the investment rate, and He and Pindyck (1992) and Cortazar and Schwartz (1993) determine two optimal control variables. This approach has been used to analyze uncertainty on many underlying assets, including exchange rates (Dixit, 1989), costs (Pindyck, 1993) and commodities (Ekern, 1988). Finally, many asset types and problems have been modeled using this approach, including natural resource investments, environmental and new technology adoption, and strategic and competitive options (Trigeorgis, 1996, 2000; Brennan and Trigeorgis, 2000; Dixit and Pindyck, 1994).

Our model follows the Brennan and Schwartz (1985) model for valuing natural resource investments. Closely related papers on the modeling of undeveloped oil fields or the software implementation approach are Paddock et al. (1988), Cortazar and Schwartz (1997), and Cortazar and Casassus (1998). Other models for valuing oil contingent claims include Smith and McCardle (1998, 1999), Lehman (1989) and Trigeorgis (1990).

9.2 THE MODEL

9.2.1 Modeling the exploration investment decision

Exploration of natural resources typically involves following several stages, each one with an investment schedule and with associated success and failure probabilities. A representation of n exploration stages is presented in Figure 9.1 with the following notation:

- X^j Value of the exploration project at the initial point in stage j
- I^j Present value of the investment during stage j
- T^j Time of exploration stage j
- p^j Probability of success stage j
- H Value of the project at the end of the exploration stages, conditional on success

We can view the exploration project X as an infinitely compounded option that may be continuously exercised as the exploration investment is undertaken. The model assumes that at any point in time investment can be stopped or resumed depending on the expected value of the project, which in turn depends on the geological and technical data, as will be explained later.

Contingent on an exploration success, the project may be developed by investing a present value of I_d^i during a development time of T_d^i. These values depend on the characteristics of the mine i found. The model considers a perpetual timing option by allowing the delay of this development investment. Once the decision to invest is made, there is no possibility of stopping the development investment.

When the development investment is concluded, the mine enters into the extraction phase. We use Brennan and Schwartz (1985) to model this phase considering opening, closing and abandonment options.

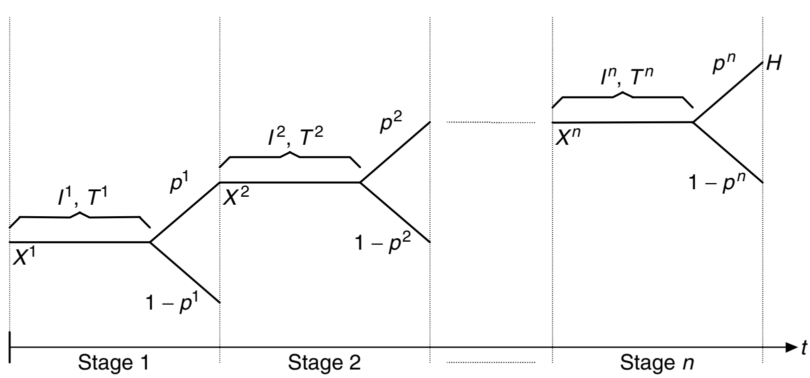

Figure 9.1 *Illustration of the n exploration stages of the project*

9.2.2 Modeling price and geological–technical risk

Exploration prospects of natural resources (e.g. oil, copper) are very risky investments because both output prices (which have been widely studied) and output quantities and development/production costs are uncertain.

Commodity price risk has long been modeled using no-arbitrage finance models in a continuous-time setting. Even though recent multi-factor price models are very promising for explaining commodity price behavior (Schwartz, 1997; Cortazar et al., 1999), for simplicity in this chapter we use a standard one-factor constant-convenience-yield model for risk-neutral prices:

$$\frac{\mathrm{d}S}{S} = (r - c)\mathrm{d}t + \sigma_S \,\mathrm{d}\omega_S \tag{9.1}$$

with the following notation:

- S Spot unit price of copper
- r Risk-free real rate of interest, assumed constant
- c Convenience yield on holding one unit of copper
- σ_S Instantaneous volatility of returns on holding one unit of copper
- $\mathrm{d}\omega_S$ Increments to standard Gauss–Wiener process

While price risk is constant during all three phases of the project (exploration, development and extraction), the geological–technical risk is much higher during the exploration phase. This geological–technical risk may be decomposed into two parts: one is the success or failure in finding an economically feasible mine, and the other is related to the particular characteristics of the (eventual) mine. While the first part has already been modeled using the discrete success–failure probabilities at each of the exploration stages, the second part, defined by mine reserve levels, development investments, production schedules and cost structures, is explained in what follows.

Before undertaking any exploration investment, uncertainty on the value of the expected final reserve, conditional on success, is at its greatest level. The characteristics of the eventual mine are highly unknown, ranging from modest to highly profitable. In a real options framework it is clear that exploration investments should optimally be undertaken considering both the expected mine value and its distribution.

As exploration investment is undertaken, uncertainty on the final characteristics of the mine decreases and price risk becomes comparatively more important. The Brennan and Schwartz (1985) model of a copper mine, for example, considers only price risk once development investment starts. But it is clear that an appropriate exploration investment model should add to its price risk the effect of a declining geological–technical risk.

To model geological–technical risk, different approaches may be followed. One alternative could be to define a vector of geological–technical variables that affect mine value and specify a stochastic process for each of them. In this case the value of the (already explored) mine would be a function of output price, S, as well as of a vector G of mine characteristics, $\{G_1, G_2, \ldots, G_N\}$, which could include development investment, extraction rate, costs, etc. Thus, the value of the expected mine would be defined as $H(S, G_1, G_2, \ldots, G_N)$. In this setting, model complexity increases with the dimension of the mine-characteristics vector G due to both the amount of information required to specify the multivariate process for the state variables, and the added effort of solving a multi-factor model.

We take a simpler approach that provides a reasonable approximation for many cases and helps keep the model tractable. Instead of asking geologists/mining engineers to specify the level and multivariate process for all relevant mine characteristics, we ask them only to determine a representative set of possible mine types, with their probabilities of occurrence, that could be found attending current prospect characteristics. Each mine type is defined using the parameters required by the Brennan and Schwartz (1985) model, including total reserves, development investment amounts and schedule, and production schedule (amounts and costs), with associated opening and closing costs. Thus we are able to value each of the possible mines as a function of output price using the Brennan and Schwartz (1985) model. Using the conditional probabilities for each mine type, we obtain the *expected* mine value (as a function of output price) as well as an initial empirical *distribution* of mine values that we define as the geological–technical risk.

To obtain the process for this geological–technical risk we start by defining a one-dimensional state variable G to represent this risk. We know the initial empirical *distribution* of mine values associated with this risk (obtained using the initial probability assessment) and assume that after the exploration investment concludes there will be no residual geological–technical uncertainty. In the absence of better information on the particular characteristics of the exploration process, we assume that initial geological–technical uncertainty is reduced continuously as exploration investment is undertaken.

Once we have defined mine value as a function of two state variables, output spot price, S, and geological–technical risk, G, we use the fact that both factors may be assumed to be independent and in many cases may be collapsed into one state variable, Z. This makes the model very simple to implement, while providing a reasonable approximation to the project value.

To formalize the model we define a geological–technical risk factor G (for example the amount of mineral in a mine) that follows a zero-drift constant volatility Brownian motion, as follows:[2]

$$\frac{\mathrm{d}G}{G} = \sigma_G \, \mathrm{d}\omega_G \tag{9.2}$$

This geological–technical risk factor is assumed to be independent of output price S:

$$d\omega_S \, d\omega_G = 0 \tag{9.3}$$

Mine value, $H(S, G)$, can be modeled as a function of output price, S, and the geological–technical variable, G.

We can now define a new state variable Z, a function of S and G, such that:

$$H(Z) \equiv H(S, G) \tag{9.4}$$

and

$$Z \equiv F(S, G) \tag{9.5}$$

Applying Itô's lemma and using equations (9.1) and (9.2), we obtain:

$$\begin{aligned} dZ = &\left[F_S S(r - c) + \tfrac{1}{2} F_{SS} S^2 \sigma_S^2 + \tfrac{1}{2} F_{GG} G^2 \sigma_G^2\right] dt \\ &+ F_S S \sigma_S \, d\omega_S + F_G G \sigma_G \, d\omega_G \end{aligned} \tag{9.6}$$

We can further simplify model implementation by assuming:

$$Z = SG \tag{9.7}$$

This is equivalent to assuming that an increase in any of the two factors (S or G) has a similar effect on mine value. The process for this new state variable becomes:

$$\frac{dZ}{Z} = (r - c)dt + \sigma_S \, d\omega_S + \sigma_G \, d\omega_G \tag{9.8}$$

The new state variable, Z, can be seen as a modified commodity price with the same drift as the original one, S, but with an increased volatility:

$$\sigma_Z = \sqrt{\sigma_S^2 + \sigma_G^2} \tag{9.9}$$

For many projects equation (9.7) represents a very convenient approximation; for others it is not only a good approximation, but holds perfectly. This is the case, for example, if G represents total mine reserves, A_1 is a constant that depends on a fixed extraction rate[3] and A_2 is a fixed cost. Then project value H is defined by:

$$H(S, G) = A_1 SG + A_2 \quad \text{or} \quad H(Z) = A_1 Z + A_2 \tag{9.10}$$

and equation (9.7) holds.

Reducing model complexity at the exploration phase by collapsing all geological–technical risk into one factor, with an appropriate (increased) volatility, is convenient because it allows for representing many geological–technical factors by their joint effect on mine value. The fact that all these factors are orthogonal to the market factor (represented in this model by commodity price) allows for this reduction in the state space with little loss of accuracy, while retaining the possibility of being individually considered during the extraction phase. Should there be any additional knowledge on the way exploration investment modifies geological–technical risk for any particular case, adjustments to the above model should be undertaken.

9.2.3 Estimation of geological–technical volatility

As we have stated above, all of the geological–technical risk is represented by the initial distribution of the expected mine value at the beginning of the development stage. Before initiating the exploration stage we have an expected mine value and the variance for this distribution as a function of output price. The estimation of the geological–technical volatility σ_G needs to be consistent with the variance of the expected mine value. By applying Itô's lemma to the expected value of the mine for any given output price S (such that all the volatility is due to the geological–technical risk), we obtain the relation to pin down σ_G.

9.2.4 Valuing the exploration investment project

In order to value the exploration project we start by valuing the alternative mines that could be obtained if exploration is successful. Then, we value the development investment decision, and conclude by valuing the exploration phase.

Before exploration investments begin, there is a set of alternative mines that could eventually be obtained, should the exploration phases be successful. Each one of the M possible mines is assumed to have an associated probability of occurrence, conditional on exploration success, α^i, such that:

$$\sum_{i=1}^{M} \alpha^i = 1 \tag{9.11}$$

9.2.4.1 Value of a developed mine

Each of the M possible mines is valued using the Brennan and Schwartz (1985) model:[4]

$$\max_{q_p^i} \left[\tfrac{1}{2} V_{SS}^i S^2 \sigma_S^2 + (r-c) S V_S^i - q_p^i V_Q^i + q_p^i (S - a^i) - (r+\lambda) V^i \right] = 0 \tag{9.12}$$

$$\tfrac{1}{2} W_{SS}^i S^2 \sigma_S^2 + (r-c) S W_S^i - m^i - (r+\lambda) W^i = 0 \tag{9.13}$$

Subject to:

$$W^i(S_0^{*i}, Q) = 0 \tag{9.14}$$

$$V^i(S_1^{*i}, Q) = \max[W^i(S_1^{*i}, Q) - K_1^i, 0] \tag{9.15}$$

$$W^i(S_2^{*i}, Q) = V^i(S_2^{*i}, Q) - K_2^i \tag{9.16}$$

$$W_S^i(S_0^{*i}, Q) = 0 \tag{9.17}$$

$$V_S^i(S_1^{*i}, Q) = \begin{cases} W_S^i(S_1^{*i}, Q) & \text{if } W^i(S_1^{*i}, Q) - K_1^i \geq 0 \\ 0 & \text{if } W^i(S_1^{*i}, Q) - K_1^i < 0 \end{cases} \tag{9.18}$$

$$W_S^i(S_2^{*i}, Q) = V_S^i(S_2^{*i}, Q) \tag{9.19}$$

$$V^i(S, 0) = 0 \tag{9.20}$$

$$W^i(S, 0) = 0 \tag{9.21}$$

$$V^i(0, Q) = 0 \tag{9.22}$$

$$W^i(0, Q) = 0 \tag{9.23}$$

$$\lim_{S \to \infty} V_{SS}(S, Q) = 0 \tag{9.24}$$

$$\lim_{S \to \infty} W_{SS}^i(S, Q) = 0 \tag{9.25}$$

We use the following notation:

V^i Value of mine i when it is optimal to be open
W^i Value of mine i when it is optimal to be closed
Q Reserves
q_p^i Production rate of mine i
S_0^{*i} Critical price below which it is optimal to abandon
S_1^{*i} Critical price below which it is optimal to close
S_2^{*i} Critical price above which it is optimal to open
K_1^i Cost of closing the mine i
K_2^i Cost of opening the mine i
m^i Annual maintenance cost of keeping a mine closed
λ Country risk (or probability of expropriation)

9.2.4.2 Value of an undeveloped mine

We first assume the development investment decision has been made, but is still under way. The value of the mine must now satisfy the following equation:

$$\tfrac{1}{2}U_{SS}^i S^2 \sigma_S^2 + (r - c)SU_S^i - U_T^i - (r + \lambda)U^i = 0 \tag{9.26}$$

Subject to:

$$U^i(0, T) = 0 \tag{9.27}$$

$$\lim_{S\to\infty} U^i_{SS}(S, T) = 0 \tag{9.28}$$

$$U^i(S, 0) = V^i(S, Q) \tag{9.29}$$

Then, we can define H as the value of the mine before the development investment is made, which must satisfy:

$$\tfrac{1}{2} H^i_{SS} S^2 \sigma_S^2 + (r - c) S H^i_S - (r + \lambda) H^i = 0 \tag{9.30}$$

Subject to:

$$H^i(0) = 0 \tag{9.31}$$

$$H^i(S) = \begin{cases} H^i(S) & \text{if } S \le S_d^{*i} \\ U^i(S, T_d^i) - I_d^i & \text{if } S > S_d^{*i} \end{cases} \tag{9.32}$$

We use the following notation:

I_d^i Development investment
T_d^i Time of development stage (after the investment has been done)
S_d^{*i} Critical price above which it is optimal to invest

Once each of the mines are valued, we can obtain an expected mine value by multiplying each of the values by its probability α^i:

$$H(S) = \sum_{i=1}^{M} \alpha^i H^i(S) \tag{9.33}$$

9.2.4.3 Value of an exploration project

Finally, we solve for the value of the exploration project. We consider that while the project is under way and exploration investment is undertaken, the value of the project is X, and while the project is optimally stopped, the value is Y.

Notice that while exploration investment is under way (X), relevant volatility is higher than when it is temporally stopped (Y),[5] because geological–technical information is only obtained with investment.

It is possible to solve for each stage j, with $j = 1, \ldots, n$. To do this we start by solving for stage $j = n$ and work our way backwards until $j = 1$. We now present the equations for an intermediate stage j:

$$\max_{q_i^j} \left[\tfrac{1}{2} X^j_{ZZ} Z^2 \sigma_Z^2 + (r - c) Z X^j_Z + q_i^j X^j_I - q_i^j - (r + \lambda + \gamma^j) X^j \right] = 0 \tag{9.34}$$

$$\tfrac{1}{2} Y^j_{ZZ} Z^2 \sigma_S^2 + (r - c) Z Y^j_Z - (r + \lambda) Y^j = 0 \tag{9.35}$$

Subject to:

$$X^j(0, I) = 0 \tag{9.36}$$

$$Y^j(0, I) = 0 \tag{9.37}$$

$$\lim_{Z\to\infty} X^j_{ZZ}(Z, I) = 0 \tag{9.38}$$

$$X^j(Z, I^j) = X^{j+1}(Z, 0) \qquad \text{if} \quad Z \geq Z^{*j} \tag{9.39}$$

$$Y^j(Z, I^j) = Y^{j+1}(Z, 0) \qquad \text{if} \quad Z < Z^{*j} \tag{9.40}$$

For the final stage these last two boundary conditions are replaced by:

$$X^n(Z, I^n) = H(Z) \quad \text{if} \quad Z \geq Z^{*n} \tag{9.41}$$

$$Y^n(Z, I^n) = H(Z) \quad \text{if} \quad Z < Z^{*n} \tag{9.42}$$

We use the following notation:

I Accumulated exploration investment at that stage
q_i^j Investment rate stage j
I^j Present value of the investment during stage j
T^j Time of exploration stage j
p^j Probability of success stage j
Z^{*j} Critical price for investing at stage j
γ^j Poisson probability of success stage j[6]

9.3 RESULTS

The model can be applied to oil and other commodity exploration prospects. We have applied it to several exploration prospects available to a copper company. To our knowledge there is no analytical solution to this compound option system of equations even if we consider infinite resource profiles during the extraction stage. In what follows we present a specific case and report the results of solving the above exploration model using implicit finite-difference numerical methods.

The prospect we present considers that before exploration starts the geologists have an exploration investment plan with four exploration stages, each one with its own investment schedule and probabilities of success (see Table 9.1). In case of failure at one stage the exploration is abandoned. In case of success the project could be stopped or resumed, depending on the expected value of the mine. This expected value varies due to both changes in prices and/or the expected geological–technical characteristics of the mine.

If the exploration stage is successful, the project enters into the development stage. In the specific case we are evaluating, we consider 11 different development/extraction plans, each one represented by a mine profile. Each

Table 9.1 *Description of the exploration stages of the project*

Exploration stage	Investment (MUS$)	Time of exploration (years)	Success probability
1	3	1	0.2
2	15	3	0.3
3	12	2	0.3
4	12	2	0.8

Table 9.2 *Investments of the development stage for mine 1*

Year	Investment (MUS$)
1	123.2
2	246.4
3	123.2

Table 9.3 *Costs and production schedule of the extraction stage for mine 1*

Maintenance costs (MUS$/Year) 0.45
Closure costs (MUS$) 70.4
Opening costs (MUS$) 48.3

Year	Production (Mlb)	Operating costs (MUS$)
1	248	81.1
2	248	81.9
3	248	97.5
4	248	83.6
5	248	84.4
6	248	100
7	248	86.1
8	248	87
9	248	102.6
10	248	88.7
11	248	89.6
12	166.4	60.7

of these possible mines has particular development and production schedules. For example, Tables 9.2 and 9.3 describe one of the mine profiles (‘mine 1’ in Figure 9.2) that has a 15% probability of occurrence.

Figure 9.2 shows the value of all mines contingent on output price using the Brennan and Schwartz (1985) model. The expected deposit is the expected mine value obtained by multiplying each of the values by its occurrence probability.

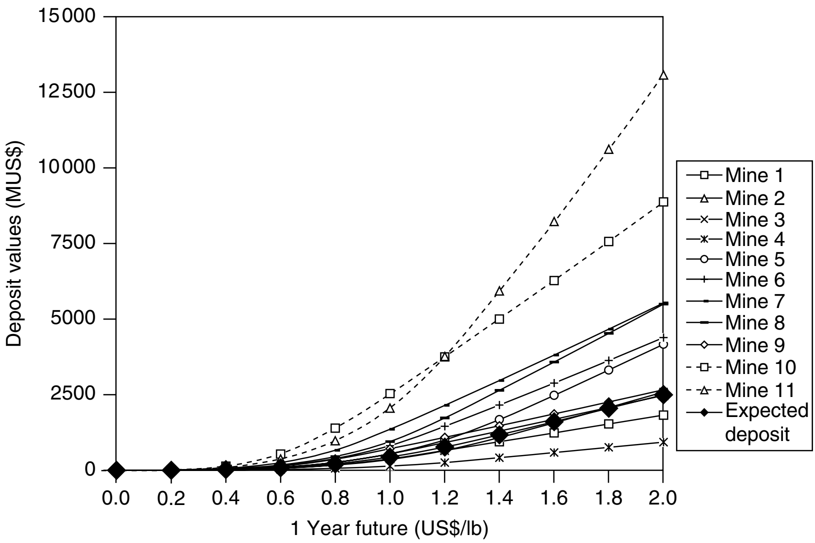

Figure 9.2 *Values before development investment for the different mine profiles and the expected deposit contingent on output price using the Brennan and Schwartz (1985) model*

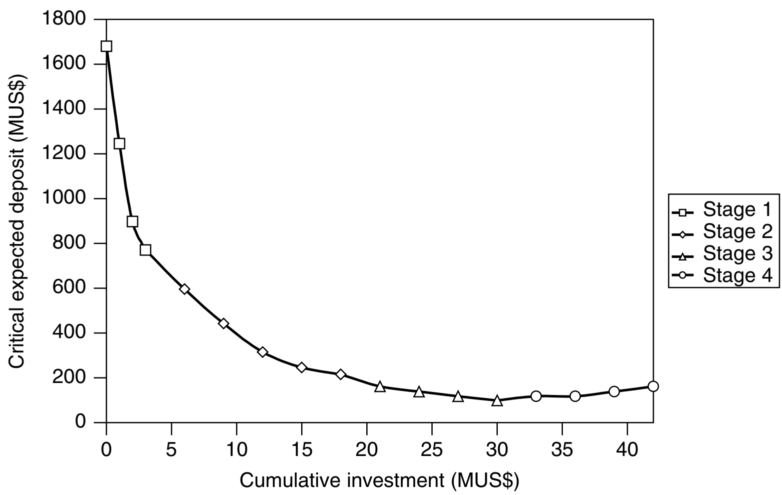

Figure 9.3 *Optimal investing policy for the four exploration stages contingent on cumulative investment. If the expected deposit is above the critical value, it is optimal to invest*

Figure 9.3 presents the optimal exploration investment schedule. If the expected mine value exceeds the critical expected mine value, then exploration investment should proceed; if not, then it should be stopped.

Finally, Table 9.4 shows the value of the exploration project at the beginning of the exploration stage and how it can be decomposed in terms of option

Table 9.4 *Sources of value of the exploration project in MUS\$, when $r = 2.8\%$, $c = 6\%$, $\sigma_S = 40\%$ and $\lambda = 2\%$*

Expected deposit (MUS\$)	Value without options	Operational options	Development option	Exploration option	Total value
500	−11.44	6.68	2.94	3.19	1.37
1000	−6.29	5.46	2.25	1.58	3.01
1500	−1.95	4.71	1.82	0.60	5.17
2000	2.26	4.15	1.49	0.20	8.09

values associated with each stage. The optimal value of the exploration project (total value) is decomposed into: (a) the value of the project if there is no flexibility (or there is no volatility) at any stage (*value without options*); (b) the value added for optimally opening, closing or abandoning the mine during the extraction stage (*operational options*); (c) the extra value for optimally deferring the development investment (*development options*); and (d) the additional value for optimally investing during the exploration stage (*exploration option*). The results are presented contingent on the value of the expected mine before development investment is made.

For example, suppose that at the beginning of the exploration stage our best estimation for the value of the mine that we can potentially develop is 500. In this value we consider the output price and estimations regarding possible reserves, investments, costs and productions during development and extraction phases. If we don't consider any flexibility during the life of the project, the value of the exploration project is -11.44. To calculate this value, we evaluate the project considering that the investment rate during exploration stage j is always q_i^j even when it is optimal to wait, that the development investment is done immediately the exploration stage is finished (in all mine profiles), and that the production rate during the extraction stage is always q_p^i (in all mine profiles), even if it is optimal to stop or abandon the mine. If we only consider operative options during the extraction phase, the value of the exploration project is -4.76 $(=-11.44 + 6.68)$. To calculate this value we allow for an optimal operation of the mine [see equations (9.12) to (9.25)]. If in addition we consider the development option the value increases to -1.82 $(=-11.44 + 6.68 + 2.94)$. To get this value we optimally defer the development investment [see equations (9.30) to (9.32)]. Finally to get the total value of the exploration project (and the exploration option value) we consider that we can optimally invest during the exploration stage [see equations (9.34) to (9.42)]. This will give a value for the exploration project of 1.37 and an exploration option value of 3.19.

Table 9.4 also shows that option values decrease when the value of the expected deposit increases. This occurs because when the value of the expected deposit is low, the option to postpone investment or close the mine is valuable. When the expected value of the mine is high, it will be optimal to invest or extract the commodity as fast as possible, and the option value will be small.[7]

Figure 9.4 gives the values of the 'delta' of the exploration project as a function of the one-year copper futures price. This represents the partial derivative of the project value with respect to changes in the futures price. The second derivative of the project value with respect to the futures price, the 'gamma' of the project, is shown in Figure 9.5. As expected the sensitivity of the delta,

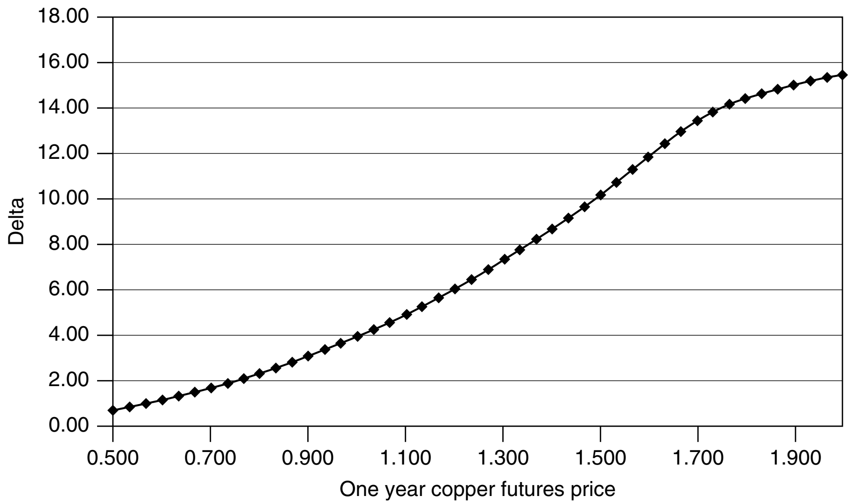

Figure 9.4 *Delta of exploration project*

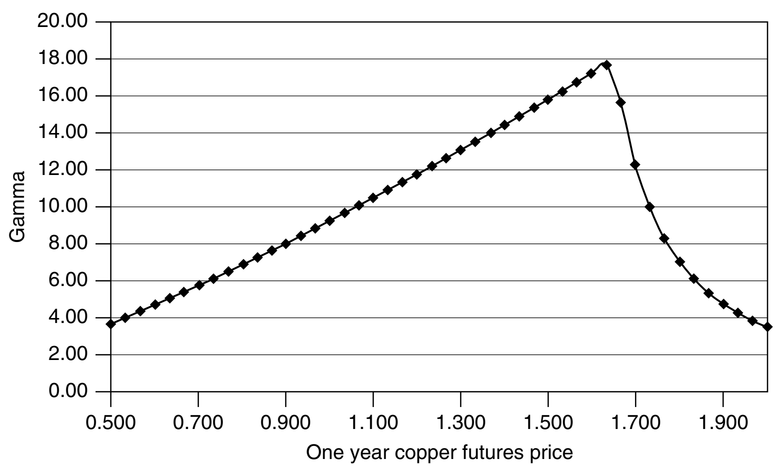

Figure 9.5 *Gamma of exploration project*

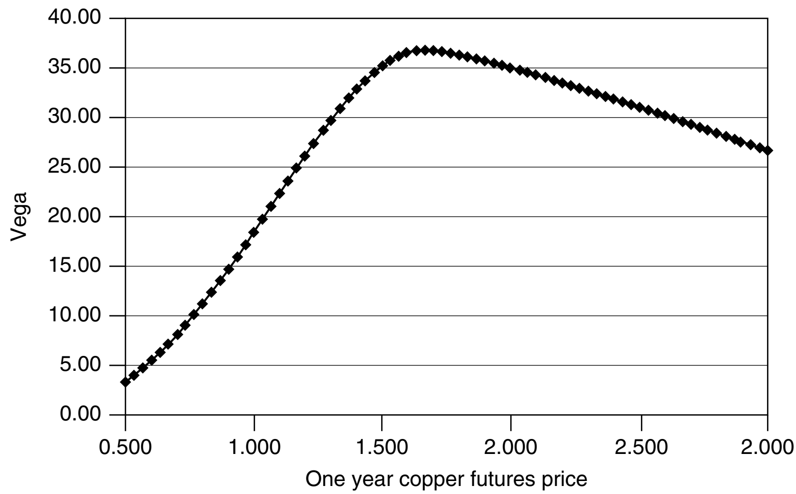

Figure 9.6 *Vega of exploration project*

or equivalently the gamma, reaches a maximum value at the exercise value of the relevant option, which in this case is the critical price for investment at the beginning of the exploration stage. Figure 9.6 presents the 'vega' of the exploration project, that is the sensitivity of the project to changes in volatility. Consistent with the idea that the value of the exploration project is a (call) option on the underlying commodity, we find that the deltas, gammas and vegas of the project are similar to those observed for simple call options.

9.4 CONCLUSIONS

We have presented a real options model for valuing natural resource exploration investments when there is joint price and geological–technical uncertainty. By collapsing both sources of uncertainty, price and geological–technical uncertainty, into a one-factor model for expected value we are able to maintain model simplicity, while retaining operational flexibility.

The model considers that the exploration investment schedule may be stopped and/or resumed at any moment depending on cash flow expectations, which depend on current commodity price and geological–technical expectations. Once all exploration phases are concluded the project is modeled as having the flexibility of postponing development investments and, once developed, as having the option to close or reopen production.

Results for a copper exploration prospect show that a significant fraction of total project value is due to the operative, the development and the exploration options available to project managers.

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NOTES

1. Actually the model only requires that price risk may be hedged. This is easily satisfied when there is a futures market, but also if there is a portfolio of traded assets perfectly correlated to commodity spot prices.
2. To understand why the process has zero drift, suppose for simplicity that G represents only the risk associated with the total reserves. If we think that the reserves will increase at a certain rate during the exploration stage, then our estimation of the initial G should be updated such that it reflects the reserves that we are expecting by the end of the exploration stage, keeping G as a martingale.
3. For example if all reserves can instantaneously be extracted, then $A_1 = 1$.
4. Although in this specification it is straightforward to allow for extra cash flows like taxes, we have not included them for simplicity.
5. Recall that $\sigma_Z > \sigma_S$.
6. Since the probability that no Poisson event occurs in interval $(0,T^j)$ (i.e. success of exploration stage j) is $e^{-\gamma^j T^j}$, it is easy to see that γ^j should be $-\ln(p^j)/T^j$.
7. This is a common characteristic of most real options models (see for example Brennan and Schwartz, 1985).

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Chapter 10

Investments in technological innovations under incomplete information

MONDHER BELLALAH

SUMMARY

An important question in financial economics is how frictions affect equilibrium in capital and real markets since in a world of costly information, some investors will have incomplete information. The specific features of financial and real markets often require an investment in information. Hence, the investment in technological innovations may require gathering information before deciding on the appropriate technology. Technological innovations may be stochastic in their arrival times and their profitability. In this context, recognizing the mechanism of learning by doing and the role of information gathering may explain actual firm policies when adopting new technologies. This chapter extends some results in Grenadier and Weiss (1997) by accounting for information costs. Information costs can be defined in the context of Merton's (1987) model of capital market equilibrium with incomplete information. We incorporate the most important characteristics of real-world technology markets as well as information costs to derive firm policies. Our formulas are simulated in different contexts for several technological parameters.

10.1 INTRODUCTION

Several models in financial economics have been proposed to deal with the ability to delay an investment expenditure. In general, the behavior of firms toward the adoption of innovations is variable. Some firms adopt new technologies when they are first available. Other firms delay the adoption until the technology is proved. Several authors analyze the factors that drive the differences in behavior. For a survey of this literature, the reader can refer to Pindyck (1991), Grenadier and Weiss (1997) and references cited therein. The innovation investment strategy can hence be viewed as a link in a chain of future investment options. Grenadier and Weiss (1997), hereafter 'G-W', identify four potential strategies.[1] The analysis in G-W shows that firms may choose to adopt an initial innovation even when facing more valuable innovations in the future. Using the path-dependent property, the model shows also that optimal migration strategies differ according to the previous histories of technology adoption by the firms. The model leads to interesting results regarding the adoption of different technologies in relation to the volatility of the technological innovations. In the conclusion of their paper, they suggest several extensions of their model, including increases in R&D by suppliers of new technology, which might be interpreted as information costs.

The shadow costs of incomplete information are considered in the analysis of Merton (1987, 1998), Orosel (1997), Shleifer and Vishny (1997), Basak and Cuoco (1998) among others. Information costs correspond to the costs required for gathering and processing data regarding the existing and future investment opportunities. These costs also include the costs of transmitting information from one party to another. If, for each technology or innovation, investors must pay a significant 'set-up' cost before they can process information released from time to time about the innovation, then this fixed cost will cause any one investor to follow only a subset of the available products. Even if the firm is not the only source of information available (the existence of professionals of information and a market place), the same argument used for the firm can also be applied to explain the costs incurred in making investors aware of these other sources of information.[2]

Agents can spend time and resources to gather information about the markets. For example, they may read newspapers, participate in seminars, subscribe to newsletters, join investment clubs, etc. Information in financial economics can be viewed as a commodity purchased in the market or produced in the household using both time and money as inputs. By introducing these information costs, it is possible to extend the G-W model. In order to apply the results about information costs, we recall that Merton's (1987) model is a modified capital asset pricing model where each investor can participate only in markets contained in an exogenous, investor-specific subset of all asset markets. Since the process of acquisition and dissemination of information is a central element in the investment process, Merton's (1987) model might provide some insights into the behavior of firms when they support these sunk information costs. Merton's model can be applied in the investment decisions and in the derivation of equilibrium option prices.

Merton (1987) adopts most of the assumptions of the original CAPM and relaxes the assumption of equal information across investors. Besides, he assumes that investors hold only securities of which they are aware. This assumption is motivated by the observation that portfolios held by actual investors include only a small fraction of all available traded securities.[3] Bellalah (1990) provides a valuation formula for commodity options in a context of incomplete information.

This chapter extends the G-W model to account for information costs. In Section 10.2, we justify information costs and their use in the valuation of real options. In Section 10.3, we present the model and derive the optimal migration strategy under incomplete information. In Section 10.4, we provide an explicit expression for the likelihood that a firm will choose a migration strategy. This allows us to study the impact of the speed of innovation arrival, the uncertainty of technological progress and the expected benefits of pending innovations. In Section 10.5, we give simulation results for the value of the option to upgrade. We simulate also the value of the option to purchase the current innovation (prior to adopting the current innovation) and before a future innovation arrives. Section 10.6 concludes.

10.2 ON THE FOUNDATIONS OF INFORMATION COSTS IN THE PRICING OF ASSETS AND DERIVATIVES

We introduce information costs in the pricing process of securities as a fundamental assumption of our model. Information plays a central role in the pricing of financial assets. Merton (1998) and Perold (1992) show that the cost of implementing financial strategies for institutions using derivatives can be one-tenth to one-twentieth of the cost of executing them in the underlying cash market securities. Bellalah and Riva (2002) show that information costs for very liquid assets are between 1 and 5%. Their empirical analysis reveals the importance of information costs for less 'liquid assets'. Differences in information are important in financial and real markets. They are used in several contexts to explain some puzzling phenomena like the 'home equity bias', the 'weekend effect' and 'the smile effect'.[4] Edwards and Wagner (1999) study the role of information in capturing the research advantage and show how to incorporate trading information into the decision process of active investment management. Information reduces uncertainty, but good execution is essential to investment success. They show that implementation costs make sense only when weighted against the benefit of enhanced performance.[5]

Edwards and Wagner (1999) show that managers must measure and develop confidence in the value of their research and then incorporate feedback from the market. The main distinction between Merton's model and the standard CAPM is that investors invest only in the securities about which they are 'aware'. This assumption is referred to as incomplete information. However, the more general implication is that markets are segmented.[6]

Using this assumption, Merton (1987) shows that the expected returns depend on other factors in addition to market risk. The main intuition behind this result is that the absence of a firm-specific risk component in the CAPM comes about because such risk can be eliminated (through diversification) and is not priced.[7] The introduction of information costs in our analysis is done with respect to the findings in Merton (1987). Merton (1987) advanced the investor recognition hypothesis (IRH) in a mean–variance model. This assumption explains the portfolio formation of informationally constrained investors. The IRH in Merton's context states that investors buy and hold only those securities about which they have enough information. Let us review the main results in Merton's model.[8] The key behavioral assumption of the model is that an investor or a firm considers including a technology X in his portfolio only if he has some information on this opportunity.[9]

Merton's model is an extension of the CAPM to a context of incomplete information. The model gives a general method for discounting future cash flows under uncertainty. In this model, assets with higher idiosyncratic risk are rationally priced to earn a higher expected return. It appears in this model that taking into account the effect of incomplete information on the equilibrium price of an asset or an investment opportunity is similar to applying an additional discount rate to its future cash flows. The implications of this model are studied in different contexts by several authors. In this setting, Kadlec and McConnell (1994) study the effect of market segmentation and illiquidity on asset prices.

Increasing empirical support for IRH-consistent behavior appeared in Falkenstein (1996), Huberman (1998) and Shapiro (2000) among others. Coval and Moskowitz (1999) document the economic significance of geography and attempt to uncover the effect of distance on portfolio choice.[10] The evidence presented in Coval and Moskowitz (1999) suggests that because local investors have more accurate estimates of future earnings prospects, they may expose themselves more to earnings risk factors. This means that investors are willing to place larger and riskier bets on firms they know more about.

10.3 THE MODEL

Consider a firm that faces several opportunities when investing in technology innovations. The investment decision concerns actual, current innovations or future innovations. A path dependency is invoked since the firm's decision to adopt a future technology is contingent upon its earlier decision regarding the current technology. The value of the current innovation at time 0 is denoted P_0 and its adoption cost (early adoption) is C_e. Hence, the payoff to early adoption is $(P_0 - C_e)$ with $P_0 \geq C_e$. The next generation of technology arrives at a random time T. If the firm has adopted the previous innovation, then it can either conserve it or adopt the new innovation worth P_T. If the firm decides to upgrade at a cost C_u, then the benefit of exchanging the current technology for the new one is $(P_T - P_0 - C_u)$.

Suppose that the firm bypassed the first innovation. In this context, it has two options: leapfrog to the new at a cost C_ℓ or acquire the older technology at a price C_d with $C_d \leq C_e$ and $C_\ell < C_e + C_u$. As in G-W, let us denote by $X(t)$ the process regarding the state of the technological progress. If the process rises to an upper boundary X_h, the future innovation arrives where the random arrival time T is characterized as follows: $T = \inf[t \geq 0 : X(t) \geq X_h]$. Let the dynamics of the project or the technology $X(t)$ obey the following process:

$$dX/X = (\alpha + \lambda_x)\,dt + \sigma\,dz \tag{10.1}$$

where α and σ refer to the instantaneous expected change and the standard deviation of the project per unit time, and dz is a geometric Brownian motion. The term λ_x corresponds to the additional return required in compensation for the costs incurred in the process of gathering information about the project or the technology X. Hence, markets with high levels of the growth term α will be characterized by faster innovation arrival and more information costs regarding the collection and gathering of data.

The random arrival time T is defined as in the literature modeling bond default. It can be shown as in Harrison (1985, equation 1.11) that the cumulative distribution function of the arrival time T is:

$$\begin{aligned}\Pr[T \leq t] = {} & N[-(\ln(X_h/X) + (\alpha + \lambda_x - \sigma^2/2)t)/\sigma\sqrt{t}] \\ & + (X_h/X)^{(2/\sigma^2(\alpha+\lambda_x-\sigma^2/2))} N[-(\ln(X_h/X) \\ & - (\alpha + \lambda_x - \sigma^2/2)t)/\sigma\sqrt{t}]\end{aligned}$$

In this expression $N(\cdot)$ corresponds to the cumulative standard normal distribution and X denotes the current value of the arrival state variable. When $\alpha + \lambda_x - \sigma^2/2 > 0$, then the expected arrival time $E(T)$ exists and is equal to $\ln(X_h/X)/(\alpha + \lambda_x - \sigma^2/2)$. Let us define the value of the future innovation P_T as $(P_0 + \varepsilon)$ with $\varepsilon \sim N(\mu, \nu^2)$ where ε is normally distributed with mean μ and variance ν^2 and ε and dz are independent. To incorporate learning, it is assumed that the cost $C_u < C_\ell$. It is important to note that in our dynamics of X, we have increased the term α by λ_x to account for the additional expected return required from information gathering. This means also that a firm, which has previously adopted the current innovation, can sooner utilize the future innovation.

Now, to derive the optimal innovation strategy, we follow the approach in G-W by working in a dynamic programming fashion. Assuming that the firm has adopted the current innovation at a cost C_e and receives P_0, it holds the option to upgrade or to convert from P_0 to P_T at a cost C_u at the random time T. The firm adopts this strategy if $P_T \geq P_0 + C_u$. This option is similar to the exchange option studied in Margrabe (1978), with the main difference that T is stochastic and is an American perpetuity.

Let us denote by $F(X)$ the option to upgrade from the current to the future innovation. Applying Itô's lemma gives:

$$\mathrm{d}F = \tfrac{1}{2} F_{xx}\sigma^2 X^2 \,\mathrm{d}t + (\alpha + \lambda_x) F_x X \,\mathrm{d}t + F_x X \sigma \,\mathrm{d}z \tag{10.2}$$

The total expected return on F per unit time is:

$$\alpha_F \equiv E[\mathrm{d}F/F](1/\mathrm{d}t) = [\tfrac{1}{2} F_{xx}\sigma^2 X^2 + (\alpha + \lambda_x) F_x X](1/F) \tag{10.3}$$

Since the expected return on the upgrade option must be equal to the equilibrium expected return for that asset in the presence of information costs, $(r + \lambda_f)$, we have:

$$\tfrac{1}{2} F_{xx}\sigma^2 X^2 + (\alpha + \lambda_x) F_x X - (r + \lambda_f) F = 0 \tag{10.4}$$

where r is the riskless interest rate.

As shown in Bellalah (1990), there are two terms corresponding to information costs in this equation. The first shadow cost is λ_x. It is linked to the costs incurred regarding X. The second is λ_f and is linked to the costs incurred regarding F. Note that each variable X and F is multiplied by its appropriate cost. This equation can also be derived as follows. Consider the return on the following portfolio: hold an option which is worth $F(X)$ and go short F_x units of the project where the subscript x refers to the partial derivative with respect to X. The value of this portfolio is:

$$F - F_x X \tag{10.5}$$

Over a short interval, the change in the value of X induces changes in the value of F_x and in the portfolio's value. The total return for this portfolio over a short interval of time $\mathrm{d}t$ is:

$$\mathrm{d}F - F_x \,\mathrm{d}X \,\mathrm{d}t \tag{10.6}$$

To avoid riskless arbitrage, the return on this portfolio must be the riskless rate. However, since there are information costs for the option and its underlying assets, the return must be equal to $(r + \lambda_x)$ for the project and $(r + \lambda_f)$ for the option, where λ_x and λ_f refer respectively to the information costs on the project and the option. Since the project may not have the same value for all firms, this information cost can be specific to each firm. Therefore, the costs of gathering information and data about the project and the investment opportunity are present in the discounting procedure. In this context, we have:

$$\mathrm{d}F - F_x\,\mathrm{d}X = (r + \lambda_f)F\,\mathrm{d}t + (r + \lambda_x)XF_x\,\mathrm{d}t \tag{10.7}$$

Assuming a hedged position is constructed and 'continuously' rebalanced, and since $\mathrm{d}F$ is a continuous and differentiable function, it is possible to use a Taylor series expansion to expand $\mathrm{d}F(X)$. When limiting arguments are used and second-order terms ignored, we obtain $\mathrm{d}F = \frac{1}{2}F_{xx}(\mathrm{d}X)^2 + F_x\,\mathrm{d}X$. This is just an extension of simple results to get Ito's lemma. The application of this lemma gives $\mathrm{d}F = \frac{1}{2}F_{xx}\sigma^2X^2\,\mathrm{d}t + F_x\,\mathrm{d}X$:

$$\mathrm{d}F = \tfrac{1}{2}F_{xx}\sigma^2X^2\,\mathrm{d}t + (\alpha + \lambda_x)F_xX\,\mathrm{d}t + F_xX\sigma\,\mathrm{d}z \tag{10.8}$$

Substituting this result, we get after simplification:

$$\tfrac{1}{2}\sigma^2X^2F_{xx} + (\alpha + \lambda_x)XF_x - (r + \lambda_f)F = 0 \tag{10.9}$$

which must be solved under the following conditions:

$$F(0) = 0 \tag{10.10}$$

$$F(X_\mathrm{h}) = E[\max(P_T - P_0 - C_\mathrm{u}, 0)] \tag{10.11}$$

The first condition indicates that the option is worthless when $X(t)$ falls to zero. The second condition gives the expected payoff of the upgrade option at the instant when the new technology arrives. The expectations are taken with respect to the distribution of P_T. The solution $F(X)$ can be given in a closed form as:

$$\begin{aligned} F(X) &= (X/X_\mathrm{h})^\beta[\nu \times n((C_\mathrm{u} - \mu)/\nu) \\ &\qquad + (\mu - C_\mathrm{u})N((\mu - C_\mathrm{u})/\nu)] \quad \text{if } X < X_\mathrm{h} \\ F(X) &= \nu \times n((C_\mathrm{u} - \mu)/\nu) \\ &\qquad + (\mu - C_\mathrm{u})N((\mu - C_\mathrm{u})/\nu) \quad \text{if } X \geq X_\mathrm{h} \end{aligned} \tag{10.12}$$

where:

$$\beta = (1/\sigma^2)[-(\alpha + \lambda_x - \sigma^2/2) + \sqrt{(\alpha + \lambda_x - \sigma^2/2)^2 + 2(r + \lambda_f)\sigma^2}]$$

In this expression $n(\cdot)$ stands for the standard normal density function, $N(\cdot)$ is its cumulative distribution function, and $X_{\mathrm{h}} = X \exp\{(\alpha + \lambda_x - \sigma^2/2)E[T]\}$. For the sake of convergence, it is assumed in this analysis that $r > \alpha$.

The partial derivative of $F(X)$ (if $X < X_{\mathrm{h}}$) with respect to λ_x is:

$$\frac{\partial F}{\partial \lambda_x} = \left(\frac{1}{\sigma^2}\right)\left\{\left[\left(\frac{X}{X_{\mathrm{h}}}\right)^{\frac{-d+\sqrt{d^2+2(r+\lambda_f)\sigma^2}}{\sigma^2}}\right]\left[\frac{d}{\sqrt{d^2+2(r+\lambda_f)\sigma^2}} - 1\right]\right.$$
$$\left. \times \ln\left(\frac{X}{X_{\mathrm{h}}}\right)\right\}\left[\nu \times n\left(\frac{C_{\mathrm{u}} - \mu}{\nu}\right) + (\mu - C_{\mathrm{u}})N\left(\frac{\mu - C_{\mathrm{u}}}{\nu}\right)\right]$$

where $d = \alpha + \lambda_x - \sigma^2/2$.

Now, we can move back a step to study the optimal investment strategy for the current innovation. When the firm acquires the current innovation, it has the current payoff from adoption $(P_0 - C_{\mathrm{e}})$ as well as an embedded option to upgrade. In this context, the firm holds a compound option when choosing its optimal exercise policy.

Let us denote by $G(X)$ the value of the option to purchase the current innovation. The optimal exercise and the optimal time for the investment in the current technology imply that the variable $X(t)$ falls to a lower trigger level X_ℓ which is chosen so as to maximize the value of $G(X)$. At this level, the benefits of investing correspond exactly to the benefits of waiting. The value of this option must satisfy the following system:

$$\tfrac{1}{2}\sigma^2 X^2 G_{xx} + (\alpha + \lambda_x)XG_x - (r + \lambda_g)G = 0 \tag{10.13}$$

where λ_g corresponds to the information costs on the option to purchase. The following boundary conditions are used:

$$G(X_\ell) = P_0 - C_{\mathrm{e}} + F(X_\ell) \tag{10.14}$$

$$G_x(x_\ell) = F_x(X_\ell) \tag{10.15}$$

$$G(X_{\mathrm{h}}) = E[\max(P_T - C_\ell, P_0 - C_{\mathrm{d}})] \tag{10.16}$$

The first condition indicates that the option payoff upon exercise is given by the net benefits of investment plus the option to upgrade. It is a familiar value-matching condition. The second condition corresponding to the classic smooth-pasting ensures optimality of the exercise at X_ℓ. The last condition indicates the expected payoff if the option is not exercised when the future technology arrives. The solution to the above system is:

$$\begin{array}{ll} G(X) = P_0 - C_e + F(X_\ell) & \text{if } X \leq X_\ell \\ G(X) = A_1 X^{-\beta_1} + A_2 X^{\beta_2} & \text{if } X_\ell < X < X_h \\ G(X) = K_2 & \text{if } X \geq X_h \end{array} \tag{10.17}$$

where:

$$X_\ell = [(\beta_1/(\beta_1 + \beta_2))((P_0 - C_e)/\omega)]^{(1/\beta_2)}$$

$$A_1 = c_1 \omega^{(\beta_1/\beta_2)}$$

$$A_2 = \omega + K_1 X_h^{-\beta_2}$$

$$K_1 = \nu \times n((C_u - \mu)/\nu) + (\mu - C_u) N((\mu - C_u)/\nu)$$

$$K_2 = \nu \times n((C_\ell - C_d - \mu)/\nu) + (C_\ell - C_d - \mu) N((C_\ell - C_d - \mu)/\nu) + P_0 - C_\ell + \mu$$

$$\beta_1 = (1/\sigma^2)[(\alpha + \lambda_x - \sigma^2/2) + (\alpha + \lambda_x - \sigma^2/2)^2 + 2(r + \lambda_g)\sigma^2]^{1/2} > 0$$

$$\beta_2 = (1/\sigma^2)[-(\alpha + \lambda_x - \sigma^2/2) + (\alpha + \lambda_x - \sigma^2/2)^2 + 2(r + \lambda_g)\sigma^2]^{1/2} > 1$$

$$c_1 = (\beta_2/\beta_1)[(\beta_1/(\beta_1 + \beta_2))(P_0 - C_e)]^{((\beta_1 + \beta_2)/\beta_1)}$$

and ω is the solution to the following equation:

$$c_1 X_h^{-\beta_1} \omega^{(-\beta_1/\beta_2)} + \omega X_h^{\beta_2} = K_2 - K_1 \tag{10.18}$$

In this context, the optimal migration strategy is similar to that reported in the context of complete information. However, the values of the embedded options are modified with respect to the information costs. In fact, the critical exercise level X_ℓ and the option value are modified according to the levels adopted for information costs.

10.4 LIKELIHOOD OF MIGRATION STRATEGIES AND INNOVATION ADOPTION BEHAVIOR IN DIFFERENT ENVIRONMENTS

The above model can be applied to predict the innovation adoption choices of a firm in the future. In fact, when the optimal migration strategy is known, it is possible to derive explicit expressions for the probability that a firm will pursue any of the possible four migration strategies. In the same vein, it is possible to obtain the expected time at which a firm first invests in an innovation. In this setting, the analysis is very similar to that in G-W.

Let us define the first passage of time $X(t)$ to the early adoption trigger by:

$$T_{\mathrm{e}} = \inf[t \geq 0 : X(t) \leq X_{\ell}] \tag{10.19}$$

We denote the first passage of time to the value that triggers the arrival of the future innovation by:

$$T = \inf[t \geq 0 : X(t) \geq X_{\mathrm{h}}] \tag{10.20}$$

The definitions of T_{e} and T in equations (10.19) and (10.20) refer also to the stopping time. In this context, the probability of early adoption is $\Pr[T_{\mathrm{e}} < T]$, which means that the firm will adopt the current innovation only if $T_e < T$. Following this line of reasoning, it is possible to provide an explicit derivation of the probabilities that a firm will fall into any one of the four possible categories. It turns out that these probabilities depend on the shadow costs of incomplete information. This is because uncertainty reflected in the volatility is outweighed or reduced with respect to the available information.

10.4.1 Case 1: a firm pursuing a compulsive strategy

Consider the first case of a firm pursuing a compulsive strategy and denote by $\mathrm{PC}(X)$ the probability of this strategy, conditional upon $X(0) = X$. A firm pursues this strategy when it adopts early ($T_{\mathrm{e}} < T$), and exercises the upgrade option at time $T (P_T - P_0 - C_{\mathrm{u}} \geq 0)$. For this case:

$$\mathrm{PC}(X) \equiv \Pr[T_{\mathrm{e}} < T, P_T - P_0 - C_{\mathrm{u}} \geq 0] \tag{10.21}$$

Using a change of variables and the analysis in Harrison (1985, section 3.2), we can show as in G-W that:

$$\mathrm{PC}(X) = H(X)[1 - N((C_{\mathrm{u}} - \mu)/\nu)] \tag{10.22}$$

where:

$$\begin{aligned} &H(X) = 1 && \text{if } X \leq X_{\ell} \\ &H(X) = (X^{-\gamma} - X_{\mathrm{h}}^{-\gamma})/(X_{\ell}^{-\gamma} - X_{\mathrm{h}}^{-\gamma}) && \text{if } X_{\ell} < X < X_{\mathrm{h}} \\ &H(X) = 0 && \text{if } X \geq X_{\mathrm{h}} \end{aligned}$$

and[11]

$$\gamma = (2/\sigma^2)(\alpha + \lambda_x - \sigma^2/2)$$

When there is no information uncertainty, the term λ_x is zero and the expressions of the probabilities collapse to those in G-W.

10.4.2 Case 2: a firm pursuing a buy-and-hold strategy

Consider the second case of a firm pursuing a buy-and-hold strategy and denote by $\mathrm{PB}(X)$ the probability of this strategy, conditional upon $X(0) = X$. A firm pursues this strategy when it adopts early and does not exercise the upgrade option at time $T(P_T - P_0 - C_u < 0)$. For this case:

$$\mathrm{PB}(X) \equiv \Pr[T_e < T, P_T - P_0 - C_u < 0] \tag{10.23}$$

or

$$\mathrm{PB}(X) = H(X)N((C_u - \mu)/\nu) \tag{10.24}$$

10.4.3 Case 3: a firm pursuing a leapfrog strategy

Consider the third case of a firm pursuing a leapfrog strategy and denote by $\mathrm{PL}(X)$ the probability of this strategy, conditional upon $X(0) = X$. A firm pursues this strategy when it does not adopt early ($T_e \geq T$) and adopts the future innovation when it arrives ($P_T - C_\ell \geq P_0 - C_d$). For this case:

$$\mathrm{PL}(X) \equiv \Pr[T_e \geq T, P_T - C_\ell \geq P_0 - C_d] \tag{10.25}$$

or

$$\mathrm{PL}(X) = [1 - H(X)][1 - N((C_\ell - C_d - \mu)/\nu)] \tag{10.26}$$

10.4.4 Case 4: a firm pursuing a laggard strategy

Consider the fourth case of a firm pursuing a laggard strategy and denote by $\mathrm{PG}(X)$ the probability of this strategy, conditional upon $X(0) = X$. A firm pursues this strategy when it does not adopt early ($T_e \geq T$) and adopts the older innovation when the future innovation arrives rather than leapfrogging ($P_T - C_\ell < P_0 - C_d$). For this case:

$$\mathrm{PG}(X) \equiv \Pr[T_e \geq T, P_T - C_\ell < P_0 - C_d] \tag{10.27}$$

or

$$\mathrm{PG}(X) = [1 - H(X)]N((C_\ell - C_d - \mu)/\nu) \tag{10.28}$$

The model can be applied to characterize the speed at which a firm invests in technology. In this case, the definitions of stopping times in equations (10.19) and (10.20) show that the time it takes a firm to adopt an innovation is $\min[T_e, T]$. In the same vein, the expected time of initial adoption corresponds to:

$$E[\min[T_e, T]|X(0) = X] \tag{10.29}$$

This expression can be computed as in Harrison (1985, section 3.2) and G-W to obtain the following expected time of future adoption.

When $(\alpha + \lambda_x \neq \sigma^2/2)$:

$$\begin{aligned} &E[\min[T_e, T]|X(0) = X] \\ &\quad = [\ln(X_h/X_\ell)/(\alpha + \lambda_x - \sigma^2/2)][(X_\ell^{-\gamma} - X^{-\gamma})/(X_\ell^{-\gamma} - X_h^{-\gamma})] \\ &\qquad - [\ln(X/X_\ell)/(\alpha + \lambda_x - \sigma^2/2)] \qquad \text{if } X \in (X_\ell, X_h) \\ &E[\min[T_e, T]|X(0) = X] = 0 \qquad \text{otherwise} \end{aligned} \tag{10.30}$$

When $(\alpha + \lambda_x = \sigma^2/2)$:

$$E[\min[T_e, T]|X(0) = X] = [\ln(X_h/X)\ln(X/X_\ell)]/\sigma^2 \qquad \text{if } X \in (X_\ell, X_h)$$

$$E[\min[T_e, T]|X(0) = X] = 0 \qquad \text{otherwise}$$

It is important to note that the term corresponding to the shadow cost of incomplete information plays a central role in the above expressions regarding the expected time of initial adoption. In fact, since it appears often in the denominator, the values attributed to this parameter can affect substantially the expected time of future innovation. Information cost is the counterpart of the volatility as it appears in the condition $(\alpha + \lambda_x = \sigma^2/2)$ determining the expected time of future adoption. This analysis allows one to study the impact of the speed of innovation arrival, the uncertainty of technological progress, and the expected benefits of pending innovations in the presence of information costs.

Consider the impact of innovation arrival. High technology environments are often regarded as having a rapid pace of innovation, while more mature technologies are characterized by minor cosmetic changes. The main question is how do firms' optimal adoption behaviors differ according to the speed of innovation arrival? In our analysis, the model reveals that for markets with rapid innovations, firms will probably postpone their innovation investments until the future innovation arrives. Hence, the leapfrog and laggard strategies are most probable in markets with rapid innovation arrival. The compulsive and buy-and-hold strategies are probable in markets with slow innovations. A market with rapid (slow) innovation is characterized by low (high) $E(T)$.

The solutions (10.22), (10.24), (10.26) and (10.28) can be represented as a function of the expected arrival time $E(T)$. In the analysis of G-W, $E(T)$ varies from zero to five years by increasing the arrival trigger parameter X_{h}. This assumption is consistent with many technology-based industries. The resulting curves can move upside and downside as a function of the shadow cost of incomplete information. The solution depends on all the parameters used.

The significance of the improvements characterizes also technological environments. The model allows one to increase the expected profitability of the future innovation, which increases the likelihood that the second innovation will be adopted. The probabilities of the various strategies can be represented as a function of the expected profitability of the future innovation μ. By increasing μ, the compulsive and leapfrog strategies increase. In the same context, when the future innovation becomes very profitable, the laggard and buy-and-hold strategies become less likely. Again, all the above results and implications of the model are a function of the shadow cost of incomplete information.

Another characteristic of technological markets concerns the level of uncertainty surrounding future innovations. The timing and impact of future innovations are difficult to forecast. In general, results in G-W reveal that volatility can retard the adoption of a current innovation, prompting firms to postpone the investment decisions into the future. The increased willingness of managers to delay the investment decision is a known result in real options pricing models. The value of the option to wait increases in more volatile environments. In this case, firms become more reluctant to follow compulsive or buy-and-hold strategies. Similar results can be derived in the presence of shadow costs of incomplete information.

10.5 SENSITIVITY ANALYSIS AND SIMULATION RESULTS

We use plausible default parameter values provided in the study of G-W, even though they do not simulate option values. We consider plausible values of information costs for the variable X, the option to upgrade and the option to purchase the current innovation. The simulation results for the first formula (10.12) are slightly easier than those of the second formula (10.17). In fact, simple numerical techniques must be used in the second case for the computation of critical levels.

For the simulations of formula (10.12), the values of the arrival trigger parameter X_{h} are set in accordance with the various levels of X given the expected arrival time of the future innovation. When X reaches X_{h}, the holder of the option has the possibility to upgrade to the realized value of the new technology P_{T}.

Table 10.1 reports the main results regarding the optimal migration strategy. The simulations assume that the firm has adopted the current innovation and

Table 10.1 *Simulation of the value of the option to upgrade $F(X)$ from the current to the future innovation as a function of current state of technological progress X using equation (10.12): the effect of information costs relative to the technological progress on the option value.* $\alpha = 0.05$, $\sigma = 0.05$, $r = 0.07$, $P_0 = 1$, $\mu = 1$, $\nu = 1$, $C_u = 0.85$, $\lambda_f = 1\%$

$E(T)$	$\lambda_x = 1\%$		$\lambda_x = 2\%$		$\lambda_x = 3\%$	
	$X(\text{h}) - X$	$F(X)$	$X(\text{h}) - X$	$F(X)$	$X(\text{h}) - X$	$F(X)$
0.00	0.0000	0.4784	0.0000	0.4784	0.0000	0.4784
0.50	0.0119	0.4602	0.0140	0.4600	0.0161	0.4600
1.00	0.0363	0.4426	0.0427	0.4424	0.0492	0.4422
1.50	0.0737	0.4257	0.0869	0.4254	0.1003	0.4251
2.00	0.1247	0.4095	0.1474	0.4090	0.1706	0.4087
2.50	0.1899	0.3939	0.2250	0.3933	0.2611	0.3929
3.00	0.2698	0.3788	0.3207	0.3782	0.3731	0.3778
3.50	0.3653	0.3644	0.4353	0.3636	0.5078	0.3632
4.00	0.4768	0.3505	0.5698	0.3497	0.6665	0.3491
4.50	0.6052	0.3371	0.7251	0.3362	0.8506	0.3357
5.00	0.7512	0.3242	0.9025	0.3233	1.0616	0.3227
5.50	0.9155	0.3119	1.1029	0.3109	1.3010	0.3102
6.00	1.0988	0.3000	1.3275	0.2989	1.5704	0.2983
6.50	1.3021	0.2885	1.5776	0.2875	1.8716	0.2868
7.00	1.5261	0.2775	1.8543	0.2764	2.2063	0.2757

In this table, α is the instantaneous conditional expected percentage change in X, σ is the instantaneous conditional standard deviation per unit time, μ is the mean of the incremental improvement, and C_u the upgrade cost. The values of the arrival trigger parameter X_h are set in accordance with the various levels of X given the expected arrival time $E(T)$ of the future innovation. When X reaches X_h the holder of the option has the possibility to upgrade to the realized value of the new technology P_T.

holds the option to upgrade to the future innovation. The firm also has the valuable option to convert from the current innovation P_0 to the future innovation P_T at an upgrade cost C_u at a random arrival time T. Information costs λ_x take the values of 1%, 2% and 3%. The table shows that higher information costs lead to higher trigger levels X_h relative to X, for the same $E(T)$. It indicates that higher information costs lead also to lower values of the option to upgrade. Hence, the option to upgrade is less valuable when information is 'rare' and investors suffer more sunk costs in collecting and analyzing the market environment.

For the simulations of formula (10.17), the values of the arrival trigger parameter X_h are set in accordance with the various levels of X given the expected arrival time of the future innovation. The optimal time at which the firm invests in the current innovation is when $X(t)$ falls to the trigger value X_ℓ. This value maximizes the option value $G(X)$. At the optimal exercise trigger level X_ℓ, the benefits of investing in the current innovation are equal to the marginal benefits of waiting.

We distinguish two cases. The first uses information costs relative to the variable X. The second uses information costs regarding the option itself $G(X)$. Table 10.2 reports the main results regarding the optimal technological migration strategy in the presence of information costs related to the variable X. The table gives simulation results for the value of the option to purchase the current innovation $G(X)$ using equation (10.17). It studies the effect of information costs relative to the technological progress. Since the firm has an embedded option to upgrade, it holds a compound option and must choose an optimal exercise policy.

The simulations give the values of X_ℓ, X_h and the value of the option $G(X)$. The value of ω is found using an iterative procedure. Information costs λ_x take the values of 0% to 6%. The table shows that higher information costs lead to lower trigger levels of X_ℓ. However, higher information costs lead to higher levels of X_h. The value of the critical ω is a slightly decreasing function of information costs.

The value of the option $G(X)$ to purchase the current innovation (prior to adopting the current innovation and before the future innovation arrives) is also a slightly decreasing function of information costs. Hence, the value of this option is less valuable when information is 'rare' and the firm suffers more sunk costs in collecting and analyzing market conditions.

Table 10.3 gives simulation results for the value of the option $G(X)$ using equation (10.17). It shows the effect of option information costs λ_g relative

Table 10.2 *Simulation of the value of the optimal technological migration strategy and the value of the option to purchase the current innovation G(X): the effect of information costs relative to the technological progress.* $\alpha = 0.05$, $\sigma = 0.05$, $r = 0.07$, $P_0 = 1$, $\mu = 1$, $\nu = 1$, $C_\mathrm{e} = 0.825$, $C_\mathrm{d} = 0.8$, $C_\ell = 1.65$, $C_\mathrm{u} = 0.85$, $\lambda_\mathrm{g} = 0\%$, $E(T) = 1$

$\lambda_x(\%)$	X_ℓ	X_h	ω	$G(X)$
0	0.9308	1.0499	0.1868	0.6340
1	0.9263	1.0605	0.1867	0.6336
2	0.9208	1.0711	0.1867	0.6333
3	0.9147	1.0819	0.1866	0.6331
4	0.9082	1.0928	0.1866	0.6330
5	0.9014	1.1039	0.1866	0.6329
6	0.8944	1.1149	0.1865	0.6329

In this table, α is the instantaneous conditional expected percentage change in X, σ is the instantaneous conditional standard deviation per unit time, μ is the mean of the incremental improvement, P_0 is the value of the current innovation to the firm, C_e the cost of early adoption of the current innovation, C_u the upgrade cost, C_ℓ the cost of leapfrog and C_d the price of the older innovation. The values of the arrival trigger parameter X_h are set in accordance with the various levels of X given the expected arrival time $E(T)$ of the future innovation. The optimal time at which the firm invests in the current innovation is when $X(t)$ falls to the trigger value X_ℓ. This value maximizes the option value $G(X)$. At the optimal exercise trigger level X_ℓ, the benefits of investing in the current innovation are equal to the marginal benefits of waiting.

Table 10.3 *Simulation of the value of the optimal technological migration strategy and the value of the option to purchase the current innovation G(X): the effect of information costs relative to the option.* $\alpha = 0.05$, $\sigma = 0.05$, $r = 0.07$, $P_0 = 1$, $\mu = 1$, $\nu = 1$, $C_e = 0.825$, $C_d = 0.8$, $C_\ell = 1.65$, $C_u = 0.85$, $\lambda_x = 1\%$, $E(T) = 1$

$\lambda_g(\%)$	X_ℓ	X_h	ω	$G(X)$
1	0.9395	1.0605	0.1850	0.6276
2	0.9499	1.0605	0.1833	0.6217
3	0.9583	1.0605	0.1815	0.6159
4	0.9654	1.0605	0.1798	0.6102
5	0.9713	1.0605	0.1782	0.6045

In this table, α is the instantaneous conditional expected percentage change in X, σ is the instantaneous conditional standard deviation per unit time, μ is the mean of the incremental improvement, P_0 is the value of the current innovation to the firm, C_e the cost of early adoption of the current innovation, C_u the upgrade cost, C_ℓ the cost of leapfrog and C_d the price of the older innovation. The values of the arrival trigger parameter X_h are set in accordance with the various levels of X given the expected arrival time $E(T)$ of the future innovation.

to the option value. The simulations give the values of X_ℓ, X_h and the value of the option $G(X)$. Information costs λ_g take the values of 1% to 5%. The table shows that higher information costs regarding the option lead to higher trigger levels of X_ℓ. However, higher option information costs do not affect the level of X_h. The value of the critical ω is slightly decreasing with information costs. The value of the option $G(X)$ to purchase the current innovation is also decreasing with information costs on that option.

Finally, the existence of information costs appears to have a significant effect on the probability of migration strategies as a function of the expected arrival time. As shown in Figure 10.1 (compared to figure 1 in Grenadier and Weiss,

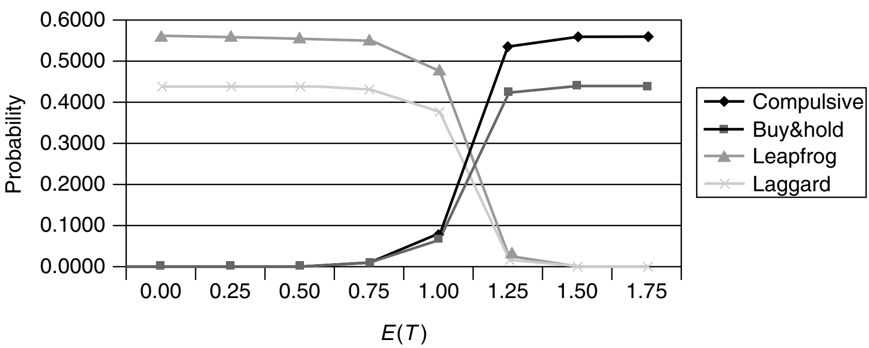

Figure 10.1 *Simulation of the effect of the speed of arrival on the probability of migration strategies, with information costs, using equations (10.22), (10.24), (10.26) and (10.28).* $\alpha = 0.05$, $\sigma = 0.05$, $r = 0.07$, $P_0 = 1$, $\mu = 1$, $\nu = 1$, $C_e = 0.825$, $C_d = 0.8$, $C_\ell = 1.65$, $C_u = 0.85$, $\lambda_x = \lambda_g = 5\%$

1997, p. 408), the probability of all strategies is 'shifted' towards extremes at earlier arrival times. Leapfrog and laggard strategies have a minimum likelihood and compulsive and buy-and-hold strategies have a maximum likelihood with earlier expected innovation arrival times. Thus with environments with slower innovation, the compulsive and buy-and-hold strategies are more likely the higher the information costs.

10.6 CONCLUSION

Information plays a central role in the pricing of financial and real assets and in the process of financial and real innovations. This appears in the work of Scholes (1998) and Merton (1998). In this spirit, the study of the analogy between the adoption of innovations and the exercise strategy of a stream of embedded options in the presence of incomplete information allows the implementation of option pricing theory to derive the firm's optimal migration strategy under technological uncertainty. We present a model which accounts for information costs and extends the main results in G-W. Our definition of information costs is similar to that in Merton (1987), who provides a simple capital market equilibrium model with incomplete information. Merton's (1987) model shows that asset returns are an increasing function of their beta risk, residual risk and size, and a decreasing function of the available information for these assets. Several empirical tests by Kadlec and McConnell (1994), Falkenstein (1996), Huberman (1999), Shapiro (2000), Bellalah and Riva (2002) among others seem to support Merton's model. The information costs correspond to the expenses in research and development which are experienced by the users and the suppliers of a new technology. The formulas derived in this chapter are simulated. The effects of changing different parameters on the options values are studied. The simulations of both analytical formulas regarding the option to upgrade and the optimal technological migration strategy reveal the effects and the role of incomplete information in the computation of option values. Our results suggest that future empirical and theoretical studies should distinguish between different technological environments as characterized by information costs and timing.

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NOTES

1. These strategies are: (i) a compulsive strategy of purchasing every innovation; (ii) a leapfrog strategy of skipping an early innovation; (iii) a buy-and-hold

strategy of purchasing only an early innovation; and (iv) a laggard strategy of waiting until a new generation of innovation arrives before purchasing the previous innovation.

2. The story of information costs applies in varying degrees to the adoption in practice of new structural models of evaluation such as real option pricing models. It applies also to the diffusion of innovations for several products and technologies. The recognition of the different speeds of information diffusion is particularly important in explaining the behavior of different firms.
3. In Merton's model, the expected returns increase with systematic risk, firm-specific risk, and relative market value. The expected returns decrease with relative size of the firm's investor base, referred to in Merton's model as the 'degree of investor recognition'.
4. See the models in Bellalah (1990).
5. They recognize that the most valuable commodity in the market is information that reduces uncertainty. In this spirit, trading cost information is part of the research that gives a manager active advantage.
6. Merton's model is based on the assumption that there are several factors in addition to incomplete information that may explain this behavior for individuals and institutions. Hence, the presence of prudent-investing laws and traditions and other regulatory constraints can rule out investment in a particular firm by some investors.
7. It appears from Merton's model that the effect of incomplete information on expected returns is greater the higher the firm's specific risk and the higher the weight of the asset in the investor's portfolio. The effect of Merton's non-market risk factors on expected returns depends on whether the asset is widely held or not.
8. Merton's model is a two-period model of capital market equilibrium in an economy where each investor (or firm) has information about only a subset of the available investment opportunities.
9. Merton's model may be stated as follows: $R_x - r = \beta_x[R_{\mathrm{m}} - r] + \lambda_x - \beta_x\lambda_{\mathrm{m}}$, where $R_x =$ the equilibrium expected return on an asset X, $R_{\mathrm{m}} =$ the equilibrium expected return on the market portfolio, $r =$ the riskless rate of interest, $\beta_x = \mathrm{cov}(R_x/R_{\mathrm{m}})/\mathrm{var}(R_{\mathrm{m}})$, $\lambda_x =$ the equilibrium aggregate 'shadow cost' for the asset X (of the same dimension as the expected rate of return on this asset X), $\lambda_{\mathrm{m}} =$ the weighted average shadow cost of incomplete information over all assets.
10. They find that local equity preference is strongly related to firm size, leverage and output tradability. Their results suggest an information-based explanation for local equity. Since a manager cannot follow all publicly traded firms, investors select specific firms for which they incur 'receiver set-up costs'. These costs are larger in small firms. In the analysis of Coval and Moskowitz (1999), proximity may lower this fixed cost, and local investors may have a larger comparative advantage informatively trading in small firms.
11. Correcting the sign given in Grenadier and Weiss (1997, p. 415).

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Chapter 11

The effect of first-mover's advantages on the strategic exercise of real options

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SUMMARY

When real investment opportunities are open to competing firms in the same line of business, strategic considerations become extremely important in determining investment/entry policies. We develop an equilibrium framework for strategic (real) option exercise where the focus is on the effect of first-mover's advantages. The generality of our framework stems from the fact that we allow such advantages to be either temporary or permanent in nature. When the latter is true, economically identical competing firms might end up investing at very distinct times simply due to the effect of first-mover's advantages. First-mover advantages are found to have an asymmetric effect on rival firms' values. If the advantages are substantial and permanent, the rival entry times are drawn further apart as uncertainty increases.

11.1 INTRODUCTION

The application of option pricing techniques to real-world investment decisions has changed the way academics and corporate practitioners think about capital budgeting and project valuation. Originating in the work of Brennan and Schwartz (1985) and McDonald and Siegel (1985, 1986), this stream of literature – collectively termed *real options* – has stressed that under conditions of uncertainty, there is an option value to waiting before making irreversible decisions. This option value of remaining uncommitted, which is not captured by traditional discounted cash flow techniques like the net present value (NPV) criterion, can be substantial in magnitude and highly important in investment appraisal decisions (see Kester, 1984).

In the case of traditional financial options, optimal exercise strategies can be derived without consideration of the strategic interactions across option holders.[1] However, this is not the case in real options: real investment opportunities are rarely held by one firm in isolation. Most projects or markets are open to more than one firm in the same industry or line of business. In such cases, where investment opportunities are not proprietary to one firm, strategic considerations become important. Optimal (real) option exercise strategies will have to strike a balance between the value of waiting for uncertainty to be resolved and the fear of pre-emption, i.e. the possibility that a rival firm with access to the project/market may seize an advantage by acting first.

In this chapter, we provide a game-theoretic approach to real options exercise in an effort to highlight the importance of strategic considerations. We focus on the *advantage of being first* in a market, and our aim is to quantify the effects of such an advantage on investment valuation and exercise policies. We allow the magnitude of *first-mover's advantages* or *degree of pre-emption* to be fully parameterized in the model, which allows us to assess their effect on entry decisions and game equilibria.

In our model, two competing firms have the option to enter a market with uncertain profitability. The option to enter is an American-style call option with an exercise price equal to the investment cost, and the underlying security is the (net) profitability derived from operating in the market. However, the exercise of the option to enter by one firm has repercussions on the value of both competing firms' options. The firm to enter first (the leader) has to sink the investment cost earlier, but can benefit from securing a higher market share than the competitor. An appealing feature of our framework is that it admits the possibility that this first-mover advantage can be *temporary* or *permanent*.

The other firm (the follower) must then decide when it is optimal to sink the cost of investment so as to claim a share of the market lower than that of the first entrant. When the follower decides to enter, the underlying game ends and the resulting market structure is a duopoly where market sharing is an explicit function of the degree of pre-emption parameter. Once the value functions of being the leader or the follower in the market are determined, we proceed in identifying and analyzing the possible equilibria for this strategic entry option game and assessing the effect of the magnitude of first-mover's advantages.

The main implication is that first-mover's advantages not only guarantee a higher market share for the leading firm in the market, but also deter optimal rival entry, thus augmenting the time period that the leader can act as a monopolist. Because of this dual role that they play, first-mover advantages, when permanent in nature, have an asymmetric effect on competing firms' value functions. Another interesting finding is that the effect of volatility on entry strategies depends positively on the magnitude of first-mover advantages. This dependence can actually reverse if such advantages are not substantial.

Our model draws on the work of Smets (1993) in the context of foreign direct investment, and the subsequent adoption of his model by Grenadier (1996) in the real-estate market. Like their work, our model admits the possibility of both simultaneous and sequential exercise equilibria depending on initial conditions and the magnitude of the first-mover's advantages. However, by explicitly accounting for the *magnitude* and *nature* (temporary or permanent) of such advantages, our model can better assess their effect on value functions, exercise strategies and game equilibria. Other related papers would include the study of capital budgeting in settings with pre-emption and learning (Spatt and Sterbenz, 1985), investment with strategic competition (Kulatilaka and Perotti, 1998), the effect of incomplete information on pre-emptive investment (Lambrecht and Perraudin, 1997) and technological uncertainty in R&D competition (Weeds, 2000).

The rest of the chapter is organized as follows. Section 11.2 presents the basic setting of the underlying game, as well as the main assumptions made. Section 11.3 derives the value functions of the leader and the follower, while Section 11.4 determines the equilibrium set of exercise strategies for both firms. Section 11.5 assesses the effect of changing parameters on exercise strategies through comparative statics, while Section 11.6 concludes.

11.2 THE BASIC MODEL

Two competing firms are contemplating entry into a new market where operating profitability is stochastic. The decision to enter the market is assumed to be completely irreversible.

Operating the market yields a revenue flow x_t, which is assumed to evolve exogenously according to a geometric Brownian motion with drift given by the following expression:

$$dx_t = \mu x_t \, dt + \sigma x_t \, dW_t \tag{11.1}$$

where $\mu \in [0, r)$ is the drift parameter, measuring the expected growth rate of x_t, $\sigma > 0$ is the instantaneous standard deviation or volatility parameter, and dW_t is the increment of a standard Wiener process, $dW_t \sim N(0, dt)$.

Note that geometric Brownian motion is a Markov process with continuous sample paths. The probability distribution for the value of the process at any future date depends only on its own current value, i.e. it is unaffected either by past values of the process or by any other current information. Thus, to make a best estimate of the future value of the process, all that is needed is its current level, along with the parameter values μ and σ.

The assumption in equation (11.1) that the state variable is lognormally distributed, even though standard in the option pricing literature, might be questionable in our context. Indeed, the fact that $\Pr(x_t > 0 | x_0 > 0) = 1$ implicit in equation (11.1) is not easily justified for the profitability of an industry or a market. Alternatively, one could assume that x_t follows an arithmetic Brownian motion, i.e.

$$dx_t = \mu\, dt + \sigma\, dW_t \tag{11.2}$$

or even a mean-reverting Ornstein–Uhlenbeck process, both of which admit the possibility of negative realizations for future values of x. The choice made will ultimately have to do with the industry under review (product market, internet or biochemical, R&D intensive, etc.) and its long-run characteristics. Subject to this critique, our exposition uses equation (11.1) simply for reasons of compatibility and comparability with the literature. All notions and findings of this chapter are qualitatively unaffected by the state variable assumption, but any quantitative implications drawn for a specific industry should be made with the above caveat in mind.

Entry by any firm entails a fixed investment cost. To avoid favoring one of the firms and to concentrate on the effect of first-mover's advantages, we impose symmetry and assume that both rivals face the same investment cost, denoted by K.

Careful readers would note that our model implicitly assumes that firms, once active, can service the market and earn revenues without incurring any more (i.e. operating) costs apart from the fixed entry cost K. The easiest way to accommodate operating costs in the model would be to interpret x_t as *net* operating profitability. Explicitly introducing operating costs would give firms the extra flexibility to temporarily cease operations and/or abandon the market in states of very low revenues, which would complicate the model without offering any additional insights on entry decisions. Abstracting from the possibility of operating costs has no effect on the conclusions of the chapter since our focus is on the effect of first-mover's advantages on strategic investment.

The underlying game is that of *Stackelberg leader–follower*: the first firm to enter the market (the leader, L) sinks the investment cost K in order to receive the monopolistic revenue flow $x_t\, dt$ for as long as it operates alone in the market. However, when the second firm enters the market (the follower, F) the order of entry determines the magnitude of duopolistic revenue flows for the two competing firms. If there is an advantage of being first in a market, the leader should have a higher market fraction than the follower, even when both firms operate. To allow that, we assume that when both firms enter the market, the leader reserves a fraction $a \in (\frac{1}{2}, 1]$ of market profitability, leaving $(1-a)x_t$ for the follower.[2] This exogenous parameter a measures the degree of pre-emption or the magnitude of first-mover's advantages: being the first to enter the market guarantees, at a minimum, ax_t of market profitability. Obviously, absolute pre-emption (first mover gets all) is included as a special case by setting $a = 1$ (see Lambrecht and Perraudin, 1997 for an absolute pre-emption duopoly game with incomplete information).

In the following sections, we derive a set of equilibrium exercise strategies. Specifically, we derive a pair of Markovian exercise strategies that form a subgame perfect equilibrium, i.e. at each point of the game, each firm's exercise strategy is optimal conditional on the rival's exercise strategy. Derivation of equilibrium strategies draws from the methodology of *stochastic stopping-time games*, in particular the results of Dutta and Rustichini (1993). Their formulation allows for the possibility that the stochastic process continues to evolve after the leader's action, and the follower still has a move to play, as is the case in our model.

Lastly, firms are assumed to use *stationary Markovian* strategies. A stationary Markovian strategy consists of actions that depend only on the current state and the strategy formulation itself does not explicitly depend on time. Because x_t follows a Markov process, Markovian strategies incorporate all payoff-relevant factors in this game. Furthermore, if one player uses a Markovian strategy then its rival has a best response which is Markovian as well. Hence a Markovian equilibrium remains an equilibrium when history-dependent strategies are also permitted, although other non-Markovian equilibria may arise.[3] With the Markovian restriction, a player's strategy is a stopping rule specifying a critical trigger value or 'threshold' of the state variable at which the firm invests.

Before turning to the derivation of the leader–follower value functions, a brief comment about the option valuation framework is in order. In traditional financial option pricing models, the approach to valuation is based on arbitrage. Namely, the fact that one can instantaneously trade the underlying asset and a riskless asset allows precise replication of the option under valuation. In the case of real assets however, where short-selling may not be possible and assets cannot be traded in infinitesimal increments, such an arbitrage technique is highly unrealistic. The alternative valuation technique used here is an equilibrium approach. For simplicity, in what follows we assume risk-neutrality. This seemingly restrictive assumption is only made for convenience and can easily be relaxed by adjusting the drift term μ in equation (11.1) to account for a risk premium in the manner of Cox and Ross (1976).

11.3 THE VALUE OF THE LEADER AND THE FOLLOWER

In this section we derive the value functions for pursuing the role of the leader and the follower in the market. For the moment, those roles are taken as exogenously assigned. In the next section, the equilibrium set of exercise strategies that determine the identities of the leader and the follower are derived.

As usual, such dynamic games are solved backwards in a dynamic programming fashion. Thus, assume – without loss of generality – that one firm has already invested in the market and first consider the optimal investment strategy of the follower.

11.3.1 The follower's value function

Denote by $V_0^{\mathrm{F}}(x)$ the value of the follower in the *continuation region* (values of the state variable x_t for which it is not yet optimal to invest). Prior to entry, the follower only holds the option to invest and claim $1 - a$ of the market. It earns no cash flows but experiences a capital gain or loss in the value of its option depending on the evolution of the market profitability. Hence, in this region the equilibrium return condition for the value of the follower is given by:

$$rV_0^{\mathrm{F}}(x)\,\mathrm{d}t = E\{\mathrm{d}V_0^{\mathrm{F}}(x)\}$$

Expanding the right-hand side of the equation using Itô's lemma and substituting from equation (11.1) yields the following ordinary differential equation that the value of the following firm must satisfy:

$$\tfrac{1}{2}\sigma^2 x^2 V_0''^{\mathrm{F}}(x) + \mu x V_0'^{\mathrm{F}}(x) - rV_0^{\mathrm{F}}(x) = 0 \tag{11.3}$$

The differential equation is solved subject to the boundary condition:

$$\lim_{x \to 0_+} V_0^{\mathrm{F}}(x) = 0 \tag{11.4}$$

which recognizes that zero is an absorbing barrier for the process of x_t in equation (11.1), i.e. if the state variable ever gets to zero, it stays there forever, thus the option to enter will be worthless. Solving equation (11.3) subject to (11.4) yields the value of the follower before investment:

$$V_0^{\mathrm{F}}(x) = Bx^{\lambda} \tag{11.5}$$

where $B \geq 0$ is a constant whose value is yet to be determined and λ is the positive root of the characteristic equation:

$$\tfrac{1}{2}\sigma^2 \lambda(\lambda - 1) + \mu\lambda - r = 0 \tag{11.6}$$

which is equal to:

$$\lambda = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1 \tag{11.7}$$

Next consider the value of the follower in the *stopping region* (values of the state variable x_t for which it is optimal to undertake the investment at once), which we denote $V_1^{\mathrm{F}}(x)$. In this region, the follower has invested the fixed cost K in order to receive perpetually the flow $(1-a)x$, thus the equilibrium return condition in this region will be:

$$rV_1^{\mathrm{F}}(x)\,\mathrm{d}t = E\{V_1^{\mathrm{F}}(x)\} + (1-a)x\,\mathrm{d}t$$

where the last term simply implies that the active follower gets a revenue flow of $(1-a)x$ every instant.[4] Expanding and rearranging yields:

$$\tfrac{1}{2}\sigma^2x^2V_1''^{\mathrm{F}}(x) + \mu xV_1'^{\mathrm{F}}(x) - rV_1^{\mathrm{F}}(x) + (1-a)x = 0 \qquad (11.8)$$

Recognize that since investment is irreversible, the value of the follower in this region $V_1^{\mathrm{F}}(x)$ is given by the expected value of revenue flows alone, with no option value terms, i.e.

$$V_1^{\mathrm{F}}(x) = \frac{1-a}{r-\mu}x \qquad (11.9)$$

The boundary between the continuation and the stopping region is given by a critical value of the stochastic process or trigger point such that continued delay (immediate investment) for the follower is optimal for values of x below (above) this level. Let $\bar{x}_{\mathrm{F}}$ denote this critical value of the state variable. At the boundary between regions, the critical value $\bar{x}_{\mathrm{F}}$ must satisfy the following conditions by arbitrage:

$$V_0^{\mathrm{F}}(\bar{x}_{\mathrm{F}}) = V_1^{\mathrm{F}}(\bar{x}_{\mathrm{F}}) - K \qquad (11.10)$$

$$V_0'^{\mathrm{F}}(\bar{x}_{\mathrm{F}}) = V_1'^{\mathrm{F}}(\bar{x}_{\mathrm{F}}) \qquad (11.11)$$

The first boundary condition is commonly termed the *value-matching condition*. It simply reflects the fact that, upon exercise, the follower gives up the option to enter $V_0^{\mathrm{F}}(\bar{x}_{\mathrm{F}})$ for the value of being active in the market, $V_1^{\mathrm{F}}(\bar{x}_{\mathrm{F}})$, minus the entry cost K. The second boundary condition is known as the 'high-contact' or *smooth-pasting* condition.[5] Essentially, this condition ensures that $\bar{x}_{\mathrm{F}}$ is the trigger that maximizes the value of the follower's option.

Substituting equations (11.5) and (11.9) into the boundary conditions (11.10) and (11.11) yields a system of two equations, which uniquely determine the two unknowns, B and $\bar{x}_{\mathrm{F}}$. Then, the value of the follower is given by:

$$V^{\mathrm{F}}(x) = \begin{cases} \left[\dfrac{1-a}{r-\mu}\bar{x}_{\mathrm{F}} - K\right]\left(\dfrac{x}{\bar{x}_{\mathrm{F}}}\right)^{\lambda} & \text{if } x < \bar{x}_{\mathrm{F}} \\[2ex] \dfrac{1-a}{r-\mu}x - K & \text{if } x \geq \bar{x}_{\mathrm{F}} \end{cases} \qquad (11.12)$$

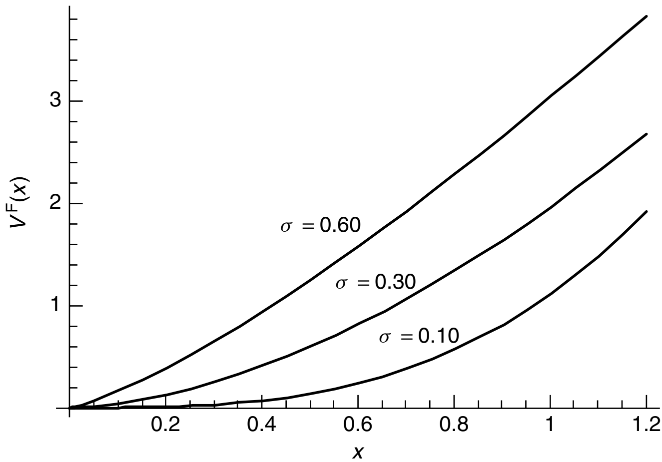

Figure 11.1 *The value of the follower's option to enter the market, $V^F(x)$, as a function of x for different values of volatility σ. The rest of the parameters are: $\mu = 0.02$, $K = 5$, $r = 0.09$ and $a = 0.60$*

where:

$$\bar{x}_F = \frac{\lambda}{\lambda - 1}(r - \mu)K\frac{1}{1 - a} \tag{11.13}$$

Figure 11.1 plots the follower's value function in equation (11.12) as a function of x for three different volatility values. The usual positive dependence of real option values on volatility is evident. Figure 11.2 shows $\bar{x}_F$, the optimal entry threshold of the follower in equation (11.13), as a function of σ for different magnitudes of first-mover advantages. The unwillingness to commit by investing ('value of waiting') as volatility increases, which has been reported in the real options literature, is apparent. Note that this unwillingness increases as a, the market share that the leader can command under duopolistic competition, increases. We return to this point in Section 11.5, where we formally assess the effect of first-mover's advantages on the entry strategies of competing firms.

Thus, the optimal follower strategy is summarized in the following proposition.

Proposition 11.1 *Conditional on the leader having entered the market, the optimal follower strategy is to invest in the market the first moment that x_t reaches $\bar{x}_F$, as defined in equation (11.13), from below. That is, the optimal entry time for the follower, T_F, can be written as:*

$$T_F = \inf\left\{t \geq T_L : x_t \geq \frac{\lambda}{\lambda - 1}(r - \mu)K\frac{1}{1 - a}\right\}$$

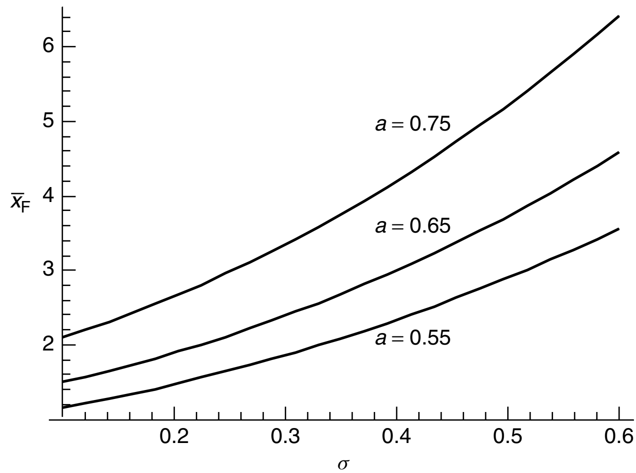

Figure 11.2 *The trigger value of the follower, $\bar{x}_F$, as a function of σ for different values of first-mover advantages a. The rest of the parameters are: $\mu = 0.02$, $K = 5$ and $r = 0.09$*

where T_L, the optimal entry time of the leader, is to be derived in the next subsection.

Note that the optimal strategy of the follower is *independent* of the point at which the leader invests: given that it invests later, the follower simply optimizes its entry time, irrespective of the precise location of the leader's trigger point.

11.3.2 The leader's value function

We now consider the value of becoming the leader in the market, given that neither firm has entered yet, and that the follower will act optimally in the future in accordance with the stopping rule described in Proposition 11.1.

Once the leader has sunk the investment cost K, it has no further action to take. It enjoys monopolistic revenues, x_t, for as long as the follower has not entered (i.e. for $t < T_F$); however, its expected value will be affected by the possible action of the rival firm investing later at T_F. Bearing that in mind, the leader's post-investment payoff can be written as:

$$V^L(x) = E_t\left[\int_t^{T_F} e^{-r\tau} x \, d\tau\right] + E_t[e^{-rT_F}]\frac{a\bar{x}_F}{r-\mu} - K \qquad (11.14)$$

In equation (11.14), the first expectation represents the period of time that the leader earns monopolistic rents. Before T_F the leader operates alone in the market, earning the full $x\,dt$ of market profitability per instant. The second expectation, however, implies that once the follower enters at T_{F}, the leading firm settles for the fraction of the market guaranteed by first-mover's advantages.

We compute the expectations (in Section 11.7.1) and substitute to get the value of the leader in the market:

$$V^{\mathrm{L}}(x) = \begin{cases} \dfrac{x}{r-\mu} + Dx^{\lambda} + \dfrac{a\bar{x}_{\mathrm{F}}}{r-\mu}\left(\dfrac{x}{\bar{x}_{\mathrm{F}}}\right)^{\lambda} & \text{if } x < \bar{x}_{\mathrm{F}} \\ \dfrac{a}{r-\mu}x - K & \text{if } x \geq \bar{x}_{\mathrm{F}} \end{cases} \tag{11.15}$$

where $D = -\bar{x}_{\mathrm{F}}^{1-\lambda}/(r-\mu) \leq 0$. Note that when $x < \bar{x}_{\mathrm{F}}$, the option term $Dx^{\lambda} \leq 0$ captures the *negative effect* that entry by the follower will have on the leader's value.[6] If the follower invests, the leader essentially stops being a monopolist in the market and loses $(1-a)$ of the market to the follower.

Finally, consider the relative values of the leader and the follower in the market in equations (11.12) and (11.15). Depending on the level of market profitability, the leader's value may be greater or less than that of the follower. The following proposition describes the relative valuations.

Proposition 11.2 *There exists a unique point,* $\bar{x}_{\mathrm{L}} \in (0, \bar{x}_{\mathrm{F}})$, *with the following properties:*

$$V^{\mathrm{L}}(x) < V^{\mathrm{F}}(x) \quad \text{for } x < \bar{x}_{\mathrm{L}}$$
$$V^{\mathrm{L}}(x) = V^{\mathrm{F}}(x) \quad \text{for } x = \bar{x}_{\mathrm{L}}$$
$$V^{\mathrm{L}}(x) > V^{\mathrm{F}}(x) \quad \text{for } x > \bar{x}_{\mathrm{L}}$$

Proof: See appendix.

Proposition 11.2 demonstrates that there is a unique value of $x \in (0, \bar{x}_{\mathrm{F}})$, denoted $\bar{x}_{\mathrm{L}}$, at which the payoffs to both the leading and the following firm are equal. At any point below (above) $\bar{x}_{\mathrm{L}}$, each firm would prefer being the follower (leader). At $\bar{x}_{\mathrm{L}}$, the benefits of pre-emption (first-mover's advantages) just equal the costs of making the investment cost earlier. Figure 11.3 graphically confirms the above by plotting the values of being the leader and the follower in a market with $a = 0.60$ pre-emption advantages.

From Proposition 11.2 the following corollary, which formalizes the leader's optimal strategy, comes naturally.

Corollary 11.1 *The optimal leader strategy is to invest in the market the first moment that* x_t *reaches* $\bar{x}_{\mathrm{L}}$, *as defined in Proposition 11.2, from below. That is, the optimal entry time for the leader,* T_{L}, *can be written as*:

$$T_{\mathrm{L}} = \inf\{t \geq 0 : x_t \geq \bar{x}_{\mathrm{L}}\}$$

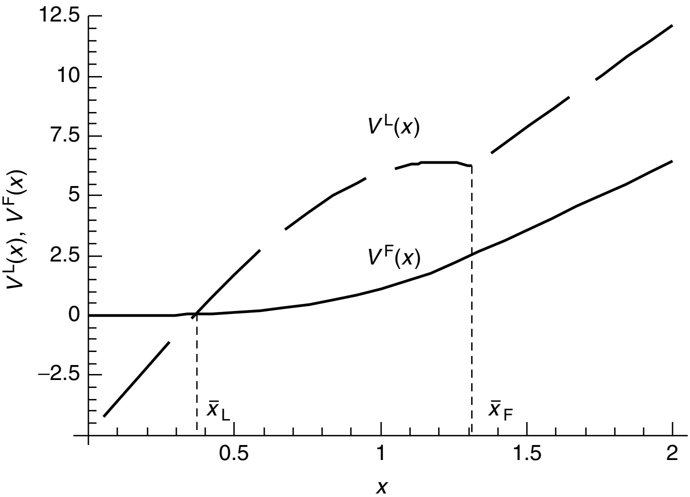

Figure 11.3 *The value functions of the leader $V^L(x)$ and the follower $V^F(x)$ for first-mover advantages of $a = 0.60$. Optimal trigger values are $\bar{x}_L = 0.3651$ and $\bar{x}_F = 1.3125$ for the leader and the follower, respectively. The rest of the parameters are: $\mu = 0.02$, $K = 5$, $r = 0.09$ and $\sigma = 0.10$*

11.4 SOLVING FOR THE EQUILIBRIUM STRATEGIES

Moving back to the beginning of the game, the two competing firms have to choose their equilibrium entry strategies. We derive a pair of symmetric, subgame perfect equilibrium entry strategies in which each firm's exercise strategy, conditional upon the other's exercise strategy, is value-maximizing. The nature of equilibrium depends (a) on the *level of the initial market operating profitability*, x_0 and (b) on the *magnitude* and *nature* of first-mover's advantages, a.

When first-mover's advantages are permanent ($a > \frac{1}{2}$), pre-emptive incentives are strong and equilibrium entry will depend on x_0. If $x_0 < \bar{x}_F$, equilibrium entry will be *sequential*: once a firm enters the market, a period of time will pass before its rival finds it optimal to enter. If $x_0 \geq \bar{x}_F$, however, equilibrium entry will be *simultaneous*: one firm enters instantly after the other. Lastly, we comment on the nature of equilibrium entry strategies when first-mover's advantages are temporary in nature.

11.4.1 Sequential equilibrium: $x_0 < \bar{x}_F$

If the investment game begins with an initial level of x which is less than $\bar{x}_F$, a pair of symmetric, subgame perfect equilibrium strategies is for each firm to act as follows:

If your competitor has not entered the market yet, invest the first time that x_t equals or exceeds $\bar{x}_L$. If your competitor has already entered the market, then wait until x_t rises to $\bar{x}_F$ before investing.

Given the above strategies, the equilibrium will appear as follows. If $x_0 < \bar{x}_L$, one firm waits until the trigger $\bar{x}_L$ is reached, and the other waits and invests at $\bar{x}_F$. At $\bar{x}_L$, the competing firms will be indifferent between leading or following in the light of Proposition 11.2 $[V^L(\bar{x}_L) = V^F(\bar{x}_F)]$. Since firms are economically identical, there are essentially two equilibria of that form, where the identities of leader and follower are interchanged between firms. Moreover, if $x_0 \in [\bar{x}_L, \bar{x}_F)$, both firms will race to enter since $V^L(x) > V^F(x)$ (see Proposition 11.2). We impose the slight technical assumption that in such a case one, randomly selected firm wins the race (e.g. through a toss of a coin or any other arbitrary consideration). The random winner will invest immediately, and the loser will get the follower's role, entering later at $\bar{x}_F$.

It is relatively simple to demonstrate that the above set of policies constitute an equilibrium. From Proposition 11.2, it is suboptimal to enter before x rises to a level of $\bar{x}_L$. Thus, if its rival follows the equilibrium strategy, the best that a firm can do is also pursue the equilibrium strategy by attempting to enter at $\bar{x}_L$. Moreover, Proposition 11.1 demonstrates that if a firm enters second, it does so optimally at $\bar{x}_F$.

11.4.2 Simultaneous equilibrium: $x_0 \geq \bar{x}_F$

Now, consider the case where $x_0 \geq \bar{x}_F$. In this range, any equilibrium will be characterized by simultaneous entry by both rivals. If one firm invests at a level of profitability greater than $\bar{x}_F$, its rival will invest immediately thereafter. This is obvious in the light of Proposition 11.1, which depicts the optimal follower response to invest the first moment that $\bar{x}_F$ is reached or exceeded. Thus, in this region, the equilibrium will be characterized by simultaneous entry.[7]

From Proposition 11.2, since $V^L(x) > V^F(x)$ for any $x > \bar{x}_F > \bar{x}_L$, there is a strong pre-emptive incentive to be the leader in the market. Both firms will again race to enter. The (random) winner will secure a fraction a of the market, while the loser will invest immediately thereafter and settle for $(1 - a)x$.

However, since x_0 could be any value greater than $\bar{x}_F$, there are essentially an infinite number of Markovian, subgame perfect equilibria over this region. Thus, when $x_0 \geq \bar{x}_F$, $a \in (\frac{1}{2}, 1]$ the equilibrium strategies of competing firms can be summarized in the following:

Invest in the market immediately to gain first-mover advantages. If your competitor wins the race to enter first, then enter immediately thereafter.

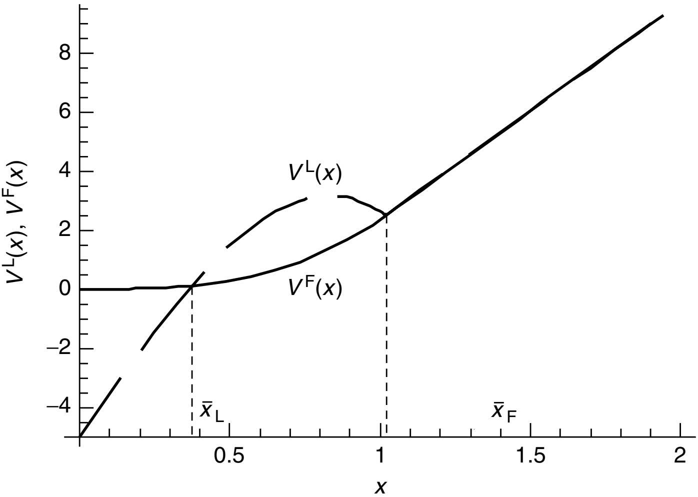

Figure 11.4 *The value functions of the leader $V^L(x)$ and the follower $V^F(x)$ for the special case of equal post-entry market sharing $a = 0.50$. Optimal trigger values are $\bar{x}_L = 0.3843$ and $\bar{x}_F = 1.0500$ for the leader and the follower, respectively. The rest of the parameters are: $\mu = 0.02$, $K = 5$, $r = 0.09$ and $\sigma = 0.10$*

Before concluding this section, a comment on the special case where first-mover's advantages are temporary. This translates to setting $a = \frac{1}{2}$, i.e. post-entry duopolistic rivals share the market equally. In this special case, any pre-emption advantage is temporary since the first-mover enjoys higher profitability only as long as it acts as a monopolist. Once the follower enters, payoffs are the same for both firms. Figure 11.4 demonstrates this graphically. Note that for $x \geq \bar{x}_F$ the value functions of leader and follower coincide. Even when advantages are temporary, equilibrium entry by rivals will still be either sequential or simultaneous, depending on initial industry profitability conditions, x_0.

11.5 COMPARATIVE STATICS

In this section, we examine the effect of changes in underlying parameters on the value functions and entry thresholds of competing firms. A major concern is to determine the effect of parameter changes on the timing of investment by leading and following firms. Towards that end, we assume throughout that $x_0 < \bar{x}_F$. Thus, we focus on the sequential equilibrium, where entry options are exercised at distinct stopping times.

First, concentrate on the effect of pre-emption incentives. We are interested in determining the dependence of leader and follower values and exercise strategies on the degree of first-mover's advantages. Differentiating equations (11.12) and

(11.15) with respect to a yields an intuitive and interesting result:

$$\frac{\partial V^{\mathrm{F}}(x)}{\partial a} = \begin{cases} -\dfrac{x}{r-\mu}\left(\dfrac{(1-a)(\lambda-1)x}{\lambda K(r-\mu)}\right)^{\lambda-1} < 0 & \text{if } x < \bar{x}_{\mathrm{F}} \\ -\dfrac{x}{r-\mu} < 0 & \text{if } x \geq \bar{x}_{\mathrm{F}} \end{cases} \tag{11.16}$$

$$\frac{\partial V^{\mathrm{L}}(x)}{\partial a} = \begin{cases} \dfrac{\lambda x}{r-\mu}\left(\dfrac{(1-a)(\lambda-1)x}{\lambda K(r-\mu)}\right)^{\lambda-1} > 0 & \text{if } x < \bar{x}_{\mathrm{F}} \\ \dfrac{x}{r-\mu} > 0 & \text{if } x \geq \bar{x}_{\mathrm{F}} \end{cases} \tag{11.17}$$

A change in the magnitude of first-mover's advantages has an *opposing effect* on the value of the leader and the follower: the higher the market share that pre-emptive entry guarantees, the higher (lower) the value of pursuing the leader's (follower's) role. Intuitively, since competition between rival firms over market fractions is direct, a higher degree of pre-emption implies that what the leading firm wins by acting first is the follower's loss. This is obvious in the above partial derivatives, especially when both firms are active ($x \geq \bar{x}_{\mathrm{F}}$): the effect of a is symmetric in magnitude but of opposite sign.

Interestingly, if not surprisingly, the effect of changes in a is *asymmetric* on $V^{\mathrm{F}}(x)$, $V^{\mathrm{L}}(x)$ prior to entry ($x < \bar{x}_{\mathrm{F}}$). It is directly observable from equation (11.16) and (11.17) that:

$$\frac{\dfrac{\partial V^{\mathrm{L}}(x)}{\partial a}}{-\dfrac{\partial V^{\mathrm{F}}(x)}{\partial a}} = \lambda > 1$$

for $x < \bar{x}_{\mathrm{F}}$, i.e. an increase in first-mover's advantages has a more pronounced (positive) effect on the leader's value compared to the (negative) effect on the follower's. The reason is that the magnitude of first-mover's advantages has a *dual effect* on the leader's payoff before follower entry: *a higher value of a increases the market share that the leader retains after follower entry but also augments the period of time that the leader earns monopolistic rents, by delaying optimal follower entry*. This is because $\partial T_{\mathrm{F}}/\partial a > 0$. Thus, before entry by the follower, the (positive) dependence of $V^{\mathrm{L}}(x)$ on a is greater in magnitude than the (negative) one of $V^{\mathrm{F}}(x)$, but it is then symmetric once the follower becomes active in the market. This is graphically confirmed in Figure 11.5, which plots the dependence of leader and follower value functions on first-mover advantages, $\partial V^{\mathrm{L}}(x)/\partial a$ and $\partial V^{\mathrm{F}}(x)/\partial a$, as a function of the state variable x under two 'regimes': before (panel a) and after (panel b) entry in the market by

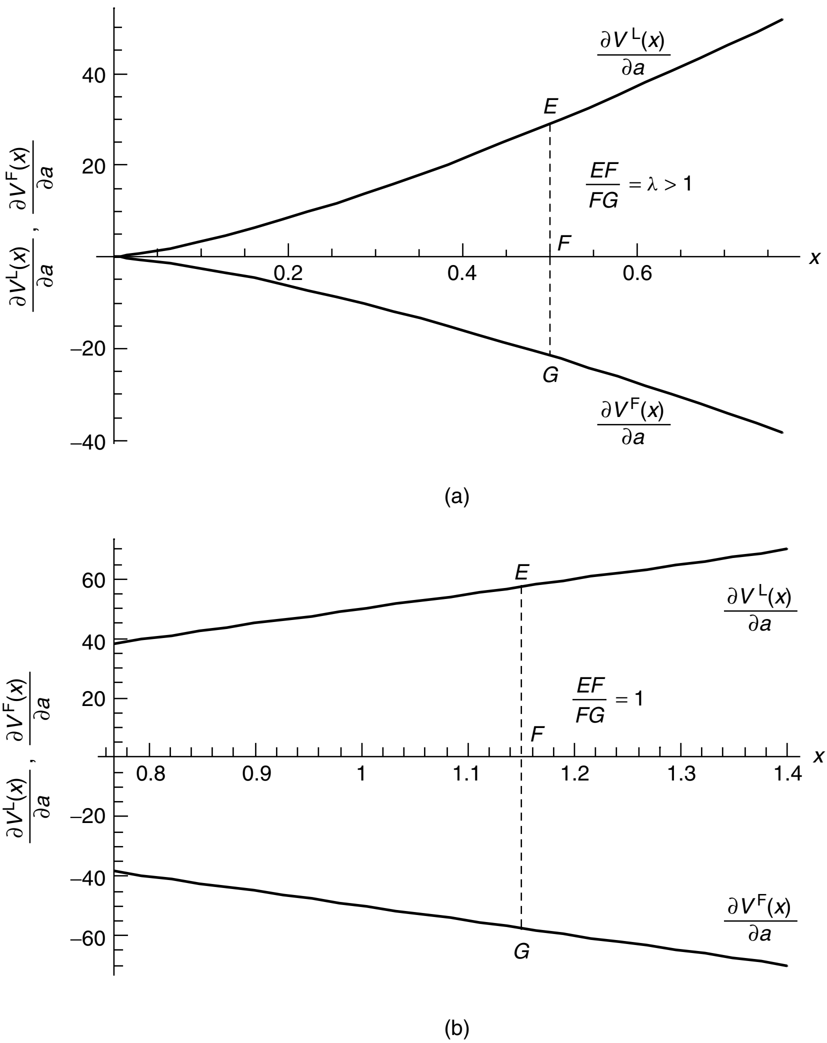

Figure 11.5 *The partial derivatives of leader and follower value functions with respect to first-mover advantages a, as a function of the state variable x prior to ($x < \bar{x}_F$, panel a) and after ($x \geq \bar{x}_F$, panel b) follower entry. The follower trigger value and the parameters used are: $\bar{x}_F = 0.7676$, $\mu = 0.05$, $K = 4$, $r = 0.07$, $a = 0.60$ and $\sigma = 0.10$*

the follower. It is evident that $\forall x < \bar{x}_F$ the effect of a change in a is more pronounced on the leader's value function by a factor λ. Once the follower enters the market ($x \geq \bar{x}_F$, panel b) the effect is symmetric: what the leading firm wins from a change in a is the follower's loss.

Next, we turn to assess the effect of volatility, where the leader–follower value functions and trigger points are complex functions of σ. A well-established result from the theory of real options is that increasing uncertainty increases the unwillingness to commit to investment. Thus, conditional on any initial level of market profitability, this result will tend to postpone entry into the market. On the other hand, increases in uncertainty have an opposing effect as well: increasing volatility makes it more probable that substantial changes in profitability can take place in shorter periods of time. Thus, changes in σ alter both the optimal entry strategies and the stochastic properties of the underlying profitability.

Figure 11.6 demonstrates that the net effect of increasing volatility is an increase in the ratio of entry thresholds of the two firms, i.e. *more volatile market profitability makes rivals enter at more separated times.* The intuition of this result is as follows. As volatility increases, it is more likely that the time it takes operating profitability to rise from $\bar{x}_L$ to $\bar{x}_F$ will fall, bringing the ratio $\bar{x}_F/\bar{x}_L$ down. On the other hand, increases in uncertainty will increase the option value of waiting, increasing both $\bar{x}_L$ and $\bar{x}_F$ since firms are less eager to invest. However, *this option to wait does not affect the leader and the follower equally since their claims are on different fractions of the underlying market*. When first-mover's advantages are substantial and permanent as in our setting, leader's and follower's entry options are on different fractions of the underlying, thus their value of waiting is differently affected by increases in σ. As Figure 11.6 confirms, it is this second effect that dominates: $\bar{x}_F$ increases more than $\bar{x}_L$,

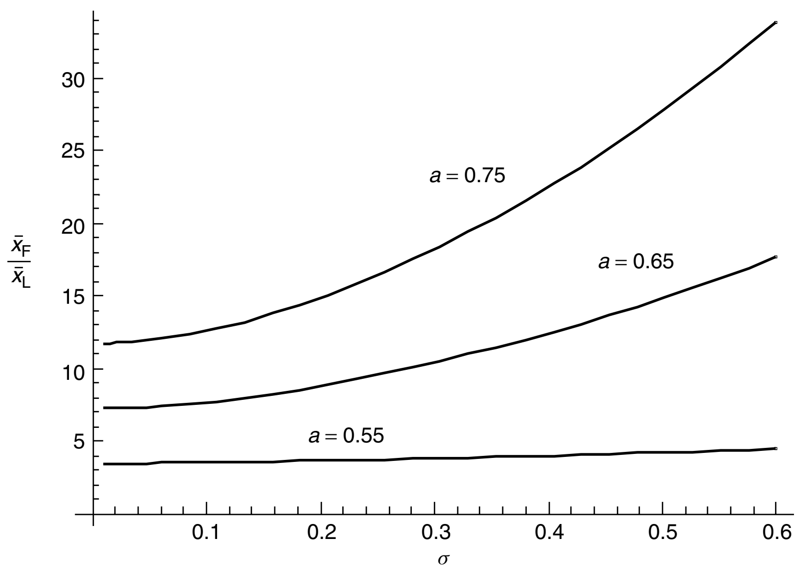

Figure 11.6 *The ratio of entry trigger values $\bar{x}_F/\bar{x}_L$ as a function of volatility σ for different values of first-mover advantages a. The rest of the parameters are: $\mu = 0.05$, $K = 4$ and $r = 0.07$*

thus their ratio is positively related to σ, and this positive dependence is more pronounced the greater the first-mover's advantages (higher a).

Turning to the dependence of the value functions on the state variable, we can differentiate equations (11.12) and (11.15) with respect to x to get:

$$\frac{\partial V^{\mathrm{F}}(x)}{\partial x} = \begin{cases} \dfrac{\lambda K}{(\lambda-1)x}\left(\dfrac{(1-a)(\lambda-1)x}{\lambda K(r-\mu)}\right)^{\lambda} & > 0 \quad \text{if } x < \bar{x}_{\mathrm{F}} \\ \dfrac{1-a}{r-\mu} & > 0 \quad \text{if } x \geq \bar{x}_{\mathrm{F}} \end{cases} \tag{11.18}$$

$$\frac{\partial V^{\mathrm{L}}(x)}{\partial x} = \begin{cases} \dfrac{1}{r-\mu} - \dfrac{\lambda^2 K}{(\lambda-1)x}\left(\dfrac{(1-a)(\lambda-1)x}{\lambda K(r-\mu)}\right)^{\lambda} & \begin{matrix}\geq 0 \\ \leq 0\end{matrix} \quad \text{if } x < \bar{x}_{\mathrm{F}} \\ \dfrac{a}{r-\mu} & > 0 \quad \text{if } x \geq \bar{x}_{\mathrm{F}} \end{cases} \tag{11.19}$$

Equations (11.18) and (11.19) could be thought of as the leader–follower deltas. Interestingly, the leader's value can be a *decreasing* function of the state variable x in the region prior to entry by the follower. If λ is large enough (i.e. for higher r, lower μ and σ), $V^{\mathrm{L}}(x)$ will actually peak to the left of $\bar{x}_{\mathrm{F}}$, and then approach $\bar{x}_{\mathrm{F}}$ with a *negative* slope. Figure 11.7 plots equations (11.18) and (11.19) as a function of the first-mover advantages a. Intuitively, the larger the fraction that

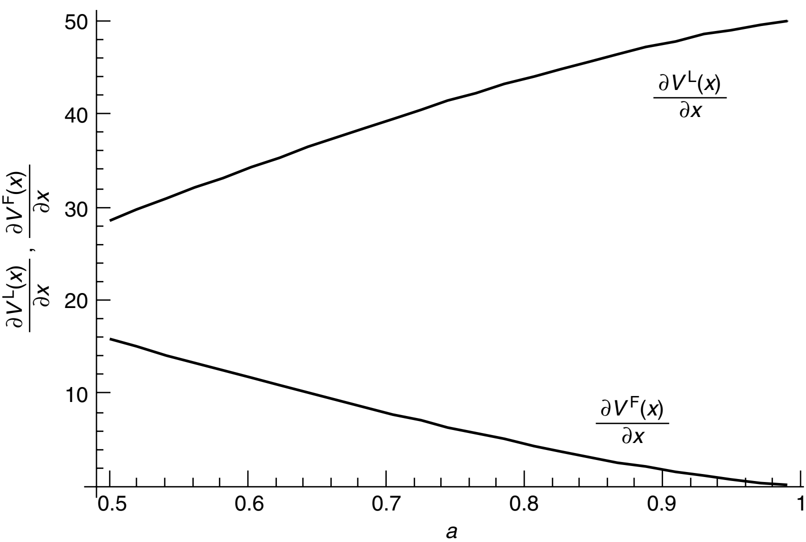

Figure 11.7 *The partial derivatives of leader and follower value functions with respect to the state variable x, as a function of first-mover advantages a. The rest of the parameters are: $\mu = 0.05$, $K = 4$, $r = 0.07$, $x = 0.17$ and $\sigma = 0.10$*

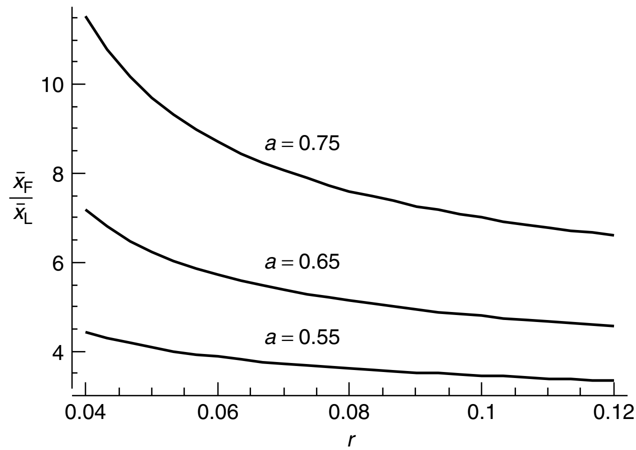

Figure 11.8 *The ratio of entry trigger values $\bar{x}_F/\bar{x}_L$ as a function of the interest rate r for different values of first-mover advantages a. The rest of the parameters are: $\mu = 0.02$, $K = 4$ and $\sigma = 0.10$*

first entry guarantees, the more (less) sensitive the leader's (follower's) value functions become to the underlying market profitability.

Finally, and in accordance with standard option pricing results, an increase in the risk-free interest rate, r, postpones option exercise by increasing both $\bar{x}_L$ and $\bar{x}_F$. However, as Figure 11.8 demonstrates, the effect on the trigger values is not symmetric. A higher interest rate brings the ratio of entry times down and this negative dependence is steeper the higher the first-mover advantages.

11.6 CONCLUSION

Real-world competitive investment situations do not allow firms to choose exercise strategies in isolation. Optimal investment policies have to take into account the threat of pre-emption and the advantages that may accrue to the first entrant in the new market. These calculations cannot be conducted separately, but must be done as part of a strategic equilibrium.

In this chapter we provide a model of strategic entry option exercise where the threat of pre-emption is severe. Namely, unlike previous research, we allow the degree of pre-emption motives to be a parameter in the model and account for the case where the advantages that accrue to the first entrant in the market can be sustained permanently.

We demonstrate that equilibrium entry into a new market can be sequential or simultaneous, depending on initial market conditions and the magnitude of first-mover's advantages. Firm values are shown to depend differently on such advantages, and the dependence is of different direction for leading and following firms. Moreover, first-mover advantages are shown to have a dual effect on the first entrant's decision, making its value more sensitive to changes in such advantages. Finally, the analysis of equilibrium exercise strategies implies that more volatile industries should experience more separated firm entries over time the more substantial the first-mover privileges.

11.7 APPENDIX

11.7.1 Expectations calculations

Here we calculate the expectations stated in Section 11.3.2 for the value of the leading firm in the market. Let:

$$f(x) = E[\mathrm{e}^{-rT_\mathrm{F}}] \quad \text{and} \quad g(x) = E\left[\int_0^{T_\mathrm{F}} \mathrm{e}^{-r\tau} x \, \mathrm{d}\tau\right]$$

where x follows the geometric Brownian motion in equation (11.1) and T_F is defined in Proposition 11.1. A more general approach to the calculation of such expectations can be found in Harrison (1985, p. 42) or Karlin and Taylor (1975, p. 362).

For $t < T_\mathrm{F}$, over any infinitesimal interval of time $\mathrm{d}t$, we can write the first expectation as a dynamic-programming recursive expression:

$$f(x) = \mathrm{e}^{-r\,\mathrm{d}t} E[f(x + \mathrm{d}x)]$$

Expanding the right-hand side using Itô's lemma, this becomes:

$$f(x) = [1 - r\,\mathrm{d}t + o(\mathrm{d}t)][f(x) + \mu x f'(x)\mathrm{d}t + \tfrac{1}{2}\sigma^2 x^2 f''(x)\mathrm{d}t + o(\mathrm{d}t)]$$

where $o(\mathrm{d}t)$ collectively accounts for all terms which are asymptotically of order less than $\mathrm{d}t$. Simplifying the above and letting $\mathrm{d}t \to 0$ yields the differential equation:

$$\tfrac{1}{2}\sigma^2 x^2 f''(x) + \mu x f'(x) - r f(x) = 0$$

This has a general solution:

$$f(x) = C_1 x^\lambda + C_2 x^\kappa$$

where λ is as in equation (11.7) and κ is the negative root of the fundamental quadratic in equation (11.6).

The constants C_1 and C_2 are determined by boundary conditions:

$$\lim_{x \to \bar{x}_{\mathrm{F}-}} f(x) = 1$$

$$\lim_{x \to 0_+} f(x) = 0$$

i.e. as x approaches $\bar{x}_F$, T_F is likely to be small and e^{-rT_F} will tend to 1. Moreover, as x tends to zero, T_F will be large and e^{-rT_F} close to zero. These boundary conditions imply $C_2 = 0$ and $C_1\bar{x}_F^{\lambda} = 1$, so:

$$f(x) = \left(\frac{x}{\bar{x}_F}\right)^{\lambda}$$

Similarly, $g(x)$ satisfies the following differential equation:

$$\tfrac{1}{2}\sigma^2 x^2 g''(x) + \mu x g'(x) - rg(x) + x = 0$$

with general solution:

$$g(x) = D_1 x^{\lambda} + D_2 x^{\kappa} + \frac{x}{r-\mu}$$

and boundary conditions:

$$\lim_{x\to\bar{x}_{F-}} g(x) = 0$$

$$\lim_{x\to 0_+} g(x) = 0$$

Therefore, $D_2 = 0$ and $D_1 = -\bar{x}_F^{1-\lambda}/(r-\mu)$. Substituting the expectations in equation (11.14) and rearranging yields equation (11.15) in the text.

11.7.2 Proof of Proposition 11.2

Define the function $Q(x) = V^L(x) - V^F(x)$ where $V^L(x)$ and $V^F(x)$ are defined in equations (11.15) and (11.12), respectively.

To establish *existence* of a root for $Q(x)$ in $(0, \bar{x}_F)$, first establish that:

$$Q(0) = -K \qquad < 0$$

$$Q(\bar{x}_F) = \frac{\lambda(2a-1)K}{(1-a)(\lambda-1)} > 0$$

Since $Q(x)$ is continuous in $(0, \bar{x}_F)$, it has at least one root in this interval.

To prove *uniqueness*, one merely needs to demonstrate strict concavity of $Q(x)$ over this interval. Differentiating $Q(x)$ twice yields:

$$Q''(x) = -\frac{\lambda(\lambda+1)K}{x^2}\left(\frac{(1-a)(\lambda-1)x}{\lambda K(r-\mu)}\right)^{\lambda} < 0$$

Thus, the root is unique.

11.7.3 The effect of follower entry on the market profitability

In our basic setting, entry by the follower is assumed to have repercussions only on the sharing of the market between rivals and not on the size or the profitability of the market as a whole. In this section we briefly outline two possible ways of allowing follower entry to influence market profitability.

There is nothing in the exposition that restricts the drift μ and/or the volatility σ of the market profitability from being deterministic functions instead of constants. Thus, one could write equation (11.1) as:

$$\mathrm{d}x_t = \mu(t)x_t\,\mathrm{d}t + \sigma(t)x_t\,\mathrm{d}W_t$$

where $\mu(t)$ and $\sigma(t)$ are deterministic functions of time and replicate the whole analysis. This formulation can be very useful in cases where life cycle issues are important for a market. For example, allowing $\mu(t)$ to be a downward sloping curve can accommodate cases where there is a downward shift in profitability (decreasing rents) as the market becomes more mature.

Another possibility is to allow for a 'structural break' in market profitability after entry by the follower. Assume that equation (11.1) is replaced by:

$$\mathrm{d}x_t = \mu_j x_t\,\mathrm{d}t + \sigma_j x_t\,\mathrm{d}W_t$$

for $j = 1, 2$ (the number of active firms in the market). Intuition would suggest that $\mu_1 \geq \mu_2$, i.e. operating a monopoly is more profitable than a duopoly. The relative magnitudes of the volatility parameters are not so clear.

This formulation allows us to accommodate cases where entry by a rival (follower) makes market profitability returns lower (even negative) for both firms and more (or less) volatile. Allowing this, equations (11.7), (11.12) and (11.13) should be replaced by:

$$\lambda = \frac{1}{2} - \frac{\mu_1}{\sigma_1^2} + \sqrt{\left(\frac{\mu_1}{\sigma_1^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma_1^2}} > 1$$

$$V^{\mathrm{F}}(x) = \begin{cases} \left[\dfrac{1-a}{r-\mu_2}\bar{x}_{\mathrm{F}} - K\right]\left(\dfrac{x}{\bar{x}_{\mathrm{F}}}\right)^{\lambda} & \text{if } x < \bar{x}_{\mathrm{F}} \\[2ex] \dfrac{1-a}{r-\mu_2}x - K & \text{if } x \geq \bar{x}_{\mathrm{F}} \end{cases}$$

$$\bar{x}_{\mathrm{F}} = \frac{\lambda}{\lambda - 1}(r - \mu_2)K\frac{1}{1-a}$$

respectively, and Figures 11.1–11.5 will have the same shape but with lower values. Interestingly, optimal exercise strategies will *only* depend on σ_1 and *not* on the post-entry profitability volatility σ_2.

ACKNOWLEDGMENTS

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NOTES

1. Notable exceptions would be warrants and convertible securities where exercising requires firms to issue new shares of common stock, thus influencing the value of the underlying and of remaining unexercised contracts. On the strategic exercise of warrants and convertible securities see Emanuel (1983), Constantinides (1984) and Spatt and Sterbenz (1988).
2. The restriction $a > \frac{1}{2}$ is necessary for first-mover's advantages. Mathematically, there is nothing in the exposition that restricts a from being any real number.
3. For further explanation see Fudenberg and Tirole (1985, 1986, 1991).
4. It is implicitly assumed that entry by the follower affects only the way the market is shared by rivals and not the market profitability or size *per se* (i.e. the process of x_t). This seemingly restrictive assumption can be easily relaxed to allow for different profitability patterns in monopolistic and duopolistic competitive eras. See the appendix (Section 11.7.3) for a discussion along these lines.
5. See Merton (1973) and Dumas (1991) for a clear presentation of the smooth-pasting and high-contact conditions or Dixit and Pindyck (1994, chapter 4, appendix C) for a less technical explanation.
6. In a complete information setting, all parameters are common knowledge to all agents/rivals, thus value functions, which are forward looking, anticipate future optimal (re)actions.
7. The concept of one firm acting 'instantaneously' after its rival is not problem-free in continuous-time stochastic games. However, Simon and Stinchcombe (1989) provide a framework that resolves difficulties such as this, by specifying pure strategies in continuous time that conform as closely as possible to their discrete-time analog.

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Chapter 12

Leader/follower real value functions if the market share follows a birth/death process

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SUMMARY

For a duopoly environment, we model the leader and follower real value functions assuming that the leader's 'market share' evolves according to an immigration (birth) and death process. We derive analytical solutions for the follower and leader options to invest, and numerical solutions for the leader's optimal investment timing. Then we calculate the partial derivatives of the leader and follower value functions to market share, birth/death parameters and market profitability. This model is possibly more realistic than that proposed by some other authors studying the advantages of being first (and also being a follower).

We show that over certain ranges of parameter values, the leader and follower real options to wait to invest, and not to wait to invest, are sometimes surprising, but possibly on reflection plausible. The follower's value function is usually less sensitive than (and of opposite sign to) the leader's value function to market share or the rate of customer arrivals/departures until the expected revenue exceeds the follower's trigger investment level. However, the sensitivity is dependent on the relative parameters, particularly the revenue and trigger. The follower's trigger increases with market share, the immigration/death ratio and revenue volatility. The leader's value function 'deltas' are highly sensitive and unstable as revenues approach the follower's trigger, confirming the adage, if you're ahead, 'watch the competition'.

12.1 INTRODUCTION

There is a developing real option literature that considers firms not to be in a monopoly setting and focuses on the option of not waiting. Fearing the move of competitors, many times firms act in order to achieve the advantages of being first, balanced against the advantages of the option to wait. In a duopoly setting where one of the firms is the first to enter, from now on defined as the leader, and the other one the follower, there are some advantages and disadvantages for assuming either of the roles. The leader normally has advantages in distribution, product line breadth, product line and especially market share (Tellis and Golder, 1996). The follower can have lower adoption costs and a reduction in uncertainty (Hope, 2000), through 'learning from the leader's mistakes'. An adequate model to determine investment/entry timing should consider the strategic policies of each firm and consequently include the advantages and disadvantages of each role.

The advantages of leaders establishing a dominant market share have been documented using the PIMS[1] database. More than 70 % of current market leaders are market pioneers. Tellis and Golder (1996) argue that although being first does not necessarily induce an advantage, it certainly creates an opportunity. When the pioneer is alone in the market, the leader enjoys the revenues of a monopolist; when other firms enter, the pioneer can continue to be the leader or not and that will depend on his ability to satisfy customers and innovate.

Spatt and Sterbenz (1985) consider learning and pre-emption. Smets (1991) considers a strategic setting where firms can act under the fear of pre-emption. Grenadier (1996) applies the model to the real-estate market. The effect of incomplete information is analyzed by Lambrecht and Perraudin (1997); strategic competition in Kulatilaka and Perotti (1998); the advantage of being first with the network advantage of adopting with others is in Mason and Weeds (2000); R&D competition in Weeds (2000); and Tsekrekos (2002) studies the sensitivity of the leader and follower value function to market share, assumed to be constant after the follower enters.

We relax the constant market share assumption to reflect a possibly more realistic environment, where the market and the market share reflect new customers arriving (birth or immigration process) and old customers departing (death process). Section 12.2 develops this model. Section 12.3 derives the partial derivatives of the follower and leader value functions to changes in market share, birth/death parameters, volatility and market profitability. Section 12.4 concludes.

12.2 MARKET SHARE AND BIRTH/DEATH PROCESSES MODEL

In common with Smets (1991), Weeds (2000) and Tsekrekos (2002), we develop a model where two competing firms have the option to enter the market; the leader will invest earlier and will benefit from securing a higher market share than its competitors. Operating the market will yield a net revenue flow x_t that evolves according to a geometric Brownian motion given by:

$$dx_t = \mu x_t \, dt + \sigma x_t \, dw_t \tag{12.1}$$

where μ is the drift parameter, σ is the standard deviation and dw_t is the increment of a standard Wiener process. We assume lognormality of revenues and do not consider operating costs explicitly, so there is no option to abandon. The underlying game is a Stackelberg leader–follower: the leader receives a monopolistic revenue flow $x_t \, dt$ when alone in the market. When the follower enters, that revenue will be shared with the leader having a higher market share, a. The market share is presented in a deterministic setting; the leader will have a revenue flow of ax_t and the follower of $(1 - a)x_t$, which might be adjusted by a multiplier.

12.2.1 The follower's value function

Let $V_0^F(x)$ represent the value of the follower in the region where it is not yet optimal to invest. This option gives the follower a capital gain or loss according to the evolution of the market. In the continuation region the value of the follower is given by:

$$rV_0^F(x) \, dt = E[dV_0^F(x)] \tag{12.2}$$

where r is the riskfree interest rate.

Using Itô's lemma, we obtain the differential equation:

$$\tfrac{1}{2}\sigma^2 x^2 V''^F_0(x) + \mu x V'^F_0(x) - rV_0^F(x) = 0 \tag{12.3}$$

This has the general solution:

$$V_0^F(x) = Ax^{\beta_1} + Bx^{\beta_2} \tag{12.4}$$

where A and B are constants, and β_1 and β_2 are:

$$\beta_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1 \tag{12.5}$$

$$\beta_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} < 0 \tag{12.6}$$

We assume that if our state variable reaches zero it will stay there forever, meaning that zero is an absorbing barrier, and so we obtain the following boundary condition:

$$V_0^{\mathrm{F}}(0) = 0 \tag{12.7}$$

Since as the state variable goes to zero the function has to decrease, B in equation (12.4) has to be equal to zero, so our solution becomes:

$$V_0^{\mathrm{F}}(x) = Ax^{\beta_1} \tag{12.8}$$

Let $V_1^{\mathrm{F}}(x)$ denote the value of the follower in the stopping region, the region where it is optimal to invest. In this region the follower invests a fixed cost K in order to receive perpetually a proportion of the net market revenue that will be determined by the market share.

Consider now that the initial market share a evolves according to a random process, more specifically an immigration–death model where new clients arrive according to a Poisson process and once there, can leave at any time. The lifetime of each individual as a client has an exponential distribution. This model gives a realistic representation of many practical situations, like molecules of gas in a given space. New particles can enter at any time and the Poisson process represents their arrival pattern. Once inside the molecules can leave at any time, so the exponential distribution also provides a good model for the time spent inside the space.[2] This process can also be used to model the number of people in a shop, or use of a telephone system, adoption of 3G mobile facility, or net new internet banking customers. In the UK, football pay-TV viewing developed first in one media (B-SkyB) and some customers were expected to migrate to another (ITV Digital). (In this case, the follower's expectations on customer migration rates from the leader turned out to be irrationally exuberant, as ITV Digital failed.)

The immigration–death process has an equilibrium distribution (the distribution of the population size at time t approaches a limiting distribution as t increases). This equilibrium distribution gives the expected proportion of time spent in each state in the long run. Consider an immigration–death model where the individuals join the population according to a Poisson process at rate λ and the lifetime distribution of each individual is exponentially distributed, $M(\nu)$. Let $X(t)$ equal the population size at time t, $X(t)$ is asymptotically Poisson distributed with parameter $\rho = \lambda/\nu$ (see the appendix, Section 12.5) where birth (or immigration) has the parameter λ and death ν:

$$P(X(t) = n) \to \frac{\rho^n}{n!}\mathrm{e}^{-\rho} \tag{12.9}$$

Now assume that the 'market share' is not constant, so some new clients will arrive and others will leave, and that the immigration–death model is appropriate for this phenomenon. We define 'market share' broadly as the multiplier for a standard revenue x. The multiplier is itself adjusted over time by a parameter ρ, which is immigration (λ) divided by death (ν) (new customers adjusted for old and new customers leaving). In the stopping region, the follower receives perpetually the expected value of the active project with no option value, so the expected value will be given by:

$$V_1^{\mathrm{F}}(x_t)\mathrm{d}t = \int_0^{\infty} (1 - a\rho)x\,\mathrm{e}^{-\gamma t}\,\mathrm{d}t \tag{12.10}$$

where $\gamma = r - \mu + \rho$.

Solving equation (12.10) we obtain the function for the follower in the stopping region:

$$V_1^{\mathrm{F}}(x) = \frac{x(1 - a\rho)}{\gamma} \tag{12.11}$$

As usual, the optimal investment rule is found by solving for the boundary between the continuation and the stopping regions. The boundary is the trigger point x_{F}. If the value of the state variable is smaller than the trigger, the optimal decision for the investor is not to invest, i.e. to continue in the continuation region; if it exceeds the trigger, then the follower should invest. At the boundary two conditions must be satisfied: the value-matching condition requires that when the state variable reaches the trigger the investor will invest so that:

$$V_0^{\mathrm{F}}(x_{\mathrm{F}}) = V_1^{\mathrm{F}}(x_{\mathrm{F}}) - K \tag{12.12}$$

and the smooth-pasting condition requires that the derivatives of the functions match at the boundary:

$$V'^{\mathrm{F}}_0(x_{\mathrm{F}}) = V'^{\mathrm{F}}_1(x_{\mathrm{F}}) \tag{12.13}$$

Conditions (12.12) and (12.13) imply:

$$x_{\mathrm{F}} = \frac{K\beta_1\gamma}{(1 - a\rho)(\beta_1 - 1)} \tag{12.14}$$

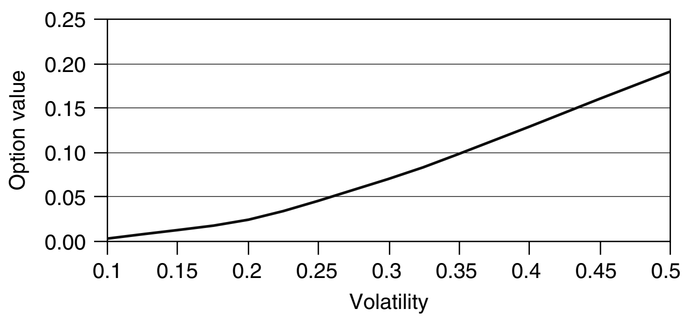
and

$$A = \frac{K\left(\dfrac{K\beta_1\gamma}{(1 - a\rho)(\beta_1 - 1)}\right)^{-\beta_1}}{\beta_1 - 1} \tag{12.15}$$

Putting equations (12.8), (12.11), (12.14) and (12.15) together we obtain the value function of the follower:

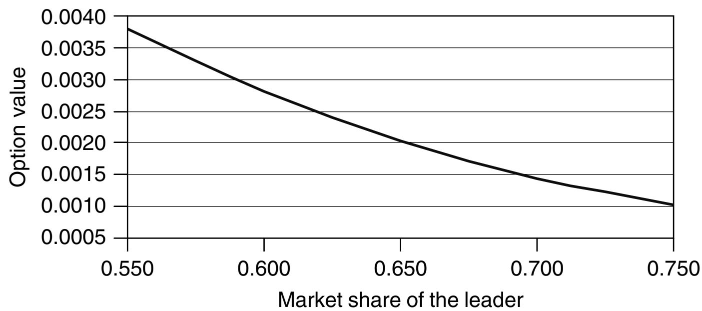
$$V_F(x) = \begin{cases} \dfrac{K}{\beta_1 - 1}\left(\dfrac{x}{x_F}\right)^{\beta_1} & \text{if } x < x_F \\ \dfrac{x(1-a\rho)}{\gamma} - K & \text{if } x \geq x_F \end{cases} \tag{12.16}$$

Figure 12.1 shows the sensitivity of the follower's option to enter to volatility, and as expected, that the option increases with volatility. Figure 12.2 shows the sensitivity of the follower's option to the initial market share, and that as the market share of the follower diminishes so does the value of the option to enter the market. Note that the sensitivity rate declines as the leader's market share increases. Figure 12.3 shows the sensitivity of the follower's option to the parameter ρ. Since ρ explains the evolution of the market share of the leader, increases in ρ imply an increase in the leader's future market share. Thus as



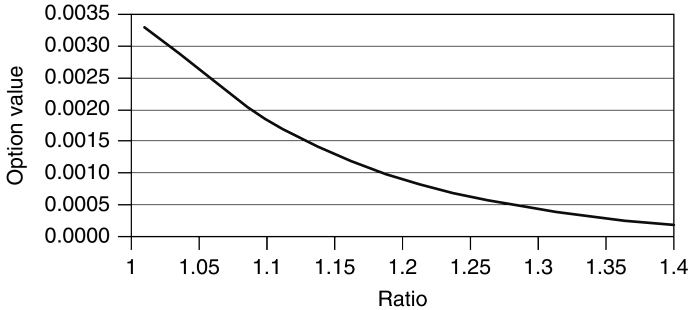
The parameters are: $\mu = 0.02$, $K = 5$, $r = 0.09$, $a = 0.55$, $\rho = 1.01$ and $x = 2$.

Figure 12.1 *Sensitivity of the follower's option to wait to volatility*



The parameters are: $\mu = 0.02$, $K = 5$, $r = 0.09$, $\sigma = 0.1$, $\rho = 1.01$ and $x = 2$.

Figure 12.2 *Sensitivity of the follower's option to wait to market share*



The parameters are: $\mu = 0.02$, $K = 5$, $r = 0.09$, $\sigma = 0.1$, $a = 0.55$ and $x = 2$.
Ratio = immigration/death rates.

Figure 12.3 *Sensitivity of the follower's option to wait to the immigration/death parameter*

a ratio greater than one increases, due to an increase in net immigration, the option value of the follower will decrease, because the probability of obtaining those clients is decreasing.

According to our model the optimal strategy for the follower is stated in Proposition 12.1.

Proposition 12.1 *In a duopoly setting where the market share evolves according to an immigration–death model, the optimal entry time for the follower conditional on a previous entrance of the leader is given by:*

$$T_{\mathrm{F}} = \inf \left\{ t \geq T_{\mathrm{L}} : \; x_t \geq \frac{K}{(\beta_1 - 1)} \left(\frac{x}{x_{\mathrm{F}}} \right)^{\beta_1} \right\} \tag{12.17}$$

where T_{L} is the trigger time for the leader.

12.2.2 The leader's value function

Until the follower enters the market, the leader's decision either to enter the market or to wait may seem identical to the single setting framework. So the basic idea, following Dixit and Pindyck (1994), would be that there exists an optimal time to enter that will maximize the firm's value. Until that moment the firm should wait to invest and its value is explained by the option to wait. When that moment is reached, the firm should invest and its value function is given by the present value of the revenues in perpetuity. The possible problem with the option to wait is that it excludes the case where companies do not have the possibility of waiting, and also that not waiting can itself be an important option.

First-mover advantage should make pre-emption attractive, and pre-emption should lead to early adoption by the leader. Examples where the value of being the first can become very important are: the location of a building because this can determine how profitable it will be and once it is built you cannot change its location; and the decision of companies to have a website. A first-mover company might buy cheaper domain names and obtain lower staff costs and better access to resources. However, the value of the clients obtained by being first is offset by not learning by others' mistakes.

Often, if a company does not make an investment immediately, it loses either the investment opportunity, or the chance of success is diminished. Our model is not concerned with what happens to the leader prior to investment. We are assuming that the fear of pre-emption leads to a possible early entrance into the market, or in other words that the option to wait is nullified by the fear of not achieving the advantage of being first.

Once entering the market, the leader has no further action to take. It will enjoy monopolistic revenues until the moment that the follower enters the market and will share them with the follower afterwards. The value function of the leader, before the follower enters the market, can be explained by the following equation:

$$V^{\mathrm{L}}(x) = E\left[\int_0^{T_{\mathrm{F}}} \mathrm{e}^{-r\tau} x \,\mathrm{d}\tau\right] + E_t[\mathrm{e}^{-rT_{\mathrm{F}}}]\frac{a\rho x_{\mathrm{F}}}{\gamma} - K \tag{12.18}$$

The first function of equation (12.18) represents the monopolistic revenues received by the leader until the follower enters the market.

Let the expectations terms of equation (12.18) be respectively $f(x) = E[\mathrm{e}^{-rT_{\mathrm{F}}}]$ and $g(x) = E\big[\int_0^{T_{\mathrm{F}}} \mathrm{e}^{-r\tau} x \,\mathrm{d}\tau\big]$, where x follows a geometric Brownian motion as described in equation (12.1). Over the time interval $\mathrm{d}t$ we can write the first expectation as:

$$f(x) = \mathrm{e}^{-r\,\mathrm{d}t} E[f(x + \mathrm{d}x)] \tag{12.19}$$

Using Itô's lemma we obtain the following partial differential equation:

$$\tfrac{1}{2}\sigma^2 x^2 f''(x) + \mu x f'(x) - rf(x) = 0 \tag{12.20}$$

with a general solution:

$$f(x) = Cx^{\beta_1} + Dx^{\beta_2} \tag{12.21}$$

where β_1 and β_2 are as defined previously in equations (12.5) and (12.6), respectively, and $f(x)$ represents the expectation of the discounted term at the risk-free rate, during the time T_{F}. So, we can submit equation (12.21) to two boundaries: as our state variable x tends to the trigger price of the follower x_{F}, the optimal time to invest T_F will be very small; obviously the moment that it reaches zero our function $f(x)$ will be one so $\lim_{x \to x_F} f(x) = 1$. The other boundary is defined as x goes to zero. If the revenues go to zero, the follower will not enter in the short run, implying that its optimal time to enter will be very large, so that $f(x)$ will tend to zero, $\lim_{x \to 0} f(x) = 0$. This last boundary implies that D in equation (12.21) has to equal zero, and the first one implies that $C = 1/x_F^{\beta_1}$. So, our equation (12.21) becomes:

$$f(x) = \left(\frac{x}{x_F}\right)^{\beta_1} \tag{12.22}$$

In the same way, $g(x)$ satisfies the following partial differential equation:

$$\tfrac{1}{2}\sigma^2 x^2 g''(x) + \mu x g'(x) - r g(x) + x = 0 \tag{12.23}$$

with a general solution:

$$g(x) = E x^{\beta_1} + F x^{\beta_2} + \frac{x}{r - \mu} \tag{12.24}$$

and subject to two boundary conditions: as x goes to zero, $g(x)$ will tend to zero implying that F in equation (12.24) has to be zero; on the other hand as x tends to x_F, the monopoly revenues will also tend to zero because the follower will enter the market. This last boundary condition implies that: $E = -x_F^{(1-\beta_1)}/(r - \mu)$. So equation (12.24) becomes:

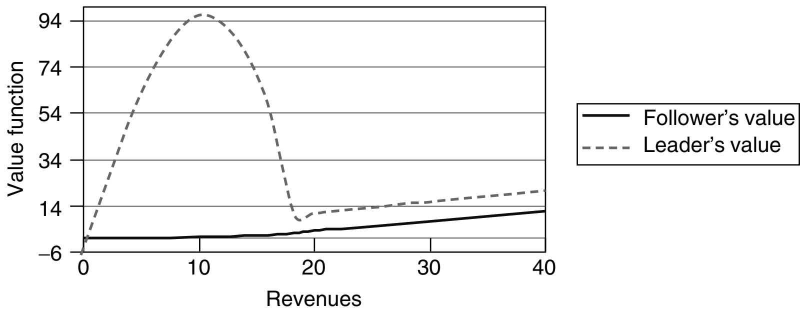
$$g(x) = -\frac{x_F^{1-\beta_1}}{r - \mu} x^{\beta_1} \tag{12.25}$$

Substituting $f(x)$ and $g(x)$ back into equation (12.18) we obtain:

$$V^L(x) = \begin{cases} \dfrac{x}{r - \mu} + E x^{\beta_1} + \dfrac{a\rho x_F}{\gamma}\left(\dfrac{x}{x_F}\right)^{\beta_1} - K & \text{if } x < x_F \\ \dfrac{a\rho x}{\gamma} & \text{if } x \geq x_F \end{cases} \tag{12.26}$$

where $E x^{\beta_1}$ is an option-like term that captures the negative effect that the entry of the follower will have on the leader's value function.

The value functions of the leader and of the follower are shown in Figure 12.4. The value function of the leader is almost always higher than that of the follower. It is possible, as can be seen in the figure, for the follower to have a higher value function when the revenues are very low. In this case the follower has not yet entered the market while the leader has already invested.



The parameters are: $\mu = 0.02$, $K = 5$, $r = 0.09$, $\sigma = 0.1$, $a = 0.55$ and $\rho = 1.01$.

Figure 12.4 *The follower's and leader's value functions*

We can also observe that when the follower enters the two functions almost meet tangentially.[3] Dixit and Pindyck (1994) describe this as a smooth-pasting-like property of present values.

The value function of the leader is more complicated than that of the follower. It is concave until the trigger time of the follower is reached and at that precise moment its slope is discontinuous. This happens because the follower's decision changes discontinuously at x_{F} (Dixit and Pindyck, 1994). The two curves meet at a point that we will designate as x_{L}; this point should be the trigger point of the leader since until that point its value function is negative, following the equalization principle of Fudenberg and Tirole (1985).[4]

Although we cannot obtain an explicit general expression for x_{L}, we can prove that this expression has a root strictly below x_{F}.[5] If we evaluate $V(x)$ at $x = 0$ using our value functions for the leader and the follower, we obtain:

$$V(0) = -K < 0$$

and evaluating $V(x)$ at x_{F}:

$$V(x_{\mathrm{F}}) = \frac{(2a\rho - 1)K\beta_1}{(1 - a\rho)(\beta_1 - 1)} > 0$$

Since $V(x)$ is continuous on the interval $(0, x_{\mathrm{F}})$, it has at least one root in that interval. Uniqueness of the root x_{F} can be proved while demonstrating strict concavity of $V(x)$ over the same interval. The second derivative of $V(x)$ is:

$$V''(x) = -\frac{\beta(\beta - 1)\left(\dfrac{K}{\beta - 1} + \dfrac{x_{\mathrm{F}}}{r - \mu} - \dfrac{a\rho x_{\mathrm{F}}}{\gamma}\right)\left(\dfrac{x}{x_{\mathrm{F}}}\right)}{x_{\mathrm{F}}^2} < 0$$

So, the root is unique, with $V(x) < 0$ for $V(x) \in (0, x_{\mathrm{L}})$ and $V(x) > 0$ for $V(x) \in (x_{\mathrm{L}}, x_{\mathrm{F}})$. Thus we have shown that there exists a single point belonging to the interval $(0, x_{\mathrm{F}})$ at which the leader and the follower have the same value. At any point below that interval the follower has a higher value, meaning that the only motive that can explain a rational leader entering the market is fear of pre-emption. After passing the trigger point of the leader, the leading firm benefits from the advantage of being the leader, in this special case from a higher market share that evolves according to an immigration–death process. In Figure 12.4 we can see that until the leader trigger point is achieved[6] the leader incurs losses while the follower has a positive value function. The stopping time for the leader is given in the following proposition.

Proposition 2.2 *The optimal leader strategy is to invest as soon as the revenues reach x_{L}. In other words the optimal time for the leader to invest is*:

$$T_{\mathrm{L}} = \inf\{t \geq 0 : \ x \in [x_{\mathrm{L}}, x_{\mathrm{F}}]\} \tag{12.27}$$

12.3 VALUE FUNCTION PARTIAL DERIVATIVES

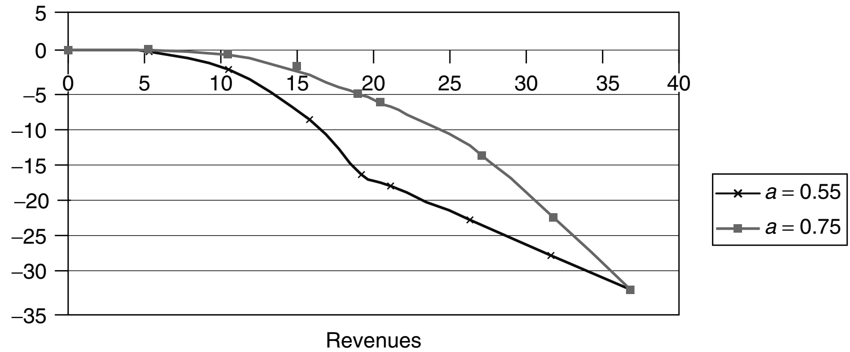
In studying the behavior of our value functions, we derive in this section some partial derivatives, namely we study the sensitivity of our value functions to changes in the market share, in the immigration/death ratio, the revenues and also the sensitivity of the trigger function of the follower to volatility.

The partial derivatives of the value functions to market share ('MS Δ') are:

$$\frac{\partial V_{\mathrm{F}}}{\partial a} = \begin{cases} -\dfrac{\rho x \left(\dfrac{x}{x_{\mathrm{F}}}\right)^{\beta_1 - 1}}{\gamma} < 0 & \text{if } x < x_{\mathrm{F}} \\[2ex] -\dfrac{\rho x}{\gamma} < 0 & \text{if } x \geq x_{\mathrm{F}} \end{cases} \tag{12.28}$$

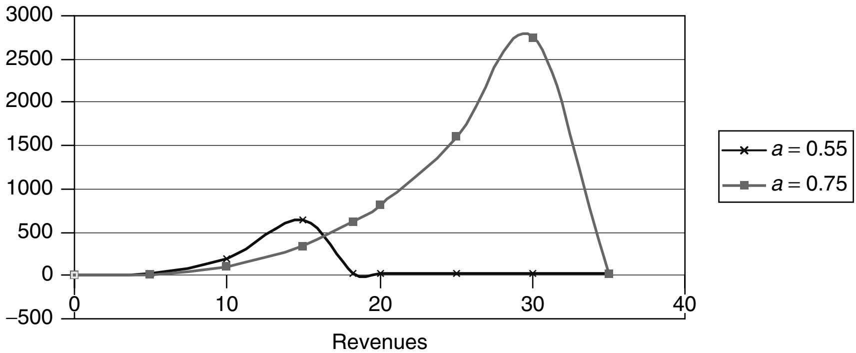
$$\frac{\partial V_{\mathrm{L}}}{\partial a} = \begin{cases} \dfrac{\beta_1 K \rho[\beta_1(r-\mu) + \rho(-1 + \beta_1(1 - ar + a\mu))]}{(\beta_1 - 1)(a\rho - 1)^2(r-\mu)} \left(\dfrac{x}{x_{\mathrm{F}}}\right)^{\beta_1} > 0 \\ \qquad \text{if } x < x_{\mathrm{F}} \\[2ex] \dfrac{\rho x}{\gamma} > 0 \quad \text{if } x \geq x_{\mathrm{F}} \end{cases} \tag{12.29}$$

Figures 12.5 and 12.6 show that the MS Δ of the follower and leader have contrasting reactions to different revenue levels. Note that in our model we are assuming that the initial market share is shared by two parties. Since a



The parameters are: $\mu = 0.02$, $K = 5$, $r = 0.09$ and $\rho = 1.01$.

Figure 12.5 *Sensitivity of the follower's MS Δ to revenues*



The parameters are: $\mu = 0.02$, $K = 5$, $r = 0.09$ and $\rho = 1.01$.

Figure 12.6 *Sensitivity of the leader's MS Δ to revenues*

represents the leader's initial share, the sign of the partial derivatives is consistent with the leader's value increasing with a, prior to the follower's entry. Consequently, an increase in the initial market share of the leader will imply a decrease in the market share of the follower, so the value function of the follower has to decrease with the market share of the leader. The slope of Figure 12.2 is negative, and the curve concave; the slope of Figure 12.5 is negative, and convex (at least before the trigger). Another interesting, though expected conclusion, is that since an increase in the market share implies an increase in the value function of the leader, and a decrease in that of the follower, and since market share appears in the model as an advantage of the leader, pre-emption is obvious and seems to justify what is described in the literature as the fear of pre-emption.

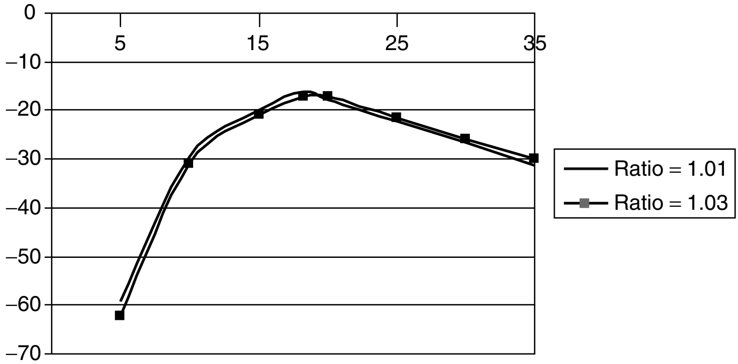
The market share as a pre-emption advantage is further pronounced prior to the entrance of the follower. After the follower enters, the leader continues to benefit from increases in its market share, and the follower continues to have a decrease in its value function, but the magnitude of changes in the market share is exactly the same for both, obviously of different sign.[7] But prior to the follower's entrance the difference between the two functions is not only in sign but also in magnitude. Although the market share as a pre-emptive factor will always constitute an advantage over the follower, the higher advantage relative to the follower will occur during the time interval that the follower is inactive. The optimal time for the follower to invest increases with a and consequently the leader will enjoy monopolistic revenues for longer.[8]

The partial derivatives of the value functions to the immigration/death ratio ('Ratio Δ') are:

$$\frac{\partial V_F}{\partial \rho} = \begin{cases} \dfrac{\beta_1 K \left(\dfrac{x}{x_F}\right)^{1-\beta_1} \left(-\dfrac{x}{x_F \gamma} - \dfrac{a(\beta_1 - 1)x}{\beta K \gamma}\right)}{\beta_1 - 1} < 0 & \text{if } x < x_F \\ -\dfrac{(1 - a\rho)x}{\gamma^2} - \dfrac{ax}{\gamma} < 0 & \text{if } x \geq x_F \end{cases} \tag{12.30}$$

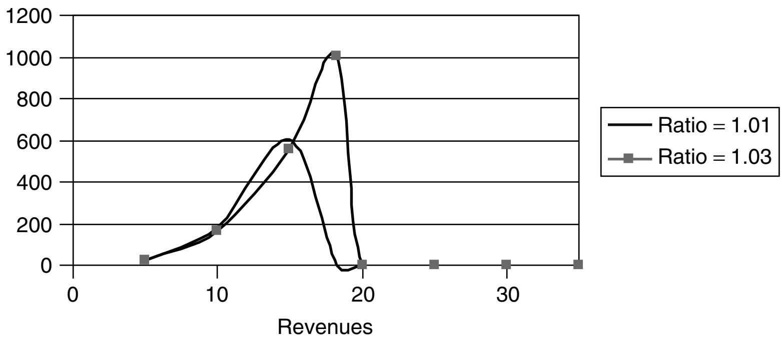
$$\frac{\partial V_L}{\partial \rho} = \begin{cases} \dfrac{\left(\dfrac{x}{x_F}\right)^{\beta_1} \{\beta_1 K[\rho[(-1 - \beta_1(-1 + a^2(r - \mu)^2)] + (r - \mu)(-1 + \beta_1 + a\beta_1 r - a\beta_1 \mu)]\}}{(\beta_1 - 1)(a\rho - 1)^2(r - \mu)\gamma} > 0 & \text{if } x < x_F \\ \dfrac{ax}{\gamma} - \dfrac{a\rho x}{\gamma^2} > 0 & \text{if } x \geq x_F \end{cases} \tag{12.31}$$

Figures 12.7 and 12.8 show that the Ratio Δ's are similar to the MS Δ's, after the follower's trigger. An increase in the ratio, that is the number of new clients divided by the ones that leave, signifies an increase in the evolving market share of the leader and consequently leads to the same conclusions as for parameter a. When $x < x_F$, the sensitivity of the follower's and the leader's value functions to changes in ρ will depend on the parameter values, particularly (x/x_F) times some variables divided by ρ for the follower, and the same ratio times some variables multiplied and divided by ρ for the leader. For illustrative parameters herein, $K = 5$, $r = 0.09$, $\mu = 0.02$, $\sigma = 0.10$, $a = 0.55$, over a range of $\rho = 0.97$ to 1.07, the follower's value function is less sensitive than the leader's value function to changes in ρ when x is slightly less than x_F and more sensitive to changes in ρ when x is slightly greater than x_F but always of opposite sign.



The parameters are: μ = 0.02, K = 5, r = 0.09 and a = 0.55.

Figure 12.7 *Sensitivity of the follower's ratio Δ to revenues*



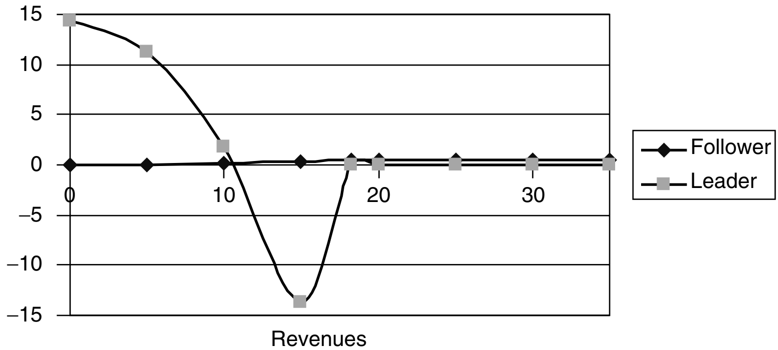
The parameters are: μ = 0.02, K = 5, r = 0.09 and a = 0.55.

Figure 12.8 *Sensitivity of the leader's ratio Δ to revenues*

The partial derivatives of the value functions to the revenues (delta) are:

$$\frac{\partial V_F}{\partial x} = \begin{cases} \dfrac{(1-a\rho)\left(\dfrac{x}{x_F}\right)^{\beta_1-1}}{\gamma} > 0 & \text{if } x < x_F \\ \dfrac{1-a\rho}{\gamma} > 0 & \text{if } x \geq x_F \end{cases} \tag{12.32}$$

$$\frac{\partial V_L}{\partial x} = \begin{cases} \dfrac{1}{r-\mu} - \dfrac{\beta_1\left(\dfrac{x}{x_F}\right)^{\beta_1-1}}{r-\mu} + \dfrac{a\beta_1\rho\left(\dfrac{x}{x_F}\right)^{\beta_1-1}}{\gamma} >< 0 & \text{if } x < x_F \\ \dfrac{a\rho}{\gamma} > 0 & \text{if } x \geq x_F \end{cases} \tag{12.33}$$



The parameters are: $\mu = 0.02$, $K = 5$, $r = 0.09$, $a = 0.55$ and $\rho = 1.01$.

Figure 12.9 *Sensitivity of the follower's and leader's delta to revenues*

The value function of the follower behaves as expected; delta is always positive. As the revenues increase, so does the follower's value function. The value function of the leader decreases as the state variable increases, as shown in Figure 12.9, while $x < x_{\mathrm{F}}$. There is a trade-off between monopoly revenue enjoyed by the leader and the likelihood that the monopoly will end with the follower's entry.

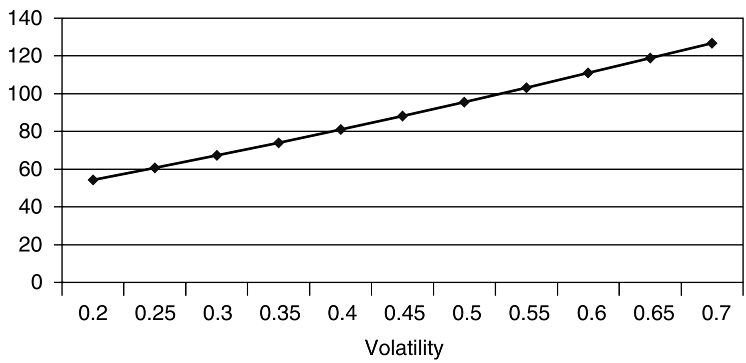
Thus the leader's value function MS Δ, Ratio Δ and delta are highly sensitive (and change signs) to expected revenues slightly below the follower's trigger revenues. In a broad sense, 'delta' hedging of the leader's value function would be very complex, and probably confounded by transaction costs.

Finally, volatility is one of the most important parameters in option pricing. From the literature, we expect an increase in real option value with volatility. Since the value functions and trigger points are complex functions of volatility we computed only the 'vega' of the trigger function of the follower, which is:

$$\frac{\partial x_{\mathrm{F}}}{\partial \sigma} = \frac{4K\gamma\left[\mu\sigma^2\left(1+\sqrt{1+\dfrac{4\mu^2}{\sigma^4}+\dfrac{8r}{\sigma^2}-\dfrac{4\mu}{\sigma^2}}\right)-2\mu^2-2r\sigma^2\right]}{(a\rho-1)\sigma\left[2\mu+\sigma^2-2\sigma^2\sqrt{\left(\dfrac{\mu}{\sigma^2}-\dfrac{1}{2}\right)^2+\dfrac{2r}{\sigma^2}}\right]^2 \times\sqrt{\left(\dfrac{\mu}{\sigma^2}-\dfrac{1}{2}\right)^2+\dfrac{2r}{\sigma^2}}} > 0 \tag{12.34}$$

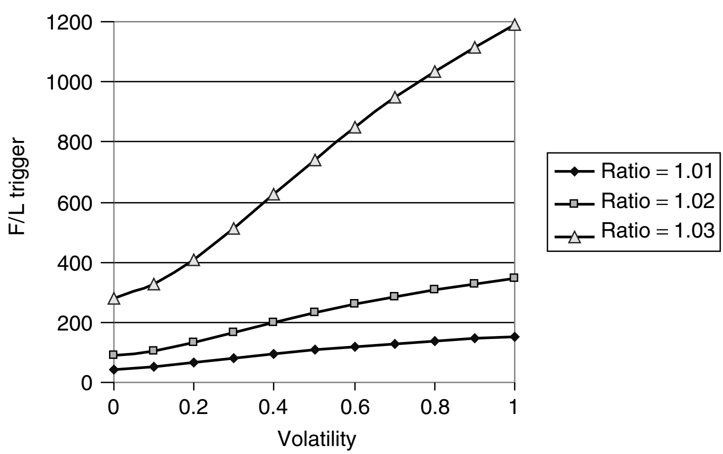
The vega behaves as expected, that is increases in volatility lead to increases in the trigger value function of the follower, because the option value of waiting until some of the uncertainty will be resolved increases. This result is confirmed in Figures 12.1 and 12.10. Figure 12.10 shows that the follower's trigger increases (almost) linearly with volatility.

Figure 12.11 shows the behavior of the trigger function of the follower divided by that of the leader as volatility and the immigration–death parameters increase. Volatility increases induce an increase of higher magnitude in the trigger function of the follower compared to the leader. This result leads to the conclusion that the follower's decision to invest is more affected by volatility than that of the leader, probably because the follower is the one being pre-empted. Notice that as the advantage of being the first increases, so does the



The parameters are: $\mu = 0.02$, $K = 5$, $r = 0.09$, $a = 0.55$, $\rho = 1.01$ and $x = 2$.

Figure 12.10 *Sensitivity of the partial derivative of the follower's trigger function to volatility, as a function of volatility*



The parameters are: $\mu = 0.02$, $K = 5$, $r = 0.09$, $a = 0.55$ and $x = 5$.

Figure 12.11 *Sensitivity of the follower's/leader's trigger functions to volatility and the immigration/death ratio*

ratio of the follower/leader triggers. When the advantage of being first is very high, the follower will desire even more that the uncertainty is resolved before investing, meaning that the follower will attribute higher value to the option to wait. In contrast, although there is a higher value for the leader's option to wait while facing higher volatility, first-mover advantage gives a lower value relative to the follower.

12.4 CONCLUSION

For a duopoly environment, we model the leader and follower value functions assuming that the leader's 'market share' evolves according to an immigration (birth) and death process. We define 'market share' broadly as the multiplier for a standard revenue x. The multiplier is itself adjusted over time by a parameter ρ, which is immigration (λ) divided by death (ν) (new customers adjusted for old and new customers leaving). We derive analytical solutions for the options to invest, and numerical solutions for the leader's optimal investment trigger. Then we calculate the partial derivatives of the leader and follower value functions to market share, birth/death parameters and market profitability. This model is possibly more realistic than that proposed by some other authors studying the advantages of being first (and also being a follower).

We show that over certain ranges of the parameter values, the leader and follower real options to wait to invest, and not to wait to invest, are sometimes surprising and not immediately intuitive. The follower's value function is less sensitive than the leader's value function to market share or the rate of customer arrivals/departures, until the expected revenue exceeds the follower's trigger investment level. The follower's trigger increases with market share, the immigration/death ratio and revenue volatility.

The leader's value function 'deltas' are highly sensitive and unstable as revenues approach the follower's trigger, confirming the adage, if you're ahead, 'watch the competition'.

12.5 APPENDIX: THE IMMIGRATION–DEATH MODEL

Immigration: individuals join the population according to a Poisson process at rate λ.

Death: the lifetime distribution of each individual is exponential, $M(\nu)$.

The overall birth and death rates of this process are:

$$\beta_x = \lambda, \quad \nu_x = \nu x$$

The Kolmogorov forward equation is:

$$\frac{\mathrm{d}}{\mathrm{d}t} p_x(t) = \beta_{x-1} p_{x-1}(t) + \nu_{x+1} p_{x+1}(t) - (\beta_x + \nu_x) p_x(t) \tag{A12.1}$$

So the differential-difference equations for the immigration–death model are:

$$\frac{\mathrm{d}}{\mathrm{d}t}p_x(t) = \lambda p_{x-1}(t) + \nu(x+1)p_{x+1}(t) - (\lambda + \nu x)p_x(t) \quad x = 1, 2, \ldots \tag{A12.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}p_x(t) = \nu p_1(t) - \lambda p_0(t) \quad x = 0 \tag{A12.3}$$

12.5.1 The equilibrium distribution

The equilibrium distribution, if it exists, is found by putting $\mathrm{d}p_x(t)/\mathrm{d}t = 0$ in the forward equations, and solving:

For $x = 0$, $\quad \nu p_1 - \lambda p_0 = 0, \quad$ so $p_1 = \dfrac{\lambda}{\nu}p_0$

For $x = 1$, $\quad \lambda p_0 + 2\nu p_2 - (\lambda + \nu)p_1 = 0, \quad$ so $p_2 = \dfrac{\lambda}{2\nu}p_1,\ p_1 = \dfrac{\lambda^2}{2\nu^2}p_0$

For $x = 2$, $\quad \lambda p_1 + 3\nu p_3 - (\lambda + 2\nu)p_2 = 0, \quad$ so $p_3 = \dfrac{1}{3!}\left(\dfrac{\lambda}{\nu}\right)^3 p_0$

For general x:

$$p_x = \frac{1}{x!}\left(\frac{\lambda}{\nu}\right)^x p_0$$

For a proper probability distribution, $\sum_{x=0}^{\infty} p_x = 1$:

$$\sum_{x=0}^{\infty} p_x = p_0 \sum_{x=0}^{\infty} \frac{1}{x!}\left(\frac{\lambda}{\nu}\right)^x = p_0\,\mathrm{e}^{\lambda/\nu}, \qquad p_0 = \mathrm{e}^{-\lambda/\nu}$$

The equilibrium is:

$$p_x = \frac{\mathrm{e}^{-\lambda/\nu}\left(\dfrac{\lambda}{\nu}\right)^x}{x!}, \qquad x = 0, 1, \ldots$$

This is a Poisson distribution with parameter $\rho = \lambda/\nu$.

Assuming that the conditional probability of the market share having a certain value in a short interval of length $\mathrm{d}t$ is $\rho\,\mathrm{d}t$, and the density function of the time that the company takes to achieve a certain market share a is given by $\rho\,\mathrm{e}^{-\rho t}$, then equation (12.9) follows.

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NOTES

1. Profit Impact of Market Strategies.
2. See Karlin and Taylor (1975, chapter 4).
3. The leader always has a higher value function of $x \geq x_{\mathrm{L}}$ because we are assuming a first-mover advantage.
4. At the leader's investment point the expected payoff of the two firms must be equal. If this were not the case, one firm would have an incentive to deviate and the proposed outcome would not be an equilibrium.
5. This implies that the trigger point exists and is unique.
6. The trigger point for the leader for $r = 0.09$, $\mu = 0.02$, $K = 5$, $\sigma = 0.1$, $a = 0.55$ and $\rho = 1.01$ is 0.35 (calculated numerically).
7. Note that after the follower enters, the partial derivatives of the value functions are exactly the same, except for the sign.
8. The relative higher advantage of the value function of the leader compared to the follower prior to the entrance of the latter can also be seen in Figure 12.4.

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Chapter 13

R&D investment decision and optimal subsidy

JYH-BANG JOU AND TAN LEE

SUMMARY

This chapter assumes that a firm facing technological uncertainty must decide when to purchase R&D capital. R&D capital exhibits both irreversibility and externality through the learning-by-doing effect. The combination of irreversibility and uncertainty drives agents to be more prudent, i.e. the maxim ‘better safe than sorry’ applies. This maxim is more important if uncertainty is greater, technology progresses at a lower pace, the externality is stronger, or a catastrophic event is less likely to occur. A firm ignoring the externality will both invest later and disinvest earlier than a social planner who internalizes the externality. An equal rate of investment tax credits should be given to both costlessly reversible investments and irreversible ones, and the same rate of taxation should be imposed on disinvestment.

13.1 INTRODUCTION

Research and development (R&D) activities have three major characteristics. First, R&D outlays are usually irreversible. Second, the future rewards from R&D activities are usually uncertain. Finally, R&D activities may exhibit positive externalities. Previous literature fails to combine these characteristics. The literature on the ‘endogenous growth’ theory (see Romer, 1986) emphasizes the last one while abstracting from the first two. In contrast, real options literature (see Dixit and Pindyck, 1994; Schwartz and Moon, 2000) emphasizes the first two while ignoring the last one. This chapter will introduce externalities into real options literature so as to examine the issues regarding both R&D investment decisions and the optimal R&D subsidy.

We assume the industry considered has a fixed number of identical firms and its demand function has constant elasticity. Each firm's production is in a Cobb–Douglas form with respect to its employed workers, its employed R&D capital, the industry's average amount of R&D capital, and a multiplicative technology-shift factor. The last one, which evolves as a mix of a Poisson process and a geometric Brownian motion, captures the uncertainty faced by all firms in the industry. The combination of irreversibility and uncertainty induces agents to be more prudent. Given that firms may either invest or disinvest R&D capital when the payoff is uncertain, this implies that agents will both invest later and disinvest earlier, compared to the case either when there is no uncertainty or when there is complete reversibility.

Externalities due to investments may result in inefficient market solutions, because a firm ignoring the externality will both invest later and disinvest earlier than a social planner. Accordingly, there is room for governmental intervention. This chapter will focus on both tax credits on investment and taxation on disinvestment while abstracting from any other policy instrument. This chapter will show that an equal rate of investment tax credits should be given to both costlessly reversible investments and irreversible ones. In addition, this optimal rate coincides with the optimal rate of taxation on disinvestment.

Before proceeding, it is important to place our analysis in the context of both real options and optimal subsidy literature. This chapter abstracts from two main aspects of R&D investment that have received attention in real options literature:[1] sequential nature and strategic interactions between firms. On the one hand, Roberts and Weitzman (1981) suggest that the investment in R&D is usually an 'exploratory' one that discloses information for the later-stage investment such as marketing or commercialization. Several articles on real options have incorporated this suggestion, see Bar-Ilan and Strange (1998), Dixit and Pindyck (1994, chapter 10), Grenadier and Weiss (1997), Lambrecht (2000), Pindyck (1993) and Schwartz and Moon (2000). In contrast, this chapter, like Weeds (1999), assumes that R&D capital is input for producing final goods. On the other hand, Beath et al. (1989), Fudenberg et al. (1983) and Harris and Vickers (1985) have allowed firms to compete for R&D investment. This has been incorporated into real options literature by Dixit and Pindyck (1994, chapter 9), Lambrecht (2000) and Weeds (2000). In contrast, this chapter assumes that competition between firms is of a Cournot–Nash type, thus abstracting from the Stackelberg-type competition addressed by those articles.

Several articles examine the issue of optimal subsidy on capital while assuming that capital exhibits costless reversibility, e.g. Romer (1986) and Judd (1997). Romer considers that human capital rather than physical capital exhibits externalities through the 'learning-by-doing' effect. He finds that the optimal tax credit on human capital depends on the relative magnitude between the external and the internal effect of capital if the production technology is in a Cobb–Douglas form with respect to an individual firm's human capital input and the industry's aggregate stock of human capital.[2] His results are similar to ours. Judd considers imperfect rather than perfect competitive markets. He shows that subsidies on capital should be higher if the gap between price and marginal cost is higher. His analysis thus justifies the investment tax credit policy commonly employed in the US which favors equipment over structures. In contrast, this chapter abstracts from imperfect competitive markets and finds that investment tax credits should be equal across industries.

Section 13.2 develops the model for optimal individual and then social profit with respect to R&D investment. Section 13.3 derives the optimal individual and social investments, along with the optimal tax and tax credits. Section 13.4 provides numerical examples, and Section 13.5 concludes.

13.2 THE MODEL

Dixit (1991) builds a model to examine the impact of price ceilings on irreversible investment decisions. This chapter introduces externalities into his model while abstracting from price ceilings. The industry under consideration is composed of N identical risk-neutral firms, indexed by 1 to N, which face a demand function with a constant elasticity ε (≥ 0), i.e.

$$Q(t) = P(t)^{-\varepsilon} \tag{13.1}$$

where $Q(t)$ is quantity and $P(t)$ is price. The output of each i firm depends on its employed labor force, denoted by $l_i(t)$, its employed capital stock, denoted by $k_i(t)$, the industry's aggregate capital stock, denoted by $K_{\mathrm{a}}(t)$ $(= \sum_{j=1}^{N} k_j(t))$, and a technology-shift factor, denoted by $Z(t)$. More precisely, the production technology is given by:

$$q_i(t) = l_i(t)^{\gamma}(Z(t)k_i(t))^{1-\gamma}K_{\mathrm{a}}(t)^{\lambda} \tag{13.2}$$

Two sources of technology uncertainty are considered. First, following Schwartz and Moon (2000) and Weeds (1999), a catastrophic event that suddenly drives $Z(t)$ to zero is assumed to follow a Poisson process with a hazard rate μ (which is the probability per unit of time that drives $Z(t)$ to zero). Second, in the case where a catastrophic event does not happen, $Z(t)$ evolves as a geometric Brownian motion:

$$\mathrm{d}Z(t) = \eta Z(t)\mathrm{d}t + \sigma Z(t)\mathrm{d}\Omega(t) \tag{13.3}$$

where the drift parameter η is the expected growth rate of $Z(t)$ and the parameter σ (>0) is the instantaneous volatility of the growth rate of $Z(t)$, and $\mathrm{d}\Omega(t)$ is an increment to a standard Wiener process, with $E\{\mathrm{d}\Omega(t)\} = 0$ and $E\{\mathrm{d}\Omega(t)\}^2 = \mathrm{d}t$. For a high-technology environment such as DRAM production, $Z(t)$ may exhibit a higher expected growth rate. Accordingly, the expected unit production cost tends to decline over time. A high-technology environment may also have a higher volatility of $Z(t)$, and therefore may have a wide variation of unit production costs. Furthermore, the specification in equation (13.3) also indicates that information about the evolution of each individual firm's output that arrives in time is independent of its decision to invest. This contrasts with that of both Pindyck (1993) and Schwartz and Moon (2000), where uncertainty can be reduced over time through learning.

The production function given by equation (13.2) suggests that R&D capital exhibits externality because the output of firm i will be higher not only when the firm itself installs more capital, but also when any other firm in the industry installs more capital. This kind of externality may result from the 'learning-by-doing' effect, first addressed by Arrow (1962) and later widely employed by 'endogenous growth' models (e.g. Romer, 1986; Lucas, 1988). In equation (13.2), the effect of $k_i(t)$ on $q_i(t)$ is internal, and the term $1 - \gamma$ measures its size. In contrast, the effect of $K_\mathrm{a}(t)$ on $q_i(t)$ is external with λ measuring its size.

Suppose that $q(t) = (q_1(t), \ldots, q_N(t))$ and $k(t) = (k_1(t), \ldots, k_N(t))$. Denote by w the wage rate. From equation (13.2), firm i's short-run variable cost, $wl_i(t)$, will then be given by:

$$C_i(q(t), k(t), Z(t)) = wq_i(t)^g (Z(t)k_i(t))^{1-g} K_\mathrm{a}(t)^{-\lambda g} \tag{13.4}$$

where $g = 1/\gamma > 1$. Differentiating equation (13.4) with respect to $q_i(t)$ yields firm i's short-run marginal cost as given by:

$$MC_i(q(t), k(t), Z(t)) = wgq_i(t)^{g-1} (Z(t)k_i(t))^{1-g} K_\mathrm{a}(t)^{-\lambda g} \tag{13.5}$$

The operating flow profit of firm i, denoted by $\pi_i(q(t), k(t), Z(t))$, is equal to its revenue, $P(t)q_i(t)$, net of its operating cost, $C_i(\cdot)$ in equation (13.4), thus yielding:

$$\pi_i(q(t), k(t), Z(t)) = P(t)q_i(t) - C_i(q(t), k(t), Z(t)) \tag{13.6}$$

In (Cournot–Nash) short-run equilibrium, firm i will take the other firms' production strategies as given while choosing an amount of output, denoted by $q_i(t)^*$, to maximize its operating flow profit, $\pi_i(\cdot)$ given by equation (13.6). Consequently, $q_i(t)^*$ is derived by setting the derivative of $\pi_i(\cdot)$ with respect to $q_i(t)$ equal to zero. This yields the equality of the marginal revenue, $1 - q_i(t)/(\varepsilon Q(t))$ multiplied by $P(t)$ defined in equation (13.1), with the short-run marginal cost, $MC_i(\cdot)$ given by equation (13.5). This equality, together with the equilibrium condition $q_i(t) = q_i(t)^*$ $(j = 1, \ldots, N)$ yields $q_i(t)^*$ and its corresponding $P(t)^*$ as respectively given by:

$$q_i(t)^* = \left[\frac{w}{\gamma} N^e \left(1 - \frac{e}{N}\right)^{-1}\right]^{\frac{-1}{(e+g-1)}} [(Z(t)k_i(t))^{1-\gamma} K_{\mathrm{a}}(t)^{\lambda}]^{\frac{g}{(e+g-1)}} \quad (13.7)$$

and

$$P(t)^* = \left[\frac{w}{\gamma} \left(1 - \frac{e}{N}\right)^{-1}\right]^{\frac{e}{(e+g-1)}} N^{\frac{-e(g-1)}{(e+g-1)}} [(Z(t)k_i(t))^{1-\gamma} K_{\mathrm{a}}(t)^{\lambda}]^{\frac{-eg}{(e+g-1)}} \quad (13.8)$$

where $e = 1/\varepsilon$, which is required to be smaller than the number of firms N to ensure that $q_i(t)^*$ and $P(t)^*$ are both positive.

For ease of exposition, define $k_{\mathrm{a}}(t)(= K_{\mathrm{a}}(t)/N)$ as the average capital stock of the industry. Evaluating $\pi_i(\cdot)$ given by equation (13.6) at $q_i(t) = q_i^*(t)$ $(j = 1, \ldots, N)$ yields the optimized value of firm i's private flow profit, denoted by $\pi_i^1(k(t), Z(t))$, as shown by equation (13.9) below. Define $f = eg/(e + g - 1)$, then:

$$\pi_i^1(k(t), Z(t)) = d_0 Z(t)^{(1-f)} k_i(t)^{(1-f)} k_{\mathrm{a}}(t)^{\frac{(1-e)f\lambda}{e}} \quad (13.9)$$

where:

$$d_0 = \left[1 - \left(1 - \frac{e}{N}\right)\gamma\right] N^{-f\left(1 - \frac{(1-e)\lambda}{e}\right)} \left(1 - \frac{e}{N}\right)^{\frac{(1-e)}{(e+g-1)}} \left(\frac{w}{\gamma}\right)^{\frac{(e-1)}{(e+g-1)}}$$

In contrast, a social planner will internalize the external effect of R&D capital before investing. The social planner understands that firm i's capital stock, $k_i(t)$, is equal to the industry's average level of capital stock, $k_{\mathrm{a}}(t)$, because all firms are identical. Substituting this equality into the right-hand side of equation (13.9) yields the optimized value of the social flow profit, denoted by $\pi_i^2(k_i(t), Z(t))$, as given by:

$$\pi_i^2(k_i(t), Z(t)) = d_0 Z(t)^{(1-f)} k_i(t)^{\left(1-f+\frac{(1-e)f\lambda}{e}\right)} \quad (13.10)$$

13.3 OPTIMUM INVESTMENT TAX CREDITS

For ease of exposition, we ignore capital depreciation and assume that the purchasing and installation price of capital, denoted by P_K, is constant over time. The resale price of capital is denoted by θP_K, where $1 \geq \theta \geq 0$. When $\theta = 1$, then capital exhibits complete reversibility; otherwise, capital exhibits at least partial irreversibility.[3]

When R&D capital exhibits externality, as suggested by equation (13.2), then market outcomes will be inefficient. The policy to correct this includes an investment tax credit, deposit refund schemes, and accelerated depreciation (Hassett and Hubbard, 1996). In the following, we will focus on the first one. We assume there is a simplified tax system in which a government collects lump-sum taxes from firms, and then returns the proceeds to them in the form of an investment tax credit once they invest. The rate of this credit is denoted by h so that the purchase price of capital faced by firms is given by $(1-h)P_K$. Similarly, we will assume that a government imposes a tax once firms disinvest and then returns the proceeds from taxation to them in the form of lump-sum transfers. The tax rate is denoted by u so that the net revenue firms receive from selling per unit of capital is given by $(1-u)\theta P_K$.

13.3.1 Costlessly reversible investment

Suppose that investment is costlessly reversible, then $k_i(t)$ will be a choice variable rather than a state variable. Let ρ denote a given discount rate. The R&D capital stock chosen by firm i at each instant is found by equating the private marginal return to capital with the user cost of capital (Jorgenson, 1963), i.e.

$$\frac{\partial \pi_i^1(k(t), Z(t))}{\partial k_i(t)} = (1-h)(\rho+\mu)P_K \tag{13.11}$$

The left-hand side of equation (13.11) indicates that the marginal return to firm i's capital is equal to the derivative with respect to $k_i(t)$ of its optimized private flow profit, $\pi_i^1(\cdot)$ given by equation (13.9). The right-hand side of equation (13.11) indicates that the user cost is equal to the rental cost of capital, $(1-h)\rho P_K$, plus capital loss, $(1-h)\mu P_K$; the latter arises because the probability of a catastrophic event can be interpreted as a 'tax rate' μ on the value of capital since on average a fraction of μ of this value will be lost per unit of time (Brennan and Schwartz, 1985). In (Cournot–Nash) industry equilibrium, each firm will choose an equal amount of capital stock, denoted by $k_{\mathrm{f}1}(\cdot, h, t)$, where the first subscript 'f' denotes the case for a frictionless world, and the second subscript '1' denotes the case for a decentralized economy. Substituting this equilibrium condition into equation (13.11), and rearranging yields:

$$k_{\mathrm{f}1}(\cdot, h, t) = \left[\frac{d_0(1-e)f}{e(1-h)(\rho+\mu)P_K}\left(1-\gamma+\frac{\lambda}{N}\right)Z(t)^{(1-f)}\right]^{\frac{1}{f\left(1-\frac{(1-e)\lambda}{e}\right)}} \tag{13.12}$$

To ensure that $k_{\mathrm{f}1}(\cdot, h, t)$ is decreasing with P_K, here and in what follows, we will assume that $e > (1-e)\lambda$. This inequality is more likely to hold if either demand elasticity is lower (ε is lower) or the external effect is less significant (λ is lower).

Similarly, the capital stock chosen by a social planner at each instant, denoted by $k_{f2}(\cdot, t)$ (the second subscript '2' denotes the case for a centralized economy), is found by equating the social marginal return to capital, the derivative with respect to $k_i(t)$ of the optimized social flow profit, $\pi_i^2(\cdot)$ given by equation (13.10), with the user cost of capital, i.e.[4]

$$\frac{\partial \pi_i^2(k_i(t), Z(t))}{\partial k_i(t)} = (\rho + \mu) P_K \tag{13.13}$$

Evaluating the left-hand side of equation (13.13) at $k_i(t) = k_{f2}(\cdot, t)$ and then rearranging yields:

$$k_{f2}(\cdot, t) = \left[\frac{d_0(1-e) f(1-\gamma+\lambda)}{e(\rho+\mu) P_K} Z(t)^{(1-f)}\right]^{\frac{1}{f\left(1-\frac{(1-e)\lambda}{e}\right)}} \tag{13.14}$$

The role of externality is to raise the optimal stock of R&D capital. In other words, in the absence of any investment tax credits ($h = 0$), the social marginal return to capital, the term on the left-hand side of equation (13.13), will outweigh the private marginal return to capital, the term on the left-hand side of equation (13.11). Consequently, the R&D capital stock chosen by a social planner, $k_{f2}(\cdot, t)$ given by equation (13.14), will be higher than that chosen by an individual firm, $k_{f1}(\cdot, 0, t)$ given by equation (13.12). An investment tax credit can abolish this wedge, as suggested by Proposition 13.1.

Proposition 13.1 *Suppose that h_f^* denotes the optimal rate of investment tax credits for R&D capital which exhibits complete reversibility, then*:

$$h_f^* = \frac{\lambda}{(1-\gamma+\lambda)}\left(1 - \frac{1}{N}\right) \tag{13.15}$$

Proof: h_f^* is the h that satisfies the equality:

$$k_{f1}(\cdot, h, t) = k_{f2}(\cdot, t) \tag{13.16}$$

Replacing $k_{f1}(\cdot, h, t)$ given by equation (13.12) with $k_{f2}(\cdot, t)$ given by equation (13.14) yields h_f^* as shown in equation (13.15).

13.3.2 Irreversible investment

When R&D capital exhibits irreversibility, then in the long run, through Cournot–Nash competition, firm i will maximize the expected discounted private flow profit, net of the costs from purchasing capital or plus the proceeds from selling capital, taking the strategies of the other firms as given (Baldursson, 1998; Dixit, 1991; Lucas and Prescott, 1971). Consequently, the Bellman value function of firm i, denoted by $V_1(k(t), Z(t))$, is given by:

$$V_1(\cdot) = \max_{\{k_i(\tau)\}} E_t \int_t^{\infty} \mathrm{e}^{-\rho(\tau-t)}[\pi_i^1(k(\tau), Z(\tau))\mathrm{d}\tau - 1_{[\mathrm{d}k_i(\tau)>0]} \times P_K \, \mathrm{d}k_i(\tau) + 1_{[\mathrm{d}k_i(\tau)<0]}\theta P_k \, \mathrm{d}k_i(\tau)] \quad (13.17)$$

where $\pi_i^1(\cdot)$ is given by equation (13.9), ρ is a given discount rate, $k(\tau) = (k_1(\tau), \ldots, k_N(\tau))$, $1_{[\cdot]}$ is an indicator function which is equal to one if the condition within $[\cdot]$ is satisfied, and zero otherwise, and $E_t\{\cdot\}$ denotes conditional expectation taken at time t. The maximization problem faced by firm i amounts to choosing the optimal path for $k_i(t)$. There are $N+1$ state variables in this maximization problem, $k(t)$ and $Z(t)$; $k_{-i}(t) = (k_1(t), \ldots, k_{i-1}(t), k_{i+1}(t), \ldots, k_N(t))$ and $Z(t)$ are exogenously given, while $k_i(t)$ is subject to control. However, all $N-1$ elements of $k_{-i}(t)$ should be equal to $k_i(t)$ in industry equilibrium.

As is well known in real options literature (e.g. Dixit and Pindyck, 1994), the interaction of the stochastic evolution of $Z(t)$ and capital irreversibility indicates that firm i solves a problem of two-sided instantaneous control of Brownian motion. The optimal policy is to regulate the state variable $k_i(t)$ at both an upper barrier, denoted by $k_i(t)^*$, and a lower barrier, denoted by $k_i(t)_*$ (Harrison and Taksar, 1985). In other words, as long as $k_i(t)^* > k_i(t) > k_i(t)_*$, no action will be taken (i.e. $\mathrm{d}k_i(t) = 0$). However, when $k_i(t)$ hits $k_i(t)^*$, a minimum amount of capital to be sold is chosen to prevent $k_i(t)$ from rising above $k_i(t)^*$ (i.e. $\mathrm{d}k_i(t) < 0$). Similarly, when $k_i(t)$ hits $k_i(t)_*$, a minimum amount of capital stock to be purchased is chosen to prevent $k_i(t)$ from falling below $k_i(t)_*$ (i.e. $\mathrm{d}k_i(t) > 0$), the 'desired' capital stock coined by Bertola and Caballero (1994).

When $k_i(t)^* > k_i(t) > k_i(t)_*$, the private marginal gain from increasing the capital stock, $v_i(\cdot) = \partial V_1(\cdot)/\partial k_i(t)$, is given by (see appendix, Section 13.6.1):

$$v_1(\cdot) = A_1 \left[Z(t)^{(1-f)} k_i(t)^{(-f)} k_{\mathrm{a}}(t)^{\frac{(1-e)f\lambda}{e}-1} \left(\theta k_{\mathrm{a}}(t) + \frac{\lambda k_i(t)}{N} \right) \right]^{\beta} + \frac{d_0(1-e)f}{\phi(1)e} Z(t)^{(1-f)} k_i(t)^{(-f)} k_{\mathrm{a}}(t)^{\frac{(1-e)f\lambda}{e}-1} \left(\theta k_{\mathrm{a}}(t) + \frac{\lambda k_i(t)}{N} \right) \quad (13.18)$$

where A_1 is a constant to be determined, β is the larger root of τ in the quadratic equation given by (A13.4), and $\phi(1)$ is obtained by setting $\tau = 1$ in (A13.5). On the right-hand side of equation (13.18), the first term (which is negative since A_1 is negative) is the private value of the option to install one more incremental unit of capital, while the second term is the private value of the last incremental unit of installed capital. Two optimal conditions must be satisfied at $k_i(t) = k_i(t)_*$

(Bertola and Caballero, 1994; Pindyck, 1988). First, the private marginal gain from increasing the capital stock must equal its marginal costs at the instant of investing, i.e.

$$v_1(\cdot) = (1-h)P_K \tag{13.19}$$

This is the value-matching condition. Second, condition (13.19) must be satisfied at the states both just before and just after the investment, thus yielding:

$$\frac{\partial v_1(\cdot)}{\partial Z(t)} = 0 \tag{13.20}$$

This is called the smooth-pasting condition.

Denote by $k_{s1}(\cdot, h, t)$ (the first subscript 's' denotes the case where the investment costs are sunk) the $k_i(t)_*$ chosen by firm i when investment is irreversible. This $k_{s1}(\cdot, h, t)$ is solved by the following procedures. First, the equilibrium condition $k_j(t) = k_{s1}(\cdot, h, t)$ $(j = 1, \ldots, N)$ is substituted into both conditions (13.19) and (13.20). Second, equation (13.20) is multiplied by $Z(t)/[(1-f)\beta]$, and then the result is added into equation (13.19), along with the value of $\phi(1)$ defined in (A13.5). As a result, we obtain:

$$k_{s1}(\cdot, h, t) = m_1 k_{f1}(\cdot, h, t) \tag{13.21}$$

where:

$$m_1 = \left[\frac{\alpha}{1+\alpha}\right]^{f\left(1-\frac{(1-e)\lambda}{e}\right)} \quad (<1)$$

$-\alpha$ is the smaller root of τ in the quadratic equation given by (A13.4) and $k_{f1}(\cdot, h, t)$ is given by equation (13.12). Proposition 13.2 will then follow.

Proposition 13.2 *The wedge between the desired capital stock with irreversible investment, $k_{s1}(\cdot, h, t)$, and the choice of capital stock with costlessly reversible investments, $k_{f1}(\cdot, h, t)$, will be expanded (i.e. the multiple m_1 will be lower) in the following cases: (i) uncertainty is greater (σ is higher); (ii) technology shifts at a lower pace of growth (η is smaller); (iii) a catastrophic event is less likely to occur (μ is lower); and (iv) the external effect of capital is higher (λ is larger).*

Proof: See appendix, Section 13.6.2.

The intuition behind Proposition 13.2 is as follows. As is well known in real options literature, the combination of irreversibility and uncertainty will induce an individual firm to be more prudent, i.e. the maxim 'better safe than sorry' applies. Given that a firm invests in R&D capital when the payoff is uncertain, the implication of this maxim is that the firm's optimal stock of R&D capital will be lower compared to the case when either there is no uncertainty or there is complete reversibility. As either uncertainty becomes greater or technology progresses at a lower rate, the 'desired' capital stock for costlessly reversible investments will remain unchanged. In contrast, the desired capital stock for irreversible ones will be reduced because the option to install one more incremental unit of capital is raised by more than is the private value of the last incremental unit of installed capital (see e.g. Pindyck, 1988). Accordingly, this maxim will become more important. As either a catastrophic event becomes less likely to occur, or the externality is stronger, the 'desired' capital stock for costlessly reversible investments will be raised by a proportion more than it will be for irreversible ones. Accordingly, this maxim will also become more important.

Consider the optimal conditions when disinvestment occurs. The value-matching and smooth-pasting conditions which must be satisfied at $k_i(t) = k_i(t)^*$ are respectively given by:

$$v_1(\cdot) = (1-u)\theta P_K \tag{13.22}$$

$$\frac{\partial v_1(\cdot)}{\partial Z(t)} = 0 \tag{13.23}$$

Denote by $k_{\mathrm{d}1}(\cdot, u, t)$ (the first subscript 'd' denotes the case for disinvestment) the $k_i(\cdot, t)^*$ chosen by firm i when investment is irreversible. Solving conditions (13.22) and (13.23) simultaneously, and then imposing the equilibrium condition $k_j(t)^* = k_{\mathrm{d}1}(\cdot, u, t)$ $(j = 1, \ldots, N)$ yields:

$$k_{\mathrm{d}1}(\cdot, u, t) = m_2 k_{\mathrm{s}1}(\cdot, u, t) \tag{13.24}$$

where:

$$m_2 = \theta^{\overline{\frac{-1}{f\left(1-\frac{(1-e)\lambda}{e}\right)}}} \quad (>1)$$

The equality given by (13.24) indicates that the ratio of the capital stock that triggers disinvestment over that which triggers investment, i.e. the factor m_2, will be expanded as the size of capital irreversibility, i.e. $1-\theta$, becomes larger. Following Pindyck (1988), this implies that disinvestment will be less likely to occur as capital investment exhibits a higher irreversibility.

Now consider the optimal investment and disinvestment decisions for a social planner. In the long run, the social planner will internalize the external effect when choosing an optimal path of $k_i(t)$ to maximize the expected discounted social flow profit, net of the investment costs or plus the proceeds from selling capital. Suppose that $V_2(k(t), Z(t))$ denotes the Bellman value function of the social planner, which is given by:

$$\begin{aligned} V_2(\cdot) = \max_{\{k_i(\tau)\}} E_t \int_t^{\infty} \mathrm{e}^{-\rho(\tau-t)}[\pi_i^2(k_i(\tau), Z(\tau))\mathrm{d}\tau - 1_{[\mathrm{d}k_i(\tau)>0]} \\ \times P_k\, \mathrm{d}k_i(\tau) + 1_{[\mathrm{d}k_i(\tau)<0]} P_k\, \mathrm{d}k_i(\tau)] \end{aligned} \tag{13.25}$$

where $\pi_i^2(\cdot)$ is given by equation (13.10). The maximization problem faced by the social planner has two state variables, $Z(t)$ and $k_i(t)$; $Z(t)$ is exogenous, while $k_i(t)$ is subject to control. Note that $k_i(t)^*$ and $k_i(t)_*$ have been defined as the respective upper and lower barriers of the capital stock. When $k_i(t)^* > k_i(t) > k_i(t)_*$, the social marginal gain from increasing the capital stock, $v_2(\cdot) = \partial V_2(\cdot)/\partial k_i(t)$, is given by (see appendix, Section 13.6.1):

$$v_2(\cdot) = A_2\left[Z(t)^{1-f}k_i(t)^{f\left(\frac{(1-e)\lambda}{e}-1\right)}\right]^{\beta} + \frac{d_0(1-e)f(1-\gamma+\lambda)}{\phi(1)e} \times Z(t)^{1-f}k_i(t)^{f\left(\frac{(1-e)\lambda}{e}-1\right)} \tag{13.26}$$

where A_2 is a constant to be determined. The value-matching and smooth-pasting conditions applied to $v_2(\cdot)$ are respectively given by:

$$v_2(\cdot) = P_K \tag{13.27}$$

and

$$\frac{\partial v_2(\cdot)}{\partial Z(t)} = 0 \tag{13.28}$$

These two equations must be satisfied at $k_i(t) = k_i(t)_*$.

Suppose that $k_{s2}(\cdot, t)$ denotes the 'desired' capital stock $k_i(t)_*$ chosen by a social planner when investment is irreversible. This $k_{s2}(\cdot, t)$ is solved by the following procedures. First, the left-hand side of both conditions (13.27) and (13.28) is evaluated at $k_i(t) = k_{s2}(\cdot, t)$. Second, equation (13.28) is multiplied by $Z(t)/[-(1-f)\beta]$, and then the result is added into equation (13.27), along with the value of $\phi(1)$ defined in (A13.5). As a result, we obtain:

$$\frac{d_0(1-e)f(1-\gamma+\lambda)\alpha}{e\rho(\alpha+1)} Z(t)^{1-f}k_{s2}(\cdot, t)^{f\left(1-\frac{(1-e)\lambda}{e}\right)} = P_K \tag{13.29}$$

Solving for $k_{s2}(\cdot, t)$ in equation (13.29) yields:

$$k_{s2}(\cdot, t) = m_1 k_{f2}(\cdot, t) \tag{13.30}$$

where m_1 is given by equation (13.21) and $k_{f2}(\cdot, t)$ is given by equation (13.14).

In the absence of any regulations, the social marginal gain from increasing the capital stock, the left-hand side of equation (13.27), will outweigh its private marginal gain, the left-hand side of equation (13.19). Consequently, a social planner will choose a higher 'desired' capital stock than that chosen by an individual firm. Both an investment tax credit given to firms and a tax on disinvestment will be optimal if they cause both the decentralized and the command economy to have identical upper and lower barriers of capital stocks.[5] Proposition 13.3 derives the optimal rate of investment tax credits.

Proposition 13.3 *Denote the optimal rate of investment tax credits for irreversible investments as h_s^*, then:*

$$h_s^* = \frac{\lambda}{(1-\gamma+\lambda)}\left(1-\frac{1}{N}\right) \tag{13.31}$$

Proof: h_s^* is the h that satisfies:

$$k_{s1}(\cdot, h, t) = k_{s2}(\cdot, t) \tag{13.32}$$

Replacing $k_{s1}(\cdot, h, t)$ given by equation (13.21) with $k_{s2}(\cdot, t)$ given by equation (13.30) yields h_s^* as shown in equation (13.31).

Comparing h_f^* given by equation (13.15) with h_s^* given by equation (13.31) yields Corollary 13.1.

Corollary 13.1 *An equal rate of tax credits should be given to both costlessly reversible investments and irreversible ones.*

The intuition behind Corollary 13.1 is as follows. Evaluating equation (13.21) at $h = 0$, and then dividing each side of equation (13.30) by its counterpart of equation (13.21) yields $k_{f2}(\cdot, t)/k_{f1}(\cdot, 0, t) = k_{s2}(\cdot, t)/k_{s1}(\cdot, 0, t)$. In other words, in the absence of any regulations, the ratio between the social and private optimal stocks of R&D capital when investment exhibits complete reversibility is equal to its counterpart when investment exhibits irreversibility. Consequently, irreversibility is irrelevant to the optimal rate of investment tax credits.

Consider the optimal conditions for a social planner who sells installed capital. The value-matching and smooth-pasting conditions required to be satisfied at $k_i(t) = k_i(t)^*$ are respectively given by:

$$v_2(\cdot) = \theta P_K \tag{13.33}$$

$$\frac{\partial v_2(\cdot)}{\partial Z(t)} = 0 \tag{13.34}$$

Denote by $k_{d2}(\cdot, t)$ the $k_i(t)^*$ chosen by the social planner when investment is irreversible. Solving conditions (13.33) and (13.34) simultaneously, and then imposing $k_i(t)^* = k_{d2}(\cdot, t)$ yields:

$$k_{d2}(\cdot, t) = m_2 k_{s2}(\cdot, t) \tag{13.35}$$

where m_2 is given by equation (13.24).

In the absence of any regulations, an individual firm will sell installed capital earlier than socially desirable, as suggested by $k_{d2}(\cdot, t) > k_{d1}(\cdot, 0, t)$. Consequently, a tax should be imposed, and its optimal rate is given by Proposition 13.4.

Proposition 13.4 *Denote the tax rate that should be imposed on firms selling capital that exhibits irreversibility as u^*, then*:

$$u^* = \frac{\lambda}{(1 - \gamma + \lambda)} \left(1 - \frac{1}{N}\right) \tag{13.36}$$

Proof: u^* is the u that satisfies $k_{d1}(\cdot, u, t) = k_{d2}(\cdot, t)$. Substituting $k_{d1}(\cdot, u, t)$ given by equation (13.24) and $k_{d2}(\cdot, t)$ given by equation (13.35) into this equality yields u^* as shown in equation (13.36).

Comparing Proposition 13.3 with Proposition 13.4 yields Corollary 13.2.

Corollary 13.2 *The rate of taxation that should be imposed on disinvestment, u^*, is equal to the rate of tax credits required to subsidize investment. This common rate will be higher as either $\lambda/(1 - \gamma)$ is higher or N is larger.*

The intuition behind Corollary 13.2 is as follows. Evaluating equation (13.24) at $u = 0$, and then dividing each side of equation (13.35) by its counterpart of equation (13.24) yields $k_{s2}(\cdot, t)/k_{s1}(\cdot, 0, t) = k_{d2}(\cdot, t)/k_{d1}(\cdot, 0, t)$. In other words, in the absence of any regulations, the ratio between the social and private lower barriers of R&D capital, $k_{s2}(\cdot, t)/k_{s1}(\cdot, 0, t)$, coincides with the ratio between the social and private upper barriers of R&D capital, $k_{d2}(\cdot, t)/k_{d1}(\cdot, 0, t)$. Consequently, the optimal subsidy rate on investment will be equal to the optimal penalty rate on disinvestment. This common rate will be higher as either the ratio between the external and the internal effect of capital, $\lambda/(1 - \gamma)$, becomes higher, or the industry is composed of more firms; this is because in both cases, the inefficiencies caused by ignoring externality become more significant.

13.4 NUMERICAL EXAMPLES

We establish a set of central values for the parameters and then investigate a wide variation around this. The benchmark parameter values are as follows: the (risk-adjusted) discount rate $\rho = 8\%$ per year, the price of capital $P_K = 1$, the wage rate $w = 0.8$, demand elasticity $\varepsilon = 2$, the size of the internal effect $1 - \gamma = 0.4$, the size of the external effect $\lambda = 0.1$, the number of firms $= 100$, the technology-shift factor $Z(t) = 1$, the hazard rate $\mu = 1\%$ per year, the drift parameter $\eta = 0.1$, the volatility parameter $\sigma = 20\%$ per year, and the size of irreversibility $1 - \theta = 0.1$.[6] Given these central cases of parameter values, Table 13.1 shows that factor $m_1 = 0.92$, factor $m_2 = 1.18$, and that the rate of tax credits for costlessly reversible investments (h_{f}^*), for irreversible investments (h_{s}^*), and the penalty rate on disinvestment (u^*) should be equal to 19.8%.

Table 13.1 also reports the effects of an increase in σ in a region over (0%, 40%), η in a region over (6%, 14%), μ in a region over (0%, 2%), λ in a region over (0, 0.2), and h in a region over (0%, 40%) on $k_{\mathrm{f1}}(\cdot)$, $k_{\mathrm{f2}}(\cdot)$, $k_{\mathrm{s1}}(\cdot)$, $k_{\mathrm{s2}}(\cdot)$, as well as the option value multiple, m_1 and m_2. The results of Table 13.1 (see also Figures 13.1–13.4) show that regardless of whether the economy is decentralized or centralized, investment is more likely to occur, that is, the 'desired' capital stock when investment is irreversible is larger, as (i) uncertainty (σ) is smaller; (ii) technology shifts at a higher pace of growth (η is larger); (iii) a catastrophic event is less likely to happen (μ is lower); and (iv) the external effect of capital is more significant (λ is higher). Table 13.1 (see Figure 13.5) also shows that investment tax credits are effective regardless of whether investment is irreversible or not because a more generous tax credit policy raises both the choice of capital stock when investment is costlessly reversible and also the 'desired' capital stock when investment is irreversible.

Table 13.1 also yields results that conform to those in the last section (see also Figures 13.1–13.5). First, in the absence of any regulations, the ratio between the desired capital stock with irreversible investments, $k_{\mathrm{s1}}(\cdot, 0, t)$, and the choice of capital stock with costlessly reversible investments, $k_{\mathrm{f1}}(\cdot, 0, t)$, will be lower (i.e. m_1 will be lower) in the following cases: (i) uncertainty is greater (σ is higher); (ii) technology progresses at a lower pace of growth (η is smaller); (iii) a catastrophic event is less likely to occur (μ is lower); and (iv) the external effect of capital is stronger (λ is higher). This confirms the result of Proposition 13.2. Second, $k_{\mathrm{f2}}(\cdot, t)/k_{\mathrm{f1}}(\cdot, 0, t) = k_{\mathrm{s2}}(\cdot, t)/k_{\mathrm{s1}}(\cdot, 0, t)$. In other words, in the absence of any regulations, the following two ratios will be equal: (i) the ratio between the social and private choices of capital stocks when investment is costlessly reversible; and (ii) the ratio between the social and private 'desired' capital stocks when investment is irreversible. Finally, the common ratio stated above leads to a common value of the optimal rate of investment tax credits for costlessly reversible investments and that for irreversible investments, as suggested by Corollary 13.1. This common rate is irrelevant to variations in the volatility parameter σ, the drift parameter η, and the hazard rate μ. In contrast, this common rate will be higher as the size of externality λ becomes more significant. For example, given the parameter values of the benchmark case, except that λ is raised from its central value 0.1 to 0.2, this common rate will be raised from 19.8% to 33%. This confirms the result of Corollary 13.2.

Table 13.1 *Capital stock, option value multiple and optimal investment tax credit rate. Central case:* $\rho = 0.08$, $P_K = 1$, $w = 0.8$, $\varepsilon = 2$, $1 - \gamma = 0.4$, $\lambda = 0.1$, $N = 100$, $Z(t) = 1$, $\mu = 0.01$, $\eta = 0.1$, $\sigma = 0.2$, $1 - \theta = 0.1$; $m_1 = 0.9159$, $m_2 = 1.1781$, $h_f^* = h_s^* = u^* = 19.8\%$

	Variation in σ				
	0%	10%	20%	30%	40%
$k_{f1}(\cdot, 0, t)$	0.0121	0.0121	0.0121	0.0121	0.0121
$k_{f2}(\cdot, t)$	0.0171	0.0171	0.0171	0.0171	0.0171
$k_{s1}(\cdot, 0, t)$	0.0121	0.0119	0.0111	0.0100	0.0087
$k_{s2}(\cdot, t)$	0.0171	0.0167	0.0156	0.0141	0.0122
m_1	1.0000	0.9781	0.9159	0.8237	0.7161
	Variation in η				
	6%	8%	10%	12%	14%
$k_{f1}(\cdot, 0, t)$	0.0121	0.0121	0.0121	0.0121	0.0121
$k_{f2}(\cdot, t)$	0.0171	0.0171	0.0171	0.0171	0.0171
$k_{s1}(\cdot, 0, t)$	0.0101	0.0109	0.0111	0.0112	0.0114
$k_{s2}(\cdot, t)$	0.0150	0.0154	0.0156	0.0158	0.0160
m_1	0.8803	0.9005	0.9159	0.9277	0.9369
	Variation in μ				
	0%	0.5%	1%	1.5%	2%
$k_{f1}(\cdot, 0, t)$	0.0146	0.0132	0.0121	0.0111	0.0103
$k_{f2}(\cdot, t)$	0.0205	0.0187	0.0171	0.0157	0.0145
$k_{s1}(\cdot, 0, t)$	0.0133	0.0121	0.0111	0.0102	0.0094
$k_{s2}(\cdot, t)$	0.0188	0.0171	0.0156	0.0144	0.0133
m_1	0.9145	0.9152	0.9159	0.9165	0.9172
	Variation in λ				
	0	0.05	0.1	0.15	0.2
$k_{f1}(\cdot, 0, t)$	0.0118	0.0120	0.0121	0.0123	0.0125
$k_{f2}(\cdot, t)$	0.0118	0.0142	0.0171	0.0206	0.0251
$k_{s1}(\cdot, 0, t)$	0.0109	0.0110	0.0111	0.0112	0.0113
$k_{s2}(\cdot, t)$	0.0109	0.0131	0.0156	0.0188	0.0228
m_1	0.9240	0.9201	0.9159	0.9111	0.9059
m_2	1.1590	1.1680	1.1781	1.1895	1.2025
$h_f^* = h_s^* = u^*(\%)$	0	11	19.8	27	33
	Variation in h				
	0%	10%	19.8%	30%	40%
$k_{f1}(\cdot, h, t)$	0.0121	0.0143	0.0171	0.0211	0.0268
$k_{f2}(\cdot, t)$	0.0171	0.0171	0.0171	0.0171	0.0171
$k_{s1}(\cdot, h, t)$	0.0111	0.0131	0.0156	0.0193	0.0246
$k_{s2}(\cdot, t)$	0.0156	0.0156	0.0156	0.0156	0.0156

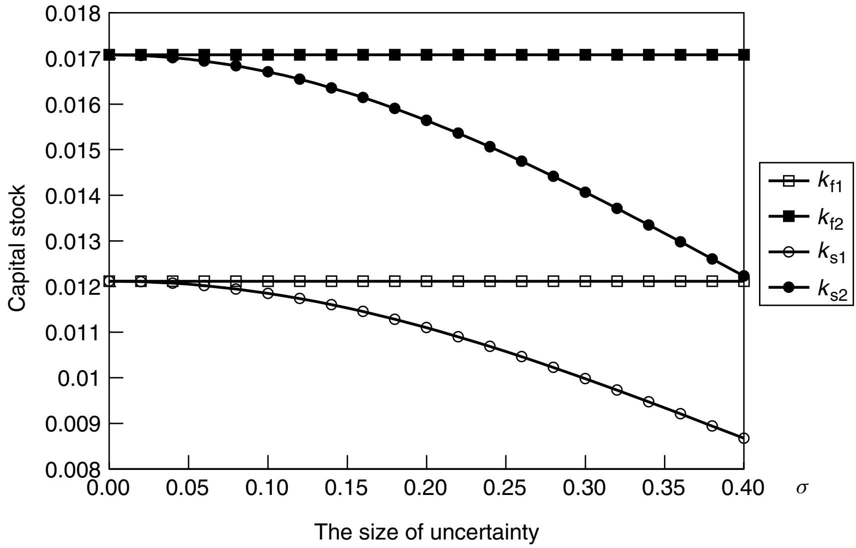

Figure 13.1 *The effect of a change in the size of uncertainty*

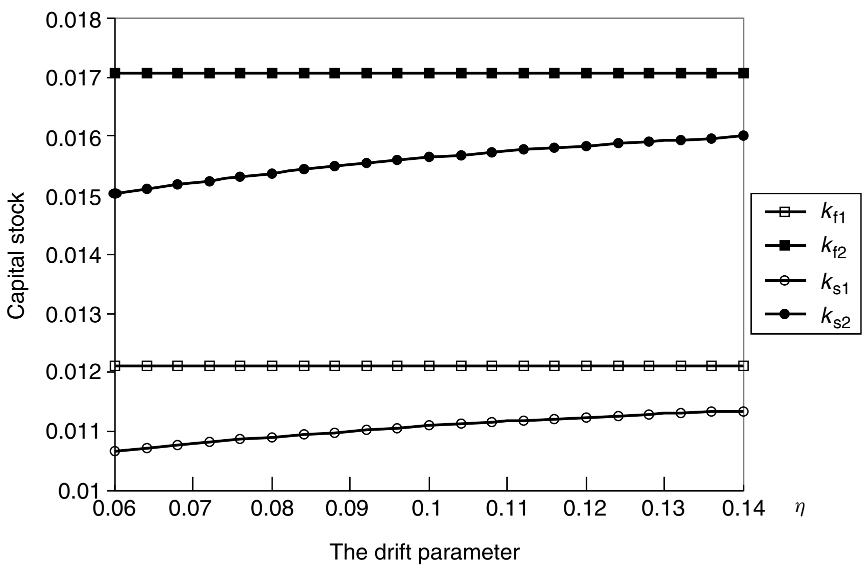

Figure 13.2 *The effect of a change in the drift parameter*

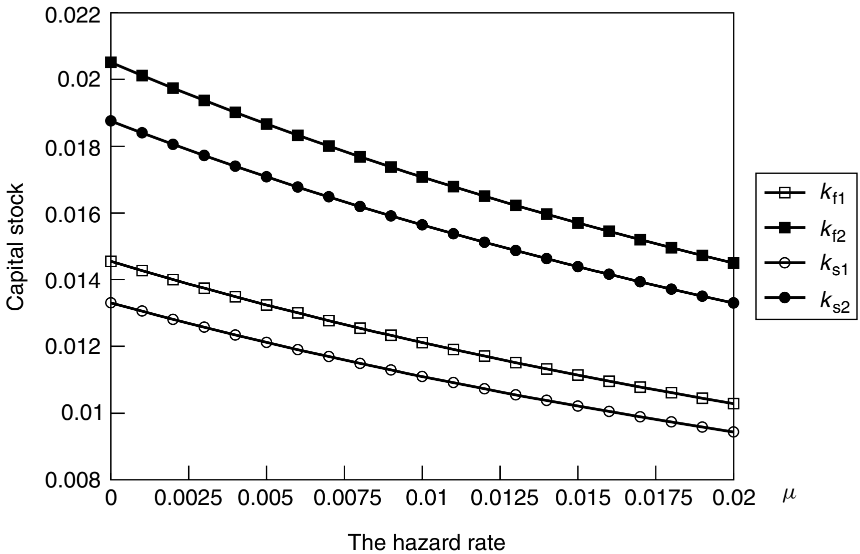

Figure 13.3 *The effect of a change in the hazard rate*

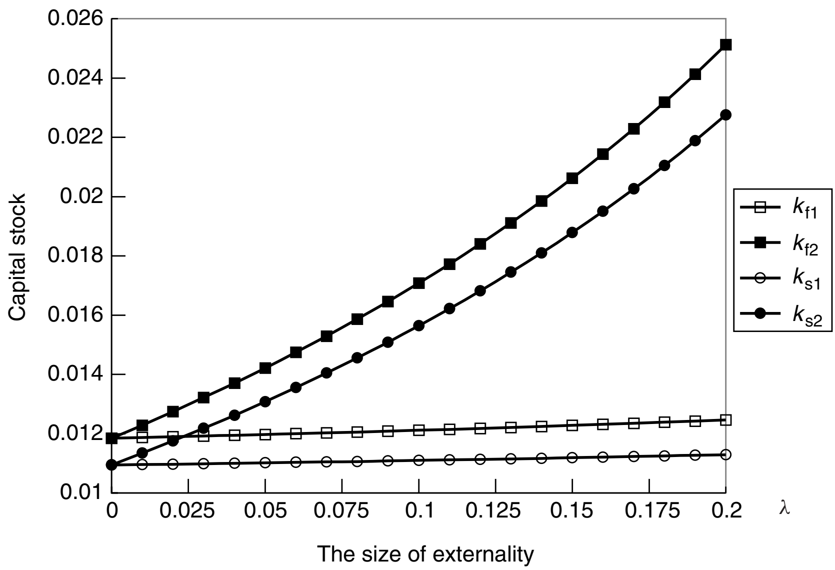

Figure 13.4 *The effect of a change in the size of externality*

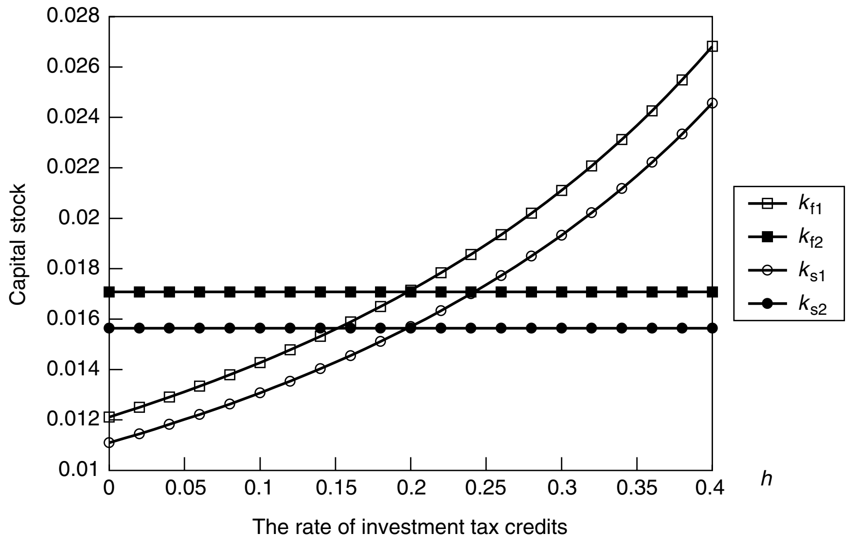

Figure 13.5 *The effect of a change in the rate of investment tax credits*

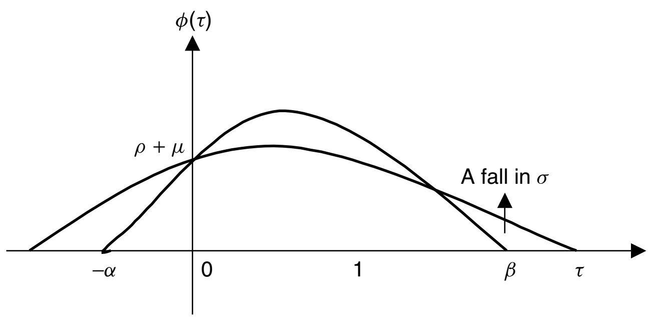

Figure 13.6 *$\phi(\tau)$ versus τ*

13.5 CONCLUSION

This chapter assumes that a firm facing technological uncertainty must decide whether to purchase R&D capital at each instant. R&D capital exhibits both irreversibility and externality through the learning-by-doing effect. The combination of irreversibility and uncertainty drives agents to be more prudent, i.e. the maxim 'better safe than sorry' applies. This maxim is more important as uncertainty is greater, technology progresses at a lower pace, the externality is stronger, or a catastrophic event is less likely to occur. A firm ignoring the externality will both invest later and disinvest earlier than a social planner who internalizes the externality. An equal rate of investment tax credits should be given to both costlessly reversible investments and irreversible ones, and the same rate of taxation should be imposed on disinvestment.

Corollary 13.1 indicates that asset characteristics such as irreversibility and uncertainty are irrelevant to the optimal tax incentives. It can be shown that asset durability is also unrelated to the optimal tax incentives if we allow capital to be depreciating at a constant exponential rate. Nevertheless, it is common for a government to give more generous tax credits to either (i) short-lived assets such as equipment than long-lived assets such as structures (Gravelle, 1994) or (ii) high-technology industries that use capital assets with either a higher expected growth pace of technology or greater degree of technological uncertainty. Our result thus questions these discriminatory R&D subsidy policies. Nevertheless, it deserves investigating whether Corollary 13.2 is robust enough by relaxing several main assumptions which include (i) demand is constant elastic; (ii) production technology is of a Cobb–Douglas type; and (iii) production externality arises from the 'learning-by-doing' effect.

Corollary 13.2 indicates that a tax reduction for purchasing R&D capital should be accompanied with the same tax rate on sales of R&D capital. Currently, in most countries sales of R&D are either free of taxation or taxed at a lower rate than the subsidy rate for purchasing R&D capital. Our result thus questions such a lenient policy towards sales of R&D capital.

Future research may also relax several other assumptions of this chapter. First, firms may be better informed on their R&D technology (e.g. Gaudet et al., 1998). Second, R&D decisions may be characterized in a sequence (e.g. Bar-Ilan and Strange, 1998; Grenadier and Weiss, 1997). Third, capital investment may take time to build (e.g. Majd and Pindyck, 1987; Bar-Ilan and Strange, 1998). Finally, strategic interactions between firms in making R&D investments may be important (Lambrecht, 2000; Weeds, 2000).

13.6 APPENDIX

13.6.1 Derivation of $v_1(\cdot)$ and $v_2(\cdot)$

We will solve for $v_1(\cdot)$ given by equation (13.18) first, and later for $v_2(\cdot)$ given by equation (13.26). Suppose that $k_i(t)^* > k_i(t) > k_i(t)_*$. Treating $V_1(\cdot)$ as an asset value, using equation (13.3) and applying Itô's lemma yields its expected capital gain as:

$$E_t \frac{\mathrm{d}V_1(\cdot)}{\mathrm{d}t} = \frac{1}{2}\sigma^2 Z(t)^2 \frac{\partial^2 V_1(\cdot)}{\partial Z(t)^2} + \eta Z(t) \frac{\partial V_1(\cdot)}{\partial Z(t)} \tag{A13.1}$$

This expected capital gain plus the dividend $d_0[Z(t)k_i(t)]^{(1-f)}k_{\mathrm{a}}(t)^{(1-e)f\lambda/e}$, $\pi_i^1(\cdot)$ given by equation (13.9), should be equal to the normal return $(\rho +$

$\mu)V_1(\cdot)$ to prevent any arbitrage profits from arising. This yields the differential equation:

$$\frac{1}{2}\sigma^2 Z(t)^2 \frac{\partial^2 V_1(\cdot)}{\partial Z(t)^2} + \eta Z(t) \frac{\partial V_1(\cdot)}{\partial Z(t)} - (\rho + \mu)V_1(\cdot) + d_0[Z(t)k_i(t)]^{1-f} k_{\mathrm{a}}(t)^{\frac{(1-e)f\lambda}{e}} = 0 \tag{A13.2}$$

Let $\partial V_1(\cdot)/\partial k_i(t) = v_i(\cdot)$. Differentiating equation (A13.2) term by term with respect to $k_i(t)$ yields:

$$\frac{1}{2}\sigma^2 Z(t)^2 \frac{\partial^2 v_1(\cdot)}{\partial Z(t)^2} + \eta Z(t) \frac{\partial v_1(\cdot)}{\partial Z(t)} - (\rho + \mu)v_1(\cdot) + \frac{d_0(1-e)f}{e} \times Z(t)^{(1-f)} k_i(t)^{-f} k_{\mathrm{a}}(t)^{\frac{(1-e)f\lambda}{e}-1} \left[\theta k_{\mathrm{a}}(t) + \frac{\lambda k_i(t)}{N}\right] = 0 \tag{A13.3}$$

By Bertola and Caballero (1994, appendix), the term $[Z(t)^{(1-f)} k_i(t)^{(-f)} \times k_{\mathrm{a}}(t)^{(1-e)f\lambda/e-1}(\theta k_{\mathrm{a}}(t) + \lambda k_i(t)/N)]$ solves the homogeneous part of the quadratic equation (A13.3). Substituting this into (A13.3) yields the quadratic equation:

$$\phi(\tau) = -\tfrac{1}{2}\sigma^2(1-f)\tau[(1-f)\tau - 1] - (1-f)\eta\tau + (\rho + \mu) = 0 \tag{A13.4}$$

Denote β and $-\alpha$ respectively as the larger and smaller roots in the quadratic equation given by (A13.4). Following Dixit (1991), equation (A13.4) can be rewritten as:

$$\phi(\tau) = \frac{1}{2}(1-f)^2\sigma^2(\alpha + \tau)(\beta - \tau) = \frac{(\rho + \mu)(\alpha + \tau)(\beta - \tau)}{\alpha\beta} \tag{A13.5}$$

where $\phi(\tau) > 0$ if $-\alpha < \tau < \beta$. Figure 13.6 depicts $\phi(\tau)$ as a function of τ.

One particular solution from the non-homogeneous part of equation (A13.3) is given by:

$$v_{1P}(\cdot) = \frac{d_0(1-e)f}{\phi(1)e} Z(t)^{(1-f)} k_i(t)^{(-f)} k_{\mathrm{a}}(t)^{\frac{(1-e)f\lambda}{e}-1} \left[\theta k_{\mathrm{a}}(t) + \frac{\lambda k_i(t)}{N}\right] \tag{A13.6}$$

Since the value function $V_1(\cdot)$ must approach zero as $Z(t)$ approaches zero, only the positive root of equation (A13.4) should be considered. The general solution of equation (A13.3), which is composed of solutions from both the homogeneous and non-homogeneous parts of equation (A13.3), is shown in equation (13.18). Following similar procedures as above yields $v_2(\cdot) = \partial V_2(\cdot)/\partial k_i$ as shown in equation (13.26).

13.6.2 Proof of Proposition 13.2

Let $E = \alpha/(1+\alpha) < 1$ and $G = 1/f(1-(1-e)\lambda/e) > 0$, then $m_1 = E^G$. Differentiating m_1 with respect to η, σ, μ and λ yields the following results:

$$\frac{\partial m_1}{\partial \sigma} = \frac{m_1 G}{\alpha(1+\alpha)} \frac{\partial \alpha}{\partial \sigma} < 0 \tag{A13.7}$$

where:

$$\frac{\partial \alpha}{\partial \sigma} = \frac{\partial \phi(-\alpha)/\partial \sigma}{\partial \phi(-\alpha)/\partial \tau} < 0$$

since $\partial \phi(-\alpha)/\partial \sigma = -\sigma[(1-f)(-\alpha)][(1-f)(-\alpha)-1] < 0$ and $\partial \phi(-\alpha)/\partial \tau > 0$ from Figure 13.6;

$$\frac{\partial m_1}{\partial \eta} = \frac{m_1 G}{\alpha(1+\alpha)} \frac{\partial \alpha}{\partial \eta} > 0 \tag{A13.8}$$

where:

$$\frac{\partial \alpha}{\partial \eta} = \frac{\partial \phi(-\alpha)/\partial \eta}{\partial \phi(-\alpha)/\partial \tau} > 0$$

since $\partial \phi(-\alpha)/\partial \eta = (1-f)\alpha > 0$ and $\partial \phi(-\alpha)/\partial \tau > 0$;

$$\frac{\partial m_1}{\partial \mu} = \frac{m_1 G}{\alpha(1+\alpha)} \frac{\partial \alpha}{\partial \mu} > 0 \tag{A13.9}$$

where:

$$\frac{\partial \alpha}{\partial \mu} = \frac{\partial \phi(-\alpha)/\partial \mu}{\partial \phi(-\alpha)/\partial \tau} > 0$$

since $\partial \phi(-\alpha)/\partial \mu = 1$ and $\partial \phi(-\alpha)/\partial \tau > 0$;

$$\frac{\partial m_1}{\partial \lambda} = m_1 \ln \frac{\alpha}{(1+\alpha)} \frac{\partial G}{\partial \lambda} < 0 \tag{A13.10}$$

since $\partial G/\partial \lambda = G^2 f(1-e)/e > 0$

ACKNOWLEDGMENTS

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NOTES

1. Some studies focus solely on valuation of R&D investment, e.g. Pennings and Lint (1997) and Lint and Pennings (1999).

2. See also Saint-Paul (1992) who emphasizes that an investment tax subsidy rather than public debt can be used to solve the externality problem.
3. Partial irreversibility may come from firm-level or industry-level specific assets, the lemons problem, or government regulation (see Dixit and Pindyck, 1994).
4. The term e is required to be smaller than one to assure that the private and social marginal returns to capital are both positive. We will assume this holds in what follows.
5. See Pindyck (1988) for a proof and also Xepapadeas (1999).
6. In Dixit (1989), some capital costs arise from depreciation and are more thought of as recurrent, and some costs are recoverable when disinvestment occurs. Accordingly, a ratio of $w : \rho P_K = 10 : 1$ seems plausible.

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Chapter 14

Optimal R&D investment tax credits under mean reversion return

JYH-BANG JOU AND TAN LEE

SUMMARY

This chapter assumes that the return to R&D capital is driven by a technological factor that follows a mean-reverting process. R&D capital also exhibits both irreversibility and externality through the learning-by-doing effect. The optimal paths for R&D capital under both the decentralized and the centralized economy are derived and then compared. It is found that an equal rate of investment tax credits should be given to both costlessly reversible investments and irreversible ones, and this common rate is unrelated to the parameters that characterize the mean-reverting process.

14.1 INTRODUCTION

Our previous article (Jou and Lee, 2001) considers the issue of optimal subsidy for R&D in an oligopoly industry that has a fixed number of identical firms. Each firm in the industry installs R&D capital that exhibits production externalities through the learning-by-doing effect (Arrow, 1962). The return to R&D is driven by a technology-shift factor that evolves as a mix of a Poisson process and a geometric Brownian motion. The costs associated with installing R&D capital are partially sunk. It is then found that an equal rate of investment tax credits should be given to both costlessly reversible investments and irreversible ones. In addition, this optimal rate coincides with the optimal rate of taxation on disinvestment.

While considering the same issue as our previous article, this chapter differs in the following respects. First, it considers a competitive industry that allows free entry and exit rather than an oligopoly industry. Second, it considers the polar case where the costs of purchasing R&D capital are fully sunk, thus abstracting from the issue regarding optimal taxation on disinvestment. Last, but most importantly, it models the technology-shift factor that affects the return to R&D capital as a mean-reverting process. This is more realistic because in our previous article, the return to R&D capital can be infinite given that the underlying process of the technological-shift factor can diverge over time. But an infinite return to capital seems to be incompatible with the assumption that all firms in the oligopoly industry (and also the competitive industry) can access R&D capital.

This chapter shows that the mean-reverting process matters for R&D investment decisions: no matter whether the external effect is internalized or not, as the speed of mean reversion increases, the incentive to invest is raised because the long-run variance of the technology-shift factor is dampened. However, similar to our previous article, we find that an equal rate of investment tax credits should be given to both costlessly reversible investments and irreversible ones, and this common rate is unrelated to the parameters that characterize the mean-reverting process. Consequently, the optimal subsidy on capital is unaffected by the use of a mean-reverting process rather than a geometric Brownian motion process.

The literature on real options that investigates the mean-reverting process includes Dixit and Pindyck (1994, chapter 5), Metcalf and Hassett (1995) and Biekpe et al. (2001).[1] Dixit and Pindyck heuristically derive the investment option exercise policy given that the value of the firm follows that process. In contrast, Metcalf and Hassett assume that the output price follows that process, and point out that the cumulative investment is generally unaffected by the use of a mean-reverting rather than a geometric Brownian motion process. Finally, Biekpe et al. show that closed-form expressions for the value of a firm's investment opportunities exist when the firm's cash flow is generated by the mean-reverting process.

The remaining sections are organized as follows. Section 14.2 develops the model for optimal private and then social flow surplus with respect to R&D investment. Section 14.3 derives the optimal private and social investments, along with the optimal tax credits. Section 14.4 provides numerical examples, and Section 14.5 concludes.

14.2 THE MODEL

Dixit (1991) models the impact of price ceilings on irreversible investment decisions. This chapter introduces externalities into his model while abstracting from price ceilings, and considers a competitive industry that is composed of $N(t)$ identical risk-neutral firms. The industry as a whole faces a demand function with a constant elasticity ε (≥ 0), i.e.

$$Q(t) = P(t)^{-\varepsilon} \tag{14.1}$$

where $Q(t)$ is quantity and $P(t)$ is price. The output of each firm i depends on its employed labor force, denoted by $l_i(t)$, its employed capital stock, denoted by $k_i(t)$, the industry's aggregate stock of capital, denoted by $K_{\mathrm{a}}(t)$, and a technology-shift factor, denoted by $Z(t)$. More precisely, the production technology is given by:

$$q_i(t) = l_i(t)^{\gamma}(Z(t)k_i(t))^{1-\gamma}K_{\mathrm{a}}(t)^{\lambda} \tag{14.2}$$

We assume that $Z(t)$ follows a mean-reverting process given by:

$$\mathrm{d}Z(t) = [\eta(\bar{Z} - Z(t))]Z(t)\mathrm{d}t + \sigma Z(t)\mathrm{d}\Omega(t) \tag{14.3}$$

The parameter η (>0) measures the speed of reversion. The parameter $\bar{Z}$ is the normal level of $Z(t)$, i.e. the level to which $Z(t)$ tends to revert. Note that the expected growth rate of $Z(t)$ depends on the difference between $Z(t)$ and $\bar{Z}$. The parameter σ (>0) is the instantaneous volatility of the growth rate of $Z(t)$, and $\mathrm{d}\Omega(t)$ is an increment to a standard Wiener process, with $E\{\mathrm{d}\Omega(t)\} = 0$ and $E\{\mathrm{d}\Omega(t)\}^2 = \mathrm{d}t$.

The production function given by equation (14.2) suggests that R&D capital exhibits externality from the 'learning-by-doing' effect (Arrow, 1962). In equation (14.2), the effect of $k_i(t)$ on $q_i(t)$ is internal, and the term $1-\gamma$ measures its size. In contrast, the effect of $K_{\mathrm{a}}(t)$ on $q_i(t)$ is external, with λ measuring its size.

Multiplying both sides of equation (14.2) by $N(t)$ yields:

$$Q(t) = L(t)^{\gamma}(Z(t)K(t))^{1-\gamma}K_{\mathrm{a}}(t)^{\lambda} \tag{14.4}$$

where $Q(t)$ is the industry's aggregate output, $L(t)$ is the total amount of labor employed by the industry, and $K(t)$ denotes the aggregate stock of capital. The difference between $K(t)$ and $K_{\mathrm{a}}(t)$ is as follows. When making short-run output and long-run investment decisions, a competitive firm will ignore the external effect. Accordingly, the firm will take the industry's stock of capital $K_{\mathrm{a}}(t)$ as given. In contrast, a social planner will internalize the externality, and will therefore impose $K_{\mathrm{a}}(t) = K(t)$ when making these two decisions.

We follow Dixit (1991) to solve the short-run output decision faced by the competitive industry as a whole, which acts as if to choose an aggregate amount of output to maximize the private flow surplus. Define the area under the inverse demand curve as:

$$G(Q(t)) = \int_0^{Q(t)} q(t)^{-e} \mathrm{d}q(t) = \frac{Q(t)^{1-e}}{(1-e)} \tag{14.5}$$

where e is equal to $1/\varepsilon$ and is required to be smaller than one. Denote by w a given wage rate, then from equation (14.4) the short-run variable cost perceived by the competitive industry as a whole, $wL(t)$, is given by:

$$C(Q(t), K(t), K_\mathrm{a}(t), Z(t)) = wQ(t)^g (Z(t)K(t))^{1-g} K_\mathrm{a}(t)^{-\lambda g} \tag{14.6}$$

where $g = 1/\gamma$. The private flow surplus is defined as:

$$S(Q(t), K(t), K_\mathrm{a}(t), Z(t)) = G(Q(t)) - C(Q(t), K(t), K_\mathrm{a}(t), Z(t)) \tag{14.7}$$

Setting the derivative of $S(Q(t), K(t), K_\mathrm{a}(t), Z(t))$ with respect to $Q(t)$ equal to zero yields the choice of aggregate output as given by:

$$Q^*(t) = [wg(Z(t)K(t))^{1-g} K_\mathrm{a}(t)^{-\lambda g}]^{\frac{-1}{(e+g-1)}} \tag{14.8}$$

Define $f = eg/(e + g - 1)$. Substituting $Q(t) = Q^*(t)$ into equation (14.8) yields the optimal level of private flow surplus as given by:

$$S_1^*(K(t), K_\mathrm{a}(t), Z(t)) = d_0(Z(t)K(t))^{1-f} K_\mathrm{a}(t)^{\frac{(1-e)f\lambda}{e}} \tag{14.9}$$

where:

$$d_0 = \left(\frac{1}{1-e} - \frac{1}{g}\right)(wg)^{\frac{e-1}{e+g-1}}$$

In contrast, a social planner will internalize the external effect of R&D capital before investing. Accordingly, the social planner will impose $K_\mathrm{a}(t) = K(t)$ into equation (14.9), thus yielding the optimized value of the social flow surplus as given by:

$$S_2^*(K(t), Z(t)) = d_0 Z(t)^{1-f} K(t)^{1-f+\frac{(1-e)f\lambda}{e}} \tag{14.10}$$

14.3 OPTIMUM INVESTMENT TAX CREDITS

For ease of exposition, we abstract from capital depreciation and assume that the purchasing and installation price of capital is constant over time, denoted by

P_K. We also assume that the investment costs are either completely reversible or completely irreversible.

When R&D capital exhibits externality, as suggested by equation (14.2), then market outcomes will be inefficient. We focus on an investment tax policy to correct this. We assume that a regulator collects lump-sum taxes from competitive firms, and then returns the proceeds to them in the form of an investment tax credit once they invest. The rate of this credit is denoted by h so that the purchase price of capital faced by competitive firms is given by $(1-h)P_K$.

14.3.1 Costlessly reversible investment

Suppose that investment is costlessly reversible, then $K(t)$ will be a choice variable rather than a state variable. Let ρ denote a given (risk-adjusted) discount rate. At each instant, the aggregate R&D capital stock chosen by the competitive industry as a whole is found by equating the private marginal return to capital with the user cost of capital, taking $K_{\mathrm{a}}(t)$ as exogenously determined (Jorgenson, 1963), i.e.

$$\frac{\partial S_1^*(K(t), K_{\mathrm{a}}(t), Z(t))}{\partial K(t)} = (1-h)\rho P_K \tag{14.11}$$

The left-hand side of equation (14.11) indicates that the marginal return to capital is equal to the derivative with respect to $K(t)$ of the optimized private flow surplus, $S_1(\cdot)$ given by equation (14.9). The right-hand side of equation (14.11) indicates that the user cost is equal to the rental cost of capital, $(1-h)\rho P_K$. In competitive industry equilibrium, it is required that the solution path $K(t)$ for equation (14.11) coincides with the given path $K_{\mathrm{a}}(t)$ so that actual and expected behavior are the same, i.e. $K(t) = K_{\mathrm{a}}(t)$. Substituting this equality into (14.11) and calling the resulting $K(t)$ $K_{\mathrm{f1}}(\cdot, h, t)$ (the first subscript 'f' denotes the case for a frictionless world, and the second subscript '1' denotes the case for a decentralized economy) yields:

$$K_{\mathrm{f1}}(\cdot, h, t) = \left[\frac{d_0(1-f)Z(t)^{1-f}}{(1-h)\rho P_K}\right]^{\frac{1}{f\left(1-\frac{(1-e)\lambda}{e}\right)}} \tag{14.12}$$

To ensure that $K_{\mathrm{f1}}(\cdot, h, t)$ is decreasing with P_K and increasing with $Z(t)$, here and in what follows, we will assume that $e > (1-e)\lambda$. In other words, we will assume that as either the cost of R&D is higher or technology shifts upward, demand for capital stock (i.e. the choice of capital stock) will also be higher.

Similarly, the capital stock chosen by a social planner at each instant, denoted by $K_{\mathrm{f2}}(\cdot, t)$ (the second subscript '2' denotes the case for a centralized economy), is found by equating the social marginal return to capital, the derivative with respect to $K(t)$ of the optimized social flow surplus, $S_2(\cdot)$ given by equation (14.10), with the user cost of capital, i.e.[2]

$$\frac{\partial S_2^*(K(t), Z(t))}{\partial K(t)} = \rho P_K \tag{14.13}$$

Evaluating the left-hand side of equation (14.13) at $K(t) = K_{f2}(\cdot, t)$, and then rearranging yields:

$$K_{f2}(\cdot, t) = \left[\frac{d_0 \left(1 - f + \frac{(1-e) f \lambda}{e} \right) Z(t)^{1-f}}{\rho P_K} \right]^{\frac{1}{f\left(1 - \frac{(1-e)\lambda}{e}\right)}} \tag{14.14}$$

The role of externality is to raise the optimal stock of R&D capital. In other words, in the absence of any investment tax credits ($h = 0$), the social marginal return to capital, the term on the left-hand side of equation (14.13), will outweigh the private marginal return to capital, the term on the left-hand side of equation (14.11). Consequently, the aggregate R&D capital stock chosen by the social planner, $K_{f2}(\cdot, t)$ given by equation (14.14), will be higher than that chosen by the competitive industry as a whole, $K_{f1}(\cdot, 0, t)$ given by equation (14.12). An investment tax credit can abolish this wedge, as suggested by Proposition 14.1.

Proposition 14.1 *Suppose that h_f^* denotes the optimal rate of investment tax credits for R&D capital that exhibits complete reversibility, then*:

$$h_f^* = \frac{\lambda}{(1 - \gamma + \lambda)} \tag{14.15}$$

Proof: h_f^* is the h that satisfies the equality:

$$K_{f1}(\cdot, h, t) = K_{f2}(\cdot, t) \tag{14.16}$$

Replacing $K_{f1}(\cdot, h, t)$ in equation (14.12) with $K_{f2}(\cdot, t)$ in equation (14.14) yields h_f^* as shown in equation (14.15).

14.3.2 Irreversible investment

When R&D capital exhibits complete irreversibility, then in the long run, through competition, the competitive industry as a whole will maximize the expected discounted private flow surplus, net of the costs from purchasing capital (Dixit, 1991; Lucas and Prescott, 1971). Consequently, the Bellman value function for the competitive industry as a whole, denoted by $V_1(K(t), Z(t))$, is given by:

$$V_1(\cdot) = \max_{\{K(\tau)\}} E_t \int_t^{\infty} \mathrm{e}^{-\rho(\tau-t)}[S_1^*(K(\tau), K_\mathrm{a}(\tau), Z(\tau))\mathrm{d}\tau - 1_{[\mathrm{d}K(\tau)>0]} P_K \,\mathrm{d}K(\tau)] \tag{14.17}$$

where $S_1^*(\cdot)$ is given by equation (14.9), ρ is a given discount rate, $1_{[\cdot]}$ is an indicator function that is equal to one if the condition within $[\cdot]$ is satisfied, and zero otherwise, and $E_t\{\cdot\}$ denotes conditional expectation taken at time t. The maximization problem faced by the competitive industry as a whole amounts to choosing the optimal path for $K(t)$. There are three state variables in this maximization problem, $K(t)$, $K_\mathrm{a}(t)$ and $Z(t)$; $Z(t)$ is exogenous, while $K(t)$ is subject to control. However, $K_\mathrm{a}(t)$ should be equal to $K(t)$ in industry equilibrium.

As is well known in the real options literature (e.g. Dixit and Pindyck, 1994), the interaction of the stochastic evolution of $Z(t)$ and capital irreversibility indicates that the competitive industry as a whole solves a one-sided instantaneous control problem. The optimal policy is to regulate the state variable $K(t)$ at a lower barrier, denoted by $K(t)_*$ (see Harrison and Taksar, 1985). In other words, as long as $K(t) > K(t)_*$, no action will be taken (i.e. $\mathrm{d}K(t) = 0$). However, when $K(t)$ hits $K(t)_*$, a minimum amount of capital stock to be purchased is chosen to prevent $K(t)$ from falling below $K(t)_*$ (i.e. $\mathrm{d}K(t) > 0$), the 'desired' capital stock coined by Bertola and Caballero (1994).

When $K(t) > K(t)_*$, the private marginal gain from increasing the capital stock, $\nu_1(\cdot) = \partial V_1(\cdot)/\partial K(t)$, is given by (see appendix, Section 14.6):

$$\nu_1(\cdot) = a_0 Z(t)^{\delta_2} A + d_0(1-f)K(t)^{-f} K_\mathrm{a}(t)^{\frac{(1-e)f\lambda}{e}} B \tag{14.18}$$

where $A = H(2\eta Z(t)/\sigma^2, \delta_2, Y(\delta_2))$ [see equation (A14.8)],
$B = \sum_{i=1}^{\infty} c_i Z(t)^{i-f}$

$$c_1 = \frac{d_0(1-f)}{\rho - \eta\bar{Z}(1-f) + \sigma^2(1-f)f/2},$$

$$c_i = \frac{2\eta(i-1-f)c_{i-1}}{\sigma^2(i-f-\delta_2)(i-f-\delta_1)} \quad (i > 1)$$

a_0 is a constant to be determined, δ_2 and δ_1 are respectively the larger and smaller roots of δ in the quadratic equation given by (A14.5), and $Y(\delta)$ is defined in (A14.7). On the right-hand side of equation (14.18), the first term is the private value of the option to install an additional unit of capital, while the second term is the private value of the last incremental unit of installed capital. Two optimal conditions must be satisfied at $K(t) = K(t)_*$ (Bertola and Caballero, 1994; Pindyck, 1988). First, the private marginal gain from increasing the capital stock must be equal to its marginal costs at the instant of investing, i.e.

$$\nu_1(\cdot) = (1-h)P_K \tag{14.19}$$

This is the value-matching condition. Second, condition (14.19) must be satisfied at the states both just before and just after the investment, thus yielding:

$$\frac{\partial \nu_1(\cdot)}{\partial Z(t)} = 0 \tag{14.20}$$

This is called the smooth-pasting condition. The explicit functional form for equation (14.20) is:

$$a_0 Z(t)^{\delta_2 - 1} D + d_0(1-f)Z(t)^{-1}K(t)^{-f}K_{\mathrm{a}}(t)^{\frac{(1-e)f\lambda}{e}} F = 0 \tag{14.20$'$}$$

where $D = \delta_2 H(2\eta Z(t)/\sigma^2, 1+\delta_2, Y(\delta_2))$ and $F = \sum_{i=1}^{\infty} c_i(i-f)Z(t)^{i-f}$.

The $K(t)_*$ chosen by the competitive industry as a whole when the investment is completely irreversible, denoted by $K_{\mathrm{s}1}(\cdot, h, t)$ (the first subscript 's' denotes the case where the investment costs are fully sunk), is solved as follows. First, the equilibrium condition $K(t) = K_{\mathrm{a}}(t)(j = 1, \ldots, N)$ is substituted into both conditions (14.19) and (14.20′). Second, equation (14.20′) is multiplied by $-Z(t)A/D$ and then the result is added into equation (14.19). As a result, we obtain:

$$K_{\mathrm{s}1}(\cdot, h, t) = \left[\frac{d_0(1-f)(B - AF/D)}{(1-h)P_K}\right]^{\frac{1}{f\left(1-\frac{(1-e)\lambda}{e}\right)}} \tag{14.21}$$

In equation (14.21), as technology shifts upward, i.e. $Z(t)$ is higher, the 'desired' capital will also be higher. In other words, a firm's manager will find that the firm's current optimal stock is too low as technology receives a positive shock. Accordingly, the flow of R&D should increase. Alternatively, the firm may hedge this by changing a long position of $\Delta K_{\mathrm{s}1}(\cdot, h, t) \times \beta \times$ NASDAQ index, where β is the regression coefficient between the firm's R&D value and an index of high-technology stocks.

Now consider the optimal investment decision for a social planner. In the long run, the social planner will internalize the external effect when choosing an optimal path of $K(t)$ to maximize the expected discounted social flow surplus, net of the investment costs. The Bellman value function for the social planner, denoted as $V_2(K(t), Z(t))$, is given by:

$$V_2(\cdot) = \max_{\{K(\tau)\}} E_t \int_t^{\infty} \mathrm{e}^{-\rho(\tau-t)} [S_2^*(K(\tau), Z(\tau))\mathrm{d}\tau - 1_{[\mathrm{d}K(\tau)>0]} P_K \,\mathrm{d}K(\tau)] \tag{14.22}$$

where $S_2^*(\cdot)$ is shown in equation (14.10). The maximization problem faced by the social planner has two state variables, $Z(t)$ and $K(t)$; $Z(t)$ is exogenous, while $K(t)$ is subject to control. Note that $K(t)_*$ has previously been defined as the lower barrier of the capital stock. When $K(t) > K(t)_*$, the social marginal gain from increasing the capital stock, $\nu_2(\cdot) = \partial V_2(\cdot)/\partial K(t)$, is given by (see appendix, Section 14.6):

$$\nu_2(\cdot) = b_0 Z(t)^{\delta_2} A + d_0 \left(1 - f + \frac{(1-e) f \lambda}{e}\right) K(t)^{\frac{(1-e)f\lambda}{e} - f} B \tag{14.23}$$

where b_0 is a constant to be determined. The value-matching and smooth-pasting conditions applied to $\nu_2(\cdot)$ are respectively given by:

$$\nu_2(\cdot) = P_K \tag{14.24}$$

and

$$\frac{\partial \nu_2(\cdot)}{\partial Z(t)} = 0 \tag{14.25}$$

The explicit form of equation (14.25) is given by:

$$b_0 Z(t)^{\delta_2 - 1} D + d_0 \left(1 - f + \frac{(1-e) f \lambda}{e}\right) Z(t)^{-1} K(t)^{\frac{(1-e)f\lambda}{e} - f} F = 0 \tag{14.25$'$}$$

Equations (14.24) and (14.25′) must be satisfied at $K(t) = K(t)_*$.

The 'desired' capital stock $K(t)_*$ chosen by a social planner when investment is irreversible, denoted by $K_{s2}(\cdot, t)$, is solved as follows. First, the left-hand side of both conditions (14.24) and (14.25′) is evaluated at $K(t) = K_{s2}(\cdot, t)$. Second, equation (14.25′) is multiplied by $-Z(t)A/D$ and then the result is added into equation (14.24). Then, we obtain:

$$K_{s2}(\cdot, t) = \left[\frac{d_0 \left(1 - f + \frac{(1-e) f \lambda}{e}\right)\left(B - \frac{AF}{D}\right)}{P_K} \right]^{\frac{1}{f\left(1 - \frac{(1-e)\lambda}{e}\right)}} \tag{14.26}$$

In the absence of any regulations, the social marginal gain from increasing the capital stock, the left-hand side of equation (14.24), will outweigh its private marginal gain, the left-hand side of equation (14.19). Consequently, a social planner will choose a higher 'desired' capital stock than that chosen by the competitive industry as a whole. An investment tax credit given to competitive firms will be optimal if it causes both the decentralized and the centralized economy to have identical lower barriers of capital stocks. Proposition 14.2 derives this optimal rate of investment tax credits.

Proposition 14.2 *The optimal rate of investment tax credits for irreversible investments, denoted as* h_s^*, *is given by*:

$$h_s^* = \frac{\lambda}{(1-\gamma+\lambda)} \tag{14.27}$$

Proof: h_s^* is the h that satisfies:

$$K_{s1}(\cdot, h, t) = K_{s2}(\cdot, t) \tag{14.28}$$

Replacing $K_{s1}(\cdot, h, t)$ given by equation (14.21) with $K_{s2}(\cdot, t)$ given by equation (14.26) yields h_s^* as shown in equation (14.27).

In equation (14.27), the optimal investment tax credits for irreversible investments depend on the ratio $\lambda/(1-\gamma)$, i.e. the relative magnitude between the external and internal effects of capital on production, but not on the parameters that characterize the mean-reverting process, i.e. the speed of mean reversion (η) and the instantaneous volatility (σ). The reason is as follows. No matter whether the economy is centralized or not, a higher speed of reversion (η is higher) will dampen the long-run variance of the technology-shift factor (see Metcalf and Hassett, 1995), and therefore it will exhibit a similar qualitative effect as a lower uncertainty (σ is lower). That is, the incentive to invest will be raised, or equivalently, the 'desired' capital will be lower. However, the proportional decline of the 'desired' capital is the same for both the decentralized and centralized economies. Accordingly, the required tax credit rate on investment is irrelevant to both η and σ, which characterize the mean-reverting process.

Comparing h_f^* in equation (14.15) with h_s^* in equation (14.27) yields Corollary 14.1.

Corollary 14.1 *An equal rate of tax credits should be given to both costlessly reversible investments and irreversible ones.*

The intuition behind Corollary 14.1 is as follows. Evaluating equation (14.21) at $h = 0$, and then dividing each side of equation (14.26) by its counterpart of equation (14.21) yields $K_{f2}(\cdot, t)/K_{f1}(\cdot, 0, t) = K_{s2}(\cdot, t)/K_{s1}(\cdot, 0, t)$. In other words, in the absence of any regulations, the ratio between the social and private optimal stocks of R&D capital when investment exhibits complete reversibility is equal to its counterpart when investment exhibits complete irreversibility. Consequently, irreversibility is irrelevant to the optimal rate of investment tax credits.

14.4 NUMERICAL EXAMPLES

We establish a set of central values for the parameters and then investigate a wide variation around this. The benchmark parameter values are as follows: the (risk-adjusted) discount rate $\rho = 8\%$ per year, the price of capital $P_K = 1$, the wage rate $w = 0.8$, demand elasticity $\varepsilon = 2$, the size of the internal effect $1 - \gamma = 0.4$, the size of the external effect $\lambda = 0.1$, the technology-shift factor $Z(t) = 1$, $\bar{Z} = 1$, the speed of reversion $\eta = 10\%$ per year, and the instantaneous volatility $\sigma = 20\%$ per year.[3] Given these parameter values, Table 14.1 shows that $K_{f1}(\cdot, 0, t) = 10.092$, $K_{f2}(\cdot, t) = 14.28$, $K_{s1}(\cdot, 0, t) = 1.697$, $K_{s2}(\cdot, t) = 2.402$, and both the rate of tax credits for costlessly reversible investments and irreversible ones (h^*) are equal to 20%. It is interesting to compare the above results with the polar case where the technological-shift factor is driven by a geometric Brownian motion rather than the mean-reverting process, i.e. η is equal to zero rather than 10%, while the other parameters are held at their benchmark values. Under this polar case, both $K_{s1}(\cdot, 0, t)$ and $K_{s2}(\cdot, t)$ decline to 1.542 and 2.182, respectively. These results, which are not shown in Table 14.1, indicate that increasing mean reversion encourages

Table 14.1 *Capital stock and optimal investment tax credit rate. Central case: $\rho = 0.08$, $P_K = 1$, $w = 0.8$, $\varepsilon = 2$, $1 - \gamma = 0.4$, $\lambda = 0.1$, $\bar{Z} = 1$, $Z(t) = 1$, $\eta = 0.1$, $\sigma = 0.2$, $h_f^* = h_s^* = 20\%$*

	Variation in σ				
	0%	10%	20%	30%	40%
$K_{f1}(\cdot, 0, t)$	10.092	10.092	10.092	10.092	10.092
$K_{f2}(\cdot, t)$	14.280	14.280	14.280	14.280	14.280
$K_{s1}(\cdot, 0, t)$	2.002	1.862	1.697	1.510	1.307
$K_{s2}(\cdot, t)$	2.833	2.635	2.402	2.136	1.849
	Variation in η				
	6%	8%	10%	12%	14%
$K_{f1}(\cdot, 0, t)$	10.092	10.092	10.092	10.092	10.092
$K_{f2}(\cdot, t)$	14.280	14.280	14.280	14.280	14.280

(continued overleaf)

Table 14.1 *(continued)*

	Variation in η				
	6%	8%	10%	12%	14%
$K_{s1}(\cdot, 0, t)$	1.649	1.675	1.697	1.716	1.732
$K_{s2}(\cdot, t)$	2.334	2.370	2.402	2.428	2.451
	Variation in λ				
	0	0.05	0.1	0.15	0.2
$K_{f1}(\cdot, 0, t)$	8.009	8.936	10.092	11.563	13.474
$K_{f2}(\cdot, t)$	8.009	10.630	14.280	19.536	27.394
$K_{s1}(\cdot, 0, t)$	1.610	1.651	1.697	1.751	1.813
$K_{s2}(\cdot, t)$	1.610	1.964	2.402	2.959	3.687
$h^*(\%)$	0	11.11	20	27.27	33.33
	Variation in $(1 - \gamma)$				
	0.3	0.35	0.4	0.45	0.5
$K_{f1}(\cdot, 0, t)$	6.082	7.877	10.092	12.855	16.333
$K_{f2}(\cdot, t)$	9.215	11.483	14.280	17.762	22.133
$K_{s1}(\cdot, 0, t)$	0.850	1.214	1.697	2.346	3.226
$K_{s2}(\cdot, t)$	1.288	1.770	2.402	3.241	4.371
$h^*(\%)$	25	22.22	20	18.18	16.67
	Variation in ρ				
	4%	6%	8%	10%	12%
$K_{f1}(\cdot, 0, t)$	29.666	15.789	10.092	7.133	5.371
$K_{f2}(\cdot, t)$	41.977	22.340	14.280	10.092	7.600
$K_{s1}(\cdot, 0, t)$	4.851	2.623	1.697	1.212	0.920
$K_{s2}(\cdot, t)$	6.865	3.711	2.402	1.715	1.302
	Variation in ε				
	1.5	1.75	2	2.25	2.5
$K_{f1}(\cdot, 0, t)$	6.974	8.347	10.092	12.335	15.255
$K_{f2}(\cdot, t)$	9.244	11.422	14.280	18.083	23.219
$K_{s1}(\cdot, 0, t)$	1.848	1.773	1.697	1.622	1.546
$K_{s2}(\cdot, t)$	2.450	2.426	2.402	2.378	2.353
	Variation in h				
	0%	10%	20%	30%	40%
$K_{f1}(\cdot, h, t)$	10.092	11.890	14.280	17.577	22.340
$K_{f2}(\cdot, t)$	14.280	14.280	14.280	14.280	14.280
$K_{s1}(\cdot, h, t)$	1.697	2.000	2.402	2.956	3.757
$K_{s2}(\cdot, t)$	2.402	2.402	2.402	2.402	2.402

investment by dampening the long-run volatility of the technological-shift factor that affects the return to capital.

Table 14.1 also shows the effects of an increase in σ in a region over (0%, 40%), η in a region over (6%, 14%), λ in a region over (0, 0.2), $(1-\gamma)$ in a region over (0.3, 0.5), ρ in a region over (4%, 12%), ε in a region over (1.5, 2.5) and h in a region over (0%, 40%) on $K_{f1}(\cdot)$, $K_{f2}(\cdot)$, $K_{s1}(\cdot)$ and $K_{s2}(\cdot)$. The results of Table 14.1 (see also Figures 14.1–14.6) show that no matter whether the economy is decentralized or centralized, the optimal

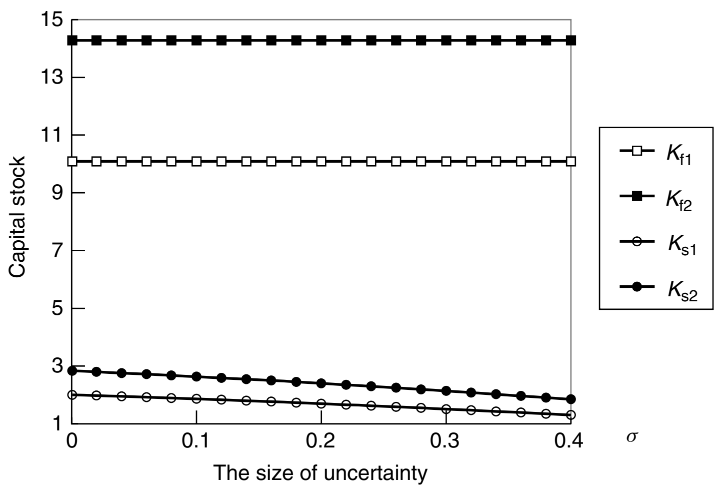

Figure 14.1 *The effect of a change in the size of uncertainty*

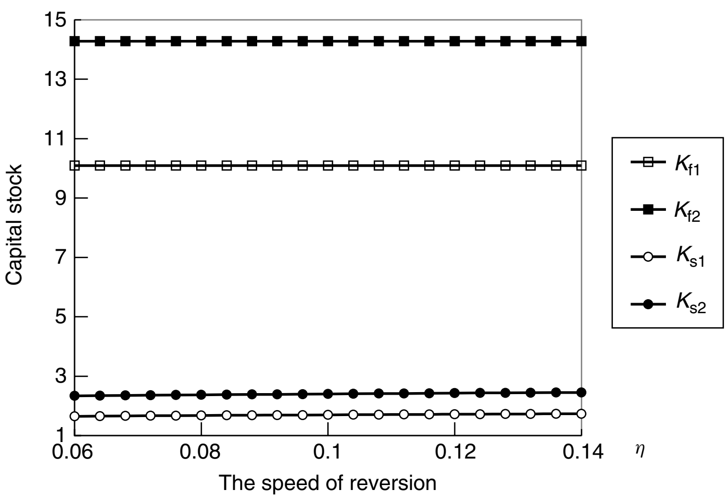

Figure 14.2 *The effect of a change in the speed of reversion*

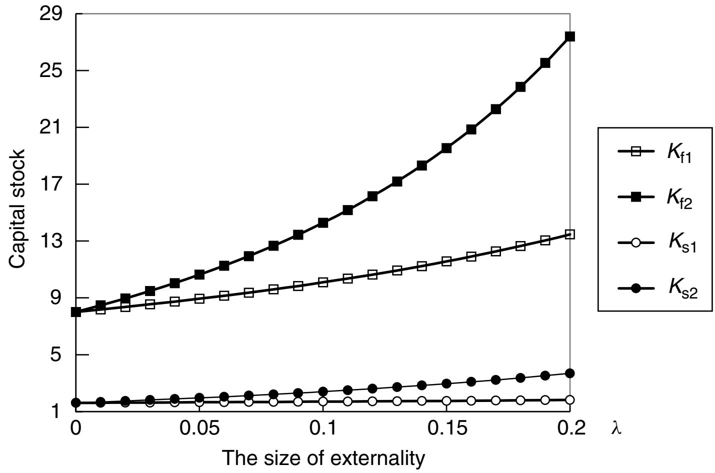

Figure 14.3 *The effect of a change in the size of externality*

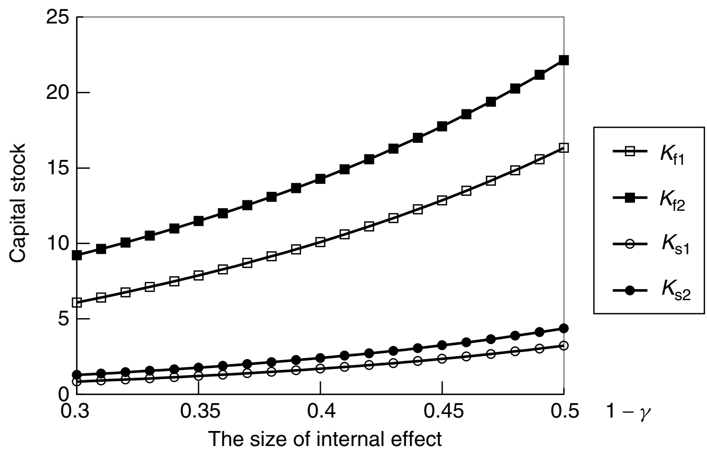

Figure 14.4 *The effect of a change in the size of internal effect*

R&D capital stock when R&D investment is costlessly reversible, i.e. $K_{f1}(\cdot)$ or $K_{f2}(\cdot)$, will be larger as (i) the external effect of capital is more significant (λ is higher); (ii) the size of internal effect is greater [$(1-\gamma)$ is larger]; (iii) the (risk-adjusted) discount rate is lower (ρ is smaller); or (iv) demand becomes more elastic (ε is larger). We reach similar results as (i)–(iii) when R&D investment is completely irreversible. However, the 'desired' capital stock, when the economy is decentralized or centralized, i.e. $K_{s1}(\cdot)$ or $K_{s2}(\cdot)$, is greater as uncertainty (σ) is smaller, the speed of mean reversion is faster (η is larger), or demand elasticity is smaller.

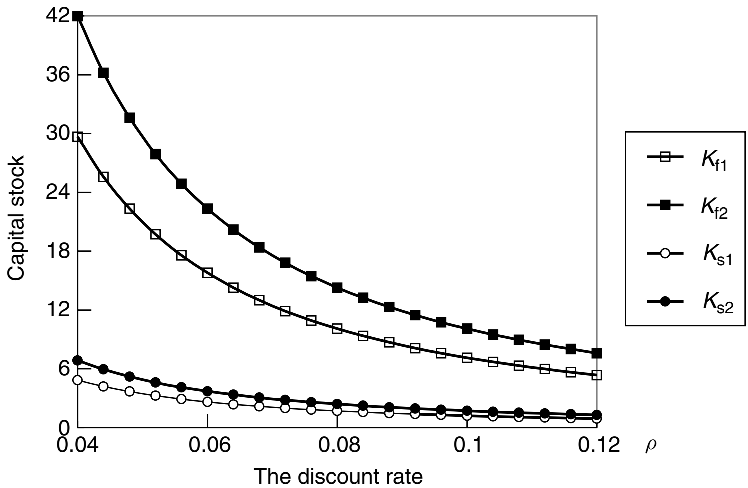

Figure 14.5 *The effect of a change in the discount rate*

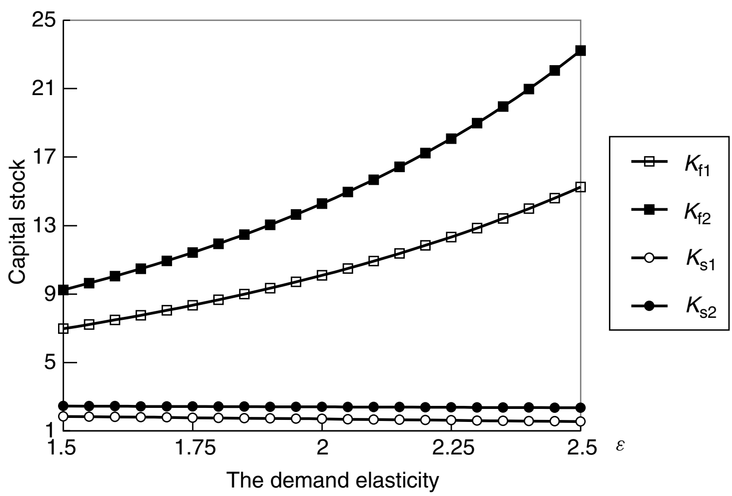

Figure 14.6 *The effect of a change in the demand elasticity*

Table 14.1 also shows that the optimal investment tax credit is not related to either σ, η, ρ or ε, but is increasing with λ, while decreasing with $(1 - \gamma)$, thus confirming Proposition 14.2. Finally, firms should invest more as they receive more investment tax credits regardless of whether investment is irreversible or not. As shown in Table 14.1, a more generous tax credit policy (higher h) raises both the choice of capital stock when investment is costlessly reversible, and the 'desired' capital stock when investment is irreversible (see also Figure 14.7).

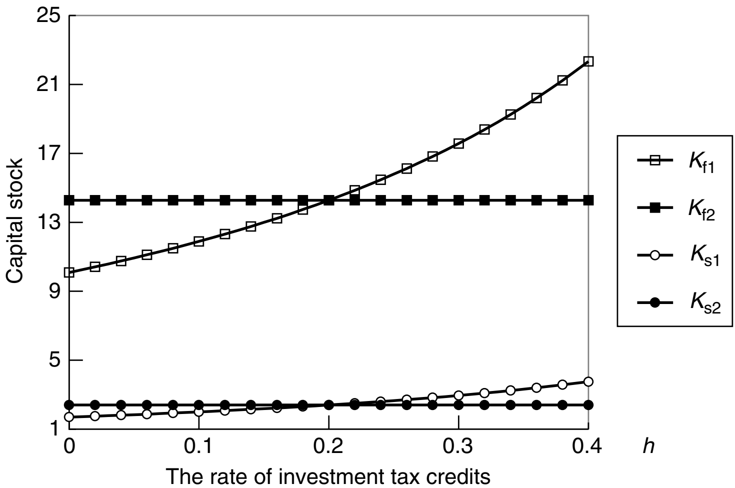

Figure 14.7 *The effect of a change in the rate of investment tax credits*

14.5 CONCLUSION

This chapter assumes that the return to R&D capital is driven by a technological factor that follows a mean-reverting process. R&D capital also exhibits both irreversibility and externality through the learning-by-doing effect. The optimal paths for R&D capital under both the decentralized and centralized economies are derived and then compared. It is found that an equal rate of investment tax credits should be given to both costlessly reversible investments and irreversible ones, and this common rate is unrelated to the parameters that characterize the mean-reverting process.

This chapter provides some examples which show how a firm's investment decision and how a regulator's investment tax credit policy should respond to changes of economic environments. In practical terms, these responses depend on the value of parameters such as the discount rate (ρ), the demand elasticity (ε), the internal effect of capital ($1 - \gamma$), the external effect of capital (λ), the speed of mean reversion (η), and the instantaneous volatility (σ). Among them, only the term λ is unobservable (and is thus subjective), and all the other terms may be calculated (or inferred) from long-term historical data.

Our previous article (Jou and Lee, 2001) allows both investment and disinvestment and models the source of technology uncertainty as a geometric Brownian motion. It is then found that the rate required to subsidize capital investment is the same as the rate required to penalize disinvestment and this common rate is irrelevant to investment irreversibility and the parameters that characterize the geometric Brownian process. This chapter strengthens the results of our previous article because these results also apply to the more generalized process, i.e. the mean-reverting process.

14.6 APPENDIX

We follow Metcalf and Hassett (1995) to solve for $\nu_1(\cdot)$ given by equation (14.18) first, and later solve for $\nu_2(\cdot)$ given by equation (14.23). Suppose that $K(t) > K(t)_*$. Treating $V_1(\cdot)$ as an asset value, using equation (14.3) and applying Itô's lemma yields the expected capital gain of this asset as:

$$E_t \frac{\mathrm{d}V_1(\cdot)}{\mathrm{d}t} = \frac{1}{2}\sigma^2 Z(t)^2 \frac{\partial^2 V_1(\cdot)}{\partial Z(t)^2} + \eta(\bar{Z} - Z(t))Z(t)\frac{\partial V_1(\cdot)}{\partial Z(t)} \tag{A14.1}$$

This expected capital gain plus the dividend $d_0[Z(t)K(t)]^{(1-f)}K_{\mathrm{a}}(t)^{(1-e)f\lambda/e}$, $S_1^*(\cdot)$ given by equation (14.9), should be equal to the normal return $\rho V_1(\cdot)$ to prevent any arbitrage profits from arising. This yields the differential equation:

$$\begin{aligned}&\frac{1}{2}\sigma^2 Z(t)^2 \frac{\partial^2 V_1(\cdot)}{\partial Z(t)^2} + \eta(\bar{Z} - Z(t))Z(t)\frac{\partial V_1(\cdot)}{\partial Z(t)}\\ &\quad - \rho V_1(\cdot) + d_0[Z(t)K(t)]^{1-f}K_{\mathrm{a}}(t)^{\frac{(1-e)f\lambda}{e}} = 0\end{aligned} \tag{A14.2}$$

Let $\partial V_1(\cdot)/\partial K(t) = \nu_1(\cdot)$. Differentiating equation (A14.2) term by term with respect to $K(t)$ yields:

$$\begin{aligned}&\frac{1}{2}\sigma^2 Z(t)^2 \frac{\partial^2 \nu_1(\cdot)}{\partial Z(t)^2} + \eta(\bar{Z} - Z(t))Z(t)\frac{\partial \nu_1(\cdot)}{\partial Z(t)}\\ &\quad - \rho \nu_1(\cdot) + d_0(1-f)Z(t)^{(1-f)}K(t)^{-f}K_{\mathrm{a}}(t)^{\frac{(1-e)f\lambda}{e}} = 0\end{aligned} \tag{A14.3}$$

A power series solution of the form $\nu_{1h}(\cdot) = \sum_{i=0}^{\infty} a_i Z(t)^{i+\delta}$ provides a solution to the homogeneous part of the quadratic equation (A14.3). Substituting this into (A14.3) yields:

$$\begin{aligned}&Z(t)^{\delta}a_0\left[\tfrac{1}{2}\sigma^2\delta(\delta-1) + \bar{Z}\eta\delta - \rho\right]\\ &\quad + Z(t)^{\delta+1}\left\{\left[\tfrac{1}{2}\sigma^2\delta(\delta+1) + \bar{Z}\eta(\delta+1) - \rho\right]a_1 - \eta\delta a_0\right\}\\ &\quad + Z(t)^{\delta+2}\left\{\left[\tfrac{1}{2}\sigma^2(\delta+1)(\delta+2) + \bar{Z}\eta(\delta+2) - \rho\right]a_2 - \eta\delta a_1\right\} + \cdots = 0\end{aligned} \tag{A14.4}$$

We choose δ as the roots of the quadratic equation:

$$Q(\delta) = \frac{\sigma^2}{2}\delta(\delta-1) + \bar{Z}\eta\delta - \rho = 0 \tag{A14.5}$$

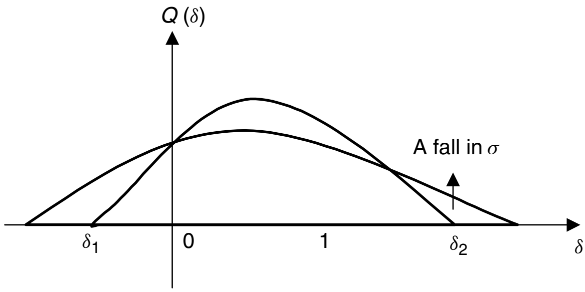

Figure 14.8 *Q(δ) versus δ*

Denote $\delta_1(<0)$ and $\delta_2(>0)$ as the smaller and larger roots in the quadratic equation given by (A14.5). Figure 14.8 depicts $Q(\delta)$ as a function of δ.

Noting that $-\sigma^2\delta(\delta-1)/2 = \bar{Z}\eta\delta - \rho$, we obtain the recurrence relation:

$$a_n = \frac{(2\eta/\sigma^2)(\delta+n-1)}{n(2\delta+n-1+(2\bar{Z}\eta/\sigma^2))}a_{n1}, \quad n \geq 1 \tag{A14.6}$$

Defining $Y(\delta) = 2\delta + (2\bar{Z}\eta/\sigma^2)$, we get:

$$a_n = \frac{(2\eta/\sigma^2)^n\delta(\delta+1)\dots(\delta+n-1)}{n!Y(\delta)(Y(\delta)+1)\dots(Y(\delta)+n-1)}a_0 \tag{A14.7}$$

where a_0 is determined as a constant of integration. Let $H(x, a, b)$ be the confluent hypergeometric function:

$$H(x, a, b) = 1 + \frac{a}{b}x + \frac{a(a+1)}{2!b(b+1)}x^2 + \frac{a(a+1)(a+2)}{3!b(b+1)(b+2)}x^3 + \cdots \tag{A14.8}$$

then a solution for $\nu_{1h}(\cdot)$ is given by:

$$\nu_{1h}(\cdot) = A_1Z(t)^{\delta_1}H\left(\frac{2\eta Z(t)}{\sigma^2}, \delta_1, Y(\delta_1)\right) + A_2Z(t)^{\delta_2}H\left(\frac{2\eta Z(t)}{\sigma^2}, \delta_2, Y(\delta_2)\right) \tag{A14.9}$$

Note that $Z(t)$ equals zero is an absorbing state so that $\nu_{1h}(\cdot) = 0$ as $Z(t) = 0$. Hence A_1 must be equal to zero since $\delta_1 < 0$. Thus:

$$\nu_{1h}(\cdot) = a_0Z(t)^{\delta_2}H\left(\frac{2\eta Z(t)}{\sigma^2}, \delta_2, Y(\delta_2)\right) \tag{A14.10}$$

For the particular solution of (A14.3), we try a power series of the form:

$$\nu_{1p}(\cdot) = \sum_{i=0}^{\infty} c_iZ(t)^{(i-f)}d_0(1-f)K(t)^{-f}K_{\mathrm{a}}(t)^{\frac{(1-e)f\lambda}{e}} \tag{A14.11}$$

We take the derivative of the power series, and then substitute the result into (A14.3), and finally group powers of $Z(t)$. This yields:

$$\begin{aligned}
&\left[\tfrac{1}{2}\sigma^2 f(1+f) - \eta\bar{Z}f - \rho\right]c_0 \\
&\quad + \left[\eta f c_0 + \left(\tfrac{1}{2}\sigma^2(f-1)f + \eta\bar{Z}(1-f) - \rho\right)c_1 + d_0(1-f)\right]Z(t) \\
&\quad + \sum_{i=2}^{\infty}\left\{\left[\tfrac{1}{2}\sigma^2(i-f)(i-1-f) + \eta\bar{Z}(i-f) - \rho\right]c_i \right. \\
&\quad \left. -\eta(i-1-f)c_{i-1}\right\}Z(t)^i = 0
\end{aligned} \tag{A14.12}$$

Equation (A14.12) should be satisfied for any $Z(t)$. Therefore:

$$\begin{aligned}
&c_0 = 0, \quad c_1 = \frac{d_0(1-f)}{\rho - \eta\bar{Z}(1-f) + \sigma^2(1-f)f/2}, \\
&c_i = \frac{2\eta(i-1-f)c_{i-1}}{\sigma^2(i-f-\delta_2)(i-f-\delta_1)}, \quad i = 2, 3, \ldots
\end{aligned} \tag{A14.13}$$

Summing up, the solution for the value function $\nu_1(\cdot)$ is given by $\nu_1(\cdot) = \nu_{1h}(\cdot) + \nu_{1p}(\cdot)$, which is also shown by equation (14.18). Following similar procedures as above yields the solution for $\nu_2(\cdot) = \partial V_2(\cdot)/\partial K(t)$ as shown by equation (14.23).

ACKNOWLEDGMENTS

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NOTES

1. See also the article by Booth et al. (2002) which assumes that a firm's productivity follows the mean-reverting process and shows that the degree to which quit rates affect hiring and training depend on the ratio of firing to hiring costs.
2. The term e is required to be smaller than one such that the private and social marginal returns to capital will both be positive. We assume this holds in what follows.
3. As suggested by Dixit (1989), some capital costs arise from depreciation and are more thought of as recurrent, and some costs are recoverable when disinvestment occurs. Accordingly, a ratio of $w : \rho P_K = 10 : 1$ seems plausible.

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Chapter 15

Genzyme Biosurgery: a virtual real R&D option case

DEAN A. PAXSON

15.1 INTRODUCTION

Kate Hudson was finishing her MBA education and hoping to work for Genzyme Corporation in Boston. Having a degree in biochemistry, Kate could relate to the products that Genzyme developed, and was keen to understand a little more about the corporate structure, financial results and investment decisions that she might face on a day-to-day basis.

15.2 OVERVIEW OF GENZYME CORPORATION

Genzyme Corporation is a biotechnology company that develops and markets innovative products and services designed to address significant unmet medical needs. The corporation is comprised of three divisions, each with its own 'tracking stock'.

Genzyme General develops and markets therapeutic products and diagnostic products and services. This division has four therapeutic products on the market and a strong pipeline of therapeutic products in development, focused on the treatment of genetic disorders and other chronic debilitating diseases. Genzyme General also manufactures and markets diagnostic products, genetic testing services and pharmaceutical intermediates.

Genzyme Biosurgery was formed in December 2000 by combining two Genzyme divisions, Genzyme Surgical Products and Genzyme Tissue Repair with Biomatrix, Inc. This division develops and markets a portfolio of devices, advanced biomaterials and biotherapeutics, primarily for cardiothoracic, orthopedic and general surgery markets.

Genzyme Molecular Oncology is developing a new generation of cancer products, focusing on cancer vaccines and angiogenesis inhibitors. This division of Genzyme is attempting to integrate its gene discovery, gene therapy, small molecule drug discovery, protein therapeutic and genetic diagnostic efforts.

15.2.1 Tracking stocks

The equity structure of Genzyme Corporation is relatively unusual. There is no series of stock that represents Genzyme Corporation as a whole, but rather three separate series of stock which are known as 'tracking stocks' because they are designed to reflect the value of each division to the shareholders. A tracking stock was first issued by General Motors in 1984, followed by USX Corporation, US West, Sprint, AT&T, Georgia Pacific Group, PE Corporation and The Walt Disney Company. Some authors have previously noted that tracking stocks (and similar financing arrangements) are appropriate for biotechnology firms (see Michael E. Solt, 'SWORD financing of innovation in the biotechnology industry', *Financial Management*, Summer 1993, 173–187 and Michael E. Raynor, 'Tracking stocks and the acquisition of real options', *Journal of Applied Corporate Finance*, Summer 2000, 74–83).

None of Genzyme's divisions maintains an ownership interest in any other division of the corporation. Genzyme Corporation has a single board of directors that has an equal duty to shareholders in each of Genzyme's three tracking-stock divisions to act in good faith and in a manner it reasonably believes to be in the best interests of the company as a whole. Genzyme Corporation files consolidated financial statements with the Securities and Exchange Commission, and also files separate financial statements for each of its three divisions. Each of Genzyme's three divisions reports financial results quarterly and publishes its own annual report to shareholders.

Each of Genzyme's divisions operates with its own financial resources. In addition, each division has access to the shared resources of the corporation, such as research and development staff and technology, manufacturing facilities, intellectual property, and clinical and regulatory personnel. The cost of these resources is allocated to each division based on utilization.

From company information, Kate gathered that Genzyme's corporate structure provided some benefits that support the development needs of its divisions.

It allows each Genzyme business to focus on a specific sector of the health care marketplace. This strategy allows Genzyme to concentrate its resources to take advantage of opportunities in four very significant markets: chronic debilitating diseases, cardiothoracic disease, bio-orthopedics and cancer.

It facilitates the growth of Genzyme's development-stage businesses. Genzyme recognizes that the financial and strategic objectives of its three businesses differ according to their stages of development. Genzyme's corporate structure allows its development-stage and emerging businesses (Genzyme Biosurgery and Genzyme Molecular Oncology) to invest in R&D, while preserving the capacity of its more established business (Genzyme General) to generate earnings growth.

It broadens the investor base and provides some financing flexibility. Genzyme is able to draw on a diversity of investors by offering investments that focus on differing businesses with varied risk profiles and potential returns. This diversification enables investors to select the investment that most closely fits their risk and investment preferences. Having its own separate tracking stock also allows each Genzyme division to raise capital to fund development activities, or issue stock as an acquisition currency, without diluting the value of shares in the other divisions.

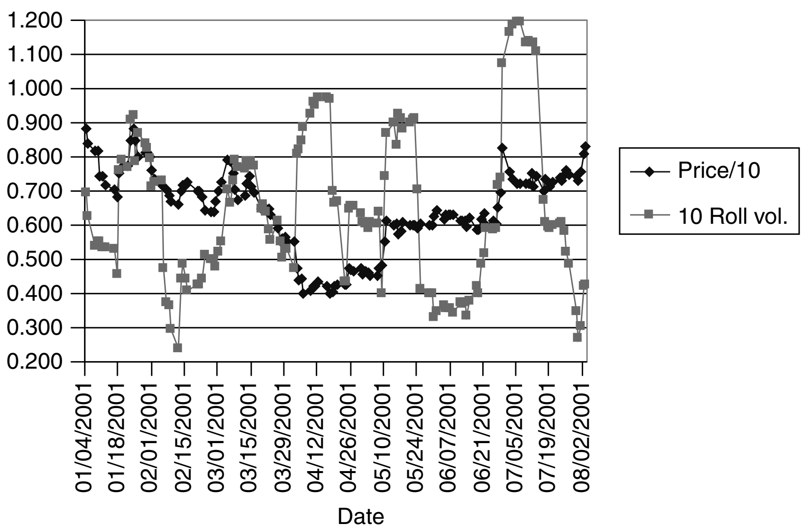
It may foster an entrepreneurial environment. The organizational structure that results from having three separate divisions may encourage an entrepreneurial culture that enables employees of a particular division to make important contributions to the division's performance and growth. In addition, the use of tracking-stock options enables Genzyme to reward employees with a share of the value they may help to produce within a particular division.

It facilitates sharing central resources. By sharing resources among three divisions, Genzyme may be able to realize the operating efficiencies of a larger entity, spread the value of its technology and expertise, and provide its development-stage divisions with access to a broad range of resources that they might find too costly if they were small, independent companies.

It offers tax benefits. By filing a consolidated corporate tax return, Genzyme is able to immediately use any tax benefits from the losses that may arise from investing in R&D in any division. These benefits reduce the current taxes in any profitable divisions, rather than deferring the benefits until the division with the current losses is profitable.

Whilst the tracking stocks are designed to track the financial performance of a specified subset of the business operations and its allocation of assets, each tracking stock is actually a common stock of Genzyme Corporation, not of a division; each division is not a company or legal entity and therefore cannot issue stock. Consequently, holders of a series of tracking stock have no specific rights to assets allocated to the corresponding division, but are subject to the risks of investing in the business, assets and liabilities of Genzyme as a whole. In respect of voting rights, Genzyme General Stock is entitled to one vote per share, which is never adjusted. However, the votes per share of the other series of common stock are adjusted every two years based on that stock's market value divided by the market value of a share of Genzyme General Stock. As of end 2000, each share of Genzyme Biosurgery and Molecular Oncology stock is entitled to 0.14 votes, respectively.

Genzyme Biosurgery stock ('Price/10' = market price divided by 10) had been quite volatile over the past year, with a 10-day rolling volatility ('10 Roll vol') as high as nearly 120% per annum, and a low of around 24%, with an average volatility of 64.6% in the first seven months of 2001.



Biosurgery: stock price and volatility

15.3 FINANCIAL OVERVIEW

At present, the Genzyme General division is the main generator of both revenues and profits. The majority of products in both the Genzyme Biosurgery and Genzyme Molecular Oncology divisions are in the early stages of development (either in the research or preclinical stages) and both divisions are loss-making.

Financial Summary for Genzyme Corporation

		1998 Actual US$'000	1999 Actual US$'000	2000 Actual US$'000
Revenue				
	Genzyme General	569 319	635 366	752 483
	Genzyme Biosurgery	121 075	132 353	145 214
	Genzyme Molecular Oncology	19 407	4 619	5 671
		709 801	772 338	903 368
Net income/(loss)				
	Genzyme General	133 052	142 077	85 956
	Genzyme Biosurgery	(90 242)	(78 077)	(162 217)
	Genzyme Molecular Oncology	(19 107)	(28 832)	(23 096)
	Adjustments/eliminations	38 864	35 813	36 867
		62 567	70 981	(62 490)

Genzyme General net income is after deducting R&D, which amounted to around 14.5% of revenue; adjusted net income is around 28% of revenue. Based on the above allocation of net income/(loss) to each division, basic income/(loss) per share for each class of tracking stock, compared with the average share price for each year, is as follows:

Basic diluted income per share	1998	1999	2000
Genzyme General stock	$1.48	$1.71	$1.35
Share price	*$24.19*	*$21.91*	*$44.97*
Biosurgery stock	–	–	$(2.40)
Share price			*$8.98*
Molecular Oncology stock	$(3.81)	$(2.25)	$(1.60)
Share price	*$3.25*	*$6.63*	*$9.19*
Surgical Products stock	–	$(1.38)	$(3.67)
Tissue Repair stock	$(1.99)	$(1.26)	$(0.69)

On the day of Kate's first look at Genzyme, 5 October 2001, the shares were quoted at $48.27, $5.10 and $8.15, which valued the Genzyme General, Biosurgery and Molecular Oncology divisions at $4330 m, $186 m and $130 m, respectively.

15.4 REAL OPTIONS

Following some articles, and her classroom education, Kate believed real options could be applied to Genzyme. The holder of a real option has the right, although not an obligation, to take a decision at one or more points in the future. Real option analysis applies some of the theories of financial options to help a company or investor to decide (i) how much money they should spend to acquire the particular economic opportunity, and (ii) when (if ever) they should commit to one of the available decisions.

Kate was aware that R&D was a well-known example of a real option, whereby a company embarking upon R&D has the option to take a decision further down the line as to whether to develop or launch a new product, or conduct further R&D to develop perhaps a secondary product. It was clear that Genzyme's management faced a number of real options on a regular basis. Genzyme effectively regularly purchases a stream of options, including growth, value-creating investment options and 'scale-down' and abandonment options. Drug development in particular represents 'options on options', or a series of compound options. Kate was aware that mathematical modeling was required in the evaluation of some of these real options, and she started to study some plausible theories.

15.5 SIX REAL R&D OPTION MODELS

Kate had studied the Margrabe exchange option in the first year at graduate business school. A simple real option example assumes that all new product development investments $D + I$ (D = development, I = product launch costs) are at the date the option expires (assumed to be product launch date). (See Sections 15.8.1 for an illustration and 15.8.7 (Table 15.1) for Excel formulas of the Margrabe exchange option, standardized present value of the R&D discovery product cash flows, V, and standard D and I.)

A year ago, Kate had attended a Real R&D Options Symposium where several alternative models for 'compound'-type real options were presented, where first there was an R&D expenditure, then a new product development (NPD) expenditure. Lint and Pennings (2001) showed that paying for R&D is similar to a premium paid for a forward start option (with timing decided by management) of an NPD with deterministic investment costs and uncertain eventual future cash flows, continuing in perpetuity (see Sections 15.8.2 and 15.8.7 (Table 15.2)).

Telecommunication practitioners had presented a compound options model, where there is a three-phase life cycle, consisting of research, development and deployment. Jensen and Warren (2001) adapted the compound option formula, (see R. Geske, 'The valuation of compound options. *Journal of*

Financial Economics, 7, 63–81) using the bivariate normal distribution (see Sections 15.8.3 and 15.8.7 (Table 15.3)).

Recently, Martzoukas (2002) noted that real option models should recognize that management can and will add value through (endogenous) actions, which are similar to jump processes with upward (but stochastic) values (see Sections 15.8.4 and 15.8.7 (Table 15.4)).

Since it is not clear that there is full current information on the cost of biotechnology R&D or the value of possible discoveries, Bellalah (2002) provided for explicit information costs for an R&D project and for options on the project (see Sections 15.8.5 and 15.8.7 (Table 15.5)).

Finally, an application of a real option model to value speculative developments, where both investment cost and subsequent product value are stochastic, was provided by Quigg (1993) (see Sections 15.8.6 and 15.8.7 (Table 15.6)).

15.6 APPLYING REAL OPTIONS TO GENZYME BIOSURGERY

Kate had read the following passage in Howell et al. (2001) *Real Options: Evaluating Corporate Investment Opportunities in a Dynamic World* (pp. 197–198) 'One interesting evolution will take place inside real options itself, namely that nature will increasingly imitate art. That is to say, businesses will increasingly structure their deals in order to have them make sense in real option terms ... Firms may well float individual business projects (as well as on-going business projects) on stock markets.' Kate believed Genzyme had done exactly that, but she was puzzled as to how the tracking stocks in the loss-making divisions should be valued by investors and investment analysts. Kate was particularly interested in the Genzyme Biosurgery division, which had a number of products in the pipeline, but was still a loss-making operation. The Genzyme Biosurgery division, which had a loss of $160 million, had a market capitalization of $186 million. The summary results for the Genzyme Biosurgery division for the three years ended 31 December 2000 are as follows:

Financial Summary for Genzyme Biosurgery

	1998 Actual US$'000	1999 Actual US$'000	2000 Actual US$'000
Revenue	121 075	132 353	145 214
growth		*9%*	*10%*
Operating loss(*)	(81 311)	(77 762)	(159 371)
Net cash flow from operating activities	(73 907)	(66 110)	(54 818)
% of revenues	*−61%*	*−50%*	*−38%*
R&D costs	29 080	36 075	37 000
Net cash flow from operating activities pre R&D costs	(44 827)	(30 035)	(17 818)

(*)Operating loss in 2000 includes $82 m non-recurring write-off of R&D purchased.

The balance sheet of the division is set out below. As can be seen, the net assets of the division ($511 m) are primarily represented by intangible assets, such as goodwill and patents. The division also has debt of $211 m which is due for repayment after 2004.

Balance Sheet as at 31 December 2000

	US$'000
Cash and cash equivalents	78 163
Net receivables	38 952
Inventory	61 574
Other current assets	9 543
Total current assets	188 232
Fixed assets	57 409
Intangible assets	562 635
Investments in equities	1 603
Other non-current assets	1 721
Total assets	811 600
Liabilities	
Accounts payable	6 074
Due to Genzyme General	18 645
Other current liabilities	64 694
Long-term debt	211 004
Other liabilities	77
Total liabilities	300 494
Net assets	511 106
Retained earnings	
Share capital	
Total equity	511 106

Genzyme Biosurgery provided some limited forecasts for the next few years on the 'cost to R&D completion' for certain products, while warning about the inherent uncertainty of forecasting biotechnological and pharmaceutical products. Genzyme Biosurgery currently had 19 new products in various stages of development from the late research stage through phase 3 of clinical trials (see appendix, Section 15.9).

Kate assumed that total R&D costs to completion to the product development and launch stage (if successful) for the existing portfolio of 19 products would be $70 m, spread over the next four years. Revenues from the division's two current primary profitable products were expected to increase by approximately 10% per annum. Overall, it was expected that the division would move into a cash-neutral position by 2004, and that operating cash flow from existing products (excluding R&D and NPD costs) would reach around $20 million in

2005, increasing by \$10 million thereafter for another nine years. The division had limited cash available, and it was likely that some costs would need to be funded by further rights issues to the shareholders.

Given the history of the biotechnology industry, Kate made several other assumptions: (a) that the average success rate of a new product advancing from the research/preclinical stage through FDA approval for marketing was 12%; (b) that, on average, the new products were 60% through the average overall development period of 10 years; (c) that 50% of all successful new products would be 'blockbusters', worth at least twice the present value of an 'ordinary successful product', with revenue streams lasting 10 years after launch; (d) that around \$50 m of intangible assets in the accounts represented patents and intellectual capital that are not part of the current (disclosed) pipeline; and (e) that the 38% discount rate used by Biosurgery for cash flows from each future project once it reached technological feasibility is appropriate (halved for existing products). These assumptions enabled Kate to focus on 'blockbuster equivalents' (BE), which in this case would be 19 pipeline new products times 12% probability of success = 2.28 successes divided by 1.33 = 1.71 BE.

An example of a blockbuster was Renagel, a new product in the Genzyme General division for which the company had strong expectations. The overall launch costs ($D + I$) (including marketing and advertising costs, manufacturing set-up, packaging and brand development) of this drug were assumed to be \$100 m, spread over the next four years. The hypothetical market size (and Renagel share of the market) is shown in the appendix, Section 15.10. There are various forecasts of Renagel annual revenues ranging from \$33 m (1999), \$70 m (2002) of Patrick E. Flannigan III and Jonathan R. Moran, 'GelTex', SG Cowen, 19 November 1999 to \$164 m (2001), \$457 m (2004) of Ronald C. Renaud, 'Genzyme General', Bear Stearns, Inc., 8 October 2001.

Investment analysts have focused on other real options within the Bio-surgery pipeline, such as currently profitable products like Synvics (synovial fluid treatment for knees) expansion into Europe, and the Sepra anti-adhesion products (see Robin R. Young, 'Research Notes', Stephens Inc., 22 October 2001).

The more Kate thought about the nature of the Biosurgery tracking stock, the more she realized that the shareholders were likely to be risk-takers, and were investing in potential future revenues arising from successful product developments. In addition, these shareholders knew it was likely that investors would have to invest further cash in the division to allow further R&D, development of specific products or launch of a new product. In fact, the tracking stock itself appeared to be a real option for a potential investor, allowing the investor a right to a future business decision (i.e. the decision to inject further cash if a new successful product looked likely).

15.7 CASE QUESTIONS

1. How should the real options in the Genzyme Biosurgery division be valued, using the six real option-pricing models? Which model is most appropriate? How sensitive are real option values to Kate's assumptions?
2. Is the real option value reflected in the current market capitalization? If not, what is an appropriate corporate finance strategy for the Genzyme group?
3. Are there further real options within the business that should be considered and valued, especially put options which reflect the downside risks of large R&D expenditures, funding problems and product guarantees and complications?
4. See www.genzyme.com.

15.8 APPENDIX 1: SIX REAL OPTION MODELS FOR GENZYME BIOSURGERY

15.8.1 Margrabe exchange options

Margrabe, W. (1978) The value of an option to exchange one asset for another. *Journal of Finance*, 33, 177–186.

Assume that bio development project present value is V, and the cost of the investment in year two is $I + D$, and that both are stochastic. Margrabe (1978) showed such a European exchange option has the value $F(V, I, D, \sigma_V, \sigma_{I+D}, \rho, r, \delta_V, t_2) = V\,\mathrm{e}^{-\delta_V t_2} N(d_1) - (I + D)N(d_2)$, where V = value of developed system, I = development cost, D = second-phase R&D expenditure, t_2 = time of the development, σ_V = instantaneous standard deviation of V, σ_{I+D} = instantaneous standard deviation of $I + D$, ρ = correlation between V and $I + D$, r = risk-free interest rate, $\delta_V = 0$ = income rate of developed system V, $\sigma = \sqrt{\sigma_V^2 - 2\rho\sigma_V\sigma_{I+D} + \sigma_{I+D}^2}$, $N(\cdot)$ = cumulative standard normal distribution function, $d_1 = \ln(V/(I + D)) + (-\delta_V + 0.5\sigma^2)t_2/\sigma\sqrt{t_2}$, $d_2 = d_1 - \sigma\sqrt{t_2}$. The European option model assumes that I and D cannot occur until t_2.

15.8.2 Forward start options

Lint, O. and Pennings, E. (2001) An option approach to the new product development process: a case study at Philips Electronics. *R&D Management*, 31(2), 163–172.

When the possibility of product launch has been created, the value of the timing option is:

$$F(V(T_L)) = \begin{cases} AV^{\beta}(T_L) & V(T_L) < V^* \\ V(T_L) - I & V(T_L) \geq V^* \end{cases}$$

with:

$$\beta = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}, \quad A = \left(\frac{(\beta-1)^{\beta-1}}{\beta^{\beta} I^{\beta-1}}\right),$$

$$V^* = I\frac{\beta}{\beta-1}$$

r = riskless rate, δ = dividend yield, σ_2 = volatility of V for $t < T_L$, σ_1 = volatility of V for $t > T_L$, μ = value drift, T_L is time of NPD launch, I is launch investment cost. The value of the product launch option $O_L(t_{RD})$ as of the initial R&D decision is the discounted expected value of the timing option:

$$\begin{aligned} O_L(t_{RD}) &= AV^{\beta}(t_{RD}) \exp\left(\beta\omega_{\mu} + \tfrac{1}{2}\beta^2\omega_{\sigma}^2\right) \Phi(\kappa_1) \\ &\quad + V(t_{RD}) \exp(\mu(T_L - t_{RD}))\Phi(\kappa_2) - I\Phi(\kappa_3) \end{aligned}$$

where $\Phi(\cdot)$ denotes the cumulative probability distribution function of a standard normal variable, $x^* = [\ln(V^*) - \ln(V(t_{RD})) - \omega_{\mu}]/\omega_{\sigma}$, $\omega_{\mu} = (\mu - \frac{1}{2}\sigma_1^2)(T_L - t_{RD})$ and $\omega_{\sigma} = \sigma_1\sqrt{T_L - t_{RD}}$, $\kappa_1 = x^* - \beta\omega_{\sigma}$, $\kappa_2 = -x^* + \omega_{\sigma}$ and $\kappa_3 = -x^*$.

15.8.3 Compound options

Jensen, K. and Warren, P. (2001) The use of options theory to value research in the service sector. *R&D Management*, 31(2), 173–180.

The Geske (1979) formula for a compound option states that the option value is:

$$\begin{aligned} G &= M\left[d_1\left(\frac{V}{V^*\exp(-rt_1)}, t_1\right), d_1\left(\frac{V}{I\exp(-rt_2)}, t_2\right), \rho\right] \\ &\quad - I\exp(-rt_2)M\left[d_2\left(\frac{V}{V^*\exp(-rt_1)}, t_1\right), d_2\left(\frac{V}{I\exp(-rt_2)}, t_2\right), \rho\right] \\ &\quad - D\exp(-rt_1)N\left[d_2\left(\frac{V}{V^*\exp(-rt_1)}, t_1\right)\right] \end{aligned}$$

where the functions d_1 and d_2 are defined by: $d_1(x,t) = (\ln(x) + \sigma^2 t/2)/\sigma\sqrt{t}$, $d_2(x,t) = d_1(x,t) - \sigma\sqrt{t}$ and G = Geske compound option value, V = estimated current present value of launch cash flows, I = investment required at launch stage, D = investment required at development stage, t_1 = time of development decision, t_2 = time of launch decision, σ = volatility associated with the launch value V, r = risk-free interest rate, N = cumulative normal distribution, M = bivariate cumulative normal distribution, $\rho = \sqrt{t_1/t_2}$ = correlation coefficient in bivariate distribution, V^* = value of V at which the option at t_1 (development) should be exercised, determined by solving the standard Black–Scholes equation for the option maturing at t_2, i.e. V^* is the solution to the equation: $F_{\mathrm{BS}}(V^*, I, \sigma, t_2 - t_1, r) = 0$ where $F_{\mathrm{BS}}(V, I, \sigma, t, r)$ is the value of a simple European call option on an asset currently valued at V with exercise price I, volatility σ, time to expiry t and risk-free interest rate r. The maximum justified value of research expenditure is then given by $R_{\max} = G$.

15.8.4 Mixed jump diffusion options

Martzoukas, S.H. (2003) Real R&D options with endogenous and exogenous learning. In D. Paxson (ed.), *Real R&D Options*. Oxford: Butterworth-Heinemann, 111–129.

Suppose that there is the possibility that alert management is capable of affecting an upward jump of the value of an R&D discovery, with a mean jump proportion of 10%, and a volatility of 50%; otherwise the underlying value of the discovery has a gBm at termination, with a volatility of 50%. A European call option C conditional on activation of i random controls at $t = 0$ equals:

$$C_{\mathrm{cond}}(V, I, T, \sigma, \delta, r, \gamma_i, \sigma_i) = \mathrm{e}^{-rT} E[(V_T - I)^+ \mid i \text{ controls}]$$

The discounted risk-neutral expectation, derived along the lines of the Black–Scholes model but conditional on control activation equals:

$$\mathrm{e}^{-rT} E[(V_T - I)^+ \mid i \text{ controls}] = V \exp\left[-\delta T + \sum_{i=1}^{n} (\gamma_i)\right] N(d_1) - I\,\mathrm{e}^{-rT} N(d_2)$$

where:

$$d_1 \equiv \frac{\ln(V/I) + (r-\delta)T + \sum_{i=1}^{n}(\gamma_i) + 0.5\sigma^2 T + \sum_{i=1}^{n}(0.5\sigma_i^2)}{\left[\sigma^2 T + \sum_{i=1}^{n}(\sigma_i^2)\right]^{1/2}}$$

and

$$d_2 \equiv d_1 - \left[\sigma^2 T + \sum_{i=1}^{n}(\sigma_i^2)\right]^{1/2}$$

with $N(d)$ denoting the cumulative standard normal density evaluated at d.

15.8.5 Incomplete information options

Bellalah, M. (2003) On irreversibility, sunk costs and investment under incomplete information. In D. Paxson (ed.), *Real R&D Options*. Oxford: Butterworth-Heinemann, 11–29.

The value of the investment timing option $C(V)$, which has an information uncertainty cost of λ_V for the R&D discovery value and λ_c for the information cost of the option, is: $C(V) = aV^{\beta}$, where $\beta = 0.5 - (r - \delta + \lambda_V)/\sigma^2 + \{[(r - \delta + \lambda_V)/\sigma^2 - 0.5]^2 + 2(r + \lambda_c)/\sigma^2\}^{0.5}$, $V^* = \beta I/(\beta - 1)$ and $a = (V^* - I)/(V^{*\beta})$. V^* corresponds to an optimal timing of the investment.

15.8.6 Stochastic value and cost options

Quigg, L. (1993) Empirical testing of real option-pricing models. *Journal of Finance*, June, 621–640.

Assume that both the cost and value of the R&D program are stochastic, the value is a perpetuity (or nearly so, with a long patent life), there is an agreed risk premium for both the investment cost and the value, a constant volatility for both value and cost, and a constant correlation between the two stochastic processes. Assume there is a ratio (z^*) of value to costs at which it is optimal to invest, and there are certain other boundary conditions.

One solution is offered by Quigg (1993), among others:

$$C(V, I) = I(Az^j + k)$$

where:

$$j = \omega^{-2}(0.5\omega^2 + \nu_I - \nu_V + [\omega^2(0.25\omega^2 - \nu_V - \nu_I + 2r) + (\nu_I - \nu_V)^2]^{0.5})$$

$$A = (z^* - 1 - k)(z^*)^{-j}$$

$$z^* = j(1 + k)/(j - 1)$$

$$k = \beta z/(r - \nu_x)$$

$$\omega^2 = \sigma_I^2 - 2\rho\sigma_I\sigma_V + \sigma_V^2$$

15.8.7 Tables of Excel formulas

Table 15.1

	A	B	C
1	**MARGRABE 1978**		
2			
3	**EUROPEAN EXCHANGE OPTION**		
4			
5	**INPUT**		
6	DEVELOPMENT TIME	2	
7	INVESTMENT TIME	2	
8	INTEREST RATE	0.050	
9	VALUE VOLATILITY	0.500	
10	COST VOLATILITY	0.500	
11	CORRELATION	0.500	
12	I	50.000	
13	D	50.000	
14	I+D	100.000	
15	V (PRESENT VALUE)	100.000	
16	V-D-I NET PRESENT VALUE	9.516	
17			
18	**OUTPUT**		
19	**REAL EXCHANGE OPTION VALUE**	**26.025**	`B15*B23-B14*B24`
20	EXCHANGE VOLATILITY	0.500	`SQRT(B9^2+B10^2-2*B11*B9*B10)`
21	d1	0.707	`(LN(B15/B14)+(0.5*B20*B7))/`
			`(B20*SQRT(B7))`
22	d2	0.000	`B21-B20*SQRT(B7)`
23	N1	0.760	`NORMSDIST(B21)`
24	N2	0.500	`NORMSDIST(B22)`
25			
26	The first six inputs are the D and I timing estimates, the interest rate,		
27	and the value and investment cost volatility and correlation.		
28	The next three inputs are V, I and D estimates.		
29	The Margrabe exchange option values assume I+D is at t2.		

Table 15.2

	A	B	C
1	**LINT & PENNINGS 2001**		
2			
3	**STOCHASTIC R&D, AND NPD**		
4			
5	**INPUT**		
6	t(R)	0	
7	t(D)	1	
8	T	2	
9	V	100.000	
10	I	100.000	
11	σ_1	0.500	
12	σ_2	0.250	
13	r	0.050	
14	δ	0.000	
15	μ	0.050	
16			
17	**OUTPUT**		
18	**REAL OPTION VALUE**	**65.801**	`B30*B34*B31+B9*B35*B32-B10*B33`
19	$\omega(\mu)$	−0.150	`(B15-0.5*B11^2)*(B8-B6)`
20	$\omega(\sigma)$	0.707	`B11*SQRT(B8-B6)`
21	x*	2.519	`(LN(B27)-LN(B9)-B19)/B20`
22	$\kappa 1$	1.639	`B21-B29*B20`
23	$\kappa 2$	−1.811	`-B21+B20`
24	$\kappa 3$	−2.519	`-B21`
25	F(V)	54.075	`IF(B9<B27,B28*(B9^B29),B9-B10)`
26	V-I	0.000	`B9-B10`
27	V*	510.850	`(B29/(B29-1))*B10`
28	A	0.176	`((B29-1)^(B29-1))/((B29^B29)*B10^(B29-1))`
29	$\beta 1$	1.243	`0.5-(B13-B14)/(B11^2)+SQRT(((B13-B14)/(B11^2)+0.5)^2 + 2*B13/(B11^2))`
30	$AV^{\beta 1}$	54.075	`B28*(B9^B29)`
31	$\phi(\kappa 1)$	0.949	`NORMSDIST(B22)`
32	$\phi(\kappa 2)$	0.035	`NORMSDIST(B23)`
33	$\phi(\kappa 3)$	0.006	`NORMSDIST(B24)`
34	O(L)1	1.221	`EXP(B29*B19+0.5*(B29^2)*(B20^2))`
35	O(L)2	1.051	`EXP(B15*(B8-B7))`

Table 15.3

	A	B	C
1			**GESKE 1979**
2			
3			**EUROPEAN COMPOUND CALL OPTION**
4			
5	**INPUT**		
6	DEVELOPMENT TIME	1	
7	INVESTMENT TIME	2	
8	INTEREST RATE	0.050	
9	VALUE VARIANCE	0.250	
10	VALUE VOLATILITY	0.500	
11	V (PRESENT VALUE)	100.000	
12	I	50.000	
13	D	50.000	
14	V-D-I NET PRESENT VALUE	7.197	`B11-B13*EXP(-B8*B6)-B12*EXP(-B8*B7)`
15			
16	**OUTPUT**		
17	**REAL OPTION VALUE**	**23.001**	`B11*B32-B12*EXP(-B8*B7)*B33-B13*`
			`EXP(-B8*B6)*B31`
18	t2-t1	1.000	`B7-B6`
19	V*	96.381	
20	d1	1.663	`(LN(B19/B12)+((B8+0.5*B9)*B18))/`
			`(SQRT(B9*B18))`
21	d2	1.163	`B20-SQRT(B9*B18)`
22	N1	0.952	`NORMSDIST(B20)`
23	N2	0.877	`NORMSDIST(B21)`
24	FB′	50.000	`B19*B22-B12*EXP(-B8*B18)*B23`
25	FB-D	0.000	`B24-B13`
26	ρ(t1/t2)	0.707	`SQRT(B6/B7)`
27	d1,t1	0.424	`(LN(B11/(B19*EXP(-B8*B6)))+(0.5*B9*B6))/`
			`(SQRT(B9*B6))`
28	d1,t2	1.475	`(LN(B11/(B12*EXP(-B8*B7)))+(0.5*B9*B7))/`
			`(SQRT(B9*B7))`
29	d2,t1	−0.076	`B27-SQRT(B9*B6)`
30	d2,t2	0.768	`B28-SQRT(B9*B7)`
31	N2	0.470	`NORMSDIST(B29)`
32	M1	0.657	`BivariateNormal(B27,B28,B26)`
33	M2	0.449	`BivariateNormal(B29,B30,B26)`
34			
35	The first five inputs are the D and I timing estimates, the interest rate, and the		
36	value variance and volatility from comparable securities.		
37	The next three inputs are V, I and D estimates.		
38	The call option value assumes V* is the value above which the call option will be exercised at t1.		
39	USE TOOLS/SOLVER, SETTING B25 = 0 BY CHANGING B19.		

Table 15.4

	A	B	C
1	**MARTZOUKAS 2003**		
2			
3	**MIXED JUMP DIFFUSION**		
4			
5	**INPUT**		
6	V	100.000	
7	I+D	100.000	
8	T	2	
9	r	0.050	
10	δ	0.000	
11	σ	0.250	
12	***POISSON***		
13	JUMP	1	
14	σ_J	0.250	
15	E[k]	0.105	`EXP(B16)-1`
16	γ	0.100	
17			
18	**OUTPUT**		
19	**REAL OPTION VALUE**	**29.013**	`B6*B22*EXP((-B10)*B8+`
			`B16)-B7*B23*EXP(-B9*B8)`
20	d1	0.678	
21	d2	0.245	`B20-(SQRT((SUM(B14^2))`
			`+(B11^2)*B8))`
22	N1	0.751	`NORMSDIST(B20)`
23	N2	0.597	`NORMSDIST(B21)`
24	`d1=(LN(B6/B7)+SUM(B16)+0.5*SUM(B14^2)+(B9-B10+0.5*(B11^2))*`		
25	`(B8))/(SQRT(SUM(B14^2)+((B11^2)*B8)))`		
26			
27	Assumes one upward jump (R&D discovery) of 10%, with 25% volatility,		
28	otherwise 25% volatility ignoring jumps.		

Table 15.5

	A		B	C
1	**BELLALAH 2003**			
2				
3	**R&D UNDER INCOMPLETE INFORMATION**			
4				
5	**INPUT**			
6	V		100.000	
7	I		100.000	
8	r		0.050	
9	σ		0.500	
10	μ		0.100	
11	α		0.100	
12	δ		0.100	
13	λ1		0.020	
14	λ2		0.010	
15				
16	**OUTPUT**			
17	**REAL OPTION VALUE**		**36.501**	`B18*B6^B19`
18	a		0.029	`(B22-B7)/(B22^B19)`
19	β		1.549731	`0.5-B20+SQRT((B20-0.5)^2+B21)`
20		1	−0.120	`(B8-B12+B13)/B9^2`
21		2	0.480	`2*(B8+B14)/B9^2`
22	**V****		281.907	`B19*B7/(B19-1)`

Table 15.6

	A	B	C
1	**QUIGG 1993**		
2			
3	**REAL DEVELOPMENT OPTION**		
4			
5	**INPUT**		
6	INVESTMENT COST	100.000	
7	DEVELOPMENT VALUE	100.000	
8	COST DRIFT	0	
9	COST VOLATILITY	0.500	
10	VALUE DRIFT	0	
11	VALUE OTHER INCOME	0	
12	VALUE VOLATILITY	0.500	
13	CORRELATION	0.500	
14	RISK I	0.100	
15	RISK V	0.100	
16	INTEREST RATE	0.050	
17	ASSET CARRY COST	0.000	
18			
19	**OUTPUT**		
20	**REAL OPTION VALUE**	**37.475**	`B6*(B27*B21^B28+(B25))`
21	z	1.000	`B7/B6`
22	ν_I	−0.050	`B8-B14*B9`
23	ν_V	−0.050	`B10-B11-B15*B12`
24	ω^2	0.250	`B9^2-2*B13*B9*B12+B12^2`
25	k	0.000	`B17*B21/(B16-B22)`
26	z*	2.906	`(B28*(1+B25))/(B28-1)`
27	A	0.375	`(B26-1-B25)*(B26)^(-B28)`
28	j	1.525	
29	`j=(1/B24)*(0.5*B24+B22-B23+SQRT(B24*(0.25*B24-B23-B22+2*B16)+(B22-B23)^2))`		

15.9 APPENDIX 2: GENZYME BIOSURGERY PRODUCT PIPELINE

			Clinical Trials		
Cardiothoracic	Research	Preclinical	Phase 1	Phase 2	Phase 3
Seprafilm™ II for adhesions			●		
Gene Therapy – peripheral vascular disease			●		
Gene Therapy – heart bypass surgery			●		
Gene Therapy – congestive heart failure	●				
Gene Therapy – restenosis	●				
Cell Therapy – ventricular restoration		●			
Drug Delivery – atrial fibrillation	●				
Bio-Orthopedic					
Synvisc for hip			in development		
Synvisc for other joints		●			
QuickTock		●			
Carticel II		●			
Small Molecule for osteoarthritis		●			
Biosurgical Specialties					
Seprapack for sinus surgery			cleared for marketing		
Sepragel Sinus			cleared for marketing		
Seprafilm – safety outcomes study			post marketing		
Seprafilm II					●
Sepragel Spine					●
Sepragel – for abdominal and pelvic		●			
TGF-B	●				

15.10 APPENDIX 3: RENAGEL, MARKET SIZE AND SHARE

Increase in dialysis pop	0.00%											
Cost increase	5.00%											
	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
US dialysis pop	220 000	220 000	220 000	220 000	220 000	220 000	220 000	220 000	220 000	220 000	220 000	220 000
Worldwide	600 000	600 000	600 000	600 000	600 000	600 000	600 000	600 000	600 000	600 000	600 000	600 000
Cost per patient ($k)	1.27	1.33	1.40	1.47	1.54	1.62	1.70	1.79	1.88	1.97	2.07	2.17
Percentage of US market	8.00%	12.00%	30.00%	50.00%	50.00%	50.00%	50.00%	50.00%	50.00%	50.00%	50.00%	50.00%
Percentage of ROW market	0.00%	2.00%	7.00%	10.00%	20.00%	30.00%	40.00%	40.00%	40.00%	40.00%	40.00%	40.00%
US revenues	22 352	35 204	92 412	161 720	169 806	178 297	187 211	196 572	206 401	216 721	227 557	238 934
ROW revenues	0	16 002	58 807	88 211	185 243	291 758	408 461	428 884	450 328	472 845	496 487	521 311
Annual revenues	22 352	51 206	151 219	249 931	355 049	470 054	595 673	625 456	656 729	689 565	724 044	760 246

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Chapter 16

Selective review of real R&D options literature

DEAN A. PAXSON

SUMMARY

Real R&D options are perceived processes in R&D that offer flexibility to the researchers, in terms of timing, commitment, expenditures and research procedures, which can be valued as options. The literature on real R&D options is long and distinguished, starting shortly after the literature on financial options in the 1980s. Articles are cited according to the assumed diffusion process of the R&D project, by the dynamics of R&D over time and the empirical basis of illustrated valuations. Real R&D options have been applied to biotechnology, energy, defense and telecommunication research.

In the future, real R&D options may become integrated with capital market prices, so internal and external market valuation using option theory may be more or less consistent and sometimes useful in making R&D capital allocation and other corporate finance decisions.

16.1 INTRODUCTION

'Real R&D options' are opportunities (and possibly implicit commitments) to acquire or develop or dispose of real assets related to R&D at an investment and implementation cost determined (or estimated) in the present with the benefits delivered in the future. Like financial options, there is conceptually an underlying asset, or liability, that determines the option value at termination. However, unlike financial options, real R&D options are not (yet) commonly traded, are often difficult to identify, with possibly few comparables and limited public information, and may involve complex methods for valuation. Many R&D projects are not proprietary (until perhaps patented), so competition and first-mover advantages/disadvantages must be considered.

Some of the differences between financial and real options have diminished as R&D 'tracking stocks' and 'synthetic real options' are traded, sometimes linked to specific identifiable indices and valued using common option pricing methodology.

Real R&D option theory has been applied to a wide variety of characteristic aspects of projects, including timing of investment expenditures in monopoly and competitive environments, choices in R&D budgets, sequential alternative actions, follow-on investment opportunities, and flexibility in R&D project development.

Real R&D *call options* are opportunities for the holder to benefit from the upside, while only suffering the loss of *premium* (equals 'unrecoverable' R&D costs) as a downside. *Put options* are opportunities for the holder to benefit from the downside, such as guaranteed reimbursement of R&D expenses. Written R&D put options may involve real or implicit warranties of the value of R&D, as well as required future expenditures such as further clinical trials, or liabilities from harmful products.

Use of real option theory in everyday practices is as old as early Greek philosophy. Around 550 BC, Thales of Miletus is said (by Aristotle, around 334 BC) to have shown that philosophers can easily be rich if they like. Thales predicted there would be an abundant harvest of olives in the coming year (similar to forecasting high future volatility, or lots of upside). Having a little money, he gave deposits for the use of all olive presses in Miletus (this is similar to paying an option premium). At the next harvest, which was indeed abundant, he rented the presses at high prices (this is similar to exercising an option), capitalizing on pre-emption.

More recently, the University of Manchester economics professor, Jevons (1871) identified real (environmental) options in the prospective use of a commons, which 'might be allowed to perish at any moment, without harm, if we could have it re-created with equal ease at a future moment, when need of it arises'.

Today, many academics and practitioners have shown that option analysis is an appropriate valuation technique for a firm's growth opportunities (call options) including timing (*exercising* options) for future investments. Indeed almost every project competes with itself postponed, in view of the uncertainty in interest rates. Resources that have *abandonment* or *switch use* possibilities are equivalent to put options. Sick (1989), Dixit and Pindyck (1994), Trigeorgis (1996), Paxson (1997), Amram and Kulatilaka (1999), Brennan and Trigeorgis (2000), Copeland and Antikarov (2001), Grenadier (2001), Howell et al. (2001) and Schwartz and Trigeorgis (2001) provide surveys of real option valuation and applications in property, energy, manufacturing and R&D.

Since most R&D consists of some flexibility over time in terms of budgets, the number and quality of research personnel, and the nature and direction of research strategy, real R&D option theory has a wide and varied range of applications. However, there are usually problems in perceiving that the R&D processes involve options. First, it is necessary to identify and test the goodness of fit for the diffusion process of the 'underlying R&D value'. R&D does not often result in quantified (and market) values, available in time series.

There is a rich literature on modeling real R&D options, although these have not always been recognized as real options. Section 16.2 shows some of the assumed diffusion processes for R&D. Section 16.3 considers some of the dynamic models for R&D, in particular Roberts and Weitzman (1981), Dixit (1989), Grenadier and Weiss (1997), Bar-Ilan and Strange (1998) and Childs et al. (1998). Section 16.4 discusses several pre-emption articles. Section 16.5 surveys some of the empirical inputs provided for real R&D option illustrations. Section 16.6 discusses some of the practical uses of real R&D option models, and concludes with some comments regarding potential new models, new solutions, new empirical applications and new uses of real R&D option models.

16.2 DIFFUSION PROCESSES FOR R&D

The diffusion process that the value and cost of R&D is expected to follow in continuous time is critical in formulating the problem and arriving at a solution for valuation, possible timing and other problems. A general diffusion process format is:

$$\frac{\mathrm{d}X}{X} = \kappa(\mu - a\bar{X})\mathrm{d}t + b\sigma^{\xi}\,\mathrm{d}z_1 + c\lambda\,\mathrm{d}z_2$$

where X is the value or cost of R&D, $\bar{X}$ is the long-term mean, κ is the speed of reversion, μ is the drift, a is 1 for a mean-reverting process and 0 otherwise, b is 1 for a geometric Brownian motion and 0 for a deterministic process, σ is the expected volatility, ξ is the power to which volatility is raised (usually equals 1), c is 0 for a standard geometric Brownian motion and 1 for a jump process, λ is the number of jumps per unit time (usually with an expected jump size and volatility), the dz are standard Wiener processes, and ρ is the possible correlation of value and cost. Several authors believe it is necessary to take the expectation of future cash flows under a risk-adjusted probability measure, usually subtracting from the drift a risk-aversion factor times the volatility, especially where there are no traded securities appropriate as a proxy return

Table 16.1 *Assumed general diffusion processes*

dX/X	κ	μ	$a\bar{X}$	b	ξ	c	X
Samuelson 65	1	μ	0	1	1	0	Value
Merton 76	1	μ	λk	1	1	1	Price
Aase 85	1	μ	0	1	1	1	Project
Dixit 89	1	μ	0	1	1	0	Value
Ott 92	1	$-i$	0	0	0	$g(I, i)$	Investment
Willner 95	1	μ	0	0	0	$(\gamma - \delta)$	Venture PV
Newton et al. 96	1	μ	0	1	1	0	Value, cost
Grenadier & Weiss 97	1	μ	0	1	1	0	Value
Childs et al. 98	1	μ	0	1	1	0	Value
Bar-Ilan & Strange 98	1	μ	0	1	1	0	Price
Schwartz & Moon 00a	1	$-i$	0	$\beta(i, I)$	0.5	0	Investment
Schwartz & Moon 00b	κ	μ	$\bar{u}$	1	1	0	Sales growth
Schwartz & Moon 00c	κ	γ	$\bar{\gamma}$	1	1	0	Variable cost

Ott 92 — I = total R&D investment, i = rate of investment.
Willner 95 — γ = size discovery jump up, δ = size competitive jump down.
Schwartz & Moon 00a — i = rate of R&D invesment.

for the underlying process. Table 16.1 shows a selection of some of the diffusion processes assumed by authors (for illustration, since some authors choose different diffusion processes for different R&D elements).

Table 16.1 shows that most authors assume (but seldom test empirically) that the R&D eventual value, and cost to develop, follow a geometric Brownian motion, although the early models assumed a deterministic cost. An example of using two geometric Brownian processes is Newton et al. (1996). These authors assumed that both R&D values (X) and costs (I) are stochastic and possibly correlated, $dX = \mu X\,dt + \sigma X\,dz$, that is the developed R&D project values are lognormally distributed, and R&D costs (equal to exercise price) follow a similar diffusion process, $dI = \mu I\,dt + \sigma I\,dz$, with a constant correlation between the processes.

Aase (1985) provided a very general framework for R&D values, including up and down jumps. Schwartz and Moon (2000a) considered R&D benefits and costs stochastic with zero correlation, but benefits were also modeled with a Poisson (surprise success or failure) element. Weeds (2000) incorporated a stochastic innovation element (Poisson arrival) into the R&D process.

Several authors have modeled the R&D cost to completion as a stochastic process, often dependent on a maximum investment spend (as in Ott, 1992 and Pindyck, 1993). Schwartz and Moon (2000a) made a similar assumption and then related the completion cost volatility to the R&D investment spend, a type of reducing uncertainty through R&D. Schwartz and Moon (2000b,c) assumed a mean-reverting process, in the first case for the sales growth, and in the second case for the variable cost.

Other models do not necessarily fit these simple general diffusion processes. For instance, Roberts and Weitzman (1981) modeled the terminal R&D benefits as a geometric Brownian motion process, where the concern was an optimal stopping problem. Since costs were deterministic, R&D provided information learned in stages. Weitzman et al. (1981) used similar assumptions, except that costs were stochastic, and process volatility decreased over time.

Over a decade ago, Carr (1988), building on Margrabe (1978) and Geske (1979), provided the valuation of sequential exchange options, that is where there are stages at which further expenditures are considered. This has been applied to real R&D options in Carr (1995), Taudes (1997) and Childs et al. (1998). Childs et al. (1998) assumed that there is a lognormal distribution of the benefits of any real option process, but then comparable (and possibly competing) projects can be modeled as either parallel or sequential processes. Lee and Paxson (2001) extended this approach for real R&D American sequential exchange options. The analysis was based on the R&D specifications and timing of initial expenditures and a second phase of R&D expenditures and final developments, when the project values are realized.

Childs and Triantis (1999) assumed a general diffusion process for R&D development, so that the volatility and drift of R&D value can depend on a number of factors. For instance, the volatility could depend on time, either elapsed or the cumulative time expended on development. Nested within this model are assumptions such as Pindyck (1993), where technical uncertainty is resolved through time by investment (as is often characterized in natural resources), or Weeds (2000), where the possibility of a research breakthrough (or for Kulatilaka and Perotti, 1998 a pre-emption) is a Poisson process, constant through time. The Childs and Triantis (1999) solution of such a general model used a trinomial lattice, so that several alternative R&D strategies can be evaluated. Dynamic funding policies are also modeled in this context.

Berk et al. (1999) considered several sources of R&D risk, including technical uncertainty of value, cost and time, exogenous risk associated with competitors or the environment, and traditional risks of product demand and production costs. The cash flows of new ventures are modeled with two stochastic processes, one conditional on a catastrophic event (termed ‘obsolescence’) and the other conditional on no obsolescence. The firm ‘learns by doing’, so that required investments are either successful or not, before moving to subsequent stages. The authors show a closed-form solution for some limited cases, and then ‘backward iteration’ solutions for general cases.

Schwartz and Zozaya-Gorostiza (2000) assumed that there are two types of technology investments, developments and acquisitions. Developments are situations where the expected cost of completion K follows a controlled diffusion process, as in Pindyck (1993) (or more complex): $\mathrm{d}I = -i\,\mathrm{d}t + g(i, I)\mathrm{d}z$, where

i is the rate of investment and the stochastic term is dependent on both i and I, as in Schwartz and Moon (2000a). Then the full model is a second-order elliptic partial differential equation, solved by successive over-relaxation. Acquisitions are where the investment is instantaneous and the project benefits are received over time in the future. Under a risk-neutral measure, the solution is a second-order parabolic differential equation, solved by the alternating direction implicit method.

Huchzermeier and Loch (2001) provided a dynamic programming model of R&D, allowing for time-variant transition probabilities of product performance, where additional information might be incorporated into R&D management flexibility. Childs et al. (2001) considered several diffusion processes for additional noise in R&D, including mean-reverting noise, and showed the effect of noise and noise variability on the value of real options.

In summary, even early real R&D option models considered complex and combined stochastic processes, usually without first empirically testing whether the assumed diffusion processes closely fit actual, historical R&D values and/or costs. The wide range of diffusion processes considered by these many authors provide a rich menu for future modelers. Some of the most interesting past choices, and closed-form solutions, are illustrated in the next section.

16.3 DYNAMICS OF REAL R&D OPTIONS

Many of the real R&D option models above assume that the R&D options are more or less proprietary and that the R&D framework is static over time, the volatility of R&D cost, value and correlation is constant over time. Of course, these are particularly inappropriate assumptions regarding the typical R&D process, where learning occurs, competitors develop superior products, a sequence of decisions over time are required, and both costs and R&D values change over time. Some of these models can be made ‘dynamic’ by sequential changes in the parameters over time, and by inputting different volatilities and correlations for various stages of R&D.

16.3.1 Sequential decisions and deterministic project volatility

Roberts and Weitzman (1981) and Weitzman et al. (1981) (the ‘Weitzman articles’) might be regarded (along with Samuelson, 1965) as among the first to employ the insight that projects are often contingent claims and can be valued as real options. Perhaps Black and Scholes (1973) and Merton (1973) should be cited as creators of real option theory, since not all of their imagined applications were to traded assets. Brennan and Schwartz (1985) and McDonald and Siegel (1986) were early developers of theory subsequently applied to R&D and exploration and development in natural resources.

The Weitzman articles develop the theory of real R&D options in stages, and provide an empirical application to synthetic oil fuels ('synfuels'). The model of sequential development stages assumes that R&D is 'one-sided', where the R&D is required before project completion. Project volatility is assumed to be a deterministic function of R&D expenditure, reducing by the amount spent. Let $C(s)$ = costs remaining to completion at state s, $\sigma(s)$ = volatility of the project as perceived at state s, then assume $\sigma(s) = kC(s)^{\gamma}$. This is similar to Cortazar et al. (2001), who assume that technical–geological uncertainty is reduced proportionally to the exploration expenditures.

The Weitzman articles were concerned with the justified (from a social viewpoint) subsidy for the development of liquid synfuel from coal, assuming 'learning externalities associated with reducing cost uncertainty'. Assume one knows the price (P) of oil (when the coal liquefaction industry comes on line) and the estimated mean and volatility of costs to completion. With the above deterministic volatility (with respect to R&D cost) and a Wiener process with regard to the underlying project value, the problem of continuing even if NPV< 0 is an optimal stopping problem. Suppose C = ultimate cost/bbl of coal liquids, $C \approx N(EC, \sigma^2)$ is normally distributed, and EC equals the expected value of C. If there is a government subsidy for the R&D, the solution for the optimal stopping problem is: continue subsidy if $g(EC, \sigma, S, P, n, d, T, r) > 0$, else terminate, where:

$$g(EC, \sigma, S, P, n, d, T, r) = \sigma \frac{\phi((P - EC)/\sigma)}{1 - \Phi((P - EC)/\sigma)} - \frac{re^{rT}}{nd} S$$

where $\Phi(\cdot)$ = cumulative distribution function, $\phi(\cdot)$ = standard normal density, S = total subsidy to completion, T = time lag until project operation, n = factory output (bbl/day), d = days per year production and r = riskless rate.

Figure 16.1 shows that given the parameter values of a one million bbl/day plant, production for 10 years, volatility of \$10, synfuel cost of \$44 and current

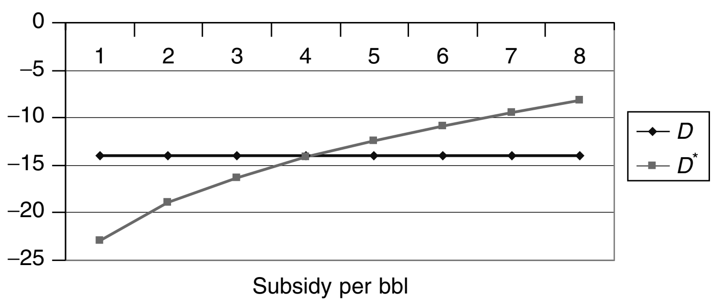


Figure 16.1 *Optimal stopping margin ($D^* = D$) that justifies terminating synthetic oil project*

imported oil price of \$30, the justified subsidy should be around \$4/bbl, where $D^* = D$ ($D = -P +$ expected cost of coal liquids, $D^* = -P +$ minimum syncost that justifies continued subsidy of R&D).

Another interpretation of this approach is to derive the minimum syncost that, for a given subsidy, is justified by project volatility. For the same parameter values, except for volatility, and a subsidy of \$5 billion, the current syncost might be up to 'min syncost', if the project volatility is as illustrated in Figure 16.2.

In general the solution shows that: (i) minimal syncost which justifies a set subsidy increases with $\sigma(s)$; (ii) similarly D^* (negative margin) increases with $\sigma(s)$; and (iii) D^* decreases with size of subsidy.

The Weitzman articles did not consider an R&D model with a term structure of oil prices and oil price volatility, possibly because at that time there was no established futures and options market for crude oil. Even so, given

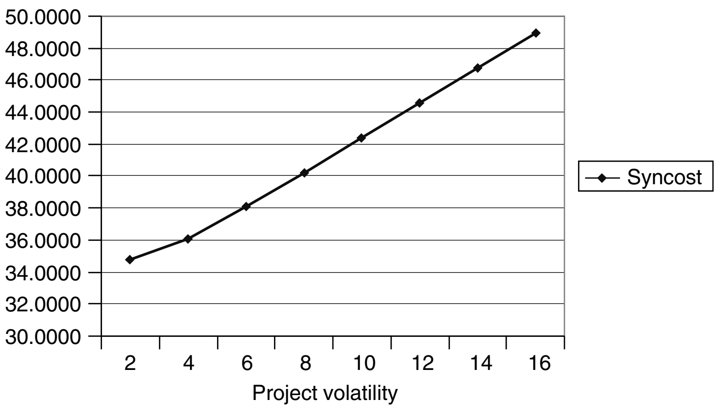


Figure 16.2 *Min syncost that is justified by project volatility*

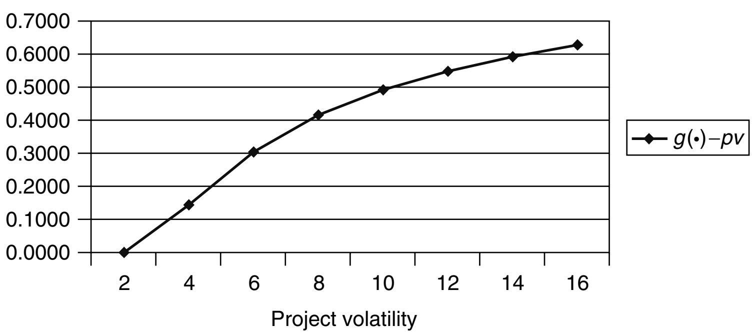


Figure 16.3 *Net project value [$g(\cdot) - pv$] as a function of project volatility*

optimal subsidy, and otherwise fixed parameters, Figure 16.3 shows that R&D net project value [$g(\cdot)$ − present value of investment cost] is a positive function of project volatility, but at a decreasing rate.

Grossman and Shapiro (1986) specifically addressed the dynamics of R&D over time, primarily assuming deterministic processes. Given progress of R&D as a deterministic function, they utilized a hazard rate function regarding the probability that success will be achieved, given that R&D has progressed along a certain distance. Under most assumptions, the optimal R&D spend should increase if the R&D value is high, the closer the R&D is to success. Grossman and Shapiro (1986) also considered R&D programs with stochastic progress and noted the importance of a game-theoretic approach, which does not neglect the aspects of rivalry.

16.3.2 Sequential or parallel R&D development

Childs et al. (1998) examined R&D investment decisions and project values, where two projects (a and b) can be developed in parallel or in sequence. They developed closed-form solutions for the value of the investment program and an optimal strategy for multiple projects (European-style), and analyzed the optimal decision and the factors that affect the choice between some sequential and parallel investments, as follows.

Parallel Development

The firm simultaneously invests in both projects at t0 by paying the combined costs for development.

t0 t1

Invest $C_a + C_b$ ⟶ Receive $(x_a, x_b)^+$

Once development is completed at t1, the firm decides whether to invest K_a or K_b to implement project a or project b, respectively.

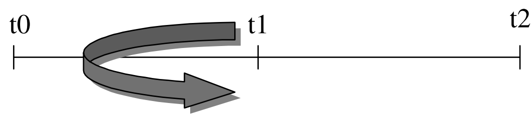
The value of the firm's investment program at t1 is equal to $(x_a, x_b)^+$, where

$$x_i = X_i - K_i$$

X_i = PV of project i's future cash flows, K_i = net implementation cost.

The General Model: Sequential Development

Four feasible sequential investment strategies at t1



V: if the firm does nothing, the cash flow will be zero

V_I: if the firm implements project *a* but does not develop project *b*, the value is simply x_a

V_D: if the firm develops project *b* but does not implement project *a*, the firm will take the maximum value of the two projects at t2: $(x_a, x_b)^+$

V_{ID}: if the firm implements project *a* and also develops project *b*, the firm may decide to abandon the project *a* at t2 if project *b* proves to be superior.

The value of the program at t0 under sequential development is

$$
\begin{aligned}
V^S = {} & e^{-rt_1}E[x_a^+] - C_a - (e^{-rt_1} - e^{-rt_2})E[x_a^+ | x_a \in I_D] \\
& + e^{-rt_1}E[e^{-r(t_2-t_1)}(x_b - (\delta X_a - \beta K_a))^+ - C_b | x_a \in I_{ID}] \\
& + e^{rt_1}E[e^{-r(t_2-t_1)}(x_b - x_a^+)^+ - C_b | x_a \in I_D]
\end{aligned}
$$

Interpretation: the first line of the RHS is the value of implementating project *a*. The next two lines express the value of a European compound option to exchange project *a* for project *b*. The exercise price of the first stage for the compound option at t1 is C_b, and the exercise price of the second stage depends on the decision for project *a* at time t1.

The parallel development strategy is desirable when the level of correlation between project values is low, development costs are low, and implementation costs are large and not easily recovered. If the project values are highly correlated, development costs are high, and implementation costs can be partially recovered, a sequential strategy would be optimal. Figures 16.4 and 16.5 show the value at t_1 for four strategies: $V =$ no action, which is along the horizontal axis of xa, the project *a* ultimate value; *VI* is implementing *a* first; *VD* is developing *b*; *VID* is implementing *a* and developing *b*. Figure 16.4 shows that when the correlation between *a* and *b* is positive, $VD > V > VID > VI$ at most negative project *a* values. Figure 16.5 shows that when the correlation

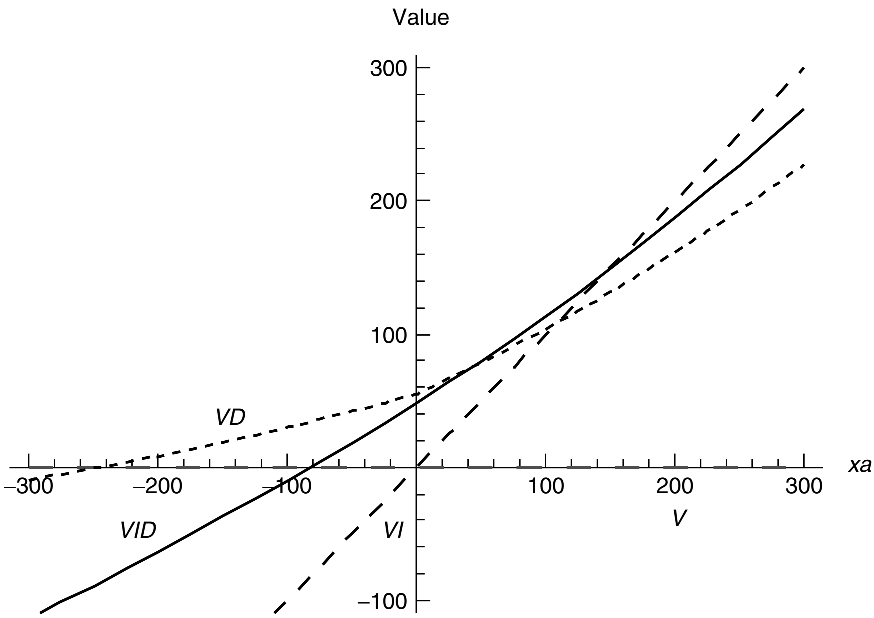


Figure 16.4 *Feasible sequential investment strategies ($\rho = 0.4$)*

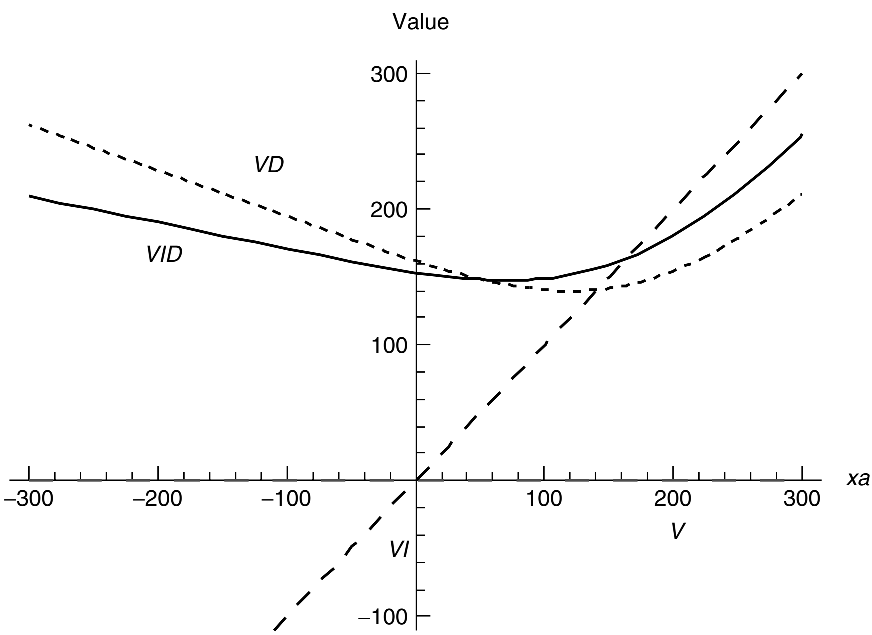


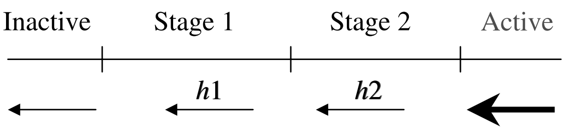
Figure 16.5 *Feasible sequential investment strategies ($\rho = -0.4$)*

is negative, at negative project a values, $VD > VID > V > VI$, that is ‘do nothing’ shifts from the second best strategy to the second worst strategy.

Bar-Ilan and Strange (1998) also considered a model of a two-stage sequential R&D investment, and presented closed-form solutions for a firm’s optimal sequential investment: (1) without suspension; (2) with suspension; and (3) with costly suspension. There are two trigger prices, which induce the firm to carry out the first and second stages of the project.

Consider a project that when completed will provide one unit of output per unit of time at a variable cost w, where r = riskless rate, θ = time remaining in the stage, u = price drift and L = abandonment cost. The first stage requires a payment k_1 at the beginning of the stage. The stage is completed $h1$ years later. The second stage timing symbols are k_2 and $h2$. If the investment is completely irreversible, once paid, k_1 and k_2 can never be recovered. The price of output (P) follows a geometric Brownian motion. The solutions for value functions at various stages, before considering suspension, are:

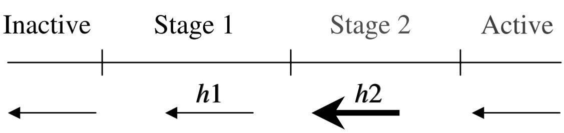
Solutions for Value Functions (1)



Assuming no abandonment after stage 2, the value of an active firm:

$$V_3(P) = \frac{P}{r - \mu} - w/r$$

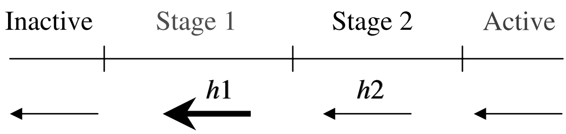
Solutions for Value Functions (2)



The value of a firm during the stage 2:

$$\begin{aligned} V_2(P, \theta_2) &= e^{-r\theta_2} E(V_3(P(\theta_2))) \\ &= e^{-r\theta_2}\left(\frac{Pe^{\mu\theta_2}}{r - \mu} - \frac{w}{r}\right) \end{aligned}$$

Solutions for Value Functions (3)

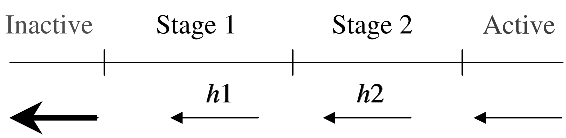


The value of a firm during the stage 1 depends on the value of a firm during the stage 2:

$$V_1(P, \theta_1) = e^{-r\theta_1} E \max[V_0(P(\theta_1)) - L,\ V_2(P(\theta_1), h1) - k_2]$$

The value is composed of two contingent parts: if the project is abandoned at L and if the investment k_2 is made at $h2$.

Solutions for Value Functions (4)



For $P \le P_1$, the firm will remain inactive, and its value is the option to begin the investment project, which is a well-known ODE solution:

$$V_0(P) = B_0 P^{\beta}$$

For $P > P_1$, the firm begins the project, and the value of an inactive firm equals a firm beginning stage 1:

$$V_0(P) = V_1(P, h1) - k_1$$

With the ability to suspend a project, a firm may be more willing to begin the project; without suspension the firm's incentives would be much weaker. Thus clinical trials might be initiated sooner, if they can be delayed or canceled as unfavorable results emerge. With suspension, first-stage triggers are relatively low, which encourages 'exploratory' investments. A small change in the suspension cost can have a large effect on investment incentives.

16.3.3 Decisions to enter/exit an R&D venture

Even in a monopoly situation, or with proprietary options, a firm should base decisions to commence or cease R&D ventures on more than the sign of the net present value of expected future cash flows, including the R&D expenditures. Dixit (1989) provided an elegant approach to the subject, and Grenadier and Weiss (1997) provided analytical models for the readiness of firms to adopt innovations (enter a business) against inclinations to delay adoption (or instead adopt earlier innovations), perhaps learning in the process and obtaining lower costs.

Dixit (1989) noted that an idle and an active firm are call options on each other, since exercising the option to invest creates another option, namely to abandon the investment and revert to the original situation. Entry/exit decisions are often not based on present value, resulting in hysteresis, the failure of an effect to reverse itself as its underlying cause is reversed.

Consider a single discrete project with ρ the rate of interest, κ the sunk investment (R&D that has no 'salvage' value) and ω the avoidable operating cost per unit time. Define the output flow of the project as a unit, so the revenue for the project is simply the output price P. The optimal decision rule consists of two triggers P_{H} and P_{L}, with $P_{\mathrm{H}} > P_{\mathrm{L}}$, such that the investment should be made if P rises above P_{H} and abandoned if P falls below P_L. Suppose that the firm does not have an investment in place and that it believes that P will never change. It will make the investment if $P > \omega + \rho\kappa$. The right-hand side is the annualized full cost of making and operating the investment. Alternatively, suppose that a firm has such an investment in place and that the price falls to a new level P, where the firm believes it will persist forever. The firm will abandon the investment if $P < \omega$.

Now consider uncertainty, where the current price is $\omega + \rho\kappa$ but thereafter the price will change in equal up or down steps with equal probabilities. If the firm invests now and continues forever, the NPV will be zero. Suppose that it waits one period. If at the end of that period the price has gone up, the firm can invest, having a positive NPV. If it goes down, then it will not invest, so the expected present value of waiting one period is positive. So at the price P_{H} the investment is an option that is only just in-the-money, and it is not optimal to exercise unless it goes deeper in-the-money.

Suppose the market price evolves exogenously over time as a geometric Brownian motion, with drift μ and volatility σ, and in addition the firm can suspend operations at a cost λ. Let $V_0(P)$ be the value of starting operations, and $V_1(P)$ be the value for the active state. The general solution for the idle firm is:

$$V_0 = A_0 P^{-\alpha} + B_0 P^{\beta}$$

where:

$$\beta = \frac{(1-m) + [(1-m)^2 + 4r]^{0.5}}{2} \quad \text{and}$$

$$-\alpha = \frac{(1-m) - [(1-m)^2 + 4r]^{0.5}}{2}$$

with $m = 2\mu/\sigma^2$ and $r = 2\rho/\sigma^2$.

The general solution for the active firm is:

$$V_1 = A_1 P^{-\alpha} + B_1 P^{\beta} + \left(\frac{P}{\rho - \mu} - \frac{\omega}{\rho} \right)$$

The third term of the equation represents the expected present value of the investment, so that the rest of the equation is the option to abandon. Similarly, since the idle firm has no current operating profit, V_0 must be the value of the option to become active.

If P is very low, the value of the option to activate is worthless, so $A_0 = 0$ and the option to activate is $V_0 = B_0 P^{\beta}$. As P increases, the value of the option to abandon has to decrease so $B_1 = 0$. Thus the option to abandon is $V_1 = A_1 P^{-\alpha}$. The firm pays κ to exercise the option to invest and gets an asset of value V_1, so: $V_0(P_\mathrm{H}) = V_1(P_\mathrm{H}) - \kappa$. The high-order and smooth-pasting condition is $\mathrm{d}V_0/\mathrm{d}P_\mathrm{H} = \mathrm{d}V_1/\mathrm{d}P_\mathrm{H}$. The price P_L triggers exit. The firm pays the exit cost and obtains the option to invest: $V_1(P_\mathrm{L}) = V_0(P_\mathrm{L}) - \lambda$. The smooth-pasting condition is $\mathrm{d}V_1/\mathrm{d}P_\mathrm{L} = \mathrm{d}V_0/\mathrm{d}P_\mathrm{L}$. These four conditions result in four equations with four unknowns A, B, P_L and P_H.

Hysteresis is:

$$V_1(P) - V_0(P) = AP^{-\alpha} - BP^{\beta} + \left(\frac{P}{\rho - \mu} - \frac{\omega}{\rho} \right)$$

Consistence with hysteresis is $P_\mathrm{H} > \omega + \rho\kappa = W_\mathrm{H}$ and $P_\mathrm{L} < \omega - \rho\lambda = W_\mathrm{L}$. This is the effect of uncertainty. The Marshallian (NPV) trigger prices for investment and abandonment are W_H and W_L. The former is the usual full cost but the latter differs from the variable cost ω, because there are now exit costs. At a price between these limits, an idle firm does not invest and an active firm does not exit. Uncertainty widens this Marshallian range of inaction. Some additional results are that as both κ and λ tend to zero, P_H and P_L tend to the common limit ω. If $\lambda \geq \omega/\rho$, the project is never abandoned. If $\sigma \to 0$, $P_\mathrm{H} \to W_\mathrm{H}$, $P_\mathrm{L} \to W_\mathrm{L}$. As ω increases, both P_H and P_L increase. As κ increases, P_L decreases and P_H increases, as shown in Figure 16.6, where the R&D investment costs κ are taken over a range of 0 to 8. Thus R&D might be continued even if unprofitable, but not necessarily initiated if expected NPV is positive.

Grenadier and Weiss (1997) cast the innovation investment strategy as a sequence of embedded options. First they identify four potential migration strategies. Then they compute the probability that a firm will take each of the four strategies in different technological environments, i.e. the expected arrival time and the expected profitability of the future innovation. Finally they provide closed-form solutions for the optimal migration strategies. This is relevant for the modeling of customers of innovation due to R&D, and also for industrial patterns of innovating.

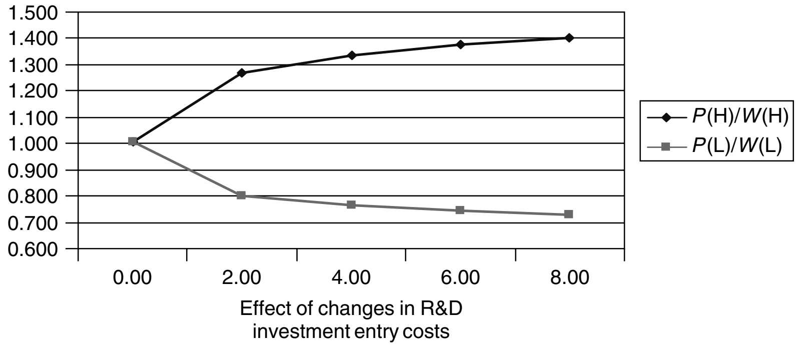


Figure 16.6 *Optimal exercise prices*

Assume the state of technological progress $X(t)$ follows a gBm. The random arrival time of the future innovation, T, is the first passage time of $X(t)$ to the boundary X_h. For $\alpha - \sigma^2/2 > 0$, α = drift, σ = volatility, the expected arrival time $E(T)$ is equal to:

$$\frac{\ln\left(\frac{X_h}{X}\right)}{\alpha - \sigma^2/2}$$

There are four potential migration strategies: a *compulsive* strategy of purchasing every innovation; a *leapfrog* strategy of skipping an early innovation; a *buy-and-hold* strategy of only purchasing an early innovation; and a *laggard* strategy of waiting until a new generation of innovation arrives before purchasing the previous innovation, as shown below.

Four Potential Migration Strategies

A firm is initially confronted with an opportunity to invest in a current innovation.

The arrival of current innovation, P_0, at time zero

The arrival of future innovation, P_T, at uncertain time T

Invest at early adoption cost, C_e → *Upgrade* new technology at upgrade cost, C_u / *Hold* its current technology

Bypass and wait the arrival of future innovation → *Leapfrog* to the new innovation at a cost, C_l / *Purchase* the old innovation at a discounted cost, C_d

Probabilities of Four Possible Strategies

Current innovation, P_0 — Future innovation, P_T

Invest: compulsive → *Upgrade*
$$PC(X) = \Pr[Te < T, P_T - P_0 - C_u \geq 0]$$

Invest: buy-and-hold → *Hold*
$$PB(X) = \Pr[Te < T, P_T - P_0 - C_u < 0]$$

Bypass: leapfrog → *Leapfrog*
$$PL(X) = \Pr[Te \geq T, P_T - C_l \geq P_0 - C_d]$$

Bypass: laggard → *Purchase*
$$PG(X) = \Pr[Te \geq T, P_T - C_l < P_0 - C_d]$$

The impact of the speed of innovation arrival or the expected arrival time is shown in Figure 16.7. For markets with rapid innovation [low $E(T)$], firms are most likely to adopt the leapfrog and laggard strategies; while the compulsive and buy-and-hold strategies dominate for markets with slow innovation [high $E(T)$].

The impact of increasing the expected profitability of the future innovation (μ) is shown in Figure 16.8. For markets with greater μ, the compulsive and

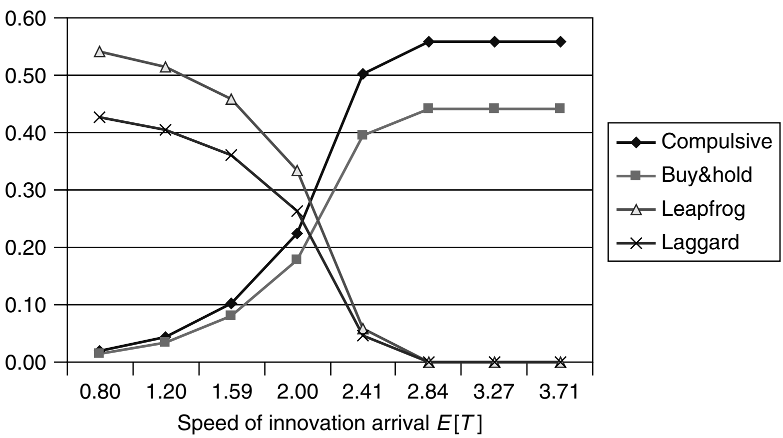


Figure 16.7 *Probability of migration strategies*

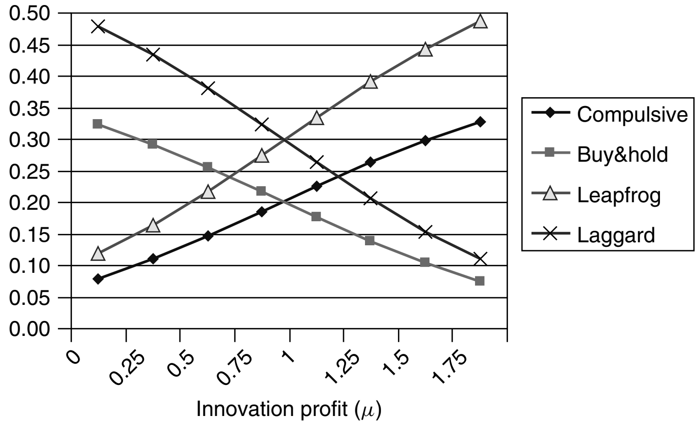


Figure 16.8 *Probability of migration strategies*

leapfrog strategies should become more likely; while the laggard and buy-and-hold strategies become less compelling. This illustrates an environment where the speed of innovation arrival is slow, so the leapfrog will be the dominant strategy if there is very high profitability, and the laggard will be dominant for low profitability.

The impact of uncertainty (the future evolution of technology) is such that higher σ increases the likelihood of the leapfrog and laggard strategies, while reducing the likelihood of the compulsive and buy-and-hold strategies. So, increasing σ prompts firms to delay investment.

A limitation of this model is that the probability of a migration strategy is highly sensitive to the expected arrival time. Strategies are sharply changed for the given parameters around the range $E(T)$ {1.5 to 2.5}, as is evident in Figure 16.7.

16.4 FIRST MOVER AND PRE-EMPTION MODELS

Most early real option models show the importance of deferring 'sunk cost' expenditures, 'what is gained by waiting to invest'. Kulatilaka and Perotti (1991, 1998) consider 'what is lost by waiting to invest'. In a competitive market, early investment may confer an advantage of a greater share of the market, a type of pre-emption, as well as early cash flows. Dixit and Pindyck (1994) present a pre-emption model that is based on Fudenberg and Tirole (1985) and Smets (1993) for an oligopolistic industry. Lambrecht and Perraudin (1997) extend standard models of irreversible investment by incorporating strategic entry by competing firms. Lambrecht (2000) models a competitive R&D stage, where there is a trade-off between the value of waiting to invest and the cost of being pre-empted, and a second stage, where there may be 'sleeping patents', inventions which are not immediately put to use.

First-mover advantages are created by (1) the technological advantages of being first; (2) patents if innovative and first; (3) the brand image of a first mover; and (4) organization and location advantage. However, being first may not be the same as being the best, since it may pay to wait and learn from the first-mover's mistakes. One of the challenges is to quantify the first-mover's advantages and disadvantages. Frequent and public monitoring is sometimes feasible, for instance in the cases of public marketing of innovations, FDA applications and patent applications. However, brand loyalty and differential pricing for the first mover is not always transparent or measurable. For instance, for e-banking and internet service providers, only some companies provide frequent information on new accounts, churn and usage, and seldom provide detailed frequent information on profits.

Kulatilaka and Perotti (1998) assume a lognormal random demand, and a deterministic production cost shift, upon first-mover investment. They also assume that under imperfect competition there will be a 'lower industry profit' after subsequent investments, since new competitors may have 'lower costs' which will result in reduced profits for the first mover after the follower enters the scene. They assume $P(Q) = \theta - Q$, where Q = total supply, θ = random demand, both the leader and the follower(s) may have the same production cost K at $t = 0$, both firms invest only in period 0 or 1, and share the market equally after entry. If $\theta \geq K$, then $Q_1^N = (\theta - K)/3$ and profits $= (1/9)(\theta - K)^2$ for each firm.

The first mover invests early, and the initial investment I reduces unit costs to κ, where $\kappa < K$, due to learning and other improvements. If $\theta < K$, there is no production. If $\theta > K$, and the leader invests I, it will choose output so that profits $= (1/9)(\theta - K - 2\kappa)^2$. Point of entry for a competitor is higher, where $\theta > 2K - \kappa$. The value of not investing $[V^N]$ is:

$$V^N(\theta_0, \sigma) = \tfrac{1}{9}[\theta_0^2 e^{\sigma^2} N(d_1) - 2K\theta_0 N(d_2) + K^2 N(d_3)]$$

where:

$$d_1 = \frac{2\ \ln(\theta_0/K) + 3\sigma^2}{2\sigma}, \quad d_2 = d_1 - \sigma, \quad d_3 = d_2 - \sigma$$

The first mover (pre-emption) value of not waiting to invest $[V^I]$ is:

$$V^I(\theta_0, \sigma) = \tfrac{1}{9}[2K\theta_0 N(d_5) + K^2 N(d_6)] - I + \theta_0^2\, e^{\sigma^2}/4\left[1 - \tfrac{5}{9}N(d_4)\right]$$

where:

$$d_4 = \frac{2\ \ln(\theta_0/2K) + 3\sigma^2}{2\sigma}, \quad d_5 = d_4 - \sigma, \quad d_6 = d_5 - \sigma$$

Both V^N and V^I increase with increased random demand. At low demand volatility, V^I is more sensitive to increased demand than V^N, as shown in Figure 16.9.

The threshold demand for investment depends on capacity cost advantage $(K - \kappa)$. Suppose $\kappa = 0$, so that unit cost falls to 0 if there is a first-mover investment. Then the threshold initial demand that motivates I is θ, so that $V(N) = V(I)$. Figure 16.10 shows that the random demand θ which justifies a first-mover's investment decision declines with an increase of the capacity cost advantage.

Suppose there is systematic risk aversion, so that increased uncertainty over θ is not diversifiable. Then the 'market price of systematic risk' times the volatility should be incorporated in $V(N)$ and $V(I)$. At high volatility, $V(I)$ relative to $V(N)$ is reversed as market price of risk increases, as shown in Figure 16.11.

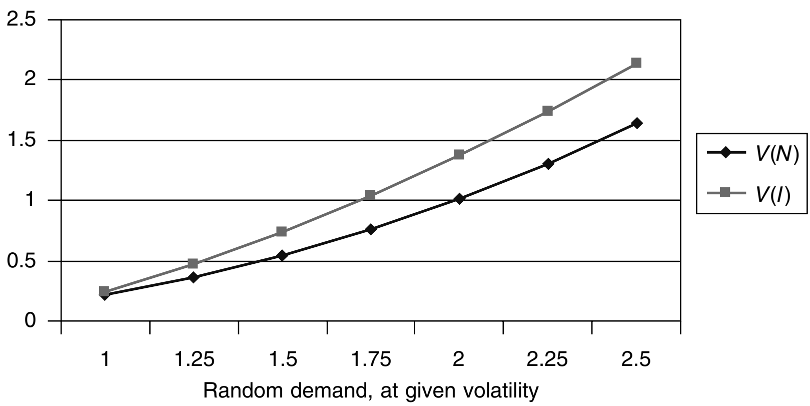


Figure 16.9 *Effect of increased random demand on V(N), V(I) under imperfect competition*

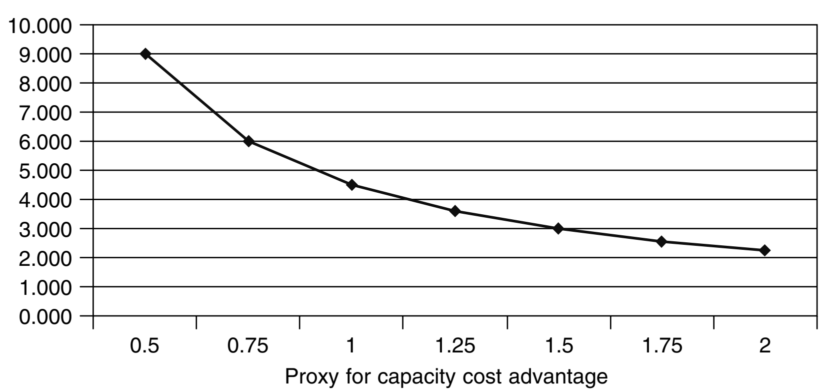


Figure 16.10 *Sensitivity of θ (threshold)*

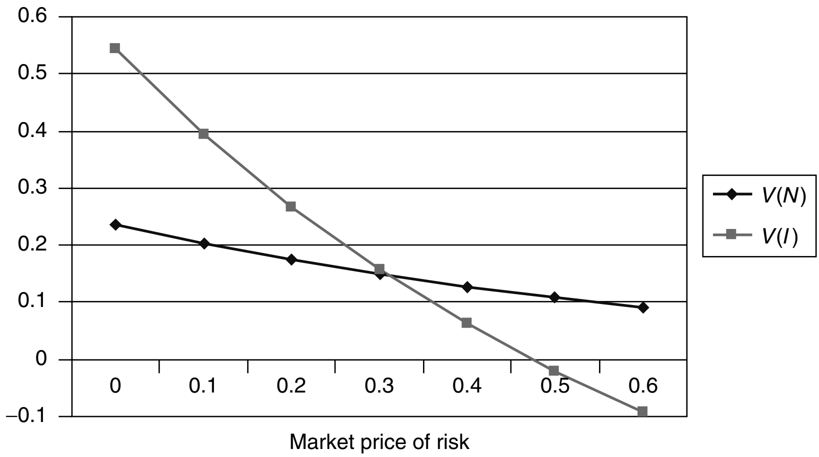


Figure 16.11 *Pre-emption value, given systematic risk*

In Dixit and Pindyck (1994) there are two firms, where the leader after investment of a sunk cost I will have a profit flow of x_1, until a follower also makes the same sunk cost investment I, when both firms have a profit flow of x_2.[1] (Other authors modify the profit flow to reflect market sharing and market growth.) The real option value for the leader and the follower is established by finding the optimal T (stopping region), when it is optimal for those firms to invest.

Let $V_1^{\mathrm{F}}(x)$ denote the value of the follower in the stopping region, the region where it is optimal to invest, r = riskless rate and μ = drift rate, and $V_0^{\mathrm{F}}(x)$ denote the value of the follower prior to investment. The optimal investment rule is found by solving for the boundary between the continuation and the stopping regions. The boundary is the trigger point x_{F}. If the value of the state variable is smaller than the trigger, the optimal decision for the follower is not to invest, i.e. to remain in the continuation region. If it exceeds the trigger, then the follower should invest. At the trigger point, two conditions must be satisfied: (1) the value-matching that makes explicit that when the state variable reaches the trigger the follower will invest so that $V_0^{\mathrm{F}}(x_{\mathrm{F}}) = V_1^{\mathrm{F}}(x_{\mathrm{F}}) - I$ and (2) the smooth-pasting condition, that requires that the derivatives of the functions match at the boundary, $V'^{\mathrm{F}}_0(x_{\mathrm{F}}) = V'^{\mathrm{F}}_1(x_{\mathrm{F}})$. These conditions imply that x_{F} satisfies:

$$x_2 = \frac{I\beta_1(r-\mu)}{(\beta_1 - 1)}$$

where:

$$\beta_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1$$

Thus we obtain the value function of the follower:

$$V^{\mathrm{F}}(x)=\begin{cases}\left(\dfrac{x_2}{r-\mu}-I\right)\left(\dfrac{x}{x_{\mathrm{F}}}\right)^{\beta_1} & \text{if } x<x_{\mathrm{F}}\\[2ex] \dfrac{x_2}{r-\mu}-I & \text{if } x\geq x_{\mathrm{F}}\end{cases}$$

Until the follower enters the market, the leader's decision either to enter the market or to wait is identical to the monopoly framework. So following Dixit and Pindyck (1994), there exists an optimal time to enter that will maximize the firm's value. Until that moment the firm should wait to invest and its value is explained by the option to wait. When that moment is reached, the firm should invest and its value function is given by the present value of the revenues in perpetuity. (One problem with the option to wait is that it excludes the case where companies do not have the possibility of waiting.)

First-mover advantage should make pre-emption attractive, and pre-emption should lead to early adoption by the leader. There are many examples in R&D of the advantages of being first. The first company in an industry to have a website might buy cheaper domain names and obtain lower staff costs and better access to resources. In orphan drug status, the FDA may award the first mover a pre-emptive advantage, even prior to the completion of clinical trials. In general, once a patent is obtained as a result of R&D, the first mover may have an advantage for the number of years the patent is valid, if not in perpetuity.

Having entered the market, the leader will enjoy monopolistic revenues until the moment that the follower enters the market, and will share revenues with the follower afterwards. The general solution for the leader is:

$$V^{\mathrm{L}}(x)=\begin{cases}\dfrac{x_1}{r-\mu}\left[1-\left(\dfrac{x}{x_{\mathrm{F}}}\right)^{\beta_1-1}\right]+\dfrac{x_2}{r-\mu}\left(\dfrac{x}{x_{\mathrm{F}}}\right)^{\beta_1}-I & \text{if } x<x_{\mathrm{F}}\\[2ex] \dfrac{x_2}{r-\mu}-I & \text{if } x\geq x_{\mathrm{F}}\end{cases}$$

Figure 16.12 shows that the follower's critical profitability (x_{F}) is higher, which justifies investment, and the real option value (V) is lower, the higher the investment cost I. Figure 16.13 shows that both the follower's critical profitability and real option value increase with profit volatility.

The Lambrecht and Perraudin (1997) model is dynamic and continuous, and evaluates the 'waiting value' versus 'fear of pre-emption'. Without pre-emption the follower's real option value is similar to the Dixit and Pindyck (1994) equation above for V^{F}. With pre-emption, the follower's real option value is the previous real option value (ROV) adjusted for the pre-emption fear. Firm i conjectures that firm j will invest when x_t crosses some level x, which is

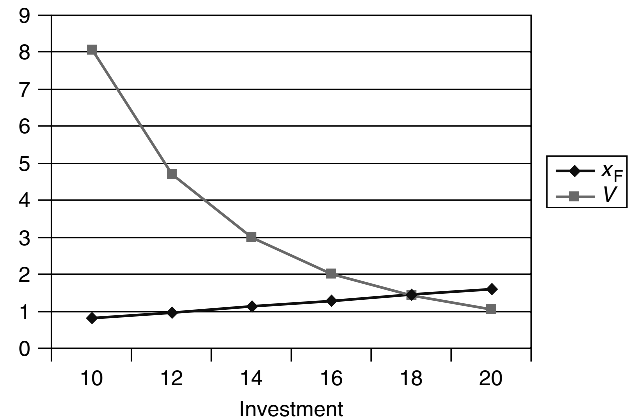


Figure 16.12 *Real R&D option value (V) and critical profit level (x_F)*

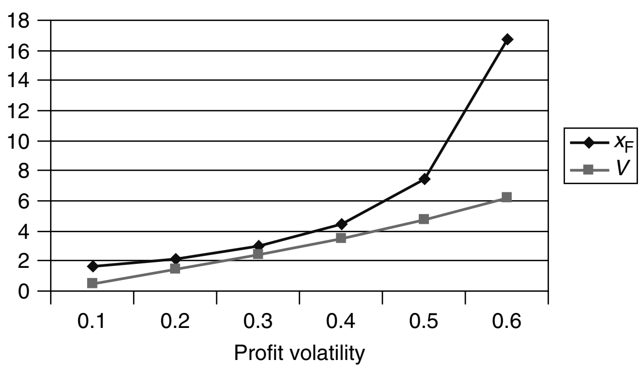


Figure 16.13 *Critical expected profit (x_F) and real R&D option value (V)*

an independent draw from a distribution $F_j[x_j]$ (assumed to be Pareto), with a density function $F'_j[x_j]$. The conjectured first crossing $x(is)$ is the previous optimal x_F, adjusted for the hazard rate ($= x^*$), so that the adjustment is ROV* $\{[1 - F(x(is))]/[1 - F(x)]\}$.

Figure 16.14 shows some comparisons of the cost trigger maps, for NPV $[x(m)]$, ROV with (x^* PRE) and without (x^*) pre-emption. For any level of the investment cost k, the profit level which triggers investment is lowest for NPV, and highest for x^*, as shown. The spread between these optimal triggers increases with the investment cost.

There are several other models of pre-emption and hysteresis (see Weeds, 1999a,b, 2000). Some models consider jump processes for new competitive entry. Other plausible models have a stochastic market share and/or profitability. Followers and leaders might learn from each other, which may affect

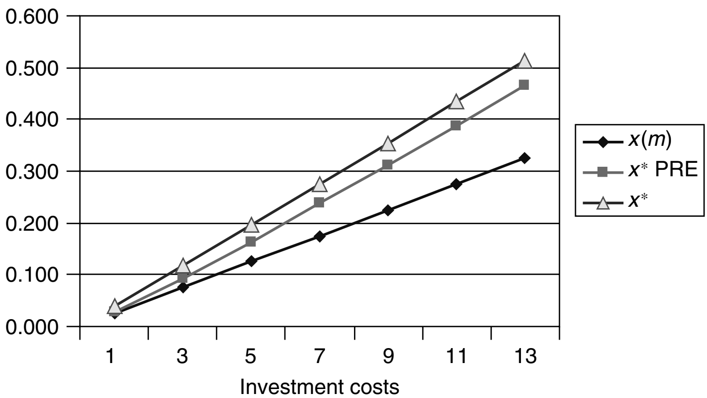


Figure 16.14 *Critical investment triggers: NPV (x(m)), with pre-emption (x* PRE), without (x*)*

either pricing, or market share, or production cost. Possible pre-emption models might consider adjusting real option value for convenience yield, or assume higher/lower implementation costs, as firms enter the market. In general, pre-emption will be less likely if cash flows naturally are mean-reverting. Huisman (2001) considers a game-theoretical real options approach, with the introduction of multiple new technologies. Other applications of pre-emption models include where creditors may foreclose first, informed investors buy or sell first, and where some contingent claim holders exercise their options first.

Most imperfect competition real option models are appropriate in an environment where global presence can be established from any country, such as now is the case for electronic commerce or internet activities. These markets are not constrained by national boundaries. On the other hand, the ease of global presence implies that extreme market penetration achieved possibly by first movers means extreme profitability. 'Winner takes all' models will probably be based on extreme distributions.

16.5 EMPIRICAL VARIABLES FOR REAL R&D OPTION MODELS

One of the most difficult questions in valuing real R&D options concerns the empirical basis of the parameters. Many authors including Newton and Pearson (1994), Newton et al. (1996), Grenadier and Weiss (1997), Childs et al. (1998), Berk et al. (1999), Childs and Triantis (1999) and Schwartz and Moon (2000a) simply provided parameter inputs 'by assumption'.

However, other authors have obtained data from surveys of 'experts', engineering estimates, time series of commodity prices, and lately from stock and traded options markets for R&D-intensive enterprises. Information on R&D values and cost are often hard to obtain from public sources, and there are large transaction and search costs in obtaining private information, which can in any case be noisy.

Weitzman et al. (1981) used capital cost estimates from process engineers, operating and maintenance costs from research engineers, and feedstock costs and conversion efficiencies from scientists; variances were derived from ‘energy specialists’ for a synthetic crude oil model. Ott (1992) used several deterministic inputs for current and projected cost of producing electricity over the estimated life of a power plant, along with maximum R&D spend on a project per year. He projected US demand for total electricity and relative prices from economic studies, and based growth rates and new and existing technology variances on assumptions. Pindyck (1993) obtained expected construction cost (and variance) for a kilowatt of nuclear generating capacity from ‘survey data’ on actual individual plant costs over several years.

Lint and Pennings (1997) conducted semi-structured interviews with R&D and marketing departments with quantitative questions on future prices, quantities, costs, rate of acceptance of both standards, and capital and marketing expenditures for creating the available product options. The uncertainty surrounding both standards and the correlations were from a cross-section analysis of the interview data. Lint and Pennings (1998) conducted in-depth interviews with managers (with statements verified by the market intelligence department) on expected market size, growth rate, expected market share and profitability of targeted market segments. ‘Unlikely subjective estimates were returned to involved managers, discussed and adjusted if necessary’. Pennings and Lint (1997) obtained estimates of future cash flows ‘from senior management’. Benaroch and Kauffman (1999) also interviewed senior managers regarding the range of potential revenues, perceived volatility and entry timing.[2] Taudes (1997) used estimates from in-house experience regarding the number of electronic transactions, growth and variance, and estimates from a software supplier for the implementation costs over various stages.

Lambrecht and Perraudin (1997) suggested using stock market quotations for a series of biotechnology companies to characterize some of the parameters for their competitive markets. Lee and Paxson (2001) utilized historical volatilities from stock prices of R&D-intensive companies in e-commerce, and the implied volatilities of traded options for some of those companies, in providing estimates for future R&D volatilities, and for the correlation of research costs and benefits. Ottoo (1998) provided an illustration (however, based on assumptions) for the type of empirical inputs that might be useful in valuing a biotechnology company, including probabilities of success for such a company in R&D and also that of its competitors.

There is a natural development in linking embedded real R&D option valuation to stock market valuation. Kellogg and Charnes (2000) valued a biotechnology company, Agouron Pharmaceuticals, using a simple binomial approach and some assumed discovery probabilities and investment cost present values, compared to the market valuation. Schwartz and Moon (2000b) considered a model for an internet company with stochastic revenues, where the volatility term is mean-reverting and the drift term is also stochastic and mean-reverting, with a convergence of the sales growth rate to a long-term average, and deterministic drift rate volatility. These authors used averages of historical sales growth and analysts' estimates of future growth, historical volatility of sales, implied volatility of stock options (as a proxy for volatility of sales growth), and balance sheet data for cash flow and burn rates for Amazon.com. The model solution is through Monte Carlo simulation. The real option value of Amazon.com market capitalization was compared to the adjusted real option value.

Schwartz and Moon (2000c) is a similar application to Exodus Communications, an internet web-page host. The variable cost function was assumed to follow a mean-reverting process. Using the implied volatility of options on Exodus stock, plus balance sheet and quarterly income statement analysis as the basis for parameter estimation, the authors simulate a contingent claim 'fundamental' valuation of the stock, compared to the market valuation.

These authors have not (yet) compared alternative per share valuations to the share price over time, as in Lehocky and Paxson (1998), who also show the fitted daily risk premiums implied by alternative per share estimates. A further development is to test whether some trading rules using the embedded real R&D option approach will result in superior share investment performance over time.

In the last few years, numerous e-commerce, internet, biotechnology and other high-technology enterprises have been securitized, and so there is a rich source of quoted shares and traded options for a wide variety of R&D values and costs, including more or less pure R&D, software development, internet facilities and e-commerce applications. Conceivably there will emerge derivative markets (both futures and options, and exotic options) for direct R&D values and costs, such as semiconductors, software and broadbands, so that empirical modeling can be (to a greater degree) based on market variables rather than on broad assumptions.

16.6 USES OF REAL R&D OPTION MODELS AND FURTHER RESEARCH

The primary uses of real R&D option models are: (i) determining the appropriate R&D strategy and overall budget; (ii) determining the optimal timing (and likely occurrence) of R&D stages, if the timing is flexible; (iii) allocating the overall budget among competing research proposals; and (iv) valuing the

R&D process, either for the purpose of investment budgeting, or for external sale, joint venture or corporate financial engineering. The applications are to different levels of working interests, from the *Mini* level, where the concern is regarding the incentives, motivations and rewards of individual researchers, to the *Micro* level, that is enterprise-level decisions, possibly in competition with other firms, and finally to the *Macro* level, that is broad industry, economic and social policies, including tax and direct subsidies for R&D. First, one always has to ask, what is required in using these real R&D option models, and then what are the possible applications, perhaps outside the original focus of the authors.

16.6.1 Expected value and cost

Most of these critical uses require both the estimation of the future cash flows arising from successful R&D results and the required R&D expenditures to obtain those results. The overall R&D budget is itself both a valuation and an optimal 'timing' issue, since valuation justifies the investment (or not) and timing is (usually) a critical aspect of the valuation. Of course, both value and timing are viewed in the context of market environments, where the fund-raising capacity (and price) will be apparent (almost) at any time, and possibly the market implied volatilities of R&D value and cost will be (more or less) available.

16.6.2 Diffusion processes for value and cost

In the standard R&D option models, there are very broad assumptions as to the stochastic processes governing R&D values and development costs. The usual presumption is that both value (and sometimes cost) are lognormally distributed and stationary. In practice, R&D values are not lognormally distributed, so that in the future, distributions such as the extreme(s), Student t or stable Paretian may be considered more appropriate and realistic. Other remaining theoretical problems include using multivariate distributions that can model several investment stages over time; volatility and correlation matrices for these sequential investments; and allowing for variable R&D income and development cost escalation assumptions over time.

16.6.3 Other parameters

Additional considerations for real R&D options are the chance of success or failure, usually assumed to be in the family of univariate discrete distributions such as Poisson or compound Poisson. Finding or devising appropriate empirical proxies for these 'discovery' or failure inputs is an important and challenging business. Also since R&D discoveries are seldom proprietary (until patented, if that is possible), parameters such as the first-mover's advantages (and disadvantages) in markets with a time-variant size, and share, have to be considered.

16.6.4 Capital market variables

At all stages of R&D, there may be several corporate finance issues. For R&D enterprises already quoted in public markets, there is the possibility of issuing equity if the R&D enterprise market capitalization exceeds the real option value adjusted for other enterprise net assets. Other possibilities include cutting R&D and repurchasing shares in the opposite case, and/or buying other enterprises (or outsourcing) if the relative real option value to market capitalization of others is high. For start-up R&D and other ventures not quoted in public markets, there are the same types of questions with regard to venture funds and social funds. For purely academic and social ventures, while future cash flows might be of less concern, there is conceptually the same approach (with perhaps R&D results measured by reputation standards, such as publications). There are similar possible actions, that is where external value exceeds R&D cost, there should be more emphasis on soliciting sponsorship, and in the opposite case, considerations of abandoning 'fruitless' R&D.

16.6.5 Model applications

The primary uses of overall strategy, optimal timing, allocation of resources among competing projects and external validation can be considered at three levels. Individuals are, of course, focused on the Mini level, where the concern is on personal and educational efforts, and the design, implementation (and acceptance) of suitable risk evaluation and reward systems for individual (or team) R&D. For instance, researchers in real R&D options will decide: (i) the time and effort to devote to the subject; (ii) whether to delay or accelerate efforts until model uncertainty is resolved and better data is available; (iii) which of the many strands of the topic to pursue, given their own competences and interests, as well as evaluation of the likely outcomes of their R&D; and (iv) whether and how to seek external funding, and/or capitalization of the fruits of research, including publications.

Research managers are focused on the Micro level, where R&D research takes place in groups or enterprises. The design, implementation (and acceptance) of suitable risk evaluation and reward systems for group efforts is the primary concern of most real R&D models. For instance, biotechnology managers will decide: (i) the (usually) annual overall R&D budget; (ii) whether to delay R&D emphasis until plausible results emerge or to speed up R&D in anticipation of competing firms achieving first-mover advantage from the fruits of early

R&D; (iii) for larger entities, allocating the (usually) annual R&D budget among competing R&D proposals; and (iv) whether and how to seek external funding, and/or joint ventures, or cancellation of fruitless, costly or unworthy R&D. Since typical start-up high-technology investments depend on raising cash in the future to support negative cash flow operations, this aspect of R&D corporate funding activity is itself a real option.

It is challenging for accountants and investment analysts to identify and disclose real R&D assets and liabilities, including potential product defects, requirements for future R&D expenditures and even first-mover advantages and disadvantages. Also perhaps one will see more 'synthetic' real R&D options (building on R&D tracking stocks), including R&D venture index futures and options, so that the synthetic vehicles can be used for 'hedging' firms' delay or commencement of certain types of R&D in the financial markets, to offset practical limitations in the physical environment.

Policy makers might be focused on the Macro level, that is how much of the economy should be devoted to R&D, in the context of evaluating the cost/benefits of changing the current R&D structure and incentives (subsidies, direct government R&D and taxes). There are the continual issues of: (i) whether to delay concentration on developments until basic research uncertainty is resolved; (ii) which of the many types of R&D to pursue; (iii) the appropriate format for R&D, private, government or academic (if there are externalities); and (iv) how to disseminate (or protect) the fruits of research.

Competition policy might be considered a Macro issue, where pre-emption might be deemed 'unfair' or harmful to investment, or alternatively to be encouraged in the case of 'orphan' drugs for rare diseases. Debates on generic Aids drugs for developing countries, the ownership of human genome research, and the auctioning of 3G licenses indicate the relevance of real R&D option models in the public arena.

16.6.6 Future developments in real R&D options

What is the future of real R&D option models? The topic is in its infancy, since published empirical applications are limited, and perhaps have hardly advanced beyond Weitzman et al. (1981). The vast amount of data from clinical trials alone is a suitable source of real R&D option information. No doubt many new models will be developed, with new analytical and numerical solutions, probably integrating various parts of stochastic control systems and engineering with real option approaches. The distributions thus far considered for R&D results and costs are quite limited. Non-normal distributions and associations between costs and values using various multivariate measures, and copulas, will be some of the challenges for future researchers.

Finally, uses of real R&D options theory have been primarily in the Micro area. However, considerations of industry adoption of innovation, value and motivation of first movers are extensions into the Macro aspects. In the future expect employment of real R&D option models in Macro debates on the amount, direction, type and evaluation of R&D, as well as extensions to Mini concerns regarding the value of individual researchers.

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NOTES

1 Each firm produces a unit output at zero variable cost, and the price follows a demand function $P = YD(Q)$, where $Q = 0, 1, 2$. So x_1, x_L is the profit function for $YD(1)$ and x_2, x_F for $YD(2)$.
2 Schwartz and Zozaya-Gorostiza (2000) use this data as empirical inputs for R&D development models.

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