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Jin Akiyama (right) and Vašek Chvátal (left)
at the conference Banquet celebrating their 60th birthdays

Preface

This volume consists of the refereed proceedings of the Kyoto Conference on Computational Geometry and Graph Theory (KyotoCGGT 2007), held at Kyoto University in Kyoto, Japan, 11–15 June 2007, to honor Jin Akiyama and Vašek Chvátal on their 60th birthdays. More than 200 participants from 20 countries attended the conference.

Akiyama and Chvátal have been good friends since they met in Tokyo in 1979. Akiyama started the conference series Japan Conference on Discrete and Computational Geometry (JCDCG) in 1997, which has been held annually since that time. In 2001, the conference venue began to alternate between Tokyo and selected Asian cities to attract and encourage Asian graph theorists and geometers. Chvátal, on the other hand, is world-renowned for his contributions to discrete mathematics.

Since it was first organized in 1997, the annual JCDCG conference has attracted a growing international participation. Earlier conferences were held in Tokyo, followed by conferences in Manila, Philippines (2001), Bandung, Indonesia (2003), and Tianjin and Xi'an, China (2005). The proceedings of JCDCG 1998, 2000, 2002, 2004, IJCCGGT 2003 and CJCDGCGT 2005 were published by Springer in the series *Lecture Notes in Computer Science* (LNCS) as volumes 1763, 2098, 2866, 3742, 3330 and 4381, respectively, while the proceedings of JCDCG 2001 were also published by Springer as a special issue of the journal *Graphs and Combinatorics*, Vol. 18, No. 4, 2002.

The organizers of KyotoCGGT 2007 gratefully acknowledge the support of the sponsors, the work of the conference secretariat and the participation of the principal speakers : William Cook, Greg Frederickson, Ferran Hurtado, Joseph O'Rourke, János Pach, Bruce Reed, Akira Saito, Kokichi Sugihara, Godfried Toussaint and Jorge Urrutia.

June 2008

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Dudeney Transformation of Normal Tiles

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Abstract. Let A and B be convex polygons. We say that A and B are D-*equivalent* if there are convex polygons $A = A_1, A_2, \ldots, A_n = B$ and Dudeney dissections of A_i to A_{i+1} $(1 \leq i \leq n-1)$. A polygon is called a *tile* if the 2-dimensional Euclidean plane can be tiled by congruent copies of the polygon. A polygon is called a *normal tile* if the plane can be tiled by congruent copies of the polygon which are obtained without turning over the polygon. The numbers of types of convex tiles and convex normal tiles are still uncertain. In this paper, we prove that all convex normal tiles with the same area that we know so far are D-*equivalent*.

1 Introduction

H. E. Dudeney proposed the following problem [3]: "Cut an equilateral triangle into four pieces and put them together to make a perfect square without turning over any piece." After he gave an answer, he wrote "I add an illustration showing the puzzle in a rather curious practical form, as it was made in polished mahogany with brass hinges for use by certain audiences. It will be seen that the four pieces form a sort of chain, and that when they are closed up in one direction they form the triangle, and when closed in the other direction they form the square." (See Fig. 1.)

We will define a Dudeney dissection following Dudeney's problem. Let A and B be convex polygons with the same area. A *Dudeney dissection* of A to B is a partition of A into a finite number of parts which can be reassembled to produce B as follows: Hinge the adjoining parts of A on points along the perimeter of A, then fix one of the parts and rotate the remaining parts about the fixed part to form B in such a way that:

(1) All of the perimeter of A is in the interior of B.
(2) The perimeter of B consists of the dissection lines in the interior of A.
(3) The pieces of A are never turned over.

Note that it is not necessary that the pieces form a sort of chain. Dudeney dissections of various convex polygons are discussed in [1] and [2]. A convex polygon A is *Dudeney dissectible* if there exists a Dudeney dissection of A to B for some convex polygon B.

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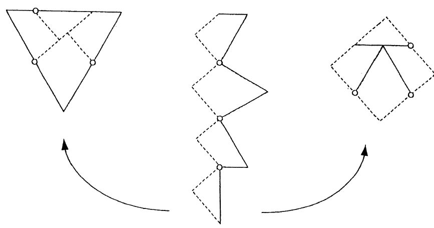


Fig. 1.

Let A and B be convex polygons. We say that A can be *transformed* to B if there are convex polygons $A_1, A_2, \ldots, A_n$, where $n \geq 2$, $A_1 = A$ and $A_n = B$, and Dudeney dissections of A_i to A_{i+1} $(1 \leq i \leq n-1)$. We denote this by $A \to B$. Let $\mathcal{P}$ be the set of all Dudeney dissectible convex polygons with a given area. Then we have, for $A, B, C \in \mathcal{P}$,

(i) $A \to A$,
(ii) $A \to B$ implies $B \to A$,
(iii) $A \to B$, $B \to C$ implies $A \to C$.

Thus the transformation is an equivalence relation of $\mathcal{P}$. We say that A and B are D-*equivalent* or that A is D-*equivalent* to B if $A \to B$.

A polygon is called a *tile* if the plane R^2 (2-dimensional Euclidean plane) can be tiled by congruent copies of the polygon, where we say that R^2 can be tiled if R^2 is covered without gaps or (2-dimensional) overlaps. A polygon is called a *normal tile* if the plane R^2 can be tiled by congruent copies of the polygon which are obtained without turning over the polygon.

It is known that if a convex n-gon is a tile, then we have $n \leq 6$ ([8]). Every triangle and every quadrilateral are tiles. Let ABCDEF be a hexagonal tile where $a =$ FA, $b =$ AB, $c =$ BC, $d =$ CD, $e =$ DE, $f =$ EF. All convex hexagonal tiles have been classified into three types: type 1: $\angle\mathrm{A} + \angle\mathrm{B} + \angle\mathrm{C} = 360°$, $a = d$, type 2: $\angle\mathrm{A} + \angle\mathrm{B} + \angle\mathrm{D} = 360°$, $a = d, c = e$, type 3: $\angle\mathrm{A} = \angle\mathrm{C} = \angle\mathrm{E} = 120°$, $a = b, c = d, e = f$. There are no other convex hexagonal tiles ([5], p494). However, the number of types of convex pentagonal tiles is still uncertain. At present, 14 types of convex pentagons are known to be tiles ([9], [10], [13]). We follow the numbering of the types given in [5] (p492) and [13].

Every triangle, every quadrilateral, hexagons of types $1, 3$, and pentagons of types $1, 3, 4, 5, 6$ are normal tiles. Pentagons and hexagons of the other types are normal tiles if they have line symmetry.

All convex normal tiles that we know so far are the following: (1) every triangle, (2) every convex quadrilateral, (3) pentagons of types $1, 3, 4, 5, 6$, (4) pentagons of types $2, 7, 8, 9, 10, 11, 12, 13, 14$ with line symmetry, (5) hexagons of types $1, 3$, (6) hexagons of type 2 with line symmetry.

In this paper, we will prove that all the above polygons with a given area are D-equivalent, that is, we will prove the following theorem.

Theorem 1. *The following polygons with the same area are all D-equivalent:*

(1) *every triangle,*
(2) *every convex quadrilateral,*
(3) *pentagons of types* 1, 3, 4, 5, 6*,*
(4) *pentagons of types* 2, 7, 8, 9, 10, 11, 12, 13, 14 *with line symmetry,*
(5) *hexagons of types* 1, 3*,*
(6) *hexagons of type* 2 *with line symmetry.*

In this paper, we consider only convex polygons, so we sometimes call them just "polygons" for simplicity.

2 Triangles and Quadrilaterals

Proposition 2.1. *All triangles and quadrilaterals with the same area are D-equivalent.*

Proof. Suppose that triangles and quadrilaterals we consider here have the area 1. The proof consists of four steps:

Step 1. Any triangle can be transformed to a parallelogram.
Step 2. Any quadrilateral can be transformed to a parallelogram.
Step 3. Any parallelogram can be transformed to a rectangle.
Step 4. Any rectangle can be transformed to a square.

If we prove Steps 1, 2, 3 and 4, then we find any triangle and any quadrilateral can be transformed to a square, thus we complete the proof.

Step 1. Let ABC be any triangle. Let L, M, N be the midpoints of AB, BC, CA, respectively. Draw dotted lines LN and AM and tile the plane with the triangle ABC as shown in Fig. 2. Cut it along the dotted lines, then we have four parts. Rotate three parts of them as shown in the figure, then we obtain a parallelogram DEFG.

Step 2. Let ABCD be any (convex) quadrilateral. Let L, M, N, K be the midpoints of AB, BC, CD, DA, respectively. Draw dotted lines LN and KM and tile the plane with the quadrilateral ABCD as shown in Fig. 3. Cut it along the dotted lines, then we have four parts. Rotate three parts of them as shown in the figure, then we obtain a parallelogram EFGH. We note that Step 1 is a degenerate case of Step 2.

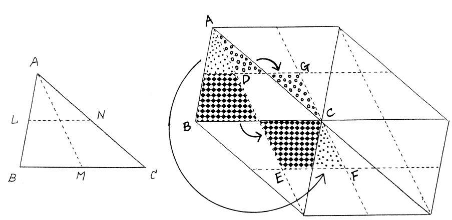


Fig. 2.

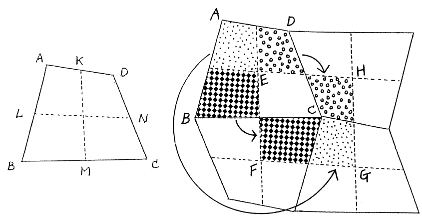


Fig. 3.

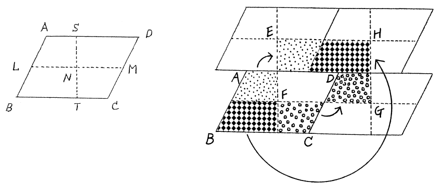


Fig. 4.

Step 3. Let ABCD be any parallelogram. We can assume AB $\leq$ BC and $\angle$B $\leq 90°$. Let L and M be the midpoints of AB and CD, respectively. Let S and T be points on the segments AD and BC, respectively, such that ST is perpendicular to LM, and the intersection point N of ST and LM is in the parallelogram ABCD. We can choose such points S, T under our assumption. Draw dotted lines LM and ST. Tile the plane with the parallelogram ABCD as shown in Fig. 4. Cut it along the dotted lines, then we have four parts. Rotate three parts of them as shown in the figure, then we obtain a rectangle EFGH.

Step 4. We will prove the following Lemmata.

Lemma 2.1. *Any rectangle can be transformed to a rectangle such that all side-lengths are less than or equal to* $\sqrt{2}$.

Proof. Let ABCD be any rectangle with AB $\geq$ BC. Put $a =$AB.

If $a > \sqrt{2}$, cut the rectangle ABCD into five pieces as shown in Fig. 5, where AE=FB=DH=GC= $a/4$ and I, J, K, L are midpoints of AD, EH, FG, BC, respectively. Rotate four pieces as shown in the figure, we obtain a new rectangle A′B′C′D′ with A′B′= $a/2$ and B′C′= $2/a$.

(i) When $\sqrt{2} < a < 2$, then $a/2 < 1 < 2/a < \sqrt{2}$. The side-lengths A′B′ and B′C′ of the new rectangle are less than $\sqrt{2}$.

(ii) When $2 \leq a \leq 2\sqrt{2}$, then $1 \leq a/2 \leq \sqrt{2}$. The side-lengths A′B′ and B′C′ of the new rectangle are less than or equal to $\sqrt{2}$.

(iii) When $a > 2\sqrt{2}$, then $a/2 > \sqrt{2}$. The side-length A′B′ of the new rectangle is still greater than $\sqrt{2}$. In this case, repeat the above Dudeney dissection until all the side-lengths of a new rectangle are less than or equal to $\sqrt{2}$.

Thus we complete the proof of Lemma 2.1. □

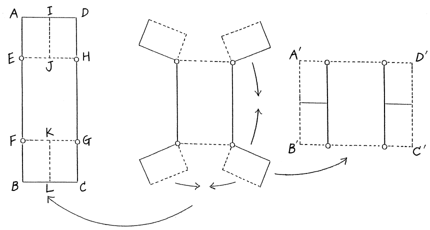


Fig. 5.

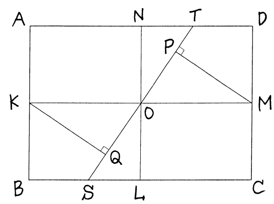


Fig. 6.

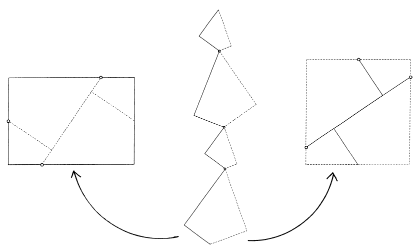


Fig. 7.

Lemma 2.2. *Any rectangle of size $a \times a^{-1}$ can be transformed to a square, where $a, a^{-1} \leq \sqrt{2}$.*

Proof. The following dissection is Taylor's dissection applied in the case of rectangles. (For Taylor's dissection, see [4] p102, [12] p89.) Let ABCD be any rectangle with AB $\leq$ BC. Put $a =$BC, then we have $1 \leq a \leq \sqrt{2}$. We may assume $a \neq 1$. Let O be the center of the rectangle ABCD, and K, L, M, N the midpoints of AB, BC, CD, DA, respectively. Let T be the point on the segment ND with OT $= 1/2$. We can choose the point T since OD $> 1/2$ and ON $< 1/2$ (because of $a > 1$). Let P be the point on the segment OT such that MP is perpendicular to OT. We can choose such a point P. In fact, draw the uper semicircle with the diameter OM, then it is inside or on the rectangle OMDN as $a \leq \sqrt{2}$. We note that when $a = \sqrt{2}$, the point P corresponds to the point T.

Similarly we draw segments OS and KQ (Fig. 6). Then cut the rectangle ABCD into four pieces along the lines TS, MP, KQ, and rotate them as shown in Fig. 7. Then we obtain a rectangle. We see that it is a square since QT + TP = 1. This completes the proof of Lemma 2.2. □

Therefore we complete Step 4 and then the proof of Prop. 2.1. □

3 Hexagons

All convex hexagonal tiles have been classified into three types: type 1, 2, 3. There are no other convex hexagonal tiles. Hexagons of type 1 and 3 and hexagons of type 2 with line symmetry are normal tiles. In this section, we prove the following proposition.

Proposition 3.1. *Hexagons of type* 1 *and* 3 *and hexagons of type* 2 *with line symmetry can be transformed to quadrilaterals.*

Proof. Let ABCDEF be a hexagon. Put a = FA, b = AB, c = BC, d = CD, e = DE, f = EF.

(1) Assume that ABCDEF is a hexagon of type 1, i.e., ∠A + ∠B + ∠C = 360°, $a = d$. First we will show that the hexagon can be transformed to a parallel hexagon. We call a hexagon *parallel* if all pairs of opposite sides are parallel and have the same length. Let K, L, M, N be the midpoints of AB, BC, DE, EF, respectively. Let P and Q be points such that KP is parallel to QM and LP is

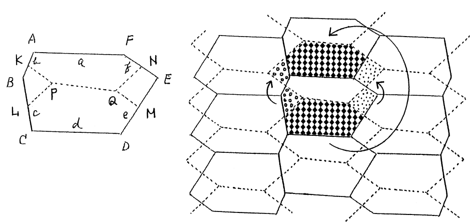
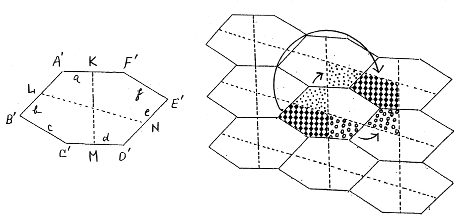


Fig. 8.



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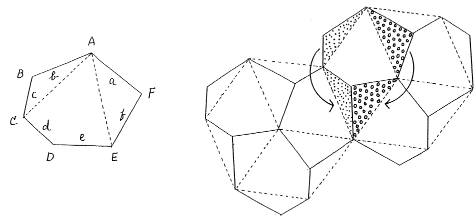


Fig. 10.

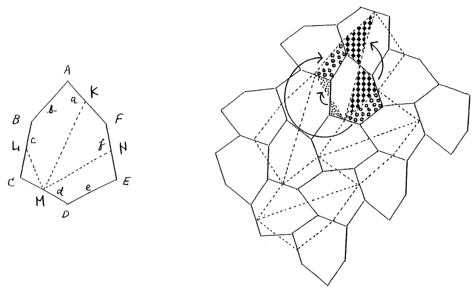


Fig. 11.

parallel to QN. In the tiling of Fig. 8, cut the hexagon into four pieces along the dotted lines and rotate three pieces, then we obtain a parallel hexagon.

Next we will show that any parallel hexagon A′B′C′D′E′F′ can be transformed to a parallelogram. Let L, M, N, K be the midpoints of A′B′, C′D′, D′E′, F′A′, respectively. As shown in Fig. 9, the parallel hexagon can be transformed to a parallelogram.

Thus hexagons of type 1 can be transformed to quadrilaterals.

(2) Assume that ABCDEF is a hexagon of type 3, i.e., ∠A = ∠C = ∠E = 120°, $a = b, c = d, e = f$. The hexagon can be transformed to a quadrilateral as shown in Fig. 10.

(3) Assume that ABCDEF is a hexagon of type 2, i.e., ∠A + ∠B + ∠D = 360°, $a = d, c = e$. When the hexagon has line symmetry, it has either a line of symmetry through opposite sides, or a line of symmetry through opposite points. In the former case, the opposite sides are parallel and have the same lengths, so the hexagon is also of type 1. So we consider only the latter case.

If the hexagon has a line of symmetry AD, then we have $a = b, c = f, d = e$, ∠B = ∠F, ∠C= ∠E. So we have $a = b = c = d = e = f$. Let L, M, N, K be midpoints of BC, CD, EF, FA. The hexagon can be transformed to a quadrilateral as shown in Fig. 11.

If the hexagon has a line of symmetry BE, then we have $b = c, e = f$, ∠A = ∠C, ∠F = ∠D. So we have $c = f$ and ∠A +∠B + ∠F = 360°. It is also of type 1.

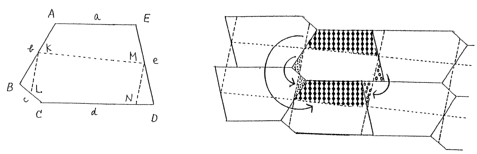


Fig. 12.

If the hexagon has a line of symmetry FC, then it is relatively the same as the hexagon with a line of symmetry AD. So it can be transformed to a quadrilateral.

Thus hexagons of type 2 with line symmetry can be transformed to a quadrilateral.

This completes the proof of Prop. 3.1. □

4 Pentagons

At the present, 14 types of convex pentagons are known to be tiles. Pentagons of types $1, 3, 4, 5, 6$ are normal tiles, and pentagons of the other types are normal tiles if they have line symmetry. In this section, we prove the following propositions.

Proposition 4.1. *A pentagon of type* $1, 3, 4, 5, 6$ *can be transformed to a triangle or a quadrilateral.*

Proof. Let ABCDE be a pentagon. Put $a =$ EA, $b =$ AB, $c =$ BC, $d =$ CD, $e =$ DE.

(1) Assume that ABCDE is a pentagon of type 1, i.e., $\angle$D + $\angle$E = 180°. Cut the pentagon into four pieces along the dotted lines and rotate three pieces as shown in Fig. 12, where K, L, M are the midpoints of AB, BC, DE, respectively, and KL and MN are parallel. Then we obtain a parallelogram.

(2) Assume that ABCDE is a pentagon of type 3, i.e., $\angle$A = $\angle$C = $\angle$D = 120°, $a = b, d = c + e$. The pentagon can be transformed to a rhombus as shown in Fig. 13.

(3) Assume that ABCDE is a pentagon of type 4, i.e., $\angle$A = $\angle$D = 90°, $a = b, d = e$. The pentagon can be transformed to a triangle as shown in Fig. 14, where M is the midpoint of BC.

(4) Assume that ABCDE is a pentagon of type 5, i.e., $\angle$A = 60°, $\angle$C = 120°, $a = b, c = d$. The pentagon can be transformed to an equilateral triangle as shown in Fig. 15, where M is the midpoint of DE.

(5) Assume that ABCDE is a pentagon of type 6, i.e., $\angle$A + $\angle$B + $\angle$D = 360°, $\angle$A = 2$\angle$C, $a = b = e, c = d$. The pentagon can be transformed to a triangle as shown in Fig. 16, where M is the midpoint of AB.

This completes the proof of Prop. 4.1. □

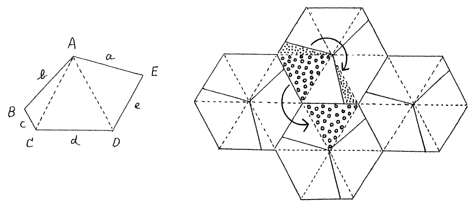
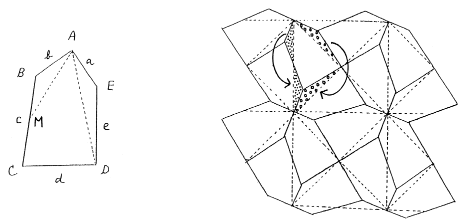


Fig. 13.



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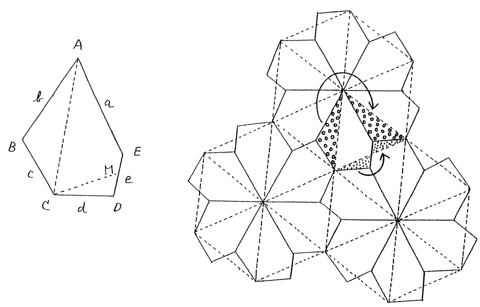

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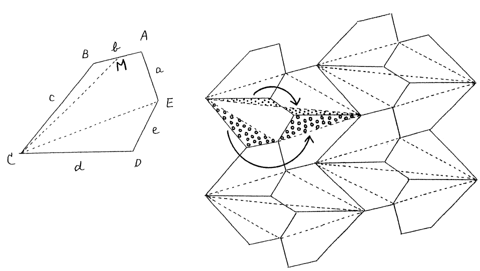

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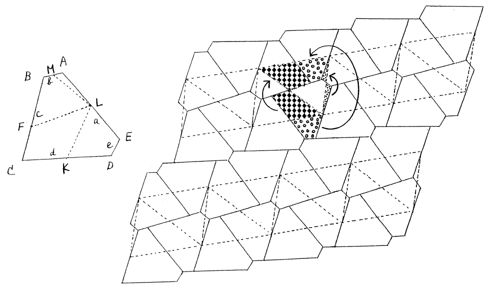

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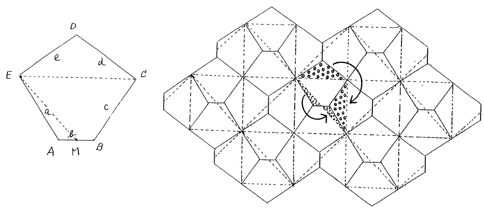

Fig. 18.

Proposition 4.2. *If a pentagon of type* 2, 7, 8, 9, 10, 11, 12, 13, 14 *has line symmetry, it can be transformed to a triangle or a quadrilateral.*

Proof. Let ABCDE be a pentagon. Put $a =$ EA, $b =$ AB, $c =$ BC, $d =$ CD, $e =$ DE.

(1) Assume that ABCDE is a pentagon of type 2, i.e., $\angle$C + $\angle$E = 180°, $a = d$.

(i) If the pentagon has a line of symmetry through point A, we have $\angle$B = $\angle$E, $\angle$C = $\angle$D, $a = b$, $c = e$. Then we have $\angle$A = 180° which is a contradiction. Thus the pentagon doesn't have a line of symmetry through point A.

(ii) If the pentagon has a line of symmetry through point B, we have $\angle$A = $\angle$C, $\angle$D = $\angle$E, $b = c$. Then we have $\angle$B = 180° which is a contradiction. Thus the pentagon doesn't have a line of symmetry through point B.

(iii) If the pentagon has a line of symmetry through point C, we have $\angle$B = $\angle$D, $\angle$A = $\angle$E, $c = d, b = e$. Then we have $\angle$B + $\angle$D = 180° + $\angle$C. If $\angle$C $\leq$ 60°, then we have $\angle$C +$\angle$D $\leq$ 180°, so we have $a < d$ which contradicts to $a = d$. If 60° < $\angle$C < 90°, then it can be transformed to a triangle as shown in Fig. 17, where K, L, M are the midpoints of CD, EA, AB, respectively. If $\angle$C = 90°, then $\angle$E = $\angle$A = 90°. It is also a pentagon of type 1. If 90° < $\angle$C < 180°, then 0° < $\angle$E $\leq$ 90°. Thus we have $a > d$ which contradicts to $a = d$.

(iv) If the pentagon has a line of symmetry through point D, then we have $\angle$A = $\angle$B, $\angle$C = $\angle$E, $a = c, d = e$. It can be transformed to a triangle as shown in Fig. 18.

(v) If the pentagon has a line of symmetry through point E, then we have $\angle$A = $\angle$D, $\angle$B = $\angle$C, $a = e, b = d$. Then we have $\angle$A + $\angle$D = 180° + $\angle$E. If $\angle$E

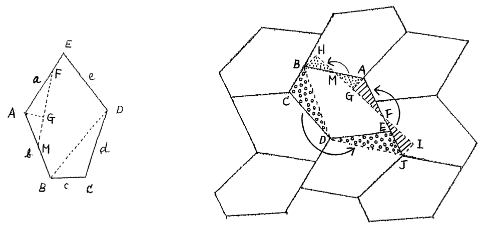

Fig. 19.

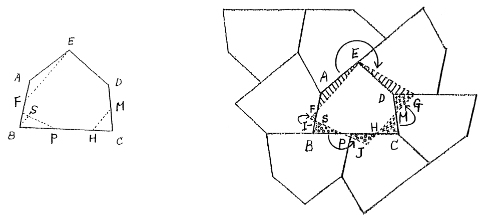

Fig. 20.

$\leq 60°$, $\angle A = \angle D \leq 120°$, so we have $\angle A + \angle E \leq 180°$ and $\angle D + \angle E \leq 180°$. It contradicts to $a = b = d = e$. If $60° < \angle E < 90°$, then the pentagon ABCDE can be transformed to a pentagon HBDJI of type 1 as shown in Fig. 19, where M is the midpoint of AB and HB, AG, IJ are parallel.

If $\angle E = 90°$, then $\angle B = \angle C = 90°$. It is also of type 1. If $90° < \angle E < 180°$, then the pentagon ABCDE can be transformed to a quadrilateral EIJG as shown in Fig. 20, where M is the midpoint of DC; EF, GH, SB are parallel and AF = DG = CH.

The proofs of the following are omitted because they are similar to the above.

(2) Assume that ABCDE is a pentagon of type 7, i.e., $2\angle B + \angle C = 2\angle D + \angle A = 360°$, $a = b = c = d$. If the pentagon has a line of symmetry through point C, then it is also of type 1. There are no pentagons of type 7 with a line of symmetry through point A, B, D, or E.

(3) Assume that ABCDE is a pentagon of type 8, i.e., $2\angle A + \angle B = 2\angle D + \angle C = 360°$, $a = b = c = d$. If the pentagon has a line of symmetry through point E, then it can be transformed to a triangle as shown in Fig. 21. There are no pentagons of type 8 with a line of symmetry through point A, B, C, or D.

(4) A pentagon of type 9 is characterized by $2\angle E + \angle B = 2\angle D + \angle C = 360°$, $a = b = c = d$. There are no pentagons of type 9 with line symmetry.

(5) Assume that ABCDE is a pentagon of type 10, i.e., $\angle A = 90°$, $\angle B + \angle E = 180°$, $2\angle D + \angle E = 2\angle C + \angle B = 360°$, $a = b = c + e$. If the pentagon has a line of symmetry through point A, then it is also of type 1. There are no pentagons of type 10 with a line of symmetry through point B, C, D, or E.

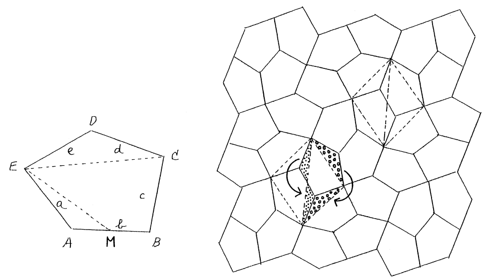

Fig. 21.

(6) A pentagon of type 11 is characterized by ∠A = 90°, ∠C + ∠E = 180°, 2∠B + ∠C = 360°, $d = e = 2a + c$. There are no pentagons of type 11 with line symmetry.

(7) A pentagon of type 12 is characterized by ∠A = 90°, ∠C + ∠E = 180°, 2∠B + ∠C = 360°, $2a = c + e = d$. There are no pentagons of type 12 with line symmetry.

(8) A pentagon of type 13 is characterized by ∠A = ∠C = 90°, 2∠B + ∠D = 2∠E + ∠D = 360°, $c = d$, $2c = e$. There are no pentagons of type 13 with line symmetry.

(9) A pentagon of type 14 is characterized by ∠A = 90°, ∠C + ∠E = 180°, 2∠B + ∠C = 360°, $d = e = 2a$, $a = c$. There are no pentagons of type 14 with line symmetry.

This completes the proof of Prop. 4.2. □

From Prop. 2.1, 3.1, 4.1 and 4.2, we obtain Theorem 1.

Acknowledgments

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Chromatic Numbers of Specified Isohedral Tilings

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Abstract. A *tile* T in the plane is a compact set whose boundary is a simple closed curve. A family of tiles in the plane is called a *tiling* if it covers the plane with no gaps and no (2-dimensional) overlaps. Let V be a doubly covered square, that is, a flat polygon consisting of two congruent square faces joined together along each of their corresponding edges. We prove that for every development (unfolding) T of V, there is a tiling of congruent copies of T whose chromatic number is at most three. By using this fact, we prove that chromatic numbers of specified isohedral tilings with half-turn symmetry are at most three. Then, we notice that self-replicating tiles with fractal boundaries derived from developments of a doubly covered square, which we studied in [3,4], are three-colorable in a sense.

1 Introduction

We consider figures in the Euclidean plane. A *tile* in the plane is a compact set whose boundary is a simple closed curve. A family of tiles, $\mathcal{T} = \{T_\alpha\}_{n\in\Lambda}$, is called a *tiling* if it covers the plane with no gaps and no (2-dimensional) overlaps. Two tiles in the tiling $\mathcal{T}$ are *adjacent* if the topological dimension of their intersection is one. A *doubly covered square* is a flat polygon consisting of two congruent square faces joined together along each of their corresponding edges. A tile T is called a *development of a doubly covered square* V if there is a map f_T from T onto V which is locally isometric and whose image has no (2-dimensional) overlaps.

Section 2 is devoted to preliminaries. We showed in [3,5] that for every development T of a doubly-covered square V there is a tiling $\mathcal{T}$ consisting of congruent copies of T. In Section 3, we prove that such tiling $\mathcal{T}$ has chromatic number $\chi(\mathcal{T}) \leq 3$, that is, there is an assignment of three colors to the tiles in $\mathcal{T}$ in such a way that any two adjacent tiles have different colors (Theorem 1). Notice that any tiling $\mathcal{T}$ of congruent copies of T satisfies $\chi(\mathcal{T}) \leq 4$ by Four Colar theorem, but it is not necessary to satisfy $\chi(\mathcal{T}) \leq 3$ in general (see Fig. 1 for an example; T is a development of V and any tiling containing six congruent copies of T in Fig. 1(b) has chromatic number four).

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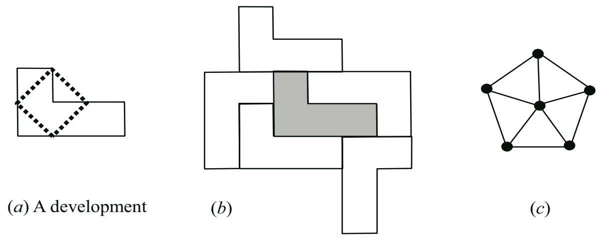


Fig. 1. An example of a tiling whose chromatic number is four

An *isohedral tiling* of the plane is a tiling of congruent copies of a single tile so that the symmetry group of the tiling acts transitively on the tiles. In Section 4, we prove that a specified isohedral tiling $\mathcal{T}$ with half-turn symmetry has chromatic number $\chi(\mathcal{T}) \leq 3$ (Theorem 2). A tile T is called *self-replicating* or *k-replicating* if there are k congruent copies of T so that interiors of any two of them are disjoint and that their union is similar to the original figure T. A simple curve in the plane is called *fractal* if its Hausdorff-dimension is more than one. In Section 5, we show sufficient conditions under which self-replicating tiles with fractal boundaries can be derived from developments of doubly-covered squares. These tiles are three-colorable in a sense (Theorem 4).

2 Preliminaries

Let T be a development of a doubly covered square V with a map f_T from T onto V which is locally isometric and whose image has no (2-dimensional) overlaps. We denote by $f_T(\partial T)$ the image of the boundary ∂T (Fig.2(b)). A point A in ∂T is called a *leaf-point* if its image is a leaf (an end-point) of $f_T(\partial T)$ (for example, A, C, D in Fig. 2(a)). A point P in ∂T is called a *singular point* if there are more than two points (including P) in ∂T whose images by f_T are identical (for example, P_1, P_2 and P_3 in Fig. 2(a)).

Proposition 1. *([6]) Let T be a development of the doubly covered square V.*

(1) *The image $f_T(\partial T)$ is a tree, that is, it is connected and has no cycle* (Fig. 2(c)).

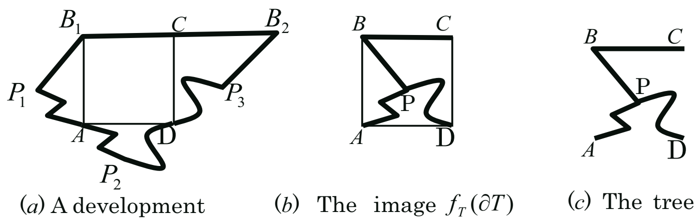


Fig. 2. A development of a doubly covered square

(2) *All vertices of* V *are in* $f_T(\partial T)$ (Fig. 2(*b*)).
(3) *Every leaf-point in* ∂T *is only one pre-image of some vertex in* V *by* f_T (Fig. 2(*a*)(*b*)).

Proposition 2. *Let* T *be a development of* V. *Then there are at least two leaf-points.*

Proof. Since any tree has at least two leaves and leaves of the tree $f_T(\partial T)$ are vertices of V, there are at least two leaf-points in ∂T. □

Proposition 3. *([3,5]) Every development* T *of* V *tiles the plane, that is, there is a family of congruent copies of* T *which is a tiling.*

Carve distinct figures on each of the faces of V, dip the faces in ink and print the figures on a blank sheet as follows: First, print the figures on one face, choose one edge of the face as an axis, turn V over along the axis and print the figure on the other face. Repeat the process continuously. The printed pattern that results is a periodic tiling of the plane. Moreover the same pattern will result regardless of the direction or order in which V is turned over. It is convenient to use orthogonal coordinates (x, y) in the plane and consider a development of V as a figure in the plane, so that faces of V corresponds to unit squares in the plane. A point $P(x, y)$ on the plane is a *lattice point* if both x and y are integers. Hence lattice points are traces of vertices of V.

Definition 1. *([1]) Two points* P *and* Q *in the plane are equivalent, denoted by* $P \equiv Q$, *if either the midpoint of the segment between them is a lattice point or if both the differences between corresponding coordinates are even integers.*

Proposition 4. *([1]) A tile* T *in the plane is a development of* V *if and only if* T *consists of all representatives of the equivalence classes defined by Definition 1 such that no two points in its interior are equivalent.*

3 Colorings of Tilings Derived from Developments

If there is a coloring with k colors for a tiling $\mathcal{T}$, the tiling is said to be k-*colorable*. The minimum of such k is called the *chromatic number* of the tiling $\mathcal{T}$, and is denoted by $\chi(\mathcal{T})$.

In this section, V is a doubly covered square and T is a development of V in the plane with orthogonal xy-coordinates. We denote by F_A the set obtained by a half-circular rotation of F about a point A in the plane, and by $F + (a, b)$ the translate $\{(x + a, y + b) : (x, y) \in F\}$. We use the symbol F_o instead of F_O.

Lemma 1. *([3]) The family*

$$\{T + (2k, 2l) : k, l \in \mathbb{Z}\} \cup \{T_A + (2k, 2l) : k, l \in \mathbb{Z}\}$$

is a tiling for any lattice point A *in the plane.*

Proof. For any point P in the plane, there is a point Q in T which is equivalent to P, which means the differences of corresponding xy-coordinates are even, or the midpoint of P and Q is a lattice point. In the former case, P is in $T+(2k,2l)$ for some $k,l \in \mathbb{Z}$.

In the latter case, the point R symmetric to Q about A is in T_A and the differences of corresponding xy-coordinates of P and R are even, so P is in $T_A+(2k,2l)$ for some $k,l \in \mathbb{Z}$.

Therefore, the family of $\{T+(2k,2l) : k,l \in \mathbb{Z}\} \cup \{T_A+(2k,2l) : k,l \in \mathbb{Z}\}$ covers the plane. Since the interior of T has no two equivalent points, the family is a tiling. □

Let A be a lattice point in the plane with xy-coordinates. Then the tiling $\mathcal{T} = \{T+(2k,2l) : k,l \in \mathbb{Z}\} \cup \{T_A+(2k,2l) : k,l \in \mathbb{Z}\}$ does not depend on the choice of A, so we call $\mathcal{T}$ *the stamp method tiling* by T.

Lemma 2. *Let A be a leaf-point of ∂T. Then the boundary $\partial(T \cup T_A)$ is a simple closed curve and the point A is an interior point of $T \cup T_A$.*

Proof. Let A be a leaf-point of ∂T. Put marks on all points which are leaf-points or singular points in ∂T. Then the point A is marked and there are two marked points P, Q which are next to the point A along the simple closed curve ∂T. Since A is a leaf-point of ∂T, P and Q are symmetric about the point A and they are equivalent (Fig.2(a)). The boundary $\partial(T \cup T_A)$ is the curve obtained by removing $\partial T \cap \partial T_A$ from $\partial T \cup \partial T_A$ and adding P, Q. Hence $\partial(T \cup T_A)$ is a simple closed curve. Then the point A is in the interior of $T \cup T_A$. □

Definition 2. *Let $A(0,0)$ and $B(m,n)$ be distinct leaf-points of ∂T, whose existence is guaranteed by Proposition 2. We denote the set of all lattice points with even coordinates by $\mathcal{W} = \{(2k, 2l : k,l \in \mathbb{Z}\}$.*

Two points $P_i(2k_i, 2l_i) \in \mathcal{W}$ $(i = 1,2)$ are called (m,n)-equivalent, denoted by $P_1 \equiv_{(m,n)} P_2$ if $(2k_1 - 2k_2, 2l_1 - 2l_2) = (tm, tn)$ for some integer t.

Since the relation $\equiv_{(m,n)}$ is an equivalence relation on $\mathcal{W}$, the set $\mathcal{W}$ is classified into the equivalence classes $\{C_\alpha\}_{\alpha \in \Lambda}$, where we denote by C_o the class with the origin $O(0,0)$.

Lemma 3. *Let $A = O(0,0)$ and $B(m,n)$ be leaf-points of ∂T. Let*

$$F_\alpha = \bigcup_{(a,b) \in C_\alpha} \{T \cup T_o + (a,b)\} \quad \text{for } \alpha \in \Lambda.$$

(1) *$F_\alpha = F_o + (a,b)$ for any $\alpha \in \Lambda$ and any $(a,b) \in C_\alpha$.*
(2) *The boundary of F_α $(\alpha \in \Lambda)$ consists of two disjoint simple curves.*
(3) *The union of all F_α $(\alpha \in \Lambda)$ covers the plane.*
(4) *The interiors of F_α and F_β are disjoint for any distinct $\alpha, \neq \beta \in \Lambda$.*

Proof. Let $A = O(0,0)$ and $B(m,n)$ be leaf-points of ∂T.

(1) By $F_o = \{T \cup T_o + (2mt, 2nt) : t \in \mathbb{Z}\}$ we have

$$F_\alpha = \bigcup \{T \cup T_o + (a,b) + (2mt, 2nt) : t \in \mathbb{Z}\} = F_o + (a,b)\ .$$

for any $(a,b) \in C_\alpha$.

(2) Since the origin $(0,0)$ is an interior point of $T \cup T_o$ by Lemma 2, all points in $C_o = \{(2mt, 2nt) : t \in \mathbb{Z}\}$ are in the interior of $F_o = \{T \cup T_o + (2mt, 2nt) : t \in \mathbb{Z}\}$. Since the point $B(m,n)$ is a leaf-point of ∂T, the point B is in the interior of $T \cup T_B = T \cup \{T_o + (2m, 2n)\}$ by Lemma 2. Hence all points in the set $\{(2t+1)m, (2t+1)n : t \in \mathbb{Z}\}$ are in the interior of F_o.

We can draw a simple curve in the interior of F_o which passes through all points $\{(mt, nt) : t \in \mathbb{Z}\}$. Using a process similar to the one used in the proof of Lemma 2, we can prove that for each family

$$\{(T \cup T_o) + (2mt, 2nt) : t = 0, \pm 1, \pm 2, \ldots, \pm n\} \ \ (n \in \mathbb{N}),$$

the boundary of its union is a simple closed curve.

Therefore, the boundary of F_o consists of two disjoint simple curves. Since $F_\alpha = F_o + (a,b)$ for $(a,b) \in C_\alpha$ $(\alpha \in \Lambda)$, the boundary of F_α consists of two disjoint simple curves.

(3) Since

$$\bigcup\{F_\alpha : \alpha \in \Lambda\} = \bigcup\{T \cup T_o + (a,b) : (a,b) \in \mathcal{W}\} = \bigcup\{T \cup T_o + (2k, 2l) : k, l \in \mathbb{Z}\}$$

and the family $\{T \cup T_o + (2k, 2l) : k, l \in \mathbb{Z}\}$ is a tiling, the union of all F_α covers the plane.

(4) Since the family $\{T \cup T_o + (2k, 2l) : k, l \in \mathbb{Z}\}$ is a tiling, the interiors of any two distinct sets F_α, F_β $(\alpha \neq \beta)$ are disjoint. □

Definition 3. *We define an infinite graph $G = G_T$ as follows:*

(1) *The set of vertices in G consists of all elements of the tiling*

$$\{T + (2k, 2l) : k, l \in \mathbb{Z}\} \cup \{T_o + (2k, 2l) : k, l \in \mathbb{Z}\}.$$

(2) *Two vertices in G are joined by an edge if and only if the intersection of corresponding congruent copies of T has the topological dimension one.*

Proposition 5. *We can draw the graph $G = G_T$ satisfying the following conditions:*

(1) *G is a geometric planar graph in the plane.*
(2) *G is symmetric about the origin.*
(3) *The translate $G + (2k, 2l)$ is identical to G for any $k, l \in \mathbb{Z}$.*

Proof. (1) holds by the definition of G. (2) holds because the tiling

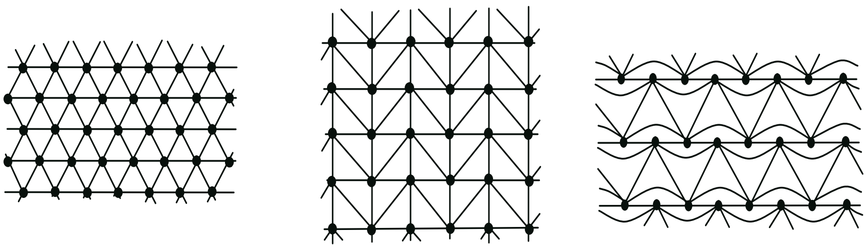
$$\{T + (2k, 2l) : k, l \in \mathbb{Z}\} \cup \{T_o + (2k, 2l) : k, l \in \mathbb{Z}\}$$

is symmetric with respect to the origin. (3) holds by Lemma 3(1). □

Let $A = O(0,0)$ and $B(m,n)$ be distinct leaf-points of ∂T. Denote by G_o the induced subgraph of G_T by the set of vertices corresponding to

$$\{T + (2km, 2ln) : k, l \in \mathbb{Z}\} \cup \{T_o + (2km, 2ln) : k, l \in \mathbb{Z}\}.$$

Fig. 3. A graph which consists of an infinite path and additional edges



(*a*) H_1 (*b*) H_2 (*c*) H_3

Fig. 4. The graphs H_1, H_2 and H_3

Proposition 6. *Let $A = O(0,0)$ and $B(m,n)$ be distinct leaf-points of ∂T. The induced subgraph G_o is either an infinite path, or the graph consisting of an infinite path and additional edges which connect any two vertices with distance two in the infinite path* (Fig. 3).

Proof. Since the interior of F_o is connected, G_o contains an infinite path. If G_o has an edge joining two points with distance greater than two in the infinite path, then some edges are crossing each other in their interiors by condition (3) in Proposition 5, which is a contradiction since G is a geometric planar graph. So, G_o is either an infinite path or a graph isomorphic to the graph in Fig. 3. □

Proposition 7. *Suppose G_o is an infinite path. Then the graph G is isomorphic to either a subgraph of the graph H_1 or a subgraph of the graph H_2 shown in* Fig. 4(a)(b).

Proof. Since G is a geometric planar graph and satisfies conditions in Proposition 5, we get the graphs H_1 and H_2 in Fig. 4(a)(b) by drawing as many edges as possible among vertices in infinitely many translates of G_o. H_1 and H_2 are maximal graphs, so the graph G is isomorphic to one of their subgraphs. □

Proposition 8. *Suppose G_o is a graph isomorphic to the graph in* Fig. 3. Then the graph G is isomorphic to a subgraph of the graph H_3 showed in Fig. 4(c) (see Fig. 5for an example).

Proof. Since G is a geometric planar graph and satisfies conditions in Proposition 5, we get the graph in Fig. 4(c) by drawing as many edges as possible. This graph H_3 is a maximal graph.

Theorem 1. *For every development T of a doubly covered square V there is an isohedral tiling $\mathcal{F}$ consisting of congruent copies of T which is three-colorable, that is, $\chi(\mathcal{F}) \leq 3$.*

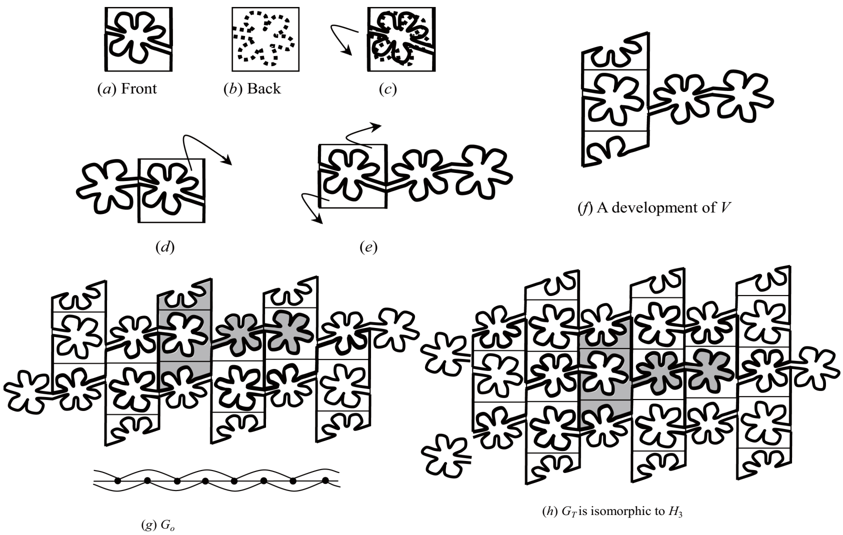


Fig. 5. An example for the graph H_3

Proof. It is easy to see that graphs H_1, H_2 and H_3 in Fig. 4 are three-colorable. Since G_T is isomorphic to a subgraph of these graphs, G_T is also three-colorable. Therefore the tiling

$$\mathcal{F} = \{T + (2k, 2l) : k, l \in \mathbb{Z}\} \bigcup \{T_o + (2k, 2l) : k, l \in \mathbb{Z}\}$$

is three-colorable.

Corollary 1. *Let T be a development of the doubly covered square V in the plane with xy-coordinates. Let $A = O(0,0)$ and $B(m,n)$ be leaf-points of the boundary ∂T. Then m and n are relatively prime.*

Proof. If there is a common divisor $s > 1$ of m and n, then $C_o = \{(2mt, 2nt : t \in \mathbb{Z}\}$ and $C_o + (2m/s, 2n/s)$ are distinct classes with respect to the equivalence relation $\equiv_{(m,n)}$. Since there is a simple curve Γ joining all points $\{mt, nt) : t \in \mathbb{Z}\}$ in the interior of $F_o = \{T \cup T_o + (2mt, 2nt) : t \in \mathbb{Z}\}$ by Lemma 3(2), the translate $\Gamma + (2m/s, 2n/s)$ is a simple curve joining all points in the class $C_{(2m/s,2n/s)}$ by Lemma 3(1)(2), which contradicts Lemma 3(4). □

4 Chromatic Numbers of Specified Isohedral Tilings

We denote by $\Pi(e_1, e_2)$ the plane with oblique coordinates determined by two independent unit vectors e_1, e_2 whose lengths are not necessary equal. We can generalize Theorem 1 as follows:

Theorem 2. *Let $\mathcal{T}$ be an isohedral tiling of a single tile T with two translations by independent vectors $2e_1, 2e_2$ and half-turn symmetry about lattice points in $\Pi(e_1, e_2)$. Then $\mathcal{F}$ is three-colorable, that is, $\chi(\mathcal{T}) \leq 3$.*

Proof. Let $\mathcal{T}$ be an isohedral tiling of a single tile T with two translations by independent vectors $2e_1, 2e_2$ and half-turn symmetry about lattice points in $\Pi(e_1, e_2)$. Let ψ be the affine transformation defined by the inverse of the matirix (e_1, e_2). Then the image of $\mathcal{T}$ is a tiling in the plane and it is represented by

$$\{\psi(T) + (2k, 2l) : k, l \in \mathbb{Z}\} \cup \{\psi(T)_o + (2k, 2l) : k, l \in \mathbb{Z}\}$$

in the plane with orthogonal xy-coordinates. By Proposition 4, the image of T by ψ is a development of a doubly covered square. Hence the image of $\mathcal{T}$ by ψ is three-colorable by Theorem 1, so $\chi(\mathcal{T}) \leq 3$. □

A doubly covered square may be considered as a tetrahedron consisting of four congruent isosceles right triangular faces.

Corollary 2. *For every development T of a tetrahedron consisting of four congruent acute triangular faces, there is an isohedral tiling $\mathcal{T}$ consisting of congruent copies of T satisfying $\chi(\mathcal{F}) \leq 3$.*

Proof. It is proved in [4,5] that every development T of a tetrahedron consisting of four congruent acute triangular faces is described in the plane with obilique coordinates and it has an isohedral tiling consisting of congruent copies of T satisfying all conditions in Theorem 2. Hence $\chi(\mathcal{F}) \leq 3$ by Theorem 2. □

Conjecture 1. The chromatic numbers of isohedral tilings of the plane are at most three.

5 Self-replicating Tiles and Fractals

We mentioned self-replicating tiles and fractals derived from a development of a doubly covered square in [3,4]. We continue the discussion of this subject in this section. Let $A(a, b)$ be a lattice point in the plane $\Pi(e_1, e_2)$ where e_1, e_2 are relatively orthogonal vectors. We denote coordinates by $(a, b)_{(e_1, e_2)}$ instead of (a, b) when necessary.

Theorem 3. *([4]) Let T be a development of a doubly covered square V in the plane with xy-coordinates so that the origin is a leaf-point of ∂T. Then the union $T \cup T_o$ is a development of a doubly covered square whose faces are congruent to the square $ABCD$, where $A(0, -1)$, $B(-1, 0)$, $C(0, 1)$, $D(1, 0)$* (Fig. 6).

Proof. Let u_1, u_2 be basic vectors in the plane with xy-coordinates. Rotate u_1, u_2 by $\pi/4$, multiply their lengths by $\sqrt{2}$ and denote the resulting vectors by e_1, e_2 respectively. Let $\Pi(e_1, e_2)$ be the plane with the origin $O(0, 0)_{(e_1, e_2)} = O(0, 1)$. Then all lattice points in $\Pi(e_1, e_2)$ are also lattice points in the plane with xy-coordinates. Since the origin is a leaf-point of ∂T, the origin $O(0, 0)$ is an interior point of $T \cup T_o$ by Lemma 2.

Let $S = T \cup T_o$. The symbol S_o stands for $S_{O_{(e_1, e_2)}}$. Since

$$\{S \cup S_o + (2k, 2l)_{(e_1, e_2)} : k, l \in \mathbb{Z}\} = \{T \cup T_o + (2k, 2l) : k, l \in \mathbb{Z}\}$$

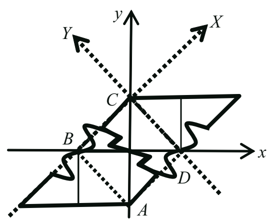


Fig. 6. $T \cup T_o$

holds and the family $\{T \cup T_o + (2k, 2l) : k, l \in \mathbb{Z}\}$ is a tiling, the family

$$\{S \cup S_o + (2k, 2l)_{(e_1, e_2)} : k, l \in \mathbb{Z}\}$$

is also a tiling. By Proposition 4, $S = T \cup T_o$ is a development of a doubly covered square whose faces are congruent to the square $ABCD$, where $A(0, -1), B(1, 0), C(0, 1), D(-1, 0)$. $\square$

We call a tile T in the plane a *k-omino* if T consists of k congruent squares and if the intersection of any two of those squares is a point, one edge or empty.

Definition 4. *Let T be a development of a doubly covered square V in the plane with xy-coordinates. Suppose T be a k-omino. We define Condition 1 as follows:*

Condition 1. *Four points* $(0, 0), (0, 1), (1, 1)$ *and* $(1, 0)$ *are leaf-points of ∂T and they are mid-points of some edges of squares in the k-omino T (Fig. 7(a)).*

Definition 5. *([4]) Let T and S be developments of V in the plane with xy-coordinates. Suppose T and S are a k_1-omino and a k_2-omino respectively, and they satisfy Condition 1. Reduce V and T by the ratio $1/\sqrt{k_2}$ and denote the resulting figures by V^* and T^* respectively. Then T^* is a development of V^*. Assign the labels a, b, c, d to the four points in ∂T^* corresponding to leaf-points of ∂T in clockwise order along the boundary curve. Let F be a square in k_2-omino S with the origin. Then F is a development of V^*. We define a set $S(T)$ as follows.*

Step (1) *Draw tilted squares by connecting midpoints of four edges in each square in the k_2-omino of S (Fig. 7(a)). Assign the labels a, b, c, d to vertices of each tilted square in clockwise order, where the origin is assigned by a. Then the assignment is unique because it is similar to part of the stamp method tiling by the development F of V^* (Fig. 7(a)).*

Step (2) *Replace each square in the k_2-omino S by a congruent copy of T^* such that four labels a, b, c, d are matched to the same labels on each square of S. We denote the resulting figure by $S(T)$ (Fig. 7(b) where $S = T$).*

Lemma 4. *Let T, S be developments of V in the plane with xy-coordinates. Suppose T, S are k_1-omino and k_2-omino respectively and they satisfy Condition 1. Then $S(T)$ is another development of V which is a $k_1 k_2$-omino and satisfies Condition* 1.

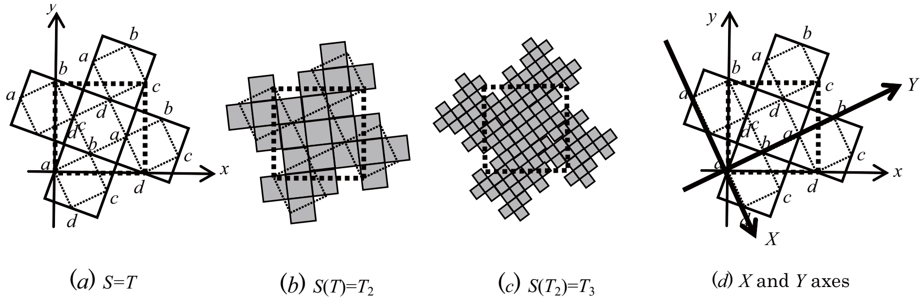


Fig. 7. The process to get $S(T)$

Proof. Let e_1, e_2 be two orthogonal vectors parallel to edges in the tilted squares in S with lengths equal to the lengths of the edges of the tilted squares (for example, unit vectors of XY-coordinates in Fig. 7(d)). Then if two points P and Q are equivalent in the plane with xy-coordinates, then P and Q are also equivalent in $\Pi(e_1, e_2)$. By the definition of $S(T)$, $S(T)$ consists of a subfamily of the stamp method tiling by T^*. Since the family $\{S \cup S_o + (2k, 2l) : k, l \in \mathbb{Z}\}$ is a tiling, forming the elements of the tiling

$$\{T^* + (2k, 2l)_{(e_1,e_2)} : k, l \in \mathbb{Z}\} \cup \{(T^*)_o + (2k, 2l)_{(e_1,e_2)} : k, l \in \mathbb{Z}\}$$

into groups by figures congruent to $S(T)$, we obtain the family $\{S(T) \cup (S(T))_o + (2k, 2l) : k, l \in \mathbb{Z}\}$, which is a tiling. By Proposition 4, $S(T)$ is a development of V. Moreover, $S(T)$ is a k_1k_2-omino and it satisfies Condition 1 by the definition of $S(T)$. □

Definition 6. *Let T be a development of a doubly-covered square V. Suppose T is a k-omino and it satisfies Condition*(1)*. Define a sequence $\{T_n\}_{n\in\mathbb{N}}$ as follows:*

(1) $T_1 = T$.
(2) *For* $n \geq 1, T_{n+1} = T(T_n)$.

A tile F is called *self-replicating* or *k-replicating* if there are k congruent copies of F such that interiors of any two of them are disjoint and that their union is similar to the original figure F. The k-replicating tile F is called *three-colorable* if there is an assignment of three colors to these k congruent copies of F so that any two adjacent copies have different colors.

Theorem 4. *([4]) Let T be a development of the doubly covered square V. Suppose T is a k-omino satisfying Condition 1. Then the sequence $\{T_n\}_{n\in\mathbb{N}}$ converges to a tile, denoted by T_∞, in the Hausdorff-metric for compact sets in the plane and T_∞ is a development of V. Moreover, T_∞ is k-replicating and three-colorable.*

Proof. Let T be a development of the doubly covered square V which is a k-omino satisfying Condition 1. By Lemma 4 the sequence $\{T_n\}_{n\in\mathbb{N}}$ is well-defined.

For $n \geq 1$ the Hausdorff-metric distance between T_n and T_{n+1} is less than $(1/\sqrt{k})^n c$ where c is a constant. So $\{T_n\}_{n \in \mathbb{Z}}$ is a Cauchy-sequence and it converges to a compact set T_∞. The sequence of their boundaries $\{\partial T_n\}_{n \in \mathbb{N}}$ also converges to a simple closed curve. Hence T_∞ is a tile and a development of V. For T_∞, there are k congruent copies $\mathcal{T} = \{F_i\}_{i=1}^k$ of T_∞ so that their union is similar to T_∞ and it is a subfamily of the stamp method tiling by T_∞. Hence $\mathcal{T}$ is three-colorable. □

A simple curve Γ in the plane is called a *fractal* if its Hausdorff-dimension is greater than one. Let T be the development of V in Fig. 6(a). In each step of the construction for the sequence $\{T_n\}_{n \in \mathbb{N}}$, enlarging the figure by the ratio $\sqrt{5}$ results in the boundary ∂T_{n+1} whose the total length is three times that of ∂T_n. Hence the Hausdorff-dimension of the boundary ∂T_∞ is $\log_{\sqrt{5}} 3$. The boundary of T_∞ is fractal.

Remark. Let a k-omino T be a development of a doubly covered square V in the plane with xy-coordinates. Define *Condition 2* as follows:

Condition 2. Four points $(0,0), (0,1), (1,1)$ and $(1,0)$ are leaf-points of ∂T and they are vertices of some squares in the k-omino, and T is symmetry by half-turn about the point $(1/2, 1/2)$.

Then all statements in this section hold for T satisfying Condition 2 instead of Condition 1. Proofs are almost the same as above, so we omit them.

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Transforming Graphs with the Same Degree Sequence

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Abstract. Let G and H be two graphs with the same vertex set V. It is well known that a graph G can be transformed into a graph H by a sequence of 2-switches if and only if every vertex of V has the same degree in both G and H. We study the problem of finding the minimum number of 2-switches for transforming G into H.

1 Introduction

A graphic sequence is the sequence of numbers that are vertex degrees of a graph. Any degree sequence whose sum is even can be realized by a multigraph having loops [3]. In this paper we consider simple graphs (graphs without loops and multiple edges). Erdős and Gallai [2] found a characterization of graphic sequences.

Theorem 1 (Erdős and Gallai [2]). *A sequence of positive numbers $d_1 \geq d_2 \geq \cdots \geq d_n$ is graphic if and only if $d_1 + d_2 + \cdots + d_n$ is even and the inequalities*

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$$

hold for every k.

Havel [4] and Hakimi [3] found another characterization of graphic sequences.

Theorem 2 (Havel [4], Hakimi [3]). *For $n \geq 1$, a sequence S of n nonnegative integers is graphic if and only if S' is graphic, where S' is the sequence of size $n-1$ obtained from S by deleting its largest element d and subtracting 1 from its d next largest elements. The only 1-element graphic sequence is $d_1 = 0$.*

The following transformation of a graph preserves the degree sequence.

Definition 1. *A* 2-switch *is the replacement of a pair of edges (a, b) and (c, d) in a simple graph by the edges (a, c) and (b, d), given that (a, c) and (b, d) did not appear in the graph originally.*

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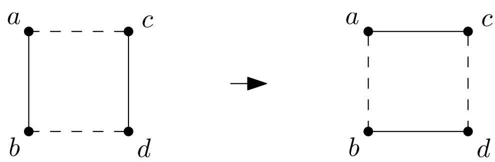

Fig. 1. 2-switch

It is clear that the degrees of the vertices remain unchanged when a 2-switch is applied to a graph. The following theorem shows that two graphs with the same graphic sequence can be transformed one to the other using 2-switches.

Theorem 3. *If G and H are two simple graphs with vertex set V, then $d_G(v) = d_H(v)$ for every $v \in V$ if and only if there is a sequence of 2-switches that transforms G into H.*

Note that the graphs G and H have the same set of vertices. It can also be viewed as two labelled graphs with the same set of labels.

Probably the earliest reference of Theorem 3 is Berge [1] stating that the 2-switch graph on the set of graphs with fixed degree sequence is connected. It also can be found in West [5, page 45]. Some feel that this is "implicit" in the work of Havel and Hakimi [6]. In the proof of Theorem 3 both G and H are reduced to a *canonical graph* with vertex set V. Each reduction uses at most $m - 1$ transformations where m is the number of edges in G (see more details in Section 2). Thus, the number of 2-switches transforming G to H is at most $2m - 2$. Finding the minimum number of 2-switches transforming given G and H is of particular interest of this paper.

Let $G = (V, E_G)$ and $H = (V, E_H)$ be two simple graphs such that $d_G(v) = d_H(v)$ for every $v \in V$. We consider a new graph $F(G, H)$ or just F defined as (V, E_F) where $E_F = E_G \cup E_H - E_G \cap E_H$. We color the edges of F with two colors as follows. An edge e is colored in (i) red if $e \in E_G - E_H$, and (ii) blue if $e \in E_H - E_G$. The number of red edges and the number of blue edges in F are equal. We denote it by $r(G, H)$.

The set of edges of F can be decomposed into *red-blue alternating walks* (note that the vertices of a walk may not be pairwise distinct). Let $p(G, H)$ be the maximum number of walks in a decomposition of a F into red-blue alternating walks. Our main result is the following theorem.

Theorem 4. *Let $G = (V, E_G)$ and $H = (V, E_H)$ be two simple graphs such that $d_G(v) = d_H(v)$ for every $v \in V$. The smallest number of 2-switches for transforming G into H is equal to $r(G, H) - p(G, H)$.*

2 Preliminaries

Let $G = (V, E_G)$ and $H = (V, E_H)$ be two simple graphs such that $d_G(v) = d_H(v)$ for every $v \in V$. We consider a new graph $F(G, H)$ or just F defined as (V, E_F)

where $E_F = E_G \cup E_H - E_G \cap E_H$. We color the edges of F with two colors as follows. An edge e is colored in

- red if $e \in E_G - E_H$,
- blue if $e \in E_H - E_G$.

We call the coloring of F *even* since, for any vertex v, equal number of red and blue edges are incident to v. It implies that the number of red edges and the number of blue edges in F are equal. We denote it by $r(G, H)$. We also consider the complete graph K_n with the set of vertices V. Clearly, F is the subgraph of K_n. We color the edges of K_n: the common edges of F and K_n are colored in red and blue as before and the other edges are colored in black and white as follows. An edge e is colored in

- black if $e \in E_H \cap E_G$,
- white if $e \notin E_G \cup E_H$.

Since the coloring of F is even, the set of edges of F can be decomposed into red-blue alternating walks. A walk is not a simple path since some vertices may be repeated. We assume that the edges in a walk are all distinct.

The proof Theorem 3 uses a *canonical graph* C with vertex set V defined inductively as follows. Let $v_1, v_2, \ldots, v_n$ be the vertices of V sorted such that their degrees form a non-increasing sequence $d(v_1) \geq d(v_2) \geq \ldots d(v_n)$. Consider the sequence

$$d(v_2) - 1, d(v_3) - 1, \ldots, d(v_{k+1}) - 1, d(v_{k+2}), \ldots, d(v_n)$$

where $k = d(v_1)$. By Theorem 2 it is a graphic sequence. Let C' be a *canonical graph* corresponding to it. Then C is obtained from C' by adding a new vertex v_1 and edges $(v_1, v_2), (v_1, v_3), \ldots, (v_1, v_{k+1})$.

The main argument in the proof of Theorem 3 is as follows. Consider two sets $S = \{v_2, v_3, \ldots, v_{k+1}\}$ and $N(v_1)$, the set of neighbors of v_1. If $S = N(v_1)$, then the theorem holds by induction hypothesis. If $S \neq N(v_1)$, then any edge connecting v_1 and a vertex $z \notin S$ can be flipped to an edge connecting v_1 and a vertex $x \in S - N(v_1)$ using a 2-switch. By repeating this step we spend at most k transformations for the induction step. With every 2-switch we insert a new edge of the canonical graph C. In the last 2-switch we add two edges of C. So, the total number of 2-switches is at most $m - 1$.

This gives an upper bound of $2m - 2$ for transformation the graph G to H. Theorem 4 implies that, if $m > 0$, then at most $(m - 1)$ 2-switches suffice since $r(G, H) \leq m$ and $p(G, H) \geq 1$.

3 Main Result

Our main result characterizes the 2-switch distance between two graphs. We denote the distance by $\psi(G, H) = r(G, H) - p(G, H)$. First, we prove lower and upper bounds for 2-switches.

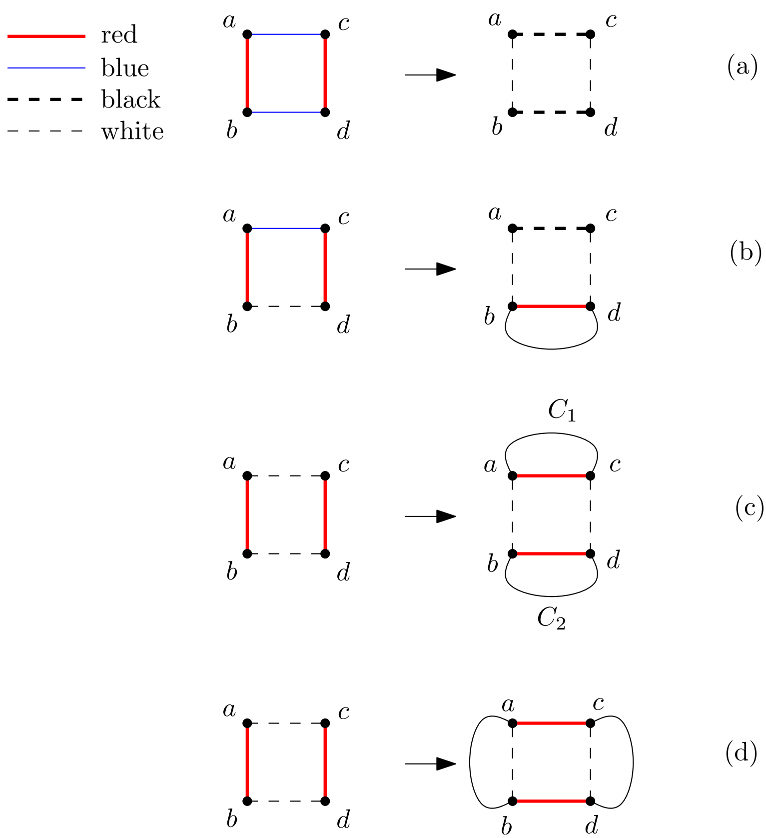

Fig. 2. Case 1 of the lower bound. Red and blue edges are shown as solid lines, bold and thin respectively. Black and white edges are shown as dashed lines, bold and thin respectively. The edges (a, b) and (c, d) are red and the edges (a, c) and (b, d) are (a) both blue, and (b) blue and white, and (c),(d) both white.

Lemma 1 (Lower Bound). *Let G' be the graph obtained by a 2-switch from G. Then*

$$\psi(G', H) \geq \psi(G, H) - 1. \tag{1}$$

Proof. Consider any 2-switch of edges ab and cd by the edges ac and bd. The edges ab and cd are colored in red or black each. The edges ac and bd are colored in blue or white each. Let $\mathcal{C}'$ be a partition of $F(G', H)$ into $p(G', H)$ alternating walks.

Case 1. Both edges (a, b) and (c, d) are red
Suppose that the colors of (a, c) and (b, d) are blue, see Fig. 2 (a). Then $r(G, H) = r(G', H) + 2$. The walks of $\mathcal{C}'$ and $abdca$ form a partition of the set of edges of $F(G, H)$ into alternating walks. Therefore $p(G, H) \geq p(G', H) + 1$. The bound (1) follows.

Suppose that the colors of (a, c) and (b, d) are blue and white respectively, see Fig. 2 (b). Then $r(G, H) = r(G', H) + 1$. One of the walks of $\mathcal{C}'$ contains (b, d). We replace (b, d) with $bacd$ to obtain the set of alternating walks for $F(G, H)$. Thus $p(G, H) \geq p(G', H)$. The bound (1) follows.

Suppose that the colors of (a, c) and (b, d) are white, see Fig. 2 (c). Then $r(G, H) = r(G', H)$. If the red edges (a, c) and (b, d) belong to different walks C_1 and C_2 of $\mathcal{C}'$, then $C_1 - \{(a, c)\}$ and $C_2 - \{(b, d)\}$ can be combined in one alternating walk with (a, b) and (c, d) in $F(G, H)$, see Fig. 2 (c). Thus $p(G, H) \geq p(G', H) - 1$. The bound (1) follows.

If the red edges (a, c) and (b, d) are connected in one alternating walk C, then $p(G, H) \geq p(G', H) + 1$, see Fig. 2 (d). The bound (1) follows.

Case 2. The edge (a, b) is red and the edge (c, d) is black
Suppose that the colors of (a, c) and (b, d) are blue, see Fig. 3 (a). Then $r(G, H) = r(G', H) + 1$. By replacing the edge (c, d) in an alternating walk from $\mathcal{C}'$ by $cabd$ we bound $p(G, H) \geq p(G', H)$. The bound (1) follows.

Suppose that the colors of (a, c) and (b, d) are blue and white respectively, see Fig. 3 (b). Then $r(G, H) = r(G', H)$. If the edges (b, d) and (c, d) are in two alternating walks C_1 and C_2 of $\mathcal{C}'$, then they can be combined in one alternating

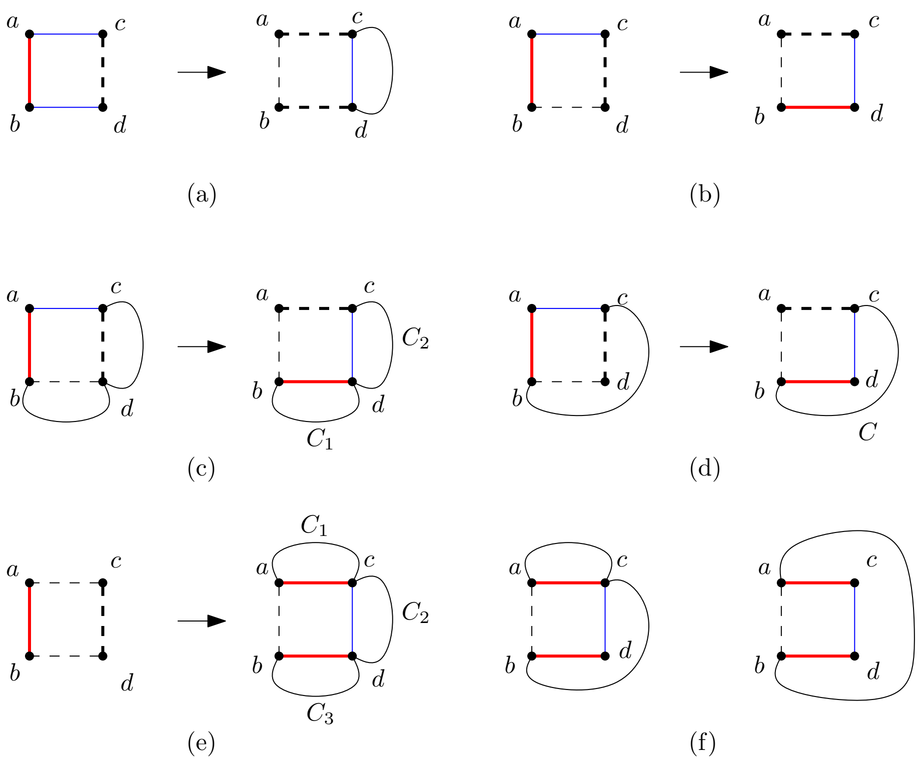

Fig. 3. Case 2 of the lower bound

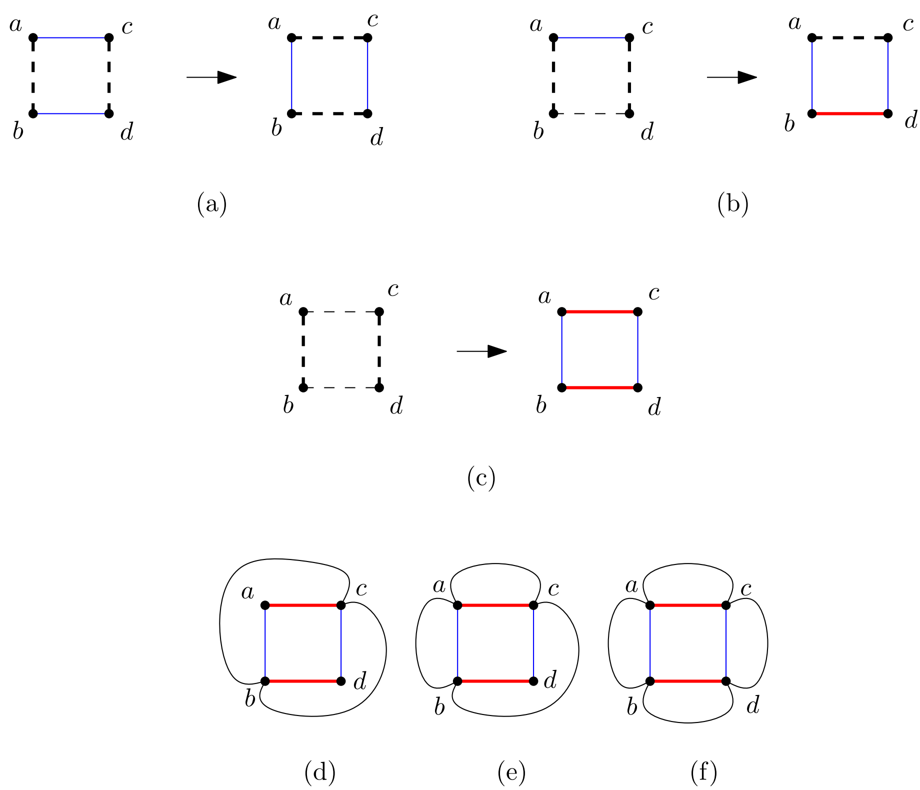

Fig. 4. Case 3 of the lower bound

walk $C_1 \cup C_2 \cup \{(a,b),(a,c)\} - \{(b,d),(c,d)\}$, see Fig. 3 (c). If the edges (b,d) and (c,d) are in a same alternating walk, then they can be replaced by (a,b) and (a,c). In both cases $p(G,H) \geq p(G',H) - 1$. The bound (1) follows.

Suppose that the colors of (a,c) and (b,d) are white, see Fig. 3 (e). Then $r(G,H) = r(G',H) - 1$. To bound $p(G,H)$ we check the walks of $\mathcal{C}'$ containing the edges (a,c), (c,d) and (b,d). If there are three walks, then they can be combined in one walk for G, see Fig. 3 (e). The number of walks can be two or one, see Fig. 3 (f). In all cases $p(G,H) \geq p(G',H) - 2$ and the bound (1) follows.

Case 3. The edges (a,b) and (c,d) are black

Suppose that the colors of (a,c) and (b,d) are blue, see Fig. 4 (a). Then $r(G,H) = r(G',H)$ and $p(G,H) \geq p(G',H) - 1$ by an argument similar to Case 1 where (a,c) and (b,d) are white. The bound (1) follows.

Suppose that the colors of (a,c) and (b,d) are blue and white respectively, see Fig. 4 (b). Then $r(G,H) = r(G',H) - 1$ and $p(G,H) \geq p(G',H) - 2$ by an argument similar to Case 2 where (a,c) and (b,d) are white. The bound (1) follows.

Suppose that the colors of (a,c) and (b,d) are white, see Fig. 4 (c). Then $r(G,H) = r(G',H) - 2$. If $abcda$ is a walk of $\mathcal{C}'$, then $p(G,H) \geq p(G',H) - 1$. If

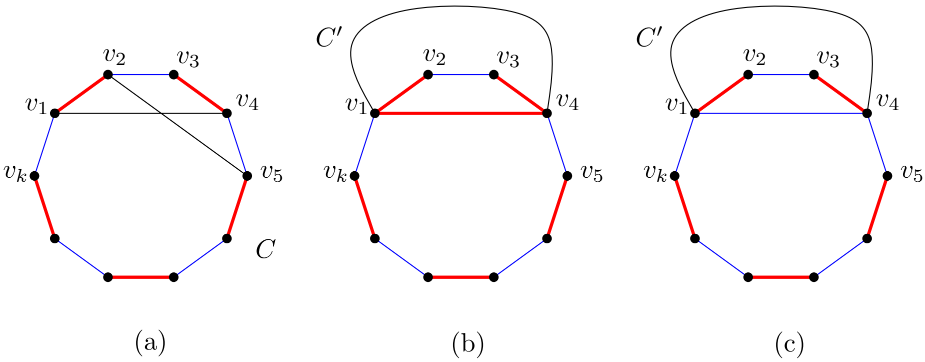

Fig. 5. Lemma 2. (a) $v_1 \neq v_4$ or $v_2 \neq v_5$. (b) (v_1, v_4) is red. (c) (v_1, v_4) is blue.

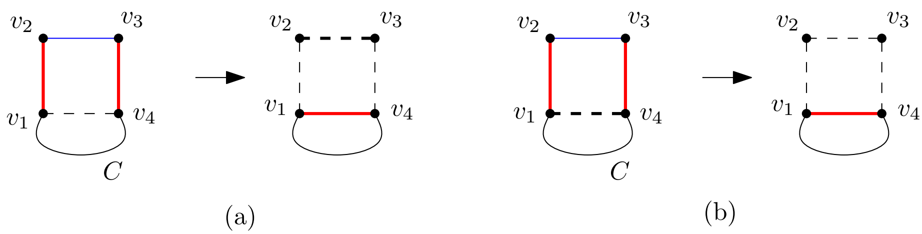

Fig. 6. Lemma 2. (a) (v_1, v_4) is white. (b) (v_1, v_4) is black.

the edges of *abcda* participate in two walks of $\mathcal{C}'$, then $p(G, H) \geq p(G', H) - 1$, see Fig. 4 (d). If the edges of *abcda* participate in three cycles of $\mathcal{C}'$, then $p(G, H) \geq p(G', H) - 2$, see Fig. 4 (e). If the edges of *abcda* participate in four cycles of $\mathcal{C}'$, then $p(G, H) \geq p(G', H) - 3$, see Fig. 4 (f). In all cases $p(G, H) \geq p(G', H) - 3$ and the bound (1) follows. □

Lemma 2 (Upper Bound). *Let $G = (V, E_G)$ and $H = (V, E_H)$ be two simple graphs such that $d_G(v) = d_H(v)$ for every $v \in V$. There exists a 2-switch in G or H that decreases the distance $\psi(G, H)$ by exactly one.*

Proof. The graph $F(G, H)$ can be partitioned into $p(G, H)$ alternating walks. From all partitions of $F(G, H)$ into $p(G, H)$ alternating walks, we select a partition $\mathcal{C}$ such that its shortest walk $C = v_1 v_2 \ldots v_k$ has minimum length.

Suppose that $|C| = 4$. We apply a 2-switch in G replacing edges (v_1, v_2) and (v_3, v_4) with (v_1, v_2) and (v_3, v_4). Let G' be the new graph. Then $r(G', H) = r(G, H) - 2$ and $p(G', H) = p(G, H) - 1$. Thus, $\psi(G', H) = \psi(G, H) - 1$.

Now suppose that $|C| \geq 6$. Then $v_1 \neq v_4$ or $v_2 \neq v_5$ since the edges $(v_1, v_2$ and (v_4, v_5) have different colors, see Fig. 5 (a). Without loss of generality we assume that $v_1 \neq v_4$. We consider 4 cases depending on the color of (v_1, v_4).

Suppose that (v_1, v_4) is red. Let C' be a walk in $\mathcal{C}$ containing (v_1, v_4). The edges of $C \cup C'$ can be partitioned into two walks so that one walk is $v_1 v_4 v_5 \ldots v_k$, see Fig. 5 (b). This walk is shorter than C'. Contradiction.

If (v_1, v_4) is blue, then again $C \cup C'$ can be partitioned into two walks so that one walk is 4-cycle $v_1v_2v_3v_4$, see Fig. 5 (c).

If (v_1, v_4) is white, then apply a 2-switch in G replacing edges (v_1, v_2) and (v_3, v_4) with (v_1, v_4) and (v_2, v_3). If (v_1, v_4) is black, then apply a 2-switch in G replacing edges (v_1, v_4) and (v_2, v_3) with (v_1, v_2) and (v_3, v_4). In both cases this reduces $v_1v_2v_3v_4$ in C to v_1v_4. Let G' be the new graph. Then $r(G', H) = r(G, H) - 1$ and $p(G', H) = p(G, H)$, see Fig. 6. The lemma follows. □

Theorem 4 simply follows from the upper and lower bounds.

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The Forest Number of (n, m)-Graphs

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Abstract. Let $G = (V, E)$ be a graph and $F \subseteq V$. Then F is called an induced forest of G if $G[F]$ is acyclic. The *forest number*, denoted by $f(G)$, of G is defined by

$$f(G) := \max\{|F| : F \text{ is an induced forest of } G\}.$$

We proved that if G runs over the set of all graphs of order n and size m, then the values $f(G)$ completely cover a line segment $[x, y]$ of positive integers. Let $\mathcal{G}(n, m)$ be the set of all graphs of order n and size m and $\mathcal{CG}(n, m)$ be the subset of $\mathcal{G}(n, m)$ consisting of all connected graphs. We are able to obtain the extremal results for the forest number in the class $\mathcal{G}(n, m)$ and $\mathcal{CG}(n, m)$.

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1 Interpolation Theorems

We consider only finite simple graphs. Let $G = (V, E)$ denote a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order and the size of G are denoted by $\nu(G) = |V|$ and $\varepsilon(G) = |E|$, respectively. The *degree* of a vertex v of a graph G is denoted by $d_G(v)$. The maximum degree and the minimum degree of a graph G is usually denoted by $\Delta(G)$ and $\delta(G)$, respectively. If $S \subseteq V(G)$, the graph $G[S]$ is the subgraph induced by S in G and we use the notation $\varepsilon(S)$ for $\varepsilon(G[S])$. For a graph G and $X \subseteq E(G)$, $G - X$ denotes the graph $(V, E - X)$. If $X = \{e\}$, we write $G - e$ for $G - \{e\}$. For a graph G and $X \subseteq V(G)$, $G - X$ is the graph obtained from G by removing all vertices in X and all edges incident with vertices in X. For a graph G and $X \subseteq E(\overline{G})$, $G + X$ denotes the graph $(V, E \cup X)$. If $X = \{e\}$, we simply write $G + e$ for $G + \{e\}$. Two graphs G and H are disjoint if $V(G) \cap V(H) = \emptyset$. For any two disjoint graphs G and H, $G \cup H$ is defined by $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$ and the union of p copies of G is denoted by pG. We can extend this definition to a finite union of pairwise disjoint graphs.

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Let $\mathcal{G}$ be the class of all simple graphs. A function $\pi : \mathcal{G} \to \mathbb{Z}$ is called a *graph parameter* if $\pi(G) = \pi(H)$ for all isomorphic graphs G and H. A graph parameter π is called an *interpolation graph parameter over* $\mathcal{J} \subseteq \mathcal{G}$ if there exist integers x and y such that

$$\{\pi(G) : G \in \mathcal{J}\} := [x, y] := \{k \in \mathbb{Z} : x \leq k \leq y\}.$$

If π is an interpolation graph parameter over $\mathcal{J}$, then $\{\pi(G) : G \in \mathcal{J}\}$ is uniquely determined by $\min(\pi, \mathcal{J}) = \min\{\pi(G) : G \in \mathcal{J}\}$ and $\max(\pi, \mathcal{J}) = \max\{\pi(G) : G \in \mathcal{J}\}$.

For positive integers n and m ($0 \leq m \leq \binom{n}{2}$), let $\mathcal{G}(n, m)$ be the class of all distinct spanning subgraphs of K_n and size m. Let $\mathcal{CG}(n, m)$ be the subset of $\mathcal{G}(n, m)$ consisting of all connected graphs. We simply write $\min(\pi; n, m)$ and $\max(\pi; n, m)$ for $\min(\pi, \mathcal{G}(n, m))$ and $\max(\pi, \mathcal{G}(n, m))$, respectively. Also we write $\mathrm{Min}(\pi; n, m)$ and $\mathrm{Max}(\pi; n, m)$ for $\min(\pi, \mathcal{CG}(n, m))$ and $\max(\pi, \mathcal{CG}(n, m))$, respectively.

Interpolation theorems for graph parameters may be divided into two parts, the first part deals with the question that given a graph parameter π and a subset $\mathcal{J}$ of $\mathcal{G}$, does π interpolate over $\mathcal{J}$? If π interpolates over $\mathcal{J}$, then $\{\pi(G) : G \in \mathcal{J}\}$ is uniquely determined by $\min(\pi, \mathcal{J})$ and $\max(\pi, \mathcal{J})$. The second part of the interpolation theorems for graph parameters is to find $\min(\pi, \mathcal{J})$ and $\max(\pi, \mathcal{J})$ for the corresponding interpolation graph parameters and this part is the extremal problems in graph theory.

Several graph parameters over the class of all graphs with the same degree sequence were proved to interpolate and were presented the two parts of the interpolation theorems by the second author in [3].

Let G be a graph and $F \subseteq V(G)$. F is called an *induced forest* of G if $G[F]$ is an acyclic graph. The maximum cardinality of an induced forest of a graph G is called the *forest number* of G and is denoted by $f(G)$. That is

$$f(G) := \max\{|F| : F \text{ is an induced forest of } G\}.$$

The second author proved in [2] the following theorems.

Theorem 1. *The forest number f is an interpolation graph parameter over $\mathcal{G}(n, m)$.* □

Theorem 2. *The forest number f is an interpolation graph parameter over $\mathcal{CG}(n, m)$.* □

2 Extremal Results

Note that f is an interpolation graph parameter over $\mathcal{G}(n, m)$ and $\mathcal{CG}(n, m)$. We now answer the second part of interpolation theorem.

Let G be a graph and X, Y be disjoint nonempty subsets of $V(G)$. Denote by $\varepsilon(X, Y)$ the number of edges in G connecting vertices in X to vertices in Y.

Let $G \in \mathcal{G}(n,m)$ and F be a maximum induced forest of G. Let $|F| = a$. Therefore $G - F$ has order $n - a$. An upper bound of m can be obtained by the following inequality.

$$m = \varepsilon(G - F) + \varepsilon(G - F, F) + \varepsilon(F) \leq \binom{n-a}{2} + a(n-a) + (a-1).$$

Let $a = n - i$ for any $i \in \{1, 2, \ldots, n-2\}$. We get $m \leq (i+1)n - \frac{i^2+3i+2}{2}$.

For an integer $i = 1, 2, \ldots, n-2$, let $\mathbf{M}_n(n-i) = (i+1)n - \frac{i^2+3i+2}{2}$. It is clear that $\mathbf{M}_n(n-i)$ is an integer. We can show that $\max(f; n, m) = n - i$ if and only if $\mathbf{M}_n(n-i+1) < m \leq \mathbf{M}_n(n-i)$ by constructing a graph $G \in \mathcal{G}(n,m)$ with $\mathbf{M}_n(n-i+1) < m \leq \mathbf{M}_n(n-i)$ and $f(G) = n - i$ as follows.

Let G be a graph with $V(G) = X \cup Y$ where $X = \{v_1, v_2, \ldots, v_{n-i}\}$, $Y = \{u_1, u_2, \ldots, u_i\}$ and $E(G) = \{v_j v_{j+1} : 1 \leq j \leq n-i-1\} \cup \{uv : u, v \in Y\} \cup \{u_j v_k : 1 \leq j \leq i-1, 1 \leq k \leq n-i\} \cup \{u_i v_k : 1 \leq k \leq m - \mathbf{M}_n(n-i+1)+1\}$. It is easy to check that $f(G) = n - i$ and $G \in \mathcal{CG}(n,m)$. Thus we have the following theorem.

Theorem 3. *Let n and m be integers satisfying $0 \leq m \leq \binom{n}{2}$. Then $\max(f; n, m) = n - i$ if and only if $\mathbf{M}_n(n-i+1) < m \leq \mathbf{M}_n(n-i)$, and $\mathrm{Max}(f; n, m) = n - i$ if and only if $m \geq n - 1$ and $\mathbf{M}_n(n-i+1) < m \leq \mathbf{M}_n(n-i)$.* □

The problem of finding $\min(f; n, m)$ is difficult and finding $\mathrm{Min}(f; n, m)$ is even more difficult. In order to obtain the values of $\min(f; n, m)$, we first find the minimum number of edges of a graph order n having the forest number a. Let $\mathcal{G}(n; f = a)$ be the set of graphs of order n having the forest number a. It is clear that $\mathcal{G}(n; f = a) \neq \emptyset$ if and only if $2 \leq a \leq n$. For integers n and a, let

$$\mathbf{m}_n(a) = \min\{\varepsilon(G) : G \in \mathcal{G}(n; f = a)\}.$$

Further, $\mathbf{m}_n(n) = 0$, $\mathbf{m}_n(n-1) = 3$ and $\mathbf{m}_n(2) = \binom{n}{2}$. It is easy to see that for a graph G of order $n \geq 2$, $f(G) = 2$ if and only if $G \cong K_n$. We now find $\mathbf{m}_n(a)$ for $2 < a < n$. The following lemma was proved in [1] which provide a characterization of graphs with forest number 3.

Lemma 1. *Let G be a graph of order $n \geq 3$ and $G \not\cong K_n$. Then $f(G) = 3$ if and only if $\overline{G}$ is a union of stars.* □

By Lemma 1 we have $\mathbf{m}_n(3) = \binom{n}{2} - n + 1$, for all $n \geq 4$. The second author proved in [1] the following lemma.

Lemma 2. *If G is a graph of order n with maximum degree $\Delta(G) = \Delta$, then $f(G) \geq \frac{2n}{\Delta+1}$.* □

Lemma 3. *If G is a graph of order n with $\Delta(G) = \Delta$ and $f(G) = 2q + 1$ for some integer q, then $n \leq (\Delta+1)q + 1$.*

Proof. We will proceed by induction on q. Suppose that $n = (\Delta + 1)q + t$ for some integer $t \geq 2$. Since $f(G) = 2q + 1$, it follows by Lemma 2 that $2 \leq t \leq \Delta$. Suppose that $q = 1$, that is $n = \Delta + 1 + t$. Thus $\delta(\overline{G}) = t \geq 2$ and hence $\overline{G}$ is not a union of stars. By Lemma 1, $f(G) \geq 4$. Therefore the result holds for $q = 1$. Suppose that there exists a graph G of order $n = (\Delta + 1)q + t$ with $f(G) = 2q + 1$ for smallest possible integer $q \geq 2$. Let v be a vertex of G of degree Δ and let H be the graph obtained from G by deleting v and its neighbors. Thus H has order $(\Delta + 1)(q - 1) + t$ and by minimality of q, there exists a maximum induced forest F in H of order at least $2(q - 1) + 2 = 2q$. Since $F_v = F \cup \{v\}$ is an induced forest of G and $|F_v| = |F| + 1$, it follows that $|F| = 2q$ and F_v is a maximum induced forest in G of order $2q + 1$. Note that if T is a maximum induced forest in a graph G and $v \in V(G - T)$, then there exists a nontrivial connected component C of T such that v is adjacent to at least 2 vertices in C. Since F_v contains at most $2q$ vertices of degree at least 1 and $G - F_v$ has order $(\Delta - 1)q + t - 1$, there are at least $2(\Delta - 1)q + 2(t - 1)$ edges from $G - F_v$ to the nontrivial components of F_v. Since F_v has at most $2q$ vertices of degree at least 1 and $\frac{2(\Delta-1)q+2(t-1)}{2q} > \Delta - 1$, it follows that there exists a vertex in F_v of degree at least $\Delta + 1$. Thus we get a contradiction. □

By Lemma 2, we have a lower bound on the maximum degree of a given graph in terms of its order and its forest number. In other words, if G is a graph of order n, then $\Delta(G) \geq \lceil \frac{2n}{f(G)} \rceil - 1$. In particular, if $f(G) = 2q$ for some integer q, then $\Delta(G) \geq \lceil \frac{n}{q} \rceil - 1$. By Lemma 3 the lower bound of $\Delta(G)$ can be improved if $f(G)$ is odd. That is, if $f(G) = 2q + 1$ for some integer q, then $n \leq (\Delta(G) + 1)q + 1$ which is equivalent to $\Delta(G) \geq \lceil \frac{n-1}{q} \rceil - 1$. We have the following corollary.

Corollary 1. *Let G be a graph of order n and q be a positive integer. If $f(G) = 2q$, then $\Delta(G) \geq \lceil \frac{n}{q} \rceil - 1$, and if $f(G) = 2q + 1$, then $\Delta(G) \geq \lceil \frac{n-1}{q} \rceil - 1$.* □

Let $\mathcal{G}^*(n; f = a) = \{G \in \mathcal{G}(n; f = a) : G$ is a union of $\lceil \frac{a}{2} \rceil$ cliques$\}$. It is clear that $\mathcal{G}^*(n; f = a) \subset \mathcal{G}(n; f = a)$. We have the following theorem.

Theorem 4. *Let G be a graph of order n with $f(G) = a$. Then there exists a graph $H \in \mathcal{G}^*(n; f = a)$ such that $\varepsilon(H) \leq \varepsilon(G)$.*

Proof. We will proceed by induction on n. The result holds for $n = 1$. Suppose that $n \geq 2$. If there exists a vertex v of G such that $f(G - v) = f(G) - 1 = a - 1$, then, by induction, there exists a graph $H' \in \mathcal{G}^*(n - 1; f = a - 1)$ such that $\varepsilon(H') \leq \varepsilon(G - v)$. Let $H = H' \cup \{v\}$. Thus $H \in \mathcal{G}^*(n; f = a)$ and $\varepsilon(H) = \varepsilon(H') \leq \varepsilon(G - v) \leq \varepsilon(G)$. Suppose that for every vertex v of G, $f(G - v) = f(G) = a$. Let v be a vertex of G of degree $\Delta(G)$. Thus, by induction, there exists a graph $H' \in \mathcal{G}^*(n - 1; f = a)$ such that $\varepsilon(H') \leq \varepsilon(G - v)$. We consider two cases.

Case 1. Suppose that $f = 2q$ for some integer q. Thus H' is a union of at least q cliques. Therefore there exists a component C' of H' such that C' is a clique of order at most $\lfloor \frac{n-1}{q} \rfloor$. By Corollary 1, $d_G(v) \geq \lceil \frac{n}{q} \rceil - 1$. Note that $\lceil \frac{n}{q} \rceil - 1 = \lfloor \frac{n-1}{q} \rfloor$. Let C be a clique obtained from C' by adding v and at most

$\lfloor \frac{n-1}{q} \rfloor$ edges from v to C'. Let $H = (H' - V(C')) \cup C$. Then $H \in \mathcal{G}^*(n; f = a)$ and $\varepsilon(H) \leq \varepsilon(H') + \lfloor \frac{n-1}{q} \rfloor \leq \varepsilon(G - v) + d_G(v) \leq \varepsilon(G)$, as required.

Case 2. Suppose that $n \geq 4$ and $f = 2q + 1$ for some integer q. Thus H' is a union of at least $q+1$ cliques one of which is an isolated vertex. Therefore there exists a component C' of H' such that C' is a nontrivial clique of order at most $\lfloor \frac{n-2}{q} \rfloor$. By Corollary 1, $d_G(v) \geq \lceil \frac{n-1}{q} \rceil - 1$. Note that $\lceil \frac{n-1}{q} \rceil - 1 = \lfloor \frac{n-2}{q} \rfloor$. Let C be a clique obtained from C' by adding v and at most $\lfloor \frac{n-2}{q} \rfloor$ edges form v to C'. Let $H = (H' - V(C')) \cup C$. Then $H \in \mathcal{G}^*(n; f = a)$ and $\varepsilon(H) \leq \varepsilon(H') + \lfloor \frac{n-2}{q} \rfloor \leq \varepsilon(G - v) + d_G(v) \leq \varepsilon(G)$, as required. □

By Theorem 4, we know the structure of graphs of order n with prescribed forest numbers. In general, for a graph $G \in \mathcal{G}(n; f = a)$, there may be many such graphs $H \in \mathcal{G}^*(n; f = a)$. We now seek for such a graph H with minimum number of edges.

It is trivial that for any integer $n \geq 6$, $\mathbf{m}_n(4) = \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2}$ and for $n \geq 7$, $\mathbf{m}_n(5) = \binom{\lceil \frac{n-1}{2} \rceil}{2} + \binom{\lfloor \frac{n-1}{2} \rfloor}{2}$.

Theorem 5 is the famous result of Turán [4] which is viewed as the origin of extremal graph theory.

The *Turán graph* $T_{n,r}$ is the complete r-partite graph of order n whose partite sets differ in size by at most 1. It is clear that the complement of the Turán graph $T_{n,r}$ is a union of r cliques.

Theorem 5. *Among the graphs of order n containing no complete subgraph of order $r+1$, $T_{n,r}$ has the maximum number of edges.* □

By Theorem 5, we know that among the graphs of order n which is a union of r cliques, the complement of the Turán graph $T_{n,r}$ has the minimum number of edges.

Turán theorem can be applied for finding $\mathbf{m}_n(a)$ for $4 \leq a \leq n-1$. We first establish certain facts and notation.

1. Let $G = p_1K_1 \cup p_3K_3 \cup p_4K_4 \cup \ldots \cup p_kK_k$. Then the order of G is $p_1 + 3p_3 + 4p_4 + \ldots + kp_k$ and $f(G) = p_1 + 2(p_3 + p_4 + \ldots + p_k)$. Suppose that $p_1 \geq 2$, $p_k \geq 1$ and $k \geq 4$. Then by replacing $2K_1 \cup K_k$ by $K_3 \cup K_{k-1}$ we obtain a graph H with $\varepsilon(H) \leq \varepsilon(G)$. Furthermore, $\varepsilon(H) = \varepsilon(G)$ if and only if $k = 4$.
2. $\mathbf{m}_n(n-1) = 3$ if $n \geq 4$. Let $G \in \mathcal{G}(n; f = n-1)$. Then $\varepsilon(G) = 3$ if and only if $n \geq 4$ and $G = (n-3)K_1 \cup K_3$.
3. Let a be an integer satisfying $\frac{2n}{3} \leq a \leq n-1$. If (p, q) is the solution of $p + 3q = n$ and $p + 2q = a$, then $G = pK_1 \cup qK_3$ satisfies $f(G) = a$.
4. Let a be an integer satisfying $\frac{2n}{3} \leq a \leq n-2$ and $G \in \mathcal{G}(n; f = a)$ such that $\varepsilon(G) = \mathbf{m}_n(a)$. Then by Theorem 4, we can choose $G = p_1K_1 \cup p_3K_3 \cup p_4K_4 \cup \ldots \cup p_kK_k \in \mathcal{G}^*(n; f = a)$ and $k \leq 4$. If $k = 4$, then $p_1 \geq 2$. Thus by the first fact above, there exists a graph $H = pK_1 \cup qK_3$ such that $p+3q = n$, $p + 2q = a$ and $\varepsilon(H) = \varepsilon(G) = \mathbf{m}_n(a)$.
5. Let a be an integer with $a < \frac{2n}{3}$ and $G \in \mathcal{G}(n; f = a)$. Then, by Lemma 2, $\Delta(G) \geq 3$. Thus if $G = p_1K_1 \cup p_3K_3 \cup p_4K_4 \cup \ldots \cup p_kK_k$, $f(G) = a < \frac{2n}{3}$ and $\varepsilon(G) = \mathbf{m}_n(a)$, then $p_1 \leq 1$ and $k \geq 4$.

6. If $n = rq + t,\ 0 \le t < r$, then $T_{n,r}$ consists of t partite sets of cardinality $\lceil \frac{n}{r} \rceil$ and $r - t$ partite sets of cardinality $\lfloor \frac{n}{r} \rfloor$.
7. $\varepsilon(T_{n,r}) = \binom{n-a}{2} + (r-1)\binom{a+1}{2}$, where $a = \lfloor \frac{n}{r} \rfloor$.
8. Let $t(n,r) = \varepsilon(T_{n,r})$. Then for a fixed n, by using elementary arithmetic, we get $t(n, r-1) < t(n,r)$ for all r, $2 \le r \le n$. In fact $t(n,r) - t(n,r-1) \ge \binom{a+1}{2}$, where $a = \lfloor \frac{n}{r} \rfloor$.

Let $\bar{t}(n,r) = \binom{n}{2} - \varepsilon(T_{n,r})$. Summarizing the results, we have the following theorems.

Theorem 6. *Let n and a be integers satisfying $2 \le a \le n-1$. Then*

1. $\mathbf{m}_n(n) = 0$,
2. $\mathbf{m}_n(n-1) = 3$ *if* $n \ge 3$ *and* $G = (n-3)K_1 \cup K_3$ *is the only graph of order* n *satisfying* $f(G) = n-1$ *and* $\varepsilon(G) = 3$,
3. $\mathbf{m}_n(n-i) = 3i$ *if* $1 \le i \le \lceil \frac{n}{3} \rceil$,
4. *Suppose* $4 \le a < \frac{2n}{3}$. *Then* $\mathbf{m}_n(a) = \bar{t}(n,q)$ *if* $a = 2q$, *and* $\mathbf{m}_n(a) = \bar{t}(n-1,q)$ *if* $a = 2q+1$, *for some integer* q, *and*
5. $\mathbf{m}_n(3) = \binom{n-1}{2}$ *if* $n \ge 3$, *and* $\mathbf{m}_n(2) = \binom{n}{2}$ *if* $n \ge 2$. □

Theorem 7. *Let n and m be integers satisfying $0 \le m \le \binom{n}{2}$. Then*

1. $\min(f; n, m) = \max(f; n, m) = n$ *if and only if* $m \in \{0, 1, 2\}$,
2. $\min(f; n, m) = \max(f; n, m) = 2$ *if and only if* $m = \binom{n}{2}$, *and*
3. *for* $3 \le a \le n-1$, $\min(f; n, m) = a$ *if and only if* $\mathbf{m}_n(a) \le m < \mathbf{m}_n(a-1)$. □

We now find the minimum number of edges of a connected graph order n having the forest number a. Let $\mathcal{CG}(n; f = a)$ be the set of connected graphs of order n having the forest number a. For integers n and a, let

$$\mathbf{cm}_n(a) = \min\{\varepsilon(G) : G \in \mathcal{CG}(n; f = a)\}.$$

Further, $\mathbf{cm}_n(n) = n-1,\ \mathbf{cm}_n(2) = \binom{n}{2}$. We now find $\mathbf{cm}_n(a)$ for $2 < a < n$.

Let $\mathcal{CG}^*(n; f = a) = \{G \in \mathcal{CG}(n; f = a) : G$ is obtained from $\lceil \frac{a}{2} \rceil$ disjoint cliques and $\lceil \frac{a}{2} \rceil - 1$ edges$\}$. We have the following theorem.

Theorem 8. *Let G be a connected graph of order n with $f(G) = a$. Then there exists a graph $H \in \mathcal{CG}^*(n; f = a)$ such that $\varepsilon(H) \le \varepsilon(G)$.*

Proof. We will proceed by induction on n. The result holds for $n = 1$. Let $n \ge 2$. Suppose that there exists a vertex v of G such that $f(G - v) = f(G) - 1 = a - 1$. Let $G_1, G_2, \ldots, G_k$ be the k components of $G - v$ having $n_1, n_2, \ldots, n_k$ vertices, respectively. Then $d_G(v) \ge k$. Thus for each $i \in \{1, 2, \ldots, k\}$, there exists a graph $H_i \in \mathcal{CG}^*(n_i; f = f(G_i))$ such that $\varepsilon(H_i) \le \varepsilon(G_i)$. Let $w_i \in V(H_i)$ and $H = (H_1 \cup H_2 \cup \ldots \cup H_k \cup \{v\}) + X$ where $X = \{vw_i : i = 1, 2, \ldots, k\}$. Then $H \in \mathcal{CG}^*(n; f = a)$ and $\varepsilon(H) = k + \sum_{i=1}^{k} \varepsilon(H_i) \le d_G(v) + \sum_{i=1}^{k} \varepsilon(G_i) \le \varepsilon(G)$.

Suppose that for every vertex v of G, $f(G - v) = f(G) = a$. Let v be a vertex of G of degree $\Delta(G)$. We consider two cases.

Case 1. $G-v$ is connected. Then there exists a graph $H' \in \mathcal{CG}^*(n-1; f=a)$ such that $\varepsilon(H') \le \varepsilon(G-v)$. If $f=2q$ for some integer q, then H' is obtained from q disjoint cliques and $q-1$ edges. Therefore there exists a clique C' of H' of order at most $\lfloor \frac{n-1}{q} \rfloor$. By Corollary 1, $d_G(v) \ge \lceil \frac{n}{q} \rceil - 1$. Note that $\lceil \frac{n}{q} \rceil - 1 = \lfloor \frac{n-1}{q} \rfloor$. Let C be a clique obtained from C' by adding v and at most $\lfloor \frac{n-1}{q} \rfloor$ edges from v to C'. Let H be a graph with $V(H)=V(G)$ and $E(H)=(E(H')-E(C')) \cup E(C)$. Then $H \in \mathcal{CG}^*(n; f=a)$ and $\varepsilon(H) \le \varepsilon(G)$, as required.

If $n \ge 4$ and $f=2q+1$ for some integer q, then H' is obtained from $q+1$ cliques and q edges where one of cliques is an isolated vertex. Therefore there exists a non-trivial clique C' of H' of order at most $\lfloor \frac{n-2}{q} \rfloor$. By Corollary 1, $d_G(v) \ge \lceil \frac{n-1}{q} \rceil - 1$. Note that $\lceil \frac{n-1}{q} \rceil - 1 = \lfloor \frac{n-2}{q} \rfloor$. Let C be a clique obtained from C' by adding v and at most $\lfloor \frac{n-2}{q} \rfloor$ edges from v to C'. Let H be a graph with $V(H)=V(G)$ and $E(H)=(E(H')-E(C')) \cup E(C)$. Then $H \in \mathcal{CG}^*(n; f=a)$ and $\varepsilon(H) \le \varepsilon(G)$, as required.

Case 2. $G-v$ is disconnected. Let $G_1, G_2, \ldots, G_k$ be the k components of $G-v$ having $n_1, n_2, \ldots, n_k$ vertices, respectively. For each $i \in \{1,2,\ldots,k\}$, we put $G'_i = G[V(G_i) \cup \{v\}]$ and $G' = G[V(G-G_i) \cup \{v\}]$. Suppose that there exists $i; 1 \le i \le k$ such that $f(G'_i) = f(G_i)+1$. Then $f(G'_j) = f(G_j)$ for all $j \ne i$. Then G' is a connected graph of order $n' = n - n_i$ such that $f(G') = f(\bigcup_{i\ne j=1}^k G_j) = \sum_{i \ne j=1}^k f(G_j)$. Thus for $w \in V(G_i), H^* = (G_i \cup G') + \{vw\}$ is a connected graph of order n such that $\varepsilon(H^*) = \varepsilon(G_i) + \varepsilon(G') + 1 \le \varepsilon(G)$ and $f(H^*) = f(G_i) + f(G') = \sum_{j=1}^k f(G_j) = f(G-v) = a$. Then there exist $H_i \in \mathcal{CG}^*(n_i; f=f(G_i))$ and $H' \in \mathcal{CG}^*(n'; f=f(G'))$ such that $\varepsilon(H_i) \le \varepsilon(G_i)$ and $\varepsilon(H') \le \varepsilon(G')$. Let $w \in V(H_i)$ and $H = (H_i \cup H') + \{vw\}$. Then $\varepsilon(H) = \varepsilon(H_i) + \varepsilon(H') + 1 \le \varepsilon(G_i) + \varepsilon(G') + 1 \le \varepsilon(G)$ and $f(H) = f(H_i) + f(H') = f(G_i) + f(G') = f(G) = a$. Thus $H \in \mathcal{CG}^*(n; f=a)$ and $\varepsilon(H) \le \varepsilon(G)$, as required.

Suppose that $f(G'_i) = f(G_i)$ for all $i \in \{1,2,\ldots,k\}$. If G_i is a complete graph for all i, then G'_i is complete. Since G_i is complete, it follows that for any $e \in E(G_i)$, $f(G_i - e) = f(G_i)+1$. Then for $w \in V(G_i), H^* = ((G_i - e) \cup G') + \{vw\}$ is a connected graph of order n such that $\varepsilon(H^*) = \varepsilon(G_i - e) + \varepsilon(G') + 1 \le \varepsilon(G)$ and $f(H^*) = f(G_i - e) + f(G') = f(G_i) + f(G') + 1 = \sum_{j=1}^k f(G_j) = f(G)$. By the same argument in the first part of Case 2, there exists $H \in \mathcal{CG}^*(n; f=f(G))$ such that $\varepsilon(H) \le \varepsilon(G-e) < \varepsilon(G)$, as required.

Assume that there exists $j; 1 \le j \le k$ such that G_j is not complete. Then for any $i \in \{1,2,\ldots,k\}$ there exists $H_i \in \mathcal{CG}^*(n_i; f=f(G_i))$ such that $\varepsilon(H_i) \le \varepsilon(G_i)$. For $w_i \in V(H_i)$, we put $H^* = \bigcup_{i=1}^k H_i + X$ where $X = \{vw_i : i = 1,2,\ldots,k\}$. Then H^* is a connected graph of order n with $\varepsilon(H^*) \le \varepsilon(G)$ and $f(H^*) = \sum_{i=1}^k f(H_i) = \sum_{i=1}^k f(G_i) = \sum_{i=1}^k f(G'_i) = f(G) = a$.

We put $H'_i = H^*[V(H_i) \cup \{v\}]$. Then there exists $i; 1 \le i \le k$ such that $f(H'_i) = f(H_i)+1$. By the same argument in the first part of Case 2, there exists $H \in \mathcal{CG}^*(n; f=a)$ such that $\varepsilon(H) \le \varepsilon(H^*) \le \varepsilon(G)$, as required. $\square$

By Theorem 8 we know that for a graph $G \in \mathcal{CG}(n; f = a)$, there may be many such graphs $H \in \mathcal{CG}^*(n; f = a)$. By applying Turán Theorem, we have the following theorems.

Theorem 9. *Let n and a be integers satisfying $2 \le a \le n-1$. Then*

1. $\mathbf{cm}_n(n) = n - 1$,
2. *Suppose* $4 \le a \le n-1$. *Then* $\mathbf{cm}_n(a) = \bar{t}(n,q) + q - 1$ *if* $a = 2q$, *and* $\mathbf{cm}_n(a) = \bar{t}(n-1,q) + q$ *if* $a = 2q+1$, *for some integer* q, *and*
3. $\mathbf{cm}_n(3) = \binom{n-1}{2} + 1$ *if* $n \ge 3$, *and* $\mathbf{cm}_n(2) = \binom{n}{2}$ *if* $n \ge 2$. □

Theorem 10. *Let n and m be integers satisfying $n - 1 \le m \le \binom{n}{2}$. Then*

1. $\mathrm{Min}(f; n, m) = \mathrm{Max}(f; n, m) = n$ *if and only if* $m = n - 1$,
2. $\mathrm{Min}(f; n, m) = \mathrm{Max}(f; n, m) = 2$ *if and only if* $m = \binom{n}{2}$, *and*
3. *for* $3 \le a \le n-1$, $\mathrm{Min}(f; n, m) = a$ *if and only if* $\mathbf{cm}_n(a) \le m < \mathbf{cm}_n(a-1)$.

□

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Computing Simple Paths on Points in Simple Polygons

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Abstract. Given a simple polygon P with m vertices, a set $X = \{x_1, x_2, ..., x_n\}$ of n points within P, a start point $s \in X$, and an end point $t \in X$, we give an $O(n^3 \log m + mn)$ time, $O(n^3 + m)$ space algorithm to find $O(n^2)$ **simple** polygonal paths from s to t that have their vertices among the points in X and stay inside P, or report that no such path exists.

1 Introduction

In this paper we study the following problem: Given a simple polygon P with m vertices, a set $X = \{x_1, x_2, ..., x_n\}$ of n points within P, a start point $s \in X$, and an end point $t \in X$, find a **simple** polygonal path K from s to t (a path without self-intersections) that has its vertices among the points in X and stays inside P, or report that no such path exists (see Figure 1). The points in X are allowed to be on the edges or at the vertices of P and a line segment of K can be tangent to a vertex of P.

Obviously, if self-intersection is allowed then an s-to-t path with vertices in X can be found by computing the visibility graph of X. However, in our problem, self-intersections are not allowed. We also make the following simple observations: (1) While a shortest s-to-t path in P always exists, it is possible no simple s-to-t path exists; (2) In general, a shortest s-to-t path in P, restricted to turn only at points in X, is not a simple path, as illustrated in Figure 1; (3) If there exists an s-to-t path that turns only at points in X then there exists a shortest s-to-t path that turns only at points in X; (4) The existence of an s-to-t path that turns only at points in X does not guarantee the existence of a **simple** s-to-t path that turns only at points in X.

The problem we study is a special case of finding a simple path on points while avoiding a set of obstacles in the plane. In our problem, the obstacle set is the union of connected line segments. The problem, both in its general

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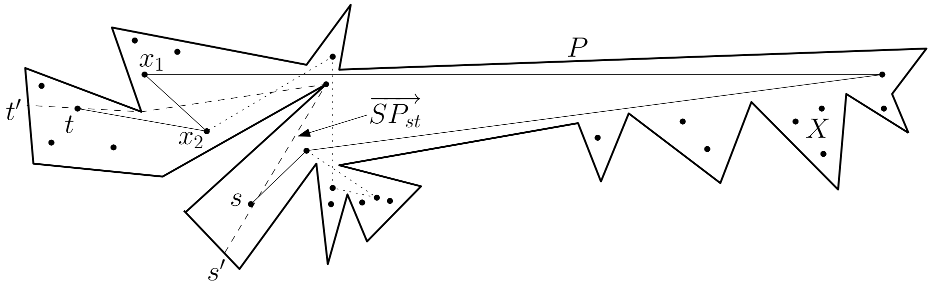

Fig. 1. A simple s-to-t path that turns on points in X (bold line), a shortest s-to-t path that turns on points in X and is not a **simple** path (dotted line), and the shortest s-to-t path in P (bold dashed line)

form and in the special form studied in this paper, has been introduced in [3]. They have motivated it by its applications to polygon generation, where given a set of points X inside a simple polygon P one wishes to generate simple, or x-monotone, polygons with vertices in X [1, 3, 7].

In [3], they consider the problem of finding a simple path on points while avoiding a set of obstacles in the plane and prove it is NP-complete for arbitrary obstacles. For the special case when the obstacles form a simple polygon P and X is in P, they give a polynomial, $O(m^2n^2)$ time and space algorithm, based on dynamic programming. Their dynamic programming solution is straightforward. First, they pair the points of X with the vertices of P based on visibility: a point is paired with a vertex if they are visible inside P. Each of the resulting pairs corresponds to a vertex in a graph G, and G can have $\Omega(mn)$ vertices in the worst case. Then, directed edges are added between vertices of G based on a simple partition rule and there can be $\Omega(m^2n^2)$ edges in the worst case. The graph is acyclic and an s-to-t path in G is the sought solution. Thus, recent results on computing the visibility graph of X within P [2] would not help in speeding up the algorithm in [3].

In contrast, for the special case of points within a simple polygon, we present a new algorithm based on the construction of a *pyramid* graph. In this graph, a node corresponds to a pair of points of X that can see each other in P. Two nodes defined by the oriented pairs (x_i, x_j) and (x_j, x_k) are connected by an edge if x_i and x_k are not visible in P; in this case, the points x_i, x_j, and x_k define a *pyramid*. For example, in Figure 1, the points x_1, x_2, and t define a pyramid. The algorithm requires $O(n^3 \log m + nm)$ time and $O(n^3 + m)$ space. Our algorithm, called $MOD - SSSP$, resembles a single source shortest path algorithm and is executed on the pyramid graph of X (it requires the computation of the visibility graph of points within a simple polygon [2, 4]). The key contribution is how to make sure that when a line segment (u, v), with $u, v \in X$, is added to the simple path constructed so far, all possible simple paths which start at s and end at u have been considered, and the line segment (u, v) does not intersect with at least one of those simple paths. The foundation of our algorithm is given by an interesting property that relates the shortest paths in P (without the restriction that they turn at points in X) with simple paths in P that turn only at points in X, that may be of more general interest. Intuitively, this property says that a simple s-to-t path travels in the same "direction" as the shortest path (see Lemma 5 and Lemma 7). Compared to the previous solution, our algorithm reduces the space from $O(m^2n^2)$ to $O(n^3)$ whenever $m = \omega(\sqrt{n})$, and improves over the running time whenever $n = o(\frac{m^2}{\log m})$. Moreover, it can report not only one but $O(n^2)$ simple paths from s to t.

Overall, our solution proves other competitive approaches are possible, and opens the door to more efficient solutions for the problem addressed in this paper.

2 Preliminaries

In this section, we give the definitions and notations that will be used in the rest of the paper. For consistency, we use the same notations as in [3] whenever possible.

Let P be a simple polygon and let ∂P be the boundary of P. Let X be a set of n points inside P. Let $[x, y]$ denote the line segment with endpoints x and y. For a polygonal path K, $K[x, y]$ denotes the sub-path of K between x and y.

Definition 1. *A polygonal path $K = x_1x_2 \ldots x_k$, with vertices x_1,x_2,...,x_k, is called* simple *if it is homeomorphic to a line segment (in other words, K does not intersect itself). When X is a finite set of points and $s = x_1, x_2, \ldots, x_k = t \in X$, K is called an (s, X, t)-path.*

Definition 2. *A path $K = x_1x_2 \ldots x_k$ is called* 3-shortcut-free *if for any three consecutive vertices x_{i-1}, x_i, and x_{i+1} of K, $2 \leq i \leq k-1$, x_{i-1} and x_{i+1} are visible to x_i but x_{i-1} and x_{i+1} can not see each other in P.*

For example, the simple (s, X, t) path in Figure 1 is a 3-shortcut-free path.

Definition 3. *For three points x_l, x_p, and x_q such that $K = x_lx_px_q$ is a 3-shortcut-free path, $[x_l, x_p] \bigcup [x_p, x_q]$ is called a* pyramid*, denoted as $\widehat{lpq}$ or $x_l\widehat{x_p}x_q$; the pyramid $\widehat{lpq}$ is directed from l-to-p-to-q, and thus $\widehat{lpq} \neq \widehat{qpl}$.*

For example, the points x_1, x_2, and t in Figure 1 define the pyramid $x_1\widehat{x_2}x_3$. It will become clear later on (see Definition 8 and Lemma 5) that the orientation of the pyramids is crucial in solving the problem.

For any two points a, b in P, we denote the shortest (directed) path from a to b inside P as $\overrightarrow{SP_{ab}}$. Notice that this path is not restricted to turn at points in X. We assume that s and t are on a horizontal line, with s to the left of t. Consider the shortest path $\overrightarrow{SP_{st}}$ inside P and extend the first and last line segments of SP_{st} until they intersect the boundary ∂P of P at points s' and t', respectively (see Figure 1). The points s' and t' separate ∂P into an upper part, ∂P_U, and a lower part, ∂P_L, while $[s's] \cup \overrightarrow{SP_{st}} \cup [tt']$ splits P accordingly into two parts, P_U and P_L.

For any two points $a, b \in \partial P_U$ (a, b may not be vertices of P), denote the sub-path of ∂P_U between a and b as $\partial P_U[a, b]$. Similarly, for any two points a and b on ∂P_L denote the sub-path of ∂P_L between a and b as $\partial P_L[a, b]$.

Definition 4. *If $a \in \partial P_U[s', b]$ we say that a is before b on ∂P_U; else, a is after b on ∂P_U. Similarly, if $a \in \partial P_L[s', b]$ we say that a is before b on ∂P_L; else, a is after b on ∂P_L.*

Definition 5. *A* separator $S_{(a,b,c)}$ *is the union of two line segments $[a, b] \cup [b, c]$ such that $[a, b] \subset P, [b, c] \subset P$, a is a vertex of ∂P_U, c is a vertex of ∂P_L, and b is a point inside or on boundary of P.*

Definition 6. *A* separator $S_{(a,b,c)}$ *separates P into two parts. If one part includes s and the other part includes t, then $S_{(a,b,c)}$ is called* st separator, *denoted as $S^{st}_{(a,b,c)}$. The part including s is denoted as $P^s_{(a,b,c)}$ and the other part including t is denoted as $P^t_{(a,b,c)}$.*

Definition 7. *For a* 3-shortcut-free *path $K = x_1x_2 \dots x_k$, if we can find a set of st separators $S^{st}_{(a_i,x_i,c_i)}$ $(i = 1, ..., k)$ such that $P^s_{(a_1,x_1,c_1)} \subset P^s_{(a_2,x_2,c_2)} \subset P^s_{(a_3,x_3,c_3)} \cdots \subset P^s_{(a_k,x_k,c_k)}$ and $P^t_{(a_k,x_k,c_k)} \subset P^t_{(a_{k-1},x_{k-1},c_{k-1})} \subset P^t_{(a_{k-2},x_{k-2},c_{k-2})}$ $\ldots \subset P^t_{(a_1,x_1,c_1)}$, then K is an* st oriented 3-shortcut-free *path. If $x_1 = s$ or $x_k = t$, then $S^{st}_{(a_i,x_i,c_i)}$, for $i = 1$ or $i = k$, can be ignored.*

Note that

$$[x_1, x_2] \subset P^t_{(a_1,x_1,c_1)} \cap P^s_{(a_2,x_2,c_2)},$$

$$[x_2, x_3] \subset P^t_{(a_2,x_2,c_2)} \cap P^s_{(a_3,x_3,c_3)},$$

$$...,$$

$$[x_{k-1}, x_k] \subset P^t_{(a_{k-1},x_{k-1},c_{k-1})} \cap P^s_{(a_k,x_k,c_k)}$$

Moreover, $P^s_{(a_1,x_1,c_1)}$, $P^t_{(a_1,x_1,c_1)} \cap P^s_{(a_2,x_2,c_2)}$,...,$P^t_{(a_{k-1},x_{k-1},c_{k-1})} \cap P^s_{(a_k,x_k,c_k)}$, $P^t_{(a_k,x_k,c_k)}$ are interior disjoint and then an *st oriented 3-shortcut-free* path $K = x_1x_2 \dots x_k$ is a simple path.

A pyramid $x_l\widehat{x_p}x_q$ is a special case of 3-shortcut-free path with only three vertices. We define the *st oriented pyramid* analogous to the definition of st oriented 3-shortcut-free path.

Definition 8. *A pyramid $x_l\widehat{x_p}x_q$ is an* st oriented pyramid *if there exist three* st separators $S_{(a_z,x_z,c_z)}(z = l, p, q)$ *such that $P^s_{(a_l,x_l,c_l)} \subset P^s_{(a_p,x_p,c_p)} \subset P^s_{(a_q,x_q,c_q)}$ and $P^t_{(a_q,x_q,c_q)} \subset P^t_{(a_p,x_p,c_p)} \subset P^t_{(a_l,x_l,c_l)}$.*

A shortest path $\overrightarrow{SP_{x_ix_j}} = x_if_1f_2...f_kx_j$ is also a special 3-shortcut-free path, having all internal vertices among the vertices of P. We define the *st oriented shortest path* analogous to the definition of st oriented 3-shortcut-free path with small modifications.

Definition 9. *A shortest path* $\overrightarrow{SP_{x_ix_j}} = x_if_1f_2...f_kx_j$ *is an* st oriented shortest path *if there exists a set of st separators* $S^{st}_{(a_i,x_i,c_i)}$, $S^{st}_{(a_l,f_l,c_l)}(l = 1,...,k)$, $S^{st}_{(a_j,x_j,c_j)}$ *such that* $f_l = a_l$ *or* c_l, $P^s_{(a_i,x_i,c_i)} \subset P^s_{(a_1,f_1,c_1)} \subset P^s_{(a_2,f_2,c_2)} \cdots \subset P^s_{(a_j,x_j,c_j)}$ *and* $P^t_{(a_j,x_j,c_j)} \subset P^t_{(a_k,f_k,c_k)} \subset P^t_{(a_{k-1},f_{k-1},c_{k-1})} \cdots \subset P^t_{(a_i,x_i,c_i)}$.

Note that for all internal vertices of $\overrightarrow{SP_{x_ix_j}}$, the st separator passing through each internal vertex is a line segment since $f_l = a_l$ or c_l.

Consider a pyramid $\widehat{lpq}$, and let $f_1, f_2, \ldots, f_k$ be the interior vertices of $\overrightarrow{SP_{x_lx_q}}$. Notice that $\overrightarrow{SP_{x_lx_q}}$ is a convex chain.

Definition 10. *If all* $f_1, f_2, \ldots, f_k$ *are on* ∂P_U *then* $\widehat{lpq}$ *is an* upper pyramid. *If all* $f_1, f_2, \ldots, f_k$ *are on* ∂P_L *then* $\widehat{lpq}$ *is a* lower pyramid. *If some of* $f_1, f_2, \ldots, f_k$ *are on* ∂P_U *and the others are on* ∂P_L, *then* $\widehat{lpq}$ *is a* hybrid pyramid.

We can prove that hybrid pyramids do not exist in a simple 3-shortcut-free (s, X, t)-path (see Lemma 3).

3 Properties

Lemma 1. *Let* Q *be a simple polygon in* P *with vertices* $v_1, v_2, \ldots, v_k$, $k > 3$. *Then, there exist at least two sets of three consecutive vertices* v_{i-1}, v_i, v_{i+1}, *and* v_{j-1}, v_j, v_{j+1} *of* Q *which do not form a pyramid, and such that* v_i *and* v_j *are not adjacent vertices of* Q.

Proof. Follows directly from Meister's Two-Ears theorem: every polygon of n vertices has at least two non-overlapping ears [5]. □

Lemma 2 below extends Lemma 1 in [3] to 3-shortcut-free paths.

Lemma 2. *Let* $K = x_1x_2 \ldots x_k$ *be a simple 3-shortcut-free path in* P, *and let* $x, y \in P$ *be two points that are visible to each other in* P. *Let* $z_1, z_2, \ldots, z_j$ *be the intersection points of* K *with* $[x, y]$, *listed in the order in which they appear on* K. *Then* $z_1, z_2, \ldots, z_j$ *must also appear on* $[x, y]$ *in order, that is,* z_i *is between* z_{i-1} *and* z_{i+1} *for all* $i = 2, 3, \ldots, j - 1$.

Proof. Suppose the first intersection point which is not in order is z_b, and the intersection point before z_b on K is z_a. $K[z_a, z_b] \cup [z_a, z_b]$ forms a simple polygon Q. There are two cases:

1. z_b is before z_1 on $[x, y]$. There are two subcases:
   (a) x_1 is outside Q (see Figure 2). It is easy to see there exists a vertex x_i on $K[z_1, z_a]$ and another vertex x_j on $K[z_a, z_b]$ such that x_i and x_j are visible to each other. $K[x_i, x_j] \cup [x_i, x_j]$ forms a simple polygon. From Lemma 1, we know that at least one vertex of $x_{i+1}, \ldots, x_{j-1}$ can not form a pyramid with its neighboring vertices. This contradicts that K is a simple 3-shortcut-free path.

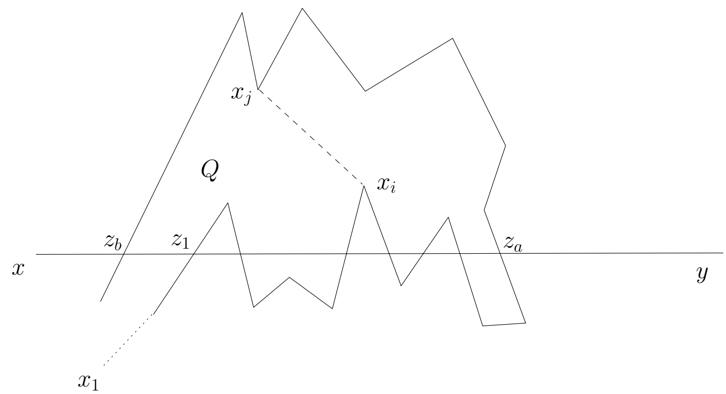

Fig. 2. Case 1(a): z_b is before z_1 on $[x, y]$ and x_1 is outside Q

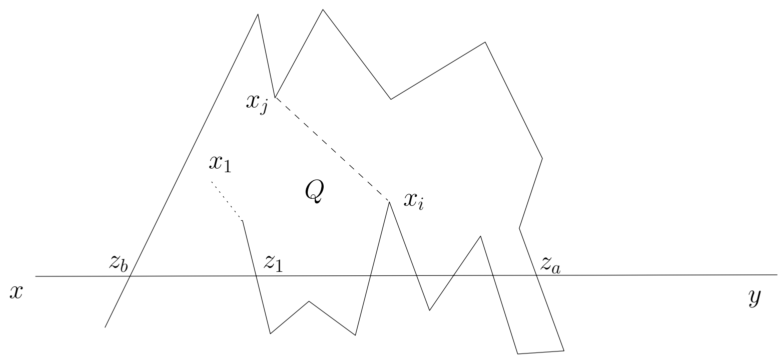

Fig. 3. Case 1(b): z_b is before z_1 on $[x, y]$ and x_1 is inside Q

(b) x_1 is inside Q (see Figure 3). It is easy to see there exists a vertex x_i on $K[x_1, z_a]$ and another vertex x_j on $K[z_a, z_b]$ such that x_i and x_j are visible to each other. $K[x_i, x_j] \cup [x_i, x_j]$ forms a simple polygon. From Lemma 1, we know at least one vertex of $x_{i+1}, \ldots, x_{j-1}$ can not form a pyramid with its neighboring vertices. This contradicts that K is a simple 3-shortcut-free path.

2. z_b is after z_1 on $[x, y]$. Let z_c be the intersection point before z_b on $[x, y]$ and before z_b on K. It is easy to see there exists a vertex x_i on $K[z_c, z_a]$ and another vertex x_j on $K[z_a, z_b]$ such that x_i and x_j are visible to each other (see Figure 4). $K[x_i, x_j] \cup [x_i, x_j]$ forms a simple polygon. From Lemma 1, we know that at least one vertex of $x_{i+1}, \ldots, x_{j-1}$ can not form a pyramid with its neighboring vertices. This contradicts that K is a simple 3-shortcut-free path. If there are only two vertices x_i, x_{i+1} on $K[z_c, z_b]$ (see Figure 5), then we can find a vertex x_j which is on $K[z_b, x_k]$ and inside the triangle $\Delta z_c x_i z_a$ such that x_i and x_j are visible to each other. $K[x_i, x_j] \cup [x_i, x_j]$ forms a simple polygon. From Lemma 1, we know that at least one vertex

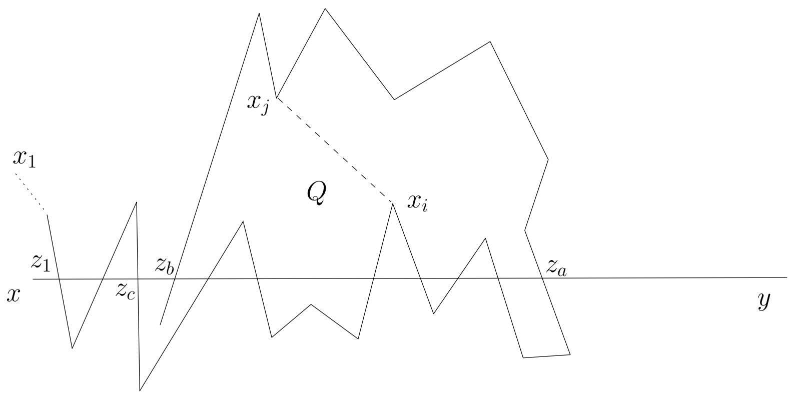

Fig. 4. z_b is after z_1 on $[x, y]$ and there are more than two vertices on $K[z_c, z_b]$

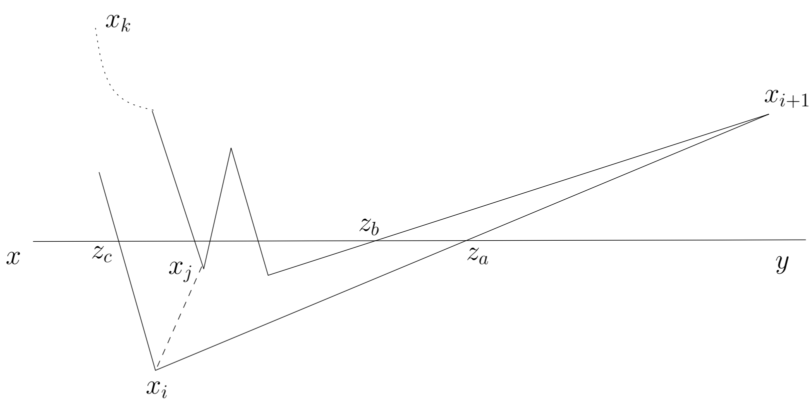

Fig. 5. z_b is after z_1 on $[x, y]$ and there are only two vertices on $K[z_c, z_b]$

of $x_{i+1}, \ldots, x_{j-1}$ can not form a pyramid with its neighbor vertices, which contradicts that K is a simple 3-shortcut-free path.

This concludes the proof of the lemma. □

Lemma 3. *A simple 3-shortcut-free (s, X, t)-path $K = sx_1x_2...x_kt$ has no hybrid pyramids.*

Proof. Suppose $x_{i-1}\widehat{x_i}x_{i+1}$ is a hybrid pyramid. Then one of x_{i-1} and x_{i+1} should be in P_U and the other one in P_L. Suppose x_{i-1} is in P_U and x_{i+1} is in P_L. $\overrightarrow{SP_{x_{i-1}x_{i+1}}}$ is a convex chain $x_{i-1}f_1f_2...f_jx_{i+1}$ and it is separated by $\overrightarrow{SP_{st}}$ into two parts (see Figure 6). Suppose $f_1, f_2, ..., f_l$ are on ∂P_U and $f_{l+1}, f_{l+2}, ..., f_j$ are on ∂P_L. s should be in the simple polygon formed by $\partial P_U[s', f_l] \cup \partial P_L[s', f_{l+1}] \cup [f_l, f_{l+1}]$. $\overrightarrow{SP_{x_{i-1}x_{i+1}}} \cup x_{i-1}\widehat{x_i}x_{i+1}$ forms a pseudo triangle with two straight edges and one convex chain and this pseudo triangle is inside P. Then the subpath

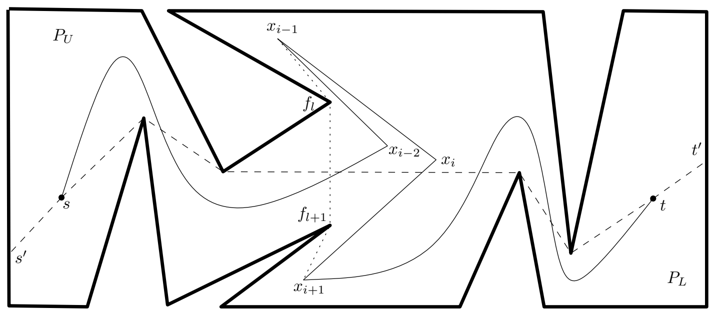

Fig. 6. Illustration of Lemma 3

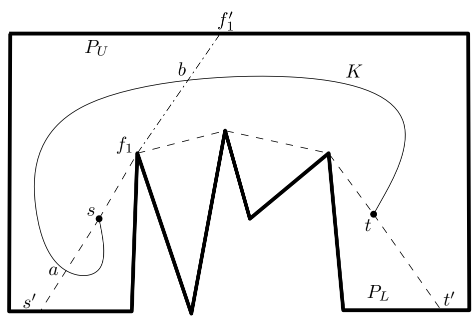

Fig. 7. $sf_1f_2...f_kt$ is a convex chain

$K' = sx_1x_2...x_{i-1}$ enters into the pseudo triangle through $[f_l, f_{l+1}]$ and x_{i-2} is inside the pseudo triangle, which means x_{i-2} can see x_i. This contradicts the fact that K is a 3-shortcut-free path. □

From Lemma 3, we know that hybrid pyramids can not appear in a simple 3-shortcut-free (s, X, t)-path. We only need focus on lower and upper pyramids in the remaining part of this paper.

Lemma 4. *Let $K = sx_1 \ldots x_kt$ be a simple 3-shortcut-free path in P. Then K can not cross $[s, s']$.*

Proof. Let $\overrightarrow{SP_{st}}$ be $sf_1f_2...f_kt$. There are two cases:

1. $sf_1f_2...f_kt$ is a convex chain (see Figure 7). Suppose $f_1, f_2, ..., f_k \in \partial P_L$ (similar if $f_1, f_2, ..., f_k \in \partial P_U$). Shoot a ray $\overrightarrow{s'f_1}$ until it hits a point $f_1' \in \partial P_U$. Suppose K crosses $[s, s']$ at point a. Since $K[a, t]$ need reach t, it has to cross $[f_1, f_1']$. Suppose the cross point is b. Then s, a, and b do not appear on $[s', f_1']$ in order. This contradicts with Lemma 2.
2. $f_1, f_2, ..., f_i \in \partial P_L$ and $f_{i+1} \in \partial P_U$ (see Figure 8). Then $sf_1f_2...f_if_{i+1}$ is a convex chain. Shoot a ray $\overrightarrow{s'f_1}$ until it hits a point $f_1' \in \partial P_U$. Suppose K

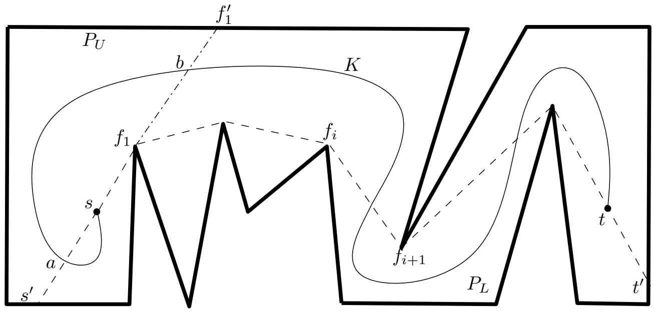

Fig. 8. $f_1, f_2, ..., f_i \in \partial P_L$ and $f_{i+1} \in \partial P_U$

crosses $[s, s']$ at point a. Note that $K[a, t]$ has to cross $[f_i, f_{i+1}]$ in order to reach t. But it has to cross $[f_1, f_1']$ to reach $[f_i, f_{i+1}]$. Let the cross point with $[f_1, f_1']$ be b. Then s, a, and b do not appear on $[s', f_1']$ in order. This contradicts with Lemma 2.

Lemma 5. *Let $K = sx_1 \ldots x_k t$ be a simple 3-shortcut-free path in P. Then K is an st oriented 3-shortcut-free path and every pyramid on K is st oriented.*

Proof. In order to prove that K is an st oriented 3-shortcut-free path, we need to find a sequence of st separators $S^{st}_{(a_i,x_i,c_i)}$ $(i = 1, ..., k)$ such that $[s, x_1] \subset P^s_{(a_1,x_1,c_1)} \subset P^s_{(a_2,x_2,c_2)} ... \subset P^s_{(a_k,x_k,c_k)}$ and $[t, x_k] \subset P^t_{(a_k,x_k,c_k)} \subset P^t_{(a_{k-1},x_{k-1},c_{k-1})} ... \subset P^t_{(a_1,x_1,c_1)}$.

For the first pyramid $s\widehat{x_1 x_2}$, we can find an st separator $S^{st}_{(a_1,x_1,c_1)}$ as follows. Let $\overrightarrow{x_1 s}$ be the ray originating from x_1. Rotate ray $\overrightarrow{x_1 s}$ around x_1, both clockwise and counterclockwise, until it hits the first vertices on ∂P_U and ∂P_L. Let a_1 be the first vertex on ∂P_U and c_1 the first vertex on ∂P_L. We next prove that $[s, x_1] \subset P^s_{(a_1,x_1,c_1)}$ and $K[x_1, t] \subset P^t_{(a_1,x_1,c_1)}$.

From the construction of $S^{st}_{(a_1,x_1,c_1)}$, we know that $[s, x_1] \subset P^s_{(a_1,x_1,c_1)}$ and $[s, s'] \subset P^s_{(a_1,x_1,c_1)}$. Suppose $K[x_1, t] \not\subset P^t_{(a_1,x_1,c_1)}$, which means at least part of $K[x_1, t]$ is inside $P^s_{(a_1,x_1,c_1)}$. We argue that at least one vertex of $K[x_1, t]$ is inside $P^s_{(a_1,x_1,c_1)}$. Suppose no vertex of $K[x_1, t]$ is inside $P^s_{(a_1,x_1,c_1)}$ but a line segment $[x_i, x_{i+1}]$, $i \in [2..k]$, of $K[x_1, t]$ passes through $P^s_{(a_1,x_1,c_1)}$. Since K is a simple path, the only possible way that $[x_i, x_{i+1}]$ passes through $P^s_{(a_1,x_1,c_1)}$ is that $[x_i, x_{i+1}]$ crosses $[s, s']$ which contradicts Lemma 4.

Now suppose the first vertex of $K[x_1, t]$ inside $P^s_{(a_1,x_1,c_1)}$ is x_i, for some $i \in \{2, 3, \ldots, k\}$. From Lemma 4, we know that K can not cross $[s, s']$. Then x_i is inside the simple polygon formed by $[x_1, s] \cup [s, s'] \cup \partial P_U[s', a_1] \cup [a_1, x_1]$ (similar if x_i is inside the simple polygon formed by $[x_1, s] \cup [s, s'] \cup \partial P_L[s', c_1] \cup [c_1, x_1]$). There are two cases:

1. $x_i = x_2$. Since K is a 3-shortcut-free path, s can not see x_2. Only $\partial P_U[s', a_1]$ can block the view from s to x_2. There is at least one vertex $b \in \partial P_U[s', a_1]$

such that b is visible to x_1. Then when we rotate the ray $\overrightarrow{x_1 s}$, it hits b before a_1. It contradicts the fact that a_1 is the first vertex of ∂P_U that the ray $\overrightarrow{x_1 s}$ hits during rotation.

2. $x_i \neq x_2$. There are two subcases:

   (a) x_i is visible to x_1. Then $[x_1, x_i] \cup K[x_1, x_i]$ forms a simple polygon. According to Lemma 1, we know at least one vertex of $x_1, \ldots, x_i$ can not form a pyramid with its neighboring vertices. This contradicts that K is a simple 3-shortcut-free path.

   (b) x_i is not visible to x_1. Only $\partial P_U[s', a_1]$ can block the view from x_1 to x_i. There is at least one vertex $b \in \partial P_U[x_1, x_i]$ such that b is visible to x_1. Then, when we rotate the ray $\overrightarrow{x_1 s}$, it hits b before a_1, contradicting the fact that a_1 is the first vertex of ∂P_U that the ray $\overrightarrow{x_1 s}$ hits during rotation.

We can find all other st separators $S^{st}_{(a_i,x_i,c_i)}(i = 2, ..., k)$ as follows. Rotate ray $\overrightarrow{x_i x_{i-1}}$ around x_i, both clockwise and counterclockwise, until it hits the first vertices on $\partial P_U[a_{i-1}, t']$ and $\partial P_L[c_{i-1}, t']$. Let a_i be the first vertex on $\partial P_U[a_{i-1}, t']$ and c_i be the first vertex on $\partial P_L[c_{i-1}, t']$. Clearly, a_i (resp., (c_i)) must exist since only $\partial P_U[a_{i-1}, t']$ (resp., $(\partial P_L[c_{i-1}, t'])$) can block the view from x_i to a_i (resp., (c_i)). Using a similar argument as above we can prove that $[x_{i-1}, x_i] \subset P^s_{(a_i,x_i,c_i)} \cap P^t_{(a_{i-1},x_{i-1},c_{i-1})}$ and $K[x_i, t] \subset P^t_{(a_i,x_i,c_i)}$.

Since we can find a sequence of st separators $S^{st}_{(a_i,x_i,c_i)}$ $(i = 1, ..., k)$ such that $[s, x_1] \subset P^s_{(a_1,x_1,c_1)} \subset P^s_{(a_2,x_2,c_2)} ... \subset P^s_{(a_k,x_k,c_k)}$ and $[t, x_k] \subset P^t_{(a_k,x_k,c_k)} \subset P^t_{(a_{k-1},x_{k-1},c_{k-1})} ... \subset P^t_{(a_1,x_1,c_1)}$, we have that K is an st oriented 3-shortcut-free path. From the construction above, it follows that for each pyramid $x_{i-1}\widehat{x_i}x_{i+1}$, the exist three separators $S_{(a_j,x_j,c_j)}$ $(j = i-1, i, i+1)$ such that $P^s_{(a_{i-1},x_{i-1},c_{i-1})} \subset P^s_{(a_i,x_i,c_i)} \subset P^s_{(a_{i+1},x_{i+1},c_{i+1})}$ and $P^t_{(a_{i+1},x_{i+1},c_{i+1})} \subset P^t_{(a_p,x_p,c_p)} \subset P^t_{(a_{i-1},x_{i-1},c_{i-1})}$, and thus every pyramid on K is st oriented. □

Lemma 6. *For a shortest path $\overrightarrow{SP_{x_i x_j}}$, if there exist two nonintersecting st separators $S^{st}_{(a_i,x_i,c_i)}$ and $S^{st}_{(a_j,x_j,c_j)}$ such that $s \in P^s_{(a_i,x_i,c_i)}$ and $t \in P^t_{(a_j,x_j,c_j)}$, then $\overrightarrow{SP_{x_i x_j}}$ is st oriented.*

Proof. First we prove that $\overrightarrow{SP_{x_i x_j}} \subset P^t_{(a_i,x_i,c_i)} \cap P^s_{(a_j,x_j,c_j)}$. Suppose $\overrightarrow{SP_{x_i x_j}}$ enters $P^s_{(a_i,x_i,c_i)}$ by crossing $[x_i, a_i]$. $[x_i, a_i]$ already intersects $\overrightarrow{SP_{x_i x_j}}$ at x_i. This contradicts the fact that a shortest path in P can not intersect a line segment in P more than once. Similarly, we can argue that $\overrightarrow{SP_{x_i x_j}}$ can not enter $P^t_{(a_j,x_j,c_j)}$. Therefore $\overrightarrow{SP_{x_i x_j}} \subset P^t_{(a_i,x_i,c_i)} \cap P^s_{(a_j,x_j,c_j)}$.

Let $\overrightarrow{SP_{x_i x_j}} = x_i f_1 f_2 ... f_k x_j$. We can find all st separators $S^{st}_{(a_l,f_l,c_l)}(l = 1, ..., k)$ as follows: if $f_l \in \partial P_L$ (similar if $f_l \in \partial P_U$), then let $c_l = f_l$ and a_l be the first vertex on $\partial P_U[a_{l-1}, a_j]$ $(a_0 = a_i)$ which is visible to f_l. Such a_l must exist since only $\partial P_U[a_{l-1}, a_j]$ can block the view from f_l to a_l. Also $\overrightarrow{SP_{f_l x_j}}$ can not enter $P^s_{(a_l,f_l,c_l)}$ since it already intersects the line segment $[f_l, a_l]$ at f_l. So we find all st separators $S^{st}_{(a_l,f_l,c_l)}(l=1,...,k)$ which separate $\overrightarrow{SP_{f_l x_j}}$ in a way that assures $\overrightarrow{SP_{x_i x_j}}$ is st oriented. □

Lemma 7. *For two points x_i and x_j that cannot see each other in P, if there exists an st oriented 3-shortcut-free path $K = x_i x_{i+1} \ldots x_j$, then $\overrightarrow{SP_{x_i x_j}}$ is st oriented.*

Proof. Since K is st oriented, there exists two disjoint st separators $S^{st}_{(a_i,x_i,c_i)}$ and $S^{st}_{(a_j,x_j,c_j)}$ such that $s \in P^s_{(a_i,x_i,c_i)}$ and $t \in P^t_{(a_j,x_j,c_j)}$. According to Lemma 6, $\overrightarrow{SP_{x_i x_j}}$ is st oriented. □

For a pyramid $x_{k-2}\widehat{x_{k-1}}x_k$, there could be many possible simple st oriented 3-shortcut-free paths from s to x_k through $x_{k-2}\widehat{x_{k-1}}x_k$. For each such st oriented 3-shortcut-free path $sx_1x_2...x_{k-2}x_{k-1}x_k$, all st separators $S^{st}_{(a_i,x_i,c_i)}$ $(i = 1,...,k)$ are found as in Lemma 5. Then, either a_k or c_k is the last internal vertex of $\overrightarrow{SP_{x_{k-2}x_k}}$.

Lemma 8. *For any st oriented 3-shortcut-free path from s to x_k such that the last pyramid is $x_{k-2}\widehat{x_{k-1}}x_k$, a_k or c_k is decided only by the pyramid $x_{k-2}\widehat{x_{k-1}}x_k$.*

Proof. Suppose $x_{k-2}\widehat{x_{k-1}}x_k$ is a lower pyramid and the last internal vertex of $\overrightarrow{SP_{x_{k-2}x_k}}$ is $f \in \partial P_L$. First we prove that the ray $\overrightarrow{x_k f}$ intersects $[x_{k-2}, x_{k-1}]$ before it intersects another part of $sx_1x_2...x_{k-2}x_{k-1}x_k$. Suppose $\overrightarrow{x_k f}$ intersects $sx_1x_2...x_{k-2}$ first at d and then intersects $[x_{k-2}, x_{k-1}]$ at e, then d is between e and x_k on $[e, x_k]$ which is not the order they appear on $sx_1x_2...x_{k-2}x_{k-1}x_k$. This contradicts Lemma 2. Second, we prove that f is the first vertex on ∂P_L that is visible to x_k. Again, let e be the intersection point of ray $\overrightarrow{x_k f}$ with $[x_{k-2}, x_{k-1}]$. Since $\overrightarrow{SP_{x_{k-2}x_k}}$ is a convex chain, no point of ∂P_L is in the triangle formed by e, x_{k-1}, x_k. Therefore, f is the first vertex on $\partial P_L[s, f]$ that is visible to x_k. Finally, we prove that $[x_k, f]$ does not intersect $[x_{k-1}, c_{k-1}]$ which is the lower part of $S^{st}_{(a_{k-1},x_{k-1},c_{k-1})}$. Suppose $[x_k, f]$ intersects $[x_{k-1}, c_{k-1}]$ at e. Then, c_{k-1} is after f on ∂P_L. Since no part of $sx_1x_2...x_{k-2}$ appears in the triangle formed by f, x_{k-1} and x_k, $[x_{k-1}, f]$ does not intersect any part of $sx_1x_2...x_{k-2}$, which means we can at least move c_{k-1} to f. From the argument above it follows c_k is f, which is the last internal vertex of $\overrightarrow{SP_{x_{k-2}x_k}}$, and is decided only by the pyramid $x_{k-2}\widehat{x_{k-1}}x_k$. □

4 The Algorithm

Our algorithm, $MOD{-}SSSP$, resembles a single source shortest path algorithm. First, we construct the visibility graph $G = (X, E)$, where E corresponds to pairs of vertices (points in X) that can see each other in P. The edges are directed so that between two visible points there are two corresponding edges in E. Second, we construct another graph $G' = (V', E')$ where each node $v' \in V'$ corresponds to an edge in E. Each edge $e' \in E'$ corresponds to an st oriented pyramid $x_i\widehat{x_j}x_k$. So G' has $O(n^2)$ vertices and $O(n^3)$ edges. G' could be a cyclic graph.

From Lemma 7 it follows that if x_i, x_j cannot see each other in P, are on a cycle in G', and $\overrightarrow{SP_{x_ix_j}}$ is st oriented, then there is no st oriented 3-shortcut-free path which starts at x_j and ends at x_i. Otherwise $\overrightarrow{SP_{x_jx_i}}$ is also st oriented, which is impossible since $\overrightarrow{SP_{x_ix_j}}$ is st oriented. Therefore, we can create another directed acyclic graph $G'' = (X, E'')$ such that $e'' = (x_i, x_j) \in E''$ if $\overrightarrow{SP_{x_ix_j}}$ is st oriented. After topological sort on G'', we can assign a number SN_{x_i} to each node $x_i \in X$ according to that order. For each vertex $v' = [x_i, x_j]$ in G', let the "value" of v' be $Value[v'] = SN_{x_j}$. Note that we can ignore vertices that appear before s in the topological sort order.

We then perform a traversal of G' in the order given by SN values. For each vertex $v'_{ij} = [x_i, x_j]$, we define the "distance" $Dist(v'_{ij})$ from s to v'_{ij} as follows: initially, $Dist(v'_{ij}) = \Phi$. If $x_i = s$, rotate ray $\overrightarrow{x_js}$ around x_j, both clockwise and counterclockwise, until it hits the first vertices on ∂P_U and ∂P_L. Let a_j be the first vertex on ∂P_U and c_j be the first vertex on ∂P_L. If the separator $S_{(a_j,x_j,c_j)}$ is an st separator, then $Dist(v'_{ij}) = P^s_{(a_j,x_j,c_j)}$. Otherwise, we do not need to update $Dist(v'_{ij})$. We define $Dist(v'_{ij}) < Dist(v'_{kl})$ if $P^s_{(a_j,x_j,c_j)} \subset P^s_{(a_l,x_l,c_l)}$, $Dist(v'_{ij}) > Dist(v'_{kl})$ if $P^s_{(a_l,x_l,c_l)} \subset P^s_{(a_j,x_j,c_j)}$ and $Dist(v'_{ij}) = Dist(v'_{kl})$ if $P^s_{(a_j,x_j,c_j)} \cap P^s_{(a_l,x_l,c_l)} \neq \Phi$. If $x_i \neq s$ and there exists a simple st oriented 3-shortcut-free path $sx_1x_2...x_i$ with $Dist(v_{i-1i}) = P^s_{(a_i,x_i,c_i)}$ already computed, then rotate the ray $\overrightarrow{x_jx_i}$ around x_j, both clockwise and counterclockwise, until it hits the first vertices on $\partial P_U[a_i, t']$ and $\partial P_L[c_i, t']$. Let a_j be the first vertex on $\partial P_U[a_i, t']$ and c_j be the first vertex on $\partial P_L[c_i, t']$. If the separator $S_{(a_j,x_j,c_j)}$ is an st separator, then $Dist(v'_{ij}) = P^s_{(a_j,x_j,c_j)}$. Note that there are could be many st oriented paths from s to x_j. We need a list $DistList(v'_{ij})$ to record all possible distances from s to v'_{ij}. Initially, $DistList(v'_{ij})$ is empty. When a new $Dist(v'_{ij})$ is computed, if it is less than some distances in $DistList(v'_{ij})$, we add it into $DistList(v'_{ij})$ and delete all distances greater than this new distance from $DistList(v'_{ij})$. If it is greater than some distances in $DistList(v'_{ij})$, then do nothing. If it equals all distances in $DistList(v'_{ij})$, then we add it into $DistList(v'_{ij})$. In this way, all distances in $DistList(v'_{ij})$ are equal and all corresponding st separators are arranged in an interleaved way (see Fig. 9). Assume x_k precedes x_i on the path computed so far. To compute $Dist(v'_{ij})$, we select the distance from $DistList(v'_{ki})$ such that $Dist(v'_{ij})$ is as close as possible to s' (on ∂P_U or ∂P_L). How can we achieve this? According to Lemma 8, if pyramid $x_k\widehat{x_i}x_j$ is a lower pyramid, then c_j is fixed no matter which distance we selected from $DistList(v'_{ki})$. In order to make a_j as close to s' as possible, we can select the separator $S_{(a^l_i,x_i,c^l_i)}$ such that $[x_i, c^l_i]$ is the closest one (when rotating the ray $\overrightarrow{x_ix_j}$ towards s') to $[x_i, x_j]$ and $[x_i, x_j] \in P^t_{(a^l_i,x_i,c^l_i)}$. The distance selected from $DistList(v'_{ki})$ corresponds to an edge $e'_i = x_{i-2}\widehat{x_{i-1}}x_i$ in G'. We set the predecessor $\pi[e'_j] = e'_i$, where $e'_j = x_k\widehat{x_i}x_j$.

Theorem 1. *The algorithm MOD−SSSP can find $O(n^2)$ simple (s, X, t) paths or report no such path exists in $O(n^3 \log m + nm)$ time and $O(n^3 + m)$ space.*

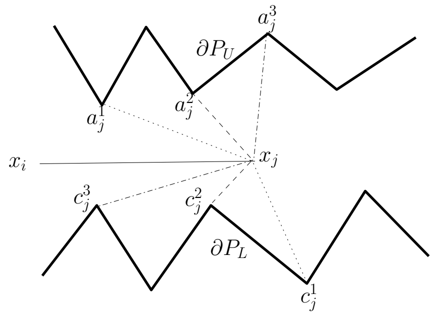

Fig. 9. Illustration of interleaved $DistList(v'_{ij}$

Proof. Lemma 7 ensures that after a node v' is selected for processing later selections do not impact $DistList_{v'}$. To see this, suppose a node $v' = [x_i, x_j]$ is processed and later on a node $w' = [x_k, x_l]$ is selected. Since we traverse G' in the order given by SN values, $SN_{x_j} < SN_{x_l}$ which means $\overrightarrow{SP_{x_j x_l}}$ is not st oriented and there is no st oriented 3-shortcut-free path from x_l to x_j according to Lemma 7. So $DistList_{w'}$ does not impact $DistList_{v'}$ since we never visit v' from w'. When the edge from v' to some other vertex u' is processed, the appropriate "distance" selected from $DistList_{v'}$ ensures that the extended edge does not intersect at least one simple path that starts from s and ends at v'. Furthermore, $Dist_{u'}$ is computed from $DistList_{v'}$ such that no future candidate edge originating at u' is excluded.

For the running time,

1. We can compute the visibility graph of the points in X in $O(m + n \log n \log mn + n^2)$ time and $O(n + m)$ space [2]. The construction of G takes $O(n^2)$ time.
2. From G, we compute G' in $O(n^3)$ time; G' has $O(n^2)$ nodes and $O(n^3)$ edges. Each node $v' \in G'$ is assigned a value $Value(v')$ according to the topological sort order for G''.
3. To construct G'', we need decide whether the shortest path between any two points of X is st oriented. We can construct the visibility graph of the points of X to the vertices of P in $O(nm)$ time [6]. This visibility graph consists of n star shaped trees. The centers of those star trees are the points of X and the other vertices in the star trees are vertices of P. For each point x_i, the vertices of P in the tree centered at x_i are in clockwise order. Then, all vertices on ∂P_U are neighbors and all vertices on ∂P_L are neighbors. We can list those vertices in order: $p_{iU}^1, p_{iU}^2, ..., p_{iU}^q, p_{iL}^{q+1}, p_{iL}^{q+2}, ..., p_{iL}^k$, where $k = O(m)$. These lists, for all points in X, can be computed in $O(nm)$ time. From lemma 6, to decide whether $\overrightarrow{SP_{x_i x_j}}$ is st oriented, we only need to find two nonintersecting st separators $S_{(a_i,x_i,c_i)}^{st}$ and $S_{(a_j,x_j,c_j)}^{st}$ such that $s \in P_{(a_i,x_i,c_i)}^s$ and $t \in P_{(a_j,x_j,c_j)}^t$. There are several subcases (See Fig. 10):

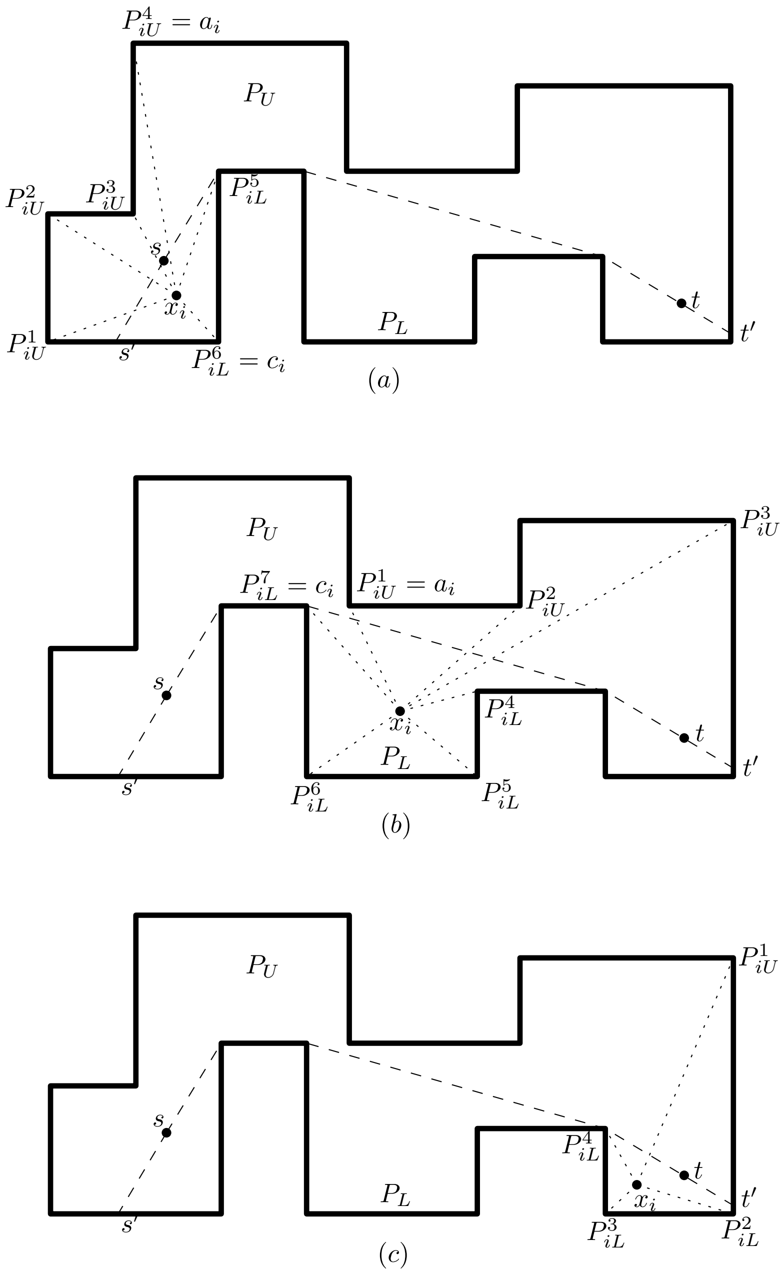

Fig. 10. (a) s is located inside the simple polygon formed by $[x_i, p_{iU}^3], [x_i, p_{iU}^4]$ and $\partial P_U[p_{iU}^3, p_{iU}^4]$. (b) s is located inside the simple polygon formed by $[x_i, p_{iU}^1], [x_i, p_{iL}^7]$, $\partial P_U[s', p_{iU}^1]$, and $\partial P_L[s', p_{iL}^7]$. (c) s is located inside the simple polygon formed by $[x_i, p_{iU}^1], [x_i, p_{iL}^2]$, $\partial P_U[s', p_{iU}^1]$, and $\partial P_U[s', p_{iL}^2]$.

(a) s is located inside a simple polygon formed by $[x_i, p^l_{iU}], [x_i, p^{l+1}_{iU}]$ and $\partial P_U[p^l_{iU}, p^{l+1}_{iU}]$, $1 \le l \le q-1$, as illustrated in Fig. 10 (a) (similar if inside a simple polygon formed by $[x_i, p^l_{iL}], [x_i, p^{l+1}_{iL}]$, and $\partial P_L[p^l_{iL}, p^{l+1}_{iL}]$, $q+1 \le l \le k-1$). Set $a_i = p^{l+1}_{iU}$, $c_i = p^k_{iL}$.

(b) s is located inside the simple polygon formed by $\partial P_L[s', p^k_{iL}]$, $[x_i, p^1_{iU}]$, $[x_i, p^k_{iL}]$, and $\partial P_U[s', p^1_{iU}]$. Set $a_i = p^1_{iU}$, $c_i = p^k_{iL}$.

(c) s is located inside the simple polygon formed by $[x_i, p^q_{iU}], [x_i, p^{q+1}_{iL}]$, $\partial P_U[s', p^q_{iU}]$, and $\partial P_L[s', p^{q+1}_{iL}]$. There is no *st separator* for x_i.

For x_j, we can find a_j and c_j in the same way. For one point of X, it takes $O(\log m)$ time to find an st separator if one exists or report no st separator exists. For all n points of X, the total time is $O(n \log m)$. After that we can decide whether $\overrightarrow{SP_{x_i x_j}}$ is st oriented in constant time. There are $O(n^2)$ pairs of points of X. The total time for constructing G'' is $O(n^2 + nm)$.

4. Traversal of G' takes $O(n^3 \log m)$, since

(a) Each node of G' is visited $O(n)$ times and each edge of G' is visited once.

(b) The processing of an edge $e' = (u', v') \in E'$, where $u' = [x_i, x_j]$ and $v' = [x_j, x_k]$, can be done in $O(\log m)$ time (finding the appropriate "distance" in sorted $DistList_{x_i x_j}$ takes $O(\log m)$ time and inserting the newly created $Dist_{x_j x_k}$ into sorted $DistList_{x_j x_k}$ takes $O(\log m)$ time since $|DistList_{x_j x_k}| = O(m)$).

Therefore, the total running time is $O(n^3 \log m + nm)$. The space used by $MOD - SSSP$ is $O(n^3 + m)$ since for each edge in G', it creates at most one "distance" entry.

To report a simple path (if one exists), start from any pyramid $e' = \widehat{x_j x_k} t$ which ends at t and trace back by using $\pi[e']$ and so on. In this way, we can find $O(n^2)$ simple paths from s to t since there can be $O(n^2)$ pyramids which end at t. $\square$

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Deflating the Pentagon

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Abstract. In this paper we consider deflations (inverse pocket flips) of n-gons for small n. We show that every pentagon can be deflated after finitely many deflations, and that any infinite deflation sequence of a pentagon results from deflating an induced quadrilateral on four of the vertices. We describe a family of hexagons that deflate infinitely for a specific deflation sequence, yet induce no infinitely deflating quadrilateral. We also review the known understanding of quadrilateral deflation.

1 Introduction

A *deflation* of a simple planar polygon is the operation of reflecting a subchain of the polygon through the line connecting its endpoints such that (1) the line intersects the polygon only at those two polygon vertices, (2) the resulting polygon is simple (does not self-intersect), and (3) the reflected subchain lies inside the hull of the resulting polygon. A polygon is *deflated* if it does not admit any deflations, i.e., every pair of polygon vertices either defines a line intersecting the polygon elsewhere or results in a nonsimple polygon after reflection.

Deflation is the inverse operation of pocket flipping. Given a nonconvex simple planar polygon, a *pocket* is a maximal connected region exterior to the polygon and interior to its convex hull. Such a pocket is bounded by one edge of the convex hull of the polygon, called the *pocket lid*, and a subchain of the polygon, called the *pocket subchain*. A *pocket flip* (or simply *flip*) is the operation of reflecting the pocket subchain through the line extending the pocket lid. The

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result is a new, simple polygon of larger area with the same edge lengths as the original polygon. A convex polygon has no pocket and hence admits no flip.

In 1935, Erdős conjectured that every nonconvex polygon convexifies after a finite number of flips [5]. Four years later, Nagy [2] claimed a proof of Erdős's conjecture. Recently, Demaine et al. [3,4] uncovered a flaw in Nagy's argument, as well as other claimed proofs, but fortunately correct proofs remain.

In the same spirit of finite flips, Wegner conjectured in 1993 that any polygon becomes deflated after a finite number of deflations [8]. Eight years later, Fevens et al. [6] disproved Wegner's conjecture by demonstrating a family of quadrilaterals that admit an infinite number of deflations. They left an open problem of characterizing which polygons deflate infinitely. Ballinger [1] closed the problem for quadrilaterals by proving that all infinitely deflating simple quadrilaterals are in the family defined by Fevens et al. [6].

This paper attempts to advance the understanding of deflating n-gons beyond $n = 4$. We prove that every pentagon admitting an infinite number of deflations induces an infinitely deflating quadrilateral on four of its vertices. Then we show our main result: unlike quadrilaterals, every pentagon can be deflated after finitely many (well-chosen) deflations. Finally, we construct a family of infinitely deflatable hexagons that induce no infinitely deflating quadrilateral; however, they deflate infinitely only according to a specific deflation sequence.

2 Definitions and Notation

Let $P = \langle v_0, v_1, \ldots, v_{n-1}\rangle$ be a polygon together with a clockwise ordering of its vertices. Let $P^k = \langle v_0^k, v_1^k, \ldots, v_{n-1}^k\rangle$ denote the polygon after k arbitrary deflations, and P^* denote the limit of P^k, when it exists, having vertices v_i^*. Thus, the initial polygon $P = P^0$. The *turn angle* of a vertex v_i is the signed angle $\theta \in (-180°, 180°]$ between the two vectors $v_i - v_{i-1}$ and $v_i - v_{i+1}$. A vertex of a polygon is *straight* if the angle between its incident edges is 180°, i.e., forming a turn angle of 0°. A *flat polygon* is a polygon with all its vertices collinear. A *hairpin* vertex v_i is a vertex whose incident edges overlap each other, i.e., forming a turn angle of 180°.[1] A polygon vertex is *sharpened* when its absolute turn angle decreases.

3 Deflation in General

In this section, we prove general properties about deflation of arbitrary simple polygons. Our first few lemmata are fairly straightforward, while the last lemma is quite intricate and central to our later arguments.

Lemma 1. *Deflation only sharpens angles.*

This result follows from an analogous result for pocket flips, which only flatten angles (see, e.g., [7]). For completeness, we provide a proof.

[1] This terminology was introduced in [4] where it plays a role in pocket flips.

Proof. Consider the chain $v_i, v_{i+1}, \ldots, v_j$ that is to be deflated across line ℓ passing through v_i and v_j. The two vertices v_{i+1} and v_{i-1} are on different sides of ℓ. After deflating the chain $v_i, v_{i+1}, \ldots, v_j$, v_{i+1} is reflected across ℓ and its reflection is v'_{i+1}. Consider the two triangles $v_{i-1}v_iv_{i+1}$ and $v_{i-1}v_iv'_{i+1}$. The sides v_iv_{i+1} and $v_iv'_{i+1}$ have the same length (deflation preserves edge lengths). Because v_{i+1} and v'_{i+1} have the same distance from ℓ, and v'_{i+1} is on the same side of ℓ as v_{i-1}, then the length of $v_{i-1}v'_{i+1}$ is less than the length of $v_{i-1}v_{i+1}$. This implies that the angle opposite edge $v_{i-1}v'_{i+1}$ is smaller than the angle opposite edge $v_{i-1}v_{i+1}$ (by Euclid's Propositions I.24 and I.25). Thus, the angle at vertex v_i sharpens. □

Corollary 1. *Any n-gon with no straight vertices will continue to have no straight vertices after deflation, even in an accumulation point P^*.*

Lemma 2. *In any infinite deflation sequence $P^0, P^1, P^2, \ldots$, the absolute turn angle $|\tau_i|$ at any vertex v_i has a (unique) limit $|\tau_i^*|$.*

Proof. By Lemma 1, $|\tau_i|$ never increases. Also, $|\tau_i|$ is bounded in the range $[0, 360°)$. Hence, $|\tau_i|$ has a limit $|\tau_i^*|$. □

Corollary 2. *In any infinite deflation sequence $P^0, P^1, P^2, \ldots$, v_i^* is a hairpin vertex in some accumulation point P^* if and only if v_i^* is a hairpin vertex in all accumulation points P^*.*

Lemma 3. *Any n-gon with n odd and having no straight vertices cannot flatten in an accumulation point of an infinite deflation sequence.*

Proof. Suppose for contradiction that there is a flat accumulation point. By Lemma 1, this limit has no straight vertices, so all vertices must be hairpins. Hence, the edges of the polygon alternate left and right. Because the edges form a closed cycle, when the first edge goes left, the last edge has to come back right in order to close the cycle. Hence, the number of edges of a flat polygon must be even. Therefore, any polygon with an odd number of vertices cannot flatten. □

Lemma 4. *For any infinite deflation sequence $P^0, P^1, P^2, \ldots$, there is a subchain $v_i, v_{i+1}, \ldots, v_j$ (where $j - i \geq 2$) that is the pocket chain of infinitely many deflations.*

Proof. Label each time t with (i, j) if the t-th deflation has pocket chain $v_i, v_{i+1}, \ldots, v_j$ (with $j - i \geq 2$). There are only finitely many labels, but infinitely many deflations, so some label must appear infinitely often. This label (i, j) corresponds to the desired subchain $v_i, v_{i+1}, \ldots, v_j$. □

We conclude this section with a challenging lemma showing that infinitely deflating pockets flatten:

Lemma 5. *Assume $P = P^0$ has no straight vertices. If P^* is an accumulation point of the infinite deflation sequence $P^0, P^1, P^2, \ldots$, and subchain $v_i, v_{i+1}, \ldots, v_j$ (where $j - i \geq 2$) is the pocket chain of infinitely many deflations, then*

$v_i^*, v_{i+1}^*, \ldots, v_j^*$ are collinear and $v_{i+1}^*, \ldots, v_{j-1}^*$ are hairpin vertices. Furthermore, if $v_{i+1}^*, \ldots, v_{j-1}^*$ extends beyond v_j^*, then v_j^* is a hairpin vertex; and if $v_{i+1}^*, \ldots, v_{j-1}^*$ extends beyond v_i^*, then v_i^* is a hairpin vertex. In particular, if $j-i=2$, then either v_i^* or v_j^* is a hairpin vertex.

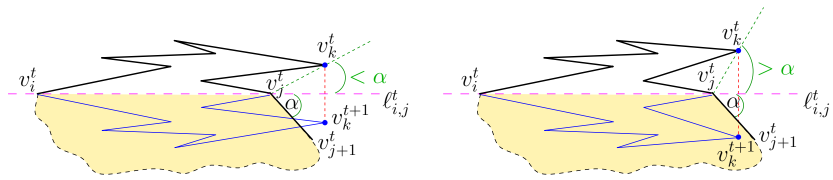
Proof. Because $P^0 \supseteq P^1 \supseteq P^2 \supseteq \cdots$, we have $\mathrm{hull}(P^0) \supseteq \mathrm{hull}(P^1) \supseteq \mathrm{hull}(P^2) \supseteq \cdots$, and in particular $\mathrm{area}(\mathrm{hull}(P^0)) \geq \mathrm{area}(\mathrm{hull}(P^1)) \geq \mathrm{area}(\mathrm{hull}(P^2)) \geq \cdots \geq 0$. Thus, $\sum_{t=1}^{\infty}[\mathrm{area}(\mathrm{hull}(P^t)) - \mathrm{area}(\mathrm{hull}(P^{t-1}))] \leq \mathrm{area}(\mathrm{hull}(P^0))$, so $\mathrm{area}(\mathrm{hull}(P^t)) - \mathrm{area}(\mathrm{hull}(P^{t-1})) \to 0$ as $t \to \infty$. Hence, for any $\epsilon > 0$, there is a time T_ϵ such that, for all $t \geq T_\epsilon$, $\mathrm{area}(\mathrm{hull}(P^t)) - \mathrm{area}(\mathrm{hull}(P^{t-1})) \leq \epsilon$. As a consequence, for all $t \geq T_\epsilon$, $\mathrm{hull}(P^{t-1}) \subseteq \mathrm{hull}(P^t) \oplus D_{\epsilon/\ell}$ where $\oplus$ denotes Minkowski sum, D_x denotes a disk of radius x, and ℓ is the length of the longest edge in P, which is a lower bound on the perimeter of $\mathrm{hull}(P^t)$.

Let $t_1, t_2, \ldots$ denote the infinite subsequence of deflations that use $v_i, v_{i+1}, \ldots, v_j$ as the pocket subchain, where P^{t_r} is the polygon immediately after the rth deflation of the pocket chain $v_i, v_{i+1}, \ldots, v_j$. Consider any vertex v_k with $i < k < j$. If $t_r \geq T_\epsilon$, then $v_k^{t_r-1} \in \mathrm{hull}(P^{t_r}) \oplus D_{\epsilon/\ell}$. Also, $v_k^{t_r-1}$ is in the halfplane H_r exterior to the line of support of P^{t_r} through $v_i^{t_r}$ and $v_j^{t_r}$. Now, the region $(\mathrm{hull}(P^{t_r}) \oplus D_{\epsilon/\ell}) \cap H_r$ converges to a subset of the line $\ell_{i,j}^{t_r}$ through $v_i^{t_r}$ and $v_j^{t_r}$ as $\epsilon \to 0$ while keeping $t_r \geq T_\epsilon$. Thus, for any accumulation point P^*, v_k^* is collinear with v_i^* and v_j^*, for all $i < k < j$. In other words, $v_{i+1}^*, \ldots, v_{j-1}^*$ lie on the line $\ell_{i,j}^*$ through v_i^* and v_j^*. By Corollary 1, $v_{i+1}^*, \ldots, v_{j-1}^*$ are not straight, so they must be hairpins.

By Lemma 2, the absolute turn angle $|\tau_j|$ of vertex v_j has a limit $|\tau_j^*|$. If $|\tau_j^*| > 0$ (i.e., v_j^* is not a hairpin in all limit points P^*), then by Lemma 1, $|\tau_j^t| \geq |\tau_j^*| > 0$. For sufficiently large t, v_{j-1}^t approaches the line $\ell_{i,j}^t$. To form the absolute turn angle $|\tau_j^t| \geq |\tau_j^*| > 0$ at v_j, v_{j+1}^t must eventually be bounded away from the line $\ell_{i,j}^t$: after some time T, the minimum of the two angles between $v_j^t v_{j+1}^t$ and $\ell_{i,j}^t$ must be bounded below by some $\alpha > 0$. Now suppose that some $v_k^{t_r-1}$ were to extend beyond $v_j^{t_r-1}$ in the projection onto the line $\ell_{i,j}^{t_r-1}$ for some $t_r - 1 > T$. As illustrated in Figure 1, for the deflation of the chain $v_i^{t_r-1}, v_{i+1}^{t_r-1}, \ldots, v_j^{t_r-1}$ to not cause the next polygon P^{t_r} to self-intersect, the minimum of the two angles between $v_j^{t_r-1} v_k^{t_r-1}$ and $\ell_{i,j}^{t_r-1}$ must also be at least α.

But this is impossible for sufficiently large t, because v_k^t accumulates on the line $\ell_{i,j}^t$. Hence, in fact, v_k^t must not extend beyond v_j^t in the $\ell_{i,j}^t$ projection for sufficiently large t. In other words, when v_j^* is not a hairpin, each v_k^* must not extend beyond v_j^* on the line $\ell_{i,j}^*$. A symmetric argument handles the case when v_i^* is not a hairpin.

Finally, suppose that $j - i = 2$. In this case, because $v_{i+1}^* = v_{j-1}^*$ is a hairpin, it must extend beyond one of its neighbors, v_i^* or v_j^*. By the argument above, in the first case, v_i^* must be a hairpin, and in the second case, v_j^* must be a hairpin. Thus, as desired, either v_i^* or v_j^* must be a hairpin. □



(a) The angle between $v_k^t v_j^t$ and $\ell_{i,j}^t$ is less than α, hence in the next deflation step the chain $v_i^t \dots v_j^t$ will intersect the polygon.

(b) The angle between $v_k^t v_j^t$ and $\ell_{i,j}^t$ is greater than α, so the polygon will not self-intersect in the next deflation step.

Fig. 1. Because v_j^t is not a hairpin, the minimum angle α between $v_j^t v_{j+1}^t$ and $\ell_{i,j}^t$ is strictly positive. If any vertex v_k^t of the chain $v_i^t, v_{i+1}^t, \dots, v_j^t$ extends beyond v_j^t, then the minimum angle between $v_k^t v_j^t$ and $\ell_{i,j}^t$ must be at least α for the next deflation step P^{t+1} to not self-intersect. The dotted curve represents the rest of the polygon chain and the shaded area is the polygon interior below line $\ell_{i,j}^t$.

4 Deflating Quadrilaterals

We briefly review facts about quadrilateral deflation proved by Fevens et al. [6] and Ballinger [1]. For completeness, we also show how to prove these results using, in particular, our new Lemma 5.

Lemma 6. [1] *Any accumulation point of an infinite deflation sequence of a quadrilateral is flat and has no straight vertices.*

Proof. First we argue that all quadrilaterals $P^1, P^2, \dots$ (excluding the initial quadrilateral P^0) have no straight vertices. Because deflations are the inverse of pocket flips, and pocket flips do not exist for convex polygons, deflation always results in a nonconvex polygon. Thus all quadrilaterals P^t with $t > 0$ must be nonconvex. Hence no P^t with $t > 0$ can have a straight vertex, because then it would lie along an edge of the triangle of the other three vertices, making the quadrilateral convex. By Corollary 1, there are also no straight vertices in any accumulation point P^*.

By Lemma 4, there is a subchain $v_i, v_{i+1}, \dots, v_j$, where $j - i \geq 2$, that is the pocket chain of infinitely many deflations. In fact, $j - i$ must equal 2, because reflecting a longer (4-vertex) pocket chain would not change the polygon. Applying Lemma 5 to $P^1, P^2, \dots$ (where there are no straight vertices), for any accumulation point P^*, v_{i+1}^* is a hairpin and either v_i^* or $v_j^* = v_{i+2}^*$ is a hairpin. Hairpin v_{i+1}^* implies that v_i^*, v_{i+1}^*, v_{i+2}^* are collinear, while hairpin v_i^* or v_{i+2}^* implies that the remaining vertex $v_{i+3}^* = v_{i-1}^*$ lie on that same line. Therefore, any accumulation point P^* is flat. □

Theorem 1. [6,1] *A simple quadrilateral with side lengths $\ell_1, \ell_2, \ell_3, \ell_4$ is infinitely deflatable if and only if*

1. *opposite edges sum equally, i.e., $\ell_1 + \ell_3 = \ell_2 + \ell_4$; and*
2. *adjacent edges differ, i.e., $\ell_1 \neq \ell_2$, $\ell_2 \neq \ell_3$, $\ell_3 \neq \ell_4$, $\ell_4 \neq \ell_1$.*

Furthermore, every such infinitely deflatable quadrilateral deflates infinitely independent of the choice of deflation sequence.

Proof. Fevens et al. [6] proved that every quadrilateral satisfying the two conditions on its edge lengths is infinitely deflatable, no matter which deflation sequence we make. Thus the two conditions are sufficient for infinite deflation.

To see that the first condition is necessary, we use Lemma 6. Because deflation preserves edge lengths, so do accumulation points of an infinite deflation sequence, so the flat limit configuration from Lemma 6 is a flat configuration of the edge lengths $\ell_1, \ell_2, \ell_3, \ell_4$. By a suitable rotation, we may arrange that the flat configuration lies along the x axis. By Lemma 6, no vertex is straight, so every vertex must be a hairpin. Thus, during a traversal of the polygon boundary, the edges alternate between going left ℓ_i and going right ℓ_i. At the end of the traversal, we must end up where we started. Therefore, $\pm(\ell_1 - \ell_2 + \ell_3 - \ell_4) = 0$, i.e., $\ell_1 + \ell_3 = \ell_2 + \ell_4$.

To see that the second condition is necessary, suppose for contradiction that $\ell_1 = \ell_2$ (the other contrary cases are symmetric). By the first condition, $\ell_1 + \ell_3 = \ell_2 + \ell_4$, so $\ell_3 = \ell_4$. Thus, the polygon is a kite, having two pairs of adjacent equal sides. (Also, all four sides might be equal.) Every kite has a chord that is a line of reflectional symmetry. No kite can deflate along this line, because such a deflation would cause edges to overlap with their reflections. If a kite is convex, it may deflate along its other chord, but then it becomes nonconvex, so it can be deflated only along its line of reflectional symmetry. Therefore, a kite can be deflated at most once, so any infinitely deflatable quadrilateral must have $\ell_1 \neq \ell_2$ and symmetrically $\ell_1 \neq \ell_2$, $\ell_2 \neq \ell_3$, $\ell_3 \neq \ell_4$, and $\ell_4 \neq \ell_1$. □

5 Deflating Pentagons

First we observe that the pentagon problem is relatively simple if we allow a straight vertex: we can subdivide the long edge of an infinitely deflating quadrilateral.

Theorem 2. *There is a simple pentagon with a straight vertex that deflates infinitely for all deflation sequences, exactly like the quadrilateral on the nonflat vertices.*

Proof. See Figure 2. We start with an infinitely deflating quadrilateral $\langle v_0, v_1, v_2, v_3\rangle$ according to Theorem 1, and add a straight vertex v_4 along the edge v_3v_0. As long as we never deflate along a line passing through the straight vertex v_4, the deflations act exactly like the quadrilateral, and thus continue infinitely no matter which deflation sequence we choose. To achieve this property, we set the length of segment v_3v_0 to 1, with v_4 at the midpoint; we set the lengths of edges v_0v_1 and v_2v_3 to 2/3; and we set the length of edge v_1v_2 to 1/3. Then we deflate the quadrilateral until the vertices are so close to being hairpins that v_4 cannot see the nonadjacent convex vertex and the line through v_4 and the reflex vertex intersects the pentagon at another point. Thus no line of deflation passes through v_4, so we maintain infinite deflation as in the induced quadrilateral. □

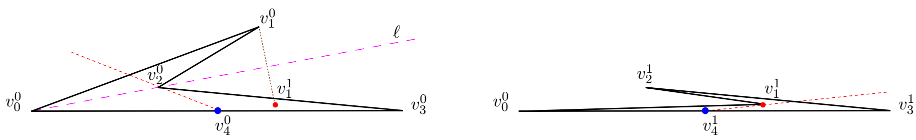


Fig. 2. An infinitely deflatable pentagon that induces an infinitely deflatable quadrilateral (left) and its configuration after the first deflation (right)

Finally we show that any infinitely deflating pentagon induces an infinitely deflating quadrilateral.

Theorem 3. *Every simple pentagon with no straight vertices can be deflated by a finite sequence of (well-chosen) deflations. Furthermore, any infinite deflation sequence in such a pentagon induces an infinitely deflating quadrilateral.*

Proof. Let P be a pentagon with no straight vertices, and assume for the sake of contradiction that P deflates infinitely. Consider any accumulation point P^* of an infinite deflation sequence $P^0, P^1, P^2, \ldots$. By Lemma 4, there is an infinitely deflating pocket chain, say $v_0, v_1, \ldots, v_j$, where $j \geq 2$. By Lemma 5, $v_1^*, \ldots, v_{j-1}^*$ are hairpin vertices. Because the pentagon has only five vertices, $j \leq 4$. In fact, $j \leq 3$: if $j = 4$, this pocket chain would encompass all five vertices, making P^* collinear, which contradicts Lemma 3. If $j = 3$, then v_1^* and v_2^* are hairpins. If $j = 2$, then by Lemma 5, either v_0^* or v_2^* must be a hairpin; assume by symmetry that it is v_2^*. Thus, in this case, again v_1^* and v_2^* are hairpins. Hence, in all cases, v_1^* and v_2^* are hairpins, so $v_0^*, v_1^*, v_2^*, v_3^*$ are collinear, while by Lemma 3 the fifth vertex v_4^* must be off this line. In particular, v_0^*, v_3^*, and v_4^* are not hairpins.

By Lemma 5, any infinitely deflating chain is flat in the accumulation point P^*, so the only possible infinitely deflating chains are v_0, v_1, v_2; v_1, v_2, v_3; and v_0, v_1, v_2, v_3 (Figure 3). Let T denote the time after which only these three chains deflate. Thus, after time T, v_0, v_3, and v_4 stop moving, so in particular, v_4's angle and the length of the edge v_0v_3 take on their final values. Therefore, after time T, the vertices v_0, v_1, v_2, v_3 induce a quadrilateral that deflates infinitely, except that the chain v_0, v_1, v_2, v_3 might deflate. However, if at some time $t > T$ the chain $v_0^t, v_1^t, v_2^t, v_3^t$ deflates along the line through v_0^t and v_3^t into the triangle $v_0^t v_3^t v_4^t$, then the convex hull of P^{t+1} is $v_0^{t+1} v_3^{t+1} v_4^{t+1}$, which is fixed, so no further deflations are possible, resulting in a finite deflation sequence. Therefore the infinite deflation sequence can deflate only the chains v_0, v_1, v_2 and v_1, v_2, v_3 after time T. Indeed, after time T the sequence must alternate between deflating these two chains, because no chain can deflate twice in a row.

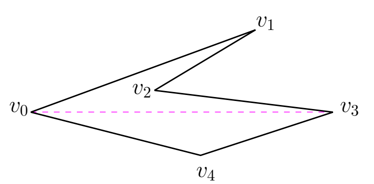


Fig. 3. A pentagon with an induced infinitely deflating quadrilateral, which is infinitely deflatable if we deflate only the subchain v_0, v_1, v_2, v_3

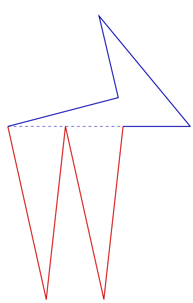
We claim that v_1^* and v_2^* lie along the segment $v_0^*v_3^*$. Because v_1^* and v_2^* are hairpins, the only other possibilities are that v_1^* extends beyond v_3^* or that v_2^* extends beyond v_0^*. If v_1^* extended beyond v_3^*, then applying Lemma 5 to v_1, v_2, v_3 would imply that v_3^* is a hairpin, which is a contradiction. Therefore, v_1^* must lie along the segment $v_0^*v_3^*$, and similarly v_2^* must lie along the segment $v_0^*v_3^*$. By Theorem 1, no two adjacent edges of the quadrilateral have the same length, so in fact v_1^* and v_2^* must be strictly interior to the segment $v_0^*v_3^*$. Hence, for sufficiently large $t > T$, $v_0^t, v_1^t, v_2^t, v_3^t$ are arbitrarily close to collinear with v_1^t and v_2^t projecting to the relative interior of segment $v_0^tv_3^t$. Also, v_1^t and v_2^t must be outside the triangle $v_0^tv_3^tv_4^t$ because the quadrilateral v_0, v_1, v_2, v_3 remains deflatable. As a consequence, for sufficiently large $t > T$, we can deflate the chain $v_0^t, v_1^t, v_2^t, v_3^t$, which prevents all further deflations as argued above. Thus we obtain an alternate, finite deflation sequence. □

6 Larger Polygons and Well-Chosen Deflations

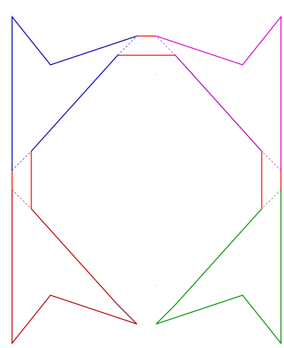
It is easy to construct n-gons with $n \geq 6$ that deflate infinitely, no matter which deflation sequence we choose. See Figure 4(a) for the idea of the construction. We can add any number of spikes to an infinitely deflating quadrilateral to obtain n-gons with $n \geq 6$ and even. For $n \geq 7$ and odd, we can shave off the tip of one of the spikes. Thus, $n = 5$ is the only value for which every n-gon with no straight vertices can be finitely deflated.

None of the infinitely deflating polygons of Figure 4 are particularly satisfying because their accumulation points are not flat. Are there any n-gons, $n > 4$, that have no straight vertices and always deflate infinitely to flat accumulation points?

If we require that the n-gon is infinitely deflatable to a flat accumulation point only for at least one deflation sequence, then we can construct such a hexagon by taking two infinitely deflating quadrilaterals v_0, v_1, v_2, v_3 and v_3, v_4, v_5, v_0



(a) An infinitely deflating octagon constructed by adding long spikes to an infinitely deflating quadrilateral.



(b) An infinitely deflating 18-gon constructed from four infinitely deflating quadrilaterals.

Fig. 4. Infinitely deflating polygons by combining infinitely deflating quadrilaterals

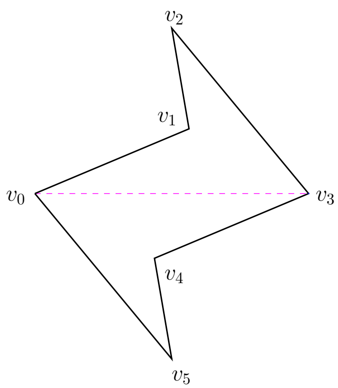


Fig. 5. An infinitely deflating hexagon constructed by joining two infinitely deflating quadrilaterals along their longest edge

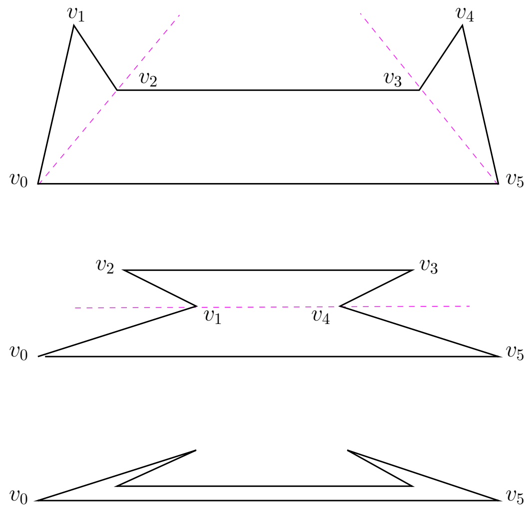


Fig. 6. A hexagon that deflates infinitely for a well-chosen deflation sequence but induces no infinitely deflating quadrilateral

(with their longest edge having the same length) and joining them along their longest edge; removing this edge will leave us with hexagon $v_0, v_1, v_2, v_3, v_4, v_5$. See Figure 5. This hexagon will deflate infinitely if we deflate only the two subchains v_0, v_1, v_2, v_3 and v_3, v_4, v_5, v_0 independently, and never deflate across the line through v_0 and v_3. This hexagon has an infinitely deflating quadrilateral as a subpolygon, and indeed its infinite deflation sequences are interleavings of the two such quadrilaterals.

Next we present a family of hexagons that deflate infinitely to a flat accumulation point for some deflation sequence but do not induce an infinitely deflating quadrilateral. Figure 6 shows an example.

Theorem 4. *A simple hexagon $H = \langle v_0, v_1, v_2, v_3, v_4, v_5 \rangle$ with side lengths $\ell_i = |v_{i-1}v_i|$ (where $v_6 = v_0$) has an infinite deflation sequence with flat accumulation points if it satisfies the following five properties:*

1. *opposite edges sum equally, i.e., $\ell_1 + \ell_3 + \ell_5 = \ell_2 + \ell_4 + \ell_6$;*
2. *adjacent edges differ, i.e., $\ell_1 \neq \ell_2$, $\ell_2 \neq \ell_3$, $\ell_3 \neq \ell_4$, $\ell_4 \neq \ell_5$, $\ell_5 \neq \ell_6$, $\ell_6 \neq \ell_1$;*
3. *$\frac{1}{2}\ell_1 < \ell_2 < \ell_1$;*
4. *$\ell_6 > 3\ell_1$; and*
5. *the hexagon is symmetric about the perpendicular bisector of the edge v_0v_5. (In particular, $\ell_1 = \ell_5$ and $\ell_2 = \ell_4$, and v_0v_5 is parallel to v_2v_3.)*

Proof. Consider a hexagon H satisfying the five properties. Assume by suitable rotation and reflection that v_0v_5 (and hence v_2v_3) is horizontal, v_0 is left of v_5, and v_2 (and hence v_3) is above the horizontal line through v_0 and v_5.

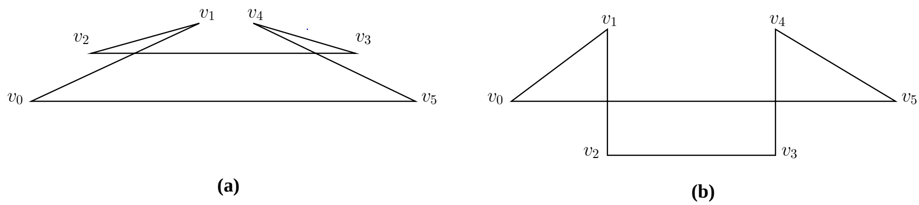


Fig. 7. The two possible configurations of H if it self-intersects

We argue that any such hexagon H is simple. Obviously, the parallel edges v_2v_3 and v_0v_5 do not cross. If v_0v_1 (and hence v_4v_5) intersects v_2v_3, as in Figure 7(a), then by the planar quadrilateral uncrossing lemma, $\ell_1+\ell_3 > \ell_2+|v_0v_3|$ and $\ell_5+|v_0v_3| > \ell_4+\ell_6$, which sum to $\ell_1+\ell_3+\ell_5+|v_0v_3| > \ell_2+\ell_4+\ell_6+|v_0v_3|$, contradicting Property 1. Similarly, if v_1v_2 (and hence v_3v_4) intersects v_0v_5, as in Figure 7(b) or its reflection, then $\ell_2+\ell_6 > \ell_1+|v_2v_5|$ and $\ell_4+|v_2v_5| > \ell_3+\ell_5$, which sum to $\ell_2+\ell_4+\ell_6+|v_2v_5| > \ell_1+\ell_3+\ell_5+|v_2v_5|$, contradicting Property 1. In projection onto the horizontal line through v_0v_5, v_1 can reach at most ℓ_1 to the right of v_0 and v_4 can reach at most $\ell_5 = \ell_1$ to the left of v_5. By Property 4, this travel is small enough that v_1 must be left of v_4. Thus, in particular, v_0v_1 cannot cross v_4v_5. If v_2 were right of v_3, then $|v_0v_2|+|v_3v_5| > \ell_3+\ell_6$, so by the triangle inequality, $\ell_1+\ell_2+\ell_4+\ell_5 > \ell_3+\ell_6$, so by Property 4, $\ell_1+\ell_2+\ell_4+\ell_5 > \ell_3+3\ell_1$, i.e., $\ell_2+\ell_4 > \ell_3+\ell_1$, so by Property 1, $\ell_6 < \ell_5$, contradicting Property 4. Hence, v_2 is left of v_3. Thus v_0, v_1, and v_2 are left of v_3, v_4, and v_5, so v_0v_1 and v_1v_2 cannot cross v_3v_4 or v_4v_5. Hence no pairs of edges can cross. Property 2, together with Properties 1 and 5, forbids edges from overlapping and forbids nonadjacent edges from touching. Therefore H must be simple.

Next we claim that v_1 (and hence v_4), like v_2 and v_3, is above the horizontal line through v_0v_5, implying that v_0 (and hence v_5) is convex. Because $\ell_1 = \ell_5$ and $\ell_2 = \ell_4$, Property 1 can be rewritten as $2\ell_1 + \ell_3 = 2\ell_2 + \ell_6$. By Property 3, $\ell_2 < \ell_1$, so $\ell_3 < \ell_6$. Thus v_2 is above and to the right of v_0. Because $\ell_2 < \ell_1$, if v_1 were not also above v_0, the edge v_1v_2 could not reach a point above and to the right of v_0 without crossing v_0v_5. But we showed that H is simple, so v_1 must in fact be above v_0.

Now we claim that the hexagon H deflates infinitely by repeating the following three-step sequence ad infinitum: first deflate across the line passing through v_0 and v_2, second across the line through v_3 and v_5, and third across the line through v_2 and v_3. Exactly where we begin this infinite sequence depends on the initial hexagon H: if v_2 (and hence v_3) is reflex, we start on the first step; otherwise, we start on the third step. In general, the first step will be executed when v_2 (and v_3) is reflex, the second step will be executed when just v_3 is reflex, and the third step will be executed when v_2 and v_3 are convex. We also maintain the invariant that the hexagon is symmetric about the perpendicular bisector of v_0v_5 (Property 5) after every execution of the second and third steps. We need to show that (1) no deflation step introduces crossings, and (2) every line of deflation intersects the hexagon only at the two vertices defining it.

We have already shown that the hexagon is simple after any execution of the second or third step, because then the hexagon satisfies Property 5. We can argue simplicity after the execution of the first step by comparing with the hexagon that was just before the first step and with the hexagon that will be just after the next second step. Therefore the hexagon is simple at all stages.

It remains to show that every line of deflation hits the hexagon boundary just at its two defining vertices. The argument for the first step, deflating across v_0v_2, is below. The argument for the second step is similar to simplicity: Properties 3 and 4 guarantee that v_1 is always left of v_3, and in this case v_1 is below the horizontal line through v_3, while v_0 is below and right of v_3, so the line through v_3 and v_5 cannot hit v_0v_1 or v_1v_2. The argument for the third step is easy: the line through v_1 and v_4 cannot hit any of the incident edges (v_0v_1, v_1v_2, v_3v_4, and v_4v_5), and by Property 5 the line is horizontal, so it cannot hit the two remaining horizontal edges (v_2v_3 and v_0v_5).

Finally we consider deflating across v_0v_2, where it suffices to prove that v_4 is to the right of the line from v_0 to v_2. Assume by suitable translation that vertex v_0 is at the origin, and let θ be the interior angle at v_0. Then v_1 has coordinates $\langle \ell_1 \cos\theta, \ell_1 \sin\theta \rangle$ and $v_4 = \langle \ell_6 - \ell_1 \cos\theta, \ell_1 \sin\theta \rangle$. The x coordinate of v_2 is $\frac{1}{2}\ell_6 - \frac{1}{2}\ell_3$, which by adding half of Property 1 is $\ell_1 - \ell_2$. Now consider the right triangle v_1v_2x, where x is the point below v_1 and horizontal with v_2. The hypotenuse is ℓ_2, and the horizontal edge has length $(\ell_1 - \ell_2) - \ell_1\cos\theta = \ell_1(1 - \cos\theta) - \ell_2$, so the vertical edge has length $\sqrt{\ell_2^2 - (\ell_1(1-\cos\theta) - \ell_2)^2}$. Thus, $v_2 = \langle \ell_1 - \ell_2, \ell_1 \sin\theta - \sqrt{\ell_2^2 - (\ell_1(1-\cos\theta)-\ell_2)^2} \rangle$. Note that, for v_2 to have a valid (noncomplex) solution, we must have $2\ell_2 > \ell_1$, which is part of Property 3.

Now, v_4 is to the right of the line from v_0 to v_2 if and only if the signed area of the triangle $v_0v_2v_4$ is negative. Thus we desire the following inequality:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_5 & y_5 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ \ell_1 - \ell_2 & \ell_1 \sin\theta - \sqrt{\ell_2^2 - (\ell_1(1-\cos\theta) - \ell_2)^2} & 1 \\ \ell_6 - \ell_1\cos\theta & \ell_1 \sin\theta & 1 \end{vmatrix} < 0.$$

After significant simplification, this inequality becomes

$$\ell_1(\cos\theta - 1)(\ell_1 - \ell_2)[\ell_1^2(1 + 3\cos\theta + 4\cos^2\theta) - \ell_6\ell_1(2 + 6\cos\theta) + 2\ell_6^2 - \ell_1\ell_2(1 + \cos\theta)] < 0.$$

Because θ is between 0 and π, and $\ell_2 < \ell_1$, this inequality is equivalent to

$$\ell_1^2(1 + 3\cos\theta + 4\cos^2\theta) - \ell_6\ell_1(2 + 6\cos\theta) + 2\ell_6^2 - \ell_1\ell_2(1 + \cos\theta) > 0.$$

Also, because $\ell_2 < \ell_1$, it is enough to show

$$\ell_1^2(2\cos\theta + 4\cos^2\theta) - \ell_6\ell_1(2 + 6\cos\theta) + 2\ell_6^2 > 0.$$

If $\ell_6 = \alpha\ell_1$, then the inequality becomes

$$(\cos\theta + 2\cos^2\theta) - \alpha(1 + 3\cos\theta) + \alpha^2 > 0.$$

The maximum lower bound on α that satisfies this inequality occurs at $\theta = 0$; in this case, we obtain $3 - 4\alpha + \alpha^2 = 0$, which has solution $\alpha = 3$. Therefore, Condition 4 that $\ell_6 > 3\ell_1$ suffices.

We can easily show that every accumulation point of our deflation sequence is flat: because each of the chains v_0, v_1, v_2; v_3, v_4, v_5; and v_1, v_2, v_3, v_4 deflate infinitely, then by Lemma 5, in every accumulation point, the vertices of each of the chains are collinear, forcing all six vertices to be collinear. □

7 Open Problems

It remains open whether there exist n-gons, $n \geq 6$, that have no straight vertices and deflate infinitely for every deflation sequence to flat accumulation points. Also, does every infinite deflation sequence have a (unique) limit? Our proofs would likely simplify if we knew there were only one accumulation point.

Is there an efficient algorithm to determine whether a given polygon P has an infinite deflation sequence? What about detecting whether all deflation sequences are infinite? Even given a (succinctly encoded) infinite sequence of deflations, can we efficiently determine whether the sequence is valid?

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Enumeration of Polyominoes, Polyiamonds and Polyhexes for Isohedral Tilings with Rotational Symmetry

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Abstract. We describe computer algorithms that can enumerate and display, for a given $n > 0$ (in theory, of any size), all n-ominoes, n-iamonds, and n-hexes that can tile the plane using only rotations; these sets necessarily contain all such tiles that are fundamental domains for p4, p3, and p6 isohedral tilings. We display the outputs for small values of n. This expands on earlier work [3].

1 Polyominoes, Polyiamonds, Polyhexes and Isohedral Tilings

Polyominoes, polyiamonds and polyhexes are among the simplest shapes for tiles and are easily produced by computer or by hand. A polyomino (or n-omino) is a planar tile made up of n congruent squares joined at their edges. A polyiamond (or n-iamond) is a planar tile made up of n congruent equilateral triangles joined at their edges. A polyhex (or n-hex) is a planar tile made up of n congruent regular hexagons joined at their edges.

There is a rich store of problems concerning these tiles. Most of the problems about these tiles are about their tiling properties. In this work, we focus on isohedral tilings of the plane by these tiles in which the tilings have 3-, 4-, or 6-fold rotational symmetry. An isohedral tiling of the plane is one in which congruent copies of a single tile fill the plane without gaps or overlaps, and the symmetry group of the tiling acts transitively on the tiles. In our discussion, we assume knowledge of the lattice of rotation centers of the symmetry groups **p3**, **p4**, and **p6** [1,2].

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In our previous article [3], we discussed a method to generate polyominoes that can tile the plane using only 90° rotations with centers on the boundaries of the tiles, and consequently we could enumerate, for a given n, all n-ominoes that are fundamental domains for **p4** isohedral tilings, that is, minimal regions which, when acted on by the symmetry group of the tiling, generate the whole tiling. In this work, we present a new algorithm for polyominoes and extend it to polyiamonds and polyhexes that can tile the plane using only 120° rotations, and to polyiamonds that can tile the plane using 60° and 120° rotations. As a result, we shall enumerate, for small values of n, all n-ominoes, n-iamonds and n-hexes that are fundamental domains for **p4**, **p3** or **p6** isohedral tilings. We use a shortened terminology for the tiles produced by our algorithms, and call them **p4** polyominoes, **p3** polyiamonds and polyhexes, and **p6** polyiamonds.

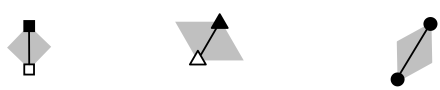
2 p4 Polyominoes

In our previous article [3], we introduced a line segment to represent a unit shape as shown in Fig. 1 (a)–(c) for **p4** polyominoes, **p3** polyiamonds, and **p6** polyiamonds, respectively. For **p4** polyominoes, the square is a unit shape (that is, every n-omino is a union of such congruent shapes) and so we could enumerate all polyominoes that are fundamental domains for **p4** isohedral tilings. On the other hand, for **p3** or **p6** polyiamonds, the diamonds are not unit shapes since unions of these can only produce n-iamonds for n even. This is a main reason why we could not enumerate all polyiamonds that are fundamental domains for **p3** or **p6** isohedral tilings.

In our new algorithm, instead of using a line segment to represent a unit shape, we use the unit shape itself to generate polyominoes, polyiamonds and polyhexes, i.e., a unit square, unit equilateral triangle, and unit regular hexagon, respectively.

We first consider the **p4** polyominoes. Underlying every edge-to-edge tiling of the plane by congruent n-ominoes is a lattice of the unit squares that make up each n-omino tile (Fig. 2). If the tiling is isohedral and has **p4** symmetry, then two 4-fold rotations acting on one tile can generate the whole tiling. If the n-omino tile is a fundamental domain for the tiling, then necessarily the centers of these 4-fold rotations are on the boundary of the tile [4] and hence they must lie on lattice points of the square lattice underlying the tiling.

To produce polyominoes that can tile the plane isohedrally by 4-fold rotations, we begin with a lattice of unit squares and place the first 4-fold center at a lattice



(a) (b) (c)

Fig. 1. Line segments in [3] represent a unit shape for **(a) p4** polyominoes, **(b) p3** polyiamonds, and **(c) p6** polyiamonds

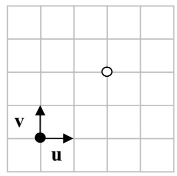


Fig. 2. Square lattice

point (the black circle in Fig. 2) and call this center the *origin*. Next, we choose a second lattice point to be the other 4-fold center (the white circle in Fig. 2); this can be written as $x\mathbf{u} + y\mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are the horizontal and vertical unit vectors of the lattice shown in Fig. 2, respectively, and $x, y = 0, 1, 2, \ldots$. We call this second center the *terminus*.

Successive rotations about these two 4-fold centers will generate the **p4** symmetry group. Moreover, the area of a fundamental domain for an isohedral tiling with this symmetry group is

$$S = \frac{x^2 + y^2}{2}$$

square units. Note that to produce an n-omino that is a fundamental domain, we must have $n = S$ and S must be an integer, i.e.,

$$n = 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, \ldots$$

The **p4** symmetry group generated by the two designated centers acts on all the unit squares in the lattice, partitioning them into n equivalence classes (orbits). Thus we can construct **p4** n-ominoes having area S by the following procedure, in which the unit shape is a unit square in the square lattice.

Procedure

1. Choose the first unit shape to have one vertex at the origin.
2. Choose a new unit shape so that it has an edge in common with at least one of the unit shapes already chosen, and so that the new unit shape belongs to an equivalence class different from all equivalence classes of unit shapes already chosen.
3. Repeat 2 until n unit shapes are chosen and one of the unit shapes has a vertex at the terminus.
4. Record the tile produced and continue 1–3 recursively, not allowing the same sequence of choices in 1–3 that produced any previously recorded tiles.
5. Eliminate equivalent tiles which yield the same tiling pattern. Here two tiles are equivalent if and only if they are congruent, with their rotation centers in the same positions on their boundaries. (It is necessary to permute the three types of centers (black, white, and gray) in order to compare the positions of rotation centers on two congruent tiles.)

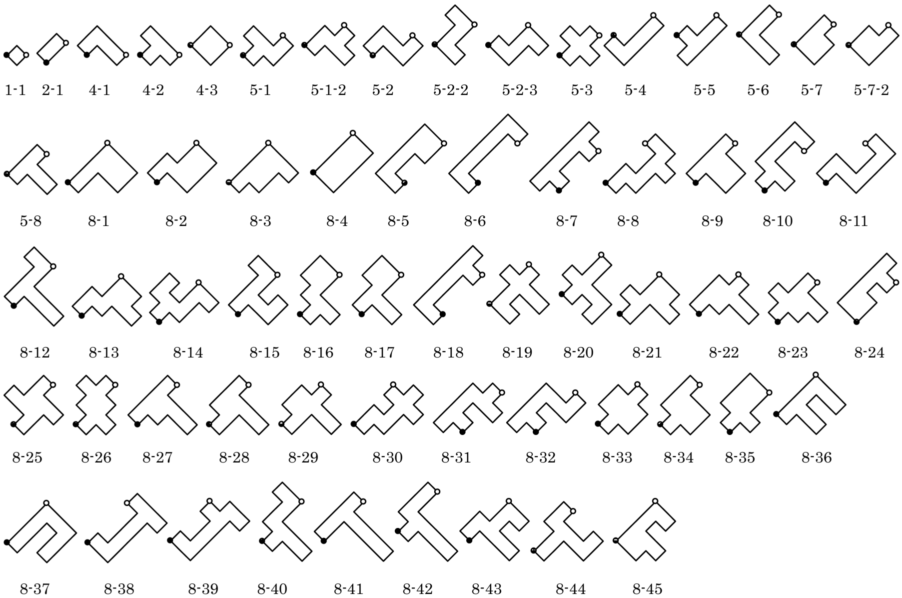


Fig. 3. List of **p4** n-ominoes for $n \leq 8$. The labels indicate n and the tile number for that n.

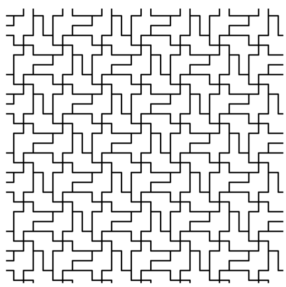


Fig. 4. An isohedral **p4** tiling with an 8-omino as fundamental domain

Fig. 3, reproduced from Fig. 4 in reference [3], is the result of the above procedure. The first number in the 2-number label identifies n and the second number identifies the number of the tile for that n. For example, 8-17 identifies the 17^{th} 8-omino listed by the computer. If the same polyomino can have different 4-fold rotation centers, a third number is added after the 2-number label. For example, 5-1 and 5-1-2 are the same polyomino but have differently-placed rotation centers.

For each tile, its corresponding **p4** tiling is obtained by using the black and white circles as 4-fold rotation centers. For example, tile 8-22 produces the tiling shown in Fig. 4. For $n > 8$ we obtain 80(82) **p4** 9-ominoes and 277(300) **p4** 10-ominoes, where the numbers in parentheses denote the number of polyominoes when those with differently-placed rotation centers are distinguished in counting.

When the tiling generated by rotations about the black and white centers marked on a tile has as its symmetry group the group generated by these rotations, the tile is a usually a fundamental domain for that tiling. When the tiling has a larger symmetry group, or the tile has symmetry, often only a fraction of the tile will be a fundamental domain. Our earlier article [3] gives some examples of tiles that are not fundamental domains for the tiling generated by rotations about the two centers on their boundaries, and other examples are given in section 5 of this article. However, the set of all tiles that are generated by our algorithm will contain all those that are fundamental domains for the tilings they generate.

3 p3 and p6 Polyiamonds

The algorithm introduced in the previous section is also applicable to polyiamonds and polyhexes. In this section, we consider **p3** and **p6** polyiamonds.

Any edge-to-edge tiling of the plane by congruent n-iamonds will have an underlying regular tiling by the unit equilateral triangles that make up the n-iamonds; this triangular lattice is shown in Fig. 5. If the tiling is isohedral and has **p3** or **p6** symmetry, then two rotations acting on a single tile can generate the whole tiling. If the polyiamond is a fundamental domain for the tiling, then in the **p3** case, two inequivalent 3-fold rotation centers will be on the tile's boundary (these produce a third inequivalent 3-fold center on the tile's boundary). In the **p6** case, the tile will have a 6-fold and a 3-fold center on its boundary. [4] These rotation centers will necessarily occur at lattice points of the underlying triangular lattice.

To produce a **p3** or **p6** polyiamond, we follow a sequence of steps similar to those in the last section, this time beginning with a lattice of unit triangles. For a **p3** n-iamond, we place the first 3-fold center (the black circle in Fig. 5) and call it the origin. For a **p6** n-iamond, we first place the 6-fold center (the black circle in Fig. 5) and call it the origin.

The terminus in both cases is a 3-fold center (the white circle in Fig. 5), which can be written as $x\mathbf{u} + y\mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are the vectors from the origin along the edges of the unit triangle as shown in Fig. 5 and $x, y = 0, 1, 2, \ldots$. Rotations

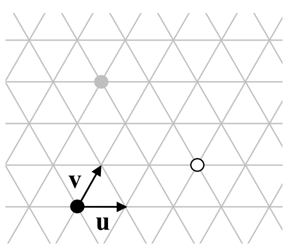


Fig. 5. Triangular lattice

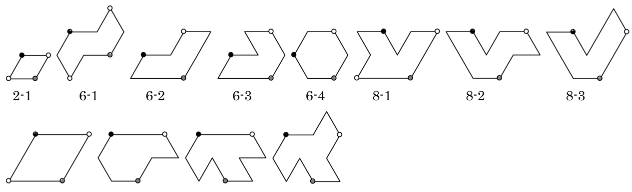


Fig. 6. List of **p3** n-iamonds for $n \leq 8$. The labels indicate n and the tile number for that n.

about these centers will generate the **p3** or **p6** symmetry groups, respectively. In the case of the **p3** symmetry group, the gray circle in Figure 5 indicates the location of the third kind of 3-fold center.

If we let the area of a unit triangle in the lattice be 1, then the area of a fundamental domain for a **p3** isohedral tiling with symmetry group generated by rotations about the designated centers is given by

$$S = 2(x^2 + y^2 + xy),$$

i.e., our n-iamonds must have $S = n = 2, 6, 8, 14, 18, 24, 26, 32, \ldots$, and the area of a fundamental domain for a **p6** isohedral tiling with symmetry group generated by rotations about the black and white circles is given by

$$S = x^2 + y^2 + xy,$$

i.e., our n-iamonds must have $S = n = 1, 3, 4, 7, 9, 12, 13, 16, 19, 21, \ldots$.

For given x, y and n, the unit triangles in the triangular lattice are partitioned into n equivalence classes by the action of the **p3** or **p6** symmetry groups, and so we can construct all **p3** and **p6** n-iamonds with area S using the 'procedure' in section 2, taking a unit equilateral triangle in the lattice as the unit shape.

Fig. 6 and Fig. 7 display the list of all **p3** n-iamonds for $n \leq 8$ and all **p6** n-iamonds for $n \leq 9$, respectively, obtained by the above procedure. The 2 or 3-number labeling system in Fig. 3 is also used in Fig. 6 and Fig. 7.

For each **p3** polyiamond, its corresponding **p3** tiling is obtained by rotating the tile 120° repeatedly about the inequivalent 3-fold centers on its boundary. For example, tile 6-3 in Fig. 6 produces the tiling shown in Fig. 8 (a). For $n > 8$ we obtain 288(306) **p3** 14-iamonds, where the number in parentheses denotes the number of polyiamonds when those with differently-placed rotation centers are distinguished in counting.

For each **p6** polyiamond, its corresponding **p6** tiling is obtained by repeated 60° and 120° rotations about the 6-fold and 3-fold centers on its boundary (shown as the black and white circles in Fig. 7, respectively). For example, tile 7-2 in Fig. 7 produces the tiling shown in Fig. 8 (b). For $n > 9$ we obtain 191(195) **p6** 12-iamonds, 472(504) **p6** 13-iamonds and 1487(1493) **p6** 16-iamonds, where the numbers in parentheses denote the number of polyiamonds when those with differently-placed rotation centers are distinguished in counting.

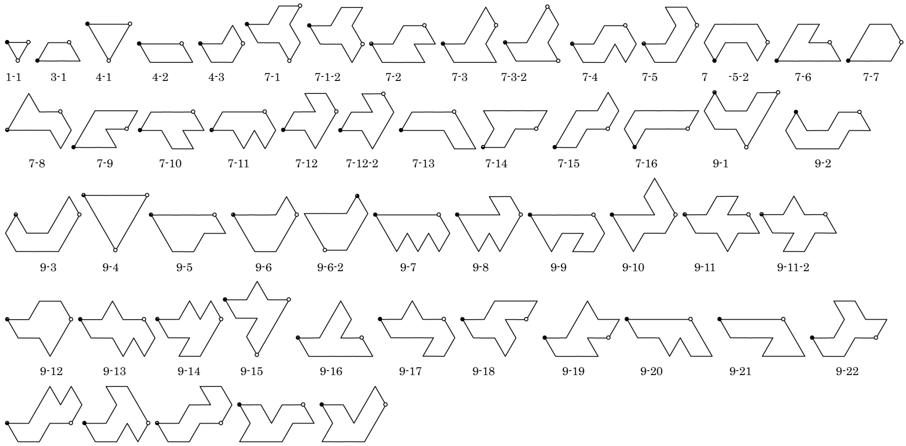


Fig. 7. List of **p6** n-iamonds for $n \leq 9$. The labels indicate n and the tile number for that n.

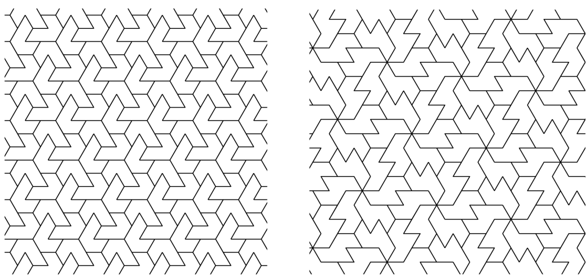


Fig. 8. (a) An isohedral **p3** tiling with a 6-iamond as fundamental domain and (b) an isohedral **p6** tiling with a 7-iamond as fundamental domain

4 p3 Polyhexes

Finally we consider polyhexes. We first note that no polyhex can be a fundamental domain for a **p6** isohedral tiling since a 6-fold center of the tiling can never occur on the boundary of a polyhex. Thus our final consideration is of **p3** polyhexes. To produce these, we begin with the regular hexagonal tiling in Fig. 9; this underlies every edge-to-edge tiling by polyhexes. As before (for **p3** polyiamonds), if the polyhexes we produce are to possibly be fundamental domains for **p3** isohedral tilings, then they must have two inequivalent 3-fold centers on their boundaries. These centers necessarily will be positioned at vertices of the underlying tiling by unit hexagons. We choose a vertex of one hexagon in our regular hexagonal tiling as origin (the black circle as shown in Fig. 9), then choose a second vertex of this tiling as terminus (shown as the white circle in Fig. 9). The

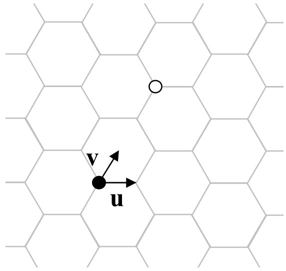


Fig. 9. Regular hexagonal pattern

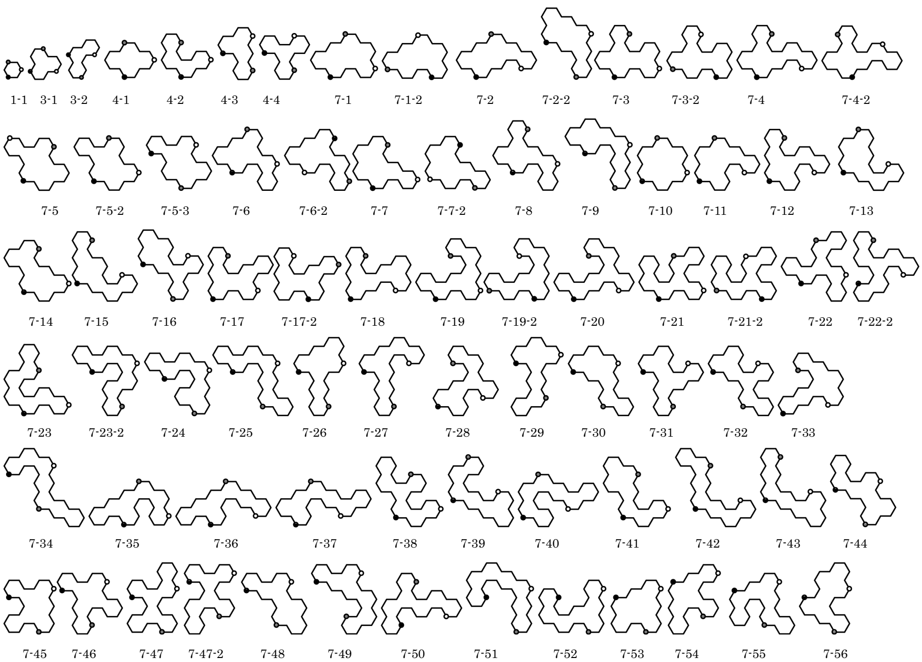


Fig. 10. List of all **p3** n-hexes for $n \leq 7$. The labels indicate n and the tile number for that n.

terminus can be written as $x\mathbf{u} + y\mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are vectors from the origin, the first directed along an edge and the second towards the center of the unit hexagon as shown in Fig. 9, with $x, y = 0, 1, 2, \dots$ and $x \neq y + 2 \pmod 3$.

Rotations of 120° about these two centers will generate the **p3** symmetry group, and the area of a fundamental domain for an isohedral tiling having this symmetry group is given by

$$S = \frac{x^2 + y^2 + xy}{3}$$

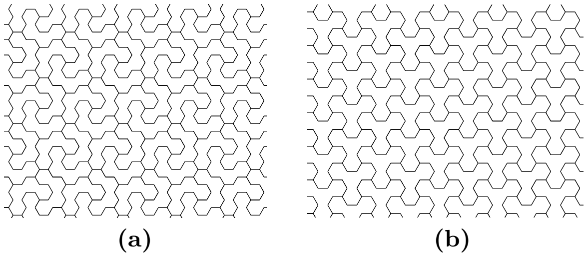


Fig. 11. (a) An isohedral **p3** tiling with a 4-hex as fundamental domain. The mirror symmetry of the tile is not a symmetry of the tiling. (b) An isohedral **p31m** tiling by a 4-hex in which the symmetries of the tile are symmetries of the tiling; a fundamental domain is 1/6 of the tile.

hexagonal units. Thus to be a fundamental domain, an n-hex must have area $n = S$, which must be an integer, i.e.,

$$n = 1, 3, 4, 7, 9, 12, 13, 16, 19, 21, \ldots$$

For such a given n, we can construct **p3** n-hexes by the 'procedure' in section 2 by taking the unit shape as a regular hexagon in the hexagonal tiling shown in Fig. 9.

Fig. 10 is the list of all **p3** n-hexes for $n \leq 7$ obtained by the above procedure. For each **p3**-polyhex, its corresponding **p3** tiling is obtained by 120° rotations about the three circles on its boundary (as shown in Fig. 10). For example, tiles 4-2 and 4-4 in Fig. 10 produce the tilings shown in Fig. 11 (a) and (b), respectively. For $n > 7$ we obtain 155(157) **p3** 9-hexes, where the number in parentheses denotes the number of polyhexes when those with differently-placed rotation centers are distinguished in counting. We observe that each **p3** n-hex is also a **p3** $6n$-polyiamond, distinguished by the property that each of its interior angles is either 120° or 240°. The list of all **p3** polyhexes is a small subset of the set of all **p3** polyiamonds.

5 Summary and Discussion

We have developed computer programs to obtain very fundamental and important tiles, that is, **p4** polyominoes, **p3** and **p6** polyiamonds, and **p3** polyhexes according to our improved algorithm. As a result, we could enumerate a list that contains all polyominoes, polyiamonds and polyhexes of a given size that are fundamental domains for **p4**, **p3** or **p6** isohedral tilings.

The running time of our algorithm is exponential in the length of the output; however it is polynomial to determine that a given tile has a particular property. The actual computer time, using an Intel Core Duo 1.83GHz CPU, to obtain the lists in Fig. 3, Fig. 6, Fig. 7 and Fig. 10 was less than two seconds each.

In listing these tiles, it would be interesting to distinguish which of the associated tilings have symmetries not contained in the rotation symmetry group that

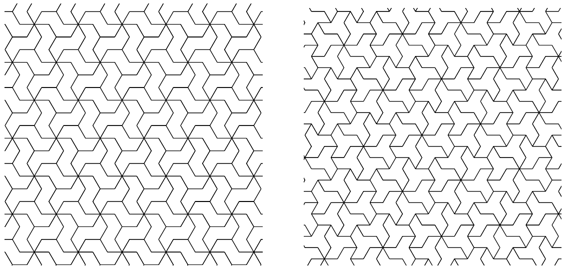


Fig. 12. (a) Isohedral **p6** tiling by a **p3** 6-iamond whose 180° rotation center is also a rotation center of the tiling. Half the tile is a fundamental domain. (b) Isohedral **p6** tiling with a **p6** 7-iamond as fundamental domain. The 120° rotation center of the tile is not a rotation center of the tiling.

produced the tiling. For **p4** tiles, the tiling might have symmetry group **p4g** or **p4m**; for **p3** tiles, the tiling might have symmetry group **p3m1**, **p31m**, **p6** or **p6m**; and for **p6** tiles, the tiling might have symmetry group **p6m**. For example, 6-1 in Fig. 6 and 4-4 in Fig. 10 are **p3** tiles, but the tilings produced by 120° rotations about their designated 3-fold centers have additional symmetry, as seen in Fig. 12 (a) and Fig. 11 (b), respectively. In both cases, the tiles themselves are symmetric (6-iamond 6-1 has 180° rotational symmetry and 4-hex 4-4 has 120° rotational symmetry and mirror symmetry) and their symmetries are present in their associated tilings. As a result, the tiles are not fundamental domains for their associated tilings; rather, a fraction of each tile is a fundamental domain with respect to the full symmetry group of the tiling.

A symmetric tile does not always produce a tiling having that same symmetry; an example is shown in Fig. 11 (a), in which the mirror symmetry of tile 4-2 in Fig. 10 is not a symmetry of the tiling. Here the tiling has **p3** symmetry group, and the 4-hex is a fundamental domain for the tiling. Another example is tile 7-1-2 in Fig. 7. This 7-iamond has 3-fold rotational symmetry, but the tile's 3-fold center is not a rotation center for the associated tiling shown in Fig. 12 (b), and the tile is a fundamental domain for the tiling.

These examples illustrate the fact that to distinguish tiles in our list that are not fundamental domains for their associated tilings is not as simple as identifying those tiles that are symmetric. Indeed, to accomplish this, we would have to investigate polyominoes, polyiamonds and polyhexes that are fundamental domains for **p4g**, **p4m**, **p3m1**, **p31m** and **p6m** from the beginning.

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Solvable Trees

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Abstract. A *state* of a simple graph G is an assignment of either a 0 or 1 to each of its vertices. For each vertex i of G, we define the *move* $[i]$ to be the switching of the state of vertex i, and each neighbor of i, from 0 to 1, or from 1 to 0. The given initial state of G is said to be *solvable* if a sequence of moves exists such that this state is transformed into the 0-state (all vertices have state 0.) If every initial state of G is solvable, we call G a *solvable graph*. We shall characterize here the solvable trees.

1 Introduction

The *lights out puzzle* is played on a rectangular grid, usually 5×5. The cells in the rectangular grid represent rooms. Initially, some cells are lighted and some are not. When the switch in a cell is toggled, the state of the cell, and that of each neighboring cell, is switched from lights on to lights off, or vice versa. If there is a way to toggle some switches until all lights are off, we say that the given initial state is solvable. R. Cowen and J. Kennedy [1] describe this game using Mathematica.

The usual lights out puzzle can be viewed as a game on a special graph called the planar grid. J. Goldwasser and W. Klostermeyer [3] considered playing the game on an arbitrary graph. Since a move switches the state of a vertex, and the state of every vertex adjacent to it, then we can only play the lights out puzzle on graphs without loops. We shall consider here the lights out puzzle played on a tree.

Definition 1. *A state s of a graph G is a function $s\colon V(G) \to Z_2 = \{0,1\}$.*

Let G be a graph of order n and $i \in V(G)$. We denote by s_i or $s(i)$ the image of i under s. We shall call s_i the *state of vertex* i. We will find it useful to denote s by the vector

$$s = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

For each vertex i we define the *move* $[i]$ to be the switching of the state of vertex i, as well as the state of each neighbor of i, from 0 to 1 or from 1 to 0.

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We may think of $[i]$ as a vector in Z_2^n where the j^{th} component is 1 if $j = i$ or j is a neighbor of i; else the j^{th} component is 0. Then the new state arising from move $[i]$ is just $s + [i]$, where addition is taken modulo 2. The sequence of moves $[v_1], [v_2], \ldots, [v_k]$ transforms s to the 0-state (lights out) if and only if

$$s + [v_1] + [v_2] + \cdots + [v_k] = 0$$

If $B(G)$ is the matrix whose columns are $[1], [2], \ldots, [n]$, then s can be transformed into the 0-state if and only if s is a linear combination of the columns of B. A state that can be transformed into the 0-state is called a *solvable state.* A graph is *solvable* if every initial state of the graph is solvable. Thus, every initial state is solvable if and only if the columns of $B(G)$ form a basis for Z_2^n. Let us observe that for any graph G, the solvable states are precisely the linear combinations of the columns of $B(G)$.

2 Preliminary Results

From the remark at the end of the preceding section, we obtain the following characterization of solvable graphs.

Theorem 1. *A graph G is solvable if and only if* $\det B(G) \not\equiv 0 \pmod 2$.

If G is a graph with components $G_1, G_2, \ldots, G_k$, then $\det B(G) = \prod \det B(G_i)$. Hence, G is solvable if and only if each G_i is solvable.

Let us observe that if $A(G)$ is the adjacency matrix of a graph G of order n, then $B(G) = A(G) + I_n$, where I_n is the identity matrix of order n.

The symbol P_n denotes a path of order n. In this paper we shall write $x_1 x_2 \cdots x_n$ to denote the path with vertices x_i and edges $x_i x_{i+1}$, $1 \le i < n$. The symbol $N(x)$ denotes the set of all vertices in a graph that are adjacent to x. Elements of $N(x)$ are called *neighbors of x*. The *degree* of a vertex x in a simple graph, denoted by $\deg(x)$, is the number $|N(x)|$. If $\deg(x) = 1$, we call x a *pendant vertex.* An edge of the form xx is called a *loop* and we say that there is a loop at x. If there is a loop at x, we define $\deg(x) = |N(x)| + 1$.

An important result, established in [2] for loopless graphs, also holds for graphs with loops. So we state and prove a more general result here. This is some sort of a reduction formula.

Lemma 1. *Let G be a graph (possibly with loops) and let x and y be vertices in G such that $N(x) \subseteq N(y)$. If G' is the graph obtained from G by deleting all the edges of the form yz, where $z \in N(x)$, then* $\det A(G') = \det A(G)$.

Proof. For simplicity, let 1, 2, $\ldots$, n be the vertices of G, and, without loss of generality, let $N(1) \subseteq N(2)$. We may assume further that $N(1) = \{p, p{+}1, \ldots, u\}$ and $N(2) = \{q, q+1, \ldots, v\}$ where $1 \le q \le p \le u \le v$. Consider the adjacency matrix $A(G)$. Subtract row 1 from row 2 and then subtract column 1 from column 2. The result is a new matrix which is easily seen to be the adjacency matrix of the graph obtained from G by deleting all edges of the form $2i$, where $i \in N(1)$. □

3 Graph Pruning Operations

In theory, given a graph G, we can decide whether or not it is solvable by applying Theorem 1. In practice, this is not easy to do if the graph involved is large in terms of the number of vertices or the number of edges. It would be nice if we can say something like "G is solvable if and only if H is solvable" where H is a smaller graph derived from G by an easy rule or process.

We now define type I, type II, and type III graph pruning operations. Please refer to Figure 1.

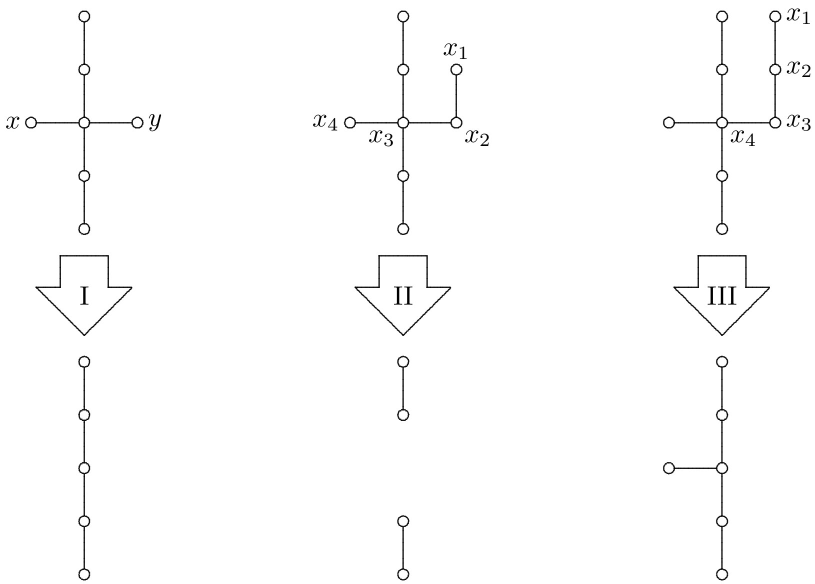

Fig. 1. The three types of graph pruning operations

I Let x and y be pendant vertices in a graph G having a common neighbor. Delete x and y to obtain a new graph G'.

II Let $P_4 = x_1x_2x_3x_4$ be a path in G such that x_1 and x_4 are pendant vertices, $\deg(x_2) = 2$, $\deg(x_3) > 2$. Delete x_1, x_2, x_3, x_4 to obtain a new graph G''.

III Let $P_4 = x_1x_2x_3x_4$ be a path in G such that x_1 is a pendant vertex, and $\deg(x_2) = \deg(x_3) = 2$. Delete x_1, x_2, x_3 to obtain a new graph G'''.

Theorem 2. *Let G be any graph. If G', G'', and G''' are graphs obtained from G by a type I, type II, or type III pruning operation respectively, then*

$$\det B(G') \equiv \det B(G) \pmod 2,$$
$$\det B(G'') = -\det B(G), \text{ and}$$
$$\det B(G''') = -\det B(G).$$

Proof. We observe first that $B(G)$ is the adjacency matrix of the graph $\widehat{G}$ obtained from G by putting a loop at every vertex of G.

(1) Let x and y be pendant vertices of G having common neighbor v. In $\widehat{G}$, $N_{\widehat{G}}(x) = \{x, v\} \subseteq N_{\widehat{G}}(v)$. Likewise, $N_{\widehat{G}}(y) = \{y, v\} \subseteq N_{\widehat{G}}(v)$. Without loss of generality, let the first three rows of $B(G)$ correspond to x, y, v respectively. Then $B(G)$ has the form

$$B(G) = \begin{bmatrix} 1\,0\,1\,0 \cdots & 0 \\ 0\,1\,1\,0 \cdots & 0 \\ 1\,1 & \\ 0\,0 \quad B(G') & \\ \vdots & \\ 0\,0 & \end{bmatrix}$$

where G' is the graph obtained from G by deleting x and y. By adding rows 1 and 2 to row 3 $\pmod 2$ we immediately get $\det B(G) \equiv \det B(G') \pmod 2$.

(2) Let $x_1x_2x_3x_4$ be a path in G where x_1 and x_4 are pendant vertices, $\deg x_2 = 2$ and $\deg x_3 > 2$. We consider these four vertices in $\widehat{G}$. By Lemma 1, we can remove the edges x_1x_2, x_2x_2, x_3x_4 and x_3x_3. In the resulting graph, the vertex x_2 is a pendant vertex. Aside from x_2, let $v_1, v_2, \ldots, v_k$ be the vertices adjacent to x_3. Note that $N(x_2) = \{x_3\} \subseteq N(v_i)$ for each i. By Lemma 1, we can remove the edges $v_1x_3, v_2x_3, \ldots, v_kx_3$. The adjacency matrix of the resulting graph has determinant still equal to $\det B(G)$. But this resulting graph consists of $\widehat{G''}$, two nonadjacent vertices, namely x_1 and x_4, each with a loop, and $P_2 = x_2x_3$. Since $\det B(P_1) = 1$ and $\det A(P_2) = -1$, then $\det B(G) = -\det A(\widehat{G''}) = -\det B(G'')$.

(3) Let $x_1x_2x_3x_4$ be a path in G with $\deg(x_1) = 1$, $\deg(x_2) = \deg(x_3) = 2$. Consider these vertices in $\widehat{G}$. By Lemma 1, we can remove the edges x_1x_2 and x_2x_2 since $N(x_1) \subseteq N(x_2)$. In the resulting graph, $N(x_2) = \{x_3\} \subseteq N(x_4)$. Therefore by Lemma 1, we can remove the edge x_3x_4. This graph now consists of one isolated vertex x_1 with a loop, one $P_2 = x_2x_3$ with a loop at x_3, and $\widehat{G'''}$. The graph x_1 with a loop has adjacency matrix with determinant 1. The graph x_2x_3 with a loop at x_3 has adjacency matrix with determinant -1. Therefore, $\det B(G) = -\det B(G''')$. □

The following Corollary is an easy consequence of Theorem 2.

Corollary 1. *Let G be a graph and let G^* be any graph obtained from G by a sequence of pruning operations. Then, G is solvable if and only if G^* is solvable.*

Definition 2. *A graph G is prune-irreducible if no pruning operation can be applied to G. If some pruning operation applies to G, we say that G is prune-reducible.*

A connected graph without cycles is called a *tree*. Some examples of prune-irreducible trees are shown in Figure 2.

The third example in Figure 2 belongs to a class of trees which we shall now define. For convenience, a vertex of degree greater than 2 will be called a *joint vertex*.

Fig. 2. Prune-irreducible trees

Definition 3. *A tree having at least one joint vertex is called a* hard tree *if it satisfies the following conditions.*

1. *Every joint vertex is adjacent to at most one pendant vertex.*
2. *Every joint vertex that is 2 units away from a pendant vertex is not adjacent to any pendant vertex.*
3. *Every pendant vertex is either 1 or 2 units away from the nearest joint vertex.*

The next Lemma determines all the prune-irreducible trees.

Lemma 2. *The only prune-irreducible trees are P_1, P_2 and the hard trees.*

Proof. It is easily verified that P_1, P_2, and hard trees are prune-irreducible. Let T be any tree which is prune-irreducible. Consider the following cases.

Case 1. T is a path of order n. If $n = 3$, then T is prune-reducible by type I operation. If $n > 3$, then T is prune-reducible by type III operation. Therefore, T is either P_1 or P_2.

Case 2. T is not a path. Then T contains some joint vertices. We claim that T is a hard tree. Suppose not. Then one of the three conditions in the definition of a hard tree does not hold. If (1) does not hold, there is a joint vertex adjacent to at least two pendant vertices. But then a type I pruning operation can be applied to T. If (2) does not hold, then there is a pendant vertex that is at least 3 units away from the nearest joint vertex. But then a type III pruning operation can be applied to T. If (3) does not hold, then there is a joint vertex 2 units away from a pendant vertex and which is also adjacent to a pendant vertex. But then a type II pruning operation can be applied to T. Hence, T is reducible. This is a contradiction and so T must be a hard tree. □

4 Solvable Trees

By *forest* we mean a graph whose components are trees. We now state and prove our main result.

Theorem 3. *A tree is solvable if and only if it can be pruned to a forest containing only isolated vertices.*

Proof. A tree is either prune-irreducible or can be pruned to a forest whose components are prune-irreducible. Since a graph is solvable if and only if each of its components is solvable, we need to determine only the solvable prune-irreducible trees. Obviously, P_1 is solvable while P_2 is not. Let T be a hard tree. Then there is at least one joint vertex, say v, that is 2 units away from each of two distinct pendant vertices, say x and y. Now, $B(T)$ is the adjacency matrix of the graph $\widehat{T}$ obtained by putting a loop at every vertex of T, Applying Lemma 1 to $\widehat{T}$, we may remove the edges xx', $x'x'$, yy', $y'y'$ where x' is the unique neighbor of x and y' is the unique neighbor of y. Then, x' and y' have a common neighbor set $\{v\}$. It follows that the adjacency matrix of the resulting graph has determinant 0. Consequently, $|B(T)| \equiv 0 \pmod 2$, and T is not solvable. Thus, among the irreducible trees, only P_1 is solvable. □

The last theorem is equivalent to the following.

Theorem 4. *A tree is not solvable if and only if it can be pruned to a forest with some component isomorphic to P_2 or to a hard tree.*

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Ramsey Numbers on a Union of Identical Stars Versus a Small Cycle*

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Abstract. The Ramsey number for a graph G versus a graph H, denoted by $R(G,H)$, is the smallest positive integer n such that for any graph F of order n, either F contains G as a subgraph or $\overline{F}$ contains H as a subgraph. In this paper, we investigate the Ramsey numbers for stars versus small cycle. We show that $R(S_8, C_4) = 10$ and $R(kS_{1+p}, C_4) = k(p+1)+1$ for $k \geq 2$ and $p \geq 3$.

Keywords: Ramsey number, star, cycle.

1 Introduction

Throughout this paper, all graphs are finite and simple. Let G be any graph with the vertex set $V(G)$ and the edge set $E(G)$. The graph $\overline{G}$, the *complement* of G, is obtained from the complete graph on $|V(G)|$ vertices by deleting the edges of G. A graph $F = (V', E')$ is a *subgraph* of G if $V' \subseteq V(G)$ and $E' \subseteq E(G)$. For $S \subseteq V(G)$, $G[S]$ represents the *subgraph induced* by S in G. For $v \in V(G)$ and $S \subset V(G)$, the *neighborhood* $N_S(v)$ is the set of vertices in S which are adjacent to v. Furthermore, we define $N_S[v] = N_S(v) \cup \{u\}$. If $S = V(G)$, then we use $N(v)$ and $N[v]$ instead of $N_{V(G)}(v)$ and $N_{V(G)}[v]$, respectively. The *degree* of a vertex v in G is denoted by $d_G(v)$. The *order* of G, denoted by $|G|$, is the number of its vertices. Let S_n be a *star* on n vertices and C_m be a *cycle* on m vertices. *Cocktail-party graph* H_s is the graph which is obtained by removing s disjoint edges from K_{2s}. We denote the *complete bipartite* whose partite sets are of order n and p by $K_{n,p}$. A *windmill* graph M_n is a graph on $2n+1$ vertices obtained from n disjoint triangles by identifying precisely one vertex of every triangle.

Given two graphs G and H, the *Ramsey number* $R(G,H)$ is defined as the smallest natural number n such that for any graph F on n vertices, either F contains G or $\overline{F}$ contains H. Chvátal and Harary [3] established a useful and general lower bound on the exact Ramsey numbers $R(G,H)$ as follows.

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Theorem A. [3] *Let G and H be two graphs (not necessarily different) with no isolated vertices. Then the following lower bound holds*:

$$R(G,H) \geq (\chi(G)-1)(n(H)-1)+1,$$

where $\chi(G)$ is the chromatic number of G and $n(H)$ is the number of vertices in the largest component of H.

This result of Chvátal and Harary has motivated various authors to determined the Ramsey numbers $R(G,H)$ for many combinations of graphs G and H, see the nice survey paper [10].

Corollary 1. $R(S_{1+p},C_4) \geq (\chi(C_4)-1)(V(S_{1+p})-1)+1 = p+1.$

Some results about the Ramsey numbers for stars versus cycle have obtained. For instance, Lawrence [7] showed that $R(S_{16},C_4)=20$ and

$$R(S_{1+n},C_m) = \begin{cases} m \text{ if } m \geq 2n, \\ 2n+1 \text{ if } m \text{ is odd and } \; m \leq 2n+1 \end{cases}$$

Parsons in [9] considered about the Ramsey numbers for S_{1+p} versus C_4 as presented in Theorem B.

Theorem B. (Parsons's upper bound [9]) For $p \geq 2$, $R(S_{1+p},C_4) \leq p+\sqrt{p}+1$.

Recently, Hasmawati et al. [4] and [5] proved that $R(S_6,C_4)=8$, and $R(S_6, K_{2,m})=13$ for $m=5$ or 6 respectively.

Recently, Baskoro et al. [1] determined the Ramsey numbers for multiple copies of a star versus a wheel and for a forest versus a complete graph. Their results are given in the following three theorems.

Theorem C. [1] *If m is odd and $5 \leq m \leq 2n-1$, then $R(kS_n,W_m) = 3n-2+(k-1)n$.*

Theorem D. [1] *For $n \geq 3$,*

$$R(kS_n,W_4) = \begin{cases} (k+1)n & \textit{if } n \textit{ is even} \;\; \textit{and} \;\; k \geq 2, \\ (k+1)n-1 & \textit{if } n \textit{ is odd and } k \geq 1. \end{cases}$$

Theorem F. [1] *Let $n_i \geq n_{i+1}$ for $i=1,2,\ldots,k-1$. If m is such that $n_i > (n_i-n_{i+1})(m-1)$ for every i, then $R(\bigcup_{i=1}^{k} T_{n_i},K_m) = R(T_{n_k},K_m)+\sum_{i=1}^{k-1} n_i$.*

In this paper, we study the Ramsey numbers for multiple copies of stars versus small cycle. We determine the Ramsey numbers $R(S_8,C_4)$ and $R(kS_{1+p},C_4)$ for $p \geq 3$ and $k \geq 2$.

2 Main Results

The results are presented in the next two theorems.

Theorem 1. $R(S_8,C_4)=10$.

Proof. Consider $F := H_4 \cup K_1$. Clearly, F has nine vertices and contains no S_8. Its complement is isomorphic with M_4. Thus it's clear that M_4 contains no C_4.

Hence, we have $R(S_8, C_4) \geq 10$. By Parsons's upper bound in [9], $R(S_8, C_4) \leq 8 + \sqrt{7}$. Therefore, we have $R(S_8, C_4) \leq 10$. Thus, $R(S_8, C_4) = 10$. $\square$

Lemma 1. *For $k \geq 2$ and $p \geq 3$, $R(kS_{1+p}, C_4) \geq k(p+1) + 1$*

Proof. Let $p \geq 3$ and $k \geq 2$. Consider $F := K_{k(p+1)-1} \cup K_1$. F has $k(p+1)$ vertices, however it contains no kS_{1+p}. It is easy to see that $\overline{F}$ is isomorphic with $K_{1,k(p+1)-1}$. So, $\overline{F}$ contains no C_4. Hence, $R(kS_{1+p}, C_4) \geq k(p+1) + 1$.

Theorem 2. *For $p \geq 3$, $R(2S_{1+p}, C_4) = 2(p+1) + 1$.*

Proof. Let F_1 be a graph of order $2(p+1)+1$ for $p \geq 3$. Suppose $\overline{F}_1$ contains no C_4. By Parsons's upper bound, we have $|F_1| \geq R(S_{1+p}, C_4)$ for $p \geq 3$. Thus $F_1 \supseteq S_{1+p}$. Let $V(S_{1+p}) = \{v_0, v_1, \ldots, v_p\}$ with center v_0. Write $A = F_1 \backslash S_{1+p}$ and $T = F_1[A]$. Thus $|T| = p + 2$. If there exists $v \in T$ with $d_T(v) \geq p$, then T contains S_{1+p}. Hence F_1 contains $2S_{1+p}$. Therefore, we assume that for every vertex $v \in T$, $d_T(v) \leq (p-1)$.

Let u be any vertex in T. Write $Q = T \backslash N_T[u]$. Clearly, $|Q| \geq 2$. Observe that if there exists $s \in F_1$ where $s \neq u$ which is not adjacent to at least two vertices in Q, then $\overline{F}[\{s, u\} \cup Q]$ will contains C_4, a contradiction. Hence, for the remaining of the proof we will use the following assumption.

Assumption 1. Every vertex $s \in F_1$, $s \neq u$ is not adjacent to at most one vertex in Q.

Let u be adjacent to at least $p - |N_T(u)|$ vertices in $S_{1+p} \backslash \{v_0\}$, call them $v_1, \ldots, v_{p-|N_T(u)|}$. Observe that $p - |N_T(u)| = |Q| - 1$. By Assumption 1, vertex v_0 is adjacent to at least $|Q| - 1$ vertices in Q, namely $q_1, \ldots, q_{p-|N_T(u)|}$. Then we have two new stars, namely S'_{1+p} and S''_{1+p}, where

$$V(S'_{1+p}) = (S_{1+p} \backslash \{v_1, \ldots, v_{p-|N_T(u)|}\}) \cup \{q_1, \ldots, q_{p-|N_T(u)|}\}$$

with v_0 as the center and

$$V(S''_{1+p}) = N_T[u] \cup \{v_1, \ldots, v_{p-|N_T(u)|}\}$$

with u as the center. Hence, we have $F_1 \supseteq 2S_{1+p}$.

Now, we assume that u is adjacent to at most $p - |N_T(u)| - 1$ vertices in $S_{1+p} \backslash \{v_0\}$. This means u is not adjacent to at least $|N_T(u)| + 1$ vertices in $S_{1+p} \backslash \{v_0\}$. Let $Y = \{y \in S_{1+p} \backslash \{v_0\} : yu \notin E(F_1)\}$. Then $|Y| \geq |N_T(u)| + 1 \geq 1$.

It will be shown that there is $y' \in Y$ so that y' is adjacent to all vertices in $N_T(u)$ (see Fig. 1).

Suppose for every $y \in Y$, there exists $r \in N_T(u)$ such that $yr \notin E(F_1)$. Since $|N_T(u)| < |Y|$, then there exists $r_0 \in N_T(u)$ so that r_0 is not adjacent to at least two vertices in Y, say y_1 and y_2. This implies, $\overline{F_1}[u, r_0, y_1, y_2]$ forms a C_4, a contradiction. Hence, there exists $y' \in Y$ so that y' is adjacent to all vertices in $N_T(u)$. Furthermore, by Assumption 1 we have that $|N_T(y')| \geq |N_T(u)| + |Q| - 1 = |T| - 2 = p$.

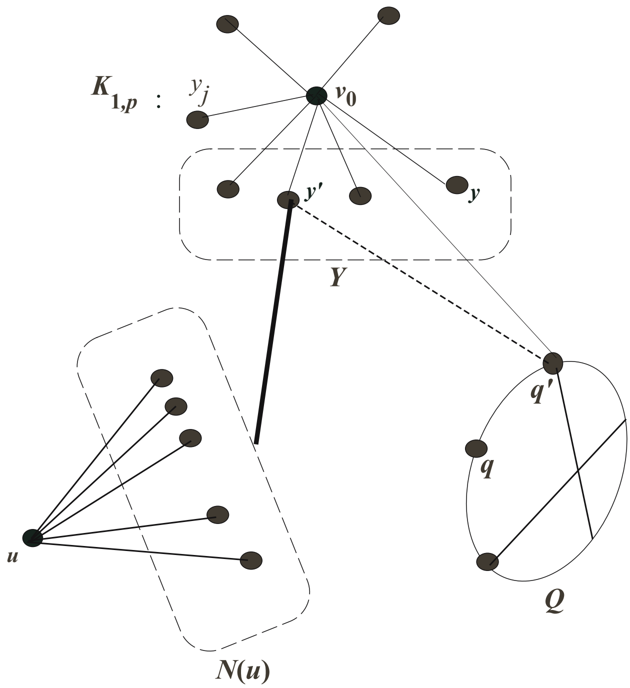

Fig. 1. An illustration of proof of Theorem 2

Let q' be the vertex in Q which is not adjacent with y'. If $v_0u \notin E(F_1)$, then v_0 must be adjacent to q'. (Otherwise $\overline{F}$ would contain C_4 formed by $\{v_0, y', q', u\}$.) Now we have two new stars, namely S^1_{1+p} and S^2_{1+p}, where $V(S^1_{1+p}) = N_T[y']$ with y' as the center and $V(S^2_{1+p}) = (S_{1+p}\backslash\{y'\}) \cup \{q'\}$. If $v_0u \in E(F_1)$, then we also have two new stars. The first one is S^1_{1+p} as in the previous case and the second one is S^3_{1+p} where $V(S^3_{1+p}) = (S_{1+p}\backslash\{y'\}) \cup \{u\}$ with v_0 as the center. In case that y' is adjacent with all vertices in Q, then the first star is $N_T[y']\backslash\{q\}$ and the second star is S^4_{1+p} where $V(S^4_{1+p}) = (S_{1+p}\backslash\{y'\}) \cup \{q\}$, $q \in Q$ with v_0 as the center. The fact that $v_0q \in E(F)$ is guaranteed by Assumption 1.

Therefore, we have $F_1 \supseteq 2S_{1+p}$. Thus $R(2S_{1+p}, C_4) \leq 2(p+1)+1$. By Lemma 1 we have $R(2S_{1+p}, C_4) = 2(p+1)+1$. The proof is now complete. □

Theorem 3. *For $p \geq 3$ and $k \geq 3$, $R(kS_{1+p}, C_4) = k(p+1)+1$.*

To obtain the Ramsey number we use induction on k. We assume the theorem holds for every $2 \leq r < k$. Let F_2 be a graph of order $k(p+1)+1$. Suppose $\overline{F}_2$ contains no C_4. We will show that $F_2 \supseteq kS_{1+p}$. By induction hypothesis, $F_2 \supseteq (k-1)S_{1+p}$. Write $B = F_2\backslash(k-2)S_{1+p}$ and $T' = F_2[B]$. Thus $|T'| = 2(p+1)+1$. Since $\overline{T}'$ contains no C_4 and its follows from Theorem 2 that T' contains $2S_{1+p}$. Hence F_2 contains $(k-2)S_{1+p} \cup 2S_{1+p} = kS_{1+p}$. Thus we have $R(kS_{1+p}, C_4) \leq k(p+1)+1$. On the other hand, we have $R(kS_{1+p}, C_4) \geq k(p+1)+1$ (by Lemma 1). The assertion follows. □

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A Minimal Planar Point Set with Specified Disjoint Empty Convex Subsets

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Abstract. For a planar point set P in general position, an empty convex k-gon or a k-hole of P is a convex k-gon H such that the vertices of H are elements of P and no element of P lies inside H. Let $n(k_1, k_2, \cdots, k_l)$ be the smallest integer such that any set of $n(k_1, \cdots, k_l)$ points contains a k_i-hole for each i, $1 \leq i \leq l$, where the holes are pairwise disjoint. We evaluate such values. In particular, we show that $n(1, 2, 3, 4, 5) = 15$.

1 Introduction

We deal with only a finite number of planar point set P and assume that no three elements in P are on a line throughout the paper. For $Q \subseteq P$ with $ch(Q) \cap P = Q$, we distinguish the *vertices* which lie on the convex hull boundary from the remaining *interior points*, where ch stands for the convex hull. Let $V(Q)$ be a set of the vertices and $I(Q)$ be a set of the interior points of Q. We call a subset Q with k elements of P an *empty convex k-gon* or a *k-hole* if $V(Q)$ determines a k-gon and $I(Q) = \emptyset$.

In 1975, Erdős [3] asked the following problem: Find the smallest integer $n(k)$ such that any set of $n(k)$ points contains an empty convex k-gon. Klein found $n(4) = 5$ in [4]. Harborth [6] determined $n(5) = 10$, where he gave the configuration of 9 points with no empty convex pentagons. Moreover, Horton [7] constructed sets of arbitrarily many points with no empty convex heptagons, i.e., $n(k)$ does not exist for $k \geq 7$. For the remaining case of $k = 6$, Overmars exhibited a set of 29 points, the largest known, with no empty convex hexagons in [10]. Most recently, Gerken [5] showed that any set containing a convex 9-gon also contains an empty convex hexagon. Hence, using the upper bound of Erdős-Szekeres theorem [11], $n(6) \leq 1717$ holds. That is, the current record of $n(6)$ is $30 \leq n(6) \leq 1717$.

A family of holes $\{H_i\}_{i \in I}$ is said to be *vertex disjoint* if $V(H_i) \cap V(H_j) = \emptyset$ for every pair $\{i, j\} \subseteq I$. If, moreover, their convex hulls are pairwise disjoint, i.e., $ch(H_i) \cap ch(H_j) = \emptyset$, we simply say that these holes are *disjoint*. In this paper, we study the existence of a few disjoint holes in a given point set: Determine the smallest integer $n(k_1, k_2, \cdots, k_l)$ such that any set of $n(k_1, k_2, \cdots, k_l)$ points in the plane, no three collinear, contains a k_i-hole for every i, $1 \leq i \leq l$, where the holes are disjoint. Here, a 1-hole is a point and a 2-hole is a line segment. Clearly, every 6 point set is partitioned into two disjoint 3-holes, which implies

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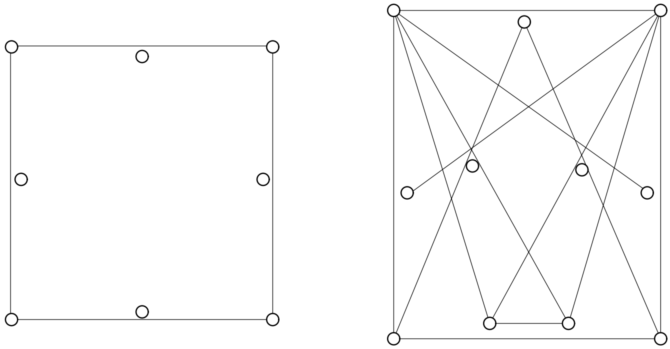


(a) (b)

Fig. 1.

that $n(3,3) = 6$. By Horton's construction, $n(k_1, k_2) = \infty$ for $k_1 \geq 7$ or $k_2 \geq 7$. In [12], the problem of partitioning a planar point set into disjoint holes is discussed, and it was shown that every set of 7 points is partitioned into a 3-hole and a disjoint 4-hole, implying that $n(3,4) = 7$. Using this result, we can always partition any 10 point set into a 1-hole, a 2-hole, a 3-hole and a 4-hole which are disjoint, i.e., $n(1,2,3,4) = 10$. We showed that every set of 9 points contains two disjoint 4-holes in [8], and the configuration of 8 points in Fig. 1(a) does not contain two disjoint 4-holes, i.e., $n(4,4) = 9$. In [9], we introduced the concept of $n(k_1, k_2)$ and estimated the following values:

1. $n(3,5) = 10$: Every set of 10 points contains a 3-hole and a disjoint 5-hole. $n(3,5) \geq 10$ follows immediately from $n(5) = 10$.
2. $12 \leq n(4,5) \leq 14$: Every set of 14 points contains a 4-hole and a disjoint 5-hole. There exists a configulation of 11 points in Fig. 1(b) without these two holes.
3. $16 \leq n(5,5) \leq 20$: There exists a configulation of 15 points without two disjoint 5-holes, and the upper bound is easily shown by $n(5) = 10$.

The next section gives two new results for $n(k_1, k_2)$ and we discuss in Section 3 disjoint partitions of any 15 point set into distinct five holes. We conclude this paper with some open problems.

Terminologies. We represent a k-hole H by $H = (p_1 p_2 \cdots p_k)_k$ if $V(H) = \{p_1, p_2, \cdots, p_k\}$ is in counter-clockwise order. Let $l(a,b)$ be the line passing through a and b. Denote the closed half-plane with $l(a,b)$, which contains c or does not contain c by $H(c; ab)$ or $H(\overline{c}; ab)$, respectively. Let $S(ab; cd)$ be the closed region between $l(a,b)$ and $l(c,d)$, containing $\{a,b,c,d\}$ when the segments $\overline{ab}$ and $\overline{cd}$ do not intersect. Let R be a region in the plane. An *interior point* of R is an element of a given point set P in the interior, and if R contains no interior points, R is said to be *empty*.

Denote the *convex cone* by $\gamma(a; b, c)$ with apex a, determined by a, b and c. If $\gamma(a; b, c)$ is not empty, we define an *attack point* from the half-line ab to ac, denoted by $\alpha(a;b,c)$ as the interior point in $\gamma(a;b,c)$ such that $\gamma(a;b,\alpha(a;b,c))$ is empty. For $\delta = b$ or c of $\gamma(a;b,c)$, let δ' be a point such that a is on the line segment $\overline{\delta\delta'}$.

2 Two Disjoint Holes

In this section we prove the next results:

Theorem 1. *Any set P of 13 points in the plane, no three collinear, contains a 4-hole and a disjoint 5-hole, that is, $n(4,5) \leq 13$.*

Theorem 2. *There exists a configuration of 16 points in the plane, no three collinear, without two disjoint 5-holes, that is, $n(5,5) \geq 17$.*

We first improve on the upper bound of $n(4,5)$. In [1], Aichholzer *et al.* gave a complete data base of all combinatorial different sets of up to 11 points and showed several combinatorial results. To show Theorem 1, we use their result as Lemma A.

Lemma A. *Any set of 11 points in the plane, no three collinear, contains either a 6-hole or a 4-hole and a disjoint 5-hole.*

Proof (of Theorem 1). We assume by Lemma A that P contains a 6-hole; $K = (v_1v_2v_3v_4v_5v_6)_6$. Consider the convex regions of $E_1 = \gamma(v_1;v_4',v_3) \cap \gamma(v_3;v_1,v_6')$, $E_2 = \gamma(v_3;v_6',v_5) \cap \gamma(v_5;v_3,v_2')$ and $E_3 = \gamma(v_5;v_2',v_1) \cap \gamma(v_1;v_5,v_4')$. We remark that if we have a 5-hole and a disjoint convex region with at least 5 points of P, then we are done by $n(4) = 5$.

Let Q_i be a point set of $P \backslash V(K)$ in E_i for any i. If $|Q_1| \geq 4$, i.e., we have at least 5 points of $Q_1 \cup \{v_2\}$ in E_1, we have $(v_1v_3v_4v_5v_6)_5$ and a disjoint 4-hole in E_1. Thus we may assume that $|Q_1| \leq 3$. Similarly, we may also assume that $|Q_i| \leq 3$ for $i = 2,3$. Hence, it suffices to consider the following two cases by symmetry.

(I) $|Q_1| = |Q_3| = 3$ and $|Q_2| = 1$:
Let $Q_2 = \{p\}$. Assume that p is in $H(v_3;v_1v_4)$ by symmetry. If p is in $S(v_2v_3;v_1v_4)$, we have $(v_1v_2v_3pv_4)_5$ disjoint from a 4-hole of 5 points of $Q_3 \cup \{v_5,v_6\}$. Assume that p is in $\gamma(v_3;v_2',v_6')$. Since $E_1 \cap S(v_2v_3;v_1v_4)$ is also empty by the same way, we have 5 points of $Q_1 \cup \{v_2,p\}$ in $H(\overline{v_1};v_2v_3)$ disjoint from $(v_1v_3v_4v_5v_6)_5$.

(II) $|Q_1| = |Q_3| = 2$ and $|Q_2| = 3$:
We assume that $S(v_2v_5;v_1v_6)$ is empty since, otherwise, we have 5 points of $Q_2 \cup \{v_3,v_4\}$ and a 5-hole by $\{v_1,v_2,v_5,v_6,p\}$ for $p \in S(v_2v_5;v_1v_6)$. $S(v_2v_1;v_3v_6)$ is also empty by the same reason. See Fig. 2.

Consider the region $R = \gamma(v_1;v_2',v_4')$. We have three subcases according to the number of points of Q_3 in R. If R contains exactly 2 points, we have $(v_2v_3v_4v_5v_6)_5$ and 5 points of $Q_1 \cup Q_3 \cup \{v_1\}$ in $H(\overline{v_6};v_1v_2)$.

If R has exactly 1 point p, the other point p' in Q_3 is in $\gamma(v_6;v_1',v_3')$. Again, if $q \in Q_2 \cup \{p'\}$ exists in $S(v_3v_6;v_4v_5)$, we have a 5-hole by $\{v_3,v_4,v_5,v_6,q\}$ and

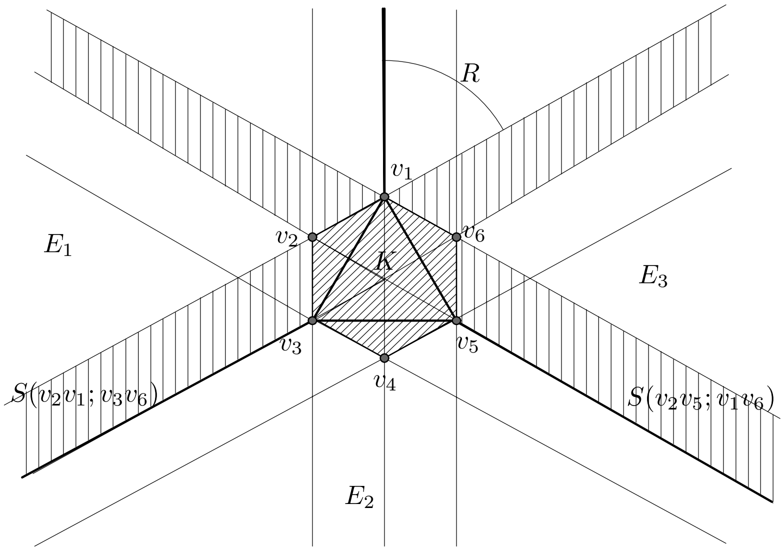


Fig. 2.

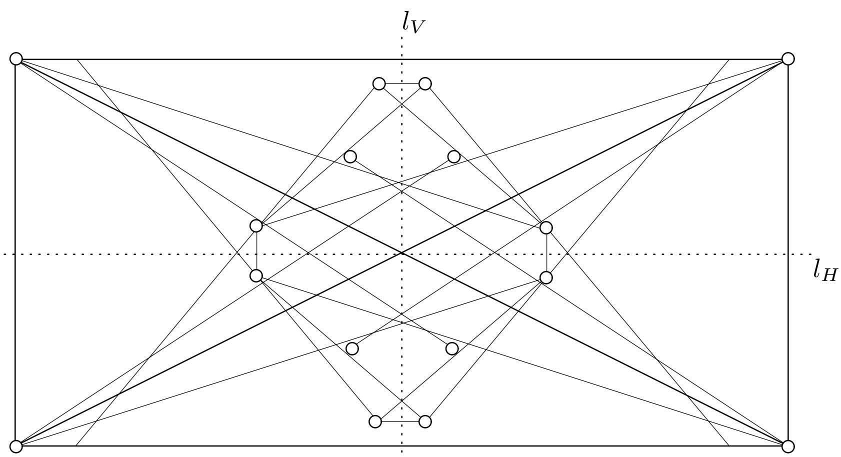


Fig. 3. $n(5,5) \geq 17$

5 points of $Q_1 \cup \{v_1, v_2, p\}$. Thus we may assume that $S(v_3v_6; v_4v_5)$ is empty. Then $H(\overline{v_6}; v_4v_5)$ contains 5 points of $Q_2 \cup \{v_4, p'\}$ disjoint from $(v_1v_2v_3v_5v_6)_5$.

Suppose that R is empty, i.e., Q_3 is in $\gamma(v_6; v_1', v_3')$. By the same way, assume that Q_1 is in $\gamma(v_2; v_1', v_5')$. Note that $H(v_2; v_1v_4)$ contains at least 2 points Q_2' in Q_2 by symmetry. Therefore, we have at least 5 points of $Q_1 \cup Q_2' \cup \{v_3\}$ disjoint from $(v_1v_2v_4v_5v_6)_5$. $\square$

Concerning Theorem 2, let us remark the following. To improve on the lower bound of $n(5,5)$, we give a symmetric configuration of 16 points P without two disjoint 5-holes. See Fig. 3. Observe that there does not exist a 5-hole of the 9 point set in any triangle determined by three vertices of P. Moreover, each half-plane bounded by the line $l_V(l_H)$ contains no 5-holes of P. If we take any 5-hole of P, then each subset of the remaining 11 points in a convex region disjoint from the 5-hole is in one of the above regions. We leave the detailed proof to the reader.

3 Several Disjoint Holes

In this section, we study the problem of partitioning a point set into disjoint holes. Clearly, $n(1,2) = 3$, $n(1,3) = n(2,2) = 4$ and $n(2,3) = 5$. It is not difficult to show that every set of 6 points is partitioned into a line segment and a disjoint empty convex quadrilateral, i.e., $n(2,4) = 6$. We have introduced $n(1,2,3,4) = 10$. In fact, we could determine the several values for $n(k_1, k_2, \cdots, k_l)$. We now show the following main result.

Theorem 3. *Any set P of 15 points in the plane, no three collinear, is partitioned into a 1-hole, a 2-hole, a 3-hole, a 4-hole and a 5-hole which are disjoint, that is, $n(1,2,3,4,5) = 15$.*

To prove this result, we use the following proposition.

Proposition 1. $n(2,3,5) = 11$.

Proof. To prove the lower bound, we give a configuration of 10 points as shown in Fig. 4, which does not contain a 2-hole, a 3-hole and a 5-hole which are disjoint. Thus, $n(2,3,5) \geq 11$ holds.

We show that any set P of 11 points contains a 2-hole, a 3-hole and a 5-hole which are disjoint. By $n(5) = 10$, P contains a 5-hole, denoted by $F = (v_1v_2v_3v_4v_5)_5$. Consider a convex cone $E_i = \gamma(v_i; v_{i+1}, v'_{i-1})$ for $1 \leq i \leq 5$. Let Q_i be a point set of $P \backslash V(F)$ in E_i for any i. We assume that $|Q_1| \geq |Q_i|$ for

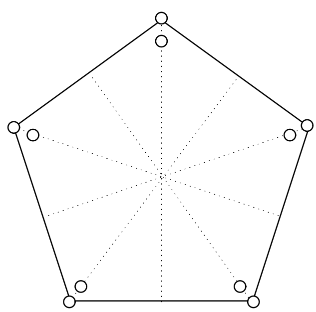


Fig. 4. $n(2,3,5) \geq 11$

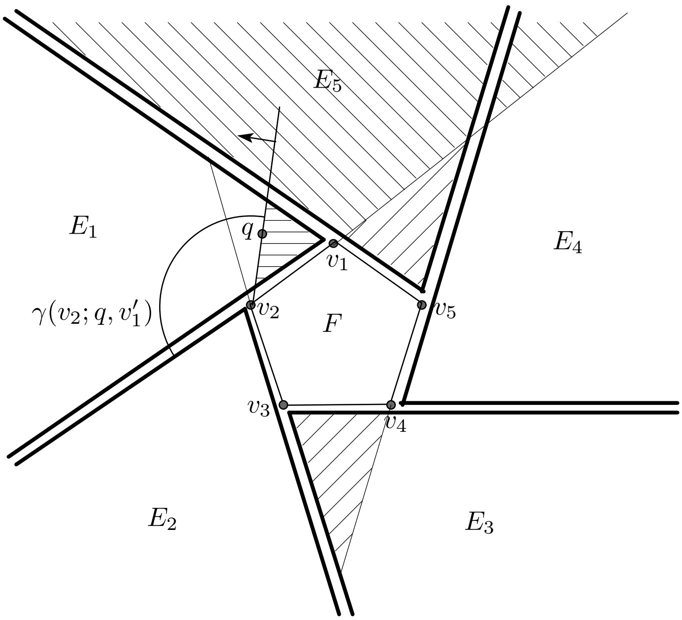


Fig. 5.

any $i \neq 1$ w.l.o.g. If $|Q_1| \geq 5$, E_1 contains a 2-hole and a disjoint 3-hole by $n(2,3) = 5$, and we have a 5-hole F disjoint from these holes. Next, we consider the other cases.

(I) $|Q_1| = 4$: If $\gamma_1 = \gamma(v_1; v_2', v_5')$ is not empty, we have F and at least 5 points in $H(\overline{v_3}; v_1v_2)$. Assume that γ_1 is empty. If E_5 is not empty, there exists $p = \alpha(v_5; v_1, v_4')$. Then we have a 5-hole of $(v_1v_3v_4v_5p)_5$ and exactly 5 points of $Q_1 \cup \{v_2\}$. Thus, E_5 is also empty. By the same reason, we assume that $E_3 \cap H(v_3; v_4v_5)$ is empty.

If $q = \alpha(v_2; v_1, v_3')$ exists as shown in Fig. 5, $\gamma(v_2; q, v_1')$ contains exactly 3 interior points, i.e., a 3-hole exists in it. As for the other two holes, if there exists a point r in E_2, we have the line segment $\overline{v_3r}$ and $(v_2v_4v_5v_1q)_5$. If E_2 is empty, we have a 5-hole F and the remaining 2 points in $H(\overline{v_3}; v_4v_5)$.

Suppose that $\gamma(v_2; v_1, v_3')$ is empty. Again, if E_2 is not empty, $H(\overline{v_1}; v_2v_3)$ contains at least 5 points disjoint from F. If E_2 is empty, we obtain F, a 3-hole in Q_1 and a 2-hole in $H(\overline{v_3}; v_4v_5)$.

(II) $|Q_1| = 3$: We consider the location of the remaining 3 points. If any other E_i contains at least 2 points, we have a 2-hole of Q_i, a 3-hole of Q_1 and a 5-hole F. We now assume that $E_2 = \emptyset$ and $|Q_i| = 1$ for $i = 3, 4, 5$. Let $Q_3 = \{p\}$ and $Q_4 = \{q\}$. If p is in $H(\overline{v_3}; v_4v_5)$, there exists the line segment $\overline{pq}$ disjoint from F and a 3-hole of Q_1. Suppose that p is in $H(v_3; v_4v_5)$. Then there exists a 5-hole $(pv_4v_1v_2v_3)_5$ disjoint from $\overline{qv_5}$. We argue for any other cases by the same way.

(III) $|Q_1| = 2$: We need three subcases according to the location of the remaining 4 points.

(a) $|Q_i| = 1$ for each $i = 2, 3, 4, 5$: Let $Q_i = \{p_i\}$ for any $i \neq 1$.

Suppose first that p_2 is in $H(v_2; v_3v_4)$. If p_4 is in $H(\overline{v_4}; v_1v_5)$, we have a 2-hole of Q_1, $(p_4p_5v_5)_3$ and $(p_2v_3v_4v_1v_2)_5$. Thus, p_4 is in $H(v_4; v_1v_5)$. Then if p_3 is in $H(v_5; v_1v_4)$ or $H(\overline{v_5}; v_1v_4)$, we have $(p_3p_4v_4)_3$ and $(p_2v_3v_5v_1v_2)_5$, or $(p_3v_4v_3)_3$ and $(p_2p_4v_5v_1v_2)_5$, respectively.

Suppose that p_2 is in $H(\overline{v_2}; v_3v_4)$. We consider the location of p_5. If p_5 is in $H(v_5; v_1v_2)$, we have a 2-hole of Q_1, $(p_2p_3v_3)_3$ and $(p_5v_1v_2v_4v_5)_5$. If not so, we have a 3-hole of $Q_1 \cup \{p_5\}$ disjoint from $\overline{p_2p_3}$ and F.

(b) $|Q_2| = 2$: If $\gamma(v_1; v_2', v_5')$ is not empty, we have a 3-hole in $H(\overline{v_3}; v_1v_2)$, a 2-hole of Q_2 and F. Thus, $\gamma(v_1; v_2', v_5')$ is empty. Then if E_5 is not empty, i.e., there exists $p = \alpha(v_5; v_1, v_4')$, we have a 2-hole of Q_1, a 3-hole of $Q_2 \cup \{v_3\}$ and $(pv_1v_2v_4v_5)_5$. Thus, E_5 is empty. By the same reason, $E_3 \cap H(v_3; v_4v_5)$ is empty.

Let K be the 2-hole in $E_3 \cup E_4$. If $q = \alpha(v_2; v_1', v_3')$ exists in E_1, we have a 3-hole of $Q_2 \cup \{q\}$ disjoint from F and K. If $\gamma(v_2; v_1', v_3')$ is empty, we have K, a 3-hole of $Q_2 \cup \{v_3\}$ and $(rv_2v_4v_5v_1)_5$ for $r = \alpha(v_2; v_1, v_3')$.

We remark that we are immediately done when we have such two consecutive convex cones E_i's, each of which contains exactly 2 interior points.

(c) $|Q_4| = 2$: We assume that Q_5 is empty. In fact, if not so, there exists $p = \alpha(v_5; v_1, v_4')$. If p is in $H(v_5; v_1v_2)$, we have $(pv_1v_3v_4v_5)_5$, a 3-hole of $Q_1 \cup \{v_2\}$ and a 2-hole of Q_4. If not, there exists a 3-hole of $Q_1 \cup \{p\}$ disjoint from a 2-hole of Q_4 and F. By the same way, $E_2 \cap H(v_2; v_3v_4)$ is empty. See Fig. 6. Here, we can assume that $|Q_2| = |Q_3| = 1$ by (III-b).

If $q = \alpha(v_2; v_1, v_3')$ exists in E_1, we have $(qv_2v_3v_4v_1)_5$, a 3-hole of $Q_4 \cup \{v_5\}$ and a 2-hole of $Q_2 \cup Q_3$. If $E_1 \cap \gamma(v_2; v_1, v_3')$ is empty, we have a 3-hole of $Q_1 \cup Q_2$ disjoint from F and a 2-hole of Q_4. $\square$

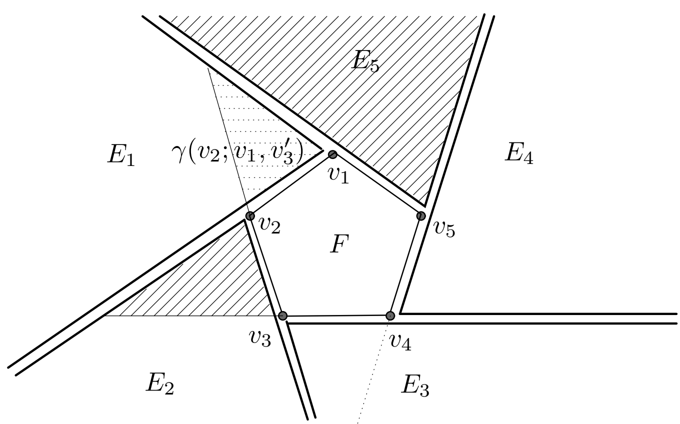


Fig. 6.

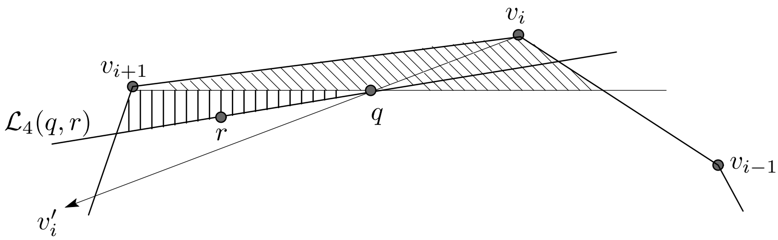


Fig. 7.

Theorem 3. $n(1, 2, 3, 4, 5) = 15$.

Proof. It is trivial that $n(1, 2, 3, 4, 5) \geq 15$. For the upper bound, it suffices to show $n(2, 3, 4, 5) \leq 15$, i.e., any set P of 15 points contains a 2-hole, a 3-hole, a 4-hole and a 5-hole which are disjoint.

If we find a 4-hole and a disjoint convex region with the remaining 11 points, we are done by Proposition 1. Then there exists a straight line which separates the 4-hole from the other 11 points. We call such a line a *cutting line* through two elements u and v in P, denoted by $\mathcal{L}_4(u, v)$. We remark that $n(2, 3, 4) = 9$ by $n(3, 4) = 7$. Therefore, if there exists a cutting line $\mathcal{L}_5(u, v)$ which separates a 5-hole from the remaining at least 9 points, we also find the desired sets.

Let $V(P) = \{v_1, v_2, \cdots, v_l\}$ in counter-clockwise order. We consider any edge $\overline{v_i v_{i+1}}$ of $V(P)$. Let $\triangle abc = ch(\{a, b, c\})$ for elements a, b and c of P. If $\triangle v_{i-1}v_i v_{i+1}$ is empty, we have a cutting line $\mathcal{L}_4(v_{i+1}, p)$ for $p = \alpha(v_{i+1}; v_{i-1}, v'_i)$, which separates $(pv_{i-1}v_i v_{i+1})_4$ from the remaining 11 points. Suppose that $\triangle v_{i-1}v_i v_{i+1}$ is not empty. Then $q = \alpha(v_{i+1}; v_i, v_{i-1})$ exists. If $\gamma(q; v_{i+1}, v'_i)$ is not empty, we also have a cutting line $\mathcal{L}_4(q, r)$ as shown in Fig. 7. Therefore, we can assume that there is a point p_i of P for each $\overline{v_i v_{i+1}}$ s.t. $\gamma(v_i; v_{i+1}, p_i) \cup \gamma(v_{i+1}; v_i, p_i) = \emptyset$. We call this p_i a *friend* of $\overline{v_i v_{i+1}}$. We remark that $p_i \neq p_j$ for $i \neq j$, i.e., $|P| \geq 2|V(P)|$.

Consider any consecutive edges, say $\overline{v_1 v_2}$ and $\overline{v_2 v_3}$ with their friends p_1, p_2. Let T be a subset with $V(T) = \{v_2, p_2, p_1\}$. We have the cases according to the number of the interior points of T. Let $I(R)$ be the interior points of a region R.

(I) $I(T) = \emptyset$: If $\triangle p_1 p_2 v_3$ is not empty, we have a cutting line $\mathcal{L}_4(p_1, q)$ for $q = \alpha(p_1; p_2, v_3)$. Thus, we can assume that $\triangle p_1 p_2 v_3$ is empty. If $\triangle v_1 p_1 v_3$ is not empty, we also have a cutting line $\mathcal{L}_4(v_3, \alpha(v_3; p_1, v_1))$. Thus, we can assume that $\triangle v_1 p_1 v_3$ is also empty, i.e., $|V(P)| \geq 4$. Then we have a cutting line $\mathcal{L}_5(v_3, r)$ for $r = \alpha(v_3; v_1, v_4)$, separating $(rv_1 p_1 p_2 v_3)_5$ from the remaining 9 points. See Fig. 8.

(II) $|I(T)| = 1$: Let $I(T) = \{q\}$. If $\triangle qp_2 v_3$ is not empty, we have $\mathcal{L}_4(q, \alpha(q; p_2, v_3))$. Thus, we can assume that $\triangle qp_2 v_3$ is empty. If $\triangle qv_3 p_1$ is empty, we also have $\mathcal{L}_5(p_1, \alpha(p_1; v_3, v'_2))$ separating $(p_1 q p_2 v_3 \alpha(p_1; v_3, v'_2))_5$. Hence, we assume that $\triangle qv_3 p_1$ is not empty, and there exists a point $r = \alpha(v_3; q, p_1)$ in it. We remark that r is in $\gamma(v_2; q, p_1)$ since, otherwise, $\mathcal{L}_4(r, v_3)$ separates $(rqv_2 p_2)_4$.

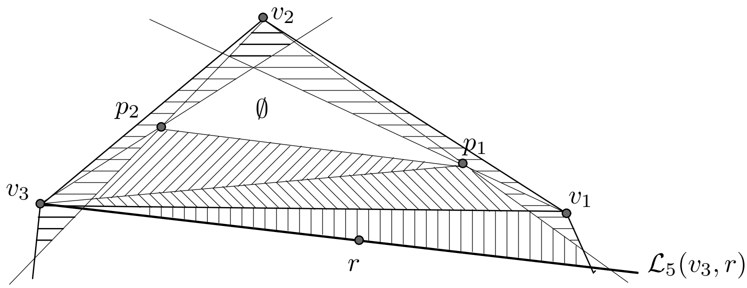


Fig. 8.

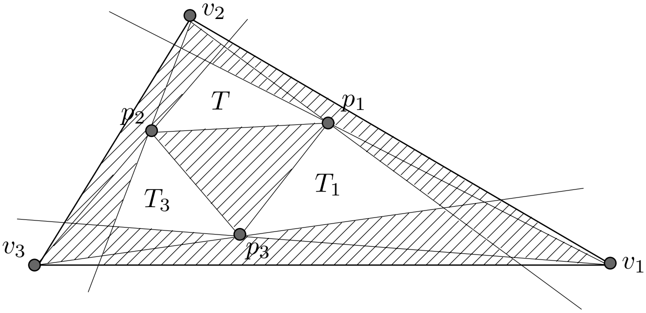


Fig. 9. Shaded area is empty

Again, if $\gamma(r; v_3, q')$ is not empty, we have $\mathcal{L}_5(r, \alpha(r; v_3, q'))$ separating $(rqp_2v_3\alpha)_5$. If it is empty, $\mathcal{L}_4(q, r)$ separates $(rqp_2v_3)_4$.

(III) $|I(T)| = 2$: Let $T' = T\backslash\{v_2\}$, where $|V(T')| = 3, 4$. For $|V(T')| = 3$, let $q \in V(T')$ and $I(T') = \{r\}$. Assume that r is in $\gamma(v_2; p_2, q)$ by symmetry, then we have $\mathcal{L}_4(p_2, r)$ separating $(v_2p_2rq)_4$. For $|V(T')| = 4$, if $\triangle p_1p_2v_3$ is empty, $\mathcal{L}_5(p_1, v_3)$ separates a 5-hole of $T' \cup \{v_3\}$, and if not empty, we also have $\mathcal{L}_5(p_1, \alpha(p_1; p_2, v_3))$.

(IV) Otherwise: Let T_i be a triangle with $V(T_i) = \{v_i, p_i, p_{i-1}\}$ for any i where $T = T_2$. Since $|P| = 15$ and $|I(T_i)| \geq 3$ for every i, we assume that $|V(P)| = 3$ and $|I(T_i)| = 3$ for $i = 1, 2, 3$ and $\triangle p_1p_2p_3$ is empty. See Fig. 9.

Again, consider $T' = T\backslash\{v_2\}$. If $|V(T')| = 5$, $\mathcal{L}_5(p_1, p_2)$ separates a 5-hole of T'. For $|V(T')| = 3$, let $q \in V(T')$ and $r = \alpha(p_2; q, p_1)$. For the location of r, since we have $\mathcal{L}_4(p_2, r)$ if r is in $\gamma(v_2; q, p_2)$, we can assume that r is in $\gamma(v_2; q, p_1)$. Then if $\gamma(r; p_2', p_1)$ is empty, $\mathcal{L}_4(r, p_1)$ separates $(qrp_1v_2)_4$, and if not, we also have $\mathcal{L}_4(r, \alpha(r; p_2', p_1))$.

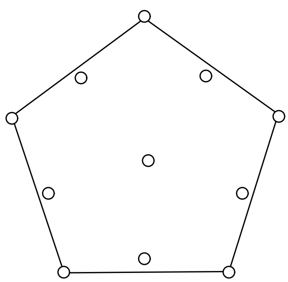
Suppose that $|V(T')| = 4$. Let $V(T') = \{p_2, q, r, p_1\}$ in clockwise order and $I(T') = \{s\}$. If s is in $\triangle qp_2p_3$, we have the line segment $\overline{v_1v_2}$, a 3-hole of $I(T_1)$, $(rqsp_3p_1)_5$ and a 4-hole of the remaining 5 points of $I(T_3) \cup \{p_2, v_3\}$ by $n(4) = 5$. Since the same situation occurs if s is in $\triangle rp_3p_1$, we can assume that s is in

$\triangle qp_3r$. Then if v_2 is in $\gamma(p_3; q, r)$, $\mathcal{L}_4(p_1, p_2)$ separates $(qsrv_2)_4$. Thus, v_2 is in $\gamma(p_3; p_2, q)$ by symmetry and consider $t = \alpha(p_2; p_3, v_3)$. Then we finally obtain $(p_3qv_2p_2t)_5$ and a 2-hole of $I(T_3)\backslash\{t\}$, and $I(H(s; qp_3))$ contains a 3-hole and a 4-hole by $n(3, 4) = 7$. □

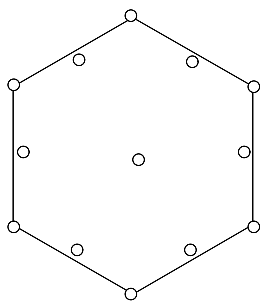
4 Final Remarks

We first remark the case for $n(2, k)$. We introduced $n(2, 4) = 6$ in Section 3, and $n(2, 5) = 10$ holds by $n(3, 5) = 10$ and $n(5) = 10$. In Section 2, we established new bounds for $n(4, 5)$ and $n(5, 5)$. Although $n(4, 5) = 12$ or 13, we have not determined a non-trivial upper bound of $n(5, 5)$. We do not argue the value of $n(k, 6)$ since the valid value is not known yet. For $n(k_1, k_2, k_3)$, we can inductively show the several values. For example,

1. $n(2, 3, 4) \leq n(3) + n(2, 4) = 9$ implies $n(2, 3, 4) = 9$.
2. $n(3, 4, 4) \leq n(4) + n(3, 4) = 12$, so $n(3, 4, 4) = 12$ (See Fig. 10(a)).
3. $n(4, 4, 4) \leq n(4) + n(4, 4) = 14$, so $n(4, 4, 4) = 14$ (See Fig. 10(b)).



(a) $n(3, 4, 4) \geq 12$



(b) $n(4, 4, 4) \geq 14$

Fig. 10.

It is not so difficult to prove the following (i), (ii) and (iii). Our open problems are to determine their exact values.

(i) $12 \leq n(3, 3, 5) \leq 13$. (ii) $13 \leq n(3, 4, 5) \leq 14$. (iii) $15 \leq n(4, 4, 5) \leq 17$.

In [8], we introduced the following: Given natural numbers k and n, let $F_k(n)$ denote the maximum number of pairwise disjoint empty convex k-gons that can be found in every n point set in the plane, no three collinear. As for $F_5(n)$ we have $2\lfloor n/n(5, 5)\rfloor \leq F_5(n)$. Concerning the upper bound, the configuration in Fig. 3 shows that $F_5(n) \leq 1$ for $n \leq 16$. In this connection, Bárány and Károlyi showed that $F_5(n) < n/6$ in [2].

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Fast Skew Partition Recognition

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Abstract. Chvátal defined a *skew partition* of a graph G to be a partition of its vertex set into two non-empty parts A and B such that A induces a disconnected subgraph of G and B induces a disconnected subgraph of $\overline{G}$. Skew partitions are important in the characterization of perfect graphs. De Figuereido et al. presented a polynomial time algorithm which given a graph either finds a skew partition or determines that no such partition exists. It runs in $O(n^{101})$ time. We present an algorithm for the same problem which runs in $O(n^4m)$ time.

1 Introduction

A *skew partition* of a graph $G = (V, E)$ is a partition of V into two nonempty sets A and B such that $G[A]$ is not connected and $\overline{G[B]}$ is not connected. If such a partition exists, B is called a *skew cutset*. Clearly, a skew partition (A, B) of G yields a skew partition (B, A) of $\overline{G}$. It is this self-complementarity which first suggested that these partitions might be important to an understanding of the structure of perfect graphs. A graph is *perfect* if each induced subgraph has chromatic number equal to the size of its largest clique. A graph is *Berge* if it contains neither an induced odd chordless cycles of length at least five or the complement of such a cycle. Berge introduced these two classes of graph and proposed the *Strong Perfect Graph Conjecture* that, in fact, they are identical [1].

Speculating that skew partitions might play a key role in a decomposition theorem for Berge graphs which would imply the Strong Perfect Graph Conjecture, Chvátal [3] introduced skew partitions and conjectured that no minimal imperfect graph permits a skew partition. Both his speculation and his conjecture were accurate. Indeed, Chudnovsky, Robertson, Seymour, and Thomas [2] recently proved every Berge graph either:

(a) Is in one of five basic classes of perfect graphs (line graphs of bipartite graphs, their complements, bipartite graphs, their complements, or double split graphs), or

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(b) Permits one of three partitions (a proper 2-join, a homogeneous pair, or a special type of skew partition which they call balanced[1].

It was known that the first two of these three partitions could not occur in a minimal imperfect graph (see [5] and [4]). Chudnovsky et al. also proved that balanced skew partitions cannot occur in a smallest minimal imperfect Berge graph. These results taken together imply the Strong Perfect Graph Conjecture.

This paper presents a polynomial time algorithm to test if a graph has a skew partition. This is not the first such algorithm. In [6], de Figuereido, Klein, Kohayakawa and Reed present one whose running time is $O(n^{101})$. In this paper we present an algorithm with running time $O(n^4 m)$. We note that in related work, Trotignon has concurrently and independently developed an $O(n^9)$ algorithm for determining if a Berge graph has a balanced skew cutset and proves that determining if an arbitrary graph has such a cutset is NP-hard [11]. In §2, we present a preliminary discussion of our method for finding skew partitions. In §3, we present our $O(n^4 m)$ time algorithm to find a skew partition. We assume the reader is familar with the standard definitions and notations of perfect graph theory which can be found in [8]. We warn the reader that, following the conventions of that field, by a subgraph we mean an induced subgraph.

2 The Idea

Our algorithm breaks the problem up into two subproblems. We first check if the graph has a special kind of skew partition known as a T-cutset. We then look for skew partitions which are not T-cutsets. T-cutsets are easy to handle, so we treat them first.

Hoàng [7] defined a *T-cutset* as a skew cutset B where there exist two vertices x and y, such that x and y are in different components of $G - B$ and some component of $\overline{G[B]}$ is contained in $N(x) \cap N(y)$. For a subset of the vertex set $S = \{s_1, s_2, \ldots, s_i\}$, let $N(S) = \bigcap_{j=1}^{i} N(s_j)$.

Lemma 1. *There is an $O(n^3 m)$ time algorithm to decide whether a graph G contains a T-cutset.*

Proof. Consider every pair of vertices x, y and component C of $\overline{G[N(x) \cap N(y)]}$. If B is a T-cutset separating x from y such that C is a component of $\overline{G[B]}$ then $B \subseteq C \cup N(C) - x - y$. Moreover, $\overline{G[C \cup N(C) - x - y]}$ is disconnected. So $C \cup N(C) - x - y$ is also a T-cutset separating x from y. Given x, y and C we can test if $C \cup N(C) - x - y$ is a T-cutset in $O(m)$ time (by testing the connectivity of $G[V - (C \cup N(C) - x - y)]$). So, we can test if any T-cutset corresponding to such a triple exists in $O(n^3 m)$.

[1] A skew partition (A, B) is *balanced* if every path $P = \{v_1, v_2, ..., v_{k-1}, v_k\}$ in G of length at least 2 such that v_1 and v_2 are in B and $v_2, ..., v_{k-1}$ are in A has even length, and if every path $\overline{P} = \{v_1, v_2, ..., v_{k-1}, v_k\}$ in $\overline{G}$ of length at least 2 such that v_1 and v_2 are in A and $v_2, ..., v_{k-1}$ are in B has even length.

Our approach to looking for skew cutsets which are not T-cutsets is motivated by Reed's algorithm [9] for finding skew cutsets in bipartite graphs (in which case they are complete bipartite subgraphs). So, we first briefly sketch the ideas of this algorithm.

Suppose we have a bipartite graph with bipartition (S, T). Reed's approach was to ask, for each k, if there is a skew cutset B with $|B \cap T| = k$. Since we know G has no T-cutsets, we can restrict our attention to skew cutsets B with $|N(x) \cap N(y)| < k$ for every pair of vertices x and y of S in different components of $G - B$. On the other hand, if x and y are two vertices of $B \cap S$ then the intersection of their neighbourhoods contains $B \cap T$ and thus has at least k vertices. Hence, $B \cap S$ is a clique cutset in the auxiliary graph whose vertex set is S and where two vertices are adjacent if their neighbourhoods intersect in at least k vertices.

Therefore, we need only check if this auxiliary graph contains a clique cutset which corresponds to a skew cutset of G. As not every clique cutset of the auxiliary graph corresponds to a skew cutset, we use an algorithm due to Tarjan [10] to construct a structure known as a clique cutset tree, and then do a bit more work, to determine if the auxiliary graph has the desired special clique cutset. We omit further details in this preliminary discussion.

Bipartite graphs are rather special in that any skew cutset B consists of exactly two stable sets. Furthermore, every vertex sees vertices in only one stable set of B. So, bounding the size of the stable sets of B bounds the size of $N(x) \cap N(y)$ for x and y in different components of $G - C$. This is not true in general graphs which complicates our cutset finding algorithm.

For each vertex r of the graph and every pair of integers k_1 and k_2, with $k_2 \leq k_1$ we ask if there is a skew cutset B such that

(a) $r \in B$,
(b) some largest component L of $\overline{G[B]}$ has size k_1, $r \notin L$, and
(c) the component U of $\overline{G[B]}$ containing r has size k_2.

Given such a skew cutset B, for any two nonadjacent vertices x and y of B, either $N(x) \cap N(y)$ contains $V(L)$ and hence

(i) some component of $\overline{G[N(x) \cap N(y)]}$ has size at least k_1,

or x and y are in L and hence $V(U)$ is contained in $N(x) \cap N(y)$, so

(ii) some component of $\overline{G[N(x) \cap N(y)]}$ contains r and has size at least k_2.

On the other hand, if x and y are in different components of $G - B$ then they are not adjacent and $N(x) \cap N(y) \subseteq B$. The key lemma is the following:

Lemma 2. *Suppose B is a skew cutset of G which is not a T-cutset and B satisfies (a)–(c). If x, y are vertices of different components of $V - B$, then neither (i) nor (ii) holds for x, y.*

As in the bipartite case, we let H be an auxiliary graph with vertex set V where vertices x and y are adjacent if they are adjacent in G or if they satisfy either

(i) or (ii). Lemma 2 implies that such a skew cutset B is a clique cutset of H. As we describe in the next section, we use these ideas to develop an algorithm for finding the desired skew cutset.

3 The Details

This section presents the details of the skew partition algorithm for general graphs sketched in the previous section. In particular, we prove our main result:

Theorem 1. *Let G be a graph. There exists an $O(n^4 m)$ algorithm to find a skew cutset in G or decide that no such cutset exists.*

By Lemma 1, we can check if G contains a T-cutset in $O(n^3 m)$ time. Henceforth, we assume G contains no T-cutset. For each vertex r and each pair of integers $0 \leq k_2 \leq k_1 \leq n - k_2$, we search for a skew cutset B satisfying (a)–(c). In Lemma 4, we show for each such triple (k_1, k_2, r) an $O(nm)$ algorithm to check if the desired skew cutset exists or return that no such cutset exists. Thus, our algorithm to find a skew cutset in G or decide that no such cutset exists takes $O(n^4 m)$ time.

We create an auxiliary graph H whose vertex set is $V(G)$ and for which xy is an edge of H if either xy is an edge of G or one of (i) and (ii) holds for x, y. The previous section showed B induces a clique cutset in H. As not every clique cutset of H is a skew cutset of G, we need to check if H contains a clique cutset whose vertices induce a skew cutset of G. To do so, we use an auxiliary structure known as a *clique cutset tree*.

A clique cutset tree for a graph F consists of a rooted tree T with root r such that (I) every node t of T is labelled with a subgraph F_t of F, in particular, $F_r = F$, (II) if node t has children $s_1, \ldots, s_i$, then $F_t = F_{s_1} \cup ... \cup F_{s_i}$ and $F_{s_1} \cap ... \cap F_{s_i}$, which we denote K_t, is a clique cutset of F_t, and (III) if l is a leaf of T then F_l has no clique cutset. We note that clique cutset trees have been well studied in the context of perfect graphs and refer the interested reader to [8] for more details.

Let T be a clique cutset tree for H. For the root w of T, trivially, H_w contains every clique cutset of H. For any leaf l of T, by definition, H_l does not contain any clique cutset. On the other hand, every clique is contained in some leaf of the clique cutset tree. Thus, for every clique cutset C there exists a node s of T such that C is contained in H_s, C is not a cutset of H_s and C is a cutset of H_t for the parent t of s. Thus, our clique cutset K, if it exists, corresponds to a skew cutset of G for which conditions (a)–(c) from §2 hold for the triple (r, k_1, k_2), and such that for some node s of T

(d) K is contained in H_s,
(e) K is not a cutset of H_s, and
(f) K is a cutset of H_t, where t is the parent of s.

In looking for skew cutsets satisfying (d)–(f) for a node s with parent t in T, the following will be useful:

Lemma 3. *Let K be a clique cutset in H whose vertices induce a skew cutset B in G such that conditions (a)–(f) are satisfied for some node s of T with parent t. Then s has a sibling u in T such that $H_s - K$ and $H_u - K_t$ are in different components of $G - B$.*

Proof. Let $R = H_s - K$. By (e), R is connected in H. We now show R is contained in the same component of $G - B$. Assume this is not the case and let C_1 and C_2 be two components of $G - B$ such that $C_1 \cap R \neq \emptyset$ and $C_2 \cap R \neq \emptyset$. Let c_1 be in $C_1 \cap R$ and c_2 be in $C_2 \cap R$. As R is connected, there exists some path $P_H = p_1 p_2 ... p_k$ in R where $p_1 = c_1$ and $p_k = c_2$. Let p_i be the first vertex not contained in C_1. But, as $p_{i-1}p_i \notin E(G)$ and $p_{i-1}p_i \in E(H)$, p_{i-1} and p_i must therefore satisfy (i) or (ii) and contradict Lemma 2. This contradiction shows that R is indeed in a single component of $G - B$.

Let the other children of t be $u_1, \ldots, u_k$, and for $i = 1, \ldots, k$ let $R_i = H_{u_i} - K_t$. Every component R_i is disjoint from H_s and, by (d), disjoint from K. By an identical argument presented for R, R_i is contained in a single component of $G - B$. Finally, as K is a clique cutset of H_t and R is connected in $G - B$ there exists some i for which R_i and R are in different components of $H_t - K$. As $E(G) \subseteq E(H)$ and K corresponds to B, it follows that R_i and R are in different components of $G - B$.

This claim leads to the following algorithm.

Lemma 4. *There is an $O(nm)$ time algorithm to decide whether there exists a clique cutset K of H corresponding to a skew cutset in G satisfying (a)–(c).*

Proof. We begin by building a clique cutset tree T for H, which as shown by Tarjan [10], can be constructed in $O(nm)$ time. Tarjan's algorithm constructs a clique cutset tree with at most $n - 1$ leaf nodes and such that every internal node of this tree has exactly two children.

For each node s where H_s is a clique, we first check if H_s induces a skew cutset in G in $O(n + m)$ time and henceforth assume that if K exists it is properly contained in H_s for some s. We then use an $O(n + m)$ time algorithm which decides for a fixed sibling pair s, u in T with parent t if there exists a clique cutset K corresponding to a skew cutset in G satisfying (a)–(f) such that $H_s - K$ and $H_u - K_t$ are in different components of $G - B$. Either we find the desired K or we check all sibling pairs, in which case Lemma 3 implies G contains no skew cutset. As there are at most $n - 1$ sibling pairs, we have the desired $O(nm)$ time algorithm. All that remains to be shown is the $O(n + m)$ algorithm.

For fixed siblings s, u in T, let t be their parent. Our algorithm maintains two vertex sets: the set R' containing vertices which must be in the same component of $G - B$ as $H_u - K$ and the set K' containing vertices of H_s which must be in K. Initially let R' and K' be empty. We use the standard depth first search algorithm with the following changes: 1) we start the search from any unseen node in $H_u - K_t$, 2) for each node seen which is not contained in H_s we add it to R' and continue the search as normal from this node, and 3) for each node seen in H_s we add it to K' and do not continue searching any deeper from this node. If our search ends, then if $H_u - (K_t \cup R') \neq \emptyset$ we restart the search at any node in $H_u - (K_t \cup R')$, otherwise we terminate the search. These modification maintain the $O(n+m)$ running time of depth first search.

After the search if $K' = H_s$ then the desired skew cutset does not exist. If not then the set K' is a cutset and we need only check if it is contained in a skew cutset in G. If $\overline{G[K']}$ is disconnected then K' is itself a skew cutset. Otherwise, if K' is part of a skew cutset then K' must be contained in one of its components. Let $J = N(K') \cap H_s$. If $J = \emptyset$, then the desired skew cutset does not exist. If $(K' \cup J) \neq H_s$, then $(K' \cup J)$ is the desired skew cutset. If $(K' \cup J) = H_s$ and $|J| = 1$ then the desired skew cutset does not exist. Otherwise, if $|J| > 1$ then for any vertex v of J, we have that $K' \cup \{v\}$ is the desired skew cutset. It follows these steps take $O(n+m)$ running time, so together with the search we have the desired $O(n+m)$ algorithm.

4 Concluding Remarks

Reed's algorithm for finding a clique cutset in a bipartite graph [9] does not use Tarjan's algorithm [10] for finding a clique cutset. We can improve the run time of this algorithm to $O(n^2 m)$ by using the ideas presented in Section 2 and Lemma 4. This follows from the proof of Reed's algorithm as its run time is $O(n \times (f_1 + f_2))$ where f_1 is the run time for constructing a clique cutset tree and f_2 is run time for the algorithm presented in Lemma 4.

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Some Results on Fractional Graph Theory

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Abstract. We will convert integer-based definitions and invariants into their fractional analogues. Some results on fractional factors, fractional Hamiltonian graphs, fractional (g, f)-factors and fractional colorings are presented. The relationships of the programming and the graph theory are discussed. In particular, some new results related to fractional (g, f)-factors obtained by us are given. Furthermore, some open problems are presented.

1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex x of G the degree of x in G is denoted by $d_G(x)$ and set $\delta(G) = \min_{x \in V(G)} d_G(x)$. Let g and f be two integer-valued functions defined on $V(G)$ such that $0 \leq g(x) \leq f(x)$ for every $x \in V(G)$. Then a (g, f)-factor of G is a spanning subgraph H of G satisfying $g(x) \leq d_H(x) \leq f(x)$ for all $x \in V(H)$. In particular, if G itself is a (g, f)-factor, then G is called a (g, f)-graph. Let a and b be two non-negative integers. If $g(x) = a$ and $f(x) = b$ for every $x \in V(G)$, then a (g, f)-factor is called an $[a, b]$-factor. If $g(x) = f(x)$ for all $x \in V(G)$, then a (g, f)-factor is called an f-factor. If $a = b = k$, an $[a, b]$-factor is called a k-factor. In particular, a 1-factor is also called a perfect matching. Let G be a connected graph and $S \subset V(G)$. Denote the number of components of $G - S$ by $\omega(G - S)$. The toughness of G is defined to be

$$t(G) = \min\{\frac{|S|}{\omega(G - S)} : S \subset V(G), \omega(G - S) \geq 2\}.$$

The binding number of a graph G is defined to be

$$bind(G) = \min\{\frac{N_G(X)}{|X|} : \phi \neq X \subset V(G), N_G(x) \neq V(G)\}$$

where $N_G(x) = \{y : yx \in E(G), x \in X\}$. Let G be a graph and let S and T be two disjoint subsets of $V(G)$. Then $E_G(S, T)$ denotes the set of edges in G joining S and T and $e_G(S, T) = |E_G(S, T)|$. A component C of $G - (S \cup T)$ is odd or even according to whether $e_G(T, V(C)) + \sum_{x \in V(C)} f(x)$ is odd or even. The number of odd components of $G - (S \cup T)$ is denoted by $h_G(S, T)$. For convenience, sometimes we write $f(S) = \sum_{x \in S} f(x)$ and $d_G(S) = \sum_{x \in S} d_G(x)$.

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In the following we give some basic results on factors which are very important for our fractional factors.

Theorem 1.1. [21] *A graph G has a 1-factor if and only if for any $S \subset V(G)$, $q(G-S) \leq |S|$ where $q(G-S)$ is the number of odd components of $G-S$.*

Theorem 1.2. [13] *Let G be a graph and let g, f be two integer-valued functions defined on $V(G)$ such that $0 \leq g(x) \leq f(x)$ for all $x \in V(G)$. Then G has a (g,f)-factor if and only if for all disjoint subsets S and T of $V(G)$*

$$\delta_G(S,T) = \sum_{x \in T} (d_{G-S}(x) - g(x)) - h_G(S,T) + \sum_{x \in S} f(x) \geq 0.$$

Theorem 1.3. [8] *A graph G has an $[a,b]$-factor if and only if for any $S \subseteq V(G)$,*

$$\sum_{j=0}^{a-1} (a-j) P_j(G-S) \leq b|S|$$

where $P_j(G-S) = |\{x : d_{G-S}(x) = j\}|$.

Many other results on factors can be found in [1].

In this paper we will convert integer-based definitions and invariants into their fractional analogues. Find the relationships of the programming and the graph theory.Some results on fractional factors, fractional Hamilton graphs, fractional (g,f)-factors and fractional colorings are presented. In particular, our some new results on fractional (g,f)-factors are given. Furthermore, some open problems are presented.

2 Some Results on Fractional Matching

A fractional matching is a function h that assigns to each edge of a graph a number in $[0,1]$ so that, for each vertex x , we have $\sum h(e) \leq 1$ where the sum is taken over all edges incident to x. If $h(e) \in \{0,1\}$ for every edge e, then h is just a matching, or more precisely, the indicator function of a matching. The fractional matching number $\mu_h(G)$ of a graph G is the supremum of $\sum\limits_{e \in E(G)} h(e)$ over all fractional matchings h. It is clear that $\mu_h(G) \geq \mu(G)$ where $\mu(G)$ is the matching number of G. The following results are based on the work of Balas, Balinski, Bourjolly, Lovász, Plummuer, Pulleyblank, and Uhry and so on.

Theorem 2.1. [20] $\mu_h(G) \leq \frac{1}{2}|V(G)|$.

Theorem 2.2. [9] *If G is bipartite, then $\mu_h(G) = \mu(G)$.*

Theorem 2.3. [3] *For any graph G, $2\mu_h(G)$ is an integer. Moreover, there is a fractional matching h for which*

$$\sum_{e \in E(G)} h(e) = \mu_h(G)$$

such that $h(e) \in \{0, 1/2, 1\}$ for every edge e.

A fractional transversal of graph G is a function $p : V(G) \to [0,1]$ satisfying $\sum_{x\in e} p(x) \geq 1$ for every $e \in E(G)$. The fractional transversal number is the infimum of $\sum_{x\in V(G)} p(x)$ taken over all fractional transversals p of G.

Theorem 2.4. [20] *For every graph G, there is a fractional transversal p for which*

$$\sum_{x\in V(G)} p(x) = \mu_h(G)$$

such that $p(x) \in \{0, 1/2, 1\}$ for every vertex x.

Suppose that h is a fractional matching. Then h is a fractional 1-factor (or perfect matching) if and only if for every $x \in V(G)$, $\sum_{e\ni x} h(e) = 1$.

Theorem 2.5 (Fractional Tutte's Theorem). [14] *A graph G has a fractional 1-factor (or perfect matching) if and only if*

$$i(G-S) \leq |S|$$

for every set $S \subseteq V(G)$ where $i(G-S)$ is the number of isolated vertices of $G-S$.

Theorem 2.6 (Fractional Berge's Theorem). [19] *For any graph G,*

$$\mu_h(G) = \frac{1}{2}(|V(G)| - \max\{i(G-S) - |S|\})$$

where the maximum is taken over all $S \subseteq V(G)$.

The above results can be proved by graph theory method or programming method and they are basic for our fractional factor theory. Now we give some new results on fractional matching which obtained by us. At first we introduce two concepts. The first one is introduced by author. The isolated toughness of G is defined as

$$I(G) = \min\{\frac{|S|}{i(G-S)} : S \subset V(G), i(G-S) \geq 2\}.$$

if G is not complete. Otherwise, $I(G) = \infty$[22]. If graph G has a k-matching (a matching with k edges) and for any k-matching M graph G has a fractional 1-factor containing M, then we say that G is fractional k-extendable.

Theorem 2.7. [15] *Let G be a graph and k be a non-negative integer. If the connectivity $\kappa(G) \geq 2k+1$, $I(G) \geq k+1$ and $|V(G)| \geq 2k+1$, then G is fractional k-extendable.*

Theorem 2.8. [17] *Let G be a graph with $|V(G)| \geq 2k+4$ and let F be an arbitrary 1-factor of G. If $G - \{u,v\}$ is fractional k-extendable for each $e = uv \in F$, then G is fractional k-extendable.*

Theorem 2.9. [23] *A graph G has a fractional 1-factor if and only if $bind(G) \geq 1$.*

Theorem 2.10. [12] *Let G be a graph. Then $\mu_h(G) = \mu(G)$ if and only if $D(G)$ is an independent set where $D(G)$ is the set of all vertices in G which are missed by at least one maximum matching of G.*

Finally we present the following problems.

Problem 2.1. *Find the relationship between binding number and fractional k-extendable of a graph.*

Problem 2.2. *Give an algorithm for determining whether a graph is fractional k-extendable.*

Many problems on fractional graph theory can be solved in polynomial time by a linear programming. The constraint matrix in fractional matching number problem has size $|V(G)| \times |E(G)|$. Thus polynomial linear programming solutions for this problem exist. Therefore the fractional matching number of a graph can be computed in polynomial time [20].

3 Fractional Hamiltonian Graphs

A graph G is called fractionally Hamiltonian if there is a function $h : E(G) \to [0, 1]$ such that the following two conditions hold

$$\sum_{e \in E(G)} h(e) = |V(G)|$$

and for all $\emptyset \subset S \subset V(G)$

$$\sum_{e \in [S, \bar{S}]} h(e) \geq 2$$

where $[S, \bar{S}]$ for the set of all edges with exactly one end in S. We called such a function h a fractional Hamiltonian cycle. For example, Petersen's graph is fractionally Hamiltonian , but is not Hamiltonian. Now we give some results on this topic.

Theorem 3.1. [7] *Let h be a fractional Hamiltonian cycle for a graph G and let x be any vertex of G. Then*

$$\sum_{e \ni x} h(e) = 2.$$

Theorem 3.2. [7] *If G is fractionally Hamiltonian , then G has a fractional 1-factor.*

From Theorem 3.2 and Theorem 2.5 we obtain the following result.

Theorem 3.3. [20] *If G is fractionally Hamiltonian and $|V(G)| \geq 3$, then $I(G) \geq 1$.*

Now we give a linear programming for the fractional Hamiltonicity of a graph. Let G be a graph. We consider $h(e)$ to be the "weight" of edge e, which we also denote w_e. Consider the following Linear programming which is called $FHLP$.

$FHLP:$

$$\min \sum_{e \in E(G)} w_e$$

$$subject \quad to \sum_{e \in [S,\bar{S}]} w_e \geq 2 \quad for \quad all \quad S, \emptyset \subset S \subset V(G)$$

$$w_e \geq 0, \quad for \quad all \quad e \in E(G)$$

We have the following theorem which shows the relationship between the graph theory and programming.

Theorem 3.4. [20] *Let G be a graph on at least 3 vertices. Then G is fractional Hamiltonian if and only if the value of the $FHLP$ is exactly $|V(G)|$.*

We have know that the best known necessary condition for a graph G to be Hamiltonian is $t(G) \geq 1$. This fact is true for fractional Hamiltonicity.

Theorem 3.5. [4] *If graph G is fractional Hamiltonian, then $t(G) \geq 1$.*

Theorem 3.6. [4] *Let $t < \frac{3}{2}$. Then there is a graph G with $t(G) \geq t$ that is not fractional Hamiltonian.*

Note that there is a graph G with $t(G) \geq 2$ that is not Hamiltonian. But the following conjecture was presented in [20].

Conjecture 3.1. *If $t(G) \geq 2$, then G is fractional Hamiltonian.*

There are many problems on graphs which are fractional Hamiltonian can be considered. For example, we can consider the structures and properties of fractional Hamiltonian graphs.

Problem 3.1. *Find the necessary and sufficient conditions for a graph to be fractional Hamiltonian.*

Problem 3.2. *Find the relationship between the binding number and fractional Hamiltonian graphs.*

The decision problem of determining whether a graph is Hamiltonian is NP-complete. But the fractional Hamiltonicity can be tested in polynomial time. We can solve this problem using linear programming.

4 Fractional (g, f)-Factors

Let G be a graph and let g and f be two integer-valued functions defined on $V(G)$ such that $0 \leq g(x) \leq f(x)$ for all x in $V(G)$. Let $d_G^h(x) = \sum_{e \in x} h(e)$. A fractional (g, f)-factor is a function h that assigns to each edge of a graph a number in $[0,1]$ so that, for each vertex x of G we have $g(x) \le d_G^h(x) \le f(x)$. If $g(x) = f(x)$ for all $x \in V(G)$, then a fractional (g,f)-factor is called a fractional f-factor. Let a and b be two non-negative integers. If $g(x) = a$ and $f(x) = b$ for every $x \in V(G)$, then a fractional (g,f)-factor is called a fractional $[a,b]$-factor. If $a = b = k$, a fractional $[a,b]$-factor is called a fractional k-factor. A fractional $[0,1]$-factor is also called a fractional matching. At first we introduce the following main theorem which was first given by Astee in [2] and for which the author gave a new proof in [10].

Theorem 4.1 (Fractional Lovász's Theorem). [10] *A graph G has a fractional (g,f)-factor if and only if for any subset S of $V(G)$*

$$g(T) - d_{G-S}(T) \le f(S)$$

where $T = \{x : x \in V(G) \setminus S,\ d_{G-S}(x) \le g(x)\}$.

Recently Liu, Zhang and Ma studied the properties of fractional (g,f)-factors and the relationship between isolated toughness and fractional factors of graphs. The following theorems were obtained.

Theorem 4.2. [26] *A graph G has a fractional k-factor if and only if for any subset S of $V(G)$*

$$\sum_{j=0}^{k-1}(k-j)p_j(G-S) \le k|S|$$

where $p_j(G-S)$ denotes the number of vertices in $G-S$ with degree j.

Theorem 4.3. [10] *Let G be a bipartite graph or $g(x) \ne f(x)$ for all $x \in V(G)$. Then G has a fractional (g,f)-factor if and only if G has a (g,f)-factor.*

Theorem 4.4. [22] *Let G ba a graph. If for every pair of vertices x , y of G,*

$$g(y)d_G(x) \le f(x)d_G(y),$$

then G has a fractional (g,f)-factor.

Theorem 4.5. [10] *There are polynomial algorithms for finding a fractional (g,f)-factor and a maximum fractional (g,f)-factor in graphs.*

Theorem 4.6. [15, 18] *Let G be a connected graph and $k > 0$ be an integer. If $\delta(G) \ge k$ and $I(G) \ge k$, then G has a fractional k-factor.*

Theorem 4.7. [18] *Let G be a graph and let a and b be two integers such that $a \le b$. If $I(G) \ge a - 1 + \frac{a}{b}$ and $\delta(G) \ge I(G)$, then G has a fractional $[a,b]$-factor.*

Theorem 4.8. [25] *Let G be a graph with $|V(G)| \ge k+1$ where $k \ge 2$. Then G has a fractional k-factor if $t(G) \ge k - \frac{1}{k}$.*

Theorem 4.9. [25] *If graph G has a fractional (g,f)-factor, then G has a factional (g,f)-factor h such that $h(e) \in \{0, 1, \frac{1}{2}\}$.*

Other results on fractional factors can be found in [5, 11, 22]. In the following we give some conjectures and problems. Conjecture 4.2 improves the results in Theorem 4.6, Theorem 4.7 and Theorem 4.8.

Conjecture 4.1. *Let G be a graph. If $t(G) \geq k$, then G has a connected fractional k-factor where $k \geq 2$.*

Conjecture 4.2. *Let G be a graph and let a and b be two integers such that $a \leq b$. If $I(G) \geq a - 1 + \frac{a-1}{b}$, then G has a fractional $[a, b]$-factor.*

Problem 4.1. *Find the relationship between fractional factors and the factors in graphs.*

Problem 4.2. *Find the relationship between fractional factors and the binding number of a graph.*

Problem 4.3. *Find the sufficient conditions for a graph to have connected fractional factors.*

It is not difficult to see that the fractional (g, f)-factor problem can be formulated as a linear programming and can also be solved in polynomial time. In another way the polynomial algorithms for solving the problem of fractional (g,f)-factors by finding the increasing chains in a graph were given in [10].

5 The Fractional Covering and Packing of Hypergraphs

In this section we study the fractional covering and packing of hypergraphs. Some problems on fractional graph theory can be solved as the problems on fractional covering and packing of hypergraphs. A hypergraph $\mathcal{H}$ is a pair (S, X), where S is a finite set and X is a family of subsets of S. The set S is called the vertex set of the hypergraph, and so we sometimes write $V(\mathcal{H})$ for S. The elements of X are called huperedges or sometimes just edges. A covering of $\mathcal{H}$ is a collection of hyperedges $X_1, X_2, \cdots, X_j$ so that $S \subset X_1 \cup X_2 \cup \cdots X_j$. The least j for which this is possible is called the covering number of $\mathcal{H}$, and is denoted by $k(\mathcal{H})$. The covering problem can be formulated as an integer program (IP). To each set $X_i \in X$ associate a $0, 1$-variable x_i. The vector $\mathbf{x}$ is an indicator of the sets we have selected for the cover. Let M be the vertex-hyperedge incidence matrix of $\mathcal{H}$. The condition that the indicator vector $\mathbf{x}$ corresponds to a covering is simply $M\mathbf{x} \geq 1$ (that is, every coordinate of $M\mathbf{x}$ is at least 1). Thus $k(\mathcal{H})$ is the value of the integer program minimize $\mathbf{I}'\mathbf{x}$ subject to $M\mathbf{x} \geq 1$ and $\mathbf{x} \geq 0$ where $\mathbf{I}$ represents a vector of all ones.

A packing of a hypergraph $\mathcal{H}$ is a subset $Y \subset S$ with the property that no two elements of Y are together in the same member of X. The packing number $p(\mathcal{H})$ is defined to be the largest size of a packing. The packing number of a graph is its independence number. There is a corresponding IP formulation. Let y_i be a $0, 1$-indicator variable that is 1 just when $s_i \in Y$. The condition that Y is a packing is simply $M'\mathbf{y} \leq 1$ where M is as above. Thus $p(\mathcal{H})$ is the value of the integer program maximize $\mathbf{I}'\mathbf{y}$ subject to $M'\mathbf{y} \leq 1$ and $\mathbf{y} \geq 0$.

This is the dual IP to the covering problem and the following result is true.

Theorem 5.1. [20] *For a hypergraph $\mathcal{H}$, we have $p(\mathcal{H}) \leq k(\mathcal{H})$.*

Note that many graph theory concepts can be seen as hypergraph covering or packing problems. For instance, the chromatic number and matching number. Now we consider *fractional covering and packing.*

We define the fractional covering number and fractional packing number of $\mathcal{H}$, denoted $k_f(\mathcal{H})$ and $p_f(\mathcal{H})$, respectively, to be the values of the dual linear program " minimize $\mathbf{I}'\mathbf{x}$ subject to $M\mathbf{x} \geq 1$ and $\mathbf{x} \geq 0$" and "maximize $\mathbf{I}'\mathbf{y}$ subject to $M'\mathbf{y} \leq 1$ and $\mathbf{y} \geq 0$." By duality we have $k_f(\mathcal{H}) = p_f(\mathcal{H})$. There is a second way to define the fractional covering and packing numbers. We begin by define t-fold covering and t-fold covering number of $\mathcal{H}$ where t is a positive integer. A t-fold covering of $\mathcal{H}$ is a multiset $\{X_1, X_2, \cdots, X_j\}$ where $X_i \in X$ with the property that each $s \in S$ is in at least t of the $X_i's$. The smallest cardinality (least j) of such multiset is called the t-fold covering number of $\mathcal{H}$ and denoted by $k_t(\mathcal{H})$. Clearly, $k_1(\mathcal{H}) = k(\mathcal{H})$. Note that $k_{s+t}(\mathcal{H}) \leq k_s(\mathcal{H}) + k_t(\mathcal{H})$. Therefore we define the fractional covering number of $\mathcal{H}$ to be

$$k_f(\mathcal{H}) = \lim_{t \longrightarrow \infty} \frac{k_t(\mathcal{H})}{t} = \inf_t \frac{k_t(\mathcal{H})}{t}.$$

We can prove that the above two definitions are the same and $k_f(\mathcal{H}) \leq k(\mathcal{H})$.

In the same way, we define a t-fold packing of $\mathcal{H}$ to be a multiset Y of the vertex set with the property that for every $X_i \in X$ we have $\sum_{s \in X} m(s) \leq t$ where m is the multiplicity of $s \in S$ in Y. The t-fold packing number of $\mathcal{H}$, denoted by $p_t(\mathcal{H})$, is the largest cardinality of a t-fold packing. Observe that $p_1(\mathcal{H}) = p(\mathcal{H})$. We define the fractional packing number of $\mathcal{H}$ to be

$$p_f(\mathcal{H}) = \lim_{t \longrightarrow \infty} \frac{p_t(\mathcal{H})}{t}.$$

We also have $p_f(\mathcal{H}) \geq p(\mathcal{H})$. It is easy to see that the following result holds.

Theorem 5.2. [20] *If $\mathcal{H}$ has no exposed vertices, then $k_f(\mathcal{H})$ is a rational number and there exists a positive integer s for which $k_f(\mathcal{H}) = k_s(\mathcal{H})/s$.*

6 Fractional Coloring Problems

The fractional chromatic number of a graph is defined as follows. A b-fold coloring of a graph G assigns to each vertex of G a set of b colors so that adjacent vertices receive disjoint sets of colors. We say that G is $a : b$-colorable if it has a b-fold coloring in which the colors are drawn from a palette of a colors. We also call this coloring an $a : b$-coloring. The least a for which G has a b-fold coloring is the b-fold coloring number of G, denoted $\chi_b(G)$. Note that $\chi_1(G) = \chi(G)$, where $\chi(G)$ is the chromatic number of G. Thus the fractional chromatic number to be

$$\chi_f(G) = \lim_{b \longrightarrow \infty} \frac{\chi_b(G)}{b} = \inf_b \frac{\chi_b(G)}{b}.$$

On the other hand we can describe the fractional chromatic number as follows. Given graph G, we construct a hypergraph $\mathcal{H}$ such that the vertex set of $\mathcal{H}$ is the vertex set of G and the hyperedges of $\mathcal{H}$ are the independent sets of G. Then it ia easy to see that $k(\mathcal{H}) = \chi(G)$, $\chi_f(G) = k_f(G)$ and $\chi_f(G) \leq \chi(G)$. It is easy to see that the following results hold.

Theorem 6.1. *For any graph G, $\chi_f(G) \geq |V(G)|/\alpha(G)$, where $\alpha(G)$ is the independent number of G.*

Theorem 6.2. *Let C_{2m+1} be a cycle with $2m+1$ vertices. Then $\chi_f(G) = 2 + (1/m)$.*

In an $a : b$-coloring of a graph, we assign a set of b colors to each vertex, with adjacent vertices receiving disjoint color sets; the colors are selected from a master palette of a colors. We say a graph G is $a : b$-choosable if for every a-palette P of G we can assign to each vertex v of G a b-set of colors $C(v) \subseteq P(v)$ so that $C(v) \cap C(w) = \emptyset$ when v is adjacent to w. The b-fold list chromatic number of G, denoted $\chi_b^l(G)$, is the least a so that G is $a : b$-choosable. We define the fractional list chromatic number of G to be

$$\chi_f'(G) = \lim_{b \longrightarrow \infty} \frac{\chi_b^l(G)}{b} = \inf_b \frac{\chi_b^l(G)}{b}.$$

Note that if a graph is $a : b$-choosable, then it must be $a : b$-colorable. Thus

$$\chi_f(G) = \lim_{b \longrightarrow \infty} \frac{\chi_b}{b} \leq \lim_{b \longrightarrow \infty} \frac{\chi_b^l}{b} = \chi_b^l(G).$$

Theorem 6.3. [20] *The fractional chromatic number of a graph equals its fractional list number.*

Similarly, we can define the fractional edge coloring of a graph. we can describe the fractional edge chromatic number $\chi_f'(G)$ as follows. Given graph G, we construct a hypergraph $\mathcal{H}$ such that the vertex set of $\mathcal{H}$ is the edge set of G and the hyperedges of $\mathcal{H}$ are the matchings of G. Then it is easy to see that $\chi_f'(G) = k_f(G)$. and $\chi_f'(G) \leq \chi'(G)$ where $\chi'(G)$ is the edge chromatic number of G. Let H be any induced subgraph of G such that $|V(H)|$ is odd and at lest 3. we define

$$\Lambda(G) = \max_H \frac{2|E(G)|}{|V(H)| - 1}.$$

The main result on fractional edge colorings is the following result the proof of which is very difficult.

Theorem 6.4. [20] *For any loopless multigraph G,*

$$\chi_f'(G) = \max\{\Delta(G), \Lambda(G)\}$$

where $\Delta(G)$ is the maximum degree of G.

Now we present some open problems.

Problem 6.1. *When we have $\chi_f(G) = \chi(G)$ and $\chi'_f(G) = \chi'(G)$?*

Problem 6.2. *Given graph G how to find $\chi_f(G)$ and $\chi'_f(G)$?*

A graph G is called fractional vertex-critical if for any edge e of G, $\chi_f(G) < \chi_f(G-e)$ and called fractional edge-critical if for any edge e $\chi'_f(G) < \chi'_f(G-e)$.

Problem 6.3. *Find the properties of factional vertex-critical graphs and fractional edge-critical graphs.*

Note that computing χ'_f can be done in polynomial time. But computing χ_f can not be done in polynomial time although the linear program can be solve by polynomial time. This is because of the linear program may has exponentially many (in the number of vertices) variables. One for each maximal independent set in the graph. Some other new results on fractional factors and colors can be found in [5, 11, 24, 25].

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Seven Types of Random Spherical Triangle in $\mathbf{S}^n$ and Their Probabilities

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Abstract. Spherical triangle in the n-dimensional unit sphere $\mathbf{S}^n$ is classified into seven types according to side and angle. In this paper, we consider random spherical triangles in $\mathbf{S}^n$, and calculate the probabilities of seven types to which random spherical triangles belong. Each probability monotone converges to a certain value as the dimension n tends to infinity. As an application, we can estimate the expectation of numbers of division on acute triangulation of a random spherical triangle in $\mathbf{S}^2$.

1 Introduction

In Euclidean geometry, triangle is roughly classified into two types with respect to angle: acute triangles and obtuse triangles. On the other hand, in spherical geometry, there are several varieties of spherical triangle, because a spherical triangle $\triangle ABC$ on the unit sphere centered at O has six angles: three vertex angles $\angle A$, $\angle B$ and $\angle C$ and three arc angles (sides) $\angle AOB(=\overline{AB}=c)$, $\angle BOC(=\overline{BC}=a)$ and $\angle COA(=\overline{CA}=b)$.

In this paper, we show that there are seven types of spherical triangle in the n-dimensional unit sphere according to whether three sides and three angles are acute or obtuse. Three random points in $\mathbf{S}^n$ determine a spherical triangle by connecting these points as the minor arcs. We calculate the probabilities of seven types to which random spherical triangles belong as in Table 1. As the dimension n of the unit sphere increases, each probability monotone converges to a certain limit value. This simple phenomenon is geometrically understood by considering a small perturbation of right regular spherical triangle.

This paper is made up of four sections. In Section 2, we give a classification of possible spherical triangles in seven types according to side and angle. Section 3 gives the probabilities of seven types, and the asymptotic behavior is studied as the dimension of the sphere tends to infinity in Section 4. We finally provide an application for acute triangulations of triangles on the sphere in Section 5.

2 Seven Types of Spherical Triangle

We denote by ac.(resp. ob.) that the length of side is less(resp. more) than $\pi/2$ respectively, for example, by 2ac.(=1ob.) the spherical triangle with two acute sides and one obtuse side. In the same way, we denote by Ac.(resp. Ob.) that

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Table 1. Probabilities of random triangles

			I_n	$\frac{1}{2}$	$\frac{\pi}{8}+\frac{1}{2\pi}$	$\frac{5}{12}$	$\frac{\pi}{8}+\frac{2}{3\pi}$	$\nearrow$	$\frac{\pi}{4}$
type	side	angle	probability	$\mathbf{S}^2$	$\mathbf{S}^3$	$\mathbf{S}^4$	$\mathbf{S}^5$	$\cdots$	$\mathbf{S}^\infty$
I	3 ac.	3 Ac.	$-\frac{1}{8}+\frac{I_n}{\pi}$	$-\frac{1}{8}+\frac{1}{2\pi}$	$\frac{1}{2\pi^2}$	$-\frac{1}{8}+\frac{7}{12\pi}$	$\frac{2}{3\pi^2}$	$\nearrow$	$\frac{1}{8}$
II	3 ac.	2 Ac.	$\frac{3}{8}-\frac{3I_n}{2\pi}$	$\frac{3}{8}-\frac{3}{4\pi}$	$\frac{3}{16}-\frac{3}{4\pi^2}$	$\frac{3}{8}-\frac{7}{8\pi}$	$\frac{3}{16}-\frac{1}{\pi^2}$	$\searrow$	0
III	2 ac.	2 Ac.	$\frac{3I_n}{2\pi}$	$\frac{3}{4\pi}$	$\frac{3}{16}+\frac{3}{4\pi^2}$	$\frac{7}{8\pi}$	$\frac{3}{16}+\frac{1}{\pi^2}$	$\nearrow$	$\frac{3}{8}$
IV	1 ac.	2 Ac.	$\frac{3}{4}-\frac{3I_n}{\pi}$	$\frac{3}{4}-\frac{3}{2\pi}$	$\frac{3}{8}-\frac{3}{2\pi^2}$	$\frac{3}{4}-\frac{7}{4\pi}$	$\frac{3}{8}-\frac{2}{\pi^2}$	$\searrow$	0
V	1 ac.	1 Ac.	$-\frac{3}{8}+\frac{3I_n}{\pi}$	$-\frac{3}{8}+\frac{3}{2\pi}$	$\frac{3}{2\pi^2}$	$-\frac{3}{8}+\frac{7}{4\pi}$	$\frac{2}{\pi^2}$	$\nearrow$	$\frac{3}{8}$
VI	1 ac.	0 Ac.	$\frac{3}{8}-\frac{3I_n}{2\pi}$	$\frac{3}{8}-\frac{3}{4\pi}$	$\frac{3}{16}-\frac{3}{4\pi^2}$	$\frac{3}{8}-\frac{7}{8\pi}$	$\frac{3}{16}-\frac{1}{\pi^2}$	$\searrow$	0
VII	0 ac.	0 Ac.	$\frac{I_n}{2\pi}$	$\frac{1}{4\pi}$	$\frac{1}{16}+\frac{1}{4\pi^2}$	$\frac{7}{24\pi}$	$\frac{1}{16}+\frac{1}{3\pi^2}$	$\nearrow$	$\frac{1}{8}$

the vertex angle is less(resp. more) than $\pi/2$, for example, by 3Ac.(=0Ob.) the spherical triangle with three acute vertex angles. In the following argument, we only consider spherical triangle in $\mathbf{S}^n (n \geq 2)$ such that its three sides and three angles are not 0, $\pi/2$ nor π. This exception has no influence on the probabilities of random spherical triangle.

Theorem 1. *Spherical triangle in $\mathbf{S}^n (n \geq 2)$ is classified into the following seven types:*
I(3ac. 3Ac.), II(3ac., 2Ac.), III(2ac., 2Ac.), IV(1ac., 2Ac.),
V(1ac.,1Ac.), VI(1ac., 0Ac.), VII(0ac., 0Ac.)
as in Table 2.

Proof. As for $n \geq 3$, spherical triangle $\triangle ABC$ in $\mathbf{S}^n$ can be reduced to that in $\mathbf{S}^2$ by considering the 2-subsphere spanned by the three points A, B, and C. In this way, let us consider spherical triangle in $\mathbf{S}^2$. The important tool to solve this problem is the spherical trigonometry, especially, the following two laws of cosines for sides and angles (e.g., [1],p286, [3],p54 and p59):

$$\cos a = \cos b \cos c + \sin b \sin c \cos A, \tag{1}$$
$$\cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

Since the signs of $\sin b, \sin c, \sin B$ and $\sin C$ are all positive,

$$sign(\cos A) = sign(\cos a - \cos b \cos c), \tag{2}$$
$$sign(\cos a) = sign(\cos A + \cos B \cos C). \tag{3}$$

From (3), 3Ac. implies 3ac., so the possibilities of (2ac., 3Ac.), (1ac., 3Ac.) and (0ac., 3Ac.) are denied. In the similar way, from (2), 3ob.(=0ac.) implies 3Ob.(=0Ac.), so the possibilities of (0ac., 2Ac.) and (0ac., 1Ac.) are denied. In addition, from (3), (A, B, C)=(Ac., Ob., Ob.) implies (a, b, c)=(ac., ob., ob.), hence the possibilities of (3ac., 1Ac.) and (2ac., 1Ac.) are denied. In the similar way, from (2), (a, b, c)=(ac., ac., ob.) implies (A, B, C)=(Ac., Ac., Ob.), hence the possibility of (2ac., 0Ac.) is denied. Finally, we will show that the possibility of (3ac., 0Ac.) is also denied. The condition 3ac. implies that $\triangle ABC$ is inside of a right regular triangle with the area $\pi/2$. However, 0Ac.(=3Ob.) implies that the area of $\triangle ABC$ is greater than $\pi/2$. It is a contradiction. We have thus proved the theorem. □

Table 2. Seven types of spherical triangle

side \ angle	3Ac. (0Ob.)	2Ac. (1Ob.)	1Ac. (2Ob.)	0Ac. (3Ob.)
3ac. (0ob.)	I	II	×	×
2ac. (1ob.)	×	III	×	×
1ac. (20b.)	×	IV	V	VI
0ac. (3ob.)	×	×	×	VII

3 Probabilities of Seven Types

In this section, we will calculate probabilities of these seven types to which random spherical triangles belong. Without loss of generality, assume that points $A = (1, 0, 0, \ldots, 0)$ and $B = (\cos\theta, \sin\theta, 0, \ldots, 0)$ where $\theta \in (0, \pi)$ is a random variable with a certain density function depending on the dimension n.

Figure 1 shows the relation between the third point C and seven types on the hemisphere of $\mathbf{S}^2$. Point A and B are on the equator. According to whether the arc length $\overline{AB}$ is acute or obtuse, each hemisphere is divided into several regions. If point C is in a region, the spherical triangle $\triangle ABC$ belongs to the label of the region. Note that almost all boundaries of these regions are given as the intersection of a hyperplane and the sphere. There are a few curves, for example, the boundary between type I and type II in Figure 1(left), however, we will calculate the probabilities without considering these curves.

For $n \geq 3$, the divided regions are also composed of the intersection of a hyperplane and $\mathbf{S}^n$, hence, the argument is parallel to that of $\mathbf{S}^2$. Only one difference among them is the density function of the arc length between two random points.

Theorem 2. *The probability density function $f_n(\theta)$ of the arc length θ between two random points $\mathbf{x}, \mathbf{y}$ in $\mathbf{S}^n (n \geq 2)$ is given as*

$$f_n(\theta) = \frac{(\sin\theta)^{n-1}}{B\left(\frac{1}{2}, \frac{n}{2}\right)} \qquad (0 \leq \theta \leq \pi)$$

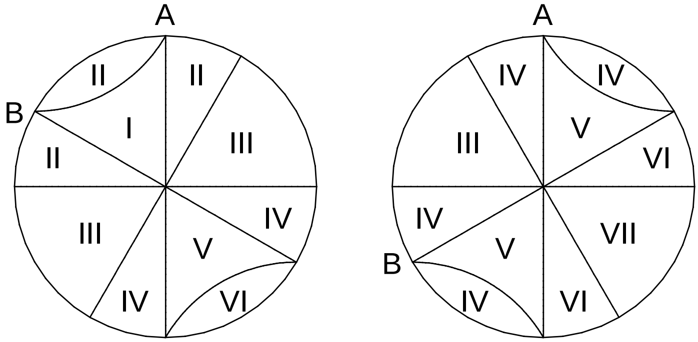

Fig. 1. Position of C and 7 types: $\overline{AB} < \pi/2$ (left) and $\overline{AB} > \pi/2$ (right)

where $B\,(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ *is the beta function.*

Proof. Let $\mathbf{x} = (x_0, x_1, \ldots, x_n)$ be a random point in $\mathbf{S}^n$. First, we will show that the random variable x_0^2 has the beta distribution

$$\mathrm{Beta}\left(\frac{1}{2}, \frac{n}{2}\right) = \frac{x^{-\frac{1}{2}}(1-x)^{\frac{n}{2}-1}}{B\left(\frac{1}{2}, \frac{n}{2}\right)} \qquad (0 \le x \le 1).$$

Let $z_0, z_1, \ldots, z_n$ be independent normal variables with zero mean and unit variance, i.e., the density function of z_i is given by $e^{-z^2/2}/\sqrt{2\pi}$. Then the point $\mathbf{z} = (z_0, z_1, \ldots, z_n) \in \mathbf{R}^{n+1}$ has the probability density function

$$\frac{1}{(2\pi)^{(n+1)/2}} e^{-(z_0^2+z_1^2+\ldots+z_n^2)/2} = \frac{1}{(2\pi)^{(n+1)/2}} e^{-|\mathbf{z}|^2/2}.$$

Therefore, the point $\mathbf{z}/|\mathbf{z}|$ is uniformly distributed on the surface of the unit sphere $\mathbf{S}^n$. In this way, the random point $\mathbf{x} = (x_0, x_1, \ldots, x_n)$ is realized as $\mathbf{z}/|\mathbf{z}|$, i.e.,

$$x_i = \frac{z_i}{\sqrt{z_0^2 + z_1^2 + \ldots + z_n^2}} \qquad (i = 0, 1, \ldots, n).$$

Two random variables z_0^2 and $z_1^2 + z_2^2 + \ldots + z_n^2$ are independent and have the χ^2 distributions with degree of freedom 1 and n, respectively. Now recall the fact that if X and Y are independent random variables having the χ^2 distributions with degrees of freedom a and b, then the random variable $X/(X+Y)$ has the beta distribution Beta$(a/2, b/2)$(see, e.g., [4], p.64 and [5], p.187). Using this fact, x_0^2 has the distribution Beta$(1/2, n/2)$.

Now we can calculate the distribution function $F_n(\theta)$ of $f_n(\theta)$. We can assume that $\mathbf{y} = (1, 0, \ldots, 0)$ and $\mathbf{x} = (x_0, x_1, \ldots, x_n)$. If $0 \le \theta \le \pi/2$,

$$F_n(\theta) = \Pr(\overline{\mathbf{xy}} \le \theta) = \frac{1}{2} \int_{\cos^2\theta}^{1} \frac{x_0^{-\frac{1}{2}}(1-x_0)^{\frac{n}{2}-1}}{B\left(\frac{1}{2}, \frac{n}{2}\right)} dx_0$$

$$= \int_0^\theta \frac{(\sin\varphi)^{n-1}}{B\left(\frac{1}{2},\frac{n}{2}\right)} d\varphi,$$

where $x_0 = \cos^2\varphi$. Therefore, the density function $f_n(\theta)$ is given as

$$f_n(\theta) = \frac{(\sin\theta)^{n-1}}{B\left(\frac{1}{2},\frac{n}{2}\right)} \tag{4}$$

for $0 \le \theta \le \pi/2$. From the symmetry of $\mathbf{S}^n$, $f_n(\theta) = f_n(\pi - \theta)$ implies that (4) is valid for $0 \le \theta \le \pi$. This completes the proof of Theorem 2. □

Before the calculation of the probabilities, we prepare the following definite integral I_n:

$$I_n = \int_0^{\frac{\pi}{2}} \theta f_n(\theta) d\theta = \int_0^{\frac{\pi}{2}} \theta \frac{(\sin\theta)^{n-1}}{B\left(\frac{1}{2},\frac{n}{2}\right)} d\theta.$$

It is easy to check that I_n satisfies the recurrence formula

$$I_{n+2} - I_n = \frac{1}{B\left(\frac{1}{2},\frac{n}{2}\right) n(n+1)} \qquad (n \ge 1), \qquad I_1 = \frac{\pi}{8}, \quad I_2 = \frac{1}{2}, \tag{5}$$

which is convenient for the following calculations.

Theorem 3. *The probabilities of seven types in $\mathbf{S}^n (n \ge 2)$ are given as*
$\Pr(\mathrm{I}) = -\frac{1}{8} + \frac{I_n}{\pi}$, $\Pr(\mathrm{II}) = \frac{3}{8} - \frac{3I_n}{2\pi}$, $\Pr(\mathrm{III}) = \frac{3I_n}{2\pi}$, $\Pr(\mathrm{IV}) = \frac{3}{4} - \frac{3I_n}{\pi}$,
$\Pr(\mathrm{V}) = -\frac{3}{8} + \frac{3I_n}{\pi}$, $\Pr(\mathrm{IV}) = \frac{3}{8} - \frac{3I_n}{2\pi}$, $\Pr(\mathrm{VII}) = \frac{I_n}{2\pi}$,
where I_n is defined as (5).

Proof. As in Figure 1, the probability of each type is simply calculated by the expectation of the ratio of the region to $\mathbf{S}^n$ with the weight $f_n(\theta)$.

$$\begin{aligned}\Pr(\mathrm{I} \cup \mathrm{II}) &= \int_0^{\frac{\pi}{2}} \Pr\left(\triangle ABC \in \mathrm{I} \cup \mathrm{II} \mid \overline{AB} = \theta\right) f_n(\theta) d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{\pi - \theta}{2\pi} f_n(\theta) d\theta = \frac{1}{4} - \frac{1}{2\pi}\int_0^{\frac{\pi}{2}} \theta f_n(\theta) d\theta.\end{aligned}$$

The probability of Type II is given as three times of the case that $\angle ABC \in (\pi/2, \pi)$, hence

$$\Pr(\mathrm{II}) = 3 \times \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2} - \theta}{2\pi} f_n(\theta) d\theta = \frac{3}{8} - \frac{3}{2\pi}\int_0^{\frac{\pi}{2}} \theta f_n(\theta) d\theta,$$

$$\Pr(\mathrm{I}) = -\frac{1}{8} + \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \theta f_n(\theta) d\theta.$$

The probability of Type III is given as three times of the case that $\overline{AB} \in (\pi/2, \pi)$, hence, using $f_n(\pi - \theta) = f_n(\theta)$,

$$\Pr(\mathrm{III}) = 3 \times \int_{\frac{\pi}{2}}^{\pi} \frac{\pi - \theta}{2\pi} f_n(\theta) d\theta = \frac{3}{2\pi}\int_0^{\frac{\pi}{2}} \theta f_n(\theta) d\theta.$$

The total probability of Types IV, V and VI is given as three times of the case that $\overline{AB} \in (0, \pi/2)$, hence

$$\Pr(\mathrm{IV} \cup \mathrm{V} \cup \mathrm{VI}) = 3 \times \int_0^{\frac{\pi}{2}} \frac{\pi - \theta}{2\pi} f_n(\theta) d\theta = \frac{3}{4} - \frac{3}{2\pi} \int_0^{\frac{\pi}{2}} \theta f_n(\theta) d\theta.$$

The probability of Type VI is given as three times of the case that $\overline{AB} \in (\pi/2, \pi)$ and $\overline{BC} \in (0, \pi/2)$, hence

$$\Pr(\mathrm{VI}) = 3 \times \int_{\frac{\pi}{2}}^{\pi} \frac{\theta - \frac{\pi}{2}}{2\pi} f_n(\theta) d\theta = \frac{3}{8} - \frac{3}{2\pi} \int_0^{\frac{\pi}{2}} \theta f_n(\theta) d\theta.$$

The total probability of Types V and VI is given as three times of the case that $\overline{AB} \in (0, \pi/2)$, hence

$$\Pr(\mathrm{V} \cup \mathrm{VI}) = 3 \times \int_0^{\frac{\pi}{2}} \frac{\theta}{2\pi} f_n(\theta) d\theta = \frac{3}{2\pi} \int_0^{\frac{\pi}{2}} \theta f_n(\theta) d\theta.$$

Therefore,

$$\Pr(\mathrm{V}) = -\frac{3}{8} + \frac{3}{\pi} \int_0^{\frac{\pi}{2}} \theta f_n(\theta) d\theta, \qquad \Pr(\mathrm{IV}) = \frac{3}{4} - \frac{3}{\pi} \int_0^{\frac{\pi}{2}} \theta f_n(\theta) d\theta.$$

Finally, as for Type VII,

$$\Pr(\mathrm{VII}) = \int_{\frac{\pi}{2}}^{\pi} \frac{\pi - \theta}{2\pi} f_n(\theta) d\theta = \frac{1}{2\pi} \int_0^{\frac{\pi}{2}} \theta f_n(\theta) d\theta.$$

This completes the proof of Theorem 3. □

Remark 1. There are several simple relations among these seven probabilities:

$$\begin{aligned} \Pr(\mathrm{III}) = 3\Pr(\mathrm{VII}),\ \Pr(\mathrm{IV}) &= 2\Pr(\mathrm{II}), \\ \Pr(\mathrm{V}) = 3\Pr(\mathrm{I}),\quad \Pr(\mathrm{VI}) &= \Pr(\mathrm{II}). \end{aligned} \tag{6}$$

4 Limit Values of Seven Types

In this section, we will investigate the asymptotic behavior of seven probabilities as the dimension n tends to infinity.

Theorem 4. *The seven probabilities monotone converge as the dimension n of $\mathbf{S}^n$ tends to infinity, and the limit values are given as the follows:*
$\Pr(\mathrm{I}) \nearrow 1/8$, $\Pr(\mathrm{II}) \searrow 0$, $\Pr(\mathrm{III}) \nearrow 3/8$, $\Pr(\mathrm{IV}) \searrow 0$,
$\Pr(\mathrm{V}) \nearrow 3/8$, $\Pr(\mathrm{VI}) \searrow 0$, $\Pr(\mathrm{VII}) \nearrow 1/8$.

Proof. It is enough to show that $I_n \nearrow \pi/4$. First, since the function

$$2 f_n(\theta) = \frac{(\sin\theta)^{n-1}}{B\left(\frac{1}{2}, \frac{n}{2}\right)/2}$$

is a continuous probability density function inside the interval $(0, \pi/2)$,

$$\int_0^{\frac{\pi}{2}} \theta\,(2f_n(\theta))\,d\theta = 2I_n.$$

This density function concentrates on $\theta = \pi/2$ as n increases, hence the monotoneity of I_n is trivial and $\lim_{n\to\infty} I_n \le \pi/4$.

On the other hand, I_{2m} is related to the Taylor expansion of arcsine. In fact, for $|z| < 1$,

$$(1-z)^{-\frac{1}{2}} = 1 + \frac{1}{2}z + \frac{1\cdot 3}{2\cdot 4}z^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}z^3 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}z^4 + \cdots.$$

Substitution of x^2 in z and integration yield the Taylor expansion of arcsine:

$$\arcsin x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\frac{x^9}{9} + \cdots \quad (|x| < 1).$$

From the recurrence formula (5),

$$I_{2m} = \frac{1}{2} + \frac{1}{2}\frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{1\cdot 3}{2\cdot 4}\frac{1}{5} + \frac{1}{2}\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{7} + \cdots + \frac{1}{2}\frac{1\cdot 3\cdots(2m-3)}{2\cdot 4\cdots(2m-2)}\frac{1}{2m-1}.$$

For $|x| < 1$,

$$\lim_{m\to\infty} I_{2m} > \frac{1}{2}\arcsin x,$$

therefore, $\lim_{m\to\infty} I_{2m} \ge \lim_{x\to 1} 1/2 \arcsin x = \pi/4$. This completes the proof of Theorem 4. □

Remark 2. From the fact $I_{2m+1} \nearrow \pi/4$ $(m \to \infty)$, we get

$$\frac{\pi^2}{8} = \frac{1}{1\cdot 2} + \frac{2}{1}\frac{1}{3\cdot 4} + \frac{2\cdot 4}{1\cdot 3}\frac{1}{5\cdot 6} + \frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5}\frac{1}{7\cdot 8} + \cdots = \sum_{m=0}^{\infty}\frac{(m!)^2 2^{2m}}{(2m+2)!}.$$

Remark 3. Let us explain these limit values from the geometrical point of view. The more the dimension n increases, the more the density function $f_n(\theta)(0 \le \theta \le \pi)$ concentrates on $\pi/2$. Therefore, we can assume that random spherical triangle is almost a regular right triangle. Three sides a, b and c are nearly equal to $\pi/2$, i.e., $a = \pi/2 + \varepsilon_a, b = \pi/2 + \varepsilon_b, c = \pi/2 + \varepsilon_c$ where $\varepsilon_a, \varepsilon_b$ and ε_c are infinitesimally small. In the same way, three vertex angles A, B and C are nearly equal to $\pi/2$, i.e., $A = \pi/2 + \varepsilon_A, B = \pi/2 + \varepsilon_B, C = \pi/2 + \varepsilon_C$ where $\varepsilon_A, \varepsilon_B$ and ε_C are also infinitesimally small. Applying these values to the spherical cosine law for sides (1),

$$-\sin\varepsilon_a = \sin\varepsilon_b \sin\varepsilon_c - \cos\varepsilon_b \cos\varepsilon_c \sin\varepsilon_A,$$

hence we get $\sin\varepsilon_a \sin\varepsilon_A > 0$, i.e., the signs of ε_a and ε_A are the same. This fact implies that for sufficiently large dimension, random spherical triangle tends to belong to the types I, III, V and VII. Assuming that the signs of ε_a, ε_b and ε_c are random and independent, we can get the desired result.

5 Acute Triangulation of a Random Spherical Triangle

In [2], Itoh and Zamfirescu studied acute triangulations of spherical triangles. They classified the spherical triangles into the following three types.

(i) At most one edge length is larger than $\pi/2$ and precisely one angle is obtuse or right.
(ii) No angle is obtuse, or all 3 edge lengths are larger than $\pi/2$, or precisely 2 edges have lengths larger than $\pi/2$ and both opposite angles are obtuse or right.
(iii) Precisely two edges have lengths larger than $\pi/2$ and one of the opposite angles is acute.

In Case (i), every non-acute spherical triangle can be triangulated with 7 acute triangles. And in Cases (ii) and (iii), any spherical triangle is triangulable with at most 10 acute triangles. Using this result, we can estimate the upper bound of the expectation of the number of acute triangles in acute triangulation. Case (i) corresponds to Types II and III in our classification. Case (iii) corresponds to Type IV, and other types belong to Case (ii). Of course, triangles in Type I is acute, hence,

$$E(\#(\text{acute triangles})) \leq 1 \times \left(-\frac{1}{8} + \frac{1}{2\pi}\right) + 7 \times \frac{3}{8} + 10 \times \left(\frac{3}{4} - \frac{1}{2\pi}\right),$$

this value is nearly equal to 8.6.

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(3,2)-Track Layout of Bipartite Graph Subdivisions

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Abstract. A $(3,2)$-*track layout* of a graph G consists of a 2-*track assignment* of G and an edge 3-coloring of G with no monochromatic X-*crossing*. This paper studies the problem of $(3,2)$-track layout of bipartite graph subdivisions. Recently Dujmović and Wood showed that every graph G with n vertices has a $(3,2)$-track subdivision of G with $4\lceil\log qn(G)\rceil + 3$ division vertices per edge, where $qn(G)$ is the queue number of G. This paper improves their result for the case of complete bipartite graphs, and shows that every complete bipartite graph $K_{m,n}$ has a $(3,2)$-track subdivision of $K_{m,n}$ with $2\lceil\log qn(K_{m,n})\rceil + 1$ division vertices per edge, where m and n are numbers of vertices of the partite sets of $K_{m,n}$ with $m \geq n$.

1 Introduction

A graph $G_{m,n}$ is a *bipartite graph* having *partite sets* A with m vertices and B with n vertices if $V(G) = A \cup B$, $A \cap B = \phi$ and each edge joins a vertex of A to a vertex of B. A bipartite graph $G_{m,n}$is *complete* if $G_{m,n}$ contains all edges joining vertices in distinct sets. A complete bipartite graph is denoted by $K_{m,n}$.

An *ordering* of a set S is a total order $<_\sigma$ on S. It will be convenient to interchange "σ" and $<_\sigma$ when there is no ambiguity. A *vertex ordering* of a graph G is an ordering σ of the vertex set $V(G)$.

A *vertex t-coloring* of a graph G is a partition $\{V_1, V_2\}$ of $V(G)$ such that for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$ then $i \neq j$. Suppose that each color class V_i is ordered by $<_i$. Then the ordered set $(V_i, <_i)$ is called a *track*, and $\{(V_1, <_1), (V_2, <_2)\}$ is called a *2-track assignment* of G.

An X-*crossing* in a track assignment consists of two edges vw and xy such that $v <_i x$ and $y <_j w$, for distinct colors i and j. An *edge 3-coloring* of G is simply a partition $\{E_1, E_2, E_3\}$ of $E(G)$. An edge $vw \in E_i$ is said to be *colored* i. A $(3,2)$-*track layout* of G consists of a 2-track assignment of G and an edge 3-coloring of G with no monochromatic X-crossing. A graph admitting a $(3,2)$-track layout is called a $(3,2)$-*track graph*. Track layouts were introduced by Dujmović, Morin, and Wood [1].

If $e = uv$ is an edge of G, then e is *subdivided* when it is replaced by the edges uw and wv. We call the new vertex w the *division vertex*. If every edge of G is

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subdivided, the resulting graph is the *subdivision graph*. Note that a graph is also considered to be a subdivision of itself.

This paper studies (3,2)-track layouts of graph subdivisions. Recently Dujmović and Wood [2] showed the following proposition:

Proposition 1. *(Dujmović and Wood* [**2**]*) Every graph G with n vertices has a $(3,2)$-track subdivision of G with $4\lceil \log qn(G)\rceil + 3$ division vertices per edge, where $qn(G)$ is the queue number of G.*

The definition of the *queue number* is as follows. In a vertex ordering σ of a graph G, let $L(e)$ and $R(e)$ denote the endpoints of each edge $e \in E(G)$ such that $L(e) <_\sigma R(e)$. Consider two edges $e, f \in E(G)$ with no common endpoint such that $L(e) <_\sigma L(f)$. If $L(e) <_\sigma L(f) <_\sigma R(f) <_\sigma R(e)$ then e and f *nest*. A *queue* is a set of edges $E \subset E(G)$ such that no two edges in E nest. For an integer $d > 0$, a *d-queue layout of G* consists of a vertex ordering σ of G and a partition $\{E_\ell : 1 \leq \ell \leq d\}$ of $E(G)$, such that each E_ℓ is a queue in σ. The *queue number* $qn(G)$ of a graph G is the minimum d such that there is a d-queue layout of G. As for queue layout, see [3,4] etc. There is a summary of bounds on the queue number for various kinds of graph family in [5].

As for Proposition 1, Dujmović and Wood [2] also showed that the order of the number of division vertices is optimal. Thus, to find a track layout with fewer tracks, edge-colorings and division vertices for various kinds of graph family become an interesting problem.

This paper deals with the number of division vertices of bipartite graphs and shows the following theorem:

Theorem 1. *Every bipartite graph $G_{m,n}$ has a $(3,2)$-track subdivision with $2\lceil \log n\rceil - 1$ division vertices per edge, where $m \geq n$.*

For the queue number of a complete bipartite graph $K_{m,n}$, L. S. Heath and A. L. Rosenberg [**4**] showed the following proposition:

Proposition 2. *(L. S. Heath and A. L. Rosenberg* [**4**]*)*

$$qn(K_{m,n}) = \min\left(\lceil m/2\rceil, \lceil n/2\rceil\right).$$

Applying Proposition 2 to Theorem 1, we have the following Theorem:

Theorem 2. *Every complete bipartite graph $K_{m,n}$ $(m \geq n)$ has a $(3,2)$-track subdivision with*

$$2\lceil \log qn(K_{m,n})\rceil + 1$$

division vertices per edge.

This Theorem 2 improves the Dujmović and Wood's result (Proposition 1) for the case of complete bipartite graphs. The proof of Theorem 1 is similar to that of their result [2], however, it becomes simpler by capitalizing the character of bipartite graphs.

2 Proof of Theorem 2

First, we define a breadth-first ordering of number-string sets.

Let $S = \{0, 1\}$ be the binary alphabet and S^* the set of all strings over S. If $s \in S^*$ has length k $(k > 0)$, then write $s = s_1 s_2 \dots s_k$ where s_i is the character of s in position i. Order the elements of S by $0 < 1$. Define a *breadth-first ordering* $<_*$ on S^* as follows: Suppose $s, t \in S^*$.

1. If s is shorter than t, then $s <_* t$.
2. Suppose s and t have the length k. If i is the first position where s and t differ and $s_i < t_i$, then $s <_* t$.

For a number n $(n > 1)$, define $k = \lceil \log n \rceil$. A number s $(0 \le s \le n-1)$ has a unique representation as a string in S^k using the binary representation, where S^k is the set of all elements of length k. For a number s, use the representation $s_1 \dots s_k$ for its binary representation, where s_1 is the highest-order digit. For a string $s = s_1 \dots s_k$ in S^k let $s(i)$ be the string consisting of the first i letters of s, that is, $s(i) = s_1 \dots s_i$ and $s(0)$ be the empty string ϵ.

Next, we will construct a $(3, 2)$-track layout of $G_{m,n}$ $(m \ge n)$ with $2\lceil \log n \rceil - 1$ division vertices per edge.

Define $k = \lceil \log n \rceil$. Consider a subdivision $G^*_{m,n}$ of $G_{m,n}$ made by subdividing each edge

$$(a_s, b_t) \in E(G_{m,n}) \quad (a_s \in A, b_t \in B, 0 \le s < m, 0 \le t < n)$$

by adding vertices labeled as follows:

$$V(G^*_{m,n}) = V_1 \cup V_2,$$

$$V_1 = \{(a_s, b_t; i) | (a_s, b_t) \in E(G_{m,n}), 0 \le i \le k, 0 \le s < m, 0 \le t < n\},$$

$$V_2 = \{(a_s, b_t; i-1, i) | (a_s, b_t) \in E(G_{m,n}), 0 < i \le k, 0 \le s < m, 0 \le t < n\},$$

where $(a_s, b_t; 0)$ is identified with a_s and $(a_s, b_t; k)$ is identified with b_t.

We will construct a $(3, 2)$-track layout of $G^*_{m,n}$; first we define the vertex orderings σ of V_1 and π of V_2, respectively. Then add 3 numbers, 0, 1, 2 to edges of $G^*_{m,n}$ so that there is no monochromatic X-crossing.

Theorem 3. *There exists a* $(3, 2)$*-track layout of the subdivision* $G^*_{m,n}$ *of a bipartite graph* $G_{m,n}$.

Proof. First, we define the vertex ordering σ of V_1. Two division vertices $(a_s, b_t; i)$, $(a_p, b_q; j) \in V_1$ are ordered $(a_s, b_t; i) <_\sigma (a_p, b_q; j)$ if one of the following three conditions holds:

1. $t(i) <_* q(j)$.
2. $t(i) = q(j)$ and $s < p$.
3. $t(i) = q(j)$ and $s = p$, $t < q$.

For example, if we consider a complete bipartite graph $K_{4,4}$ (i.e., $m = n = 4$), the vertex ordering σ of $K_{4,4}$ is as follows:

$$a_{00}, a_{01}, a_{10}, a_{11}, (a_{00}, b_{00}; 1), (a_{00}, b_{01}; 1), (a_{01}, b_{00}; 1), (a_{01}, b_{01}; 1),$$
$$(a_{10}, b_{00}; 1), (a_{10}, b_{01}; 1), (a_{11}, b_{00}; 1), (a_{11}, b_{01}; 1),$$
$$(a_{00}, b_{10}; 1), (a_{00}, b_{11}; 1), (a_{01}, b_{10}; 1), (a_{01}, b_{11}; 1),$$
$$(a_{10}, b_{10}; 1), (a_{10}, b_{11}; 1), (a_{11}, b_{10}; 1), (a_{11}, b_{11}; 1), b_{00}, b_{01}, b_{10}, b_{11}.$$

Next, we define the vertex ordering π of V_2. Two division vertices $(a_s, b_t; i - 1, i), (a_p, b_q; j - 1, j) \in V_2$ are ordered $(a_s, b_t; i - 1, i) <_\pi (a_p, b_q; j - 1, j)$ if one of the following two conditions holds:

1. $(a_s, b_t; i - 1) <_\sigma (a_p, b_q; j - 1)$.
2. $(a_s, b_t; i - 1) = (a_p, b_q; j - 1)$ and $(a_s, b_t; i) <_\sigma (a_p, b_q; j)$.

Note that both σ and π are total orderings. As for the adjacency relations of vertices in $V(G^*_{m,n})$, connect $(a_s, b_t; i - 1)$ and $(a_s, b_t; i - 1, i)$ and connect $(a_s, b_t; i - 1, i)$ and $(a_s, b_t; i)$ $(0 < i \leq k)$ if there is the edge $a_s b_t \in E(G_{m,n})$.

Next, we define the color of each division edges of $G^*_{m,n}$ as follows: Edges

$$E_1 = \{((a_s, b_t; i - 1), (a_s, b_t; i - 1, i)) : 0 < i \leq k\}$$

are colored 2. Edges

$$E_2 = \{((a_s, b_t; i - 1, i), (a_s, b_t; i)) : 0 < i \leq k\}$$

are colored t_i.

Finally, we show that this track layout is legal, i.e., no two edges in this track assignment $\{(V_1, <_\sigma), (V_2, <_\pi)\}$ form monochromatic X-crossing.

First, consider two edges

$$((a_s, b_t; i-1), (a_s, b_t; i-1, i)), ((a_p, b_q; j-1), (a_p, b_q; j-1, j)) \in E_1 \quad (0 < i, j \leq k).$$

Although they receive the same color 2, these two edges don't cross each other because of the definition of the vertex ordering π of V_2.

Next, consider two edges

$$((a_s, b_t; i - 1), (a_s, b_t; i - 1, i)) \in E_1 \quad \text{and} \quad ((a_p, b_q; j - 1, j), (a_p, b_q; j)) \in E_2$$

$(0 < i, j \leq k)$. The edge $((a_s, b_t; i - 1), (a_s, b_t; i - 1, i))$ receives color 2 while the edge $((a_p, b_q; j - 1, j), (a_p, b_q; j))$ receives the color 0 or 1. Thus they don't form monochromatic X-crossing.

Finally consider two edges

$$((a_s, b_t; i - 1, i), (a_s, b_t; i)), ((a_p, b_q; j - 1, j), (a_p, b_q; j)) \in E_2 \quad (0 < i, j \leq k).$$

Let the two edges form X-crossing. We may assume that the endpoints of the two edges are laid out in the order $(a_s, b_t; i-1, i) <_\pi (a_p, b_q; j-1, j)$ and $(a_p, b_q; j) <_\sigma (a_s, b_t; i)$. We want to show that the two division edges receive different color.

By the above assumption and the definition of the division vertex ordering, we have $t(i-1) \leq_* q(j-1) <_* q(j) \leq_* t(i)$. From the definition of the breadth-first ordering, this inequality holds only when $i = j$.

Suppose $t(i-1) <_* q(i-1)$, then by the definition of the division vertex ordering we have $t(i) <_* q(i)$ which contradicts the assumption. Thus we have $t(i-1) = q(i-1)$ and $q(i) \leq_* t(i)$.

Suppose $q(i) = t(i)$ then, by the definition of vertex ordering, we have $(a_s, b_t; i) <_\sigma (a_p, b_q; j)$ which contradicts the assumption. Therefore $t(i-1) = q(i-1)$ and $q_i < t_i$. In this case, $q_i = 0$ and $t_i = 1$, thus the two edges have different colors.

Thus we have proved that this track layout is leagal.

In the proof of Theorem 3, each edge (a_s, b_t) of $G_{m,n}$ is divided by adding $2k-1$ division vertices in the subdivision $G^*_{m,n}$, where $k = \lceil \log n \rceil$. Thus, we have Theorem 1.

3 Conclusion

This paper improves the Dujmović and Wood's result in [2] for the case of complete bipartite graphs $K_{m,n}$ having two partite sets with m and n vertices respectively ($m \geq n$) and shows that $K_{m,n}$ has a $(3,2)$-track subdivision with $2\lceil \log qn(K_{m,n}) \rceil + 1$ division vertices per edge. We don't know whether this result is best possible or not. To find better track layout for complete bipartite graphs is still an interesting problem.

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Bartholdi Zeta Functions of Branched Coverings of Digraphs

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Abstract. In this paper, we consider branched (di)graph coverings or graphs with semi-free action. A (di)graph with semi-free action of a group Γ is a (di)graph such that a sub(di)graph is fixed by Γ while its complement carries a free action. A branched regular covering of a (di)graph is a (di)graph, where vertices are either regular (free orbits) or totally ramified (fixed vertices). Deng, Sato and Wu treated the characteristic polynomial of a branched covering of digraph, where a subdigraph is an irregular covering of some digraph and its complement is totally ramified.

We give a decompostion formula for the Bartholdi zeta function of a branched covering of a digraph D which treated by Deng, Sato and Wu. As a corollary, we obtain a decomposition formula for the Bartholdi zeta function of a graph having a semi-free action.

1 Introduction

Graphs and digraphs treated here are finite. Let $G = (V(G), E(G))$ be a connected graph (possibly multiple edges and loops) with the set $V(G)$ of vertices and the set $E(G)$ of unoriented edges uv joining two vertices u and v. For $uv \in E(G)$, an arc (u, v) is the oriented edge from u to v. Set $D(G) = \{(u, v), (v, u) \mid uv \in E(G)\}$. For $e = (u, v) \in D(G)$, set $u = o(e)$ and $v = t(e)$. Furthermore, let $e^{-1} = (v, u)$ be the *inverse* of $e = (u, v)$.

A *path* P *of length* n in G is a sequence $P = (e_1, \ldots, e_n)$ of n arcs such that $e_i \in D(G)$, $t(e_i) = o(e_{i+1})(1 \leq i \leq n-1)$. Set $\mid P \mid= n$, $o(P) = o(e_1)$ and $t(P) = t(e_n)$. Also, P is called a $(o(P), t(P))$*-path*. We say that a path $P = (e_1, \ldots, e_n)$ has a *backtracking* (or a *bump* at $t(e_i)$) if $e^{-1}_{i+1} = e_i$ for some $i(1 \leq i \leq n-1)$. A (v, w)-path is called a v*-cycle* (or v*-closed path*) if $v = w$. The *inverse cycle* of a cycle $C = (e_1, \ldots, e_n)$ is the cycle $C^{-1} = (e^{-1}_n, \ldots, e^{-1}_1)$.

We introduce an equivalence relation between cycles. Such two cycles $C_1 = (e_1, \ldots, e_m)$ and $C_2 = (f_1, \ldots, f_m)$ are called *equivalent* if there exists k such that $f_j = e_{j+k}$ for all j, where indices are treated *mod* n. The inverse cycle of C is in general not equivalent to C. Let $[C]$ be the equivalence class which contains a cycle C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *power* of B. A cycle C is *reduced* if both C and C^2 have no backtracking. Furthermore, a cycle C is *prime* if it is not a power of a strictly smaller cycle. Note that each equivalence class of prime, reduced cycles

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of a graph G corresponds to a unique conjugacy class of the fundamental group $\pi_1(G, v)$ of G at a vertex v of G.

The *(Ihara) zeta function* of a graph G is defined to be a function of $u \in \mathbf{C}$ with $\mid u \mid$ sufficiently small, by $\mathbf{Z}(G, u) = \mathbf{Z}_G(u) = \prod_{[C]}(1 - u^{|C|})^{-1}$, where $[C]$ runs over all equivalence classes of prime, reduced cycles of G(see [8]).

Zeta functions of graphs started from zeta functions of regular graphs by Ihara [8]. In [8], he showed that their reciprocals are explicit polynomials. A zeta function of a regular graph G associated with a unitary representation of the fundamental group of G was developed by Sunada [17,18]. Hashimoto [7] treated multivariable zeta functions of bipartite graphs. Bass [2] generalized Ihara's result on the zeta function of a regular graph to an irregular graph, and showed that its reciprocal is again a polynomial.

Theorem 1 (Bass). *Let G be a connected graph. Then the reciprocal of the zeta function of G is given by*

$$\mathbf{Z}(G, u)^{-1} = (1 - u^2)^{r-1} \det(\mathbf{I} - u\mathbf{A}(G) + u^2(\mathbf{D} - \mathbf{I})),$$

where r and $\mathbf{A}(G)$ are the Betti number and the adjacency matrix of G, respectively, and $\mathbf{D} = \mathbf{D}_G = (d_{ij})$ is the diagonal matrix with $d_{ii} = \deg_G v_i$ where $V(G) = \{v_1, \ldots, v_n\}$.

Stark and Terras [16] gave an elementary proof of Theorem 1, and discussed three different zeta functions of any graph. Furthermore, various proofs of Bass's Theorem were given by Foata and Zeilberger [5], Kotani and Sunada [10].

Mizuno and Sato [11] gave a determinant expression for the zeta function of a regular covering of a graph.

Let G be a connected graph. Then the *cyclic bump count* $cbc(\pi)$ of a cycle $\pi = (\pi_1, \ldots, \pi_n)$ is $cbc(\pi) =\mid \{i = 1, \ldots, n \mid \pi_i = \pi_{i+1}^{-1}\} \mid$, where $\pi_{n+1} = \pi_1$. Then the *Bartholdi zeta function* of G is defined to be a function of $u, t \in \mathbf{C}$ with $\mid u \mid, \mid t \mid$ sufficiently small, by $\zeta_G(u, t) = \zeta(G, u, t) = \prod_{[C]}(1 - u^{cbc(C)} t^{|C|})^{-1}$, where $[C]$ runs over all equivalence classes of prime cycles of G(see [1]). If $u = 0$, then the Bartholdi zeta function of G is the (Ihara) zeta function of G.

Bartholdi [1] gave a determinant expression of the Bartholdi zeta function of a graph.

Theorem 2 (Bartholdi). *Let G be a connected graph with n vertices and m unoriented edges. Then the reciprocal of the Bartholdi zeta function of G is given by*

$$\zeta(G, u, t)^{-1} = (1 - (1 - u)^2 t^2)^{m-n} \det(\mathbf{I} - t\mathbf{A}(G) + (1 - u)(\mathbf{D} - (1 - u)\mathbf{I})t^2).$$

In the case of $u = 0$, Theorem 2 implies Theorem 1.

Mizuno and Sato [12] presented a decomposition formula for the Bartholdi zeta function of a regular covering of a graph.

Let D be a connected simple digraph with n vertices and m arcs which is not a symmetric digraph. Cycles, reduced cycles and prime cycles in a simple digraph which is not symmetric are defined similarly to the case of a symmetric digraph.

Then the *Bartholdi zeta function* of D is defined to be a function of $u, t \in \mathbf{C}$ with $\mid u \mid, \mid t \mid$ sufficiently small, by $\zeta_D(u,t) = \zeta(D,u,t) = \prod_{[C]}(1 - u^{cbc(C)} t^{|C|})^{-1}$, where $[C]$ runs over all equivalence classes of prime cycles of D. If D is the symmetric digraph corresponding to a connected graph G, then the Bartholdi zeta function of D is the Bartholdi zeta function of G.

Let $m_1 =\mid \{(u,v) \in A(D) \mid (v,u) \in A(D)\} \mid /2$. Then we define an $n \times n$ matrix $\mathbf{A}_1 = \mathbf{A}_1(D) = (a_{uv})$ as follows:

$$a_{uv} = \begin{cases} 1 \text{ if } (u,v) \text{ and } (v,u) \in A(D), \\ 0 \text{ otherwise.} \end{cases}$$

Furthermore, let $\mathbf{A}_0 = \mathbf{A}_0(D) = \mathbf{A}(D) - \mathbf{A}_1$.

Let $V(D) = \{v_1, \ldots, v_n\}$. Then an $n \times n$ matrix $\mathbf{S} = \mathbf{S}_D = (s_{ij})$ is the diagonal matrix defined by $s_{ii} =\mid \{(v_i, v_j) \in A(D) \mid (v_j, v_i) \in A(D)\} \mid$. Note that $\sum_{i=1}^n s_{ii} = 2m_1$. Set $s_D(v_i) = s_{ii},\ 1 \leq i \leq n$.

Two $m \times m$ matrices $\mathbf{B} = (b_{ef})_{e,f \in A(D)}$ and $\mathbf{J} = (t_{ef})_{e,f \in A(D)}$ are defined as follows:

$$b_{ef} = \begin{cases} 1 \text{ if } t(e) = o(f), \\ 0 \text{ otherwise} \end{cases}, t_{ef} = \begin{cases} 1 \text{ if } f = e^{-1}, \\ 0 \text{ otherwise.} \end{cases}$$

Mizuno and Sato [13] presented a decomposition formula for the Bartholdi zeta function of a digraph which is not symmetric.

Theorem 3 (Mizuno and Sato). *Let D be a connected digraph with n vertices and m arcs which is not symmetric, and m_1 the number of pairs of symmetric arcs. Then the reciprocal of the Bartholdi zeta function of D is*

$$\zeta(D,u,t)^{-1} = \det(\mathbf{I}_m - (\mathbf{B} - (1-u)\mathbf{J})t)$$

$$= (1-(1-u)^2t^2)^{m_1-n} \det(\mathbf{I}_n - t\mathbf{A}_1 - (1-(1-u)^2t^2)t\mathbf{A}_0 + (1-u)t^2(\mathbf{S} - (1-u)\mathbf{I}_n)).$$

If D is the symmetric digraph corresponding to a connected graph G, then Theorem 3 implies Theorem 2.

Sato [14] introduced a regular covering of a digraph, and presented a decomposition formula for the Bartholdi zeta function of it.

In Section 2, we introduce a covering of a digraph D and give a characterization of it by permutation voltage assignments. In Section 3, we consider a branched covering of D, and consider its ajacency matrix. In Section 4, we give a decompostion formula of the Bartholdi zeta function of a branched covering of D. In Section 5, we obtain a decomposition formula for the Bartholdi zeta function of a graph having a semi-free action.

2 The Permutation Voltage Digraph Construction

A graph H is called a *covering* of a graph G with projection $\pi : H \longrightarrow G$ if there is a surjection $\pi : V(H) \longrightarrow V(G)$ such that $\pi|_{N(v')} : N(v') \longrightarrow N(v)$ is a bijection for all vertices $v \in V(G)$ and $v' \in \pi^{-1}(v)$. The projection $\pi : H \longrightarrow G$ is an *n-fold covering* of G if π is n-to-one. A covering $\pi : H \longrightarrow G$ is said to be *regular* if there is a subgroup B of the automorphism group $Aut\ H$ of H acting freely on H such that the quotient graph H/B is isomorphic to G.

Let G be a graph and S_n the symmetric group on the set $N = \{1, 2, \ldots, n\}$. Then a mapping $\alpha : D(G) \longrightarrow S_n$ is called a *permutation voltage assignment* if $\alpha(v, u) = \alpha(u, v)^{-1}$ for each $(u, v) \in D(G)$. The pair (G, α) is called a *permutation voltage graph.* The *derived graph* G^α of the permutation voltage graph (G, α) is defined as follows: $V(G^\alpha) = V(G) \times N$ and $((u, h), (v, k)) \in D(G^\alpha)$ if and only if $(u, v) \in D(G)$ and $k = \alpha(u, v)(h)$. The *natural projection* $\pi_\alpha : G^\alpha \longrightarrow G$ is defined by $\pi_\alpha(u, h) = u$. The graph G^α is called a *derived graph covering* of G with voltages in S_n or an *n-covering* of G. Note that the n-covering G^α is an n-fold covering of G. Futhermore, every n-fold covering of a graph G is an n-covering G^α of G for some permutation voltage assignment $\alpha : D(G) \longrightarrow S_n$(see [6]).

Let G be a graph and Γ a finite group. Then a mapping $\alpha : D(G) \longrightarrow \Gamma$ is called an *ordinary voltage assignment* if $\alpha(v, u) = \alpha(u, v)^{-1}$ for each $(u, v) \in D(G)$. The pair (G, α) is called an *ordinary voltage graph.* The *derived graph* G^α of the ordinary voltage graph (G, α) is defined as follows: $V(G^\alpha) = V(G) \times \Gamma$ and $((u, h), (v, k)) \in D(G^\alpha)$ if and only if $(u, v) \in D(G)$ and $k = h\alpha(u, v)$. The *natural projection* $\pi_\alpha : G^\alpha \longrightarrow G$ is defined by $\pi_\alpha(u, h) = u$. The graph G^α is called a *derived graph covering* of G with voltages in Γ or a *Γ-covering* of G. The natural projection π_α commutes with the right multiplication action of the $\alpha(e), e \in D(G)$ and the left action of Γ on the fibers: $g(u, h) = (u, gh), g \in \Gamma$, which is free and transitive. Thus, the Γ-covering G^α is a $\mid \Gamma \mid$-fold regular covering of G with covering transformation group Γ. Futhermore, every regular covering of a graph G is a Γ-covering of G for some group Γ(see [6]).

We can generalize the notion of a covering of a graph to a simple digraph. Let D be a connected simple digraph. A digraph H is called a *covering* of D with projection $\pi : H \longrightarrow D$ if there is a surjection $\pi : V(H) \longrightarrow V(D)$ such that both $\pi|_{N^+(v')} : N^+(v') \longrightarrow N^+(v)$ and $\pi|_{N^-(v')} : N^-(v') \longrightarrow N^-(v)$ are bijections for all vertices $v \in V(D)$ and $v' \in \pi^{-1}(v)$, where $N^+(v) = N^+_D(v) = \{w \in V(D) \mid (v, w) \in A(D)\}$ and $N^-(v) = N^-_D(v) = \{w \in V(D) \mid (w, v) \in A(D)\}$, etc. The projection $\pi : H \longrightarrow D$ is an *n-fold covering* of D if π is n-to-one.

We can generalize the notion of a covering of a graph by a permutation voltage assignment to a simple digraph. Let D be a connected digraph and S_n the symmetric group on $N = \{1, 2, \ldots, n\}$. Let $A(D)$ be the set of arcs in D. Then a mapping $\alpha : A(D) \longrightarrow S_n$ is called a *pseudo permutation voltage assignment* if $\alpha(v, u) = \alpha(u, v)^{-1}$ for each $(u, v) \in A(D)$ such that $(v, u) \in A(D)$. The pair (D, α) is called a *permutation voltage digraph.* The *derived digraph* D^α of the permutation voltage digraph (D, α) is defined as follows: $V(D^\alpha) = V(D) \times N$ and $((u, h), (v, k)) \in A(D^\alpha)$ if and only if $(u, v) \in A(D)$ and $k = \alpha(u, v)(h)$. The digraph D^α is called an *n-covering* of D. The *natural projection* $\pi_\alpha : D^\alpha \longrightarrow S_n$ is defined by $\pi_\alpha(u, h) = u$. Note that an n-covering of the symmetric digraph corresponding to a graph G is an n-covering of G.

Lemma 1. *Let D be a connected digraph, S_n the symmetric group on $\{1, 2, \ldots, n\}$ and $\alpha : A(D) \longrightarrow S_n$ a pseudo permutation voltage assignment. Then the natural projection $\pi_\alpha : D^\alpha \longrightarrow D$ is a covering.*

Proof. The proof is straightforward by the definition of an n-covering and a covering of D. Q.E.D.

Theorem 4. *Let $\pi : \tilde{D} \longrightarrow D$ be an n-fold covering of a connected digraph D. Suppose that the preimage of each symmetric arcs of D is symmetric arcs. Then there exists a pseudo permutation voltage assignment $\alpha : A(D) \longrightarrow S_n$ such that the n-covering D^α is isomrphic to $\tilde{D}$.*

Proof. At first, assume that $\tilde{D}$ is connected.

Let u be any vertex of D. Then we label the n vertices in the fibre $\pi^{-1}(u)$ in any way as $u_1, \ldots, u_n$. If $e = (u, v)$ is an arc of D, then the fibre $\pi^{-1}(e)$ is a family of n arcs which originate at one of the vertices u_i and terminate at one of the verices v_i, because the restrictions $\pi|_{N^+(u_i)} : N^+(u_i) \longrightarrow N^+(u)$ and $\pi|_{N^-(v_i)} : N^-(v_i) \longrightarrow N^-(v)$ of π are bijections.

Let e_i be the arc originating at u_i and $v_j = t(e_i)$. Furthermore, let the permtation η of the set $\{1, \ldots, n\}$ be defined by $\eta(i) = j$, and let $\alpha : A(D) \longrightarrow S_n$ be defined by $\alpha(e) = \eta$. Then the labeling of $\tilde{D}$ gives the isomorphism $\tilde{D} \longrightarrow D^\alpha$.

By the hypothesis that the preimage of each symmetric arcs of D is symmetric arcs, we have $\alpha(v, u) = \alpha(u, v)^{-1}$ for each $(u, v) \in A(D)$ such that $(v, u) \in A(D)$. Q.E.D.

3 Branched Coverings of Digraphs

We can generalize the notion of a branhced covering of a graph to a simple digraph. Let D be a connected simple digraph. Then a digraph H is called a *branched covering* of D with projection $\pi : H \longrightarrow D$ if there are a surjection $\pi : V(H) \longrightarrow V(D)$ and a subset $B \subset V(D)$ such that $\pi|_{N^+(v')} : N^+(v') \longrightarrow N^+(v)$ is a bijection and $\pi|_{N^-(v')} : N^-(v') \longrightarrow N^-(v)$ is a bijection for all vertices $v \in V(D) - B$ and $v' \in \pi^{-1}(v)$ (c.f., [6]). Note that $\pi \mid_{H - \pi^{-1}(B)} : H - \pi^{-1}(B) \longrightarrow D - B$ is a covering. The set B is called a *branch set*. Furthermore, the set B is called be of *index 1* if $\mid \pi^{-1}(v) \mid = 1$ for any $v \in B$. The projection $\pi : H \longrightarrow D$ is an *n-fold branched covering* of D if $\pi \mid_{H - \pi^{-1}(B)}$ is n-to-one.

Let D be a connected digraph, $B \subset V(D)$, $N = \{1, 2, \ldots, n\}$ and $\alpha : A(D - B) \longrightarrow S_n$ a pseudo permutation voltage assignment. The subdigraph $< B > = < B >_D$ of D *induced by* B is a digraph with vertex set B and arc set $\{(u, v) \in A(D) \mid u \in B, v \in B\}$. Let $(B, \bar{B}) = \{(u, v) \in A(D) \mid u \in B, v \in \bar{B} = V(D) - B\}$. Then a *branched n-covering* D^α_B with branch set B of index 1 is defined as follows: $V(D^\alpha_B) = (V(D - B) \times N) \cup B$ and $A(D^\alpha_B) = A(< B >_D) \cup \{(u, (v, i)) \mid (u, v) \in (B, \bar{B}), i \in N\} \cup \{((u, i), v)) \mid (u, v) \in (\bar{B}, B), i \in N\} \cup \{(u, i), (v, j))) \mid (u, v) \in A(D - B), j = \alpha(u, v)(i)\}$.

By Lemmas 5.1 and 5.2 in Deng, Sato and Wu [4], the following result holds.

Theorem 5 (Deng, Sato and Wu). *Let $\pi : \tilde{D} \longrightarrow D$ be an n-fold branched covering of a connected digraph D which has the branch set B with index 1. Suppose that the preimage of each symmetric arcs of D is symmetric arcs. Then there exists a pseudo permutation voltage assignment $\alpha : A(D-B) \longrightarrow S_n$ such that the branched n-covering D^{α}_B is isomrphic to $\tilde{D}$.*

Next, we consider the adjacency matrix of an n-fold branched covering D^{α}_B of D with branch set B of index 1, where $\alpha : A(D-B) \longrightarrow S_n$ is a pseudo permutation voltage assignment.

Let $D_1 = D - B$, $B = \{v_1, \dots, v_m\}$ and $V(D-B) = \{u_1, \dots, u_\nu\}$. Let

$$\mathbf{A}(D) = \begin{bmatrix} \mathbf{A}(< B >_D) & \mathbf{F} \\ \mathbf{K} & \mathbf{A}(D-B) \end{bmatrix},$$

where $\mathbf{F}$ is a $m \times \nu$ matrix and $\mathbf{K}$ is a $\nu \times m$ matrix. Arrange vertices of D^{α}_B in $n+1$ blocks: $v_1, \dots, v_m$; $(u_1, 1), \dots, (u_\nu, 1); \dots ; (u_1, n), \dots, (u_\nu, n)$. We consider the adjacency matrix $\mathbf{A}(D^{\alpha}_B)$ under this order. Then we have

$$\mathbf{A}(D^{\alpha}_B) = \begin{bmatrix} \mathbf{A}(< B >_D) & \mathbf{J} \bigotimes \mathbf{F} \\ {}^t\mathbf{J} \bigotimes \mathbf{K} & \mathbf{A}((D-B)^{\alpha}) \end{bmatrix},$$

where $\mathbf{J} = (1 \cdots 1)$ is the $1 \times n$ matrix. The *Kronecker product* $\mathbf{A} \bigotimes \mathbf{B}$ of matrices of $\mathbf{A}$ and $\mathbf{B}$ is considered as the matrix $\mathbf{A}$ having the element a_{ij} replaced by the matrix $a_{ij}\mathbf{B}$.

Now, let $\Gamma =< \{\alpha(e) \mid e \in A(D-B)\} >$ be the subgroup of S_n generated by $\{\alpha(e) \mid e \in A(D-B)\}$. For $h \in \Gamma$, the matrix $\mathbf{P}_h = (p^{(h)}_{ij})$ is defined as follows:

$$p^{(h)}_{ij} = \begin{cases} 1 \text{ if } h(i) = j, \\ 0 \text{ otherwise.} \end{cases}$$

If $(u, v) \in A(D-B)$ and $\alpha(u, v) = h$, then $j = \alpha(u, v)(i) = h(i)$, i.e., $((u, i), (v, j)) \in A((D-B)^{\alpha})$. Thus we have

$$\mathbf{A}((D-B)^{\alpha}) = \sum_{h \in \Gamma} \mathbf{P}_h \bigotimes \mathbf{A}_h,$$

where the matrix $\mathbf{A}_h = (a^{(h)}_{uv})$ is given by

$$a^{(h)}_{uv} := \begin{cases} \mid \{e \in A(D-B) \mid e = (u, v), \alpha(e) = h\} \mid & \text{if there exists an arc } (u, v) \in A(D-B), \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 1.

$$\mathbf{A}(D^{\alpha}_B) = \begin{bmatrix} \mathbf{A}(< B >_D) & \mathbf{J} \bigotimes \mathbf{F} \\ {}^t\mathbf{J} \bigotimes \mathbf{K} & \sum_{h \in \Gamma} \mathbf{P}_h \bigotimes \mathbf{A}_h \end{bmatrix}.$$

4 Bartholdi Zeta Functions of Branched Coverings of Digraphs

Let D be a connected digraph and $N = \{1, 2, \ldots, n\}$. Furthermore, let D^{α}_{B} be a branched n-covering of D with branch set B of index 1, where $\alpha : A(D-B) \longrightarrow S_n$ is a pseudo permutation voltage assignment.

Next, let $\Gamma =< \{\alpha(e) \mid e \in A(D-B)\} >$, and let $\rho_1 = 1, \rho_2, \ldots, \rho_k$ be the inequivalent irreducible unitary representations of Γ, and f_i the degree of ρ_i for each i, where $f_1 = 1$. Furthermore, let $P : \Gamma \rightarrow GL(n, \mathbf{C})$ be a permutation representation of Γ such that $P(g) = \mathbf{P}_g$, and m_i the multiplicity of ρ_i in P for each $i = 1, \ldots, k$, that is, P is equivalent to a representation $m_1 \oplus m_2 \circ \rho_2 \oplus \cdots \oplus m_k \circ \rho_k$. Assume that $(D-B)^{\alpha}$ is connected. Then we have $m_1 = 1$(see [15]).

Let $\mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_s$ be the block diagonal sum of square matrices $\mathbf{M}_1, \ldots, \mathbf{M}_s$. If $\mathbf{M}_1 = \mathbf{M}_2 = \cdots = \mathbf{M}_s = \mathbf{M}$, then we write $s \circ \mathbf{M} = \mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_s$.

By the fact of group representation, there exists a nonsigular matrix $\mathbf{U}$ such that

$$\mathbf{U}^{-1} P(h) \mathbf{U} = (1) \oplus m_2 \circ \rho_2(h) \oplus \cdots \oplus m_k \circ \rho_k(h) \tag{1}$$

for each $h \in \Gamma$(see [15]).

Since $(D-B)^{\alpha}$ is connected, [6, Theorem 2.5.2] implies that Γ acts transitively on $\{1, 2, \ldots, n\}$. We use Lemma 6.2 in Deng, Sato and Wu [4] (c.f., [3, Lemma 4.2]).

Lemma 2 (Deng, Sato and Wu). *The matrix* $\mathbf{U}$ *in (1) can be chosen to satisfy*

$$\mathbf{J}\mathbf{U} = (\sqrt{n} 0 \cdots 0) \text{ and } \mathbf{U}^{-1 t}\mathbf{J} = {}^t(\sqrt{n} 0 \cdots 0).$$

For an induced subdigraph K of a connected digraph D, let the matrix $\mathbf{S}_{D;K}$ be a $\mid V(K) \mid \times \mid V(K) \mid$ diagonal matrix with diagonal elements $s_D(w_1), \ldots, s_D(w_m)$, where $V(K) = \{w_1, \ldots, w_m\}$.

Theorem 6. *Let* D *be a connected digraph with* ν *vertices and* ϵ *arcs,* $N = \{1, 2, \cdots, n\}$ *and* D^{α}_{B} *a branched* n*-covering of* D *with branch set* B *of index 1, where* $\alpha : A(D-B) \longrightarrow S_n$ *is a pseudo permutation voltage assignment.*

Set $\epsilon_1 =\mid \{e \in A(< B >_D) \mid e^{-1} \in A(< D >_B)\} \mid /2$, $\epsilon_2 =\mid \{e \in A(D-B) \mid e^{-1} \in A(D-B)\} \mid /2$, $\epsilon_3 =\mid \{e \in (B, \bar{B}) \mid e^{-1} \in (\bar{B}, B)\} \mid /2$, $\nu_1 =\mid B \mid$, $\nu_2 =\mid V(D-B) \mid$ *and* $\Gamma =< \{\alpha(e) \mid e \in A(D-B)\} >$. *Furthermore, let* $\rho_1 = 1, \rho_2, \ldots, \rho_k$ *be the inequivalent irreducible representations of* Γ*, and* f_i *the degree of* ρ_i *for each* i*, where* $f_1 = 1$*. Let* $P : \Gamma \rightarrow GL(n, \mathbf{C})$ *be a permutation representation of* Γ *such that* $P(g) = \mathbf{P}_g$*. Assume that* $(D-B)^{\alpha}$ *is connected and* $P = 1 \oplus m_2 \circ \rho_2 \oplus \cdots \oplus m_k \circ \rho_k$*.*

For $g \in \Gamma$*, the matrix* $\mathbf{A}_{0,g} = (a^{(g)}_{uv})$ *is defined as follows:*

$$a^{(g)}_{uv} := \begin{cases} 1 \text{ if } (u,v) \in A(D-B), (v,u) \notin A(D-B) \text{ and } \alpha(u,v) = g, \\ 0 \text{ otherwise.} \end{cases}$$

Furthermore, the matrix $\mathbf{A}_{1,g} = (b^{(g)}_{uv})$ *is defined as follows:*

$$b^{(g)}_{uv} := \begin{cases} 1 \text{ if } (u,v), (v,u) \in A(D-B) \text{ and } \alpha(u,v) = g, \\ 0 \text{ otherwise.} \end{cases}$$

Then the reciprocal of the Bartholdi zeta function of D^{α}_B is

$$\zeta(D^{\alpha}_B,u,t)^{-1}=(1-(1-u)^2t^2)^{\epsilon_1-\nu_1+n\epsilon_3}\det(\mathbf{I}_{\nu}-t\mathbf{A}'_1-(1-(1-u)^2t^2)t\mathbf{A}'_0$$

$$+(1-u)t^2(\mathbf{S}'_D-(1-u)\mathbf{I}_{\nu}))\prod_{i=2}^{k}\{(1-(1-u)^2t^2)^{(\epsilon_2-\nu_2)f_i}\det(\mathbf{I}_{\nu_2 f_i}-t\sum_{h\in\Gamma}\rho_i(h)\bigotimes\mathbf{A}_{1,h}$$

$$-(1-(1-u)^2t^2)t\sum_{h\in\Gamma}\rho_i(h)\bigotimes\mathbf{A}_{0,h}+(1-u)t^2(\mathbf{I}_{f_i}\bigotimes(\mathbf{S}_{D;D-B}-(1-u)\mathbf{I}_{\nu_2}))\}^{m_i},$$

where

$$\mathbf{A}'_i=\begin{bmatrix}\mathbf{A}_i(<B>_D) & n\mathbf{F}_i\\ \mathbf{K}_i & \mathbf{A}_i(D-B)\end{bmatrix}\ \textit{when}\ \mathbf{A}_i=\begin{bmatrix}\mathbf{A}_i(<B>_D) & \mathbf{F}_i\\ \mathbf{K}_i & \mathbf{A}_i(D-B)\end{bmatrix}(i=0,1),$$

and $\mathbf{S}'_D=\mathbf{S}_{D^{\alpha}_B;<B>}\oplus\mathbf{S}_{D;D-B}$.

Proof. By Theorem 3, we have

$$\zeta(D^{\alpha}_B,u,t)^{-1}=(1-(1-u)^2t^2)^{(\epsilon_2-\nu_2)n+n\epsilon_3+\epsilon_1-\nu_1}$$

$$\times\det(\mathbf{I}_{\nu_2 n+\nu_1}-t\mathbf{A}_1(D^{\alpha}_B)-(1-(1-u)^2t^2)t\mathbf{A}_0(D^{\alpha}_B)+(1-u)(\mathbf{S}_{D^{\alpha}_B}-(1-u)\mathbf{I}_{\nu_2 n+\nu_1})t^2).$$

At first, we have $\mathbf{S}_D=\mathbf{S}_{D;<B>}\oplus\mathbf{S}_{D;D-B}$ and $\mathbf{S}_{D^{\alpha}_B}=\mathbf{S}_{D^{\alpha}_B;<B>}\oplus\mathbf{S}_{D^{\alpha}_B;(D-B)^{\alpha}}$ $=\mathbf{S}_{D^{\alpha}_B;<B>}\oplus(\mathbf{I}_n\bigotimes\mathbf{S}_{D;D-B})$.

Let

$$\mathbf{A}_i(D)=\begin{bmatrix}\mathbf{A}_i(<B>_D) & \mathbf{F}_i\\ \mathbf{K}_i & \mathbf{A}_i(D-B)\end{bmatrix}(i=0,1).$$

Note that $\mathbf{A}(<B>_D)=\mathbf{A}_0(<B>_D)+\mathbf{A}_1(<B>_D)$ and $\mathbf{A}(D-B)=\mathbf{A}_0(D-B)+\mathbf{A}_1(D-B)$.

But, we have

$$\mathbf{A}_i(D-B)=\sum_{h\in\Gamma}\mathbf{P}_h\bigotimes\mathbf{A}_{i,h}\ (i=0,1)\ \textit{and}\ \mathbf{A}(D^{\alpha}_B)=\mathbf{A}_0(D^{\alpha}_B)+\mathbf{A}_1(D^{\alpha}_B).$$

By Proposition 1, we have

$$\mathbf{A}_i(G^{\alpha}_B)=\begin{bmatrix}\mathbf{A}_i(<B>_D) & \mathbf{J}\bigotimes\mathbf{F}_i\\ {}^t\mathbf{J}\bigotimes\mathbf{K}_i & \sum_{h\in\Gamma}\mathbf{P}_h\bigotimes\mathbf{A}_{i,h}\end{bmatrix}(i=0,1).$$

Now, let

$$f(K)=f_L(K)=\mathbf{I}_{|V(K)|}-t\mathbf{A}_1(K)-(1-(1-u)^2t^2)t\mathbf{A}_0(K)+(1-u)(\mathbf{S}_{L;K}-(1-u)\mathbf{I}_{|V(K)|})t^2$$

for an induced subdigraph K of a connected digraph L. Then we have

$$f(D^{\alpha}_B)=\begin{bmatrix}f(<B>_{D^{\alpha}_B}) & \mathbf{J}\bigotimes(-t\mathbf{F}_1-(1-(1-u)^2t^2)t\mathbf{F}_0)\\ {}^t\mathbf{J}\bigotimes(-t\mathbf{K}_1-(1-(1-u)^2t^2)t\mathbf{K}_0) & f((D-B)^{\alpha})\end{bmatrix}.$$

Here $f(< B >_{D^{\alpha}_B}) = \mathbf{I}_{\nu_1} - t\mathbf{A}_1(< B >_D) - (1-(1-u)^2t^2)t\mathbf{A}_0(< B >_D) + (1-u)(\mathbf{S}_{D^{\alpha}_B;<B>} - (1-u)\mathbf{I}_{\nu_1})t^2$ and $f((D-B)^{\alpha}) = \mathbf{I}_{n\nu_2} - t\mathbf{A}_1((D-B)^{\alpha}) - (1-(1-u)^2t^2)t\mathbf{A}_0((D-B)^{\alpha}) + (1-u)(\mathbf{I}_n \bigotimes (\mathbf{S}_{D;D-B} - (1-u)\mathbf{I}_{\nu_2}))t^2$.

Let $\rho_1 = 1, \rho_2, \ldots, \rho_k$ be the irreducible representations of Γ, and f_i the degree of ρ_i for each i, where $f_1 = 1$. Furthermore, let $P : \Gamma \to GL(n, \mathbf{C})$ be a permutation representation of Γ such that $P(g) = \mathbf{P}_g$, and let $P = 1 \oplus m_2 \circ \rho_2 \oplus \cdots \oplus m_k \circ \rho_k$. By Lemma 2, there exists a nonsingular matrix $\mathbf{U}$ such that $\mathbf{JU} = (\sqrt{n}0 \cdots 0)$, $\mathbf{U}^{-1\,t}\mathbf{J} = {}^t(\sqrt{n}0 \cdots 0)$ and $\mathbf{U}^{-1}P(h)\mathbf{U} = (1) \oplus m_2 \circ \rho_2(h) \oplus \cdots \oplus m_k \circ \rho_k(h)$ for each $h \in \Gamma$.

Putting $\mathbf{Y} = (\mathbf{U}^{-1} \bigotimes \mathbf{I}_{\nu_2})(\mathbf{A}_1((D-B)^{\alpha}) + (1-(1-u)^2)t^2\mathbf{A}_0((D-B)^{\alpha}))(\mathbf{U} \bigotimes \mathbf{I}_{\nu_2})$, we have

$$\mathbf{Y} = \sum_{h \in \Gamma} \{(1) \oplus m_2 \circ \rho_2(h) \oplus \cdots \oplus m_k \circ \rho_k(h)\} \bigotimes (\mathbf{A}_{1,h} + (1-(1-u)^2t^2)\mathbf{A}_{0,h}).$$

Note that $\mathbf{A}_i(D-B) = \sum_{h \in \Gamma} \mathbf{A}_{i,h}, i = 0, 1$.

Next, let

$$\mathbf{X} = \begin{bmatrix} \mathbf{I}_{\nu_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{U} \bigotimes \mathbf{I}_{\nu_2} \end{bmatrix}.$$

Since $1 + m_2f_2 + \cdots + m_kf_k = n$, it follows that

$$\mathbf{X}^{-1}f(D^{\alpha}_B)\mathbf{X} = \begin{bmatrix} f(< B >_{D^{\alpha}_B}) & \sqrt{n}(-t\mathbf{F}_1 - (1-(1-u)^2t^2)t\mathbf{F}_0) \\ \sqrt{n}(-t\mathbf{K}_1 - (1-(1-u)^2t^2)t\mathbf{K}_0) & f(D-B) \end{bmatrix}$$

$$\oplus(\oplus_{i=2}^{k} m_i \circ \{\mathbf{I}_{\nu_2 f_i} - t \sum_{h \in \Gamma} \rho_i(h) \bigotimes \mathbf{A}_{1,h}$$

$$-(1-(1-u)^2t^2)t \sum_{h \in \Gamma} \rho_i(h) \bigotimes \mathbf{A}_{0,h} + (1-u)t^2(\mathbf{I}_{f_i} \bigotimes (\mathbf{S}_{D;D-B} - (1-u)\mathbf{I}_{\nu_2}))\}).$$

But, we have

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & 1/\sqrt{n}\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{B}_1 & \sqrt{n}\mathbf{B}_2 \\ \sqrt{n}\mathbf{B}_3 & \mathbf{B}_4 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \sqrt{n}\mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 & n\mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{B}_4 \end{bmatrix}.$$

Therefore, it follows that

$$\zeta(D^{\alpha}_B, u, t)^{-1} = (1-(1-u)^2t^2)^{\epsilon_1 - \nu_1 + n\epsilon_3} \det(\mathbf{I}_{\nu} - t\mathbf{A}'_1 - (1-(1-u)^2t^2)t\mathbf{A}'_0$$

$$+(1-u)t^2(\mathbf{S}'_D - (1-u)\mathbf{I}_{\nu})) \prod_{i=2}^{k} \{(1-(1-u)^2t^2)^{(\epsilon_2 - \nu_2)f_i} \det(\mathbf{I}_{\nu_2 f_i} - t \sum_{h \in \Gamma} \rho_i(h) \bigotimes \mathbf{A}_{1,h}$$

$$-(1-(1-u)^2t^2)t \sum_{h \in \Gamma} \rho_i(h) \bigotimes \mathbf{A}_{0,h} + (1-u)t^2(\mathbf{I}_{f_i} \bigotimes (\mathbf{S}_{D;D-B} - (1-u)\mathbf{I}_{\nu_2}))\}^{m_i},$$

where

$$\mathbf{A}'_i = \begin{bmatrix} \mathbf{A}_i(< B >_D) & n\mathbf{F}_i \\ \mathbf{K}_i & \mathbf{A}_i(D-B) \end{bmatrix} (i = 0, 1) \textit{ and } \mathbf{S}'_D = \mathbf{S}_{D^{\alpha}_B;<B>} \oplus \mathbf{S}_{D;D-B}.$$

Q.E.D.

In the case of $\epsilon_3 = 0$, Theorem 6 imples Corollary 1.

Corollary 1. *Under the same conditions as Theorem 6, assume that* $\epsilon_3 = 0$. *Then the reciprocal of the Bartholdi zeta function of* D^{α}_B *is*

$$\zeta(D^{\alpha}_B, u, t)^{-1} = (1-(1-u)^2t^2)^{\epsilon_1 - \nu_1} \det(\mathbf{I}_{\nu} - t\mathbf{A}_1(D) - (1-(1-u)^2t^2)t\mathbf{A}'_0$$

$$+(1-u)t^2(\mathbf{S}_D - (1-u)\mathbf{I}_{\nu})) \prod_{i=2}^{k} \{(1-(1-u)^2t^2)^{(\epsilon_2-\nu_2)f_i} \det(\mathbf{I}_{\nu_2 f_i} - t\sum_{h\in\Gamma} \rho_i(h) \bigotimes \mathbf{A}_{1,h}$$

$$-(1-(1-u)^2t^2)t\sum_{h\in\Gamma} \rho_i(h) \bigotimes \mathbf{A}_{0,h} + (1-u)t^2(\mathbf{I}_{f_i} \bigotimes (\mathbf{S}_{D-B} - (1-u)\mathbf{I}_{\nu_2}))\}^{m_i},$$

where

$$\mathbf{A}'_0 = \begin{bmatrix} \mathbf{A}_0(< B >_D) & n\mathbf{F} \\ \mathbf{K} & \mathbf{A}_0(D-B) \end{bmatrix} \text{ when } \mathbf{A}_0 = \begin{bmatrix} \mathbf{A}_0(< B >_D) & \mathbf{F} \\ \mathbf{K} & \mathbf{A}_0(D-B) \end{bmatrix}.$$

Proof. Since $\epsilon_3 = 0$, we have $\mathbf{S}'_D = \mathbf{S}_D = \mathbf{S}_{<B>} \oplus \mathbf{S}_{D-B}$, $\mathbf{S}_{D;D-B} = \mathbf{S}_{D-B}$ and

$$\mathbf{A}'_1 = \mathbf{A}_1(D) = \begin{bmatrix} \mathbf{A}_1(< B >_D) & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_1(D-B) \end{bmatrix}.$$

Thus, the result follows. Q.E.D.

5 The Bartholdi Zeta Function of a Graph Having a Semi-Free Action

We consider a graph having a semi-free action(see [3,9]).

Let G be a connected graph and Γ a subgroup of the automorphism group $Aut\ G$. Furthermore, let $\pi : G \longrightarrow G/\Gamma$ be the projection defined by $\pi(v) = [v]$ and $\pi(e) = [e]$ for any $v \in V(G)$ and $e \in D(G)$, where $[v]$ is the Γ-orbit of v, etc. Then Γ *acts semi-freely on* G if there exists a subset B of $V(G)$ such that Γ is trivial on $< B >_G$ and acts freely on $G - B$. Note that $\pi \mid_{G-\pi^{-1}(B)} : G - \pi^{-1}(B) \longrightarrow G/\Gamma - B$ is a regular covering, and so there exists an ordinary voltage assignment $\alpha : D(K-B) \longrightarrow \Gamma$ such that $K^{\alpha} \cong G - B$, where $K = G/\Gamma$.

Let K be a connected graph, $B \subset V(K)$, Γ a finite group and $\alpha : D(K-B) \longrightarrow \Gamma$ an ordinary voltage assignment. Then we define a *branched* Γ*-covering* K^{α}_B with branch set B of index 1 similarly to a branched n-covering of a digraph with branch set of index 1. That is, a branched Γ-covering of K with branch set B of index 1 is a branched $\mid \Gamma \mid$-covering of K with branch set of index 1. which the complement of the totally ramified part is a Γ-covering of some graph.

By the above fact and the proof of Theorem 5, the following result holds.

Proposition 2. *Let* $\pi : G \longrightarrow K = G/\Gamma$ *be a graph with a group* Γ *of automorphisms of* G *acting semi-freely on* G *and* $B \subset V(G)$ *the trivial part of* G. *Then there exists an ordinary voltage assignment* $\alpha : D(K-B) \longrightarrow \Gamma$ *such that* K^{α}_B *is isomrphic to* G.

Thus, a graph with a semi-free action is a branched $\mid \Gamma \mid$-covering of a symmetric digraph, and so we obtain the following result by a similar method of Theorem 6.

For an induced subgraph K of a connected graph G, let the matrix $\mathbf{D}_{G;K}$ be a $\mid V(K) \mid \times \mid V(K) \mid$ diagonal matrix with diagonal elements $\deg_G w_1, \dots, \deg_G w_m$, where $V(K) = \{w_1, \dots, w_m\}$.

Theorem 7. *Let G be a connected graph, Γ a subgroup of Aut g acting semi-freely on G and $B \subset V(G)$ the trivial part of G. Furthermore, let $\alpha : D(K-B) \longrightarrow \Gamma$ be an ordinary voltage assignment such that $(K-B)^{\alpha} \cong G-B$, where $K = G/\Gamma$.*

Set $\epsilon_1 =\mid E(< B >_K) \mid$, $\epsilon_2 =\mid E(K-B) \mid$, $\epsilon_3 =\mid \{e = vw \in E(K) \mid v \in B, w \in \bar{B}\} \mid$, $\nu_1 =\mid B \mid$ and $\nu_2 =\mid V(K-B) \mid$. Furthermore, let $\rho_1 = 1, \rho_2, \dots, \rho_k$ be the inequivalent irreducible representations of Γ, and f_i the degree of ρ_i for each i, where $f_1 = 1$.

Then the reciprocal of the Bartholdi zeta function of G is

$$\zeta(G,u,t)^{-1} = (1-(1-u)^2t^2)^{\epsilon_1-\nu_1+n\epsilon_3} \det(\mathbf{I}_{\nu} - t\mathbf{A}'_1 + (1-u)t^2(\mathbf{D}'_K - (1-u)\mathbf{I}_{\nu}))$$
$$\prod_{i=2}^{k} \{(1-(1-u)^2t^2)^{(\epsilon_2-\nu_2)f_i} \det(\mathbf{I}_{\nu_2 f_i} - t\sum_{h\in\Gamma} \rho_i(h) \bigotimes \mathbf{A}_{1,h} + (1-u)t^2(\mathbf{I}_{f_i} \bigotimes (\mathbf{D}_{K;K-B} - (1-u)\mathbf{I}_{\nu_2}))\}^{f_i},$$

where

$$\mathbf{A}'_1 = \begin{bmatrix} \mathbf{A}(< B >_K) & \mid \Gamma \mid \mathbf{F} \\ {}^t\mathbf{F} & \mathbf{A}(K-B) \end{bmatrix} \ if \ \mathbf{A}(K) = \begin{bmatrix} \mathbf{A}(< B >_K) & \mathbf{F} \\ {}^t\mathbf{F} & \mathbf{A}(K-B) \end{bmatrix},$$

and $\mathbf{D}'_K = \mathbf{D}_{G;<B>_G} \oplus \mathbf{D}_{K;K-B}$.

Proof. In Theorem 6, $\mathbf{A}'_0 = \mathbf{0}$ and $\mathbf{A}_{0,h} = \mathbf{0}$ for each $h \in \Gamma$. Futhermore, we have $\mathbf{S}'_K = \mathbf{D}_{G;<B>_G} \oplus \mathbf{D}_{K;K-B} = \mathbf{D}'_K$ and $\mathbf{S}_{K;K-B} = \mathbf{D}_{K;K-B}$. Thus, the result follows. Q.E.D.

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On Super Edge-Magic Strength and Deficiency of Graphs

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Abstract. A graph G is called *super edge-magic* if there exists a one-to-one mapping f from $V(G) \cup E(G)$ onto $\{1, 2, 3, \cdots, |V(G)| + |E(G)|\}$ such that for each $uv \in E(G)$, $f(u) + f(uv) + f(v) = c(f)$ is constant and all vertices of G receive all smallest labels. Such a mapping is called *super edge-magic labeling* of G. The *super edge-magic strength* of a graph G is defined as the minimum of all $c(f)$ where the minimum runs over all super edge-magic labelings of G. Since not all graphs are super edge-magic, we define, the *super edge-magic deficiency* of a graph G as either minimum n such that $G \cup nK_1$ is a super edge-magic graph or $+\infty$ if there is no such n. In this paper, the bound of super edge-magic strength and the super edge-magic deficiency of some families of graphs are obtained.

1 Introduction

All graphs considered are finite and simple. The graph G has the vertex-set $V(G)$ and edge-set $E(G)$ where $|V(G)| = p$ and $|E(G)| = q$. A bijection $f : V(G) \cup E(G) \to \{1, 2, 3, \cdots, p+q\}$ is called *edge-magic labeling* of G if for every edge xy of G, $f(x) + f(xy) + f(y)$ is a constant $c(f)$ (magic constant of f). The graph that admits such a labeling is called an *edge-magic graph*. An edge-magic labeling f is called *super edge-magic* if $f(V(G)) = \{1, 2, 3, \cdots, p\}$ and a graph that admits a super edge-magic labeling is called *super edge-magic*. The edge-magic and super edge-magic concepts were first introduced by Kotzig and Rosa [9] and Enomoto, Lladó, Nakamigawa and Ringel [5], respectively.

In their papers, Kotzig and Rosa [9], and Enomoto et al. [5] conjectured that every tree is edge-magic and every tree is super edge-magic, respectively. These conjectures have become very popular in the area of graph labeling. Some classes of trees have been proved to admit a (super) edge-magic labeling, however these conjectures still remain open.

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In this paper, we discuss two related concepts with super edge-magic labeling, namely the super edge-magic strength and super edge-magic deficiency of a graph. Some of our results give supporting examples to these conjectures.

In proving the main results, the following lemma will be frequently used.

Lemma 1. *[6] A graph G with p vertices and q edges is super edge-magic if and only if there exists a bijective function $f : V(G) \rightarrow \{1, 2, \cdots, p\}$ such that the set $S = \{f(x) + f(y)|xy \in E(G)\}$ consists of q consecutive integers. In such a case, f extends to a super edge-magic total labeling of G with the magic constant $c = p + q + min(S)$.*

By this lemma, it suffices to exhibit the vertex labeling of a super edge-magic graph.

Lemma 2. *[5] If a graph G with p vertices and q edges is super edge-magic, then $q \leq 2p - 3$.*

In light of Lemma 2, the minimum degree of a super edge-magic graph is at most 3. In particular for a triangle free graph, Figueroa-Centeno et al. [6] gave a better bound than the bound in Lemma 2 as follows.

Lemma 3. *[6] Let G be a triangle free super edge-magic graph with p (≥ 4) vertices and q edges. Then, $q \leq 2p - 5$.*

2 The Super Edge-Magic Strength of Graphs

The concept of the magic strength of a graph was first introduced by Avadayappan et.al [1]. They define the *magic strength* of a graph G, $m(G)$, as the minimum of all $c(f)$ where the minimum is taken over all edge-magic labelings f of G. That is, $m(G) = \min\{c(f) : f \text{ is an edge-magic labeling of } G\}$. Furthermore, the concept of the super edge-magic strength of a graph was also introduced by Avadayappan et.al [2]. The *super edge-magic strength* of G, $sm(G)$, is defined as the minimum of all $c(f)$ where the minimum is taken over all super edge-magic labelings f of G. That is, $sm(G) = \min\{c(f) : f \text{ is a super edge-magic labeling of } G\}$. In [2], the super edge-magic strength of some classes of graphs such as for paths P_n, stars $K_{1,n}$, odd cycles C_{2n+1}, P_n^2, and a union of odd number copies of P_2 were determined. In this paper, we determine the bounds for the super edge-magic strength of some particular type of trees.

The first type of tree which we consider is a fire cracker. In [4], Chen, Li, and Yeh defined a *fire cracker* as a tree obtained from the concatenation of stars by linking one leaf from each. Swaminathan and Jeyanthi [12] gave the bound for the super edge-magic strength of a fire cracker with equal number of pendant edges for each star. In this section, we give the bounds for the super edge-magic strength of a fire cracker with each star has a different number of pendant edges.

First, let G_1 be a fire cracker defined as follows.

$$V(G_1) = \{u_i, u_{i,i} : 1 \leq i \leq n\} \cup \{u_{i,i}^j : 1 \leq i \leq n, 1 \leq j \leq m_i\},$$

and

$$E(G_1) = \{u_i u_{i+1} : 1 \le i \le n-1\} \cup \{u_i u_{i,i} : 1 \le i \le n\} \\ \cup \{u_{i,i} u_{i,i}^j : 1 \le i \le n, 1 \le j \le m_i\},$$

where $m_1 < m_2 < m_3 < \cdots < m_n$.

Clearly, G_1 has $2n + \sum_{i=1}^n m_i$ vertices. Among these vertices, two vertices has degree 2, $n-2$ vertices have degree 3, one vertex of degree $m_i + 1$ for each $i = 1, 2, 3, \cdots, n$, and $\sum_{i=1}^n m_i$ vertices have degree 1. To present our first result, we introduce the constant α defined as follows.

$$\alpha = \begin{cases} \frac{n-1}{2}, \text{ for odd } n, \\ \frac{n-2}{2}, \text{ for even } n. \end{cases}$$

Theorem 1. *If G_1 is a fire cracker defined as above, then G_1 is a super edge-magic with*

$$\beta \le sm(G_1) \le 2p + n + 1 + \sum_{h=0}^{\alpha} m_{2h+1},$$

where

$$\beta = \begin{cases} \frac{1}{p-1}\left(3n^2 - n + 1 + p(2p-1) + \sum_{i=1}^{n-2} i m_{n+1-i}\right), & \text{if } m_1 = 1, \quad m_2 = 2, \\ \frac{1}{p-1}\left(3n^2 - 3n + 3 + p(2p-1) + \sum_{i=1}^{n-1} i m_{n+1-i}\right), & \text{if } m_1 = 1, \quad m_2 \ge 3, \\ \frac{1}{p-1}\left(3n^2 - n + 1 + p(2p-1) + \sum_{i=1}^{n-1} i m_{n+1-i}\right), & \text{if } m_1 = 2, \\ \frac{1}{p-1}\left(3n^2 - 3n + 1 + p(2p-1) + \sum_{i=1}^{n} i m_{n+1-i}\right), & \text{if } m_1 \ge 3. \end{cases}$$

Proof. Let f be any super edge-magic labeling of G_1 with the magic constant $c(f)$. We consider four following cases:

Case 1. $m_1 = 1$ and $m_2 = 2$.
In this case, G_1 has $\sum_{i=1}^n m_i$ vertices of degree 1, three vertices of degree 2, $n-1$ vertices of degree 3, and one vertex of degree $m_i + 1$ for $3 \le i \le n$. By assigning the smaller labels to the vertices of higher degrees, and by setting $q = p - 1$, we have

$$\begin{aligned} c(f) &\ge \tfrac{1}{p-1}\big[1(m_n+1) + 2(m_{n-1}+1) + 3(m_{n-2}+1) + \cdots + (n-2)(m_3+1) \\ &\quad + 3\textstyle\sum_{i=n-1}^{2n-3} i + 2\sum_{i=2n-2}^{2n} i + \sum_{i=2n+1}^{p} i + \sum_{i=p+1}^{2p-1} i\big] \\ &= \tfrac{1}{p-1}\big[\textstyle\sum_{i=1}^{n-2} i m_{n+1-i} + 2\sum_{i=n-1}^{2n-3} i + \sum_{i=2n-2}^{2n} i + \sum_{i=1}^{2p-1} i\big] \\ &= \tfrac{1}{p-1}\big[3n^2 - n + 1 + p(2p-1) + \textstyle\sum_{i=1}^{n-2} i m_{n+1-i}\big]. \end{aligned}$$

Case 2. $m_1 = 1$ and $m_2 \ge 3$.
In this case, G_1 has $\sum_{i=1}^n m_i$ vertices of degree 1, three vertices of degree 2, $n-2$ vertices of degree 3, and one vertex of degree $m_i + 1$ for $2 \le i \le n$. As in the Case 1, we have

$$\begin{aligned} c(f) &\ge \tfrac{1}{p-1}\big[1(m_n+1) + 2(m_{n-1}+1) + 3(m_{n-2}+1) + \cdots + (n-1)(m_2+1) \\ &\quad + 3\textstyle\sum_{i=n}^{2n-3} i + 2\sum_{i=2n-2}^{2n} i + \sum_{i=2n+1}^{p} i + \sum_{i=p+1}^{2p-1} i\big] \\ &= \tfrac{1}{p-1}\big[\textstyle\sum_{i=1}^{n-1} i m_{n+1-i} + 2\sum_{i=n}^{2n-3} i + \sum_{i=2n-2}^{2n} i + \sum_{i=1}^{2p-1} i\big] \\ &= \tfrac{1}{p-1}\big[3n^2 - 3n + 3 + p(2p-1) + \textstyle\sum_{i=1}^{n-1} i m_{n+1-i}\big]. \end{aligned}$$

Case 3. $m_1 = 2$.
In this case, G_1 has $\sum_{i=1}^{n} m_i$ vertices of degree 1, two vertices of degree 2, $n-1$ vertices of degree 3, and one vertex of degree m_i+1 for $2 \le i \le n$. By the similar argument as in the Case 1, we have

$$\begin{aligned}
c(f) &\ge \tfrac{1}{p-1}\big[1(m_n+1)+2(m_{n-1}+1)+3(m_{n-2}+1)+\cdots+(n-1)(m_2+1) \\
&\quad + 3\textstyle\sum_{i=n}^{2n-2} i + 2\sum_{i=2n-1}^{2n} i + \sum_{i=2n+1}^{p} i + \sum_{i=p+1}^{2p-1} i\big] \\
&= \tfrac{1}{p-1}\big[\textstyle\sum_{i=1}^{n-1} i m_{n+1-i} + 2\sum_{i=n}^{2n-2} i + \sum_{i=2n-1}^{2n} i + \sum_{i=1}^{2p-1} i\big] \\
&= \tfrac{1}{p-1}\big[3n^2 - n + 1 + p(2p-1) + \textstyle\sum_{i=1}^{n-1} i m_{n+1-i}\big].
\end{aligned}$$

Case 4. $m_1 \ge 3$.
In this case, G_1 has $\sum_{i=1}^{n} m_i$ vertices of degree 1, two vertices of degree 2, $n-2$ vertices of degree 3, and one vertex of degree m_i+1 for $1 \le i \le n$. It is easy to verify that the following formula is hold.

$$\begin{aligned}
c(f) &\ge \tfrac{1}{p-1}\big[1(m_n+1)+2(m_{n-1}+1)+3(m_{n-2}+1)+\cdots+n(m_1+1) \\
&\quad + 3\textstyle\sum_{i=n+1}^{2n-2} i + 2\sum_{i=2n-1}^{2n} i + \sum_{i=2n+1}^{p} i + \sum_{i=p+1}^{2p-1} i\big] \\
&= \tfrac{1}{p-1}\big[\textstyle\sum_{i=1}^{n} i m_{n+1-i} + 2\sum_{i=n+1}^{2n-2} i + \sum_{i=2n-1}^{2n} i + \sum_{i=1}^{2p-1} i\big] \\
&= \tfrac{1}{p-1}\big[3n^2 - 3n + 1 + p(2p-1) + \textstyle\sum_{i=1}^{n} i m_{n+1-i}\big].
\end{aligned}$$

To prove the upper bound, consider the vertex labeling $f : V(G_1) \to \{1, 2, 3, \cdots, p\}$ defined as follows.

$$f(u_i) = \begin{cases} m_1 + i + \sum_{h=1}^{\frac{i-1}{2}} m_{2h}, & \text{if } \ i \text{ is odd}, \\ n + i + \sum_{h=0}^{\alpha} m_{2h+1} + \sum_{h=1}^{\frac{i-2}{2}} m_{2h+1}, & \text{if } \ i \text{ is even}. \end{cases}$$

The vertices of degree $m_i + 1$ for $1 \le i \le n$, we label as follows.

$$f(u_{i,i}) = \begin{cases} n + i + \sum_{h=0}^{\alpha} m_{2h+1} + \sum_{h=1}^{\frac{i-1}{2}} m_{2h}, & \text{if } \ i \text{ is odd}, \\ i + \sum_{h=0}^{\frac{i-2}{2}} m_{2h+1}, & \text{if } \ i \text{ is even}. \end{cases}$$

The remaining vertices are labeled as follows.

$$f(x) = \begin{cases} j, & \text{if } \ x = u_{1,1}^{j}, \ \text{ for } 1 \le j \le m_1, \\ i + j - 1 + \sum_{h=0}^{\frac{i-3}{2}} m_{2h+1}, & \text{if } \ x = u_{i,i}^{j}, \ \text{ for odd } i \ge 3, \ 1 \le j \le \gamma, \\ i + j + \sum_{h=0}^{\frac{i-3}{2}} m_{2h+1}, & \text{if } \ x = u_{i,i}^{j}, \ \text{ for odd } i \ge 3, \ 1 + \gamma \le j \le m_i, \\ n + j + 2 + \sum_{h=0}^{\alpha} m_{2h+1}, & \text{if } \ x = u_{2,2}^{j}, \ \text{ for } 1 \le j \le m_2, \\ \epsilon - 1, & \text{if } \ x = u_{i,i}^{j}, \ \text{ for even } i \ge 4, \ 1 \le j \le \delta, \\ \epsilon, & \text{if } \ x = u_{i,i}^{j}, \ \text{ for even } i \ge 4, \ 1 + \delta \le j \le m_i, \end{cases}$$

where $\gamma = \sum_{h=1}^{\frac{i-1}{2}} m_{2h} - \sum_{h=1}^{\frac{i-3}{2}} m_{2h+1}$, $\delta = \sum_{h=1}^{\frac{i-2}{2}} m_{2h+1} - \sum_{h=1}^{\frac{i-2}{2}} m_{2h}$, and $\epsilon = n + i + j + \sum_{h=0}^{\alpha} m_{2h+1} + \sum_{h=1}^{\frac{i-2}{2}} m_{2h}$.

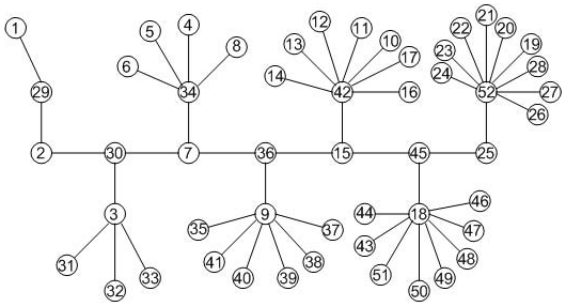

Fig. 1. A super edge-magic fire cracker

It is easy to verify that $S = \{f(x) + f(y) : xy \in E(G_1)\}$ is a set of consecutive integers with $max(S) = p+n+\sum_{h=0}^{\alpha} m_{2h+1}$. By Lemma 1, f extends to a super edge-magic labeling with the magic constant $c(f) = 2p + n + 1 + \sum_{h=0}^{\alpha} m_{2h+1}$. Hence, $sm(G_1) \leq 2p + n + 1 + \sum_{h=0}^{\alpha} m_{2h+1}$. □

Figure 1 shows a super edge-magic labeling of a fire cracker in case $n = 7, m_1 = 1, m_2 = 3, m_3 = 4, m_4 = 6, m_5 = 7, m_6 = 8$, and $m_7 = 9$.

Notice that, the upper bound also works for the case of $m_1 \leq m_2 \leq m_3 \leq \cdots \leq m_n$.

The next tree which we consider is a lobster G_2 having a form as in Figure 2.

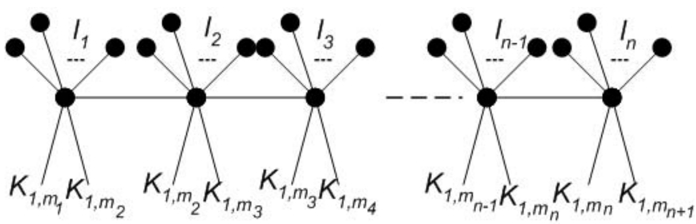

Fig. 2. A special type of lobster

We define G_2 as a lobster having

$$\begin{aligned} V(G_2) = &\{u_i : 1 \leq i \leq n\} \cup \{v_{i,j} : 1 \leq i \leq n, j = 1, 2\} \\ &\cup \{v_{i,1}^k : 1 \leq i \leq n, 1 \leq k \leq m_i\} \cup \{v_{i,2}^k : 1 \leq i \leq n, 1 \leq k \leq m_{i+1}\} \\ &\cup \{w_{i,h} : 1 \leq i \leq n, 1 \leq h \leq l_i\}, \end{aligned}$$

and

$$\begin{aligned} E(G_2) = &\{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_{i,j} : 1 \leq i \leq n, j = 1, 2\} \\ &\cup \{v_{i,1} v_{i,1}^k : 1 \leq i \leq n, 1 \leq k \leq m_i\} \cup \{v_{i,2} v_{i,2}^k : 1 \leq i \leq n, 1 \leq k \leq m_{i+1}\} \\ &\cup \{u_i w_{i,h} : 1 \leq i \leq n, 1 \leq h \leq l_i\}. \end{aligned}$$

Theorem 2. *If G_2 is a lobster defined as above, then G_2 is super edge-magic.*

Proof. First, we define two constants λ and μ as follows.

$$\lambda = \lfloor \frac{3n}{2} \rfloor + \sum_{t=1}^{\mu} m_t + \sum_{t=1}^{\lfloor \frac{n}{2} \rfloor} l_{2t},$$

and

$$\mu = \begin{cases} n+1, & \text{for odd } n, \\ n, & \text{for even } n. \end{cases}$$

Next, consider a vertex labeling $f : V(G_2) \to \{1, 2, 3, \cdots, p\}$ defined as follows.

$$f(u_i) = \begin{cases} \frac{1}{2}(3i-1) + \sum_{t=1}^{i} m_t + \sum_{t=1}^{\frac{i-1}{2}} l_{2t}, & \text{if } i \text{ is odd}, \\ \lambda + \frac{3i}{2} + \sum_{t=2}^{i} m_t + \sum_{t=0}^{\frac{i-2}{2}} l_{2t+1}, & \text{if } i \text{ is even}. \end{cases}$$

$$f(v_{i,1}) = \begin{cases} \lambda + \frac{1}{2}(3i-1) + \sum_{t=2}^{i} m_t + \sum_{t=0}^{\frac{i-3}{2}} l_{2t+1}, & \text{if } i \text{ is odd}, \\ \frac{1}{2}(3i-2) + \sum_{t=1}^{i} m_t + \sum_{t=1}^{\frac{i-2}{2}} l_{2t}, & \text{if } i \text{ is even}. \end{cases}$$

$$f(v_{i,2}) = f(v_{i,1}) + l_i + 1, \text{ for all } i.$$

We label all pendant vertices with the following formula.

$$f(v_{i,1}^j) = \begin{cases} \frac{3}{2}(i-1) + j + \sum_{t=1}^{i-1} m_t + \sum_{t=1}^{\frac{i-1}{2}} l_{2t}, & \text{if } i \text{ is odd}, 1 \le j \le m_i, \\ \lambda + \frac{1}{2}(3i-2) + j + \sum_{t=2}^{i-1} m_t + \sum_{t=0}^{\frac{i-2}{2}} l_{2t+1}, & \text{if } i \text{ is even}, 1 \le j \le m_i. \end{cases}$$

$$f(v_{i,2}^j) = \begin{cases} \frac{1}{2}(3i-1) + j + \sum_{t=1}^{i} m_t + \sum_{t=1}^{\frac{i-1}{2}} l_{2t}, & \text{if } i \text{ is odd}, 1 \le j \le m_{i+1}, \\ \lambda + \frac{3i}{2} + j + \sum_{t=2}^{i} m_t + \sum_{t=0}^{\frac{i-2}{2}} l_{2t+1}, & \text{if } i \text{ is even}, 1 \le j \le m_{i+1}. \end{cases}$$

$$f(w_{i,h}) = f(v_{i,1}) + h, \quad \text{for } 1 \le i \le n, \text{ and } 1 \le h \le l_i.$$

Finally, one can verify that $S = \{f(x) + f(y) : xy \in E(G_2)\}$ is a set of consecutive integers with $max(S) = p + \lambda$. By Lemma 1, f extends to a super edge-magic labeling of G_2 with the magic constant $2p + 1 + \lambda$. □

Figure 3 shows an illustration of Theorem 2 for $n = 5$, $m_1 = m_5 = 3$, $m_2 = m_4 = 1$, $m_3 = 2$, $m_6 = 4$, and $l_1 = l_5 = 3$, $l_2 = l_4 = 2$, $l_3 = 4$.

Corollary 1. *If $m_1 = m_2 = \cdots = m_{n+1} = l_1 = \cdots = l_n = m$ then*

$$\frac{1}{2p-2}\big[p(4p+3n-1) - (6n+4n-2)\big] \le sm(G_2) \le \frac{1}{2}(5p+2), \textit{for even } n,$$

and

$$\frac{1}{2p-2}\big[p(4p+3n-1) - (6n+4n-2)\big] \le sm(G_2) \le \frac{1}{2}(5p+m+1), \textit{for odd } n.$$

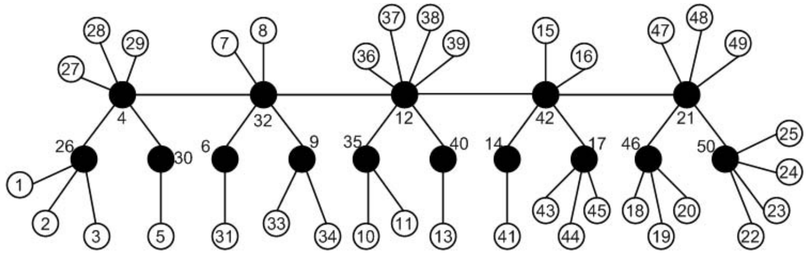

Fig. 3. A super edge-magic labeling on a special type of lobster

Proof. If $m_1 = m_2 = \cdots = m_{n+1} = l_1 = \cdots = l_n = m$, then G_2 has $3nm$ vertices of degree 1, $2n$ vertices of degree $m+1$, two vertices degree $m+3$, and $n-2$ vertices of degree $m+4$. Let f be any super edge-magic labeling of G_2, then

$$\begin{aligned} c(f) &\geq \tfrac{1}{q}\big[(m+4)\textstyle\sum_{i=1}^{n-2} i + (m+3)(n-1+n) + (m+1)\sum_{i=n+1}^{3n} i \\ &\quad + \textstyle\sum_{i=3n+1}^{p} i + \sum_{i=p+1}^{p+q} i\big] \\ &= \tfrac{1}{q}\big[3\textstyle\sum_{i=1}^{n-2} i + 2(2n-1) + m\sum_{i=1}^{3n} i + \sum_{i=1}^{p+q} i\big] \\ &= \tfrac{1}{2p-2}\big[p(4p+3n-1) - (6n+4n-2)\big]. \end{aligned}$$

By Theorem 2, G_2 is super edge-magic with the magic constant $\frac{1}{2}(5p+2)$ for even n, and $\frac{1}{2}(5p+m+1)$ for odd n. □

3 The Super Edge-Magic Deficiency of Graphs

The *edge-magic deficiency* of a graph G, $\mu(G)$, is the minimum nonnegative integer n such that $G \cup nK_1$ has an edge-magic labeling. Kotzig and Rosa [9] proved that the edge-magic deficiency of a graph is always finite.

Motivated by the Kotzig and Rosa's concept, Figueroa-Centeno et al. [7] defined the notion of super edge-magic deficiency of a graph. The *super edge-magic deficiency* of a graph G, $\mu_s(G)$, is the minimum nonnegative integer n such that $G \cup nK_1$ has a super edge-magic labeling or $+\infty$ if there exists no such n. Unlike the edge-magic deficiency, not all graphs have finite super edge-magic deficiency. The examples of such graphs can be found in [7]. As a direct consequence of the above two definitions, we have that for every graph G, $\mu(G) \leq \mu_s(G)$.

Figueroa-Centeno et al. in two separate papers [7,8] provided the exact values of (super) edge-magic deficiencies of several classes of graphs, such as complete graphs, some classes of forests, 2-regular graphs, and complete bipartite graphs $K_{2,m}$. They also provided an upper bound of the super edge-magic deficiency of complete bipartite graphs $K_{m,n}$.

In [11], Ngurah, Baskoro and Simanjuntak proved that the double fan $F_{n,2} \cong P_n + 2K_1$ is super edge-magic if and only if $n \leq 2$. Additionally, they proved the following result.

Theorem 3. *[11] The super edge-magic deficiency of* $F_{n,2}$ *satisfies* $\lfloor\frac{n-1}{2}\rfloor \leq \mu_s(F_{n,2}) \leq n-2$ *for all* $n \geq 2$.

In the next theorem, we show that for even n the super edge-magic deficiency of $F_{n,2}$ is equal to its lower bound.

Theorem 4. *For even* n *and* $n \geq 4$, $\mu_s(F_{n,2}) = \frac{n-2}{2}$.

Proof. Let $G \cong F_{n,2} \cup \frac{n-2}{2}K_1$ be a graph having

$$V(G) = \{x, y, z_i : 1 \leq i \leq n\} \cup \{w_i : 1 \leq i \leq \frac{n-2}{2}\},$$

and

$$E(G) = \{xz_i : 1 \leq i \leq n\} \cup \{yz_i : 1 \leq i \leq n\} \cup \{z_iz_{i+1} : 1 \leq i \leq n-1\}.$$

Now, define a vertex labeling $f : V(G) \to \{1, 2, 3, \cdots, \frac{3n+2}{2}\}$ as follows.

$$f(u) = \begin{cases} 1, & \text{if } \; u = x, \\ \frac{1}{2}(3n+2), & \text{if } \; u = y, \\ \lfloor\frac{1}{2}(3i+1)\rfloor, & \text{if } \; u = z_i, \; \text{for } 1 \leq i \leq n, \end{cases}$$

and

$$f(\{w_i : 1 \leq i \leq \frac{1}{2}(n-2)\}) = \{4, 7, 10, \cdots, \frac{1}{2}(3n-4)\}.$$

It can be checked that $S = \{f(u)+f(v) : uv \in E(V(G)\}$ is a set of consecutive integers with $max(S) = 3n+1$. By Lemma 1, f extends to a super edge-magic labeling of G with the magic constant $\frac{1}{2}(9n+6)$. By this fact and Theorem 3, we conclude $\mu_s(F_{n,2}) = \frac{n-2}{2}$. □

In [11], we showed that $\mu_s(F_{3,2}) = 1$, $\mu_s(F_{5,2}) = 2$, and $\mu_s(F_{7,2}) = 3$. Here, we prove that $\mu_s(F_{9,2}) = 4$, $\mu_s(F_{11,2}) = 5$, and $\mu_s(F_{13,2}) = 6$ by labeling the vertices $\{x, y; z_1, z_2, \cdots, z_n\}$ of $F_{n,2}$ for $n = 9, 11,$ and 13 as follows: $\{4, 9; 5, 3, 1, 2, 11, 14, 12, 15, 13\}$, $\{1, 18; 3, 6, 4, 7, 5, 14, 12, 15, 13, 16, 2\}$, and $\{2, 21; 10, 3, 1, 6, 8, 7, 9, 17, 15, 18, 16, 19, 4\}$. Based on this fact, we propose the following conjecture.

Conjecture 1. *For odd* n, $n \geq 3$, $\mu_s(F_{n,2}) = \frac{n-1}{2}$.

In the following results, we consider the super edge-magic deficiency of mK_{n_1,n_2}. We start with $mK_{2,2}$.

If $mK_{2,2}$ has a super edge-magic labeling f, then the magic constant is

$$\begin{aligned} c(f) &= \tfrac{1}{4m}\left(2\sum_{i=1}^{4m} i + \sum_{i=4m+1}^{8m} i\right) \\ &= \tfrac{1}{4m}\left(\sum_{i=1}^{4m} i + \sum_{i=1}^{8m} i\right) \\ &= \tfrac{1}{2}(16m^2 + 6m + 1). \end{aligned}$$

However $c(f)$ cannot be integer for all values of m. Hence, we have the following lemma.

Lemma 4. *For every integer $m \geq 1$ the graph $mK_{2,2}$ is not super edge-magic.* □

Theorem 5. *For every integer $m \geq 1$,*

$$\mu_s(mK_{2,2}) \leq \begin{cases} m, & \text{for odd } m, \\ m-1, & \text{for even } m. \end{cases}$$

Proof. For $1 \leq i \leq m$, let V_1^i and V_2^i be the partite sets of the i^{th}-component of $mK_{2,2}$. Next, define a vertex labeling g as follows.

$$g(V_1^i) = \{i, m+i\},$$

$$g(V_2^i) = \begin{cases} \{\frac{1}{2}(5m+2i+1), \frac{1}{2}(9m+2i+1) & \text{for odd } m \text{ and } 1 \leq i \leq \frac{m-1}{2}, \\ \{\frac{1}{2}(3m+2i+1), \frac{1}{2}(7m+2i+1)\}, & \text{for odd } m \text{ and } \frac{m+1}{2} \leq i \leq m, \\ \{\frac{1}{2}(7m-6i+4), \frac{1}{2}(7m-6i+2)\}, & \text{for even } m \text{ and } 1 \leq i \leq \frac{m}{2}, \\ \{\frac{1}{2}(13m-6i+4), \frac{1}{2}(13m-6i+2)\}, & \text{for even } m \text{ and } \frac{m+2}{2} \leq i \leq m. \end{cases}$$

It is easy to verify that $S = \{g(x)+g(y) : xy \in E(mK_{2,2})$ is a set of consecutive integers with $max(S) = \lfloor \frac{1}{2}(13n+1) \rfloor$. We can also verify that the labels are all positive and no labels are repeated. However, the largest vertex label used is $5m$ if m is odd or $5m-1$ if m is even, and there exist m labels if m is odd or $m-1$ labels if m is even that are not utilized. So, for each of the numbers between 1 and the largest vertex label that has not been used as a label, we introduce a new vertex with that number as its label, which gives m or $m-1$ new isolated vertices depend on parity of m. By Lemma 1, g extends to a super edge-magic labeling of $mK_{2,2} \cup mK_1$ or $mK_{2,2} \cup (m-1)K_1$ with magic constant $\frac{1}{2}(23m+3)$ for odd m or $\frac{23m}{2}$ for even m. This completes the proof. □

Open Problem 6. *Find a better upper bound of $\mu_s(mK_{2,2})$.*

By a similar argument in Lemma 4, we have the following fact.

Lemma 5. *For every even integer $m \geq 2$ the graph $mK_{3,3}$ is not super edge-magic.* □

Theorem 7. *For every integer $m \geq 1$,*

$$\mu_s(mK_{3,3}) \leq 3m+1$$

Proof. Consider a vertex labeling h defined as follows.

$$h(W) = \begin{cases} \{4i-3, 4i-2, 4i-1\}, & \text{if } W = V_1^i \text{ for } 1 \leq i \leq m, \\ \{4m+5i-5, 4m+5i-2, 4m+5i+1\}, & \text{if } W = V_2^i \text{ for } 1 \leq i \leq m, \end{cases}$$

where V_1^i and V_2^i are the partite sets of the i^{th}-component of $mK_{3,3}$.

One can verify that the maximum vertex label used under the labeling h is $9m+1$. Also, it is no difficult to verify that $S = \{h(u)+h(v) : uv \in E(mK_{3,3})\}$ is a set of consecutive integers with $max(S) = 13m$. By Lemma 1, h extends to a super edge-magic labeling of $mK_{3,3} \cup (3m+1)K_1$ with the magic constant $22m+2$. □

Open Problem 8. *For even m, find a better upper bound of $\mu_s(mK_{3,3})$.*

For $m \geq 3$ odd, we have not found any super edge-magic labeling of $mK_{3,3}$. Therefore, we propose the following open problem.

Open Problem 9. *For $m \geq 3$ odd, determine if there is a super edge-magic labeling of $mK_{3,3}$.*

Theorem 10. *For every integers $n_1, n_2 \geq 4$, and $m \geq 1$,*

$$\lfloor \frac{mn_1n_2}{2} \rfloor - m(n_1+n_2) + 3 \leq \mu_s(mK_{n_1,n_2}) \leq m(n_1n_2 - n_1 - n_2) + 1.$$

Proof. This graph is not super edge-magic by Lemma 2. Without lost of generality, we assume that $n_1 \leq n_2$. For $1 \leq i \leq m$, let V_1^i and V_2^i be the partite sets of the i^{th}- component of mK_{n_1,n_2}. Now, define a vertex labeling f such that $f(V_1^i) = \{a_1, a_1+1, a_1+2, a_1+3, \cdots, a_1+(n_1-1)\}$, and $f(V_2^i) = \{a_2, a_2+n_1, a_2+2n_1, a_2+3n_1, \cdots, a_2+(n_2-1)n_1\}$, where $a_1 = n_1(i-1)+i$, and $a_2 = m(n_1+1)+(i-1)(n_1n_2-n_1-1)$. One can verify that the maximum vertex label used is mn_1n_2+1, and $S = \{f(u)+f(v) : uv \in E(mK_{n_1,n_2})\}$ is a set of consecutive integers with $max(S) = m(n_1n_2+n_1+1)$. By Lemma 1, f extends to a super edge-magic labeling of $mK_{n_1,n_2} \cup (m(n_1n_2-n_1-n_2)+1)K_1$ with the magic constant $m(2n_1n_2+n_1+1)+2$.

The lower bound is obtained from Lemma 2, since bipartite graphs contain no odd cycles. □

Open Problem 11. *Find a better lower and upper bounds of $\mu_s(mK_{n_1,n_2})$ for $n_1,\ n_2 \geq 4$.*

The above results concern on super edge-magic deficiency of disconnected graphs whose components are isomorphic. The following result present super edge-magic deficiency of disconnected graphs whose components are not isomorphic.

Theorem 12. *If $G \cong \cup_{i=0}^{m} G_i$, where $G_i \cong K_{4+i,4+i}$, $0 \leq i \leq m$. Then, $\lfloor \frac{1}{12}(2m^3+15m^2+13m) \rfloor + 3 \leq \mu_s(G) \leq \frac{1}{6}(m+3)(m+4)(2m+7)-(m+5)$.*

Proof. This graph has minimum degree 4, by Lemma 2, it is not a super edge-magic graph. Let V_1^i and V_2^i be the partite sets of G_i, $0 \leq i \leq m$. Consider a vertex labeling g such that

$g(U) = \{b_1, b_1+1, b_1+2, \cdots, b_1+i+3\}$ if $U = V_1^i$ for $0 \leq i \leq m$, and $g(U) = \{b_2, b_2+(4+i), b_2+2(4+i), \cdots, b_2+(i+3)(4+i)\}$ if $U = V_2^i$ for $0 \leq i \leq m$,

where $b_1 = \frac{1}{2}(i^2+9i+2)$, and $b_2 = \frac{1}{2}(m+1)(m+8)+(m+1)+\frac{1}{3}(i^3+9i^2+23i)$. By Lemma 1, g extends to a super edge-magic labeling of $G \cup (\frac{1}{6}(m+3)(m+4)(2m+7)-(m+5))K_1$. As in Theorem 10, the lower bound is obtained from Lemma 2. □

Open Problem 13. *Find a better lower and upper bounds of $\mu_s(G)$ where G defined as in Theorem 12.*

In above results, we present the upper bound of the super edge-magic deficiency of mK_{n_1,n_2} for some values of n_1 and n_2. However, these results do not include cases for $mK_{1,n}$, $mK_{2,n}$, and $mK_{3,n}$. In fact, we are not able to determine whether these graphs are super edge-magic or not. However, for graph $mK_{1,n}$ in case $n = 1, 2$, we have the following results. Notice that, it is well known that $K_{1,n}$ is a super edge-magic and 3-colorable graph. So, $mK_{1,n}$ is super edge-magic for odd m, that is, $\mu_s(mK_{1,n}) = 0$, for odd m. For even m, we only known that $\mu_s(mK_{1,1}) = 1$ (see [7]), and $\mu_s(mK_{1,2}) = 0$ (see [3]). Hence, we have the following open problems.

Open Problem 14. *For even $m \geq 2$ and $n \geq 3$, determine if there is a super edge-magic labeling of $mK_{1,n}$.*

Open Problem 15. *For $m \geq 2$ and $n \geq 3$, determine if there is a super edge-magic labeling of $mK_{2,n}$.*

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The Number of Flips Required to Obtain Non-crossing Convex Cycles

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Abstract. In this paper, we consider Hamiltonian cycles of vertices in convex position on the plane, where, in general, these cycles contain crossing edges. We give several results concerning the minimum number of operations that delete two crossing edges, add two other edges and preserve hamiltonicity in transforming these cycles to non-crossing Hamiltonian cycles.

1 Introduction

In this paper, we consider Hamiltonian cycles on the plane. Let S be a set of n vertices on the plane, and let the vertices be labeled from 1 to n. We denote a Hamiltonian cycle passing through vertices $i_1, i_2, \cdots, i_n$ by $(i_1, i_2, \cdots, i_n)$. For example, we write $(1, 3, 5, 4, 7, 2, 8, 6)$ for the cycle in Fig. 1. We regard the Hamiltonian cycles $(i_1, i_2, \cdots, i_n)$, $(i_2, \cdots, i_n, i_1)$ and $(i_n, i_{n-1}, \cdots, i_1)$ to be equivalent. An operation that deletes two crossing edges and adds two other edges while preserving hamiltonicity will be termed a *flip* in this paper.

The present study considers the following problem: For a given Hamiltonian cycle, find the minimum number of flips required to obtain a non-crossing Hamiltonian cycle. Note that the repetition of flips is similar to 2-OPT, which is one of the most famous local search methods for the Traveling Salesman Problem:

For a Hamiltonian cycle C on the plane, if a Hamiltonian cycle C' is constructed from C by deleting two edges and adding two other edges such that the total Euclidean distance of C' is less than that of C, the transformation from C to C' is called *a feasible 2-change*. The procedure that repeats feasible 2-change operations until no further feasible 2-changes are possible is called *2-OPT*. Recently, Englert et al. proved the following theorem:

Theorem A (Englert, Röglin and Vöcking, 2007 [2]). *There exists a set of n vertices S on the plane together with a Hamiltonian cycle C of S such that we need $\Omega(2^{\frac{n}{8}})$ feasible 2-change operations to reach a local optimal solution from C.*

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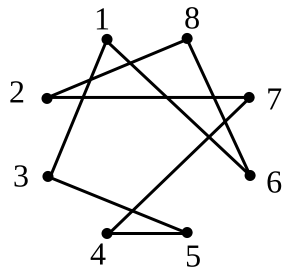

Fig. 1. A convex cycle

Regarding flip operations on the plane, van Leeuwen and Schoone presented the following theorem:

Theorem B (van Leeuwen and Schoone, 1980 [7]). *Let S be a set of n vertices arranged in general positions on the plane, that is, no three vertices are collinear. Any Hamiltonian cycle of S can be transformed into a non-crossing Hamiltonian cycle by $O(n^3)$ flips.*

We note that van Leeuwen and Schoone proved a slightly stronger theorem than the one above, that is, the vertices in S are not supposed to be in general positions, but they admitted two other types of operations that untangle edges or paths of three or more collinear vertices. In this paper, we shall consider a set of vertices in convex position. *A convex cycle* is a Hamiltonian cycle whose vertices are arranged in convex position, and these vertices are supposed to be labeled from 1 to n in cyclic order (see Fig. 1 again). Let $\mathcal{A}_n$ be the set of convex cycles on S of order n and let $\delta(C)$ be the minimum number of flips required to obtain a non-crossing convex Hamiltonian cycle of the form $I_n := (1, 2, 3, \cdots, n)$ from a convex cycle C. This study considers the following problem:

Problem. *Find the value $\delta(n)$ such that*

$$\delta(n) := \max_{C \in \mathcal{A}_n} \delta(C).$$

In this paper, we give several results for the above problem. In Section 2, we give lower and upper bounds on the minimum number of flips and, moreover, we describe some restricted sets of convex cycles for which an exact value for the minimum number of flips can be obtained. In Section 3, some computational results will be presented. The related problems are discussed in [3,4,5,8]. Pach and Tardos [8] presented a different type of transformation; in each step we can arbitrarily relocate one of its vertices on the plane. They gave upper and lower bounds on the minimum number of steps required to obtain a non-self-intersecting closed polygon from an arbitrary self-intersecting closed polygon. In [3], the authors gave several results concerning the operations that delete two edges, add two other edges and preserve hamiltonicity, where we do not care if these two edges are crossing or non-crossing. Ito, Uehara and Yokoyama [4] and Ito [5] gave several properties of convex cycles which will be discussed in Section 2.

2 Main Results

We first give the following upper bound for general convex cycles.

Theorem 1. *For any integer $n \geq 4$,*

$$\delta(n) \leq 2n - 7.$$

For convenience, we give orientations for each edge of a convex cycle, so such a cycle may be regarded as a directed cycle. Since a flip preserves hamiltonicity, the following fact is easy to understand (see Fig. 2).

Fact 1. *Upon applying a flip to the pair of arcs (u_1, v_1) and (u_2, v_2) of a convex cycle, the new added edges are (u_1, u_2) and (v_1, v_2), i.e. one of the added edges connects the tails u_1 and u_2 of the deleted arcs, while the other connects the heads v_1 and v_2.*

Let $e = (u, v)$ be an edge with $u < v$. The depth of e is defined as the minimum of $v - u$ and $u - v + n$, which means the minimum difference of the labels of u and v modulo n.

Lemma 1. *Any convex cycle C can be transformed into a convex cycle containing a boundary edge by at most one flip.*

Proof. Let e be an edge of minimum depth in C. We may suppose that the depth m of e is more than 1, and that the tail of e is vertex 1 and the head of e is vertex $m+1$ by symmetry. Let e' be the edge whose tail is vertex 2. Then, e and e' are crossing because of the minimality of the depth of e. With reference to Fact 1, the flip on the pair $\{e, e'\}$ therefore produces the new boundary edge $(1, 2)$. □

Proof (of Theorem 1). We use induction on n. It is easy to check that $\delta(4) = 1$. Assume $n \geq 5$. By Lemma 1, at most one flip is required to produce a new convex cycle C' containing a boundary edge $e = (u, u+1)$ (u may be considered in modulo n) from an arbitrary convex cycle C. We then contract the edge e of C' and denote the resulting cycle with $n-1$ vertices by $\tilde{C}'$. By the induction hypothesis, it holds that $\delta(\tilde{C}') \leq 2(n-1) + 7$. Hence, after at most $\delta(\tilde{C}') + 1$ flips, the convex cycle C can be transformed into I_n or a cycle of the form

$$(\cdots, u-1, u+1, u, u+2, \cdots).$$

For the latter case, by operating one more flip on the pair of edges $\{(u-1, u+1), (u, u+2)\}$, we obtain I_n. Therefore, we have

$$\delta(n) \leq \delta(\tilde{C}') + 1 + 1 \leq \{2(n-1) + 7\} + 1 + 1 = 2n - 7. \qquad \square$$

Next, we give lower bounds on $\delta(n)$.

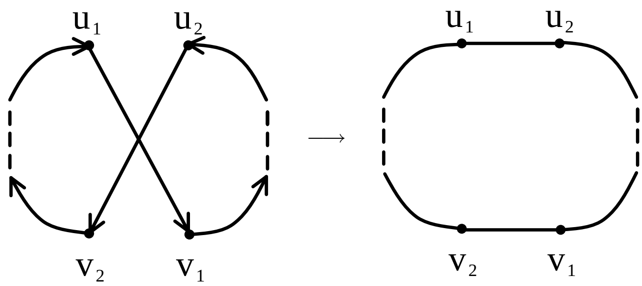

Fig. 2. A flip on two arcs

Theorem 2. *For any integer $n \geq 3$,*

$$\delta(n) \geq \begin{cases} n-3 & \textit{if } n = 3, 4, 6, \\ n-2 & \textit{if } n = 5 \textit{ or } n \geq 7. \end{cases}$$

In order to prove Theorem 2 it is convenient to define some particular convex cycles. Let $C_{n,p}$ be the convex cycle traversing every p steps in cyclic order. For example, Fig. 3 shows the convex cycle $C_{8,3}$. We may assume that n and p are mutually prime, since otherwise $C_{n,p}$ does not form a Hamiltonian cycle. Set

$$\delta(n, p) := \delta(C_{n,p}).$$

We prove the following theorem.

Theorem 3. *For any odd integer $n \geq 5$,*

$$\delta(n, 2) = n - 2.$$

Here we introduce some definitions. A Hamiltonian cycle is called *pyramidal* if the cycle is the form of $1 = v_0, v_1, \cdots, v_r = n, v_{r+1}, \cdots, v_{n-1}, v_n = 1$ such that $v_0 < v_1 < \cdots < v_r > v_{r+1} > \cdots > v_{n-1} > v_n$. The concept of pyramidal cycles is frequently used for special cases of the Traveling Salesman Problem (see [1,6]). The following fact is fundamental.

Fact 2. *Let C be a convex pyramidal directed cycle. For any pair of crossing arcs (i, j) and (k, l) with $i < k$, it holds that either $i < l < j < k$ or $l < i < k < j$.*

Proof. If neither $i < l < j < k$ nor $l < i < k < j$ holds, then either $i < k < j < l$ or $j < l < i < k$. However, both these cases contradict the pyramidal property of C. □

Proof (of Theorem 3). It is easy to see that $n - 2$ flips on $\{(1, n-1), (2, n)\}$, $\{(1, 3), (2, 4)\}$, $\{(1, 4), (3, 5)\}$, $\{(1, 5), (4, 6)\}, \cdots$, $\{(1, n-1), (n-2, n)\}$ from $C_{n,2}$ results in the non-crossing convex cycle I_n. So, it holds that $\delta(n, 2) \leq n - 2$.

Next, we shall show that $\delta(n, 2) \geq n - 2$. By symmetry, one flip results in a unique convex cycle if we ignore the labeling on vertices. Without loss of generality, we may assume that the cycle is the form of $(1, 2, 4, 6, \cdots, n-1, n,$

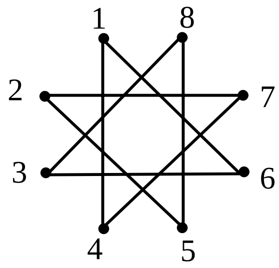

Fig. 3. The convex cycle $C_{8,3}$

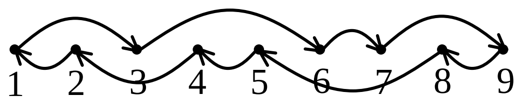

Fig. 4. A pyramidal cycle

$n-2, n-4, \cdots, 3, 1)$. Set $C^+_{n,2}$ to be this convex cycle (see Fig. 5). It is clear that the convex cycle $C^+_{n,2}$ is pyramidal.

Claim. Let C be a convex pyramidal cycle. Any convex cycle obtained from C by any flip is pyramidal.

Proof. We give orientations for all the edges of C so that it becomes a directed cycle. By Fact 2 and symmetry, we may assume that C has a pair of crossing arcs (i, j) and (k, l) with $i < l < j < k$. By Fact 1, the flip on that pair of edges produces the convex cycle $C \cup \{(i, k), (j, l)\} - \{(i, j), (k, l)\}$, which is pyramidal. □

We note that $C^+_{n,2}$ is a convex pyramidal cycle and has two boundary edges $(1, 2)$ and $(n-1, n)$. Upon applying a flip, the number of boundary edges $(i, i+1)$ with $2 \leq i \leq n-2$ increases by at most 1 according to Facts 1 and 2, and the resulting cycle is also convex and pyramidal by the above claim. By repeating this argument, we know that at least $n-3$ flips must be applied to $C^+_{n,2}$ in order to obtain a cycle containing boundary edges $(i, i+1)$ with $1 \leq i \leq n-1$. Therefore, we have $\delta(n, 2) \geq 1 + (n-3) = n-2$. Thus, Theorem 3 follows. □

Next, we define the convex cycle D_n of n vertices for any even integer n as follows:

$$D_n = (1, 3, 5, \cdots, n-5, n-2, n, n-3, n-1, 2, 4, 6, \cdots, n-4).$$

For example, D_8 and D_{10} are in Fig. 6.

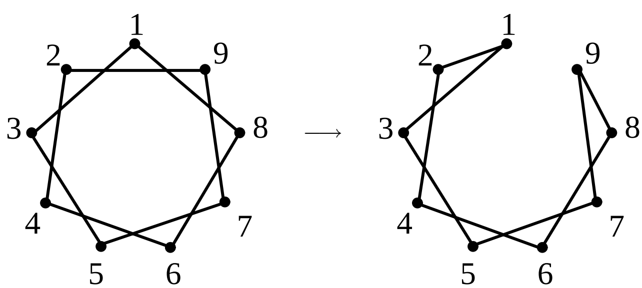

Fig. 5. The first flip from $C_{9,2}$ to $C^+_{9,2}$

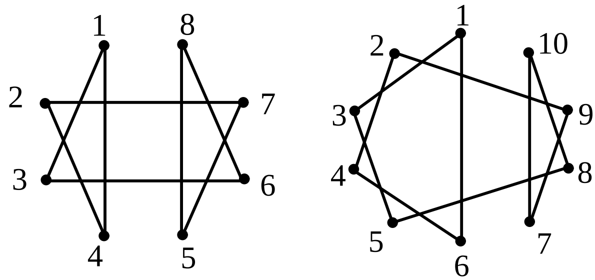

Fig. 6. The convex cycles D_8 and D_{10}

Theorem 4. *For any even integer* $n \geq 8$,

$$\delta(D_n) \geq n - 2.$$

Proof. Define two subpaths of D_n as follows:

$$X_{n-2} = (y_1, 2, 4, \cdots, n-4, 1, 3, \cdots, n-5, y_2)$$
$$Y = (x_1, n-2, n, n-3, n-1, x_2),$$

where $y_1 = n-1$, $y_2 = n-2$, $x_1 = n-5$ and $x_2 = 2$ at this point. We note that X_{n-2} and Y contain two edges $(n-1, 2)$ and $(n-5, n-2)$ in common.

In order to obtain I_n from D_n, it is clearly necessary to transform X_n to the path $(y'_1, 1, 2, \cdots, n-4, y'_2)$ and to transform Y to the path $(x'_1, n-3, n-2, n-1, n, x'_2)$, where $n-3 \leq y'_i \leq n$ and $1 \leq x'_i \leq n-4$ for $i = 1, 2$. It is easy to see that if we regard y_1 and y_2 as the same vertex in X_{n-2}, then the graph $\tilde{X}_{n-2}$ is isomorphic to $C_{n-3,2}$. Similarly, if we regard x_1 and x_2 as the same vertex in Y, then the graph $\tilde{Y}$ is isomorphic to $C_{5,2}$. We note that any flip on the pair $\{(x, y), (x', x'')\}$ with $1 \leq x, x', x'' \leq n-4$ and $y = y_1$ or y_2 has no effect on the path of vertices in $\{n-3, n-2, n-1, n\}$ and that any flip on the pair $\{(x, y), (y', y'')\}$ with $x = x_1$ or x_2 and $n-3 \leq y, y', y'' \leq n$ has no effect on the path of vertices in $\{1, 2, \cdots, n-4\}$. So, we have

$$\delta(D_n) \geq \delta(C_{n-3,2}) + \delta(C_{5,2}) = \{(n-3) - 2\} + (5-2) = n - 2.$$

□

Now, we prove Theorem 2.

Proof (of Theorem 2). It is easy to check that $\delta(3) = 0$, $\delta(4) = 1$, $\delta(5) = 3$ and $\delta(6) = 3$. For any integer $n \geq 7$ Theorems 3 and 4 imply Theorem 2. □

Next, we prove the following theorem for $\delta(n, 3)$.

Theorem 5. *For any integer* $n \geq 7$,

$$\delta(n,3) = \begin{cases} \dfrac{n+3}{2} & \textit{if } n \equiv 1, 5 \pmod 6, \\ \dfrac{n}{2} & \textit{if } n \equiv 2, 4 \pmod 6. \end{cases}$$

Proof. First of all, we show

$$\delta(n,3) \geq \begin{cases} \dfrac{n+3}{2} & \text{if } n \equiv 1,5 \pmod 6, \\ \dfrac{n}{2} & \text{if } n \equiv 2,4 \pmod 6. \end{cases} \tag{1}$$

It is clear that the number of boundary edges that arise due to a flip is at most two. Thus, we have $\delta(n,3) \geq \left\lceil \frac{n}{2} \right\rceil$. It remains to show that $\delta(n,3) > \frac{n+1}{2}$ if n is odd. Suppose that $\delta(n,3) = \frac{n+1}{2}$. For this case, by the same reasoning as above, exactly one new boundary edge is produced by a single flip F, while two new boundary edges are produced by each of the other $\frac{n+1}{2} - 1$ flips. When two new boundary edges are produced by a flip, the depths of the deleted two edges must be the same. Since the last flip that operates to obtain I_n produces two new boundary edges, flip F cannot be the last one. Since any edge of $C_{n,3}$ has depth 3, any flip before F deletes two edges of depth 3 and adds two (boundary) edges of depth 1. Furthermore, the depths of the two edges that arise due to flip F are 1 and 5 by Facts 1 and 2. Let e be the edge of depth 5. Any flip that follows F produces two new boundary edges. One of these has to use e, but this is impossible since after F the convex cycle consists of e and edges of depth 1 or 3. Therefore, we have $\delta(n,3) > \frac{n+1}{2}$ if n is odd. Thus, the inequality (1) holds.

Next, we shall prove

$$\delta(n,3) \leq \begin{cases} \dfrac{n+3}{2} & \text{if } n \equiv 5 \pmod 6, \\ \dfrac{n}{2} & \text{if } n \equiv 2 \pmod 6. \end{cases} \tag{2}$$

Fig. 7 shows us that $\delta(8,3) \leq 4$. Similarly, it holds that $\delta(11,3) \leq 7$ by the following steps:

$$\begin{aligned} C_{11,3} = (1,4,7,10,2,5,8,11,3,6,9) &\xrightarrow{(1,9),(2,10)} C^{+}_{11,3} = (1,2,5,8,11,3,6,9,10,7,4) \\ &\xrightarrow{(2,5),(3,6)} (1,2,3,11,8,5,6,9,10,7,4) \xrightarrow{(7,10),(8,11)} (1,2,3,11,10,9,6,5,8,7,4) \\ &\xrightarrow{(1,4),(3,11)} (1,2,3,4,7,8,5,6,9,10,11) \xrightarrow{(4,7),(6,9)} (1,2,3,4,6,5,8,7,9,10,11) \\ &\xrightarrow{(4,6),(5,8)} (1,2,3,4,5,6,8,7,9,10,11) \\ &\xrightarrow{(6,8),(7,9)} I_{11} = (1,2,3,4,5,6,7,8,9,10,11). \end{aligned}$$

Assume $n \geq 14$. By the flip on the pair $\{(1,n-2),(2,n-1)\}$ for $C_{n,3}$, we have the convex cycle

$$(1,2,5,\cdots,n-6,n-3,n,3,6,\cdots,n-5,n-2,n-1,n-4,n-7,\cdots,4).$$

Denote the above cycle by $C^{+}_{n,3}$. By three flips on the pairs $\{(n-2,n-5),(n-3,n-6)\}$, $\{(n-4,n-7),(n-5,n-8)\}$ and $\{(n-3,n),(n-4,n-1)\}$ we have the cycle E_n given by

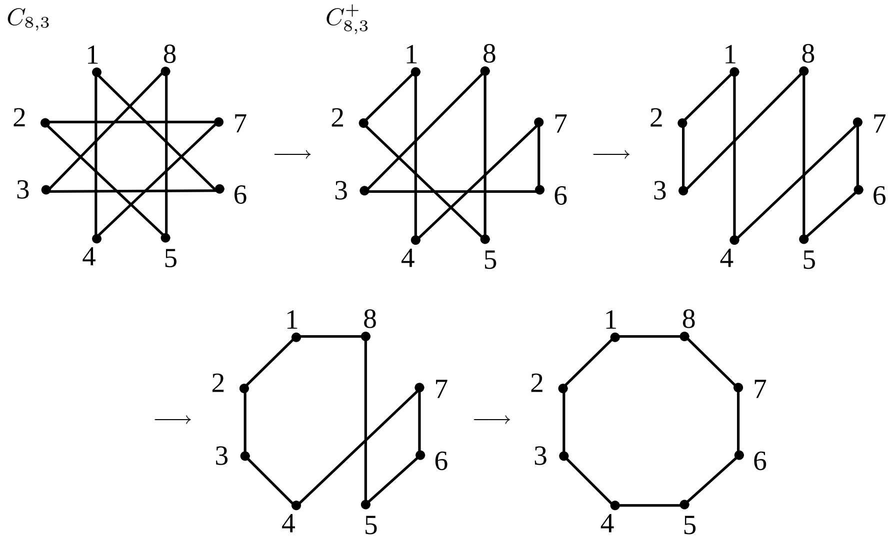

Fig. 7. The result of four flip operations applied successively to $C_{8,3}$.

$$(1,2,5,\cdots,n-9,n-6,n-5,n-4,n-3,n-2,n-1,n,3,6,\\ \cdots,n-8,n-7,n-10,\cdots,4).$$

Let

$$E'_n = (1,2,5,\cdots,n-9,n-6,3,6,\cdots,n-8,n-7,n-10,\cdots,4),$$

which is obtained from E_n by replacing the path $(n-6,n-5,n-4,n-3,n-2,n-1,n)$ with the vertex $n-6$. Then, $E'_n \simeq C^+_{n-6,3}$. By using induction, we have

$$\begin{aligned}\delta(n,3) &\le \delta(n-6,3)-1+4\\ &\le \begin{cases} \dfrac{(n-6)+3}{2}+3=\dfrac{n+3}{2} & \text{if } n\equiv 5 \pmod 6,\\[2mm] \dfrac{n-6}{2}+3=\dfrac{n}{2} & \text{if } n\equiv 2 \pmod 6.\end{cases}\end{aligned}$$

Hence, we have the inequality (2).

Finally, we shall show

$$\delta(n,3) \le \begin{cases} \dfrac{n+3}{2} & \text{if } n\equiv 1 \pmod 6,\\[2mm] \dfrac{n}{2} & \text{if } n\equiv 4 \pmod 6.\end{cases} \tag{3}$$

We obtain $\delta(7,3)\le 5$ by the following 5 flips:

$$\begin{aligned}C_{7,3}&=(1,4,7,3,6,2,5)\xrightarrow{(1,5),(2,6)}(1,2,5,6,3,7,4)\xrightarrow{(4,7),(3,6)}(1,2,5,6,7,3,4)\\ &\xrightarrow{(1,4),(2,5)}(1,2,4,3,7,6,5)\xrightarrow{(2,4),(3,7)}(1,2,3,4,7,6,5)\\ &\xrightarrow{(1,5),(4,7)} I_7=(1,2,3,4,5,6,7).\end{aligned}$$

Thus, we may assume $n \geq 10$. Here we shall prove the following claim.

Claim. For any integer $n \geq 10$ with $n \equiv 1 \pmod 3$,

$$\delta(n,3) \leq \delta(n-2,3) + 1.$$

Proof. By the flip on the pair $\{(1, n-2), (2, n-1)\}$ for $C_{n,3}$, we have the convex cycle

$$(1, 2, 5, \cdots, n-2, n-1, n-4, n-7, \cdots, 3, n, n-3, \cdots, 4).$$

Let $C^{+}_{n,3}$ be the above cycle. Moreover, by the next flip on the pair $\{(n, n-3), (n-1, n-4)\}$, we have the cycle E_n

$$(1, 2, 5, \cdots, n-5, n-2, n-1, n, 3, 6, \cdots, n-4, n-3, n-6, \cdots, 4).$$

Let

$$E'_n = (1, 2, 5, \cdots, n-5, n-2, 3, 6, \cdots, n-4, n-3, n-6, \cdots, 4),$$

which is obtained from E_n by replacing the path $(n-2, n-1, n)$ with the vertex $n-2$. Then, $E'_n \simeq C^{+}_{n-2,3}$. By using induction, we have

$$\delta(n,3) \leq \delta(n-2,3) - 1 + 2 = \delta(n-2,3) + 1.$$

Thus, the claim follows. □

By the above claim, in case of $n \equiv 1 \pmod 6$ it holds that $\delta(n,3) \leq \delta(n-2,3) + 1 \leq \frac{(n-2)+3}{2} + 1 = \frac{n+3}{2}$ since $n-2 \equiv 5 \pmod 6$. Also, in the case that $n \equiv 4 \pmod 6$ we have $\delta(n,3) \leq \delta(n-2,3) + 1 \leq \frac{n-2}{2} + 1 = \frac{n}{2}$ because $n-2 \equiv 2 \pmod 6$. Therefore, inequality (3) is proved.

This completes the proof of Theorem 5. □

We have also derived upper bounds for cases in which p is even, $p = 4, 6$ and 8. We have proved that $\delta(n,4) \leq \frac{2}{3}n + O(1)$, $\delta(n,6) \leq \frac{3}{5}n + O(1)$ and $\delta(n,8) \leq \frac{4}{7}n + O(1)$ for any sufficiently large n, but do not present the proofs here.

3 Conclusion

Finally, we briefly describe some computational results for $\delta(n)$ and $\delta(n,p)$. We verified computationally that $\delta(n) = n-2$ for any integer n from 7 to 10, and present the list of convex cycles that require $n-2$ flips to obtain I_n in Table 1. We give the following conjecture:

Conjecture 1. For any integer $n \geq 7$,

$$\delta(n) = n - 2.$$

Table 1. The list of convex cycles that require $n-2$ flips (up to symmetry) to be transformed to non-crossing convex cycles

n	Convex cycles
7	$(1,3,5,7,2,4,6) = C_{7,2}$
	$(1,4,7,3,6,2,5) = C_{7,3}$
8	$(1,3,6,8,5,7,2,4) = D_8$
	$(1,3,7,5,8,6,2,4)$
	$(1,3,6,2,7,5,8,4)$
9	$(1,3,5,7,9,2,4,6,8) = C_{9,2}$
	$(1,4,2,6,3,8,5,9,7)$
10	$(1,3,5,8,10,7,9,2,4,6) = D_{10}$
	$(1,3,5,9,7,10,8,2,4,6)$
	$(1,4,8,10,7,9,3,6,2,5)$
	$(1,4,9,7,10,8,3,6,2,5)$
	$(1,4,9,2,7,10,5,8,3,6)$

Table 2. Computed results for $\delta(n,4)$, $\delta(n,5)$ and $\delta(n,6)$ ($''\cdot''$ means that $C_{n,p}$ is not defined while $''-''$ means that we have not computed this value)

p\n	7	8	9	10	11	12	13	14	15	16	17	18	19
3	5	4	·	5	7	·	8	7	·	8	10	·	11
4			6	·	6	·	9	·	10	·	10	·	13
5					8	6	9	7	·	8	12	–	–
6							10	·	·	·	–	·	–

We have also obtained $\delta(n,4)$, $\delta(n,5)$ and $\delta(n,6)$ for small n as presented in Table 2. From the table it is clear that for fixed p the value of $\delta(n,p)$ does not increase monotonically with increasing n, but the values exhibit a strong dependence on modulo p. A precise expression of $\delta(n,p)$ for general values of n and p would thus appear to remain an open problem.

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Divide and Conquer Method for k-Set Polygons

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Abstract. The k-sets of a set V of points in the plane are the subsets of k points of V that can be separated from the rest by a straight line. In order to find all the k-sets of V, one can build the so called k-set polygon whose vertices are the centroids of the k-sets of V. In this paper, we extend the classical convex-hull divide and conquer construction method to build the k-set polygon.

1 Introduction

Given a finite set V of $|V| = n$ points in the Euclidean plane (no three of them being collinear) and an integer k $(0 < k \leq n)$, the k-set polygon $g^k(V)$ of V is the convex hull of the centroids $g(T)$ of all the subsets T of k elements of V. Andrzejak and Fukuda [1] showed that the vertices of $g^k(V)$ are the centroids of the k-sets of V, i.e. of the subsets of k points of V that can be strictly separated from the rest by a straight line. Thus, determining k-sets comes down to constructing k-set polygons. Counting and constructing k-sets is an important problem in computational geometry. Cole, Sharir, and Yap [4] have given an algorithm to determine the k-sets by generalizing the convex hull algorithm of Jarvis [7]. Their algorithm can be implemented to run in $O(n \log n + m \log^2 k)$ time, where m is the size of the output. In [6] we extended the incremental convex hull construction algorithm to construct k-set polygons. The algorithm performance was in $O(n \log n + k(n-k) \log^2 k)$ time and we showed that incremental algorithms constructing k-set polygons may have to construct $\Omega(k(n-k))$ edges.

In this paper we extend another classical convex hull construction method to the k-set polygon, namely the divide and conquer method. Our algorithm works similarly in that it starts by dividing the set of points V recursively into subsets of relatively equal size, then recursively constructs their k-set polygons, and finally merges the two polygons. We first characterize the edges to remove, which form two connected lines on the k-set polygons to merge. We also show that the edges to create can be obtained by considering k-set polygons of only $2k$ points. This leads to an algorithm that constructs the k-set polygon of n points in $O(n \log n + m \log^2 k \log(n/k))$ time, where m is the worst case size of the output.

2 Preliminaries

Throughout this paper we will consider the boundary of the k-set polygon $g^k(V)$ of V to be oriented in counter clockwise direction. We denote by st the closed

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oriented line segment with s as startpoint and t as endpoint, by (st) the oriented straight line generated by st, and by $(st)^+$ (resp. $(st)^-$) the open half plane on the left (resp. right) of (st). $\overline{(st)}^+$ and $\overline{(st)}^-$ denote the closures of $(st)^+$ and $(st)^-$.

Let us first recall two important properties of the vertices and edges of k-set polygons given by Andrzejak and Fukuda [1], and by Andrzejak and Welzl [2].

Proposition 1. *The centroid $g(T)$ of a subset T of k points of V is a vertex of $g^k(V)$ if and only if T is a k-set of V. Moreover, the centroids of distinct k-sets are distinct vertices.*

Proposition 2. *$g(T)g(T')$ is an oriented edge of $g^k(V)$ if and only if there exist two points s and t of V and a subset P of $k-1$ points of V such that $T = P \cup \{s\}$, $T' = P \cup \{t\}$, and $V \cap (st)^- = P$.*

Such an oriented edge will be denoted by $e_P(s,t)$; $g(P \cup \{s\})$ will be called its start vertex, and $g(P \cup \{t\})$ its end vertex. Note that $e_P(s,t)$ is parallel to (st) (see Fig. 1).

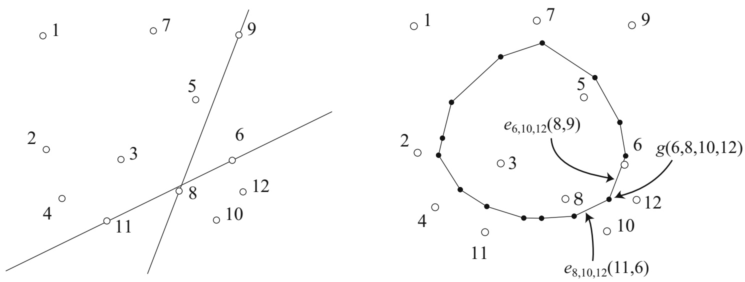

Fig. 1. Edges and vertices of a 4-set polygon of 12 points

From the above propositions we can easily obtain the following:

Proposition 3. *If $e_P(s,t)$ and $e_{P'}(s',t')$ are two consecutive edges of $g^k(V)$, then the line segments st and $s't'$ intersect.*

Propositions 1 and 2, lead to an efficient data structure to store k-set polygons. Indeed, the boundary of $g^k(V)$ can be stored in a circular list L whose elements represent the edges of $g^k(V)$. To any element e of L, which represents an edge $e_P(s,t)$, are associated the two elements of L that represent the predecessor and the successor of $e_P(s,t)$ on the boundary of $g^k(V)$, as well as the two points s and t of V. Note that, from Proposition 2, the k-sets defining two consecutive vertices of $g^k(V)$ differ from each other by one point and thus it suffices to know one k-set T of V and one edge with endpoint $g(T)$ to be able to generate the whole k-sets of V while traversing L. It follows that a k-set polygon with c edges can be stored in a data structure of size $O(c+k)$. In this paper we will store in the data structure the two k-sets whose centroids are the leftmost and the rightmost vertices of $g^k(V)$.

3 Edge Removal

For the sake of simplicity of the exposition, we will assume that no two points of V belong to a same vertical line and that no four distinct points of V belong to two parallel lines.

Suppose now that the points of V are sorted with respect to the x-axis (from left to right). Suppose also that we are given two subsets V_l and V_r of at least k points of V each, having at most $k-1$ common points, and such that there exists a vertical strip containing $V_l \cap V_r$. $V_l \setminus V_r$ and $V_r \setminus V_l$ are respectively on the left and on the right side of the strip. More precisely, there exist two vertical straight lines Δ_l and Δ_r oriented upwards such that $\Delta_l \subset \Delta_r^+$, $V_r \subset \Delta_l^-$, $V_l \subset \Delta_r^+$, and $V_l \cap V_r = (\Delta_l^- \cap \Delta_r^+) \cap (V_l \cup V_r)$. Suppose furthermore that $g^k(V_l)$ and $g^k(V_r)$ are given and that we want to construct $g^k(V_l \cup V_r)$. As in the construction of the classical convex hull, the method consists in finding the edges to remove on the given k-set polygons $g^k(V_l)$ and $g^k(V_r)$ and afterwards determining the new edges to create in order to get $g^k(V_l \cup V_r)$ (see Fig. 2).

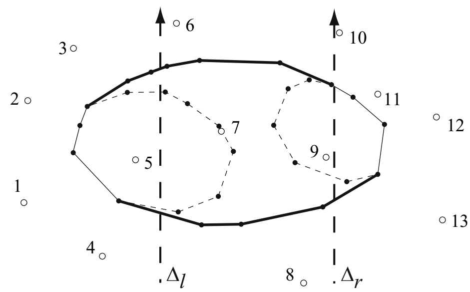

Fig. 2. Merging the 5-set polygons $g^5(1, ..., 9)$ and $g^5(6, ..., 13)$

In this section we focus on the edges to remove. We notably characterize the two connected lines that they form on $g^k(V_l)$ and $g^k(V_r)$.

Lemma 1. *(i) If $g^k(V_l)$ (resp. $g^k(V_r)$) is not reduced to a unique vertex, at least one of the edges incident in its rightmost (resp. leftmost) vertex is to be removed.*

(ii) The leftmost vertex of $g^k(V_l)$ and the rightmost vertex of $g^k(V_r)$ are vertices of $g^k(V_l \cup V_r)$.

From now on we will consider $g^k(V)^+$ (resp. $g^k(V)^-$) to be the oriented polygonal line of the edges of $g^k(V)$ that connects the rightmost to the leftmost (resp. leftmost to rightmost) vertex of $g^k(V)$ in counter clockwise direction.

Obviously, an edge st precedes an edge $s't'$ in counter clockwise direction on $g^k(V)^+$ if and only if the angle $\theta(st)$ of the oriented line (st) with the x-axis (oriented from left to right) is smaller than the angle $\theta(s't')$ of $(s't')$ with the x-axis. In the remainder of this paper the three following notations will be equivalent: $\theta(st) < \theta(s't')$, $(st) <_\theta (s't')$, and $st <_\theta s't'$.

Lemma 2. *Let $e_P(s,t)$ and $e_{P'}(s',t')$ be two edges of the line $g^k(V_l)^+$ such that $e_P(s,t) <_\theta e_{P'}(s',t')$ and let r be a point of $V_r \setminus V_l$. If $r \in (s't')^-$ then $r \in (st)^-$.*

Proof. If $e_P(s,t)$ and $e_{P'}(s',t')$ are two consecutive edges of $g^k(V_l)^+$, then from Proposition 3, the line segments st and $s't'$ intersect and, since $V_l \subset \Delta_r^+$, their intersection point belongs to Δ_r^+. Moreover, since $st <_\theta s't'$, $(s't')^- \cap \Delta_r^- \subset (st)^-$. It follows that, if a site r of $V_r \setminus V_l$ belongs to $(s't')^-$, it also belongs to $(st)^-$. By an elementary induction, the result holds for any edges $e_P(s,t)$ and $e_{P'}(s',t')$ of $g^k(V_l)^+$ such that $e_P(s,t) <_\theta e_{P'}(s',t')$. □

Similar results hold for $g^k(V_l)^-$, $g^k(V_r)^+$, and $g^k(V_r)^-$. Thus, using Proposition 2:

Theorem 1. *The edges to remove on $g^k(V_l)$ (resp. $g^k(V_r)$) form a connected line which contains the rightmost vertex of $g^k(V_l)$ (resp. leftmost vertex of $g^k(V_r)$).*

Denote respectively by $\mathcal{D}^+_{left}$ and $\mathcal{D}^+_{right}$ the lines to remove on $g^k(V_l)^+$ and $g^k(V_r)^+$, oriented in counter clockwise direction (see Fig. 3). That is, the rightmost vertex of $g^k(V_l)$ and the leftmost vertex of $g^k(V_r)$ are respectively the start vertex of $\mathcal{D}^+_{left}$ and the end vertex of $\mathcal{D}^+_{right}$.

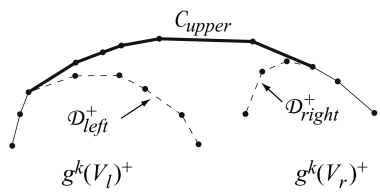

Fig. 3. The upper lines handled in Algorithms 1 and 2

We show now that the edges of $\mathcal{D}^+_{left}$ can be found efficiently by only traversing the edges to remove on the k-set polygons.

Given an oriented straight line Δ, we say that a set T is Δ-separable from V if T is a subset of V such that $\Delta^- \cap V = T$. Moreover, T is said to be $//_\Delta$-separable from V if there exists a straight line Δ', parallel to Δ and oriented as Δ, such that T is Δ'-separable from V.

Lemma 3. *An edge $e_P(s,t)$ of $g^k(V_l)^+$ is also an edge of $g^k(V_l \cup V_r)$ if and only if the k-element set T_r which is $//_{(st)}$-separable from V_r is such that $T_r \setminus V_l \subset (st)^+$.*

Proof. If $(T_r \setminus V_l) \cap (st)^- \neq \emptyset$ then, from Proposition 2, $e_P(s,t)$ is not an edge of $g^k(V_l \cup V_r)$. Conversely, if $T_r \setminus V_l \subset (st)^+$, let Δ be a straight line parallel to (st), oriented as (st), and such that T_r is Δ-separable from V_r. Since $|T_r \setminus V_l| \geq 1$, at least one point of T_r belongs to $(st)^+$. It follows that $\Delta \subset (st)^+$, that is, $V_r \setminus T_r \subset (st)^+$. Hence, from Proposition 2, $e_P(s,t)$ is an edge of $g^k(V_l \cup V_r)$. □

Thus, finding the edges of $\mathcal{D}^+_{left}$ comes down to finding the first edge $e_P(s,t)$ of $g^k(V_l)^+$ that verifies Lemma 3. Now, given a straight line Δ, the k-element set $//_\Delta$-separable from V_r can be found thanks to the following lemma:

Lemma 4. *If the k-set polygon $g^k(V_r)$ is not reduced to a unique vertex, let $g(T_0)$, ..., $g(T_m)$ and $e_{P_1}(s_1,t_1)$, ..., $e_{P_m}(s_m,t_m)$ be the vertices and the edges of $g^k(V_r)^+$, given in counter clockwise direction. Let Δ be an oriented straight line with $\theta(\Delta) \in [\pi/2, 3\pi/2]$. T_i is $//_\Delta$-separable from V_r if and only if,*
- either $i = 0$ and $\Delta <_\theta (s_1t_1)$,
- or $i \in \{1,\dots,m-1\}$ and $(s_it_i) <_\theta \Delta <_\theta (s_{i+1}t_{i+1})$,
- or $i = m$ and $(s_mt_m) <_\theta \Delta$.

Proof. Since $g(T_0)$ is the rightmost vertex of $g^k(V_r)$, T_0 can be separated from V_r with a straight line Δ_0 such that $\theta(\Delta_0) = \pi/2$. The same, T_m can be separated from V_r with a straight line Δ_{m+1} such that $\theta(\Delta_{m+1}) = 3\pi/2$. For every $i \in \{1,\dots,m\}$, let $\Delta_i = (s_it_i)$. Then, from Proposition 2, for every $i \in \{0,\dots,m\}$, $T_i \subset \overline{\Delta_i^-} \cap \overline{\Delta_{i+1}^-}$ and $V_r \setminus T_i \subset \overline{\Delta_i^+} \cap \overline{\Delta_{i+1}^+}$. For every straight line Δ such that $\Delta_i <_\theta \Delta <_\theta \Delta_{i+1}$, the line Δ', parallel to Δ, oriented as Δ, and passing through $x = \Delta_i \cap \Delta_{i+1}$ is such that $\overline{\Delta_i^-} \cap \overline{\Delta_{i+1}^-} \subset \overline{\Delta'^-}$ and $\overline{\Delta_i^+} \cap \overline{\Delta_{i+1}^+} \subset \overline{\Delta'^+}$. It follows that T_i is $//_\Delta$-separable from V_r (if $x \in V_r$, it suffices to move slightly Δ' parallely to itself such that it strictly separates T_i from V_r).

The converse follows directly from the fact that, for a given straight line Δ, there exists at most one k-set T_i $//_\Delta$-separable from V_r. $\square$

To avoid dealing with the special cases $i = 0$ and $i = m$ in the algorithm, we add two anchor-edges to the upper line $g^k(U)^+$ of the k-set polygon of any subset U of V in the following way: If $g(T)$ is the rightmost vertex of $g^k(U)^+$, insert an anchor-edge $e_P(s,t)$ with end vertex $g(T)$, such that t is the leftmost point of T and s is any point in the plane having the same x-coordinate as t and a smaller y-coordinate (note that, from the assumptions on V, $s \notin V$). Clearly, $\theta(st) = \pi/2$. In the same way, if $g(T)$ is the leftmost vertex of $g^k(U)^+$, insert an anchor-edge $e_P(s,t)$ with start vertex $g(T)$, such that s is the rightmost point of T and t is any point in the plane having the same x-coordinate as s and a smaller y-coordinate (i.e. $\theta(st) = 3\pi/2$). Note that if $g^k(U)^+$ is reduced to a unique vertex, this vertex will be incident to both anchor-edges.

The edges of $\mathcal{D}^+_{left}$ can then be found by the following algorithm:

Algorithm 1 – Find $\mathcal{D}^+_{left}$

let $e_P(s,t)$ be the edge of $g^k(V_l)^+$ with start vertex the rightmost vertex of $g^k(V_l)^+$
let $g(T)$ be the leftmost vertex of $g^k(V_r)$
let $e_{P'}(s',t')$ be the edge of $g^k(V_r)^+$ with end vertex $g(T)$ (i.e. $P' \cup \{t'\} = T$)
(1) *while $T \setminus V_l \not\subset (st)^+$*
    $e_P(s,t) \longleftarrow$ *successor of $e_P(s,t)$ on $g^k(V_l)$*
(2) *while $st <_\theta s't'$*
    $e_{P'}(s',t') \longleftarrow$ *predecessor of $e_{P'}(s',t')$ on $g^k(V_r)$*
    (3) *while $(P' \cup \{t'\}) \setminus V_l \not\subset (st)^+$*
        $e_P(s,t) \longleftarrow$ *successor of $e_P(s,t)$ on $g^k(V_l)$*
return $e_P(s,t)$

For any polygonal line $\mathcal{L}$, let $|\mathcal{L}|$ be the number of edges of $\mathcal{L}$.

Proposition 4. *(i) At the end of the algorithm, $g(P \cup \{s\})$ is the end vertex of $\mathcal{D}^+_{left}$.*

(ii) $\mathcal{D}^+_{left}$ can be found in $O(k + |\mathcal{D}^+_{left}| \log k + |\mathcal{D}^+_{right}|)$ time.

Proof. (i.1) Every edge $e_P(s,t)$ of $g^k(V_l)$ traversed by the algorithm, except possibly the last one, is to be removed since at least one of the points of $V_r \setminus V_l$ belongs to $(st)^-$, as checked in loops (1) and (3) conditions. Note that these loops necessarily stops at the latest on an anchor-edge.

(i.2) If $e_P(s,t)$ and $e_{P'}(s',t')$ are the last edges encountered by the algorithm respectively on $g^k(V_l)^+$ and on $g^k(V_r)^+$ then, from loop (2) condition, there exist two consecutive edges $e_{P'_1}(s'_1,t'_1)$ and $e_{P'_2}(s'_2,t'_2)$ of $g^k(V_r)^+$ such that $s't' \leq_\theta s'_1t'_1 <_\theta st <_\theta s'_2t'_2$. From Lemma 4, $P'_1 \cup \{t'_1\}$ is then $//_{(st)}$-separable from V_r. Moreover, from loops (1) and (3) conditions, there exists an edge $e_{P_1}(s_1,t_1)$ of $g^k(V_l)^+$ such that $s_1t_1 \leq_\theta st$ and $(P'_1 \cup \{t'_1\}) \setminus V_l \subset (s_1t_1)^+$. From Lemma 2, $(P'_1 \cup \{t'_1\}) \setminus V_l \subset (st)^+$ and, from Lemma 3, $e_P(s,t)$ is an edge of $g^k(V_l \cup V_r)$. Since $\mathcal{D}^+_{left}$ is connected, it follows from (i.1) that $g(P \cup \{s\})$ is the end vertex of $\mathcal{D}^+_{left}$.

(ii.1) From (i.1), within a margin of one, only the edges to remove are traversed on $g^k(V_l)^+$. Moreover, for every edge $e_{P'}(s',t')$ of $g^k(V_r)$ traversed by the algorithm, except for the last one, there exists an edge $e_P(s,t)$ of $g^k(V_l)^+$ such that $st <_\theta s't'$ (loop (2) condition) and $(P' \cup \{t'\}) \setminus V_l \subset (st)^+$ (loops (1) and (3) conditions). Since, $(P' \cup \{t'\}) \setminus V_l$ contains at least one point and since this point belongs to $\Delta^-_r \cap (st)^+$, $(st) \cap (s't') \in \Delta^-_r$. Thus $\overline{(st)^-} \cap \Delta^+_l \subset (s't')^-$ and, since $\overline{(st)^-} \cap \Delta^+_l$ contains at least one point of $(P \cup \{s,t\}) \setminus V_r$, $e_{P'}(s',t')$ is not an edge of $g^k(V_l \cup V_r)$. It follows that, within a margin of one, only the edges to remove are traversed on $g^k(V_r)^+$.

(ii.2) In loop (1), $g(T)$ is the leftmost vertex of $g^k(V_r)$ and, by hypothesis, T is stored in the data structure containing $g^k(V_r)$. To check whether $T \setminus V_l \subset (st)^+$, it suffices to compute the straight line passing through s, tangent to the convex hull $conv(T \setminus V_l)$ at a point r, and such that $conv(T \setminus V_l) \subset \overline{(rs)^+}$. $(T \setminus V_l) \subset (st)^+$ is then equivalent to $r \in (st)^+$. Since the points of V are sorted from left to right, $T \setminus V_l$ can be obtained in $O(k)$ time and $conv(T \setminus V_l)$ can also be computed in $O(k)$ time. Any tangent to $conv(T \setminus V_l)$ can then be found in $O(\log k)$ time (see for example [9]). The time complexity of loop (1) can thus be bounded by $O(k + |\mathcal{D}^+_{left}| \log k)$.

(ii.3) From (ii.1), the test $(P' \cup \{t'\}) \setminus V_l \not\subset (st)^+$ in loop (3) condition is done at most $|\mathcal{D}^+_{left}| + |\mathcal{D}^+_{right}| + 2$ times. From Lemma 2, given a set $P' \cup \{t'\}$, if an edge $e_P(s,t)$ of $g^k(V_l)^+$ is such that $(P' \cup \{t'\}) \setminus V_l \subset (st)^+$, then all the successors of $e_P(s,t)$ verify the same inclusion. Furthermore, if $e_{P''}(s'',t'')$ is the predecessor of an edge $e_{P'}(s',t')$ of $g^k(V_r)$, then $P'' \cup \{t''\} = (P' \cup \{t'\}) \setminus \{t'\} \cup \{s'\}$, from Proposition 2. It follows that, for two consecutive passes in loop (2), the considered sets $(P' \cup \{t'\}) \setminus V_l$ differ from each other by at most one point and the test $(P' \cup \{t'\}) \setminus V_l \not\subset (st)^+$ can be achieved in constant time. It is the same with all the other instructions in loop (2), which are all together in $O(|\mathcal{D}^+_{left}| + |\mathcal{D}^+_{right}|)$; hence the result. $\square$

Obviously, $\mathcal{D}^+_{right}$ can be found in a symmetric way and it is the same with the lines to remove on $g^k(V_l)^-$ and $g^k(V_r)^-$. Hence the theorem:

Theorem 2. *The edges to remove on $g^k(V_l)$ and $g^k(V_r)$ can be found in $O(k + d \log k)$ time, where d is the total number of edges to remove.*

4 Edge Construction

In the preceding section we have seen that the edges to remove form two connected lines on the left and on the right k-set polygons. Since the k-set polygon to construct is convex, the edges to create form also two connected lines, an upper and a lower one (see Fig. 2 and Fig. 3). We denote by $\mathcal{C}_{upper}$ the oriented upper line to create and by $\mathcal{C}_{lower}$ the lower line. $\mathcal{C}_{upper}$ connects the start vertex of $\mathcal{D}^+_{right}$ to the end vertex of $\mathcal{D}^+_{left}$ (obtained by Algorithm 1).

We show now that the edges of $\mathcal{C}_{upper}$ can be found by considering k-set polygons of at most $2k$ points of $V_l \cup V_r$.

Lemma 5. *$e_P(s,t)$ is an edge of $\mathcal{C}_{upper}$ if and only if the k-element sets T_l and T_r, which are $//_{(st)}$-separable from V_l and V_r respectively, are such that:*
– $e_P(s,t)$ is an edge of $g^k(T_l \cup T_r)^+$,
– $g(T_l)$ is a vertex of $\mathcal{D}^+_{left}$ and $g(T_r)$ is a vertex of $\mathcal{D}^+_{right}$.

Proof. (i) If $e_P(s,t)$ is an edge of $\mathcal{C}_{upper}$, $P \cup \{s,t\}$ contains at least one point u of $V_l \setminus V_r$ (otherwise it would be an edge of $g^k(V_r)$). If there were a point v of $V_r \setminus T_r$ in $\overline{(st)^-}$ then, since there exists an oriented straight line Δ parallel to (st) such that T_r is Δ-separable from V_r, we would have $\Delta \subset \overline{(st)^-}$ and thus $T_r \subset \overline{(st)^-}$. Hence $T_r \cup \{u,v\} \subset \overline{(st)^-}$. This is impossible, from Proposition 2, and thus $V_r \setminus T_r \subset (st)^+$. In the same way, $V_l \setminus T_l \subset (st)^+$. It follows that $P \cup \{s,t\} \subseteq T_l \cup T_r$ and that $e_P(s,t)$ is an edge of $g^k(T_l \cup T_r)$. More precisely, since $e_P(s,t)$ is an edge of $g^k(V_l \cup V_r)^+$, it is also an edge of $g^k(T_l \cup T_r)^+$.

Moreover, if $g(T_r)$ is the leftmost vertex of $g^k(V_r)$, from Lemma 1, it belongs to $\mathcal{D}^+_{right}$. Otherwise, from Lemma 4, the edge $e_{P'}(s',t')$ of $g^k(V_r)$ with start vertex $g(T_r)$ is such that $st <_\theta s't'$. Then $e_{P'}(s',t')$ cannot be an edge of $g^k(V_l \cup V_r)$ since it should precede $e_P(s,t)$ on $g^k(V_l \cup V_r)^+$. It follows that $g(T_r)$ is a vertex of $\mathcal{D}^+_{right}$. In the same way, $g(T_l)$ is a vertex of $\mathcal{D}^+_{left}$.

(ii) Conversely, let T_l and T_r be two k-element sets $//_\Delta$-separable from V_l and V_r respectively (with a same straight line Δ) and such that $g^k(T_l \cup T_r)^+$ admits an edge $e_P(s,t)$ parallel to Δ. Since $|T_l| = k$ and $|P| = k-1$, at least one point of T_l belongs to $\overline{(st)^+}$. It follows that if Δ' is a straight line parallel to Δ such that T_l is Δ'-separable from V_l, we have $(st) \subset \Delta'^-$. Hence $V_l \setminus T_l \subset \Delta'^+ \subset (st)^+$. In the same way, $V_r \setminus T_r \subset (st)^+$. From Proposition 2, $e_P(s,t)$ is then an edge of $g^k(V_l \cup V_r)$. Moreover, since $e_P(s,t)$ belongs to $g^k(T_l \cup T_r)^+$, it also belongs to $g^k(V_l \cup V_r)^+$. Furthermore, since $g(T_r)$ belongs to $\mathcal{D}^+_{right}$, every edge $e_{P'}(s',t')$ of $g^k(V_r)^+$ that belongs to $g^k(V_l \cup V_r)$ is such that $s't' <_\theta st$. The same, since $g(T_l)$ belongs to $\mathcal{D}^+_{left}$, every edge $e_{P''}(s'',t'')$ of $g^k(V_l)^+$ that belongs to $g^k(V_l \cup V_r)$ is such that $st <_\theta s''t''$. It follows that $e_P(s,t)$ is an edge of $\mathcal{C}_{upper}$. □

It follows from this proposition that, to construct $\mathcal{C}_{upper}$, we have to consider all the couples of vertices $(g(T_l), g(T_r))$, where $g(T_l)$ and $g(T_r)$ belong to $\mathcal{D}^+_{left}$ and $\mathcal{D}^+_{right}$ respectively and such that T_l and T_r are $//_\Delta$-separable from V_l and V_r with a same straight line Δ. Then it suffices, for each of these couples, to compute the k-set polygon of $T_l \cup T_r$ and to extract some of its edges. We give now an algorithm that generates these couples efficiently, by using the result of Lemma 4.

If $e_{P_l}(s_l, t_l)$ and $e_{P'_l}(s'_l, t'_l)$ are the edges of $g^k(V_l)^+$ respectively starting and ending in $g(T_l)$ and if $e_{P_r}(s_r, t_r)$ and $e_{P'_r}(s'_r, t'_r)$ are the edges of $g^k(V_r)^+$ respectively starting and ending in $g(T_r)$, we denote by $\theta(g(T_l), g(T_r))$ the interval $[max(\theta(s'_l t'_l), \theta(s'_r t'_r)), min(\theta(s_l t_l), \theta(s_r t_r))]$.

We suppose that the upper line of any k-set polygon is completed with the same two anchor-edges as in Algorithm 1.

Algorithm 2 – Construct $\mathcal{C}_{upper}$

let $e_{P_{min}}(s_{min}, t_{min})$ *be the edge of* $g^k(V_r)^+$ *ending at the start vertex of* $\mathcal{D}^+_{right}$
let $e_{P_{max}}(s_{max}, t_{max})$ *be the edge of* $g^k(V_l)^+$ *starting at the end vertex of* $\mathcal{D}^+_{left}$
let $e_{P_l}(s_l, t_l)$ *be the edge of* $g^k(V_l)^+$ *starting at the rightmost vertex of* $\mathcal{D}^+_{left}$
(1) *while* $s_l t_l <_\theta s_{min} t_{min}$
        $e_{P_l}(s_l, t_l) \longleftarrow$ *successor of* $e_{P_l}(s_l, t_l)$ *on* $g^k(V_l)^+$
$T_l \longleftarrow P_l \cup \{s_l\}$
$e_{P_r}(s_r, t_r) \longleftarrow$ *successor of* $e_{P_{min}}(s_{min}, t_{min})$ *on* $g^k(V_r)^+$
$T_r \longleftarrow P_r \cup \{s_r\}$
$T \longleftarrow T_r$
(2) *do*
        let Θ *be the interval* $\theta(g(T_l), g(T_r))$
        (3) *let* $e_P(s, t)$ *be the edge of* $g^k(T_l \cup T_r)$ *starting at* $g(T)$
        (4) *while* $\theta(st) \in \Theta$
                insert $e_P(s, t)$ *in* $g^k(V_l \cup V_r)^+$ *such that it starts at* $g(T)$
                (5) $T \longleftarrow P \cup \{t\}$
                let $e_P(s, t)$ *be the edge of* $g^k(T_l \cup T_r)$ *starting at* $g(T)$
        (6) *if* $s_l t_l <_\theta s_r t_r$
                $e_{P_l}(s_l, t_l) \longleftarrow$ *successor of* $e_{P_l}(s_l, t_l)$ *on* $g^k(V_l)$
                $T_l \longleftarrow P_l \cup \{s_l\}$
        else
                $e_{P_r}(s_r, t_r) \longleftarrow$ *successor of* $e_{P_r}(s_r, t_r)$ *on* $g^k(V_r)$
                $T_r \longleftarrow P_r \cup \{s_r\}$
while $\theta(s_{max} t_{max}) \notin \Theta$

Proposition 5. *The algorithm constructs* $\mathcal{C}_{upper}$ *and can be implemented to run in* $O((k + |\mathcal{D}^+_{right}| + |\mathcal{D}^+_{left}| + |\mathcal{C}_{upper}|) \log^2 k)$ *time.*

Proof. (i) We first show that the algorithm constructs $\mathcal{C}_{upper}$. Since $g^k(V_l \cup V_r)^+$ is convex, the angles of the edges of $\mathcal{C}_{upper}$ with the x-axis belong to the interval $[\theta(s_{min}, t_{min}), \theta(s_{max}, t_{max})]$. Now, the intervals $\theta(g(T_l), g(T_r))$ considered in loop (2) partition $[\theta(s_{min}, t_{min}), \theta(s_{max}, t_{max})]$. It follows, from Lemmas 4 and 5, that the edges of $\mathcal{C}_{upper}$ are the edges $e_P(s,t)$ of $g^k(T_l \cup T_r)^+$ such that $\theta(s,t) \in \theta(g(T_l), g(T_r))$, for all couples (T_l, T_r) treated in loop (2). Moreover, since the intervals $\theta(g(T_l), g(T_r))$ are treated in increasing angular order, the edges extracted from two consecutive k-set polygons $g^k(T_l \cup T_r)$ will appear consecutively on $\mathcal{C}_{upper}$.

In order to show that the algorithm works, it remains to prove that instruction (3) is valid, that is, that $g(T)$ is really a vertex of the considered k-set polygon $g^k(T_l \cup T_r)$ (even if none of its edges belongs to $g^k(V_l \cup V_r)^+$). By construction, $g(T)$ is the end vertex of the already constructed part of $\mathcal{C}_{upper}$. Denote now by $e_{P_1}(s_1, t_1)$ and $e_{P_2}(s_2, t_2)$ the edges of $g^k(V_l \cup V_r)^+$ respectively ending and starting in $g(T)$. On the one hand, T_l and T_r are such that, when instruction (3) is executed, $\theta(s_1 t_1)$ is smaller than the lower bound θ_{min} of $\theta(g(T_l), g(T_r))$, since $e_{P_1}(s_1, t_1)$ has been constructed before the couple (T_l, T_r) was considered. On the other hand, $\theta(s_2 t_2)$ is greater than θ_{min} since $e_{P_2}(s_2, t_2)$ has still to be constructed. Let Δ be a straight line such that $\theta(\Delta) \in \theta(g(T_l), g(T_r))$ and $\theta(\Delta) < \theta(s_2 t_2)$ (Δ can always be chosen in such a way that it is not parallel to any straight line passing through two points of V). Since $\theta(s_1 t_1) < \theta(\Delta) < \theta(s_2 t_2)$, T is $//_\Delta$-separable from $V_l \cup V_r$, by Lemma 4. Let now $g(T')$ be the vertex of $g^k(T_l \cup T_r)$ such that T' is $//_\Delta$-separable from $T_l \cup T_r$. If Δ' is the oriented straight line parallel to Δ such that T' is Δ'-separable from $T_l \cup T_r$, then the same reasoning as in proof of Lemma 5 shows that no point of $V_l \setminus T_l$ and of $V_r \setminus T_r$ belongs to Δ'^-. It follows that $g(T')$ is also a vertex of $g^k(V_l \cup V_r)$. More precisely, $g(T')$ is the vertex of $g^k(V_l \cup V_r)^+$ such that T' is $//_\Delta$-separable from $V_l \cup V_r$, that is, $g(T') = g(T)$. Hence, $g(T)$ is really a vertex of $g^k(T_l \cup T_r)$.

(ii) The essential step of the algorithm, given a vertex $g(T)$ of $g^k(T_l \cup T_r)$, is to determine the edge $e_P(s,t)$ of $g^k(T_l \cup T_r)$ starting at $g(T)$. If the convex hulls of T and $(T_l \cup T_r) \setminus T$ are given, it suffices to find the common oriented tangent Δ of these convex hulls such that $T \subset \overline{\Delta^-}$, $(T_l \cup T_r) \setminus T \subset \overline{\Delta^+}$, and $s = T \cap \Delta$ precedes $t = ((T_l \cup T_r) \setminus T) \cap \Delta$ on Δ. Indeed, from Proposition 2, $e_{T\setminus\{s\}}(s,t)$ is then the edge of $g^k(T_l \cup T_r)$ starting at $g(T)$. We have thus to maintain the convex hulls of T and of $(T_l \cup T_r) \setminus T$ all along the algorithm.

At the beginning of the algorithm, $g(T) = g(T_r)$ is the start vertex of $\mathcal{D}^+_{right}$. The convex hull of T can then be obtained in the following way: If $g(T')$ is the leftmost vertex of $g^k(V_r)$, the convex hull of T' can be directly computed since the points of T' are stored in the data structure containing $g^k(V_r)$. Let CH be the data structure containing this convex hull. $\mathcal{D}^+_{right}$ can then be traversed from $g(T')$ to $g(T)$ and, for each traversed edge $e_P(s,t)$, t is removed from CH and s is inserted in CH. When arriving in $g(T)$, CH contains the convex hull of T. Using the fully dynamic convex hull data structure of Overmars and van Leeuwen [8], the convex hull of T' can be stored in CH in $O(k \log^2 k)$ time and every insertion or deletion in CH can be done in $O(\log^2 k)$ time (the same operations can be achived in $O(\log k)$ amortized time with the data structure of [3]). The convex hull of the set T at the beginning of the algorithm can thus be obtained in $O((k + |\mathcal{D}^+_{right}|) \log^2 k)$ time. Since, at the beginning of the algorithm, $T = T_r$, the convex hull of $(T_l \cup T_r) \setminus T = T_l \setminus T$ can be computed in the same way (in a dynamic structure CH') while traversing $\mathcal{D}^+_{left}$ in loop (1). In order to place in CH' only the points of T_l that are not in T, it suffices to mark the points of T (for example, while constructing their convex hull). During the execution of the algorithm, the set T is only modified by instruction (5). To update CH, we have just to remove s and to insert t. Since instruction (5) happens exactly once per edge created on $\mathcal{C}_{upper}$, the overall complexity of all the updates of CH is $O(|\mathcal{C}_{upper}| \log^2 k)$. The set $(T_l \cup T_r) \setminus T$ is modified by instructions (5) and (6). In the same way, for each of these instructions, at most one point is removed from CH' and at most one point is inserted (a point that already belongs to CH is neither removed nor inserted in CH'). Since the total number of passes in loop (2) is at most $|\mathcal{D}^+_{right}| + |\mathcal{D}^+_{left}|$ and since the total number of passes in loop (4) is equal to the number of edges of $\mathcal{C}_{upper}$, it follows that the overall complexity of the updates of CH' is $O((|\mathcal{D}^+_{right}| + |\mathcal{D}^+_{left}| + |\mathcal{C}_{upper}|) \log^2 k)$. Since a common tangent of T and $(T_l \cup T_r) \setminus T$ can also be found in $O(\log^2 k)$ time using CH and CH', $\mathcal{C}_{upper}$ can be constructed in $O((k + |\mathcal{D}^+_{right}| + |\mathcal{D}^+_{left}| + |\mathcal{C}_{upper}|) \log^2 k)$ time. □

Obviously, the lower polygonal line can be constructed similarly and we get the following result:

Theorem 3. *The edges to construct while merging $g^k(V_l)$ and $g^k(V_r)$ can be found in $O((k + d + c) \log^2 k)$ time, where d and c are the numbers of edges to delete and to create.*

The divide and conquer construction of the k-set polygon of a set V of at least k points is then as follows:

Algorithm 3 – Construct $g^k(V)$

if $|V| \leq k + 1$
  construct directly $g^k(V)$
else
  if $|V| < 2(k + 1)$
    divide V into two non-disjoint subsets V_l and V_r of k or $k+1$ points each such that $V_l \cap V_r$ belong to a vertical strip separating $V_l \setminus V_r$ and $V_r \setminus V_l$
  else
    divide V into two disjoint subsets V_l and V_r of $\lceil |V|/2 \rceil$ and $\lfloor |V|/2 \rfloor$ points separable by a vertical straight line
  construct recursively $g^k(V_l)$ and $g^k(V_r)$
  merge $g^k(V_l)$ and $g^k(V_r)$ with Algorithms 1 and 2

Theorem 4. *Algorithm 3 constructs the k-set polygon of n points in $O(n \log n + m \log^2 k \log(n/k))$ time, where m is the worst case size of the output.*

Proof. If $n \leq k + 1$, the algorithm directly constructs the k-set polygon of V. If $n = k$, this k-set polygon is reduced to the centroid $g(V)$. If $n = k + 1$, from

Proposition 1, a vertex of the k-set polygon of V is the centroid of a subset of k points of V separable from the last one by a straight line. This last point is then a vertex of the convex hull of V and it follows that constructing the k-set polygon of V comes to constructing its convex hull. This can be done in $O(k)$ time since V is sorted.

If $n > k+1$, V is divided into two subsets V_l and V_r such that $|V_l| = \lceil n/2 \rceil$ and $|V_r| = \lfloor n/2 \rfloor$ if $n \geq 2(k+1)$, and $|V_l| \leq k+1$ and $|V_r| \leq k+1$ otherwise. The k-set polygons of V_l and V_r are then recursively constructed and, finally, merged with Algorithms 1 and 2 in $O((k+d+c)\log^2 k)$ time, where d and c are the total numbers of edges deleted and constructed in the merging step.

Now, Dey [5] and Tóth [10] have shown that the size of a k-set polygon of n points is in $O(n\beta(k))$, with $2^{\Omega(\sqrt{\log k})} \leq \beta(k) \leq O(k^{1/3})$. It follows that d and c are bounded by $O(n\beta(k))$ and that the complexity of the merging is $O(n\beta(k)\log^2 k)$. Hence the induction relation that gives the complexity $T(n)$ of the algorithm (without the sorting step):

$$\begin{array}{ll} T(n) \leq T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + O(n\beta(k)\log^2 k) & \text{if } n \geq 2(k+1) \\ T(n) \leq 2T(k+1) + O(n\beta(k)\log^2 k) & \text{if } k+1 < n < 2(k+1) \\ T(n) = O(k) & \text{if } n \leq k+1 \end{array}$$

Solving this relation, we get $T(n) = O(n\beta(k)\log^2 k \log(n/k))$. Thus, the overall complexity of Algorithm 3, including sorting, is $O(n\log n + m\log^2 k \log(n/k))$, where $m = O(n\beta(k))$ is the worst case size of a k-set polygon of n points. □

5 Conclusion

In this paper we have applied a classical algorithmic method to the construction of k-set polygons in the plane: The divide and conquer method. To this aim we have characterized the edges that are removed and the ones that are created when two k-set polygons are merged.

In the worst-case time complexity of the final divide and conquer algorithm appears an additional $\log(n/k)$ factor, in comparison with the algorithm of [4]. We suspect that this factor comes from over-estimates in the complexity computation. Indeed, in this computation, we suppose that the number of edges removed and created at each merging step is linear with the worst case sizes of the merged k-set polygons. Applied to the recursive convex hull construction (i.e. for $k = 1$), this way of computing leads to a total number of $O(n\log n)$ created edges whereas it is well known that this algorithm only constructs $O(n)$ edges in all.

The aim now is to find a finer analysis method of our algorithm to remove the $\log(n/k)$ factor. A second aim is to extend other classical convex hull algorithms to the construction of k-set polygons and namely the Quick-Hull algorithm in order to check if its good practical performances hold for k-set polygon construction.

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Coloring Axis-Parallel Rectangles

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Abstract. For every k and r, we construct a finite family of axis-parallel rectangles in the plane such that no matter how we color them with k colors, there exists a point covered by precisely r members of the family, all of which have the same color. For $r = 2$, this answers a question of S. Smorodinsky [S06].

1 Introduction

Given a set of points P and a family of regions $\mathcal{R}$ in the plane, in a natural way one can associate two hypergraphs with them, dual to each other. Let $H(P, \mathcal{R})$ denote the hypergraph on the vertex set P, whose hyperedges are all subsets of P that can be obtained by intersecting P with a member of $\mathcal{R}$. The hypergraph $H^*(P, \mathcal{R})$ is defined by swapping the roles of $\mathcal{R}$ and P: its vertex set is $\mathcal{R}$, and for each $p \in P$ it has a hyperedge consisting of all regions in $\mathcal{R}$ that contain p.

Let H be a hypergraph with vertex set $V(H)$. The *chromatic number* $\chi(H)$ of H is the smallest number of colors in a coloring of $V(H)$ such that no hyperedge with at least two vertices is monochromatic. Let H_r (and $H_{\geq r}$) denote the hypergraph on the vertex set $V(H)$, consisting of all r-element (at least r-element) hyperedges of H. By definition, we have $\chi(H) = \chi(H_{\geq 2})$.

In the special case when $\mathcal{R}$ is the family of all *axis-parallel rectangles* and P is a set of n points in the plane, the problem of bounding $\chi(H(P, \mathcal{R}))$ reduces to estimating the chromatic number of the graph $G(P, \mathcal{R}) := H_2(P, \mathcal{R})$ consisting of all *two-element* (hyper)edges of $H(P, \mathcal{R})$. Kříž and Nešetřil [KN91] gave an explicit construction showing that the chromatic number of $G(P, \mathcal{R})$ cannot be bounded by an absolute constant. Chen, Pach, Szegedy, and Tardos [CPST07] proved by a probabilistic argument that there exist n-element point sets P such that the chromatic numbers of $G(P, \mathcal{R})$ grow as fast as at least $\Omega\left(\frac{\log n}{\log^2 \log n}\right)$. On the other hand, Ajwani, Elbassioni, Govindarajan, and Ray [AEGR07] proved that $\chi(H(P, \mathcal{R})) = \chi(G(P, \mathcal{R})) \leq O\left(n^{.383}\right)$.

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It was also shown in [CPST07] that, for a fixed $r \geq 2$, a randomly and uniformly selected set P of n points in the unit square almost surely satisfies

$$\chi(H_r(P, \mathcal{R}) = \Omega\left(\frac{\log^{1/(r-1)} n}{\log^2 \log n}\right),$$

as n tends to infinity.

Concerning the dual question, S. Smorodinsky [S06] proved that if $\mathcal{R}$ is a family of n open axis-parallel rectangles and P is a set of points in the plane, then $\chi(H^*(P, \mathcal{R})) = O(\log n)$. In other words, the rectangles in $\mathcal{R}$ can be colored by at most constant times $\log n$ colors so that, for any point $p \in P$ covered by more than one rectangle, at least two rectangles containing p have different colors. Smorodinsky asked whether there always exists such a coloring with a bounded number of colors. In the present note, we answer this question in the negative, in the following stronger form

Theorem 1. *Let $n \geq r \geq 2$ be integers. There exists a family $\mathcal{R}$ of n axis-parallel rectangles in the plane such that for any coloring of the rectangles with $k \leq C\frac{\log n}{r \log r}$ colors, one can find a point covered by exactly r members of $\mathcal{R}$, all of the same color. Here $C > 0$ is an absolute constant.*

Using our notation, we have that $\chi(H_r^*(P, \mathcal{R})) \geq C\frac{\log n}{r \log r}$, where $P = \mathbb{R}^2$ (or a suitable finite subset of $\mathbb{R}^2$). For $r = 2$, the above mentioned result of Smorodinsky [S06] shows that Theorem 1 is not far from being optimal.

A family of axis-parallel rectangles is said to form a *r-fold covering* of the plane if every point $p \in \mathbb{R}^2$ is contained in at least r members of the family. It is called *locally finite* if no point of the plane belongs to infinitely many rectangles. We say that a covering has a *k-split* if the family can be partitioned into k parts such that the union of any $k-1$ parts form a (1-fold) covering. Theorem 1 yields

Corollary 1. *For every $r, k \geq 2$, there is a locally finite r-fold covering of the plane with axis-parallel rectangles that does not have a k-split.*

In the case $k = 2$, Corollary 1 states that there are locally finite r-fold coverings of the plane with axis-parallel rectangles that cannot be partitioned into two coverings. A very simple direct construction proving this can be found in [PTT07]. Some positive results with half-planes, disks, translates of a convex polygon, etc., in the place of rectangles, were established in [P80], [MP87], [P86], [TT07], [K07], and [ACCLS07]. There is an intimate relationship between questions of this type and the notion of *conflict-free* colorings, introduced by Even, Lotker, Ron, and Smorodinsky [ELRS03]; see also [HS05]. Many similar problems on colorings are discussed in [BMP05] and [MP06].

2 The Construction and its Basic Properties

First we have to introduce some notations.

For any two integers $c \geq 2$ and $k \geq 0$, let $[c] := \{0, 1, \ldots, c-1\}$ and let $[c]^k$ stand for the set of strings of length k over the alphabet $[c]$. For $x \in [c]^k$, let x_j denote the jth digit of x $(1 \leq j \leq k)$, so that we have $x = x_1 \ldots x_k$.

Let $\overleftarrow{x}$ denote the reverse of x, that is, $\overleftarrow{x} = x_k \dots x_1$. An initial segment of x is a string $x_1 \dots x_j$ for some $0 \le j \le k$. Expanding x as a c-ary fraction, we obtain a number $\overline{x} := \sum_{j=1}^{k} x_j / c^j$.

Let $c \ge 2$ and $d \ge 1$ be integers. For any $0 \le k \le d$, $u \in [c]^k$, and $v \in [c]^{d-k}$, define an open axis-parallel rectangle $R^k_{u,v}$ in the plane as follows:

$$R^k_{u,v} := (\overline{u}, \overline{u} + c^{-k}) \times (\overline{v}, \overline{v} + c^{k-d}).$$

Now we are in a position to define the family of rectangles $\mathcal{R}$ meeting the requirements of Theorem 1. Let $\mathcal{R}$ consist of the rectangles in

$$\left\{ R^k_{u,v} \;\mid\; 0 < k < d, u \in [c]^k, v \in [c]^{d-k}, u_k = v_{d-k} \right\}$$

together with the rectangles in

$$\left\{ R^0_{\varepsilon,v} \;\mid v \in [c]^d, v_d = 0 \right\} \cup \left\{ R^d_{u,\varepsilon} \;\mid\; u \in [c]^d, u_d = 0 \right\},$$

where ε stands for the empty string and $\overline{\varepsilon} = 0$. For convenience, we slightly change the notation. For any $z \in [c]^{d-1}$, $0 < k < d$, set $S^k_z := R^k_{u,v}$, where u is the initial segment of z of length k, and v is the initial segment of $\overleftarrow{z}$ of length $d-k$. Further, for any $z \in [c]^{d-1}$, set $S^0_z := R^0_{\varepsilon,v}$ and $S^d_z := R^d_{u,\varepsilon}$, where u is obtained from z by appending to it a 0 as its last digit, and v is obtained from $\overleftarrow{z}$ in the same way. Using this notation, we have

$$\mathcal{R} = \mathcal{R}(c,d) = \{ S^k_z \mid 0 \le k \le d, z \in [c]^{d-1} \}.$$

Clearly, we have $|\mathcal{R}| = (d+1)c^{d-1}$. Finally, let $H^* = H^*(c,d) = H^*(\mathbb{R}^2, \mathcal{R}(c,d))$ denote the hypergraph on the vertex set $\mathcal{R} = \mathcal{R}(c,d)$, whose hyperedges are all nonempty subsets of $\mathcal{S} \subseteq \mathcal{R}$ for which there is a point in the plane covered by the elements of $\mathcal{S}$, but by no other element of $\mathcal{R}$.

The most important property of our construction is the following.

Theorem 2. *Let $d > 0$, $2 \le r < c$, and let $H^* = H^*(c,d)$ denote the hypergraph defined above. If a subset $I \subseteq \mathcal{R}(c,d)$ contains no hyperedge of H^* of size r, then we have*

$$|I| \le \frac{c^{d-1}}{\frac{1}{r-1} - \frac{1}{c-1}}.$$

Let $G^* := G^*(c,d)$ denote the graph H^*_2, consisting of all two-element hyperedges of $H^* = H^*(c,d)$. In view of Theorem 2, the chromatic number $\chi(H^*(c,d))$ satisfies the following.

Corollary 2. *(1) The graph $G^*(2,d)$ is bipartite.*

(2) The chromatic number of $G^*(3,d)$ is at least $\frac{d+1}{2}$.

(3) For $c \ge d+3$, the chromatic number of $G^*(c,d)$ is $d+1$.

Proof. To establish (1), color the rectangles $S^k_z \in \mathcal{R}(2,d)$ according to the parity of $k + \sum_{i=1}^{d} z_i$. Clearly, this is a proper coloring of the vertex set of $G^*(2,d)$.

Applying Theorem 2 with $r = 2$, we obtain that the size of every independent set in $G^*(c, d)$ is at most $c^{d-1}/(1 - 1/(c - 1))$. As the total number of vertices of $G^*(c, d)$ is $(d + 1)c^{d-1}$, the chromatic number of $G^*(c, d)$ is at least $(d + 1)$ $(1-1/(c-1))$. In case $c = 3$, this gives the bound claimed in (2). For $c \geq d+3$, we have $(d + 1)(1 - 1/(c - 1)) > d$, so that $\chi(G^*(c, d)) \geq d + 1$. This is tight, since the vertices $S^k_z \in V(G^*(c, d)) = \mathcal{R}(c, d)$ can be colored according to the value k, $0 \leq k \leq d$. □

Theorem 1 immediately follows from

Corollary 3. *Let $k, r \geq 2$ be fixed. There exists a family of $k(2r)^{2kr}$ axis-parallel rectangles in the plane such that for any coloring of these rectangles with k colors, one can find a point covered by exactly r rectangles, all of which have the same color.*

Proof. Consider the family $\mathcal{R}(c, d)$ with $c = 2r$, $d = 2kr - 1$. For any k-coloring of the members of $\mathcal{R}(c, d) = \mathcal{R}(2r, 2kr - 1)$, the size of the largest color class is at least $(d+1)c^{d-1}/k = c^d$, which is larger than the bound in Theorem 2. Thus, the largest color class contains a hyperedge of $H^*(c, d)$ of size r. Note that the slightly smaller choices $c = 2r - 1$, $d = 2kr - 2k$ would also suffice for the proof. □

It remains to establish Theorem 2.

3 Proof of Theorem 2

Let $d \geq 1$ and $c \geq 2$ be fixed. Let $\overline{\mathcal{R}} = \overline{\mathcal{R}}(c, d)$ denote the family of all rectangles $R^k_{u,v}$ with $0 \leq k \leq d$, $u \in [c]^k$, $v \in [c]^{d-k}$. Let $\overline{H}^* = \overline{H}^*(c, d)$ be the corresponding hypergraph, that is, let the vertices of $\overline{H}^*$ be the members of $\overline{\mathcal{R}}$, and let its hyperedges be all sets of the form $\{R \in \overline{\mathcal{R}} \mid p \in R\}$, where $p \in \mathbb{R}^2$.

First we study the structure of $\overline{H}^*$. We define $R^k_{u,v} \leq R^l_{w,z}$ if $k \leq l$, u is an initial segment of w, and z is an initial segment of v. Note that $\leq$ is a partial order on $\overline{\mathcal{R}}$. We show that the hyperedges of $\overline{H}^*$ form intervals in this partial order.

Lemma 1. *The hyperedges of $\overline{H}^*$ are exactly the sets $\{e \in \overline{\mathcal{R}} \mid a \leq e \leq b\}$, where $a \leq b$ are elements of $\overline{\mathcal{R}}$.*

Proof. Let $a = R^k_{u,v}$ and $b = R^l_{w,z}$ be two elements of $\overline{\mathcal{R}}$ with $a \leq b$. Let $p_{a,b} = (\overline{w}+c^{-l-1}, \overline{v}+c^{k-d-1})$, a point of the plane. Let $e = R^m_{x,y} \in \overline{\mathcal{R}}$ arbitrary. We have $e = (\overline{x}, \overline{x}+c^{-m}) \times (\overline{y}, \overline{y}+c^{m-d})$. So $p_{a,b} \in e$ if and only if $\overline{x} < \overline{w}+c^{-l-1} < \overline{x}+c^{-m}$ and $\overline{y} < \overline{v} + c^{k-d-1} < \overline{y} + c^{m-d}$. The first pair of inequalities is satisfied if and only if $m \leq l$ and x is an initial segment of w, while the second pair is satisfied if and only if $k \leq m$ and y is an initial segment of v. Both pairs are satisfied if and only if $a \leq e \leq b$. Therefore, the point $p_{a,b}$ shows that $\{e \in \overline{\mathcal{R}} \mid a \leq e \leq b\}$ is an edge of $\overline{H}^*$.

To see that $\overline{H}^*$ has no additional hyperedges, it is sufficient to prove the following two claims.

(1) Any two rectangles $a, b \in \overline{\mathcal{R}}$ are disjoint, unless $a \leq b$ or $b \leq a$.
(2) For any three rectangles $a \leq e \leq b$, we have $a \cap b \subseteq e$.

Here (1) shows that the vertices of any hyperedge of $\overline{H}^*$ are linearly ordered by $\leq$. If a is the minimal vertex of a hyperedge f and b is its maximal vertex, then $f \subseteq \{e \in \overline{\mathcal{R}} \mid a \leq e \leq b\}$, and by (2) equality must hold.

To verify (1), let $a = R^k_{u,v} = a_1 \times a_2$ and $b = R^l_{w,z} = b_1 \times b_2$. Here a_1, a_2, b_1, and b_2 are open intervals. We may assume $k \leq l$, by symmetry. If for some $i \leq k$ we have $u_i \neq w_i$, then a_1 and b_1 are disjoint. If for some $i \leq d-l$, we have $v_i \neq z_i$, then a_2 and b_2 are disjoint. So if a and b intersect, we must have $a \leq b$. Notice that in this case we have $b_1 \subseteq a_1$ and $a_2 \subseteq b_2$.

To verify (2), let $a \leq e \leq b$ with $a = a_1 \times a_2$ and $b = b_1 \times b_2$. Using the last observation of the previous paragraph, we have $a \cap b = b_1 \times a_2$ and by the same observation again, this is contained in e. □

The *type* of a rectangle $R^k_{u,v} \in \overline{\mathcal{R}}$ is k. It is easy to see that if $a \leq b$ are rectangles in $\overline{\mathcal{R}}$ of type k and l, respectively, then for all $k \leq m \leq l$, there exists precisely one $d \in \overline{\mathcal{R}}$ of type m that satisfies $a \leq d \leq b$. Consider now the two-coloring of $\overline{\mathcal{R}}$, where the color of a rectangle is determined by the parity of its type. The last observation shows that in any hyperedge of $\overline{H}^*$ the number of vertices of the two color classes differ by at most one. In particular, no edge of size at least two is monochromatic.

Next we describe the structure of the subfamily $\mathcal{R} = \mathcal{R}(c, d)$ of $\overline{\mathcal{R}}$. The elements of $\mathcal{R}$ are partially ordered by $\leq$. Note that for any $a = S^k_z$ and $b = S^l_t$, we have $a \leq b$ if and only if $k \leq l$ and $z_i = t_i$ for all $1 \leq i \leq k$ and $l \leq i \leq d-1$. For any $a \leq b$ in $\mathcal{R}(c, d)$, we define the *interval* $[a, b] = \{e \in \mathcal{R} \mid a \leq e \leq b\}$. For any $a = S^k_z$ and $b = S^l_t$ with $a \leq b$, the interval $[a, b]$ contains one element of type m for each index m such that $k \leq m \leq l$ and $z_m = t_m$. Here the type of the rectangle S^m_w (inherited from $\overline{\mathcal{R}}$) is m.

Corollary 4. *The hyperedges of $H^*(c, d)$ are exactly the intervals $[a, b]$, where $a \leq b$ are vertices of $H^*(c, d)$.*

Proof. The hyperedges of $H^*(c, d)$ are the sets $e \cap \overline{\mathcal{R}}$ where e is a hyperedge of $\overline{H}^*(c, d)$. The assertion follows from Lemma 1. □

Corollary 5. *Two vertices S^k_z and S^l_t are connected in $G^*(c, d)$ by an edge if and only if*
(1) $k \neq l$,
(2) for all indices i strictly between k and l, we have $z_i \neq t_i$, and
(3) for all other indices i, we have $z_i = t_i$. □

Proof of Theorem 2. Let us fix $d \geq 1$, $2 \leq r < c$ and a set $I \subseteq \mathcal{R} = \mathcal{R}(c, d)$ such that no edge of $H^*(c, d)$ of size r is contained in I. For any vertex $a \in \mathcal{R}$ of type $0 \leq i < d$, we define the *next* vertex $N(a)$ of type $i+1$ as follows. Let $N(S^i_x) = S^{i+1}_x$ if $i = 0$ or $S^i_x \in I$. If $0 < i < d$ and $S^i_x \notin I$, then let $N(S^i_x) = S^{i+1}_y$, where $y_j = x_j$ for all indices $1 \leq j \leq d-1$, $j \neq i$, and $y_i = (x_i + s) \bmod c$. We choose s to be the smallest positive integer such that $S_y^i \notin I$. In other words, we obtain y from x by shifting the ith digit cyclically upward until we reach a value y with $S_y^i \notin I$. Note that the choice $s = c$ makes $y = x$ and $S_y^i \notin I$. We call the vertex S_x^i *bad* if $s = c$ is the minimal choice, that is, if $1 \le i \le d-1$, $S_x^i \notin I$ and $N(S_x^i) = S_x^{i+1}$. Clearly, S_x^i is bad if and only if $S_x^i \notin I$ but $S_y^i \in I$ for all the $c-1$ strings y that differ from x only at the position i. Therefore, for the set B of bad vertices, we have $|B| \le |I|/(c-1)$.

It is easy to see that for $0 \le i \le d-1$, the function $N : \mathcal{R}(c,d) \to \mathcal{R}(c,d)$ is a one-to-one mapping from the vertices of type i to the vertices of type $i+1$. We call a sequence $a_0, a_1, \dots, a_d$ of vertices of $\mathcal{R}$ a *cluster* if $N(a_i) = a_{i+1}$ for $0 \le i \le d-1$. Note that a_i is of type i for every i. There are c^{d-1} clusters (one for each a_0 of type 0) and they form a partition of $\mathcal{R}$ into subsets of size $d+1$.

The main observation is the following. If in a cluster $a_0, a_1, \dots, a_d$ we have $a_i, a_j \in I$ for some indices $0 \le i \le j \le d$, then $a_i \le a_j$ and $[a_i, a_j] = \{a_k \mid i \le k \le j,\ a_k \in I \cup B\}$. This follows readily from the definition of the relation $\le$.

Now we use that intervals like $[a_i, a_j]$ are edges of the hypergraph $H^*(c,d)$ (Corollary 4), and no edge of size r is contained in I. Therefore, a cluster contains at most $r-1$ elements of I not separated by bad vertices. If there are b bad vertices in a cluster, then the cluster contains at most $(r-1)(b+1)$ elements of I. Summing over all clusters, we conclude that

$$|I| \le (r-1)|B| + (r-1)c^{d-1}.$$

Using the inequality $|B| \le |I|/(c-1)$ and rearranging the terms, the desired bound on $|I|$ follows. $\square$

4 Concluding Remarks

For any $c \ge 2, d \ge 1$, define a graph $G' = G'(c,d)$ on the vertex set $\{v_z^k \mid 0 \le k \le d, z \in [c]^d\}$ as follows. Connect two vertices v_z^k and v_t^l with $k \le l$ by an edge if and only if

(1) $k < l$,
(2) for all indices i with $k < i \le l$, we have $z_i \ne t_i$, and
(3) for all other indices i, we have $z_i = t_i$.

Clearly, the definition of $G'(c,d)$ is very close to the description of $G^*(c,d)$ given in Corollary 5. Let us compare the two graphs.

It is easy to see that G' contains a complete graph on $d+1$ vertices. For example, let $z(i)$ be a string of i zeros followed by $d-i$ ones. Notice that the vertices $v_{z(i)}^i$, $0 \le i \le d$, form a complete subgraph. Thus, the chromatic number of G' is at least $d+1$, and this is tight, as shown by the coloring that assigns color i to all vertices of the form v_z^i. Moreover, the vertex set of G' can be partitioned into $(d+1)$-element sets so that each set induces a complete graph in G'. To see this, let $x = x_1 \dots x_d \in [c]^d$, and for any $0 \le i \le d$, define $x(i) \in [c]^d$ as follows: let $(x(i))_j = x_j$ if $j > i$, and let $(x(i))_j = (x_j + 1) \bmod c$ if $j \le i$. Clearly, the vertices $v_{x(i)}^i$ induce a complete subgraph in G', and for the different choices of x, they form a vertex partition. This shows that no independent set in G' contains more than a fraction of $1/(d+1)$ of the vertices.

On the other hand, the graph G^* is triangle-free. To verify this, consider three vertices $a = S_z^k$, $b = S_t^l$, and $e = S_w^m$ with $k \le l \le m$. If a and b are connected by an edge in G^*, we have $k < l$ and $z_l = t_l$. Analogously, if b and e are connected, then $l < m$ and $t_l = w_l$. This yields that $k < l < m$ and $z_l = w_l$, which implies that a and e are not connected. Despite this difference, the proof of Theorem 2 was inspired by the similarity between G^* and G', and by the simple argument above, which provides an upper bound on the size of the largest independent set in G'.

The fact that G^* is triangle-free is not merely a coincidence. Consider any family $\mathcal{S}$ of axis-parallel rectangles in the plane, and construct a graph $G = (V, E)$ on the vertex set $V = \mathcal{S}$ by connecting two rectangles if they have a point in common that is not covered by any other member of $\mathcal{S}$. Recall that G^* was also constructed in this manner. While G may contain triangles and even subgraphs isomorphic to K_4, it cannot have a complete subgraph on *five* vertices. Furthermore, we can partition the edge set of G into two subsets $E = E_1 \cup E_2$ as follows. Let us put an edge $\{v_1, v_2\} \in E$ in E_1 if the boundaries of the rectangles v_1 and v_2 cross in *four* points. Otherwise, put the edge $\{v_1, v_2\}$ in E_2. It is easy to see that $G_1 := (V, E_1)$ is triangle-free. It seems that G_1 is the interesting part of G, as the chromatic number of the graph $G_2 := (V, E_2)$ is small. Perhaps G_2 is even *Δ-degenerate* for an appropriate absolute constant Δ, that is, every subgraph of G_2 has a vertex of degree at most Δ. To prove this, it would be sufficient to show that $|E_2| \le \Delta |V|/2$.

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Domination in Cubic Graphs of Large Girth

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Abstract. We prove that connected cubic graphs of order n and girth g have domination number at most $0.32127n + O\left(\frac{n}{g}\right)$.

The domination number $\gamma(G)$ of a (finite, undirected and simple) graph $G = (V, E)$ is one of the most well-studied graph parameters [4] and is defined as the minimum cardinality of a set $D \subseteq V$ of vertices such that every vertex in $V \setminus D$ has a neighbour in D.

Initially motivated by Reed's [10] disproved [6] conjecture that every connected cubic graph of order n has domination number at most $\lceil \frac{n}{3} \rceil$, several authors recently studied the domination number of cubic graphs of large girth where the girth is the length of a shortest cycle in G.

Kawarabayashi, Plummer and Saito [5] proved $\gamma(G) \leq \left(\frac{1}{3} + \frac{1}{9k+3}\right) n$ for every 2-edge connected cubic graph G of order n and girth at least $3k$ for some $k \in \mathbb{N}$ and Kostochka and Stodolsky [7] proved $\gamma(G) \leq \left(\frac{1}{3} + \frac{8}{3g^2}\right) n$ for every connected cubic graph G of order $n > 8$ and girth g. While these two bounds tend to $n/3$ for $g \to \infty$, Löwenstein and Rautenbach [8] recently showed that one actually gets below $n/3$ for sufficiently large girth by proving $\gamma(G) \leq \left(\frac{44}{135} + \frac{82}{135g}\right) n \approx 0.3259n + O\left(\frac{n}{g}\right)$ for every cubic graph of order n and girth $g \geq 5$. In the present paper we will slightly improve the constant in this upper bound.

Since for fixed d and g the numbers of cycles in random cubic graphs of fixed lengths $r < g$ are asymptotically distributed as independent Poisson variables [2] with mean $(d-1)^r/2r$, the number of cycles in a random cubic graph of length smaller than g is asymptotically almost surely bounded. Therefore, the asymptotic bounds for the domination number of random cubic graphs carry over to cubic graph of large girth and indicate how much more the above constant could be improved. Molloy and Reed [9] proved that the domination number γ of a random cubic graph of order n asymptotically almost surely satisfies $0.2636n \leq \gamma \leq 0.3126n$ and Duckworth and Wormald [3] improved the upper bound to $0.2794n$.

We immediately proceed to our main result.

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Theorem 1. *If $G = (V, E)$ is a connected, cubic graph of order n, girth g and domination number γ, then*

$$\gamma \leq 0.32127n + O\left(\frac{n}{g}\right).$$

Proof. Let the graph $G = (V, E)$ be as in the statement of the theorem.

Clearly, we may assume $g \geq 7$. We will first prove the existence of a matching that covers all but $O\left(\frac{n}{g}\right)$ vertices. Therefore, for some set $S \subseteq V$ let $q_1(S)$ denote the number of components of $G[V \setminus S]$ of odd order that are joined to S by only one edge and let $q_2(S)$ denote the number of components of $G[V \setminus S]$ of odd order that are joined to S by at least 2 edges. Since G is cubic, all of the $q_2(G)$ odd components are joined to S by at least 3 edges and we have $q_2(G) \leq |S|$. Furthermore, every component of $G[V \setminus S]$ of odd order that is joined S by only one edge must contain a cycle which implies that it has order at least g and $q_1(S) \leq \frac{n}{g}$. Altogether, we obtain $q_1(S) + q_2(S) - |S| \leq \frac{n}{g}$ and by the Tutte-Berge formula, there is a matching M in G such that the set D_0 of vertices not incident to an edge in M satisfies $|D_0| \leq \frac{n}{g}$.

Since there are at most $3|D_0| \leq \frac{3n}{g}$ edges between D_0 and $V \setminus D_0$, the graph $G[V \setminus D_0] - M$ consists of cycles and at most $\frac{3n}{2g}$ paths. Let $G[V \setminus D_0] - M$ contain k_1 cycles and $(k - k_1)$ components which are paths of order more than $\frac{g}{3}$. Note that the cycles are all of order at least g. Let the orders of these k cycles and long paths be $n_1, n_2, \ldots, n_k > \frac{g}{3}$. Clearly, $n_1 + n_2 + \cdots + n_k \leq n$ and $k \leq \frac{3n}{g}$. If we decompose each of these k cycles and long paths into paths of length $\lfloor \frac{g}{3} \rfloor$ and possibly one shorter path, then we obtain a total number of

$$\sum_{i=1}^{k} \left\lceil \frac{n_i}{\lfloor \frac{g}{3} \rfloor} \right\rceil \leq \sum_{i=1}^{k} \left\lceil \frac{n_i}{\frac{g}{4}} \right\rceil \leq \sum_{i=1}^{k} \left(\frac{n_i}{\frac{g}{4}} + 1 \right) \leq \frac{4n}{g} + \frac{3n}{g} = O\left(\frac{n}{g}\right)$$

paths. Hence there is a collection $\mathcal{P}$ of $O\left(\frac{n}{g}\right)$ vertex disjoint paths which are all of lengths at most $\frac{g}{3}$ and contain all vertices in $V \setminus D_0$.

Since $\frac{g}{3} < \frac{g-2}{2}$, these paths are induced and no two of these paths are joined by more than one edge.

Let H denote the graph with vertex set $V \setminus D_0$ whose edges are the edges of the paths in $\mathcal{P}$ together with the matching M. Note that M is a perfect matching of H. We will describe a probabilistic procedure for constructing a small dominating set D_1 of H in five phases.

Phase 1

We select a random subset $\mathcal{P}_0$ of $\mathcal{P}$ by assigning each $P \in \mathcal{P}$ to $\mathcal{P}_0$ independently at random with probability p for some $0 \leq p \leq 1$ to be specified later. Let $\mathcal{P}_1 = \mathcal{P} \setminus \mathcal{P}_0$.

Phase 2

For every path $P : x_1 x_2 \ldots x_l \in \mathcal{P}_0$ we choose independently at random a parity

$j \in \{0, 1, 2\}$ each with probability $\frac{1}{3}$ and set

$$D(P) = \{x_i \mid 1 \leq i \leq l, i \equiv j \pmod 3\}.$$

Phase 3

For every path $P : x_1 x_2 \ldots x_l \in \mathcal{P}_1$ we will determine a set $D(P) \subseteq \{x_1, x_2, \ldots, x_l\}$ by the following procedure:

(1) We set $D(P) := \emptyset$ and $i := 1$.
(2) We query whether x_i is adjacent in H to a vertex in $\bigcup_{Q \in \mathcal{P}_0} D(Q)$.
  - If the answer to the query is 'yes' and $i \leq l - 1$, then we set $i := i + 1$ and go to (2).
  - If the answer to the query is 'yes' and $i = l$, then we terminate.
  - If the answer to the query is 'no' and $i \leq l - 3$, then we set $D(P) := D(P) \cup \{x_{i+1}\}$, $i := i + 3$ and go to (2).
  - If the answer to the query is 'no' and $i \geq l - 2$, then we set $D(P) := D(P) \cup \left\{x_{\min\{i+1,l\}}\right\}$ and terminate.

Phase 4

For every edge $uv \in M$ such that $u \in D(P)$, $v \in D(Q)$ with $P, Q \in \mathcal{P}_0$ we delete at most one vertex, say u, from $D(P) \cup D(Q)$, if the neighbour(s) of u on P are adjacent in H to a vertex in $\bigcup_{R \in \mathcal{P}_0} D(R) \setminus D(P)$. (Note that in H the vertex u has no neighbour on a path in $\mathcal{P}_1$.)

Phase 5

Let D' denote the set of endvertices of the paths in $\mathcal{P}_0$ and let $D_1 = D' \cup \bigcup_{P \in \mathcal{P}} D(P)$. This terminates the last phase.

It is obvious from the construction that the set D_1 is a dominating set of H. We will now estimate the expected value of $|D_1|$. Therefore, let $n_1 = n - |D_0|$.

The expected number of vertices added to $\bigcup_{P \in \mathcal{P}_0} D(P)$ in the Phase 2 is $\frac{pn_1}{3}$, because the probability that a path in $\mathcal{P}$ is in $\mathcal{P}_0$ is p and subject to this the probability of a vertex on P to belong to $D(P)$ is $\frac{1}{3}$.

Now we proceed to the Phase 3. For some path $P : x_1 x_2 \ldots x_l$ in $\mathcal{P}_1$ let q denote the total number of queries and let q_y denote the number of queries with answer 'yes' during the construction of $D(P)$. Note that while the answers to previous queries influence for which vertex we ask the next query, the answer to every query is independent of the answers to previous queries, because no two paths are joined by more than one edge and the random choices for different paths are independent.

If we query the corresponding adjacency for some vertex, then the path Q containing its neighbour outside of P lies in $\mathcal{P}_0$ with probability p and subject to this the neighbour lies in $D(Q)$ with probability $1/3$. Therefore, the probability of a positive answer to an individual query is $p/3$ and the expected values for q_y and q satisfy $E[q_y] = \frac{pE[q]}{3}$.

Since with the exception of queries within distance $O(1)$ of the end of P, for every positive anser to a query the index i is incremented by 1 and for every negative anser to a query the index i is incremented by 3, we have $q = \frac{l+2q_y}{3} + O(1)$. Therefore, $E[q] = \frac{l+2E[q_y]}{3} + O(1)$ which together with $E[q_y] = \frac{pE[q]}{3}$ implies that $E[q] = \frac{3}{9-2p}l + O(1)$.

Since for every query with a negative answer, one vertex is added to $D(P)$ we have

$$E[|D(P)|] = E[q - q_y] = E[q] - E[q_y] = (1 - p/3)E[q] = \frac{3-p}{9-2p}l + O(1).$$

Finally, since every of the $O\left(\frac{n}{g}\right)$ paths in $\mathcal{P}$ belongs to $\mathcal{P}_1$ with probability $(1-p)$, we obtain, by linearity of expectation, that

$$\begin{aligned} E\left[\left|\bigcup_{P\in\mathcal{P}_1} D(P)\right|\right] &= \frac{3-p}{9-2p}(1-p)n_1 + O\left(\frac{n}{g}\right) \\ &= \frac{(1-p)n_1}{3} + \frac{p(p-1)n_1}{3(9-2p)} + O\left(\frac{n}{g}\right). \end{aligned}$$

We proceed to Phase 4. The expected number of edges among the total $\frac{n_1}{2}$ edges in M which join two paths in $\mathcal{P}_0$ and for which we remove one vertex from $\bigcup_{P\in\mathcal{P}_0} D(P)$ in the fourth phase equals

$$\frac{n_1}{2}p^2\frac{1}{9}\left(2\left(\frac{p}{3}\right)^2 - \left(\frac{p}{3}\right)^4\right).$$

(The two paths containing the endpoints of an edge uv in M lie in $\mathcal{P}_0$ with probability p^2 and the two vertices u and v lie in $\bigcup_{P\in\mathcal{P}_0} D(P)$ as constructed in the second phase with probability $\frac{1}{9}$. The term $\left(2\left(\frac{p}{3}\right)^2 - \left(\frac{p}{3}\right)^4\right)$ is the probability that the neighbour(s) of u and v are still adjacent in H to a vertex in $\bigcup_{R\in\mathcal{P}_0} D(R)$ after removing either u or v.)

Since the set D' in the fifth phase is of order $O\left(\frac{n}{g}\right)$, we obtain altogether

$$\begin{aligned} E[|D_1|] &= \frac{pn_1}{3} + \frac{(1-p)n_1}{3} + \frac{p(p-1)n_1}{3(9-2p)} + O\left(\frac{n}{g}\right) - \frac{p^2n_1}{18}\left(2\left(\frac{p}{3}\right)^2 - \left(\frac{p}{3}\right)^4\right) \\ &= \frac{n_1}{3} - n_1\left(\frac{p(1-p)}{3(9-2p)} + \frac{p^2}{18}\left(2\left(\frac{p}{3}\right)^2 - \left(\frac{p}{3}\right)^4\right)\right) + O\left(\frac{n}{g}\right). \end{aligned}$$

Over the interval $[0,1]$ the function

$$f(p) = \frac{p(1-p)}{3(9-2p)} + \frac{p^2}{18}\left(2\left(\frac{p}{3}\right)^2 - \left(\frac{p}{3}\right)^4\right)$$

has maximum value $f(0.74379) > 0.012117$. Since $D_0 \cup D_1$ is a dominating set of G, we obtain $\gamma \leq |D_0| + (n - |D_0|)\left(\frac{1}{3} - 0.012117\right) + O\left(\frac{n}{g}\right) = 0.32127n + O\left(\frac{n}{g}\right)$ and the proof is complete.

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Chvátal–Erdős Theorem: Old Theorem with New Aspects

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Abstract. The Chvátal–Erdős Theorem states that a 2-connected graph is hamiltonian if its independence number is bounded from above by its connectivity. In this short survey, we explore the recent development on the extensions and the variants of this theorem.

1 Introduction

Hamiltonian cycles in graphs have been one of the central topics in graph theory. From a computational point of view, to check whether a given graph is hamiltonian is an NP-complete problem. Therefore, it seems to be impossible to obtain a criterion for a graph to be hamiltonian which implies a polynomial-time algorithm. Thus, the main focus has been taken on sufficient conditions for hamiltonicity. We believe that the more sufficient conditions we find, the deeper insight we have on the structure of hamiltonian graphs.

Probably, one of the most famous sufficient conditions for hamiltonicity is Ore's Theorem. For a vertex x in a graph G, let $\deg_G x$ denote the degree of x in G.

Theorem 1 (Ore's Theorem [28]). *Let G be a graph of order $n \geq 3$. If $\deg_G x + \deg_G y \geq n$ holds for every pair of nonadjacent vertices x and y in G, then G is hamiltonian.*

Since [28], many results have been obtained on the relationship between hamiltonicity and degrees of graphs. These results give so-called "degree conditions", which generally claim that a graph is hamiltonian if the degree of almost every vertex is large. For general hamiltonian problems including degree conditions, we refer the reader to the excellent surveys by Gould [17] and [19].

In 1972, Chvátal and Erdős gave a sufficient condition for a graph to be hamiltonian. For a graph G, let $\alpha(G)$ and $\kappa(G)$ denote the independence number and the connectivity of G, respectively.

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Theorem 2 (The Chvátal–Erdős Theorem [8]). *Every 2-connected graph G with $\alpha(G) \leq \kappa(G)$ is hamiltonian.*

The Chvátal–Erdős Theorem has a number of unique aspects. For example, it is not a "degree condition", in the sense that the degree does not appear in the statement. On the other hand, as Bondy remarked in [3], it implies Ore's Theorem.

Theorem 3 ([3]). *If a graph G of order $n \geq 3$ satisfies $\deg_G x + \deg_G y \geq n$ for every pair of nonadjacent vertices x and y in G, then G is 2-connected and $\alpha(G) \leq \kappa(G)$.*

The Chvátal–Erdős Theorem has opened a new vista in the research of hamiltonian graphs. Since [8], there have been obtained a large number of sufficient conditions for a graph to satisfy a certain cycle-related or path-related property, which are mainly described in terms of the independence number and the connectivity. We call them "Chvátal–Erdős conditions". Jackson and Ordaz [20] published an excellent survey on Chvátal–Erdős conditions.

The purpose of this paper is to give an overview of the recent progress in the research of Chvátal–Erdős conditions. We mainly discuss the results which have been obtained since [20] was published and give an update to the status of the research. We also deal with several topics not addressed in [20].

2 Pancyclicity

A graph of order $n \geq 3$ is said to be *pancyclic* if it contains a cycle of every length between three and n. Many degree conditions for hamiltonicity have been extended to sufficient conditions for a graph to be pancyclic. For example, Bondy [2] extended Ore's Theorem in the following way.

Theorem 4 ([2]). *A hamiltonian graph of order n with at least $\frac{1}{4}n^2$ edges is either pancyclic or the balanced complete bipartite graph. In particular, if G satisfies the assumption of Ore's Theorem, then G is either pancyclic or a balanced complete bipartite graph.*

On the other hand, the extension of The Chvátal–Erdős Theorem to pancyclicity has so far been of little success. In particular, the following natural conjecture, made by Jackson and Ordaz [20] is still open.

Conjecture 1 ([20]). Every graph G with $\alpha(G) < \kappa(G)$ is pancyclic.

Amar, Fournier and Germa [1] investigated the distribution of the lengths of cycles in a 2-connected graph G with $\alpha(G) \leq \kappa(G)$. They made one conjecture, which was later proved by Lou [23].

Theorem 5 ([23]). *A 2-connected triangle-free graph G of order n with $\alpha(G) \leq \kappa(G)$ has a cycle of every length between four and n unless G is a balanced complete bipartite graph or C_5.*

Flandrin et al. [13] have proved that Conjecture 1 holds if the order of the graph is bounded from below by a function of the independence number.

Theorem 6 ([13]). *A 2-connected graph G of order n with $\alpha(G) = \alpha$ and $\kappa(G) = \kappa$ is pancyclic if $\alpha \le \kappa$ and $n \ge 2 \cdot r(4\alpha, \alpha + 1)$, where $r(m, n)$ is the Ramsey number.*

Theorem 6 is a nice step toward Conjecture 1. However, we cannot simply say that it solves the conjecture for sufficiently large graphs since the lower bound of the order is not a constant.

3 Cycle Covering

Let $C_1, \ldots, C_m$ be cycles in a graph G. Then we say that $C_1, \ldots, C_m$ *cover* G if $V(G) = \bigcup_{i=1}^{m} V(C_i)$. Note that we do not require $C_1, \ldots, C_m$ to be disjoint. In other words, $C_1, \ldots, C_m$ may not form a 2-factor of G.

By the definition, G is hamiltonian if and only if G is covered by one cycle. This interpretation leads us to consider a sufficient condition for a graph to be covered by a specified number of cycles. Kouider [22] proved the following beautiful extension of the Chvátal–Erdős Theorem, which was first conjectured by Fournier [15].

Theorem 7 ([22]). *Every 2-connected graph G with $\alpha(G) = \alpha$ and $\kappa(G) = \kappa$ is covered by $\lceil \alpha/\kappa \rceil$ cycles.*

We see that the number $\lceil \alpha/\kappa \rceil$ of cycles to cover G is sharp by considering complete bipartite graphs.

4 Long Cycles

For a graph G with at least one cycle, the length of a longest cycle in G is called the *circumference* of G, and is denoted by $\mathrm{circ}(G)$. A graph G of order n is hamiltonian if and only if $\mathrm{circ}(G) = n$. If a graph G does not satisfy a sufficient condition for hamiltonicity, we cannot guarantee the existence of a hamiltonian cycle. But if G is close to satisfying the condition, we may hope to find some "hamiltonian-like" structure. One candidate of such structures is a long cycle. From this point of view, a number of researches have been carried out to give a lower bound on the circumference. When we extend The Chvátal–Erdős Theorem into this direction, the lower bounds are expected to be described in terms of the order, the connectivity and the independence number.

Kouider's Theorem in the previous section immediately gives one bound. By taking a longest cycle in the cycle covering of Theorem 7, we have the following.

Corollary 1. *Every 2-connected graph G of order n has a cycle of length at least n/m, where $m = \lceil \alpha(G)/\kappa(G) \rceil$.*

Fouquet and Jolivet [14] conjectured that a slightly better bound may hold.

Conjecture 2 ([14]). A 2-connected graph of order n with $\alpha(G) = \alpha$ and $\kappa(G) = \kappa$ has a cycle of length at least $\min\left\{n, \frac{\kappa(n+\alpha-\kappa)}{\alpha}\right\}$.

Though this conjecture has been solved for $\alpha = 2$, 3, $\kappa + 1$ and $\kappa + 2$ (see [15], [16] and [24]), it is still open for other values of α.

Both Corollary 1 and Conjecture 2 are sharp for general graphs. However, if we restrict ourselves to triangle-free graphs, we obtain a much better bound. The following bound was obtained by Enomoto et al. [11].

Theorem 8 ([11]). *Let G be a 2-connected triangle-free graph of order n with $\alpha(G) = \alpha$ and $\kappa(G) = \kappa$. If $\alpha \leq 2\kappa - 2$, then G has a cycle of length at least $\min\{n, n - \alpha + \kappa\}$.*

In [11], they proved the sharpness of the bound $\min\{n, n - \alpha + \kappa\}$. On the other hand, they fail to show the sharpness of the condition $\alpha \leq 2\kappa - 2$. However, they constructed an example which shows that the conclusion no longer holds if $\alpha > 5\kappa - 6$.

5 Dominating Cycles

A set of vertices S in a graph G is said to be *dominating* if $N_G(x) \cap S \neq \emptyset$ for each $x \in V(G) - S$, where $N_G(x)$ is the neighborhood of x in G. A cycle in G is called a *dominating cycle* if its vertex set is a dominating set of G. Trivially, a hamiltonian cycle is a dominating cycle. From this point of view, several sufficient conditions for hamiltonicity have been extended to those which guarantee the existence of a dominating cycle.

The notion of dominating set has also been extended in a number of ways. One extension is a distance domination. The definition of a dominating set can be paraphrased in terms of distance. For $S \subset V(G)$ and $x \in V(G)$, the distance $d_G(x, S)$ between S and x is defined by $d_G(x, S) = \min\{d_G(x, s) \colon s \in S\}$, where $d_G(u, v)$ denote the distance between two vertices u, v in a graph G. Then S is a dominating set if and only if $d_G(x, S) \leq 1$ for every vertex x in G. Extending this version of the definition, for a nonnegative integer m, we define an *m-dominating* set S as a set of vertices S satisfying $d_G(x, S) \leq m$ for every vertex x in G. A cycle is said to be an *m-dominating cycle* if $V(C)$ is an m-dominating set in G. By the definition, a hamiltonian cycle is a 0-dominating cycle.

Broersma [5] and Fraisse [18] independently gave a sufficient condition for a graph to have an m-dominating cycle. A set of vertices S in a graph G is said to be *k-independent* if $d_G(x, y) > k$ for every pair of distinct vertices x, y in S. The order of a largest k-independent set is called a *k-independence number*, and denoted by $\alpha_k(G)$. By the definition, $\alpha_1(G) = \alpha(G)$.

Theorem 9 ([5], [18]). *A 2-connected graph G with $\alpha_{2k+1}(G) < \kappa(G)$ has a k-dominating cycle.*

The Chvátal–Erdős Theorem corresponds to the case $k = 0$ of the above theorem.

Theorem 9 was further generalized in [29]. Let f be an integer-valued function defined on $V(G)$. Then a set of vertices S is called an *f-dominating* set if it satisfies $d_G(x, S) \leq f(x)$ for every vertex x in G. An *f-dominating cycle* is a cycle whose vertex set is an f-dominating set. If f is a constant function taking value m, an f-dominating cycle coincides with an m-dominating cycle.

One advantage of an f-dominating cycle over an m-dominating cycle is that it extends the notion of cyclability as well as hamiltonicity. For a given set of vertices X in a graph G of order n, set $f\colon V(G) \to \boldsymbol{Z}$ as

$$f(x) = \begin{cases} 0 & \text{if } x \in X \\ n & \text{if } x \in V(G) - X. \end{cases}$$

Then an f-dominating cycle coincides with a cycle containing X. Therefore, an f-dominating cycle is a common generalization of a hamiltonian cycle and a cycle through specified vertices.

For $f\colon V(G) \to \boldsymbol{Z}$, a set $S \subset V(G)$ is said to be an *f-independent* set if $d_G(x, y) \geq f(x) + f(y)$ for each pair of distinct vertices x, y in S. The order of a largest f-independent set is called the *f-independence number*, and is denoted by $\alpha_f(G)$. If f is a constant function taking value m, then an f-independent set coincides with a $(2m-1)$-independent set, and $\alpha_f(G) = \alpha_{2m-1}(G)$. The following theorem was proved in [29]. For a function $f\colon V(G) \to \boldsymbol{Z}$ and an integer constant c, define a function $f + c$ by $(f + c)(x) = f(x) + c$ $(x \in V(G))$.

Theorem 10 ([29]). *Let G be a 2-connected graph and let f be an integer-valued function defined on $V(G)$. If $\alpha_{f+1}(G) \leq \kappa(G)$, then G has an f-dominating cycle.*

6 2-Factors

For a graph G and a positive integer k with $k \leq \alpha(G)$, we define $\sigma_k(G)$ by

$$\sigma_k(G) = \min \left\{ \sum_{x \in S} \deg_G x \colon S \text{ is an independent set of } G \text{ of order } k \right\}.$$

If $k > \alpha(G)$, we define $\sigma_k(G) = +\infty$. Using this parameter, we can paraphrase Ore's Theorem in the following way.

Theorem 11. *A graph G of order $n \geq 3$ with $\sigma_2(G) \geq n$ is hamiltonian.*

Like hamiltonian cycles, factors in graphs are one of the main topics in graph theory. When we look at a hamiltonian cycle from the theory of factors, we see that a hamiltonian cycle is a connected 2-factor. Therefore, a sufficient condition for hamiltonicity can be interpreted as a sufficient condition for a graph to have a connected 2-factor, or a 2-factor with exactly one component. From this observation, we naturally hope that sufficient conditions for hamiltonicity may admit an extension to conditions for a graph to have a 2-factor with a specified number of components. Actually, Brandt et al. [4] proved that Ore's Theorem admits such an extension.

Theorem 12 ([4]). *Let k be a positive integer, and let G be a graph of order n. If $n \geq 4k$ and $\sigma_2(G) \geq n$, then G has a 2-factor with exactly k components.*

Note that the balanced complete bipartite graph $K_{2k-1,2k-1}$ satisfies the assumption on $\sigma_2(G)$ and has order $4k-2$, but does not have a 2-factor with k components. Therefore, the bound $4k$ of the order in the assumption is almost sharp. Later Enomoto [10] relaxed the assumption "$n \geq 4k$" to "$n \geq 4k-1$" and filled the gap.

If we look at Theorem 12, we notice that the condition on $\sigma_2(G)$ is the same as that in Ore's Theorem. Actually, this condition is sharp for any positive integer k. For example, $K_{m,m+1}$ has no 2-factor. In other words, if a graph G of order n satisfies $\sigma_2(G) \geq n$, then the existence of a 2-factor with k components is guaranteed for each integer k between 1 and $\frac{1}{4}(n+1)$, while if $\sigma_2(G) = n-1$, we cannot even guarantee the mere existence of a 2-factor.

As we mentioned in Section 1, The Chvátal–Erdős Theorem implies Ore's Theorem. And we have seen in this section that Ore's Theorem admits an extension to a sufficient condition for a graph to have a 2-factor with a specified number of components. Therefore, we may expect that The Chvátal–Erdős Theorem also admits a similar extension.

Here we remark that when we consider this problem, we have to take the order of a graph into account. Clearly, a graph having a 2-factor with k components must have order at least $3k$. This suggests that when we consider a sufficient condition, we must always impose a condition on the order of a graph.

Chen et al. [7] posed the following conjecture.

Conjecture 3 ([7]). For each positive integer k, there exists a positive integer $f(k)$ such that every 2-connected graph G of order at least $f(k)$ satisfying $\alpha(G) \leq \kappa(G)$ has a 2-factor with exactly k components.

Kaneko and Yoshimoto [21] tackled the conjecture for $k = 2$, and claimed that they solved it for 4-connected graphs.

Theorem 13 ([21]). *Every 4-connected graph G of order at least six with $\alpha(G) \leq \kappa(G)$ has a 2-factor with two components.*

Recently, Egawa [9] reported that the proof of Theorem 13 misses one case to be considered, which leaves the correctness of the theorem in question at the time when the author is writing this survey.

Like Theorem 6, if we allow the independence number to come into the bound of the order, the conclusion of Conjecture 3 holds.

Theorem 14 ([7]). *Let G be a positive integer, and let G be a 2-connected graph of order n with $\alpha(G) = \alpha$ and $\kappa(G) = \kappa$. Suppose $\alpha \leq \kappa$. Then*

(1) *if $n \geq k \cdot r(\alpha+4, \alpha+1)$, then G has a 2-factor with exactly k components, and*
(2) *if $n \geq r(2\alpha+3, \alpha+1) + 3(k-1)$, then G has a 2-factor with exactly k components, $k-1$ of which have order exactly three.*

Note that the above theorem is not a complete solution of Conjecture 3 since the bound of the order in the conjecture does not involve the independence number.

7 Spanning Trees

As an immediate corollary of The Chvátal–Erdős Theorem, we have the following sufficient condition for a graph to have a hamiltonian path.

Corollary 2. *Every graph G with $\alpha(G) \leq \kappa(G) + 1$ has a hamiltonian path.*

Just as there are many ways to extend hamiltonian cycles, we can generalize the notion of a hamiltonian path by looking at it from a different point of view.

One may think that a path is a tree with at most two endvertices, where we define an endvertex to be a vertex of degree at most one. This observation leads us to investigate spanning trees with a bounded number of endvertices. For a positive integer m, an *m-ended tree* is a tree with at most m endvertices. Thus, a spanning 2-ended tree is a hamiltonian path.

Win [30] proved the following nice theorem, which naturally extends Corollary 2.

Theorem 15 ([30]). *Let m be a positive integer and let G be a 2-connected graph. If $\alpha(G) \leq \kappa(G) + m - 1$, then G has a spanning m-ended tree.*

Another extension of a hamiltonian path focuses on the maximum degree. We say that a tree T is an *m-tree* if $\Delta(T) \leq m$, where $\Delta(G)$ is the maximum degree of G. Then a hamiltonian path is a spanning 2-tree. Neumann-Lara and Rivera-Campo [26] proved a Chvátal–Erdős type theorem concerning the existence of a spanning m-tree.

Theorem 16 ([26]). *Let m be an integer with $m \geq 2$, and let G be a graph. If $\alpha(G) \leq 1 + (m-1)\kappa(G)$, then G has a spanning m-tree.*

Recently, Matsuda and Matsumura [25] gave a further generalization. They considered the existence of a spanning m-tree which contains specified vertices as its endvertices.

Theorem 17 ([25]). *Let m and s be positive integers with $m \geq 2$, and let G be a $(s+1)$-connected graph. If $\alpha(G) \leq 1 + (m-1)(\kappa(G) - s)$, then for every set of s vertices S, there exists a spanning m-tree which contains all the vertices in S as its endvertices.*

By putting $k = m = 2$, we see that Theorem 17 not only extends Theorem 16, but also extends the following variant of The Chvátal–Erdős Theorem on hamiltonian connectedness.

Theorem 18 ([8]). *Every 3-connected graph G with $\alpha(G) \leq \kappa(G) - 1$ is hamiltonian-connected.*

Recently, Enomoto and Ozeki [12] have considered a yet further generalization. Let f be an integer-valued function defined on the vertex set of a graph G. Then a spanning tree T of G is said to be an *f-tree* if $\deg_T v \leq f(v)$ holds for every vertex v in G. If f is a constant function taking value m, then an f-tree of G is its spanning m-tree. Moreover, if we put $f(v) = 1$ for several vertices v in

G, an f-tree has those vertices as its endvertices. Therefore, an f-tree gives a unified view to m-trees and those with specified endvertices. Note that a tree of order n has $n-1$ edges and hence the sum of the degrees of its vertices is $2n-2$. Therefore, a graph G of order n has an f-tree, then it must satisfy $\sum_{v \in V(G)} f(v) \geq 2n-2$.

Enomoto and Ozeki made the following conjecture

Conjecture 4 ([12]). Let G be a k-connected graph of order n and let f be an integer-valued function defined on $V(G)$, which satisfies $f(v) \geq 1$ for each $v \in V(G)$ and $\sum_{v \in V(G)} f(v) \geq 2(n-1)$. If

$$\alpha(G) \leq \min \left\{ \sum_{v \in R} \bigl(f(v)-1\bigr) \colon R \subset V(G), |R| = k \right\} + 1,$$

then G has an f-tree.

They gave a partial answer to the conjecture.

Theorem 19 ([12]). *Let G be a k-connected graph of order n and let f be an integer-valued function defined on $V(G)$, which satisfies $f(v) \geq 1$ for each $v \in V(G)$ and $\sum_{v \in V(G)} f(v) \geq 2(n-1)$. Let $S_i(f) = \{v \in V(G) \colon f(v) = i\}$. If $|S_1(f)| + |S_2(f)| \leq n-1$ and*

$$\alpha(G) \leq \min \left\{ \sum_{v \in R} \bigl(f(v)-1\bigr) \colon R \subset V(G), |R| = k \right\} + 1,$$

then G has an f-tree.

8 Localization of Chvátal–Erdős Theorem

In this section, we consider the localization of the Chvátal–Erdős condition, and pose a conjecture as a possible new direction of research.

A graph G is said to be claw-free if it does not contain $K_{1,3}$ as an induced subgraph. A number of sufficient conditions for hamiltonicity have been relaxed in the class of 2-connected claw-free graphs. For example, while Ore's Theorem gives a sharp bound of $\sigma_2(G)$ which guarantee the existence of a hamiltonian cycle for general graphs, Zhang [31] proved that 2-connected claw-free graphs admit a better bound.

Theorem 20 ([31]). *For an integer $k \geq 2$, a k-connected claw-free graph of order n with $\sigma_{k+1}(G) \leq n-k$ is hamiltonian.*

Oberly and Sumner [27] introduced the notion of locally connected graphs. A vertex x in a graph G is said to be *locally connected* if $N_G(x)$ induces a connected subgraph in G. And G is said to be *locally connected* if every vertex of x is locally connected. Oberly and Sumner proved the following theorem.

Theorem 21 ([27]). *Every connected, locally connected claw-free graph of order at least three is hamiltonian.*

The closed neighborhood $N_G[x]$ of a vertex x in G is the union of x and its neighborhood: $N_G[x] = N_G(x) \cup \{x\}$. In other words, $N_G[x]$ is the set of vertices which are distance at most one from x. Let B_x be the subgraph of G induced by $N_G[x]$. It is sometimes called the ball of radius one with center x.

A graph is claw-free if and only if $\alpha(B_x) \leq 2$ for each $x \in V(G)$, and $x \in V(G)$ is locally connected if and only if $\kappa(B_x) \geq 2$. Therefore, we can paraphrase Theorem 21 in the following way.

Theorem 22. *Let G be a connected graph. If $\alpha(B_x) \leq 2 \leq \kappa(B_x)$ holds for each $x \in V(G)$, then G is hamiltonian.*

This theorem says that if B_x satisfies the Chvátal–Erdős condition in a special manner for each $x \in V(G)$, then G is hamiltonian. But we can naturally ask whether the threshold of two between $\alpha(B_x)$ and $\kappa(B_x)$ is necessary. Therefore, at the end of this survey, we make the following conjecture.

Conjecture 5. Let G be a connected graph. If $\alpha(B_x) \leq \kappa(B_x)$ holds for each $x \in V(G)$, then G is hamiltonian.

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Computer-Aided Creation of Impossible Objects and Impossible Motions

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Abstract. There is a class of pictures called "pictures of impossible objects". These pictures generate optical illusion; when we see them, we have impression of solid objects and at the same time we feel that such solids cannot be realized. Although they are called "impossible", they are not necessarily impossible; some of them can be realized as real solids. This is because a single picture does not convey enough information about depth. Using the same trick, we can also design what may be called "impossible motions". That is, we can construct a shape of a solid which looks like an ordinary solid but which admits physical motions that look like impossible. A computer-aided method for creating impossible objects and impossible motions is presented with examples.

1 Introduction

Optical illusion is one of the most interesting research topics in visual psychology, and has been studied extensively [10,14]. However, there are a number of different types of optical illusion and unified study seems difficult; each type of optical illusion is studied individually. Among them, a class of pictures called "pictures of impossible objects" forms a special type of illusion in that it is related to three-dimensional structures while other optical illusions are primarily related to two-dimensional structures.

The pictures of impossible objects are the pictures which, when we see them, give us some ideas of three-dimensional object structures but at the same time give us the impression that they cannot be realized in the three-dimensional space. A typical old example of the impossible objects is the Penrose triangle [13] found in 1958.

In addition to the area of visual psychology, the impossible objects have also been studied from a mathematical point of view. One of the pioneers is Huffman, who characterized impossible objects from a viewpoint of computer interpretation of line drawings [11]. Clowes [2] also proposed a similar idea in a different manner. Cowan [3,4] and Térouanne [20] characterized a class of impossible objects that are topologically equivalent to a torus. Draper studied pictures of impossible objects through the gradient space [6]. Sugihara classified pictures of impossible objects according to his algorithm for interpretation of line drawings [15,17].

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Artists also use impossible objects as material for artistic work. One of the most famous examples is the endless loop of stairs drawn by Dutch artist M. C. Escher in his work titled "Klimmen en dalen (Ascending and descending)" [8]. Other examples include painting by Mitsumasa Anno [1] and drawings by Sandro del Prete [7], to mention a few.

While those activities are stories about two-dimensional pictures, several tricks have also been found for realization of impossible objects as actual three-dimensional structures. The first trick is to use curved surfaces for faces that look planar; Mathieu Hamaekers generated the Penrose triangle by this trick [7]. The second trick is to generate hidden gaps in depth; Shigeo Fukuda used this trick and generated a solid model of Escher's "Waterfall" [7].

This paper shows that some "impossible" objects can be realized as three-dimensional solids even if those tricks are not employed; in other words, "impossible" objects can be realized under the conditions that faces are made by planar (non-curved) polygons and that object parts are actually connected whenever they look connected in the picture plane. For example, Figure 1(a) shows a line drawing representing a V-shape solid with a horizontal bar. Exchanging the visible parts and hidden parts, we get the line drawing in (b), which belongs to the class of pictures of impossible objects. However, this impossible object can be realized as a solid in the three-dimensional space as shown in Figure 1(c).

Such solid models can generate visual illusions in the sense that although we are looking at actual objects, we feel that those objects can not exist.

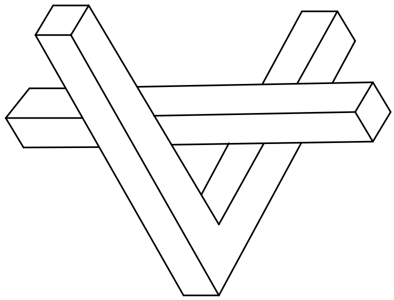
The same trick can also be used to generate a new class of visual illusion called "impossible" physical motions. The basic idea is as follows. Instead of pictures of impossible objects, we choose pictures of ordinary objects around us, and reconstruct solid models whose shapes are different from the original objects. The resulting solid models are unusual in their shapes although they look ordinary. Because of this gap between the perceived shape and the actual shape, we can add actual physical motions that look like impossible.

The organization of the paper is as follows. In Section 2, we review the basic method for judging the realizability of a solid from a given picture, and in Section 3, we review the robust method for reconstructing objects from pictures. In Section 4, we study algorithmic aspect of our robust method. We show examples of the three dimensional realization of impossible objects and impossible motions in Section 5, and give conclusions in Section 6.

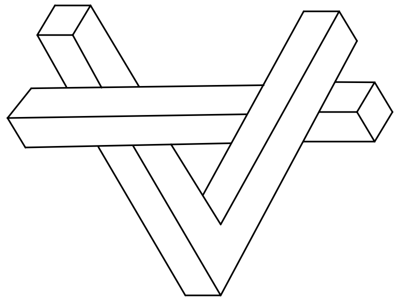
2 Ambiguity in Depths

In this section we briefly review the algebraic structure of the freedom in the choice of the polyhedron represented by a picture [17]. This gives the basic tool with which we construct our algorithm for designing impossible motions.

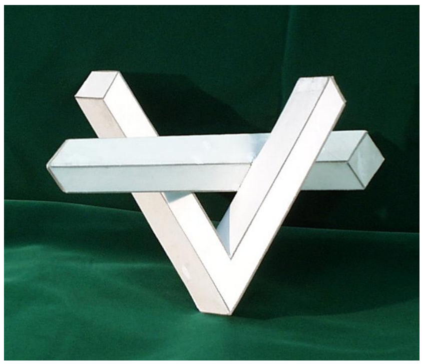
As shown in Figure 2, suppose that an (x, y, z) Cartesian coordinate system is fixed in the three-dimensional space, and a given polyhedral object P is projected by the central projection with respect to the center at the origin $\mathrm{O} = (0, 0, 0)$ onto the picture plane $z = 1$. Let the resulting picture be denoted by D. If the



(a)



(b)



(c)

Fig. 1. Three-dimensional realization of impossible V and bar: (a) ordinary picture; (b) picture of an impossible object; (c) solid model realizing the picture in (b).

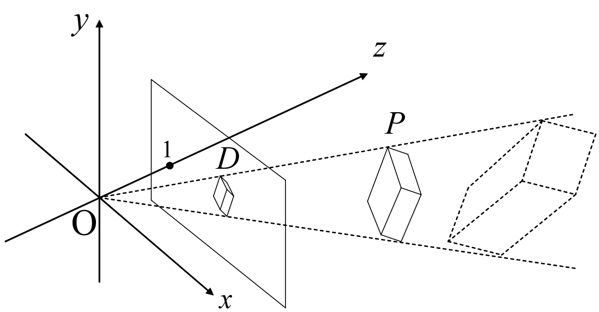


Fig. 2. Different solids can generate the same picture

polyhedron P is given, the associated picture D is uniquely determined. On the other hand, if the picture D is given, the associated polyhedron is not unique; there is large freedom in the choice of the polyhedron whose projection coincides with D. The algebraic structure of the degree of freedom can be formulated in the following way.

For a given polyhedron P, let m be the number of vertices, $V = \{v_1, v_2, \ldots, v_m\}$ be the set of all the vertices of P, n be the number of faces, $F = \{f_1, f_2, \ldots, f_n\}$ be the set of all the faces of P, and R be the set of all pairs (v_i, f_j) of vertices v_i $(\in V)$ and faces f_j $(\in F)$ such that v_i is on f_j. We call the triple $I = (V, F, R)$ the *incidence structure* of P.

Let (x_i, y_i, z_i) be the coordinates of the vertex v_i $(\in V)$, and let

$$a_j x + b_j y + c_j z + 1 = 0 \tag{1}$$

be the equation of the plane containing the face f_j $(\in F)$. The central projection $v_i' = (x_i', y_i', z_i')$ of the vertex v_i onto the picture plane $z = 1$ is given by

$$x_i' = x_i/z_i, \quad y_i' = y_i/z_i, \quad z_i' = 1. \tag{2}$$

Suppose that we are given the picture D and the incidence structure $I = (V, F, R)$, but we do not know the exact shape of P. Then, the coordinates of the projected vertices x_i' and y_i' are given constants, while z_i $(i = 1, 2, \ldots, m)$ and a_j, b_j, c_j $(j = 1, 2, \ldots, n)$ are unknown variables. Let us define

$$t_i = 1/z_i, \tag{3}$$

and treat t_i as unknown variable instead of z_i. Then, we get from (2)

$$x_i = x_i'/t_i, \quad y_i = y_i'/t_i, \quad z_i = 1/t_i. \tag{4}$$

Assume that $(v_i, f_j) \in R$. Then, the vertex v_i is on the face f_j, and hence

$$a_j x_i + b_j y_i + c_j z_i + 1 = 0 \tag{5}$$

should be satisfied. Substituting (4), we get

$$x_i' a_j + y_i' b_j + c_j + t_i = 0, \tag{6}$$

which is linear in the unknowns a_j, b_j, c_j and t_i because x_i' and y_i' are known constants.

Collecting the equations of the form (6) for all $(v_i, f_j) \in R$, we get the system of linear equations, which we denote by

$$A\boldsymbol{w} = 0, \tag{7}$$

where $\boldsymbol{w} = (t_1, \ldots, t_m, a_1, b_1, c_1, \ldots, a_n, b_n, c_n)$ is the vector of unknown variables and A is a constant matrix.

The picture D also gives us information about the relative depth between a vertex and a face. Suppose that a visible face f_j hides a vertex v_i. Then, f_j is nearer to the origin than v_i, and hence we get

$$x_i' a_j + y_i' b_j + c_j + t_i < 0. \tag{8}$$

If the vector v_i is nearer than the face f_j, then we get

$$x_i' a_j + y_i' b_j + c_j + t_i > 0. \tag{9}$$

Collecting all of such inequalities, we get a system of linear inequalities, which we denote by

$$B\boldsymbol{w} > 0, \tag{10}$$

where B is a constant matrix.

The linear constraints (7) and (10) specify the set of all possible polyhedron represented by the given picture D. In other words, the set of all $\boldsymbol{w}$'s that satisfy the equations (7) and the inequalities (10) represents the set of all possible polyhedrons represented by D. Actually the next property holds [17].

Property 1 (correctness of a picture). Picture D represents a polyhedron if and only if the system of linear equations (7) and inequalities (10) has a solution. ∎

Hence, to reconstruct a polyhedron from a given picture D is equivalent to choose a vector $\boldsymbol{w}$ that satisfies (7) and (10). (Refer to [17] for the formal procedure for collecting the equation (7) and the inequalities (10) and for the proof of this property.)

3 Removal of Superstrictness

As seen in the last section, we can characterize the set of all polyhedrons represented by a given picture in terms of linear constraints. However, these constraints are too strict if we want to apply them to actual reconstruction procedure. This can be understood by the next example.

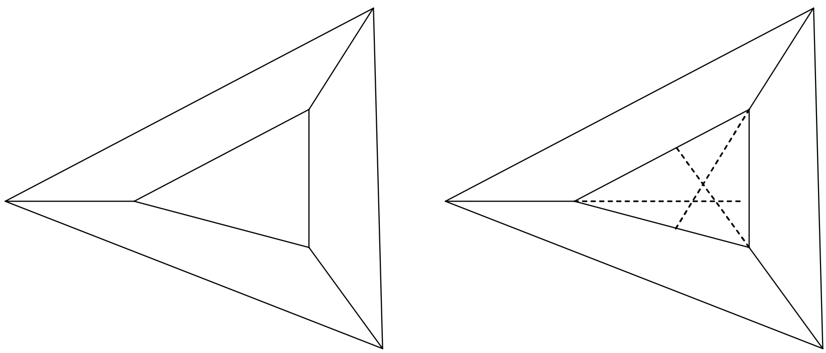
Consider the picture shown in Figure 3(a). We, human beings, can easily interpret this picture as a truncated pyramid seen from above. However, if we search by a computer for the vectors w that satisfy the constraints (7) and (10), the computer usually judges that the constraints (7) and (10) are not satisfiable and hence the picture in Figure 3(a) does not represent any polyhedron.

This judgment is mathematically correct because of the following reason.

Suppose that Figure 3(a) represents a truncated pyramid seen from above. Then, the three side faces should have a common point of intersection when they are extended. Since this point is also on the common edge of two side faces, it should also be the common point of intersection of the three side edges of the truncated pyramid. However, as shown by the broken lines in Figure 3(b), the three edges are not concurrent. Therefore, this picture is not a projection of any truncated pyramid. The object can be reconstructed only when we use curved faces instead of planar faces.

By this example, we can understand that the satisfiability of the constraints (7) and (10) does not give a practical solution of the problem of recognizing the reconstructability of polyhedra from a picture. Indeed, numerical errors cannot be avoided when the pictures are represented in a computer, and hence the picture of a truncated pyramid becomes almost always incorrect even if one carefully draws it in such a way that the three edges meet at a common point.

This superstrictness of the constraints comes from redundancy of the equations. When the vertices of the truncated pyramid were placed at strictly correct



(a) (b)

Fig. 3. Picture of a truncated pyramid: (a) a picture which we can easily interpret as a truncated pyramid; (b) incorrectness of the picture due to lack of the common point of intersection of the three side edges that should be the apex of the pyramid.

positions in the picture plane, the associated coefficient matrix A is not of full rank. When those vertices contain digitization errors, the rank of A increases and consequently the set of constraints (7) and (10) becomes inconsistent.

To make a practical method for judging the reconstructability of polyhedra, we should remove redundant equations from (10). For this purpose, the next property is helpful. Assume that a picture with the incidence structure $I = (V, F, R)$ is given. For subset $X \subset F$, let us define

$$V(X) \equiv \{v \in V \mid (\{v\} \times X) \cap R \neq \emptyset\}, \tag{11}$$

$$R(X) \equiv (V \times X) \cap R, \tag{12}$$

that is, $V(X)$ $(\subset V)$ denotes the set of vertices that are on at least one face in X, and $R(X)$ $(\subset R)$ denotes the set of incidence pairs (v, f) such that $f \in X$. For any finite set X, let $|X|$ denote the number of elements in X. Then the next property holds [17].

Property 2 (nonredundancy of equations). The set of equations (10) is nonredundant if and only if

$$|V(X)| + 3|X| \geq |R(X)| + 4 \tag{13}$$

for any subset $X \subset F$ such that $|X| \geq 2$. ∎

Refer to [Sugihara 1986] for the strict meaning of "nonredundant" and for the proof.

Let us see how this property works by an example. The picture in Figure 3(a) has 6 vertices and five faces (including the rear face) and hence $|V| + 3|F| = 21$. On the other hand, this picture has 2 triangular faces and 3 quadrilateral faces, and hence has $|R| = 2 \times 3 + 3 \times 4 = 18$ incidence pairs in total. Therefore, the inequality (13) is not satisfied and consequently we can judge that the associated equations are redundant. Property 2 also tells us that if we remove any one equation from (10), the resulting equation becomes nonredundant.

In this way, we can use this property to judge whether the given incidence structure generates redundant equations, and also to remove redundancy if redundant.

Using Properties 1 and 2, we can design a robust method for reconstructing a polyhedron from a given picture in the following way.

Suppose that we are given a picture. We first construct the equations (7) and the inequalities (10). Next, using Property 2, we judge whether (7) is redundant, and if redundant, we remove equations one by one until they become nonredundant. Let the resulting equations be denoted by

$$A'w = 0, \tag{14}$$

where A' is a submatrix of A obtained by removing the rows corresponding to redundant equations. Finally, we judge whether the system of (10) and (14) has solutions. If it has, we can reconstruct the solid model corresponding to an arbitrary one of the solutions. If it does not, we judge that the picture does not represent any polyhedron.

With the help of this procedure, Sugihara found that actual solid models can be reconstructed from some of pictures of impossible objects [18,19].

4 Efficient Recognition of Superstrictness

The next problem is how to check the condition in Property 2 efficiently. There are $\mathrm{O}(2^n)$ subsets of F, and hence if we check the inequality (13) for each subset one by one, it would take an exponential time. We have to construct an efficient method in order to use the property for practical use. To this goal, we change the inequality in the following way. Suppose that we are given an incidence structure $I = (V, F, R)$. For subset $Y \subset R$, let us define

$$V(Y) = \{v \in V \mid (\{v\} \times F) \cap Y \neq \emptyset\}, \tag{15}$$

$$F(Y) = \{f \in F \mid (V \times \{f\}) \cap Y \neq \emptyset\}. \tag{16}$$

That is, $V(Y)$ is the set of vertices included in at least one equation in Y, and $F(Y)$ is the set of faces included in at least one equation in Y.

Then, the next property holds [17].

Property 3. For an incidence structure $I = (V, F, R)$, the following (i) and (ii) are equivalent.

(i) the inequality (13) is satisfied for any $X \subset F$ such that $|X| \geq 2$.
(ii) the inequality

$$|Y| + 4 \leq |V(Y)| + 3|F(Y)| \tag{17}$$

is satisfied for any $Y \subset R$ such that $|F(Y)| \geq 2$. ∎

Refer to Sugihara [17] for the proof of this property.

Let us consider intuitive meaning of the inequality (17). The first term of the left-hand side of (17) represents the number of equations in Y, and the second term represents the minimum degree of freedom for us to reconstruct a solid object from its projection. Recall that we have to give the z values of three vertices to fix the plane passing them, and have to give at least one more z value of a vertex to specify the thickness of the solid.

On the other hand, the right-hand side of (17) represents the number of unknowns included in Y, because each vertex has one unknown and each face has three unknowns.

Therefore, the inequality (17) says that for any subset Y of equations related to two or more faces, Y should contain at least $|Y| + 4$ unknowns. This is a general condition for a system of equations with nonzero degrees of freedom to be structurally consistent.

Let $G = (R, V \cup F, W)$ be the bipartite graph having the left node set R and the right node set $V \cup F$ and the arc set $W \subset R \times (V \cup F)$ such that $(r, v), (r, f) \in W$ if and only if the equation $r \in R$ represents that the vertex v should be on the face f. From G we construct network N in the following way. As shown in Figure 4, we add source node s and sink node t in such a way that s is connected to all nodes in R by directed arcs with capacity 1, all nodes in V are connected to t by directed arcs with capacity 1, and all nodes in F are connected to t by directed arcs with capacity 3. Furthermore, we regard the arcs of G as directed arcs from R to $V \cup F$ with capacity ∞. We consider flow from s to t along arcs such that the amount of flow coming in is equal to that going out at every node in $R \cup V \cup F$ and the amount of flow along an edge does not exceed the associated capacity.

Then, the next property in well known [9,12].

Property 4. The maximum flow from s to t in the network N coincides with $|R|$ if and only if

$$|X| \leq |V(X)| + 3|F(X)| \tag{18}$$

is satisfied for any $X \subset R$. ∎

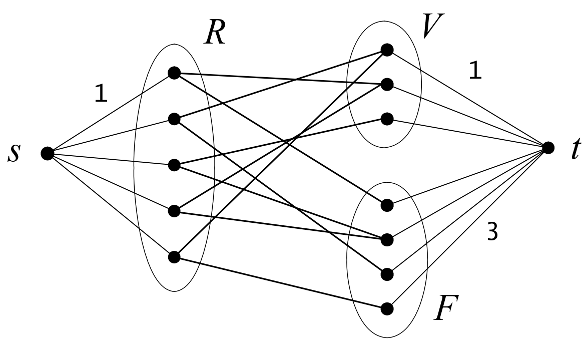


Fig. 4. Network associated with the incidence structure $I = (V, F, R)$

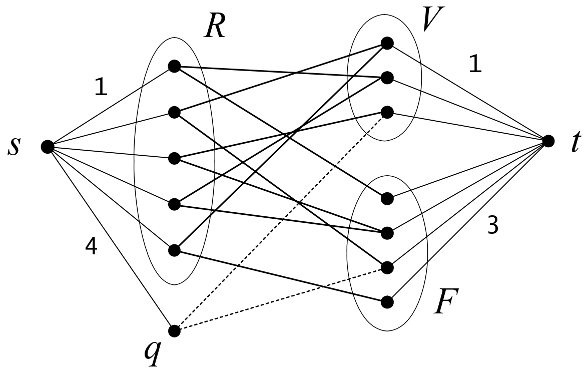


Fig. 5. Augmented network for checking the superstructures of a picture

Using maximum flow algorithms, we can construct the maximum flow of N efficiently, and hence can check the inequality (18) for all $X \subset R$ efficiently.

Note that our inequality (17) is similar to (18); the only difference is the term "4" in the left-hand side. In order to check the inequality (17), we modify the network N by adding one left node q and arc (s, q) with capacity 4, as shown in Figure 5. Next, we add two arcs from q, one to F and the other to $V \cup F$, as shown by broken lines in Figure 5. For every possible connection we construct the maximum flow and see if its value is equal to $|R| + 4$, and thus check the condition (ii) stated in Property 3 efficiently. Actually we can check it in $\mathrm{O}(n^2)$ time by some additional tricks. Refer to [16] for details.

Thus, we can check the superstrictness of the picture, and remove redundant equations from (17) efficiently. Combining this with Property 1, we can check the correctness of line drawings efficiently, and consequently can find realizable impossible objects.

5 Examples

The first examples of the realization of the impossible object shown in Figure 1(c) was constructed in the following manner. First, we constructed the system of equations (7) for the picture in Figure 1(b), then, removed redundant equations using Theorem 2 and got a non-redundant system (14) of equations. Next, we got a solution of eq. (14), which represents a specific shape of the three-dimensional solid. Finally, we computed the figure of the unfolded surface of this solid, and made a paper model by hands.

Figure 6 shows another view of this solid. As we can understand from this figure, the faces are not rectangular.

Figure 7(a) shows another example of an impossible object constructed in a similar manner. In this object, the near-far relations of the poles seem inconsistent; the left pole is nearer than the other on the lower floor while it is farther at the upper floor. This inconsistent structure is essentially similar to

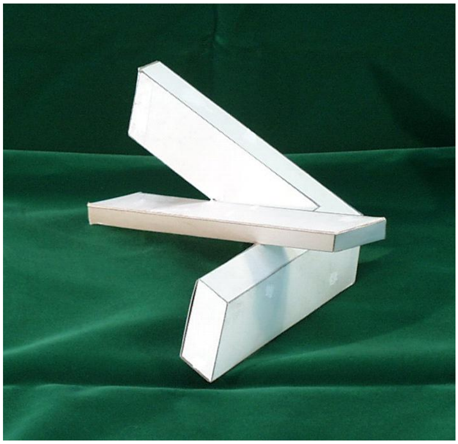
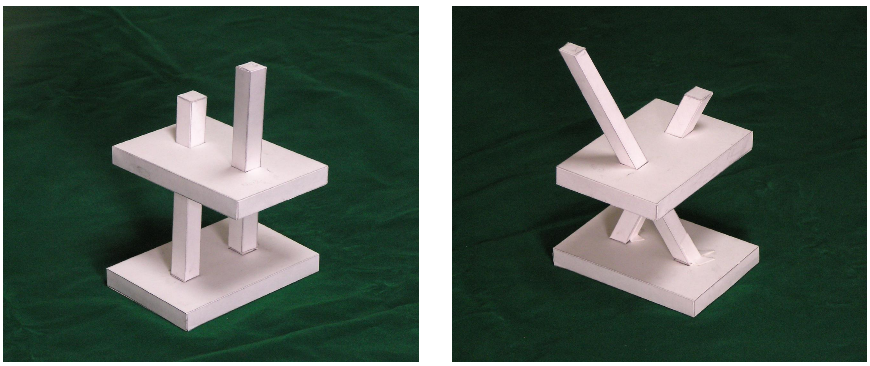


Fig. 6. The same object as shown in Figure 1(c) seen from a different viewpoint



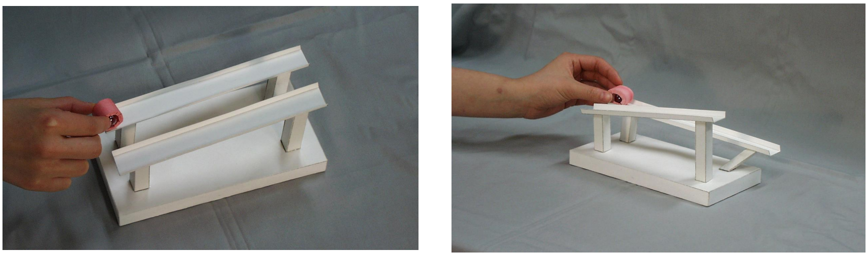
(a) (b)

Fig. 7. "Impossible" columns: (a) shows an impossible structure which is similar to Escher's lithograph "Belvédère"; (b) shows another view of the same solid.

that represented by Escher's lithograph "Belvédère" in 1958. Figure 5(b) shows the same solid seen from a different direction.

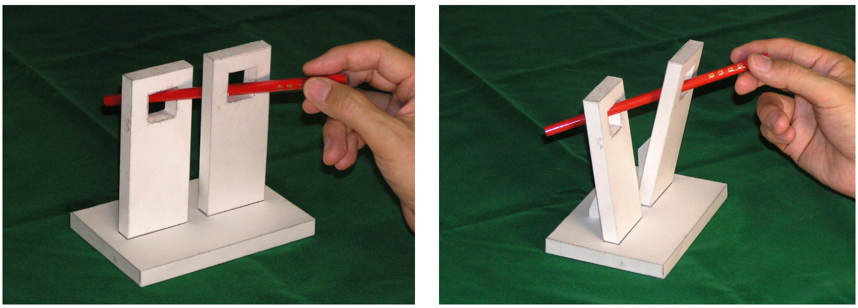
Next, let us consider "impossible" physical motions. A typical example of impossible motions is represented in Escher's lithograph "Waterval" in 1961, in which water is running uphill through the water path and is falling down at the waterfall, and is running uphill again. This motion is really impossible because otherwise an eternal engine could be obtained but that contradicts the physical law.

However, this impossible motion is realizable partially in the sense that material looks running uphill a slope. An example of this impossible motion is shown in Figure 8. Figure 8(a) shows a solid consisting of two parallel slopes, both of which go down from the right to the left. If we put a ball on the left edge of



(a) (b)

Fig. 8. Impossible motion of a ball along "Antigravity Parallel Slopes": (a) a ball climbing up a slope; (b) another view of the same situation.



(a) (b)

Fig. 9. "Soft Walls": (a) a straight bar passing throw the two windows in an unusual manner; (b) another view.

the farther slope, as shown in Figure 8(a), the ball moves climbing up the slope from the left to the right; thus the ball admits an impossible motion.

The actual shape of this solid can be understood if we see Figure 8(b), which is the photograph of the same solid as in Figure 8(a) seen from another direction. From this figure, we can see that actually the ball is just rolling down the slope according to the natural properties of the ball and the slope.

Still another example is shown in Figure 9. In this figure, there are two walls with windows, but a straight bar passes through them in an unusual way.

6 Concluding Remarks

We have presented a computer-aided method for creating "impossible" objects and "impossible" motions. We formulated the design of a solid admitting impossible objects and motions as a search for feasible solutions of a system of linear equations and inequalities. The resulting method enables us to realize impossible objects in the three-dimensional space.

The impossible objects and motions obtained by this method can offer a new type of optical illusion. When we see these objects and motions, we have a strange impression; we feel they are impossible although we are looking at them. One of our future work is to study this type of optical illusion from a view point of visual psychology.

Other problems for future include (1) to collect other variations of impossible objects and impossible motions, (2) to find criteria for choosing the best shape of an impossible solid among infinitely many possible shapes, and (3) to apply impossible objects to space art such as sculptures and buildings.

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The Hamiltonian Number of Cubic Graphs

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Abstract. A Hamiltonian walk in a connected graph G of order n is a closed spanning walk of minimum length in G. The Hamiltonian number $h(G)$ of a connected graph G is the length of a Hamiltonian walk in G. Thus h may be considered as a measure of how far a given graph is from being Hamiltonian. We prove that if G runs over the set of connected cubic graphs of order n and $n \neq 14$ then the values $h(G)$ completely cover a line segment $[a, b]$ of positive integers. For an even integer $n \geq 4$, let $\mathcal{C}(3^n)$ be the set of all connected cubic graphs of order n. We define $\min(h, 3^n) = \min\{h(G) : G \in \mathcal{C}(3^n)\}$ and $\max(h, 3^n) = \max\{h(G) : G \in \mathcal{C}(3^n)\}$. Thus for an even integer $n \geq 4$, the two invariants $\min(h, 3^n)$ and $\max(h, 3^n)$ naturally arise. Evidently, $\min(h, 3^n) = n$. The exact values of $\max(h, 3^n)$ are obtained in all situations.

Keywords: Hamiltonian walk, Hamiltonian number, cubic graph.

AMS Subject Classification: 05C12, 05C45

1 Introduction

If G is a connected graph of size m, there is always a closed spanning walk of length at most $2m$. Goodman and Hedetniemi [6] introduced the concept of a *Hamiltonian walk* in a connected graph G. It is defined as a closed spanning walk of minimum length in G. They denoted the length of a Hamiltonian walk of G by $h(G)$. Therefore, for a connected graph G of order $n \geq 3$, it follows that $h(G) = n$ if and only if G is Hamiltonian. Thus h may be considered as a measure of how far a given graph is from being Hamiltonian. Hamiltonian walks were studied further by Asano, Nishizeki, and Watanabe [2,3], Bermond [4], and Vacek [11]. Since every connected 2-regular graph is Hamiltonian, it is reasonable to investigate the Hamiltonian number in the class of cubic graphs. In [10], it was shown that all cubic graphs of order n for $n = 4, 6, 8$ are Hamiltonian. Let $P(k, m)$ be a generalized Petersen graph such that $V(P(k, m)) = \{u_i, v_i : i = 0, 1, \ldots, k-1\}$ and $E(P(k, m)) = \{u_i u_{i+1}, v_i v_{i+m}, u_i v_i : i = 0, 1, \ldots, k-1\}$ where addition is taken modulo k and $m \leq \frac{k}{4}$. Alspach [1] completed the determination of the parameters k, m for which $P(k, m)$ is Hamiltonian. He proved that the

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generalized Petersen graph $P(k,m)$ is non-Hamiltonian if and only if $m = 2$ and $k \equiv 5 (\bmod 6)$. We showed in [8] that $h(P(5,2)) = 11$ and $h(k,m) = 2k+1$ for all integers $k \equiv 5 (\bmod 6)$ and $m = 2$.

The following result is known in [5] and will be applied throughout the paper.

Theorem 1. *Let G be a connected graph and $B_1, B_2, \ldots, B_k$ be the blocks of G. Then $h(G) = \sum_{i=1}^{k} h(B_i)$.* □

2 Graph Notations

For a graph $G = (V, E)$ and $X \subseteq E$, by $G - X$ we denote the graph obtained from G by removing all edges of X. If $X = \{e\}$, we write $G - e$ for $G - \{e\}$. For a graph G and $X \subseteq V$, $G - X$ is the graph obtained by removing all vertices of X and all edges incident with these vertices from G. If $X \subseteq E(\overline{G})$, we denote $G + X$ the graph obtained from G by adding all edges from X. Similarly, if $X = \{e\}$, we simply write $G + e$ for $G + \{e\}$. Two graphs G and H are disjoint if $V(G) \cap V(H) = \emptyset$. For any two disjoint graphs G and H, we define their union, $G \cup H$, by $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. Since the operation "$\cup$" is associative, we can extend this definition to a finite union of pairwise disjoint graphs. If $G_1 \cong G_2 \cong \ldots \cong G_k \cong G$ and $G_1, G_2, \ldots, G_k$ are pairwise disjoint graphs, then we will use kG for $G_1 \cup G_2 \cup \ldots \cup G_k$. A *cubic graph* is a graph in which every vertex has degree 3. Let $\mathcal{C}(3^n)$ be the set of all connected cubic graphs of order n. It is well known that, for $n \geq 10$, $\mathcal{C}(3^n)$ can be partitioned as $\mathcal{C}_1(3^n) \cup \mathcal{C}_2(3^n) \cup \mathcal{C}_3(3^n)$, where $\mathcal{C}_1(3^n)$ is the set of all connected cubic graphs of order n containing a cut edge, $\mathcal{C}_2(3^n)$ is the set of all 2-connected cubic graphs of order n containing a set of two cut edges, and $\mathcal{C}_3(3^n)$ is the set of all 3-connected cubic graphs of order n. It is also well known that the edge-connectivity and the vertex-connectivity of a given cubic graph are equal.

The following graph constructions and notations will be used from now on.

1. Let G be a cubic graph and $v \in V(G)$. We denote $G * v$ to be the graph obtained from G by replacing v by a triangle, matching the vertices of the triangle to the former neighbors of v. Thus $G * v$ is also a cubic graph containing a triangle.

2. Let G be a cubic graph. We denote G^+ for a graph obtained from G by subdivision an edge of G. Thus K_4^+ is unique. When the subdivision vertex is specified, say u, we use $G(u)$ for a graph with $V(G(u)) = V(G) \cup \{u\}$ and $E(G(u)) = (E(G) - xy) \cup \{xu, uy\}$, where $xy \in E(G)$.

3. Let G and H be vertex disjoint cubic graphs. Let u, v be new vertices. We denote $\langle G(u), H(v) \rangle$ for a connected cubic graph of minimum order containing $G(u) \cup H(v)$ as its induced subgraph. Note that the graphs $G(u)$ and $H(v)$ are not unique but the graph $\langle G(u), H(v) \rangle$ is uniquely determined by $G(u)$ and $H(v)$.

4. Let G, H and K be pairwise vertex disjoint cubic graphs. Let x, y, z be new vertices. A connected cubic graph of minimum order containing $G(x) \cup$

$H(y) \cup K(z)$ as its induced subgraph is denoted by $\langle G(x), H(y), K(z) \rangle$. Note that the graph $\langle G(x), H(y), K(z) \rangle$ is uniquely determined by $G(x), H(y)$ and $K(z)$ and $|\langle G(x), H(y), K(z) \rangle| = |G(x)| + |H(y)| + |K(z)| + 1$. More generally, if $G_1, G_2, \ldots, G_k$ are pairwise disjoint graphs and for all $i = 1, 2. \ldots, k$, vertices of G_i are of degree 2 or 3 with at least one vertex of degree 2, then $\langle G_1, G_2, \ldots, G_k \rangle$ denotes a connected cubic graph of minimum order containing $G_1, G_2, \ldots, G_k$.

5. We use K_4^- for the graph obtained from K_4 by removing an edge.

3 Cycles in 2-Connected Cubic Graphs

A *factor* of a graph G is a spanning subgraph of G. A k-*factor* is a spanning k-regular subgraph. In particular, a 1-factor of a graph G is a 1-regular spanning subgraph of G and a 2-factor is a 2-regular spanning subgraph of G. Let G be a connected cubic graph containing a 1-factor F_1 and F be a graph with $V(F) = V(G)$ and $E(F) = E(G) - E(F_1)$. Then F is a 2-factor of G.

By a well-known theorem of Petersen [7], every 2-connected cubic graph G has a 2-factor. Thus if G is a 2-connected cubic graph, then the edge set of G can be partitioned into a 1-factor and a 2-factor. The following theorem due to Schönberger [9] and it is considered as an extension of the Petersen theorem.

Theorem 2. *Let G be a 2-connected cubic graph and $e \in E(G)$. Then G has a 1-factor containing e.* □

As a consequence of Theorem 2, if G is a 2-connected cubic graph and e, f are two incident edges of G, then G has a 2-factor containing both e and f.

In [10], it was shown that there are two 2-connected cubic graphs of order 6 and five 2-connected cubic graphs of order 8. All of those graphs are Hamiltonian. For an integer $m \geq 3$, let C_m denote an m-cycle, a cycle of order m.

Let G be a 2-connected cubic graph of order $n \geq 10$ and $F = \bigcup_{i=1}^{k} C_{p_i}$ be a 2-factor of G. If the girth of G is at least 5, then for each $i = 1, 2, \ldots, k$, $p_i \geq 5$.

Let $T = \{x, y, z\}$ be a triangle of a cubic graph G. Then T is called a *pure triangle* if x, y, z have no other neighbor in common. If G contains a pure triangle $T = \{x, y, z\}$, then we define G' to be the graph obtained from G by identifying the vertices x, y, z to a new vertex v, and joining v to the third neighbors of x, y and z. Thus G' is a 2-connected simple cubic graph of order $n - 2$.

Theorem 3. *Let G be a 2-connected graph of order $n \geq 6$. Then G has a 2-factor $F = \bigcup_{i=1}^{k} C_{p_i}$ of G such that for all $i = 1, 2, \ldots, k$, $p_i \geq 4$. Moreover, if $n = 4q$, for some integer q, then $k < \frac{n}{4}$.*

Proof. Let G be a 2-connected graph of order $n \geq 6$. The result of this theorem holds if the girth of G is at least 5. Thus we can assume that the girth of G is at most 4.

Suppose that G contains a pure triangle $T = \{x, y, z\}$. Let x', y', z' be the third neighbors of x, y, z, respectively. Then we define G' to be the graph obtained from G by identifying the vertices x, y, z to a new vertex v, matching v to the neighbors of x, y, z not in $V(T)$. Thus G' is 2-connected simple cubic graph of order $n-2$. By induction, there is a 2-factor $F' = \bigcup_{i=1}^{m} C_{q_i}$ of G' such that for all $i = 1, 2, \ldots, m$, $q_i \geq 4$. Let C' be the cycle in F' containing v and suppose without loss of generality that C' contains x', v, z' as its consecutive vertices. By replacing v by x, y, z yields a cycle C of G. Thus we obtain a 2-factor $F = (F' - \{C'\}) \cup \{C\}$ of G satisfying the desired property.

Since K_4^- consists of two triangles with a common edge, any triangle of K_4^- can not belong to any 2-factor of G. Therefore, if G does not contain a pure triangle, then any 2-factor of G is a union of cycles of length at least 4.

Suppose that $n = 4q$, for some integer q. Suppose further that G does not contain a pure triangle and all 2-factors of G is a union of 4-cycles. Let $F = \bigcup_{i=1}^{k} C_{q_i}$ be a 2-factor of G such that $q_1 = q_2 = \ldots = q_k = 4$. Let C be a 4-cycle in F, $V(C) = \{x, y, z, w\}$, $E(C) = \{xy, yz, zw, wx\}$ and x', y', z', w' are the third neighbors of x, y, z, w, respectively. If $xz \notin E(G)$ and $yw \notin E(G)$, then xx', yy', zz', ww' are independent and hence x', y', z', w' are pairwise distinct. By Theorem 2, let F' be a 2-factor of G containing $x'x, xy$. Then, by assumption that all cycles in F' are 4-cycles, either $\{x', x, y, y'\}$ or $\{x', x, y, z\}$ induces a 4-cycle in G. Since $x' \neq z'$, it follows that $\{x', x, y, y'\}$ induces a 4-cycle in G. Similarly, each of $\{z', z, w, w'\}, \{x', x, w, w'\}, \{y', y, z, z'\}$ also induces a 4-cycle in G. Thus G is a Hamiltonian graph of order 8. Suppose that all 4-cycles in F induce K_4^- in G. Then G is Hamiltonian. Therefore, G has a 2-factor $F = \bigcup_{i=1}^{k} C_{p_i}$ of G with $k < \frac{n}{4}$, as required. □

By using the result in Theorem 2, we obtain the following stronger result :

Theorem 4. *Let G be a 2-connected graph of order $n \geq 6$ and $e \in E(G)$. Then G has a 2-factor $F = \bigcup_{i=1}^{k} C_{p_i}$ such that for all $i = 1, 2, \ldots, k$, $p_i \geq 4$ and $e \notin E(F)$. Moreover, if $n = 4q$, for some integer q, then $k < \frac{n}{4}$.*

Proof. By Theorems 2 and 3, the result of this theorem holds where G does not contain a pure triangle as its subgraph. The result also holds if $n = 6, 8$. Suppose that G contains a pure triangle $T = \{x, y, z\}$, $n \geq 10$, and let G' be the graph as described earlier. Thus G' is of order $n-2$. By induction, for every edge $e \in G'$, G' has a 2-factor $F' = \bigcup_{i=1}^{m} C_{q_i}$ such that for all $i = 1, 2, \ldots, m$, $q_i \geq 4$ and $e \notin E(F)$. Let C' be the cycle in F' containing v. If $e \notin \{vx', vy', vz'\}$, then, without loss of generality, we may assume that y', v, z' are as consecutive vertices in C'. Let C be a cycle obtained from C' by replacing v by y, x, z. Thus $F = (F' - \{C'\}) \cup \{C\}$ is a 2-factor of G with the desired property. If $e \in \{vx', vy', vz'\}$, say $e = vx'$, then y', v, z' are as consecutive vertices on C'. Let C be a cycle obtained from C' by replacing v by y, x, z. Thus $F = (F' - \{C'\}) \cup \{C\}$ is a 2-factor of G with the desired property. If $e \in \{xy, yz, xz\}$, say $e = yz$, then there exists a 2-factor $F' = \bigcup_{i=1}^{m} C_{q_i}$ of G' for all $i = 1, 2, \ldots, m$, $q_i \geq 4$, $vx' \notin E(F')$. Thus there exists a cycle C' in F' such that C' contains y', v, z' as its consecutive vertices. Thus we can extend F' to a 2-factor F of G as described above. Thus the proof is complete. □

4 The Range of the Hamiltonian Numbers

For an even integer $n \geq 4$, we have already denoted that $h(3^n) = \{h(G) : G \in \mathcal{C}(3^n)\}$. We put $\min(h, 3^n) = \min\{h(G) : G \in \mathcal{C}(3^n)\}$ and $\max(h, 3^n) = \max\{h(G) : G \in \mathcal{C}(3^n)\}$. It is well-known that for any even integer $n \geq 4$, there exists a Hamiltonian cubic graph of order n. Thus $\min(h, 3^n) = n$. It is also well-known that $\max(h, 3^n) = \min(h, 3^n) = n$ if and only if $n = 4, 6, 8$. For $n = 10, 12$, we have $\max(h, 3^n) = n + 2$. Observe that for $n = 10, 12$, a connected cubic graph G having $h(G) = \max(h, 3^n)$ is a graph with a cut edge. The problem of determining $\max(h, 3^n)$ is more challenging. The following facts are useful.

1. If a connected graph G contains an edge e such that $G - e$ is connected, then $h(G) \leq h(G - e)$.

2. If G is a 2-connected cubic graph of order n and $F = \bigcup_{i=1}^{k} C_{p_i}$ is a 2-factor of G, then there exists a set $X = \{e_1, e_2, \ldots, e_{k-1}\} \subseteq E(G) - E(F)$ such that $F + X$ is connected. Thus $h(G) \leq h(F + X) = n + 2(k - 1)$. In particular, if G is a 2-connected cubic graph of order n, then, by Theorem 3, $h(G) \leq n + 2(k - 1)$, where $k \leq \lfloor \frac{n-2}{4} \rfloor$.

3. For an integer $n \geq 10$, let $H = \langle (q-2)K_4^-, 2K_4^+ \rangle$ if $n = 4q + 2$ and $H = \langle (q-2)K_4^-, K_4^+, K \rangle$ if $n = 4(q+1)$, where K is a graph obtained from a cubic graph of order 6 and a subdivision of an edge. Then, by Theorem 1, $h(H) = n + 2(q - 1)$.

As a consequence of above observations, we obtain the following lemma.

Lemma 1. *Let G be a 2-connected cubic graph of order $n \geq 10$. Then there exists a graph $H \in \mathcal{C}_1(3^n)$ such that $h(G) \leq h(H)$.*

Proof. By using above observation and Theorem 3, we have $k \leq q$. Thus $h(G) \leq n + 2(k - 1) \leq n + 2(q - 1) = h(H)$. Note that $H \in \mathcal{C}_1(3^n)$. □

By Lemma 1 and Theorem 4, we obtain the following result.

Corollary 1. *Let G be a 2-connected cubic graph of order $n \geq 10$ and $e = xy \in E(G)$. Then there exist a closed spanning walk W of G such that x, y appear as consecutive vertices on W exactly once and a graph $H \in \mathcal{C}_1(3^n)$ such that $|W| \leq h(H)$.* □

Thus in order to find the value of $\max(h, 3^n)$, by Lemma 1, it is enough to seek for a connected cubic graph $G \in \mathcal{C}_1(3^n)$ with $h(G) = \max(h, 3^n)$. Let $G \in \mathcal{C}_1(3^n)$ with $h(G) = \max(h, 3^n)$. Let $B_1, B_2, \ldots, B_k$ be the blocks of G with $|B_i| \geq 3$, for all $i = 1, 2, \ldots, k$. Thus there are $k - 1$ blocks of G of order 2. A block of G of order 2 is called *trivial*, otherwise, it is called *nontrivial.* A nontrivial block B is called a *leaf block* if B has exactly one vertex of degree 2. It is clear that G has at least two leaf blocks. Thus by a nontrivial block B, we mean in this paper, a 2-connected graph of order at least 3, $2 \leq d(x) \leq 3$, for all $x \in V(B)$, and B contains at least one vertex of degree 2. Similarly, by a leaf block, we mean a nontrivial block with exactly one vertex of degree 2.

Lemma 2. *Let B be a leaf block of order b and $5 \leq b \leq 9$. Then B is Hamiltonian.*

Proof. Since B is a leaf block of order b, b is odd and $b \geq 5$. Let v be the vertex of degree 2 of B. Then B can be pictured as in Figure 1. Let $B_1 = B_1' + u_r v_r$. Then B_1 is a 2-connected cubic graph of order at most 8 and hence B_1 Hamiltonian. By Theorem 2, B_1 has a Hamiltonian cycle C' containing $u_r v_r$. The cycle C' can be extended to a Hamiltonian cycle C of B. Thus B is Hamiltonian. □

Let B be a nontrivial block. If $W : x_1, x_2, \ldots, x_{t+1} = x_1$ is a closed spanning walk of B and $e = xy \in E(B)$, then we say that x, y appear as consecutive vertices in W if there exists i, $1 \leq i \leq t$, such that $\{x, y\} = \{x_i, x_{i+1}\}$. If x, y, x appear in W as consecutive vertices, then we say that x, y appear twice in W.

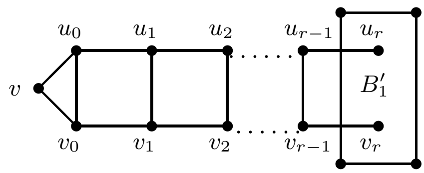

Fig. 1.

Lemma 3. *Let B be a leaf block of order $b \geq 11$. Then there exist a closed spanning walk $W : x_1, x_2, \ldots, x_{t+1} = x_1$ of B and a connected graph H with a cut edge such that B and H are of the same degree sequence, and $|W| \leq h(H)$.*

Proof. Let B be a leaf block of G of order $b \geq 11$. Thus B can be pictured as shown in Figure 1. If B is Hamiltonian of order $b \geq 11$, then we can choose a Hamiltonian walk W of B and a graph H obtained from a connected cubic graph of order $b-1$ with a cut edge e and a subdivision to the edge e. Thus $|W| \leq h(H)$ and H and B are of the same degree sequence. Suppose that B is not Hamiltonian. Let $B_1 = B_1' + u_r v_r$. Thus B_1 is 2-connected cubic graph of order $b_1 \leq b-1$. If $b_1 \leq 8$, then B is Hamiltonian and the result follows. Suppose that $b_1 \geq 10$. By Corollary 1, there exists a closed spanning walk W_1 of B_1 such that u_r, v_r appear as consecutive vertices in W_1 by exactly once and a graph $H_1 \in \mathcal{C}_1(3^n)$ such that $|W_1| \leq h(H_1)$, where B_1 and H_1 are of the same degree sequence. Thus W_1 and H_1 can be easily extended to W and H with desired result. Thus the proof is complete. □

For a 2-connected cubic graph G, we have constructed a closed spanning walk W of G traveling along the cycles in a given 2-factor of G and $k-1$ edges connecting between k cycles in the 2-factor. It turns out that for every edge on the k cycles appears exactly once in W while every edge that connects to the cycles appears exactly twice in W. Lemma 2 showed similar result for a leaf block.

Let B be a block of order $b \geq 3$. If $b = 3, 4$, then B is Hamiltonian. If $b = 5$ and B is not Hamiltonian, then $B \cong K_{2,3}$ and $h(B) = 6$. Note that the graph $K_{2,3}$ has a property that for every edge $e = xy$, there is a Hamiltonian walk containing x, y as its consecutive vertices exactly once.

If $b \in \{6, 8\}$, B contains exactly two vertices of degree 2, and the two vertices x, y are not adjacent, then the graph $B' = B + xy$ is a cubic graph of order b. Thus B' is Hamiltonian. By Theorem 4, B' has a Hamiltonian cycle not containing xy. Thus B is Hamiltonian. Suppose that the two vertices are adjacent. Let B' be the graph obtained from B by removing the two vertices of degree 2. Thus $B' \cong K_4^-$ if $b = 6$, and $B' \cong K$ if $b = 8$, where K is obtained from a cubic graph of order 6 by removing one edge. Thus B' is Hamiltonian and consequently B is Hamiltonian.

If $b \in \{7, 9\}$ and B contains exactly three vertices of degree 2, then the three vertices are not pairwise adjacent. Thus there are two vertices x, y of degree 2 in B that are not adjacent and $B' = B + xy$ is a leaf block of order b. By Lemma 2, B' is Hamiltonian.

Theorem 5. *Let B be a nontrivial block of order $b \geq 6$. Then B is Hamiltonian or there exist a closed spanning walk $W : x_1, x_2, \ldots, x_{t+1} = x_1$ of B and a connected graph H with a cut edge satisfying the following conditions.*

1. *For every edge xy of B, x, y appear as its consecutive vertices in W by at most twice,*

2. *H and B are of the same degree sequence, and*

3. *$t \leq h(H)$.*

Proof. Let B be a nontrivial block of order $b \geq 6$. Suppose that B is a non-Hamiltonian graph. If $b = 6$, then B contains exactly four vertices of degree 2 and $h(B) = 7$. Let H be a graph of order 6 obtained from two disjoint triangles and one edge connecting from one vertex of a triangle to one vertex of the other triangle. Thus H contains a cut edge and $h(H) = 8$. Thus the result follows if $b = 6$. Suppose $b \geq 7$, B can be pictured as in Figure 1. Let B' be the subgraph of B induced by $\{v, u_0, v_0, u_1, v_1, \ldots, u_{r-1}, v_{r-1}\}$. Thus $V(B) = V(B') \cup V(B_1')$ and $E(B) = E(B') \cup E(B_1') \cup \{u_{r-1}u_r, v_{r-1}v_r\}$. Let $B_1 = B_1' + u_r v_r$. Then B_1 is a block of order at most $b - 1$. If B_1 is of order 5, then either B_1 is Hamiltonian or $B_1 \cong K_{2,3}$. Since every block of order 5 has a property that every edge is contained in its closed spanning walk of length 6, it follows that $h(B) \leq b + 1$. It is easy to construct a graph H with the desired property. Suppose that B_1 is of order at least 6. By induction, there exist a closed spanning walk W_1 of B_1 in which for every edge xy of B_1, x, y appear as its consecutive vertices at most twice and a connected graph H_1 with an edge cut such that $|W_1| \leq h(H_1)$, where B_1 and H_1 are of the same degree sequence. We now construct a closed spanning walk W of B according to the following properties of W_1.

1. If u_r, v_r appear exactly once as consecutive vertices on W_1, then we obtain a closed spanning walk W of B from W_1 by replacing u_r, v_r by $u_r, u_{r-1}, \ldots, u_0, v, v_0, v_1, \ldots, v_{r-1}, v_r$. Thus $|W| = |W_1| - 1 + 2r + 2 = |W_1| + 2r + 1$.

2. If u_r, v_r appear exactly twice as consecutive vertices on W_1, then we obtain a closed spanning walk W of B from W_1 by replacing u_r, v_r by $u_r, u_{r-1}, \ldots, u_0, v, v_0,$

$v_1, \ldots, v_{r-1}, v_r$ for the first pair and replace u_r, v_r by $u_r, u_{r-1}, v_{r-1}, v_r$ for the second pair. Thus $|W| = |W_1| - 1 + 2r + 2 - 1 + 3 = |W_1| + 2r + 3$.

3. If u_r, v_r, u_r appear as consecutive vertices on W_1, then we obtain a closed spanning walk W of B from W_1 by replacing u_r, v_r, u_r by $u_r, u_{r-1}, \ldots, u_0, v, v_0, v_1, \ldots, v_{r-1}, v_r, v_{r-1}, u_{r-1}, u_r$. Thus $|W| = |W_1| - 1 + 2r + 2 - 1 + 3 = |W_1| + 2r + 3$.

4. If u_r, v_r do not appear as consecutive vertices on W_1, then we obtain a closed spanning walk W of B from W_1 by replacing u_r by $u_r, u_{r-1}, \ldots, u_0, v, v_0, v_1, \ldots, v_{r-1}, u_{r-1}, u_r$. Thus $|W| = |W_1| + 2r + 2 + 1 = |W_1| + 2r + 3$.

Thus W is a closed spanning walk of B and $|W| \leq |W_1| + 2r + 3$ and for every edge $e = xy \in E(B)$, x, y appear as consecutive vertices in W at most twice. On the other hand, let B' be a graph with $V(B') = V(B) - V(B_1)$ and $E(B') = E(B) - E(B_1)$. Since H_1 contains a cut edge $e = pq$, it follows that a graph H, where $V(H) = V(B') \cup V(H_1)$ and $E(H) = E(B') \cup E(H_1 - e) \cup \{u_{r-1}p, v_{r-1}q\}$, contains a cut edge. Thus $h(H) = h(B') + h(H_1) - 2 + 4 = h(H_1) + 2r + 1 + 2 = h(H_1) + 2r + 3 \geq |W|$. Thus the proof is complete. □

Let $G \in \mathcal{C}_1(3^n)$ such that $n \geq 14$ and $h(G) = \max(h, 3^n)$. Then, by Lemmas 2 and 3, there exists a leaf block of G that is isomorphic to K_4^+, and by Theorem 5, all nontrivial blocks of G are $K_{2,3}$ or Hamiltonian. The following lemma is easily obtained.

Lemma 4. $\max(h, 3^{14}) = 18$, $\max(h, 3^{16}) = 21$ *and* $\max(h, 3^{18}) = 24$.

Proof. We now construct graphs G_{14}, G_{16} and G_{18} with $h(G_{14}) = 18$, $h(G_{16}) = 21$ and $h(G_{18}) = 24$. Let $G \in \mathcal{C}_1(3^{14})$ such that $h(G) = \max(h, 3^{14})$ and G contains K_4^+ as a leaf block. Thus $G = \langle K_4^+, G_1 \rangle$, where G_1 is a connected graph of order 9 containing 8 vertices of degree 3 and a vertex of degree 2. If G_1 is 2-connected, then G_1 is Hamiltonian and hence $h(G) = 16 < 18 = h(G_{14})$. Thus G_1 contains a cut edge. Therefore $G \cong G_{14}$. Thus $\max(h, 3^{14}) = 18$. Let $G \in \mathcal{C}_1(3^{16})$ and $h(G) = \max(h, 3^{16})$. If G has only two trivial blocks, then $h(G) \leq 16 + 4 = 20 < 21 = h(G_{16})$. Thus G must have at least three trivial blocks. Since the order of G is 16 and G has at least two leaf block of order 5, $G \cong G_{16}$. Let $G \in \mathcal{C}_1(3^n)$ and $h(G) = \max(h, 3^n)$. If G contains a nontrivial block of order 3, then the block is a pure triangle. We can form a new graph G' of order $n - 2$ by identifying the triangle to a new vertex, matching this new vertex to the former neighbors of the triangle and $h(G) = h(G') + 3$. Let $G \in \mathcal{C}_1(3^{18})$ and $h(G) = \max(h, 3^{18})$. Thus if G has a block of order 3, then $G \cong G_{18}$ and $h(G) = 24$. If G does not have a nontrivial block of order 3, then $G \cong \langle 2K_4^+, 2K_4^- \rangle$. Thus $\max(h, 3^{18}) = 24$. □

Let $G \in \mathcal{C}_1(3^n)$ and $e = xy$ be a cut edge of G. A graph $G(K_4(v))$ is a graph obtained from G and $K_4(v)$ by deleting xy and adding edges xz, zy, zv, where z is a new vertex. By Theorem 1, it follows that $h(G(K_4(v)) = h(G) + 9$. Note that the graph $G(K_4(v)) \in \mathcal{C}_1(3^{n+6})$. Let $G_{14} = \langle 2K_4^+, K_4^- \rangle, G_{16} = \langle 3K_4^+ \rangle$ and $G_{18} = \langle 3K_4^+, K_3 \rangle$. Thus G_{14}, G_{16} and G_{18} are connected cubic graphs of order

14, 16, and 18, respectively, and $h(G_{14}) = 18, h(G_{16}) = 21$ and $h(G_{18}) = 24$. Note that each of the graphs G_{14}, G_{16} and G_{18} contains a cut edge. Thus for an integer $n_i = 14 + 2i$, $i \geq 3$, a graph $G_{n_i} = G_{n_{i-3}}(K_4(v)) \in \mathcal{C}_1(3^{n_i})$ and $h(G_{n_i}) = 18 + 3i$. Thus $\max(h, 3^{14+2i}) \geq 18 + 3i$, for all non-negative integer i. We show in the next theorem that the graph G_{n_i} satisfies $h(G_{n_i}) = \max(h, 3^{n_i})$.

Theorem 6. *Let i be a non-negative integer and $n_i = 14 + 2i$. Then* $\max(h, 3^{14+2i}) = 18 + 3i$.

Proof. We have already mentioned that $\max(h, 3^{14+2i}) \geq 18 + 3i$, for all non-negative integer i. We have also obtained $\max(h, 3^{14+2i}) = 18+3i$, for $i = 0, 1, 2$. Suppose $i \geq 3$ and $G \in \mathcal{C}_1(3^{14+2i})$ with $h(G) = \max(h, 3^{14+2i})$. Let B be a nontrivial block of G of order b which is not a leaf block and B is of minimum order. The following five cases are considered.

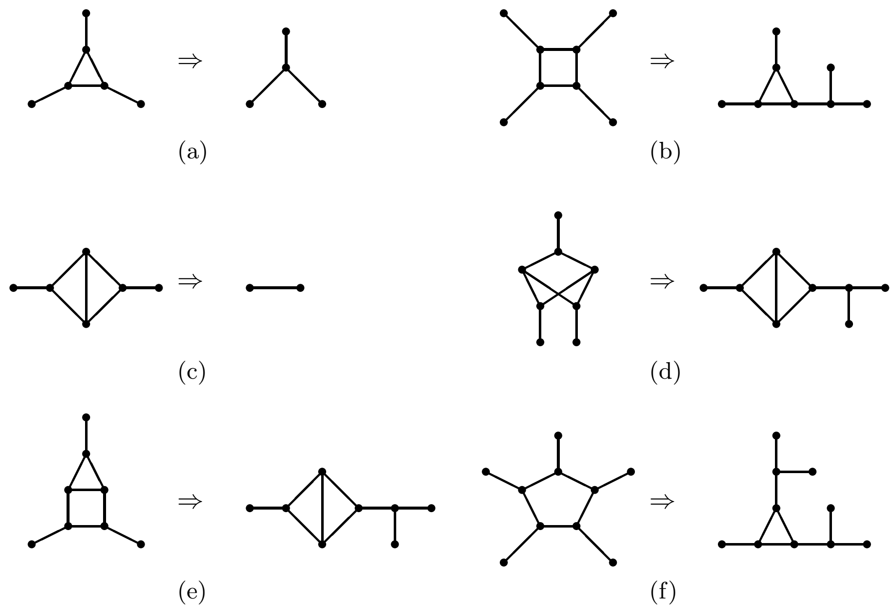

Fig. 2.

Case 1. If $b = 3$, then B is a pure triangle. Let G' be a graph obtained from G by identifying B to a new vertex v, matching v to the former neighbors of B (see Figure 2(a)). Clearly $G' \in \mathcal{C}_1(3^{14+2(i-1)})$ and by induction, $h(G) = h(G') + 3 \leq 18 + 3(i-1) + 3 = 18 + 3i$.

Case 2. If $b = 4$ and the induced subgraph of B in G is a 4-cycle, then we can replace B by a graph in Figure 2(b) and the result follows by Case 1.

Case 3. If $b = 4$ and the induced subgraph of B in G is a K_4^-, then let G' be a graph obtained from G by deleting the K_4^- and connecting the two neighbors of K_4^- (see Figure 2(c)). Then $G' \in \mathcal{C}_1(3^{14+2(i-2)})$ and by induction, $h(G) = h(G') + 6 \leq 18 + 3(i-2) + 6 = 18 + 3i$.

Case 4. If $b = 5$, then B is one of the graphs shown in Figure 2((d), (e), (f)) and the result follows by Case 1, Case 2 or Case 3.

Case 5. If $b \geq 6$, then, by Theorem 5, B is Hamiltonian. By an arrangement of blocks of G, we may assume that B is adjacent to a leaf block K_4^+. Let v be a vertex of degree 2 of B such that v is adjacent to the vertex of degree 2 of K_4^+. Since $b \geq 6$, we may assume that the neighbors x, y of v in B are not adjacent. Let G' be the graph obtained from G by removing $V(K_4^+) \cup \{v\}$ and adding an edge xy. Thus $G' \in \mathcal{C}_1(3^{14+2(i-3)})$. Since $h(G) = h(G') + 1 + 7 = h(G') + 8$, by induction, $h(G) = h(G') + 8 \leq 18 + 3(i-3) + 8 < 18 + 3i$. Thus the proof is complete. □

Recall that the range of the Hamiltonian numbers of connected cubic graphs of order n is $h(3^n) = \{h(G) : G \in \mathcal{C}(3^n)\}$. It is easy to obtain $h(3^n)$ for small values of n. In other words, $h(3^n) = \{n\}$ if and only if $n = 4, 6, 8$ and $h(3^n) = \{n, n+1, n+2\}$ if and only if $n = 10, 12$. By using Theorems 1, 6 and Lemma 4, it is not difficult to show that $h(3^{14}) = \{14, 15, 16, 18\}$, $h(3^{16}) = \{16, 17, 18, 19, 20, 21\}$, $h(3^{18}) = \{18, 19, 20, 21, 22, 23, 24\}$ and $h(3^{20}) = \{20, 21, 22, 23, 24, 25, 26, 27\}$. Observe that for even integers n, $4 \leq n \leq 20$, $h(3^n)$ completely covers all integers from $\min(h, 3^n)$ to $\max(h, 3^n)$, except $n = 14$.

Let j be a non-negative integer. A connected graph G of order n is called a j-*Hamiltonian graph* if $h(G) = n + j$. Thus a 0-Hamiltonian graph is Hamiltonian. A 1-Hamiltonian graph is called an *almost Hamiltonian graph*. We proved in [8] that for an even integer $n \geq 10$, there exists an almost Hamiltonian cubic graph of order n. Furthermore, we proved that an almost Hamiltonian cubic graph is 2-connected. Thus for an even integer $n \geq 10$, $\{n, n+1\} \subseteq h(3^n)$.

Let n be an even integer and $n \geq 10$ and H be a Hamiltonian cubic graph of order $n - 6$. Then $G = \langle K_4(u), H(v)\rangle$ exists and $h(G) = n + 2$. Thus for an even integer $n \geq 10$, $\{n, n+1, n+2\} \subseteq h(3^n)$.

Let n be an even integer where $n \geq 16$. Consider the following elementary facts.

1. The graph $G = \langle P(u), K(v)\rangle$ satisfies $h(G) = n + 3$, where P is the Petersen graph of order 10 and $K(v)$ is a Hamiltonian graph of order $n - 11$ obtained from a Hamiltonian cubic graph K of order $n - 12$. Thus $\{n, n+1, n+2, n+3\} \subseteq h(3^n)$.

2. The graph $G = \langle 2K_4^+, K - e\rangle$ satisfies $h(G) = n + 4$, where K is a Hamiltonian cubic graph of order $n - 10$ and $e \in E(K)$. Thus $\{n, n+1, n+2, n+3, n+4\} \subseteq h(3^n)$.

3. The graph $G = \langle 2K_4^+, K(v)\rangle$ satisfies $h(G) = n + 5$, where $K(v)$ is a Hamiltonian graph of order $n - 11$ obtained from a Hamiltonian cubic graph K of order $n - 12$. Thus $\{n, n+1, n+2, n+3, n+4, n+5\} \subseteq h(3^n)$.

4. Since $\max(h, 3^{14+2i}) = 18 + 3i$, the maximum value of j of which there exists a j-Hamiltonian graph in $\mathcal{C}(3^{14+2i})$ is $4 + i$.

5. If $j = 4 + i$, then there exists $G \in \mathcal{C}_1(3^{14+2i})$ such that G is a j-Hamiltonian graph in which all nontrivial blocks are Hamiltonian.

6. If $6 \leq j \leq 4 + i$ and $G \in \mathcal{C}_1(3^{14+2i})$ such that G is a j-Hamiltonian graph containing a Hamiltonian leaf block B and $v \in V(B)$, then $G * v \in \mathcal{C}_1(3^{14+2(i+1)})$ and $G * v$ is a j-Hamiltonian graph with a Hamiltonian leaf block.

We have the following lemma.

Lemma 5. *If $n \geq 10$ and $\mathcal{C}(3^n)$ contains a j-Hamiltonian graph, then $\mathcal{C}(3^{n+2})$ contains a j-Hamiltonian graph.* □

By Lemma 5, if $h(3^n) + 2 = \{h(G) + 2 : G \in \mathcal{C}(3^n)\}$, then $h(3^n) + 2 \subseteq h(3^{n+2})$. Thus by Theorem 6, we have the following theorem.

Theorem 7. *For an even integer $n \geq 4$ and $n \neq 14$. There exists an integer b such that $h(3^n) = \{k \in \mathbb{Z} : n \leq k \leq b\}$. Moreover, an explicit formula for the integer b is given by the following.*

1. *$b = n$ if and only if $n = 4, 6, 8$.*

2. *$b = n + 2$ if and only if $n = 10, 12$.*

3. *If $n = 14 + 2i$ and $i \geq 1$, then $b = 18 + 3i$.* □

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SUDOKU Colorings of the Hexagonal Bipyramid Fractal

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Abstract. The hexagonal bipyramid fractal is a fractal in three dimensional space, which has fractal dimension two and which has six square projections. We consider its 2nd level approximation model, which is composed of 81 hexagonal bipyramid pieces. When this object is looked at from each of the 12 directions with square appearances, the pieces form a 9×9 grid of squares which is just the grid of the SUDOKU puzzle. In this paper, we consider colorings of the 81 pieces with 9 colors so that it has a SUDOKU solution pattern in each of the 12 appearances, that is, each row, each column, and each of the nine 3×3 blocks contains all the 9 colors in each of the 12 appearances. We show that there are 140 solutions modulo change of colors, and, if we identify isomorphic ones, we have 30 solutions. We also show that SUDOKU coloring solutions exist for every level $2n$ approximation models ($n \geq 1$).

1 Introduction

The Sierpinski tetrahedron is a well-known fractal in three-dimensional space. When A is a regular tetrahedron with the vertices c_1, c_2, c_3, c_4, it is defined as the fixed point of the iteration function system (IFS) $\{f_1, f_2, f_3, f_4\}$ with f_i the dilation with the ratio $1/2$ and the center c_i ($i = 1, 2, 3, 4$). It is self-similar in that it is equal to the union of four half-sized copies of itself, and it is two dimensional with respects to fractal dimensions like the similarity dimension and the Hausdorff dimension. We refer the reader to [3] and [4] for the theory of fractals.

Fig. 1 shows some computer graphics images of the Sierpinski tetrahedron. As Fig. 1 (b) shows, it has a solid square image when projected from an edge, and there are three orthogonal directions in which the projection images become square. Here, we count opposite directions once. It is true both for the mathematically-defined pure Sierpinski tetrahedron and for its level n approximation model, which is obtained by applying the IFS n times starting with the tetrahedron A, and composed of 4^n regular tetrahedrons ($n \geq 0$).

While studying about generalizations of the Sierpinski tetrahedron, the author found a fractal in three-dimensional space which has *six* square projections [2]. This fractal and its finite approximation models are shown in Fig. 2. We start with a hexagonal bipyramid Fig. 2 (A, B, C). This dodecahedron is the intersection of a cube with its 60-degree rotation along a diagonal, and each of

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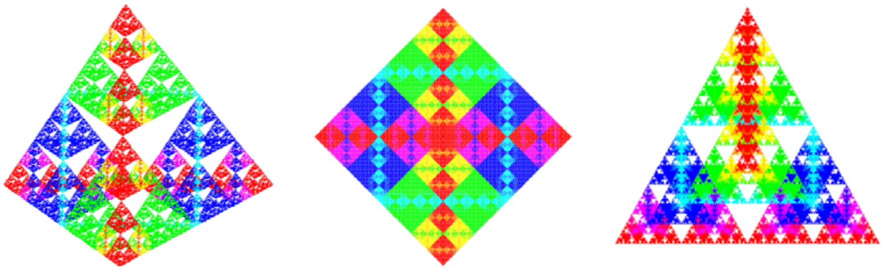


(A) (B) Edge view (C) Vertex view

Fig. 1. Three views of the Sierpinski tetrahedron ([2])

its face is an isosceles triangle whose height is 3/2 of the base. As (C) shows, it has square projections in six directions and it has square appearances when it is viewed from each of the 12 faces. We consider the IFS which is composed of nine dilations with the ratio 1/3 and the centers the 8 vertices and the center point of a hexagonal bipyramid. The pictures (D, E, F, G) and (H, I, J, K) are the 1st and 2nd level approximation models, respectively, and (L, M, N, O) are computer graphics of the hexagonal bipyramid fractal. As (G, K, O) shows, this fractal and its finite approximation models have six square projections.

In this paper, we consider the 2nd level approximation model. It consists of 81 hexagonal bipyramid pieces and, as (K) shows, it has 12 square appearances each of which is composed of $9 \times 9 = 81$ squares, which are divided into nine 3×3 blocks (P). This is nothing but the grid of the puzzle called SUDOKU or Number Place. The objective of this puzzle is to assign digits from 1 to 9 to the 81 squares so that each column, each row, and each of the nine 3x3 block contains all the nine digits. In SUDOKU puzzle, the digits do not have the meaning and we can use, for example, nine colors instead.

The goal of this paper is to find assignments of nine colors to the 81 pieces of the 2nd level approximation model of the hexagonal bipyramid fractal so that it has SUDOKU solution patterns in all the 12 square appearances. We also study SUDOKU coloring problem of level $2n$ approximation models $n \geq 1$.

Notation

When Δ is an alphabet and f is a function from Δ to Δ, we denote by $\mathrm{map}_n(f) : \Delta^n \to \Delta^n$ the component-wise application of f.

When $p \in \Delta^n$ and $q \in \Delta^m$ are sequences, we denote by $p \cdot q \in \Delta^{n+m}$ the concatenation of p and q, and by $q_{[i,j]}$ the subsequence of q from index i to j.

We use two alphabets $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ and $\Gamma = \{a, b, c, d, e, f, g, h, i\}$. Sequences of Σ are used for addresses, and sequences of Γ are used for colors.

2 Mathematical Formulation

We can define SUDOKU coloring not only on the level 2 object but also on level $2n$ objects ($n \geq 1$). A level $2n$ object consists of 9^{2n} pieces and when it is viewed

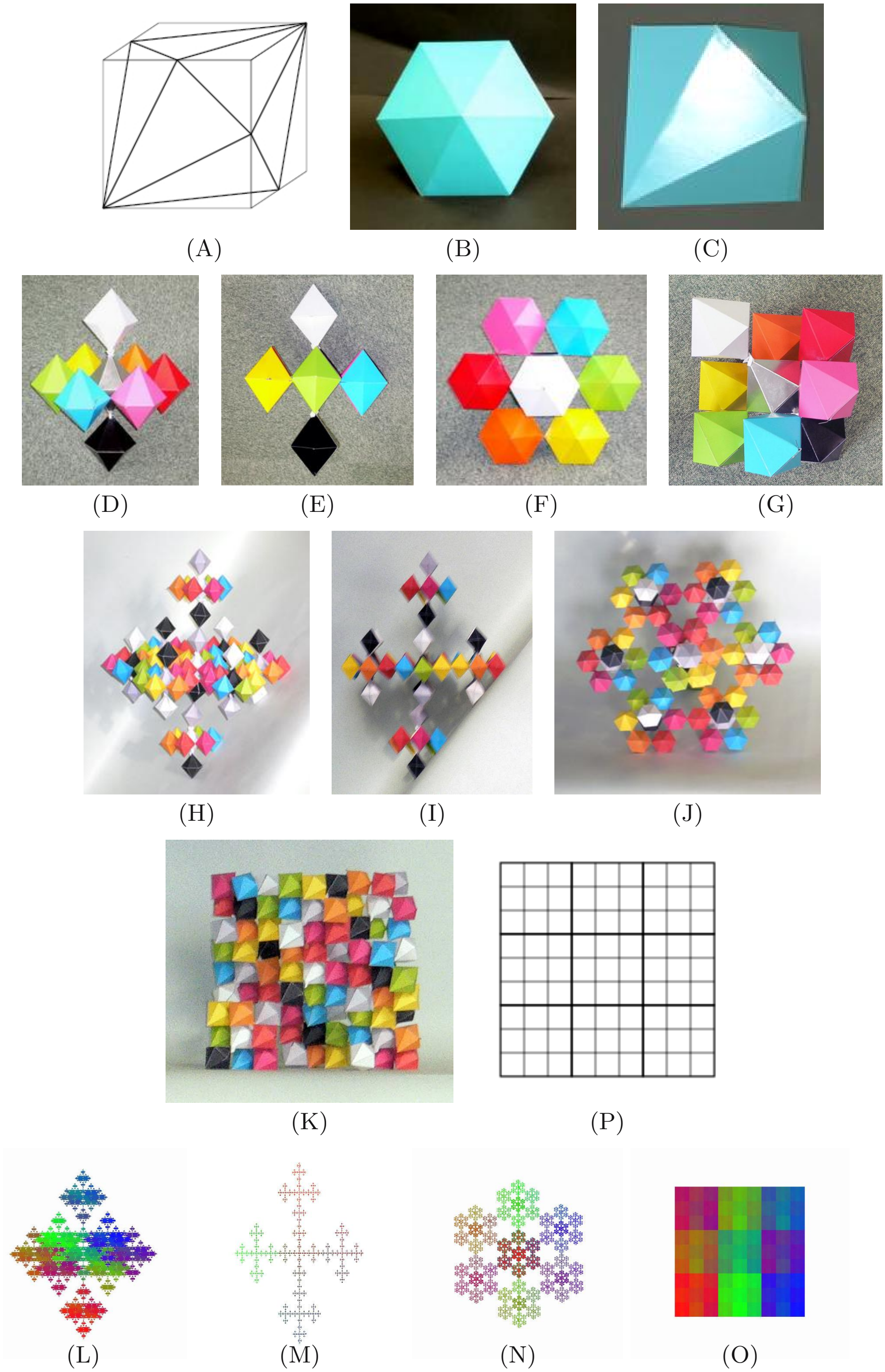


Fig. 2. The hexagonal bipyramid fractal and its approximations ([2])

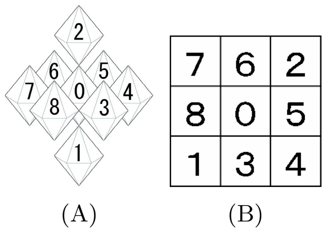


Fig. 3. Addressing of the level 1 object

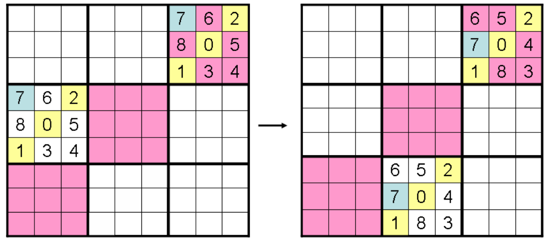


Fig. 4. Rotation of the level 2 object

from each of its face, we have a $9^n \times 9^n$ grid of squares which consists of $3^n \times 3^n$-blocks arranged in a $3^n \times 3^n$ grid. Therefore, we have the coloring problem to assign 9^n colors so that each column, row, and $3^n \times 3^n$-block contain all the 9^n colors. We will first formalize this SUDOKU coloring problem symbolically as a problem of words.

We give addressing of the 1st level object with Σ as in Fig. 3 (A). We call the pieces 0, 1, 2 the axis pieces and 3, 4,..., 8 the ring pieces. The address of the level 2 object is given as pairs (a, b) for $a, b \in \Sigma$, where a specifies a block and b specifies a piece in the block. We call the blocks 0, 1, 2 the axis blocks and 3, 4,...,8 the ring blocks. Similarly, we give the addressing of the level $2n$ object with a tuple in Σ^{2n}. We sometimes fix a viewpoint so that the nine pieces of the 1st level object are arranged as Fig. 3 (B), and we specify a rotation of the object with a permutation on Σ^{2n}, instead of a changing of the viewpoint.

The symmetry group of a hexagonal bipyramid is the dihedral group D_6 of order 12. It is also true for its fractal and the finite approximation models. This group is composed of a rotation σ of 60 degree around the axis between block 1 and 2 and a rotation τ of 180 degree around the axis between block 4 and 7.

When σ is applied to the 1st level object, the axis pieces are fixed and the ring pieces are shifted to the next position. Therefore, it causes the permutation $\sigma_1 = (3\ 4\ 5\ 6\ 7\ 8)$ on Σ. A rotation on the 2nd level object is the combination of a revolution around the axis blocks and rotation inside each block, as Fig. 4 shows. In general, σ causes on the n-th level object a component-wise application of σ_1, which is $\sigma_n = \mathrm{map}_n(\sigma_1)$. We also consider the permutation corresponding to τ, which is defined for the 1st level object as $\tau_1 = (1\ 2)(6\ 8)(3\ 5)$ and for the level n object $\tau_n = \mathrm{map}_n(\tau_1)$. Thus, the symmetry group of the level n object is represented as $\langle \sigma_n, \tau_n \rangle$. We also define reflection $\upsilon_1 = (1\ 2)$ on the first level object and $\upsilon_n = \mathrm{map}_n(\upsilon_1)$ on the level n object.

Definition 1. *A coloring of the level $2n$ object is a function from Σ^{2n} to Γ^n.*

We first define a SUDOKU coloring of a $9^n \times 9^n$-grid and then define a SUDOKU coloring of the level $2n$ object.

Definition 2. *A coloring $\gamma : \Sigma^{2n} \to \Gamma^n$ of the level $2n$ object is a one-face SUDOKU coloring when*
(1) The restriction of γ to $a_1a_2\dots a_n\Sigma^n$ is surjective for every $a_1a_2\dots a_n \in \Sigma^n$.
(2) For $A_1 = \{1,3,4\}, A_2 = \{8,0,5\}, A_3 = \{7,6,2\}$, the restriction of γ to $A_{d(1)}A_{d(2)}\dots A_{d(2n)}$ is surjective for every function $d : \{1,\dots,2n\} \to \{1,2,3\}$.
(3) For $B_1 = \{1,8,7\}, B_2 = \{3,0,6\}, B_3 = \{4,5,2\}$, the restriction of γ to $B_{d(1)}B_{d(2)}\dots B_{d(2n)}$ is surjective for every function $d : \{1,\dots,2n\} \to \{1,2,3\}$.

In condition (1), $a_1\dots a_n$ specifies a $3^n \times 3^n$-block and this condition says that every $3^n \times 3^n$-block contains all the 9^n colors. In condition (2) and (3), a function $d : \{1,\dots,2n\} \to \{1,2,3\}$ specifies a row or a column and these conditions say that every row and every column contains all the 9^n colors.

Definition 3. *A coloring $\gamma : \Sigma^{2n} \to \Gamma^n$ is a SUDOKU coloring of the level $2n$ object if $\gamma \circ {\sigma_n}^k$ is a one-face SUDOKU coloring for $k = 0,1,\dots,5$.*

Note that $\gamma \circ \tau_n$ is always a one-face SUDOKU coloring when γ is, and we do not need to consider it in this definition.

Definition 4. *(1) Two colorings δ and η are change of colors when $\delta = p \circ \eta$ for a permutation p on Γ^n.*

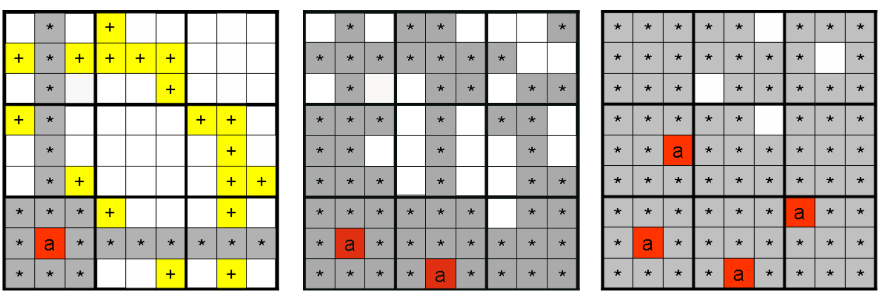
(2) Two colorings δ and η are isomorphic when δ and $\eta \circ r$ are change of colors for r an element of the dihedral group D_6 generated by $\langle \sigma_{2n}, \tau_{2n} \rangle$.

(3) A coloring δ is a reflection of η when δ and $\eta \circ r$ are change of colors for $r \in \langle \sigma_{2n}, \tau_{2n}, \upsilon_{2n} \rangle$.

We identify colorings obtained by change of colors, and in this definition, we define isomorphism and reflection modulo change of colors.

3 SUDOKU Coloring of the 2nd Level Object

In this section, we determine all the SUDOKU colorings of the 2nd level object. Consider the assignment of color 'a' to the piece $(1,0)$ as in Fig. 5 (A). In ordinary



(A) (B) (C)

Fig. 5. Possible arrangement of one color 'a'

SUDOKU, we cannot assign the same color 'a' to pieces marked with $*$. In our SUDOKU coloring, we cannot assign it to pieces with $+$ either, because they come to places with $*$ by σ^k (k $= 1, \ldots, 5$). Note that $(1, 0)$ is on the axis of the axis block and therefore fixed by σ. As this example shows, the constraint we need to consider is very tight. Fig. 5 (B) shows the places 'a' can be assigned after it is assigned to $(1, 0)$ and $(3, 3)$. This figure shows that there is only one piece left in block 4 and 8, and when we have such an assignment, there is no piece left on block 5 (Fig. 5 (C)). Thus, the assignment of 'a' to $(3, 3)$ will cause a conflict. Similarly we cannot assign 'a' to $(3, 6)$. Therefore, $(3, 1)$ and $(3, 2)$ are the only pieces to which we can assign 'a', and they determine the left of the assignments of 'a' as in Fig. 6. Therefore, there are only two solutions for the assignment of 'a'. It is also the case for the colors assigned to $(0, 1)$ and $(0, 2)$, each of which has only two possible configurations, which are color 'b' and color 'c' in Fig. 6, respectively. As a consequence, we have only two colorings of the axis pieces, given in Fig. 6. Note that they switch by the application of σ, and also by the application of υ, modulus of change of colors.

It also shows that the SUDOKU coloring problem is the product of two independent coloring problems, one is the coloring of the 27 axis pieces with three colors 'a', 'b', and 'c', and the other one is the coloring of the 54 ring pieces with six colors 'd', 'e', 'f', 'g', 'h', and 'i'. We have shown that there are two isomorphic solutions to the former one.

Now, we study the coloring of the ring pieces. First, we study the two blocks 1 and 2. In Fig. 7, the four pieces marked with $+$ (yellow) move to the places with * (gray) by σ^3, and pieces with the marks $3'$ and $4'$ move to 3 and 4 by σ^3. It means that the two colors assigned to $3''$ and $4''$ are different from the four colors assigned to the pieces marked with $+$, and thus equal to the two colors assigned to 3 and 4 after the application of σ^3, which are the colors of 3' and 4'. The same consideration applies to every pair of adjacent pieces on the rings of blocks 1 and 2, and therefore, we can conclude that the color arrangements on the rings of these two blocks are the same. Since the coloring of the axis of these blocks are also the same in both colorings of the axis pieces, block 1 and 2 have the same coloring.

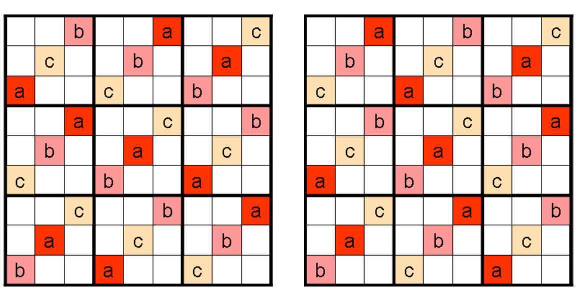


Fig. 6. Two solutions of the coloring of the axis pieces

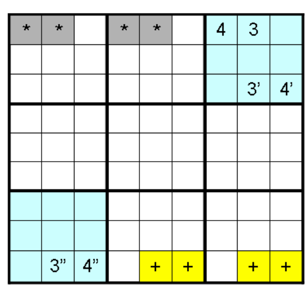
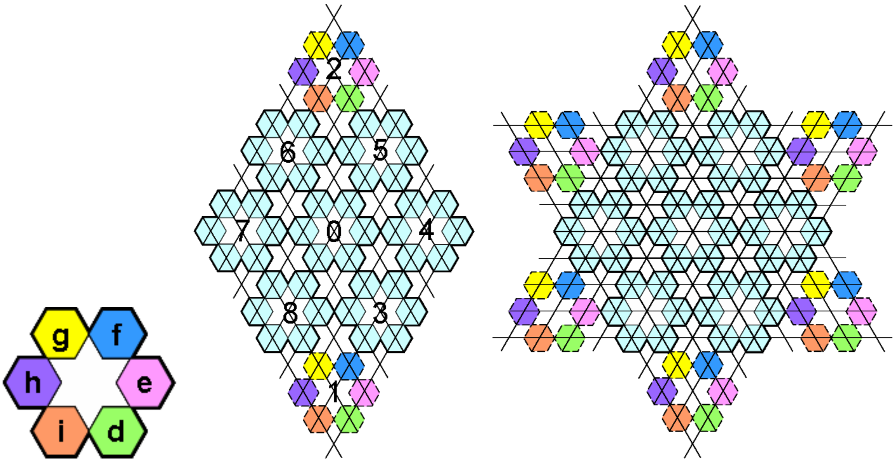


Fig. 7. Colorings of block 1 and 2



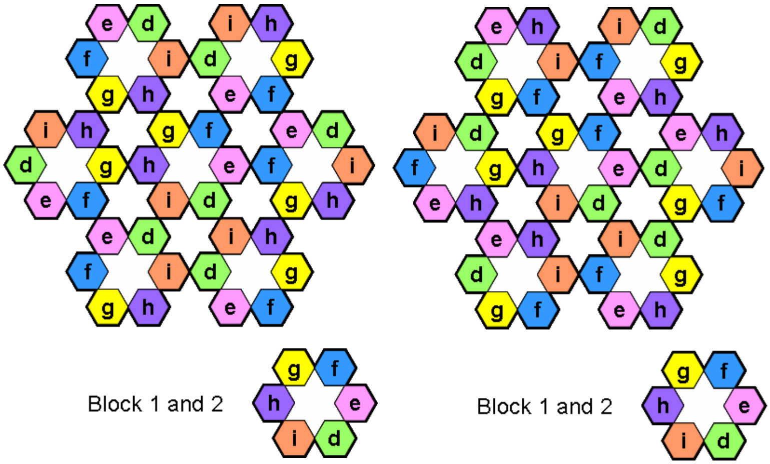
(A) (B) (C)

Fig. 8. Coloring constraint of the ring pieces of block 0, 3–8

Next, we study the color arrangements of the ring pieces of the other seven blocks. In the followings, we fix the colors assigned to the pieces $(1, 3), \ldots (1, 8)$ to be 'd', … 'i', respectively, as Fig. 8 (A) shows. In this figure, pieces $3, 4, 5, \ldots$ are located from the lower-right corner anticlockwise. Since block 1 and 2 have the same coloring, the colors of the pieces $(2, 3)$ to $(2, 8)$ are also 'd', to 'i'. Each of the 18 lines of Fig. 8 (B) shows ring pieces which form the same row or column from the viewpoint of Fig. 3 (B). Fig. 8 (C) shows the rows and the columns of all the viewpoints in one figure, with the colorings of blocks 1 and 2 copied around the other blocks. In this figure, there are 27 lines. This figure shows the constraint we need to solve. That is, our goal is to assign 6 colors to the 42 pieces so that each of the 7 blocks and each of the 27 lines contain all the 6 colors.

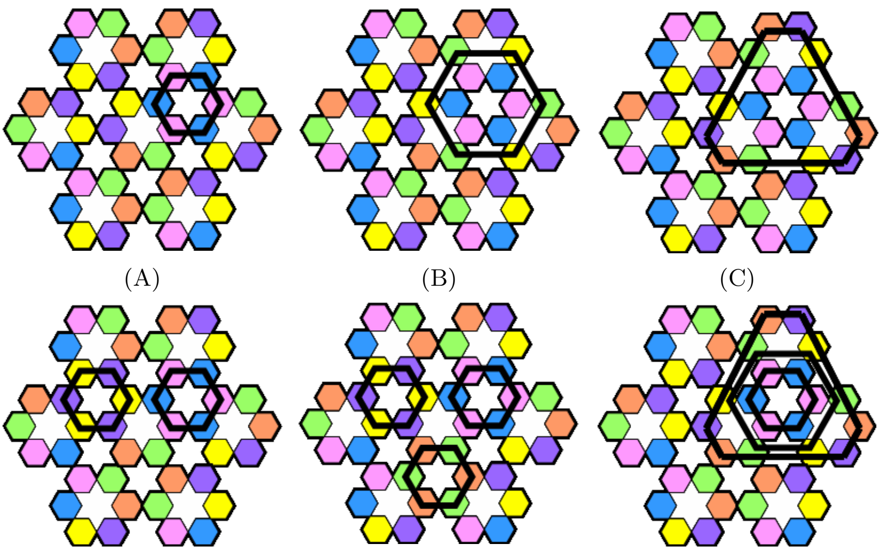
Before studying the general case, we consider the case the center block (block 0) also has the same color assignment as block 1 and 2. In this case, according to the constraint, colors 'd', 'f', 'h' can appear on the pieces 3, 5, 7, and 'e', 'g', 'i' can appear on the pieces 4, 6, 8 of each block, respectively. Therefore, this coloring problem is the product of two coloring problems each of which is a coloring of 18 pieces with 3 colors. One can easily check that there are two solutions to each of them and we have 4 solutions as their composition. Since the two solutions alter by the application of σ, we have two solutions modulus of isomorphism (Fig. 9). Solution A has only one coloring pattern of a block, which is rotated by 120 degree to form three block patterns, which is assigned to blocks numbered with $(0, 1, 2)$, $(3, 5, 7)$, and $(4, 6, 8)$, respectively. Solution B has three block coloring patterns each of which is assigned to three blocks as Solution A.

When block 0 is allowed to have different coloring from block 1 and 2, we have more solutions. Solution A in Fig. 9 has hexagonally arranged six pieces with two alternating colors which range over three blocks as Fig. 10 (A) shows. If we switch the two colors on these six pieces, the result also satisfy the constraint



Solution A Solution B

Fig. 9. Two ring colorings for the case the three axis blocks have the same coloring



(A) (B) (C) (D) (E) (F)

Fig. 10. Six pieces of Solution A, whose coloring can be switched

in Fig. 8 (C). There are three kinds of such hexagonal six pieces as Fig. 10 (A, B, C) shows. In these three figures, the vertices of each hexagon have two colors which can be switched. For each of these three, there are three places that the same kind of switching may occur. Since there is no overlapping, two or three switching may occur simultaneously as Fig. 10 (D, E) shows. Therefore, there are 3×3 non-isomorphic patterns of this kind. Patterns in Fig. 10 (A, B, C) can occur simultaneously only when they have the same center as in Fig. 10 (F), because there exist overlapping pieces for each of the other cases. Therefore, four kinds of combinations: (A, B), (B, C), (A, C), and (A, B, C) exist. Therefore, when we identify isomorphic colorings, we have $3 \times 3 + 4 = 13$ solutions except for solutions A and B.

Next, we count colorings which are isomorphic to one of them. Three solutions like (E) is mapped to itself by the application of σ^2 and τ, and therefore there are two non-isomorphic colorings, which switch by the application of σ. For the other ten patterns, the application of σ^k $(k = 0, 1.., 5)$ are all different and τ will map a coloring to one of them. Therefore, in all, there are $10 \times 6 + 3 \times 2 = 66$ solutions if we do not identify isomorphic ones.

As the result, we have $13 + 2 = 15$ solutions on the coloring of the rings modulus of isomorphism, and $66 + 4 = 70$ solutions if we do not identify isomorphic ones. Since there are two solutions on the coloring of the axis, we have 30 SUDOKU colorings of the level 2 model if we identify isomorphic ones, and 140 SUDOKU colorings if we do not identify isomorphic ones.

From the constraint of Fig. 8 (B), we can show that they are all the SUDOKU colorings of the level 2 object. From Fig. 8 (B), each ring piece of a ring block has three coloring possibilities as in Fig. 11. If we assume the color of (0, 3) (red piece in Fig. 11) to be 'f', then the possible colors of (3, 3) and (6, 3) are both 'h' and 'g', and therefore, 'h' is assigned to at least one of them. In the same way, 'h' must be assigned to one of (4, 3) and (7, 3), and one of (5, 3) and (8, 3). Therefore, 'h' is assigned to three pieces out of these six pieces, and one can show through some calculation that it causes a conflict. Therefore, it is not allowed to assign 'f' to (0, 3). In the same way, we cannot assign 'h' to (0, 3). Therefore, only 'd', 'e', 'g', 'i' are allowed to (0, 3). Among them, 'd' is the color assigned in Solution A and B. When 'e' is assigned to (0, 3), it is easily shown

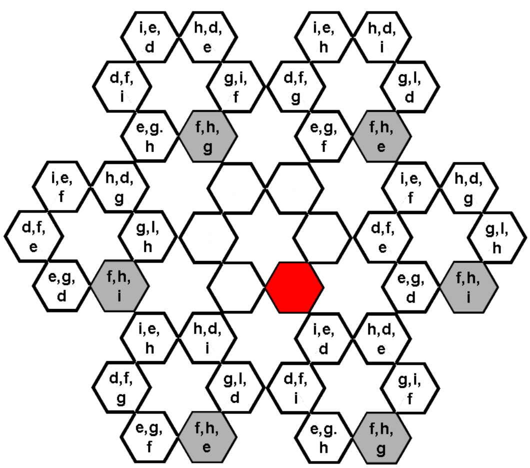


Fig. 11. The other 13 solutions on the ring

that the color of (0, 4) must be 'd'. Similarly, when 'g' is assigned to (0, 3), the color of (0, 6) must be 'd', and when 'i' is assigned to (0, 3), the color of (0, 8) must be 'd', In this way, we can show that the possible configurations of block 0 are only those obtained from that of Solution A through the switching listed in Fig. 10. Then, for each of the color assignment of block 0 obtained in this way, we can calculate one or two color assignments of the other pieces, which are all among the solutions we have explained.

By the application of v, the two axis colorings switch as we have noted, and the ring coloring is fixed because ring colorings on block 1 and 2 are the same. Therefore, the switching of the two axis-colorings causes reflection of the coloring.

Theorem 5. *The level 2 object has (1) 140 SUDOKU colorings if we identify change of colors, (2) 30 SUDOKU colorings if we identify isomorphic ones, (3) 15 SUDOKU colorings if we identify reflections.*

4 SUDOKU Coloring of the Level $2n$ Object

In this section, we show that SUDOKU colorings of the level $2n$ object exist for every $n \geq 1$.

Definition 6. *Let $\gamma : \Sigma^{2n} \to \Gamma^n$ and $\delta : \Sigma^{2m} \to \Gamma^m$ be colorings of the level $2n$ and the level $2m$ object, respectively. We define a coloring $\mathrm{comp}(\gamma, \delta) : \Sigma^{2(n+m)} \to \Gamma^{n+m}$ of the level $2(n+m)$ object as follows.*

$$\mathrm{comp}(\gamma, \delta)(p) = \gamma(p_{[m+1,m+2n]}) \cdot \delta(p_{[1,m]} \cdot p_{[m+2n+1,2m+2n]}) \,.$$

Proposition 7. *Suppose that γ and δ are SUDOKU colorings of the level $2n$ and the level $2m$ object, respectively. Then, $\mathrm{comp}(\gamma, \delta,)$ is a SUDOKU coloring of the level $2(n+m)$ object.*

Proof. We need to show that $\mathrm{comp}(\gamma, \delta) \circ {\sigma_{2(n+m)}}^k$ is a one-face SUDOKU coloring of level $2(n+m)$ for $k = 0, 1, \ldots, 5$. However, since $\sigma_{2(n+m)}$ is the application of σ_1 to each component, we have $\sigma_{a+b}(p \cdot q) = \sigma_a(p) \cdot \sigma_b(q)$ for $p \in \Sigma^a$ and $q \in \Sigma^b$. Therefore,

$$\begin{aligned}
&\mathrm{comp}(\gamma, \delta)(\sigma_{2(n+m)}(p)) \\
&\qquad = \mathrm{comp}(\gamma, \delta)(\sigma_m(p_{[1,m]}) \cdot \sigma_{2n}(p_{[m+1,m+2n]}) \cdot \sigma_m(p_{[m+2n+1,2m+2m]})) \\
&\qquad = \gamma(\sigma_{2n}(p_{[m+1,m+2n]})) \cdot \delta(\sigma_m(p_{[1,m]}) \cdot \sigma_m(p_{[m+2n+1,2m+2n]})) \\
&\qquad = \gamma(\sigma_{2n}(p_{[m+1,m+2n]})) \cdot \delta(\sigma_{2m}(p_{[1,m]} \cdot p_{[m+2n+1,2m+2n]})) \\
&\qquad = \mathrm{comp}(\gamma \circ \sigma_{2n}, \delta \circ \sigma_{2m})(p).
\end{aligned}$$

From our assumption, $\gamma \circ \sigma_{2n}$ and $\delta \circ \sigma_{2m}$ are one-face SUDOKU colorings of the level $2n$ and the level $2m$ object, respectively. Therefore, we only need to show that when γ and δ are one-face SUDOKU colorings of the level $2n$ and level $2m$ objects, respectively, then $\mathrm{comp}(\gamma, \delta)$ is a one-face SUDOKU coloring of the level $2(n+m)$ object.

Fig. 12. A SUDOKU Sculpture

Conditions (2) and (3) of Definition 2 hold because they are independent of the permutation of the coordinates. We show that condition (1) holds. Suppose that $p_1 \in \Sigma^n$, $p_2 \in \Sigma^m$, $r_1 \in \Gamma^n$, and $r_2 \in \Gamma^m$. We need to show that $r_1 \cdot r_2 = \mathrm{comp}(\gamma, \delta)(p_1 \cdot p_2 \cdot q)$ for some $q \in \Sigma^{n+m}$. Since γ and δ satisfy condition (1), there exists q_1 and q_2 such that $\gamma(p_1 \cdot q_1) = r_1$ and $\gamma(p_2 \cdot q_2) = r_2$ hold. Then, $\mathrm{comp}(\gamma, \delta)(p_2 \cdot p_1 \cdot q_1 \cdot q_2) = \gamma(p_1 \cdot p_2) \cdot \delta(q_1 \cdot q_2) = r_1 \cdot r_2$.

We proved in Section 3 that there are 30 SUDOKU colorings of level 2 object. Therefore, by induction we have the following.

Theorem 8. *There exist SUDOKU colorings of level* $2n$ *object for every* $n \geq 1$*.*

Usually, there are two ways of constructing level $n+1$ approximation models of a fractal from level n approximation models. One is to fix the size of the fundamental piece and construct a larger model by combining copies of the level n models. The other one is to fix the total size and replace the fundamental pieces of a level n model with level 1 approximation models. Correspondingly, there are two induction schemes to prove properties of approximation models of a fractal. In our proof of Proposition 7, we used both of them simultaneously. Let C_0 be a hexagonal bipyramid we start with. We consider the following construction of the level $2n+2$ approximation model C_{n+1} from the level $2n$ approximation model C_n. We replace each fundamental piece of C_n with a level 1 approximation model to form a level $2n+1$ approximation model B_n, then make 8 copies of B_n and locate them on the vertices of B_n so that B_n becomes the block 0 of the level $2n+2$ object C_{n+1}. Then, the size of the $3^n \times 3^n$-SUDOKU "blocks" of C_n is equal to the size of C_0 for every n. Our proof for the case $m = 1$ constructs the SUDOKU coloring of C_{n+1} from that of C_n and a SUDOKU coloring of the 1st level object.

Corresponding to this construction, it is more natural to shift the index of the address space from $[0, 2n]$ to $[-n+1, n] = \{-(n-1), -(n-2), \ldots, 0, 1, 2, \ldots, n\}$. Then, the address grows in both directions when the level of the model increases, and the 9^n SUDOKU "blocks" are addressed with the index $[-(n-1), 0]$ and inside each block, each piece is addressed with the index $[1, n]$.

5 Conclusion

We showed that there are 140 SUDOKU colorings of the level 2 approximation of the hexagonal bipyramid fractal if we identify change of colors, and 30 if we identify isomorphic ones. This result is also verified by a computer program. Note that the ordinary SUDOKU has $18,383,222,420,692,992$ solutions if we identify change of colors [1]. Our SUDOKU problem has a strict constraint compared with the ordinal SUDOKU and therefore it is natural that we do not have many solutions. It is interesting to note that, as we have seen, all the solutions are symmetric to some extent and are logically constructed. It is in contrast to the lot of random-looking solutions of the ordinal SUDOKU.

Among the 30 SUDOKU coloring solutions, Solution A is most beautiful in that the order of the colors of the rings of the nine blocks are the same. The paper model in Figure 2 (H, I, J, K) has this coloring. This object is displayed at the Kyoto University Museum, with an acrylic resin frame object with 12 square colored faces. The 12 square faces are located so that this object looks square with SUDOKU pattern when it is viewed through these faces.

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